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Some Applications of Markov Additive Processes as Models in Insurance and Financial Mathematics

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SUMMARY

This thesis consists mainly of three papers concerned with Markov additive processes, Lévy processes and applications on finance and insurance.

The first chapter is an introduction to Markov additive processes (MAP) and a presentation of the ruin problem and basic topics of Mathematical Finance. The second chapter contains the paper *Lévy Systems and the Time Value of Ruin for Markov Additive Processes* [**7**] written with Manuel Morales and that is published in the *European Actuarial Journal*. This paper studies the ruin problem for a Markov additive risk process. An expression of the expected discounted penalty function is obtained via identification of the Lévy systems. It is a generalization of results available in the literature for spectrally-negative Lévy risk processes and Markov-additive risk processes with phase-type jumps.

The third chapter contains the paper *On a Generalization of the Expected Discounted Penalty Function to Include Deficits at and Beyond Ruin* [**6**] that is submitted for publication. This paper presents an extension of the expected discounted penalty function in a setting involving aggregate claims modelled by a subordinator, and Brownian perturbation. This extension involves a sequence of expected discounted functions of successive minima reached by a jump of the risk process after ruin. It has important applications in risk management and in particular, it is used to compute the expected discounted value of capital injection.

Finally, the fourth chapter contains the paper *The Minimal Entropy Martingale Measure (MEMM) for a Markov-Modulated Exponential Lévy Model* [**50**] written with Romuald Hérvé Momeya and that is published in the journal *Asia Pacific Financial Market*. It presents new results related to the incompleteness problem in a financial market, where the risky asset is driven by Markov additive exponential model. These results characterize the martingale measure satisfying the entropy criterion. This measure is used to compute the price of the option and the portfolio of hedging in an exponential Markov-modulated Lévy model.

Key words: Minimal entropy martingale measure, exponential financial models, Markov additive processes, Lévy systems, ruin theory, Gerber-Shiu function, risk models.

RÉSUMÉ

Cette thèse est principalement constituée de trois articles traitant des processus markoviens additifs, des processus de Lévy et d'applications en finance et en assurance.

Le premier chapitre est une introduction aux processus markoviens additifs (PMA), et une présentation du problème de ruine et de notions fondamentales des mathématiques financières. Le deuxième chapitre est essentiellement l'article *Lévy Systems and the Time Value of Ruin for Markov Additive Processes* [**7**] écrit en collaboration avec Manuel Morales et publié dans la revue *European Actuarial Journal*. Cet article étudie le problème de ruine pour un processus de risque markovien additif. Une identification de systèmes de Lévy est obtenue et utilisée pour donner une expression de l'espérance de la fonction de pénalité actualisée lorsque le PMA est un processus de Lévy avec changement de régimes. Celle-ci est une généralisation des résultats existant dans la littérature pour les processus de risque de Lévy et les processus de risque markoviens additifs avec sauts *phase-type*.

Le troisième chapitre contient l'article *On a Generalization of the Expected Discounted Penalty Function to Include Deficits at and Beyond Ruin* [**6**] qui est soumis pour publication. Cet article présente une extension de l'espérance de la fonction de pénalité actualisée pour un processus subordinateur de risque perturbé par un mouvement brownien. Cette extension contient une série de fonctions escomptée éspérée des minima successives dus aux sauts du processus de risque après la ruine. Celle-ci a des applications importantes en gestion de risque et est utilisée pour déterminer la valeur espérée du capital d'injection actualisé.

Finallement, le quatrième chapitre contient l'article *The Minimal Entropy Martingale Measure (MEMM) for a Markov-modulated Exponential Lévy Model* [**50**] écrit en collaboration avec Romuald Hervé Momeya et publié dans la revue *Asia-Pacific Financial Market*. Cet article présente de nouveaux résultats en lien avec le problème de l'incomplétude dans un marché financier où le processus de prix de l'actif risqué est décrit par un modèle exponentiel markovien additif. Ces résultats consistent à charactériser la mesure martingale satisfaisant le critère de l'entropie. Cette mesure est utilisée pour calculer le prix d'une option, ainsi que des portefeuilles de couverture dans un modèle exponentiel de Lévy avec changement de régimes.

Mots clés: Mesure martingale minimisant l'entropie, modèles exponentiels en finance, processus markoviens additifs, systèmes de Lévy, théorie de la ruine, fonction de Gerber-Shiu, modèles du risque.

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To my father !

CONTENTS

INTRODUCTION

Le capital d'une compagnie d'assurance et le prix de nombreux actifs financiers comportent des mouvements imprévisibles continus et de grande amplitude qui ne peuvent pas s'expliquer seulement avec les modèles classiques et continus. La prise en compte de cette caractéristique a conduit au développement de modèles stochastiques basés sur des processus de Lévy comportant les modèles aux distributions non-gaussiennes, asymétriques, à queues épaisses, etc (voir Schoutens [**60**]). Ceci a mené à étudier les propriétés des trajectoires du processus de Lévy et des variables associées, ainsi qu'au developpement de la théorie de la fluctuation sur ce sujet. Cette classe de processus est à la fois riche pour décrire la réalité des marchés et relativement simple pour permettre un traitement rigoureux et des calculs explicites. En revanche, ces processus ne peuvent pas expliquer à eux seuls l'influence des évènements macroéconomiques sur les prix des actifs et les réserves des compagnies d'assurance. En effet, en raison de survenance de certains évènements ou à l'arrivée de nouveaux renseignements importants, le marché pourrait subir un changement structurel selon une diversité de scénarios. Ce changement définit différents états ou régimes qui ont un impact non négligeable sur le rendement et le risque.

Dès lors, la recherche de modèles plus adéquats qui prennent en considération de tels changements structurels s'est imposée comme une problématique principale dans la littérature en théorie du risque et comme conséquence, l'on note une apparition de modèles en assurance et en finance avec changement de régimes (*regime-switching*). Dans ces modèles, la dynamique et les paramètres de la variable étudiée sur le marché dépendent du régime dans lequel ils se trouvent.

Des recherches récentes en finance et en science actuarielle ont commencé à prêter attention aux modèles avec changements de régime. Hamilton [**45**] a introduit les modèles à changement de régime markoviens pour modéliser les évènements macroéconomiques qui influent sur le prix des actifs. Di Masi *et al* [**19**] ont étudié les options européennes pour un modèle de Black-Scholes lorsque l'économie du marché subit un changement selon un nombre fini d'états. Buffington et Elliot [**3**] ont discuté des options américaines dans le même contexte. Yang et Yin [**63**], et Lu et Tsai [**55**] ont considéré le modèle avec changement de régimes markovien pour modéliser le processus de surplus en assurance. Ces modèles peuvent capter les politiques de financement et d'assurance qui peuvent s'adapter aux changements des environements économiques ou politiques.

Pour modéliser un tel changement de régime d'une façon à ce que la flexibilité, la réalité et la richesse dûs aux propriétés du processus de Lévy soient conservés, la famille de processus markoviens additifs (PMAs) s'est imposée comme un outil mathématique indispensable. Cette famille de processus est le concept mathématique général permettant d'analyser rigoureusement les phénomènes aléatoires avec changements de régimes.

Qu'est ce qu'un processus markovien additif? Il s'agit intuitivement d'un couple de processus dont le premier est un processus de Markov à espace d'états fini ou dénombrable, le deuxième est un processus qui se comporte comme étant un processus de Lévy ou plus généralement un processus *additif* (processus à accroissements indépendants) conditionnellement aux trajectoires du premier. Les premiers travaux sur l'existence et l'étude du PMA remontent aux années 1970. Plus particulièrement, en 1972, Çinlar publiait *Markov Additive processes* I et II ([**16**] et [**17**]) portant sur l'existence et la construction de ce processus. En 1975, Çinlar [**18**] et Grigelionis [**38**] ont étudié les propriétés de trajectoire de cette famille et ont donné beaucoup de résultats importants qui généralisent ceux du processus additif. Les travaux qui ont suivi ont porté sur les études des trajectoires et des fluctuations du processus de Lévy avec changement de régime, un cas particulier très important du PMA qui est adopté récemment dans la modélisation avec changement de régime. Nous suggérons au lecteur de consulter le livre de Asmussen [**1**] où se trouve une étude détaillée de cet exemple du PMA.

Évidemment ce type de PMA se réduit à un processus de Lévy dès qu'on suppose qu'il y a un seul état de nature. Les résultats de cette thèse développés pour les modèles du PMA, se projettent sur les modèles unidimensionnels décrits par différentes classes de processus de Lévy.

Conséquemment, le PMA a commencé à étendre son influence dans le domaine d'applications d'intérêt pour cette thèse, soit la théorie du risque et les mathématiques financières.

La présente thèse contient quatre chapitres, dont trois sont des articles publiés, acceptés ou soumis pour publication. Le premier chapitre présente un survol des notions fondamentales connues de la théorie des mathématiques financières et actuarielles.

Le premier chapitre est une introduction aux processus markoviens additifs (PMA), et une présentation du problème de ruine et de notions fondamentales des mathématiques financières. On y présente la définition du PMA dans sa forme générale intoduite par Çinlar [**16**] et [**17**] ainsi que quelques exemples, en particulier le processus de Lévy avec changement de régime.

Ce premier chapitre est suivi de l'article *Lévy Systems and the Time Value of Ruin for Markov Additive Processes* [**7**], écrit en collaboration avec Manuel Morales et qui paraîtra prochainement dans la revue *European Actuarial Journal*. Cet article étudie le problème de ruine pour un processus de risque markovien additif. Nous donnons une expression pour l'espérance de la fonction de pénalité actualisée, par l'extension de résultats disponibles dans la littérature. En particulier, nous généralisons certains résultats dans [**40**], [**41**], [**11**] et [**12**] qui sont obtenus dans le modèle de risque utilisant un processus de Poisson composé, un processus de Poisson composé perturbé par un mouvement brownien et plus généralement, un processus de Lévy spectralement négatif. Cette extension est possible grâce à l'identification de *systèmes de Lévy* d'un PMA introduite dans [**18**]. Ceci nous a permis d'étendre des résultats connus pour les processus de Lévy à une grande famille de processus additifs de Markov. Nous discutons aussi comment l'expression de l'espérance de la fonction de pénalité actualisée peut être obtenue en utilisant la notion de la *scale matrix* d'un processus de Markov additif récemment introduite par Klusik et Palmowski [**52**].

Le troisième chapitre contient essentiellement l'article *On a generalization of the expected discounted penalty function to include deficits at and beyond ruin*, récemment soumis pour publication. Dans cet article, nous proposons un concept élargi de l'espérance de la fonction de pénalité actualisée qui prend en compte de nouvelles variables aléatoires liées à la ruine. Nous ajoutons à cette fonction de pénalité classique introduite par Gerber et Shiu [**40**], la série de fonctions actualisées espérées des maxima dus aux sauts du processus de risque après la ruine. Inspiré par les résultats de Huzak et al [**46**], et les développements dans la théorie des fluctuations pour les processus de Lévy spectralement négatifs, nous fournissons une caractérisation pour cette fonction de pénalité généralisée pour un modèle de risque utilisant un subordinateur perturbé par un mouvement brownien. Nous illustrons comment cette fonction de pénalité généralisée peut être utilisée pour calculer la valeur espérée du capital d'injections actualisé pour un modèle de risque perturbé par un mouvement brownien. Ceci donne en particulier une forme explicite de la valeur espérée qui devrait être injectée aux moments de déficits, permettant entre autres à la compagnie d'assurance de survivre et de poursuivre ses activités.

Le quatrième chapitre contient l'article *The Minimal entropy martingale measure (MEMM) for a Markov-modulated exponential Lévy model* [**50**] écrit en collaboration avec Romuald Hervé Momeya et publié dans la revue *Asia-Pacific Financial Markets.* Cet article présente de nouveaux résultats en lien avec le problème de l'incomplétude dans un marché financier où le processus de prix de l'actif risqué est décrit par un modèle exponentiel markovien additif. Ces résultats consistent à caractériser la mesure martingale satisfaisant le critère de l'entropie. Cette mesure est utilisée pour calculer le prix d'une option, ainsi que des portefeuilles de couverture dans un modèle exponentiel de Lévy avec changement de régime. Ce modèle est caractérisé par la présence d'un processus d'arrière-plan qui décrit les mouvements des prix des actifs risqués entre les différents régimes ou les environnements du marché. Cela permet de souligner la forte dépendance entre les prix des actifs financiers et les changements structurels dans les conditions du marché. Les résultats de ce chapitre généralisent des travaux antérieurs dans la littérature traitant le modèle exponentiel de Lévy et et le modèle exponentiel additif dans [**34**] et[**35**].

Notons que le premier et le troisième article sont l'aboutissement de quelques années de travail sur la résolution de certains problèmes classiques en finance et en assurance dans le cadre de changement de régime, lorsque la dynamique de prix d'actifs et l'avoir d'une compagnie d'assurance sont décrits par un PMA. Ensuite, lors de l'étude du comportement du processus de risque de Lévy après la ruine, un projet qui ménera à l'article *On a generalization of the expected discounted penalty function to include deficits at and beyond ruin.* Nous souhaitons pouvoir prolonger le résultat de cet article à un modéle de risque décrit par un PMA dans les projets qui viendront.

Chapter 1

PRELIMINARIES

In this preliminary chapter, we give a brief overview of Markov Additive Processes (MAPs) which are used in Chapter 2 and 3 to describe respectively the stock price process and the risk process under different regimes or market environments. Then we present a summary discussion of the ruin problem involving the Expected Discounted Penalty Function (EDPF). In particular, we present the most important results characterizing the EDPF for different risk models driven by a Lévy process and MAP which have been studied recently. Finally, we give a presentation of the general theory of Mathematical Finance that gives the context our last contribution in Chapter 4 where we discuss the existence and characterization of Minimal Entropy Martingale Measure (MEMM) for exponential models.

1.1. A survey of Markov Additive Processes (MAP)

Recent research in finance and actuarial science have started paying attention to Markov-modulated (or regime-switching) models. The Markov-modulated models were originally introduced to model the macroeconomic events which influence the asset price (Hamilton, 1989). Di Masi et al (1994) considered the European options under the Black-Scholes formulation of the market in which the underlying economy switches among a finite number of states. Buffington and Elliot (2001) discussed the American options under this set-up. Yang and Yin (2002) considered the Markovian regime switching formulation to model the insurance surplus process. These models can capture the economical or political environment changes that can impact the financial and insurance processes. To model such a regime change, we use the Markov additive processes (MAPs) which are the mathematical structures behind the regime-switching models. Our interest in this section is to give some preliminary tools on MAPs and present some examples which have important applications to regime-switching models in finance and insurance.

Before introducing the notion of MAP, let us recall the basic concepts from the theory of Markov processes as found in Blumenthal and Getoor (1969) to make this presentation clear.

1.1.1. Markov Processes

Let $(E, \mathcal{E})$ be a measurable space, where E is a locally compact separable metric space and $\mathcal{E}$ is the Borel σ-algebra (σ-field) that contains all open subsets of E. Note that $(\mathbb{R}^m, B(\mathbb{R}^m))$ is the Euclidean space of dimension $m \geq 1$. Let $\mathcal{T} = [0, T)$ or $\mathcal{T} = \{0, 1, 2, ...\}$ be the time parameter space; where $T \in (0, \infty]$.

Definition 1.1.1 (Transition probability measure). *A function $P_{s,t}(x, A)$ defined for $s \leq t \in \mathcal{T}$, $x \in E$, $A \in \mathcal{E}$ and taking its values in $[0, 1]$ is a **transition probability measure** on $(E, \mathcal{E})$ if*

- *$A \to P_{s,t}(x, A)$ is a probability measure on $\mathcal{E}$, for any $(s, t, x) \in \mathcal{T} \times \mathcal{T} \times E$ fixed;*
- *$(t, x) \to P_{s,t}(x, A)$ is a measurable function, for each $A \in \mathcal{E}$ and $(s, t) \in \mathcal{T} \times \mathcal{T}$ fixed;*
- *$P_{s,s}(x, A) = \delta_x(A)$ for $s \in \mathcal{T}$, where $\delta_x(A) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{otherwise} \end{cases}$*
- *for $s \leq u \leq t$ in $\mathcal{T}$ and for $x \in E$, $A \in \mathcal{E}$*

$$P_{s,t}(x, A) = \int P_{s,u}(x, dy) P_{u,t}(y, A). \tag{1.1.1}$$

A transition probability measure $P_{s,t}(x, A)$ on $(E, \mathcal{E})$ is *temporally homogeneous* if there exists a measurable function $P_t(x, A)$ defined for $t > 0$, $x \in E$, and $A \in \mathcal{E}$ such that

$$P_{s,t}(x, A) = P_{t-s}(x, A) \quad \text{for any } s, t, x \text{ and } A. \tag{1.1.2}$$

In this case $P_t(x, A)$ is called a *temporally homogeneous transition probability measure* on $(E, \mathcal{E})$.

Definition 1.1.2 (Markov process). *Let $\mathcal{F}_t^J := \sigma(J_s : 0 \leq s \leq t)$ be the natural filtration of a stochastic process J augmented with $\mathbb{P}-$null sets of Ω.*

1) J is a Markov process if

$$\mathbb{P}\Big[J_t \in A \Big| \mathcal{F}_s^J\Big] = \mathbb{P}\Big[J_t \in A \Big| \sigma(J_s)\Big], \quad \text{for all } s < t \in \mathcal{T} \text{ and } A \in \mathcal{E}. \tag{1.1.3}$$

2) If $\{\mathcal{G}_t : t \in \mathcal{T}\}$ is a filtration with $\mathcal{F}_t^J \subset \mathcal{G}_t, \forall t \in \mathcal{T}$, J is a Markov process with respect to $\{\mathcal{G}_t : t \in \mathcal{T}\}$ if (1.1.3) holds with $\mathcal{F}_t^J$ replaced by $\mathcal{G}_t$.

Remark 1.1.1. *Property 1.1.3 is generally known as the* Markov property*.*

Definition 1.1.3. *The process J is a Markov process with transition probability measure $P_{s,t}(x, A)$ if*

$$E^{\mathbb{P}}\Big[f \circ J_t \Big| \mathcal{F}_s^J\Big] = \int f(y) P_{s,t}(J_s, dy) \tag{1.1.4}$$

for any $s < t \in \mathcal{T}$ and f a bounded test function defined on E.

1.1.2. Definition of Markov Additive Processes (MAPs)

Markov additive processes (MAPs) are a class of Markov processes. The state space of a MAP is bidimensional, so the state space can be split into two components $E \times F$, one of which, say E, is the Markov component and the other, F, is the additive component. The corresponding stochastic process can be denoted by (J, X). Generally speaking, a MAP (J, X) is a Markov process whose transition probability measure is translation invariant in the additive component X. To understand the definition clearly, let us introduce

- $(\Omega, \mathcal{M}, \{\mathcal{M}\}_{t \in \mathcal{T}}, \mathbb{P})$: the probability space, where $\mathcal{M} = \{\mathcal{M}\}_{t \in \mathcal{T}}$ is an incresaing family of σ-algebras on Ω;
- $\{J_t, t \in \mathcal{T}\}: \quad \Omega \longmapsto E$ is a stochastic process on a measurable space $(E, \mathcal{E})$;
- $\{X_t, t \in \mathcal{T}\}: \quad \Omega \longmapsto F$ is a stochastic process on a measurable space $(F, \mathcal{F})$, where $(F, \mathcal{F}) = (\mathbb{R}^m, \mathcal{R}^m)$ is a m-dimensional Euclidean space.

Let $(J, X) = \{(J_t, X_t), t \in \mathcal{T}\}$ be a Markov process on $(E \times F, \mathcal{E} \times \mathcal{F})$ with respect to $\{\mathcal{M}_t, t \in \mathcal{T}\}$, where the transition function is

$$P_{s,t}(i, y; A \times B),$$

for any $s < u < t, s, u, t \in \mathcal{T}, i \in E, y \in F, A \in \mathcal{E}, B \in \mathcal{F}$.

Let $\{Q_{s,t}, s < t, s, t \in \mathcal{T}\}$ be a family of transition probabilities from $(E, \mathcal{E})$ to $(E \times F, \mathcal{E} \times \mathcal{F})$.

Definition 1.1.4 (Çinlar (1972a)).

If the transition function satisfies the following conditions:

(1) The family of transition probabilities $\{Q_{s,t}, s < t, s,t \in \mathcal{T}\}$ *is a semi-Markov transition function on* $(E \times F, \mathcal{E} \times \mathcal{F})$*, i.e.*

$$Q_{s,t}(i, A \times B) = \int_{E\times F} Q_{s,u}(y, A \times (B - z))Q_{u,t}(i, dy \times dz), \tag{1.1.5}$$

for any $s < u < t, s, u, t \in \mathcal{T}, i, y \in E, z \in F, A \in \mathcal{E}, B \in \mathcal{F}$ *and* $B - z = \{b - z, b \in B\}$.

(2) If the Markov transition function $\{P_{s,t}\}$ *satisfies*

$$P_{s,t}(i, y, A \times B) = Q_{s,t}(i, A \times (B - y)), \tag{1.1.6}$$

then (J, X) *is a MAP with respect to* $\{\mathcal{M}_t, t \in \mathcal{T}\}$.

The above condition implies that

$$P_{s,t}(i, y, A \times B) = P_{s,t}(i, 0, A \times (B - y)). \tag{1.1.7}$$

Under the condition that the information of the Markov process J (Markov chain on the state E) in any time interval $(s, t]$ is known, the increment of $X_t - X_s$ is independent of X_s. Definition 1.1.4 becomes:

Definition 1.1.5 (Grigelionis (1978)).

Let $(J, X) = \{(J_t, X_t), t \in \mathbb{T}\}$*. Then* (J, X) *is a* Markov additive process *with respect to the filtration* $\{\mathcal{F}_t, t \in \mathbb{T}\}$ *if*

$$\mathbb{P}\Big[J_t \in A, X_t - X_s \in B\Big|\mathcal{F}_s\Big] = \mathbb{P}\Big[J_t \in A, X_t - X_s \in B\Big|J_s\Big] \quad \mathbb{P}\text{-}a.s. \tag{1.1.8}$$

for all $0 \le s \le t \in \mathbb{T}$ *and* $A \in \mathcal{E}$, $B \in \mathcal{F}$.

In many cases, the state space of the Markov component is discrete. For this special case,

- The transition probability should satisfy the condition

$$\begin{aligned}\mathbb{P}[J_{s+t} = k, X_{s+t} \in A|J_s = j, X_s = y] &= \mathbb{P}[J_{s+t} = k, X_{s+t} - X_s \in A - y|J_s = j] \\ &= \mathbb{P}[J_{s+t} = k|J_s = j] \times \ \mathbb{P}[X_{s+t} - X_s \in A - y|J_s = j, J_{s+t} = k].\end{aligned}$$

  In some sense, the first term of the second line gives the joint distribution of the Markov component, while the second term gives the conditional distribution of the additive component.

- We should remark that because (J,X) is Markov, it follows easily from the above equation that J is Markov and that X has conditionally independent increments. Since, in general, X is non-Markovian, this is the reason why we call J the Markov component and X the additive component of the MAP (J,X).

1.1.3. Examples of (E,T)-MAP

We give here some examples of MAPs. These examples are defined by specifying some conditions on one or the other component of a MAP. Most of the subjects of this sub-section are based on the book of Pacheco et *al.*(2009).

1.1.3.1. *E is a single point*

In this case ($E=\{i\}$):

$$\begin{aligned}\mathbb{P}[X_{s+t}\in A|J_s=i,X_s=y) &= \mathbb{P}(J_{s+t}=i,X_{s+t}\in A|J_s=i,X_s=y] \\ &= \mathbb{P}[J_{s+t}=i,X_{s+t}-X_s\in A-y|J_s=i] \\ &= \mathbb{P}[X_{s+t}-X_s\in A-y]. \qquad (1.1.9)\end{aligned}$$

This means X is an independent increment process and then, the MAP can be seen as the extension of the independent increment process. In general, the additive component of the MAP does not have independent increments.

1.1.3.2. *$\mathcal{T}$ is discrete*

In this case ($\mathcal{T}=\mathbb{N}$), the MAP is a Markov random walk (MRW). A stochastic process $\{(J_n,X_n),n\in\mathbb{N}\}$ on $E\times\mathbb{R}^m$ is called MRW if its transition probability measure has the property:

$$\begin{aligned}\mathbb{P}[J_{m+n}=i,X_{m+n}\in A|J_m=j,X_m=y) &= \mathbb{P}(J_{m+n}=i,X_{m+n}-X_m\in A-y|J_m=j] \\ &= \mathbb{P}[X_{m+n}-X_m\in A-y|J_m=j,J_{m+n}=i] \\ &\quad \times \mathbb{P}[J_{m+n}=i|J_m=j].\end{aligned}$$

We note that the additive component can be written as $X_n=\sum_{k=1}^n Y_k-Y_{k-1}$, which is like a random walk in the usual sense.

1.1.3.3. *E is finite and $\mathcal{T}$ is discrete*

In this case (E is finite, $\mathcal{T} = \mathbb{N}$), the state space of a MAP is $E \times \mathbb{R}^m$ ($m \geq 1$) and the MAP is specified by the measure-valued matrix (kernel) $F(dx)$ with the (i,j)-th element given by

$$F_{i,j}(dx) = \mathbb{P}_{i,0}(J_1 = j, X_1 \in dx | J_0 = i, X_0 = 0).$$

This provides a convenient way of simulating a MAP. Since we can simulate the Markov chain first, and then $Z_1, Z_2, ...$ by generating $Z_n = X_n - X_{n-1}$ according to $F_{i,j}$ when $J_{n-1} = i$ and $J_n = j$. In a particular case, if $m = 1$ and $F_{i,j}$ is concentrated on $(0, \infty)$, the MAP (J_n, X_n) is a Markov Renewal Process (MRP), in which $Z_n = X_n - X_{n-1}$ can be interpreted as interarrival times.

More generally, MRPs are thus discrete versions of MAPs with the additive part taking values in $\mathbb{R}_+^m (m \geq 1)$ and are called a Markov subordinator.

1.1.3.4. *E is finite and T is continuous*

In this case ($E = \{1, \cdots, N\}$ is finite with N elements and $\mathcal{T}$ is continuous), the most important example that is the most used in literatures is the *Markov-modulated Lévy process*. This MAP is defined by (J, X), where J is a Markov process specified by its intensity matrix $Q = (q_{i,j})_{i,j \in E}$ and the increments of X are governed by J in the sense that

$$\mathbb{E}_{i,0}\Big[f(X_{t+s} - X_t) g(J_{t+s}) \Big| \mathcal{F}_t\Big] = \mathbb{E}_{J_t,0}\Big[f(X_s) g(J_s)\Big] \, , \tag{1.1.10}$$

for any two bounded measurable functions f, g. Here, $\mathbb{E}_{i,0}(\cdot)$ denotes the expectation under the probability $\mathbb{P}_{i,0}(\cdot) = \mathbb{P}(\cdot | J_0 = i, X_0 = 0)$.

- On an interval $[s, s+t)$ where $J_s = i$, X_s evolves like a Lévy process (with stationary independent increments) with characteristic triplet $(\mu_i, \sigma_i^2, \nu_i(dx))$ depending on the state i, i.e. its Lévy exponent is

$$\psi^{(i)}(\alpha) = \alpha \mu_i + \alpha^2 \frac{\sigma_i^2}{2} + \int_{-\infty}^{+\infty} (e^{\alpha y} - 1 - \alpha y 1_{|y| \leq 1}) \nu_i(dy). \tag{1.1.11}$$

- A jump of J from i to $j \neq i$ has probability $\pi_{i,j}$ ($\pi_{i,i} = 0$) of giving rise to a jump of X at the same time; the distribution of which has some distribution $B_{i,j}$.

The process X can be written as the sum of two independent processes

$$X_t = X_t^{(1)} + X_t^{(2)} \ . \tag{1.1.12}$$

We will specify each term in (1.1.12). The process $(X_t^{(1)})$ behaves in law like a Lévy process with the Laplace exponent given by (1.1.11), when $J_t = i$.

Now, as for the second term in (1.1.12), let $\{U_n^{(i,j)}\}_{i,j\in E}$ be i.i.d. random variables (with $U_n^{(i,i)} = 0$), independent of J_t and with a distribution function $B_{i,j}$. Moreover, let us denote the jump times of J_t by $\{T_n\}_{n\in\mathbb{N}}$ (with $T_0 = 0$).

The jump process $X^{(2)}$ is described by

$$X_t^{(2)} = \sum_{n\geq 1}\sum_{i,j\in E} U_n^{(i,j)} 1_{\{J(T_{n-1})=i, J(T_n)=j, T_n\leq t\}}. \tag{1.1.13}$$

According to the path decomposition in (1.3.17), such a MAP which is also called *Markov-modulated Lévy process* has two types of behavior. It follows a Lévy process with characteristic triplet (μ_i, σ_i, ν_i) while J_t remains in state i; and at times when J_t jumps from state i to state j, it jumps according to a random variable $U^{(i,j)}$. Clearly, this process has only jumps that can come either from a Lévy process or from the random variables $U^{(i,j)}$ with distribution $B_{i,j}$. We notice that the processes $X^{(1)}$ and $X^{(2)}$ are fully specified by the characteristics

$$(q_{i,j}, B_{i,j}, \mu_i, \sigma_i, \nu_i)_{i,j\in E}. \tag{1.1.14}$$

For a basic review on Markov-modulated Lévy process, see Asmussen (2003), chapter XI. The basic references to MAPs are still Çinlar (1972a and 1972b), and Blumenthal and Getoor (1969). For recent work on MAPs up to 1991, see Prabhu (1991). A survey of MRWs and some new results up to 1991 have been given in Prabhu, Tang and Zhu (1991). The major reference to MRPs is Çinlar (1969). For a review of the literature on MRPs before 1991, see Prabhu (1991). Markov subordinators are reviewed in Prabhu and Zhu (1998), where applications to queueing systems are considered.

1.1.4. Matrix Moment Generating Function

We offen suppose that the state space E of the Markov component is finite, which makes sense in terms of modelling to have a finite set of possible scenarios. To reduce the complexity due to the two-dimensional structure of the MAP, we will use the matrix form and then, we shall understand $\mathbb{E}_y(V; J(\delta))$ to be the matrix with (i,j)-th element $\mathbb{E}_{i,y}(V; J(\delta) = j)$ for any random variable V and random time δ. For an event $\mathcal{G}$,

$\mathbb{P}_y(\mathcal{G}; J(\delta))$ will be understood in a similar sense as the matrix with (i,j)-th element, $\mathbb{P}_{i,y}(\mathcal{G}; J(\delta) = j)$. This matrix notation simplifies the study of the problem of first passage time of MAP which is needed in Chapter 2. For simplicity, we shall follow the usual notations that $\mathbb{E}_0(\cdot) = \mathbb{E}(\cdot)$ and $\mathbb{P}_0(\cdot) = \mathbb{P}(\cdot)$.

The MAP can be completely characterized by its Moment Generating Function which will be expressed below as a matrix. Thus, for a *Markov-modulated Lévy process* (J, X), let us consider the matrix form of the generating function, i.e. the matrix $\widehat{G}_t[\alpha]$ with $(i,j)-$th element

$$\widehat{G}_t[\alpha]_{i,j} = \mathbb{E}_i[e^{\alpha X_t}; J_t = j].$$

Theorem 1.1.1 (Asmussen (2003))**.**
For a MAP (J, X),

(1) If time is discrete, then $\widehat{G}_n[\alpha] = (\widehat{G}[\alpha])^n$ where

$$\widehat{G}[\alpha] = \widehat{G}_1[\alpha] = (\mathbb{E}_i[e^{\alpha X_1}; J_t = j])_{i,j\in E} = (\widehat{G}[\alpha]_{i,j})_{i,j\in E} = (p_{i,j}\widehat{H}[\alpha]_{i,j})_{i,j\in E}.$$

(2) If time is continuous, then the matrix $\widehat{G}_t[\alpha]$ is given by $e^{tF[\alpha]}$, where

$$F[\alpha] = Q + diag(\psi^{(1)}(\alpha), ..., \psi^{(N)}(\alpha)) + [q_{ij}(\, p_{ij}\widehat{B}_{i,j}[\alpha] - 1)]_{i,j\in E}. \tag{1.1.15}$$

Since we suppose $p_{ij} = 1$ for any $i \neq j$ (see Asmussen and Kella (2000)), the previous formula can be reduced as

$$F[\alpha] = Q \circ \widehat{B}[\alpha] + diag(\psi^{(1)}(\alpha), ..., \psi^{(N)}(\alpha)), \tag{1.1.16}$$

where $Q \circ \widehat{B}[\alpha] = (q_{ij}\widehat{B}_{ij}[\alpha])_{i,j\in E}$. Where $\widehat{H}[\alpha]_{i,j}$ and $\widehat{B}[\alpha]_{i,j}$ are respectively the Laplace transforms of $\widehat{H}_{ij}$ and $\widehat{B}_{ij}$.

1.1.5. Lévy systems for MAP

The concept of Lévy systems plays a fundamental role in studying the jump structure of a MAP. This notion is intimately related to the infinitesimal generator of such processes. We refer to Çinlar (1975) and Maisonneuve (1977) for a basic review on the existence of Lévy systems. See also Ben Salah and Morales (2012) for more details on Lévy systems for Markov-modulated Lévy processes.

We now state the following result [Çinlar (1975)] that guarantees the existence of a Lévy system for a MAP [see also Maisonneuve (1977)].

Theorem 1.1.2. *Let $\{(J_t, X_t) : t \in \mathcal{T}\}$ be a Markov additive process with J having a finite state space E and X, a quasi-left continuous process, taking values in the space*

$(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. *There exists a continuous increasing functional* $(H_t)_{t\in\mathcal{T}}$ *of* J *and a transition kernel* L *from* $(E, \mathcal{E})$ *into* $(E\times\mathbb{R}, \mathcal{E}\times\mathcal{B}(\mathbb{R}))$ *such that, for any non-negative measurable function* f *defined on* $(E \times \mathbb{R})$,

$$\begin{aligned} & \mathbb{E}_i\Big[\sum_{s\le T} f(J_{s-}, J_s, X_{s-}, X_s)1_{\{J_{t-}\neq J_t\}\cup\{X_t\neq X_{t-}\}}\Big] \\ = \; & \mathbb{E}_i\Big[\int_0^T dH_s \int_{E\times\overline{\mathbb{R}}} L(J_s, dz, du) f(J_s, z, X_s, X_s+u)\Big], \end{aligned} \tag{1.1.17}$$

for all $i \in E$.

The couple (H, L) *is said to be a* Lévy system *for the MAP* (J, X). *The kernel* L *is often referred to as the Lévy kernel of* (J, X) *and for any* $i \in E$, *it satisfies*

$$L(i, \{(i, 0)\}) = 0, \tag{1.1.18}$$

$$\int_{\mathbb{R}} L(i, \{i\} \times dy)(|y|^2 \wedge 1) < \infty. \tag{1.1.19}$$

Theorem 2.2.1 generalizes some results on the existence of Lévy systems for Markov processes which are identified for certain classes. In fact, if we set

$$K(i, B) = L(i, (B \setminus \{i\} \times \mathbb{R})) \,, \quad B \in \mathcal{E}, \tag{1.1.20}$$

then (H, K) is a Lévy system for the Markov process J, that is

$$\mathbb{E}_i\Big[\sum_{s\le T} g(J_{s-}, J_s)1_{\{J_{s-}\neq J_s\}}\Big] = \mathbb{E}_i\Big[\int_0^T dH_s \int_E K(J_s, dy) g(J_s, y)\Big] \,, \tag{1.1.21}$$

for any non-negative measurable function g defined on $E \times \mathbb{E}$. The existence of the transition kernel K for the Markov process has been shown in Watanabe (1964).

Remark 1.1.2. *Since* J *is a regular step process with a discrete state space* E, *then* H_t *can be taken to be* $H_t = t$ *[Çinlar (1975)]. Without loss of generality, we can set* H_t *to be equal to* t.

For certain classes of *additive processes* and *Lévy processes*, Lévy systems are identified and given explicitly in terms of Lévy measures. Incidentally, these are examples which motivate the terminology Lévy kernel, Lévy systems. In the following paragraphs we give explicit expressions of Lévy systems for special cases.

- Recall from Example 1.1.3 that since J is a singleton and X is simply an *additive process* (i.e. with independent increments); the continuous additive functionnal H can be taken to be $H_t = t$, and the Lévy kernel then becomes $L(J_s, dz, dy) = \nu_s(dy)$ where $\nu_s(.)$ is the Lévy measure of the process verifying

$$\int_0^T dt \int (|y|^2 \wedge 1)\nu_t(dy) < \infty. \tag{1.1.22}$$

For a good review on this subject, see Sato(1999) and Itô (2004).

- If in addition X is a *Lévy process* (i.e. with stationnary independent increments), then the Lévy kernel becomes $L(J_s, dz, dy) = \nu(dy)$, where ν is the Lévy measure of X verifying

$$\int (|y|^2 \wedge 1)\nu(dy) < \infty. \tag{1.1.23}$$

  Identity (1.1.17) is reduced to the so-called *compensation formula* [see Kyprianou (2006), Theorem 4.4 and Bertoin (2006)].

When (J, X) is a *Markov-modulated Lévy process*, we provide in Chapter 3 the identification of Lévy systems and give its expression in terms of its characteristic given by (1.1.14). A key concept that we have used is that of Lévy systems which is introduced in Section 1.1.5 [see Çinlar (1975), Maisonneuve (1977), and Ben Salah and Morales (2012)] and, indeed, one of the contributions of this chapter is to identify the Lévy system of a particular case when the MAP is a Markov-modulated Lévy process.

In the following theorem, we shall give a characterization of the transition kernel L for Markov-modulated Lévy process (J, Y) defined in Example 1.1.3.4. This is the first main result of Chapter 2.

Theorem 1.1.3. *(Ben Salah and Morales (2012)) Consider the Markov additive process (J, X) given in Example 1.1.3.4 of Section 1.1.3 and let $(q_{i,j}, B_{i,j}, \sigma_i, a_i, \nu_i)_{i,j\in E}$ be the characteristics of such a process. Then the following transition kernel L, from $(E, \mathcal{E})$ into $(E \times \mathbb{R}_+, \mathcal{E} \times \mathcal{B}(\mathbb{R}))$,*

$$L(i, \{j\}, du) = \nu_i(du)1_{\{i=j\}} + q_{i,j}B_{i,j}(du)1_{\{i\neq j\}}\ , \qquad i, j \in E\ , \quad \text{and } u \in \mathbb{R}\ , \tag{1.1.24}$$

is the Lévy kernel of (J, Y) in the sense of Theorem 1.1.2, where $H_t = t$ for $t \geq 0$.

1.2. Risk Models and Expected Discounted Penalty Function

Different formulations are used in literature to model the insurance surplus process and capture certain properties of the insurance portfolio. In the classical insurance risk model, the premium is assumed to be a constant and the claim process is assumed to follow a compound Poisson process where the claim sizes are *i.i.d.* random variables and the number of claims is assumed to be Poisson process. This model is far from being realistic. There is a large amount of papers in the literature devoted to the generalization of the classical model in different ways. For more detailed discussions on the ruin problems under classical risk models and various extensions, see Gerber (1979), Grandell (1991, 1997), Rolski *et al.* (1999), Asmussen (2000), Biffis and Morales (2010), Biffis and Kyprianou (2010), Breuer and Badescu (2012) and the references therein.

Three important questions of interest in the classical ruin problem are the time of ruin, the deficit at ruin, and the surplus immediately before ruin. From a mathematical point of view, a crucial role is played by the functional involving these three quantities. This functional defines the Expected Discounted Penalty Function (EDPF) introduced in the classical papers of Gerber and Shiu (1997, 1998a). Since the EDPF operates on a function of the deficit at ruin and surplus prior to ruin, applications in the context of insurance and financial mathematics are quite natural. For example, the EDPF can be used to determine the initial capital required by an insurance company to avoid insolvency. Similarly, the EDPF can be used as a pricing device for American options or reset guarantees [e.g. Gerber and Shiu (1998b)]. First, we introduce the risk model, the time of ruin and give the definition of EDPF. Then we provide the ruin probability and the characterization of EDPF for classical risk model and Lévy risk model. Next, we introduce the Lévy-modulated risk models and discuss some particular examples recently used. Finally, we describe the Markov additive risk process which is used in Chapter 2.

1.2.1. Risk process

A *risk process* $(R_t)_{t\geq 0}$, as defined in broad terms, is a model for the time evolution of the reserves of an insurance company. We denote the initial reserve by $x = R_0 \geq 0$. The probability $\theta(x)$ of ultimate ruin is the probability that the reserve ever drops below zero,

$$\theta(x) = \mathbb{P}[\inf_{t\geq 0} R_t < 0 | R_0 = x]. \tag{1.2.1}$$

The probability of ruin before time T is

$$\theta(x,T) = \mathbb{P}[\inf_{0\leq t\leq T} R_t < 0 | R_0 = x]. \tag{1.2.2}$$

We also refer to $\theta(x)$ and $\theta(x,T)$ as ruin probabilities with infinite and finite horizon, respectively.

For mathematical purposes, it is frequently more convenient to work with the *claim surplus process* (also called the aggregate loss process) $(Y_t)_{t\geq 0}$ defined by $Y_t = x - R_t$ and then the risk process can be represented as

$$R_t = x - Y_t, \text{ for } t \geq 0. \tag{1.2.3}$$

Letting

$$\tau_x = \inf\{t \geq 0, R_t < 0\} = \inf\{t \geq 0, Y_t > x\}, \tag{1.2.4}$$

be the time of the ruin, the ruin probabilities can then alternatively be written as

$$\theta(x) = \mathbb{P}\big[\tau_x < \infty\big], \tag{1.2.5}$$

$$\theta(x,T) = \mathbb{P}\big[\tau_x < T\big]. \tag{1.2.6}$$

1.2.2. Expected Discounted Penalty Function EDPF

Let us start by introducing some quantities related to the problem of ruin, the deficit at ruin $-R_{\tau_x} = Y_{\tau_x} - x$ and the surplus prior to ruin $R_{\tau_x^-} = x - Y_{\tau_x^-}$. The combination of these two quantities with the time of ruin τ_x defines the *Expected Discounted Penalty Function* (EDPF) at ruin as

$$P(q,x) = \mathbb{E}\Big[e^{-q\tau_x} w(Y_{\tau_x} - x, x - Y_{\tau_x^-}); \tau_x < \infty\Big], \tag{1.2.7}$$

where the penalty w is a non-negative function of the surplus prior to ruin and of the deficit at ruin.

The concept of EDPF has been introduced by Gerber and Shiu (1997) and (1998a). This so-called Gerber-Shiu function is a functional of the ruin time, the surplus prior to ruin, and the deficit at ruin. The EDPF can be used to determine the initial capital required by an insurance company to avoid insolvency with a minimum level of confidence and for fixed penalization of the ruin event. Similary, the EDPF can be used as a pricing device for American options [Gerber and Shiu (1998b)].

The expression $P(q,x)$ is usually referred to as the *Gerber-Shiu function*. Clearly, for $w = 1$ and $q = 0$, (1.2.7) reduces to the ruin probability $\theta(x)$, and for $w = 1$ and $q > 0$ one arrives at the Laplace transform of the time to ruin τ_x. Alternatively, if $q = 0$ and w is the bivariate Dirac-delta function, (1.2.7) represents the joint density of the surplus prior to ruin and of the deficit at ruin. The parameter q can be interpreted either as a discount rate or as the Laplace transform argument.

The EDPF has been extensively studied and generalized to various scenarios. In the following sections, we will introduce the classical risk prosess and the Lévy risk model for which expressions of the EDPF are available.

1.2.3. Compound Poisson model

The risk process (1.2.3) reduces to the *Compound Poisson model* or the *Classical risk model* when the premium is assumed to be a constant c and the aggregate claim process is assumed to follow a compound Poisson process where the claim sizes are *i.i.d.* random variables. The number of claims is assumed to follow an homogeneous Poisson process $(N_t)_{t\geq 0}$ with intensity λ. We denote $(Z_n)_{n\geq 1}$ the claim sizes which are independent of $(N_t)_{t\geq 0}$, positive and *iid* random variables with distribution function F and first moment μ. The insurance surplus process is given by

$$R_t = x + ct - \sum_{i=1}^{N_t} Z_i, \tag{1.2.8}$$

where x is the initial capital and $c < \lambda\mu$. The premium rate c is assumed to satisfy the net profit condition used to avoid the possibility that R becomes negative almost surely, that is precisely $c < \lambda\mu$.

