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Les systèmes super intégrables d'ordre trois séparables en coordonnées paraboliques

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Ce mémoire intitulé:

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présenté par:

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Véronique Hussin,	président-rapporteur
Pavel Winternitz,	directeur de recherche
Alfred Michel Grundland,	membre du jury

Mémoire accepté le: ....

RÉSUMÉ

Ce mémoire est une poursuite de l'étude de la superintégrabilité classique et quantique dans un espace euclidien de dimension deux avec une intégrale du mouvement d'ordre trois. Il est constitué d'un article. Puisque les classifications de tous les Hamiltoniens séparables en coordonnées cartésiennes et polaires sont déjà complétées, nous apportons à ce tableau l'étude de ces systèmes séparables en coordonnées paraboliques.

Premièrement, nous dérivons les équations déterminantes d'un système en coordonnées paraboliques et ensuite nous résolvons les équations obtenues afin de trouver les intégrales d'ordre trois pour un potentiel qui permet la séparation en coordonnées paraboliques.

Finalement, nous démontrons que toutes les intégrales d'ordre trois pour les potentiels séparables en coordonnées paraboliques dans l'espace euclidien de dimension deux sont réductibles. Dans la conclusion de l'article nous analysons les différences entre les potentiels séparables en coordonnées cartésiennes et polaires d'un côté et en coordonnées paraboliques d'une autre côté.

Mots clés: intégrabilité, superintégrabilité, mécanique classique, mécanique quantique, Hamiltonien, séparation de variable, commutation.

ABSTRACT

This thesis is a contribution to the study of classical and quantum superintegrability in a two-dimensional Euclidean space involving a third order integral of motion. It consists of an article. Because the classifications of all separable hamiltonians into Cartesian and polar coordinates are already complete, we bring to this picture the study of those systems in parabolic coordinates. First, we derive the determinating equations of a system into parabolic coordinates, after which we solve the obtained equations in order to find integrals of order three for potentials, which allow the separations of variables into the parabolic coordinates. Finally, we prove that all the third order integrals for separable potentials in parabolic coordinates in the Euclidean space of dimension two are reducible. In the conclusion of this article, we analyze the differences between the separable potentials in Cartesian and polar coordinates and the separable potentials in parabolic coordinates.

Keywords: integrability, superintegrability, classical mechanics, quantum mechanics, Hamiltonian, separation of variables, commutation.

TABLE DES MATIÈRES

(dédicace) à ma fille Anca : je t'aime infiniment

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CHAPITRE 1

INTRODUCTION

Définition : On dit d'un système hamiltonien classique en n dimensions qu'il est intégrable s'il possède n constantes du mouvement (intégrales) (y compris le hamiltonien) indépendantes et en involution. Autrement dit, le système est intégrable si l'on peut trouver des fonctions qui satisfont :

1) $\{X_i = X_i(\overrightarrow{x}, \overrightarrow{p})\}_{i=1,..,n}$ bien définies dans l'espace de phases M,

2) Poisson-commutent entre elles et avec le hamiltonien

$\frac{dX_i}{dt} = \{X, X_i\} = 0, \quad \{X_i, X_j\} = 0, \forall i,\, j,$

3) Fonctionnellement indépendantes, avec

$rang\frac{\partial(X_1,\ldots,X_n)}{\partial(x_1,\ldots,x_n,p_1,..,p_n)} = n.$

Le théorème de Liouville affirme, entre autres, qu'un tel système peut être résolu par quadratures, c'est-à-dire en effectuant uniquement des opérations algébriques et des intégrales.

Théorème (Liouville) : Soit$X_1, , X_n$, n fonctions bien définies qui respectent la définition d'intégrabilité dans un espace de phase M à 2n dimensions. Alors :

1)$M = \{(x, p) \in M : X_i = c_i\}$ est une variété invariante sous le flot induit par le hamiltonien $H = X_1, \forall i$;

2) Les équations $\frac{\partial H}{\partial p_i} = \frac{dx_i}{dt}, \frac{\partial H}{\partial x_i} = \frac{dp_i}{dt}$ peuvent être résolues par quadratures ;

3) Si M est compacte et connexe, alors celle-ci est difféomorphe à un tore de dimension n.

Les systèmes intégrables possèdent des comportements très réguliers. De plus, les trajectoires de ces systèmes ne s'étalent pas complétement dans tout l'espace de configuration. L'étude de ces systèmes s'avère d'une pertinence considérable à la caractérisation de la dynamique des systèmes chaotiques. L'intégrabilité propose de fortes restrictions à la physique de tels systèmes. L'intégrabilité est considérée comme la contrainte du chaos. Autrement dit, les systèmes non-intégrables sont souvent chaotiques, tandis que les systèmes intégrables ne le sont jamais. L'étude des systèmes intégrables est depuis longtemps une branche très importante de la physique - mathématique. Plusieurs recherches et classifications des systèmes intégrables ont été faites et complétées à ce jour. De cette façon, une classification de tous les systèmes intégrables classiques et quantiques possédant une intégrale d'ordre deux dans les impulsions fut complétée dans un espace euclidien de dimension deux et trois. L'existence d'une intégrale d'ordre deux indépendante de hamiltonien dans un espace euclidien de dimensions 2 entraîne la séparation des variables dans l'équation d'Hamilton - Jacobi en mécanique classique et respectivement l'équation de Schrödinger en mécanique quantique dans un des systèmes des coordonnées suivants : cartésiens, polaires, paraboliques et elliptiques. Cependant, il existe des systèmes hamiltoniens qui admettent plus de quantités conservées que de degrés de libertés. Ceux-ci sont soumis à des contraintes encore plus strictes et ils ont des comportements plus réguliers que les systèmes intégrables. Ainsi, furent établies les bases de la théorie des systèmes superintégrables.

Définition : On dit qu'un système hamiltonien intégrable est superintégrable s'il possède, en plus des intégrales X_i, i = 1,..,n, un ensemble des intégrales supplémentaires $\{Y_j\}_{i=1,..,k}$. Les intégrales X_i, Y_j doivent être fonctionnellement indépendantes, mais on n'exige pas que les intégrales Y_j soient en involution entre elles ou avec les X_i. Les intégrales X_i, Y_j satisfont :

1) les fonctions X_i et Y_j sont bien définies dans l'espace des phases ;

2) les intégrales Y_j Poisson-commutent avec le Hamiltonien : $\frac{dY_j}{dt} = \{H, Y_j\} = 0$, mais pas nécessairement entre elles ou avec les intégrales X_i ;

3) l'ensemble des intégrales est fonctionnellement indépendant :

$rang\frac{\partial(X_1,...,X_{n-1},Y_1,..,Y_k)}{\partial(x_1,...,x_n,p_1,..,p_n)} = n+k.$

Un tel système devient maximalement superintégrable si k=n-1.

En mécanique quantique les définitions d'intégrabilité et de superintégrabilité sont trés semblables aux définitions classiques. Les intégrales X_i deviennent des opérateurs qui commutent avec le hamiltonien et le crochet de Poisson est remplacé par le commutateur. Toutefois, il n'existe pas d'équivalent quantique du fameux théorème de Liouville. Les notions d'indépendance entre les intégrales du mouvement quantique restent contestées et contestables. Les systèmes intégrables et superintégrables ont été beaucoup étudiés dans les dernières années. Des recherches complètes ont été faites pour les intégrales d'ordre 1 et 2. Les systèmes superintégrables les plus connus sont le système Kepler - Coulomb avec le potentiel $V(r) = \frac{\alpha}{r}$ et l'oscillateur harmonique avec $V(r) = \alpha r$. Dans les deux cas les intégrales X_i correspondent au moment angulaire. Pour $V(r) = \frac{\alpha}{r}$ les intégrales Y_j correspondent au vecteur Laplace-Runge-Lenz et pour l'oscillateur harmonique ils correspondent au tenseur de quadripôle. Le théorème de Bertrand démontre [2] que ces potentiels sont les seuls potentiels à symétrie sphérique pour lesquelles toutes les trajectoires finies sont fermées. Dans les années soixante, les systèmes superintégrables avec deux intégrales du mouvement quadratiques dans les impulsions ont été étudiés dans un espace euclidien de deux ou trois dimensions et complétement classifiés. Dans l'article de 1967, P. Winternitz et ses collaborateurs ont trouvé 4 types de potentiels qui possédaient deux intégrales quadratiques dans E2 [6, 16, 22]. Ces articles constituent le point de départ d'une recherche des systèmes superintégrables. On va donner par la suite quelques raisons pour lesquelles on dit que les systèmes superintégrables possèdent des propriétés très intéressantes de point de vue mathématique et physique.

En mécanique classique :

1) toutes les trajectoires sont contraintes à une variété de dimension n-k dans l'espace de phase. Pour un système maximalement superintégrable (2n-1 intégrales), toutes les trajectoires bornées sont fermées et le mouvement est périodique [19] ;

2) les systèmes quadratiquement superintégrables sont multiséparables, c'est-à-dire que l'équation d'Hamilton- Jacobi permet la séparation de variable dans plus d'un système de coordonnées ;

3) les intégrales du mouvement ont une structure non-abélienne sous le crochet de Poisson. Cette structure peut être une algèbre de Lie de dimension finie, une algèbre de Kac-Moody ou une algèbre polynomiale [3, 7].

En mécanique quantique :

1) les niveaux d'énergie sont dégénérés ;

2) les systèmes quadratiquement superintégrables sont multiséparables, c'est-à-dire que l'équation de Schrödinger permet la séparation de variable dans plus d'un système de coordonnées ;

3) tous les exemples de systèmes maximalement superintégrables dans un espace euclidien sont exactement résolubles [18]. Il existe une conjecture qui dit que tous les systèmes superintégrables sont toujours exactement résolubles.

4) les intégrales du mouvement forment une algèbre non-abélienne sous le commutateur de Lie. Elles peuvent être des algèbres de Lie de dimension finie ou infinie ou une algèbre polynomiale [1, 5, 14] ;

5) dans les systèmes quadratiquement superintégrables, les mêmes potentiels sont superintégrables en mécanique classique et quantique. Toutefois, cette propriété n'est pas valable pour les systèmes avec intégrales d'ordre plus élevé [12].

Quant aux intégrales d'ordre plus élevé, l'étude de leur existence a été exposée une première fois récemment. Dans ces travaux, on a prouvé que pour une intégrale polynomiale d'ordre n en les impulsions, en mécanique classique et quantique les termes d'ordre pair et impair doivent commuter indépendamment avec le hamiltonien et les termes dominants doivent être dans l'algèbre enveloppante de l'algèbre d'isométrie de l'espace. Plus récemment, des raisons mathématiques et physiques ont mené à la recherche des potentiels superintégrables avec les intégrales d'ordre plus élevé que deux.

On peut donc se demander s'il existe d'autres hamiltoniens ayant de telles propriétés et qui pourraient modéliser certains phénomènes de la physique. L'étude des systèmes avec une intégrale d'ordre trois est plus récente. La recherche a débuté avec Drach en 1935 [4, 10]. Il a étudié le hamiltonien classique dans l'espace complexe E_2 et il a trouvé dix types de potentiel pour lesquels il existe une intégrale de mouvement d'ordre trois. Un peu plus récemment, Ranada et Tsiganov ont prouvé que parmi les dix systèmes de Drach, sept sont réductibles[17, 21]. Ils sont, en effet, des systèmes superintégrales d'ordre deux. Ainsi, les intégrales d'ordre trois trouvées sont des Poisson-commutateur entre deux intégrales d'ordre deux.

Dans les années quatre-vingt, Hietarinta a discuté les intégrales d'ordre trois de même que certains potentiels périodiques particulier unidimensionnels [13]. Il a démontré que dans ce cas-ci, l'intégrale d'ordre trois n'est pas indépendante de l'hamiltonien. De plus, il a démontré que l'existence d'une intégrale d'ordre trois ou quatre, dans un espace à deux dimensions, en mécanique quantique, faisait apparaître des différences importantes entre l'intégrabilité quantique et classique [11, 12].

Une étude systématique des intégrales d'ordre trois pour les systèmes superintégrables classiques et quantiques dans l'espace $E_2(\Re)$ a commencé en 2004. Ces systèmes ont été étudiés par S.Gravel et P. Winternitz [8, 9]. Dans les dernières années des études ont été faites pour les systèmes superintégrables avec intégrales d'ordre trois séparables en coordonnées cartésiens et polaires tant classique que quantique. Il existe des ouvrages qui donnent tous les potentiels séparables en coordonnées cartésiennes [9] et polaires[20] pour les systèmes qui admettent des intégrales d'ordre trois.

Ainsi, ce mémoire se veut une continuité des travaux entamés, pour le cas où le hamiltonien est séparable en coordonnées paraboliques. Autrement dit, à partir des conditions d'existence d'une intégrale d'ordre trois, nous les avons résolu et nous avons prouvé que tous les systèmes qui se séparent en coordonnées paraboliques sont réductibles.

