Université de Montréal

et

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# Aspects géométriques et intégrables des modèles de matrices aléatoires

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**Aspects géométriques et intégrables des modèles de matrices aléatoires**

présentée par:

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# RÉSUMÉ

Cette thèse traite des aspects géométriques et d'intégrabilité associés aux modèles de matrices aléatoires. Son but est de présenter diverses applications des modèles de matrices aléatoires allant de la géométrie algébrique aux équations aux dérivées partielles des systèmes intégrables. Ces différentes applications permettent en particulier de montrer en quoi les modèles de matrices possèdent une grande richesse d'un point de vue mathématique.

Ainsi, cette thèse abordera d'abord l'étude de la jonction de deux intervalles du support de la densité des valeurs propres au voisinage d'un point singulier. On montrera plus précisément en quoi ce régime limite particulier aboutit aux équations universelles de la hiérarchie de Painlevé II des systèmes intégrables. Ensuite, l'approche des polynômes (bi)-orthogonaux, introduite par Mehta pour le calcul des fonctions de partition, permettra d'énoncer des problèmes de Riemann-Hilbert et d'isomonodromies associés aux modèles de matrices, faisant ainsi le lien avec la théorie de Jimbo-Miwa-Ueno. On montrera en particulier que le cas des modèles à deux matrices hermitiens se transpose à un cas dégénéré de la théorie isomonodromique de Jimbo-Miwa-Ueno qui sera alors généralisé. La méthode des équations de boucles avec ses notions centrales de courbe spectrale et de développement topologique permettra quant à elle de faire le lien avec les invariants symplectiques de géométrie algébrique introduits récemment par Eynard et Orantin. Ce dernier point fera également l'objet d'une généralisation aux modèles de matrices non-hermitien ($\beta$ quelconque) ouvrant ainsi la voie à la "géométrie algébrique quantique" et à la généralisation de ces invariants symplectiques pour des courbes "quantiques". Enfin, une dernière partie sera consacrée aux liens étroits entre les modèles de matrices et les problèmes de combinatoire. En particulier, l'accent sera mis sur les aspects géométriques de la théorie des cordes topologiques avec la construction explicite d'un modèle de matrices aléatoires donnant le dénombrement des invariants de Gromov-Witten pour les variétés de Calabi-Yau toriques de dimension complexe trois utilisées en théorie des cordes topologiques.

L'étendue des domaines abordés étant très vaste, l'objectif de la thèse est de présenter

de façon la plus simple possible chacun des domaines mentionnés précédemment et d'analyser en quoi les modèles de matrices peuvent apporter une aide précieuse dans leur résolution. Le fil conducteur étant les modèles matriciels, chaque partie a été conçue pour être abordable pour un spécialiste des modèles de matrices ne connaissant pas forcément tous les domaines d'application présentés ici.

**Mots-clés : géométrie algébrique, équations de boucles, invariants symplectiques, théorie des cordes topologiques, isomonodromies, polynômes orthogonaux.**

## ABSTRACT

This thesis deals with the geometric and integrable aspects associated with random matrix models. Its purpose is to provide various applications of random matrix theory, from algebraic geometry to partial differential equations of integrable systems. The variety of these applications shows why matrix models are important from a mathematical point of view.

First, the thesis will focus on the study of the merging of two intervals of the eigenvalues density near a singular point. Specifically, we will show why this special limit gives universal equations from the Painlevé II hierarchy of integrable systems theory. Then, following the approach of (bi) orthogonal polynomials introduced by Mehta to compute partition functions, we will find Riemann-Hilbert and isomonodromic problems connected to matrix models, making the link with the theory of Jimbo, Miwa and Ueno. In particular, we will describe how the hermitian two-matrix models provide a degenerate case of Jimbo-Miwa-Ueno's theory that we will generalize in this context. Furthermore, the loop equations method, with its central notions of spectral curve and topological expansion, will lead to the symplectic invariants of algebraic geometry recently proposed by Eynard and Orantin. This last point will be generalized to the case of non-hermitian matrix models (arbitrary $\beta$) paving the way to "quantum algebraic geometry" and to the generalization of symplectic invariants to "quantum curves". Finally, this set up will be applied to combinatorics in the context of topological string theory, with the explicit computation of an hermitian random matrix model enumerating the Gromov-Witten invariants of a toric Calabi-Yau threefold.

Since the range of the applications encountered is large, we try to present every domain in a simple way and explain how random matrix models can bring new insights to those fields. The common element of the thesis being matrix models, each part has been written so that readers unfamiliar with the domains of application but familiar with matrix models should be able to understand it.

**Keywords: algebraic geometry, loop equations, symplectic invariants, topological string theory, isomonodromies, orthogonal polynomials.**

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Je tiens à remercier celles et ceux sans lesquels cette thèse n'aurait jamais vu le jour. Tout d'abord mes deux directeurs pour leur grande patience avec une mention spéciale pour les mini-cours personnalisés de Bertrand qui m'ont fait gagner un temps précieux. Je remercie également Michel Bergère pour la voiture rouge et les repas partagés plein de discussions. Egalement, un grand merci à mes amis qui ont toujours su me supporter et me donner de la motivation, du réconfort et du plaisir. Mes pensées sont particulièrement destinées à Guillaume pour les après-midi consoles endiablés, Romain pour m'avoir fait découvrir le squash, Cécilie pour tous les bons moments passés à jouer au volley-ball ensemble, Sophie pour ses dîners savoureux, Hélène pour avoir supporté le début de le thèse, Lucien pour les discussions d'informatique (entre autres), Camille pour m'avoir fait découvrir à distance de lointaines contrées, Mathilde et Elodie pour les randonnées Grenobloises, ainsi que tous mes partenaires de volley-ball pour m'avoir offert des moments de détente. Finalement, mes pensées les plus fortes vont envers mes deux frères et mes parents à qui je dois énormément pour leur soutien constant, surtout dans les périodes les plus difficiles. Enfin, une pensée spéciale va à Audrey qui verra l'aboutissement de cette thèse et je l'espère ce qui suivra ensuite.

## TABLE DES MATIÈRES

## LISTE DES ARTICLES UTILISÉS POUR CETTE THÈSE

Les articles mentionnés ici sont disponibles dans les annexes numérotés de V à X.

[**I**] O. Marchal, M. Cafasso, "Double scaling limits of random matrices and minimal $(2m, 1)$ models : the merging of two cuts in a degenerate case", *arXiv* :1002.3347v2 [math-ph] (Egalement en annexe V)

[**II**] M. Bertola, O. Marchal "The partition function of the two-matrix model as an isomonodromic tau-function", *J. Math. Phys.* **50**, 013529, 2009. (Egalement en annexe VI)

[**III**] B. Eynard, O. Marchal, "Topological expansion of the Bethe ansatz, and non-commutative algebraic geometry", *JHEP* 0903 :094, 2009, arXiv :0809.3367 [math-ph]. (Egalement en annexe VII)

[**IV**] L. Chekov, B. Eynard, O. Marchal, "Topological expansion of the Bethe ansatz, and quantum algebraic geometry", *arXiv* :0911.1664v2 [math-ph] (Egalement en annexe VIII)

[**V**] B. Eynard, A. Kashani-Poor, O. Marchal, "A matrix model for the topological string I : Deriving the matrix model", *arXiv* :1003.1737v2 [hep-th] (Egalement en annexe IX)

[**VI**] B. Eynard, A. Kashani-Poor, O. Marchal "A matrix model for the topological string II : The spectral curve and mirror geometry", *arXiv* :1007.2194v1 [hep-th] (Egalement en annexe X)

## LISTE DES ANNEXES

# NOTATION

| | |
|---|---|
| $\mathbb{N}$ | Ensemble des nombres naturels |
| $\mathbb{Q}$ | Ensemble des nombres rationels |
| $[a,b]$ | Intervalle fermé des nombre réels compris entre $a$ et $b$ avec $a \leq b$ |
| $(a,b)$ ou $]a,b[$ | Intervalle ouvert des nombre réels compris entre $a$ et $b$ avec $a \leq b$ |
| $\mathbb{R}$ | Ensemble des nombres réels |
| $\mathbb{C}$ | Ensemble des nombres complexes |
| $\mathscr{C}^\infty(E)$ | Ensemble des fonctions infiniment dérivables sur $E$ |
| $\mathscr{C}_c(E)$ | Ensemble des fonctions continues à support compact sur $E$ |
| $\mathbb{H}_n$ | Ensemble des matrices carrées hermitiennes de taille $n$ |
| $U(n)$ | Ensemble des matrices unitaires de taille $n$ |
| $Sp(2n)$ | Ensemble des matrices symplectiques de taille $2n$ |
| $O(n)$ | Ensemble des matrices orthogonales de taille $n$ |
| $Sym(n)$ | Ensemble des matrices réelles symétriques de taille $n$ |
| $\lambda$ | Vecteur composé de $\lambda_1, ...\lambda_n$. Le nombre de composantes ne sera precisé qu'en cas d'ambiguïté |
| $\Delta(\lambda)$ | Déterminant de Vandermonde des valeurs $\lambda_i$ : $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$ |
| $\mathrm{Tr}(A)$ | Trace de la matrice $A$ |
| $\det(A)$ | Déterminant de la matrice $A$ |
| $\mathrm{Re}(z)$ | Partie réelle du nombre complexe $z$ |
| $\mathrm{Im}(z)$ | Partie imaginaire du nombre complexe $z$ |

| | |
|---|---|
| $\frac{\partial}{\partial x} f$ | Dérivée partielle de la fonction $f$ par rapport à la variable $x$ |
| $\frac{d}{dx} f = f'(x)$ | Dérivée de la fonction $f$ par rapport à son unique variable réelle $x$ |
| $exp(x) = e^x$, $ln(x)$ | Fonctions exponentielles, logarithme Népérien usuelles. La variable $x$ peut être réelle ou complexe. (Dans le cas de $ln$ la coupure est supposée implicitement sur l'axe $\mathbb{R}^-$) |
| $cos(x)$, $sin(x)$, $tan(x)$ | Fonctions trigonométriques usuelles |
| $\delta(x)$ | Distribution de Dirac vérifiant $\forall f \in \mathscr{C}_c^\infty(\mathbb{R})$ : $\int_{\mathbb{R}} f(x)\delta(x)dx = f(0)$ |
| $E(x)$ | Partie entière du nombre réel $x$ |
| Pol$(f(z))$ | Partie polynomiale du développement en série de la fonction $f(z)$ au voisinage de l'infini. |

# CHAPITRE 1

# INTRODUCTION

## 1.1 Définition d'une matrice aléatoire

Historiquement [1, 2], dans le domaine des matrices de corrélation en statistiques multi-variables, le développement de la théorie des matrices aléatoires a connu sa première avancée majeure dans les annéees 1930 grâce à Wishart. A l'époque, leur essor était encore relativement faible et consistait à s'intéresser aux valeurs propres et aux vecteurs propres de certaines matrices dont les entrées obéissaient à différentes distributions de probabilités. L'introduction de la théorie des matrices aléatoires en physique nucléaire eut lieu en 1951 avec Wigner [3] qui eut l'idée d'utiliser ces matrices pour caractériser la statistique des spectres d'excitation des noyaux lourds. Citons ensuite les travaux de Dyson [4] dans les années 1950-1960 puis de Mehta [19] de 1960 aux années 2000 qui ont contribué à faire avancer la théorie des matrices aléatoires jusqu'à leur niveau actuel. Ainsi, dans les années 1930, Wishart s'intéressa aux matrices :

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{n,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \tag{1.1.1}$$

où les composantes $a_{i,j}$ sont des variables aléatoires réelles indépendantes et identiquement distribuées données par une loi de probabilités $p(x)$ :

$$Prob(a_{i,j} \in [a,b]) = \int_a^b p(x)dx \tag{1.1.2}$$

On peut alors définir une mesure de probabilités sur l'ensemble des matrices réelles $A$ en prenant le produit des lois de probabilités des composantes indépendantes :

$$Prob(A/a_{i,j} \in [\alpha_{i,j}, \beta_{i,j}]) = \prod_{i,j=1}^{n} \int_{\alpha_{i,j}}^{\beta_{i,j}} p(x)dx \tag{1.1.3}$$

Le premier calcul de Wigner a été de montrer que pour des entrées Gaussiennes, la répartition des valeurs propres (divisées par $\sqrt{n}$) ainsi que la loi de la plus grande (ou plus petite) valeur propre de ces matrices vont tendre asymptotiquement dans la limite $n \to \infty$ vers des lois de probabilités explicites qui dépendent du type de symétrie de la matrice. Ainsi, historiquement on distingue trois ensembles de matrices différents : les matrices hermitiennes (qui possèdent $n^2$ composantes réelles indépendantes), qui sont invariantes sous l'action du groupe unitaire $U(n)$, les matrices réelles symétriques (qui possèdent $\frac{n(n+1)}{2}$ composantes réelles indépendantes) qui sont invariantes sous l'action du groupe orthogonal $O(n)$ et les matrices quaternioniques réelles self-duales (qui possèdent $n(2n-1)$ composantes réelles indépendantes, Cf. appendice III) qui sont invariantes sous l'action du groupe symplectique $Sp(2n)$. Ces trois ensembles possèdent la propriété que les matrices y sont toujours diagonalisables avec des valeurs propres réelles et que la mesure induite sur l'espace des valeurs propres (c'est à dire apres intégration sur le groupe d'invariance correspondant) peut être mise sous la forme commune :

$$Z = \int_{\mathbb{R}^n} d\lambda_1 \dots d\lambda_n \left( \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^{2\beta} \right) e^{-\beta \sum_{i=1}^{n} \lambda_i^2} \tag{1.1.4}$$

avec $\beta = 1, 1/2, 2$ pour respectivement l'ensemble hermitien, symétrique et quaternionique self-dual. Néanmoins beaucoup d'autres ensembles peuvent être également envisagés : matrices unitaires, orthogonales, ou normales avec des valeurs propres localisées sur un contour fixé.

Une deuxième contribution a ensuite été d'observer que pour des entrées non gaussiennes et potentiellement corrélées, certaines lois obtenues pour le cas d'entrées i.i.d. gaussiennes se maintiennent dans la limite $n \to \infty$ sous certaines hypothèses concernant les lois de probabilités des entrées (décroissance exponentielle, indépendance, etc.). A l'heure actuelle, beaucoup de personnes cherchent à affaiblir les restrictions imposées sur les entrées des matrices (entrées corrélées, distribution de probabilité avec des longues queues, etc.) ou de trouver d'autres lois pour les valeurs propres lorsque les entrées sont distribuées suivant d'autres conditions. On citera entre autres les travaux récents de L. Erdos, A. Guionnet et de K. Johansson ([14–17]) sur ces sujets. Ainsi, on sait désormais

que si les entrées indépendantes sont i.i.d. de moyenne nulle et de variance $\sigma^2$ finie, alors la distribution de probabilités des valeurs propres normalisées aura comme limite la loi du demi-cercle de Wigner lorsque la taille $n$ des matrices tend vers l'infini. On peut illustrer ce résultat avec des entrées gaussiennes :

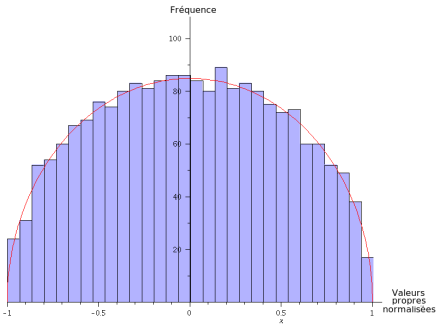


Figure 1 : Histogramme des valeurs propres (divisées par $\sqrt{n}$) d'une matrice $100 \times 100$. La courbe rouge représente la loi théorique du demi-cercle de Wigner.

D'un point de vue mathématique, le résultat peut être exprimé ainsi par :

$$\lim_{n\to\infty} \mu_{A\in \mathrm{Sym}(n)}(x) = \frac{2}{\pi}\sqrt{1-x^2} \tag{1.1.5}$$

où $\mu_{A\in \mathrm{Sym}(n)}(x)$ est la mesure de probabilité empirique des valeurs propres :

$$\mu_{A\in \mathrm{Sym}(n)}(x) = \frac{1}{n}\sum_{i=1}^{n} \delta(x - \frac{\lambda_i}{2\sqrt{n}}) \tag{1.1.6}$$

En physique, il est souvent plus intéressant de regarder la répartition entre les valeurs propres (niveaux d'énergie) consécutives dans le coeur de la distribution. Cette répartition est connue théoriquement comme nous le verrons par la suite (Chapitre 2, section 6, équation 2.6.4) pour les modèles hermitiens, symétriques réels et quaternioniques self-duaux et donne lieu à un phénomène d'universalité. Pour le cas des matrices symétriques réelles, on obtient la figure suivante :

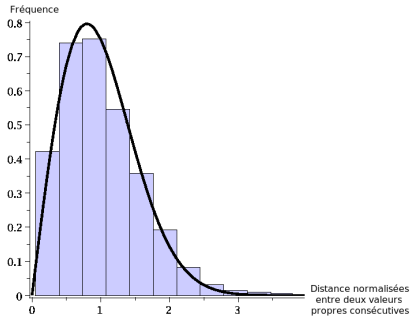


Figure 2 : Histogramme des écarts ($\sqrt{n}(\lambda_{i+1} - \lambda_i)$) entre les valeurs propres consécutives d'une matrice symétrique réelle gaussienne $1000 \times 1000$. La courbe noire représente la loi théorique.

Cela dit, si la loi de Wigner regroupe beaucoup de lois de distribution pour les entrées, elle n'est aucunement universelle puisque des entrées avec des lois de probabilité n'obéissant pas aux règles énoncées ci-dessus vont donner des distributions de valeurs propres bien différentes. Par exemple, si les entrées sont des variables de Cauchy (dont la loi est $\frac{1}{\pi(x^2+1)}$) la distribution est supportée sur $\mathbb{R}$ tout entier, et peut être illustrée par :

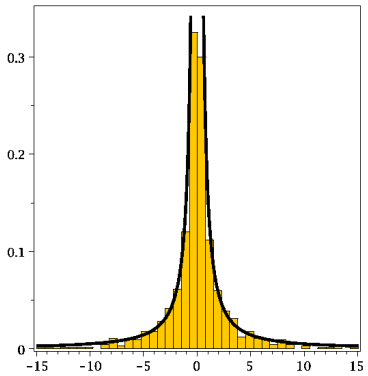


Figure 3 : Histogramme des valeurs propres (normalisées par $1/n$) d'une matrice symétrique $100 \times 100$ avec des i.i.d. suivant la distribution de Cauchy. La courbe noire représente la loi $x \mapsto \frac{1}{2\pi x^{3/2}}$.

On voit donc que l'on est très loin de la loi du demi-cercle, en particulier, la distribution limite obtenue n'est pas normalisable sur $\mathbb{R}$, ne définissant pas mathématiquement une mesure de probabilités.

## 1.2 Lien historique entre les matrices aléatoires et la physique nucléaire

Les matrices aléatoires ont été introduites par Wigner dans les années 1960 pour expliquer le spectre des noyaux lourds d'uranium qui apparait incroyablement complexe et difficilement résoluble de façon analytique au vu de la complexité et du grand nombre d'interactions présentes au sein d'un noyau. Ainsi pour l'uranium, qui contient plus de 200 protons et neutrons obéissant aux règles complexes des interactions nucléaires, un calcul des différents états d'énergie est impossible analytiquement et compliqué numériquement (surtout en 1960). En revanche, dès les années 1950, la construction d'accélérateurs de particules de hautes énergies permettait l'exploration partielle expérimentale de ces niveaux en bombardant un atome d'uranium avec un neutron accéléré, et des résultats expérimentaux étaient déjà disponibles. A l'époque, la grande majorité des physiciens pensait que les différents écarts entre niveaux de résonance consécutifs devaient être répartis selon une distribution de Poisson :

$$x = \rho S = \frac{S}{< S >} \qquad , \qquad p(x)dx = e^{-x}dx \tag{1.2.1}$$

où $\rho$ représente la densité d'état et $< S >$ l'écart moyen. Mais les mesures expérimentales imprécises ne permettaient pas de valider ou d'invalider une telle distribution. Wigner eut alors l'idée de proposer son modèle aléatoire qui donne lieu dans le cas des matrices réelles symétriques à la loi approchée :

$$x = \rho S = \frac{S}{< S >} \qquad , \qquad p(x)dx \sim \frac{\pi}{2} x e^{-\frac{\pi}{4}x^2} dx \tag{1.2.2}$$

Vers les années 1960, l'amélioration des accélérateurs permit des expériences plus précises qui tranchèrent entre les différentes lois proposées et donna raison aux modèles matriciels de Wigner. Cela est illustré dans le graphique suivant, tiré de [5] et de [6].

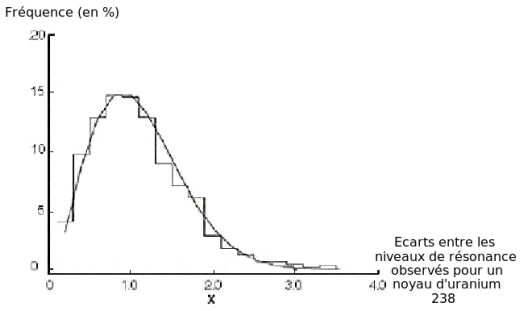


Figure 4 : Une distribution de Wigner avec la distribution des écarts entre niveaux de résonances d'un atome d'uranium 238 pour des énergies allant jusqu'à 20 keV. En abscisse se trouve l'énergie considérée et en ordonnées la probabilité d'avoir une résonance pour cette énergie donnée.

Depuis Wigner, il est connu que suivant le type de symétries du système étudié, la distribution provient de différents ensembles de matrices. Par exemple, des systèmes présentant une invariance par rotation et une invariance par renversement du temps se verront attachés aux modèles de matrices réelles symétriques (Gaussian orthogonal ensemble) tandis que ceux pour lesquels l'invariance par renversement du temps n'est pas valable sont attachés aux modèles de matrices hermitiennes (Gaussian unitary ensemble). Notons que beaucoup d'autres ensembles ont depuis été étudiés, comme par exemple les ensembles des matrices unitaires ou orthogonales dont les lois de distribution des valeurs propres sont également bien connues.

## 1.3 Matrices aléatoires et autres domaines des mathématiques

Bien qu'historiquement introduites pour la physique nucléaire, les matrices aléatoires n'ont cessé de se retrouver dans un nombre croissant de domaines à la fois appliqués et théoriques. Coté applications, on peut ainsi mentionner le repliement des brins d'ARN ou de protéines ([9], [10], [11], [13]) et de nombreuses applications en traitement du signal [12] et dans la théorie des cordes topologiques (Cf. chapitre 5). Dans le domaine des mathématiques, les intégrales matricielles qui font l'objet de cette thèse sont reliées à de nombreux problèmes : systèmes intégrables, polynômes orthogonaux, problèmes de

Riemann-Hilbert, combinatoires de cartes et même de manière assez surprenante théorie des nombres. En effet, il semblerait qu'il existe un lien incompris entre les matrices aléatoires et les zéros non triviaux de la fonction $\zeta$ de Riemann. Ce lien est illustré par la ressemblance frappante entre la distribution de Wigner 1.2 et celles des zéros de la fonction de Riemann sur la droite $\mathrm{Re}(z) = 1/2$ :

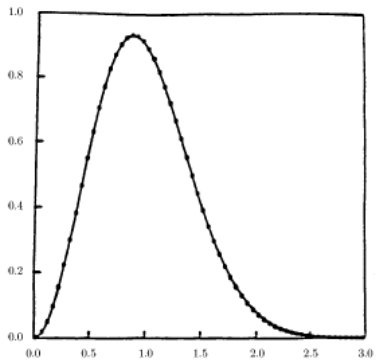


Figure 5 : Distribution des écarts de 70 millions de zéros consécutifs de la fonction $\zeta(s)$ de Riemann (partant du $1020^{ime}$) zéro). Graphique extrait de [5], [7], [8]

Ainsi, il semblerait que la distribution des zéros non triviaux de la fonction de Riemann obéissent à une loi des matrices aléatoires bien que le lien entre les deux théories soit encore aujourd'hui mystérieux. Citons également d'autres domaines dans lesquels des lois de matrices aléatoires ont pu être observées : la percolation, l'atome d'hydrogène dans un champ magnétique intense, le dénombrement de certaines familles de graphes, la chromodynamique quantique (QCD), l'étude des partitions planes,... Ainsi les matrices aléatoires, par leurs applications dans de nombreux domaines connaissent à l'heure actuelle un développement important et dans de nombreuses directions. Cette thèse sera le reflet de cette diversité puisqu'elle abordera plusieurs méthodes spécifiques permettant de traiter le problème complexe des intégrales de matrices aléatoires.

# CHAPITRE 2

# INTÉGRALES DE MATRICES ET DENSITÉ DE VALEURS PROPRES

## 2.1 Définition des intégrales de matrices

Dans l'introduction, nous avons vu que les valeurs propres de matrices aléatoires de certains ensembles obéissent à des lois simples lorsque la taille de la matrice devient grande. Cette première étape intéressante est néanmoins limitée par le fait qu'en physique statistique, les configurations d'un système sont souvent contraintes par un potentiel d'interaction. Il est donc logique, par analogie avec la physique statistique de Boltzmann, d'introduire les intégrales de matrices suivantes ou fonctions de partition :

$$Z_N(T) = \int_{\mathbb{E}_N} dM e^{-\frac{N}{T}\mathrm{Tr}(V(M))} \tag{2.1.1}$$

où $\mathbb{E}_N$ désigne un ensemble de matrices de taille $N$ (par exemple hermitiennes, symétriques réelles,...), $T$ désigne la température du modèle, $dM$ correspond aux produits des mesures de Lebesgue des entrées réelles indépendantes et $V(x)$ est le potentiel associé au modèle de matrice étudié.

D'un point de vue probabiliste, toutes les matrices $M$ n'ont désormais plus la même probabilité d'apparition, les matrices telles que $\mathrm{Tr}(V(M))$ est minimal devenant ainsi bien plus probables que les autres, correspondant ainsi à une centralisation préférentielle autour des configurations d'énergies minimales. Par analogie avec les systèmes statistiques, la quantité $\frac{N}{T}\mathrm{Tr}(V(M))$ est appelée l'action du modèle et les contributions les plus importantes à l'intégrale sont donc les matrices qui minimisent cette action. On peut ainsi résumer les probabilités comme :

$$\mathrm{Prob}\left(M/m_{i,j} \in [a_{i,j}, a_{i,j} + da_{i,j}]\right) = \frac{1}{Z_N} e^{-\frac{N}{T}\sum_{i=1}^{N}(V(A))_{i,i}} \left(\prod_{i,j} da_{i,j}\right) \tag{2.1.2}$$

ou encore :

$$\text{Prob}\left(M/m_{i,j} \in [a_{i,j}, b_{i,j}]\right) = \frac{1}{Z_N} \int_{a_{1,1}}^{b_{1,1}} \dots \int_{a_{n,n}}^{b_{n,n}} dm_{1,1} \dots dm_{n,n} e^{-\frac{N}{T} \sum_{i=1}^{N} (V(M))_{i,i}} \quad (2.1.3)$$

Notons que selon l'ensemble des matrices considéré, le nombre de composantes réelles indépendantes sur lesquelles on réalise l'intégration peut changer.

## 2.2 Diagonalisation, problème aux valeurs propres

Dans le cas des ensembles de matrices hermitiennes, symétriques réelles ou quaternioniques self-duales, l'invariance de l'action (grâce à la présence de la trace) sous le groupe unitaire, orthogonal ou symplectique permet d'effectuer l'intégration sur les variables "angulaires" et de ramener le problème à celui des valeurs propres des matrices. Cette diagonalisation n'est cependant pas complètement triviale car le Jacobien de la transformation n'est pas évident a priori. Ces diagonalisations, connues depuis longtemps, peuvent être trouvés dans [19] pour chacun des trois cas et peuvent se résumer ainsi :

$$M = U\Lambda U^{-1} \Rightarrow Z_N \propto \int_{\mathbb{R}^N} d\lambda_1 \dots d\lambda_N \, \Delta(\lambda)^{2\beta} e^{-\frac{N}{T} \sum_{i=1}^{N} V(\lambda_i)} \quad (2.2.1)$$

où $\Lambda = diag(\lambda_1, \dots, \lambda_N)$, $U$ est une matrice unitaire, orthogonale ou symplectique suivant l'ensemble initial considéré (respectivement hermitiens, symétriques réels et quaternioniques self-duaux). Le paramètre $\beta$ vaut respectivement 1, $\frac{1}{2}$ et 2 suivant les ensembles initiaux considérés (respectivement hermitien, symétrique réel et quaternionique). Enfin, les coefficients de proportionnalité peuvent être exprimés facilement dans les trois cas en prenant le potentiel $V(x)$ quadratique. Ces coefficients ne dépendent que de $N$ et peuvent être trouvés explicitement dans [19] par l'intermédiaire de formules exactes dans le formalisme des polynômes (skew) orthogonaux (le cas hermitien se réduisant ainsi aux polynômes de Hermite). Rappelons également que $\Delta(\lambda)$ désigne le déterminant de Vandermonde associé aux valeurs propres $(\lambda_i)_{i=1\dots N}$.

Note : Nous utiliserons dans cette thèse les conventions dites du "gaz de Coulomb"

pour l'exposant $\beta$, utilisées notamment par Laughlin. Cette convention diffère ainsi d'un facteur 2 par rapport à la notation plus courante de Wigner et de Mehta dans la littérature. Ainsi, dans la notation de Wigner, l'exposant du déterminant de Vandermonde n'est pas précédé d'un facteur 2, le cas hermitien correspondant alors à $\beta_{\text{lit}} = 2$. L'intérêt principal de notre convention apparaîtra plus tard lors de l'étude des modèles de matrices où le paramètre $\beta$ est quelconque.