We denote by δ_q the unique nonnegative root of the so-called *Lundberg fundamental equation*

$$q + \lambda - c\xi = \lambda \int_0^\infty e^{-\xi y} F(dy). \tag{1.2.9}$$

Theorem 1.2.1 (Gerber and Shiu (1998))**.**
The Gerber-Shiu function $P(q,x)$ in the compound Poisson model given by (1.2.8), satisfies the defective renewal equation

$$P(q,x) = \int_0^x P(q, x-y)g(y)dy + h(x), \tag{1.2.10}$$

where

$$g(y) = \frac{\lambda}{c} e^{\delta_q x} \int_y^\infty e^{-\delta_q u} F(du) \tag{1.2.11}$$

and

$$h(x) = \frac{\lambda}{c} e^{\delta_q x} \int_x^\infty \int_u^\infty e^{-\delta_q v} w(u, v-u) F(dv) du. \tag{1.2.12}$$

Note that for two integrable functions f and g defined on $[0,\infty)$, the convolution of f and g is the function

$$f * g(x) = \int_0^x f(y)g(x-y)dy, \quad x \geq 0.$$

Equation (1.2.10) can be written more concisely as

$$P(q,x) = P(q,x) * g + h. \tag{1.2.13}$$

The solution of (1.2.13) can be expressed as an infinite series of functions (Neumann series),

$$P(q,x) = \sum_{n=0}^{\infty} g^{*n} * h(x), \quad x \geq 0, \tag{1.2.14}$$

where g^{*n} $(n \geq 1)$ denotes the n-fold convolution of g with itself and and g^{*0} is the distribution function corresponding to the Dirac measure at zero. See Gerber and Shiu (1998) for more details.

The definition of $P(q,x)$ and the derivation of its properties for compounded Poisson model go back to Gerber and Shiu (1997 and 1998).

1.2.4. Lévy risk model

The risk process (1.2.3) reduces to the *Lévy risk process* with no positive jumps when the claim surplus process (Y_t) is assumed to be a *spectrally positive* Lévy process (Stochastic process with independent and stationary increments that can only have positive jumps). with characteristic triplet (μ, σ^2, ν), i.e. its *Laplace exponent* is given by

$$\begin{aligned} \psi(\alpha) &= \log \mathbb{E}[\exp(-\alpha Y_1)] \\ &= -\alpha\mu + \alpha^2 \frac{\sigma^2}{2} + \int_{(0,\infty)} [e^{-\alpha y} - 1 + \alpha y 1_{\{0<y\leq 1\}}]\nu(dy)\ , \end{aligned} \tag{1.2.15}$$

where $\int_{(0,\infty)}(1 \wedge |y|^2)\nu(dy) < \infty$.

The net profit condition, that is to say a necessary and sufficient condition to insure that Y_t drifts to ∞, is precisely $\mathbb{E}[Y_1] < 0$, which necessarily requires that

$$\int_{(1,\infty)} y\nu(dy) < \mu, \tag{1.2.16}$$

and then the process Y has a negative drift such that $\mathbb{E}[Y_1] < 0$ in order to avoid the possibility that R becomes negative almost surely. This condition is often expressed in terms of a safety loading. Indeed, it is standard to write the drift component within Y as a loaded premium.

Many risk processes are in fact special Lévy risk processes with no positive jumps. The classical compound Poisson risk process perturbed by a Brownian motion is one of them. More generally, some models have used the classical compound Poisson risk process perturbed by a Lévy process as their risk process. [See for instance Furrer (1998), Yang and Zhang (2001), Huzak *et al.* (2004), Garrido and Morales (2006), and Biffis and Morales (2010)].

Before discussing the ruin problem for the Lévy risk model, we will first introduce breifly the concept of *scale function* and show that will be needed to characterize the EDPF.

1.2.4.1. *Scale function*

For the right inverse of ψ we shall write ϕ on $[0, \infty)$, that is to say, for each $q \geq 0$,

$$\phi(q) = \sup\{\alpha \geq 0 : \psi(\alpha) = q\}. \tag{1.2.17}$$

Note that the properties of the Laplace exponent ψ given by (1.2.15) of the spectrally positive Lévy process Y, imply that $\phi(q) > 0$ for $q > 0$. Furthermore, $\phi(0) = 0$, since $\psi'(0) = -c + \int_{(0,\infty)} y\nu(dy) \leq 0$.

Note that we may define the probability measure $\mathbb{P}^{\phi(q)}$ by

$$\frac{d\mathbb{P}^{\phi(q)}}{\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{-\phi(q)Y_t - qt} \tag{1.2.18}$$

where $\phi(q)$ is the right inverse of ψ (the positive solution of the Lundberg equation $\psi(\alpha) = q$) defined in (1.2.17). Under the measure $\mathbb{P}^{\phi(q)}$, the process Y introduced in (1.2.15) is still a spectrally positive Lévy process, and still drifts to $-\infty$. We denote by $\nu_{\phi(q)}$ the Lévy measure of Y under the measure change $\mathbb{P}^{\phi(q)}$ and then,

$$\nu_{\phi(q)}(du) = e^{-\phi(q)u}\nu(du), \text{ for } u > 0. \tag{1.2.19}$$

The change of measure above, known as the *Esscher transform*, has the important property that the process Y under $\mathbb{P}^{\phi(q)}$ is still a spectrally positive Lévy process. This fact will play a crucial role in the analysis Lévy risk models. For more details about measure change of Lévy process, see for example Asmussen and Albrecher (2010), and Kyprianou (2006).

We now define the so-called *scale functions* $\{W^{(q)};\ q \geq 0\}$ of the spectrally negative process $-Y$. For every $q \geq 0$, there exists a function $W^{(q)} : \mathbb{R} \longrightarrow [0, \infty)$ such that

$W^{(q)}(y) = 0$ for all $y < 0$, absolutely continuous on $(0, \infty)$ satisfying

$$\int_0^\infty e^{-\lambda y} W^{(q)}(y) dy = \frac{1}{\psi(\lambda) - q}, \quad \text{for} \quad \lambda > \phi(q), \tag{1.2.20}$$

where $\phi(q)$ is the largest solution of $\psi(\beta) = q$ defined in (1.2.17). $W^{(q)}$ is called the *q-scale function* and for short, we shall write $W^{(0)} = W$. The 0-scale function under $\mathbb{P}^{\phi(q)}$, which we write as $W_{\phi(q)}$, is related to the q-scale function under $\mathbb{P}$, that is to say $W^{(q)}$, via the relation

$$W^{(q)}(y) = e^{\Phi(q)y} W_{\phi(q)}(y). \tag{1.2.21}$$

The reader is otherwise referred to the classical references for Lévy process [Bertoin (1996) and Sato (1999), and Kyprianou (2006)].

The theorem below (see Asmussen and Albrecher (2010)), provides an analytical characterization of the EDPF given by 1.2.7 in terms of scale functions.

Theorem 1.2.2. *Suppose that Y is a spectrally positive Lévy process. The EDPF defined by (1.2.7), is given by*

$$P(q, x) = \int_0^\infty \int_0^\infty w(v, u) (e^{-\phi(q)v} W^{(q)}(x) - W^{(q)}(x - v)) \nu(du + v) dv \tag{1.2.22}$$

where w is a bounded measurable function satisfying $w(\cdot, 0) = 0$.

Remark 1.2.1. *Note that the assumption $w(\cdot, 0) = 0$ simply restricts the EDPF to the case where ruin happens through jumps. If ruin is caused by diffusion, then it is known from the general phenomenon of* creeping *of Lévy processes that $Y_{\tau_x} = Y_{\tau_x^-} = x$ and this occurs with probability*

$$\mathbb{E}\Big[e^{-q\tau_x}; Y_{\tau_x} = x\Big] = \frac{\sigma^2}{2} [W^{(q)\prime}(x) - \phi(q) W^{(q)}(x)] \tag{1.2.23}$$

[see Pistorius (2005)].

The previous theorem is given in more general form in Biffis and Kyprianou (2010), when the EDPF also includes the size of the last minimum before ruin, see also Biffis and Morales (2010) for a convolution type approach. In fact, Biffis and Morales (2010) extended the EDPF given by (1.2.7) to include the last minimum of the surplus before ruin. The new EDPF obtained is given by

$$\widetilde{P}(q, x) = \mathbb{E}\Big[e^{-q\tau_x} w(Y_{\tau_x} - x, x - Y_{\tau_x^-}, x - \overline{Y}_{\tau_x^-}); \tau_x < \infty\Big], \tag{1.2.24}$$

where $\overline{Y}_t = \sup_{s \leq t} Y_t$ and w is a non-negative bounded measurable function such that $w(0, \cdot, \cdot) = 0$. A defective renewal equation for (1.2.24) is obtained in Biffis and Morales (2010) for a subordinator risk model perturbed by a spectrally negative Lévy process.

More generally, when the risk process is driven by a spectrally negative Lévy process, Biffis and Kyprianou (2010) provided an explicit characterization of (1.2.24) in terms of q-scale functions. The generalized EDPF (1.2.24) is given by the following theorem.

Theorem 1.2.3 (Biffis and Kyprianou (2010))**.** *Suppose that Y is a spectrally positive Lévy process. The EDPF defined by (1.2.24), is given by*

$$\widetilde{P}(w,q,x) = \int_{(0,\,\infty)^3} 1_{\{v \geq y\}} w(u,v,y) K_x^{(q)}(du,dv,dy) \tag{1.2.25}$$

where $K_x^{(q)}(du,dv,dy) = e^{-\phi(q)v} W^{(q)}(x-dy)\nu(du+v)dv.$

1.2.5. Markov additive risk model

We find that Markov additive risk models have been recently studied in the literature on insurance risk although not at the same level of generality. In this section we give a brief presentation of existing MAP models in the risk theory literature.

1.2.5.1. *Markov additive risk model with phase-type*

The first application of the class of MAP in risk theory seems to be due to Breuer (2010). He considered the risk model of the form (1.2.3) where the net aggregate claims process is a MAP with phase-type jumps. The risk reserve $R = \{x - Y_t; t \in \mathcal{T}\}$ is modelled by a Markov-modulated Brownian motion with phase-type claims (Y, J) (See the case (1.1.3.4) of Section 1.1.3), where J is an irreductible Markov process with finite state space E and x is the initial surplus.

Each phase $i \in E$ signifies a certain state of market conditions which may affect the intensity and severity of claims. Claims may occur in two ways. First, when J is in phase i, claims occur at constant rate $\lambda_i \geq 0$. The size of such a claim shall have a phase-type (PH) distribution with parameters $\alpha^{(ii)}$ and $T^{(ii)}$. Second, at time of phase changes from i to $j \neq i$, a claim may occur with probability p_{ij}. The size of such a claim shall have a PH distribution with parameters $\alpha^{(ij)}$ and $T^{(ij)}$. Note that $\eta^{(ij)} = -T^{(ij)}\overline{1}$ is the exit rate vector of the PH$(\alpha^{(ij)}, T^{(ij)})$ distribution, where $\overline{1}$ denotes the vector with entries 1. We refer to Breuer (2008 and 2010), and Badescu and Breur (2011) for additional details on this result and related definitions.

The claim surplus Y is supposed to be a MAP with phase-type jumps with characteristic triplet $(c_i, \sigma_i^2, \nu_i(dx))$ depending on the state i of J; i.e. Y evolves like a Lévy process with Laplace exponent

$$\kappa^{(i)}(\alpha) = -\alpha c_i + \alpha^2 \frac{\sigma_i^2}{2} + \int_0^{+\infty} (e^{\alpha x} - 1 - \alpha y 1_{\{0<y\leq 1\}}) \nu_i(dy)$$

during intervals when the phase equals $i \in E$, where the Lévy measure

$$\nu_i(dy) = \lambda_i 1_{\{y>0\}} \alpha^{(ii)} e^{T^{(ii)} y} \eta^{(ii)} dy. \tag{1.2.26}$$

[See the Example1.1.3.4 of Section 1.1.3].

For this risk model described above, Breuer (2010) determined the joint distribution of the surplus prior to ruin (undershoot), the deficit at ruin (overshoot), the maximal level before ruin, the time of attaining this maximum and the time between the maximum and ruin. Breuer and Badescu (2012) applied this result to derive a generalized EDPF, in particular, they gave an explicit formula for the generalized expected discounted measure

$$\mathbb{E}\Big[e^{-\alpha \overline{G}_{\tau_x} - \beta(\tau_x - \overline{G}_{\tau_x})}; Y_{\tau_x} - x \in du, x - Y_{\tau_x -} \in dv, x - \overline{Y}_{\tau_x -} \in dy\Big],$$

where τ_x is the first time passage over $x \geq 0$ (time of ruin), $\alpha, \beta \geq 0$ are the time discounting factors, $\overline{Y}_{\tau_x -} = \sup_{0 \leq t < \tau_x} Y_t$ is the supremum of Y before ruin and $\overline{G}_{\tau_x}$ is the time of attaining this supremum.

1.2.5.2. *Spectrally positive Markov additive risk model*

The family of Markov additive risk processes introduced in Section 1.2.5.1 [see Breuer (2010), and Breuer and Badescu (2012)] have been extended in Chapter 2 [see Ben Salah and Morales (2012)]. Indeed, we have studied the EDPF under a general model of the form (1.2.3) where the net aggregate claims process is a *spectrally positive* MAP (MAP that can only have positive jumps). This more general setting allows for the introduction of long-term market conditions that change over time following a finite-state continuous-time Markov process modeling different environment scenarios.

We consider a very general setup that generalizes the previous models. The model discussed in Chapter 2 is

$$R_t := x - Y_t \ , \qquad t \geq 0 \ , \tag{1.2.27}$$

where $x \geq 0$ is the initial surplus and $Y = \{Y_t; t \in \mathcal{T}\}$ is a right-continuous spectrally positive Markov-modulated Lévy process with modulating process $J = \{J_t; t \in \mathcal{T}\}$ taking values on a finite state space E. Y represents the claim process of an insurance company and J is a background process that describes the environment in which claims occur. Let us denote by $Y^{(1)}$ and $Y^{(2)}$ the path components of Y, as defined in (1.1.12), and let $(q_{i,j}, B_{i,j}, c_i, \sigma_i, \nu_i)_{i,j\in\mathbb{S}}$ be the characteristics of such processes (See the case (1.1.3.4) of Section 1). We assume the process Y to have a negative drift $a_i < 0$ for all $i \in E$.

Recall from (1.1.12) that the claim surplus process Y_t can be written as the sum of two independent processes

$$Y_t = Y_t^{(1)} + Y_t^{(2)} . \tag{1.2.28}$$

The first term $\{Y_t^{(1)}\}$ in (1.2.28) evolves like a spectrally positive Lévy process with Laplace exponent given by

$$\psi^{(i)}(\alpha) \quad = \quad \alpha c_i + \alpha^2 \frac{\sigma_i^2}{2} + \int_{(0,\infty)} [e^{-\alpha y} - 1 + \alpha y 1_{\{0<y\leq 1\}}]\nu_i(dy) \ , \tag{1.2.29}$$

where $\int_{(0,\infty)}(1 \wedge |y|^2)\nu_i(dy) < \infty$.

Now, as for the second term in (1.2.28), let $\{U_n^{(i,j)}\}_{i,j\in E}$ be i.i.d. positive random variables (with $U_n^{(i,i)} = 0$) which are independent of J_t and have a distribution function $B_{i,j}(\cdot)$ with support on $[0, \infty)$. Moreover, let us denote the jump times of J_t by $\{T_n\}_{n\in\mathbb{N}}$ (with $T_0 = 0$). The jump process $\{Y_t^{(2)}\}$ is described by

$$Y_t^{(2)} = \sum_{n\geq 1} \sum_{i,j\in E} U_n^{(i,j)} 1_{\{J(T_{n-1})=i, J(T_n)=j, T_n\leq t\}}. \tag{1.2.30}$$

During the time intervals where J is in state i, claims occur with the positive jumps of Lévy component $Y^{(1)}$ with characteristic triplet (a_i, σ_i, ν_i) introduced previously. At time of state changes from i to $j \neq i$ a claim occurs with size $U^{(i,j)}$ which has a distribution function $B_{i,j}(\cdot)$.

We study in Chapter 2 [Ben Salah and Morales (2012)] the ruin problem for Markov additive risk models given by (1.2.27). In the context of fluctuations and exit problems, this family of the Markov additive risk processes has been recently studied in Kyprianou and Palmowski (2008), Klusik and Palmowski (2011), and Ivanovs and Palmowski (2011). We give in Chapter 3 a characterization of the EDPF for the Markov additive risk process defined by (1.2.7). A key concept that we have used is that of Lévy systems which is introduced in Section 1.1.5 [see Çinlar (1975), Maisonneuve (1977), and Ben Salah and Morales (2012)] and, indeed, one of the contributions of this chapter is to identify the Lévy system of a particular case when the MAP is a Markov-modulated Lévy process. A second concept that is key to our analysis is that of a scale matrix [see Kyprianou and Palmowski (2008) and Ivanovs and Palmowski (2011)] that generalizes the well-studied notion of scale function for spectrally-positive Lévy processes. These two elements allow us to give a characterization of the EDPF that generalizes the results developed in Biffis and Kyprianou (2010) and Biffis and Morales (2010) for Lévy insurance risk processes. Moreover, our approach provides a connection with some of the concepts introduced in Kyprianou and Palmowski (2008) where they have partial answers to the same problems discussed in Chapter 3. We also generalize in Chapter 3 the results from Breuer and Badescu (2012) where a similar problem is solved for MAPs with phase-type jumps. More explicitly, the expressions for the EDPF obtained in Chapter 3 are given in terms not only of the Lévy system of the process but also in terms of the so-called q-potential measure of the risk process killed at exit.

1.3. No-Arbitrage, Fundamental Theorem and Minimal Entropy Martingale Measure

In this subsection, we present the main concept of the theory of arbitrage following Delbaen and Schachermayer (2006), and Cont and Tankov (2004).

1.3.1. Financial Model and No-Arbitrage

The notions of arbitrage and of risk-neutral valuation are crucial to modern financial theory. It is the corner-stone of the option pricing theory due to Black and Scholes (1973), and Merton (1973). An arbitrage opportunity is the possibility to make a profit in a financial market without risk and without net investment of capital. The arbitrage-free argument is a mathematical assumption that is needed in order to develop the modern theory of mathematical finance. It turns out that it has an economical interpretation and that economically speaking, the idea of arbitrage is related to the question of information and the theory changes completely when the different agents have different information. These arguments should convince the reader that *no-arbitrage principle* is economically very appealing. Hence a mathematical model of a financial market should be designed in such a way that it does not permit arbitrage.

The mathematical tool formalizing the *no-arbitrage (NA)* concept is the general theory of stochastic analysis and martingale theory. The central piece of the theory that turns the *NA* arguments into a comprehensive theory is the so-called *Fundamental Theorem of Asset Pricing* which is the central result of the theory of pricing and hedging by *NA*. The proof of this theorem is due to Harison and Pliska (1981) and more rigorously to Delbaen and Shachermayer (1994).

In this section we will attempt to explain the fundamental concepts behind the absence of arbitrage, incompleteness of market, risk-neutral pricing and equivalent martingale measures.

Consider a market whose possible evolution between 0 and T is described by a probability space $(\Omega, \mathcal{F})$ where $\mathcal{F}$ contains all statements which can be made about behavior of the prices on the period $[0, T]$. We refer to Delbaen and Schachermayer (2006) for more details on description of financial model.

The stochastic asset value can be described by a process

$$S : [0, T] \times \Omega \longrightarrow \mathbb{R}^{d+1}$$

$$(t, \omega) \longmapsto (S_t^0(\omega), S_t^1(\omega), ..., S_t^d(\omega)),$$

where $S_t^i(\omega)$ represents the value of asset i at time t in the market scenario ω, for $0 \leq i \leq d+1$. Note that S_t^0 is a numéraire and $S_t^0 = \exp(rt)$ is a typical example of a cash account with interest rate r.

We denote by $(\mathcal{F}_t)_{0 \leq t \leq T}$ the information generated by the history of all assets up to t. $\mathcal{F}_0$ contains no information and $\mathcal{F}_T = \mathcal{F}$ is the history of all assets up to T.

Definition 1.3.1. *A **model of financial market** is an $\mathbb{R}^{d+1}$-valued stochastic process $(S_t^0, S_t^1, ..., S_t^d)_{t \in [0,T]}$, based on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$. We shall assume that the zero coordinate S^0 satisfies $S_t^0 > 0$ for all $t \in [0, T]$ and $S_0^0 = 1$.*

Definition 1.3.2. *A trading strategy $H = (H_t)_{t \in [0,T]} = (H_t^0, H_t^1, ..., H_t^d)_{t \in [0,T]}$ is an $\mathbb{R}^{d+1}$-valued process which is predictable, i.e. H_t is $\mathcal{F}_{t-}$-measurable.*

The interpretation is that during the infinitesimal interval $[t, t+dt]$, the agent holds a quantity equal to H_{t+dt}^i of asset i. The decision is taken at time t and therefore, H_t is required to be $\mathcal{F}_t$-measurable. We represent the capital accumulated between 0 and t by following the strategy H, by the stochastic integral

$$\int_0^t H_u dS_u,$$

where S_t is the $\mathbb{R}^{d+1}$-vector given by $(S_t^0, S_t^1, ..., S_t^d)$, $t \in [0, T]$.

A **portfolio** is a vector describing the amount of each asset held by the investor: (H, S). The value of a such portfolio at time t is then given by $V = (V_t)_{t \in [0,T]}$, where

$$V_t = (H \cdot S)_t = \sum_{i=0}^{d} H_t^i S_t^i.$$

A strategy $H = (H_t)_{t\in[0,T]}$ is said to be **self-financing** if for every $t \in [0,T]$, the value V_t of portfolio is equal to the initial capital value plus the capital gain between 0 and t

$$V_t = \int_0^t H_u dS_u = (H \cdot S)_0 + \int_{(0,t]} H_u dS_u. \tag{1.3.1}$$

The equation (1.3.1) means that the only source of variation of the portfolio's value is the variation of asset values: by changing the portfolio from H_t to H_{t+dt}, there is no input/outflow of money.

1.3.2. Martingale Measures and the Fundamental Theorem

Let us introduce the discounted risky asset values $\widetilde{S} = (\widetilde{S}_t)_{t\in[0,T]} = (\widetilde{S}_t^1, ..., \widetilde{S}_t^d)$ where

$$\widetilde{S}_t^i = \frac{\widetilde{S}_t^i}{\widetilde{S}_t^0} \text{ for } i = 1, ..., d. \tag{1.3.2}$$

Hence the discounted value $\widetilde{V}_t$ of portfolio is

$$\widetilde{V}_t = \frac{V_t}{S_t^0} = (\frac{V_t^0}{S_t^0}, \frac{V_t^1}{S_t^0}, ..., \frac{V_t^d}{S_t^0}) \text{ for } i = 0, ..., d \text{ and } t \in [0,T].$$

Recall that on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, $\mathbb{Q}$ and $\mathbb{P}$ are said to be equivalent measures ($\mathbb{Q} \sim \mathbb{P}$), if they define the same set of possible and impossible events. Mathematically, this means

$$\mathbb{Q} \sim \mathbb{P}: \text{ for any } A \in \mathcal{F}, \quad \mathbb{Q}(A) = 0 \text{ if and only if } \mathbb{P}(A) = 0.$$

Definition 1.3.3. *A probability measure $\mathbb{Q}$ is called an* ***equivalent martingale measure*** *for $\widetilde{S}$, if $\mathbb{Q} \sim \mathbb{P}$ and $\widetilde{S}$ is a martingale under $\mathbb{Q}$, i.e., $\mathbb{E}_{\mathbb{Q}}[\widetilde{S}_t|\mathcal{F}_s] = \widetilde{S}_s$ for $0 \le s \le t \le T$.*

We denote by $\mathcal{M}^e(\widetilde{S})$ the set of equivalent martingale measures. After having fixed some preliminary results, we may give the first crucial result of the theory of pricing and hedging by NA, often referred to as the *Fundamental Theorem of Asset Pricing* [see Delbaen and Schachermayer (1998 and 2006)].

Theorem 1.3.1 (Fundamental Theorem of Asset Pricing)**.**
*A financial market model defined by $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and asset prices $(S_t)_{t\in[0,T]}$ is arbitrage-free (*NA*) if and only if there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that the discounted assets $(\widetilde{S}_t)_{t\in[0,T]}$ are martingales with respect to $\mathbb{Q}$.*

We refer to Harrison and Pilska (1983), and Delbaen and Schachermayer (1998 and 2006) for more details on this topic. This theorem establishes an equivalence between the financial concept of NA and the mathematical notion of equivalent martingale measure. As equally important to this theorem is the fact that in order to price a contingent claim in this market, we need to take expectations under the equivalent martingale measure Q and not P.

1.3.3. Market completeness

Besides the absence of arbitrage, another important concept originating in the Black-Scholes model is the concept of *perfect hedge:* a self-financing strategy (H_t) is said to be perfect hedge (or a replication strategy) for a contingent claim C (i.e. a stochastic variable that only depends on the value S_T of the stock price at the final time) if

$$C = V_0 + \int_0^T H_t dS_t \quad \mathbb{P}\text{- a.s. .} \tag{1.3.3}$$

By absence of arbitrage, if a replicating strategy exists, then V_0 is unique since two replicating strategies with different initial capital could lead to an arbitrage.

Definition 1.3.4.
A financial market is said to be ***complete*** *if any contingent claim C admits a replicating portfolio, i.e. for any contingent claim C there exists a self-financing strategy (H_t) such that (1.3.3) holds.*

If (1.3.3) holds, it also holds $\mathbb{Q}$-a.s. for any equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$. The discounted value then verifies

$$\widetilde{C} = V_0 + \int_0^T H_t d\widetilde{S}_t \quad \mathbb{Q}\text{- a.s. .} \tag{1.3.4}$$

Taking expectations with respect to $\mathbb{Q}$ and assuming that $(\int_0^t H_u d\widetilde{S}_u)_{t\in[0,T]}$ is a martingale (for example, (H_t) is bounded), we obtain

$$\mathbb{E}^{\mathbb{Q}}[\widetilde{C}] = V_0 \tag{1.3.5}$$

The relation (1.3.5) is sometimes called a *risk-neutral pricing formula*: the initial capital of hedging strategy is given by the discounted expectation of pay-off under $\mathbb{Q}$. Since this is true for any equivalent martingale measure $\mathbb{Q}$, we conclude that in a complete market there is only one way to define the value of a contingent claim: the value of any contingent claim is given by the initial capital to get up a perfect hedge for C.

Let us now give the second result of the theory of pricing and hedging by NA, sometimes called: the *second Fundamental Theorem of Asset Pricing.*

Theorem 1.3.2 (Second Fundamental Theorem of Asset Pricing).
A financial market defined by $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathcal{P})$ *and asset prices* $(S_t)_{t\in[0,T]}$ *is complete if and only if there is a unique martingale measure* $\mathbb{Q}$ *equivalent to* $\mathbb{P}$.

This theorem establishes an equivalence between the financial notion of market completeness and the uniqueness of equivalent martingale measure. A rigourous formulation of this theorem can be found in Delbaen and Schachermayer (1998).

1.3.4. Minimal Entropy Martingale Measure

In recent years, there has been many results in the area of characterization of martingales measures in incomplete markets. The mainstream of this research is concerned with the projection-based methods in which one looks at the "closest" (in some sense) martingale measure to the physical or real world probability measure relative. For example, Föllmer and Sondermann (1986), Föllmer and Schweizer (1991) and Schweizer (1991,1996) use quadratic or L^2-distance; Chan (1999), Miyahara (1999) and Frittelli (2000) use relative entropy which leads to the so-called *Minimal Entropy Martingale Measure* (MEMM).

Many arguments play in favor of the MEMM. Firstly, due to the fact that it comes from the minimization of relative entropy with respect to the real probability measure it retains every information we know about the randomness structure underlying the market thus it is consistent with the efficient market hypothesis. Secondly, the well-known duality relationship [See Fritelli (2000) and references therein] between minimization of the relative entropy and maximization of exponential utility makes the minimal entropy martingale measure economically meaningful. Thirdly, if the minimal entropy martingale exists, it is always equivalent to the objective probability measure unlike some other martingale measures such as the minimal variance martingale measure which may not be equivalent to the objective probability measure. Many authors have studied this equivalent martingale measure in different contexts. For example, Chan (1999) has studied the problem of pricing contingent claims in a Lévy model and provided a solution based on the MEMM. Frittelli (2000) has looked at the problem of existence and uniqueness of the MEMM in a general incomplete market model and he has provided its economical interpretation in terms of exponential utility functions. Miyahara (1999), Fujiwara and Miyahara (2003) have obtained some results on the characterization of the MEMM in the geometric Lévy models. Fujiwara (2009) has extended these results to the case where the geometric Lévy process is replaced by an exponential additive process. Recently, Momeya and Ben Salah (2012) have extended the result of Fujiwara (2009) to a general Markov-modulated exponential Lévy model whose main feature is the presence of a modulator factor which changes the characteristic of the dynamics of the risky asset under different regimes. In particular, they characterized the MEMM generated by an exponential Lévy model and an exponential additive model; then they discussed it for a Markov-modulated exponential Lévy model.

In this section, we introduce the notion of *relative entropy* which is often used as measure of proximity of two equivalent probability measures. In particular, we recall its definition and give the characterization of the MEMM generated by an exponential Lévy model and an exponential additive model; we then discuss it for a Markov-modulated exponential Lévy model.

Recall that $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space and let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$. We denote by $\mathcal{P}$ the set of probability measures on $(\Omega, \mathcal{G})$.

Definition 1.3.5. *For $\mathbb{Q} \in \mathcal{P}(\Omega, \mathcal{G})$, the* ***relative entropy*** *of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined as:*

$$\mathbb{H}_{\mathcal{G}}(\mathbb{Q}, \mathbb{P}) := \begin{cases} \mathbb{E}^{\mathbb{Q}}\left[\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}}\right)\right] & \text{if} \quad \mathbb{Q} << \mathbb{P} \quad \text{on} \quad \mathcal{G} \\ +\infty & \text{otherwise} \end{cases} \tag{1.3.6}$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{G}}$ denotes the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{G}$. Moreover, it verifies the following

- $\mathbb{H}_{\mathcal{G}}(\mathbb{Q}, \mathbb{P}) \geq 0$ *and* $\mathbb{H}_{\mathcal{G}}(\mathbb{Q}, \mathbb{P}) = 0$ *if and only if* $\mathbb{Q} = \mathbb{P}$*;*
- *The functional* $\mathbb{Q} \longmapsto \mathbb{H}_{\mathcal{G}}(\mathbb{Q}, \mathbb{P})$ *is strictly convex.*

Definition 1.3.6. *The* minimal entropy martingale measure *(MEMM) is a probability measure $\mathbb{Q}^\star \in \mathcal{M}^e(\widetilde{S})$ such that*

$$\mathbb{H}_{\mathcal{G}_T}(\mathbb{Q}^\star, \mathbb{P}) = \min_{\mathbb{Q} \in \mathcal{M}^e(\widetilde{S})} \mathbb{H}_{\mathcal{G}_T}(\mathbb{Q}, \mathbb{P}). \tag{1.3.7}$$

where $\mathcal{M}^e(\widetilde{S})$ is the set of equivalent martingale measures and $\mathcal{M}^a(\widetilde{S})$ the set of local martingale measures.

If the MEMM exists, by definition it is unique. Moreover [see Fritelli 2001, Theorem 2.2], under the assumption

$$\inf_{\mathbb{Q} \in \mathcal{M}^e(\widetilde{S})} \mathbb{H}_{\mathcal{G}_T}(\mathbb{Q}, \mathbb{P}) < \infty, \tag{1.3.8}$$

it is equivalent to $\mathbb{P}$.

The MEMM has been extensively studied and generalized to various scenarios and there is now a wide range of models for which expressions of the MEMM are available. All of these models incorporate different levels of complexity into the picture. In the following, we study the MEMM for different exponential models. When it exists, we give an expression of MEMM for the exponential Lévy model and exponential additive model. We discuss the existence and the characterization of the MEMM for the Markov-modulated exponential Lévy model which is the main contribution of Chapter 4 [Momeya and Ben Salah (2012)].

1.3.4.1. *Exponential Lévy model*

Let $S = (S_t)_{t\in[0,T]}$, be a geometric Lévy process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is, S is a stochastic process of the form

$$S_t = e^{Y_t} \tag{1.3.9}$$

where $S_0 > 0$ is a constant and $Y = (Y_t)_{t\in[0,T]}$ is a one-dimensional Lévy process with $Y_0 = 0$. Let (σ^2, ν, b) be the characteristic triplet of Y. By Lévy-Itô decomposition, Y has the representation:

$$Y_t = \sigma W_t + bt + \int_{(0,t]}\int_{\{|x|\leq 1\}} x\widetilde{N}(du\, dx) + \int_{(0,t]}\int_{\{|x|>1\}} xN(du\, dx), \tag{1.3.10}$$

where (W_t) is a standard Brownian motion, $N(du\, dx)$ is the Poisson random measure and $\widetilde{N}(du\, dx)$ is the compensated measure of $N(du\, dx)$ defined by

$$\widetilde{N}(du\, dx) = N(du\, dx) - \nu(dx)du. \tag{1.3.11}$$

For a detailed study of the theory of Lévy processes and that of stochastic calculus based on Poisson random measures, we refer to Sato (1999) and Jacod and Shirayev (2003).

As a financial model, we consider (S_t) as the price of a stock. For a constant $r \in \mathbb{R}$, we set $B_t = e^{rt}$ which denotes the price of a bank account with interest r.

We will show that, under the assumption (C) below, the MEMM exists for the geometric Lévy process $\widetilde{S}$. We will see also that the density process of the MEMM with respect to $\mathbb{P}$ can be represented explicitly.

Before giving a characterization of MEMM for model (1.3.9), let us state the following condition:

Condition 1.3.3. *There exists a constant $\beta^* \in \mathbb{R}$ that satisfies the following*

(i)

$$\int_{\{x>1\}} e^x e^{\beta^*(e^x-1)} \nu(dx) < \infty; \tag{1.3.12}$$

(ii)

$$b + (\frac{1}{2} + \beta^*)\sigma^2 + \int_{\{|x|\leq 1\}} ((e^x - 1)e^{\beta^*(e^x-1)} - x)\nu(dx) \\ + \int_{\{|x|>1\}} (e^x - 1)e^{\beta^*(e^x-1)} \nu(dx) = r. \tag{1.3.13}$$

Theorem 1.3.4 (Fujiwara and Miyahara (2003))**.**
Suppose that Condition 1.3.3 holds. Then the MEMM denoted by $\mathbb{P}^*$ exists and has a density process defined by:

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t} := e^{\beta^* \widehat{Y}_t - b^* t}, \tag{1.3.14}$$

where

$$\widehat{Y}_t = Y_t + \frac{1}{2}\sigma^2 t + \int_{(0,t]} \int_{\mathbb{R}\{0\}} (e^x - 1 - x) N(dudx) \tag{1.3.15}$$

and

$$b^* = \frac{\beta^*}{2}(1+\beta^*)\sigma^2 + \beta^* b + \int_{\mathbb{R}\{0\}} (e^{\beta^*(e^x-1)} - 1 - \beta^* x 1_{\{|x|\leq 1\}})\nu(dx). \tag{1.3.16}$$

The previous theorem is the main result of Fujiwara and Miyahara (2003) [Theorem 3.1], which gives the existence and the characterization of the MEMM for exponential Lévy process, and involves an explicite representation of the density process.

1.3.4.2. *Exponential additive model*

We will show in this section how the previous result (Theorem 1.3.4) on exponential Lévy processes can be extended to those on exponential additive processes. Let $Y = (Y_t)_{t\in[0,T]}$ be an additive process like those discussed in Section 1.1.3, that is, a stochastic process which has independent increments, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that additive processes are quasi-left continuous semimartingales with value 0 at time 0.

Let $(A_t, \eta(du\, dx), C_t)$ be the characteristic triplet of Y. Note that (C_t) is continuous and $\eta(\{t\}, dx) = 0$ for all $t \geq 0$, since we assume that Y is quasi-left continuous. The additive process Y have an explicit representation called the canonical decomposition:

$$Y_t = G_t + C_t + \int_{(0,t]} \int_{\{|x|\leq 1\}} x\widetilde{N}(du\ dx) + \int_{(0,t]} \int_{\{|x|>1\}} xN(du\ dx). \tag{1.3.17}$$

Here, (G_t) is a continuous local martingale with quadratic variation A_t, $N(du\ dx)$ denotes the counting measure of the jump of (Y_t)

$$N((0,t], A) = \#\{u \in (0,t]; \Delta Y_u = Y_u - Y_{u-} \in A\} \text{ for } A \in \mathcal{B}(\mathbb{R}\{0\}), \tag{1.3.18}$$

where $\mathcal{B}(\mathbb{R}\{0\})$ is the Borel σ-field on $\mathbb{R}\{0\}$. Note that $N(du\ dx)$ is a Poisson measure with the compensator (intensity measure) $\eta(du\ dx)$ and that

$$\int_{(0,t]} \int_{\mathbb{R}\{0\}} (|x|^2 \wedge 1)\eta(du\ dx) < \infty.$$

Also, $\widetilde{N}(du\ dx) := N(du\ dx) - \eta(du\ dx)$. We refer to Itô (2004) and Sato (1999) for basic concepts about additive processes and Jacod and Shiryaev (2003) for those about semimartingales.

As a typical example of additive processes, we will consider the following one, in which the characteristics are absolutely continuous with respect to Lebesgue measure du:

$$A_t = \int_0^t a_u du, \quad C_t = \int_0^t c_u du, \quad \eta((0,t], dx) = \int_0^t \nu_u du.$$

If all of a_u, c_u and ν_u are independent of u, then Y is reduced to a Lévy process.

As in Section 1.3.4.1, let $(S_t = e^{Y_t})_{t\in[0,T]}$ be an exponential additive process on the additive process Y with charateristics $(A_t, \eta(du\ dx), C_t)$.We assume that S_0 is a positive constant. We denote by (S_t) the price process of a risky asset. As the price of a non-risky asset, we consider $B_t = e^{R_t}$, where (R_t) is a continuous function with finite variation.

Before moving to the main result, we propose the following condition 1.3.5 for $\widetilde{S}$, which is described by the characteristic triplet $(A_t, n(dudx), C_t)$ of the additive process (Y_t) of (1.3.17) and (R_t).

Condition 1.3.5. *There exists a càglàd function $\beta^*_u, u \in (0,T]$, that satisfies the following conditions:*

(i)

$$\int_{(0,T]} \int_{\{x>1\}} e^x e^{\beta^*_u(e^x-1)} \eta(du\ dx) < \infty; \tag{1.3.19}$$

(ii)

$$C_t + \int_{(0,t]} (\frac{1}{2} + \beta_u^*) dA_u + \int_{(0,t]} \int_{\mathbb{R}\{0\}} [(e^x - 1)e^{\beta_u^*(e^x-1)} - x_{\{|x|\leq 1\}}]\eta(du\ dx) + = R_t. \tag{1.3.20}$$

Theorem 1.3.6 (Fujiwara (2009))**.**

Suppose that Condition 1.3.5 holds. Then the MEMM denoted by $\mathbb{P}^*$ *exists and has a density process defined by:*

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t} := \exp\Big[\int_{(0,t]} \beta_u^* d\widehat{Y}_t - \widehat{K}_t(\beta_u^*)\Big], \tag{1.3.21}$$

where

$$\widehat{Y}_t = G_t + \widehat{C}_t + \int_{(0,t]} \int_{\{|x|\leq 1\}} (e^x - 1)\widetilde{N}(du\ dx) + \int_{(0,t]} \int_{\{|x|>1\}} (e^x - 1)N(du\ dx) \tag{1.3.22}$$

with

$$\widehat{C}_t = \frac{1}{2}A_t + C_t + \int_{(0,t]} \int_{\{|x|\leq 1\}} (e^x - 1 - x)\eta(du\ dx), \tag{1.3.23}$$

and

$$\begin{aligned}\widehat{K}_t(\beta_u^*) &= \frac{1}{2}\int_{(0,t]} (\beta_u^*)^2 dA_u + \frac{1}{2}\int_{(0,t]} \beta_u^* dA_u + \int_{(0,t]} \beta_u^* dB_u \\ &+ \int_{(0,t]} \int_{\mathbb{R}\{0\}} \big[e^{\beta_u^*(e^x-1)} - 1 - \beta_u^* x 1_{\{|x|\leq 1\}}\big] n(dudx).\end{aligned} \tag{1.3.24}$$

The previous theorem is the main result of Fujiwara (2009) [Theorem 3.1], which gives the existence and a characterization of the MEMM for an exponential additive process.

1.3.4.3. *Markov-modulated exponential Lévy model*

In this section, we will present a general model which can be viewed as an extension of the exponential-additive model described in Section 1.3.4.2, the Markov-modulated exponential Lévy model. As in the previous section, we consider a financial market with two primary securities, the money market account B and the stock price S which are traded continuously over the time horizon $[0, T]$. Furthermore, we will add to this set-up a filtration which specifies the flow of information available to the investors.

Let $J := \{J_t : t \in [0, T]\}$ denote an irreducible homogeneous continuous-time Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite state space $E = \{e_1, e_2, ..., e_M\} \subset \mathbb{R}^M$ and characterized by a rate (or intensity) matrix $Q := \{q_{ij} : 1 \leq i, j \leq M\}$. The entry q_{ij} of matrix Q represents the transition rate at which the process J jumps from state i to state j [See Example 1.1.3.4 of Section 1.1.3]. We can identify $\mathbb{S}$ with the basis set of the linear space $\mathbb{R}^M$ and we set $e_i = \mathbf{e}_i := (0, 0, ..., \underbrace{1}_{i-th}, ..., 0)$.

Let r_t denote the instantaneous interest rate of the money market account B at time t. We suppose that $r_t := r(t, J_t) = \langle \underline{r}, J_t \rangle$ where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^M$ and $\underline{r} = (r_1, r_2, ..., r_M) \in {\mathbb{R}_+}^M$. The price dynamics of B is given by

$$B_t = B_0 \exp\Big(\int_0^t r_s ds\Big), \ B_0 = 1; \quad t \in [0, T]. \tag{1.3.25}$$

Let μ_t and σ_t denote the appreciation rate and the volatility of the stock S at time t; we suppose that

$$\mu_t = \langle \underline{\mu}, J_t \rangle, \quad \sigma_t = \langle \underline{\sigma}, J_t \rangle,$$

where $\underline{\mu} = (\mu_1, \mu_2, ..., \mu_M) \in \mathbb{R}^M$ and $\underline{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_M) \in {\mathbb{R}_+}^M$.

The stock price process S is described by the following Markov modulated exponential Lévy process

$$S_t = S_0 \exp(Y_t), \quad S_0 > 0, \tag{1.3.26}$$

with

$$\begin{aligned} Y_t = \int_0^t \Big(\mu_s - \frac{1}{2}\sigma_s^2\Big) ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} x \widetilde{N}^X(ds, dx) \\ - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^x - 1 - x\mathbb{I}_{|x|<1}) \rho^X(dx) ds. \end{aligned} \tag{1.3.27}$$

In expression (4.2.3), we have defined

$$\widetilde{N}^J(dt, dx) := \begin{cases} N^J(dt, dx) - \rho^X(dz)dt & \text{if } |x| < 1, \\ N^J(dt, dx) & \text{if } |x| \geq 1, \end{cases} \tag{1.3.28}$$

with $N^J(dt, dx)$ denoting the differential form of a Markov-modulated random measure on $[0, T] \times \mathbb{R}\setminus\{0\}$. We recall from Elliott and Osakwe (2006) and Elliott and Royal (2006) that a Markov-modulated random measure on $\mathcal{T} \times \mathbb{R}\setminus\{0\}$ is a family $\{N^J(dt, dx; \omega) : \omega \in \Omega\}$ of non-negative measures on the measurable space $([0, T] \times \mathbb{R}\setminus\{0\}, \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}\setminus\{0\}))$, which satisfy $N^J(\{0\}, \mathbb{R}\setminus\{0\}; \omega) = 0$ and has the following compensator, or dual predictable projection

$$\rho^J(dx)dt := \sum_{i=1}^M \langle J_{t^-}, e_i \rangle \rho_i(dx) dt. \tag{1.3.29}$$

$\rho_i(dx)$ is the Lévy measure for the jump size when the Markov chain X is in state $\mathbf{e}_i$, i.e a σ-finite Borel measure on $\mathbb{R}\backslash\{0\}$ with the property

$$\int_{\mathbb{R}\backslash\{0\}} \min(1, x^2) < \infty. \tag{1.3.30}$$

Let $W := (W_t)_{t\in[0,T]}$ denote the standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ which is supposed to be independent of J and N^J. However, we will assume that a switch of J from state $\mathbf{e}_j$ to state $\mathbf{e}_k$ and a jump of Y do not happen simultaneously, a.s. This assumption is made to simplify the model structure. Otherwise, one should specify the nature (and distribution) of jumps of Y which are concomitant with those of the Markov chain J and this would increase the number of calculations.

This model (Markov-modulated exponential Lévy model) is characterized by the presence of a background process modulating the risky asset price movements between different regimes or market environments. This allows to stress the strong dependence of financial assets price with structural changes in the market conditions.