CHAPITRE 2

DESCRIPTION DE L'ARTICLE

Le prochain chapitre du mémoire est un article dont les auteurs sont : I. Popper, S. Post and P. Winternitz. On y présente les résultats sur les systèmes superintégrables avec intégrales de mouvement d'ordre trois séparables en coordonnées paraboliques.

Cet article a été soumis au Journal of Mathematical Physics. Voici le titre et la référence sur les archive :

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Third-order superintegrable systems separable in parabolic coordinates

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In this paper, we investigate superintegrable systems which separate in parabolic coordinates and admit a third-order integral of motion. We give the corresponding determining equations and show that all such systems are multi-separable and so admit two second-order integrals. The third-order integral is their Lie or Poisson commutator. We discuss how this situation is different from the Cartesian and polar cases where new potentials were discovered which are not multi-separable and which are expressed in terms of Painlevé transcendents or elliptic functions.

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Keywords: Integrability, Superintegrability, Separation of Variables, Classical and Quantum Mechanics

I. INTRODUCTION

This article is part of a research program the aim of which is to identify all third-order superintegrable systems in two-dimensional Euclidean space. We recall that a superintegrable system is one that has more integrals of motion than degrees of freedom. We consider a classical or quantum Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(x_1, x_2) \tag{1}$$

with two integrals of motion

$$X_a = \sum_{0 \le j+k \le n} f_{a,jk}(x_1, x_2) p_1^j p_2, \qquad a = 1, 2, \tag{2}$$

where p_1, p_2 are components of the momentum $\vec{p}$. In classical mechanics, the integrals $X_{1,2}$ Poisson commute with H, are well defined functions on phase space and the three functions $\{H, X_1, X_2\}$ are functionally independent. The functions X_1 and X_2 do not Poisson commute with each other; instead they generate a non-Abelian algebra, usually a polynomial one. In quantum mechanics, H and X_a are Hermitian operators in the enveloping algebra of the Heisenberg algebra (or some generalization of the enveloping algebra). The operators X_a Lie commute with H, but not with each other. Instead of functional independence, we assume that H, X_1 and X_2 are algebraically independent. More specifically, we assume that no Jordan polynomial in the operators H, X_1 and X_2 vanishes. As indicated in (2), we assume that X_a are polynomials in the momentum. The "order of superintegrability" is the highest order of these polynomials.

The best known superintegrable systems are the Kepler-Coulomb system[1,7] with potential $V = \alpha/r$ and the harmonic oscillator[15,28] with $V = \alpha r^2$. As a matter of fact, these are the only rotationally invariant superintegrable systems in n-dimensions ($n \ge 2$) (in agreement with Bertrand's theorem[2,9]). Both of these systems are quadratically superintegrable in that the Laplace-Runge-Lenz vector for $V = \alpha/r$ and the Fradkin (or quadropole) tensor for $V = \alpha r^2$ are both second-order in the momenta.

Until recently, most studies of superintegrability concentrated on the second-order case[6,8,24,33,38]. At least in two-dimensional spaces, second-order superintegrable systems are well understood. All such systems in Euclidean spaces, spaces of constant curvature, and spaces of non-constant curvature with at least two second-order Killing tensors (Darboux spaces) have been classified[16–18,22].

Recently, infinite families of superintegrable systems with integrals of motion of arbitrary order have been discovered and investigated[4,19–21,23,26,29,30,32,34,35].

A systematic search for third-order superintegrable systems was started in 2002, both in classical and quantum mechanics[11]. However, the first (to our knowledge), article on third-order integrals of motion was written considerably earlier by Drach[5]. He considered the case of one third-order integral of motion (in addition to the Hamiltonian) in flat two-dimensional complex space in classical mechanics. He found 10 different complex potentials which allow a third-order integral. Later it was shown that 7 of them are actually quadratically superintegrable and the third-order integral is reducible, i.e. is the Poisson commutator of two second-order integrals[31,37].

The articles in Ref.'s 10, 11 and 36 were devoted to superintegrable systems with one third-order and one first or second-order integral. A first-order integral in $E_2(\mathbb{R})$ exists only if the potential is translationally or rotationally invariant, i.e. $V = V(x)$ or $V = V(r)$. In the classical case, all such potentials are second-order superintegrable and hence known. In quantum mechanics, one class of new superintegrable potentials is obtained[10] and is expressed in terms of elliptical functions, e.g.

$$V = \hbar^2\omega^2 sn^2(\omega x, k), \tag{3}$$

where ω and k are constants and $sn(\omega x, k)$ is a Jacobian elliptic function[3]. The existence of a second-order integral implies that $V(x_1, x_2)$ allows separation of variables in Cartesian, polar, parabolic or elliptic coordinates. Cartesian and polar coordinates were considered earlier[10,36]. The study provided a number of new superintegrable potentials in the classical and, much more interestingly, in the quantum case. Indeed, quantum integrable systems with higher-order integrals of motion can be quite different from classical ones; a fact first noticed by Hietarinta[12,13]. The case of quantum superintegrable systems is much richer than that of classical ones. Third-order superintegrability with separation of variables in the Schrödinger equation in Cartesian or polar coordinates lead to potentials expressed in terms of Painlevé transcendents (P_I, P_{II} and P_{IV} for Cartesian coordinates, P_{VI} for polar ones). The potentials separable in Cartesian coordinates have been intensively studied in both classical and quantum mechanics[25–27].

The purpose of this article is to find all superintegrable systems that allow (at least) one third-order integral and a second-order integral that leads to separation in parabolic coordinates.

II. THE DETERMINING EQUATIONS

Let us now assume that the Hamiltonian allows separation of variables in parabolic coordinates

$$x_1 = \frac{1}{2}(\xi^2 - \eta^2), \qquad x_2 = \xi\eta.$$

The quantum mechanical Hamiltonian has the form

$$H = -\frac{\hbar^2}{\xi^2+\eta^2}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) + V(\xi,\eta), \tag{4}$$

$$V(\xi,\eta) = \frac{W_1(\xi)+W_2(\eta)}{\xi^2+\eta^2} \tag{5}$$

and there exists a second-order integral of the form

$$Y = L_3 p_2 + p_2 L_3 + \frac{1}{\xi^2+\eta^2}\left(\xi^2 W_2(\eta) - \eta^2 W_1(\xi)\right) \tag{6}$$

with

$$p_1 = -i\hbar\frac{\partial}{\partial x_1} = \frac{-i\hbar}{\xi^2+\eta^2}\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right),$$
$$p_2 = -i\hbar\frac{\partial}{\partial x_2} = \frac{-i\hbar}{\xi^2+\eta^2}\left(\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}\right),$$
$$L_3 = -i\hbar(x_2 p_1 - x_1 p_2) = \frac{i\hbar}{2}\left(\xi\frac{\partial}{\partial\eta} - \eta\frac{\partial}{\partial\xi}\right).$$

A third-order integral will then have the form

$$X = \sum_{j+k+\ell=3} A_{jk\ell}\left\{L_3^j, p_1^k p_2^\ell\right\} + \{g_1(x_1,x_2), p_1\} + \{g_2(x_1,x_2), p_2\}. \tag{7}$$

The brackets $\{\,,\,\}$ denote anti-commutators, $A_{jk\ell}$ are real constants and the functions V, g_1 g_2 obey the four partial differential equations presented in Ref. 11 (in Cartesian coordinates). Here we need the equations in parabolic coordinates. To rewrite them in parabolic coordinates, it is convenient to replace the unknown functions $g_1(x_1,x_2)$ and $g_2(x_1,x_2)$ by $G_1(\xi,\eta)$, $G_2(\xi,\eta)$, putting

$$g_1(x_1,x_2) = \frac{\xi G_1(\xi,\eta) - \eta G_2(\xi,\eta)}{\xi^2+\eta^2}, \qquad g_2(x_1,x_2) = \frac{\eta G_1(\xi,\eta) + \xi G_2(\xi,\eta)}{\xi^2+\eta^2}.$$

The four determining equations for the integral (7), i.e. the commutativity condition $[H, X] = 0$, can be written as

$$G_{1,\xi} - \frac{\xi G_1 - \eta G_2}{\xi^2 + \eta^2} = h_1(\xi, \eta), \tag{8}$$

$$G_{1,\eta} + G_{2,\xi} - 2\frac{\eta G_1 + \xi G_2}{\xi^2 + \eta^2} = h_2(\xi, \eta), \tag{9}$$

$$G_{2,\eta} + \frac{\xi G_1 - \eta G_2}{\xi^2 + \eta^2} = h_3(\xi, \eta), \tag{10}$$

$$G_1 V_\xi + G_2 V_\eta = \frac{\hbar^2}{4}\phi, \tag{11}$$

with

$$\begin{aligned} h_1(\xi,\eta) &= \tfrac{3F_1 V_\xi + F_2 V_\eta}{\xi^2+\eta^2} \\ h_2(\xi,\eta) &= \tfrac{2F_2 V_\xi + 2F_3 V_\eta}{\xi^2+\eta^2} \\ h_3(\xi,\eta) &= \tfrac{F_3 V_\xi + 3F_4 V_\eta}{\xi^2+\eta^2}, \end{aligned}$$

and

$$\begin{aligned} \phi = {} & \frac{F_1 V_{\xi\xi\xi} + F_2 V_{\xi\xi\eta} + F_3 V_{\xi\eta\eta} + F_4 V_{\eta\eta\eta}}{(\xi^2+\eta^2)^2} \\ & + \frac{(3\xi F_1 + 2\eta F_2 - \xi F_3) V_{\xi\xi} + (\eta F_2 - \xi F_2 - \eta F_3 + \xi F_4) V_{\xi,\eta} + (\eta F_2 - 2\xi F_3 - 3\eta F_4) V_{\eta\eta}}{(\xi^2+\eta^2)^3} \\ & - \left(\frac{3(\xi^2-\eta^2)(F_1-F_3) + 6\xi\eta(F_2-F_4)}{(\xi^2+\eta^2)^4} - 4A_{300}\eta(\xi^2+\eta^2) + 2A_{201}(\xi+\eta) \right) V_\xi \\ & - \left(\frac{6\xi\eta(F_1-F_3) - 3(\xi^2-\eta^2)(F_2-F_4)}{(\xi^2+\eta^2)^4} - 4A_{300}\xi(\xi^2+\eta^2) - 2A_{201}(\xi+\eta) \right) V_\eta. \end{aligned} \tag{12}$$

The subscripts denote partial derivatives and the expressions F_1, F_2, F_3, F_4 are polynomials in ξ and η:

$$\begin{aligned} F_1 = {} & \eta^3 A_{003} + \xi\,\eta^2 A_{012} + \xi^3 A_{030} - \frac{1}{2}\eta^3\left(\xi^2+\eta^2\right) A_{102} - \frac{1}{2}\xi\,\eta^2\left(\xi^2+\eta^2\right) A_{111} - \frac{1}{2}\eta\,\xi^2\left(\xi^2+\eta^2\right) A_{120} \\ & + \frac{1}{4}\eta^3\left(\xi^2+\eta^2\right)^2 A_{201} + \frac{1}{4}\xi\,\eta^2\left(\xi^2+\eta^2\right)^2 A_{210} - \frac{1}{8}\eta^3\left(\xi^2+\eta^2\right)^3 A_{300} + \xi^2\eta\, A_{021} \end{aligned}$$

$$\begin{aligned} F_2 = {} & 3\,\xi\,\eta^2 A_{003} - \eta\,\left(\eta^2 - 2\,\xi^2\right) A_{012} - 3\,\xi^2\eta\, A_{030} - \frac{1}{2}\xi\,\eta^2\left(\xi^2+\eta^2\right) A_{102} + \frac{1}{2}\eta^3\left(\xi^2+\eta^2\right) A_{111} \\ & + \frac{1}{2}\xi\,\left(\xi^2 + 2\,\eta^2\right)\left(\xi^2+\eta^2\right) A_{120} - \frac{1}{4}\xi\,\eta^2\left(\xi^2+\eta^2\right)^2 A_{201} - \frac{1}{4}\eta\,\left(\eta^2 + 2\,\xi^2\right)\left(\xi^2+\eta^2\right)^2 A_{210} \\ & + \frac{3}{8}\xi\,\eta^2\left(\xi^2+\eta^2\right)^3 A_{300} - A_{021}\xi\,\left(2\,\eta^2 - \xi^2\right) \end{aligned}$$

$$
\begin{aligned}
F_3 = {} & 3\,\xi^2\eta\,A_{003} - \xi\,\left(2\,\eta^2 - \xi^2\right)A_{012} + 3\,\xi\,\eta^2 A_{030} + \frac{1}{2}\,\eta\,\xi^2\left(\xi^2+\eta^2\right)A_{102} + \frac{1}{2}\,\xi^3\left(\xi^2+\eta^2\right)A_{111} \\
& -\frac{1}{2}\,\eta\,\left(\eta^2+2\,\xi^2\right)\left(\xi^2+\eta^2\right)A_{120} - \frac{1}{4}\,\eta\,\xi^2\left(\xi^2+\eta^2\right)^2 A_{201} + \frac{1}{4}\,\xi\,\left(\xi^2+2\,\eta^2\right)\left(\xi^2+\eta^2\right)^2 A_{210} \\
& -\frac{3}{8}\,\eta\,\xi^2\left(\xi^2+\eta^2\right)^3 A_{300} + \eta\,A_{021}\left(\eta^2-2\,\xi^2\right)
\end{aligned}
$$

$$
\begin{aligned}
F_4 = {} & \xi^3 A_{003} - \xi^2\eta\,A_{012} - \eta^3 A_{030} + \frac{1}{2}\,\xi^3\left(\xi^2+\eta^2\right)A_{102} - \frac{1}{2}\,\eta\,\xi^2\left(\xi^2+\eta^2\right)A_{111} + \frac{1}{2}\,\xi\,\eta^2\left(\xi^2+\eta^2\right)A_{120} \\
& +\frac{1}{4}\,\xi^3\left(\xi^2+\eta^2\right)^2 A_{201} - \frac{1}{4}\,\eta\,\xi^2\left(\xi^2+\eta^2\right)^2 A_{210} + \frac{1}{8}\,\xi^3\left(\xi^2+\eta^2\right)^3 A_{300} + \xi\,\eta^2 A_{021}.
\end{aligned}
$$

The determining equations (8-10) are the same in classical and quantum mechanics but (11) contains the Planck constant on the right hand side. The corresponding equation in classical mechanics is obtained by taking the limit $\hbar \to 0$ so (11) is greatly simplified. Hence the difference between classical and quantum integrability (and superintegrability) for third-order integrals of motion.