Si le cas des matrices hermitiennes, symétriques réelles et quaternioniques self-duales est intéressant, il ne constitue cependant pas le cas le plus général. En effet, il est facile de généraliser les cas ci-dessus pour des ensembles de matrices normales (i.e. qui commutent avec leur adjoint et qui sont donc diagonalisables sur une base orthonormale de vecteurs propres) dont les valeurs propres sont assujeties à être situées sur un contour $\mathscr{C}$ fixé du plan complexe. Enfin, on peut également choisir d'étudier directement une version de 2.2.1 dans lequel le paramètre $\beta$ est arbitraire, même si pour le cas où $\beta$ est quelconque, il n'existe pas d'ensemble de matrices simples connus à ce jour qui reproduisent une telle mesure de probabilité pour les valeurs propres. Ainsi, la version plus générale du modèle diagonalisé que nous étudierons dans cette thèse est :

$$Z_N = \int_{\mathscr{C}^N} d\lambda_1 \dots d\lambda_N \, \Delta(\lambda)^{2\beta} e^{-\frac{N}{T}\sum_{i=1}^{N} V(\lambda_i)} \tag{2.2.2}$$

## 2.3 Distribution des valeurs propres : mesure d'équilibre

La première question qui vient à l'esprit lorsque l'on regarde le cas des matrices hermitiennes :

$$Z_N = \int_{\mathbb{R}^N} d\lambda_1 \dots d\lambda_N \, \Delta(\lambda)^2 e^{-\frac{N}{T}\sum_{i=1}^{N} V(\lambda_i)} = \int_{\mathbb{R}^N} d\lambda_1 \dots d\lambda_N \, e^{-\frac{N}{T}\sum_{i=1}^{N} V(\lambda_i) + 2\sum_{i<j} \ln(|\lambda_i - \lambda_j|)} \tag{2.3.1}$$

est de se demander si la distribution des valeurs propres correctement normalisées et soumises à l'action $-\frac{N}{T}\sum_{i=1}^{N} V(\lambda_i) + 2\sum_{i<j} \ln(|\lambda_i - \lambda_j|)$ va suivre une distribution de probabilité simple lorsque $N \to +\infty$ comme dans le cas de la loi du demi-cercle de Wigner. D'un point de vue physique, l'action effective précédente subie par les valeurs propres

possède deux contributions évoluant en sens opposés : une force de type Coulombienne $\frac{1}{(\lambda_i-\lambda_j)}$ provoquant une répulsion à courte distance entre les valeurs propres (analogue à celle de charges ponctuelles de même signe en électrostatique) et un terme potentiel $-V(\lambda_i)$ poussant les valeurs propres vers le minimum ou les minima du potentiel $V$. Si intuitivement, on pressent qu'un équilibre entre l'attraction par le puit de potentiel et la répulsion à courte distance va aboutir à une configuration stable et prédéfinie, la réponse définitive à cette question a été apportée par [20] et [21] et se formule ainsi :

**Théorème 2.1.** *Soit $V(x)$ un potentiel polynômial de degré pair. Soit $d\nu_N(x) = \rho_N(x)dx$ la distribution des valeurs propres sur l'axe réel donnée par :*

$$\forall \phi(x) \in \mathscr{C}_c^\infty(\mathbb{R}) : \int_{\mathbb{R}} \phi(x) d\nu_N(x) = \int_{\mathbb{R}^N} \left[ \frac{1}{N} \sum_{j=1}^{N} \phi(\lambda_j) \right] \frac{1}{Z_N} e^{-\frac{N}{T} \sum_{i=1}^{N} V(\lambda_i) + 2 \sum_{i<j} \ln(|\lambda_i - \lambda_j|)} \prod_{i=1}^{N} d\lambda_i$$

*Soit $d\nu(x;\lambda)$ la mesure discrète localisée aux $\lambda_j$ :*

$$d\nu(x;\lambda) = \frac{1}{N} \sum_{j=1}^{N} \delta(x - \lambda_j) dx$$

*Alors ces deux mesures admettent une limite commune notée $d\nu_{eq}(x)$ lorsque $N \to \infty$. Cette mesure appelée mesure d'équilibre est supportée par un nombre fini d'intervalles $[a_i, b_i]$ et est absolument continue par rapport à la mesure de Lebesgue : $d\nu_{eq}(x) = \rho_{eq}(x)dx$ avec*

$$\rho_{eq}(x) = \frac{1}{2\pi} h(x) R^{\frac{1}{2}}(x) \prod_{i=1}^{q} \mathbb{1}_{[a_i,b_i]}(x) \ , \ R(x) = \prod_{i=1}^{q} (x - a_i)(b_i - x) \tag{2.3.2}$$

*et $h(x)$ est un polynôme de degré $deg(h) = deg(V) - q - 1$. Le support de la distribution ainsi que la densité $\rho_{eq}(x)$ sont entièrement déterminés par les contraintes :*

$$V'(z) = \underset{z\to\infty}{Pol} \left( h(z) R^{\frac{1}{2}}(z) \right) , \ \underset{z\to\infty}{\text{Res}}\, h(z) R^{\frac{1}{2}}(z) = -2 \tag{2.3.3}$$

*et*

$$\int_{b_i}^{a_{i+1}} h(z)R^{\frac{1}{2}}(z)dz = 0 \ , \ \forall 1 \leq i \leq q-1 \tag{2.3.4}$$

*où la notation* $\underset{z\to\infty}{Pol}(f(z))$ *signifie la partie polynômiale du développement en série de Laurent de la fonction* $f(z)$ *à l'infini.*

Notons que 2.3.3 est équivalent à :

$$h(z) = \underset{z\to\infty}{Pol}\left(\frac{V'(z)}{R^{\frac{1}{2}}(z)}\right) \tag{2.3.5}$$

Remarquons également que les contraintes 2.3.3 et 2.3.4 restent difficiles à utiliser en pratique. D'abord, elles ne déterminent pas le nombre d'intervalles $q$ de façon immédiate. Il faut ainsi faire une hypothèse sur la valeur de $q$ puis tenter de satisfaire 2.3.3 et 2.3.4 et si cela n'est pas possible, postuler une autre valeur de $q$ et recommencer. D'autre part, ces contraintes sont fortement non-linéaires et étant donné un potentiel $V(x)$, il est quasiment impossible de déterminer analytiquement la distribution $\rho_{eq}(x)$ ou les extrémités des intervalles. En revanche, il est très facile en utilisant 2.3.3 de trouver un potentiel associé à une distribution d'équilibre $\rho_{eq}(x)$ donnée.

Enfin, il n'est pas évident a priori que les conditions 2.3.3 et 2.3.4 aboutissent à une densité de probabilité (qui, rappelons le, doit être positive et d'intégrale totale égal à un). Il est également habituel de distinguer les cas où la mesure d'équilibre s'annule sur son support :

**Définition 2.1.** *La mesure d'équilibre* $d\nu_{eq}(x)$ *est dite régulière (sinon singulière) si elle est strictement positive sur chacun de ses intervalles* $]a_i, b_i[$ *et si* $\forall 1 \leq i \leq q$ : $\rho_{eq}(x) \underset{x\to a_i}{\sim} \sqrt{x-a_i}$ *et* $\rho_{eq}(x) \underset{x\to b_i}{\sim} \sqrt{b_i - x}$ *. Dans le cas où la mesure est singulière, le potentiel* $V(x)$ *associé est dit critique.*

## 2.4 Simulations et exemples

Le résultat précédent peut se visualiser très bien avec des simulations numériques pour le cas des matrices hermitiennes. En effet, bien qu'il soit impossible de calculer

la fonction de partition $Z_N$ pour de grandes valeurs de $N$, il est en revanche possible de simuler des tirages de valeurs propres $(\lambda_1, \ldots, \lambda_N)$ suivant la loi 2.3.1 par l'algorithme de Metropolis-Hastings ou par des méthodes de Monte-Carlo. Ainsi, on peut simuler le comportement de la densité d'équilibre associée au potentiel (le paramètre $\varepsilon$ mesure l'écart par rapport au cas quartique donné par $\varepsilon = 1/2$)

$$V(x,T,\varepsilon) = \frac{1}{T}\left(\frac{x^4}{4} - \frac{4\cos(\pi\varepsilon)x^3}{3} + \cos(2\pi\varepsilon)x^2 + 8\cos(\pi\varepsilon)x\right) \tag{2.4.1}$$

Ce potentiel a été étudié dans [22] et devient critique pour la valeur $T_c = 1 + 4\cos(\pi\varepsilon)$ où la densité est alors explicitement connue :

$$\rho_c(x) = \frac{1}{2\pi T_c}(x - 2\cos\pi\varepsilon)^2\sqrt{4-x^2} \tag{2.4.2}$$

Pour $T > T_c$, il peut être montré que la densité d'équilibre a un support réduit à un intervalle, alors que pour $T < T_c$, son support est constitué de deux intervalles disjoints. En utilisant l'algorithme présenté en annexe II sur un ordinateur portable standard équipé du logiciel Maple 13, on peut ainsi obtenir pour $\varepsilon = 1/2$ et $N = 200$ :

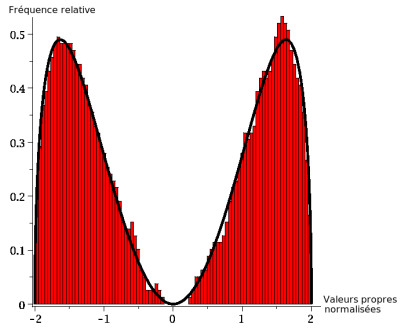


Figure 6 : Histogramme des valeurs propres obtenues par l'algorithme de Metropolis-Hastings pour le potentiel 2.4.1 à $T = T_c$. La courbe noire représente la densité 2.4.2. L'échelle de l'axe vertical est choisie pour que l'aire sous l'histogramme soit égale à un (fréquence relative).

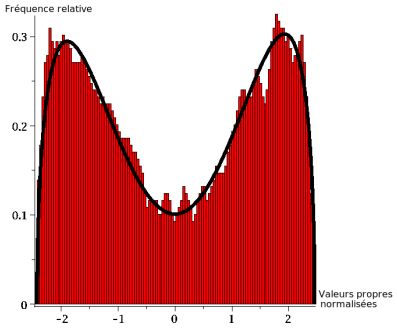


Figure 7 : Histogramme des valeurs propres obtenues par l'algorithme de Metropolis-Hastings pour le potentiel 2.4.1 à $T = 2T_c$. La courbe noire représente la densité donnée par 2.3.2 où les extrémités $a_1$ et $b_1$ sont déduites de la simulation.

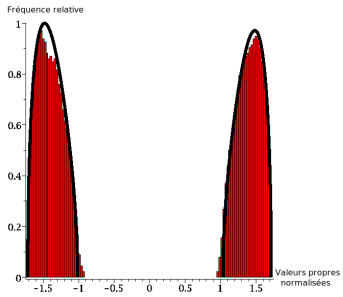


Figure 8 : Histogramme des valeurs propres obtenues par l'algorithme de Metropolis-Hastings pour le potentiel 2.4.1 à $T = 0.5T_c$. La courbe noire représente la densité donnée par 2.3.2 où les extrémités $a_1, a_2, b_1, b_2$ sont déduites de la simulation.

Cette méthode de simulation permet d'obtenir rapidement les histogrammes des valeurs propres des modèles à une matrice étant donné n'importe quel potentiel. En particulier, il est alors extrêmement facile de trouver numériquement les extrémités $a_i$ et $b_i$ des intervalles supportant la distribution et d'en déduire alors grâce à 2.3.5 la densité associée.

Note : Si l'algorithme de Metropolis-Hastings peut être utilisé pour simuler la répartition des valeurs propres, il ne peut pas être utilisé pour calculer directement la fonction de partition $Z_N$ (qui se simplifie à chaque étape de l'algorithme). Le calcul numérique de

$Z_n$ se révèle être lui particulièrement difficile dès que $N > 2$ à cause de la "malédiction des dimensions" ("curse of dimensionnality") qui demande alors une puissance de calcul très importante.

Note 2 : Dans le cas d'un potentiel quadratique $V(z) = \frac{z^2}{2}$, la densité des valeurs propres obtenue correspond à la loi du demi-cercle de Wigner.

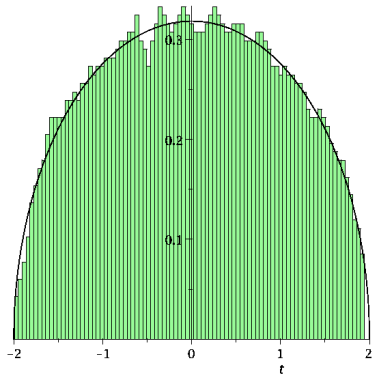


Figure 9 : Histogramme des valeurs propres obtenues par l'algorithme de Metropolis-Hastings pour le potentiel $V(z) = \frac{z^2}{2}$ et $T = 1$. En noir, est représentée la loi du demi-cercle $\frac{1}{2\pi}\sqrt{4-x^2}$

## 2.5 Les fonctions de corrélation à $n$-points et l'universalité

Au delà de la fonction de partition ou de la densité d'équilibre des valeurs propres, il est intéressant de connaître les corrélations entre les valeurs propres lorsque la taille des matrices devient grande. On définit ainsi les fonctions de corrélations à $n$-points dans le cas hermitien (Dyson 1962) par :

**Définition 2.2.** *Les fonctions de corrélation non-connexes entre les valeurs propres sont définies par :*

$$\rho_n(\lambda_1,\ldots,\lambda_n) = \frac{N!}{Z_N(N-n)!}\int_{\mathbb{R}}\ldots\int_{\mathbb{R}} d\lambda_{n+1}\ldots d\lambda_N\, e^{-\frac{N}{T}\sum_{i=1}^{N} V(\lambda_i)+2\sum_{i<j}\ln(|\lambda_i-\lambda_j|)} \tag{2.5.1}$$

Ces fonctions représentent la densité de probabilité de trouver des valeurs propres en $\lambda_1,\ldots,\lambda_n$, la position des autres valeurs propres restant non-observées (libres). En

particulier, la fonction $\rho_1(x)$ redonne la densité des valeurs propres étudiée dans la section précédente (et dont la limite $N \to +\infty$ est donnée par la mesure d'équilibre 2.3.2). Pour $n > 1$, ces fonctions traduisent les corrélations existantes entre les valeurs propres et il est intéressant de se demander si ces fonctions admettent une expression particulière dans la limite où $N \to +\infty$ ou dans la limite $n, N \to \infty$ avec $\frac{n}{N}$ fixé.

Le résultat majeur, connu sous le résultat d'**universalité** est alors le suivant :

**Théorème 2.2.** *Pour les modèles de matrices hermitiennes, symétriques réelles et quaternioniques self-duales, les fonctions de corrélations non-connexes à n-points ($n > 1$) à petite distance (i.e. d'ordre $1/N$) sont indépendantes du potentiel polynomial pair $V(z)$. En particulier, elles peuvent être calculées par le potentiel gaussien $V(z) = z^2$. Par ailleurs, la connaissance de la fonction de corrélation à 2-points est suffisante pour déterminer les autres fonctions de corrélations à l'aide de formules déterminantales. (Cf. [19] pour les formules déterminantales spécifiques des trois ensembles) Ainsi, les fonctions à 2-points non-connexes dans le coeur de la distribution sont données en notant $r = N|\lambda_1 - \lambda_2|\rho(\lambda_1)$ par :*

$$\begin{aligned}
\textit{Hermitien} : W_2(r) &= 1 - \left(\frac{\sin(\pi r)}{\pi r}\right)^2 \\
\textit{Réel symétrique} : W_2(r) &= 1 - \left(\frac{\sin(\pi r)}{\pi r}\right)^2 - \left(\int_r^\infty \frac{\sin(\pi s)}{\pi s}\, ds\right) \frac{d}{dr}\frac{\sin(\pi r)}{\pi r} \\
\textit{Quaternionique} : W_2(r) &= 1 - \left(\frac{\sin(2\pi r)}{2\pi r}\right)^2 + \left(\int_0^r \frac{\sin(2\pi s)}{2\pi s}\, ds\right) \frac{d}{dr}\frac{\sin(2\pi r)}{2\pi r}
\end{aligned} \tag{2.5.2}$$

qui peuvent être représentées graphiquement :

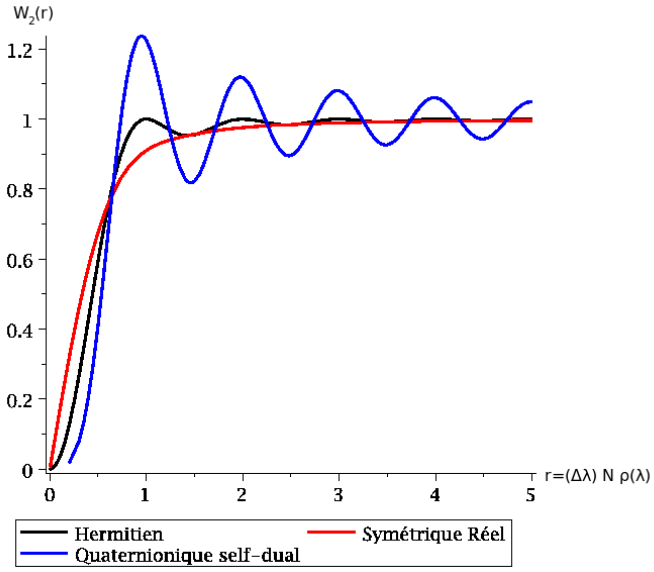


Figure 10 : Graphe des fonctions universelles à deux points pour le cas des ensembles de matrices hermitiennes, symétriques réelles et quaternioniques self-duales.

En ce qui concerne les modèles de matrices généraux où l'exposant $\beta$ est arbitraire, le théorème précédent n'est pas acquis. En particulier, l'existence d'un phénomène d'universalité ou de formules permettant de déduire les fonctions de corrélation à $n$-points à partir de celle à 2-points n'est pas connu à l'heure actuelle.

## 2.6 Universalité et premier lien avec les systèmes intégrables

Soit $E_\beta(J)$ la probabilité qu'aucune valeur propre ne soit dans l'intervalle $J$, alors dans le cas des matrices hermitiennes, symétriques réelles et quaternioniques self-duales, il est connu (Gaudin 1961 pour les matrices hermitiennes et des entrées gaussiennes, Mehta 1971 pour les matrices réelles symétriques et quaternioniques self-duales) que $E_\beta(J)$ peut s'exprimer à l'aide d'un déterminant de Fredholm (Cf. annexe IV pour la définition générale d'un déterminant de Fredholm) :

$$E_\beta(J) = \det(Id - K_\beta(J)) \tag{2.6.1}$$

où $K_\beta(J)$ est un opérateur intégral agissant sur $J$ exprimé sous la forme :

$$\frac{\phi(y)\psi(x)-\psi(y)\phi(x)}{y-x} \tag{2.6.2}$$

avec $\psi(x)$ et $\phi(x)$ les polynômes orthogonaux d'ordre $n$ et $n-1$ du système (Cf. Chapitre 3 pour la définition des polynômes orthogonaux). Notons également qu'à partir de la connaissance de $E_\beta(J)$, on peut facilement par dérivation par rapport aux bords de l'intervalle $J$ obtenir les densités de valeurs propres ainsi que diverses autres quantités à divers endroits de la distribution (le coeur ou une des extrémités de la distribution). Le second intérêt de pouvoir exprimer $E_\beta(J)$ à l'aide d'un déterminant de Fredholm est qu'il devient possible d'en extraire des limites lorsque $n$ devient grand. Ainsi, dans le cas des matrices hermitiennes, le noyau $K_\beta(J)$ tend asymptotiquement dans le coeur de la distribution vers le noyau "sinus" ("sine kernel") :

$$K_{\beta=2}(J) \underset{n\to\infty}{\to} \frac{sin\,\pi(x-y)}{\pi(x-y)} \tag{2.6.3}$$

En particulier, on peut alors obtenir la loi des écarts relatifs entre les valeurs propres (normalisées) par :

$$p_{\text{écarts relatifs}} = \frac{d^2}{ds^2}E_\beta(J=[0,s]) \tag{2.6.4}$$

connu sous le nom de distribution de Gaudin. En ce qui concerne le voisinage des extrémités du support de la distribution (par exemple la plus grande valeur propre), le noyau a cette fois-ci pour limite le noyau d'Airy. En posant :

$$\lambda_{max} = 2\sqrt{T}\sqrt{n} + \frac{\hat{\lambda}}{n^{1/6}} \tag{2.6.5}$$

alors la loi de $\hat{\lambda}$ tend asymptotiquement vers la loi dite d'Airy :

$$\text{prob}(\hat{\lambda} \le s) \underset{n\to\infty}{\to} \frac{d}{ds}\Big[\det(Id - K_{Airy}([s,+\infty[))\Big] \tag{2.6.6}$$

où $K_{Airy}(J)$ est donné par 2.6.2 avec $\psi(x) = Airy(x)$ et $\phi(x) = Airy'(x)$.

Si l'expression en termes de déterminant de Fredholm est intéressante, elle n'est en général pas facilement manipulable et se prête difficilement à des analyses numériques. Heureusement en 1980, dans leur célèbre article [67], Jimbo, Miwa, Môry et Sato ont obtenu une représentation du noyau sinus en termes de solution d'une équation de Painlevé, faisant ainsi le lien avec les systèmes intégrables. Ainsi, on a :

**Théorème 2.3.** *Représentation du noyau sinus à l'aide de l'équation de Painlevé V ([67]) :*

$$\det(Id - \lambda K_\beta([0,s[)) = exp(\int_0^{\pi s} \frac{\sigma(x,\lambda)}{x} dx) \tag{2.6.7}$$

*où $\sigma(x,\lambda)$ est l'unique solution de l'équation (cas particulier de l'équation de Painlevé V) différentielle :*

$$(x\sigma''(x,\lambda))^2 + 4\left(x\sigma'(x,\lambda) - \sigma(x,\lambda)\right)\left(x\sigma'(x,\lambda) - \sigma(x,\lambda) + (\sigma'(x,\lambda))^2\right) = 0 \tag{2.6.8}$$

*avec $\sigma(x,\lambda) \underset{x\to 0}{\to} -\frac{\lambda}{\pi}x - \frac{\lambda^2}{\pi^2}x^2$.*

De la même façon, on sait désormais que le noyau d'Airy est relié à l'équation de Painlevé II [65, 66] :

**Théorème 2.4.** *Représentation du noyau d'Airy à l'aide de l'équation de Painlevé II ([65]) :*

$$\det(Id - K_{Airy}([s,+\infty[) = exp(-\int_s^\infty (x-s)q(x)dx) \tag{2.6.9}$$

*où $\sigma(x)$ est l'unique solution de l'équation de Painlevé II :*

$$q''(x) = xq(x) + 2q(x)^3 \qquad , \qquad q(x) \underset{x\to\infty}{\sim} Ai(x) \tag{2.6.10}$$

Grâce à ces représentations différentielles, il est alors possible (bien que numériquement les équations de Painlevé soient assez instables) de comparer les lois théoriques des écarts relatifs entre les valeurs propres dans le coeur ainsi que la distribution de la plus grande valeur propre à des simulations numériques :

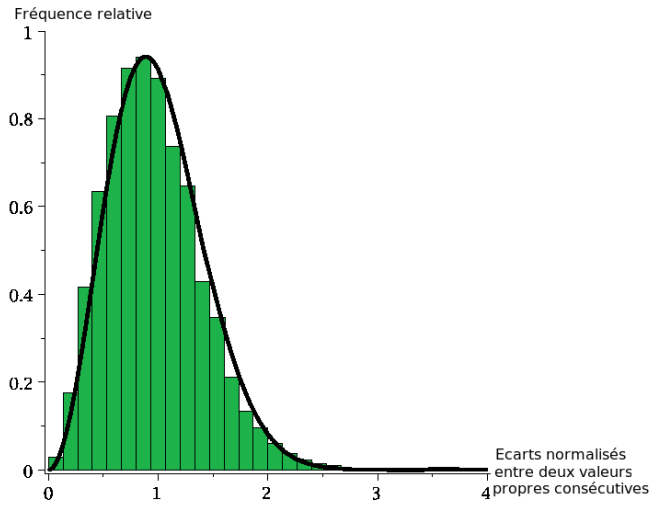


Figure 11 : Histogramme des écarts normalisés de deux valeurs propres consécutives de 500 matrices hermitiennes de taille $300 \times 300$ (entrées gaussiennes) avec la distribution théorique de Gaudin 2.6.4.

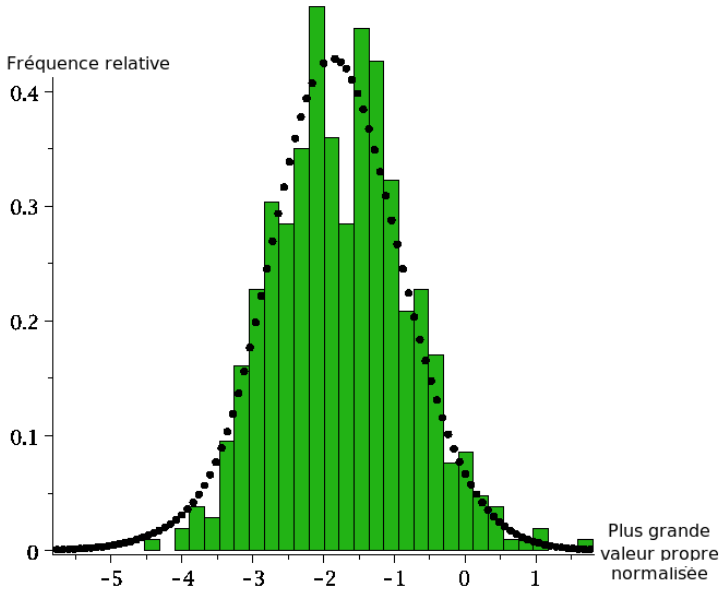


Figure 12 : Histogramme des plus grandes valeurs propres normalisées de 500 matrices hermitiennes de taille $300 \times 300$ (entrées gaussiennes) avec la distribution théorique 2.6.6.

Ou bien encore avec la fonction de répartition (pour éviter d'avoir à dériver) :

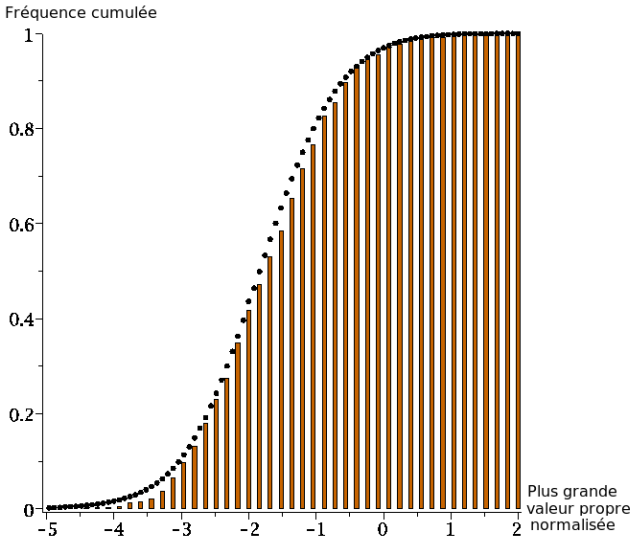


Figure 13 : Histogramme cumulé des plus grandes valeurs propres normalisées de 500 matrices hermitiennes de taille $300 \times 300$ (entrées gaussiennes) avec la fonction de répartition théorique 2.6.6.

Notons également que dans le cas des matrices réelles symétriques et quaterniones self-duales, des expressions similaires en termes de déterminant de Fredholm et de solutions de Painlevé sont connues [19, 64] donnant en particulier les densités au voisinage de la plus grande valeur propre ainsi que dans le coeur de la distribution. Enfin, le point le plus important est que ces expressions satisfont à un théorème d'universalité ([19, 64]) :

**Théorème 2.5.** *La loi de Gaudin 2.6.4 reste valable quels que soient les potentiels polynomiaux pairs $V(x)$ en tout point situé dans le coeur du support de la distribution des valeurs propres (i.e. loin des extrémités du support) où la densité de probabilité est strictement positive. La loi d'Airy reste également valide quels que soient les potentiels polynomiaux pairs $V(x)$ pour les valeurs propres situées dans un voisinage d'une des extrémités, notée a, du support de la distribution des valeurs propres, sous la condition que la densité s'y comporte localement comme $\rho(x) \underset{x\to a}{\sim} \sqrt{x-a}$ (extrémité dite régulière).*

Le cas des extrémités non régulières ou de points où la densité s'annule sont des cas singuliers et ont constitué une partie de mon travail de thèse (Cf. **[I]** à l'annexe V). Les résultats à ce sujet seront présentés dans le prochain chapitre où l'on verra que les points $a$ où la densité s'annule comme $(x-a)^{2m}$ sont en relation avec d'autres solutions d'équations de Painlevé.

## 2.7 Résolvantes et développement topologique dans le cas hermitien

Afin de calculer les fonctions de corrélations à $n$-points, il est utile de définir les *résolvantes* par :

**Définition 2.3.** *Les résolvantes sont définies par :*

$$\hat{\omega}_n(x_1,\ldots,x_n) = \left\langle \mathrm{Tr}\frac{1}{x_1-M}\ldots\mathrm{Tr}\frac{1}{x_n-M} \right\rangle = \left\langle \sum_{i_1,\ldots,i_n=1}^{N} \frac{1}{x_1-\lambda_{i_1}}\ldots\frac{1}{x_n-\lambda_{i_n}} \right\rangle \quad (2.7.1)$$

Ici, la notation $<>$ désigne la valeur moyenne selon la distribution de probabilité :

$$\forall f : \mathbb{E}_N \to \mathbb{C} \; : < f(M) >= \frac{1}{Z_N}\int_{\mathbb{E}_N} dM f(M) e^{-\frac{N}{T}\mathrm{Tr}(V(M))} \quad (2.7.2)$$

ce qui se traduit dans le problème aux valeurs propres par (pour le cas hermitien) :

$$\forall g : \mathbb{R}^N \to \mathbb{C} \; : < g(\lambda) >= \frac{1}{Z_N}\int_{\mathbb{R}^N} d\lambda_1\ldots d\lambda_N \, g(\lambda)\Delta(\lambda)^2 e^{-\frac{N}{T}\sum_{i=1}^{N} V(\lambda_i)} \quad (2.7.3)$$

Il est également intéressant d'introduire les cumulants des expressions 2.7.1 par :

**Définition 2.4.** *Les cumulants des résolvantes sont définis par :*

$$\omega_n(x_1,\ldots,x_n) = \left\langle \mathrm{Tr}\frac{1}{x_1-M}\ldots\mathrm{Tr}\frac{1}{x_n-M} \right\rangle_c \quad (2.7.4)$$

*où l'indice $_c$ désigne les cumulants de la valeur moyenne des produits, c'est à dire en notant $J=\{1,\ldots,n\}$ :*

$$< A_1 > \quad = \quad < A_1 >_c$$

$$
\begin{aligned}
< A_1 A_2 > &= < A_1 A_2 >_c + < A_1 >_c < A_2 >_c \\
< A_1 A_2 A_3 > &= < A_1 A_2 A_3 >_c + < A_1 A_2 >_c < A_3 > + < A_1 A_3 >_c < A_2 > \\
&\quad + < A_2 A_3 >_c < A_3 > + < A_1 >< A_2 >< A_3 > \\
< A_1 \dots A_n > &= < A_J >= \sum_{k=1}^{n} \sum_{I_1 \sqcup I_2 \dots \sqcup I_k = J} \prod_{i=1}^{k} < A_{I_i} >_c
\end{aligned}
\tag{2.7.5}
$$

*où la somme a lieu sur une décomposition de l'ensemble $J$ en une réunion disjointe d'ensembles non vides.*

Les densités des fonctions de corrélations $\rho_n(x_1, \dots, x_n)$ définies préalablement (2.5.1) s'obtiennent alors comme les discontinuités des résolvantes $\hat{\omega}_n(x_1, \dots, x_n)$ et inversement les résolvantes $\hat{\omega}_n(x_1, \dots, x_n)$ sont les transformées de Stieljes des densités. Par exemple la densité des valeurs propres $\rho(x) = \rho_1(x)$ s'obtient par :

$$
\hat{\omega}_1(x) = \int \frac{\rho(x')}{x - x'} dx' \Longleftrightarrow \rho(x) = \frac{1}{2i\pi} \left( \omega_1(x - i0) - \omega_1(x + i0) \right) \tag{2.7.6}
$$

**Ainsi connaître les densités de corrélations des valeurs propres est équivalent à connaître les résolvantes $\hat{\omega}_n(x_1, \dots, x_n)$ ou leurs cumulants $\omega_n(x_1, \dots, x_n)$.**

Hélas, le calcul des fonctions de corrélation ou des résolvantes n'est pas en général possible analytiquement. Une solution est alors de rechercher un développement perturbatif en série de puissances de $\frac{1}{N}$ et d'écrire :

**Définition 2.5.** *Le développement perturbatif (topologique) dans le cas des modèles de matrices hermitiennes est défini de façon formelle par [23, 137, 138] :*

$$
\begin{aligned}
F(T) = \ln(Z_N(T)) &= \sum_{g=0}^{\infty} \left( \frac{N}{T} \right)^{2-2g} F_g(T) \\
\hat{\omega}_n(x_1, \dots, x_n, T) &= \sum_{g=0}^{\infty} \left( \frac{N}{T} \right)^{2-2g-n} \hat{\omega}_n^{(g)}(x_1, \dots, x_n, T)
\end{aligned}
\tag{2.7.7}
$$

Il est important de préciser qu'un tel développement n'existe pas toujours (Si par exemple $F(T)$ possèdait un terme en $e^{-\frac{N}{T}}$, ce dernier ne contribuerait pas au dévelop-

pement formel car il est exponentiellement petit dans la limite $\frac{N}{T} \to \infty$ ) mais existe systématiquement dans des cas particuliers où le contour des valeurs propres correspond à un contour de "steepest descent" associé au potentiel $V$. De plus, cette série est toujours divergente et il ne s'agit là que d'un développement asymptotique que l'on peut manipuler de façon formelle. Quoi qu'il en soit, il est toujours possible de supposer l'existence d'un tel développement et de le manipuler de façon formelle. Cela est en particulier utile pour les problèmes de dénombrements où les $F_g$ comptent le nombre de surfaces triangulées de genre $g$ (et qui justifie le nom de développement topologique). Le lecteur intéressé pourra se référer entre autres à [23, 58]. Cela peut également être utile si l'on s'intéresse à la limite $N \to +\infty$ ou $T \to 0$ puisqu'alors seuls les premiers termes de la série contribuent de façon significative. On notera enfin que [26] permet de calculer la partie non-perturbative de la fonction de partition lorsque le développement perturbatif est connu.