The existence and characterization of the MEMM for this model will be discussed in chapter 4 [Momeya and Ben Salah (2012)]. However, we will see how the previous results on exponential Lévy processes and exponential additive processes of Fujiwara and Miyahara (2003) and Fujiwara (2009) can be extended to those on Markov-modulated exponential Lévy processes. In particular, we will give an expression, when it exists, for the MEMM for a general Markov-modulated exponential Lévy model which minimizes effectively the unconditional relative entropy. The starting point of our approach is the fact that given a MAP, it is always possible to decompose the Radon-Nikodym derivative relative to an equivalent measure as a product of two terms depending of the MAP. This enables us to work in an exponential additive setting and hence to use the result of Fujiwara (2009).

Chapter 2

LÉVY SYSTEMS AND THE TIME VALUE OF RUIN FOR MARKOV ADDITIVE PROCESSES

Abstract

In this chapter we study the ruin problem for an insurance risk processes driven by a spectrally-positive Markov additive process. Particular attention is given to the family of spectrally-positive Markov-modulated Lévy processes. We give an expression for the expected discounted penalty function by extending results available in the literature. In particular, we generalize some results in Biffis and Kyprianou (2010) to a more general setting provided by the theory of Markov additive processes. This natural extension is possible thanks to the concept of Lévy systems that allows us to generalize well-known results for Lévy processes to a larger family of Markov additive processes. We also discuss how more compact expressions for the expected discounted penalty function can be obtained using the notion of scale matrix of a Markov additive process.
Keywords: Lévy systems, scale matrices, ruin problem, spectrally-positive Markov additive processes, first-passage time, Gerber-Shiu function.

2.1. Introduction

In a now classical paper, Gerber and Shiu (1998) introduced the concept of Expected Discounted Penalty Function (EDPF). This so-called Gerber-Shiu function is a functional of the ruin time (i.e., the first time the reserve level of a firm becomes negative), the surplus prior to ruin, and the deficit at ruin. The EDPF has been extensively studied and generalized to various scenarios and there is now a wide range of models for which expressions of the EDPF are available. All of these models incorporate different levels of complexity into the picture. In this paper, we study the ruin problem for Markov additive risk models. The family of Markov additive processes have been extensively studied in the context of financial applications [see Momeya and Ben Salah (2012) and references therein]. In insurance, we find that Markov additive risk models have been recently studied in Kyprianou and Palmowski (2008) and Breuer (2008, 2010) although not at the same level of generality that we aim at in this paper. In particular, we characterize the joint distribution of surplus and deficit at ruin, and the expected discounted penalty function when the insurance risk process is driven by a Markov additive process (MAP). A key concept that we use is that of Lévy systems [see Çinlar (1975) and Maisonneuve (1977)] and, indeed one of the contributions of this paper is to identify the Lévy system of a particular case when the MAP is a Markov-modulated Lévy process. A second concept that is key to our analysis is that of a scale matrix [Kyprianou and Palmowski (2008) and Ivanovs and Palmowski (2011)] that generalizes the well-studied notion of scale function for spectrally-negative Lévy processes. These two elements allow us to give a characterization of the discounted penalty function that generalizes the results developed in Biffis and Kyprianou (2010) and Biffis and Morales (2010) for spectrally-negative Lévy insurance risk processes. Moreover, our approach provides a connection with some of the concepts introduced in Kyprianou and Palmowski (2008) where they have partial answers to the same problems discussed in this paper. We also generalize the results in Breuer (2010) where a similar problem is solved for Markov-additive processes with phase-type jumps. More explicitly, our expressions for the EDPF are given in terms not only of the Lévy system of the process but also in terms of the so-called q-potential measure of the risk process killed at exit.

The main object of study of this paper is a generalized Cramer-Lundberg model of the form,

$$R_t := x - Y_t \ , \qquad t \geq 0 \ , \tag{2.1.1}$$

where $x \geq 0$ is the initial surplus and $Y = \{Y_t; t \in \mathcal{T}\}$ is a suitable process modeling the net aggregate claims (see Asmussen and Albrecher (2010) for a thorough discussion on the ruin problem). Associated with this model we find the so-called Expected Discounted Penalty Function which is an object, originally introduced in Gerber and Shiu (1998), containing relevant information regarding how ruin occurs. If we let τ_x denote the first passage time of R below zero when $R_0 = x$, then we can write out the

EDPF as follows,

$$\phi(x;w;q) := \mathbb{E}\left[e^{-q\tau_x} w\left(R_{\tau_x -}, -R_{\tau_x}\right)\mathbb{I}_{\{\tau_x<\infty\}}|R_0 = x\right] , \tag{2.1.2}$$

where $q \geq 0$ and w is a bounded measurable function on $\mathbb{R}_+^2$ satisfying $w(0,0) = w_0 > 0$. In Gerber and Shiu (1998), the authors studied the ruin event in the compound Poisson case by analyzing the joint law of τ_x, the deficit at ruin, $-R_{\tau_x}$, and the surplus before ruin, $R_{\tau_x -}$ in one single object, the EDPF in (2.1.2). Indeed, other than the ruin time τ_x, we have at least two other quantities that characterize the first downward passage of R below zero and that contain relevant information on the ruin event from a risk management perspective, namely:

- the deficit at ruin, $-R_{\tau_x}$;
- the surplus immediately prior to ruin, $R_{\tau_x -}$.

The joint law of the above random variables has been studied in Doney and Kyprianou (2006) as part of the so-called quintuple law at first passage. We refer to Kyprianou (2006) for additional details on this result and related definitions.

All of these quantities encapsulate relevant knowledge about the ruin event. Clearly, a risk manager would be interested in gaining information regarding the probability of ruin $\mathbb{P}(\tau_x < \infty)$ which gives information on how likely the reserve is to face all claims. Even more valuable information can be found in the distribution of the deficit at ruin (depleting the risk reserve by a few dollars rather than by a couple of millions have certainly different implications for an insurance company). Information about the distribution of the reserve level just prior to ruin sheds light on the structure of the paths leading to ruin.

Following the same order of ideas, we study the EDPF under a general model of the form (2.1.1) where the net aggregate claims process is a Markov additive process. This more general setting allows for the introduction of long-term market conditions that change over time following a finite-state continuous-time Markov process modeling different environment scenarios.

The paper is organized as follows. In Section 2.2, we introduce the basic notions related with Lévy systems and MAP's. We believe this somewhat lengthy section is to a certain extent a secondary contribution of this paper. Indeed, to the best of our knowledge, it is the first time that these concepts have been brought forward to study the ruin problem. In Section 2.3, we give the definition of our model and present the main results. In particular, we define a risk process driven by a spectrally-negative MAP and we introduce a family of expected discounted penalty functions that generalize the original notion introduced in Gerber and Shiu (1998). Moreover, we provide an explicit characterization for these functions in terms of Lévy systems and q-potential measures. Indeed, one of our contributions is to identify the Lévy system for a spectrally-negative Markov risk process which in turn allows us to give expressions for the EDPF. The expressions that we develop are rather general and not fully explicit. Thus, Section 2.4 discusses how more compact expressions can be worked out. Indeed, we investigate in more detail these expressions for the EDPF which, in fact, boils down to identifying the q-potential measure associated with the process using the concept of scale matrix introduced in Kyprianou and Palmowski (2008) and Ivanovs and Palmowski (2011). A final section summarizes the conclusions.

2.2. Preliminaries

In this section, we give a brief overview of the main notions and results of the theory of *Markov additive processes* (MAP). This large class of processes give the mathematical framework for our discussion. In particular, we define these processes and give some of their fundamental properties used throughout this paper. At this point, we need to lay down the standard notation that is used in this section. Following the notation in Çinlar (1975), let $(\mathbb{F}, \mathcal{F})$ and $(\mathbb{G}, \mathcal{G})$ be two measurable spaces then we write $f \in \mathcal{F}/\mathcal{G}$ to mean that f is a mapping from $\mathbb{F}$ into $\mathbb{G}$ which is measurable with respect to $\mathcal{F}$ and $\mathcal{G}$. By transition kernel N from $(\mathbb{F}, \mathcal{F})$ into $(\mathbb{G}, \mathcal{G})$ we mean a mapping $N : \mathbb{F} \times \mathcal{G} \to \overline{\mathbb{R}_+}$ such that the mapping $A \to N(x, A)$ is a σ-finite measure on $\mathcal{G}$ for each fixed $x \in \mathbb{F}$ and that $x \to N(x, A)$ is in $\mathcal{F}/\mathcal{B}(\mathbb{R}_+)$ for each fixed $A \in \mathcal{G}$. Throughout this section we set $\mathcal{T} := [0, T]$ with $T \in (0, \infty]$.

2.2.1. Markov additive processes and Lévy systems

The mathematical theory of Markov additive processes can be traced back to the works of Çinlar (1972a), (1972b) and Grigelionis (1978). A cornerstone of this theory is found in Çinlar (1975) where the author establishes a result characterizing the Lévy system for Markov additive processes. As it turns out, a Lévy system gives all the information about the jump properties of strong Markov processes. Indeed, Benveniste and Jacod (1973) had shown the existence of a Lévy system for any quasi-left continuous strong Markov process. That is, let $\{(Z_t) : t \in \mathcal{T}\}$ be a strong Markov process which is quasi-left continuous with respect to a right-continuous filtration $\mathbf{F} := \{\mathcal{F}_t : t \in \mathcal{T}\}$

on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $Z = \{Z_t : t \in \mathcal{T}\}$ takes its values in a measurable space $(\mathbb{S}, \mathcal{S})$. Then there exists, both a transition kernel K into itself having $K(y, \{y\}) = 0$ for all $y \in \mathbb{S}$ and an (increasing) continuous additive functional H of Z, such that for any non-negative Borel measurable function f on $\mathbb{S} \times \mathbb{S}$,

$$\mathbb{E}\Big[\sum_{s \le t} f(Z_{s-}, Z_s)\, 1_{\{Z_{s-} \neq Z_s\}}\Big] = \mathbb{E}\Big[\int_0^t dH_s \int_{\mathbb{S}} K(Z_s, dy) f(Z_s, y)\Big] , \tag{2.2.1}$$

for all $t \ge 0$.

The kernel K is called a Lévy kernel and the pair (H, K) is called a Lévy system for Z. Intuitively, when time is reckoned according to the random clock H, so that the clock reads H_t when the time is t, then $K(y, A)$ can be seen as the expected number per unit time of the jumps Z makes from y into $A \in \mathbb{S}$. The name Lévy system is motivated by the fact that if Z is a process with independent and stationary increments then H_t can be taken to be equal to t and $K(z, dy) = \nu(dy - z)$ where ν is the jump measure of the process Z [see Çinlar (1975) and Maisonneuve (1977) for a detailed discussion].

In Çinlar (1975), we find a result that characterizes a Lévy system for a Markov additive process. This will be crucial to our discussion and so we include here a brief presentation of the family of Markov additive processes.

Intuitively, a Markov additive process is a bivariate process (J, X) such that J is a Markov process and X is a process with conditionally independent increments given the paths of J. This is formally stated as follows [see Grigelionis (1978)].

Definition 2.2.1. *Consider a bivariate stochastic process $\{(J_t, X_t) : t \in \mathcal{T}\}$ adapted to a right-continuous filtration $\mathbf{F} := \{\mathcal{F}_t : t \in \mathcal{T}\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This pair is such that the component $X = \{X_t : t \in \mathcal{T}\}$ takes its values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and is continuous from the right with limits on the left; whereas the component $J = \{J_t : t \in \mathcal{T}\}$ takes its values in a measurable space $(\mathbb{S}, \mathcal{S})$. This pair is called a Markov additive process (or MAP) with respect to $\mathbf{F}$ if for any Borel set $A \in \mathcal{B}(\mathbb{R})$, $B \in \mathcal{S}$ and $0 \le s < t \le T$ we have,*

$$\mathbb{P}\{X_t - X_s \in A, J_t \in B | \mathcal{F}_s\} = \mathbb{P}\{X_t - X_s \in A, J_t \in B | J_s\} , \quad a.s. \tag{2.2.2}$$

The concept of Lévy systems plays a fundamental role in studying the jump structure of a MAP. This notion is intimately related to the infinitesimal generator of such processes [see Çinlar (1975)]. In this paper, we only make use of a result that guarantees the existence of a Lévy system for a MAP.

We now state the following result [Çinlar (1975)] that will be key in our analysis [see also Maisonneuve (1977)].

Theorem 2.2.1. *Let $\{(J_t, X_t) : t \in \mathcal{T}\}$ be a Markov additive process with J having a measurable state space $\mathbb{S}$ and X, a quasi-left continuous process, taking values in the space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. There exists a continuous increasing functional of J and a transition kernel from $(\mathbb{S}, \mathcal{S})$ into $(\mathbb{S} \times \mathbb{R}_+, \mathcal{S} \times \mathcal{B}(\mathbb{R}))$ such that, for any non-negative function $f \in \mathcal{S} \times \mathcal{S} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$,*

$$\begin{aligned}&\mathbb{E}_i\Big[\sum_{s\le T} f(J_{s^-}, J_s, X_{s^-}, X_s)1_{\{J_{t^-}\neq J_t\}\cup\{X_{t^-}\neq X_{t^-}\}}\Big]\\&= \mathbb{E}_i\Big[\int_0^T dH_s \int_{\mathbb{S}\times\overline{\mathbb{R}}} K(J_s, dz, du) f(J_s, z, X_s, X_s+u)\Big],\end{aligned}$$

for all $i \in \mathbb{S}$.

The couple (H, K) is said to be a Lévy system for (J, X). The kernel K is often referred to as the Lévy kernel of (J, X) and for any $i \in \mathbb{S}$, it satisfies

$$K(i, \{(i, 0)\}) = 0, \tag{2.2.3}$$

$$\int_{\mathbb{R}} K(i, \{i\} \times dy)(|y|^2 \wedge 1) < \infty. \tag{2.2.4}$$

Moreover, if we set

$$\Lambda(i, B) = K(i, (B\{i\} \times \mathbb{R}))\ , \quad B \in \mathbb{S}, \tag{2.2.5}$$

then (H, Λ) is a Lévy system for J; that is

$$\mathbb{E}_i\Big[\sum_{s\le T} g(J_{s^-}, J_s)1_{\{J_{s^-}\neq J_s\}}\Big] = \mathbb{E}_i\Big[\int_0^T dH_s \int_{\mathbb{S}} \Lambda(J_s, dy) g(J_s, y)\Big]\ , \tag{2.2.6}$$

for any non-negative Borel measurable function g defined on $\mathbb{S} \times \mathbb{S}$.

Remark 2.2.1. *Since J is a regular step process with a discrete state space $\mathbb{S}$, then H_t can be taken to be $H_t = t$ [Çinlar (1975)].*

In this paper we consider only Markov additive processes (MAP) with a J component taking values within a finite state space $\mathbb{S} = \{s_1, ..., s_N\} \subset \mathbb{R}^N$. We further assume that X is quasi-left-continuous with respect to $\mathbf{F} := \{\mathcal{F}_t : t \in \mathcal{T}\}$ and satisfies $X_0 = 0$. In such a case, we set H_t to be equal to t and the defining property (2.2.2) of a MAP can be written in a different form. We can say that $\{(J_t, X_t) : t \in \mathcal{T}\}$ is a MAP (with a finite state space) if [see Asmussen (2003)],

$$\mathbb{E}_{i,0}\Big[f(X_{t+s} - X_t)\ g(J_{t+s})\Big|\mathcal{F}_t\Big] = \mathbb{E}_{J_t,0}\Big[f(X_s)g(J_s)\Big]\ , \tag{2.2.7}$$

for any two bounded $\mathcal{F}$-measurable functions f, g. Here, $\mathbb{E}_{i,0}(.)$ denotes the expectation under the probability $\mathbb{P}_{i,0}(.) = \mathbb{P}(.|J_0 = i, X_0 = 0)$. Equation (4.3.35) makes it clear why we refer to a MAP as having conditionally independent increments.

2.2.2. Spectrally-positive Markov additive process

The class of Markov additive processes as given in Definition 2.2.1 is very large. In this paper we focus on a subclass of this family often referred to as Markov-modulated Lévy processes. In this subclass, the pair (J, Y) is such that J is a finite-state Markov chain and Y is a process with conditionally independent and stationary increments. In other words, conditionally on $J_t = i$ (some i in the state space of J), the process Y behaves in law like a Lévy process. In the rest of the paper we work with the subclass of spectrally positive Markov-modulated Lévy processes (J, Y) for which now give a more detailed structure. Following the notation in Asmussen (2003), consider a MAP $\{(J_t, Y_t)\,; t \in \mathcal{T}\}$ as defined in equation (4.3.35), i.e. a quasi-left continuous process with respect to its right-continuous filtration $\mathbf{F} := \{\mathcal{F}_t : t \in \mathcal{T}\}$ and with finite-state space $\mathbb{S}$. We say that $Y = \{Y_t; t \in \mathcal{T}\}$ $(Y_0 = 0)$ is a right continuous spectrally positive Markov-modulated Lévy process with modulating process $J = \{J_t; t \in \mathcal{T}\}$ if,

- J_t is a right-continuous, ergodic, finite-space, continuous-time Markov chain, with intensity matrix $Q = (q_{i,j})_{i,j\in\mathbb{S}}$ and stationary distribution $\pi = (\pi_1, \ldots, \pi_N)$,
- the process Y can be written as the sum

$$Y_t = Y_t^{(1)} + Y_t^{(2)} \,, \tag{2.2.8}$$

  where the independent processes $Y_t^{(1)}$ and $Y_t^{(2)}$ are defined below.

Let us specify the first term in (2.2.8). For each $i \in \mathbb{S}$, let $Y_t^{\{i\}}$ be a spectrally positive Lévy process with triplet (a_i, σ_i^2, ν_i), i.e. its Laplace exponent is given by

$$\begin{aligned}\psi^{(i)}(\alpha) &= \log \mathbb{E}(\exp(-\alpha Y_1^{\{i\}})) \\ &= -\alpha a_i + \alpha^2 \frac{\sigma_i^2}{2} + \int_{(0,\infty)} [e^{-\alpha y} - 1 + \alpha y 1_{\{0<y\leq 1\}}]\nu_i(dy) \,, \end{aligned} \tag{2.2.9}$$

where $\int_{(0,\infty)}(1 \wedge |y|^2)\nu_i(dy) < \infty$. We denote by $Y_t^{(1)}$ the process which behaves in law like $Y_t^{\{i\}}$, when $J_t = i$.

Now, as for the second term in (2.2.8), let $\{U_n^{(i,j)}\}_{i,j\in\mathbb{S}}$ be i.i.d. random variables (with $U_n^{(i,i)} = 0$) which are independent of J_t and have a distribution function $B_{i,j}(.)$ with support on $[0, \infty)$. Moreover, let us denote the jump times of J_t by $\{T_n\}_{n\in\mathbb{N}}$ (with $T_0 = 0$).

The jump process $Y^{(2)}$ is described by

$$Y_t^{(2)} = \sum_{n\geq 1}\sum_{i,j\in\mathbb{S}} U_n^{(i,j)} 1_{\{J(T_{n-1})=i, J(T_n)=j, T_n\leq t\}}. \tag{2.2.10}$$

According to the path decomposition in (2.2.8), a spectrally positive Markov-modulated Lévy process has two types of behavior. It follows a spectrally positive Lévy process $Y_t^{\{i\}}$ while J_t remains in state i; and at times when J_t jumps from state i into state j, it jumps according to a positive random variable $U^{(i,j)}$. Clearly, such a process has only positive jumps that can come either from a spectrally positive Lévy process or from the positive random variables $U^{(i,j)}$ with distribution $B_{i,j}$. We notice that the processes $Y^{(1)}$ and $Y^{(2)}$ are fully specified by the characteristics $(q_{i,j}, B_{i,j}, \sigma_i, a_i, \nu_i)_{i,j\in\mathbb{S}}$.

Moreover, $(J, Y^{(1)})$ and $(J, Y^{(2)})$ are also MAP's with Lévy kernels denoted respectively by $K^{(1)}$ and $K^{(2)}$ such that for $i,j\in\mathbb{S}$, $K^{(1)}(i,\{j\},.) = K(i,\{j\},.)1_{\{i\neq j\}}$ and $K^{(2)}(i,\{j\},.) = K(i,\{i\},.)1_{\{i=j\}}$, where K is the Lévy kernel of (J,Y) introduced in Theorem 2.2.1 [see Çinlar (1975) for details].

For a thorough description and characterization of these processes we refer to the original works of Çinlar (1975) and Chapter XI in Asmussen (2003). Notice that if $N=1$, in other words if $\mathbb{S}$ is a singleton, then Y becomes a spectrally positive Lévy process and its properties reduce to those given in Bertoin (1996) and Kyprianou (2006).

Also, at this point, we would like to make a comment regarding notation. We always write vectors in their row form and write the usual M^T to mean the transpose of the matrix M.
Let $\widehat{B}$ denote the matrix of Laplace transforms of B defined by $\widehat{B}(\alpha)$ with (i,j)-th elements given by

$$\mathbb{E}[\exp(-\alpha U_n^{(i,j)})] = \int_{(0,\infty)} e^{-\alpha y} B_{i,j}(dy).$$

We now define the matrix cumulant generating function of (J,Y) as follows,

$$F(\alpha) = Q\circ\widehat{B}(\alpha) + diag(\psi^{(1)}(\alpha), ..., \psi^{(N)}(\alpha)) \ , \quad \alpha\geq 0 \ , \tag{2.2.11}$$

where the matrix (i,j)-th element of $F(\alpha)$ satisfies

$$\mathbb{E}[\exp(-\alpha Y_t) \ ; J_t = j \ |Y_0 = 0, J_0 = i] = e^{F_{i,j}(\alpha)t} \tag{2.2.12}$$

and $Q\circ\widehat{B}(\alpha)$ is the matrix with (i,j)-th elements given by $q_{i,j}\widehat{B}_{i,j}(\alpha)$.

We make use of the fact that, by Perron-Frobenius theory [see Section I.6 and Section II.4.d in Asmussen (2003)], the matrix $F(\alpha)$ has a real eigenvalue with maximal absolute value. We label this value by $\kappa(\alpha)$ in order to indicate its dependence on α.

We denote by $v(\alpha)$ and $h(\alpha)$ respectively the left and right $1 \times N$-eigenvectors of the matrix $F(\alpha)$ corresponding to the eigenvalue $\kappa(\alpha)$. Similarly, we label with $h(\alpha)$ the corresponding right $1 \times N$-eigenvector. Notice that $h(0) = \mathbf{1}$ where $\mathbf{1}$ is a $1 \times N$ vector consisting of a row of ones.

In the following, we use a subscript to denote the i-th element of a vector, i.e. $h_i(\alpha)$ is the i-th entry of vector $h(\alpha)$.

Another well-known fact [Chapter XI in Asmussen (2003)] is that the eigenvalue $\kappa(\alpha)$ is a convex function such that $\kappa(0) = 0$, $\kappa(\infty) = \infty$. Moreover, [Chapter XI in Asmussen (2003)] $\kappa'(0) > 0$ is the asymptotic drift of the process Y in (2.2.8). In other words, for each $i \in \mathbb{S}$, we have that

$$\lim_{t\to\infty} \mathbb{E}(Y_t | J_0 = i, Y_0 = 0)t = -\kappa'(0) \; . \tag{2.2.13}$$

Indeed, the sign of $\kappa'(0)$ determines the asymptotic behavior of Y. It is clear from (2.2.13) that when $\kappa'(0) > 0$ the process Y drifts to $-\infty$ whereas when $\kappa'(0) < 0$ the process Y drifts to ∞. Equation (2.2.13) allows us to establish the net-profit condition when we use a MAP risk model like (2.1.1) as we will see in Section 2.3.

We now let Φ denote the right inverse of κ on $[0, \infty)$ i.e, for each $q \geq 0$,

$$\Phi(q) = \sup\{\alpha \geq 0 : \kappa(\alpha) = q\} \; . \tag{2.2.14}$$

The convexity properties of κ [see Asmussen (2003)] imply that $\Phi(q) > 0$ for $q > 0$. Furthermore, $\Phi(0) = 0$ if and only if $\kappa'(0) \geq 0$, whereas if $\kappa'(0) < 0$ we have that $\Phi(0) > 0$. As we will now discuss, equation (3.3.21) and the values $\kappa(\alpha)$ and $h(\alpha)$ can be used to define a change of measure which turns out to be key in our analysis.

Recall first that the MAP $\{(J_t, Y_t)\,; t \in \mathcal{T}\}$, as given in equation (2.2.2), is defined with respect to a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} := \{\mathcal{F}_t : t \in \mathcal{T}\}, \mathbb{P})$. Let us denote by $\{\mathbb{P}_{i,x} : i \in \mathbb{S}, x \in \mathbb{R}\}$ the set of conditional probabilities $\{\mathbb{P}(\cdot \mid J_t = i\,, Y_t = x) : i \in \mathbb{S}, x \in \mathbb{R}\}$. Following the notation in Asmussen (2003), we can now define a new family of conditional probabilities $\{\mathbb{P}^{\gamma}_{i,x} : i \in \mathbb{S}, x \in \mathbb{R}\}$ through the following density process,

$$\frac{d\mathbb{P}^{\gamma}_{i,x}}{d\mathbb{P}_{i,x}}\Big|_{\mathcal{F}_t} = e^{-\gamma(Y_t - x) - \kappa(\gamma)t} \frac{h_{J_t}(\gamma)}{h_i(\gamma)}, \tag{2.2.15}$$

for γ such that $\kappa(\gamma) < \infty$. Recall that $\kappa(\gamma)$ is the largest eigenvalue of the matrix cumulant generating function $F(\gamma)$ in (2.2.11) and $h(\gamma)$ is the corresponding eigenvector.

Notice that in the definition of the density process (2.2.15), the function $\kappa(\gamma)$ plays a similar role as the Laplace exponent of Lévy process in an exponential (or Esscher) change of measure. In fact, (2.2.15) is a generalization of a Wald martingale normalized to have mean 1 so that it can define a probability measure $\mathbb{P}_{i,x}$ [see Asmussen (2003) for more details about this change of measure and the existence of $\mathbb{P}_{i,x}$].

This type of measure change has been previously studied in Palmowski and Rolski (2002) and Kyprianou and Palmowski (2008). And as it turns out, it can be shown that under the new probability measure $\mathbb{P}^{\gamma}_{i,x}$ induced by (2.2.15), the process Y is again a spectrally positive MAP. The associated matrix cumulant generating function matrix $F^{\gamma}(\alpha)$ is well defined and finite for $\alpha \geq -\gamma$ [see Palmowski ans Rolski (2002) and Kyprianou and Palmowski (2008)]. Moreover, if $F^{\gamma}(\alpha)$ has a largest eigenvalue denoted by $\kappa^{\gamma}(\alpha)$ and its associated eigenvector is denoted by $h^{\gamma}(\alpha)$, then the triple $(F^{\gamma}(\alpha), \kappa^{\gamma}(\alpha), h^{\gamma}(\alpha))$ is related to the original triple $(F(\alpha), \kappa(\alpha), h(\alpha))$ via

$$F^{\gamma}(\alpha) = \Delta^{-1}_{h(\gamma)} F(\alpha+\gamma) \Delta_{h(\gamma)} - \kappa(\gamma)\mathbf{I} \ , \tag{2.2.16}$$

$$\kappa^{\gamma}(\alpha) = \kappa(\alpha+\gamma) - \kappa(\gamma) \ , \tag{2.2.17}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix and $\Delta_{h(\gamma)}$ is a diagonal matrix given by

$$\Delta_{h(\gamma)} := \mathrm{diag}(h_1(\gamma), ..., h_N(\gamma)) \ . \tag{2.2.18}$$

After all the relevant elements regarding the MAP have been introduced we are now in a position to discuss our contribution. But before a word about notation is in order. In the rest of this paper we work with matrix notation, i.e. for a given random variable V and a stopping time δ, we shall understand $\mathbb{E}_x(V; J_\delta)$ to be the matrix whose (i,j)-th entry is $\mathbb{E}_{i,x}(V; J_\delta = j)$. Similarly, for some $\mathcal{G} \in \mathcal{F}$, the probability $\mathbb{P}_x(\mathcal{G}; J_\delta)$ should be understood as the matrix whose (i,j)-th elements are given by $\mathbb{P}_{i,x}(\mathcal{G}; J_\delta = j)$. For simplicity we follow the usual convention that $\mathbb{E}_0(.) = \mathbb{E}(.)$ and $\mathbb{P}_0(.) = \mathbb{P}(.)$.

2.3. Main results

We consider a very general setup that generalizes the standard Cramer-Lundberg model. The model discussed in this paper is,

$$R_t := x - Y_t \ , \qquad t \geq 0 \ , \tag{2.3.1}$$

where $x \geq 0$ is the initial surplus and $Y = \{Y_t; t \in \mathcal{T}\}$ is a right-continuous spectrally positive Markov-modulated Lévy process with modulating process $J = \{J_t; t \in \mathcal{T}\}$ taking values on a finite state space $\mathbb{S}$. The process Y represents the net aggregate cash inflow of an insurance company and J is a background process that describes the environment in which claims occur. Let us denote with $Y^{(1)}$ and $Y^{(2)}$ the path components of Y, as defined in (2.2.8), and let $(q_{i,j}, B_{i,j}, \sigma_i, a_i, \nu_i)_{i,j\in\mathbb{S}}$ be the characteristics of such processes.

We assume the process Y to have a negative drift $a_i < 0$ for all $i \in \mathbb{S}$. Moreover we must impose the condition $\mathbb{E}[Y_1] < 0$ in order to avoid the possibility that R becomes negative almost surely. This condition is often expressed in terms of a safety loading. Indeed, it is standard to write the drift component within Y as a loaded premium. For instance, notice that we can recuperate the classical Cramer-Lundberg model if the state space $\mathbb{S}$ is a singleton and $Y_t = S_t - c\,t$ where $c := (1+\theta)\mathbb{E}[S_1]$ and S is a compound Poisson process modeling aggregate claims. The drift c, with a positive safety loading $\theta > 0$, is the collected premium rate. We do not use the concept of safety loading in this paper in order to simplify notation but we stress the fact that this concept is implicitly considered within the drift of Y when we impose the condition $\mathbb{E}[Y_1] < 0$.

One of the advantages of considering a general Markov-modulated Lévy risk model like (2.3.1) is that we can use the tools developed for the larger class of Markov additive processes. Indeed, the model in (2.3.1) contains the classical Cramer-Lundberg model as a particular case allowing for a more comprehensive understanding of the ruin problem. Moreover, the process in (2.3.1) models the aggregate claims as having a fundamentally different behavior over different market and environment scenarios that are described by the Markov processes J. We recall that, unlike a Lévy risk process, the Markov-modulated model in (2.3.1) is not time homogeneous and it can be used to model situations where frequency and severity of the claims are time dependent.

Now, one of the main objects of interest in ruin theory is the *ruin time*, τ_x, representing the first passage time of R below zero when $R_0 = x$, i.e.

$$\tau_x := \inf\{t > 0 \ : \ Y_t > x\} \,, \tag{2.3.2}$$

where we set $\tau_x = +\infty$ if $R_t \geq 0$ for all $t \geq 0$.

Associated with the ruin time τ_x, we have at least two other quantities that contain relevant information on the ruin event from a risk management perspective, namely the deficit at ruin $-R_{\tau_x} = Y_{\tau_x} - x$ and the surplus immediately prior ruin $R_{\tau_x -} = x - Y_{\tau_x^-}$.

In Gerber and Shiu (1998), the authors studied the ruin event in the compound Poisson case by analyzing the joint law of all these quantities in one single object, the

EDPF. Following the same order of ideas, we study the EDPF under the general model (2.3.1) which takes a matrix form in this new context. More precisely, in this paper we set out to study the following EDPF for the model (2.3.1).

Definition 2.3.1. *Let w be a non-negative Borel-measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$ such that $w(.,0) = 0$. For $q \geq 0$, the Matrix Expected Discounted Penalty Function (MEDPF) associated with the risk process (2.3.1) is defined as*

$$\phi(x; w; q) = \mathbb{E}\Big[e^{-q\tau_x} w(x - Y_{\tau_x^-}, Y_{\tau_x} - x) 1_{\{\tau_x < \infty\}}; J_{\tau_x}\Big] \,, \tag{2.3.3}$$

where we should understand $\phi(w, x; q)$ to be the matrix with (i, j)-th entry given by

$$\mathbb{E}_i\Big[e^{-q\tau_x} w(x - Y_{\tau_x^-}, Y_{\tau_x} - x) 1_{\{\tau_x < \infty\}}; J_{\tau_x} = j\Big] \,, \quad i, j \in \mathbb{S} \,.$$

Note that the condition $w(.,0) = 0$ excludes the event $\{Y_{\tau_x} = x\}$. This possibility is known as creeping and we chose not to consider it in our analysis. For simplicity, we assume that the function w assigns a zero penalty when ruin occurs by continuously crossing over zero. Notice that for a model like (2.3.1), this only happens when ruin is caused by the Brownian motion component of the process Y.

Notice that the MEDPF in (2.3.3) contains all relevant information as to how ruin occurs. Indeed, each entry of (2.3.3) gives the expected penalty function (EDPF) of the risk process (2.3.1) for those paths where the modulating Markov chain J starts from state i, at time zero and then finds itself in state j at ruin time τ_x.

By construction, the surplus process defined in (2.3.1) encompasses previously existing models, among others, when $N = 1$, (2.3.1) becomes the Lévy insurance risk process studied in Biffis and Morales (2010) and Biffis and Kyprianou (2010) and $\phi(w; x; q)$ is the standard EDPF or Gerber-Shiu function as introduced in Gerber and Shiu (1998). Just like in these cases where the risk process is driven by a spectrally positive Lévy process, the problem of giving an expression for the MEDPF (2.3.3) can be reformulated in terms of a first-exit problem for the process Y in (2.3.1). This implies that some of the standard notions of potential theory need to be extended in this new setting where Y is a MAP.

Thus, we must define the q-bivariate potential measure matrix $U_x^{(q)}$ for a spectrally positive Markov additive process (J, Y) killed on exiting $(-\infty, x]$.

Definition 2.3.2. *The q-bivariate potential measure matrix $U_x^{(q)}$ for a Markov additive process (J, Y) is a matrix whose (i, j)-th entries are given by*

$$U_x^{(q)}(i, j, dy) = \int_0^\infty e^{-qt} dt \mathbb{P}_i(Y_t \in dy, \tau_x > t \,; J_t = j), \tag{2.3.4}$$

for $q \geq 0$.

This definition mimics the concept of q-potential measure as found in potential theory of Lévy processes [Kyprianou (2006)] and it will be key in our analysis of the MEDPF (2.3.3). For the sake of consistency, we also use the convention that $U_x^{(0)} = U_x$ is the potential measure matrix of (J, Y) killed on exiting $(-\infty, x]$.

Notice that when $\mathbb{S}$ is a singleton ($N = 1$) the process (Y_t) becomes a spectrally positive Lévy process and its properties and all above definitions reduce to those given in Bertoin (1996) and Kyprianou (2006).

Now, let us define the running supremum and the running infimum of a given process Y, respectively, by

$$\overline{Y}_t := \sup_{0 \leq s \leq t} Y_s \,, \qquad \underline{Y}_t := \sup_{0st} Y_s \,.$$

Then, just like in the single state case ($N = 1$), we can write the q-potential measure (2.3.4), for any independent exponentially distributed random variable e_q with parameter $q > 0$, as

$$U_x^{(q)}(i, j, dy) = \frac{1}{q} \mathbb{P}_i(Y_{e_q} \in dy, \overline{Y}_{e_q} \leq x \,; J_{e_q} = j) \,, \tag{2.3.5}$$

where $i, j \in \mathbb{S}$ and $y \in \mathbb{R}$.

Moreover, if a density for $U_x^{(q)}(i, j, dy)$ exists with respect to Lebesgue's measure for each $x \geq 0$, then we call it the bivariate potential density and label it $u_x^{(q)}(i, j, y)$, with $u_x^{(0)} = u_x$. Note that $u_x^{(q)}(y)$ is the potential density matrix of $U_x^{(q)}(i, j, y)$ with (i, j)-th elements given by $u_x^{(q)}(i, j, y)$. When $N = 1$ and then Y is spectrally positive Lévy process, it turns out that not only does a potential density exist, but it can be written in semi-explicit terms [see Theorem 8.7. in Kyprianou (2006)]. In general, the q-potential measure $U_x^{(q)}$ cannot be readily identified but only in a few special cases.

Analogously, we define the q-bivariate potential measure matrix without killing for (J, Y) as the matrix $U^{(q)}(.)$ with (i, j)-th entries $U^{(q)}(i, j, .)$ given by

$$U^{(q)}(i, j, dy) = \int_0^\infty e^{-qt} dt \mathbb{P}_i(Y_t \in dy \,; J_t = j). \tag{2.3.6}$$

Further, we denote by $\widehat{U^{(q)}}(\beta)$ the Laplace transform of $U^{(q)}$, given by

$$\widehat{U^{(q)}}(\beta) = \int_{\mathbb{R}} e^{-\beta y} U^{(q)}(dy), \tag{2.3.7}$$

where (2.3.7) must be understood as the matrix whose (i, j)-th entries are given by

$$\int_{\mathbb{R}} e^{-\beta y} U^{(q)}(i, j, dy) \,,$$

for $\beta \geq 0$.

We can identify the q-bivariate potential measure matrix via its Laplace transformation $\widehat{U^{(q)}}(\beta)$. Indeed, with the help of Fubini's theorem we have that

$$\int_{\mathbb{R}} e^{-\beta y} U^{(q)}(dy) = \int_0^\infty e^{-qt} dt \mathbb{E}[\exp(-\beta Y_t); J_t] = \int_0^\infty e^{-qt} e^{F(\beta)t} dt, \tag{2.3.8}$$

where F is the matrix cumulant generating function (2.2.11) of the net aggregate claims process (J, Y).

By the previous equality, we note that $\widehat{U^{(q)}}(\beta)$ is a resolvent [see Theorem 1.43 in El-Maati (2005)] and hence equal to

$$\widehat{U^{(q)}}(\beta) = (qI - F(\beta))^{-1} , \tag{2.3.9}$$

where $\beta < \phi(q)$. This gives a characterization of $U^{(q)}$ in terms of its Laplace transform. Notice that an analogous characterization is not available for the q-potential measure killed at exit $U_x^{(q)}$.

The model in (2.3.1) is a MAP and all the results from the previous section apply. This allows us to give a more detail analysis of the associated MEDPF (2.3.3). As we discuss in a previous section, the Lévy system of a MAP contains all relevant information about how jumps occur which, as it turns out, is a key element in our study of (2.3.3). Another key element is the so-called potential measure (2.3.4) of a MAP killed at exit. In the following sub-section, we give a preliminary result that specifies the Lévy system for the net aggregate claim process Y in (2.3.1). This will allow us to finally write an expression for the MEDPF (2.3.3) in terms of the Lévy system and the q-potential measure killed at exit of the net aggregate process Y in (2.3.1).

2.3.1. Lévy system for a MAP risk process

In the following theorem, we shall characterize the transition kernel L for the spectrally positive Markov additive process (J, Y) representing the aggregate claims in (2.3.1). This is the first main result of this paper.

Theorem 2.3.1. *Consider the spectrally positive Markov additive process (J, Y) given in the definition of the risk process (2.3.1) and let $(q_{i,j}, B_{i,j}, \sigma_i, a_i, \nu_i)_{i,j\in\mathbb{S}}$ be the characteristics of such a process. Then the following transition kernel L, from $(\mathbb{S}, \mathcal{S})$ into $(\mathbb{S} \times \mathbb{R}_+, \mathcal{S} \times \mathcal{B}(\mathbb{R}))$,*

$$L(i, \{j\}, du) = \nu_i(du) 1_{\{i=j\}} + q_{i,j} B_{i,j}(du) 1_{\{i\neq j\}} , \qquad i, j \in \mathbb{S} , \ \text{ and } u \in (0, \infty) , \tag{2.3.10}$$

is the Lévy kernel of (J, Y) *in the sense of Theorem 2.2.1.*

PROOF. Since (J, Y) is a MAP, by Theorem 2.2.1, we have the existence of a Lévy kernel from $(\mathbb{S}, \mathcal{S})$ into $(\mathbb{S} \times \mathbb{R}_+, \mathcal{S} \times \mathcal{B}(\mathbb{R}))$. Now, a Lévy kernel only takes into account the information contained in the jump sizes of Y and so, in order to identify it, we only need to verify that (2.3.10) is such that

$$\mathbb{E}_i\Big[\sum_{0<s\leq T} f(J_{s^-}, J_s, Y_s - Y_{s-})1_{\{J_{t^-}\neq J_t\}\cup\{Y_{t-}\neq Y_t\}}\Big]$$
$$= \mathbb{E}_i\Big[\int_0^T ds \int_{\mathbb{S}\times(0,\infty)} L(J_s, dz, du) f(J_s, z, u)\Big]\,, \quad i \in \mathbb{S}\,, \tag{2.3.11}$$

for any given function $f \in \mathcal{S} \times \mathcal{S} \times \mathcal{B}(\mathbb{R}_+)/\mathcal{B}(\mathbb{R})$.

Since $\mathbb{S}$ is finite, (2.3.11) is equivalent to

$$\mathbb{E}_i\Big[\int_0^T ds \sum_{j\in\mathbb{S}} \int_{(0,\infty)} L(J_s, \{j\}, du) f(J_s, z, u)\Big]$$
$$= \mathbb{E}_i\Big[\int_0^T ds \sum_{j\in\mathbb{S}} \int_{(0,\infty)} L(J_s, j, du) f(J_s, j, u)\Big], \tag{2.3.12}$$

where, if we allow for an abuse of notation, we write $L(i, j, du)$ to denote $L(i, \{j\}, du)$, for any $i, j \in \mathbb{S}$ and $u \in (0, \infty)$.

Now, the proof consists in identifying the kernel $L(du)$ in (2.3.12) as the matrix with (i, j)-th element given by (2.3.10). In order to do this, we make use of the path decomposition of Y in terms of $Y^{(1)}$ and $Y^{(2)}$ as given by (2.2.8).

In light of decomposition (2.2.8) we can write the process Y as the sum $Y^{(1)} + Y^{(2)}$. Now, because each jump of $Y^{(2)}$ belongs to J and they can not coincide with the jumps of $Y^{(1)}$, we can write for any $i \in \mathbb{S}$,

$$\mathbb{E}_i\Big[\sum_{0<s\leq T} f(J_{s^-}, J_s, Y_s - Y_{s-})1_{\{J_{t^-}\neq J_t\}\cup\{Y_{t-}\neq Y_t\}}\Big]$$
$$= \mathbb{E}_i\Big[\sum_{z\in\mathbb{S}}\sum_{0<s\leq T} f(J_{s^-}, z, Y_s^{(1)} - Y_{s^-}^{(1)})1_{\{J_{s^-}=z\}}1_{\{J_s=z\}}1_{\{Y_s^{(1)}-Y_{s^-}^{(1)}>0\}}\Big]$$
$$+\mathbb{E}_i\Big[\sum_{z\in\mathbb{S}}\sum_{0<s\leq T} f(J_{s^-}, z, Y_s^{(2)} - Y_{s^-}^{(2)})1_{\{J_{s^-}\neq z\}}1_{\{J_s=z\}}1_{\{Y_s^{(2)}-Y_{s^-}^{(2)}>0\}}\Big]\,. \tag{2.3.13}$$

We now analyze each term in the right-hand side of Equation (2.3.13) starting with the first term. Recall from Section 2.2.2 that the pair $(J, Y^{(1)})$ is a Markov additive process and therefore, $Y^{(1)}$ is a conditionally additive process given $\mathcal{F}^J = \sigma(J_s, s \geq 0)$. Because $Y^{(1)}$ is a conditionally additive process then first term on the right-hand of

Equation (2.3.13) can then be written as [see Section I in Itô (2004) or Section IV in Cont and Tankov (2004)],

$$\mathbb{E}_i\Big[\sum_{z\in\mathbb{S}}\Big(\int_0^T\int_{(0,\infty)} f(z,z,u)1_{\{J_{s^-}=z\}}N(ds,du)\Big], \tag{2.3.14}$$

where $N(ds,du)$ is the Poisson random measure

$$N((0,t],A)=\sum_{s\in(0,t]}1_{\{Y_s^{(1)}-Y_{s^-}^{(1)}\in A\}}\ ,\quad A\in\mathcal{B}(\mathbb{R})\ ,$$

having intensity measure given by $dt\times\nu_{J_t}(du)$ at (t,u) . In other words, this is the Poisson random measure of $Y^{(1)}$ given $\mathcal{F}_T^J=\sigma(J_s,0\le s\le T)$ which allows us to finally write the first term in Equation (2.3.13) as

$$\mathbb{E}_i\Big[\sum_{z\in\mathbb{S}}\Big(\int_0^\infty ds\int_{(0,\infty)} f(z,z,u)\nu_z(du)1_{\{J_s=z\}}\Big]\ . \tag{2.3.15}$$

As for the second term in the right-hand side of equation (3.5.20), recall from Section 2.2.2 that the pair $(J,Y^{(2)})$ is also a Markov additive process and therefore, $Y^{(2)}$ is a conditionally additive process given $\mathcal{F}^J$. Therefore, it can be written as

$$\begin{aligned}&\mathbb{E}_i\Big[\sum_{z\in\mathbb{S}}\Big(\mathbb{E}_i\big[\sum_{s\le T} f(J_{s^-},z,Y_s^{(2)}-Y_{s^-}^{(2)})1_{\{J_{s^-}\neq z\}}1_{\{J_s=z\}}1_{\{Y_s^{(2)}-Y_{s^-}^{(2)}>0\}}|\mathcal{F}^J\Big)\Big]\\ &=\mathbb{E}_i\Big[\sum_{s\le T}\int_{(0,\infty)} f(J_{s^-},J_s,u)1_{\{J_{s^-}\neq J_s\}}B_{J_{s^-},J_s}(du)\Big)\Big].\end{aligned} \tag{2.3.16}$$

The last equality in equation (2.3.16) follows from the fact that, at a given jump time ξ of J, the process $Y^{(2)}$ jumps by an amount whose conditional distribution, given $\mathcal{F}^J$, is of the form $B_{J_{s^-},J_s}(.)$.