The system (8-11) is overdetermined. The first three equations imply a linear compatibility condition for the potential

$$
\begin{aligned}
0 = {} & F_3 V_{\xi\xi\xi} + (3\,F_4 - 2\,F_2)V_{\xi\xi\eta} + (3\,F_1 - 2\,F_3)V_{\xi\eta\eta} + F_2 V_{\eta\eta\eta} \\
& + \left(2(F_{3\,\xi} - F_{2\,\eta}) - \tfrac{3\xi F_1 - 6\eta F_2 + 7\,\xi\,F_3}{\xi^2+\eta^2}\right)V_{\xi\xi} + \left(2(F_{2\,\eta} - F_{3\,\xi}) - \tfrac{3\,\eta\,F_4 - 6\,\xi\,F_3 + 7\eta F_2}{\xi^2+\eta^2}\right)V_{\eta\eta} \\
& + \left(2\,(3F_{1\,\eta} - F_{2\,\xi} - F_{3\,\eta} + 3F_{4\,\xi}) - \tfrac{21\,\eta\,F_1 - 5\,\eta\,F_3 - 5\,\xi\,F_2 + 21\,\xi\,F_4}{\xi^2+\eta^2}\right)V_{\xi\eta} \\
& + A\,V_\eta + B\,V_\xi,
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
A = {} & F_{2\,\eta\,\eta} - 2F_{3\,\eta\,\xi} + 3F_{4\,\xi\,\xi} + \frac{-7\eta\,F_{2\,\eta} - \xi\,F_{2\,\xi} + 6\xi\,F_{3\,\eta} + 6\eta\,F_{3\,\xi} - 3\eta\,F_{4\,\eta} - 21\xi\,F_{4\,\xi}}{\xi^2+\eta^2} \\
& +2\,\frac{21\,\xi^2 F_4 + F_2\xi^2 + 7\,\eta^2 F_2 - 12\,\xi\,\eta\,F_3 + 3\,F_4\eta^2}{\left(\xi^2+\eta^2\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
B = {} & 3F_{1\,\eta\,\eta} - 2F_{2\,\eta\,\xi} + F_{3\,\xi,\xi} - \frac{21\eta\,F_{1\,\eta} + 3\xi\,F_{1\,\xi} - 6\xi\,F_{2\,\eta} - 6\eta\,F_{2\,\xi} + \eta\,F_{3\,\eta} + 7\xi\,F_{3\,\xi}}{\xi^2+\eta^2} \\
& +2\,\frac{F_3\eta^2 + 3\,F_1\xi^2 + 21\,\eta^2 F_1 - 12\,\xi\,\eta\,F_2 + 7\,\xi^2 F_3}{\left(\xi^2+\eta^2\right)^2}.
\end{aligned}
$$

Compatibility between the first three determining equations (8-10) and the fourth one (11) requires three more conditions, this time nonlinear ones. Indeed, solving (11) for G_2, we have

$$
G_2 = \frac{1}{V_\eta}\left(\frac{\hbar^2}{4}\Phi - V_\xi G_1\right), \qquad V_\eta \neq 0. \tag{14}
$$

Replacing G_2 from (14) into (8-10), the system can then be solved for G_1,

$$G_1 = \frac{h_4}{h_5}, \qquad G_2 = \frac{h_5\hbar^2\Phi - 4V_\xi h_4}{4h_5V_\eta}, \qquad h_5 \neq 0, \tag{15}$$

with

$$\begin{aligned} h_4 &= (\xi^2+\eta^2)\left(4h_3V_\eta^3 + 4h_2V_\xi V_\eta^2 + 4h_1V_\xi^2V_\eta - \hbar^2V_\eta^2\Phi_\eta - \hbar^2V_\xi V_\eta\Phi_\xi - \hbar^2\left(V_{\eta\eta}V_\eta - V_{\xi\eta}V_\xi\right)\Phi\right) \\ &\quad +4\eta V_\xi^2 - 4\eta V_\eta^2 - 8\xi V_\eta V_\xi \\ h_5 &= 4(\xi^2+\eta^2)\left(V_\xi V_\eta(V_{\xi\xi} - V_{\eta\eta} + (V_\eta^2 - V_\xi^2)V_{\xi\eta}\right) + 4(\eta V_\xi - \xi V_\eta)(V_\xi^2 + V_\eta^2). \end{aligned}$$

Replacing (15), into (8-10), gives the three additional non-linear compatibility conditions on the potential, namely

$$\left(\frac{h_4}{h_5}\right)_\xi + \frac{\eta\hbar^2h_5\Phi + 4h_4\left(\xi V_\eta + \eta V_\xi\right)}{4h_5V_\eta(\xi^2+\eta^2)} = h_1 \tag{16}$$

$$\left(\frac{h_4}{h_5}\right)_\eta + \left(\frac{\hbar^2h_5\Phi - 4V_\xi h_4}{4h_5V_\eta}\right)_\xi + \frac{\xi\hbar^2h_5\Phi + 4h_4\left(\eta V_\eta - \xi V_\xi\right)}{2h_5V_\eta(\xi^2+\eta^2)} = h_2 \tag{17}$$

$$\left(\frac{\hbar^2h_5\Phi - 4V_\xi h_4}{4h_5V_\eta}\right)_\eta - \frac{\eta\hbar^2h_5\Phi + 4h_4\left(\xi V_\eta + \eta V_\xi\right)}{4h_5V_\eta(\xi^2+\eta^2)} = h_3. \tag{18}$$

III. GENERAL FORMS FOR W_1 AND W_2

In order to determine all possible potentials which separate in parabolic coordinates and admit a third-order integral of motion, we begin with the linear compatibility condition (13). Replacing V with the form as in (5), the compatibility condition can be differentiated to obtain a system of linear ordinary differential equations (ODEs) for W_1 and W_2. These equations are given in Appendix A for W_1. Notice that interchanging ξ and η has the effect of changing the sign of the coefficients to $(-1)^{j+k}A_{jk\ell}$. Thus, equations for W_2 are of the same form as (A1-A14), up to a change in sign in these constants. We begin our search for the admissible potentials by solving equations (A1-A14).

Theorem 1 *Given a Hamiltonian which admits a third-order integral of motion with a potential which separates in parabolic coordinates as in (5). Then any admissible terms in the potential are included in*

$$W_1(\xi) = \sum_{j=0}^{16} c_j\xi^j + c_{-2}\xi^{-2} + c_{-4}\xi^{-4} + c_{-6}\xi^{-6} + \frac{\alpha_1\xi\ln\left(\xi + \sqrt{\xi^2+B_1}\right) + \alpha_2\xi}{\sqrt{\xi^2+B_1}}, \tag{19}$$

$$W_2(\eta) = \sum_{j=1}^{16} d_j \eta^j + d_{-2}\eta^{-2} + d_{-4}\eta^{-4} + d_{-6}\eta^{-6} + \frac{\beta_1 \ln\left(\eta + \sqrt{\eta^2 + B_2}\right) + \beta_2 \eta}{\sqrt{\eta^2 + B_2}}. \quad (20)$$

In particular, any solution of the system (13) has the form (5) with W_1 and W_2 as in (19) and (20) with appropriately chosen constants.

Proof: Beginning with (A1), in the case that $A_{300} \neq 0$, the solutions are given by

$$W_1 = \sum_{i=0}^{14} c_i \xi^i + c_{-2}\xi^{-2} + c_{-4}\xi^{-4}. \quad (21)$$

Similarly, for (A2), in the case that $A_{210} \neq 0$, the solutions are

$$W_1 = \sum_{i=0}^{15} c_i \xi^i + c_{-2}\xi^{-2}. \quad (22)$$

Equation (A3) in addition requires that $c_{14} = c_{15} = 0$.

Now assume $A_{300} = A_{210} = 0$. This sets (A1-A3) to be satisfied identically and (A4) becomes

$$-\left(\left(\xi^2 A_{201} + 2\, A_{120}\right) \frac{d^{17}}{d\xi^{17}} + 33\xi\, A_{201} \frac{d^{16}}{d\xi^{16}} + 255 A_{201} \frac{d^{15}}{d\xi^{15}}\right) W_1 = 0. \quad (23)$$

The solutions of (23), assuming $A_{201} \neq 0$, are

$$W_1 = \sum_{i=0}^{14} c_i \xi^i + \frac{\alpha_1 \xi \ln\left(\xi + \sqrt{\xi^2 + B_1}\right) + \alpha_2 \xi}{\sqrt{\xi^2 + B_1}}. \quad (24)$$

with $A_{120} = B_1 A_{201}/2$. If $A_{201} = 0$, the solutions are

$$W_1 = \sum_{i=0}^{16} c_i \xi^i. \quad (25)$$

Now assume $A_{300} = A_{210} = A_{201} = A_{120} = 0$, (A1-A4) are then identically satisfied and (A5) becomes

$$\frac{1}{2}\left(\left(A_{102}\xi^2 + 2A_{021}\right) \frac{d^{17}}{d\xi^{17}} + 255 A_{102} \frac{d^{15}}{d\xi^{15}} + 33\xi\, A_{102} \frac{d^{16}}{d\xi^{16}}\right) W_1 = 0. \quad (26)$$

The solutions of (26) for $A_{102} \neq 0$ are as (24) with $A_{021} = 2A_{102}B_1$ and for $A_{102} = 0$ are as (25).

If we now assume $A_{300} = A_{210} = A_{201} = A_{120} = A_{102} = A_{021} = 0$, (A6) becomes

$$A_{003}\left[3\,\xi^2 \frac{d^{17}}{d\xi^{17}} + 111\xi \frac{d^{16}}{d\xi^{16}} + 969 \frac{d^{15}}{d\xi^{15}}\right] W_1 = 0, \quad (27)$$

the solutions of which are given by (21).

Now, assume $A_{300} = A_{210} = A_{201} = A_{120} = A_{102} = A_{021} = A_{003} = 0$, this implies (A1-A6) as well as (A10-A12) are identically satisfied. Equation (A7) becomes

$$\frac{\xi\left(A_{111}\xi^2 + 6\,A_{030} - 4\,A_{012}\right)}{2}\frac{d^{17}}{d\xi^{17}}W_1 + \left(\frac{51}{2}\,A_{111}\xi^2 - 36\,A_{12} + 54\,A_{030}\right)\frac{d^{16}}{d\xi^{16}}W_1$$
$$+\frac{813\xi A_{111}}{2}\frac{d^{15}}{d\xi^{15}}W_1 + 2016A_{111}\frac{d^{14}}{d\xi^{14}}W_1 = 0. \quad (28)$$

The general solutions of (28) are of the form

$$W_1 = \sum_{i=0}^{13} c_i\xi^i + c_{-2}\xi^{-2} + \frac{\alpha_1\xi\ln\left(\xi + \sqrt{\xi^2 + B_1}\right) + \alpha_2\xi}{\sqrt{\xi^2 + B_1}}, \quad (29)$$

with $A_{030} = \frac{1}{6}B_1A_{111} + \frac{2}{3}A_{012}$ for $A_{111} \neq 0$ and of the form (22) for $A_{111} = 0$.