## 2.8 Les équations de boucles du modèle hermitien

L'introduction du développement topologique (2.7.7), permet de trouver des relations entre les différentes résolvantes $\omega_n^{(g)}(x_1,\dots,x_n)$ et de résoudre par récurrence le problème. Les relations entre les différentes résolvantes sont données par la méthode des équations de boucles, connues aussi comme équations de Schwinger-Dyson qui consistent en de simples intégrations par parties judicieuses dans l'intégrale matricielle. La démonstration de ces équations est devenue classique pour le cas du modèle à une matrice et peut être trouvée par exemple dans [19], [23]. Par ailleurs dans le chapitre 4, nous présentons en détail la dérivation des équations de boucles dans le cas du modèle à deux matrices avec $\beta$ quelconque, à partir desquelles on peut facilement obtenir les équations de boucles qui nous intéressent présentement. Introduisons donc les fonctions :

**Définition 2.6.**

$$U_n(x_1;x_2\dots,x_n) = \left\langle \mathrm{Tr}\frac{V'(x_1)-V'(M)}{x_1-M}\mathrm{Tr}\frac{1}{x_2-M}\dots\mathrm{Tr}\frac{1}{x_n-M}\right\rangle_c \tag{2.8.1}$$

*ainsi que leur développement topologique (formel ou non) :*

$$U_n(x_1;x_2\ldots,x_n)=\sum_{g=0}^{\infty}\left(\frac{N}{T}\right)^{2-2g-n}U_k^{(g)}(x_1;x_2\ldots,x_n) \tag{2.8.2}$$

alors les équations de boucles donnent les relations suivantes [23] :

$$\omega_1(x_1)^2+\frac{T^2}{N^2}\omega_2(x_1,x_2)=V'(x_1)\omega_1(x_1)-U_1(x_1) \tag{2.8.3}$$

puis en définissant la notation $J=\{x_2,\ldots,x_n\}$ :

$$\begin{aligned}
&\left(V'(x_1)-2\omega_1(x_1)\right)\omega_n(x_1,\ldots,x_n)\\
&=U_n(x_1;x_2,\ldots,x_n)+\frac{T^2}{N^2}\omega_{n+1}(x_1,x_1,x_2,\ldots,x_n)\\
&+\sum_{I\subset J}\omega_{j+1}(x_1,x_I)\omega_{n-j}(x_1,x_{J/I})\\
&+\sum_{j=1}^{k}\frac{\partial}{\partial x_j}\left(\frac{\omega_{n-1}(x_1,\ldots,x_j,\ldots,x_n)-\omega_{n-1}(x_1,\ldots,x_1,\ldots,x_n)}{x_j-x_1}\right)
\end{aligned} \tag{2.8.4}$$

On peut alors projeter ces équations dans le développement topologique et identifier les puissances terme à terme. Cela donne :

$$\omega_1^{(0)}(x_1)^2=V'(x_1)\omega_1^{(0)}(x_1)-U_1^{(0)}(x_1) \tag{2.8.5}$$

et

$$\begin{aligned}
&\left(V'(x_1)-2\omega_1^{(0)}(x_1)\right)\omega_k^{(g)}(x_1,\ldots,x_k)\\
&=U_k^{(g)}(x_1;x_2,\ldots,x_k)+\omega_{k+1}^{(g-1)}(x_1,x_1,x_2,\ldots,x_k)\\
&+\sum_{I\in J}\sum_{h=0}^{g}\omega_{j+1}^{(h)}(x_1,x_I)\omega_{k-j}^{(g-h)}(x_1,x_{J/I})\\
&+\sum_{j=1}^{k}\frac{\partial}{\partial x_j}\left(\frac{\omega_{k-1}^{(g)}(x_1,\ldots,x_j,\ldots,x_k)-\omega_{k-1}^{g)}(x_1,\ldots,x_1,\ldots,x_k)}{x_j-x_1}\right)
\end{aligned} \tag{2.8.6}$$

En rebaptisant les fonctions de la façon suivante :

$$w(x) = \omega_1^{(0)}(x) \quad , \quad y(x) = w(x) - \frac{V'(x)}{2} \quad , \quad P(x) = \frac{V'(x)^2}{4} - U_1^{(0)}(x) \tag{2.8.7}$$

et

$$P_k^{(g)}(x_1;x_2,\ldots,x_k) = U_k^{(g)}(x_1;x_2,\ldots,x_k) + \sum_{j=2}^{k} \frac{\partial}{\partial x_j}\left(\frac{\omega_{k-1}^{(g)}(x_1,\ldots,x_j,\ldots,x_k) - \omega_{k-1}^{(g)}(x_1,\ldots,x_1,\ldots,x_k)}{x_j - x_1}\right) \tag{2.8.8}$$

on trouve les équations de boucles sous leur forme standard [23] :

**Théorème 2.6.** *Les équations de boucles peuvent être mises sous leur forme standard :*

$$y^2(x) = P(x) \tag{2.8.9}$$

*puis la formule de récurrence topologique :*

$$\begin{aligned} -2y(x)\omega_k^{(g)}(x,x_2,\ldots,x_k) \quad = \quad & P_k^{(g)}(x;x_2,\ldots,x_k) + \omega_{k+1}^{(g-1)}(x,x,x_2,\ldots,x_k) \\ & + \sum_{I\in J}\sum_{h=0}^{g} \omega_{j+1}^{(h)}(x,x_I)\omega_{k-j}^{(g-h)}(x,x_{J/I}) \end{aligned} \tag{2.8.10}$$

L'intérêt de la méthode des équations de boucles apparaît alors. En effet, les fonctions $x \to U_n(x;x_2,\ldots,x_n)$, $x \to P(x)$ et $x \to P_n(x;x_2,\ldots,x_n)$ n'ont de singularités qu'aux singularités du potentiel $V(x)$. En particulier lorsque ce dernier est polynômial, ces fonctions sont également polynômiales en $x$. L'intérêt de la méthode provient également du fait qu'une fois la courbe $y^2(x) = P(x)$ déterminée (c'est-à-dire une fois l'ordre dominant de la densité des valeurs propres connu, par exemple à l'aide de 2.3.2 ou de la donnée des fractions de remplissage), toutes les autres fonctions $\omega_n^{(g)}(x_1,\ldots,x_n)$ peuvent être calculées à l'aide des travaux de [23] et de résultats de géométrie algébrique résumés dans le paragraphe ci-dessous. Notons que la détermination du polynôme $P(x)$ peut varier suivant le contexte. En effet, par sa définition 2.8.7, il est clair que seule la moitié supérieure des coefficients de $P(x)$ peut être directement déterminée par le potentiel $V'(x)$. Pour la

moitié inférieure, cela depend du contexte qui est étudié : si l'on s'intéresse au modèle convergent, alors la densité des valeurs propres est donnée par 2.3.2 qui est intégralement connue par les contraintes 2.3.3 2.3.4. En revanche, on peut également s'intéresser à des modèles formels où une fraction donnée $\varepsilon_i$ (appelées couramment "fraction de remplissage") des valeurs propres se trouve dans la coupure $[a_i, b_i]$ de 2.3.2. Dans ce cas, les contraintes 2.3.4 ne sont plus exigibles mais doivent être remplacées par les conditions :

$$\varepsilon_i = \oint_{\mathscr{A}_i} y dx \tag{2.8.11}$$

où $\mathscr{A}_i$ est le $\mathscr{A}_i$-cycle contour entourant la coupure $[a_i, b_i]$.

## 2.9 Invariants symplectiques et géométrie algébrique

Les résultats concernant les modèles de matrices se généralisent en fait à des courbes algébriques quelconques [23] appelées *courbe spectrale* :

$$E(x,y) = 0 = \sum_{i=0}^{d_x} \sum_{j=0}^{d_y} E_{i,j} x^i y^j \tag{2.9.1}$$

Le cas particulier des modèles à une matrice hermitienne correspond ainsi au cas où $E(x,y)$ est de degré 2 en $y$ et est donné par 2.8.9 (et dépend donc d'un paramètre $T$). L'équation 2.9.1 définit une surface de Riemann à partir de laquelle des quantités vont être calculées (Cf. [23] pour plus de détails). Soient $\{a_i\}_{i=1..n}$ les points de branchements supposés **simples** (sinon la construction échoue) de la courbe spectrale $E(x,y) = 0$ que l'on suppose de genre $g$ et sur laquelle on a défini une base de cycles non contractibles indépendants : $(\mathscr{A}_i, \mathscr{B}_i)_{i=1..g}$ avec :

$$\mathscr{A}_i \cap \mathscr{B}_j = \delta_{i,j}$$

Sur cette courbe algébrique, on peut également définir une base de formes holomorphes indépendantes $du_i(x)$ que l'on normalise comme suit :

$$\oint_{\mathscr{A}_j} du_i = \delta_{i,j}$$

Un résultat de géométrie algébrique nous dit alors que la matrice des périodes de Riemann $\tau_{i,j} = \oint_{\mathscr{B}_j} du_i$ est symétrique.

Sur cette courbe algébrique, on peut enfin définir un noyau de Bergman $B(p,q)$ comme unique forme bilinéaire ayant un pôle double sans résidu en $p = q$ et normalisée de la façon suivante : $B(p,q) \sim \frac{dz(p)dz(q)}{(z(p)-z(q))^2}$. Il possède les propriétés d'être symétrique et de satisfaire $\oint_{q\in\mathscr{B}_i} B(p,q) = 2i\pi\, du_i(p)$ et $\oint_{q\in\mathscr{A}_i} B(p,q) = 0$

A partir de ces définitions classiques de géométrie algébrique on peut définir ($a$ est un point de base qui peut être choisi arbitrairement) :

$$\Phi(p) = \int_a^p y\,dx$$

$$W_2^{(0)}(p_1,p_2) = B(p_1,p_2)$$

$$\begin{aligned} W_{k+1}^{(g)}(p,p_K) \quad &= \quad \sum_i \operatorname*{Res}_{q\to a_i} \frac{dE_q(p)}{y(q)-y(\overline{q})} \\ &\left( \sum_{h=0}^{g} \sum_{J\in K} W_{|J|+1}^{(h)}(q,p_J) W_{k-|J|+1}^{(g-h)}(q,p_{K/J}) + W_{k+2}^{(g-1)}(q,q,p_K) \right) \end{aligned} \tag{2.9.2}$$

où $dE_q(p) = G(p,q) = \int^q B(p,q')$ est la forme de troisième espèce. Ces fonctions satisfont les relations suivantes :

1. Elles satisfont des "équations de boucles" identiques à 2.8.10. Les quantités :

$$\begin{aligned} P_k^{(g)}(x(p),p_K) \quad &= \quad \sum_i \left[ 2y(p^i) W_{k+1}^{(g)}(p^i,p_K) + W_{k+2}^{(g-1)}(p^i,p^i,p_K) \right. \\ &\left. + \sum_{J\in K} \sum_{h=0}^{g} W_{j+1}^{(h)}(p^i,p_J) W_{k-j-1}^{(g-h)}(p^i,p_{K/J}) \right] \end{aligned} \tag{2.9.3}$$

sont des fonctions rationnelles en $x(p)$ sans pôle aux points de branchements.

2. Les $W_n^{(g)}$ sont des fonctions symétriques de leurs arguments, de résidus nuls aux points de branchements et d'intégrales nulles sur les $\mathscr{A}$-cycles.

3. $\forall k \geq 1$ :

$$\operatorname*{Res}_{p_{k+1}\to a, p_1, \ldots, p_k} \Phi(p_{k+1}) W_{k+1}^{(g)}(p_K, p_{k+1}) = (2g+k-2) W_k^{(g)}(p_K) + \delta_{g,0}\delta_{k,1} y(p_1) dx(p_1) \tag{2.9.4}$$

4. Les invariants sont alors définis par extension de la relation précédente pour $k = 0$ :

$$F^{(g)} = \frac{1}{2-2g} \sum_i \operatorname*{Res}_{q\to a_i} W_1^{(g)}(q)\Phi(q) \tag{2.9.5}$$

Ils sont invariants par transformations symplectiques de la courbe spectrale.

Le lien avec les modèles de matrices est alors le suivant : les fonctions $W_n^{(g)}(x_1, \ldots, x_n)$ définies par la récurrence topologique 2.9.2 correspondent aux résolvantes $\omega_n^{(g)}(x_1, \ldots, x_n)$ des modèles de matrices hermitiennes 2.7.7. Il en est de même avec les invariants symplectiques 2.9.5 qui sont les mêmes que les énergies libres $F_g(T)$ des modèles de matrices 2.7.7. Ce résultat se généralise systématiquement lorsque la courbe spectrale 2.9.1 provient d'un modèle de matrices hermitiennes. (Les courbes de degrés plus élevés pouvant provenir de modèles à deux matrices hermitiennes). Il est à noter que la récurrence topologique 2.9.2 est facilement applicable en pratique puisqu'elle consiste uniquement à prendre des résidus aux points de branchements. **En résumé, les travaux de Eynard et Orantin [23] permettent, étant donnée la courbe spectrale, (i.e. le terme dominant de la densité des valeurs propres) de déterminer le développement perturbatif de toutes les résolvantes (i.e. de toutes les densités de corrélations des valeurs propres)**.

## 2.10 Potentiel singulier et double limite d'échelle

Lorsque le potentiel $V(x)$ est singulier (i.e. lorsque la densité limite des valeurs propres s'annule en au moins un point intérieur strictement à son support), le modèle de matrice associé présente des caractéristiques intéressantes dans le régime appelé *double*

*limite d'échelle* [24, 62, 62]. En effet, il est alors possible [22] de plonger le potentiel critique $V_c(x)$ dans un ensemble de potentiels dépendant d'un paramètre supplémentaire $s$ : $V(x,s)$ tel que, pour une certaine valeur du paramètre $s$, on obtient le potentiel critique :

$$V(x, s = s_c) = V_c(x) \tag{2.10.1}$$

Bien que le choix du plongement soit a priori arbitraire, le paramètre le plus naturel pour plonger le potentiel critique est la température $T$. Par analogie avec la physique statistique traditionnelle, le modèle de matrice est dit présenter une *transition de phase* pour la valeur critique $T = T_c$. Ainsi nous avons vu par exemple précédemment 2.4.1 que le potentiel :

$$V(x,T) = \frac{1}{T}\left(\frac{x^4}{4} - \frac{4\cos(\pi\varepsilon)x^3}{3} + \cos(2\pi\varepsilon)x^2 + 8\cos(\pi\varepsilon)x\right) \tag{2.10.2}$$

présente une transition de phase pour la valeur $T = T_c = 1 + 4\cos(\pi\varepsilon)$.

L'étude des potentiels singuliers et des transitions de phase est intéressante car elle permet de faire le lien avec les systèmes intégrables dans le cadre de la double limite d'échelle. La double limite d'échelle consiste à prendre simultanément la limite $N \to +\infty$ et $T \to T_c$ de telle sorte que pour une certaine valeur bien choisie $\alpha$, le produit $(T - T_c)N^{-\alpha}$ reste d'ordre un. D'une façon générale, les transitions de phase peuvent être étudiées pour des singularités de la densité des valeurs propres $\rho(x)$ de type $(p,q) \in \mathbb{N}^2$ :

$$\rho(x) \underset{x \to a}{\sim} (x-a)^{\frac{p}{q}} \tag{2.10.3}$$

Si $a$ est une extrémité du support de $\rho(x)$, les valeurs $(p = 1, q = 2)$ ne donnent pas lieu à un point critique. En revanche, dans tous les autres cas, la densité de valeurs propres $\rho(x)$ présente un point singulier en $x = a$. Dans le cas des modèles à une matrice hermitienne, le fait que la densité d'équilibre des valeurs propres soit donnée par une courbe hyperelliptique n'autorise que deux valeurs de $q$ : $q = 1$ et $q = 2$. Le cas $q = 1$ correspond à un point intérieur au support de la densité où cette dernière s'annule. Le cas $q = 2$ et $p \neq 1$ correspond quant à lui à une extrémité singulière où la densité ne se

comporte plus localement comme une racine carrée. Ces deux situations correspondent aux deux images suivantes :

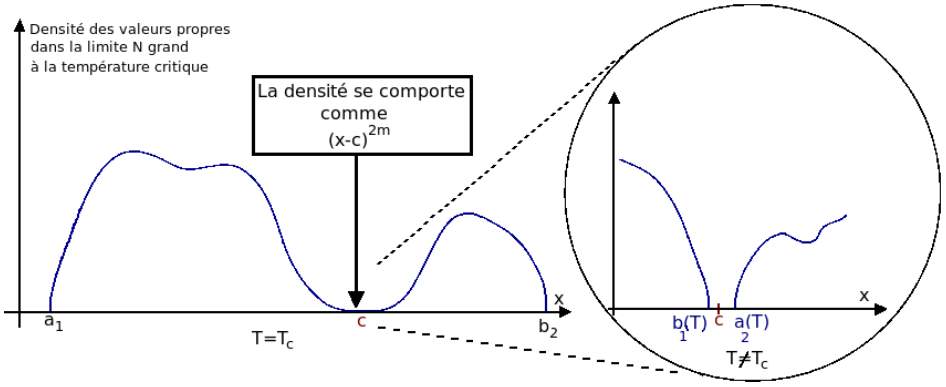


Figure 14 : Double limite d'échelle d'une densité critique dont le point critique est intérieur au support. La singularité est de type $(p = 2m, q = 1)$.

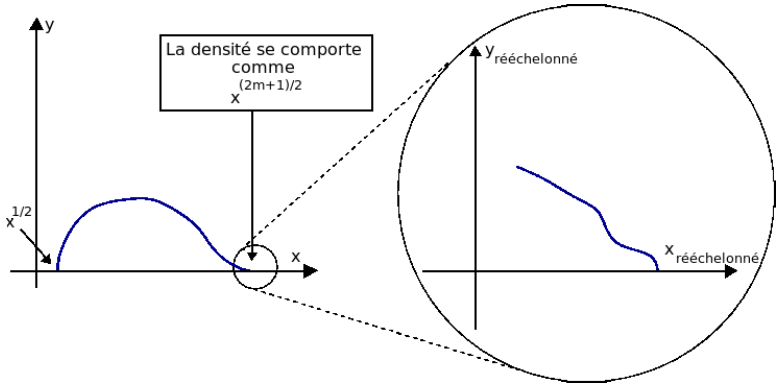


Figure 15 : Double limite d'échelle d'une densité critique dont le point critique est à une extrémité du support. La singularité est de type $(p = 2m+1, q = 2)$. [29]

De nombreux résultats concernant les doubles limites d'échelle sont connus aujourd'hui. Ainsi, d'après [23], il est connu que les doubles limites d'échelle pour une singularité de type $(p,q)$ sont reliées aux modèles minimaux $(p,q)$ de la théorie conforme (CFT). Un résultat majeur de [24] montre ainsi que si les invariants symplectiques $F_g(T)$ et les fonctions de corrélations $\omega_n^{(g)}(x_1,\ldots,x_n,T)$ peuvent être calculés pour des valeurs régulières de la température $T$, ces quantités divergent lorsque $T \to T_c$ et ne peuvent être définies par la récurrence topologique 2.9.2 habituelle pour $T = T_c$ puisque des points de branchements ne sont plus simples. En revanche, il est montré dans [23] le résultat suivant :

**Théorème 2.7.** *Sous le changement d'échelle général (point singulier $(p,q)$ en $x=a$) :*

$$x_i = a + (T-T_c)^{\frac{1}{p+q-1}}\xi_i \tag{2.10.4}$$

*tel que :*

$$y_{rescaled}(\xi) = \lim_{T\to T_c}(T-T_c)^{\frac{p+q}{p+q-1}} y(x_1,T) \tag{2.10.5}$$

*et*

$$\omega^{(g)}_{rescaled,n}(\xi_1,\dots,\xi_n) = \lim_{T\to T_c}(T-T_c)^{(2-2g-n)\frac{p+q}{p+q-1}}\omega^{(g)}_n(x_1,\dots,x_n,T) \tag{2.10.6}$$

*avec*

$$F_{rescaled,g} = \lim_{T\to T_c}(T-T_c)^{(2-2g)\frac{p+q}{p+q-1}}F_g(T) \tag{2.10.7}$$

*Les quantités, $y_{rescaled}(\xi)$, $\omega^{(g)}_{rescaled,n}(\xi_1,\dots,\xi_n)$ et $F_{rescaled,g}$ sont bien définies (au sens où la limite existe et est finie) et correspondent aux invariants symplectiques et aux résolvantes de la courbe $y_{rescaled}(\xi)$.*

Ce résultat peut être utilisé dans les deux cas provenant des modèles hermitiens à une matrice. Ainsi, le cas $(2m+1,q=2)$ a été traité par M. Bergère et B. Eynard dans [29] alors que j'ai traité le cas $(p=2m,q=1)$ dans l'article **[I]** présenté dans l'annexe V avec Mattia Cafasso. Dans notre cas, la densité critique est :

$$\rho(x,T_c) = \rho_c(x) = \frac{1}{2\pi}(x-b\varepsilon)^{2m}\sqrt{b^2-x^2} \tag{2.10.8}$$

où le point critique est $x=b\varepsilon$ de type $(2m,1)$.

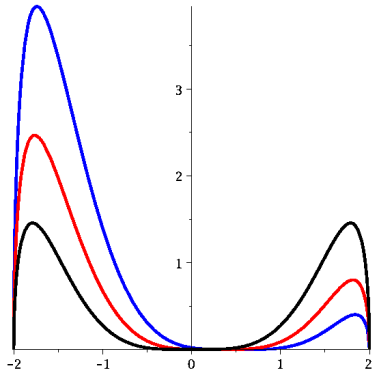


Figure 16 : Image de la fonction de densité critique donnée par (2.10.8) pour les valeurs $b = 2$, $m = 2$. La courbe noire représente le cas où $\varepsilon = 0$, la rouge celle où $\varepsilon = \frac{1}{4}$ et la bleue celle où $\varepsilon = \frac{1}{8}$.

Nous avons montré que (Cf. **[I]**, Section 2.4 de l'annexe V) cette densité critique correspond au potentiel critique suivant :

**Théorème 2.8.** *La densité critique 2.10.8 correspond au potentiel :*

$$V'(x,T) = \frac{1}{T}\Big(\sum_{j=0}^{2m+1}\Big(\binom{2m}{j-1}(-b\varepsilon)^{2m+1-j} + \sum_{n=1}^{E(\frac{2m+1-j}{2})}\binom{2m}{2n+j-1}\frac{(-1)^j(2n-2)!\varepsilon^{2(m-n)+1-j}b^{2m+1-j}}{n!(n-1)!2^{2n-1}}\Big)x^j\Big) \tag{2.10.9}$$

*avec une température critique :*

$$T_c = \frac{b^{2m+2}}{2}\sum_{n=1}^{m+1}\frac{\varepsilon^{2m-2n+2}(2m)!}{n!(2m-2n+2)!(n-1)!2^{2n-1}} \tag{2.10.10}$$

On peut alors montrer (Cf. **[I]**, Section 2.4 de l'annexe V) que le changement d'échelle $x = b\varepsilon + \xi(T-T_c)^{\frac{1}{2m}}$ donne alors la courbe réduite suivante :

**Théorème 2.9.** *Après changement d'échelle, la courbe réduite est donnée par :*

$$\boxed{y_{rescaled}(\xi) = b\pi\sqrt{1-\varepsilon^2}\sqrt{(\gamma^2-\xi^2)}\left(\xi^{2m-1}+\sum_{n=1}^{m-1}\frac{(2n)!}{(n!)^2 2^{2n}}\gamma^{2n}\xi^{2m-1-2n}\right)} \tag{2.10.11}$$

*où le paramètre $\gamma$ intervenant dans l'expression de la courbe est donné par :*

$$\gamma^{2m} = -\frac{(m!)^2 2^{2m+1}}{b^2(1-\varepsilon^2)(2m)!} \tag{2.10.12}$$

Le calcul des fonctions $\omega^{(g)}_{\text{rescaled},n}(\xi_1,\ldots,\xi_n)$ et des invariants symplectiques $F_{\text{rescaled},g}$ peut alors être effectué par la méthode générale de Eynard et Orantin pour la courbe 2.10.11. Nous avons ainsi trouvé dans **[I]** présenté dans l'annexe V que :

**Théorème 2.10.** *La fonction de corrélation à deux points est donnée par (Cf. **[I]**, Section 2.4 de l'annexe V) :*

$$\omega_2^{(0)}(\xi_1,\xi_2) = \frac{1}{4(\xi_1-\xi_2)^2}\left(-2+\sqrt{\frac{(\gamma+\xi_1)(\gamma-\xi_2)}{(\gamma-\xi_1)(\gamma+\xi_2)}}+\sqrt{\frac{(\gamma-\xi_1)(\gamma+\xi_2)}{(\gamma+\xi_1)(\gamma-\xi_2)}}\right) \tag{2.10.13}$$

Notons que l'écriture en terme des variables $\xi$ n'est pas optimale, puisqu'elle fait intervenir des racines carrées et plus généralement des fonctions multivariées. Cela provient du fait que la courbe spectrale 2.10.11 présente elle-même une racine carrée. Ainsi, il est souvent plus agréable de travailler avec une représentation paramétrique de la courbe de la forme $(\xi(z), y_{\text{rescaled}}(z))$ où $z$ est un point courant d'une surface de Riemann et $\xi(z)$ et $y_{\text{rescaled}}(z))$ sont cette fois-ci des fonctions univariées. Dans notre cas, la courbe spectrale 2.10.11 est de genre 0 donc une bonne paramétrisation est d'utiliser la transformation de Joukovski :

$$\xi = \frac{\gamma}{2}\left(z+\frac{1}{z}\right) = \frac{\gamma(z^2+1)}{2z} \Leftrightarrow z = \frac{1+\sqrt{\xi^2-\gamma^2}}{\gamma} \tag{2.10.14}$$

En plus de ce changement de paramétrisation, il est souvent préférable (et c'est égale-

ment ces quantités que le formalisme d'Eynard et Orantin calcule) de considérer des formes différentielles plutôt que des fonctions. En effet, les formes différentielles sont des objets plus intrinsèques que les fonctions dans le sens où elles ne dépendent plus du choix de la paramétrisation $\xi(z)$ retenue. Ainsi, on peut définir plus intrinsèquement les formes différentielles :

**Définition 2.7.** *Les formes différentielles associées aux résolvantes sont définies par :*

$$\mathscr{W}_n^{(g)}(z_1,\ldots,z_n)dz_1\ldots dz_n = \omega_n^{(g)}(\xi(z_1),\ldots,\xi(z_n))d\xi_1\ldots d\xi_n + \delta_{n,2}\delta_{g,0}\frac{d\xi(z_1)d\xi(z_2)}{(\xi(z_1)-\xi(z_2))^2} \tag{2.10.15}$$

Dans notre cas, on peut alors prouver que :

**Théorème 2.11.** *Après utilisation de la transformation de Joukovski 2.10.14, la 2-forme* $\mathscr{W}_2^{(0)}(z_1,z_2)dz_1dz_2$ *s'écrit :*

$$\mathscr{W}_2^{(0)}(z_1,z_2)dz_1dz_2 = \frac{dz_1dz_2}{(z_2-z_1)^2} \tag{2.10.16}$$

## 2.11 Modèles $(p,q)$ de la théorie conforme couplée à la gravité

Les modèles minimaux interviennent dans l'étude des représentations du groupe conforme en dimension 2. Pour $n \geq 2$, le groupe des transformations conforme dans $\mathbb{R}^n$ est le groupe des transformations de $\mathbb{R}^n$ dans lui-même qui conserve les angles. Ce groupe est engendré par l'inversion : $\vec{x} \mapsto \frac{\vec{x}}{||x||}$ ainsi que par le groupe de Poincaré (translations, rotations/boosts, dilatations). Il est isomorphe à $SO(n+1,1)$ et ses représentations peuvent être étudiées, y compris lorsque $\mathbb{R}^n$ est muni d'une métrique non-euclidienne. Dans le cas de la dimension 2, il est remarquable que l'algèbre de Lie du groupe conforme est de dimension infinie, ce qui n'est plus le cas dans les dimensions supérieures où le groupe conforme présente beaucoup moins d'intérêt. Il est maintenant connu que les représentations irréductibles de charge centrale $c < 1$ du groupe conforme en dimension 2 (dont l'algèbre de Lie est de dimension infinie) peuvent être classifiées par deux paramètres entiers $(p,q)$ dont la charge centrale correspondante [195] est don-

née par :

$$c = 1 - 6\frac{(p-q)^2}{pq} \tag{2.11.1}$$

L'étude de ces représentations irréductibles est d'une grande importance en physique puisque la symétrie conforme est une symétrie que l'on rencontre dans beaucoup de domaines. On citera ainsi les modèles connus suivants [197] :

1. $(1,2)$ : Airy ($c = -2$)
2. $(3,2)$ : Modèle de gravité pure ($c = 0$)
3. $(4,3)$ : Modèle d'Ising ($c = \frac{1}{2}$)
4. $(6,5)$ : Modèle de Potts à 3 états ($c = \frac{4}{5}$)

Il existe beaucoup d'approches dans la présentation des modèles minimaux $(p,q)$. En particulier, dans le cadre de cette thèse, il est utile de voir les modèles $(p,q)$ comme une réduction de la hiérarchie Kadamtsev-Petviashvili (KP) des systèmes intégrables. Ainsi, dans [22], les auteurs ont montré qu'une représentation en termes de paire de Lax du modèle $(2m,1)$ est :

**Théorème 2.12.** *Le modèle de théorie conforme couplée à la gravité* $(2m,1)$ *peut être mis sous la forme d'une paire de Lax ([22]) :*

$$\begin{aligned} \frac{1}{N}\frac{\partial}{\partial x}\Psi(x,t) &= \mathscr{D}(x,t)\Psi(x,t) \\ \frac{1}{N}\frac{\partial}{\partial t}\Psi(x,t) &= \mathscr{R}(x,t)\Psi(x,t) \end{aligned} \tag{2.11.2}$$

*où* $\Psi(x,t)$ *est une matrice* $2\times 2$ *dont les entrées peuvent être écrites comme :*

$$\Psi(x,t) = \begin{pmatrix} \psi(x,t) & \phi(x,t) \\ \tilde{\psi}(x,t) & \tilde{\psi}(x,t) \end{pmatrix} \tag{2.11.3}$$

*et satisfaisant la normalisation* $\det(\Psi(x,t)) = 1$ *(pour que le wronskien du système dif-*

*férentiel soit égal à l'unité). Les matrices $\mathscr{D}(x,t)$ et $\mathscr{R}(x,t)$ sont données par :*

$$\mathscr{R}(x,t) = \begin{pmatrix} 0 & x+u(t) \\ -x+u(t) & 0 \end{pmatrix} \tag{2.11.4}$$

*et*

$$\mathscr{D}(x,t) = \sum_{k=0}^{m} t_k \mathscr{D}_k(x,t) \tag{2.11.5}$$

*avec*

$$\mathscr{D}_k(x,t) = \begin{pmatrix} -A_k(x,t) & xB_k(x,t)+C_k(x,t) \\ xB_k(x,t)-C_k(x,t) & A_k(x,t) \end{pmatrix} \tag{2.11.6}$$

*où $A_k$, $B_k$, $C_k$ sont des polynômes en x de degrés respectifs $2k-2$, $2k-2$ et $2k$.*

Les paires de Lax sont des blocs fondamentaux des systèmes intégrables (KdV, KP, etc.) puisqu'elles permettent l'intégration explicite des équations aux dérivées partielles auxquelles elles sont reliées. En particulier, on peut montrer que les systèmes différentiels exprimés sous la forme d'une paire de Lax, possèdent une infinité de quantités conservées, ce qui rend l'intégration possible. Dans notre cas, la paire de Lax 2.11.2 permet de fournir une représentation de la hiérarchie de Painlevé II. Dans l'article **[I]**, présenté à l'annexe V), on montre ainsi à la section 3.2 :

**Théorème 2.13.** *La relation de compatiblité ($\left[\frac{1}{N}\frac{\partial}{\partial x} - \mathscr{D}(x,t), \mathscr{R}(x,t) - \frac{1}{N}\frac{\partial}{\partial t}\right] = 0$) implique que la fonction inconnue $u(x,t)$ satisfait l'équation des cordes («string equation») :*

$$\sum_{k=0}^{m} t_k \hat{R}_k(u(t)) = -tu(t) \tag{2.11.7}$$

*où les $\hat{R}_k(x)$ sont les polynômes de Gelfand-Dikii associés à la hiérarchie de Painlevé II (dont la récurrence est donnée dans **[I]** dans l'annexe V). En particulier pour $m = 1$, la fonction $u(x,t)$ doit satisfaire l'équation de Painlevé II :*

$$\frac{d^2u}{dt^2}(t) = 2u^3(t) + 4(t+t_0)u(t) \tag{2.11.8}$$

Notons qu'un travail similaire a été fait dans le cas $(p = 2m+1, q = 2)$ dans [29] à la différence près que la hiérarchie trouvée dans ce cas est celle de KdV et non celle de Painlevé II. Rappelons que les équations de Painlevé, au nombre de six, sont les seules équations différentielles ordinaires du second ordre dont les singularités possèdent la propriété de Painlevé, c'est-à-dire que les singularités mobiles (dépendant des conditions initiales) ne peuvent être que des pôles (pas de singularités essentielles mobiles). Ces équations présentent ainsi des propriétés très particulières de symétrie (transformation de Backlund) et peuvent être exprimées dans un formalisme Hamiltonien.