Now, recall from (2.2.6) of Theorem 2.2.1 that the process J has itself a Lévy kernel, this means that there exists a kernel Λ from ($\mathbb{S}$, $\mathcal{S}$) into itself satisfying,

$$\mathbb{E}_i\Big[\sum_{s\le T} g(J_{s^-},J_s)1_{\{J_{s^-}\neq J_s\}}\Big]=\mathbb{E}_i\Big[\int_0^T ds\sum_{z\in\mathbb{S}}\Lambda(J_s,\{z\})g(J_s,z)\Big]\ , \tag{2.3.17}$$

for any non-negative Borel measurable function g defined on $\mathbb{S}\times\mathbb{S}$.

By using the previous equation for $g(J_{s^-},J_s)=\int_{(0,\infty)} f(J_{s^-},J_s,u)1_{\{J_{s^-}\neq J_s\}}B_{J_{s^-},J_s}(du)$, we can write (2.3.16) as,

$$\mathbb{E}_i\Big[\sum_{z\in\mathbb{S}}\Big(\int_0^\infty ds\int_{(0,\infty)} f(J_s,z,u)1_{\{J_s\neq z\}}\Lambda(J_s,\{z\})B_{J_s,z}(du)\Big)\Big]. \tag{2.3.18}$$

The rest of the proof focuses in identifying the Lévy kernel $\Lambda(j,\{z\})$ in terms of the entries of the intensity matrix of J. In fact, we will show that the kernel $\Lambda(j,\{z\}) = q_{j,z}$ $(j, z \in \mathbb{S})$ is the Lévy kernel of J in the sense that it satisfies (2.3.17).

In order to do this, we follow a similar argument as the one in Watanabe (1964). First, we need to prove that $q_{j,z}$ $(j, z \in \mathbb{S})$ satisfies

$$\mathbb{E}_i\Big[\sum_{s\leq T} g(J_{s-}, J_s)1_{\{J_{s-}\neq J_s\}}\Big] = \mathbb{E}_i\Big[\int_0^T ds \sum_{z\in\mathbb{S}} q_{J_s,z}\, g(J_s, z)\Big] \; , \tag{2.3.19}$$

for any given non-negative Borel measurable function g, defined on $\mathbb{S}\times\mathbb{S}$, such that

$$g(y,z) = \sum_{j\in\mathbb{S}} g_j(y)1_{\{z=j\}} \; , \quad g_j \in C(\mathbb{S}) \; , \tag{2.3.20}$$

where $C(\mathbb{S})$ is the set of bounded and continuous function on $\mathbb{S}$. It is clear that any arbitrary Borel measurable function f on $\mathbb{S}\times\mathbb{S}$ can be approximated with a sequence of simpler functions of the form in (2.3.20). Then the statement in (2.3.19) can be extended for an arbitrary Borel measurable function using a standard limiting argument and dominated convergence.

Hence, let us start by defining a sequence of stopping times for any $\varepsilon > 0$,

$$\sigma_1^\varepsilon = \inf(s > 0, |J_s - J_{s-}| > \varepsilon) \; ,$$

and

$$\sigma_{n+1}^\varepsilon = \sigma_n^\varepsilon + \inf(s > \sigma_n^\varepsilon, |J_s - J_{s-}| > \varepsilon) \; ,$$

for $n \geq 1$. These are nothing but an ordered sequence of times at which the process J has jumps larger than ε and the following equality holds

$$\mathbb{E}_i\Big[\sum_{s\leq T} g(J_{s-}, J_s)1_{\{|J_s - J_{s-}|>\varepsilon\}}\Big] = \mathbb{E}_i\Big[\sum_{\sigma_n^\varepsilon\leq T} g(J_{\sigma_n^\varepsilon -}, J_{\sigma_n^\varepsilon})\Big], \tag{2.3.21}$$

for any given $\varepsilon > 0$.

Notice that when $\varepsilon \to 0$, the sequence $\{\sigma_n^\varepsilon\}_{n\geq 0}$ coincides with the jump times of J. Now, let us define a second sequence of stopping times slightly less refined than the first one. For $\eta > 0$ let $\{\tau_m^{(\eta)}\}_{m\in\mathbb{N}}$ be a sequence of stopping times for every $j \in \mathbb{S}$ defined by

$$\tau_0^{(\eta)} = 0 \; ,$$

and

$$\tau_m^{(\eta)} = \inf(t > \tau_{m-1}^{(\eta)} |g_j(J_t) - g_j(J_{\tau_{m-1}^{(\eta)}})| > \frac{1}{2\eta}).$$

Notice that this is also an ordered sequence of times at which the process J has jumps larger than $1/2\eta$ as measured by the function g. This new sequence of stopping times is such that when $\eta \to \infty$ it also coincides with the jump times of J.

It is a straight-forward exercise to show that, for every j (a.s.) and for every $s, t \in [\tau_{m-1}^{(\eta)}, \tau_m^{(\eta)})$,

$$|g_j(J_t) - g_j(J_s)| \leq \frac{1}{\eta} \; . \tag{2.3.22}$$

Using (2.4.17), we can now see that, for every j

$$\Big| \sum_{\sigma_n^\varepsilon \leq T} g_j(J_{\sigma_n^\varepsilon -}) 1_{\{J_{\sigma_n^\varepsilon}=j\}} - \sum_{\tau_m^{(\eta)} \leq T} g_j(J_{\tau_{m-1}^{(\eta)}}) \sum_{\tau_{m-1}^{(\eta)} < \sigma_n^\varepsilon \leq \tau_m^{(\eta)}} 1_{\{J_{\sigma_n^\varepsilon}=j\}} \Big| \leq \frac{1}{\eta} \sum_{\sigma_n^\varepsilon \leq T} 1_{\{J_{\sigma_n^\varepsilon}=j\}} \; . \tag{2.3.23}$$

Now, for a fix $i \in \mathbb{S}$, it follows from (2.3.23), from the asymptotic behavior of the sequences $\{\sigma_n^\varepsilon\}_{n\in\mathbb{N}}$ and $\{\tau_m^\eta\}_{m\in\mathbb{N}}$ and from the dominated convergence theorem that we can write

$$\begin{aligned}
\mathbb{E}_i\Big[\sum_{s\leq T} g(J_{s^-}, J_s) 1_{\{J_s \neq J_{s^-}\}} \Big] &= \lim_{\eta\to\infty, \varepsilon\to 0} \sum_{j\in\mathbb{S}} \mathbb{E}_i\Big[\sum_{\tau_m^{(\eta)} \leq T} g_j(J_{\tau_{m-1}^{(\eta)}}) \sum_{\tau_{m-1}^{(\eta)} < \sigma_n^\varepsilon \leq \tau_m^{(\eta)}} 1_{\{J_{\sigma_n^\varepsilon}=j\}} \Big] \\
&= \sum_{j\in\mathbb{S}} \mathbb{E}_i\Big[\lim_{\eta\to\infty} \sum_{\tau_m^{(\eta)} \leq T} g_j(J_{\tau_{m-1}^{(\eta)}}) 1_{\{J_{\tau_m^{(\eta)}}=j\}} \Big] \\
&= \sum_{j\in\mathbb{S}} \mathbb{E}_i\Big[\lim_{\eta\to\infty} \sum_{\tau_m^{(\eta)} \leq T} g_j(J_{\tau_{m-1}^{(\eta)}}) \mathbb{E}_i\Big[1_{\{J_{\tau_m^\eta}=j\}} | J_{\tau_{m-1}^{(\eta)}}\Big]\Big] \\
&= \sum_{j\in\mathbb{S}} \mathbb{E}_i\Big[\lim_{\eta\to\infty} \sum_{\tau_m^{(\eta)} \leq T} g_j(J_{\tau_{m-1}^{(\eta)}}) \mathbb{P}_i(J_{\tau_m^{(\eta)}} = j | J_{\tau_{m-1}^{(\eta)}}) \Big] \\
&= \sum_{j\in\mathbb{S}} \mathbb{E}_i\Big[\lim_{\eta\to\infty} \sum_{\tau_m^{(\eta)} \leq T} g_j(J_{\tau_{m-1}^{(\eta)}}) q_{J_{\tau_{m-1}^{(\eta)}}, j} (\tau_m^{(\eta)} - \tau_{m-1}^{(\eta)}) \Big] \; ,
\end{aligned} \tag{2.3.24}$$

for any function g of the form (2.3.20). In the last equality, we have used the description of transition probability in terms of intensity matrix elements. That is, in infinitesimal terms we have that the probability of a transition from J_s to j before $s + ds$ is $q_{J_s,j} ds$ and we can write

$$\mathbb{P}_i(J_{\tau_m^{(\eta)}} = z | J_{\tau_{m-1}^{(\eta)}} = y) = q_{y,z}(\tau_m^{(\eta)} - \tau_{m-1}^{(\eta)})$$

on $J_s \neq z$ as η goes to ∞. In standard $o(.)$ notation, this means that the probability of transition to j before $t + h$ is $q_{J_s,j} h + o(h)$ [see Section II-3 in Asmussen (2003) for details].

The term inside the expectation in equation (2.3.24) is a Riemann sum and so letting $\eta \to \infty$ we can write

$$\begin{aligned}
\mathbb{E}_i\Big[\sum_{s\leq T} g(J_{s-}, J_s)1_{\{J_s\neq J_{s-}\}}\Big] &= \sum_{j\in\mathbb{S}}\mathbb{E}_i\Big[\int_0^T g_j(J_s)q_{J_s,j}ds\Big] \\
&= \mathbb{E}_i\Big[\int_0^T \sum_{j\in\mathbb{S}} g_j(J_s)q_{J_s,j}ds\Big] \\
&= \mathbb{E}_i\Big[\int_0^T ds\sum_{z\in\mathbb{S}} q_{J_s,z}g(J_s,z)\Big], \qquad (2.3.25)
\end{aligned}$$

for any function g of the form (2.3.20) and all $i \in \mathbb{S}$. Using a standard limiting argument, equation (2.3.25) can be extended from the space of functions of the form (2.3.20) to the space of all Borel measurable functions on $\mathbb{S}\times\mathbb{S}$. In order to conclude it is enough to compare equation (2.3.25) against (2.3.17) so we can readily identify the kernel Λ for the process J

Now, by a straight-forward substitution of $\Lambda(J_s, \{z\}) = q_{J_s,z}$, equation (2.3.18) can be written as,

$$\mathbb{E}_i\Big[\sum_{z\in\mathbb{S}}\Big(\int_0^T ds\int_{(0,\infty)} f(J_s,z,u)1_{\{J_s\neq z\}}q_{J_s,z}B_{J_s,z}(du)\Big)\Big]. \qquad (2.3.26)$$

Finally, we now can substitute equations (2.3.15) and (2.3.26) into (3.5.20) yielding

$$\begin{aligned}
&\mathbb{E}_i\Big[\sum_{s\leq T} f(J_{s-}, J_s, Y_s - Y_{s-})1_{\{J_{t-}\neq J_t\}\cup\{Y_{t-}\neq Y_t\}}\Big] \\
= \ &\mathbb{E}_i\Big[\sum_{z\in\mathbb{S}}\Big(\int_0^T ds\int_{(0,\infty)} f(J_s,z,Y_s,Y_{s-}+u)\nu_z(du)1_{\{J_s=z\}}\Big] \\
&+\mathbb{E}_i\Big[\sum_{z\in\mathbb{S}}\Big(\int_0^T ds\int_{(0,\infty)} f(J_s,z,u)1_{\{J_{s-}\neq z\}}q_{J_s,z}B_{J_s,z}(du)\Big)\Big] \\
= \ &\mathbb{E}_i\Big[\sum_{z\in\mathbb{S}}\Big(\int_0^T ds\int_{(0,\infty)} f(J_s,z,u)\big[\nu_z(du)1_{\{J_s=z\}} + q_{J_s,z}B_{J_s,z}(du)1_{\{J_s\neq z\}}\big]\Big)\Big]. \\
&\qquad (2.3.27)
\end{aligned}$$

This shows that the transition kernel L in (2.3.10) can be identified to be the Lévy kernel of (J, Y); where $L(du)$ is to be understood as the matrix with (i, j)-th element given by (2.3.10). $\square$

2.3.2. Joint law under a change of measure

At this point, we recall that the main objective of this paper is to write an expression for the MEDPF in (2.3.3) and so from now on we work under an infinite-time horizon $T=\infty$. But before we can write out such an expression we need one more intermediate result that has to do with the change of measure defined through the density process in (2.2.15).

Consider a spectrally positive MAP (J,Y) representing the aggregate claims in (2.3.1). In particular, we can define a new probability measure through

$$\frac{d\mathbb{P}_i^{\Phi(q)}}{d\mathbb{P}_i}\Big|_{\mathcal{F}_t} = e^{-\Phi(q)Y_t - qt}\frac{h_{J_t}(\Phi(q))}{h_i(\Phi(q))}, \tag{2.3.28}$$

where $\Phi(q)$ the right inverse of κ defined in (3.3.21).

Moreover, recall from Section 2.2.2, that under $\mathbb{P}_i^{\Phi(q)}$ the process (J,Y) in (2.3.1), is still spectrally positive MAP drifting to $-\infty$. Let τ_x be the ruin time (3.2.5) of the MAP risk process (2.3.1). Notice that, on $\{J_{\tau_x^-} = J_{\tau_x}\}$ (a.s.), either $x - Y_{\tau_x^-} = Y_{\tau_x} - x = 0$ or $x - Y_{\tau_x^-} > 0$ and $Y_{\tau_x} - x > 0$. In the case of downward creeping across x, i.e. the case when $\{x - Y_{\tau_x^-}\} = \{Y_{\tau_x} - x = 0\}$, this only occurs when the first passage of Y across x is due to the continuous additive component of Y [see Çinlar (1975)].

In the following theorem, we shall characterize the triple law of $(J_{\tau_x^-}, x - Y_{\tau_x^-}, Y_{\tau_x} - x)$ under the probability measure $\mathbb{P}^{\Phi(q)}(.; J_{\tau_x})$ in terms in terms of q- bivariate potential measure matrix and the Lévy kernel L associated with the net aggregate claims process in (2.3.1).

Recall that in Theorem 2.3.1 we have identified the transition kernel matrix L of (J,Y) in terms of its characteristics introduced in Section 2.2.2 and that $L(du)$ should be understood as the matrix of (i,j)-th element given by (2.3.10). In light of which, the following theorem is a somewhat explicit characterization of this triple law in terms of the q-potential measure (2.3.4) and notions introduced in Section 2.2.2 for spectrally positive MAP's. This would be the second main contribution of this paper that extends similar results in Biffis and Kyprianou (2010).

Theorem 2.3.2. *Let (J,Y) the spectrally-positive Markov additive process described in 2.3.1 and let L denote its associated Lévy kernel as given in Theorem 4.2.3. Then, for $q\,, x \geq 0$ and $i \in \mathbb{S}$, the double law of $(x - Y_{\tau_x^-}, Y_{\tau_x} - x)$ under the probability measure $\mathbb{P}^{\Phi(q)}(.; J_{\tau_x})$ is given by*

$$
\begin{aligned}
&\mathbb{P}^{\Phi(q)}(x - Y_{\tau_x^-} \in dv, Y_{\tau_x} - x \in du; J_{\tau_x}) \\
= \quad &\Delta^{-1}_{h(\Phi(q))} e^{-\Phi(q)(x+u)} U^{(q)}_x(x - dv) L(v + du) \Delta_{h(\Phi(q))} \ , \qquad (2.3.29)
\end{aligned}
$$

for $u > 0$ and $v \geq 0$.

The q-bivariate potential measure $U^{(q)}_x(x - dv)$ and the Lévy kernel $L(v + du)$ are respectively the matrices with (i, j)-th entries given by

$$
U^{(q)}_x(i, j, x - dv) \ \ \text{and} \ L(i, j, du + v),
$$

for each $i, j \in \mathbb{S}$. Moreover, we write $\Delta_{h(\phi(q))}$ to denote diagonal matrix with with entries $\{h_i(\Phi(q))\}_{i\in\mathbb{S}}$.

Proof. The idea of the proof is to derive an expression for

$$
\mathbb{E}^{\Phi(q)}_i \Big[f(x - Y_{\tau_x^-}, Y_{\tau_x} - x); J_{\tau_x} = j \Big] \ ,
$$

for any given function f and from where we can readily identify the probability measure $\mathbb{P}^{\Phi(q)}(. \, ; J_{\tau_x})$.

Let f be an arbitrary positive Borel measurable function on $\mathbb{S} \times \mathbb{R}_+ \times \mathbb{R}_+$ such that $f(., 0) = 0$. Using the density process (2.3.28) we can write, for any $i, j \in \mathbb{S}$,

$$
\begin{aligned}
&\mathbb{E}^{\Phi(q)}_i \Big[f(x - Y_{\tau_x^-}, Y_{\tau_x} - x); J_{\tau_x} = j \Big] && (2.3.30) \\
= \quad &\mathbb{E}_i \Big[e^{\Phi(q)(Y_{\tau_x}) - q\tau_x} \frac{h_j(\Phi(q))}{h_i(\Phi(q))} f(x - Y_{\tau_x^-}, Y_{\tau_x} - x); J_{\tau_x} = j \Big] && (2.3.31) \\
= \quad &\mathbb{E}_i \Big[\sum_{s>0} e^{\Phi(q)(Y_{s-} + \Delta Y_s) - qs} \frac{h_j(\Phi(q))}{h_i(\Phi(q))} 1_{\{\overline{Y}_{s-} \leq x, Y_{s-} + \Delta Y_s > x\}} \\
&\qquad \times f(x - Y_{s^-}, Y_{s^-} + \Delta Y_s - x) 1_{\{J_s = j\}} \Big] \ . && (2.3.32)
\end{aligned}
$$

Now, using Theorem 2.3.1, equation (2.3.32) can be written as

$$\begin{aligned}
&\frac{h_j(\Phi(q))}{h_i(\Phi(q))}\mathbb{E}_i\Big[\int_0^\infty e^{-qs}ds\int_{(0,\infty)} 1_{\{\overline{Y}_s\le x, Y_s+y>x\}}e^{-\Phi(q)(Y_s+y)}L(J_s,j,dy)\\
&\qquad\times f(J_s,x-Y_s,Y_s+y-x)\Big] &(2.3.33)\\
=\;&\frac{h_j(\Phi(q))}{h_i(\Phi(q))}\mathbb{E}_i\Big[\int_0^\infty e^{-qs}ds\int_{(0,\infty)} 1_{\{\tau_x>s, Y_s+y>x\}}e^{-\Phi(q)(Y_s+y)}L(J_s,j,dy)\\
&\qquad\times f(x-Y_s,Y_s+y-x)\Big] &(2.3.34)\\
=\;&\frac{h_j(\Phi(q))}{h_i(\Phi(q))}\int_0^\infty e^{-qs}ds\int_{\mathbb{R}\times(0,\infty)} 1_{\{z\le x, z+y>x\}}\sum_{k\in\mathbb{S}}\mathbb{P}_i(Y_s\in dz,\tau_x>s,J_s=k)\\
&\qquad\times e^{-\Phi(q)(z+y)}L(k,j,dy)f(x-z,z+y-x)\;. &(2.3.35)
\end{aligned}$$

If we now make the following change of variables $u=z+y-x$ and $v=x-z$, then (2.3.35) can be written as

$$\begin{aligned}
&\frac{h_j(\Phi(q))}{h_i(\Phi(q))}\sum_{k\in\mathbb{S}}\int_{\mathbb{R}\times\mathbb{R}} 1_{\{z\le x, z+y-x>0\}}e^{-\Phi(q)(x+u)}U_x^{(q)}(i,k,dz)\\
&\qquad\times L(k,j,dy)f(x-z,z+y-x) &(2.3.36)\\
=\;&\int_{[0,\infty)\times\mathbb{R}} 1_{\{v\ge 0, u>0\}}f(v,u)\frac{h_j(\Phi(q))}{h_i(\Phi(q))}e^{-\Phi(q)(x+u)}\sum_{k\in\mathbb{S}}U_x^{(q)}(i,k,x-dv)\\
&\qquad\times L(k,j,du+v)\;. &(2.3.37)
\end{aligned}$$

We have shown that the expectation (2.3.30) can be written as the integral in (2.3.37) for any arbitrary function f. Since $\frac{h_j(\Phi(q))}{h_i(\Phi(q))}\sum_{k\in\mathbb{S}}U_x^{(q)}(i,k,x-dv)L(k,j,du+v)$ are the (i,j)-entries of $\Delta^{-1}_{h(\Phi(q))}U_x^{(q)}(x-dv)L(du+v)\Delta_{h(\Phi(q))}$, the statement of the theorem follows. □

2.3.3. Expected discounted penalty function for a MAP risk process

In this section, we are finally in a position to present the third and main contribution of this paper. In the following theorem, we give an expression for the MEDPF defined by (2.3.3). This expression is given in terms of the q-bivariate potential measure and the Lévy kernel associated with the Markov additive aggregate net claim process (J,Y) in (2.3.1). Recall that w is a non-negative Borel measurable function from $\mathbb{R}_+\times\mathbb{R}_+$ such that $w(.,0)=0$.

Theorem 2.3.3. *Consider the risk model in (2.3.1) where the net aggregate claims process is described by a spectrally-positive Markov additive process (J,Y). For $q\ge 0$,* *the MEDPF $\phi(w;q;x)$, as introduced in Definition 2.3.3, is given by,*

$$\int_{[0,\infty)}\int_{(0,\infty)} w(v,u)\,U_x^{(q)}(x-dv)L(du+v)\ , \qquad x\geq 0\ .$$

This is, the MEDPF itself is a matrix with the (i,j)-th entry given by,

$$\sum_{k\in\mathbb{S}}\int_{[0,\infty)}\int_{(0,\infty)} 1_{\{v\leq x\}}w(v,u)U_x^{(q)}(i,k,x-dv)\left[\nu_k(du+v)1_{\{k=j\}}+q_{k,j}B_{k,j}(du+v)1_{\{k\neq j\}}\right]\ .$$

Proof. Let w a positive Borel measurable function on $\mathbb{R}_+\times\mathbb{R}_+$ such that $w(.,0)=0$. Using the density process (2.3.28) we can write, for any $i,j\in\mathbb{S}$, the (i,j)-th entry of MEDPF as

$$\begin{aligned}
\phi(w;q;x)_{i,j} &= \mathbb{E}_i\left[e^{-q\tau_x}w(x-Y_{\tau_x^-},Y_{\tau_x}-x)1_{\{\tau_x<\infty\}};J_{\tau_x}=j\right] && (2.3.38)\\
&= \frac{h_i(\Phi(q))}{h_j(\Phi(q))}\mathbb{E}_i^{\Phi(q)}\left[e^{\Phi(q)Y_{\tau_x}}w(x-Y_{\tau_x^-},Y_{\tau_x}-x)1_{\{\tau_x<\infty\}};J_{\tau_x}=j\right] && (2.3.39)\\
&= \frac{h_i(\Phi(q))}{h_j(\Phi(q))}\int_{[0,x]\times(0,\infty)} w(v,u)e^{\Phi(q)(x+u)}\\
&\qquad\qquad \times\mathbb{P}_i^{\Phi(q)}(x-Y_{\tau_x^-}\in dv, Y_{\tau_x}-x\in du; J_{\tau_x}=j)\ .
\end{aligned}$$

If we now use Theorem 2.3.2, we can write an explicit form for the probability $\mathbb{P}^{\Phi(q)}(.;J_{\tau_x})$ yielding

$$\begin{aligned}
\phi(w;q;x)_{i,j} &= \sum_{k\in\mathbb{S}}\int_{[0,\infty)\times(0,\infty)} w(v,u)U_x^{(q)}(i,k,x-dv)L(k,j,du+v)\\
&= \int_{[0,\infty)}\int_{(0,\infty)} w(v,u)\sum_{k\in\mathbb{S}}U_x^{(q)}(i,k,x-dv)\left[\nu_k(du+v)1_{\{k=j\}}\right.\\
&+ \left. q_{k,j}B_{k,j}(du+v)1_{\{k\neq j\}}\right]\ .
\end{aligned}$$

□

2.4. Scale and potential measure matrices

The results in Theorems 2.3.1 and 2.3.3 are very general statements that give expressions for the MEDPF. However these expressions are not explicit enough since they are given in terms of the q-potential measure $U_x^{(q)}$. One of the main issues lies in identifying this q-potential measure $U_x^{(q)}$ that appears in all expressions of the MEDPF. In this section we study this question in more detail using the concept of scale matrix for MAP's introduced in Kyprianou and Palmowski (2008) and Ivanovs and Palmowski (2011).

In this section we consider a risk model defined in (2.3.1). Recall that the net aggregate claims process Y is a spectrally positive MAP with characteristics $(q_{i,j}, B_{i,j}, \sigma_i, a_i, \nu_i)_{i,j\in\mathbb{S}}$ as defined in Section 2.2.2. Formally speaking, consider a spectrally positive Markov additive processes (J, Y) as defined in Section 2.2.2 and let $Y^{(1)}$ and $Y^{(2)}$ denote the two elements in the path decomposition (2.2.8). Recall that the second term $Y^{(2)}$, as defined in (2.2.10), contains the jumps that occur at jump times of the Markov chain J and which are determined by a sequence of positive i.i.d random variables denoted by $\{U_n^{(i,j)}\}_{n>0}$ for every $i, j \in \mathbb{S}$. Let us denote with $T_n^{(i,j)}$ the n-th jump time of J from state i to state j. These form a family of sequences of jump times $\{T_n^{i,j}\}_{n>0}$ of J indexed by any two $i, j \in \mathbb{S}$.

It turns out that for this model, we can work out expressions for the q-potential matrix measure killed at exit. In this section, we characterize the bivariate potential measure given by (2.3.4) for a spectrally-positive Markov additive process. These results are based on recent contributions on fluctuations and exit problems for Markov additive processes developed in Kyprianou and Palmowski (2008), and Ivanovs and Palmowski (2011). But before we can characterize the q-potential measure matrix $U_x^{(q)}$ defined in (2.3.4) for this family of processes, we need to introduce further notation concerning the time reversal version of a MAP (J, Y) and its intensity matrix under a change of measure.

In the remaining of the section we base our discussion on results from Kyprianou and Palmowski (2008). Let $(\widehat{J}, \widehat{Y})$ denote the time-reversed version of the process (J, Y) from a fixed time t in the future when J has the stationary distribution π. Formally, for a fixed t,

$$\widehat{J}_s := J_{(t-s)-} \quad \text{and} \quad \widehat{Y}_s = Y_t - Y_{(t-s)-} \ , \quad 0 \le s \le t \ ,$$

under $\mathbb{P}_\pi = \sum_{i\in\mathbb{S}} \pi_i \mathbb{P}_i$

The time-reversed process $(\widehat{J}, \widehat{Y})$ is a MAP and we denote its characteristics by using a hat over the existing notation for the characteristics of (J, Y), i.e. $(\widehat{q}_{i,j}, \widehat{B}_{i,j}, \widehat{\sigma}_i, \widehat{a}_i, \widehat{\nu}_i)_{i,j\in\mathbb{S}}$. Moreover, we recall that the intensity matrix $\widehat{Q}$ of $\widehat{J}$ can be written in terms of the original matrix Q as follows [see Kyprianou and Palmowski (2008)]

$$\widehat{Q} = \Delta_\pi^{-1} Q^T \Delta_\pi \ ,$$

where Δ_π is the diagonal matrix whose entries are given by the vector of stationary probabilities π and Q^T denotes the transpose of the matrix Q. It is a straight-forward exercise to verify that the cumulant matrix $\widehat{F}(\alpha)$ of $\widehat{J}$ can be written in terms of the original cumulant matrix (2.2.11) as follows

$$\widehat{F}(\alpha) = \Delta_\pi^{-1} F(\alpha)^T \Delta_\pi \ .$$

Let $(-Y, J)$ denotes the spectrally negative MAP. The first passage time of $-Y$ over level x, where $x \geq 0$, is defined by

$$\widehat{\tau}_x = \inf\{t \geq 0;\ -Y_t > x\}.$$

We denote the intensity matrix of J with Λ which satisfies

$$\mathbb{P}[J_{\widehat{\tau}_x}] = e^{\Lambda x}.$$

We also know that under the change of measure defined through the density process (2.2.15), a MAP remains within the same class of processes. Thus, under $\mathbb{P}^{\Phi(q)}$, we denote the intensity matrix of J and $\widehat{J}$ respectively with Λ^q and $\widehat{\Lambda}^q$ for $q \geq 0$. For more details about properties of the intensity matrix under time-reversal and changes of measure we refer to Asmussen (2003).

2.4.1. Scale matrix

We first turn our attention to the concept of scale matrix. From Kyprianou and Palmowski (2008), there exists a unique continuous function $W : [0, \infty) \longrightarrow \mathbb{R}^N \times \mathbb{R}^N$ such that $W(x)$ is invertible for all $x > 0$,

$$\int_0^\infty e^{-\alpha x} W(x) dx = F(\alpha)^{-1} \tag{2.4.1}$$

for sufficiently large α. In addition,

$$W(x) = e^{-\Lambda x} \mathrm{L}(x) \ , \tag{2.4.2}$$

where $\mathrm{L}(x)$ is a matrix of expected occupation times at 0 up to the first passage over x. Definition and properties of $\mathrm{L}(x)$ can be found in Section 4 of Ivanovs and Palmowski (2011). Note that $\mathrm{L}(x)$ tends to L, the matrix of expected occupation times at 0, as $x \longrightarrow \infty$. $W(x)$ is called the scale matrix associated with the spectrally-negative Markov additive process $-Y$ and the probability measure $\mathbb{P}$.

We interpret $W_{\Phi(q)}(x)$ to be the scale matrix associated with $(-Y, \mathbb{P}^{\Phi(q)})$, which is characterized by its Laplace transform

$$\int_0^\infty e^{-\alpha x} W_{\Phi(q)}(x) dx = F^{\Phi(q)}(\alpha)^{-1}, \tag{2.4.3}$$

for sufficiently large α.

Let us introduce the q-scale matrix which is denoted by $W^{(q)}(x)$ and given by

$$W^{(q)}(x) = \Delta_{h(\Phi(q))} e^{\Phi(q)x} W_{\Phi(q)}(x) \Delta^{-1}_{h(\Phi(q))}. \tag{2.4.4}$$

By a simple manipulation of (2.4.3) and (2.4.4), the Laplace transform of q-scale matrix is

$$\int_0^\infty e^{-\alpha x} W^{(q)}(x)dx = (F(\alpha) - q\mathrm{I})^{-1}, \tag{2.4.5}$$

for sufficiently large α. Formally, $W^{(q)}(x)$ is a matrix analogue to the q-scale function of Lévy process. According to (2.4.2) and (2.4.4), the q-scale matrix can be written as

$$W^{(q)}(x) = \Delta_{h(\Phi(q))} e^{(\Phi(q)\mathrm{I} - \Lambda^q)x} \mathrm{L}^{(q)}(x) \Delta^{-1}_{h(\Phi(q))}, \tag{2.4.6}$$

where $\mathrm{L}^{(q)}(x)$ is the matrix of expected occupation times at 0 to the first passage $\widehat{\tau}_x$ under the probability $\mathbb{P}^{\Phi(q)}$. Note that $\mathrm{L}^{(q)}$ be the matrix of expected occupation times under $\mathbb{P}^{\Phi(q)}$, which is the limit of $\mathrm{L}^{(q)}(x)$, as $x \longrightarrow \infty$.

Let us define the second scale matrix $M^{(q)}(x)$ which is characterized by its Laplace transform

$$\int_0^\infty e^{-\alpha x} M^{(q)}(dx) = (F(\alpha) - q\mathrm{I})^{-1}(\mathrm{I} - \alpha\Delta^{-1}_{v(\Phi(q))}\big(\big[\Phi(q)I - \widehat{\Lambda}^q\big]^{-1}\big)^T \Delta_{v(\Phi(q))})(q\mathrm{I} - Q), \tag{2.4.7}$$

for sufficiently large α. For $x \geq 0$, $M^{(q)}(x)$ is given by [see Theorem 3, Kyprianou and Palmowski (2008)]

$$M^{(q)}(x) = \mathbb{E}\big[e^{-q\tau_x}\ ; J_{\tau_x}\big]. \tag{2.4.8}$$

Note that the existence and properties of scale matrices $W^{(q)}(x)$ and $M^{(q)}(x)$ are also given by Theorem 3 of Kyrianou and Palmowski (2008). In the following proposition, we give $M^{(q)}(x)$ in terms of the q-scale matrix.

Proposition 2.4.1. *The matrix $M^{(q)}(x)$ is given by*

$$M^{(q)}(x) = \mathrm{I} - \Big[\int_0^x W^{(q)}(y)dy - W^{(q)}(x)\Delta_{h(\Phi(q))}C(q)\Big](Q - q\mathrm{I}) \tag{2.4.9}$$

and then

$$M^{(q)}(dx) = \big[-W^{(q)}(x)dx + W^{(q)}(dx)C(q)\big](Q - q\mathrm{I}), \tag{2.4.10}$$

where $C(q)$ is the matrix given by

$$C(q) = \Delta_{h(\Phi(q))}(\mathrm{L}^{(q)})^{-1}(\Phi(q)\mathrm{I} - \Lambda^{(q)})^{-1}\mathrm{L}^{(q)}\Delta^{-1}_{h(\Phi(q))}. \tag{2.4.11}$$

Proof. Note that under $\mathbb{P}^{\Phi(q)}$, the process Y is still a spectrally positive MAP and $M^{(q)}(x)$ can be written as

$$\begin{aligned}
M^{(q)}(x) &= e^{\Phi(q)x}\Delta_{h(\Phi(q))}\mathbb{E}_{-x}^{\Phi(q)}\left[e^{\Phi(q)Y_{\tau_0}}\,;J_{\tau_0}\right]\Delta^{-1}_{h(\Phi(q))}\\
&= e^{\Phi(q)x}\Delta_{h(\Phi(q))}\Big[e^{-\Phi(q)x}\mathrm{I}-\int_0^x e^{\Phi(q)y}W_{\Phi(q)}(y)dy(q-Q\mathrm{I})\\
&\quad -W_{\Phi(q)}(x)\mathrm{L}^{(q)-1}(\Phi(q)\mathrm{I}-\Lambda^{(q)})^{-1}\mathrm{L}^{(q)}(q-Q\mathrm{I})\Big]\Delta^{-1}_{h(\Phi(q))}, \quad (2.4.12)
\end{aligned}$$

where in the last equality we have used Corollary 4 in Ivanovs and Palmowski (2011). By using (2.4.4), the last term reduces to

$$\mathrm{I}-\Big[\int_0^x W^{(q)}(y)dy - W^{(q)}(x)\Delta_{h(\Phi(q))}C(q)\Big](Q-q\mathrm{I})\ ,$$

where $C(q)=\Delta_{h(\Phi(q))}\mathrm{L}^{(q)-1}(\Phi(q)\mathrm{I}-\Lambda^{(q)})^{-1}\mathrm{L}^{(q)}\Delta^{-1}_{h(\Phi(q))}$.

We then finally obtain (2.4.9) and (2.4.10).

□

2.4.2. Identifying the q-potential matrix measure

We can now state a result where we give a more explicit expression for the q-potential matrix measure $U_x^{(q)}$ in terms of q-scale matrix, this is given in the form of the following theorem. Recall that e_q denotes an independent exponential random variable with parameter $q>0$.

Theorem 2.4.1. *Let (J,Y) be a spectrally-positive MAP. For $y\le x$, the bivariate potential measure matrix $U_x^{(q)}(dy)$ with (i,j)-th entries defined in (2.3.4), is given by*

$$U_x^{(q)}(dy)=\int_y^x\big[-W^{(q)}(z)dz+W^{(q)}(dz)C(q)\big]\big[K^{(q)}(dy-z)\big]^T\ , \qquad (2.4.13)$$

where $C(q)$ is the $N\times N$ matrix given by (2.4.11) and $K^{(q)}$ is the density matrix with Laplace transform

$$\int_{-\infty}^{\infty}e^{\alpha u}K^{(q)}(du)=\mathrm{I}-\alpha\Delta_{v(\Phi(q))}\big[(\Phi(q)+\alpha)\mathrm{I}-\widehat{\Lambda}(q)\big]^{-1}\Delta^{-1}_{v(\Phi(q))}\ ,\quad \alpha\in\mathbb{R}. \quad (2.4.14)$$

Proof. Let $i,j\in\mathbb{S}$ and $y\le x$. From equation (2.3.5) we can write,

$$\begin{aligned}
U_x^{(q)}(i,j,dy) &= \frac{1}{q}\mathbb{P}_i(Y_{e_q}\in dy\,,\overline{Y}_{e_q}\le x\,;J_{e_q}=j)\\
&= \frac{1}{q}\int_{[y,x]}\sum_{k\in\mathbb{S}}\mathbb{P}_i(\overline{Y}_{e_q}\in dz\,,\overline{Y}_{e_q}-Y_{e_q}\in z-dy\,|J_{\overline{G}_{e_q}}=k)\\
&\quad\times\mathbb{P}_i(J_{e_q}=j\,|J_{\overline{G}_{e_q}}=k)\mathbb{P}_i(J_{\overline{G}_{e_q}}=k)\ , \qquad (2.4.15)
\end{aligned}$$

where

$$\overline{G}_t = \sup\{s < t, \overline{Y}_s = Y_s\} \ .$$

Now, invoking the Weiner-Hopf factorization, specifically using the fact that $Y_{e_q} - \overline{Y}_{e_q}$ and $-\overline{Y}_{e_q}$ are conditionally independent on $J_{\overline{G}_{e_q}}$ [see Theorem 1 in Klusik and Palmowski (2011)], we have that

$$\begin{aligned} U_x^{(q)}(i,j,dy) &= \frac{1}{q}\sum_{k\in\mathbb{S}}\int_y^x \mathbb{P}_i(\overline{Y}_{e_q} \in dz \,|J_{\overline{G}_{e_q}})\mathbb{P}_i(\overline{Y}_{e_q} - Y_{e_q} \in z - dy \,|J_{\overline{G}_{e_q}}) \\ &\quad \times\mathbb{P}_i(J_{e_q} = j \,|J_{\overline{G}_{e_q}} = k)\mathbb{P}_i(J_{\overline{G}_{e_q}} = k) \ . \end{aligned} \tag{2.4.16}$$

By duality and analysis of the time reversed path, we can write,

$$\mathbb{P}_i(\overline{Y}_{e_q} - Y_{e_q} \in z - dy \,|J_{\overline{G}_{e_q}} = k) = \mathbb{P}_j(-\widehat{\underline{Y}}_{e_q} \in z - dy \,|\widehat{J}_{\underline{G}_{e_q}} = k) \ , \tag{2.4.17}$$

and

$$\begin{aligned} \mathbb{P}_k(J_{e_q-\overline{G}_{e_q}} = j) &= \mathbb{P}_k(J_{e_q-\overline{G}_{e_q}} = j) \\ &= \frac{\pi_k}{\pi_j}\mathbb{P}_k(\widehat{J}_{\underline{G}_{e_q}} = j) \ . \end{aligned} \tag{2.4.18}$$

Hence (2.4.16) is equal to

$$\frac{1}{q}\int_y^x \sum_{k\in\mathbb{S}}\mathbb{P}_i(\overline{Y}_{e_q} \in dz \,; J_{\overline{G}_{e_q}} = k)\pi_k\mathbb{P}_j(-\widehat{\underline{Y}}_{e_q} \in z - dy \,; J_{\underline{G}_{e_q}} = k)\frac{1}{\pi_j}, \tag{2.4.19}$$

and then the q-potential matrix can be written as

$$\begin{aligned} U_x^{(q)}(dy) &= \frac{1}{q}\int_y^x \mathbb{P}(\overline{Y}_{e_q} \in dz \,; J_{\overline{G}_{e_q}})\Delta_\pi^{-1}\mathbb{P}(-\widehat{\underline{Y}}_{e_q} \in z - dy \,; \widehat{J}_{\underline{G}_{e_q}})^T\Delta_\pi \\ &= \frac{1}{q}\int_y^x \mathbb{P}(\overline{Y}_{e_q} \in dz \,; J_{e_q})\Delta_\pi^{-1}\big[\mathbb{P}(\widehat{\underline{Y}}_{e_q} \in dy - z \,; \widehat{J}_{e_q})\widehat{I}(q)^{-1}\big]^T\Delta_\pi \ . \end{aligned} \tag{2.4.20}$$

In the last equality we have used Theorem 1, (ii) of Klusik and Palmowski (2011). In addition, by comparing (2.4.7) with $\mathbb{E}[e^{\alpha\overline{Y}_{e_q}} \,; J_{e_q}]$, we deduce that

$$\frac{1}{q}\mathbb{P}(\overline{Y}_{e_q} \in dz \,; J_{e_q}) = M^{(q)}(dz)(Q - q\mathrm{I})^{-1} \ ; \quad z \geq 0 \ , \tag{2.4.21}$$

and then

$$U_x^{(q)}(dy) = \int_y^x M^{(q)}(dz)(Q - q\mathrm{I})^{-1}\Delta_\pi^{-1}\big[\mathbb{P}(\widehat{\underline{Y}}_{e_q} \in dy - z \,; \widehat{J}_{e_q})\widehat{I}(q)^{-1}\big]^T\Delta_\pi \ . \tag{2.4.22}$$

Setting

$$K^{(q)}(dx) = \Delta_\pi\big[\mathbb{P}(\widehat{\underline{Y}}_{e_q} \in dx \ , \widehat{J}_{e_q})\widehat{I}(q)^{-1}\big]\Delta_\pi^{-1} \ , \tag{2.4.23}$$

the density matrix $K^{(q)}$ can be characterized via its Laplace transform as follows [see Theorem 1, (ii), Klusik and Palmowski (2011)]

$$\int_{-\infty}^{\infty} e^{\alpha y} K^{(q)}(dy) = \mathrm{I} - \alpha \Delta_{v(\Phi(q))} \big[(\Phi(q)+\alpha)\mathrm{I} - \widehat{\Lambda}(q)\big]^{-1} \Delta^{-1}_{v(\Phi(q)} \,. \tag{2.4.24}$$

Substituting (2.4.10) in (2.4.22), the q-potential matrix measure can be simply written as

$$U_x^{(q)}(dy) \;\; = \;\; \int_y^x \big[-W^{(q)}(z)dz + W^{(q)}(dz)C(q)\big] \big[K^{(q)}(dy-z)\big]^T \,, \tag{2.4.25}$$

and the statement of the theorem follows. □

Remark 2.4.1. *The results of Proposition 2.4.1 and Theorem 2.4.1 generalize known expressions for spectrally positive Lévy processes established in Section 8 of Kyprianou (2006). Moreover, when combined with Theorem 2.3.3, these give more compact and somewhat more explicit expressions for the EDPF in this setting.*

2.4.3. Spectrally-negative risk processes

It is interesting to point out that we can recover well-known results when the number of states is $N = 1$, i.e. when the process J is irrelevant. In this case, the process Y is simply a spectrally-positive Lévy process. Hence, $Q = 0$ and $C(q) = \frac{1}{\Phi(q)}$ and we have, with the help of Proposition 2.4.1 that,

$$M^{(q)}(z) \;\; = \;\; q\int_0^z W^{(q)}(y)dy - \frac{q}{\Phi(q)} W^{(q)}(z). \tag{2.4.26}$$

It follows that

$$M^{(q)}(dz) \;\; = \;\; qW^{(q)}(z)dz - \frac{q}{\Phi(q)} W^{(q)}(dz) \,, \tag{2.4.27}$$

where for every $q \geq 0$, $W^{(q)}(x) : \mathbb{R} \longrightarrow [0,\infty)$ denotes the so-called q-scale function for spectrally-negative Lévy process $-Y$, such that $W^{(q)}(x) = 0$ for all $x < 0$ and, otherwise, is absolutely continuous on $(0,\infty)$ satisfying,

$$\int_0^{\infty} e^{-\lambda x} W^{(q)}(x)dx = \frac{1}{\psi(\lambda) - q} \,, \qquad \text{for } \lambda > \Phi(q) \,, \tag{2.4.28}$$

where $\psi(\lambda) = \log \mathbb{E}(e^{-\lambda Y_1})$ and $\Phi(q)$ becomes simply the largest solution to the equation $\psi(\theta) = q$ [for a review on this subject, see Chapter 8 in Kyprianou (2006) and Kyprianou *et al.* (2011)].