Now assume that $A_{300} = A_{210} = A_{201} = A_{120} = A_{102} = A_{021} = A_{003} = A_{111} = 3A_{030} - 2A_{012} = 0$, this implies that (A14) has the form

$$A_{012}\left(\xi^3\frac{d^{17}}{d\xi^{17}} + 57\xi^2\frac{d^{16}}{d\xi^{16}} + 1023\xi\frac{d^{15}}{d\xi^{15}} + 5760\frac{d^{14}}{d\xi^{14}}\right)W_1 = 0. \quad (30)$$

The solutions of (30) for $A_{012} \neq 0$ are given by

$$W_1 = \sum_{i=0}^{14} c_i\xi^i + c_{-2}\xi^{-2} + c_{-4}\xi^{-4} + c_{-6}\xi^{-6}. \quad (31)$$

This is the final case, since assuming $A_{012} = 0$ in addition to $A_{300} = A_{210} = A_{201} = A_{120} = A_{102} = A_{021} = A_{003} = A_{111} = 3A_{030} - 2A_{012} = 0$ has all the A's equal 0 and so there is no longer a third order term in our constant of motion.

Thus, the most general form of the function W_1 is given by (19). By direct analogy, the most general form of the function W_2 is given by (20).

□

Corollary 1 *The potential for any 3rd-order superintegrable system which separates in parabolic coordinates satisfies a non-trivial system of linear ODEs for both W_1 and W_2.*

Proof As shown above, the compatibility conditions (13) are satisfied identically if and only if all the A_{ijk} are 0. On the other hand, if there is a non-zero A_{ijk} then the functions W_1 and W_2 in the potential will satisfy some linear ODEs.

□

IV. THE ABSENCE OF IRRATIONAL TERMS

In this section, we show that the only possible form of the potential is as a rational function of ξ and η. We shall show this by contradiction. Namely, we consider the case that either $\alpha_1 \neq 0$ or both $\alpha_2 \neq 0$ and $B_1 \neq 0$ in (19).

If we substitute the general form (19) into equations (A1-A14), it is immediate from the previous section that $A_{300} = A_{210} = 0$. We also obtain the following possible restrictions on the constants, as suggested in the previous section:

$$\left\{ \begin{array}{c} A_{003} = A_{021} = \frac{B_1(A_{102}-A_{120})}{2}, \\ A_{012} = A_{030} = \frac{A_{111}B_1}{2}, \\ A_{201} = \frac{2A_{120}}{B_1}. \end{array} \right\} \tag{32}$$

In the case that $B_1 \neq 0$, W_1 becomes

$$W_1 = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + c_5\xi^5 + c_6\xi^6 + \frac{c_{-2}}{\xi^2} + \frac{\alpha_1\xi \ln\left(\xi + \sqrt{\xi^2 + B_1}\right) + \alpha_2\xi}{\sqrt{\xi^2 + B_1}}, \tag{33}$$

with the following cases

$$\begin{array}{l} \{B_1 \neq 0, c_{-2} = c_5 = c_6 = 0\} \\ \{B_1 \neq 0, A_{201} = 0, c_{-2} = c_6 = 0\} \\ \{B_1 \neq 0, A_{111} = A_{201} = 0, c_{-2} = 0\} \\ \{B_1 \neq 0, A_{102} = A_{201} = 0.\} \end{array} \tag{34}$$

In the case that $\alpha_1 \neq 0$ and $B_1 = 0$, W_1 becomes

$$W_1 = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + c_5\xi^5 + c_6\xi^6 + \frac{c_{-2}}{\xi^2} + \frac{c_{-4}}{\xi^2} + \alpha_1 \ln(\xi), \tag{35}$$

with the following cases

$$\begin{array}{l} \{\alpha_1 \neq 0, B_1 = 0, c_{-4} = c_5 = c_6 = 0\} \\ \{\alpha_1 \neq 0, B_1 = 0, A_{201} = 0, c_{-4} = c_6 = 0\} \\ \{\alpha_1 \neq 0, B_1 = 0, A_{111} = A_{201} = 0, c_{-4} = 0\} \\ \{\alpha_1 \neq 0, B_1 = 0, A_{102} = A_{201} = 0\}. \end{array} \tag{36}$$

Note that when $B_1 = 0$, α_2 becomes an additive constant which is absorbed into c_0.

Next, we substitute these cases (34) or (36), along with the forms of W_1 (33) or (35) and W_2 (20), into the compatibility condition (13) to obtain

$$0 = M_2 ln\left(\eta^2 + \sqrt{\eta^2 + B_2}\right) + M_1\sqrt{\eta^2 + B_2} + M_0 \tag{37}$$

where the M_i are rational functions which are too long to be presented here but are available from the authors upon request. Solving $M_2 = 0$ gives 3 possibilities: either $\beta_1 = 0$, $B_2 = -B_1$ or, in the case that B_1 is not zero, B_2 may vanish if additionally $A_{111} = 0$ and $A_{102} = B_1 A_{201}/2$.

Solving $M_1 = 0$ and $M_0 = 0$ gives similar forms for W_2 : If $B_1 \neq 0$, then W_2 has the form

$$W_2 = d_1\eta + c_2\eta^2 - c_4\eta^4 + c_6\eta^6 + \frac{c_{-2}}{\eta^2} + \frac{-\alpha_1\eta \ln\left(\eta + \sqrt{\eta^2 + B_2}\right) + \beta_2\eta}{\sqrt{\eta^2 + B_2}} \tag{38}$$

and if $B_1 = 0$, then it has the form

$$W_2 = c_2\eta^2 - c_4\eta^4 + c_6\eta^6 + \frac{c_{-2}}{\eta^2} - \frac{c_{-4}}{\eta^4} - \alpha_1 \ln(\eta). \tag{39}$$

In both cases, c_3 and c_5 are also required to be 0 as well and as several additional sets of constraints which are required to completely solve (13). We shall return to some of these cases later.

To obtain a contradiction for the potentials admitting logarithmic singularities, we now turn our attention to the non-linear compatibility conditions. Beginning with Eq. (16), we clear the denominator and consider the equation

$$0 = h_5^2 V_\eta(\xi^2 + \eta^2) \left[\left(\frac{h_4}{h_5}\right)_\xi + \frac{\eta\hbar^2 h_5 \Phi + 4h_4\left(\xi V_\eta + \eta V_\xi\right)}{4h_5 V_\eta(\xi^2 + \eta^2)} - h_1 \right]. \tag{40}$$

In the case that both α_1 and B_1 are assumed non-zero, equation (40) will have polynomial dependence on the quantity $ln\left(\xi^2 + \sqrt{\xi^2 + B_1}\right)$. Substituting the forms of the potential obtained above (33) and (38) into this quantity,(40), gives

$$0 = 32\alpha^6 B_1^3 K ln^6\left(\xi^2 + \sqrt{\xi^2 + B_1}\right) + \mathcal{O}\left(ln^5\left(\xi^2 + \sqrt{\xi^2 + B_1}\right)\right), \tag{41}$$

$$\begin{aligned} K &= 9A_{111}\eta\xi^{14} + \frac{9A_{111}}{2}(16\eta^2 + 5B_1)\xi^{12} \\ &\quad + \left(2(48A_{102} - 23A_{201})\eta^2 - B_1(16A_{102} - 11A_{201})\right)\eta^2\xi^{11} + \mathcal{O}(\xi^{10}). \end{aligned} \tag{42}$$

Therefore, since it was assumed that $\alpha_1 \neq 0$ and $B_1 \neq 0$, the condition (40) requires that $A_{111} = A_{102} = A_{201} = 0$ which is a contradiction because in this case all of the $A_{jk\ell}$'s would be identically zero. In the case that $B_1 = 0$, (40) is a polynomial in $\ln\xi$ with leading order term

$$0 = \frac{K}{(\xi^2 + \eta^2)^{10}\xi^4\eta^3} \ln^4\xi + \mathcal{O}(\ln^3\xi) \tag{43}$$

$$K = 192\alpha_1^6 A_{111}\xi^{12}\eta^2 - 640\alpha_1^6 A_{102}\xi^{11}\eta^3 - 32(864 d_{-2}c_4 A_{102} + \alpha^2 A_{201})\alpha^4\xi^{13}\eta^3 + \ldots \quad (44)$$

For simplicity, we give only the most relevant terms instead of the highest order ones in K. From these terms, it can be seen that the condition (40) leads to a contradiction since they would imply A_{201}, A_{102} and A_{111} are all 0 and so every $A_{jk\ell}$ would vanish. Thus, there are no logarithmic singularities in the potential.

Next, we turn in particular to the case that $\alpha_1 = \beta_1 = 0$. In this case, we proceed slightly differently because computationally, it is more difficult for MAPLE to compute coefficients of the expression (40) with respect to $(\xi^2 + \eta^2)^{k/2}$ without first simplifying the entire expression. On the other hand, unlike in the previous section, we can solve (8-10) without much difficultly and replace the integrated forms of G_1 and G_2 into (11) to obtain the needed contradictions.

In this case, we require the complete solutions for (13). Namely, W_1 must satisfy (33) with $c_3 = c_5 = 0$ and W_2 must satisfy (38), both with $\alpha_1 = \beta_1 = 0$. Additionally, the constants satisfy one of the following cases:

$$\{B_2 = -B_1, A_{201} = A_{102} = 0, c_0 = c_1 = d_1 = c_6 = 0\} \quad (45)$$

$$\{B_2 = -B_1, A_{201} = A_{111} = 0, c_0 = c_1 = d_1 = c_{-2} = 0\} \quad (46)$$

$$\{B_2 = -B_1, A_{111} = 0, A_{102} = A_{120}, c_1 = d_1 = c_6 = c_{-2} = 0\} \quad (47)$$

$$\{B_2 = 0, A_{111} = 0, A_{102} = A_{120}, c_0 = c_1 = d_1 = c_6 = c_{-2} = 0\} \quad (48)$$

$$\{B_2 = -B_1, c_0 = c_1 = d_1 = c_6 = c_{-2} = 0\}. \quad (49)$$

There is also a complex solution for the constants

$$\{B_2 = -B_1, A_{201} = 0, A_{111} = iA_{102}, d_1 = ic_1, c_0 = c_6 = c_{-2} = 0\}. \quad (50)$$

To obtain the needed contradictions, we use the obtained sets of solutions for (13) to solve (8-10) for G_1 and G_2 and replace these solutions into (11). For example, in the case identified in (45) the relevant $G's$ are

$$G_1 = \left(\frac{-\left(-3\,B_1\xi^2 + 4\,\xi^2\eta^2 + \eta^2 B_1\right)\alpha_2}{2\sqrt{\xi^2 + B_1}\,(\xi^2 + \eta^2)} + \frac{\xi\,\eta\,\left(-\eta^2 + 2\,B_1 + \xi^2\right)\beta_2}{\sqrt{\eta^2 - B_1}\,(\xi^2 + \eta^2)} \right.$$
$$\left. -1/2\,\left(2\,\eta^4 + \eta^2 B_1 + 4\,\xi^2\eta^2 - 3\,B_1\xi^2\right)\xi\,c_4 - \frac{(\eta^2 - \xi^2 - B_1)\,c_{-2}}{\eta^2\xi} \right) A_{111}$$
$$+\frac{\eta(\xi^2 + \eta^2)}{6}k_1 + \eta k_2 + \xi k_3 \quad (51)$$

$$G_2 = \left(-\frac{\xi\,\eta\,(-\eta^2 + 2\,B_1 + \xi^2)\,\alpha_2}{\sqrt{\xi^2 + B_1}\,(\xi^2 + \eta^2)} - 1/2\,\frac{(-B_1\xi^2 + 4\,\xi^2\eta^2 + 3\,\eta^2 B_1)\,\beta_2}{\sqrt{\eta^2 - B_1}\,(\xi^2 + \eta^2)} \right.$$
$$\left. +1/2\,\eta\,\left(2\,\xi^4 - B_1\xi^2 + 4\,\xi^2\eta^2 + 3\,\eta^2 B_1\right) c_4 + \frac{(\eta^2 - \xi^2 - B_1)\,c_{-2}}{\xi^2\eta} \right) A_{111}$$
$$-\frac{\xi(\xi^2 + \eta^2)}{6} k_1 - \eta k_3 + \xi k_2, \qquad (52)$$

where k_1, k_2 and k_3 are constants of integration. These solutions for G_1 and G_2 when substituted into (11) give

$$0 = T_1\sqrt{\xi^2 + B_1}\sqrt{\eta^2 - B_1} + T_2\sqrt{\xi^2 + B_1} + T_3\sqrt{\eta^2 - B_1} + T_4. \qquad (53)$$

Again, the T_i's are rational functions of ξ and η, and can be obtained from the authors. Solving these systems, we obtain $\beta_2 = 0$ and $\alpha_2 = 0$ or $A_{111=0}$, which gives a contradiction, since in these cases either all of the $A_{jk\ell}$ are zero or the potential reduces to a rational function. By checking each case in this manner, we find that when B_1 is assumed to be non-zero the potentials reduce to rational functions.