Pour l'instant, le lecteur peut se demander le rapport entre ces équations intégrables, les modèles de matrices hermitiennes et leurs doubles limites d'échelle. En fait, le cœur de la réponse à cette question est l'existence d'une courbe spectrale naturelle associée à une paire de Lax de type (2.11.2), et de façon plus générale à n'importe quelle paire de Lax dans la limite semi-classique. En effet, la présence du facteur $\frac{1}{N}$ dans (2.11.2) permet de définir un développement en puissances de $\frac{1}{N}$ pour n'importe quelle quantité (noté ici génériquement $\mathscr{F}(x,t)$) de la forme :

$$\mathscr{F}(x,t) = \sum_{j=0}^{\infty} \frac{\mathscr{F}_j(x,t)}{N^j} \tag{2.11.9}$$

On peut alors définir la courbe spectrale associée par :

**Définition 2.8.** *La courbe spectrale naturellement associée à une paire de Lax est définie par :*

$$\det(yId - \mathscr{D}_\infty(x,t)) = 0 \tag{2.11.10}$$

*où $\mathscr{D}_\infty(x,t)$ est donc la limite $N \to +\infty$ de la matrice $\mathscr{D}$ de l'équation 2.11.5.*

Le résultat important est alors que (Cf. **[I]** Section 3.3 de l'annexe V) :

**Théorème 2.14.** *La courbe spectrale (2.11.10) coïncide avec la courbe spectrale obtenue lors de la double limite d'échelle (2.10.11) avec l'identification $\gamma = u_0(t)$.*

Ce résultat non trivial traduit un lien entre la hiérarchie de Painlevé II et les doubles limites de modèles de matrices aléatoires. Mais ce lien va au delà des simples courbes

spectrales, puisque l'on peut définir de façon naturelle dans le contexte des paires de Lax, des fonctions $W_n^{(g)}(x_1,\dots,x_n)$ qui correspondent exactement aux résolvantes $\omega^{(g)}_{\text{rescaled},n}(x_1,\dots,x_n)$ associées à la double limite d'échelle (2.10.11 et 2.10.6). La définition "naturelle" de ces fonctions dans le cadre des paires de Lax est la suivante [25] :

**Définition 2.9.** *Soit le noyau $K(x_1,x_2)$ défini à l'aide des fonctions de Baker-Akhiezer :*

$$K(x_1,x_2) = \frac{\psi(x_1)\tilde{\phi}(x_2) - \tilde{\psi}(x_1)\phi(x_2)}{x_1 - x_2} \tag{2.11.11}$$

*Les fonctions de corrélations (connexes) sont alors définies par :*

$$W_1(x) = \psi'(x)\tilde{\phi}(x) - \tilde{\psi}'(x)\phi(x) \tag{2.11.12}$$

$$W_n(x_1,\dots,x_n) = -\frac{\delta_{n,2}}{(x_1-x_2)^2} - (-1)^n \sum_{\sigma=cycles} \prod_{i=1}^{n} K(x_{\sigma(i)},x_{\sigma(i+1)}) \tag{2.11.13}$$

*Les fonctions non-connexes correspondantes prennent une forme déterminantale typique des modèles de matrices ([25]) :*

$$W_{n,n-c}(x_1,\dots,x_n) = \overset{'}{det}(K(x_i,x_j)) \tag{2.11.14}$$

*où la notation $det'$ signifie que le déterminant doit être calculé comme habituellement par une somme sur les permutations $\sigma$ de produits $(-1)^\sigma \prod_{i=1}^{n} K(x_i,K_{\sigma_i})$, à l'exception des termes $i=\sigma(i)$ et $i=\sigma(j)\,,j=\sigma(i)$ où l'on doit remplacer les termes $K(x_i,x_i)$ et $K(x_i,x_j)K(x_j,x_i)$ par respectivement $W_1(x_i)$ et $-W_2(x_i,x_j)$.*

Le théorème principal de **[I]** présenté en annexe V établit alors l'égalité des fonctions de corrélation de la double limite d'échelle de modèle de matrice singuliers de type $(2m,1)$ avec les fonctions de corrélations issues de la paire de Lax du modèle de théorie conforme couplée à la gravité $(2m,1)$ qui peut se résumer ainsi (Cf. **[I]**, Section 3.6) :

**Théorème 2.15.** *Lien entre la hiérarchie de Painlevé II et la double limite d'échelle d'un*

*modèle de matrices :*

$$\forall (n,g) : W_n^{(g)}(x_1,\ldots,x_n) = \omega_{rescaled,n}^{(g)}(x_1,\ldots,x_n) \tag{2.11.15}$$

## 2.12 Conclusion et perspectives

Dans ce chapitre, nous avons esquissé le lien profond entre les doubles limites d'échelle de type $(p = 2m, q = 1)$ et les paires de Lax des systèmes intégrables correspondant au modèle de la théorie conforme $(2m, 1)$ à tous les ordres du développement topologique. Afin de rendre ce lien complet, les calculs techniques (et longs) des preuves peuvent être trouvés dans l'article **[I]** présenté en annexe V de ce mémoire. Dans leur article [29], Bergère et Eynard montrent qu'un lien identique existe pour le cas des modèles $(p = 2m+1, q = 2)$ aussi bien pour les courbes spectrales que pour les résolvantes, à tous les niveaux du développement topologique. Des résultats similaires sont également vérifiés par Alvarez, Alonso et Medina dans [31] dans le cas de densités limites présentant un support composé de plusieurs segments. Moyennant la généralisation d'un résultat technique (le fait que les fonctions de corrélation définies par 2.11.13 satisfassent les équations de boucles du modèle à deux matrices) valable pour l'instant pour $q < 3$, il est attendu prochainement que ce type de résultats se généralise pour toutes les valeurs de $(p,q)$ avec les doubles limites d'échelle du modèle hermitien à deux matrices. D'un point de vue physique, il est intéressant de constater que dans la limite de double échelle, qui correspond à une sorte de transition de phase du modèle de matrices, on retombe, après des changements d'échelle, sur des lois universelles (KdV, équations de Painlevé,...). Ces lois universelles sont en analogie avec les exposants critiques universels des transitions de phase pour différents systèmes (modèle d'Ising, percolation, etc.). Même si à l'heure actuelle, l'étendue des phénomènes physiques ou biologiques pouvant être modélisés par des modèles de matrices aléatoires reste inconnue, l'universalité très spécifique de ces régimes limites permettra sans doute d'éclairer dans le futur cette question.

# CHAPITRE 3

# MODÈLES MATRICIELS ET POLYNÔMES ORTHOGONAUX

## 3.1 Introduction des polynômes orthogonaux

Une méthode très efficace pour calculer la fonction de partition des modèles de matrices hermitiens :

$$Z_N = \int_{\Gamma^N} d\lambda_1 \dots d\lambda_N \, \Delta(\lambda)^2 e^{-\frac{N}{T}\sum_{i=1}^{N} V(\lambda_i)} \tag{3.1.1}$$

est d'introduire la base des polynômes orthogonaux moniques $P_n(x)$ :

**Définition 3.1.** *Les polynômes orthogonaux sont définis par :*

$$\int_{\Gamma} P_n(x) P_m(x) e^{-\frac{N}{T}V(x)} dx = h_n \delta_{n,m} \tag{3.1.2}$$

*où $P_n(x)$ est un polynôme monique (i.e. $P_n(x) = x^n + \dots$).*

En général, il est aussi pratique d'introduire de véritables fonctions orthonormales pour la mesure de Lebesgue par :

**Définition 3.2.** *Définition des fonctions orthonormales :*

$$\psi_n(x) = \frac{1}{\sqrt{h_n}} p_n(x) e^{-\frac{1}{2}V(x)} \tag{3.1.3}$$

*qui vérifient :*

$$\int_{\Gamma} \psi_n(x) \psi_m(x) dx = \delta_{n,m} \tag{3.1.4}$$

Le lien entre les modèles de matrices et les polynômes orthogonaux est alors le suivant : en écrivant le déterminant de Vandermonde comme un déterminant puis en pratiquant des combinaisons linéaires sur les lignes et les colonnes, il est bien connu ([18, 19]) que la fonction de partition du modèle à une matrice hermitienne $Z_N$ peut être réécrite comme :

$$Z_N = N! \prod_{j=0}^{N-1} h_j \tag{3.1.5}$$

Le choix des matrices hermitiennes n'est pas le seul possible, on pourrait par exemple choisir d'étudier des matrices normales (i.e. commutant avec leur adjoint) dont les valeurs propres seraient imposées sur une certaine courbe $\Gamma$ du plan complexe. Notons dans ce cas que, si le support imposé des valeurs propres part à l'infini, cette direction doit être compatible avec le choix du potentiel $V(x)$ pour que l'intégrale matricielle converge (et que les polynômes orthogonaux existent).

Relier le calcul de la fonction de partition des modèles matriciels hermitiens aux polynômes orthogonaux permet d'appliquer les nombreux résultats connus sur les polynômes orthogonaux aux modèles matriciels (en particulier des cas connus comme les polynômes de Legendre, Laguerre, Hermite, Jacobi et bien d'autres). On citera en particulier les résultats :

1. Pour toute suite de polynômes orthogonaux, il existe une relation de récurrence relativement à trois indices consécutifs.

$$x\psi_n(x) = \gamma_{n+1}\psi_{n+1} + \beta_n\psi_n + \gamma_n\psi_{n-1} \tag{3.1.6}$$

   où $\gamma_n = \sqrt{\frac{h_n}{n_{n-1}}}$ et les coefficients $\beta_n$ dépendent du potentiel de départ.

2. Les zéros des polynômes orthogonaux sont toujours situés sur le contour d'intégration $\Gamma$ et les racines des polynômes se trouvent strictement entre les racines du polynôme de degré supérieur dans la suite (entrelacement).

En pratique, dès que le potentiel $V(x)$ dépasse le second ordre, le calcul analytique des polynômes orthogonaux devient difficile. Certes la relation de récurrence à trois termes 3.1.6 ramène le problème à la connaissance des coefficients $\beta_n$ et $\gamma_n$ qui peuvent être évalués numériquement. Ainsi, on sait par exemple ([42]) que si l'on définit :

$$Q = \begin{pmatrix} \beta_0 & \gamma_1 & 0 & 0 & \dots \\ \gamma_1 & \beta_1 & \gamma_2 & 0 & \dots \\ 0 & \gamma_2 & \beta_2 & \gamma_3 & \dots \\ 0 & 0 & \gamma_3 & \beta_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.1.7}$$

alors les coefficients $\beta_n$ et $\gamma_n$ obéissent aux équations :

$$\begin{aligned} \gamma_n [V'(Q)]_{n,n-1} &= \frac{nT}{N} \\ [V'(Q)]_{n,n} &= 0 \end{aligned} \tag{3.1.8}$$

Hélas, si cette derniere formule est compacte, elle donne lieu à des formules extrêmement compliquées lorsque le potentiel $V$ possède un degré élevé, si bien qu'en pratique le calcul des polynômes orthogonaux reste difficile en dehors de quelques cas connus.

## 3.2 Ecriture du problème de Riemann-Hilbert

L'écriture d'un problème de Riemann-Hilbert nécessite l'introduction des transformées de Cauchy $\tilde{\psi}_n(x)$ des polynômes orthogonaux précédents :

**Définition 3.3.** *Transformée de Cauchy des polynômes orthogonaux :*

$$\tilde{\psi}_n(x) = e^{\frac{1}{2}V(x)} \int_{\tilde{\Gamma}} dx \frac{e^{-\frac{1}{2}V(z)} \psi_n(z)}{x-z} \tag{3.2.1}$$

*Le contour $\tilde{\Gamma}$ doit être choisi dual (au sens de **[II]**) de $\Gamma$ définissant les polynômes orthogonaux.*

Le problème de Riemann-Hilbert, pour un potentiel $V(x) = \sum_{i=0}^{d} \frac{t_i}{i} x^i$, peut alors être formulé sur une matrice $2 \times 2$. Les polynômes orthogonaux et leurs transformées de Cauchy regroupés sous la forme de la matrice $2 \times 2$ :

**Théorème 3.1.** *Soit la matrice*

$$\Psi_n(x) = \begin{pmatrix} \psi_{n-1}(x) & \tilde{\psi}_{n-1}(x) \\ \psi_n(x) & \tilde{\psi}_n(x) \end{pmatrix} \tag{3.2.2}$$

*Alors $\Psi_n(x)$ est **l'unique** solution du problème de Riemann-Hilbert suivant (Cf. **[II]**, Section 2.2 de l'annexe VI) :*

1. *Régularité : $\forall z \notin \Gamma, z \mapsto \Psi_n(z)$ est analytique.*
2. *Système différentiel :*

$$\frac{d}{dx}\Psi_n(x) = \frac{d}{dx}\begin{pmatrix} \psi_{n-1} & \tilde{\psi}_{n-1} \\ \psi_n & \tilde{\psi}_n \end{pmatrix} = \mathscr{D}_n(x)\begin{pmatrix} \psi_{n-1} & \tilde{\psi}_{n-1} \\ \psi_n & \tilde{\psi}_n \end{pmatrix} \tag{3.2.3}$$

   *avec $\mathscr{D}_n(x) = \frac{1}{2}t_d x^{d-1}\sigma_3 + O\left(x^{d-2}\right)$ est une matrice polynômiale de degré $d-1$ et $\sigma_3 = diag(1,-1)$*

3. *Saut constant sur $\Gamma$ :*

$$\forall x \in \Gamma : \ \Psi_n(x)_+ = \Psi_n(x)_- J_n \tag{3.2.4}$$

   *où $J_n$ est une matrice constante (indépendante de x)*

4. *Asymptotique à l'infini : Si l'on définit la matrice $T_n(x) = (\frac{1}{2}V(x) - n\ln(x))\sigma_3$ alors*

$$\Psi_n(x) \sim C_n\left(Id + \frac{Y_{n,1}}{x} + \frac{Y_{n,2}}{x^2} + \dots\right)e^{T_n(x)} \tag{3.2.5}$$

   *où les $C_n$ et $Y_{n,i}$ sont des matrices constantes (indépendantes de x)*

Le problème de Riemann-Hilbert n'apporte pas en soi de nouvelles informations par rapport au calcul des polynômes orthogonaux. En revanche, il en constitue une reformulation pratique et standard sur laquelle beaucoup de méthodes sont applicables, en particulier la méthode de diffusion inverse («inverse scattering method») introduite par Ablowitz et Segur permettant d'obtenir des asymptotiques exacts des polynômes orthogonaux et des fonctions de partition (Cf. [42]). L'écriture d'un modèle hermitien à une matrice est un élément intéressant mais qui peut s'insérer plus généralement dans l'écriture d'un problème de Riemann-Hilbert pour un modèle à deux matrices hermitiennes. C'est dans ce contexte plus général, mais aussi plus technique, que j'ai réalisé mon travail (**[II]** présenté en annexe VI) avec Marco Bertola sur les fonctions tau-isomonodromiques. Avant de parler de ces fonctions tau qui sont un élément majeur dans la théorie de l'intégrabilité, il est préférable de définir et d'énoncer quelques propriétés du modèle à deux matrices.

### 3.3 Modèle à deux matrices hermitiennes et problème de Riemann-Hilbert associé

Le modèle à deux matrices hermitiennes se caractérise par la fonction de partition suivante :

$$Z_{2MM} = \int\int_{E_N} dM_1 dM_2 e^{-\mathrm{Tr}(V_1(M_1)+V_2(M_2)-M_1M_2)} \tag{3.3.1}$$

où $V_1(x)$ et $V_2(y)$ sont deux potentiels polynômiaux et $M_1$ et $M_2$ sont des matrices hermitiennes (ou normales) :

$$V_1(x) = \sum_{j=1}^{d_1+1} \frac{u_j}{j} x^j \ , \ V_2(y) = \sum_{j=1}^{d_2+1} \frac{v_j}{j} y^j \tag{3.3.2}$$

Ce problème peut également être rapporté à un problème aux valeurs propres moyennant l'utilisation de l'intégrale d'Itzykson-Zuber-Harish-Chandra ([139], [140]). Il se ramène alors à :

$$Z \sim \int\int_{\kappa} \left( \prod_{j=1}^{N} dx_j dy_j \right) \Delta^2(X)\Delta^2(Y) e^{-\sum_{j=1}^{N} V_1(x_j)+V_2(y_j)-x_j y_j} I(x_1,\ldots,x_N,y_1,\ldots,y_N) \tag{3.3.3}$$

où $I(x_1,\ldots,x_N,y_1,\ldots,y_N) = I(X,Y)$ est l'intégrale d'Itzykson-Zuber définie par :

$$I(X,Y) = \int_{\mathscr{U}_N} dU e^{\frac{N}{T}\mathrm{Tr}(XUYU^{-1})} = \left(\frac{N}{T}\right)^{-\frac{N(N-1)}{2}} \left( \prod_{p=1}^{N-1} p! \right) \frac{det(e^{\frac{N}{T}X_iY_j})_{i,j}}{\Delta(X)\Delta(Y)} \tag{3.3.4}$$

Le contour d'intégration des valeurs propres noté génériquement $\int\int_{\kappa}$ signifie en fait n'importe quelle combinaison linéaire de chemins admissibles (au sens où l'intégrale converge) :

$$\int\int_{\kappa} = \sum_{i,j} \kappa_{i,j} \int_{\Gamma_i} dx_i \int_{\hat{\Gamma}_j} dy_j \tag{3.3.5}$$

Comme dans le cas à une matrice, on peut alors introduire des polynômes bi-orthogonaux pour écrire cette intégrale. Ces polynômes moniques $\pi_n(x)$ et $\sigma_m(y)$ de degrés respectifs $n$ et $m$, sont définis par [18, 141] :

**Définition 3.4.** *Les polynômes bi-orthogonaux sont définis par :*

$$\int\int_{\kappa} \pi_n(x)\sigma_m(y)e^{-V_1(x)-V_2(y)+xy}dxdy = h_n\delta_{n,m} \tag{3.3.6}$$

Notons que cette fois-ci, les polynômes sont orthogonaux pour une "double intégration" ce qui rend leur calcul numérique encore plus délicat que pour les polynômes orthogonaux. La principale différence avec le cas des modèles à une matrice est que les matrices ne vont plus être de taille $2\times 2$, mais vont être de taille $d_1\times d_1$ ou $d_2\times d_2$. En cela, le cas à deux matrices est plus difficile d'un point de vue technique, mais la majeure partie des résultats du cas à une matrice s'étend pour le cas à deux matrices.

**Définition 3.5.** *Les pseudo-polynômes orthonormaux sont définis par :*

$$\psi_n(x) = \frac{1}{\sqrt{h_n}}\pi_n(x)e^{-V_1(x)}\ ,\ \phi_m(y) = \frac{1}{\sqrt{h_m}}\sigma_m(y)e^{-V_2(y)}$$

*On définit également les vecteurs :*

$$\Psi_N(x) = \begin{pmatrix} \psi_{N-d_2}(x) \\ \vdots \\ \psi_N(x) \end{pmatrix}\ ,\ \Phi_N(x) = \begin{pmatrix} \phi_{N-d_1}(x) \\ \vdots \\ \phi_N(x) \end{pmatrix} \tag{3.3.7}$$

Alors on a les résultats suivants (Cf. **[II]**, Section 3.2 de l'annexe VI) :

**Théorème 3.2.**

$$\Psi_{N+1}(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\alpha_{d_2}(N)}{\gamma(N)} & \cdots & \frac{-\alpha_1(N)}{\gamma(N)} & \frac{x-\alpha_0(N))}{\gamma(N)} \end{pmatrix}\Psi_N(x) \tag{3.3.8}$$

$$\Phi_{N+1}(y) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\beta_{d_1}(N)}{\gamma(N)} & \cdots & \frac{-\beta_1(N)}{\gamma(N)} & \frac{y-\beta_0(N))}{\gamma(N)} \end{pmatrix} \Phi_N(y) \tag{3.3.9}$$

*où, par définition, les nombres $\alpha_i(N)$, $\beta_i(N)$ et $\gamma(N)$ définissent les coefficients de la récurrence à $d_1$ et $d_2$ termes satisfaite par les $\psi_n(x)$ et les $\phi_m(y)$ :*

$$\begin{aligned} x\psi_n(x) &= \sum_{i=0}^{d_2} \alpha_i(n)\psi_{n-i}(x) + \gamma(n)\psi_{n+1}(x) \\ y\phi_n(x) &= \sum_{i=0}^{d_1} \beta_i(n)\phi_{n-i}(x) + \gamma(n)\phi_{n+1}(x) \end{aligned} \tag{3.3.10}$$

Il est également possible d'obtenir une relation matricielle reliant les dérivées :

$$\frac{d}{dx}\Psi_N(x) = \mathscr{D}_N(x)\Psi_N(x)\ ,\ \frac{d}{dx}\Phi_N(y) = \tilde{\mathscr{D}}_N(y)\Phi_N(y) \tag{3.3.11}$$

où les matrices $\mathscr{D}_N(x)$ et $\tilde{\mathscr{D}}_N(y)$ peuvent s'exprimer en fonction des coefficients $\alpha_i(N)$, $\beta_i(N)$ et $\gamma(N)$. Nous renvoyons à [33] pour une formule exacte. Comme pour le cas à une matrice, il est également possible de trouver un problème de Riemann-Hilbert satisfait par les polynômes bi-orthogonaux. Pour cela, il est nécessaire de définir l'équivalent de la transformée de Cauchy et d'obtenir des matrices de taille $d_1$ ou $d_2$. Ces résultats ont été établis dans [36] et peuvent se résumer de la manière suivante.

**Théorème 3.3.** *Définissons :*

$$\mathscr{C}_i[f](x) = \frac{1}{2i\pi}\int\int_{\hat{\kappa}} dzdy \frac{f(z)}{z-x} y^j e^{-V_1(z)-V_2(y)+zy} \tag{3.3.12}$$

*alors la matrice*

$$\Gamma_N(x) = \begin{pmatrix} \pi_N(x) & \mathscr{C}_0[\pi_N](x) & \dots & \mathscr{C}_{d_2-1}[\pi_N](x) \\ \\ \pi_{N-1}(x) & \mathscr{C}_0[\pi_{N-1}](x) & \dots & \mathscr{C}_{d_2-1}[\pi_{N-1}](x) \\ \vdots & \vdots & \vdots & \vdots \\ \pi_{N-d_2}(x) & \mathscr{C}_0[\pi_{N-d_2}](x) & \dots & \mathscr{C}_{d_2-1}[\pi_{N-d_2}](x) \end{pmatrix} \tag{3.3.13}$$

*satisfait le problème de Riemann-Hilbert suivant (résultat généralisé de [36] qui se limite à des polynômes de degré* 4 *) :*

1. *$z \mapsto \Gamma_N(z)$ est analytique sur $\mathbb{C}$ sauf sur les contours $\Gamma_j$ (introduits au départ dans la définition des polynômes orthogonaux 3.3.6) où elle présente un saut :*

$$\Gamma_{N,+}(z) = \Gamma_{N,-}(z) \begin{pmatrix} 1 & w_{j,1} & \dots & w_{j,d_2} \\ 0 & 1 & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \tag{3.3.14}$$

   *où*

$$w_{j,\nu} = e^{-V_1(x)} \sum_{k=1}^{d_2} \kappa_{j,k} \int_{\hat{\Gamma}_k} dy\, y^{\nu-1} e^{-V_2(y)+xy}$$

2. *Son asymptotique à l'infini est donné par :*

$$\Gamma_N(x) \sim \left( Id + \frac{Y_{1,N}}{x} + O\left(\frac{1}{x^2}\right) \right) \begin{pmatrix} x^N & 0 & 0 \\ 0 & x^{-m_N-1} Id_{r_N} & 0 \\ 0 & 0 & x^{-m_N} Id_{d_2-r_N} \end{pmatrix} \tag{3.3.15}$$

   *où l'on a défini $m_N$ et $r_N$ comme respectivement le quotient et le reste de la division Euclidienne de N par $d_2$ :*

$$N = m_N d_2 + r_N$$

En général, pour utiliser les techniques connues des problèmes de Riemann-Hilbert, en particulier les techniques d'isomonodromies vers lesquelles on se destine, il est pré-

férable d'avoir un problème de Riemann-Hilbert dans lequel les sauts sont constants. Dans notre cas, cela peut être fait en multipliant à droite par une matrice bien choisie. L'inconvénient est alors que l'asymptotique à l'infini devient beaucoup plus complexe. On trouve (les détails sont dans **[II]**, Section 2.2 de l'annexe VI)

**Théorème 3.4.** *Soit :* $\forall\, 1 \leq k \leq d_2$ *:*

$$\psi_m^{(k)}(x) \stackrel{def}{=} \frac{1}{2\pi i}\int_{\Gamma_k} ds \iint_{\kappa} dzdw \frac{\pi_m(z)e^{-V_1(z)}}{x-z}\frac{V_2'(s)-V_2'(w)}{s-w}e^{-V_2(w)+V_2(s)+zw-xs}, \quad 1 \leq k \leq d_2, \tag{3.3.16}$$

*et*

$$\psi_m^{(0)}(x) \stackrel{def}{=} \pi_m(x)e^{-V_1(x)} \tag{3.3.17}$$

*ainsi que la matrice de taille* $(d_2+1)\times(d_2+1)$ $\underset{N}{\Psi}(x)$ *(*$N \geq d_2$*) :*

$$\underset{N}{\Psi}(x) := \begin{bmatrix} \psi_N^{(0)}(x) & \dots & \psi_N^{(d_2)}(x) \\ \vdots & & \vdots \\ \psi_{N-d_2}^{(0)}(x) & \dots & \psi_{N-d_2}^{(d_2)}(x) \end{bmatrix} \tag{3.3.18}$$

*Le problème de Riemann-Hilbert satisfait par cette matrice est alors le suivant :*

*La matrice* $\underset{N}{\Psi}$ *est l'unique solution du problème de Riemann-Hilbert :*

1. *Régularité* $z \mapsto \underset{N}{\Psi}(z)$ *est analytique sur* $\mathbb{C}$ *sauf sur les contours* $\Gamma_j$

2. *Sauts constants :*

$$\underset{N}{\Psi}_+(x) = \underset{N}{\Psi}_-(x)\mathbf{H}^{(j)} \tag{3.3.19}$$

*avec*

$$\mathbf{H}^{(j)} := \mathbf{I} - 2\pi i \mathbf{e}_0 \kappa^T \ , \ \mathbf{e}_0 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ , \ \kappa := \begin{pmatrix} 0 \\ \kappa_{j1} \\ \vdots \\ \kappa_{jd_2} \end{pmatrix} \tag{3.3.20}$$

3. *Asymptotique à l'infini :*

$$\underset{N}{\Psi}(x) \sim \Gamma_N \begin{pmatrix} x^N e^{-V_1(x)} & 0 & 0 \\ 0 & x^{-m_N-1} Id_{r_N} & 0 \\ 0 & 0 & x^{-m_N} Id_{d_2-r_N} \end{pmatrix} \Psi_0(x) \tag{3.3.21}$$

*où*

$$\Gamma_N = Id + \frac{Y_{N,1}}{x} + ... \tag{3.3.22}$$

*et où $\Psi_0(x)$ est la solution "nue" dont l'asymptotique à l'infini peut être calculée par la méthode de «steepest descent».*

4. $\Psi'_N(x)\Psi_N^{-1} = D_N(x)$ *où $D_N(x)$ est polynômiale en x*
5. $\partial_{u_K}\Psi_N(x)\Psi_N^{-1} = U_{K,N}(x)$ *est polynômiale en x.*
6. $\partial_{v_J}\Psi_N(x)\Psi_N^{-1} = V_{J,N}(x)$ *est polynômiale en x.*
7. $\det(\Psi_{N+1}\Psi_N^{-1}) = Cste$

L'intérêt de pouvoir réécrire le problème sous la forme d'un problème de Riemann-Hilbert avec des sauts constants est qu'il permet de faire le lien avec la théorie des isomonodromies développées par Jimbo-Miwa-Ueno dans leur série d'articles [38] [39] [40].

## 3.4 Les transformations isomonodromiques

### 3.4.1 Systèmes Fuchsiens et équations de Schlesinger

Considérons un système d'équations différentielles du type :

$$\frac{dY}{dx} = AY = \sum_{i=1}^{n} \frac{A_i}{x-\lambda_i} Y \tag{3.4.1}$$

où $x \in \mathbb{C}$ et les $A_i$ sont des matrices $n \times n$ indépendantes de $x$. Les points $\lambda_i$ peuvent être vus comme des pôles de l'équation différentielle dont les matrices $A_i$ correspondantes seraient les résidus. Si l'on considère $Y$ une solution de 3.4.1, alors on peut produire

d'autres solutions en partant d'un point de base $b$ et en faisant le prolongement analytique des solutions le long d'une courbe qui entoure un des pôles $\lambda_i$. De cette façon, lorsque l'on est retourné au point de base $b$ après ce cheminement, on obtient une nouvelle solution $Y'$ différente de $Y$. Ces deux solutions sont reliées par une matrice de monodromies $M_i$ :

$$Y' = YM_i \tag{3.4.2}$$

Ainsi on peut établir un morphisme entre le groupe fondamental de $\mathbb{CP} \setminus \{\lambda_1 \dots \lambda_n\}$ (i.e. les lacets entourant les $\lambda_i$) et le groupe des matrices inversibles $GL_n(\mathbb{C})$ (les matrices de monodromies $M_i$). Il est clair que cette construction dépend du point de base $b$ d'où partent et où arrivent les lacets. Un changement de point de base correspond pour les matrices de monodromies à une conjugaison globale par une matrice fixée (caractérisant le changement de point de base). Si l'on veut donc s'affranchir du choix du point de base on ne s'intéressera donc aux matrices de monodromies qu'à une conjugaison globale près. La question légitime qui vient alors à l'esprit est de se demander ce que déterminent exactement ces matrices de monodromies. En particulier, la connaissance des matrices de monodromies suffit-elle à caractériser entièrement le système différentiel 3.4.1 ? La réponse à cette question est négative : il existe des systèmes Fuchsiens qui admettent des matrices de monodromies identiques et d'une façon générale un jeu de matrices de monodromies fixé correspond à plusieurs systèmes Fuchsiens. Notons ici que l'on ne tient pas compte des reformulations possibles d'un même système différentiel par simple changement de coordonnées (qui ne changera donc pas les monodromies), on se placera donc dans le cas où $A$ et :

$$g^{-1}(x)Ag(x) - g^{-1}(x)\frac{dg(x)}{dx} \tag{3.4.3}$$

sont considérées équivalents pour toute transformation $\tilde{x} = g(x)$ de coordonnées. Une autre question naturelle qui vient à l'esprit est de savoir si pour des matrices de monodromies données, il existe toujours un système fuchsien 3.4.1 qui redonne ces matrices. Il est connu depuis Plemelj que, sauf certains cas dégénérés dans lequel la réponse est négative, la réponse à cette question est affirmative résolvant ainsi le vingt-et-unième

problème de Hilbert.