In addition, in this case ($N = 1$), we have that $\Lambda(q) = \widehat{\Lambda}(q) = 0$ and (2.4.14) reduces to

$$\int_{\mathbb{R}} e^{-\alpha u} K^{(q)}(du) \quad = \quad \frac{\Phi(q)}{\Phi(q) + \alpha} \, . \tag{2.4.29}$$

As a consequence, $K^{(q)}$ is exponentially distributed with parameter $\Phi(q)$ and it coincides with the distribution of $-\underline{Y}_{e_q}$. By using (2.4.27), we may develop further the expression for (2.4.13) as follows,

$$\begin{aligned} U_x^{(q)}(dy) &= \Big(\int_y^x e^{-\Phi(q)z} W^{(q)}(dz) - \Phi(q) W^{(q)}(z) dz\Big) e^{\phi(q)(y)} dy \\ &= \big[e^{-\Phi(q)x} W^{(q)}(x) - e^{-\Phi(q)y} W^{(q)}(y)\big] e^{\phi(q)(y)} dy \\ &= \big[e^{-\Phi(q)(x-y)} W^{(q)}(x) - W^{(q)}(y)\big] dy. \end{aligned} \tag{2.4.30}$$

This shows that there exists a density, $u_x^{(q)}(y)$, for the measure $U_x^{(q)}(dy)$ and that it is given by

$$u_x^{(q)}(y) \quad = \quad e^{-\Phi(q)(x-y)} W^{(q)}(x) - W^{(q)}(y) \, , \tag{2.4.31}$$

as expected.

As we have seen, when the process Y is a spectrally-positive Lévy process, we can give more detailed expressions for the q-potential measure $U^{(q)}$ and for the Lévy kernel L. In this case, Theorems 2.3.2 and 2.3.3 reduce to well-known results in the literature for spectrally-negative Lévy risk processes [see Biffis and Kyprianou (2010) and Biffis and Morales for results on the EDPF for spectrally-negative Lévy risk processes].

Indeed, if $\mathbb{S}$ is a singleton then the q- potential measure defined in (2.3.4) becomes

$$U_x^{(q)}(dy) = \int_0^\infty e^{-qt} dt \mathbb{P}(Y_t \in dy, \tau_x > t) \, ,$$

and the Lévy kernel is simply given by

$$L(dy) = \nu(dy) \, ,$$

where ν is the Lévy measure of Y.

In this special case, recall that there is a version of the density of the measure $U_x^{(q)}(dy)$ that we denote with $u_x^{(q)}(s)$ given by (2.4.31). Then, the MEDPF in Theorem

3.4.1 reduces to the EDPF and can be written as

$$\begin{aligned}\phi(w,q,x) &= \mathbb{E}\Big[e^{-q\tau_x} w(x - Y_{\tau_x -}, Y_{\tau_x} - x) 1_{\{\tau_x<\infty\}}\Big] \\ &= \int_{[0,\infty]}\int_{(0,\infty)} w(v,u) U_x^{(q)}(x-dv)\nu(du+v) \\ &= \int_{[0,\infty]}\int_{(0,\infty)} w(v,u) u_x^{(q)}(x-v)\nu(du+v)dv\ . \qquad (2.4.32)\end{aligned}$$

We know that from (2.4.31) the density $u_x^{(q)}$ can be given in terms of the so-called scale function.

After substituting (2.4.31) in (2.4.32), the EDPF can be written out in terms of the q-scale functions $W^{(q)}$ as follows,

$$\phi(w,q,x) = \int_0^\infty \int_0^\infty w(v,u)\big[e^{-\phi(q)v} W^{(q)}(x) - W^{(q)}(x-v)\big]\nu(du+v)dv. \qquad (2.4.33)$$

The identity (3.4.11) is given in a more general form in Biffis and Kyprianou (2010), when the EDPF also includes the size of the last minimum before ruin $x - \overline{Y}_{\tau_x -}$.

2.5. Conclusion

In this paper we have studied a risk model driven by a spectrally-positive Markov additive process. The motivation is that such a model is no longer time-homogeneous and it allows for market conditions changing over the long term. This is modeled via a background finite-state continuous in time Markov chain that represents different macroeconomic scenarios. For such a model, we derive expressions for the Expected Discounted Penalty Function (EDPF) in terms of the characteristics and the q-potential measure of the risk processes. This is a first step towards obtaining tractable examples for which the EDPF can be computed. The main contributions of this paper are found in Theorems 2.3.1, 2.3.2 and 2.3.3. These three results give expressions for the EDPF in a Markov additive process setting. The expressions are not completely explicit and so we have studied further the problem of identifying q-potential measure. This is another contribution that can be found in Theorem 2.4.1. In all, further work is needed in order to study the numerical tractability of these results and to identify examples for which these expressions can be computed with ease.

Chapter 3

ON A GENERALIZATION OF THE EXPECTED DISCOUNTED PENALTY FUNCTION TO INCLUDE DEFICITS AT AND BEYOND RUIN

Abstract

In this chapter we propose an extended concept of the expected discounted penalty function (EDPF) that takes into account new ruin-related random variables. We add to the EDPF, which was introduced in classical papers [Gerber and Shiu (1997), (1998) and Gerber and Landry (1998)], a sequence of expected discounted functions of new record minima reached by a jump of the risk process after ruin. Inspired by results of Huzak et al. (2004) and developpements in fluctuation theory for spectrally negative Lévy processes, we provide a characterization for this extended EDPF in a setting involving a cumulative claims modelled by a subordinator, and Brownian perturbation. We illustrate how the extended EDPF can be used to compute the expected discounted value of capital injections (EDVCI) for Brownian perturbed risk model.

3.1. Introduction

The concept of Expected Discounted Penalty Function (EDPF) has been introduced in classical papers [Gerber and Shiu (1997), (1998)]. This so-called Gerber-Shiu function is a functional of the ruin time (i.e., the first time the reserve level of a firm becomes negative), the surplus prior to ruin, and the deficit at ruin. The EDPF has been extensively studied and generalized to various scenarios and there is now a wide range of models for which expressions of the EDPF are available. Since the EDPF operates on a random cashflow at ruin, where the cashflow is a function of the deficit at ruin and the surplus prior to ruin, applications in the context of insurance and finance are quite natural. For example, the EDPF can be used to determine the initial capital required by an insurance company to avoid insolvency with a minimum level of confidence and for fixed penalization of the ruin event. Similary, the EDPF can be used as a pricing device for American options [Gerber and Shiu (1998b)].

In oder to have a more valuable Gerber-Shiu function for the management of insurance risks and the monitoring of the solvency of a firm, Biffis and Morales (2010) extended the EDPF to include path-dependent penalties. In particular, they generalized the definition of EDPF to include a new random variable, the last minimum of the surplus before ruin. They obtained a defective renewal equation for this generalized EDPF. The representation is obtained for a subordinator risk model perturbed by a spectrally negative Lévy process. More generally, when the risk process is driven by a spectrally negative Lévy process, Biffis and Kyprianou (2010) provided an explicit characterization of this generalized EDPF in terms of scale functions, extending results available in literature. One of the reasons for the limited use of such EDPF is that the last minimum of the surplus before ruin, the surplus prior to ruin and the deficit at ruin, only characterize the surplus before and in a neighborhood of the ruin time. In other words, none of the arguments in the EDPF can be used as a predictive tool for successive deficit times after ruin. The situation would change if a penalty could apply after ruin, for example by acting on relevant characteristics of the paths of the risk process that may lead to successive minima after ruin, and not just on its level before, at, or immediately prior to ruin.

In this chapter we show how to extend the EDPF to include these new random variables. In particular, we generalize the EDPF to include the sequence of successive record minima reached by a jump of the risk process after ruin. We obtain a new form of EDPF which gives characteristics of the paths of the risk process after ruin and not only before, and in a neighborhood of, ruin time. There are practical applications of this extended EDPF in the context of insurance and reinsurance. For example, it can be used to determine the capital required by an insurance company to survive not only in the neighborhood of ruin, but also after ruin when the risk process continuous to jump downwards, that is it continues to pay out claims. Similary, this extended EDPF can be used by insurance company or government institutions to determine the capital which should be injected at each deficit time at and after ruin that will allow it to continue its operations. At the time of ruin, the insurance company could have access to other reserves allowing it to survive and pay to the customers claims made after ruin. It is at this time that the company might need the capital injections that will allow the net aggregate cash inflow to return to pre-ruin levels. This ruin cycle may occur several times and requires repeated interventions with capital injections. Thus, it would be interesting to add to the EDPF the expectation of a sequence of discounted functions of the successive minima reached by claims of the risk process after ruin. This extended EDPF is needed to determine the Expected Discounted Value of Capital Injection (EDVCI) for a subordinator risk model perturbed by a Brownian motion. Moreover, our approach provides a connection with some of the works in Einsenberg and Schmidli (2011) where they have studied the EDVCI for the classical risk model. Consequently, we generalize some results in Einsenberg and Schmidli (2011) where a similar problem is solved for the classical risk model.

We use the results of Huzak *et al.* (2004) and developments in fluctuation theory for spectrally negative Lévy processes to give an explicit characterization of this extended EDPF. The characterization is obtained for a subordinator risk model perturbed by a Brownian motion.

The chapter is organized as follows. In Section 3.2, we describe a perturbed subordinator risk model and define our extended EDPF. In Section 3.3 and we review and provide some results that are needed in our derivations. In particular, we briefly review some results in Huzak *et al.* (2004) about fluctuation theory for spectrally negative Lévy processes, and we give some preliminary results for first-passage times of a subordinator risk process perturbed by a Brownian motion under a change of measure. In section 4, we provide the expression of the extended EDPF in terms of convolution product of densities which are identified in Section 2. In section 5, we show how the results of section 4 can be used to give explicitly the EDVCI for a subordinator risk model perturbed by a Brownian motion, and how it can be used to recuperate the expression of the EDVCI for the classical risk model [see Einsenberg and Schmidli (2011)]. Finally, Section 6 offers some concluding remarks.

3.2. Risk model and the Expected Discounted Penalty Function

Let $(\Omega, \mathbf{F}, \mathbb{P})$ be a filtered probability space on which all random variables will be defined. Let us define $S = (S_t)_{t \geq 0}$ to be a subordinator (i.e., a Lévy process of bounded variation and nondecreasing paths) without a drift. Let ν be the Lévy measure of S; that is, ν is a σ-finite measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge y)\nu(dy) < \infty$.

We define the spectrally negative (i.e. a Lévy process with negative jumps) process $X = (X_t)_{t\geq 0}$ as

$$X_t = ct - S_t + Z_t \ , \tag{3.2.1}$$

where Z is a multiple of a standard Brownian motion, we write

$$Z_t = \sigma B_t,$$

with B a standard Brownian motion independent of S. Let $\mathcal{F} := (\mathcal{F}_t)_{t\geq 0}$ be the filtration obtained by $\sigma(S_s, B_s, s \geq 0)$.

We consider a very general setup that generalizes the standard Cramér-Lundberg model. The model discussed in this paper is,

$$R_t := x - Y_t \ , \qquad t \geq 0 \ , \tag{3.2.2}$$

where $Y = (Y_t)_{t\geq 0}$ is a spectrally positive Lévy process (i.e. a Lévy process with positive jumps) defined by $Y_t = -X_t$, where X_t is given by (3.2.1). As introduced previously, note that the risk process given by

$$R_t = x + ct - S_t + Z_t \ , \tag{3.2.3}$$

is on the same spirit as the original perturbed model in Dufresne and Gerber (1991). The constant $x > 0$ represents the initial surplus, while the process Y represents the cash outflow of an insurance company. The subordinator S represents cumulated claims, and this is why we need it to be increasing since the jumps represent claims paid out. The Brownian motion Z accounts for any fluctuations affecting the components of the risk process dynamics, such as claims arrivals, premium income and investment returns; $c\,t$ represents premium inflow over the interval of time $[0, t]$.

The premium rate c is assumed to satisfy the net profit condition, precisely $\mathbb{E}[S_1] < c$, which requires

$$\int_{(0,\infty)} y\nu(dy) < c \ . \tag{3.2.4}$$

The condition in equation (3.2.4) implies that the process Y has a negative drift in order to avoid the possibility that R becomes negative almost surely. This condition is often expressed in terms of a safety loading. Indeed, it is standard to write the drift component within Y as a loaded premium. For instance, notice that we can recuperate the classical Cramér-Lundberg model if $\sigma = 0$ where $c := (1 + \theta)\mathbb{E}[S_1]$ and S is a compound Poisson process modeling aggregate claims. The drift c, with a positive safety loading $\theta > 0$, is the collected premium rate. We do not use the concept of safety loading in this paper in order to simplify the notation but we stress the fact that this concept is implicitly considered within the drift of Y when we impose the condition in equation (3.2.4). The classical compound Poisson model can be incorporated in this framework by setting $\nu(dy) = \lambda K(dy)$, where λ is the Poisson arrival rate and K is a diffuse claim distribution.

We refer to Asmussen (2000) for an account on the classical risk model, and to Furrer and Schmidli (1994), Yang and Zhang (2001), Huzak *et al.* (2004) and Biffis and Morales (2010) for different generalizations and studies of model (3.2.2).

Now, one of the main objects of interest in ruin theory is the *ruin time*, τ_x, representing the first passage time of R_t below zero when $R_0 = x$, i.e.

$$\tau_x := \inf\{t > 0 \ : \ Y_t > x\} \ , \tag{3.2.5}$$

where we set $\tau_x = +\infty$ if $R_t \geq 0$ for all $t \geq 0$.

Associated with the ruin time τ_x, we have at least two other quantities that contain relevant information on the ruin event from a risk management perspective, namely the deficit at ruin $-R_{\tau_x} = Y_{\tau_x} - x$ and the surplus immediately prior ruin $R_{\tau_x-} = x - Y_{\tau_x^-}$.

Gerber and Shiu (1998) studied the ruin event in the compound Poisson case by analyzing the joint law of all these quantities in one single object, the EDPF. In the following, we define the EDPF under the model (3.2.2).

Definition 3.2.1. *Let w be a non-negative Borel-measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$ such that $w(.,0) = 0$. For $q \geq 0$, the EDPF associated with the risk process (3.2.2) is defined as*

$$\phi(w;x;q) = \mathbb{E}\Big[e^{-q\tau_x} w(x - Y_{\tau_x^-}, Y_{\tau_x} - x)\Big] \, . \tag{3.2.6}$$

Note that the condition $w(\cdot, 0) = 0$ excludes the event $\{Y_{\tau_x} = x\}$. This possibility is known as creeping and we chose not to consider it in our analysis. For simplicity, we assume that the function w assigns a zero penalty when ruin occurs by continuously crossing over zero. Notice that for a model like (3.2.2), this only happens when ruin is caused by the Brownian motion component of the process Y.

Following the same order of ideas, we study the EDPF under the general context which gives relevant informations on and after the ruin event. More precisely, we generalize the EDPF defined in (3.2.6) to include the quantities associated with the ruin time τ_x and times sequence of successive minima reached by a claim of the risk process (3.2.2) after ruin. This implies that some of notations related to record minima need to be introduced .

Thus, we must define the first new record time of the running supremum

$$\tau := \inf\{t > 0, Y_t > \overline{Y}_{t^-}\}\ , \tag{3.2.7}$$

and the sequence of times corresponding to new records of Y reached by a jump of S after the ruin time. More precisely, let

$$\tau^{(1)} := \tau_x, \tag{3.2.8}$$

and inductively on $\{\tau^{(n)} < \infty\}$,

$$\tau^{(n+1)} := \inf\{t > \tau^{(n)}, Y_t > \overline{Y}_{t^-}\}. \tag{3.2.9}$$

Recall from Theorem 4.1 of Huzak *et al.* (2004) that the sequence $(\tau^{(n)})_{n\geq 1}$ is discrete, and, in particular, neither time 0 non any other time is an accumulation point of those times. More precisely, $\tau > 0$ a.s. and $\tau^{(n)} < \tau^{(n+1)}$ a.s. on $\{\tau^{(n)} < \infty\}$. As a consequence, we can order the sequence $(\tau^{(n)})_{n\geq 1}$ of times when a new supremum is reached by a jump of a subordinator as $0 < \tau^{(1)} < \tau^{(2)} < \cdots$ a.s.

Let us introduce the random number N given by

$$N := \max\{n : \tau^{(n)} < \infty\}, \tag{3.2.10}$$

which represents the number of new records reached by a claim of the risk process (3.2.2). In the following, we study the EDPF in a new context involving the deficits at times $(\tau^{(n)})_{n\geq 1}$. More precisely, in this paper we set out to study the following extended EDPF for the model (3.2.2)

Definition 3.2.2. *Let $F = (F_n)_{n\geq 0}$ be a sequence of non-negative measurable functions from $\mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}$, x and $q \geq 0$. The discounted penalty associated with the risk process (3.2.2), F and q is defined as*

$$P(F, q, x) = \mathbb{E}\Big[\sum_{n=1}^{N} e^{-q\tau^{(n)}} F_n(Y_{\tau^{(n-1)}}, Y_{\tau^{(n)}}); \tau_x < \infty\Big]. \tag{3.2.11}$$

We assume in the previous definition that $\tau^{(0)} = \tau_x^-$ and

$$F_1(\cdot, x) = 0. \tag{3.2.12}$$

Note that the condition given by (3.2.12) is used to exclude from calculation the event $\{Y_{\tau_x} = x\}$. $P(F, q, x)$ is an extension of the classical EDPF defined in (3.2.6). In particular, it reduces to $\phi(w; x; q)$ since we suppose that $F_1(u, v) = w(x - u, v - x)$ and $F_n = 0$ for $n \geq 2$.

3.3. Preliminary results

In this section, we will give some preliminary results for first-passage times of the risk process defined in (3.2.2) under change of measure. These results are based on the works in Huzak *et al.* (2004) where the ruin probability has been studied for a subordinator risk model perturbed by a spectrally negative Lévy process. This allows us to give a more detailed analysis of the extended EDPF defined in (3.2.11).

Recall from Section 3.2 that S is a subordinator defined on the filtred probability space $(\Omega, \mathbf{F}, \mathbb{P})$. The Laplace exponent of S is defined by

$$\psi_S(\alpha) = \int_{(0,\infty)} [e^{\alpha y} - 1]\nu(dy) \ , \tag{3.3.1}$$

where

$$\mathbb{E}[\exp(\alpha S_t)] = \exp(t\psi_S(\alpha)) \ . \tag{3.3.2}$$

Note that

$$\mathbb{E}[S_1] = \psi_S^{'}(0^+) = \int_{(0,\infty)} y\nu(dy) = \int_0^\infty \nu(y,\infty)dy \ , \tag{3.3.3}$$

where the last equality follows from integration by parts. As explained before, we assume throughout that $\mathbb{E}[S_1] < \infty$.

The Laplace exponent ψ of X defined in (3.2.1) is defined by the relation

$$\mathbb{E}[\exp(\beta X_t)] \quad = \quad \exp(t\psi_X(\beta)), \tag{3.3.4}$$

where

$$\begin{aligned} \psi_X(\beta) \quad &= \quad c\beta + \psi_S(-\beta) + \psi_Z(\beta) \\ &= \quad \psi_S(-\beta) + \widetilde{Z}(\beta) \qquad \beta \geq 0 \ , \end{aligned}$$

where $\widetilde{Z}_t = ct + Z$ and $\psi_{\widetilde{Z}}(\beta) = c\beta + \psi_Z(\beta)$. The last equality is due to the independence of S and Z. We refer to Bertoin (1996), Sato (1999) and Kyprianou (2006) for a comprehensive account on Lévy process.

Let us introduce the distribution function G of $-\inf_{t\geq 0}(\widetilde{Z}_t) = \sup_{t\geq 0}(-ct - Z_t)$. Using a method similar to Yang and Zhang (2001) (see also Huzak *et al.* (2004)), the Laplace transform of G can be shown to be given by

$$\begin{aligned} \widehat{G}(\beta) \quad &= \quad \int_0^\infty e^{-\beta y} G(dy) \\ &= \quad \frac{c\beta}{\psi_Z(\beta)} \ . \end{aligned} \tag{3.3.5}$$

That is,

$$\widehat{G}(\beta) \quad = \quad \frac{c\beta}{c\beta + \frac{\sigma^2\beta^2}{2}} \ , \tag{3.3.6}$$

since $Z_t = \sigma B_t$ is a multiple of a standard Brownian motion. Then, G is given explicitly as an exponential distribution function with parameter $2c/\sigma^2$, i.e. G has density

$$G(dy) = \frac{2c}{\sigma^2} e^{-\frac{2c}{\sigma^2}y} dy \ . \tag{3.3.7}$$

We also introduce the parameter

$$\rho \quad := \quad \frac{\mathbb{E}[S_1]}{c} = \frac{1}{c}\int_{(0,\infty)} y\nu(dy) \in (0,1) \ . \tag{3.3.8}$$

We denote by $\overline{Y}$ the supremum given by $\overline{Y}_t = \sup_{s\geq t} Y_s$, for $t \geq 0$.

Let us introduce the density of the overshoot at time τ which is defined by

$$H(du) = \mathbb{P}(Y_\tau - \overline{Y}_{\tau^-} \in du;\ \tau < \infty) \ , \tag{3.3.9}$$

where $u > 0$. We shall give in the following proposition the density $H(\cdot)$ in terms of Lévy measure and the premium rate.

Proposition 3.3.1. *Let Y the spectrally-positive Lévy process defined in (3.2.3).*
1-The distribution of $Y_\tau - \overline{Y}_{\tau^-}$ on the set $\tau < \infty$ is given by

$$H(du) = \frac{1}{c}\int_0^\infty \nu(du + y)dy \ ; \quad u > 0 \ . \tag{3.3.10}$$

2- The distribution of Y_τ is given by

$$\mathbb{P}(Y_\tau \in du;\ \tau < \infty) = H * G(du) \ ; \quad u > 0 \ , \tag{3.3.11}$$

where $H * G(\cdot)$ denotes the convolution of $H(\cdot)$ with $G(\cdot)$ defined by

$$\int_A f(u)H * G(du) = \int_{\{y+v\in A\}} f(y+v)H(dy)G(dv) \ ,$$

for all Borel set A of $\mathbb{R}\times\mathbb{R}$.

Proof. 1- Let us suppose that f is a nonnegative bounded Borel function. We prove firstly that

$$\mathbb{E}[f(Y_\tau - \overline{Y}_{\tau^-});\ \tau < \infty] \quad = \quad \mathbb{E}[\int_0^\tau \widetilde{f}(\overline{Y}_t - Y_t)dt] \ , \tag{3.3.12}$$

where $\widetilde{f}(y) = \int_{(0,\infty)} f(u-y)1_{\{u>y\}}\nu(du)$. The proof of (3.3.12) follows by an application of the compensation formula [see Bertoin(1996) p.9 or Theorem 4.4. of Kyprianou

(2006)] applied to the Poisson random measure, with intensity measure dt , $\nu(du)$ associated to the jump of Y. We have

$$
\begin{aligned}
\mathbb{E}[f(Y_\tau - \overline{Y}_{\tau^-});\ \tau < \infty] &= \mathbb{E}[\sum_{t>0} f(Y_t - \overline{Y}_{t-}) 1_{\{Y_t > \overline{Y}_{t-}, t \le \tau\}}] \\
&= \mathbb{E}[\int_0^\infty \int_{[0,\infty)} f(u - \overline{Y}_t + Y_t) 1_{\{u > \overline{Y}_t - Y_t, t \le \tau\}}] dt \nu(du) \\
&= \mathbb{E}[\int_0^\tau \int_{(0,\infty)} f(u - (\overline{Y}_t - Y_t)) 1_{\{u > \overline{Y}_t - Y_t\}}] dt \nu(du) \\
&= \mathbb{E}[\int_0^\tau \widetilde{f}(\overline{Y}_t - Y_t) dt]. \qquad (3.3.13)
\end{aligned}
$$

Hence, from Proposition 4.3. of Huzak *et al.* (2004), the expected occupation time measure of (3.3.13) is given by

$$
\mathbb{E}[\int_0^\tau \widetilde{f}(\overline{Y}_t - Y_t) dt] = \frac{\mathbb{P}(\tau = \infty)}{c - \mathbb{E}[S_1]} \int_0^\infty \widetilde{f}(y) dy. \qquad (3.3.14)
$$

Since $\mathbb{P}(\tau = \infty) = 1 - \mathbb{P}(\tau < \infty) = 1 - \rho$ [see Corollary 4.5. of Huzak *et al.* (2004)], (3.3.14) is equal to

$$
\begin{aligned}
\frac{1-\rho}{c - \mathbb{E}[S_1]} \int_0^\infty \widetilde{f}(u) du &= \frac{1}{c} \int_0^\infty \int_{(0,\infty)} f(u-y) 1_{\{u>y\}} \nu(du) dy \\
&= \frac{1}{c} \int_{(0,\infty)} f(u) \int_0^\infty \nu(du + y) dy. \qquad (3.3.15)
\end{aligned}
$$

Equating the left-hand side of (3.3.12) and (3.3.15) implies that

$$
\mathbb{P}(Y_\tau - \overline{Y}_{\tau^-} \in du;\ \tau < \infty) = \frac{1}{c} \int_{(0,\infty)} \int_0^\infty \nu(du + y) dy\ , \qquad (3.3.16)
$$

and the statement 1 of Proposition 3.3.1 follows.

2- Using the conditional independence of $Y_\tau - \overline{Y}_{\tau^-}$ and $\overline{Y}_{\tau^-}$ given $\tau < \infty$,

$$
\begin{aligned}
\mathbb{P}(Y_\tau \in du\ ;\ \tau < \infty) &= \mathbb{P}(Y_\tau \in du |\ \tau < \infty) \mathbb{P}(\tau < \infty) \qquad (3.3.17) \\
&= \mathbb{P}(Y_\tau - \overline{Y}_{\tau^-} + \overline{Y}_{\tau^-} \in du |\ \tau < \infty) \rho \qquad (3.3.18) \\
&= H * \overline{G}(du)\ , \qquad (3.3.19)
\end{aligned}
$$

where $\overline{G}(du)$ is the conditional distribution $\mathbb{P}(\overline{Y}_{\tau^-} \in du |\ \tau < \infty)$. From corollary 4.6 of Huzak *et al.* (2004), $\overline{G}(du)$ is equal to unconditionnal distribution $\mathbb{P}(\overline{Y}_{\tau^-} \in du)$. Corollary 4.10 of Huzak *et al.* (2004) implies that

$$
\mathbb{P}(\overline{Y}_{\tau^-} \in du) = \mathbb{P}(\sup_{t \ge 0}(-ct - \sigma B_t) \in du), \qquad (3.3.20)
$$

from which it immediately follows that $\overline{G} = G$. Thus we have proved the the statement 2 of proposition. $\square$

For the right inverse of ψ, we shall write ϕ on $[0,\infty)$. That is to say, for each $q \geq 0$,

$$\Phi(q) = \sup\{\alpha \geq 0 : \psi(\alpha) = q\}. \tag{3.3.21}$$

Note that the properties of ψ for such a Lévy process X, imply that $\phi(q) > 0$ for $q > 0$. Further $\phi(0) = 0$, since $\psi'(0) = c - \mathbb{E}[S_1] \geq 0$.

Let us define the probability measure $\widetilde{\mathbb{P}}$ by the density process

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{-\Phi(q)Y_t - qt} \tag{3.3.22}$$

where $\Phi(q)$ is the right inverse of ψ defined in (3.3.21). Note that under $\widetilde{\mathbb{P}}$ the process Y introduced in (3.2.1), is still a spectrally positive Lévy process, and still drifts to $-\infty$ [see Kyprianou (2006)]. In addition, the process Y keeps the same form under $\widetilde{\mathbb{P}}$ and then, $Y_t = -\widetilde{c}t + S_t - \sigma\widetilde{B}_t$, where $\widetilde{c} = c + \sigma^2\Phi(q)$ and $\widetilde{B}_t = B_t - \sigma\Phi(q)t$ are respectively the premium rate and the standard Brownian motion under the probability measure $\widetilde{\mathbb{P}}$.

We denote by $\widetilde{\nu}$ the Lévy measure of S under the change of measure $\widetilde{\mathbb{P}}$ and then,

$$\widetilde{\nu}(du) = e^{-\Phi(q)u}\nu(du). \tag{3.3.23}$$

Let

$$\widetilde{\rho} = \frac{\widetilde{\mathbb{E}}[S_1]}{\widetilde{c}} = \frac{\int_{(0,\infty)} ye^{-\Phi(q)y}\nu(dy)}{c + \sigma^2\Phi(q)^2}\ , \tag{3.3.24}$$

then by (3.2.4), $0 < \widetilde{\rho} < 1$ and the net profit condition is well preserved under $\widetilde{\mathbb{P}}$. Note that the distribution function of $-\inf_{t\geq 0}(Z_t) = \sup_{t\geq 0}(-ct - \sigma B_t)$ under the change of measure, is denoted by $\widetilde{G}$. Recall from (3.3.6) that the density $\widetilde{G}$ is given via its Laplace transform by

$$\int_0^\infty e^{-\beta y}\widetilde{G}(dy) \quad = \quad \frac{\widetilde{c}\beta}{\widetilde{c}\beta + \frac{\sigma^2\beta^2}{2}}, \quad \beta > 0; \tag{3.3.25}$$

and then, $\widetilde{G}$ is given explicitly as an exponential distribution function with parameter $2\widetilde{c}/\sigma^2$. We denote by $\widetilde{H}$ the distribution of $Y_\tau - \overline{Y}_{\tau^-}$ under the probability measure $\widetilde{\mathbb{P}}$, i.e

$$\widetilde{H}(du) = \widetilde{\mathbb{P}}(Y_\tau - \overline{Y}_{\tau^-} \in\ du\ ;\ \tau < \infty),\ \text{for } u > 0. \tag{3.3.26}$$

Moroever, since the characteristics of the risk process are preserved under the change of measure, we can derive a result analogous to Proposition 3.3.1 under $\widetilde{\mathbb{P}}$ as:

Proposition 3.3.2. *Let Y be a spectrally positive Lévy process defined in (3.2.3).*
1-The distribution of $Y_\tau - \overline{Y}_{\tau-}$ on the set $\tau < \infty$ under the probability measure $\widetilde{\mathbb{P}}$ is given by

$$\begin{aligned} \widetilde{H}(du) &= \frac{1}{\widetilde{c}} \int_0^\infty \widetilde{\nu}(du+y)dy \\ &= \frac{e^{-\Phi(q)u}}{c+\Phi(q)\sigma^2} \int_0^\infty e^{-\Phi(q)y} \nu(du+y)dy \ ; \ u > 0. \end{aligned} \tag{3.3.27}$$

2- The distribution of Y_τ under the probability measure $\widetilde{\mathbb{P}}$ is given by

$$\widetilde{\mathbb{P}}(Y_\tau \in du \ ; \ \tau < \infty) = \widetilde{H} * \widetilde{G}(du) \ ; \ u > 0. \tag{3.3.28}$$

In the following paragraph, we shall give the ruin probability under the change of measure defined by (3.3.22).

We denote by $\widetilde{\theta}(x)$ the ruin probability under the probability measure $\widetilde{\mathbb{P}}$, that is

$$\widetilde{\theta}(x) = \widetilde{\mathbb{P}}(\sup_{t\geq 0} Y_t < x) \ , \qquad x \geq 0.$$

For $q \geq 0$, the following proposition gives the Pollazek-Hinchin formula for the survival probability under the change measure $\widetilde{\mathbb{P}}$.

Proposition 3.3.3. *The survival probability of the general perturbed risk process introduced in (3.2.2) is given by*

$$1 - \widetilde{\theta}(x) = (1-\widetilde{\rho}) \sum_{n=0}^\infty \big(\widetilde{L}^{*(n)} * \widetilde{G}^{*(n+1)}\big)(x) \widetilde{\rho}^n \ ; \tag{3.3.29}$$

where

$$\widetilde{L}(dy) = \frac{1}{\widetilde{c}} \widetilde{\nu}(y,\infty)dy 1_{\{y>0\}} \ ,$$

and $\widetilde{G}(\cdot)$ is an exponential distribution function with parameter $2\widetilde{c}/\sigma^2$, f^{*n} $(n \geq 1)$ denotes the n-fold convolution of f with itself and f^{*0} is the distribution function corresponding to the Dirac measure at zero.

Proof. Using a similar method to that in Huzak *et al.* (2004), we have by taking limits in the Laplace transform of the infimum evaluated at an independant exponential time e_q with parameter $q > 0$ [see chapter VIII in Kyprianou (2006)] ,

$$\begin{aligned} \widetilde{\mathbb{E}}[e^{\beta \underline{X}_\infty}] &= \widetilde{\mathbb{E}}[e^{-\beta \overline{Y}_\infty}] \\ &= \widetilde{\psi}'(0+) \frac{\beta}{\widetilde{\psi}(\beta)}, \ \text{for } \beta > 0. \end{aligned} \tag{3.3.30}$$

Let us now compute (3.3.30) in terms of $\widetilde{\rho}$, $\widehat{\widetilde{L}}$ and $\widehat{\widetilde{G}}$, where $\widetilde{\rho}$ is the parameter given by (3.3.24), and $\widehat{\widetilde{L}}$ and $\widehat{\widetilde{G}}$ are respectively the Laplace transforms of $\widetilde{L}$ and $\widetilde{G}$. Equation (3.3.30) is equal to

$$\frac{\widetilde{d}}{\widetilde{c}} \quad \frac{\widehat{\widetilde{G}}(\beta)}{1-\widetilde{\rho}\widehat{\widetilde{G}}(\beta)\widehat{\widetilde{L}}(\beta)} \quad = \quad (1-\widetilde{\rho})\widehat{\widetilde{G}}(\beta)\sum_{n=0}^{\infty}\left(\widetilde{\rho}\widehat{\widetilde{G}}(\beta)\widehat{\widetilde{L}}(\beta)\right)^{n}. \tag{3.3.31}$$

By inverting the Laplace transform (3.3.31), we obtain

$$\widetilde{\mathbb{P}}(\overline{Y}_{\infty} \leq x) = 1-\widetilde{\theta}(x) = (1-\widetilde{\rho})\sum_{n=0}^{\infty}\widetilde{\rho}^{n}\left(\widetilde{G}^{*(n+1)} * \widetilde{L}^{*n}\right)(x). \tag{3.3.32}$$

□

We now introduce the so-called *q-scale function* $\{W^{(q)},\ q \geq 0\}$ of the process X. For every $q \geq 0$, there exists a function $W^{(q)} : \mathbb{R} \longrightarrow [0,\infty)$ such that $W^{(q)}(y) = 0$ for all $y < 0$ and otherwise absolutely continuous on $(0,\infty)$ satisfying

$$\int_0^{\infty} e^{-\lambda y}W^{(q)}(y)dy = \frac{1}{\psi(\lambda)-q}\,, \quad \text{for} \quad \lambda > \Phi(q)\,, \tag{3.3.33}$$

where $\Phi(q)$ is the largest solution of $\psi(\beta) = q$ defined in (3.3.21).

For short, we shall write $W^{(0)} = W$. Let us introduce the 0-scale function under $\widetilde{\mathbb{P}}$, which we write as $W_{\Phi(q)}$, related to the q-scale function of X, $W^{(q)}$, via the relation

$$W^{(q)}(y) = e^{\Phi(q)y}W_{\Phi(q)}(y). \tag{3.3.34}$$

The reader is otherwise referred to Bertoin (1996) and Chapter 8 of Kyprianou (2006) for a fuller account.

In the following, we shall describe the discounted penalty function introduced in Definition 3.2.2 in terms of densities $\widetilde{G}$, $\widetilde{H}$ and q-scale function of the spectrally negative process X which we have briefly introduced above.

3.4. Extension of the Expected Discounted Penalty Function

At this point, we recall that the main objective of this paper is to write an expression for the extended EDPF in (3.2.11). But before we can write out such an expression we need one more intermediate result that has to do with the change of measure defined through the density process in (3.3.22).

Let us start by giving the standard EDPF defined in (3.2.6) in terms of the convolution product of two functions depending on Lévy measure ν and q-scale function which is defined by (3.3.33). Recall that w is a non-negative bounded measurable function on $\mathbb{R} \times \mathbb{R}$ such that $w(\cdot,0) = 0$.

Lemma 3.4.1. *Consider the risk model in (3.2.2).*

(1) The distribution of the overshoot at τ_x, $\widetilde{T}_x$, is given by

$$\widetilde{T}_x(du) = \int_0^\infty \int_0^v e^{-\Phi(q)u} \nu(du - x + v) W'_{\Phi(q)}(x-y)dydv \ , \tag{3.4.1}$$

where $\widetilde{T}_x(du) = \widetilde{\mathbb{P}}(Y_{\tau_x} \in du; \tau_x < \infty)$.

(2) For $q \geq 0$, the EDPF $\phi(w, q, x)$, as introduced in Definition 3.2.1, is given by,

$$f_1 * f_2(x) \ , \tag{3.4.2}$$

where

$$\begin{aligned} f_1(x) &= e^{\Phi(q)x} W'_{\Phi(q)}(x) \\ &= W'^{(q)}(x) - \Phi(q) W^{(q)}(x) \ , \end{aligned} \tag{3.4.3}$$

and

$$f_2(x) = e^{\Phi(q)x} \int_x^\infty e^{-\Phi(q)v} \int_{(0,\infty)} w(u,v)\nu(du+v)dv \ , \tag{3.4.4}$$

for $u > x, v > 0$ and $0 < y < x \wedge v$.

Proof. 1). Follow the method used in Biffis and Kyprianou (2010) [see also the end of Section 8.4 of Kyprianou (2006)] and recalling that X drifts to ∞ (and hence $\Phi(0) = 0$), we know that

$$\begin{aligned} \mathbb{P}(Y_{\tau_x} - x \in du, x - Y_{\tau_x^-} \in dv; \tau_x < \infty) &= \nu(du+v)[W(x) - W(x-v)]dv \\ &= \nu(du+v) \int_0^v W'(x-y)dydv \ , \end{aligned} \tag{3.4.5}$$

for $u > 0, \ v > 0$ and $0 < y < x \wedge v$, where W' is a version of the density of W. Using the previous equality under the change of measure $\widetilde{\mathbb{P}}$, we obtain the identity

$$\begin{aligned} \widetilde{\mathbb{P}}(Y_{\tau_x} - x \in du, x - Y_{\tau_x^-} \in dv \ ; \ \tau_x < \infty) &= \widetilde{\nu}(du+v) \int_0^v W'_{\Phi(q)}(x-y)dydv \\ &= e^{-\Phi(q)(u+v)} \nu(du+v) \\ &\quad \times \int_0^v W'_{\Phi(q)}(x-y)dydv \ , \end{aligned} \tag{3.4.6}$$

for $u>0, v>0$, $0<y<x\wedge v$ and then

$$\begin{aligned}
\mathbb{E}\Big[e^{-q\tau_x}w(Y_{\tau_x}-x, x-Y_{\tau_x^-});\tau_x<\infty\Big] &= \widetilde{\mathbb{E}}\Big[e^{\Phi(q)Y_{\tau_x}}w(Y_{\tau_x}-x, x-Y_{\tau_x^-});\tau_x<\infty\Big] \\
&= \int_{(0,\infty)}\int_{(0,\infty)} e^{\Phi(q)(u+x)}w(u,v) \\
&\quad \times\widetilde{\mathbb{P}}(Y_{\tau_x}-x\in du, x-Y_{\tau_x^-}\in dv;\tau_x<\infty) \\
&= \int_0^x e^{\Phi(q)(x-y)}W'_{\Phi(q)}(x-y)e^{\Phi(q)y} \\
&\quad \times\int_y^\infty e^{-\Phi(q)v}\int_{(0,\infty)} w(u,v)\nu(du+v)dvdy \\
&= f_1 * f_2(x)\,, \qquad (3.4.7)
\end{aligned}$$

where

$$f_2(x)=e^{\Phi(q)x}\int_x^\infty e^{-\Phi(q)v}\int_{(0,\infty)} w(u,v)\nu(du+v)dv. \qquad (3.4.8)$$

By using (3.3.34) we get,

$$\begin{aligned}
f_1(x) &= e^{\Phi(q)x}W'_{\Phi(q)}(x) \\
&= W'^{(q)}(x)-\Phi(q)W^{(q)}(x)\,, \qquad (3.4.9)
\end{aligned}$$

from which, statement in 1) holds.

2). From (3.4.6), we deduce (3.4.1) by writing,

$$\begin{aligned}
\widetilde{T}_x(du) &= \widetilde{\mathbb{P}}(Y_{\tau_x}-x\in du-x;\tau_x<\infty) \\
&= \int_0^\infty\int_0^v e^{-\Phi(q)u}\nu(du-x+v)W'_{\Phi(q)}(x-y)dydv, \qquad (3.4.10)
\end{aligned}$$

for $u>x, v>0$ and $0<y<x\wedge v$. □

Remark 3.4.1. *Equation (3.4.2) is equivalent to the following equality which describes the Gerber-Shiu function in terms of scale function of risk process;*

$$\begin{aligned}
\phi(w,q,x) &= \int_{(0,\infty)}\int_{(0,\infty)} e^{-\Phi(q)(x-v)}w(u,v)\nu(du+v)\big[W^{\Phi(q)}(x)-W^{\Phi(q)}(x-v)\big]dv \\
&= \int_0^\infty\int_0^\infty w(v,u)\big[e^{-\phi(q)v}W^{(q)}(x)-W^{(q)}(x-v)\big]\nu(du+v)dv\,, \qquad (3.4.11)
\end{aligned}$$

by using (3.3.34), where w is a bounded measurable function such that $w(\cdot,0)=0$.

Identity (3.4.11) is given in more general form in Biffis and Kyprianou (2010), where the EDPF also includes the size of the last minimum before ruin $x-\overline{Y}_{\tau_x-}$.

Recall that from Section 3.3 that N is the random number given by (3.2.10). We give in the following proposition the distribution of N under the probability measure defined by (3.3.22).

Proposition 3.4.1. *The distribution of N on $\{\tau_x < \infty\}$ is given by*

$$\widetilde{\mathbb{P}}(N = n, \tau_x < \infty) = (1-\widetilde{\rho})\widetilde{\rho}^n(1-\widetilde{\theta}(x)) \; ; \tag{3.4.12}$$

where $n \geq 0$.

Proof. By the strong Markov property of $\overline{Y}$ under $\widetilde{\mathbb{P}}$, we can identify the distribution of N on $\{\tau_x < \infty\}$ as

$$\begin{aligned}
\widetilde{\mathbb{P}}(N = 0, \tau_x < \infty) &= \widetilde{\mathbb{P}}(\tau^{(1)} = \infty | \tau_x < \infty)\widetilde{\mathbb{P}}(\tau_x < \infty) \\
&= \widetilde{\mathbb{P}}(\tau = \infty)(1-\widetilde{\theta}(x)) \\
&= (1-\widetilde{\rho})(1-\widetilde{\theta}(x)) \; , \qquad (3.4.13)
\end{aligned}$$

where in the last equality we have used Corollary 4.5 of Huzak *et al.* (2004) under the probability measure $\widetilde{\mathbb{P}}$.

For $n \geq 1$,

$$\begin{aligned}
\widetilde{\mathbb{P}}(N = n, \tau_x < \infty) &= \widetilde{\mathbb{P}}(\tau^{(n+1)} = \infty | \tau^{(n)} < \infty)\widetilde{\mathbb{P}}(\tau^{(n)} < \infty | \tau^{(n-1)} < \infty) \times \\
&\quad ... \times \widetilde{\mathbb{P}}(\tau^{(1)} < \infty | \tau_x < \infty)\widetilde{\mathbb{P}}(\tau_x < \infty) \\
&= \widetilde{\mathbb{P}}(\tau = \infty)\widetilde{\mathbb{P}}(\tau < \infty)^n(1-\widetilde{\theta}(x)) \\
&= (1-\widetilde{\rho})\widetilde{\rho}^n(1-\widetilde{\theta}(x)). \qquad (3.4.14)
\end{aligned}$$

□

Now, we are finally in a position to present the first and main contribution of this paper. In the following theorem, we give an expression for the extended EDPF $P(F;q;x)$ given in Definition 3.2.2. This expression is given in terms of the q-scale function, Lévy measure, and the densities $\widetilde{G}$ and $\widetilde{H}$ which are introduced in Section 3.3. Recall that $F = (F_n)_{n\geq 0}$ is a sequence of non-negative measurable functions from $\mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}$ such that $F_0(.,x) = 0$.