Thus, we have shown by contradiction that the only possible potentials which satisfy both the linear (13) and nonlinear (16-18) compatibility conditions are rational functions of ξ and η.

V. FINAL LIST OF SUPERINTEGRABLE POTENTIALS

Since there are no longer any irrational terms in the potential, it is a straightforward computation to find the admissible choices of constants which satisfy the linear compatibility condition (13), to use these choices to solve the linear partial differential equations (8-10) for G_1 and G_2 and to solve the resulting algebraic system determined by the coefficients of (11). In this section, we exhibit the possible potentials which remain, i.e. those potentials which separate in parabolic coordinates and admit a third order integral of motion. Remarkably, the only such potentials are second-order superintegrable. These are exactly the potentials which separate in parabolic coordinates as well as another orthogonal coordinate system in $E_2(\mathbb{R})$. We also obtain a potential in $E(1,1)$ which is presented in Appendix B.

It is interesting to note that, in addition to the systems which admit a single third-order integral, this method allows us to obtain potentials in the quantum case which admit more than one third-order integral. Furthermore, these additional potentials give proof that the quantum correction to (11), namely Φ, is not identically 0 for potentials which are superintegrable in both the classical and quantum cases. The quantity Φ only vanishes when, in addition, the appropriate choices of A_{jkl} are assumed. For example, for potential V_1 below, the quantity Φ vanishes only when all of the A_{jkl}'s are assumed 0 except for A_{012}.

A. Potentials which admit a single third-order integral

1. A deformation of the anisotropic oscillator potential: V_1

The following potential

$$\begin{aligned} V_1 &= \left(\eta^4 - \xi^2\eta^2 + \xi^4\right)\alpha + \beta(\xi^2 - \eta^2) + \frac{\gamma}{\xi^2\eta^2} \\ &= \alpha\left(4\,x^2 + y^2\right) + 2\,\beta\,x + \frac{\gamma}{y^2} \\ &= \alpha r^2\left(3\cos^2\theta + 1\right) + 2\,\beta\,r\cos\theta + \frac{\gamma}{r^2\sin^2\theta} \end{aligned} \tag{54}$$

admits a third-order integral with all all $A_{jk\ell} = 0$ except A_{021} and functions

$$\begin{aligned} G_1 &= 2\,\frac{A_{012}\left(-2\,\alpha\,\xi^2\eta^6 + \eta^4\alpha\,\xi^4 + \eta^4\beta\,\xi^2 + \gamma\right)}{\xi\,\eta^2} \\ G_2 &= -2\,\frac{A_{012}\left(-2\,\eta^2\alpha\,\xi^6 + \eta^4\alpha\,\xi^4 - \eta^2\beta\,\xi^4 + \gamma\right)}{\eta\,\xi^2}. \end{aligned}$$

2. A deformation of the Coulomb potential: V_2

The following potential

$$\begin{aligned} V_2 &= \frac{1}{\xi^2 + \eta^2}\left(\frac{\alpha}{\xi^2} + \frac{\beta}{\eta^2} + \gamma\right) \\ &= \frac{1}{2\sqrt{x^2 + y^2}}\left(\frac{\alpha}{x + \sqrt{x^2 + y^2}} + \frac{\beta}{x - \sqrt{x^2 + y^2}} + \gamma\right) \\ &= \frac{\alpha}{2r^2\left(\cos\theta + 1\right)} - \frac{\beta}{2r^2\left(\cos\theta - 1\right)} + \frac{\gamma}{2r} \end{aligned} \tag{55}$$

admits one third-order constant of motion with all $A_{jk\ell} = 0$ except A_{210} and functions

$$\begin{aligned} G_1 &= \frac{A_{210}\left(2\,\eta^4\gamma\,\xi^2 + 2\,\eta^4\alpha + 4\,\xi^2\beta\,\eta^2 - \eta^2\xi^2h^2 + 2\,\xi^4\beta\right)}{4\xi\,\eta^2} \\ G_2 &= -\frac{A_{210}\left(2\,\xi^4\beta + 2\,\xi^4\gamma\,\eta^2 + 4\,\xi^2\alpha\,\eta^2 - \eta^2\xi^2h^2 + 2\,\eta^4\alpha\right)}{4\eta\,\xi^2}. \end{aligned}$$

3. A second deformation of the Coulomb potential: V_3

The following potential

$$V_3 = \frac{\alpha\,\xi + \beta\,\eta + \gamma}{\xi^2 + \eta^2} \tag{56}$$
$$= \frac{1}{2\sqrt{x^2+y^2}}\left(\alpha\sqrt{x+\sqrt{x^2+y^2}} + \beta\sqrt{\sqrt{x^2+y^2}-x} + \gamma\right)$$
$$= \frac{1}{2r}\left(\alpha\sqrt{2}\cos\frac{\theta}{2} + \beta\sqrt{2}\sin\frac{\theta}{2} + \gamma\right)$$

admits one third-order integral associated with A_{102} (all the remaining $A_{jk\ell} = 0$)

$$G_1 = -\frac{1}{2}A_{102}\left(-\beta\,\xi^2 + \beta\,\eta^2 + 2\,\eta\,\alpha\,\xi + 2\,\eta\,\gamma\right)$$
$$G_2 = \frac{1}{2}A_{102}\left(\alpha\,\xi^2 - \alpha\,\eta^2 + 2\,\xi\,\gamma + 2\,\eta\,\xi\,\beta\right).$$

B. Potentials which admit more than one third-order integrals

The following potentials are sub-cases of those from the previous section, in the quantum cases. They admit at least two third-order integrals.

1. V_1 subcases

The following potential

$$V_{1,a} = \beta\left(\xi^2 - \eta^2\right) + \frac{h^2}{\xi^2\eta^2} \tag{57}$$
$$= 2\beta x + \frac{\hbar^2}{y^2}.$$

admits three linearly independent third-order integrals associated with the constants A_{012}, A_{003} and A_{102} (the remaining $A_{jk\ell}$'s are 0). The functions G_1 and G_2 are

$$G_1 = -\frac{\left(3\,h^2\eta^2 - h^2\xi^2 + 2\,\eta^4\xi^4\beta\right)A_{102}}{2\eta\,\xi^2} + 3\,\frac{h^2 A_{003}}{\eta\,\xi^2} + 2\,\frac{\left(\eta^4\xi^2\beta + h^2\right)A_{012}}{\xi\,\eta^2}$$
$$G_2 = -\frac{\left(-3\,h^2\xi^2 + h^2\eta^2 + 2\,\xi^4\eta^4\beta\right)A_{102}}{2\xi\,\eta^2} + 3\,\frac{h^2 A_{003}}{\xi\,\eta^2} + 2\,\frac{\left(\xi^4\eta^2\beta - h^2\right)A_{012}}{\eta\,\xi^2}.$$

2. V_2 subcases

The following potential

$$V_{2,a} = \frac{\gamma}{\xi^2+\eta^2} + \frac{h^2}{\xi^2\eta^2} = \frac{\gamma}{2r} + \frac{\hbar^2}{r^2\sin^2\theta}. \tag{58}$$

admits two linearly independent third-order integrals associated with the constants A_{300} and A_{210} (the remaining $A_{jk\ell}$'s are 0). The functions G_1 and G_2 are

$$G_1 = -\frac{h^2\,(\xi^2+\eta^2)\,(3\,\xi^4+2\,\xi^2\eta^2+3\,\eta^4)\,A_{300}}{8\eta\,\xi^2} + \frac{(3\,\eta^2\xi^2h^2+2\,\eta^4\gamma,\xi^2+2\,\eta^4h^2+2\,\xi^4h^2)\,A_{210}}{4\eta^2\xi}$$

$$G_2 = \frac{h^2\,(\xi^2+\eta^2)\,(3\,\xi^4+2\,\xi^2\eta^2+3\,\eta^4)\,A_{300}}{8\xi\,\eta^2} - \frac{(3\,\eta^2\xi^2h^2+2\,\xi^4h^2+2\,\eta^2\xi^4\gamma+2\,\eta^4h^2)\,A_{210}}{4\eta\,\xi^2}.$$

Another subcase of the potential V_2 which admits three linearly independent third-order integrals is given by

$$V_{2,b} = \frac{1}{\xi^2+\eta^2}\left(\gamma + \frac{h^2}{\xi^2}\right) = \frac{\gamma}{2r} + \frac{\hbar^2}{2r^2(1+\cos\theta)}. \tag{59}$$

The integrals are associated with coefficients A_{300}, A_{210} and A_{201} with the remaining $A_{jk\ell}=0$ and functions

$$G_1 = \frac{h^2\eta\,(3\,\eta^2+2\,\xi^2)\,(\xi^2+\eta^2)\,A_{300}}{8\xi^2} + \frac{(2\,h^2\eta^2+2\,\eta^2\xi^2\gamma-h^2\xi^2)\,A_{210}}{4\xi} - \frac{A_{201}\eta\,(2\,\xi^4\gamma-3\,h^2\eta^2)}{4\xi^2}$$

$$G_2 = \frac{h^2\,(3\,\eta^2+2\,\xi^2)\,(\xi^2+\eta^2)\,A_{300}}{8\xi} - \frac{\eta\,(2\,\xi^4\gamma+2\,h^2\eta^2+3\,h^2\xi^2)\,A_{210}}{4\xi^2} + \frac{A_{201}\,(2\,\xi^4\gamma-h^2\eta^2)}{4\xi}.$$

C. Potentials admitting group symmetry

The following potentials admit a Killing vector associated with group symmetry.

1. Rotational symmetry

The Coulomb potential

$$V_C = \frac{\alpha}{\xi^2+\eta^2} = \frac{\alpha}{r} \tag{60}$$

admits four linearly independent third-order integrals associated with constants $A_{300}, A_{210}, A_{201}$ and A_{102} as well as a Killing vector associated with k_1 in

$$G_1 = A_{210}\frac{\xi\,(2\,\alpha\,\eta^2 - h^2)}{4} - \eta\,A_{102}\alpha - A_{201}\frac{\eta\,(2\,\alpha\,\xi^2 + h^2)}{4} - k_1\frac{\eta\,(\xi^2 + \eta^2)}{2}$$
$$G_2 = -A_{210}\frac{\eta\,(2\,\alpha\,\xi^2 - h^2)}{4} + A_{102}\xi\,\alpha + A_{201}\frac{\xi\,(2\,\alpha\,\xi^2 - h^2)}{4} + k_1\frac{\xi\,(\xi^2 + \eta^2)}{2}.$$

2. *Translation symmetry*

The first potential is

$$V_{T,y} = \frac{\alpha}{\xi^2\eta^2} = \frac{\alpha}{y^2}. \tag{61}$$

It admits 4 linearly independent constants of the motion associated with the constants $A_{012}, A_{030}, A_{111}, A_{210}$ and a Killing vector associated with k_1, with functions

$$G_1 = \frac{\alpha\,(\eta^4 + \xi^4)\,A_{210}}{2\xi\,\eta^2} - \frac{\alpha\,(\eta - \xi)\,(\eta + \xi)\,A_{111}}{\xi\,\eta^2} + 2\,\frac{A_{012}\alpha}{\xi\,\eta^2} + k_1\,\xi$$
$$G_2 = -\frac{\alpha\,(\eta^4 + \xi^4)\,A_{210}}{2\eta\,\xi^2} + \frac{\alpha\,(\eta - \xi)\,(\eta + \xi)\,A_{111}}{\eta\,\xi^2} - 2\,\frac{A_{012}\alpha}{\eta\,\xi^2} - k_1\,\eta.$$

The potential

$$V_{T,y,\hbar} = \frac{\hbar^2}{\xi^2\eta^2} = \frac{\hbar^2}{y^2} \tag{62}$$

admits 8 linearly independent third-order constants of the motion associated with constants A_{003}, A_{012}, A_{030}, A_{102}, A_{111}, A_{201}, A_{210}, and A_{300} as well as a Killing vector associated with the constant k_1. The G's are

$$G_1 = \frac{h^2\,(3\,\eta^4 - \xi^4)\,A_{201}}{4\eta\,\xi^2} + \frac{h^2\,(\eta^4 + \xi^4)\,A_{210}}{2\xi\,\eta^2} - \frac{h^2\,(3\,\eta^6 + 5\,\xi^2\eta^4 + 5\,\xi^4\eta^2 + 3\,\xi^6)\,A_{300}}{8\eta\,\xi^2} + \xi k_1$$
$$G_2 = \frac{h^2\,(3\,\xi^4 - \eta^4)\,A_{201}}{4\xi\,\eta^2} - \frac{h^2\,(\eta^4 + \xi^4)\,A_{210}}{2\eta\,\xi^2} + \frac{h^2\,(3\,\eta^6 + 5\,\xi^2\eta^4 + 5\,\xi^4\eta^2 + 3\,\xi^6)\,A_{300}}{8\xi\,\eta^2} - \eta k_1.$$

Finally, the potential

$$V_{T,x} = \alpha(\xi^2 - \eta^2) = 2\alpha x \tag{63}$$

admits four linearly independent third-order integrals associated with constants A_{003}, A_{102}, A_{021}, A_{012} and a Killing vector associated with k_1, with

$$G_1 = -\eta^3\xi^2 A_{102}\alpha - 2\,\eta^3 A_{021}\alpha + 2\,\eta^2\xi\,A_{012}\alpha + 2\,\eta\,\xi^2 A_{021}\alpha + \eta\,k_1$$
$$G_2 = -\xi^3 A_{102}\alpha, \eta^2 + 2\,\xi^3 A_{021}\alpha + 2\,\eta\,A_{012}\xi^2\alpha - 2\,\xi\,A_{21}\alpha, \eta^2 + \xi\,k_1.$$

VI. CONCLUSIONS

The main conclusion of this article is a negative one: all third-order integrals for potentials separating in parabolic coordinates in $E_2(\mathbb{R})$ are reducible. The corresponding potentials allow separation of variables in at least two coordinate systems and are already known to be quadratically superintegrable[8,38]. Thus V_1 (54) is a deformed anisotropic harmonic oscillator, separable also in Cartesian coordinates. The potential V_2 (55) is a deformed Coulomb potential, separable also in polar coordinates. The potential V_3 (56) is a different deformation of the Coulomb potential, separable in two different parabolic coordinate systems (with the directrix of the parabolas either along the x or the y axis). Since these potentials are second-order superintegrable, they are the same in classical and quantum mechanics.