### 3.4.2 Les transformations isomonodromiques

Si pour des matrices de monodromies données il existe en général beaucoup de systèmes Fuchsiens correspondants, on peut alors se demander quelques types de transformations "isomonodromiques" permettent de connecter ces différents systèmes. Si l'on suppose que les matrices $A_i$ dépendent de la position des pôles $\lambda_j$, il a été montré en 1912 par Schlesinger que dans le cas générique, les transformations isomonodromiques (i.e. ne changeant pas les matrices de monodromies) doivent satisfaire les équations d'holonomies intégrables connues maintenant sous le nom d'équations de Schlesinger [196] :

$$\begin{aligned} \frac{\partial A_i}{\partial \lambda_j} &= \frac{[A_i, A_j]}{\lambda_i - \lambda_j} \qquad j \neq i \\ \frac{\partial A_i}{\partial \lambda_i} &= -\sum_{j \neq i} \frac{[A_i, A_j]}{\lambda_i - \lambda_j} \end{aligned} \tag{3.4.4}$$

Notons que ces équations peuvent être interprétées comme des équations de courbure nulle sur l'espace des paramètres de déformations $\lambda_j$.

### 3.4.3 Les singularités d'ordre supérieur, la contribution de l'école japonaise

Un des buts de l'école japonaise de Jimbo-Miwa-Ueno a été de généraliser les résultats précédents dans le cadre de singularités d'ordre supérieur [38–40] :

$$\frac{dY}{dx} = AY = \sum_{i=1}^{n} \sum_{j=1}^{r_i+1} \frac{A_j^{(i)}}{(x-\lambda_i)^j} Y \tag{3.4.5}$$

où les matrices $A_j^{(i)}$ sont indépendantes de $x$. Cette fois-ci les données de monodromies sont plus délicates à définir. En effet, en plus des matrices de monodromies, il faut cette fois-ci ajouter des matrices de Stokes reliant des solutions entre deux secteurs de Stokes d'un même pôle. Enfin, il faut également rajouter des matrices de connexion reliant les solutions canoniques de différents secteurs de différents pôles. Les solutions canoniques

sont définies grâce à un théorème de Birkhoff. En effet, une solution simple mais purement formelle consiste à résoudre terme à terme en $x_i = x - \lambda_i$, les équations donnant la connexion $g_i$ ($T_j^{(i)}$ et $M^{(i)}$ sont diagonales) :

$$\frac{d(g_i^{-1}Z_i)}{dx_i} = \left( \sum_{j=1}^{r_i} \frac{(-j)T_j^{(i)}}{x_i^{j+1}} + \frac{M^{(i)}}{x_i} \right) (g_i^{-1}Z_i) \tag{3.4.6}$$

qui donneraient alors localement :

$$Z_i = g_i \exp\left( M^{(i)} \log(x_i) + \sum_{j=1}^{r_i} \frac{T_j^{(i)}}{x_i^j} \right) \tag{3.4.7}$$

Malheureusement, la résolution terme à terme en puissances de $x_i$ donne en général lieu à une série divergente. Cependant, le théorème de Birkhoff assure l'existence d'une unique solution convergente $G_i$ qui est asymptotiquement équivalente à $g_i$ dans un secteur du pôle $\lambda_i$. Dès lors, la solution :

$$Z_i = G_i \exp\left( M^{(i)} \log(x_i) + \sum_{j=1}^{r_i} \frac{T_j^{(i)}}{x_i^j} \right) \tag{3.4.8}$$

est bien définie dans un des secteurs du pôle $\lambda_i$ et constitue une solution canonique de 3.4.5 dans ce secteur. Les données de monodromies consistent alors en des matrices reliant les différents secteurs d'un même pôle (matrices de Stokes) ou entre différents pôles (matrices de monodromies ou matrices de connexion).

Les transformations isomonodromiques peuvent alors être définies commes les transformations préservant les données de monodromies. Si l'on s'autorise à varier les quantités suivant la position des pôles $\lambda_i$ et suivant les résidus diagonaux $T_j^{(i)}$ alors les transformations isomonodromiques d'un système (3.4.5) caractérisé par $A$ doivent satisfaire l'équation :

$$dA + [\Omega, A] + \frac{d\Omega}{dx} = 0 \tag{3.4.9}$$

où la 1-forme $\Omega$ est définie par :

$$\Omega = \sum_{i=1}^{n} \left( Ad\lambda_i - g_i D \left( \sum_{j=1}^{r_i} T_j^{(i)} \right) g_i^{-1} \right) \tag{3.4.10}$$

et $D$ représente la dérivation extérieure sur les $T_j^{(i)}$. Notons à nouveau que 3.4.9 possède l'interprétation géométrique d'une courbure nulle. On peut alors prouver que 3.4.9 permet de montrer que 3.4.10 est une forme fermée donc localement exacte, c'est-à-dire que l'on peut définir une fonction $\tau$-isomonodromique (à une constante multiplicative près) par :

$$\Omega = d(\ln\tau) \tag{3.4.11}$$

### 3.4.4 Propriétés d'intégrabilité des transformations isomonodromiques

Une des propriétés les plus intéressantes des transformations isomonodromiques (demontrée par Malgrange dans le cas Fuchsien et par Miwa dans le cas général) est que toutes les singularités essentielles des solutions sont fixées, bien que la position des pôles $\lambda_j$ puisse bouger. En d'autres termes, cela veut dire que les solutions satisfont automatiquement la propriété de Painlevé (singularités essentielles fixées) signifiant que l'on retrouve l'aspect des systèmes intégrables.

## 3.5 Fonctions de partition des modèles matriciels et fonction tau

La principale nouveauté apportée dans l'article **[II]**, Section 3 de l'annexe VI a été de généraliser la définition d'une fonction $\tau$-isomonodromique dans le cadre dégénéré du problème de Riemann-Hilbert 3.4. En effet dans ce cas la matrice :

$$S = \begin{pmatrix} x^N e^{-V_1(x)} & 0 & 0 \\ 0 & x^{-m_N-1} Id_{r_N} & 0 \\ 0 & 0 & x^{-m_N} Id_{d_2-r_N} \end{pmatrix} \tag{3.5.1}$$

possède des valeurs propres dégénérées, et la théorie générique de Jimbo-Miwa-Ueno, n'est donc a priori plus valide. Néanmoins, en définissant convenablement une 1-forme similairement à 3.4.10, on peut prouver que cette nouvelle 1-forme reste fermée et définie donc également une fonction $\tau$-isomonodromique. En particulier dans **[II]** (annexe VI) on montre que la bonne définition de la fonction tau s'inscrit dans le cadre très général suivant :

Soit une matrice $\Psi(x)$ vérifiant l'asymptotique :

$$\Psi(x) \sim Y(x)\,\Xi(x)\ ,\ \ Y(x) := \left(\mathbf{1} + \frac{Y_1}{x} + \frac{Y_2}{x^2} + \dots\right) x^S \tag{3.5.2}$$

où $\Xi(x) = \Xi(x;\mathbf{t})$ est une expression explicite supposée connue et $S$ est une matrice diagonale indépendante des temps d'isomonodromies et dont les valeurs propres peuvent être multiples (et qui dans l'application aux modèles de matrice sera donnée par 3.5.1). Cela implique en particulier que si l'on définit la 1-forme matricielle $\mathscr{H}(x;\mathbf{t})$ par :

$$\mathscr{H}(x;\mathbf{t}) = d\Xi(x;\mathbf{t})\,\Xi(x;\mathbf{t})^{-1} \tag{3.5.3}$$

alors $\mathscr{H}(x) = \sum \mathscr{H}_a dt_a$ est solution d'équation de courbure nulle :

$$\partial_a \mathscr{H}_b - \partial_b \mathscr{H}_a = [\mathscr{H}_a, \mathscr{H}_b] \tag{3.5.4}$$

Dans ce contexte général, la fonction tau s'obtient alors par la formule :

**Définition 3.6.** *La fonction tau est la 1-forme définie par :*

$$\omega := \sum \omega_a dt^a := \sum_a \operatorname{res}\operatorname{tr}\left(Y^{-1}Y'\mathscr{H}_a\right) dt^a \tag{3.5.5}$$

Les résultats principaux de l'article ont consisté en la démonstration de la fermeture de la 1-forme et de l'obtention de l'égalité entre cette fonction tau et la fonction de partition du modèle de matrices. En particulier une partie importante de la démonstration a consisté à utiliser des transformations de Schlesinger discrètes sur le paramètre $N$ (taille des matrices) pour expliciter le rapport $\frac{\tau_{N+1}}{\tau_N}$ en fonction d'un des coefficients de $Y_{N,1}$ (Cf.

équation 3.3.22). Une fois cela établi, la définition même de ce coefficient ainsi que les propriétés d'orthogonalité des polynômes bi-orthogonaux permettent de constater que ce même coefficient est en fait le rapport $\frac{Z_{N+1}}{Z_N}$. Finalement, l'étude pour $N = 1$ permet de montrer le théorème suivant (Cf. **[II]**, Section 3.2 de l'annexe VI) :

**Théorème 3.5.** *La fonction de partition du modèle à deux matrices et la fonction tau-isomonodromique vérifie l'égalité (**[II]** présenté en annexe VI) :*

$$\forall N \in \mathbb{N} : Z_N = (v_{d_2+1})^{\frac{d_2\alpha_N(\alpha_N+1)}{2}+\alpha_N(N-d_2\alpha_N)}\,\tau_N \tag{3.5.6}$$

*avec $\alpha_N = E\left(\frac{N}{d_2}\right)$ (l'égalité étant entendue à une constante multiplicative près indépendante de N et des potentiels. $E(x)$ désigant la partie entière de x).*

En d'autres termes, à l'exception d'une puissance multiplicative en $v_{d_2+1}$ (qui provient d'une mauvaise normalisation de la fonction $\tau$) **on retrouve, après avoir généralisé la définition de Jimbo-Miwa-Ueno à un cas où l'asymptotique est dégénéré, le fait que la fonction de partition du modèle à deux matrices hermitiennes est une fonction $\tau$-isomonodromique.** Ce résultat était déjà connu pour le cas à une matrice hermitienne après le travail de M.Bertola, B.Eynard et J. Harnad [37] et permet de renforcer le lien profond entre les modèles de matrices aléatoires (au moins ceux où les matrices sont hermitiennes) et la théorie de l'intégrabilité. Nous renvoyons le lecteur intéressé par la démonstration complète à l'article **[II]** situé en annexe VI.

## 3.6 Cas des modèles non-hermitiens

Après avoir vu que les fonctions de partition des modèles à une puis deux matrices hermitiennes donnent lieu à des fonctions $\tau$-isomonodromiques, il est naturel de se demander si le résultat s'étend à d'autres ensembles de matrices non-hermitiennes. La réponse à cette question n'est pas connue à l'heure actuelle. En effet, un ingrédient crucial dès le départ a été de réduire le calcul de la fonction de partition à une intégrale sur les valeurs propres puis à un problème de polynômes (bi)-orthogonaux grâce aux propriétés du déterminant de Vandermonde (3.1.5). Mais, dans le cas où la puissance du déter-

minant de Vandermonde n'est pas deux, l'usage des polynômes (bi)-orthogonaux n'est plus possible (pour le cas symétrique réel et quaternionique self-dual il est néanmoins possible de définir des polynômes skew-orthogonaux [19]). Dans le cas où l'exposant est $\beta$-quelconque, des liens avec les polynômes de Jack ou de McDonald peuvent être espérés, mais les propriétés de ces polynômes restent quasiment inconnues et aucune reformulation en termes de problème de Riemann-Hilbert n'est connue à ce jour. Sans cette reformulation, la question de la définition d'une fonction $\tau$ et de son éventuel lien avec la fonction de partition reste sans objet. En revanche, comme nous le verrons par la suite, d'autres méthodes, en particulier la méthode des équations de boucles, se généralisent plus directement à des matrices non-hermitiennes où la puissance du déterminant de Vandermonde est arbitraire.

# CHAPITRE 4

# MODÈLES DE MATRICES POUR $\beta$-ARBITRAIRE

## 4.1 Généralisation des modèles matriciels aux "ensembles $\beta$"

Jusqu'à présent nous avons considéré des modèles de matrices hermitiennes, c'est-à-dire que l'intégrale était définie sur des matrices hermitiennes. Cependant, comme nous l'avons mentionné auparavant, d'autres ensembles de matrices peuvent avoir un intérêt physique, comme les matrices symétriques réelles ou les matrices quaternioniques self-duales. Dans ces deux cas précis, la diagonalisation du problème en un problème aux valeurs propres est connue et est donnée par :

$$Z_N \propto \int_{\mathbb{C}^N} d\lambda_1 \dots d\lambda_N \, \Delta(\lambda)^{2\beta} e^{-\frac{N}{T}\sum_{i=1}^{N} V(\lambda_i)} \tag{4.1.1}$$

où la valeur du paramètre $\beta$ vaut 1, $1/2$ ou 2 selon l'ensemble de matrice choisie (Cf. 2.2.1). Dès lors, il est intéressant de se demander si l'on ne pourrait pas étudier directement ces trois ensembles de matrices en conservant ce paramètre $\beta$ arbitraire, afin de traiter les trois modèles d'un seul coup. Cette approche constitue ce que l'on appelle dans la littérature *le modèle à une matrice avec $\beta$-quelconque*. La généralisation au modèle à deux matrices est plus délicate. En effet, pour le cas hermitien $\beta = 1$, la diagonalisation en un problème aux valeurs propres fait intervenir l'intégrale d'Itzykson-Zuber (3.3.4) sur le groupe unitaire, spécifique aux matrices hermitiennes. Il est donc nécessaire de trouver une généralisation naturelle de cette intégration. La généralisation pour $\beta$-quelconque du modèle à deux matrices est donnée par :

$$Z_\beta \overset{def}{=} \int dX dY \Delta(X)^{2\beta} \Delta(Y)^{2\beta} e^{-\frac{N\beta}{T}\left[\sum_{i=1}^{N} V_1(x_i) + \sum_{j=1}^{N} V_2(y_j)\right]} I_\beta(X,Y) \tag{4.1.2}$$

où $I_\beta(X,Y)$ est la généralisation "naturelle" de l'intégrale d'Itzykson-Zuber que l'on va maintenant décrire plus en détail.

## 4.2 L'intégrale d'Itzykson-Zuber généralisée

Dans le cas hermitien, l'intégrale d'Itzykson-Zuber est définie par :

$$I_{\text{herm}}(X,Y) = \int_{\mathscr{U}_N} dU e^{\frac{N}{T}\text{Tr}(XUYU^{-1})} \tag{4.2.1}$$

où $\mathscr{U}_N$ est le groupe unitaire équipé de la mesure de Haar. Il est également intéressant de définir les quantités suivantes :

$$M_{i,j} = \int_{\mathscr{U}_N} dU \|U_{i,j}\|^2 e^{\frac{N}{T}\text{Tr}(XUYU^{-1})} \tag{4.2.2}$$

qui peuvent être utilisées pour déterminer l'intégrale d'Itzykson-Zuber par la formule :

$$I_{\text{herm}}(X,Y) = \sum_{i=1}^{N} M_{i,j} = \sum_{j=1}^{N} M_{i,j} \tag{4.2.3}$$

Cette dernière formule est évidente puisque $\sum\limits_{i=1}^{N} \|U_{i,j}\|^2 = 1 = \sum\limits_{j=1}^{N} \|U_{i,j}\|^2$ sur le groupe unitaire. Dans le cas hermitien, les $M_{i,j}$ sont connus pour satisfaire l'équation de Dunkl [43] :

$$\forall\, 1 \leq i,j \leq N : \frac{\partial}{\partial x_i} M_{i,j} + \sum_{k\neq i} \frac{M_{i,j} - M_{k,j}}{x_i - x_k} = y_j M_{i,j} \tag{4.2.4}$$

qui sera à la base de la généralisation au cas où $\beta$ est quelconque. Dans [43], les auteurs montrent que l'on peut généraliser les $M_{i,j}$ au cas où $\beta$ est arbitraire (que l'on notera $M_{i,j}^{(\beta)}$) par les conditions suivantes :

**Définition 4.1.** *Les $M_{i,j}^{(\beta)}$ sont définis par les propriétés suivantes :*

1. *Les $M_{i,j}^{(\beta)}$ satisfont l'équation de Calogero-Moser-Dunkl généralisée :*

$$\forall\, 1 \leq i,j \leq N : \frac{\partial}{\partial x_i} M_{i,j}^{(\beta)} + \beta \sum_{k\neq i} \frac{M_{i,j}^{(\beta)} - M_{k,j}^{(\beta)}}{x_i - x_k} = \frac{N\beta}{T} y_j M_{i,j}^{(\beta)} \tag{4.2.5}$$

*(Le facteur $\frac{N\beta}{T}$, absent de [43], provient de la présence dans l'exponentielle du préfacteur $\frac{N\beta}{T}$ qui peut être absorbé par le changement de variables $Y \leftrightarrow \frac{N\beta}{T}Y$).*

2. *$M_{i,j}^{(\beta)}$ doivent être des matrices stochastiques : $I_\beta = \sum_{i=1}^{N} M_{i,j}^{(\beta)}$ doit être indépendant de $j$ et $I_\beta = \sum_{j=1}^{N} M_{i,j}^{(\beta)}$ doit être indépendant de i.*

3. *La fonction $(X,Y) \mapsto I_\beta(X,Y) = \sum_{i=1}^{N} M_{i,j}^{(\beta)}(X,Y) = \sum_{j=1}^{N} M_{i,j}^{(\beta)}(X,Y)$ doit être une fonction symétrique de ses variables.*

Ces conditions permettent de définir les $M_{i,j}^{(\beta)}$ de façon unique à l'exception d'une constante multiplicative globale sans intérêt. De plus, ces conditions sont vérifiées pour les trois cas connus : $\beta = 1, \frac{1}{2}$ et 2 pour lesquels des démonstrations spécifiques existent (utilisant les propriétés spécifiques de ces ensembles de matrices). Il est alors logique de définir l'intégrale d'Itzykson-Zuber généralisée par la formule connue pour le cas hermitien :

**Définition 4.2.** *L'intégrale d'Itzykson-Zuber généralisée est définie par :*

$$I_\beta(X,Y) \overset{def}{=} \sum_{i=1}^{N} M_{i,j}^{(\beta)}(X,Y) = \sum_{j=1}^{N} M_{i,j}^{(\beta)}(X,Y) \tag{4.2.6}$$

Les indices $i$ et $j$ dans les sommes peuvent être choisis de façon arbitraire puisque les sommes en sont indépendantes : notons qu'à nouveau, cette définition recouvre les trois cas connus $\beta = 1, \frac{1}{2}, 2$. Cette définition permet alors de montrer que $I_\beta(X,Y)$ satisfait l'équation suivante :

**Théorème 4.1.** *L'intégrale d'Itzykson-Zuber généralisée vérifie l'équation :*

$$H_X^{(\beta)} I_\beta \overset{def}{=} \sum_{i=1}^{N} \frac{\partial^2 I_\beta}{\partial x_i^2} + \beta \sum_{i \neq j} \frac{1}{x_i - x_j} \left( \frac{\partial I_\beta}{\partial x_i} - \frac{\partial I_\beta}{\partial x_j} \right) = \left( \frac{N\beta}{T} \right)^2 \left( \sum_{j=1}^{N} y_j^2 \right) I_\beta \tag{4.2.7}$$

*où $H_X^{(\beta)}$ est l'Hamiltonien de Calogero-Moser ce qui légitime le choix de la généralisation.*

La démonstration de cette identité est facile :

Preuve de 4.2.7 :

Observons d'abord que :

$$\frac{\partial I_\beta}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} M_{i,j}^{(\beta)}\right)$$

en prenant la somme sur $j$ de 4.2.5 et en observant que $\sum_{j=1}^{N}\left(M_{i,j}^{(\beta)} - M_{k,j}^{(\beta)}\right) = 0$ pour tout $i$ et $k$, on trouve que :

$$\frac{\partial I_\beta}{\partial x_i} = \frac{N\beta}{T}\sum_{j=1}^{N} y_j M_{i,j}^{(\beta)}$$

En dérivant cette égalité, on obtient alors :

$$\begin{aligned}
\frac{\partial^2 I_\beta}{\partial x_i^2} &= \frac{N\beta}{T}\sum_{j=1}^{N} y_j \frac{\partial M_{i,j}^{(\beta)}}{\partial x_i} \\
&= \frac{N\beta}{T}\sum_{j=1}^{N} y_j \left(\frac{N\beta}{T} y_j M_{i,j}^{(\beta)} - \beta \sum_{k\neq i} \frac{M_{i,j}^{(\beta)} - M_{k,j}^{(\beta)}}{x_i - x_k}\right) \\
&= \left(\frac{N\beta}{T}\right)^2 \sum_{j=1}^{N} y_j^2 M_{i,j}^{(\beta)} - \beta \sum_{k\neq i} \frac{\frac{\partial I_\beta}{\partial x_i} - \frac{\partial I_\beta}{\partial x_k}}{x_i - x_k}
\end{aligned} \tag{4.2.8}$$

qui donne exactement 4.2.7.

Ainsi, il est alors naturel de définir le modèle à deux matrices avec $\beta$-quelconque de la façon suivante :

**Définition 4.3.** *le modèle à deux matrices avec $\beta$-quelconque est défini par :*

$$Z_\beta \overset{def}{=} \int\int dX dY \Delta(X)^{2\beta} \Delta(Y)^{2\beta} e^{-\frac{N\beta}{T}\left[\sum_{i=1}^{N} V_1(x_i) + \sum_{j=1}^{N} V_2(y_j)\right]} I_\beta(X,Y) \tag{4.2.9}$$

*avec l'intégrale d'Itzykson-Zuber généralisée $I_\beta(X,Y)$ définie par (4.2.5) et (4.2.6).*

Notons en particulier que dans le cas où $V_2(y)$ est un potentiel quadratique on retombe sur le modèle à une matrice avec $\beta$-quelconque précédemment décrit.

## 4.3 Equations de boucles pour le modèle à deux matrices et $\beta$-quelconque

Maintenant que le modèle à deux matrices est généralisé pour des valeurs de $\beta$ quelconque, il faut se demander quelles méthodes employer pour le résoudre. D'après ce que l'on a vu précédemment, il est clair que pour une valeur arbitraire de $\beta$, la méthode des polynômes orthogonaux ou bi-orthogonaux ne pourra pas fonctionner car elle est spécifique de la puissance 2 du déterminant de Vandermonde. Cela dit, l'utilisation de certains types de polynômes pourrait peut être permettre la résolution de ce modèle comme c'est le cas dans le cas hermitien, mais à l'heure actuelle, aucune réponse générale n'a été trouvée bien que l'utilisation des polynômes de Jack ou de MacDonald semble être une possibilité. La seconde alternative consiste alors à utiliser l'approche des équations de boucles. Comme nous allons le voir, l'écriture des équations de boucles possède l'avantage de se généraliser relativement facilement au cas où $\beta$ est arbitraire. En revanche, la résolution de ces équations de boucles devient beaucoup plus délicate en dehors du cas hermitien dont la spécificité ressort nettement dans les équations. Cette section aura donc pour objet de décrire en détail l'obtention des équations de boucles pour le modèle à deux matrices avec $\beta$-quelconque. Le lecteur intéressé pourra alors retrouver le cas hermitien en prenant $\beta = 1$, ainsi que le cas à une matrice en prenant $V_2(y) = \frac{y^2}{2}$.

### 4.3.1 Notations

La principale difficulté dans l'écriture des équations de boucles se situe dans le nombre important d'indices et de fonctions à définir préalablement. Nous utiliserons donc les notations suivantes :

**Définition 4.4.** – *Les potentiels sont supposés polynômiaux :*

$$V_1'(x) = \sum_{k=0}^{d_1} t_k x^k \qquad , \qquad V_2'(x) = \sum_{k=0}^{d_2} \tilde{t}_k x^k \tag{4.3.1}$$

– *Les résolvantes sont définies par :*

$$W_n(z_1,\dots,z_n) = \left\langle \sum_{i_1,\dots,i_n=1}^{N} \frac{1}{z_1 - x_{i_1}} \dots \frac{1}{z_n - x_{i_n}} \right\rangle_c \tag{4.3.2}$$

*où les crochets $\langle\rangle$ indiquent que l'on prend la valeur moyenne relativement à la mesure définie par 4.1.2. L'indice $_c$ indique que l'on prend la partie connexe lors d'un produit de traces. Par exemple si l'on note : $X = (x_1,\dots,x_N)$ et Y=$(y_1,\dots,y_N)$, alors*

$$\langle A(X,Y)\rangle \stackrel{def}{=} \frac{1}{Z_\beta} \int dX dY A(X,Y) e^{-\frac{N\beta}{T}(\mathrm{tr}V_1(X)+V_2(Y))} \Delta(X)^{2\beta} \Delta(Y)^{2\beta} I_\beta(X,Y) \tag{4.3.3}$$

*et la partie connexe indique que l'on doit calculer :*

$$\langle A(X,Y)B(X,Y)\rangle_c = \langle A(X,Y)B(X,Y)\rangle - \langle A(X,Y)\rangle \langle B(X,Y)\rangle \tag{4.3.4}$$

*Pour simplifier un peu les notations, nous noterons $W(x) = W_1(x)$ pour la première résolvante qui joue un rôle particulier.*

– *Afin de fermer les équations de boucles, nous allons avoir besoin des fonctions suivantes :*

$$U_n(x,y;z_1,\dots,z_n) = \sum_{i,j,i_1,\dots,i_n=1}^{N} \left\langle \frac{1}{x-x_i} \frac{M_{i,j}^{(\beta)}}{I_\beta} \frac{V_2'(y) - V_2'(y_j)}{y-y_j} \frac{1}{z_1 - x_{i_1}} \dots \frac{1}{z_n - x_{i_n}} \right\rangle_c \tag{4.3.5}$$

*qui est un polynôme en y. Finalement nous aurons besoin également de :*

$$P_n(x,y;z_1,\dots,z_n) = \sum_{i,j,i_1,\dots,i_n=1}^{N} \left\langle \frac{V_1'(x) - V_2'(x_i)}{x-x_i} \frac{M_{i,j}^{(\beta)}}{I_\beta} \frac{V_2'(y) - V_2'(y_j)}{y-y_j} \frac{1}{z_1 - x_{i_1}} \dots \frac{1}{z_n - x_{i_n}} \right\rangle_c \tag{4.3.6}$$

*qui est un polynôme à la fois en x et en y.*

Pour résoudre les futures équations de boucles, nous allons avoir besoin d'écrire le développement topologique (identique à celui de [23]) des fonctions précédentes . Afin

de garantir l'existence de tels développements, nous nous plaçons dans le cas d'un modèle "formel" de matrices, c'est-à-dire que nous supposons l'existence d'un développement perturbatif en puissances de $\frac{1}{N}$, sans nous préocupper de la convergence des séries (considérées comme formelles).

**Définition 4.5.** *Le développement topologique des fonctions de corrélation est défini par :*

$$\begin{aligned}
W_n(x_1,\dots,x_n) &= \beta^{-\frac{n}{2}} \sum_{g=0}^{\infty} \left(\frac{N\sqrt{\beta}}{T}\right)^{2-2g-n} W_n^{(g)}(x_1,\dots,x_n) \\
U_0(x,y) &= \frac{N}{T}\left(U_0^{(0)}(x,y) - x + V_2'(y)\right) + \beta^{-\frac{1}{2}} \sum_{g=1}^{\infty} \left(\frac{N\sqrt{\beta}}{T}\right)^{1-2g} U_0^{(g)}(x,y) \\
U_n(x,y;x_1,\dots,x_n) &= \beta^{-\frac{n+1}{2}} \sum_{g=0}^{\infty} \left(\frac{N\sqrt{\beta}}{T}\right)^{2-2g-(n+1)} U_n^{(g)}(x,y;x_1,\dots,x_n) \\
P_0(x,y) &= \frac{N}{T}\left(P_0^{(0)}(x,y) + \hbar - T\right) + \beta^{-\frac{1}{2}} \sum_{g=1}^{\infty} \left(\frac{N\sqrt{\beta}}{T}\right)^{1-2g} P_0^{(g)}(x,y) \\
P_n(x,y;x_1,\dots,x_n) &= \beta^{-\frac{n+1}{2}} \sum_{g=0}^{\infty} \left(\frac{N\sqrt{\beta}}{T}\right)^{2-2g-(n+1)} P_n^{(g)}(x,y;x_1,\dots,x_n)
\end{aligned} \tag{4.3.7}$$

Une remarque importante est que nous avons choisi ici de translater les fonctions $U_0^{(0)}(x,y)$ et $P_0^{(0)}(x,y)$. Bien que cela puisse paraître étrange, cela permettra par la suite de simplifier légèrement l'écriture des équations de boucles. Par ailleurs, nous avons omis pour des raisons de simplicité d'écriture la dépendance des fonctions ($W_n$, $W_n^{(g)}$, $P_n^{(g)}$, $U_n^{(g)}$ etc.) dans le paramètre $\beta$.

Finalement, il est aussi utile de définir les nombres $F_g$ comme le développement topologique de la fonction de partition elle-même :

**Définition 4.6.** *La fonction de partition s'écrit formellement :*

$$Z_\beta = e^F \qquad , \qquad F = \sum_{g=0}^{\infty} \left(\frac{N\sqrt{\beta}}{T}\right)^{2-2g} F_g \tag{4.3.8}$$

ainsi que le paramètre $\hbar$ qui jouera un rôle crucial dans la suite et que l'on obtient à partir de $\beta$ par :

**Définition 4.7.** *Le paramètre $\hbar$ est relié au paramètre $\beta$ par la relation :*

$$\boxed{\hbar = \frac{T}{N}\left(1-\frac{1}{\beta}\right) = \frac{T}{N\sqrt{\beta}}\left(\sqrt{\beta}-\frac{1}{\sqrt{\beta}}\right)} \tag{4.3.9}$$

Tout comme pour le cas hermitien, nous introduisons également les opérateurs "d'insertion de boucles" définis par :

**Définition 4.8.** *Les opérateurs d'insertion sont définis par :*

$$\begin{aligned} \frac{\partial}{\partial V_1(x)} &:= -\sum_{k=1}^{\infty} \frac{1}{x^{k+1}} k \frac{\partial}{\partial t_{k-1}} \\ \frac{\partial}{\partial V_2(x)} &:= -\sum_{k=1}^{\infty} \frac{1}{y^{k+1}} k \frac{\partial}{\partial \tilde{t}_{k-1}} \end{aligned} \tag{4.3.10}$$

Ils possèdent les propriétés :

$$\frac{\partial V_j(x)}{\partial V_l(x')} = \delta_{j,l}\frac{1}{x-x'} \qquad , \qquad \frac{\partial V_j'(x)}{\partial V_l(x')} = -\delta_{j,l}\frac{1}{(x-x')^2} \tag{4.3.11}$$

Ces opérateurs sont particulièrement intéressants car ils permettent de passer d'une résolvante à la suivante :

$$\frac{N\beta}{T}W(x) = \frac{\partial F}{\partial V_1(x)} \qquad , \qquad \frac{\partial F_g}{\partial V_1(x)} = W_1^{(g)}(x) \tag{4.3.12}$$

et également :

$$\begin{aligned} \frac{N\beta}{T}W_n(x_1,\ldots,x_n) &= \frac{\partial W_{n-1}(x_1\ldots,x_{n-1})}{\partial V_1(x_n)} \\ \frac{N\beta}{T}U_n(x,y;x_1,\ldots,x_n) &= \frac{\partial U_{n-1}(x,y;x_1\ldots,x_{n-1})}{\partial V_1(x_n)} \end{aligned} \tag{4.3.13}$$

ce qui donne dans les développements topologiques :

$$\begin{aligned} W_n^{(g)}(x_1,\ldots,x_n) &= \frac{\partial W_{n-1}^{(g)}(x_1\ldots,x_{n-1})}{\partial V_1(x_n)} \\ U_n^{(g)}(x,y;x_1,\ldots,x_n) &= \frac{\partial U_{n-1}^{(g)}(x,y;x_1\ldots,x_{n-1})}{\partial V_1(x_n)} \end{aligned} \tag{4.3.14}$$

On voit donc que la connaisance des $F_g$ permet ensuite par simple application de ces opérateurs de dérivation de trouver tous les $W_n^{(g)}(x_1,\ldots,x_n)$ correspondants.