Theorem 3.4.1. *Consider the risk model in (3.2.2). For $q \geq 0$, the EDPF $P(F,q,x)$, as introduced in Definition 3.2.2, is given by,*

$$\begin{aligned}
\phi(w,q,x) + \sum_{n=1}^{\infty} \int_{(x,\infty)} \int_{(0,\infty)} & e^{\Phi(q)(u+v)} F_{n+1}(v, u+v) \\
& \widetilde{H} * \widetilde{G}(du)\widetilde{H}^{*n} * \widetilde{G}^{*n} * \widetilde{T}_x(dv) \; , \qquad (3.4.15)
\end{aligned}$$

where w is a mesurable Borel-function satisfying $w(u,v) = F_1(x-v, u-x)$ for $u,\ v \geq 0$ and

$$\phi(w,q,x) = \mathbb{E}\Big[e^{-q\tau_x} w(x - Y_{\tau_x^-}, Y_{\tau_x} - x)\ ; \tau_x < \infty\Big]. \tag{3.4.16}$$

PROOF. We prove the result in three steps. Let us suppose $F = (F_n)_{n\geq 1}$ is a sequence of non-negative measurable functions from $\mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}$, x and $q \geq 0$.

Step 1: We prove in this step that

$$\widetilde{\mathbb{P}}(Y_{\tau^{(k)}} - Y_{\tau^{(k-1)}} \in dy | \tau^{(k)} < \infty) = \frac{1}{\widetilde{\rho}}\, \widetilde{H} * \widetilde{G}(dy); \tag{3.4.17}$$

where $k \geq 2$ and $y \geq 0$. The proof of (3.4.17) follows by an application of Theorem 4.7, Huzak et al (2004). Recall that under the measure change $\widetilde{\mathbb{P}}$, Y has the same form and it can be written as $Y_t = -\widetilde{c}t + S_t - \sigma \widetilde{B}_t$. By using the Markov proprety of $\overline{Y}$ at $\tau^{(k)}$

$$\begin{aligned}\widetilde{\mathbb{P}}(Y_{\tau^{(k)}} - Y_{\tau^{(k-1)}} \in dy | \tau^{(k)} < \infty) &= \widetilde{\mathbb{P}}(Y_\tau \in dy | \tau < \infty) \\ &= \frac{1}{\widetilde{\rho}}\, \widetilde{H} * \widetilde{G}(dy), \end{aligned} \tag{3.4.18}$$

where in the last equality we have used Equation (3.3.28) and the identity $\widetilde{\mathbb{P}}(\tau < \infty) = \widetilde{\rho}$ [see Corollary 4.6, Huzak *et al.* (2004)].

Step 2: Next we prove that

$$\widetilde{\mathbb{P}}(Y_{\tau^{(k)}} \in dy | \tau^{(k)} < \infty) = \frac{1}{\widetilde{\rho}^k (1 - \widetilde{\theta}(x))}\, (\widetilde{H}^{*k} * \widetilde{G}^{*k} * \widetilde{T}_x)(dy), \tag{3.4.19}$$

where $k \geq 2$ and $y \geq 0$.

$$\begin{aligned}\widetilde{\mathbb{P}}(Y_{\tau^{(k)}} \in dy | \tau^{(k)} < \infty) &= \widetilde{\mathbb{P}}(Y_{\tau^{(k)}} - Y_{\tau^{(k-1)}} + ... + Y_{\tau^{(1)}} - Y_{\tau_x} + Y_{\tau_x} \in dy | \tau^{(k)} < \infty) \\ &= \frac{1}{\widetilde{\rho}^k}\, (\widetilde{H}^{*(k)} * \widetilde{G}^{*(k)} * \widetilde{T}_x)(dy) \frac{1}{1 - \widetilde{\theta}(x)}, \end{aligned} \tag{3.4.20}$$

by using the independent increments of Y and (3.4.17), where $\widetilde{T}_x(dy) = \widetilde{\mathbb{P}}(Y_{\tau_x} \in dy\ ; \tau_x < \infty)$.

Step 3: Before using the conclusions of step 1 and step 2, let us write $P(F,q,x)$ as an expansion under $\widetilde{\mathbb{P}}$,

$$\begin{aligned}
P(F,q,x) &= \sum_{n=0}^{\infty}\mathbb{E}\Big[\sum_{k=0}^{n} e^{-q\tau^{(k)}}F_{k+1}(Y_{\tau^{(k)}},Y_{\tau^{(k+1)}})\ ;N=n;\tau_x<\infty\Big] \\
&= \mathbb{E}\Big[e^{-q\tau_x}F_1(Y_{\tau_x^-},Y_{\tau_x}\ ;N=0;\tau_x<\infty)\Big] \\
&\quad+\sum_{n=1}^{\infty}\mathbb{E}\Big[\sum_{k=0}^{n} e^{-q\tau^{(k+1)}}F_{k+1}(Y_{\tau^{(k)}},Y_{\tau^{(k+1)}})\ ;N=n;\tau_x<\infty\Big] \\
&= \mathbb{E}\Big[e^{-q\tau_x}F_1(Y_{\tau_x^-},Y_{\tau_x})\ ;N=0;\tau_x<\infty\Big] \\
&\quad+\sum_{n=1}^{\infty}\Big(\mathbb{E}\Big[e^{-q\tau_x}F_1(Y_{\tau_x^-},Y_{\tau_x})\ ;N=n;\tau_x<\infty\Big] \\
&\quad+\sum_{k=1}^{n}\mathbb{E}\Big[e^{-q\tau^{(k+1)}}F_{k+1}(Y_{\tau^{(k)}},Y_{\tau^{(k+1)}})\ ;N=n\ ,\tau_x<\infty\Big]\Big),
\end{aligned}\tag{3.4.21}$$

where in the last equality we have used Proposition 3.4.1. Then $P(F,q,x)$ is equal to

$$\begin{aligned}
&\widetilde{\mathbb{E}}\Big[e^{\Phi(q)Y_{\tau_x}}F_1(Y_{\tau_x^-},Y_{\tau_x})\Big|\tau_x<\infty\Big](1-\widetilde{\rho})(1-\widetilde{\theta}(x)) \\
&+\sum_{n=1}^{\infty}\Big(\mathbb{E}\Big[e^{-q\tau_x}F_1(Y_{\tau_x^-},Y_{\tau_x})|\tau_x<\infty)\Big]+\sum_{k=1}^{n}\widetilde{\mathbb{E}}\Big[F_{k+1}(Y_{\tau^{(k)}},Y_{\tau^{(k+1)}})|\tau^{(k+1)}<\infty\Big]\Big) \\
&(1-\widetilde{\rho})\widetilde{\rho}^n(1-\widetilde{\theta}(x))
\end{aligned}\tag{3.4.22}$$

$$\begin{aligned}
&= \widetilde{\mathbb{E}}\Big[F_1(Y_{\tau_x^-},Y_{\tau_x})\Big|\tau_x<\infty)\Big](1-\widetilde{\theta}(x))(1-\widetilde{\rho})[1+\sum_{n=1}^{\infty}\widetilde{\rho}^n] \\
&\quad+\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}\widetilde{\mathbb{E}}\Big[e^{\Phi(q)Y_{\tau^{(k+1)}}}F_{k+1}(Y_{\tau^{(k)}},Y_{\tau^{(k+1)}})\Big|\tau^{(k+1)}<\infty\Big]\Big)(1-\widetilde{\rho})\widetilde{\rho}^n(1-\widetilde{\theta}(x)) \\
&= \phi(q,w,x)+\sum_{n=1}^{\infty}\Big[\sum_{k=1}^{n}\int_{(0,\infty)}\int_{(0,\infty)}e^{\Phi(q)(u+v)}F_{k+1}(v,u+v) \\
&\quad\widetilde{\mathbb{P}}(Y_{\tau^{(k+1)}}-Y_{\tau^{(k)}}\in du,Y_{\tau^{(k)}}\in dv|\tau^{(k+1)}<\infty)\Big](1-\widetilde{\rho})\widetilde{\rho}^n(1-\widetilde{\theta}(x)) \qquad (3.4.23)
\end{aligned}\tag{3.4.24}$$

$$\begin{aligned}
&= \phi(q,w,x)+\sum_{n=1}^{\infty}\int_{(0,\infty)}\int_{(0,\infty)}e^{\Phi(q)(u+v)}F_{n+1}(v,u+v) \\
&\quad\times\widetilde{\mathbb{P}}(Y_{\tau^{(n+1)}}-Y_{\tau^{(n)}}\in du\Big|\tau^{(n+1)}<\infty)\widetilde{\mathbb{P}}(Y_{\tau^{(n)}}\in dv|\tau^{(n+1)}<\infty)\Big)\widetilde{\rho}^{n+1}(1-\widetilde{\theta}(x))
\end{aligned}\tag{3.4.25}$$

$$
\begin{aligned}
&= \phi(q,w,x) + \sum_{n=1}^{\infty} \int_{(0,\infty)} \int_{(0,\infty)} e^{\Phi(q)(u+v)} F_{n+1}(v, u+v)(\widetilde{H} * \widetilde{G})(du) \\
&\quad (\widetilde{H}^{*(n)} * \widetilde{G}^{(*(n)} * \widetilde{T}_x)(dv)
\end{aligned}
\tag{3.4.26}
$$

where in the last equality we have used (3.4.17) and (3.4.19). □

Remark 3.4.2. *Let $F = (F_n)_{n\geq 1}$ be a sequence of bounded measurable functions.*

If F satisfies $F_1(u,v) = w(x-u, v-x)$ and $F_n = 0$ for $n \geq 2$, where w is a measurable function such that $w(\cdot, 0) = 0$, then $P(F,q,x)$ reduces to the classical EDPF defined by (3.4.2) and then,

$$
\begin{aligned}
P(F,q,x) &= \phi(w,q,x) \\
&= \mathbb{E}\Big[e^{-q\tau_x} w(x - Y_{\tau_x^-}, Y_{\tau_x} - x); \tau_x < \infty\Big],
\end{aligned}
\tag{3.4.27}
$$

which is completely characterized in terms of Lévy measure and scale function in Lemma 3.4.1 and Remark 3.4.1.

3.5. Capital injections

In this subsection, we introduce the Expected Discounted Value of Capital Injections (EDVCI), which are necessary to keep the reserve process R above 0. In our context, if R goes under 0 by jumping, we must apply control to prevent the process staying in $(-\infty, 0)$. In fact, we should inject capital only when the risk process becomes negative and only when, the new record infimum under 0 (undershoot) is reached by a jump of a subordinator.

Recall that in Theorem 3.4.1 we have identified the extended EDPF $P(F;q;x)$ defined by (3.2.2) in terms of the q-scale function, Lévy measure, and the densities $\widetilde{G}$ and $\widetilde{H}$ introduced in Section 3.3. Using the connection with the result of Theorem 3.4.1, we will characterize the EDVCI for the risk process defined in (3.2.2). We will study more explicitly the classical case driven by the Cramér-Lundberg risk model [see Einsenberg and Schmidli (2011)].

3.5.1. Expected discounted value of capital injections (EDVCI)

Recall that R is the risk process defined in (3.2.2) by

$$
R_t := x - Y_t\,, \qquad t \geq 0\,,
\tag{3.5.1}
$$

where $Y_t = -ct + S_t - \sigma B_t$.

We denote by C_t the cumulative capital injections up to time t. The controlled risk process R^C is given by

$$R_t^C := x - Y_t + C_t \;, \qquad t \geq 0 \;. \tag{3.5.2}$$

We have to inject the first capital when the risk process falls below zero. Let

$$\tau^{(1)} = \tau_x = \inf\{s > 0, R_s < 0\}$$

denote the time of first ruin, and

$$C_1 = R_{\tau_1} - x$$

denote the first injection. At time $\tau^{(1)}$ the controlled risk process R^C starts with initial capital equal to zero.

For $n \geq 1$, let

$$\tau^{(n+1)} = \inf\{t \geq \tau^{(n)}, \underline{R}_{t^-} < R_t\}$$

denote the time of the n-th injection. The size of the n-th injection becomes

$$\begin{aligned} C_n &= R_{\tau^{(n)}} - R_{\tau^{(n+1)}} \\ &= Y_{\tau^{(n+1)}} - Y_{\tau^{(n)}}, \text{ for } n \geq 1. \end{aligned} \tag{3.5.3}$$

The accumulated injections can be described as

$$C_t = \sum_{i=n}^{N} C_n 1_{\{\tau^{(n)} \leq t\}}.$$

Let us define the expected discounted value of capital injections (EDVCI) as

$$V(q,x) = \mathbb{E}\Big[\sum_{n=0}^{N} e^{-q\tau^{(n+1)}} C_n\Big]. \tag{3.5.4}$$

Let introduce $\kappa(q,x)$ as

$$\kappa(q,x) = \mathbb{E}\Big[e^{-q\tau_x}; \tau_x < \infty\Big], \tag{3.5.5}$$

and

$$\varphi(q,x) = \mathbb{E}\Big[e^{-q\tau_x}(Y_{\tau_x} - x); \tau_x < \infty\Big]. \tag{3.5.6}$$

From Lemma 3.4.1, we can give an expression for (3.5.5) and (3.5.6) in terms of q-scale function and then

$$\varphi(q,x) \;\; = \;\; f_1 * h(x) \tag{3.5.7}$$

and

$$\kappa(q,x) \quad = \quad f_1 * t(x); \tag{3.5.8}$$

where

$$\begin{aligned} h(x) &= e^{\Phi(q)x}\int_x^\infty e^{-\Phi(q)v}\int_{(0,\infty)} u\nu(du+v)dv \\ &= e^{\Phi(q)x}\int_x^\infty e^{-\Phi(q)v}\int_{(v,\infty)} (u-v)\nu(du)dv, \end{aligned} \tag{3.5.9}$$

and

$$t(x) = e^{\Phi(q)x}\int_x^\infty e^{-\Phi(q)v}\nu(v,\infty)dv. \tag{3.5.10}$$

Recall that in Theorem 3.4.1 we have identified the extended EDPF introduced in Definition 3.2.2 in terms of the q-scale function, Lévy measure and notions introduced in Section 3.3 for the model (3.2.2). By using connection with Theorem 3.4.1, the following theorem is an explicit characterization of the EDVCI defined by (3.5.4). This would be the second main contribution of this paper that extends similar results in Einsenberg and Schmidli (2011).

Theorem 3.5.1. *The Expected Discounted Value of Capital Injections EDVCI introduced in (3.5.4) is given by*

$$V(q,x) = \varphi(q,x) + \frac{\delta(q,\sigma)}{1-\xi(q,\sigma)}\kappa(x,\sigma)\ , \tag{3.5.11}$$

where $\xi(q,\sigma)$ and $\delta(q,\sigma)$ are given, respectively, by

$$\xi(q,\sigma) = (1+\frac{\Phi(q)\sigma^2}{2c+\Phi(q)\sigma^2})[1-\frac{q+\frac{\sigma^2}{2}\Phi(q)}{\Phi(q)(c+\frac{\sigma^2}{2}\Phi(q))}]\ , \tag{3.5.12}$$

and

$$\delta(q,\sigma) = \frac{2c}{\Phi(q)(2c+\Phi(q)\sigma^2)}\Big[\frac{2q}{\Phi(q)(2c+\Phi(q)\sigma^2)}+\rho-1\Big]. \tag{3.5.13}$$

Equation (3.5.11) gives an explicit formula of the expected value which should be injected at each deficit time at and after ruin that will allow the insurance company to survive and continue its operations when the risk process continuous to jump downwards.

Proof. Let us consider the sequence of functions $F=(F_n)_{n\geq 1}$ as defined above. Since we suppose $F_1(v,u)=u-x$ and $F_n(v,u)=u-v$ for $u\geq 0, \quad v\in\mathbb{R}$ and $n\geq 2$, then the extended EDPF associated with F and q, $P(F,q,x)$, defined in (3.2.2), is equal to $V(q,x)$. By using Theorem 3.4.1, we can easily derive (3.5.11) and then

$$\begin{aligned}
V(q,x) &= \mathbb{E}\Big[e^{-q\tau_x}(Y_{\tau_x}-x);\tau_x<\infty\Big]+\\
&\quad \sum_{n=1}^{\infty}\int_{(x,\infty)}\int_{(0,\infty)} e^{\Phi(q)(u+v)}u(\widetilde{H}*\widetilde{G})(du)(\widetilde{H}^{*n}*\widetilde{G}^{*n}*T_x^{(q)})(dv)\\
&= \varphi(q,x)+\int_{(0,\infty)} e^{\Phi(q)u}u\ \widetilde{H}*\widetilde{G}(du)\\
&\quad \sum_{n=0}^{\infty}\int_{(x,\infty)} e^{\Phi(q)v}\widetilde{H}^{*n}*\widetilde{G}^{*n}*\widetilde{T}_x(dv)\\
&= \varphi(q,x)+\int_{(0,\infty)} e^{\Phi(q)u}u\ \widetilde{H}*\widetilde{G}(du)\ \sum_{n=0}^{\infty}\int_{(x,\infty)} e^{\Phi(q)v}\widetilde{T}_x(dv)\\
&\quad \int_{(0,\infty)} e^{\Phi(q)v}\widetilde{H}^{*n}*\widetilde{G}^{*n}(dv)\\
&= \varphi(q,x)+\underbrace{\int_{(0,\infty)} e^{\Phi(q)u}u\ \widetilde{H}*\widetilde{G}(du)}_{I}\ \sum_{n=0}^{\infty}\underbrace{\int_{(x,\infty)} e^{\Phi(q)v}\widetilde{T}_x(dv)}_{II}\\
&\quad \underbrace{[\int_{(0,\infty)} e^{\Phi(q)v}\widetilde{H}*\widetilde{G}(dv)]^n}_{III}.
\end{aligned} \tag{3.5.14}$$

Recall that $\widetilde{\mathbb{P}}(Y_{\tau_x}\in dv\ \tau_x<\infty)=\widetilde{T}_x(dv)$, for $v>x$, hence by (3.3.22), (II) is equal to

$$\begin{aligned}
\widetilde{\mathbb{E}}[e^{\Phi(q)Y_{\tau_x}};\ \tau_x<\infty] &= \mathbb{E}[e^{-q\tau_x};\ \tau_x<\infty]\\
&= \kappa(q,x).
\end{aligned} \tag{3.5.15}$$

(III) is equal to

$$\int_{(0,\infty)} e^{\Phi(q)u}\ \widetilde{H}*\widetilde{G}(du)=\int_{(0,\infty)} e^{\Phi(q)u}\ \widetilde{H}(du)\ \int_{(0,\infty)} e^{\Phi(q)u}\ \widetilde{G}(du)\ , \tag{3.5.16}$$

where

$$\begin{aligned}
\int_{(0,\infty)} e^{\Phi(q)u}\ \widetilde{H}(du) &= \int_{(0,\infty)}\frac{1}{c+\Phi(q)\sigma^2}\int_0^{\infty} e^{-\Phi(q)y}\ \nu(du+y)dy\\
&= \frac{1}{c+\Phi(q)\sigma^2}\int_0^{\infty} e^{-\Phi(q)y}\ \nu(y,\ \infty)dy.
\end{aligned} \tag{3.5.17}$$

Recall that

$$\psi(\Phi(q))=q\ , \tag{3.5.18}$$

by using integration by part, (3.5.18) is equivalent to

$$c\Phi(q) + \frac{\sigma^2\Phi(q)^2}{2} - \Phi(q)\int_0^\infty e^{-\Phi(q)y}\,\nu(y,\ \infty)dy = q \qquad (3.5.19)$$

and then,

$$\int_0^\infty e^{-\Phi(q)y}\,\nu(y,\infty)dy = c + \frac{\sigma^2\Phi(q)}{2} - \frac{q}{\phi(q)}. \qquad (3.5.20)$$

We have

$$\begin{aligned}\int_{(0,\infty)} e^{\Phi(q)u}\,\widetilde{G}(du) &= \int_{(0,\infty)} \frac{2\widetilde{c}}{\sigma^2} e^{-(\frac{2\widetilde{c}}{\sigma^2}-\Phi(q))y}dy \\ &= 1 + \frac{\phi(q)\sigma^2}{2c+\phi(q)\sigma^2} = \alpha(q,\sigma). \qquad (3.5.21)\end{aligned}$$

By substituting (3.5.20) in (3.5.16), we conclude that

$$\begin{aligned}\int_{(0,\infty)} e^{\Phi(q)u}\,\widetilde{H}*\widetilde{G}(du) &= \frac{\alpha(q,\sigma)}{c+\sigma^2\Phi(q)}(c+\sigma^2\Phi(q) - \frac{q}{\Phi(q)}) \\ &= \alpha(q,\sigma)[1 - \frac{q+\frac{\sigma^2}{2}\Phi(q)}{\Phi(q)(c+\frac{\sigma^2}{2}\Phi(q))}] \\ &= \xi(q,\sigma). \qquad (3.5.22)\end{aligned}$$

(I) is equal to

$$
\begin{aligned}
\int_{(0,\infty)} e^{\Phi(q)u} u\, \widetilde{H} * \widetilde{G}(du) &= \int_{(0,\infty)} e^{\Phi(q)u} \big[\int_{(0,\infty)} (u+v) e^{\Phi(q)v} \widetilde{G}(dv) \big] \, \widetilde{H}(du) \\
&= \int_{(0,\infty)} e^{\Phi(q)u} \big[\int_{(0,\infty)} (u+v) e^{\Phi(q)v} \widetilde{G}(dv) \big] \, \widetilde{H}(du) \\
&= \int_{(0,\infty)} e^{\Phi(q)u} \big[u \int_0^\infty \frac{2\widetilde{c}}{\sigma^2} e^{-(\frac{2\widetilde{c}}{\sigma^2} - \Phi(q))v} dv \\
&\quad + \int_0^\infty \frac{2\widetilde{c}}{\sigma^2} v e^{-(\frac{2\widetilde{c}}{\sigma^2} - \Phi(q))v} dv \big] \, \widetilde{H}(du) \\
&= \int_{(0,\infty)} e^{\Phi(q)u} \big[\alpha(q,\sigma)(u + \frac{\sigma^2}{2c + \Phi(q)\sigma^2}) \big] \, \widetilde{H}(du) \\
&= \alpha(q,\sigma) \frac{1}{c + \Phi(q)\sigma^2} \Big[\frac{\sigma^2}{2c + \Phi(q)\sigma^2} \int_{(0,\infty)} e^{-\Phi(q)y} \, \nu(y,\infty) dy \\
&\quad + \int_0^\infty \int_{(0,\infty)} u e^{-\Phi(q)y} \, \nu(du + y) dy \Big] \\
&= \alpha(q,\sigma) \frac{1}{c + \Phi(q)\sigma^2} \Big[\frac{\sigma^2}{2c + \Phi(q)\sigma^2} \int_{(0,\infty)} e^{-\Phi(q)y} \, \nu(y,\infty) dy \\
&\quad + \frac{1}{\Phi(q)} \big[\int_0^\infty \nu(y,\infty) dy - \int_0^\infty e^{-\Phi(q)y} \pi(y,\infty) dy \big] \Big] \\
&= \alpha(q,\sigma) \frac{1}{c + \Phi(q)\sigma^2} \Big[\big(\frac{\sigma^2}{2c + \Phi(q)\sigma^2} - \frac{1}{\Phi(q)} \big) \\
&\quad \int_{(0,\infty)} e^{-\Phi(q)y} \, \nu(y,\infty) dy + \frac{1}{\Phi(q)} \mathbb{E}[S_1] \Big].
\end{aligned}
\tag{3.5.23}
$$

Using (3.5.20), the last equality can be written as

$$
\begin{aligned}
&\alpha(q,\sigma) \frac{1}{c + \Phi(q)\sigma^2} \Big[\frac{-2c}{\Phi(q)(2c + \Phi(q)\sigma^2)} \big(c + \frac{\sigma^2 \Phi(q)}{2} - \frac{q}{\phi(q)} \big) + \frac{c}{\Phi(q)} \rho \Big] \\
&= \frac{\alpha(q,\sigma) c}{\Phi(q)(c + \Phi(q)\sigma^2)} \Big[\frac{2q}{\Phi(q)(2c + \Phi(q)\sigma^2)} + \rho - 1 \Big] \\
&= \delta(q,\sigma)
\end{aligned}
\tag{3.5.24}
$$

and then (I) is equal to

$$
\int_{(0,\infty)} e^{\Phi(q)u} u\, \widetilde{H} * \widetilde{G}(du) = \delta(q,\sigma). \tag{3.5.25}
$$

In addition, by using the identification above of (II) and (III), (3.5.14) is equal to

$$
\begin{aligned}
&\varphi(q,x) + \delta(q,\sigma) \sum_{n=0}^{\infty} \kappa(x,\sigma) \xi(q,\sigma)^n \\
&= \varphi(q,x) + \frac{\delta(q,\sigma)}{1 - \xi(q,\sigma)} \kappa(x,\sigma)
\end{aligned}
\tag{3.5.26}
$$

and the theorem follows. □

In the next subsection, we illustrate the previous result by a specific example of Cramér-Lundenberg risk model when the Brownian component vanishes.

3.5.2. Classical risk model

Let us consider the classical risk model. The number of claims is assumed to follow a Poisson process $(N_t)_{t\geq 0}$ with intensity λ. We denote by $(Z_n)_{n\geq 1}$ the claim sizes which are independant of $(N_t)_{t\geq 0}$, positive and *iid* with distribution function K and first moment μ. The aggregate claim process is given by

$$R_t = x + ct - \sum_{i=1}^{N_t} Z_i, \tag{3.5.27}$$

where x is the initial capital and c is the premium rate.

The surplus process introduced by (3.5.2) then has the form

$$R_t^C := x + ct - \sum_{i=1}^{N_t} Z_i + C_t \ , \qquad t \geq 0 \ . \tag{3.5.28}$$

Note that $S_t = \sum_{i=1}^{N_t} Z_i$ and then $\mathbb{E}[S_1] = \lambda\mu$. However, the net profit condition (3.2.4) becomes $c > \lambda\mu$ and then $\rho = \lambda\mu/c$. Since $\sigma = 0$, it follows from Theorem 3.5.1 that the EDVCI for classical model introduced above is given by

$$V(q,x) \quad = \quad \varphi(q,x) + \frac{\delta(q,0)}{1-\xi(q,0)}\kappa(x,0), \tag{3.5.29}$$

where

$$\xi(q,0) = 1 - \frac{q}{\Phi(q)c}, \quad \text{and } \delta(q,0) = \frac{1}{\Phi(q)}\Big[\frac{q}{\Phi(q)c} + \rho - 1\Big] = \frac{q-(c-\lambda\mu)\Phi(q)}{c\Phi(q)^2}. \tag{3.5.30}$$

Since the Brownian component vanishes, Y is a process with bounded variation and then, from Kyprianou (2006), $W_{\Phi(q)}(dx) = \sum_{n=0}^{\infty} \eta^{*n}(dx)$, where

$$\begin{aligned} \eta(dx) &= \frac{1}{c}\widetilde{\nu}(x,\infty)dx \\ &= \frac{1}{c}\int_{(x,\infty)} e^{-\Phi(q)u}\nu(du)\, dx. \end{aligned} \tag{3.5.31}$$

Note that Equation (3.4.9) reduces to

$$f_1(x) \quad = \quad \frac{\lambda}{c}\sum_{n=0}^{\infty}\int_{(x,\infty)} e^{-\Phi(q)u}K(du), \tag{3.5.32}$$

when in the last equality, we have used the identity $\nu(du) = \lambda K(du)$. In addition Equations (3.5.9) and (3.5.10) reduce to

$$h(x) \quad = \quad \frac{\lambda}{c} e^{\Phi(q)x} \int_x^\infty e^{-\Phi(q)v} \int_{(v,\infty)} (u-v) K(du) dv \tag{3.5.33}$$

and

$$t(x) = \frac{\lambda}{c} e^{\Phi(q)x} \int_x^\infty e^{-\Phi(q)v} (1 - F(v)) dv. \tag{3.5.34}$$

Consequently, Equation 3.5.29 reduces to the expression of the EDVCI for classical model (3.5.27) given in Einsenberg and Schmidli (2011).

3.6. Conclusion

In this paper we have generalized the Expected Discounted Penality Function (EDPF) indroduced by Gerber and Shiu (1997, 1998) to include the successive minima reached by the risk process because claim after ruin. In addition to the surplus before ruin and the deficit at ruin, we have added to the EDPF the expectation of a sequence of discounted functions of minima in the context of subordinator risk model perturbed by a Brownian motion. By using some results in Huzak et al (2004) and developments in theory of fluctuations for spectrally negative Lévy processes, we have derived an expicit expression of this extended EDPF.

Our generalization of the EDPF includes information on the path behavior of the risk process not only in a neighborhood of the ruin time, but also after ruin. Such information is relevant for risk management process aimed at preventing successive occurences of the insolvency events. In addition to the classical EDPF introduced by Gerber and Shiu (1997, 1998), the new extension of EDPF contains a sequence of expected discounted functions of successive minima reached by jumps after ruin. This sequence of EDPF has many interesting potential applications. For example, it could be used as a predictive tool for successive deficit times after ruin. In particular, we have used this extended EDPF to derive explicitly the Expected Discounted Value of Capital Injections EDVCI which are necessary to keep the risk process above zero.

Inspired by results of Huzak et al. (2004) and developpements in fluctuation theory for spectrally negative Lévy processes, we provide a characterization for this extended EDPF in a setting involving a cumulative claims modelled by a subordinator, and spectrally negative perturbation. We illustrate how the ESDPF can be used to compute the expected discounted value of capital injections (EDVCI) for Brownian perturbed risk model. The main contributions of this paper are found in Theorems 3.4.1 and

3.5.1. These two results give expressions for the extended EDPF given by Definition 3.2.2, and the EDVCI introduced in Subsection 3.5.1. These expressions can be easily computed by considering particular examples of subordinators.

In oder to make these results as explicit as possible, we have used the risk model driven by Brownian perturbed subordinator process. Further work is needed in order to give these results for more general risk processes driven by a spectrally negative Lévy process.

Chapter 4

THE MINIMAL ENTROPY MARTINGALE MEASURE (MEMM) FOR A MARKOV-MODULATED EXPONENTIAL LÉVY MODEL

Abstract

This paper deals with the characterization problem of the minimal entropy martingale measure (MEMM) for a Markov-modulated exponential Lévy model. This model is characterized by the presence of a background process modulating the risky asset price movements between different regimes or market environments. This allows to stress the strong dependence of financial assets price with structural changes in the market conditions. Our main results are obtained from the key idea of working conditionally on the modulator-factor process. This reduces the problem to studying the simpler case of processes with independent increments. Our work generalizes some previous works in the literature dealing with either the exponential Lévy case or the exponential-additive case.

4.1. Introduction

Regime-switching models were originally introduced in order to model the macroeconomic events which influence asset prices (Hamilton 1989). These cycles are modeled by an underlying Markov chain that drives the asset prices through structurally different market scenarios (due to trades or to arrival of significant new information). In the context of derivative pricing these models lead to incomplete markets and therefore, there are in general infinitely many equivalent (local)martingale measures (EMM) or equivalently there is no unique preference-independent price for options.

In recent years there have been many papers in the area of characterization of martingales measures in incomplete markets. The mainstream of this research is concerned with the projection-based methods in which one looks at the "closest" (in some sense) martingale measure to the physical or real world probability measure relative. For example, Föllmer and Sondermann (1986), Föllmer and Schweizer (1991), Schweizer (1991,1996) use quadratic or L^2-distance; Chan (1999), Miyahara (1999) and Frittelli (2000) use Kullback-Leibler distance (or relative entropy) which leads to the so-called *minimal entropy equivalent martingale measure.*

Many arguments play in favor of the MEMM. Firstly, due to the fact that it comes from the minimization of relative entropy with respect to the real probability measure it retains every information we know about the randomness structure underlying the market thus it is consistent with the efficient market hypothesis. Secondly, the well-known duality relationship [See Fritelli (2000) and references therein] between minimization of the relative entropy and maximization of exponential utility makes the minimal entropy martingale measure economically meaningful. Thirdly, if the minimal entropy martingale exists, it is always equivalent to the objective probability measure unlike some other martingale measures such as the minimal variance martingale measure which may not be equivalent to the objective probability measure.

Many authors have studied this equivalent martingale measure in different contexts. For example, Chan (1999) has studied the problem of pricing contingent claims in a Lévy model and provided a solution based on the MEMM. Frittelli (2000) has looked at the problem of existence and uniqueness of the MEMM in a general incomplete market model and provided its economical interpretation in terms of exponential utility functions. Miyahara (1999), Fujiwara and Miyahara (2003) have obtained some results on the characterization of the MEMM in the geometric Lévy models. Recently, Fujiwara (2009) has extended these results to the case where the geometric Lévy process is replaced by an exponential additive process.

In this chapter, we extend the result of Fujiwara (2009) to a general Markov-modulated exponential Lévy model whose a main feature is the presence of a modulator factor which changes the characteristic of the dynamics of the risky asset under different regimes. There are few results in the literature for the problem of determination of MEMM for the regime-switching models in continuous time. The only at our knowledge is given by Elliott *et al.* (2005) which in the setting of a Markov-modulated Black-Scholes model showed that the equivalent martingale measure defined by a regime-switching Esscher transform minimizes the conditional relative entropy with respect to the historic measure. Our main contribution here consists in giving an expression, when it exists, for the minimal entropy equivalent measure for a general Markov-modulated exponential Lévy model which minimizes effectively the unconditional relative entropy.

The chapter is organized as follows. In Section 4.2, we describe the model set-up and give some preliminaries tools. In particular, we recall some facts on the theory of Markov additive processes (MAP) which are the mathematical structures behind regime-switching models. The main problem is discussed in Section 4.3. The starting point of our approach is the fact that given a Markov additive process defined on a probability space, it is always possible to decompose the Radon-Nykodym derivative relative to an equivalent measure as a product of two terms depending of the MAP. This enables us to work in an exponential additive setting and hence to use the result of Fujiwara (2009). Section 4.4 contains an example which illustrates the feasibility of the results obtained and Section 4.5 concludes the chapter.

4.2. Model description and preliminaries

In this section, we present a general model which can be viewed as an extension of the exponential-Lévy model described in Cont and Tankov (2003, p.283) where a factor of modulation is introduced to allow for more flexibility, especially for the time-inhomogeneity property. Also, we describe the main properties of the theory of Markov additive processes which will be useful to obtain our main result.

4.2.1. The model set-up

We consider a financial market with two primary securities, namely a money market account B and a stock S which are traded continuously over the time horizon $\mathcal{T} := [0, T]$, where $T \in (0, \infty)$ represents the maturity time for investment. To formalize this market, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the real-world probability. Further, we will add to this set-up a filtration which specifies the flow of informations available for the investors.

Let $X := \{X_t : t \in \mathcal{T}\}$ denote an irreducible homogeneous continuous-time Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite state space $\mathbb{S} = \{s_1, s_2, ..., s_M\} \subset \mathbb{R}^M$ and characterized by a rate (or intensity) matrix $\mathbb{A} := \{a_{ij} : 1 \leq i, j \leq M\}$. The entry of the a_{ij}-matrix represent the transition rates at which the process X jumps from state i to state j.

Following Elliott (1993), we can identify $\mathbb{S}$ with the basis set of the linear space $\mathbb{R}^M$. From now on, we set $s_i = \mathbf{e}_i := (0, 0, ..., \underbrace{1}_{i-th}, ..., 0)$.

Let r_t denotes the instantaneous interest rate of the money market account B at time t. We suppose that $r_t := r(t, X_t) = \langle \underline{r}, X_t \rangle$ where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^M$ and $\underline{r} = (r_1, r_2, ..., r_M) \in {\mathbb{R}_+}^M$. The price dynamics of B is given by:

$$B_t = B_0 \exp\Big(\int_0^t r_s ds\Big),\ B_0 = 1; \quad t \in \mathcal{T}. \tag{4.2.1}$$

Let μ_t and σ_t denote the appreciation rate and the volatility of the stock S at time t, we suppose respectively that:

$$\mu_t = \langle \underline{\mu}, X_t \rangle, \quad \sigma_t = \langle \underline{\sigma}, X_t \rangle,$$

where $\underline{\mu} = (\mu_1, \mu_2, ..., \mu_M) \in \mathbb{R}^M$ and $\underline{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_2) \in {\mathbb{R}_+}^M$.

The stock price process S is described by the following Markov modulated exponential Lévy process:

$$S_t = S_0 \exp(Y_t), \quad S_0 > 0, \tag{4.2.2}$$

with

$$\begin{aligned} Y_t = \int_0^t \Big(\mu_s - \frac{1}{2}\sigma_s^2\Big) ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}\backslash\{0\}} z \widetilde{N}^X(ds, dz) \\ - \int_0^t \int_{\mathbb{R}\backslash\{0\}} (e^z - 1 - z\mathbb{I}_{|z|<1}) \rho^X(dz) ds. \end{aligned} \tag{4.2.3}$$

In the expression (4.2.3) we have defined

$$\widetilde{N}^X(dt, dz) := \begin{cases} N^X(dt, dz) - \rho^X(dz)dt & \text{if } |z| < 1, \\ N^X(dt, dz) & \text{if } |z| \geq 1, \end{cases} \tag{4.2.4}$$

with $N^X(dt, dz)$ denotes the differential form of a Markov-modulated random measure on $\mathcal{T} \times \mathbb{R}\backslash\{0\}$. We recall from Elliott and Osakwe (2006) and Elliott and Royal (2006) that a Markov-modulated random measure on $\mathcal{T} \times \mathbb{R}\backslash\{0\}$ is a family $\{N^X(dt, dz; \omega) : \omega \in \Omega\}$ of non-negative measures on the measurable space $(\mathcal{T} \times \mathbb{R}\backslash\{0\}, \mathcal{B}(\mathcal{T}) \otimes \mathcal{B}(\mathbb{R}\backslash\{0\}))$, which satisfy

$N^X(\{0\}, \mathbb{R}\backslash\{0\}; \omega) = 0$ and has the following compensator, or dual predictable projection:

$$\rho^X(dz)dt := \sum_{i=1}^M \langle X_{t^-}, e_i \rangle \rho_i(dz) dt. \tag{4.2.5}$$

$\rho_i(dz)$ is the Lévy measure for the jump size when the Markov chain X is in state $\mathbf{e}_i$, i.e a σ-finite Borel measure on $\mathbb{R}\backslash\{0\}$ with the property

$$\int_{\mathbb{R}\backslash\{0\}} \min(1, z^2) < \infty. \tag{4.2.6}$$

$W := (W_t)_{t\in\mathcal{T}}$ denote the standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ which is supposed to be independent of X and N^X. However, we will assume that a switch of X from state $\mathbf{e}_j$ to state $\mathbf{e}_k$ and a jump of Y do not happen simultaneously, a.s. This assumption is made to simplify the model structure. Otherwise one should specify the nature (and distribution) of jumps of Y which are concomitant with those of the Markov chain X and this would increase the calculations.

For future calculations, we will need an another representation of S through *Doléans-Dade (or stochastic) exponential.* Indeed, by using Itô formula

$$S_t = S_0\mathcal{E}(\widehat{Y}_t), \tag{4.2.7}$$

with the process $\{\widehat{Y}_t : t \in \mathcal{T}\}$ defined by

$$\widehat{Y}_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}\backslash\{0\}} (e^z - 1)\widetilde{N}^X(ds, dz). \tag{4.2.8}$$

Also, sometimes we will use the following process $(\widetilde{R}_t)$ introduced in Fujiwara and Miyahara (2003)

$$\widetilde{R}_t := \int_0^t \frac{1}{\widetilde{S}_{s^-}} d\widetilde{S}_s, \tag{4.2.9}$$

where

$$\widetilde{S}_t := \frac{S_t}{B_t} = e^{-\int_0^t r_s ds} S_t. \tag{4.2.10}$$

From (4.2.2) and (4.2.11), it is easy to see that

$$\widetilde{R}_t = \widehat{Y}_t - \int_0^t r_s ds. \tag{4.2.11}$$

4.2.2. Some preliminaries

We review here some notions related to the *Markov additive processes* (MAP) which are the mathematical object behind Markov-modulated exponential Lévy models. In particular, we define this object and give some of its fundamental properties.

4.2.2.1. *Markov additive Processes*

The mathematical theory of Markov additive processes can be traced back to the works of Ezhov and Skorokhod (1969a,1969b), Çinlar (1972a,b) and Grigelionis (1978). We now recall some useful results from this theory. Throughout this section, $\mathcal{T} := [0, T]$ with $T < \infty$.

Definition 4.2.1.

*Consider a stochastic process $\{(J_t, Z_t) : t \in \mathcal{T}\}$ which is adapted to a right-continuous filtration $\mathbf{F} := \{\mathcal{F}_t : t \in \mathcal{T}\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the component $Z = \{Z_t : t \in \mathcal{T}\}$ takes its values on $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ and is continuous from the right with limits at the left whereas the component $J = \{J_t : t \in \mathcal{T}\}$ takes its values on a measure space $(\mathbf{D}, \mathcal{D})$. The pair (J, Z) will be called a **Markov additive process** with respect to $\mathbf{F}$ if for any Borel set $A \in \mathcal{B}(\mathbb{R}^m)$, $B \in \mathcal{D}$ and $0 \leq s < t \leq T$ we have almost surely (a.s)*

$$\mathbb{P}[Z_t - Z_s \in A, J_t \in B|\mathcal{F}_s] = \mathbb{P}[Z_t - Z_s \in A, J_t \in B|J_s], \tag{4.2.12}$$

or, more generally, for any bounded random variable η that is measurable with respect to the σ-algebra $\sigma\Big\{(Z_t - Z_s, J_t) : t \in [s, T]\Big\}$ and $s \in \mathcal{T}$ we have almost surely

$$\mathbb{E}^{\mathbb{P}}\Big[\eta\Big|\mathcal{F}_s\Big] = \mathbb{E}^{\mathbb{P}}\Big[\eta\Big|J_s\Big]. \tag{4.2.13}$$

In the following, we introduce for $0 \leq s \leq t with\ s, t \in \mathcal{T}$

- $\mathcal{F}^J_{s,t} := \sigma(J_u; u \in \mathcal{T}, s \leq u \leq t)$;
- $\mathcal{F}^J_t := \mathcal{F}^J_{0,t}$;
- $\mathcal{F}^Z_t := \sigma(Z_u; u \in \mathcal{T}, 0 \leq u \leq t)$;
- $\mathcal{G}_t := \mathcal{F}^J_t \vee \mathcal{F}^Z_t$;
- $\overline{\mathcal{G}}_t = \mathcal{F}^J_T \vee \mathcal{F}^Z_t$.

Remark 4.2.1. *The two filtrations $\mathbf{G} := \{\mathcal{G}_t : t \in \mathcal{T}\}$ and $\overline{\mathbf{G}} := \{\overline{\mathcal{G}}_t : t \in \mathcal{T}\}$ are of particular importance for our model. The filtration $\overline{\mathbf{G}}$ could be seen as the information set given the present and the future of the Markov chain J.*

We now state a fundamental property of Markov additive process $\{(J_t, Z_t) : t \in \mathcal{T}\}$.

Theorem 4.2.1 (Grigelionis 1978)**.** *The component Z is a process with conditional independent increments with respect to the σ-algebra $\mathcal{F}^J_T$, i.e., for any $A \in \mathcal{B}(\mathbb{R}^m)$ and*

$0 \leq s \leq t$, *with* $s, t \in \mathcal{T}$ *we have almost surely*

$$\mathbb{P}[Z_t - Z_s \in A | \overline{\mathcal{G}}_s] = \mathbb{P}[Z_t - Z_s \in A | \mathcal{F}_T^J], \tag{4.2.14}$$

or equivalently, for $0 \leq s \leq t$ *with* $s, t \in \mathcal{T}$ *and* $y \in \mathbb{R}^m$ *we have almost surely*

$$\mathbb{E}^{\mathbb{P}}\Big[\exp i(y, Z_t - Z_s)\Big|\overline{\mathcal{G}}_s\Big] = \mathbb{E}^{\mathbb{P}}\Big[\exp i(y, Z_t - Z_s)\Big|\mathcal{F}_T^J\Big]. \tag{4.2.15}$$

Proof. The proof of this theorem is obtained by using lemma 1 in Grigelionis (1978).