This is in stark contrast with the results obtained in the case of potentials separating in Cartesian[10] or polar coordinates[36]. There the results were much richer, mainly because of the role played by the linear compatibility condition (13).

Indeed, let us first consider a potential separating in Cartesian coordinates $V = W_1(x) + W_2(y)$. The linear compatibility conditions for this potential, written in Cartesian coordinates is given by

$$\begin{aligned} -F_3W_{1,xxx} + 2(F_{2,y} - F_{3,x})W_{1,xx} - (3F_{1,yy} - 2F_{2,xy} + F_{3,xx})W_{1,x} = \\ F_2W_{2,yyy} + (F_{2,y} - F_{3,x})W_{2,yy} + (F_{2,yy} - 2F_{3,xy} + 3F_{4,xx})W_{2y} \end{aligned} \quad (64)$$

with

$$\begin{aligned} F_1 &= A_{300}y^3 + A_{210}y^2 - A_{120}y + A_{030} \\ F_2 &= 3\,A_{300}xy^2 - 2\,A_{210}xy + A_{201}y^2 + A_{120}x - A_{111}y + A_{021} \\ F_3 &= -3\,A_{300}x^2y + A_{210}x^2 - 2\,A_{201}xy + A_{111}x - A_{102}y + A_{012} \\ F_4 &= A_{300}x^3 + A_{201}x^2 + A_{102}x + A_{003}. \end{aligned}$$

Differentiating (64) twice with respect to x gives two linear ODEs for W_1:

$$\left(3\,A_{300}x^2 + 2\,A_{201}x + A_{102}\right)W_1^{(5)} + (36\,A_{300}x + 12\,A_{201})\,W_1^{(4)} + 84\,A_{300}W_1^{(3)} = 0, \quad (65)$$

$$\left(-A_{111}x - A_{210}x^2 - A_{012}\right)W_1^{(5)} + (-12\,A_{210}x - 6\,A_{111})\,W_1^{(4)} - 28\,A_{210}W_1^{(3)} = 0. \quad (66)$$

Differentiating (64) twice with respect to y gives two linear ODEs for W_2:

$$\left(3\,A_{300}y^2 - 2\,A_{210}y + A_{120}\right)W_2^{(5)} + (36\,A_{300}y - 12\,A_{210})\,W_2^{(4)} + 90\,A_{300}W_1^{(3)} = 0 \quad (67)$$

$$\left(A_{201}y^2 - A_{111}y + A_{021}\right) W_2^{(5)} + (12\, A_{201}y - 6\, A_{111})\, W_2^{(4)} + 30\, A_{201}W_2^{(3)} = 0. \quad (68)$$

Interesting potentials in quantum mechanics, involving elliptic functions or Painlevé trascendents are obtained if either of these pairs of linear equations are satisfied trivially, i.e. if the coefficients in (65-66) or (67-68) vanish identically. Both systems of equations (65-68) vanish identically if and only if the only non-zero coefficients $A_{jk\ell}$ are A_{030} and A_{003}. The potential in this case is for instance[10]

$$V(x,y) = \hbar^2 \left[\omega_1^2 P_I(\omega_1 x) + \omega_2^2 P_I(\omega_1 y)\right],$$

where P_I is the first Painlevé transcendent[14]. If, for example, (67-68) are satisfied trivially but (65-66) are not, this leads to potentials of the form[10]

$$V(x,y) = ay + (2\hbar^2 b^2)^{1/3} \left[P_{II}' \left(-\left(\frac{4b}{\hbar^2}\right)^{1/3} x, k \right) + P_{II}^2 \left(-\frac{4b}{\hbar^2} x, k \right) \right],$$

where P_{II} is the second Painlevé transcendent[14].

Now, let us consider potentials allowing separation of variables in polar coordinates[36]

$$V = R(r) + \frac{1}{r^2} S(\theta). \quad (69)$$

The linear compatibility condition reduces to

$$\begin{aligned}
0 &= r^4 F_3 R''' + (2r^4 F_{3,r} - 2r^2 F_{2,\theta} + 3r\,(2\,r^2 F_3 - F_1))\, R'' \\
&+ (r^4 F_{3,rr} + 2\,r^2 (3F_{3,r} + 3F_3 - F_{2,r\theta}) - 4\,r F_{2,\theta} + 3F_{1,\theta\theta})\, R' \\
&+ \tfrac{1}{r^2} F_2 S''' - \tfrac{1}{r^3}\,(2r^3 F_{3,r} - 2r F_{2,\theta} + 6F_1)\, S'' \\
&+ \tfrac{1}{r^3}\,(3r^5 F_{4,rr} + 6r^4 F_{4,r} - 2r^3 F_{3,r\theta} + 3r^2 F_{2,r} + r(F_{2,\theta\theta} - 2F_2) - 12F_{1,\theta})\, S' \\
&- \tfrac{1}{r^3}\,(2r^4 F_{3,rr} - 12r^3 F_{3,r} + 4r^2(3F_{3,r} - F_{2,r\theta}) + 4r F_{2,\theta} + 6F_{1,\theta\theta} + 18F_1)\, S
\end{aligned} \quad , \quad (70)$$

where here the F_i's are given by

$$\begin{aligned}
F_1 &= A_1 \cos 3\theta + A_2 \sin 3\theta + A_3 \cos\theta + A_4 \sin\theta \\
F_2 &= \tfrac{-3A_1 \sin 3\theta + 3A_2 \cos 3\theta - A_3 \cos\theta + A_4 \sin\theta}{r} + B_1 \cos 2\theta + B_2 \sin 2\theta + B_0 \\
F_3 &= \tfrac{-3A_1 \cos 3\theta - 3A_2 \sin 3\theta + A_3 \cos\theta + A_4 \sin\theta}{r^2} + \tfrac{-2B_1 \sin 2\theta + 2B_2 \cos 2\theta}{r} + C_1 \cos\theta + C_2 \sin\theta \\
F_4 &= \tfrac{A_1 \sin 3\theta - A_2 \cos 3\theta - A_3 \sin\theta + A_4 \cos\theta}{r^3} - \tfrac{B_1 \cos 2\theta + B_2 \sin 2\theta + B_0}{r^2} - \tfrac{C_1 \sin\theta - C_2 \cos\theta}{r} + D_0,
\end{aligned}$$

with

$$\begin{aligned}
&A_1 = \tfrac{A_{030} - A_{012}}{4}, \quad A_2 = \tfrac{A_{021} - A_{003}}{4}, \quad A_3 = \tfrac{3A_{030} + A_{012}}{4}, \\
&A_4 = \tfrac{3A_{003} + A_{021}}{4}, \quad B_1 = \tfrac{A_{120} - A_{102}}{2}, \quad B_2 = \tfrac{A_{111}}{2}, \quad B_0 = \tfrac{A_{120} + A_{102}}{2}, \\
&C_1 = A_{210}, \quad C_2 = A_{201}, \quad D_0 = A_{300}.
\end{aligned}$$

Equation (70) is satisfied trivially if all the $A_{jk\ell} = 0$ except A_{300}. This leads to the potential[36]

$$V(r,\theta) = R(r) + \frac{\hbar^2}{r^2} P(\theta, t_2, t_3),$$

where $R(r)$ is arbitrary and $P(\theta, t_2, t_3)$ is the Weierstrass elliptic function[3]. This potential allows a third-order integral, however it is algebraically related to the second-order one and the system is hence not superintegrable. If however we consider the subcase $R(r) = 0$, Eq. (70) simplifies. It being satisfied trivially allows further constants in the integral to be nonzero, namely A_{300}, A_{210} and A_{201}. This leads to a superintegrable potential expressed in terms of the sixth Painlevé transcendent[14], $P_{VI}(\sin\theta/2)$.

In section III, we have shown that the linear compatibility condition (13) is never satisfied trivially for a potential separating in parabolic coordinates. This rules out any dependence of W_1 and W_2 on elliptic functions or Painlevé transcendents.

In addition to the real potentials presented in section V, we have recovered a complex one (see Appendix B) which is known to be superintegrable[22]. It can be transformed into a real form; however, it will live on the pseudo-Euclidean plane $E(1,1)$ rather than the Euclidean one.

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Appendix A: System of linear ODEs for W_1 and W_2

The linear compatibility condition (13), implies (by differentiation) two systems of 14 ODEs for W_1 and W_2, respectively. Those for W_1 are given below and those for W_2 can be obtained from these under the interchange $\xi \to \eta$ and $A_{jk\ell} \to (-1)^{j+k} A_{jk\ell}$.

$$0 = A_{300}\left(\xi^2 W_1^{(17)} + 37\xi W_1^{(16)} + 323 W_1^{(15)}\right) \tag{A1}$$

$$0 = A_{210}\left(\xi\, W_1^{(17)} + 18 W_1^{(16)}\right) \tag{A2}$$

$$0 = A_{210}\left(11\,\xi^3 W_1^{(17)} + 503\xi^2 W_1^{(16)} + 7145\,\xi W_1^{(15)} + 31360 W_1^{(14)}\right) \tag{A3}$$

$$\begin{aligned} 0 = &-9A_{300}\left(\xi^4 {W_1}^{(17)} + 65\xi^3 {W_1}^{(16)} + 1485\xi^2 {W_1}^{(15)} + 14070\xi {W_1}^{(14)} + 46410 {W_1}^{(13)}\right) \\ &+(A_{201}\xi^2 - 2A_{120}) W_1^{(17)} - 33\xi\, A_{201} W_1^{(16)} - 255 A_{201} W_1^{(15)}. \end{aligned} \tag{A4}$$

$$\begin{aligned} 0 = &-\tfrac{45}{4} A_{300}\left(\xi^6 {W_1}^{(17)} + 93\xi^5 {W_1}^{(16)} + 3375\xi^4 {W_1}^{(15)} + 60900\xi^3 {W_1}^{(14)} + 573300\xi^2 {W_1}^{(13)}\right. \\ &\left.+2653560\xi {W_1}^{(12)} + 4684680\, {W_1}^{(11)}\right) \\ &-A_{201}\left(\tfrac{5\xi^4}{2} {W_1}^{(17)} + \tfrac{301\xi^3}{2} {W_1}^{(16)} + \tfrac{6345\xi^2}{2} {W_1}^{(15)} + 27615\xi {W_1}^{(14)} + 83265\, {W_1}^{(13)}\right) \\ &-A_{120}\left(6\xi^2 {W_1}^{(17)} + 169\xi {W_1}^{(16)} + 1109 {W_1}^{(15)}\right) \\ &+\left(\xi^2 A_{102} + 2A_{021}\right) {W_1}^{17} + 33\xi\, A_{102} {W_1}^{(16)} + 255 A_{102} {W_1}^{(15)}. \end{aligned} \tag{A5}$$