Enfin, afin d'avoir des notations plus compactes, nous introduisons les fonctions translatées :

**Définition 4.9.** *Soit la fonction translatée :*

$$Y(x) := (V_1'(x) - W_1^{(0)}(x)) \tag{4.3.15}$$

*On définit la courbe spectrale par :*

$$E(x,y) := (V_1'(x)-y)(V_2'(y)-x) - P_0^{(0)}(x,y) \tag{4.3.16}$$

Notons que la plupart de nos fonctions étant polynômiales en $y$, nous pouvons les développer sur la base des $y^k$ de la façon suivante (en prenant en compte le degré) :

**Définition 4.10.** *Le développement en puissances de $y^k$ donnent les identités formelles suivantes :*

$$\begin{aligned} U_n(x,y;x_1,\ldots,x_n) &= \sum_{k=0}^{d_2} U_{n,k}(x;x_1,\ldots,x_n)y^k \\ P_n(x,y;x_1,\ldots,x_n) &= \sum_{k=0}^{d_2-1} P_{n,k}(x;x_1,\ldots,x_n)y^k \\ E(x,y) &= \sum_{k=0}^{d_2+1} E_k(x)y^k \\ U_n^{(g)}(x,y;x_1,\ldots,x_n) &= \sum_{k=0}^{d_2} U_{n,k}^{(g)}(x;x_1,\ldots,x_n)y^k \end{aligned}$$

$$P_n^{(g)}(x,y;x_1,\dots,x_n) \quad = \quad \sum_{k=0}^{d_2-1} P_{n,k}^{(g)}(x;x_1,\dots,x_n)y^k \tag{4.3.17}$$

*D'une façon générale, un indice $_k$ supplémentaire et l'absence de variable y signifie que l'on a pris la projection sur $y^k$.*

Une fois toutes ces notations introduites, nous pouvons passer à l'écriture des équations de boucles. Celle-ci se fera en deux temps par l'écriture de deux intégrales triviales.

### 4.3.2 Etape une : un résultat préliminaire

Intéressons-nous tout d'abord à l'intégrale nulle (car on peut intégrer d'abord sur $y_j$ une dérivée totale et le contour d'intégration est supposé sans bords) suivante :

$$0 = \sum_{i,j=1}^{N} \int dX dY \, \frac{\partial}{\partial y_j} \left( e^{-\frac{N\beta}{T}\mathrm{tr}(V_1(X)+V_2(Y))} \Delta(X)^{2\beta} \Delta(Y)^{2\beta} \frac{1}{x-x_i} M_{i,j}^{(\beta)} \right) \tag{4.3.18}$$

On peut faire agir la dérivée sur chaque terme, ce qui donne trois contributions différentes :

– Agissant sur l'exponentielle on trouve :

$$-\frac{N\beta}{T} \sum_{i,j=1}^{N} \left\langle V_2'(y_j) \frac{1}{x-x_i} \frac{M_{i,j}^{(\beta)}}{I_\beta} \right\rangle \tag{4.3.19}$$

– Agissant sur le déterminant de Vandermonde, on trouve :

$$2\beta \left\langle \sum_{i,j=1}^{N} \sum_{k\neq i} \frac{1}{y_j - y_k} \frac{1}{x-x_i} \frac{M_{i,j}^{(\beta)}}{I_\beta} \right\rangle \tag{4.3.20}$$

– Finalement, agissant sur $M_{i,j}^{(\beta)}$ et en utilisant l'équation différentielle satisfaite par

les $M_{i,j}^{(\beta)}$ (4.2.5) on trouve :

$$\left\langle \sum_{i,j=1}^{N} \frac{1}{x-x_i} \left( \frac{N\beta}{T} x_i M_{i,j}^{(\beta)} - \beta \sum_{k\neq j} \frac{M_{i,j}^{(\beta)} - M_{i,k}^{(\beta)}}{y_j - y_k} \right) \frac{1}{I_\beta} \right\rangle \tag{4.3.21}$$

On voit alors que 4.3.20 s'annule avec la dernière partie de 4.3.21 ce qui donne :

$$\sum_{i,j=1}^{N} \left\langle V_2'(y_j) \frac{1}{x-x_i} \frac{M_{i,j}^{(\beta)}}{I_\beta} \right\rangle = \sum_{i,j=1}^{N} \left\langle \frac{x_i}{x-x_i} \frac{M_{i,j}^{(\beta)}}{I_\beta} \right\rangle \tag{4.3.22}$$

Comme $\sum_{j=1}^{N} M_{i,j}^{(\beta)} = I_\beta$ on trouve finalement :

$$\boxed{\sum_{i,j=1}^{N} \left\langle V_2'(y_j) \frac{1}{x-x_i} \frac{M_{i,j}^{(\beta)}}{I_\beta} \right\rangle = -N + xW(x)} \tag{4.3.23}$$

concluant ainsi la première étape.

### 4.3.3 Etape deux : les équations de boucles

La deuxième étape consiste à regarder l'intégrale nulle (pour des raisons similaires à la précédente) suivante :

$$0 = \sum_{i,j=1}^{N} \int dX dY \, \frac{\partial}{\partial x_i} \left( e^{-\frac{N\beta}{T}\mathrm{tr}(V_1(X)+V_2(Y))} \Delta(X)^{2\beta} \Delta(Y)^{2\beta} \frac{1}{x-x_i} M_{i,j}^{(\beta)} \frac{V_2'(y) - V_2'(y_j)}{y-y_j} \right) \tag{4.3.24}$$

Le lecteur remarquera que cette intégrale est très similaire aux définitions des fonctions $U_0(x,y)$ et $P_0(x,y)$ (4.3.5 et 4.3.6). A nouveau, on peut faire agir la dérivation sur chacun des termes du produit. Il y a cette fois-ci quatre contributions :

– Agissant sur l'exponentielle on trouve :

$$-\frac{N\beta}{T}\sum_{i,j=1}^{N}\left\langle \frac{V_1'(x_i)}{x-x_i}\frac{M_{i,j}^{(\beta)}}{I_\beta}\frac{V_2'(y)-V_2'(y_j)}{y-y_j}\right\rangle (i) \tag{4.3.25}$$

– Agissant sur le déterminant de Vandermonde, on trouve :

$$2\beta\sum_{i,j=1}^{N}\left\langle \sum_{k\neq i}\frac{1}{x_i-x_k}\frac{1}{x-x_i}\frac{M_{i,j}^{(\beta)}}{I_\beta}\frac{V_2'(y)-V_2'(y_j)}{y-y_j}\right\rangle (ii) \tag{4.3.26}$$

– Agissant sur $\frac{1}{x-x_i}$ on trouve :

$$\sum_{i,j=1}^{N}\left\langle \frac{1}{(x-x_i)^2}\frac{M_{i,j}^{(\beta)}}{I_\beta}\frac{V_2'(y)-V_2'(y_j)}{y-y_j}\right\rangle (iii) \tag{4.3.27}$$

– Enfin, agissant sur $M_{i,j}^{(\beta)}$ et en utilisant de nouveau 4.2.5 on trouve :

$$\sum_{i,j=1}^{N}\left\langle \frac{1}{x-x_i}\frac{1}{I_\beta}\left(\frac{N\beta}{T}y_jM_{i,j}^{(\beta)}-\beta\sum_{k\neq i}\frac{M_{i,j}^{(\beta)}-M_{k,j}^{(\beta)}}{x_i-x_k}\right)\frac{V_2'(y)-V_2'(y_j)}{y-y_j}\right\rangle (iv) \tag{4.3.28}$$

Maintenant, il convient de remarquer les identités suivantes. Tout d'abord, dans 4.3.25, on peut séparer $V_1'(x_i)=V_1'(x_i)-V_1'(x)+V_1'(x)$ de telle sorte que :

$$(i)\Leftrightarrow -\frac{N\beta}{T}\left(V_1'(x)U_0(x,y)-P_0(x,y)\right) \tag{4.3.29}$$

Deuxièmement, on peut couper $(ii)$ de la façon suivante :

$$\begin{aligned}(ii) \quad = \quad & \beta\sum_{i,j=1}^{N}\left\langle \sum_{k\neq i}\frac{1}{x_i-x_k}\frac{1}{x-x_i}\frac{M_{i,j}^{(\beta)}}{I_\beta}\frac{V_2'(y)-V_2'(y_j)}{y-y_j}\right\rangle \\ & +\beta\sum_{i,j=1}^{N}\left\langle \sum_{k\neq i}\frac{1}{x-x_i}\frac{1}{x-x_k}\frac{M_{i,j}^{(\beta)}}{I_\beta}\frac{V_2'(y)-V_2'(y_j)}{y-y_j}\right\rangle \\ & +\beta\sum_{i,j=1}^{N}\left\langle \sum_{k\neq i}\frac{1}{x-x_k}\frac{1}{x_i-x_k}\frac{M_{i,j}^{(\beta)}}{I_\beta}\frac{V_2'(y)-V_2'(y_j)}{y-y_j}\right\rangle\end{aligned}$$

$$
\begin{aligned}
= \quad & \beta \sum_{i,j=1}^{N} \left\langle \sum_{k\neq i} \frac{1}{x-x_i}\frac{1}{x-x_k}\frac{M_{i,j}^{(\beta)}}{I_\beta}\frac{V_2'(y)-V_2'(y_j)}{y-y_j} \right\rangle (ii)' \\
& +\beta \sum_{i,j=1}^{N} \left\langle \sum_{k\neq i} \frac{M_{i,j}^{(\beta)}-M_{k,j}^{(\beta)}}{(x_i-x_k)(x-x_i)}\frac{V_2'(y)-V_2'(y_j)}{y-y_j} \right\rangle (ii)''
\end{aligned}
\tag{4.3.30}
$$

Remarquons alors que $(ii)''$ est identique au dernier terme de $(iv)$ ce qui provoquera leurs annulations respectives. Ensuite, on peut couper $(ii)'$ en une somme sur $i,k$ moins le cas où $i=k$ qui est quant à lui identique à $(iii)$, à un facteur $\beta$ près. Ainsi, on peut regrouper $(ii)$, $(iii)$ et $(iv)$ pour obtenir :

$$
\begin{aligned}
(ii)+(iii)+(iv) \quad = \quad & (1-\beta)\sum_{i,j=1}^{N} \left\langle \frac{1}{(x-x_i)^2}\frac{M_{i,j}^{(\beta)}}{I_\beta}\frac{V_2'(y)-V_2'(y_j)}{y-y_j} \right\rangle (1) \\
& +\beta \sum_{i,j,k=1}^{N} \left\langle \frac{1}{x-x_i}\frac{1}{x-x_k}\frac{M_{i,j}^{(\beta)}}{I_\beta}\frac{V_2'(y)-V_2'(y_j)}{y-y_j} \right\rangle (2) \\
& +\frac{N\beta}{T}\sum_{i,j=1}^{N} \left\langle \frac{y_j}{x-x_i}\frac{M_{i,j}^{(\beta)}}{I_\beta}\frac{V_2'(y)-V_2'(y_j)}{y-y_j} \right\rangle (3)
\end{aligned}
\tag{4.3.31}
$$

Observons maintenant l'identité :

$$
(1) = (1-\beta)\sum_{i,j=1}^{N} \left\langle \frac{1}{(x-x_i)^2}\frac{M_{i,j}^{(\beta)}}{I_\beta}\frac{V_2'(y)-V_2'(y_j)}{y-y_j} \right\rangle = (\beta-1)\frac{\partial}{\partial x}U_0(x,y) \tag{4.3.32}
$$

et également :

$$
\begin{aligned}
\frac{T}{N\beta}\left\langle \frac{\partial}{\partial V_1(x)}U_0(x,y) \right\rangle_c \quad = \quad & \sum_{i,k,j=1}^{N} \left\langle \frac{1}{x-x_i}\frac{1}{x-x_k}\frac{M_{i,j}^{(\beta)}}{I_\beta}\frac{V_2'(y)-V_2'(y_j)}{y-y_j} \right\rangle \\
& -\sum_{i,j=1}^{N} \left\langle \frac{1}{x-x_i}\frac{M_{i,j}^{(\beta)}}{I_\beta}\frac{V_2'(y)-V_2'(y_j)}{y-y_j} \right\rangle \left\langle \frac{1}{x-x_k}\frac{M_{i,j}^{(\beta)}}{I_\beta} \right\rangle \\
& = \sum_{i,k,j=1}^{N} \left\langle \frac{1}{x-x_i}\frac{1}{x-x_k}\frac{M_{i,j}^{(\beta)}}{I_\beta}\frac{V_2'(y)-V_2'(y_j)}{y-y_j} \right\rangle \\
& -W(x)U_0(x,y)
\end{aligned}
\tag{4.3.33}
$$

On reconnait ici le second terme de 4.3.31 :

$$\begin{aligned}(2) \quad &\Leftrightarrow \quad \frac{T}{N}\left\langle \frac{\partial}{\partial V_1(x)} U_0(x,y) \right\rangle_c + \beta W(x) U_0(x,y) \\ &\Leftrightarrow \quad \beta U_1(x,y;x) + \beta W(x) U_0(x,y) \end{aligned} \tag{4.3.34}$$

Finalement, il nous reste à traiter $(3)$. On peut alors réécrire $y_j \leftrightarrow y_j - y + y$ et le couper en deux pour avoir :

$$\begin{aligned}(3) \quad \Leftrightarrow \quad & \frac{N\beta}{T} y U_0(x,y) - \frac{N\beta}{T} V_2'(y) \sum_{i,j=1}^{N} \left\langle \frac{1}{x-x_i} \frac{M_{i,j}^{(\beta)}}{I_\beta} \right\rangle \\ & + \frac{N\beta}{T} \sum_{i,j=1}^{N} \left\langle \frac{V_2'(y_j)}{x-x_i} \frac{M_{i,j}^{(\beta)}}{I_\beta} \right\rangle \end{aligned} \tag{4.3.35}$$

Mais rappelons que $\sum_{j=1}^{N} M_{i,j}^{(\beta)} = I_\beta$. et que d'après notre résultat préliminaire nous avons 4.3.23 ce qui donne finalement :

$$(3) \Leftrightarrow \frac{N\beta}{T} y U_0(x,y) - \frac{N\beta}{T} V_2'(y) W(x) + \frac{N\beta}{T}(-N + xW(x)) \tag{4.3.36}$$

En regroupant ensemble toutes les contributions, nous arrivons donc à l'équation :

$$\begin{aligned} 0 \quad = \quad & -\frac{N\beta}{T}\left(V_1'(x) U_0(x,y) - P_0(x,y)\right) - (1-\beta)\frac{\partial}{\partial x} U_0(x,y) \\ & + \beta U_1(x,y;x) + \beta W(x) U_0(x,y) + \frac{N\beta}{T} y U_0(x,y) \\ & - \frac{N\beta}{T} V_2'(y) W(x) + \frac{N\beta}{T}(-N + xW(x)) \end{aligned} \tag{4.3.37}$$

qui peut être réécrite (en multipliant par $-\frac{T}{N\beta}$) pour donner *l'équation de boucle maîtresse* :

**Théorème 4.2.** *La fonction $W(x)$ satisfait l'équation maîtresse suivante :*

$$\boxed{\left(y - V_1'(x) + \frac{T}{N}W(x) + \hbar\partial_x\right)U_0(x,y) = (V_2'(y) - x)W(x) - P_0(x,y) + N - \frac{T}{N}U_1(x,y;x)} \tag{4.3.38}$$

Afin de résoudre cette équation de boucle maîtresse, on peut la projeter sur le développement topologique et obtenir le théorème suivant :

**Théorème 4.3.** *Les fonctions de corrélations satisfont les équations de boucles à $\beta$ quelconque :*

*Equation de boucles à l'ordre dominant :*

$$\boxed{(y - Y(x) + \hbar\partial_x)\,U_0^{(0)}(x,y) = E(x,y)} \tag{4.3.39}$$

*Equations de boucles aux ordres supérieurs :*

$$\boxed{\begin{aligned}(y - Y(x) + \hbar\partial_x)\,U_0^{(g)}(x,y) \;=\; & -W_1^{(g)}(x)U_0^{(0)}(x,y) - \sum_{h=1}^{g-1} W_1^{(g-h)}(x)U_0^{(h)}(x,y) \\ & - P_0^{(g)}(x,y) - U_1^{(g-1)}(x,y;x)\end{aligned}} \tag{4.3.40}$$

### 4.3.4 Analyse des équations de boucles : singularité du cas hermitien

Les équations de boucles (4.3.39) et (4.3.40) permettent d'obtenir le cas à une matrice (prendre $V_2'(y) = y$) et/ou le cas hermitien (prendre $\beta = 1 \Leftrightarrow \hbar = 0$). Il est alors évident de constater que le cas hermitien constitue un cas très particulier, puisqu'alors les équations de boucles deviennent purement algébriques (le facteur $\hbar$ devant les dérivées devenant nul). Cette simplification spécifique a permis à B. Eynard et N. Orantin [23] de résoudre ces équations de boucles par des méthodes de géométrie algébrique, et même de généraliser, dans le cadre d'une courbe algébrique quelconque (appelée courbe

spectrale) $E(x,y)=0$ la définition d'invariants symplectiques $F_g$ qui résolvent les équations de boucles hermitiennes (Cf. chapitre 2). En revanche, dans le cas où $\beta$ est quelconque (i.e. $\hbar \neq 0$), la nature des équations de boucles change, puisqu'elles deviennnent non plus algébriques, mais différentielles. Dès lors, la résolution, qui à ce jour n'est pas encore complètement explicite, change de nature également. Cela dit, un point important est que la limite $\hbar \to 0$ doit redonner les résultats du cas hermitien, c'est-à-dire de la théorie des invariants symplectiques correspondants. Le travail réalisé en collaboration avec B. Eynard et L. Chekhov a été de développer un formalisme de résolution de ces équations de boucles dans le cas où $\hbar \neq 0$ en réalisant une "déformation quantique" de la théorie des invariants symplectiques de B. Eynard et N. Orantin. Cette résolution, encore partielle à l'heure actuelle, consiste à s'intéresser à une "courbe quantique" :

$$P(x,y)=0 \qquad , \qquad [y,x]=\hbar \tag{4.3.41}$$

et à définir à partir de cette "courbe" des fonctions qui résolvent les équations de boucles 4.3.39 et 4.3.40. Bien que certains résultats concernant le modèle à deux matrices soient en cours de réalisation, nous nous contenterons dans cette thèse de ne traiter que des modèles à une matrice présentés dans les articles **[III]** et **[IV]** présentés respectivement en annexe VII et VIII.

## 4.4 Le modèle à une matrice pour $\beta$-quelconque et la géométrie algébrique quantique

Le cas du modèle à une matrice possède l'avantage d'être beaucoup plus simple d'un point de vue technique que le cas à deux matrices. En effet, dans le cas du modèle à une matrice, les équations de boucles se réécrivent sous la forme :

$$W_1^{(0)}(x)^2 - V_1'(x)W_1^{(0)}(x) + \hbar\,\partial_x W_1^{(0)}(x) = -P_1^{(0)}(x) \tag{4.4.1}$$

et en posant $J = (x_1, \ldots, x_n)$ :

$$\begin{aligned} P_{n+1}^{(g)}(x; x_1 \ldots, x_n) &= \left(2W_1^{(0)}(x) - V_1'(x)\right) \overline{W}_{n+1}^{(g)}(x, x_1, \ldots, x_n) + \hbar \partial_x \overline{W}_{n+1}^{(g)}(x, x_1 \ldots, x_n) \\ &\quad + \sum_{I \subset J} \overline{W}_{|I|+1}^{(h)}(x, x_I) \overline{W}_{n-|I|+1}^{(g-h)}(x, J/I) + \overline{W}_{n+2}^{(g-1)}(x, x, J) \\ &\quad + \sum_j \partial_{x_j} \left( \frac{\overline{W}_n^{(g)}(x, J/\{j\}) - \overline{W}_n^{(g)}(x_j, J/\{j\})}{(x - x_j)} \right) \end{aligned} \tag{4.4.2}$$

où la notation $\overline{W}_n^{(g)}$ signifie :

$$\overline{W}_n^{(g)}(x_1, \ldots, x_n) = W_n^{(g)}(x_1, \ldots, x_n) - \frac{\delta_{n,2}\delta_{g,0}}{2} \frac{1}{(x_1 - x_2)^2} \tag{4.4.3}$$

Comme nous allons le voir, le modèle à une matrice est relié à l'équation de Schrödinger, c'est-à-dire à une équation différentielle ordinaire de degré 2. Dans le cas du modèle à deux matrices, l'équation différentielle est de degré $d_2$, (degré du potentiel $V_2(y)$) ce qui rend les calcus plus compliqués.

### 4.4.1 Lien entre les équations de boucles et la géométrie algébrique quantique

**Définition 4.11.** *On définit la fonction $\psi(x)$ par :*

$$\hbar \frac{\psi'(x)}{\psi(x)} = W_1^{(0)}(x) - \frac{V_1'(x)}{2} \tag{4.4.4}$$

On remarque que la première équation de boucles se réécrit (Cf. **[IV]** en annexe VIII) :

$$\hbar^2 \psi''(x) = U(x)\psi(x) \qquad , \qquad U(x) = \frac{V'(x)^2}{4} - \hbar \frac{V''(x)}{2} - P_1^{(0)}(x) \tag{4.4.5}$$

c'est-à-dire que la fonction $\psi(x)$ satisfait une équation de Schrödinger. Dans le cas du modèle à deux matrices, cette équation se généralise à une équation d'ordre plus élevée :

$$W_1^{(0)}(x) = \hbar \frac{\psi'(x)}{\psi(x)}$$

$$0 \quad = \quad \left(V_1'(x) - \hbar\partial_x\right)^{d_2+1} \psi(x) - \sum_{k=0}^{d_2} \left(V_1'(x) - \hbar\partial_x\right)^k \left(E_k(x)\psi(x)\right) \quad (4.4.6)$$

où les $E_k(x)$ sont donnés par $E(x,y) = \sum_{k=0}^{d_2+1} E_k(x)y^k$ et où nous rappelons que $E(x,y)$ est la courbe spectrale donnée par 4.3.16. (Notons que l'on retrouve bien le cas à une matrice en prenant $d_2 = 1$ et $E_0(x) = -2V_1'(x)$ comme prévu). Ainsi on peut récrire ces deux modèles sous la forme suivante :

$$\boxed{\hat{y} = V_1'(x) - \hbar\partial_x \qquad , \qquad E(x,\hat{y})\psi(x) \stackrel{\text{def}}{=} \sum_{k=0}^{d_2+1} \hat{y}^k E_k^{(0)}(x)\psi(x) = 0} \quad (4.4.7)$$

avec $[\hat{y},x] = \hbar$. Notons en particulier, que les variables $x$ et $\hat{y}$ ne commutant plus, il est nécessaire de préciser la position de l'une par rapport à l'autre (les variables $\hat{y}$ se retrouvant toujours à gauche). Dans le cas à une matrice, cela se réécrit avec les notations spécifiques 4.4.4 comme :

$$\hat{y} = \hbar\partial_x \qquad , \qquad \left(\hat{y}^2 - U(x)\right)\psi(x) = 0 \quad (4.4.8)$$

On voit donc émerger une courbe "quantique" (au sens où les variables ne commutent plus : $[\hat{y},x] = \hbar$) donnée par $E(x,\hat{y})\psi(x) = 0$, où $\frac{\psi'(x)}{\psi(x)}$ représente $W_1^{(0)}(x)$ à une translation triviale par $V_1'(x)$ près. A noter que dans le cas hermitien, on retombe sur une courbe algébrique "classique" $E(x,y) = 0$ où $x$ et $y$ commutent de nouveau. On voit donc toute la singularité du cas hermitien, puisque l'on passe alors du domaine différentiel au domaine algébrique ou de façon équivalente de variables non-commutantes à des variables commutantes. Dans le cas hermitien, les travaux de B. Eynard et N. Orantin [23], [58], [56] permettent de construire à partir de la courbe algébrique $E(x,y) = 0$ toutes les autres fonctions de corrélation $W_n^{(g)}$ ainsi que les invariants symplectiques $F_g$. Cette construction, présentée brièvement au chapitre 2 utilise des notions avancées de géométrie algébrique : genre, noyau de Bergmann, intégration sur une surface de Riemann, formes holomorphes, etc.

Notre démarche a alors été de partir de la courbe "quantique" et d'une solution $\psi(x)$ associée, et de généraliser les notions développées par B. Eynard et N. Orantin pour le cas des courbes algébriques "classiques". En particulier, nous nous sommes intéressés à la généralisation de la notion de genre, de formes holomorphes, de noyau de Bergman sur notre courbe "quantique" dans le but de résoudre les équations de boucles.

### 4.4.2 La géométrie algébrique "quantique" dans le cas d'équations hyper-elliptiques

Donnons-nous donc une courbe quantique hyper-elliptique (i.e. de degré 2 en $y$) de la forme :

**Définition 4.12.** *Une courbe quantique hyper-elliptique consiste en la donnée d'une équation différentielle et d'une solution $\psi$ :*

$$E(x,\hat{y})\psi(x) = \sum_{j=0}^{2} \hat{y}^j E_j(x)\psi(x) = 0 \tag{4.4.9}$$

En divisant par le coefficient dominant devant $\psi''(x)$ et en translatant convenablement la fonction $\psi(x)$, on peut se ramener à une équation de type Schrödinger :

$$\hbar^2\psi''(x) = U(x)\psi(x) \tag{4.4.10}$$

Il est à noter que les solutions $\psi(x)$ de cette équation différentielle ne sont pas uniques, tout comme les solutions d'une équation algébrique $y^2 = U(x) \Leftrightarrow u = \pm\sqrt{U(x)}$ ne le sont pas également. Néanmoins, il est immédiat de constater que la dimension de l'espace vectoriel des solutions (ici 2) correspond toujours au degré en $y$ de la courbe, c'est-à-dire également au nombre de solutions de l'équation algébrique classique associée. En supposant que $U(x)$ est un polynôme de degré $2d$, on peut alors définir :

**Définition 4.13.** *Le potentiel associée à la courbe est défini par :*

$$V'(x) = 2\,(\sqrt{U})_+ = \sum_{k=0}^{d} t_{k+1}\,x^k \tag{4.4.11}$$

*où* $_+$ *désigne la partie polynômiale au voisinage de l'infini. On peut également définir :*

$$P(x) = \frac{V'^2(x)}{4} - U(x) - \hbar\frac{V''(x)}{2} \tag{4.4.12}$$

*qui est un polynôme de degré* $d-1$*. On définit également le paramètre* $t_0$ *par :*

$$t_0 = \lim_{x\to\infty} \frac{xP(x)}{V'(x)} \tag{4.4.13}$$

Traditionnellement, les cœfficients $t_i$ sont appelés les "Casimirs" et les autres cœfficients de $P(x)$ les "charges" qui jouent un rôle particulier. La fonction $\psi(x)$ étant solution d'une équation de type Schrödinger et $U(x)$ étant supposée polynômiale, elle présente donc un phénomène de Stokes, c'est-à-dire que bien qu'étant analytique sur $\mathbb{C}$, son asymptotique à l'infini possède des discontinuités (singularité essentielle) suivant certaines directions :

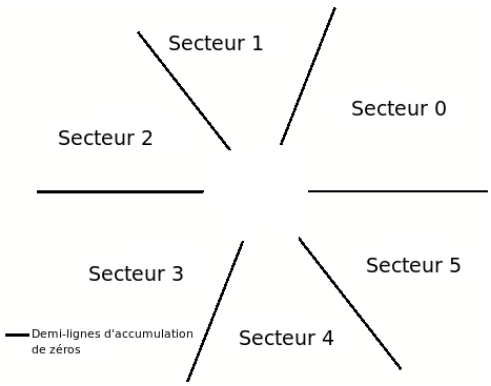


Figure 17 : Exemple de secteurs pour un potentiel de degré $\deg V = 3$, i.e. $d = 2$. Dans le cas général où $\deg V = d+1$, il y a $2d+2$ secteurs.

Dans chaque secteur $S_k$, l'asymptotique de $\psi(x)$ est donné sous la forme :

$$\psi(x) \underset{S_k}{\sim} \mathrm{e}^{\pm\frac{1}{2\hbar}V(x)}\, x^{C_k}\, (A_k + \frac{B_k}{x} + \dots) \tag{4.4.14}$$

Une fois ces considérations prises en compte, il est alors possible de généraliser des notions de géométrie algébrique dans le cadre de notre courbe "quantique" :

1. **Les différents feuillets**

Dans le cadre d'une équation algébrique du second degré, il existe deux solutions distinctes qui correspondent à deux feuillets en géométrie algébrique. Ces feuillets sont reliés par des points de branchements correspondants aux points où les solutions sont identiques. Dans le cadre de notre courbe quantique, notre solution $\psi(x)$ présente deux comportements asymptotiques différents par le choix du signe $\pm$ dans la formule 4.4.14. Dès lors, nous pouvons séparer les secteurs de Stokes en deux feuillets : le feuillet "physique", où $\psi(x)$ se comporte comme $\psi(x) \sim_{S_k} \mathrm{e}^{+\frac{1}{2\hbar}V(x)} x^{C_k}$ et le feuillet non-physique, où $\psi(x) \sim_{S_k} \mathrm{e}^{-\frac{1}{2\hbar}V(x)} x^{C_k}$. Notons que les secteurs de Stokes n'ont de sens que dans un voisinage de l'infini, ce qui rend leur interprétation plus délicate que dans le cas algébrique. Par ailleurs, afin de fixer la solution $\psi(x)$, nous choisirons la solution qui est exponentiellement décroissante dans le secteur 0. A noter que dans des cas très spécifiques, l'équation de Schrödinger 4.4.10 peut avoir des solutions polynômiales ne présentant pas de phénomène de Stokes. Ces cas singuliers ont été traités dans l'article **[III]** présenté en annexe VII.

## 2. Coupures et points de branchement

Dans le cadre de la géométrie algébrique, une équation du second degré peut être vue comme une surface de Riemann de genre $g_{\mathrm{alg}}$, c'est-à-dire comme deux copies du plan complexe reliées par $g_{\mathrm{alg}}+1$ coupures. Dans le cas où la courbe est donnée par

$$y^2 = U(x) = \prod_{i=1}^{2d}(x-a_i)$$

avec des $a_i$ distincts, les points de branchement sont les racines $a_i$ et les coupures peuvent être prises comme reliant $\bigcup_{p=1}^{d}[a_{2p-1}, a_{2p}]$ :

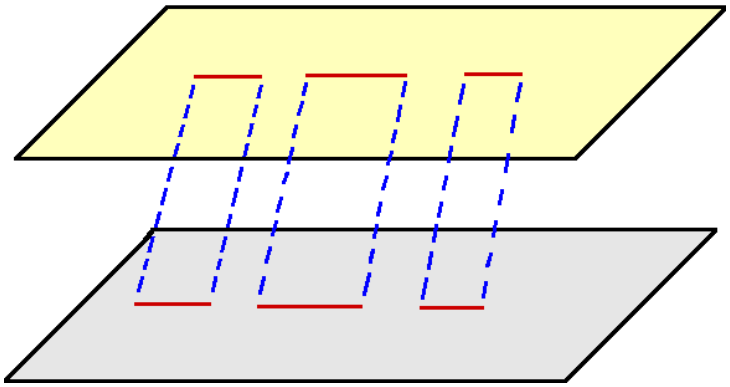

Figure 18 : Exemple de coupures dans le cas d'une équation algébrique.

Lorsque deux $a_i$ coïncident, le point de branchement devient alors dégénéré et le nombre de coupures diminue. Dans le cas général, on peut récrire :

$$y^2 = U(x) = Q^2(x)\prod_{i=1}^{2m}(x-a_i)$$

où $Q(x)$ est un polynôme. Les points de branchements sont de nouveau les $a_i$ restants et les coupures peuvent être choisies comme : $\bigcup_{p=1}^{m}[a_{2p-1},a_{2p}]$. Le genre de la courbe reste quant à lui toujours donné par $m-1$. Il est alors facile de remarquer que $0 \leq g_{\text{alg}} \leq d-1$.