□

Remark 4.2.2. *This last property highlights a feature of the MAP which will be very useful in the sequel. Indeed, if we work on a suitable stochastic basis* $(\Omega, \mathcal{F}, \mathbb{P})$ *allowing the existence of some regular version of the conditional probability* $\mathbb{P}[\cdot|\mathcal{F}_T^J]$ *which will be noted* $\widehat{\mathbb{P}}(\omega, \cdot)$ *for an* $\omega \in \Omega$, *then one can switch for the new probability space* $(\Omega, \mathcal{F}, \widehat{\mathbb{P}}(\omega, \cdot))$ *under which the component* Z *is an additive process (i.e., has independent increments).*

From now on, we assume the above mentioned setting and thus we set Ω to be the canonical space $\mathbb{D}_T(\mathbb{R}^m)$ of càdlàg functions

$$\beta : t \in [0, T] \rightarrow \beta(t) \in \mathbb{R}^m,$$

endowed with the Skorohod topology such that $\mathcal{D}(\mathbb{R}^m)$ be its Borel σ-field. The filtration taken on that space is the family $\mathbf{D}(\mathbb{R}^m) = \{\mathcal{D}_t(\mathbb{R}^m) = \bigcap_{u>t} \mathcal{D}_u^0(\mathbb{R}^m)\}$, where $\mathcal{D}_u^0(\mathbb{R}^m)$ denotes the σ-field generated by all maps : $\beta \rightsquigarrow \beta(s)$ for $s \leq t$.

Let $(J, Z) = \{(J_t, Z_t) : t \in \mathcal{T}\}$ be a Markov additive process defined on $(\Omega, \mathcal{D}(\mathbb{R}^m), \mathbf{D}(\mathbb{R}^m), \mathbb{P})$ with Z denoting its additive part. Ω defined as $\mathbb{D}_T(\mathbb{R}^m)$ is a Polish space, hence we have this lemma

Lemma 4.2.1 (Çinlar 1972a)**.**

There exists a regular version $\widehat{\mathbb{P}}$ *of the conditional probability* $\mathbb{P}\{\cdot|\mathcal{F}_T^J\}$ *on the* σ*-algebra* $\overline{\mathcal{G}_t}$ *i.e. for* $\mathbb{P}-$*almost all* $\omega \in \Omega$, $\widehat{\mathbb{P}}(\omega, \cdot)$ *is a probability measure on* $\overline{\mathcal{G}}_T$ *and* $\widehat{\mathbb{P}}(\omega, A)$ *is a version of* $\mathbb{P}[A|\mathcal{F}_T^J]$ *for every* $A \in \overline{\mathcal{G}}_T$.

Proof. See Çinlar (1972a). □

The following proposition, formulated by Grigelionis (1978) gives a characterization criterion of a Markov additive process:

Theorem 4.2.2 (Grigelionis 1978)**.**

A stochastic process $\{(J_t, Z_t) : t \in \mathcal{T}\}$ *adapted to a complete right-continuous filtration* $\mathbf{F} := \{\mathcal{F}_t : t \in \mathcal{T}\}$ *is a Markov additive process (MAP) if and only if the following conditions are satisfied:*

(1) The component Z *has an expansion* $Z_t = Z_t^{'} + Z_t^{''}$ *where* $Z^{'}$ *is continuous from the right with limits at the left and such that for all* $t \in \mathcal{T}$, $Z_t^{'}$ *is* $\mathcal{F}_T^J$*-measurable and* $Z^{''}$ *is an* $(\overline{\mathbf{G}}, \mathbb{P})$*-semimartingale whose the spot-characteristics triplet* (A_t, ν_t, γ_t) *are* $\mathcal{F}_T^J$*-measurable;*

(2) The component J *has the Markov property with respect to the filtration* $\mathbf{G} = \{\mathcal{G}_t : t \in \mathcal{T}\}$ *i.e.„ for any* $0 \leq s < t \leq T$ *and* $B \in \mathcal{D}$ *we have almost surely*

$$\mathbb{P}[J_t \in B|\mathcal{G}_s] = \mathbb{P}[J_t \in B|J_s], \tag{4.2.16}$$

With all the ingredients above we now state the first result of this contribution that allows us, when we have a MAP, to decompose the density process associated to a change of measure into some useful components related to this MAP.

Theorem 4.2.3.

Let $\mathbb{Q}$ *be an arbitrary probability measure such that* $\mathbb{Q} \overset{loc}{\ll} \mathbb{P}$ *i.e.,* $\mathbb{Q}|_{\overline{\mathcal{G}}_t} \ll \mathbb{P}|_{\overline{\mathcal{G}}_t}$ *for each* $t \in \mathcal{T}$*. We denote by* $\Lambda^{J,Z}$ *the Radon-Nikodym density process defined by*

$$\Lambda_t^{J,Z} = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_t}. \tag{4.2.17}$$

Then the following holds $\mathbb{P}$*-almost surely:*

$\Lambda^{J,Z}$ *admits the unique(up to* $\mathbb{P}$*-null sets) decomposition*

$$\Lambda_t^{J,Z} = \Lambda_t^{Z}.\ \Lambda_T^{J}, \quad for \quad t \in \mathcal{T}, \tag{4.2.18}$$

where

- $\Lambda^Z = \left\{\frac{d\widehat{\mathbb{Q}}(\omega,\cdot)}{d\widehat{\mathbb{P}}(\omega,\cdot)}\Big|_{\overline{\mathcal{G}}_t}; t \in \mathcal{T}\right\}$ *is the Radon-Nikodym density process of a regular version* $\widehat{\mathbb{Q}}(\omega,\cdot)$ *of* $\mathbb{Q}[\cdot|\mathcal{F}_T^J]$ *with respect to a regular version* $\widehat{\mathbb{P}}(\omega,\cdot)$ *of* $\mathbb{P}[\cdot|\mathcal{F}_T^J]$*;*
- $\Lambda_T^J = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_0} \left(= \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T^J}\right)$ *is the Radon-Nikodym derivative characterizing a change of measure under which the Markov chain is modified.*

Proof. 1)Let $\mathbb{Q}$ be an arbitrary probability locally absolutely continuous with respect to $\mathbb{P}$. Let $\Lambda_t^{J,Z}$ be a version of the Radon-Nikodym derivative of $\mathbb{Q}|_{\overline{\mathcal{G}}_t}$ w.r.t $\mathbb{P}|_{\overline{\mathcal{G}}_t}$. By

lemma (4.2.1) we have that for $\mathbb{P}-$almost all $\omega \in \Omega$, there exists a regular version of $\mathbb{P}[\cdot|\mathcal{F}_T^J]$ and $\mathbb{Q}[\cdot|\mathcal{F}_T^J]$ which we will denote respectively by $\widehat{\mathbb{P}}(\omega,\cdot)$ and $\widehat{\mathbb{Q}}(\omega,\cdot)$.
Since $\mathbb{Q} \overset{loc}{\ll} \mathbb{P}$, we have that for $\mathbb{P}-$almost all $\omega \in \Omega$, $\widehat{\mathbb{Q}}(\omega,\cdot)$ is also locally absolutely continuous with respect to $\widehat{\mathbb{P}}(\omega,\cdot)$.
Indeed, for each $A \in \overline{\mathcal{G}}_t$ satisfying $\widehat{\mathbb{P}}(\omega, A) = 0$ we have

$$\begin{aligned}
\widehat{\mathbb{Q}}(\omega, A) &:= \mathbb{E}^{\mathbb{Q}}[1_A|\mathcal{F}_T^J](\omega) \\
&= \mathbb{E}^{\mathbb{P}}[\Lambda_t^{J,Z}.1_A|\mathcal{F}_T^J](\omega) \times \frac{1}{\mathbb{E}^{\mathbb{P}}[\Lambda_t^{J,Z}|\mathcal{F}_T^J](\omega)} \quad \text{(by Bayes' rule)} \\
&= \mathbb{E}^{\widehat{\mathbb{P}}(\omega,\cdot)}\left[\Lambda_t^{J,Z}.1_A\right] \times \frac{1}{\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^{J,Z}\middle|\mathcal{F}_T^J\right](\omega)} \quad \text{(by definition of } \mathbb{E}^{\widehat{\mathbb{P}}(\omega,\cdot)}[\cdot]) \\
&= \frac{1}{\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^{J,Z}\middle|\mathcal{F}_T^J\right](\omega)} \times \int_\Omega \Lambda_t^{J,Z}(\omega')1_A(\omega')\widehat{\mathbb{P}}(\omega, d\omega') \\
&= 0.
\end{aligned} \tag{4.2.19}$$

Since $\widehat{\mathbb{Q}}(\omega,\cdot)|_{\overline{\mathcal{G}}_t}$ is absolutely continuous with respect to $\widehat{\mathbb{P}}(\omega,\cdot)|_{\overline{\mathcal{G}}_t}$, by theorem III-3.4 of Jacod and Shiryaev (2003) there exists for $\mathbb{P}-$almost all $\omega \in \Omega$, a density process Λ_Z such that

$$\Lambda_t^Z = \frac{d\widehat{\mathbb{Q}}(\omega,\cdot)|_{\overline{\mathcal{G}}_t}}{d\widehat{\mathbb{P}}(\omega,\cdot)|_{\overline{\mathcal{G}}_t}}, \ \forall 0 \le t \le T. \tag{4.2.20}$$

By definition, Λ^Z is a $\overline{\mathbf{G}}$-martingale under $\widehat{\mathbb{P}}(\omega,\cdot)$ also under $\mathbb{P}$.

For each $A \in \overline{\mathcal{G}_t}$ we have for $\mathbb{P}-$almost all $\omega \in \Omega$

$$\begin{aligned}
\widehat{\mathbb{Q}}(\omega, A) &= \mathbb{E}^{\widehat{\mathbb{P}}(\omega,\cdot)}[\Lambda_t^Z.1_A] \\
&= \mathbb{E}^{\mathbb{P}}[\Lambda_t^Z.1_A|\mathcal{F}_T^J](\omega),
\end{aligned} \tag{4.2.21}$$

otherwise,

$$\begin{aligned}
\widehat{\mathbb{Q}}(\omega, A) &= \mathbb{E}^{\mathbb{Q}}\left[1_A\middle|\mathcal{F}_T^J\right](\omega) \\
&= \mathbb{E}^{\mathbb{P}}\left[\Lambda_t^{J,Z}.1_A\middle|\mathcal{F}_T^J\right](\omega)\frac{1}{\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^{J,Z}\middle|\mathcal{F}_T^J\right](\omega)} \quad \text{(by Bayes' rule).}
\end{aligned} \tag{4.2.22}$$

By setting

$$\Lambda_T^J = \mathbb{E}^{\mathbb{P}}\left[\Lambda_t^{J,Z}\middle|\mathcal{F}_T^J\right], \tag{4.2.23}$$

and taking the above two expressions of $\widehat{\mathbb{Q}}(\omega, A)$, we obtain that either

$$\mathbb{E}^{\mathbb{P}}\Big[\Lambda_t^Z\Lambda_T^J.1_A\Big|\mathcal{F}_T^J\Big](\omega) = \mathbb{E}^{\mathbb{P}}\Big[\Lambda_t^{J,Z}.1_A\Big|\mathcal{F}_T^J\Big](\omega), \tag{4.2.24}$$

or

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\Lambda_t^Z\Lambda_T^J - \Lambda_t^{J,Z}\Big).1_A\Big|\mathcal{F}_T^J\Big](\omega) = 0. \tag{4.2.25}$$

Taking expectation in the both sides, we have for $\mathbb{P}$ a.s

$$\Lambda_t^{J,Z} = \Lambda_t^Z\Lambda_T^J \quad \forall t \in [0,T]. \tag{4.2.26}$$

Now we have to show that Λ_T^J is the Radon-Nikodym derivative ($\mathcal{F}_T^J$-measurable) characterizing a change of measure under which the Markov chain is modified.
Indeed, by (4.2.23)

$$\begin{aligned}
\Lambda_T^J &:= \mathbb{E}^{\mathbb{P}}\Big[\Lambda_t^{J,Z}\Big|\mathcal{F}_T^J\Big] = \mathbb{E}^{\mathbb{P}}\Big[\Lambda_t^{J,Z}\Big|\overline{\mathcal{G}}_0\Big] \\
&= \Lambda_0^{J,Z} \quad \text{(by martingale property for } \Lambda^{J,Z}\text{)}, \qquad (4.2.27)\\
&= \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_0} = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T^J}. \qquad (4.2.28)
\end{aligned}$$

By setting

$$\Lambda_t^J := \mathbb{E}^{\mathbb{P}}\Big[\Lambda_T^J\Big|\mathcal{F}_t^J\Big] \quad \forall t \in [0,T], \tag{4.2.29}$$

we define a $(\mathcal{F}^J, \mathbb{P})$ martingale process such that $\mathbb{E}^{\mathbb{P}}[\Lambda_t^J] = 1$ hence Λ^J is a density process under which the Markov chain would be eventually modified. $\square$

Remark 4.2.3. *The process (X, Y) defined in Section 4.2.1 is a Markov additive process. In particular the log-return process Y is a conditional additive process given the σ-algebra $\mathcal{F}_T^X$; hence, from (4.2.3) we have this canonical decomposition for Y_t associated with the truncation function $h(x) := x\mathbb{I}_{\{|x|\le 1\}}$:*

$$\begin{aligned}
Y_t = \int_0^t \Big(\mu_s - \frac{1}{2}\sigma_s^2\Big)ds - \int_0^t\int_{\mathbb{R}\setminus\{0\}}(e^z - 1 - h(z))\rho^X(dz)ds + \int_0^t \sigma_s dW_s \\
+ \int_0^t\int_{\mathbb{R}\setminus\{0\}} h(z)\widetilde{N}^X(ds,dz) + \int_0^t\int_{\mathbb{R}\setminus\{0\}}(z - h(z))N^X(ds,dz). \qquad (4.2.30)
\end{aligned}$$

Hence its spot-characteristic triplet (A_t, ν_t, γ_t) is given by

$$A_t := \int_0^t \sigma_s^2 ds, \tag{4.2.31}$$

$$\nu_t(dy) := \int_0^t \rho^X(dy)ds, \tag{4.2.32}$$

and

$$\gamma_t := \int_0^t \Big(\mu_s - \frac{1}{2}\sigma_s^2\Big) ds - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - h(z))\rho^X(dz)ds. \tag{4.2.33}$$

4.3. Minimal entropy equivalent martingale measure for Markov-modulated Lévy process

In this section, we address the main aim of our study: existence and characterization of the minimal entropy equivalent martingale measure for the Markov-modulated Lévy model. It is known that the financial market described by equations (4.2.2) and (4.2.1) is incomplete, so neither existence nor uniqueness of equivalent martingale measures are assured. Nevertheless, we show in this section that under mild conditions there is a MEMM and its uniqueness results from the strict convexity of the relative entropy operator.

4.3.1. Preliminaries

Let $\mathcal{G}$ be a sub σ-algebra of $\overline{\mathcal{G}}_T$ and $\mathcal{P}$ the set of probability measures on $(\Omega, \mathcal{G})$.

Definition 4.3.1. *For $\mathbb{Q} \in \mathcal{P}(\Omega, \mathcal{G})$, the **relative entropy** of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined as:*

$$\mathbb{H}_{\mathcal{G}}(\mathbb{Q}, \mathbb{P}) := \begin{cases} \mathbb{E}^{\mathbb{Q}}\Big[\log\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}}\Big)\Big] & \text{if} \quad \mathbb{Q} << \mathbb{P} \quad \text{on} \quad \mathcal{G} \\ +\infty & \text{otherwise} \end{cases} \tag{4.3.1}$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{G}}$ denotes the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{G}$. Moreover, it verifies the following

- $\mathbb{H}_{\mathcal{G}}(\mathbb{Q}, \mathbb{P}) \geq 0$ *and* $\mathbb{H}_{\mathcal{G}}(\mathbb{Q}, \mathbb{P}) = 0$ *if and only if* $\mathbb{Q} = \mathbb{P}$*;*
- *The functional* $\mathbb{Q} \longmapsto \mathbb{H}_{\mathcal{G}}(\mathbb{Q}, \mathbb{P})$ *is strictly convex.*

Now we recall a lemma (See Fujiwara and Miyahara (2003)) which is useful in what follows.

Lemma 4.3.1. *Let $\mathcal{K}$ be a sub σ-field of a σ-algebra $\overline{\mathcal{G}}_T$ and $\mathbb{Q}$ an element of $\mathcal{P}(\Omega, \mathcal{G})$ which is absolutely continuous with respect to $\mathbb{P}$. Then we have the following equivalence $(a) \sim (b) \sim (c)$*

- *(a)* $\mathbb{H}_{\mathcal{G}}(\mathbb{Q}, \mathbb{P}) \geq 0$*;*

- *(b)Let $\mathbb{P}_1 \in \mathcal{P}(\Omega, \mathcal{G})$ be equivalent to $\mathbb{P}$ on $\mathcal{G}$ and $\log\left(\frac{d\mathbb{P}_1}{d\mathbb{P}}\Big|_{\mathcal{G}}\right)$ be integrable with respect to $\mathbb{Q}$, then*

$$\mathbb{H}_{\mathcal{G}}(\mathbb{Q}, \mathbb{P}) \geq \int_{\Omega} \log\left(\frac{d\mathbb{P}_1}{d\mathbb{P}}\Big|_{\mathcal{G}}\right) d\mathbb{Q}.$$

- *(c)If $\mathcal{K} \subset \mathcal{G}$, then*

$$\mathbb{H}_{\mathcal{K}}(\mathbb{Q}, \mathbb{P}) \leq \mathbb{H}_{\mathcal{G}}(\mathbb{Q}, \mathbb{P}).$$

In particular, from statement (c) the relative entropy $\mathbb{H}_{\mathcal{K}}(\mathbb{Q}, \mathbb{P})$ is nondecreasing on the elements of a filtration.

Now we define several spaces that will be useful in the sequel.

- $\mathcal{M}^a(\widetilde{S}) := \left\{\mathbb{Q} \overset{loc}{\ll} \mathbb{P} : (\widetilde{S}_t) \text{ is a } (\overline{\mathbf{G}}, \mathbb{Q})\text{-local martingale}\right\}$;
- $\mathcal{M}^e(\widetilde{S}) := \left\{\mathbb{Q} \overset{loc}{\sim} \mathbb{P} : (\widetilde{S}_t) \text{ is a } (\overline{\mathbf{G}}, \mathbb{Q})\text{- martingale}\right\}$;

Elements of $\mathcal{M}^e(\widetilde{S})$ are called *equivalent martingale measures (EMMs)* for $\widetilde{S}$.

Definition 4.3.2. *The* minimal entropy martingale measure *is a probability measure $\mathbb{Q}^\star \in \mathcal{M}^e(\widetilde{S})$ such that*

$$\mathbb{H}_{\mathcal{G}_T}(\mathbb{Q}^\star, \mathbb{P}) = \min_{\mathbb{Q} \in \mathcal{M}^a(\widetilde{S})} \mathbb{H}_{\mathcal{G}_T}(\mathbb{Q}, \mathbb{P}). \tag{4.3.2}$$

If the MEMM exists, by definition it is unique. Moreover [see Fritelli 2001, Theorem 2.2], under the assumption

$$\inf_{\mathbb{Q} \in \mathcal{M}^e(\widetilde{S})} \mathbb{H}_{\mathcal{G}_T}(\mathbb{Q}, \mathbb{P}) < \infty \tag{4.3.3}$$

it is equivalent to $\mathbb{P}$.

4.3.2. Main results

Before moving on with our discussion and main results, we need to state and label the following condition:

Condition (H)

There exists a $\mathcal{F}_t^X$-adapted process with càglàd path $(\theta_t^X)_{\{t \in \mathcal{T}\}}$ that satisfies the following

(i)

$$\int_0^T \int_{\{z>1\}} e^z e^{\theta_s^X (e^z - 1)} \rho^X(dz) ds < \infty \quad \mathbb{P}\ (a.s), \tag{4.3.4}$$

(ii) for almost all $t \in \mathcal{T}$,

$$\mu_t + \theta_t^X \sigma_t^2 + \int_{\mathbb{R}\backslash\{0\}} \left(e^{\theta_s^X (e^z - 1)} - 1\right)(e^z - 1)\rho^X(dz)ds = r_t \quad \mathbb{P}\ (a.s). \tag{4.3.5}$$

In view of Theorem 4.2.3 and some previous results in the literature about the MEMM, we claim that for the Markov-modulated exponential Lévy model the MEMM will be constructed as product of two terms. Precisely, we have this definition

Definition 4.3.3. *Let $\{\theta_t^X : t \in \mathcal{T}\}$ be a process satisfying the condition (H). We define a probability measure $\mathbb{Q}^{\theta^X}$ on $(\Omega, \overline{\mathcal{G}}_T)$ such that*

$$\forall A \in \overline{\mathcal{G}}_t, \quad \mathbb{Q}^{\theta^X}(A) := \mathbb{E}^{\mathbb{P}}[L_t . \mathbb{I}_A], \tag{4.3.6}$$

where $L_t = \Lambda_t^{\theta^X} .\, U_T$ with

$$\Lambda_t^{\theta^X} := \frac{\exp\left(\int_0^t \theta_s^X d\widetilde{R}_s\right)}{\mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_0^t \theta_s^X d\widetilde{R}_s\right)\Big|\mathcal{F}_T^X\right]} \tag{4.3.7}$$

and U_T be a $\mathcal{F}_T^X$-measurable positive function such that $\mathbb{E}^{\mathbb{P}}[U_T] = 1$.

Remark 4.3.1. *The probability measure $\mathbb{Q}^{\theta^X}$ defined above is absolutely continuous with respect to $\mathbb{P}$ and by theorem (4.2.3) it admits the decomposition*

$$\frac{d\mathbb{Q}^{\theta^X}}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_t} = \frac{d\widehat{\mathbb{Q}}^{\theta^X}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_t} .\, U_T, \tag{4.3.8}$$

where

$$\frac{d\widehat{\mathbb{Q}}^{\theta^X}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_t} := \frac{\exp\left(\int_0^t \theta_s^X d\widetilde{R}_s\right)}{\mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_0^t \theta_s^X d\widetilde{R}_s\right)\Big|\mathcal{F}_T^X\right]}. \tag{4.3.9}$$

Proposition 4.3.1.

The probability measure $\mathbb{Q}^{\theta^X}$ defined above is an element of $\mathcal{M}^e(\widetilde{S})$.

PROOF. We only need to show that $(\widetilde{S}_t)$ is a $(\overline{\mathbf{G}}, \mathbb{Q})$- martingale.
So, using Bayes' rule we have $\forall\ 0 \leq s \leq t \leq T$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^{\theta^X}}\left[\widetilde{S}_t\Big|\overline{\mathcal{G}}_s\right] &= \frac{\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^{\theta^X} U_T \widetilde{S}_t\Big|\overline{\mathcal{G}}_s\right]}{\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^{\theta^X} U_T\Big|\overline{\mathcal{G}}_s\right]}, \\
&= \mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda_t^{\theta^X}}{\Lambda_s^{\theta^X}} e^{-\int_0^t r_u du} S_s e^{\int_s^t dY_u}\Bigg|\overline{\mathcal{G}}_s\right] \\
&= \widetilde{S}_s \mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda_t^{\theta^X}}{\Lambda_s^{\theta^X}} e^{-\int_s^t r_u du} e^{\int_s^t dY_u}\Bigg|\overline{\mathcal{G}}_s\right]. \qquad (4.3.10)
\end{aligned}$$

The proposition is proved if under the conditions (4.3.4)-(4.3.5) we have

$$\mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda_t^{\theta^X}}{\Lambda_s^{\theta^X}} e^{-\int_s^t r_u du} e^{\int_s^t dY_u}\Bigg|\overline{\mathcal{G}}_s\right] = 1, \quad \forall 0 \leq s \leq t \leq T. \qquad (4.3.11)$$

From proposition VIII-22 in Cont and Tankov (2005),

$$Y_t = \widehat{Y}_t - \frac{1}{2}\int_0^t \sigma_s^2 ds - \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left(e^z - 1 - z\right) \widetilde{N}^X(ds, dz) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left(e^z - 1 - z\right) \rho^X(dz) ds, \qquad (4.3.12)$$

Then

$$\begin{aligned}
&\frac{\Lambda_t^{\theta^X}}{\Lambda_s^{\theta^X}} e^{-\int_s^t r_u du} e^{\int_s^t dY_u} = \\
&\underbrace{\exp\left\{\widehat{K}_s(\theta^X) - \widehat{K}_t(\theta^X) - \int_s^t r_u du - \frac{1}{2}\int_s^t \sigma_u^2 du - \int_s^t \int_{\mathbb{R}\setminus\{0\}} \left(e^{\theta_u^X (e^z - 1)} - 1 - \theta_u^X (e^z - 1)\right) \rho^X(dz) du\right\}}_{I} \\
&\underbrace{\times \exp\left\{\int_s^t (\theta_u^X + 1)\mu_u du + \int_s^t (\theta_u^X + 1)\sigma_u dW_u + \int_s^t \int_{\mathbb{R}\setminus\{0\}} \left(z + \theta_u^X (e^z - 1)\right) \widetilde{N}^X(du; dz)\right\}}_{II}
\end{aligned} \qquad (4.3.13)$$

where [See Appendix A]

$$\widehat{K}_t(\theta^X) := \int_0^t \left[\theta_s^X \mu_s + \frac{1}{2}(\theta_s^X)^2 \sigma_s^2\right] ds + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left(e^{\theta_s^X (e^z - 1)} - 1 - \theta_s^X (e^z - 1)\right) \rho^X(dz) ds, \quad \forall\, t \in [0, T]. \qquad (4.3.14)$$

Noting that the term I in (4.3.13) is $\overline{\mathcal{G}}_s$-measurable, we obtain

$$\mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda_t^{\theta^X}}{\Lambda_s^{\theta^X}}e^{-\int_s^t r_u du}e^{\int_s^t dY_u}\middle|\overline{\mathcal{G}}_s\right] = \mathbb{E}^{\mathbb{P}}\Bigg[\exp\Bigg\{\int_s^t(\theta_u^X+1)\mu_u du + \int_s^t(\theta_u^X+1)\sigma_u dW_u$$

$$+\int_s^t\int_{\mathbb{R}\setminus\{0\}}\Big(z+\theta_u^X(e^z-1)\Big)\widetilde{N}^X(du;dz)\Bigg\}\Bigg|\overline{\mathcal{G}}_s\Bigg]$$

$$\times\exp\left\{\widehat{K}_s(\theta^X)-\widehat{K}_t(\theta^X)-\int_s^t r_u du-\frac{1}{2}\int_0^t\sigma_u^2 du-\int_0^t\int_{\mathbb{R}\setminus\{0\}}\Big(e^{\theta_u^X(e^z-1)}-1-\theta_u^X(e^z-1)\Big)\rho^X(dz)du\right\}. \tag{4.3.15}$$

After some algebra and using theorem (4.2.1) we have

$$\mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda_t^{\theta^X}}{\Lambda_s^{\theta^X}}e^{-\int_s^t r_u du}e^{\int_s^t dY_u}\middle|\overline{\mathcal{G}}_s\right] = \exp\Bigg\{\int_s^t(\mu_u - r_u + \theta_u^X\sigma_u^2)du$$

$$+\int_s^t\int_{\mathbb{R}\setminus\{0\}}\Big(e^{\theta_u^X(e^z-1)}-1\Big)(e^z-1)\rho^X(dz)du\Bigg\}, \tag{4.3.16}$$

Hence, using the condition (4.3.5) we have the result. □

Remark 4.3.2. *Condition (4.3.4) is useful to assure integrability of some expressions used in the proof. In the literature, (4.3.5) is known as* Martingale condition *because the probability measure $\mathbb{Q}^{\theta^X}$ defined from the process θ^X is a martingale measure for the discounted price process of risky asset.*

For pricing purposes, we need to know the dynamics of price process of the risky asset under the martingale probability measure $\mathbb{Q}^{\theta^X}$. The following proposition states a result in this direction.

Proposition 4.3.2.

Under risk-neutral probability measure $\mathbb{Q}^{\theta^X}$,

(1) the log-return process Y is expressed as

$$Y_t = \widehat{\gamma}_t+\widehat{M}_t+\int_0^t\int_{\mathbb{R}\setminus\{0\}}h(z)\underline{\widetilde{N}}^X(ds,dz)+\int_0^t\int_{\mathbb{R}\setminus\{0\}}(z-h(z))N^X(ds,dz) \quad \text{for} \quad t\in\mathcal{T}, \tag{4.3.17}$$

where

- *$(\widehat{\gamma}_t)$ is a $\mathcal{F}_T^X$-measurable continuous process defined by*

$$\widehat{\gamma}_t := \int_0^t\Big(r_s-\frac{1}{2}\sigma_s^2\Big)ds - \int_0^t\int_{\mathbb{R}\setminus\{0\}}(e^z-1-h(z))\underline{\rho}^{\theta^X}(dz)ds, \tag{4.3.18}$$

- $(\widehat{M}_t)$ *is a continuous Gaussian process with independent increments under* $\widehat{\mathbb{Q}}^{\theta^X}(\omega,\cdot)$, *defined by*

$$\widehat{M}_t := \int_0^t \sigma_s dW_s^{\theta^X}, \tag{4.3.19}$$

with W^{θ^X} *denoting the standard Brownian motion under* $\mathbb{Q}^{\theta^X}$*;*
- $\underline{\widetilde{N}}^X$ *is the compensated measure of* N^X *under* $\mathbb{Q}^{\theta^X}$ *defined by*

$$\underline{\widetilde{N}}^X(ds,dz) = N^X(ds,dz) - \underline{\rho}^{\theta^X}(dz)ds, \tag{4.3.20}$$

with $\underline{\rho}^{\theta^X}(dz) = e^{\theta^X(e^z-1)}\rho^X(dz)$.

Also, $\widehat{Y}$ *admits the following expansion*

$$\widehat{Y}_t = \int_0^t r_s ds + \int_0^t \sigma_s dW_s^\theta + \int_0^t \int_{\mathbb{R}\backslash\{0\}} (e^z-1)\underline{\widetilde{N}}^X(ds,dz) \quad \textit{for} \quad t \in \mathcal{T}. \tag{4.3.21}$$

(2) *Let* $U_t := \frac{d\mathbb{Q}^{\theta^X}}{d\mathbb{P}}\Big|_{\mathcal{F}_t^X}$ *for all* $0 \le t \le T$*. If* (U_t) *is a multiplicative functional (See Asmussen (2003)) and* $U_0 = 1$ *then* X *has the Markov property with respect to the filtration*
$\mathbf{G} = \{\mathcal{G}_t, \quad 0 \le t \le T\}$.

Proof. 1) For $\mathbb{P}$ a.s. $\omega \in \Omega$, $\Lambda_t^{\theta^X} = \frac{d\widehat{\mathbb{Q}}^{\theta^X}(\omega,\cdot)}{d\widehat{\mathbb{P}}(\omega,\cdot)}|_{\overline{\mathcal{G}}_t} = e^{D_t^{\theta^X}}$ where $D_t^{\theta^X} = \int_0^t \theta_s^X d\widehat{Y}_s - \log \widehat{K}_t(\theta)$ i.e.,

$$D_t^{\theta^X} = \int_0^t \sigma_s\theta_s^X dW_s - \frac{1}{2}\int_0^t \sigma_s^2(\theta_s^X)^2 ds + \int_0^t\int_{\mathbb{R}\backslash\{0\}} \theta_s^X(e^z-1)\widetilde{N}^X(ds,dz) \tag{4.3.22}$$

$$- \int_0^t\int_{\mathbb{R}\backslash\{0\}} \Big(e^{\theta_s^X(e^z-1)} - 1 - \theta_s^X(e^z-1)\Big)\rho^{X_s}(dz)ds. \tag{4.3.23}$$

By Itô's formula,

$$\Lambda_t^{\theta^X} = e^{D_t^{\theta^X}} = 1 + \int_0^t \Lambda_{s-}^{\theta^X}\theta_s^X\sigma_s dW_s - \int_0^t\int_{\mathbb{R}\backslash\{0\}} \Lambda_{s-}^{\theta^X}\Big(e^{\theta_s^X(e^z-1)} - 1\Big)\widetilde{N}^X(ds,dz) \tag{4.3.24}$$

Thus, by setting $M_t := \int_0^t \sigma_s dW_s$ which is a local martingale under $\mathbb{P}$ we have

$$\begin{aligned} [M,\Lambda^{\theta^X}]_t &= \int_0^t \Lambda_{s-}^{\theta^X}\theta_s^X\sigma_s^2 ds \\ &= \int_0^t \Lambda_s^{\theta^X}\theta_s^X\sigma_s^2 ds, \end{aligned} \tag{4.3.25}$$

so from Protter (2003, thm III-39) we have that the process $\widehat{M}$ defined by

$$\widehat{M}_t = M_t - \int_0^t \frac{d[M, \Lambda^{\theta^X}]_s}{\Lambda_s^{\theta^X}} = \int_0^t \sigma_s dW_s - \int_0^t \theta_s^X \sigma_s^2 ds, \tag{4.3.26}$$

is a continuous local martingale under $\widehat{\mathbb{Q}}^{\theta^X}(\omega, \cdot)$ and we could write $\widehat{M}_t = \int_0^t \sigma_s dW_s^{\theta^X}$ where $W_t^{\theta^X} = W_t - \int_0^t \theta_s^X \sigma_s ds$.

Also, from the theorem 4.2.3 we have the decomposition

$$\frac{d\mathbb{Q}^{\theta^X}}{d\mathbb{P}}|_{\overline{\mathcal{G}}_t} = \Lambda_t^{\theta^X} U_T,$$

and

$$[W, \Lambda^{\theta^X} U_T]_t = \int_0^t \Lambda_s^{\theta^X} U_T \theta_s^X \sigma_s ds.$$

Hence, using theorem III-39 of Protter (2003) $W_t^{\theta^X} = W_t - \int_0^t \frac{d[W, \Lambda^{\theta^X} U]_s}{\Lambda_t^{\theta^X} U_T}$ is a locally martingale under $\mathbb{Q}^{\theta^X}$.

Also

$$[W^{\theta^X}, W^{\theta^X}]_t = [W, W]_t = t$$

and by the Lévy characterization theorem, W^{θ^X} is the standard Brownian motion under $\mathbb{Q}^{\theta^X}$ and consequently $\widehat{M}_t = \int_0^t \sigma_s dW_s^{\theta^X}$ is a continuous Gaussian process with independent increments under $\widehat{\mathbb{Q}}^{\theta^X}(\omega, \cdot)$.

Now, we will construct the compensated measure of N^X with respect to the family $\overline{\mathbf{G}} = \{\overline{\mathcal{G}}_t, \quad 0 \leq t \leq T\}$ and the probability measure $\mathbb{Q}^{\theta^X}$.

Let $\underline{\widetilde{N}}^X$ be the process defined for any $A \in \mathcal{B}(\mathbb{R}\backslash\{0\})$ (Fujiwara 2009) by

$$\begin{aligned} \underline{\widetilde{N}}_t^X(A) &:= N^X\Big((0,t], A\Big) - \int_0^t \int_{\mathbb{R}\backslash\{0\}} \mathbb{I}_A(z) \rho^X(dz) ds \\ &= \int_0^t \int_{\mathbb{R}\backslash\{0\}} \mathbb{I}_A(z) \widetilde{N}^X(ds, dz) \end{aligned} \tag{4.3.27}$$

we have $\Big[\underline{\widetilde{N}}^X(A), \Lambda^{\theta^X} U_T\Big]_t = \int_0^t \int_{\mathbb{R}\backslash\{0\}} \mathbb{I}_A(z) \Lambda_{s-}^{\theta^X} U_T \Big(e^{\theta_s(e^z-1)} - 1\Big) N^X(ds, dz)$.

Since $\rho^X(dz)dt$ is the predictable projection of the Poisson measure $N^X(dt, dz)$,

$$< \underline{\widetilde{N}}^X(A), \Lambda^\theta U_T >_t = \int_0^t \int_{\mathbb{R}\backslash\{0\}} \mathbb{I}_A(z) \Lambda_{s-}^\theta U_T \Big(e^{\theta_s(e^z-1)} - 1\Big) \rho^{X_s}(dz) ds \tag{4.3.28}$$

So, by theorem III-3.11 of Jacod and Shiryaev (2003), the new process $\underline{\widetilde{N}}$ defined by

$$\begin{aligned} \underline{\widetilde{N}}_t^X(A) &= \underline{\widetilde{N}}_t^X - \int_0^t \frac{d < \underline{\widetilde{N}}^X(A), \Lambda^\theta U_T >_s}{\Lambda_{s-}^\theta U_T} \\ &= N^X\Big((0,t], A\Big) - \int_0^t \int_{\mathbb{R}\backslash\{0\}} \mathbb{I}_A(z) e^{\theta_s(e^z-1)} \rho^X(dz) ds \end{aligned} \tag{4.3.29}$$

is also a locally martingale under $\mathbb{Q}^{\theta^X}$.
$\mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[[\underline{\widetilde{N}},\underline{\widetilde{N}}]_t\Big] = \mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[\int_0^t\int_{\mathbb{R}\backslash\{0\}}\mathbb{I}_A(z)e^{\theta_s(e^z-1)}\rho^X(dz)ds\Big] < \infty$ by condition (4.3.4).
Then by corollary II-3 of Protter (2003), $\underline{\widetilde{N}}^X$ is a martingale under $\mathbb{Q}^{\theta^X}$ with respect to family $\overline{\mathbf{G}} = \{\overline{\mathcal{G}}_t, \quad 0 \le t \le T\}$, hence it is the compensated measure of N^X under $\mathbb{Q}^{\theta^X}$ i.e.,

$$\underline{\widetilde{N}}^X(ds,dz) = N^X(ds,dz) - \underline{\rho}^{\theta^X}(dz)ds \tag{4.3.30}$$

where $\underline{\rho}^{\theta^X}(dz) := e^{\theta_s(e^z-1)}\rho^X(dz)$.

From decomposition 4.2.3 and using the results above and condition (4.3.5), we obtain this expression of Y under $\mathbb{Q}^{\theta^X}$:

$$Y_t = \widehat{\gamma}_t + \widehat{M}_t + \int_0^t\int_{\mathbb{R}\backslash\{0\}} h(z)\underline{\widetilde{N}}(ds,dz) + \int_0^t\int_{\mathbb{R}\backslash\{0\}}(z-h(z))N^X(ds,dz) \quad \text{for} \quad \forall t\in\mathcal{T} \tag{4.3.31}$$

where $\widehat{\gamma}$ is a continuous $\mathcal{F}_T^X$-measurable process defined by

$$\widehat{\gamma}_t := \int_0^t\Big(r_s - \frac{1}{2}\sigma_s^2\Big)ds - \int_0^t\int_{\{0<|z|\le 1\}}(e^z-1-z)\underline{\rho}^{\theta^X}(dz)ds - \int_0^t\int_{\{|z|>1\}}(e^z-1)\underline{\rho}^{\theta^X}(dz)ds \tag{4.3.32}$$

The expression of $\widehat{Y}$ is obtained straightforwardly as we do for Y.

2) Let $(\mathbb{P}_x)_{x\in\mathbb{S}}$ a family of probabilities defined on $(\Omega,\overline{\mathcal{G}}_T)$ and associated to the Markov semi-group $(P_t(x,.))$ which characterizes X under $\mathbb{P}$. We define another family of probabilities $(\mathbb{Q}_x^{\theta^X})_{x\in\mathbb{S}}$ on $(\Omega,\overline{\mathcal{G}}_T)$ such that

$$\mathbb{Q}_x^{\theta^X} = \mathbb{Q}^{\theta^X}[A|X_0=x], \quad A\in\overline{\mathcal{G}}_T. \tag{4.3.33}$$

$\mathbb{Q}_x^{\theta^X}$ is locally absolutely continuous with respect to $\mathbb{P}_x$ and

$$\frac{d\mathbb{Q}_x^{\theta^X}}{d\mathbb{P}_x}\Big|_{\mathcal{F}_t^X} = U_t, \quad t\in\mathcal{T}. \tag{4.3.34}$$

The Markov property for X under $(\mathbb{P},\mathbf{G})$ could be written as

$$\mathbb{E}^{\mathbb{P}_x}\Big[Z_t\circ\tau_s\Big|\mathcal{G}_s\Big] = \mathbb{E}^{\mathbb{P}_{X_s}}\Big[Z_t\Big], \quad \forall s\le t\le t+s\le T \tag{4.3.35}$$

for any $\mathcal{F}_t^X$-measurable Z_t and where τ_t is the shift operator (see Asmussen 2003). We want to prove that X is also Markov under $(\mathbb{Q}^{\theta^X}, \mathbf{G})$.
For any $\mathcal{F}_t^X$-measurable Z_t and $x \in \mathbb{S}$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_x^{\theta^X}}\left[Z_t \circ \tau_s \middle| \mathcal{G}_s\right] &= \frac{\mathbb{E}^{\mathbb{P}_x}\left[U_{t+s} Z_t \circ \tau_s \middle| \mathcal{G}_s\right]}{\mathbb{E}^{\mathbb{P}_x}\left[U_{t+s} \middle| \mathcal{G}_s\right]}, \quad (\text{ by Bayes' rule}) \\
&= \frac{\mathbb{E}^{\mathbb{P}_x}\left[U_s.(U_t \circ \tau_s).(Z_t \circ \tau_s) \middle| \mathcal{G}_s\right]}{\mathbb{E}^{\mathbb{P}_x}\left[U_s.(U_t \circ \tau_s) \middle| \mathcal{G}_s\right]} \\
&= \frac{\mathbb{E}^{\mathbb{P}_x}\left[(U_t.Z_t) \circ \tau_s \middle| \mathcal{G}_s\right]}{\mathbb{E}^{\mathbb{P}_x}\left[U_t \circ \tau_s \middle| \mathcal{G}_s\right]} \\
&= \frac{\mathbb{E}^{\mathbb{P}_{X_s}}\left[U_t.Z_t\right]}{\mathbb{E}^{\mathbb{P}_{X_s}}\left[U_t\right]}, \quad (\text{by } 4.3.35) \\
&= \mathbb{E}^{\mathbb{P}_{X_s}}\left[U_t.Z_t\right] \\
&= \mathbb{E}^{\mathbb{Q}_{X_s}^{\theta^X}}\left[Z_t\right].
\end{aligned} \tag{4.3.36}$$

where we have used the fact that U is a multiplicative functional which verifies $\mathbb{E}^{\mathbb{P}_x}[U_t] = 1,\ \forall x \in \mathbb{S}$. $\square$

Remark 4.3.3. *Theorem 4.2.2 and proposition 4.3.2 imply that (X, Y) remains a Markov additive process under the risk-neutral measure $\mathbb{Q}^{\theta^X}$.*

Let $\omega \in \Omega$ such that $\widehat{\mathbb{P}}(\omega, \cdot)$ denotes a regular version of the conditional probability $\mathbb{P}(.|\mathcal{F}_T^X)$. We have seen above that the log-return process Y is an additive process under $(\Omega, \mathcal{G}, \widehat{\mathbb{P}}(\omega, \cdot))$, thus we can apply the existence result of MEMM for exponential additive processes of Fujiwara (2009). The following theorem is adapted from this result.

Proposition 4.3.3.
Under condition (H) *we have that for all $\mathbb{Q} \in \mathcal{M}^a(\widetilde{S})$, if $\widehat{\mathbb{Q}}(\omega, \cdot)$ and $\widehat{\mathbb{P}}(\omega, \cdot)$ denote respectively some regular versions of the conditional probability $\mathbb{Q}(.|\mathcal{F}_T^X)$ and $\mathbb{P}(.|\mathcal{F}_T^X)$ Then*

$$\mathbb{H}_{\mathcal{G}_T}(\widehat{\mathbb{Q}}, \widehat{\mathbb{P}}) \geq \mathbb{H}_{\mathcal{G}_T}(\widehat{\mathbb{Q}}^{\theta^X}, \widehat{\mathbb{P}}) \quad \mathbb{P}\ (a.s), \tag{4.3.37}$$

where $\widehat{\mathbb{Q}}^{\theta^X}$ is defined by (4.3.9).

PROOF.

Let $\mathbb{Q} \in \mathcal{M}^a(\widetilde{S})$ be arbitrary. By theorem (4.2.3), $\mathbb{Q}$ admits the decomposition

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_t} = \Lambda_t^Y \Lambda_T^X, \quad \text{for all} \quad 0 \leq t \leq T \tag{4.3.38}$$

where we set $\Lambda_t^Y := \frac{d\widehat{\mathbb{Q}}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_t}$ with $\widehat{\mathbb{Q}}$ and $\widehat{\mathbb{P}}$ denoting the regular versions of $\mathbb{Q}(.|\mathcal{F}_T^X)$ and $\mathbb{P}(.|\mathcal{F}_T^X)$ respectively. On the other hand, we have from Theorem III-33 of Protter (2003) that $\{\widetilde{R}_t : 0 \leq t \leq T\}$ and $\{\int_0^t \theta_s^X d\widetilde{R}_s : 0 < t \leq T\}$ are $(\overline{\mathbf{G}})-$local martingale under $\mathbb{Q}$ since $\{1/\widetilde{S}_t^- : 0 \leq t \leq T\}$ and $\{\theta_t^X : t \in \mathcal{T}\}$ are càglàd. Therefore, there exists a sequence $(T_n)_{n\geq 0}$ of stopping times such that $T_n \nearrow T$ as $n \longrightarrow +\infty$ and $\{\int_0^{t\wedge T_n} \theta_s^X d\widetilde{R}_s : 0 \leq t \leq T\}$ is an $(\overline{\mathbf{G}})$-martingale under $\mathbb{Q}$.