$$\begin{aligned} 0 = &\tfrac{-15}{2} A_{300}\left(\xi^8 {W_1}^{(17)} + 121\xi^7 {W_1}^{(16)} + 5993\xi^6 {W_1}^{(15)} + 157962\xi^5 {W_1}^{(14)} + 2410590\xi^4 {W_1}^{(13)}\right. \\ &\left.+21676200\xi^3 {W_1}^{(12)} + 111351240\xi^2 {W_1}^{(11)} + 296215920\xi {W_1}^{(10)} + 309188880\, {W_1}^{(9)}\right) \\ &-A_{201}\left(\tfrac{5\xi^6}{2} {W_1}^{(17)} + \tfrac{437\xi^5}{2} {W_1}^{(16)} + \tfrac{14831\xi^4}{2} {W_1}^{(15)} + 124418\xi^3 {W_1}^{(14)}\right. \\ &\left.+1081626\xi^2 {W_1}^{(13)} + 4585308\xi {W_1}^{(12)} + 7339332\, {W_1}^{(11)}\right) \\ &-A_{120}\left(7\xi^4 {W_1}^{(17)} + 388\xi^3 {W_1}^{(16)} + 7464\xi^2 {W_1}^{15} + 58632\xi {W_1}^{(14)} + 157248 {W_1}^{(13)}\right) \\ &+2A_{102}\left(\xi^4 {W_1}^{(17)} + 58\xi^3 {W_1}^{(16)} + 1170\xi^2 {W_1}^{(15)} + 9660\xi {W_1}^{(14)} + 27300 {W_1}^{(13)}\right) \\ &+A_{021}\left(\xi^2 {W_1}^{(17)} - 11\xi {W_1}^{(16)} - 409 {W_1}^{(15)}\right) \\ &+A_{003}\left(3\xi^2 {W_1}^{(17)} + 111\xi {W_1}^{(16)} + 969\, {W_1}^{(15)}\right). \end{aligned} \tag{A6}$$

$$\begin{aligned} 0 = &A_{210}\left(\tfrac{25\xi^5}{4} {W_1}^{(17)} + \tfrac{1839\xi^4}{4} {W_1}^{(16)} + \tfrac{50451\xi^3}{4} {W_1}^{(15)} + \tfrac{320901\xi^2}{2} {W_1}^{14} + \tfrac{1881243\xi}{2} {W_1}^{(13)}\right. \\ &\quad\left.+2018016\, {W_1}^{(12)}\right) \\ &+\left(3\,\xi\, A_{030} + 1/2\,\xi^3 A_{111} - 2\,\xi\, A_{012}\right) {W_1}^{17} + \left(54\, A_{030} - 36\, A_{012} + \tfrac{51}{2}\,\xi^2 A_{111}\right) {W_1}^{(16)} \\ &+\tfrac{813}{2}\, {W_1}^{(15)} \xi\, A_{111} + 2016\, {W_1}^{(14)} A_{111}. \end{aligned} \tag{A7}$$

$$\begin{aligned} 0 = &\tfrac{1}{4} A_{210}\left(25\, {W_1}^{(17)}\xi^5 + 1839\, {W_1}^{(16)}\xi^4 + 50451\, {W_1}^{(15)}\xi^3 + 641802\, {W_1}^{(14)}\xi^2 + 3762486\, {W_1}^{(13)}\xi\right. \\ &\quad\left.+8072064\, {W_1}^{(12)}\right) \\ &+\tfrac{1}{2} A_{111}\left({W_1}^{(17)}\xi^3 + 51\, {W_1}^{(16)}\xi^2 + 813\, {W_1}^{(15)}\xi + 4032\, {W_1}^{(14)}\right) \\ &+3\, A_{030}\left({W_1}^{(17)}\xi + 18\, {W_1}^{(16)}\right) - 2\, A_{012}\left({W_1}^{(17)}\xi + 18\, {W_1}^{(16)}\right). \end{aligned} \tag{A8}$$

$$
\begin{aligned}
0 =\ & \tfrac{1}{2}\,A_{210}\Big(15\,W_1{}^{(17)}\xi^7 + 1523\,W_1{}^{(16)}\xi^6 + 61761\,W_1{}^{(15)}\xi^5 + 1289820\,W_1{}^{(14)}\xi^4 + 14889420\,W_1{}^{(13)}\xi^3 \\
& \quad + 94316040\,W_1{}^{(12)}\xi^2 + 300900600\,W_1{}^{(11)}\xi + 369008640\,W_1{}^{(10)}\Big) \\
& + 2\,A_{111}\Big(W_1{}^{(17)}\xi^5 + 76\,W_1{}^{(16)}\xi^4 + 2156\,W_1{}^{(15)}\xi^3 + 28392\,W_1{}^{(14)}\xi^2 + 172536\,W_1{}^{(13}\xi \\
& \quad + 384384\,W_1{}^{(12)}\Big) \\
& - A_{012}\Big(5\,W_1{}^{(17)}\xi^3 + 173\,W_1{}^{(16)}\xi^2 + 1475\,W_1{}^{(15)}\xi + 1024\,W_1{}^{(14)}\Big) \\
& 3\,A_{030}\Big(3\,W_1{}^{(17)}\xi^3 + 115\,W_1{}^{(16)}\xi^2 + 1249\,W_1{}^{(15)}\xi + 3392\,W_1{}^{(14)}\Big)\,.
\end{aligned} \tag{A9}
$$

$$
\begin{aligned}
0 =\ & -\tfrac{9}{4}\,A_{300}\Big(W_1{}^{(17)}\xi^{12} + 177\,W_1{}^{(16)}\xi^{11} + 13413\,W_1{}^{(15)}\xi^{10} + 572670\,W_1{}^{(14)}\xi^9 + 15257970\,W_1{}^{(13)}\xi^8 \\
& \quad + 265552560\,W_1{}^{(12)}\xi^7 + 3072429360\,W_1{}^{(11)}\xi^6 + 23597814240\,W_1{}^{(10)}\xi^5 \\
& \quad + 118118800800\,W_1{}^{(9)}\xi^4 + 370767196800\,W_1{}^{(8)}\xi^3 + 681080400000\,W_1{}^{(7)}\xi^2 \\
& \quad + 642939897600\,W_1{}^{(6)}\xi + 228843014400\,W_1{}^{(5)}\Big) \\
& - \tfrac{1}{4}\,A_{201}\Big(5\,W_1{}^{(17)}\xi^{10} + 709\,W_1{}^{(16)}\xi^9 + 42051\,W_1{}^{(15)}\xi^8 + 1365672\,W_1{}^{(14)}\xi^7 \\
& \quad + 26708136\,W_1{}^{(13)}\xi^6 + 325909584\,W_1{}^{(12)}\xi^5 + 2487204720\,W_1{}^{(11)}\xi^4 \\
& \quad + 11568997440\,W_1{}^{(10)}\xi^3 + 30849698880\,W_1{}^{(9)}\xi^2 \\
& \quad + 41565363840\,W_1{}^{(8)}\xi + 20704844160\,W_1{}^{(7)}\Big) \\
& - \tfrac{1}{2}\,A_{120}\Big(9\,W_1{}^{(17)}\xi^8 + 976\,W_1{}^{(16)}\xi^7 + 42728\,W_1{}^{(15)}\xi^6 + 978432\,W_1{}^{(14)}\xi^5 + 12689040\,W_1{}^{(13)}\xi^4 \\
& \quad + 94174080\,W_1{}^{(12)}\xi^3 + 383423040\,W_1{}^{(11)}\xi^2 + 761080320\,W_1{}^{(10)}\xi + 536215680\,W_1{}^{(9)}\Big) \\
& + 2\,A_{102}\Big(W_1{}^{(17)}\xi^8 + 108\,W_1{}^{(16)}\xi^7 + 4704\,W_1{}^{(15)}\xi^6 + 107016\,W_1{}^{(14)}\xi^5 + 1375920\,W_1{}^{(13)}\xi^4 \\
& \quad + 10090080\,W_1{}^{(12)}\xi^3 + 40360320\,W_1{}^{(11)}\xi^2 + 77837760\,W_1{}^{(10)}\xi + 51891840\,W_1{}^{(9)}\Big) \\
& - A_{021}\Big(5\,W_1{}^{(17)}\xi^6 + 393\,W_1{}^{(16)}\xi^5 + 11511\,W_1\xi^{(15)} + 157140\,W_1{}^{(14)}\xi^3 + 1016820\,W_1{}^{(13)}\xi^2 \\
& \quad + 2784600\,W_1{}^{(12)}\xi + 2325960\,W_1{}^{(11)}\Big) \\
& + 9\,A_{003}\Big(W_1{}^{(17)}\xi^6 + 77\,W_1{}^{(16)}\xi^5 + 2223\,W_1{}^{(15)}\xi^4 + 30180\,W_1{}^{(14)}\xi^3 + 196980\,W_1{}^{(13)}\xi^2 \\
& \quad + 556920\,W_1{}^{(12)}\xi + 491400\,W_1{}^{(11)}\Big)\,.
\end{aligned} \tag{A10}
$$

$$
\begin{aligned}
0 = \; & -\tfrac{45}{8}A_{300}\Big(\xi^{10}{W_1}^{(17)} + 149\xi^{9}{W_1}^{(16)} + 9339\xi^{8}{W_1}^{(15)} + 322728\xi^{7}{W_1}^{(14)} \\
& \quad +6772584\xi^{6}{W_1}^{(13)} + 89618256\xi^{5}{W_1}^{(12)} + 751710960\xi^{4}{W_1}^{(11)} + 3912068160\xi^{3}{W_1}^{(10)} \\
& \quad +11961069120\xi^{2}{W_1}^{(9)} + 19148088960\xi{W_1}^{(8)} + 11987015040\,{W_1}^{(7)}\Big) \\
& -A_{201}\Big(\tfrac{5\xi^8}{2}{W_1}^{(17)} + \tfrac{573\xi^7}{2}\,{W_1}^{(16)} + \tfrac{26733\xi^6}{2}\,{W_1}^{(15)} + 329721\xi^{5}{W_1}^{(14)} + 4672395\xi^{4}{W_1}^{(13)} \\
& \quad +38640420\xi^{3}{W_1}^{(12)} + 180360180\xi^{2}{W_1}^{(11)} + 429188760\xi{W_1}^{(10)} + 392432040\,{W_1}^{(9)}\Big) \\
& -A_{120}\Big(8\xi^{6}{W_1}^{(17)} + 657\xi^{5}{W_1}^{(16)} + 20769\xi^{4}{W_1}^{(15)} + 320964\xi^{3}{W_1}^{(14)} + 2532348\xi^{2}{W_1}^{(13)} \\
& \quad +9546264\xi{W_1}^{(12)} + 13189176{W_1}^{(11)}\Big) \\
& +3A_{102}\Big(\xi^{6}{W_1}^{(17)} + 83\xi^{5}{W_1}^{(16)} + 2653\xi^{4}{W_1}^{(15)} + 41468\xi^{3}{W_1}^{(14)} \\
& \quad +330876\xi^{2}{W_1}^{(13)} + 1260168\xi{W_1}^{(12)} + 1753752{W_1}^{(11)}\Big) \\
& -3A_{021}\Big(\xi^{4}{W_1}^{(17)} + 71\xi^{3}{W_1}^{(16)} + 1583\xi^{2}{W_1}^{(15)} + 12858\xi{W_1}^{(14)} + 29022{W_1}^{(13)}\Big) \\
& +9A_{003}\Big(\xi^{4}{W_1}^{(17)} + 57\xi^{3}{W_1}^{(16)} + 1093\xi^{2}{W_1}^{(15)} + 8070\xi{W_1}^{(14)} + 17850{W_1}^{(13)}\Big).
\end{aligned}
\tag{A11}
$$