Les notions de genre, de coupures et de points de branchements se généralisent alors de la façon suivante. Notons $s_i$ les zéros de la fonction $\psi(x)$. Alors si $\psi$ présente un phénomène de Stokes, elle possède une infinité de zéros qui ne peuvent s'accumuler que le long des demi-lignes de Stokes où l'asymptotique est discontinu. Dans le cas générique, seul le secteur 0 est singulier pour $\psi_0$ (car sinon, on se retrouve dans la situation où $\psi_0$ est également sous-dominante dans un autre secteur, disons $i$. Dans ce cas cela signifie qu'il existe une solution intégrable du secteur 0 au secteur $i$ à l'équation de Schrödinger ce qui n'est pas le cas pour une équation de Schrödinger générique), et la situation se présente sous la forme :

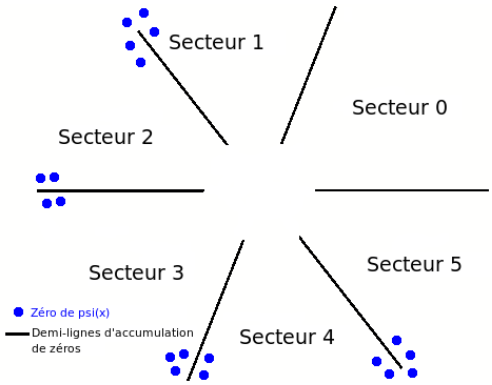


Figure 19 : Demi-lignes d'accumulation de zéros dans le cas générique

Dès lors, il est possible de définir **les coupures comme une paire de deux demi-lignes d'accumulation de zéros**. Cette appariement présente un caractère arbitraire, qui correspond dans le cas algébrique au choix de regrouper les points de branchements pour créer les coupures. Le choix le plus naturel est alors de regrouper par paires deux demi-lignes d'accumulation de zéros consécutives. Tout comme dans le cas algébrique, **le genre est alors défini comme le nombre de coupures moins un**. Dans le cas générique, toutes les demi-lignes de Stokes accumulent des zéros (sauf celle délimitant le secteur 0) et donc le genre est maximal. Cela dit, tout comme dans le cas algébrique, il se peut que l'équation de départ soit singulière et que $\psi(x)$ présente d'autres demi-lignes de Stokes n'accumulant pas de zéros. Le genre de la courbe quantique diminue alors d'une unité à chaque fois comme dans le cas algébrique. Quoiqu'il en soit, le genre de la courbe quantique satisfait les inégalités :

$$-1 \leq g \leq d-1 \tag{4.4.15}$$

Le cas où $g = -1$ correspond au cas singulier où $\psi(x)$ est polynômiale (i.e. n'a pas de phénomène de Stokes), il correspond au cas algébrique où $y^2 = Q^2(x)$ qui ne présente alors pas beaucoup d'intérêt. Néanmoins, dans le cas quantique, ce cas existe et demande un traitement particulier donné dans l'article **[III]** présenté en annexe VII. La notion de points de branchements est quant à elle plus floue dans le cas quantique. En effet, seule une des extrémités des demi-lignes d'accumulation de zéros (celle en direction de l'infini) est bien définie. Toutefois, on peut

interpréter les points de branchements comme les directions asymptotiques dans lesquelles les zéros de $\psi$ s'accumulent.

3. **Cycles**

   Dans le cadre de la géométrie algébrique, on sait que l'on peut choisir une base de $2g_{\text{alg}}$ cycles d'homologie indépendants $\mathscr{A}_i$ et $\mathscr{B}_j$ sur la surface de Riemann de genre $g_{\text{alg}}$ décrivant notre équation algébrique. Cette base vérifie les propriétés de croisement :

$$\begin{aligned} \mathscr{A}_i \cap \mathscr{B}_j &= \delta_{i,j} \\ \mathscr{A}_i \cap \mathscr{A}_j &= \emptyset \\ \mathscr{B}_i \cap \mathscr{B}_j &= \emptyset \end{aligned} \tag{4.4.16}$$

   Le choix canonique est de prendre les cycles $\mathscr{A}_i$ entourant chacune des $g-1$ premières coupures, tandis que les cycles $\mathscr{B}_i$ traversent les feuillets et se rejoignent dans la dernière coupure. Ce choix est bien sûr arbitraire, et n'importe quel autre choix de base indépendante et respectant les conditions de croisement est possible :

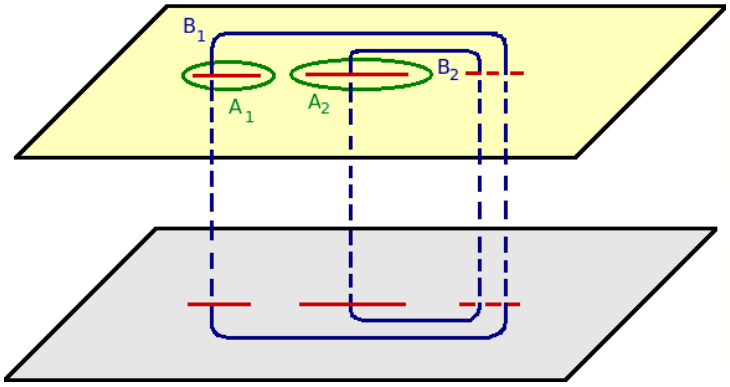


Figure 20 : Exemple de cycles dans le cas d'une équation algébrique hyperelliptique.

   Dans le cas quantique, le choix des $\mathscr{A}$-cycles est similaire : il consiste à choisir $g$ chemins reliant les différents infinis physiques entre eux. A nouveau, le choix canonique est d'entourer les coupures en restant dans le feuillet physique. Si les coupures ont été choisies consécutivement, la situation est alors décrite par :

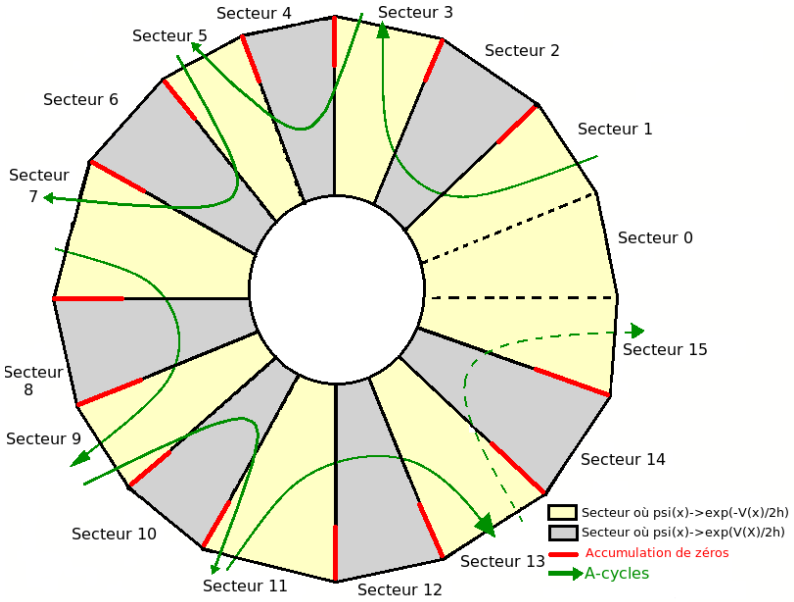


Figure 21 :Exemple de $\mathscr{A}$-cycles dans le cas d'une courbe quantique.

Le choix des $\mathscr{B}$-cycles suit la même construction que dans le cas algébrique, le chemin part de la dernière coupure, i.e. du secteur 0, traverse le $\mathscr{A}$-cycle correspondant pour passer dans l'autre feuillet, puis revient. La seule différence est qu'ici le retour impose une nouvelle fois de traverser le $\mathscr{A}$-cycle. Finalement la situation peut être visualisée comme :

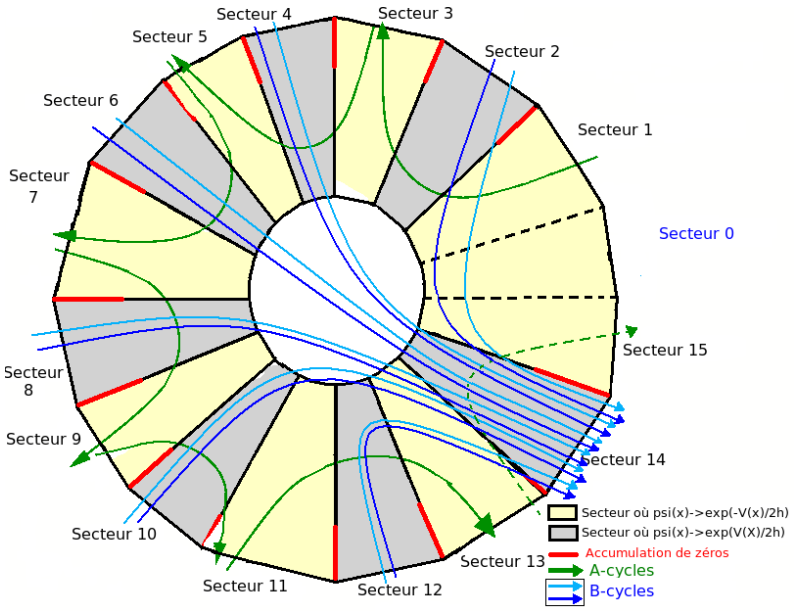

Figure 22 : Exemple de $\mathscr{B}$-cycles dans le cas d'une courbe quantique.

Notons que cette fois-ci on a les relations :

$$\begin{aligned} \mathscr{A}_i \cap \mathscr{B}_j &= 2\delta_{i,j} \\ \mathscr{A}_i \cap \mathscr{A}_j &= \emptyset \\ \mathscr{B}_i \cap \mathscr{B}_j &= \emptyset \end{aligned} \tag{4.4.17}$$

Par ailleurs, dans le cas algébrique comme dans le cas quantique, il est possible de définir des $\hat{\mathscr{A}}$-cycles et des $\hat{\mathscr{B}}$-cycles pour des coupures dégénérées (i.e. des demi-lignes sans accumulation de zéros ou des zéros multiples dans le cas algébrique). Dans le cas algébrique, ils correspondent à des cycles pincés :

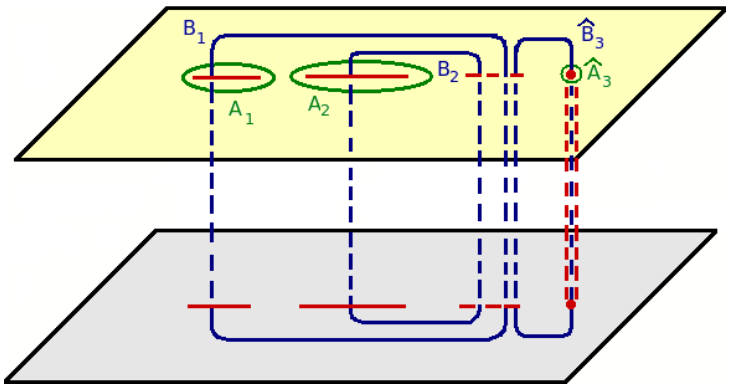


Figure 23 : Exemple de cycles pincés dans le cas d'une équation algébrique hyperelliptique.

Dans le cas quantique, les $\hat{\mathscr{A}}_i$-cycles dégénérés sont des chemins partant de l'infini 0 et allant dans le secteur dégénéré. Ils peuvent être visualisés de la manière suivante :

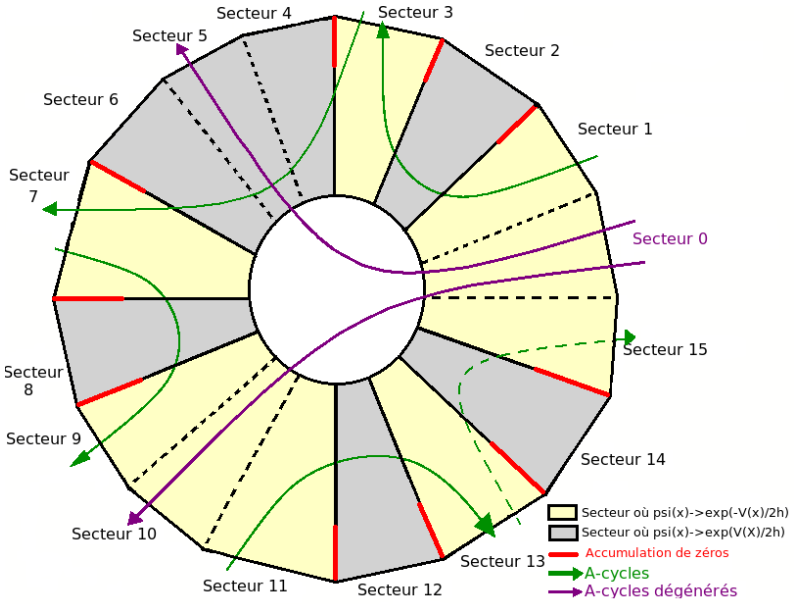


Figure 24 : Exemple de $\hat{\mathscr{A}}$-cycles dans le cas d'une courbe quantique.

Notons qu'il y a toujours exactement $g$ $\mathscr{A}$-cycles non-dégénérés et au total $d-1$ $\mathscr{A}$-cycles (dégénérés ou non) indépendants.

4. **Formes holomorphes**

En géométrie algébrique, les formes holomorphes sur une surface de Riemann de genre $g_{\text{alg}}$ sont les fonctions ne présentant aucun pôle sur la surface. Nous allons généraliser cette notion dans notre cas quantique de la façon suivante : soit $h_k(x)$ une base des polynômes de degré inférieur à $d-2$. Intéressons nous d'abord aux fonctions :

$$v_k(x) = \frac{1}{\hbar\,\psi^2(x)} \int_{\infty_0}^{x} h_k(x')\,\psi^2(x')\,dx' \qquad , \qquad \deg h_k \leq d-2. \tag{4.4.18}$$

Grâce aux propriétés dans les différents secteurs à l'infini de la fonction $\psi(x)$, on peut montrer (**[IV]** en annexe VIII) que les intégrales suivantes sont bien définies, et ne dépendent que de la classe d'homologie des contours $\mathscr{A}$.

$$I_{k,\alpha} = \oint_{\mathscr{A}_\alpha} v_k(x)\,dx \qquad , \qquad \alpha = 1,\ldots,g,\, k = 1,\ldots,d-1. \tag{4.4.19}$$

Pour les cas de cycles dégénérés $\hat{\mathscr{A}_\alpha}$, de telles intégrales divergeraient, on préférera donc prendre :

$$I_{k,\alpha} = \oint_{\hat{\mathscr{A}_\alpha}} h_k(x)\,\psi^2(x)\,dx \qquad , \qquad \alpha = g+1,\ldots,d-1, k = 1,\ldots,d-1. \tag{4.4.20}$$

La matrice $I_{k,\alpha}$ avec $k,\alpha = 1,\ldots,d-1$ est une matrice carrée donnant une connexion entre l'ensemble des chemins $\{\ \mathscr{A}_\alpha, \hat{\mathscr{A}_\alpha}\ \}$ et l'espace vectoriel des polynômes de degré inférieur à $d-2$. On peut donc choisir la base $h_k$, duale des $\mathscr{A}$-cycles, c'est-à-dire satisfaisant les relations :

$$I_{k,\alpha} = \delta_{k,\alpha}. \tag{4.4.21}$$

En choisissant cette base, on obtient alors les relations :

$$\forall\, i = 1,\ldots,g, j = 1,\ldots,d-1\,, \qquad \oint_{\mathscr{A}_i} v_j(x)\,dx = \delta_{i,j} \tag{4.4.22}$$

$$\forall i = g+1,\ldots,d-1,\ j = 1,\ldots,d-1 : \oint_{\mathscr{A}_i} h_j(x)\,\psi^2(x)\,dx = \delta_{j,i} \tag{4.4.23}$$

Par ailleurs, par sa définition même, on peut montrer aisément que les fonctions $v_k(x)$ avec $k \leq g$ se comportent comme

$$\forall k = 1,\ldots,g, \qquad v_k(x) = O(x^{-2}) \tag{4.4.24}$$

dans tous les secteurs à l'infini. Elles possèdent donc toutes les propriétés requises pour être la généralisation des formes holomorphes. Enfin, on peut facilement montrer que dans la limite $\hbar \to 0$ on retrouve :

$$v_k(x) \underset{\hbar\to 0}{\sim} \frac{\pm h_k(x)}{\sqrt{U(x)}} \tag{4.4.25}$$

qui sont effectivement les formes holomorphes de la géométrie algébrique pour la courbe $y^2 = U(x)$

5. **Matrice des périodes de Riemann**

   En géométrie algébrique, une fois les cycles et les formes holomorphes $v_k$ définis, la matrice des périodes de Riemann de taille $g \times g$ est définie par :

$$\forall 1 \leq i,j \leq g : \tau_{i,j} \stackrel{\text{def}}{=} \oint_{\mathscr{B}_i} v_j(x)\, dx. \tag{4.4.26}$$

   En effet, les formes holomorphes étant normalisées sur les $\mathscr{A}$-cycles, (4.4.22), il est naturel de s'intéresser à leurs intégrales sur les cycles duaux $\mathscr{B}_j$. Le théorème de Riemann sur les surfaces de Riemann énonce alors que la matrice des périodes $\tau$ est symétrique. Notons qu'étant donnée la définition de la matrice $\tau$, ce résultat n'est pas du tout évident. Dès lors, si notre généralisation quantique se veut correcte, elle se doit de maintenir un tel résultat. Dans l'article **[IV]** (section 3.4 de l'annexe VIII), il est montré que :

   **Théorème 4.4.** *La matrice $\tau$ quantique définie par $\tau_{i,j} \stackrel{\text{def}}{=} \oint_{\mathscr{B}_i} v_j(x)\, dx$ (où les $v_j(x)$ sont définis par 4.4.18), est symétrique.*

   Notons que ce résultat est hautement non-trivial compte-tenu des définitions précédemment introduites et constitue donc un premier pas important de la théorie.

6. **Les fractions de remplissage**

   Les fractions de remplissage jouent un rôle essentiel dans la théorie des matrices aléatoires, car elles indiquent les différentes proportions de valeurs propres se retrouvant sur chaque coupure (intervalles) dans la limite $N \to \infty$ de la mesure d'équilibre (2.3.2). En dehors du cadre des modèles de matrices, elles sont définies dans le contexte de la géométrie algébrique par :

$$\forall\, 1 \leq i \leq g_{\text{alg}} : \varepsilon_i \stackrel{\text{def}}{=} \frac{1}{2i\pi} \oint_{\mathscr{A}_i} y dx \qquad , \qquad \forall\, i > g_{\text{alg}} : \varepsilon_i = 0 \tag{4.4.27}$$

   Dans notre cas quantique, les fractions de remplissage $\varepsilon_1, \ldots, \varepsilon_d$ sont définies par :

   **Définition 4.14.** *Les fractions de remplissage sont définies par :*

$$\forall\, 1 \leq \alpha \leq g : \varepsilon_\alpha = \frac{1}{2i\pi} \oint_{\mathscr{A}_\alpha} \left( \omega(x) - \frac{t_0}{x} \right) + \frac{t_0 n_\alpha}{(d+1)} \tag{4.4.28}$$

*où l'entier $n_\alpha$ compte la moitié des demi-lignes de Stokes entourées par le cycle $\mathscr{A}_\alpha$, ce qui est équivalent à dire que $\frac{n_\alpha}{d+1}$ représente la fraction angulaire du plan complexe défini par le cycle $\mathscr{A}_\alpha$.*

*Pour $g+1 \leq \alpha \leq d-1$ la définition est :*

$$g+1 \leq \alpha \leq d-1 \,:\, \varepsilon_\alpha = 0 \tag{4.4.29}$$

*et pour $\alpha = d$, nous choisissons un cycle $\mathscr{A}_d$, (non-indépendant des autres $\mathscr{A}_i$) qui entoure les autres zéros $s_i$ qui ne sont pas entourés par les cycles $\mathscr{A}_1, \ldots, \mathscr{A}_g$. Une fois ce contour choisi, nous définissons alors :*

$$\varepsilon_d = \frac{1}{2i\pi} \oint_{\mathscr{A}_d} \left( \hbar \frac{\psi'(x)}{\psi(x)} + \frac{V'(x)}{2} - \frac{t_0}{x} \right) + \frac{t_0 n_d}{(d+1)} \tag{4.4.30}$$

Ces fractions de remplissage satisfont trivialement à la relation $\sum_{\alpha=1}^{d} \varepsilon_\alpha = t_0$ qui est également vraie dans le cas algébrique. Notons cependant que dans le cas quantique, les fractions de remplissage apparaissent arbitraires car dépendantes du choix des contours $\mathscr{A}$ relativement aux zéros de $\psi(x)$. En effet, si l'on déforme le contour $\mathscr{A}_\alpha$ pour englober un zéro $s_i$ supplémentaire, la fraction de remplissage correspondante $\varepsilon_\alpha$ augmentera de $\hbar$ (mais une autre baissera de $\hbar$). Leur interprétation pour les modèles de matrices devant être indépendante du choix précis des contours, elle ne peut donc avoir de sens qu'à un multiple entier de $\hbar$ près.

7. **Le noyau de récurrence** $K(x_0, x)$

   Le noyau de récurrence $K(x_0, x)$ n'est pas une quantité standard de géométrie algébrique, mais est un ingrédient essentiel dans la définition des invariants symplectiques $F_g$ d'Eynard et Orantin. En effet, comme nous le verrons par la suite, il permet l'écriture d'une récurrence "topologique" permettant de calculer les fonctions de corrélation pour un ordre donné à partir des autres fonctions d'ordre inférieur. Pour le cas algébrique, nous avons vu précédemment que ce noyau est défini par $\frac{dE_q(p)}{y(q)-y(\bar{q})} = \frac{\int_q^{\bar{q}} B(\xi,p)}{2(y(q)-y(\bar{q}))}$ (Cf. 2.9.2), c'est-à-dire à partir (par primitivation au voisinage d'un point de branchement) du noyau de Bergman $B(\xi, p)$ de la surface de

Riemann. Dans le cas quantique, il est plus naturel de définir d'abord le noyau de récurrence puis ensuite par dérivation de construire l'équivalent du noyau de Bergman (dont il faudra naturellement vérifier les propriétés).

**Définition 4.15.** *Le noyau de récurrence est défini par :*

$$K(x,z) = \frac{1}{\hbar \psi^2(x)} \int_{\infty_0}^{x} \psi^2(x') \frac{dx'}{x'-z} - \sum_{\alpha=1}^{d-1} v_\alpha(x) C_\alpha(z) \tag{4.4.31}$$

*avec :* $\forall\, \alpha = 1, \ldots, g$ *:*

$$\hbar C_\alpha(z) = \oint_{\mathscr{A}_\alpha} \frac{dx''}{\psi^2(x'')} \int_{\infty_0}^{x''} \psi^2(x') \frac{dx'}{x'-z} \tag{4.4.32}$$

*et* $\forall\, \alpha = g+1, \ldots, d-1$ *:*

$$C_\alpha(z) = \oint_{\hat{\mathscr{A}_\alpha}} \psi^2(x') \frac{dx'}{x'-z}. \tag{4.4.33}$$

Nous renvoyons le lecteur à **[IV]** situé en annexe VIII pour les preuves d'existence des intégrales, pour les subtilités de définition des fonctions $C_\alpha(z)$ ainsi que pour les démonstrations techniques des propriétés à venir. Ce noyau possède les propriétés suivantes :

**Théorème 4.5.** *Le noyau $K(x,z)$ possède les propriété suivantes :*

• *Il possède une discontinuité le long d'un chemin reliant $\infty_0$ a x dont le saut est donné par :*

$$\delta K(x,z) = \frac{2i\pi}{\hbar} \frac{\psi^2(z)}{\psi^2(x)} \tag{4.4.34}$$

• *Pour $\alpha = 1, \ldots g$, il possède une discontinuité de $\infty_0$ à un point $P_\alpha \in \mathscr{A}_\alpha$ donnée par le saut :*

$$\delta K(x,z) = \frac{2i\pi\ \psi^2(z)}{\hbar} \oint_{\mathscr{A}_\alpha} \frac{dx''}{\psi^2(x'')} v_\alpha(x) \tag{4.4.35}$$

*et une discontinuité sur le cycle $\mathscr{A}_\alpha$ donnée par le saut :*

$$\delta K(x,z) = \frac{2i\pi\ \psi^2(z)}{\hbar} \int_{P_\alpha}^{z} \frac{dx''}{\psi^2(x'')} v_\alpha(x) \tag{4.4.36}$$

• *Pour* $\alpha = g+1,\dots,d-1$*, il possède une discontinuité sur le cycle* $\hat{\mathscr{A}_\alpha}$ *donnée par :*

$$\delta K(x,z) = 2i\pi\ \psi^2(z) v_\alpha(x) \tag{4.4.37}$$

• *Le comportement au voisinage de l'infini est donné par :*

$$\begin{aligned} K(x,z) &\underset{x\to\infty}{\sim} O(x^{-2}) \text{ dans tous les secteurs} \\ K(x,z) &\underset{z\to\infty}{\sim} O(z^{-d}) \text{ dans tous les secteurs} \end{aligned} \tag{4.4.38}$$

• *Pour* $\alpha = 1,\dots,g$*, et z du côté des cycles* $\mathscr{A}_\alpha$ *ne contenant pas* $\infty_0$ *:*

$$\oint_{\mathscr{A}_\alpha} K(x,z)\,dx = 0 \tag{4.4.39}$$

On voit donc que ce noyau possède beaucoup de lignes de discontinuité. Cela n'est pas sans rappeler le fait que dans le cas algébrique, la quantité $\frac{dE_q(p)}{y(q)-y(\overline{q})}$ n'est définie qu'au voisinage des points de branchement mais pas de façon globale. A partir de ce noyau $K(x_0,x)$, on peut définir l'équivalent des formes de 3$^{\text{ième}}$ espèce, ainsi que l'équivalent du noyau de Bergmann pour le cas quantique.

8. **Les formes de troisième espèce : le noyau** $G(x_0,x)$

**Définition 4.16.** *A partir du noyau précédent, on définit le noyau* $G(x_0,s)$ *par la formule :*

$$G(x,z) = -\hbar\,\psi^2(z)\,\partial_z \frac{K(x,z)}{\psi^2(z)} = 2\hbar\frac{\psi'(z)}{\psi(z)}K(x,z) - \hbar\partial_z K(x,z) \tag{4.4.40}$$

Ce nouveau $G(x_0,s)$ possède de meilleures propriétés de régularité que le noyau précédent $K(x_0,x)$. En effet, il est facile de voir que toutes les discontinuités proportionnelles à $\psi^2(z)$ vont être annulées lors de la dérivation.

**Théorème 4.6.** *Les propriétés du noyau* $G(x_0,s)$ *sont alors les suivantes (Cf.* ***[IV]****, section 4.2, annexe VIII) :*

• $G(x,z)$ *est une fonction analytique de x, avec un pôle simple en* $x=z$ *de résidu* $-1$*, et des pôles doubles aux* $s_j$ *(zéros de* $\psi(x)$*) sans résidu, et potentiellement une*

*singularité essentielle à l'infini.*

• *$G(x,z)$ est une fonction analytique de $z$, avec un pôle simple en $z=x$ de résidu $+1$, des pôles simples pour $z=s_j$, et une discontinuité le long des $\mathscr{A}_\alpha$-cycles pour $\alpha=1,\dots,g$.*

$$\delta G(x,z) = -2i\pi\, v_\alpha(x) \tag{4.4.41}$$

*En particulier, elle n'a pas de discontinuité le long des autres cycles $\hat{\mathscr{A}_\alpha}$.*

• *Les limites à l'infini sont données par :*

$$G(x,z) = O(1/x^2) \tag{4.4.42}$$

• *Pour $\alpha=1,\dots,g$, et $z$ du côté des cycles $\mathscr{A}_\alpha$ ne contenant pas $\infty_0$ :*

$$\oint_{\mathscr{A}_\alpha} G(x,z)\,dx = 0 \tag{4.4.43}$$

Ce noyau constitue une bonne généralisation des formes de troisième espèce en géométrie algébrique. En effet, en géométrie algébrique, ces formes possèdent les propriétés d'être analytiques partout sur la surface de Riemann (c'est-à-dire en dehors des coupures) et de posséder un unique pôle simple. Par ailleurs, ces formes sont également normalisées sur les $\mathscr{A}$-cycles. Dans notre cas "quantique", $z \mapsto G(x,z)$ est analytique partout à l'exception des $\mathscr{A}$-cycles et des $s_i$ qui constituent justement les coupures "quantiques". Enfin, elle possède également un pôle simple en $z=x$ et elle est correctement normalisée sur les $\mathscr{A}$-cycles. Par ailleurs, si l'on effectue la limite $\hbar \to 0$ (i.e. on remplace $\psi \sim e^{\frac{\pm 1}{\hbar}\int\sqrt{U}}$) on retrouve :

$$G(x,z) \sim 2\sqrt{U(z)}K(x,z) \sim \frac{1}{x-z}\frac{\sqrt{U(z)}}{\sqrt{U(x)}} - 2\sum_\alpha v_\alpha(x)\,C_\alpha(z)\sqrt{U(z)} \tag{4.4.44}$$

Dans ce cas, la forme $G(x,z)dx$ possède donc un pôle simple en $x=z$, de résidu $+1$ dans le feuillet physique, et de résidu $-1$ dans l'autre feuillet. Elle est également normalisée sur les $\mathscr{A}$-cycles : $\oint_{\mathscr{A}_i} G(x,z)dx = 0$. On retrouve ainsi les toutes les propriétés usuelles des formes de 3$^{\text{ième}}$ espèce de la géométrie algébrique.

9. **Le noyau de Bergman**

Le noyau de Bergman (ou différentielle fondamentale de deuxième espèce) est une quantité fondamentale en géométrie algébrique. En effet, il constitue une fonction intrinsèque pouvant être définie sur n'importe quelle surface de Riemann. De plus, dans la démarche de Eynard et Orantin [23], il sert à construire les deux noyaux précédents. Ici, notre démarche étant inverse, le noyau de Bergman est défini à partir du noyau $G(x,z)$ de la même façon que le noyau de Bergman est relié aux formes de troisième espèce en géométrie algébrique :

**Définition 4.17.** *Le noyau de Bergman est défini par la formule :*

$$B(x,z) \stackrel{def}{=} -\frac{1}{2}\,\partial_z G(x,z). \tag{4.4.45}$$

Pour pouvoir prétendre être une bonne généralisation du noyau de Bergman de la géométrie algébrique, il faut que ce nouveau noyau en vérifie les propriétés fondamentales. La vérification est donnée dans **[IV]** (section 4.3, annexe VIII).

**Théorème 4.7.** *Le noyau de Bergman satisfait les propriétés suivantes :*

• $B(x,z)$ *est une fonction analytique en* $x$ *et en* $z$*, avec un pôle double en* $x = z$ *sans résidu, et des pôles doubles aux* $s_j$ *sans résidu, ainsi que potentiellement une singularité essentielle à l'infini. On notera tout particulièrement qu'il n'a pas de discontinuité sur les* $\mathscr{A}$*-cycles.*

• $B(x,z)$ *est une fonction symétrique de ses variables :* $B(x,z) = B(z,x)$*. Cette propriété est hautement non-triviale compte-tenu de sa définition, et constitue une propriété fondamentale du noyau de Bergman.*

• $B(x,z)$ *se comporte dans tous les secteurs de l'infini comme :*

$$B(x,z) \underset{x\to\infty}{=} O(1/x^2) \qquad , \qquad B(x,z) \underset{x\to\infty}{=} O(1/z^2) \tag{4.4.46}$$

*dans tous les secteurs.*

• *Le noyau $B(x,z)$ est normalisé convenablement sur les cycles :*

$$\forall\, \alpha = 1,\dots,g : \oint_{\mathscr{A}_\alpha} B(x,z)\,dx = 0 \qquad , \qquad \oint_{\mathscr{A}_\alpha} B(x,z)\,dz = 0 \tag{4.4.47}$$

*et*

$$\forall\, \alpha = 1,\dots,g : \oint_{\mathscr{B}_\alpha} B(x,z)\,dz = 2i\pi v_\alpha(x) \tag{4.4.48}$$

• Enfin une dernière propriété en lien avec les modèles de matrice (et qui d'après les travaux de Eynard et Orantin [23] et également vraie en géométrie algébrique) est que ce noyau $B(x,z)$ satisfait les équations de boucles suivantes :

**Théorème 4.8.** *Le noyau de Bergman satisfait les équations (la preuve est dans **[IV]**, annexe A) :*

$$(2\frac{\psi'(x)}{\psi(x)} + \partial_x)\left(B(x,z) - \frac{1}{2(x-z)^2}\right) + \partial_z \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z} = P_2^{(0)}(x,z) \tag{4.4.49}$$

*où $P_2^{(0)}(x,z)$ est un polynôme en $x$ de degré au plus $d-2$.*

$$(2\frac{\psi'(z)}{\psi(z)} + \partial_z)\left(B(x,z) - \frac{1}{2(x-z)^2}\right) + \partial_x \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z} = \tilde{P}_2^{(0)}(z,x) \tag{4.4.50}$$

*où $\tilde{P}_2^{(0)}(z,x)$ est en polynôme en $z$ de degré au plus $d-2$.*

Ces équations de boucles nous permettront par la suite d'identifier $B(x,z)$ avec la résolvante $W_2^{(0)}(x,z)$ des modèles de matrices qui satisfait cette équation (4.4.2).