Now, by using lemma 4.3.1 we have $\mathbb{P}$ $a.s$

$$\begin{aligned}
\mathbb{H}_{\overline{\mathcal{G}}_T}(\widehat{\mathbb{Q}}|\widehat{\mathbb{P}}) &\geq \mathbb{H}_{\overline{\mathcal{G}}_{T_n}}(\widehat{\mathbb{Q}}|\widehat{\mathbb{P}}) \text{ (since } \overline{\mathcal{G}}_{T_n} \subset \overline{\mathcal{G}}_T) \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\log\Big(\frac{d\widehat{\mathbb{Q}}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_{T_n}}\Big)\Big|\mathcal{F}_T^X\Big] \\
&\geq \mathbb{E}^{\mathbb{Q}}\Big[\log\Big(\frac{d\widehat{\mathbb{Q}}^{\theta^X}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_{T_n}}\Big)\Big|\mathcal{F}_T^X\Big]
\end{aligned} \tag{4.3.39}$$

where

$$\frac{d\widehat{\mathbb{Q}}^{\theta^X}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_{T_M}} := \frac{\exp\Big(\int_0^{T_n} \theta_s^X d\widetilde{R}_s\Big)}{\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^{T_n} \theta_s^X d\widetilde{R}_s\Big)\Big|\mathcal{F}_T^X\Big]} \quad (\mathbb{P}\ a.s), \tag{4.3.40}$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\log\left(\frac{d\widehat{\mathbb{Q}}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_{T_n}}\right)\Big|\mathcal{F}_T^X\right] = \mathbb{E}^{\mathbb{Q}}\left[\int_0^{T_n} \theta_s^X d\widetilde{R}_s - \widehat{K}_{T_n}(\theta) + \int_0^{T_n} \theta_s^X r_s ds\Big|\mathcal{F}_T^X\right]. \tag{4.3.41}$$

But

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\int_0^{T_n} \theta_s^X d\widetilde{R}_s\Big|\mathcal{F}_T^X\right] &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^{T_n} \theta_s^X d\widetilde{R}_s\Big|\mathcal{G}_0\right] \\
&= 0 \quad (\mathbb{P}\ a.s),
\end{aligned}$$

since $\Big\{\int_0^{t\wedge T_n} \theta_s^X d\widetilde{R}_s : 0 \leq t \leq T\Big\}$ is a $\overline{\mathbf{G}}$-martingale under $\mathbb{Q}$. Hence because $\{r_t : 0 \leq t \leq T\}$ and $\{\widehat{K}_t(\theta) : 0 \leq t \leq T\}$ are $\mathcal{F}_T^X-$measurable, we then have $\mathbb{P}$ $a.s$

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\log\Big(\frac{d\widehat{\mathbb{Q}}^{\theta}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_{T_n}}\Big)\right] = \int_0^{T_n} \theta_s^X r_s ds - \widehat{K}_{T_n}(\theta), \quad \forall n \in \mathbb{N}. \tag{4.3.42}$$

On the other side, we have by Dominated Convergence Theorem:

$$\lim_{n\to+\infty}\left\{\int_0^{T_n}\theta_s^X r_s ds-\widehat{K}_{T_n}(\theta)\right\}=\int_0^T\theta_s^X r_s ds-\widehat{K}_T(\theta)\quad(\mathbb{P}\ a.s). \tag{4.3.43}$$

Thus from (4.3.39), for all $\mathbb{Q}\in\mathcal{M}^a(\widetilde{S})$

$$\mathbb{H}_{\overline{\mathcal{G}}_T}(\widehat{\mathbb{Q}}|\widehat{\mathbb{P}})\geq\int_0^T\theta_s^X r_s ds-\widehat{K}_T(\theta)\quad(\mathbb{P}\ a.s). \tag{4.3.44}$$

But, we have that $(\int_0^t\theta_s^X d\widetilde{R}_s)_{0\leq t\leq T}$ is a $\overline{\mathbf{G}}$-martingale under $\mathbb{Q}^{\theta^X}$ [See Appendix B] and therefore

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^{\theta^X}}\left[\int_0^T\theta_s^X d\widetilde{R}_s\middle|\mathcal{F}_T^X\right] &= \mathbb{E}^{\mathbb{Q}^{\theta}}\left[\int_0^T\theta_s^X d\widetilde{R}_s\middle|\overline{\mathcal{G}}_0\right]\\ &= 0\quad(\mathbb{P}\ a.s).\end{aligned} \tag{4.3.45}$$

Hence, we have

$$\begin{aligned}\mathbb{H}_{\overline{\mathcal{G}}_T}(\widehat{\mathbb{Q}}^{\theta^X},\widehat{\mathbb{P}}) &:= \mathbb{E}^{\mathbb{Q}^{\theta^X}}\left[\int_0^T\theta_s^X d\widetilde{R}_s+\int_0^T\theta_s^X r_s ds-\widehat{K}_T(\theta)\middle|\mathcal{F}_T^X\right]\\ &= \mathbb{E}^{\mathbb{Q}^{\theta^X}}\left[\int_0^T\theta_s^X r_s ds\middle|\mathcal{F}_T^X\right]-\mathbb{E}^{\mathbb{Q}^{\theta}}\left[\widehat{K}_T(\theta)\middle|\mathcal{F}_T^X\right]\\ &= \int_0^T\theta_s^X r_s ds-\widehat{K}_T(\theta)\quad(\mathbb{P}\ a.s),\end{aligned}$$

Thus

$$\mathbb{H}_{\overline{\mathcal{G}}_T}(\widehat{\mathbb{Q}},\widehat{\mathbb{P}})\geq\mathbb{H}_{\overline{\mathcal{G}}_T}(\widehat{\mathbb{Q}}^{\theta^X},\widehat{\mathbb{P}})\quad(\mathbb{P}\ a.s), \tag{4.3.46}$$

for all $\mathbb{Q}\in\mathcal{M}^a(\widetilde{S})$ and we can conclude that

$$\begin{aligned}\mathbb{H}_{\overline{\mathcal{G}}_T}(\widehat{\mathbb{Q}},\widehat{\mathbb{P}}) &\geq \mathbb{H}_{\overline{\mathcal{G}}_T}(\widehat{\mathbb{Q}}^{\theta^X},\widehat{\mathbb{P}})\\ &= \int_0^T\theta_s^X r_s ds-\widehat{K}_T(\theta)\quad(\mathbb{P}\ a.s).\end{aligned} \tag{4.3.47}$$

This ends the proof of proposition 4.3.3. □

In the previous proposition, we showed the optimality under certain conditions of one component of the decomposition of $\mathbb{Q}^{\theta^X}$. Specifically, this component is optimal among all "projections" on the σ-algebra $\mathcal{F}_T^X$ of probability measures $\mathbb{Q}\in\mathcal{M}^a(\widetilde{S})$. So, to obtain a global result we need to add some conditions on the other component. This is the subject of next section.

4.3.3. An optimization problem

Theorem 4.2.3 tells us that the density process related to a probability measure $\mathbb{Q} \in \mathcal{M}^a(\widetilde{S})$ can be decomposed as a product of a density process associated to the conditional probabilities given $\mathcal{F}_T^X$ and a Radon-Nikodym derivative associated to the change of measure of the Markov chain X. This decomposition is crucial because it permits to reduce with Proposition 4.3.3 the problem of finding a MEMM for a Markov-modulated Lévy model to an optimization problem involving only the characteristics of the Markov chain component X.
In particular, we need to know how the Markov chain changes through a change of measure. So, we begin by giving a procedure to characterize this change of measure, namely by using a Girsanov-type theorem. In this setting we set the optimization problem and give the conditions for the existence of a solution.

4.3.3.1. *Girsanov's change of measure for the continuous-time Markov chain*

The Markov chain X is completely characterized under $\mathbb{P}$ by $(\Pi_0, \mathbb{A})$, where Π_0 is the initial probability distribution and $\mathbb{A} := (a_{ij})_{1 \leq i,j \leq N}$ is the intensity matrix. We recall that an intensity matrix (for X) is a square matrix $\mathbb{M} := (m_{ij})_{1 \leq i,j \leq N}$ with elements m_{ij} which satisfy the following conditions

(1) $\forall i, j = 1, 2, ..., N$ with $i \neq j, \quad m_{ij} > 0$;

(2) $\sum_{i=1}^{M} m_{ij} = 0$, so $m_{ii} < 0$.

In the sequel, we suppose without loss of generality that the change of measure modifies only the intensity matrix. Now, we introduce some notations as in Dufour and Elliott (1999).

- Let $\mathbf{a} := \Big(a_{11}, a_{22}, ..., a_{NN}\Big)' \in \mathbb{R}^M$ and $\mathbb{A}_0 := \mathbb{A} - \mathbf{Diag}(\mathbf{a})$, where $\mathbf{Diag}(\mathbf{a})$ is a diagonal matrix with the elements given by the vector $\mathbf{a}$;
- Let $\mathcal{C} := \Big\{(c_{ij})_{1 \leq i,j \leq N}$ where c_{ij} verifies conditions (1)-(2) above$\Big\}$ the set of all possible intensity matrix for X;
- For $\mathbb{A}$, $\mathbb{C} \in \mathcal{C}$ we define $\mathbb{D} := \mathbb{C}/\mathbb{A}$ the matrix defined by $\mathbb{D} := (c_{ij}/a_{ij})_{1 \leq i,j \leq N}$;
- $\mathbf{1} := (1, 1, ..., 1)' \in \mathbb{R}^M$ and $\mathbb{I}$ the identity matrix.

We have assumed that X is irreducible so $a_{ij} \neq 0$ and thus $\mathbb{D}$ is well defined. Now, consider a vector of counting process $\mathbf{N} := \{\mathbf{N}_t : t \in \mathcal{T}\}$ with

$$\mathbf{N}_t = \int_0^t \Big(\mathbb{I} - \mathbf{Diag}(X_{s^-})\Big) dX_s \tag{4.3.48}$$

Its component $N_t(i)$ counts the number of times the chain X jumps to state s_i during the time interval $[0, t]$ for $i = 1, 2, ..., N$.

Dufour and Elliott (1999) showed that the process defined by

$$\widetilde{\mathbf{N}}_t := \mathbf{N}(t) - \int_0^t \mathbb{A}_0 X_{u^-} du, \quad t \in \mathcal{T} \tag{4.3.49}$$

is a $(\{\mathcal{F}_t^X\}_{t\in\mathcal{T}}, \mathbb{P})-$martingale. Hence, we can state

Proposition 4.3.4. *The process* $\Lambda^C := \{\Lambda_t^C : t \in \mathcal{T}\}$ *defined by*

$$\Lambda_t^C = 1 + \int_0^t \Lambda_{u^-}^C [\mathbb{D}_0 X_{u^-} - \mathbf{1}]' d\widetilde{\mathbf{N}}_u \tag{4.3.50}$$

is a $(\{\mathcal{F}_t^X\}_{t\in\mathcal{T}}, \mathbb{P})-$*martingale under some regularity conditions on the coefficients of* $\mathbb{D}$.

Also, by setting

$$\frac{d\mathbb{Q}^C}{d\mathbb{P}}\Big|_{\mathcal{F}_t^X} := \Lambda^C(t) \tag{4.3.51}$$

we define an absolutely continuous probability measure $\mathbb{Q}^C$ *w.r.t* $\mathbb{P}$ *under which* X *is a Markov chain characterized by the intensity matrix* $\mathbb{C}$.

For the proof, we refer to Dufour and Elliott (1999).

An explicit expression of this likelihood ratio is given by the following proposition

Proposition 4.3.5.

$$\frac{d\mathbb{Q}^C}{d\mathbb{P}}\Big|_{\mathcal{F}_t^X} := \exp\Big\{ - \int_0^t [\mathbb{D}_0 X_{s^-} - \mathbf{1}]' \mathbb{A}_0 X_{s^-} ds \Big\} \prod_{0<s\leq t} \Big(1 + [\mathbb{D}_0 X_{s^-} - \mathbf{1}]' \Delta N_s\Big) \tag{4.3.52}$$

Furthermore,

$$\frac{d\mathbb{Q}^C}{d\mathbb{P}}\Big|_{\mathcal{F}_t^X} := \prod_{1\leq i\neq j\leq M} \exp\Bigg\{ - \int_0^t (\widehat{\lambda}_s^{ij} - \lambda_s^{ij}) ds + \int_0^t \log\Big(\frac{\widehat{\lambda}_s^{ij}}{\lambda_s^{ij}}\Big) d\mathcal{N}_s^{ij} \Bigg\} \tag{4.3.53}$$

with for $i \neq j$,

- $\mathcal{N}_t^{ij} := \sum_{0<s\leq t} \langle X_{s^-}, \mathbf{e}_i\rangle \langle X_s, \mathbf{e}_j\rangle$ *the number of times where* X *jumps from* $\mathbf{e}_i$ *to* $\mathbf{e}_j$ *during the time interval* $[0, t]$ *(*$d\mathcal{N}_t^{ij}$ *denotes its differential notation);*
- $\lambda_t^{ij} := a_{ij} \langle X_{t^-}, \mathbf{e}_i\rangle \langle X_t, \mathbf{e}_j\rangle$;
- $\widehat{\lambda}_t^{ij} := c_{ij} \langle X_{t^-}, \mathbf{e}_i\rangle \langle X_t, \mathbf{e}_j\rangle$

Proof. See Appendix C. $\square$

To simplify notation, we set for $1 \leq i \neq j \leq M$ and $s \in [0,T]$ in the expression (4.3.53)

$$\alpha_s^{ij} := \log\Big(\frac{\widehat{\lambda}_s^{ij}}{\lambda_s^{ij}}\Big), \tag{4.3.54}$$

then we have

$$\begin{aligned} \Lambda_T^C &= \prod_{1\leq i\neq j\leq M} \exp\Big\{-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}}-1)ds + \int_0^T \alpha_s^{ij} d\mathcal{N}_s^{ij}\Big\} \\ &= \exp\Big\{\sum_{1\leq i\neq j\leq M}\Big(-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}}-1)ds + \int_0^T \alpha_s^{ij} d\mathcal{N}_s^{ij}\Big)\Big\}. \end{aligned} \tag{4.3.55}$$

In the sequel, we will denote Λ_T^C by Λ_T^α.

For a given process $\theta^X := \{\theta_t^X : t \in [0,T]\}$ satisfying condition (H) and $\alpha := \{\alpha_t^{ij} : 1 \leq i \neq j \leq M; t \in [0,T]\}$ we introduce the functional $F(\alpha)$ defined by

$$\begin{aligned} F(\alpha) := \mathbb{E}^{\mathbb{P}}\Bigg[\Big\{\sum_{1\leq i\neq j\leq M}\Big(-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}}-1-\alpha_s^{ij})ds\Big) + \int_0^T \theta_s^X r_s ds - \widehat{K}_T(\theta^X)\Big\} \\ \times \exp\Big\{\sum_{1\leq i\neq j\leq M}\Big(-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}}-1)ds + \int_0^T \alpha_s^{ij} d\mathcal{N}_s^{ij}\Big)\Big\}\Bigg] \end{aligned} \tag{4.3.56}$$

Let $\mathbb{Q}^{(\alpha,\ \theta^X)}$ a probability measure absolutely continuous with respect to $\mathbb{P}$ on $\overline{\mathcal{G}}_T$ and characterized by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^{(\alpha,\ \theta^X)}}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_T} := \Lambda_T^{\theta^X}\,.\,\Lambda_T^\alpha \tag{4.3.57}$$

Lemma 4.3.2. *Under the probability measure $\mathbb{Q}^{(\alpha,\ \theta^X)}$ defined above, the functional $F(\alpha)$ admits a simple expression*

$$\begin{aligned} F(\alpha) = &\sum_{1\leq i\neq j\leq M} \alpha_{ij}\mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Big[\mathcal{N}_T^{ij}\Big] \\ +&\sum_{j=1}^M \Big[\frac{1}{2}\theta_j^2\sigma_j^2 + \int_{\mathbb{R}\backslash\{0\}}\Big(e^{-\theta_j(e^z-1)} - 1 + \theta_j(e^z-1)\Big)\underline{\rho}_j^\theta(dz) - \sum_{\substack{i=1\\ i\neq j}}^M a_{ij}(e^{\alpha_{ij}}-1)\Big]\mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Big[\tau_j\Big] \end{aligned} \tag{4.3.58}$$

where we have set

- τ_i *the occupation time of the Markov chain* X *in the state* $\mathbf{e}_i$ *during the time interval* $[0,T]$*;*
- $\alpha_t^{ij} := \alpha_{ij}\langle X_{t^-}, \mathbf{e}_i\rangle\langle X_t, \mathbf{e}_j\rangle$*;*
- $\theta_t^X = \sum_{j=1}^M \theta_j \langle X_{t^-}, \mathbf{e}_i\rangle$
- $\underline{\rho}_j^{\theta}(dz) := e^{\theta_j(e^z-1)}\rho_j(dz)$*;*

Proof. See Appendix D. $\square$

Now, we are in situation to state the main result concerning the characterization of the MEMM for the Markov-modulated exponential Lévy model.

Theorem 4.3.1. *If it exists a matrix* $\widetilde{\alpha} := \{\widetilde{\alpha}_{ij} : 1 \leq i \neq j \leq M\}$ *solution of the problem*

$$\min_{\alpha} F(\alpha) \tag{4.3.59}$$

Then the minimal entropy equivalent martingale measure for the model (4.2.1)-(4.2.2) exists and is defined by

$$\frac{d\mathbb{Q}^{(\widetilde{\alpha},\ \theta^X)}}{d\mathbb{P}}\Big|_{\mathbb{Q}^{(\widetilde{\alpha},\ \theta^X)}} := \Lambda_T^{\theta^X} \,.\, \Lambda_T^{\widetilde{\alpha}} \tag{4.3.60}$$

Proof. Let $\mathbb{Q}$ be an arbitrary probability of $\mathcal{M}^a(\widetilde{S})$. By proposition 4.2.3 we can always write that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_T} = \Lambda_T^Y \Lambda_T^{\alpha} \tag{4.3.61}$$

where $\Lambda_T^{\alpha} = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T^X}$ and $\Lambda_T^{Y} = \frac{d\widehat{\mathbb{Q}}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_T}$. By an easy calculation we obtain

$$\begin{aligned}
\mathbb{H}_{\mathcal{G}_T}(\mathbb{Q}|\mathbb{P}) &= \mathbb{E}^{\mathbb{P}}\Big[\Lambda_T^{\alpha}\log(\Lambda_T^{\alpha})\mathbb{E}^{\mathbb{P}}[\Lambda_T^{Y}|\mathcal{F}_T^X] + \Lambda_T^{\alpha}\mathbb{E}^{\mathbb{P}}[\Lambda_T^{Y}\log(\Lambda_T^{Y})|\mathcal{F}_T^X]\Big] \\
&= \mathbb{E}^{\mathbb{P}}\Big[\Lambda_T^{\alpha}\log(\Lambda_T^{\alpha}) + \Lambda_T^{\alpha}\mathbb{E}^{\mathbb{P}}[\Lambda_T^{Y}\log(\Lambda_T^{Y})|\mathcal{F}_T^X]\Big] \quad \text{since} \quad \mathbb{E}^{\mathbb{P}}[\Lambda_T^{Y}|\mathcal{F}_T^X] = 1, \\
&= \mathbb{E}^{\mathbb{P}}\Big[\Lambda_T^{\alpha}\log(\Lambda_T^{\alpha}) + \Lambda_T^{\alpha}\mathbb{H}_{\mathcal{G}_T}(\widehat{\mathbb{Q}}, \widehat{\mathbb{P}})\Big] \\
&\geq \mathbb{E}^{\mathbb{P}}\Big[\Lambda_T^{\alpha}\log(\Lambda_T^{\alpha}) + \Lambda_T^{\alpha}\mathbb{H}_{\mathcal{G}_T}(\widehat{\mathbb{Q}}^{\theta^X}, \widehat{\mathbb{P}})\Big] \qquad \text{(by proposition 4.3.3)}, \\
&= \mathbb{E}^{\mathbb{P}}\Bigg[\Bigg\{\sum_{1\leq i\neq j\leq M}\Big(-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}} - 1 - \alpha_s^{ij})ds\Big) + \int_0^T \theta_s^X r_s ds - \widehat{K}_T(\theta^X)\Bigg\} \\
&\times \ \exp\Bigg\{\sum_{1\leq i\neq j\leq M}\Bigg(-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}} - 1)ds + \int_0^T \alpha_s^{ij} d\mathcal{N}_s^{ij}\Bigg)\Bigg\}\Bigg] \\
&= F(\alpha).
\end{aligned}$$

If we admit the existence of $\widetilde{\alpha} := arg\min \ F(\alpha)$, then

$$\begin{aligned}
F(\alpha) &\geq F(\widetilde{\alpha}) \\
&= \mathbb{E}^{\mathbb{Q}^{(\widetilde{\alpha},\ \theta^X)}}\Big[\log(\Lambda_T^{\widetilde{\alpha}}) + \log(\Lambda_T^{\theta^X})\Big] \\
&= \mathbb{H}_{\overline{\mathcal{G}}_T}(\mathbb{Q}^{(\widetilde{\alpha},\ \theta^X)}|\mathbb{P}).
\end{aligned}$$

$\square$

Remark 4.3.4. *The above result tells us that once the Esscher parameter θ^X is found, the search of the MEMM consists in minimizing the functional $F(\alpha)$ over the space of matrices $\alpha := \{\alpha_{ij} : 1 \leq i \neq j \leq M\}$. In order to have an explicit result, we need an expression for $\mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Big[\tau_j\Big]$ and $\mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Big[\mathcal{N}_T^{ij}\Big]$ which we do not have since T is finite. However, by considering the asymptotic case ($T \to \infty$) and with the hypothesis that X admits a stationary distribution Π under $\mathbb{Q}^{(\alpha,\ \theta^X)}$ the problem may be solved using Lagrange multipliers.*

4.4. Example

We work out some examples to illustrate how to implement the results developed in previous Sections.

4.4.1. The Regime-switching Black-Scholes model

We take the most simple example of Markov-modulated Lévy model namely the regime-switching Black-Scholes model. This model is recovered from the general setting described in Section **??** by imposing $N(.,.) = 0$ in equation (4.2.2). For ease of exposition, we take $M = 2$ (two-regimes case).

In this case, the condition (H) yields a unique process $\theta := \{\frac{r_t - \mu_t}{\sigma_t^2} : t \in \mathbb{T}\}$ and thus we have to minimize the functional

$$\begin{aligned} F(\alpha) = &\sum_{1 \leq i \neq j \leq 2} \alpha_{ij} E^{\mathbb{Q}^{(\alpha,\ \theta^X)}} \Big[\mathcal{N}_T^{ij}\Big] \\ &+ \sum_{j=1}^{2} \Big[\frac{1}{2}(\frac{r_j - \mu_j}{\sigma_j^2})^2 \sigma_j^2 - \sum_{\substack{i=1 \\ i \neq j}}^{2} a_{ij}(e^{\alpha_{ij}} - 1)\Big] E^{\mathbb{Q}^{(\alpha,\ \theta^X)}} \Big[\tau_j\Big] \end{aligned} \tag{4.4.1}$$

If we suppose that X admits a stationary distribution $\Pi = (\pi_1, \pi_2)$ under $\mathbb{Q}^{(\alpha,\ \theta^X)}$, then by the ergodic theorem we have the following approximations as $T \to \infty$

$$\begin{aligned} E^{\mathbb{Q}^{(\alpha,\ \theta^X)}} \Big[\tau_j\Big] &\approx \pi_j T, \quad j = 1, 2 \\ E^{\mathbb{Q}^{(\alpha,\ \theta^X)}} \Big[\mathcal{N}_T^{ij}\Big] &\approx a_{ij} e^{\alpha_{ij}} \pi_j T, \quad 1 \leq i \neq j \leq 2. \end{aligned} \tag{4.4.2}$$

Moreover, the optimization problem (4.3.59) is equivalent to

$$\min_{(\Pi,\ \alpha)} \left\{ \sum_{1 \leq i \neq j \leq 2} \alpha_{ij} e^{\alpha_{ij}} a_{ij} \pi_j T + \sum_{j=1}^{2} \left[\frac{1}{2}(r_j - \mu_j)^2 - \sum_{\substack{i=1 \\ i \neq j}}^{2} a_{ij}(e^{\alpha_{ij}} - 1)\right] \pi_j T \right\} \tag{4.4.3}$$

under the constraints

$$\begin{aligned} \pi_1 + \pi_2 &= 1 \\ \pi_2 a_{21} e^{\alpha_{21}} - \pi_1 a_{12} e^{\alpha_{12}} &= 0 \end{aligned} \tag{4.4.4}$$

This problem can be easily solved using Lagrangian techniques for optimization. Indeed, we can prove that the solution $(\Pi,\ \alpha)$ is such that:

$$\begin{aligned} \pi_1 &= \frac{a_{12} e^{\alpha_{12}}}{a_{12} e^{\alpha_{12}} + a_{21} e^{\alpha_{21}}} \\ \pi_2 &= \frac{a_{21} e^{\alpha_{21}}}{a_{12} e^{\alpha_{12}} + a_{21} e^{\alpha_{21}}} \end{aligned} \tag{4.4.5}$$

where $(\alpha_{12},\ \alpha_{21})$ is the solution of the system of equations

$$a_{21}^2 e^{2\alpha_{21}}\alpha_{12} + a_{12}^2 e^{2\alpha_{12}}\alpha_{21} = 0 \tag{4.4.6}$$

$$\begin{aligned} a_{21}\alpha_{21}e^{\alpha_{21}}(a_{12}e^{\alpha_{12}} + a_{21}e^{\alpha_{21}}) = \Big[&(1-\alpha_{21})a_{21}e^{\alpha_{21}} - (1-\alpha_{12})a_{12}e^{\alpha_{12}} \\ &+ a_{12} - a_{21} - \frac{1}{2}(r_1-\mu_1)^2 + \frac{1}{2}(r_2-\mu_2)^2\Big]a_{12}e^{\alpha_{12}} \end{aligned} \tag{4.4.7}$$

which can be obtained numerically.

4.4.2. A Regime-switching Jump-diffusion model

We consider a two-state Markov-modulated Lévy model where the Lévy measure ρ^X is given by $\rho_t^X(dz) = \sum_{j=1}^2 \langle X_{t^-}|\mathbf{e}_j\rangle c_j\rho_j(dz)$ with $c = (c_1, c_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and ρ_j $(j = 1, 2)$ a probability measure on $\mathbb{R}$.
For example,

$$\rho_j(dz) = \frac{1}{2}\delta_{-x_j} + \frac{1}{2}\delta_{x_j} \tag{4.4.8}$$

where $x_j > 1;\ j = 1, 2$ and δ_x is the Dirac distribution at x.

In this case, the condition (H) yields to $\theta^X = (\theta_1, \theta_2)$ as the unique solution of

$$\mu_j + \theta_j\sigma_j^2 + \frac{1}{2}c_j\Big[\Big(e^{\theta_j(e^{-x_j}-1)} - 1\Big)(e^{-x_j}-1) + \Big(e^{\theta_j(e^{x_j}-1)} - 1\Big)(e^{x_j}-1)\Big] - r_j = 0 \qquad j = 1, 2. \tag{4.4.9}$$

Which can be retrieve numerically. Then, the functional which we have to minimize is given by

$$\begin{aligned} F(\alpha) = &\sum_{1\le i\neq j\le 2} \alpha_{ij}E^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Big[\mathcal{N}_T^{ij}\Big] \\ &+ \sum_{j=1}^{2}\Bigg[\frac{1}{2}\theta_j^2\sigma_j^2 + \frac{c_j}{2}\Bigg(\Big(e^{-\theta_j(e^{-x_j}-1)} - 1 + \theta_j(e^{-x_j}-1)\Big)e^{\theta_j(e^{-x_j}-1)} \\ &+ \Big(e^{-\theta_j(e^{x_j}-1)} - 1 + \theta_j(e^{x_j}-1)\Big)e^{\theta_j(e^{x_j}-1)}\Bigg) - \sum_{\substack{i=1\\ i\neq j}}^{2} a_{ij}(e^{\alpha_{ij}}-1)\Bigg]E^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Big[\tau_j\Big] \end{aligned} \tag{4.4.10}$$

Hence, by arguing as previous in Equations 4.4.3-4.4.7 we obtain the results.

4.5. Concluding remarks

In this paper, we showed how the minimal entropy martingale measure can be determined for a general Markov-modulated Lévy model. Our technique follows those developed by Fujiwara and Miyahara (2003) for geometric Lévy processes and Fujiwara (2009) for exponential additive processes. For our case, we use the particular properties of the Markov additive processes to overcome some difficulties.

4.6. Appendix

4.6.1. Appendix A

We have to show that $\forall\ t \in [0,T]$,

$$\begin{aligned}\widehat{K}_t(\theta^X) &:= \mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^t \theta_s^X d\widehat{Y}_s\Big)\Big|\mathcal{F}_T^X\Big] \\ &= \int_0^t \Big[\theta_s^X \mu_s + \frac{1}{2}(\theta_s^X)^2\sigma_s^2\Big]ds + \int_0^t \int_{\mathbb{R}\backslash\{0\}} \Big(e^{\theta_s^X(e^z-1)} - 1 - \theta_s^X(e^z-1))\Big)\rho^X(dz)ds.\end{aligned} \tag{4.6.1}$$

Proof. Formally, we can write by using Itô's Formula:

$$\begin{aligned}\exp\Big(\int_0^t \theta_s^X d\widehat{Y}_s\Big) &= 1 + \int_0^t \exp\Big(\int_0^s \theta_s^X d\widehat{Y}_s\Big)\Big[\theta_s^X\mu_s + \int_{\mathbb{R}\backslash\{0\}} \theta_s^X(e^z-1)\widetilde{N}^X(ds;dy)\Big]ds \\ &+ \int_0^t \exp\Big(\int_0^s \theta_s^X d\widehat{Y}_s\Big)\theta_s^X\sigma_s dW_s + \frac{1}{2}\int_0^t \exp\Big(\int_0^s \theta_s^X d\widehat{Y}_s\Big)(\theta_s^X)^2\sigma_s^2 ds \\ &+ \int_0^t \int_{\mathbb{R}\backslash\{0\}} \exp\Big(\int_0^s \theta_{s^-}^X d\widehat{Y}_s\Big)\Big(e^{\theta_{s^-}^X(e^z-1)} - 1 - \theta_{s^-}^X(e^z-1)\Big)N^X(ds,dz),\end{aligned} \tag{4.6.2}$$

or,

$$\begin{aligned}\exp\Big(\int_0^t \theta_s^X d\widehat{Y}_s\Big) &= 1 + \int_0^t \exp\Big(\int_0^s \theta_s^X d\widehat{Y}_s\Big)\theta_s^X\sigma_s dW_s \\ &+ \int_0^t \int_{\mathbb{R}\backslash\{0\}} \exp\Big(\int_0^s \theta_s^X d\widehat{Y}_s\Big)\Big(e^{\theta_s^X(e^z-1)} - 1\Big)\widetilde{N}^X(ds;dy) \\ &+ \int_0^t \exp\Big(\int_0^s \theta_s^X d\widehat{Y}_s\Big)\Big[\theta_s^X\mu_s + \frac{1}{2}(\theta_s^X)^2\sigma_s^2 + \int_{\mathbb{R}\backslash\{0\}} \Big(e^{\theta_s^X y} - 1 - \theta_s^X(e^z-1)\Big)\rho^X(dz)\Big]ds.\end{aligned} \tag{4.6.3}$$

Conditioning both sides of (4.6.3) on $\mathcal{F}_T^X$,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^t \theta_s^X d\widehat{Y}_s\Big)\Big|\mathcal{F}_T^X\Big] &= \\ &1 + \int_0^t \Big[\theta_u^X\mu_u + \frac{1}{2}(\theta_u^X)^2\sigma_u^2\Big]\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^u \theta_s^X d\widehat{Y}_s\Big)\Big|\mathcal{F}_T^X\Big]du \\ &+ \int_0^t \int_{\mathbb{R}\backslash\{0\}} \Big(e^{\theta_u^X(e^z-1)} - 1 - \theta_u^X(e^z-1)\Big)\rho^X(dz)\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^u \theta_s^X d\widehat{Y}_s\Big)\Big|\mathcal{F}_T^X\Big]du.\end{aligned} \tag{4.6.4}$$

Where we have used the fact that W and $\widetilde{N}^X(.;.)$ are martingales.

Then, by solving (4.6.4) we obtain

$$\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^t \theta_s^X d\widehat{Y}_s\Big)\Big|\mathcal{F}_T^X\Big] = \exp\Bigg[\int_0^t \Big(\theta_s^X \mu_s + \frac{1}{2}(\theta_s^X)^2\sigma_s^2\Big)ds + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Big(e^{\theta_s^X(e^z-1)} - 1 - \theta_s^X(e^z-1)\Big)\rho^X(dz)ds\Bigg]. \quad (4.6.5)$$

□

So we deduce an explicit expression for

$$\Lambda_t^{\theta^X} = \frac{\exp\Big(\int_0^t \theta_s^X d\widetilde{R}_s\Big)}{\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^t \theta_s^X d\widetilde{R}_s\Big)\Big|\mathcal{F}_T^X\Big]},\ t\in[0,T] \quad (4.6.6)$$

as

$$\Lambda_t^{\theta^X} = \exp\Big[\int_0^t \theta_s^X\sigma_s dW_s - \frac{1}{2}\int_0^t (\theta_s^X)^2\sigma_s^2 ds + \int_0^t \theta_s^X(e^z-1)\widetilde{N}^X(ds,dz) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Big(e^{\theta_s^X(e^z-1)} - 1 - \theta_s^X(e^z-1)\Big)\rho^X(dz)ds\Big]. \quad (4.6.7)$$

4.6.2. Appendix B

Indeed, it follows from (4.2.11) and (4.3.21)

$$\int_0^t \theta_s^X d\widetilde{R}_s = \int_0^t \theta_s^X\sigma_s dW_s^\theta + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \theta_s^X(e^z-1)\widetilde{N}^X(ds,dz),$$

and

a) - $\Big\{\int_0^T \theta_s^X\sigma_s dW_s^\theta : 0\le t\le T\Big\}$ is a continuous local martingale with his predictable quadratic variation which verifies

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^\theta}\Big[\Big\langle\int_0^T \theta_s^X\sigma_s dW_s^\theta\Big\rangle_T\Big] &= \mathbb{E}^{\mathbb{Q}^\theta}\Big[\Big(\int_0^T \theta_s^X\sigma_s dW_s^{\theta^X}\Big)^2\Big] \\
&= \mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[\mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[\Big(\int_0^T \theta_s^X\sigma_s dW_s^\theta\Big)^2\Big|\mathcal{F}_T^X\Big]\Big] \\
&= \mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[\int_0^T (\theta_s^X)^2\sigma_s^2 ds\Big] \\
&< \infty. \quad (4.6.8)
\end{aligned}$$

so by corollary II-3 of Protter (2003) $\Big\{\int_0^T \theta_s^X\sigma_s dW_s^{\theta^X} : 0\le t\le T\Big\}$ is a $\overline{\mathbf{G}}$-continuous martingale under $\mathbb{Q}^{\theta^X}$.

b)- Also, condition (4.3.4) implies that $\forall t \in \mathcal{T}$,

$$\mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[\int_0^t\int_{\mathbb{R}\backslash\{0\}}\theta_s^X(e^z-1)\rho^X(dz)ds\Big] \;\leq\; \mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[\int_0^T\int_{\mathbb{R}\backslash\{0\}}e^z e^{\theta_s^X(e^z-1)}\rho^X(dz)ds\Big] < \infty, \tag{4.6.9}$$

hence by corollary II-3 of Protter (2003),
$\Big\{\int_0^t\int_{\mathbb{R}\backslash\{0\}}\theta_s^X(e^z-1)\widetilde{N}^X(ds,dz)) : 0 \leq t \leq T\Big\}$ is a $\overline{\mathbf{G}}$-martingale under $\mathbb{Q}^{\theta^X}$.

4.6.3. Appendix C

We give a proof for the proposition (4.3.5).

Proof. For the first part, we just have to prove that

$$\Lambda_t^C = \exp\Big\{-\int_0^t[\mathbb{D}_0X_s-\mathbf{1}]'\mathbb{A}_0X_sds\Big\}\prod_{0<s\leq t}\Big(1+[\mathbb{D}_0X_s-\mathbf{1}]'\Delta N_s\Big). \tag{4.6.10}$$

It follows from a simple application of the Doléans-Dade exponential formula (See, Dufour and Elliott (1999)). For the second part, we have to show that

$$\Lambda_t^C = \prod_{1\leq i\neq j\leq M}\exp\Bigg\{-\int_0^t(\widehat{\lambda}_s^{ij}-\lambda_s^{ij})ds+\int_0^t\log\Big(\frac{\widehat{\lambda}_s^{ij}}{\lambda_s^{ij}}\Big)d\mathcal{N}_s^{ij}\Bigg\}. \tag{4.6.11}$$

First, we remark that for $k=1,2,...,N$

$$\mathbb{D}_0X_{t^-} = \Big(\frac{c_{1k}}{a_{1k}}\langle X_{t^-},\mathbf{e}_k\rangle,...,\underbrace{0}_{k-th},...,\frac{c_{Nk}}{a_{Nk}}\langle X_{t^-},s_k\rangle\Big)',$$

and

$$\mathbb{A}_0X_{t^-} = \Big(a_{1k}\langle X_{t^-},\mathbf{e}_k\rangle,...\underbrace{0}_{k-th},...,a_{Nk}\langle X_{t^-},\mathbf{e}_k\rangle\Big)',$$

hence,

$$[\mathbb{D}_0X_{s^-}-\mathbf{1}]'\mathbb{A}_0X_{s^-} = \sum_{i=1}^M\sum_{j=1,i\neq j}^M\langle X_s,\mathbf{e}_i\rangle\Big(c_{ij}\langle X_{s^-},\mathbf{e}_j\rangle - a_{ij}\langle X_{s^-},\mathbf{e}_j\rangle\Big), \tag{4.6.12}$$

so,

$$[\mathbb{D}_0X_{s^-}-\mathbf{1}]'\mathbb{A}_0X_{s^-} = \sum_{i=1}^M\sum_{j=1,i\neq j}^M\Big(\widehat{\lambda}_s^{ij}-\lambda_s^{ij}\Big). \tag{4.6.13}$$

Also, by noting that $\Delta N_s := N_s - N_{s^-}$ we have that $\Delta N_s = (0, ..., \underbrace{1}_{j-th}, 0, ..., 0)'$ when

$$\langle X_{s^-}, \mathbf{e}_i\rangle\langle X_s, \mathbf{e}_j\rangle = 1, \qquad \text{for } i \neq j;\ i, j = 1, ..., N \text{ and } 0 < s \leq t \leq T.$$

Thus, we obtain

$$\begin{aligned} \prod_{0<s\leq t}\Big(1 + [\mathbb{D}_0 X_{s^-} - \mathbf{1}]'\Delta N_s\Big) &= \sum_{i=1}^{M}\sum_{j=1, i\neq j}^{M}\int_0^t \Big(\frac{c_{ij}\langle X_s, \mathbf{e}_i\rangle\langle X_{s^-}, \mathbf{e}_j\rangle}{a_{ij}\langle X_s, \mathbf{e}_i\rangle\langle X_{s^-}, \mathbf{e}_j\rangle}\Big) d\mathcal{N}_s^{ij} \\ &= \sum_{i=1}^{M}\sum_{j=1, i\neq j}^{M}\int_0^t \Big(\frac{\widehat{\lambda}_s^{ij}}{\lambda_s^{ij}}\Big) d\mathcal{N}_s^{ij}, \end{aligned} \tag{4.6.14}$$

and hence,

$$\prod_{0<s\leq t}\Big(1 + [\mathbb{D}_0 X_{s^-} - \mathbf{1}]'\Delta N_s\Big) = \exp\Big\{\int_0^t \log\Big(\frac{\widehat{\lambda}_s^{ij}}{\lambda_s^{ij}}\Big) d\mathcal{N}_s^{ij}\Big\}. \tag{4.6.15}$$

□

4.6.4. Appendix D

By definition,

$$\begin{aligned} F(\alpha) := \mathbb{E}^{\mathbb{P}}\Bigg[\Bigg\{\sum_{1\leq i\neq j\leq M}\Big(-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}} - 1 - \alpha_s^{ij})ds\Big) + \int_0^T \theta_s^X r_s ds - \widehat{K}_T(\theta^X)\Bigg\} \\ \times \exp\Bigg\{\sum_{1\leq i\neq j\leq M}\Bigg(-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}} - 1)ds + \int_0^T \alpha_s^{ij} d\mathcal{N}_s^{ij}\Bigg)\Bigg\}\Bigg], \end{aligned} \tag{4.6.16}$$

so, by taking into account the expressions of $\Lambda_T^{\theta^X}$, Λ_T^{α} and $\widehat{K}(\theta^X)$

$$\begin{aligned} F(\alpha) &= \mathbb{E}^{\mathbb{P}}\Bigg[\mathbb{E}^{\mathbb{P}}\Bigg[\Lambda_T^{\alpha}\Lambda_T^{\theta^X}\log\Big(\Lambda_T^{\alpha}\Lambda_T^{\theta^X}\Big)\Bigg|\mathcal{F}_T^X\Bigg]\Bigg] \\ &= \mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Bigg[\log\Big(\Lambda_T^{\alpha}\Lambda_T^{\theta^X}\Big)\Bigg] \\ &= \mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Bigg[\log\Lambda_T^{\alpha}\Bigg] + \mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Bigg[\log\Lambda_T^{\theta^X}\Bigg]. \end{aligned} \tag{4.6.17}$$

Using the expressions from (4.3.2) and (4.3.58) we have

$$\begin{aligned} \log\Lambda_T^{\alpha} &= \sum_{1\leq i\neq j\leq M}\Bigg(-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}} - 1)ds + \int_0^T \alpha_s^{ij} d\mathcal{N}_s^{ij}\Bigg) \\ &= \sum_{1\leq i\neq j\leq M}\Bigg(-a_{ij}(e^{\alpha_{ij}} - 1)\tau_j + \alpha_{ij}\mathcal{N}_T^{ij}\Bigg), \end{aligned} \tag{4.6.18}$$

$$\log \Lambda_T^{\theta^X} = \int_0^T \theta_s^X \sigma_s dW_s^{\theta^X} + \frac{1}{2}\int_0^T (\theta_s^X \sigma_s)^2 ds + \int_0^T \int_{\mathbb{R}\setminus\{0\}} \theta_{s^-}^X (e^z - 1)\underline{\widetilde{N}}^X(ds, dz)$$
$$+ \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(e^{-\theta_s^X (e^z-1)} - 1 + \theta_s^X (e^z - 1)\right)\underline{\rho}^X(dz)ds. \quad (4.6.19)$$

Thus, since $\{W_t^{\theta^X} : t \in \mathcal{T}\}$ and $\{\underline{\widetilde{N}}^X(t, \cdot) : t \in \mathcal{T}\}$ are local-martingales under $\mathbb{Q}^{(\alpha,\, \theta^X)}$ we obtain the expression of $F(\alpha)$ given by lemma 4.3.2.

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CONLUSION

A generalization of the risk process and the exponential financial process in terms of Markov additive processes (MAPs) is considered in this work. Our first goal is to use MAPs in a finance and insurance context with regime changes that the market could undergo. From this new perspective, we extend the theory related to paths of MAPs to study more general models than those used previously in finance and insurance. *Lévy Systems and the Time Value of Ruin for Markov Additive Processes* [**7**] and *The Minimal Entropy Martingale Measure (MEMM) for a Markov-Modulated Exponential Lévy Model* [**50**] illustrate the potential of Markov additive modeling as a tool yet to be fully explored in financial and insurance mathematics. A topic of future research is to explore further the consequences of introducing MAPs, for instance modeling reserves and asset prices in the presence of different macroeconomic scenarios.

In *On a Generalization of the Expected Discounted Penalty Function to Include Deficits at and Beyond Ruin* [**6**], we have studied the behavior of a family of Lévy risk processes not only in the neighborhood of the ruin time, but also after. This study allows an extension of the concept of EDPF that gives important applications in risk management and particulary, in capital injections. It also remains to generalize this study for a large family of Markov additive risk processes.

These three chapters can be viewed as individual contributions in their own. In particular, they studied MAPs to model the risk process and the asset price in a general context taking into account the regime change. We hope to have established grounds for future research that will lead to new insights in finance and insurance modeling.

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