$$
\begin{aligned}
0 = \ & \tfrac{-8}{3} A_{300} \Big(\xi^{14} {W_1}^{(17)} + 205 \xi^{13} {W_1}^{(16)} + 18215 \xi^{(12)} {W_1}^{(15)} + 925260 \xi^{11} {W_1}^{(14)} \\
& \quad + 29849820 \xi^{10} {W_1}^{(13)} + 642762120 \xi^{9} {W_1}^{(12)} + 9454044600 \xi^{8} {W_1}^{(11)} + 95610715200 \xi^{7} {W_1}^{10} \\
& \quad + 660712852800 \xi^{6} {W_1}^{(9)} + 3062396937600 \xi^{5} {W_1}^{(8)} + 9210023222400 \xi^{4} {W_1}^{(7)} \\
& \quad + 17036091072000 \xi^{3} {W_1}^{(6)} + 17762576832000 \xi^{2} {W_1}^{(5)} \\
& \quad + 8935774848000 \xi {W_1}^{(4)} + 1525620096000 {W_1}^{(3)} \Big) \\
& - \tfrac{1}{4} A_{201} \Big({W_1}^{(17)} \xi^{12} + 169\, {W_1}^{(16)} \xi^{11} + 12157\, {W_1}^{(15)} \xi^{10} + 489230\, {W_1}^{(14)} \xi^{9} + 12178530\, {W_1}^{(13)} \xi^{8} \\
& \quad + 195839280\, {W_1}^{(12)} \xi^{7} + 2063421360\, {W_1}^{(11)} \xi^{6} + 14153499360\, {W_1}^{(10)} \xi^{5} \\
& \quad + 61556695200\, {W_1}^{(9)} \xi^{4} + 161124163200\, {W_1}^{(8)} \xi^{3} + 230659228800\, {W_1}^{(7)} \xi^{2} \\
& \quad + 148929580800\, {W_1}^{(6)} \xi + 25427001600\, {W_1}^{(5)} \Big) \\
& - \tfrac{1}{2} A_{120} \Big(2\, {W_1}^{(17)} \xi^{10} + 269\, {W_1}^{(16)} \xi^{9} + 15009\, {W_1}^{(15)} \xi^{8} + 453768\, {W_1}^{(14)} \xi^{7} + 8148504\, {W_1}^{(13)} \xi^{6} \\
& \quad + 89618256\, {W_1}^{(12)} \xi^{5} + 600359760\, {W_1}^{(11)} \xi^{4} + 2355312960\, {W_1}^{(10)} \xi^{3} + 4955670720\, {W_1}^{(9)} \xi^{2} \\
& \quad + 4618373760\, {W_1}^{(8)} \xi + 1089728640\, {W_1}^{(7)} \Big) \\
& + \tfrac{1}{2} A_{102} \Big({W_1}^{(17)} \xi^{10} + 133\, {W_1}^{(16)} \xi^{9} + 7323\, {W_1}^{(15)} \xi^{8} + 217896\, {W_1}^{(14)} \xi^{7} + 3837288\, {W_1}^{(13)} \xi^{6} \\
& \quad + 41185872\, {W_1}^{(12)} \xi^{5} + 267387120\, {W_1}^{(11)} \xi^{4} + 1006125120\, {W_1}^{(10)} \xi^{3} + 1997835840\, {W_1}^{(9)} \xi^{2} \\
& \quad + 1712430720\, {W_1}^{(8)} \xi + 363242880\, {W_1}^{(7)} \Big) \\
& - A_{021} \Big(2\, {W_1}^{(17)} \xi^{8} + 191\, {W_1}^{(16)} \xi^{7} + 7171\, {W_1}^{(15)} \xi^{6} + 135846\, {W_1}^{(14)} \xi^{5} + 1385370\, {W_1}^{(13)} \xi^{4} \\
& \quad + 7502040\, {W_1}^{(12)} \xi^{3} + 19819800\, {W_1}^{(11)} \xi^{2} + 20900880\, {W_1}^{(10)} \xi + 5045040\, {W_1}^{(9)} \Big) \\
& + 3\, A_{003} \Big({W_1}^{(17)} \xi^{8} + 97\, {W_1}^{(16)} \xi^{7} + 3713\, {W_1}^{(15)} \xi^{6} + 72090\, {W_1}^{(14)} \xi^{5} + 759150\, {W_1}^{(13)} \xi^{4} \\
& \quad + 4291560\, {W_1}^{(12)} \xi^{3} + 12022920\, {W_1}^{(11)} \xi^{2} + 13693680\, {W_1}^{(10)} \xi + 3603600\, {W_1}^{(9)} \Big) .
\end{aligned}
\tag{A12}
$$

$$
\begin{aligned}
0 = \ & \tfrac{1}{2} A_{210} \Big(10\, {W_1}^{(17)} \xi^{9} + 1297\, {W_1}^{(16)} \xi^{8} + 69655\, {W_1}^{(15)} \xi^{7} + 2022230\, {W_1}^{(14)} \xi^{6} + 34761090\, {W_1}^{(13)} \xi^{5} \\
& \quad + 364236600\, {W_1}^{(12)} \xi^{4} + 2306424120\, {W_1}^{(11)} \xi^{3} + 8421613200\, {W_1}^{(10)} \xi^{2} \\
& \quad + 15881065200\, {W_1}^{(9)} \xi + 11589177600\, {W_1}^{(8)} \Big) \\
& + 3\, A_{111} \Big({W_1}^{(17)} \xi^{7} + 101\, {W_1}^{(16)} \xi^{6} + 4067\, {W_1}^{(15)} \xi^{5} + 84140\, {W_1}^{(14)} \xi^{4} + 959140\, {W_1}^{(13)} \xi^{3} \\
& \quad + 5973240\, {W_1}^{(12)} \xi^{2} + 18618600\, {W_1}^{(11)} \xi + 22102080\, {W_1}^{(10)} \Big) \\
& + 9\, A_{030} \Big({W_1}^{(17)} \xi^{5} + 59\, {W_1}^{(16)} \xi^{4} + 1227\, {W_1}^{(15)} \xi^{3} + 10946\, {W_1}^{(14)} \xi^{2} + 40838\, {W_1}^{(13)} \xi \\
& \quad + 53872\, {W_1}^{(12)} \Big) \\
& - 3\, A_{012} \Big({W_1}^{(17)} \xi^{5} + 41\, {W_1}^{(16)} \xi^{4} + 273\, {W_1}^{(15)} \xi^{3} - 6146\, {W_1}^{(14)} \xi^{2} - 78638\, {W_1}^{(13)} \xi \\
& \quad - 206752\, {W_1}^{(12)} \Big) .
\end{aligned}
\tag{A13}
$$

$$
\begin{aligned}
0 = \; & \tfrac{1}{4} A_{210} \Big(7\, {W_1}^{(17)} \xi^{11} + 1107\, {W_1}^{(16)} \xi^{10} + 74133\, {W_1}^{(15)} \xi^{9} + 2760408\, {W_1}^{(14)} \xi^{8} \\
& \quad + 63115416\, {W_1}^{(13)} \xi^{7} + 923792688\, {W_1}^{(12)} \xi^{6} + 8757180432\, {W_1}^{(11)} \xi^{5} + 53215081920\, {W_1}^{(10)} \xi^{4} \\
& \quad + 200587907520\, {W_1}^{(9)} \xi^{3} + 439575776640\, {W_1}^{(8)} \xi^{2} + 493647073920\, {W_1}^{(7)} \xi \\
& \quad + 209227898880\, {W_1}^{(6)} \Big) \\
& + 2 A_{111} \Big({W_1}^{(17)} \xi^{9} + 126\, {W_1}^{(16)} \xi^{8} + 6546\, {W_1}^{(15)} \xi^{7} + 182868\, {W_1}^{(14)} \xi^{6} + 3004092\, {W_1}^{(13)} \xi^{5} \\
& \quad + 29811600\, {W_1}^{(12)} \xi^{4} + 176576400\, {W_1}^{(11)} \xi^{3} + 592431840\, {W_1}^{(10)} \xi^{2} + 998917920\, {W_1}^{(9)} \xi \\
& \quad + 622702080\, {W_1}^{(8)} \Big) \\
& + 3\, A_{030} \Big({W_1}^{(17)} \xi^{7} + 81\, {W_1}^{(16)} \xi^{6} + 2535\, {W_1}^{(15)} \xi^{5} + 39444\, {W_1}^{(14)} \xi^{4} + 329364\, {W_1}^{(13)} \xi^{3} \\
& \quad + 1500408\, {W_1}^{(12)} \xi^{2} + 3583944\, {W_1}^{(11)} \xi + 3459456\, {W_1}^{(10)} \Big) \\
& + A_{012} \Big({W_1}^{(17)} \xi^{7} + 129\, {W_1}^{(16)} \xi^{6} + 6027\, {W_1}^{(15)} \xi^{5} + 134484\, {W_1}^{(14)} \xi^{4} + 1544004\, {W_1}^{(13)} \xi^{3} \\
& \quad + 8969688\, {W_1}^{(12)} \xi^{2} + 23633064\, {W_1}^{(11)} \xi + 20756736\, {W_1}^{(10)} \Big) .
\end{aligned} \tag{A14}
$$

Appendix B: Complex potentials and superintegrable systems in $E(1,1)$ admitting separation in parabolic coordinates

As a byproduct of this study, we have obtained a complex potential in E_2 that can be viewed as real potential in the pseudo-Euclidean place $E(1,1)$. This is the only complex potential in Ref. 22 which is separable in parabolic coordinates and does not admit a Killing vector. We also give the subcase of this potential which admits a Killing vector and its third-order integrals.

1. System admitting a single third-order integral: V_4

The following potential

$$
\begin{aligned}
V_4 &= -\alpha(\xi^2 - \eta^2) + \beta \frac{(\xi^2 + i\xi\,\eta - \eta^2)}{\xi + i\eta} + \frac{\gamma}{\xi + i\eta}, \qquad \text{(B1)} \\
&= 2\,\alpha\, x - \frac{\beta\ (2\,x + iy)}{\sqrt{x + iy}} - \frac{\gamma}{\sqrt{x + iy}} \\
&= 2\alpha r \cos(\theta) - \frac{r\,(2\,\cos\theta + i \sin\theta)\,\beta + \gamma}{\sqrt{r\,(\cos\theta + i\sin\theta)}}
\end{aligned}
$$

admits one third-order integral with $A_3 = -A_{021}$, $A_{012} = 2iA_{021}$ and the remained $A_{jk\ell} = 0$. The functions G_1 and G_2 are given by

$$G_1 = i\left(-2\,i\eta\,\alpha\,\xi^2 + 4\,\xi\,\alpha\,\eta^2 - 2\,i\eta\,\xi\,\beta + 2\,i\alpha\,\eta^3 + \eta^2\beta + \gamma\right)A_{021}$$
$$G_2 = \left(-2\,\xi\,\alpha\,\eta^2 + 4\,i\eta\,\alpha\,\xi^2 + 2\,i\eta\,\xi\,\beta + 2\,\xi^3\alpha + \beta\,\xi^2 - \gamma\right)A_{021}.$$

Putting $iy = t$, we obtain a superintegrable system in $E(1,1)$; namely,

$$H = \frac{1}{2}(p_x^2 - p_t^2) + 2\alpha x - \frac{\beta\ (2\,x + t)}{\sqrt{x+t}} - \frac{\gamma}{\sqrt{x+t}}. \qquad \text{(B2)}$$

2. System admitting group symmetry

A special case of the previous potential admits a Killing vector. The potential is:

$$V_L = \frac{\gamma}{\xi + i\eta}. \qquad \text{(B3)}$$

The third-order integral is defined by the following non-zero $A_{jk\ell}$, along with an arbitrary constant k_1 which is the coefficient of the Killing vector

$$A_{003} = A_{030} = \frac{A_{021} + 2iA_{012}}{3}, A_{120} = -iA_{111} + A_{102}, A_{102}, A_{111}, A_{021}, A_{012},$$

$$G_1 = \gamma A_{012} + i\gamma\,A_{021} + i\gamma\,(\xi + i\eta)^2 A_{102} - \frac{\gamma}{2}\left(\xi^2 - \eta^2\right)A_{111} + k_1\,(\xi + i\eta)$$
$$G_2 = i\gamma A_{012} + \gamma A_{021} + \frac{\gamma}{2}\,(\xi + i\eta)^2 A_{102} - \frac{i\gamma}{2}\left(\xi^2 - \eta^2\right)A_{111} + ik_1\,(\xi + i\eta)\,.$$

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CHAPITRE 3

CONCLUSION

Le but de ce mémoire a été de trouver les intégrales d'ordre trois pour les potentiels séparables en coordonnées paraboliques. C'est la suite des travaux achevés par Gravel pour le cas où le hamiltonien admet la séparation de variables en coordonnées cartésiennes et par Frédérick Tremblay pour le hamiltonien séparable en coordonnées polaires.

La conclusion principale est une négative : toutes les intégrales pour les potentiels séparables en coordonnées paraboliques dans l'espace réel de dimension deux sont réductibles. Les potentiels correspondants permettent une séparation dans au moins deux systèmes de coordonnées et ils sont déjà connus étant quadratiquement superintégrables. Ainsi, $V_1 = \left(\eta^4 - \xi^2\eta^2 + \xi^4\right)\alpha + \beta\left(\xi^2 - \eta^2\right) + \frac{\gamma}{\xi^2\eta^2}$ est un oscillateur harmonique anisotrope déformé, séparable aussi en coordonnées cartésiennes. Le potentiel $V_2 = \frac{1}{\xi^2+\eta^2}\left(\frac{\alpha}{\xi^2} + \frac{\beta}{\eta^2} + \gamma\right)$ est un potentiel de Coulomb déformé séparable aussi en coordonnées polaires. Le potentiel $V_3 = \frac{\alpha\xi+\beta\eta+\gamma}{\xi^2+\eta^2}$ est une déformation différente du potentiel de Coulomb qui est séparable en différents systèmes de coordonnées paraboliques (avec la directrice des paraboles soit selon l'axe des x soit selon l'axe des y). Puisque ces potentiels sont superintégrables d'ordre deux, ils sont les mêmes autant pour la mécanique classique que pour la mécanique quantique.

Les résultats obtenus pour les potentiels séparables en coordonnées cartésiennes et polaires sont beaucoup plus riches grâce au rôle de la condition de compatibilité linéaire.

Ce qui reste toujours à étudier ce sont les systèmes superintégrables avec intégrales de mouvement d'ordre trois séparables en coordonnéees elliptiques. De plus, les coordonnées liées aux sous- groupes du groupe de l'isométries de l'espace telles que les coordonnées cartésiennes et polaires jouent un rôle privilégié. La quéstion qui se pose : " est-ce que cela est vrai dans l'espace euclidien E_n pour $n \geq 3$" [15].

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