## 4.5 Une solution des équations de boucles pour $\beta$-quelconque

• Une fois la généralisation des quantités fondamentales de géométrie algébrique effectuée, il devient facile de trouver une solution des équations de boucles pour $\beta$-quelconque. Il suffit pour cela de reprendre la même formule de récurrence topologique que celle développée par Eynard et Orantin ([23]) en remplaçant chacune des quantités par sa généralisation "quantique". Ainsi, on définit les résolvantes de la façon suivante :

**Définition 4.18.** *Les résolvantes sont définies par la récurrence :*

$$W_1^{(0)}(x) = \omega(x) \qquad , \qquad W_2^{(0)}(x_1,x_2) = B(x_1,x_2) \tag{4.5.1}$$

$$\begin{aligned} W_{n+1}^{(g)}(x_0,J) \quad = \quad & \frac{1}{2i\pi}\sum_i \int_{\mathscr{C}_i} dx\, K(x_0,x)\left(\overline{W}_{n+2}^{(g-1)}(x,x,J)\right. \\ & \left. + \sum_{h=0}^{g}{}' \sum_{I\subset J}{}' W_{|I|+1}^{(h)}(x,x_I) W_{n-|I|+1}^{(g-h)}(x,J/I)\right) \end{aligned} \tag{4.5.2}$$

*où $J$ est une notation compacte pour les variables $J = \{x_1,\dots,x_n\}$, et où $\sum\sum'$ signifie que l'on exclut les termes $(h=0, I=\emptyset)$ et $(h=g, I=J)$ (pour obtenir une véritable relation de récurrence). Par ailleurs, nous avons effectué une translation des fonctions de corrélation :*

$$\overline{W}_n^{(g)}(x_1,...,x_n) = W_n^{(g)}(x_1,...,x_n) - \frac{\delta_{n,2}\delta_{g,0}}{2}\frac{1}{(x_1-x_2)^2} \tag{4.5.3}$$

*Enfin, les points $x_0$ ainsi que tous les autres $x_i$ sont supposés être du même côté des $\mathscr{A}$-cycles que $\infty_0$. Le contour $\mathscr{C}_i$ est un contour qui entoure les demi-lignes d'accumulation de zéros $L_i$ (i.e. les points de branchements quantiques), et qui est choisi de telle façon que chaque $s_j$ soit entouré exactement une seule fois et que le contour n'intersecte aucun des $\mathscr{A}_\alpha$-cycles pour $1 \le \alpha \le g$.*

Cette formule est à comparer avec le cas de la géométrie algébrique classique [23] où :

$$\Phi(p) = \int^p y\,dx$$

$$W_2^{(0)}(p_1,p_2) = B(p_1,p_2)$$

$$W_{k+1}^{(g)}(p,p_K) \quad = \quad \sum_i \operatorname*{Res}_{q\to a_i} \frac{dE_q(p)}{y(q)-y(\overline{q})}$$

$$\left(\sum_{h=0}^{g}\sum_{I\in J}{}' W^{(h)}_{|I|+1}(q,p_I)W^{(g-m)}_{k-|I|+1}(q,p_{J/I})+W^{(g-1)}_{k+2}(q,q,p_J)\right) \tag{4.5.4}$$

On constate donc que les deux formules sont identiques et sont formées d'une somme de résidus autour des points de branchements. En d'autres termes, on constate donc que la forme de la récurrence topologique est indépendante de $\hbar$. De plus, il est montré dans **[IV]**, annexe C, le théorème suivant :

**Théorème 4.9.** *Les fonctions de corrélation $W_n^{(g)}$ définies par 4.5.2 vérifient les équations de boucles du modèle à une matrice avec $\beta$-quelconque 4.4.2 et constituent donc une solution aux équations de boucles du modèle à une matrice avec $\beta$-quelconque.*

• Dans les deux cas, les démonstrations ([23]) font appel à l'identité bilinéaire de Riemann qui peut être énoncée de la façon suivante :

**Théorème 4.10.** *Identité bilinéaire en géométrie algébrique :*

*Soient $\omega_1$ et $\omega_2$ deux formes méromorphes et $\mathscr{A}_1,\dots,\mathscr{A}_g,\mathscr{B}_1,\dots,\mathscr{B}_g$ une base de cycles d'une surface de Riemann de genre g, alors en définissant la fonction $\Phi_1(p)=\int_{p_0}^{p}\omega_1$ où $p_0$ est un point arbitraire de la surface de Riemann :*

$$\operatorname*{Res}_{p\to \text{tous les poles}}\Phi_1\omega_2=\frac{1}{2i\pi}\sum_{j=1}^{g}\int_{\mathscr{A}_j}\omega_1\int_{\mathscr{B}_j}\omega_2-\int_{\mathscr{A}_j}\omega_2\int_{\mathscr{B}_j}\omega_1 \tag{4.5.5}$$

Dans le cas quantique, cette identité peut être généralisée de la manière suivante (Cf. **[IV]**, section 4.4.4 de l'annexe VIII) :

**Théorème 4.11.** *Considérons une fonction $f(x)$ de la forme :*

$$f(x)=\frac{1}{\hbar\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\,g(x')\,dx' \;\Leftrightarrow\; \hbar(2\frac{\psi'(x)}{\psi(x)}+\partial_x)\,f(x)=g(x) \tag{4.5.6}$$

*où $g(x')$ est un polynôme de degré inférieur à $d-1$. Alors on a l'identité bilinéaire de*

*Riemann généralisée :*

$$\begin{aligned} f(x) &= \frac{1}{2i\pi}\oint_{\mathscr{C}} K(x,z)\,g(z)\,dz + \sum_{\alpha} v_\alpha(x)\oint_{\mathscr{A}_\alpha} f(z)\,dz \\ &= \frac{1}{2i\pi}\oint_{\mathscr{C}} G(x,z)\,f(z)\,dz + \sum_{\alpha} v_\alpha(x)\oint_{\mathscr{A}_\alpha} f(z)\,dz \end{aligned} \tag{4.5.7}$$

*où $\mathscr{C}$ est le contour de récurrence introduit dans 4.5.2.*

Notons que cette identité généralise bien l'identité bilinéaire de Riemann traditionnelle de géométrie algébrique que l'on peut énoncer comme suit :

$$\omega(q) = \operatorname*{Res}_{p\to poles\,de\,\omega} dS_{p,p_0}(q)\omega(p) + \sum_{i=1}^{g} du_i(q)\oint_{\mathscr{A}_i}\omega \tag{4.5.8}$$

où $dS_{p,p_0}(q)$ est la forme de troisième espèce et $du_i$ sont les formes holomorphes. Dans notre formalisme, $G(x,z)$ joue le role de cette forme de troisième espèce et les $v_\alpha(x)$ jouent le role des formes holomorphes. Au final, la seule différence d'écriture se situe dans la présence d'une intégrale de contour $\mathscr{C}$ au lieu d'une somme sur les résidus.

• On peut également montrer que ces fonctions de corrélation définies par 4.5.2 satisfont les propriétés attendues pour des fonctions de corrélations. En particulier, on a le résultat suivant (Cf. **[IV]**, annexe B) :

**Théorème 4.12.** *$W_n^{(g)}(x_1,\dots,x_n)$ est une fonction analytique avec des pôles sans résidu aux $s_i$, est une fonction symétrique de ses variables et se comporte en $O\left(\frac{1}{x_i^2}\right)$ dans tous les secteurs à l'infini.*

## 4.6 Vers des invariants symplectiques $F_g$ généralisés ?

Un résultat crucial de la récurrence topologique d'Eynard et Orantin est de pouvoir inverser la récurrence topologique de manière extrêmement simple par la formule :

$$(2-2g-n)W_n^{(g)}(p_1,\dots,p_n) = \sum_i \operatorname*{Res}_{q\to a_i} \Phi(q)W_{n+1}^{(g)}(p_1,\dots,p_n,q) \tag{4.6.1}$$

qui permet “de remonter d’une variable”. En étendant cette propriété au cas $n = 0$, les auteurs ont ainsi défini les nombres $F^{(g)}_{\beta=1}$ :

$$F^{(g)}_{\beta=1}(p_1,\dots,p_n) = \frac{1}{2-2g}\sum_i \operatorname*{Res}_{q\to a_i} \Phi(q) W_1^{(g)}(q) \tag{4.6.2}$$

et ainsi pu montrer qu’il s’agissait d’invariants symplectiques, c’est-à-dire qu’ils ne changent pas sous une transformation de la courbe algébrique de départ $E(x,y)$ lors d’une transformation symplectique, (i.e. toute transformation des coordonnées $\tilde{x} = f(x,y)$ , $\tilde{y} = g(x,y)$ laissant invariant la forme $dx \wedge dy$ : $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$). Depuis leur découverte, ces invariants symplectiques ont connu de nombreuses applications en dénombrement, en théorie des cordes topologiques et permettent de définir une fonction $\tau$ et une hiérarchie intégrable [23]. Notons également que par définition, ils permettent d’écrire un développement perturbatif de la fonction de partition par la formule :

$$\ln Z_{\beta=1} = \sum_{g=0}^{\infty} \left(\frac{N}{T}\right)^{2-2g} F^{(g)}_{\beta=1} \tag{4.6.3}$$

Dès lors, il apparait intéressant de pouvoir généraliser de tels nombres au cas quantique. Malheureusement, plusieurs étapes manquent à l’heure actuelle. Tout d’abord, la généralisation de 4.6.1 n’est que partielle à ce jour pour le cas à une matrice puisque l’on a seulement (Cf. **[IV]**, section 7, présenté en annexe VIII) :

**Théorème 4.13.** *Les fonctions $W_n^{(g)}$ satisfont la formule :*

$$(2-2g-n-\hbar\,\partial_\hbar)\,W_n^{(g)} = \left(t_0\,\partial_{t_0} + \sum_{k=1}^{d+1} t_k\,\partial_{t_k} + \sum_{i=1}^{g} \varepsilon_i \partial_{\varepsilon_i}\right) W_n^{(g)} = \hat{H}.W_{n+1}^{(g)} \tag{4.6.4}$$

*où $\hat{H}$ est un opérateur linéaire qui agit comme (et dont on peut montrer qu’il redonne le membre de droite de 4.6.1 dans la limite où $\hbar \to 0$) :*

$$\hat{H}.f(x) = t_0 \int_{\infty_0}^{\infty_-} f + \sum_{j=1}^{d+1} \operatorname*{Res}_{\infty_0} \frac{t_j x^j}{j} f + \sum_{i=1}^{g} \varepsilon_i \oint_{\mathscr{B}_i} f. \tag{4.6.5}$$

La différence principale avec le cas hermitien est alors la présence supplémentaire du facteur $\hbar\partial_\hbar$ dans le membre de gauche de 4.6.4 qui ne permet pas de définir les $F_g$ de façon unique dans le cas $n = 0$. Pour pouvoir résoudre cette difficulté, il faudrait pouvoir exprimer $\hbar\partial_\hbar W_n^{(g)}$ comme un opérateur intégral agissant sur $W_{n+1}^{(g)}$ ce qui n'est pas connu à l'heure actuelle. La seconde difficulté consiste ensuite à généraliser les résultats obtenus pour le cas du modèle à une matrice avec $\beta$ quelconque au cas du modèle à deux matrices avec $\beta$ quelconque. Nous travaillons à l'heure actuelle à la réalisation de cette étape et les résultats préliminaires semblent indiquer que cette généralisation est possible bien que le formalisme devienne plus technique. Enfin, la dernière étape, conditionnée à la réussite des deux précédentes, serait, une fois la définition des $F_g$ obtenue dans le cas des modèles à une et deux matrices, de trouver l'équivalent de la propriété d'invariance symplectique et de la démontrer. Il serait assez logique que cette propriété consiste en l'invariance des $F_g$ sous n'importe quelle reparamétrisation $\tilde{x} = f(x,\hat{y})$ , $\tilde{y} = g(x,\hat{y})$ conservant le commutateur $[\hat{y},x] = \hbar$ bien qu'il ne s'agisse ici que d'une conjecture extrêmement lointaine.

## 4.7 Conclusions

Dans ce chapitre, nous avons montré comment les quantités de géométrie algébrique ainsi que la récurrence topologique d'Eynard et Orantin utilisées pour résoudre les équations de boucles du modèle hermitien, peuvent être généralisées dans ce qui pourrait devenir de la géométrie algébrique "quantique" pour le cas du modèle à une matrice avec $\beta$ quelconque. Cependant, beaucoup de choses restent à faire dans ce nouveau domaine, en particulier trouver une généralisation des invariants symplectiques, traiter le cas des modèles à deux matrices, trouver et démontrer une généralisation de la propriété d'invariance symplectique. Par ailleurs, dans la méthode proposée, les fonctions de corrélations $W_n^{(g)}$ dépendent explicitement du choix de la solution $\psi(x)$ de l'équation différentielle 4.4.7 qui constitue la courbe "quantique". Comprendre cette dépendance et le rôle de la solution $\psi(x)$ constitue donc une étape supplémentaire importante pour la résolution explicite des modèles de matrices avec $\beta$-quelconque les plus simples. Des applications

de cette théorie sont également en cours de développement, comme par exemple : le dénombrement des surfaces non-orientables (le cas hermitien donnant celui des surfaces orientables tel que données dans ([47], [48], [51], [58], [59])), et la théorie des cordes topologiques, en particulier de la fonction de partition de Nekrasov ([52] [52], [54]) et de la conjecture AGT ([55]). Des liens avec les systèmes intégrables et la théorie des équations différentielles (puisque la méthode permettrait d'associer à toute équation différentielle ordinaire linéaire (4.4.7) des invariants $F_g$) seraient également possibles bien qu'ils soient à l'heure actuelle encore hypothétiques.

# CHAPITRE 5

# THÉORIE DES CORDES TOPOLOGIQUES ET MODÈLES DE MATRICES

## 5.1 La théorie des cordes topologiques

La théorie des cordes s'est développée en physique fondamentale dans le but de concilier les deux grandes théories de la physique moderne : la mécanique quantique (et la théorie des champs qu'elle implique) et la relativité générale. En effet, durant les cinquante dernières années, de nombreuses expériences ont permis de vérifier l'exactitude des deux théories : la théorie quantique des champs permettant de faire des prédictions sur le monde microscopique (laser, collision de particules, modèle standard,...) tandis que la relativité générale permet elle de faire des prédictions sur des corps en interaction gravitationnelle (avancée du périhélie de Mercure, correction pour le GPS, etc.). Cela dit, malgré la réussite de chacune des théories pour prédire efficacement les résultats dans leur domaine respectif, les deux formalismes mathématiques sont incompatibles lorsque l'on tente de quantifier la gravitation comme les autres champs. Certes, il est possible d'adopter une position pragmatique et de n'utiliser chacune des théories que dans son domaine respectif (le monde microscopique pour la mécanique quantique et les gros objets célestes pour la relativité générale). Après tout, la science n'a pas pour but d'être "unifiée" ou d'être "belle", mais juste de fournir des modèles explicatifs et prédictifs, ce que chaque branche réalise parfaitement pour les échelles d'énergie expérimentées jusqu'ici. Toutefois, il reste que certains phénomènes impliquent les deux échelles. Ainsi, les trous noirs, l'univers primordial, l'anomalie des sondes Pioneer, ou les futures collisions de particules à très hautes énergies du LHC sont autant d'expériences qui poussent (ou pousseront) chacune des théories à ses limites. Dès lors, il est nécessaire de trouver une théorie permettant de regrouper sous un même formalisme toutes les interactions, y compris l'interaction gravitationnelle. Cette quête de la théorie de l'unification est un des sujets qui anime la physique théorique depuis plus de cinquante ans, mais qui n'a actuellement toujours pas de solution acceptable. Si la théorie des cordes consti-

tue un candidat pour réaliser l'unification, elle n'est plus aujourd'hui l'unique théorie en lice. Ainsi, la gravitation quantique à boucles («loop quantum gravity»), la géométrie non-commutative, la dynamique Newtonienne modifiée (MOND) figurent parmi les adversaires les plus avancés et sont en développement rapide. D'autres plus exotiques sont également en cours d'élaboration : temps discret, particules supplémentaires aux propriétés étranges, modification de la théorie des champs, etc. Hélas aucune d'entre elles n'a pour l'instant résolu le problème de la "grande unification" de façon satisfaisante. Parmi toutes ces théories, la théorie des cordes semble être l'un des candidats les plus sérieux. L'idée de départ repose sur le fait que ce que nous appelons particules élémentaires (électrons, quarks, neutrinos,...) ne seraient pas des particules ponctuelles mais de minuscules cordes vibrantes. Si l'idée de base semble séduisante, la cohérence mathématique impose immédiatement que le nombre de dimensions spatiales de notre univers n'est plus de 3 mais de 10. Dès lors, comment expliquer que nous ne ressentons pas les 6 dimensions spatiales supplémentaires ? La réponse à cette question est relativement simple : nous ne les ressentons pas car elles sont de tailles minuscules : échelle de Planck ou tout du moins une échelle suffisamment petite pour que les énergies accessibles par les technologies actuelles n'aient pas permis de les détecter. En revanche, lors de l'étude de phénomènes extrêmes (univers primordial), l'influence de ces dimensions supplémentaires pourrait être détectée. D'un point de vue mathématique, la question qui apparaît naturellement est de savoir comment décrire des dimensions "de taille minuscule", c'est-à-dire comment compactifier les dimensions supplémentaires ? Pour visualiser les choses, le lecteur non spécialiste peut s'imaginer qu'un segment peut être replié en un cercle si l'on joint les deux bouts. Un ruban peut quant à lui être replié en un cylindre puis un tore, mais peut également être replié pour former un ruban de Moebius puis une bouteille de Klein. Le concept naturel en mathématique pour décrire la compactification est la notion de variétés. Ainsi, la sphère, le tore, la bouteille de Klein sont des variétés compactes, et l'on pourrait imaginer que les 6 dimensions manquantes correspondent en fait à une variété de dimension 6 compacte très enroulée sur elle-même. L'image la plus utilisée pour décrire ce type de variété est la suivante :

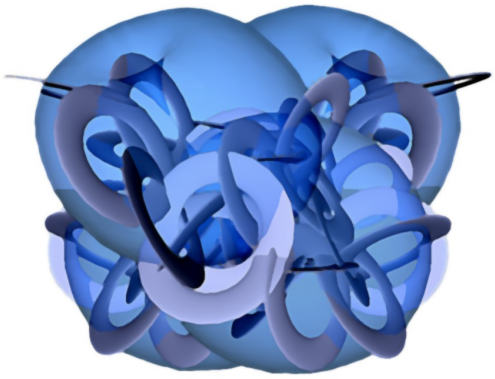

Figure 25 : Représentation imagée d'une variété de Calabi-Yau

Le choix du type de variété utilisé pour la compactification se pose alors de manière cruciale. Sans rentrer dans les détails techniques (la théorie des cordes faisant à elle seule l'objet de thèses ou de livres entiers), les conditions impliquées par la physique et le formalisme considéré imposent de choisir une variété de type Calabi-Yau que nous allons décrire rapidement dans la prochaine section.

## 5.2 Variété de Calabi-Yau

Le but de cette section n'est pas de rentrer dans les détails géométriques de la construction des variétés de Calabi-Yau. Le lecteur interessé trouvera dans [75] une excellente introduction aux variétés de Calabi-Yau. Dans ce paragraphe, nous nous contenterons de donner les définitions et quelques propriétés pour aboutir à la symétrie miroir et à la conjecture de Bouchard, Klemm, Marino et Pasquetti (BKMP) et aux modèles de matrices.

**Définition 5.1.** *Une variété de Calabi-Yau est définie comme une variété kählérienne (i.e. une variété hermitienne M, c'est-à-dire une variété complexe munie d'une métrique hermitienne h, telle que la 2-forme* $\omega = -\mathrm{Im}\, h$ *soit fermée) dont la première classe de Chern est nulle. De façon encore équivalente, un espace de Calabi-Yau de dimension complexe n (ce qui correspond à une dimension réelle* $2n$*) peut être vu comme une variété riemannienne d'holonomie réduite à* $SU(n)$ *(le groupe d'holonomie d'une variété riemannienne de dimension réelle* $2n$ *étant génériquement le groupe* $SO(2n)$*).*

Définir précisément les notions de classe de Chern et de groupe d'holonomie ainsi que leurs propriétés amènerait la discussion bien au delà du domaine des matrices aléatoires, nous nous contentons donc de citer [72] pour informations.

Un des théorèmes importants des variétés de Calabi-Yau a été la démonstration de la conjecture d'Eugène Calabi, formulée en 1957, par Shing-Tung Yau en 1977 (d'où le nom de variété de Calabi-Yau) de l'existence sur de telles variétés d'une métrique dont le tenseur de Ricci s'annule (et qui constitue un élément nécessaire pour la cohérence de la théorie physique). L'étude des variétés de Calabi-Yau en basse dimension a été également réalisée. On sait désormais que :

1. En dimension complexe 1, la seule variété Calabi-Yau est le 2-tore.
2. En dimension complexe 2, il n'existe que deux variétés Calabi-Yau à un isomorphisme près. Il s'agit du 4-tore et de l'espace $K3$. Sur ce dernier, aucune métrique Ricci-plate explicite n'est connue, bien que l'existence soit assurée par le théorème de Yau. Il en va de même pour toutes les variétés de Calabi-Yau non triviales de dimensions supérieures.
3. A partir de la dimension complexe 3 (dimension réelle 6) le nombre de variétés de Calabi-Yau devient infini et il n'existe pas encore de classification générale. On sait toutefois en construire beaucoup qui possèdent en plus la propriété d'être des variétés toriques.

La notion de variété de Calabi-Yau torique est ainsi donnée par la définition suivante :

**Définition 5.2.** *Par définition, une variété de Kähler $\mathscr{M}$, n dimensionnelle est dite "torique" s'il existe un tore maximal T inclus dans les automorphismes bi-holomorphes de $\mathscr{M}$ tel que T soit isomorphe à $(\mathbb{C}^*)^n = \mathbb{R}^n \times (\mathbb{S}^1)^n$ etque l'action du tore T sur lui même s'étendent à toute la variété $\mathscr{M}$.*

Enfin, puisque nous sommes intéressés à compactifier un espace de dimension réelle 6, nous allons donc nous intéresser à des variétés de Calabi-Yau de dimension complexe 3. Grâce à leurs applications en théorie des cordes, ces variétés ont été abondamment étudiées. En particulier, il a été découvert récemment que pour ces variétés, il existe une

symétrie “miroir”, c’est-à-dire une dualité entre familles de variétés de Calabi-Yau de dimension 3. Du point de vue mathématique, cette symétrie exprime une relation entre les nombres de courbes rationnelles sur une telle variété et les périodes des structures de Hodge sur la variété “miroir” associée. La théorie des cordes topologiques s’intéresse au dénombrement d’applications pseudoholomorphes d’une surface de Riemann de genre $g$ vers une variété de Calabi-Yau donnée (donc au premier aspect de la dualité précédente). On peut montrer ([71]) que cela revient plus ou moins à trouver les invariants de Gromov-Witten (qui sont des nombres rationnels) de cette variété de Calabi-Yau. Les invariants de Gromov-Witten étant relativement éloignés des matrices aléatoires, ils ne seront pas abordés en détail ici et nous renvoyons le lecteur à [70] pour une introduction. Lorsque la variété est torique, on peut lui associer grâce à la symétrie miroir une variété duale qui peut être décrite par une équation du type (le second aspect de la dualité précédente) :

$$H(e^x, e^y) = 0 \tag{5.2.1}$$

où $H$ est un polynôme dont les coefficients codent les propriétés géométriques de la variété torique de départ. L’intérêt de cette formulation est qu’elle permet de faire un lien avec les matrices aléatoires et en particulier le développement topologique des modèles hermitiens et qu’elle offre des meilleures perspectives de calculs pratiques. (Car d’un point de vue calculatoire, les invariants de Gromov-Witten sont en général très difficiles à calculer.) En effet, dans ce cas, on a vu dans la première partie que les modèles hermitiens donnent lieu à une courbe spectrale algébrique ainsi qu’à une collection d’invariants $F^{(g)}$ de cette courbe. La conjecture BKMP ([81]) peut alors être énoncée de la façon suivante :

**Conjecture 5.1.** *“Les invariants de Gromov Witten d’une variété torique de Calabi-Yau $\mathscr{M}$ de dimension trois sont les invariants symplectiques (au sens d’Eynard et Orantin) $F^{(g)}$ de la courbe spectrale $H(e^x, e^y) = 0$ de sa variété miroir”.*

Cette conjecture présente un double intérêt. Tout d’abord, elle permet de faire le lien entre la théorie des cordes topologiques et les modèles de matrices hermitiens, permettant en particulier d’appliquer les techniques des modèles de matrice pour la théorie des cordes topologiques. Ensuite, si la conjecture s’avère exacte, elle permettrait, dans le cas

des variétés de Calabi-Yau toriques de dimension 3, d'avoir un algorithme explicite pour calculer les invariants de Gromov-Witten par le formalisme de récurrence topologique d'Eynard et Orantin. Il est à noter que dans de nombreux cas simples ou plus compliqués ([80, 82]) cette conjecture a pu être explicitement vérifiée.

## 5.3 La formule du vertex topologique

Le point de départ pour relier la théorie des matrices aléatoires avec les invariants de Gromov-Witten est la formule du vertex topologique [76, 77]. Rappelons que les invariants de Gromov-Witten, notés $\mathscr{N}_{g,D}$, d'une variété torique de Calabi-Yau $\mathscr{M}$ de dimension 3, comptent le nombre d'applications de surfaces connexes de genre $g$ dans $\mathscr{M}$ étant donné une classe d'homologie $D = (D_1, \ldots, D_k)$. On peut alors rassembler ces invariants sous la forme d'une double série génératrice (en notant $Q^D \stackrel{def}{=} \prod_{i=1}^{k} Q_i^{D_i}$) :

$$GW(\mathscr{M}, Q, g_s) = \sum_{g=0}^{\infty} \sum_{D} Q^D g_s^{2g-2} \mathscr{N}_{g,D}(\mathscr{M}) \tag{5.3.1}$$

On peut rajouter, comme habituellement dans ce type de dénombrement, les surfaces non-connexes en prenant l'exponentielle :

$$Z_{GW}(\mathscr{M}, Q, g_s) = e^{GW(\mathscr{M}, Q, g_s)} = \sum_{g=0}^{\infty} \sum_{D} Q^D g_s^{2g-2} \mathscr{N}^*_{g,D}(\mathscr{M}) \tag{5.3.2}$$

Il a été montré [76] que cette dernière fonction de partition est identique à celle donnée par la formule du vertex topologique qui peut être exprimée dans un cas particulier dit "fiducial" (le cas général peut également être exprimé, mais comme je le mentionne plus bas, on peut toujours se ramener au cas fiducial par des opérations appelées "transitions de flop" ce qui facilite les calculs) comme :

$$Z_{vertex}(\mathscr{M}) = \sum_{j=0..n, i=1..m-1} \sum_{\alpha_{i,j}} \prod_{i=1}^{m} Z_{ligne}(\vec{\alpha}_i, \vec{\alpha}_{i+1}^T) \prod_{i,j} q^{s_{j,i}|\alpha_{i,j}|} \tag{5.3.3}$$

où

$$Z_{ligne}(\alpha;\beta^T) = \prod_{i=0}^{n}\left(\frac{[\alpha_i][\beta_i^T]}{[\beta_i,\alpha_i^T]_{Q_{\beta_i,\alpha_i}}}\right)\frac{\prod_{i<j}\left([\alpha_i,\alpha_j^T]_{Q_{\alpha_i,\alpha_j}}\right)\prod_{i<j}\left([\beta_i,\beta_j^T]_{Q_{\beta_i,\beta_j}}\right)}{\prod_{i<j}\left([\alpha_i,\beta_j^T]_{Q_{\alpha_i,\beta_j}}[\beta_i,\alpha_j^T]_{Q_{\beta_i,\alpha_j}}\right)} \tag{5.3.4}$$

et les $\alpha_i$ sont des partitions planes et les crochets appliqués à une partition plane $\gamma = (\gamma_1,\gamma_2,\dots)$ sont définis par :

$$[\gamma] = (-1)^d q^{\frac{1}{4}\kappa(\gamma)}\prod_{1\le i<j\le d}\frac{[\gamma_i-\gamma_j+j-i]}{[j-i]}\prod_{i=1}^{d}\prod_{j=1}^{\gamma_i}\frac{1}{[d+j-i]} \tag{5.3.5}$$

avec $q = e^{-g_s}$ faisant le lien avec la série génératrice des invariants de Gromov-Witten, $\kappa(\gamma) = \sum_k \gamma_k(\gamma_k - 2k + 1)$ le second Casimir de la partition plane $\gamma$ et par definition le crochet appliqué à un nombre entier est le "q-nombre" :

$$\forall n \in \mathbb{N} : \ [n] = q^{-\frac{n}{2}} - q^{\frac{n}{2}} \tag{5.3.6}$$

Ici nous utilisons les notations françaises des partitions planes décrites par la figure :

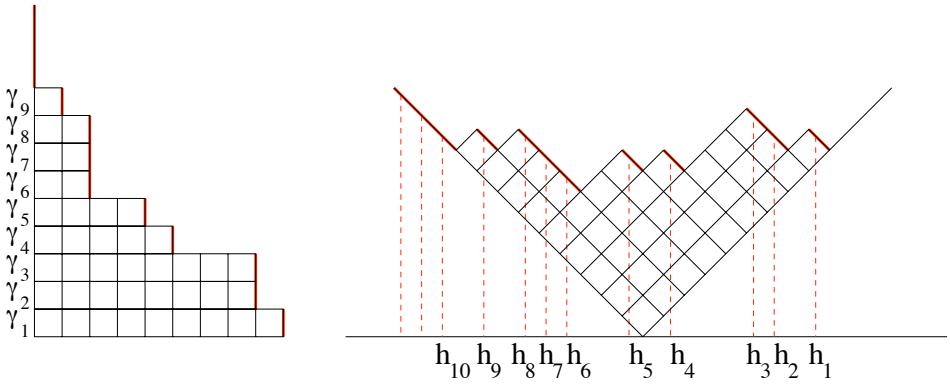


Figure 26 : Exemple de représentation de partitions planes. On remarque ainsi comment on peut passer par une simple rotation des entiers $\gamma_i$ aux entiers $h_i$.

La transposée $\gamma^T$ d'une partition $\gamma$ est définie comme la partition plane dans laquelle on a inversé les lignes et les colonnes. La notation $|\gamma|$ désigne le nombre total de boîtes

de la partition plane. Le terme de couplage entre deux partitions est donné par :

$$\begin{aligned}[\gamma,\delta^T] &= Q_{\gamma,\delta}^{-\frac{|\gamma|+|\delta|}{2}} q^{-\frac{\kappa(\gamma)-\kappa(\delta)}{4}} \prod_{i=1}^{d}\prod_{j=1}^{d} \frac{[h_i(\gamma)-h_j(\delta)]}{[a_\gamma - a_\delta + j - i]} \\ &\times \prod_{i=1}^{d}\prod_{j=1}^{\gamma_i} \frac{1}{[a_\gamma - a_\delta + j - i + d]} \prod_{i=1}^{d}\prod_{j=1}^{\delta_i} \frac{1}{[a_\gamma - a_\delta - j + i - d]} \\ &\prod_{k=0}^{\infty} g(Q_{\gamma,\delta}^{-1} q^{-k}) \end{aligned} \tag{5.3.7}$$

avec $h_i(\gamma) = \gamma_i - i + d + a$ et $g(x) = \prod_{n=1}^{\infty}\left(1 - \frac{1}{x}q^n\right)$. Le paramètre $Q_{\gamma,\delta}$ reflète les paramètres de Kähler de notre variété de départ à partir desquels on peut définir les paramètres $a_\gamma$ de façon équivalente (dans le formalisme du vertex topologique il est beaucoup plus pratique d'utiliser les paramètres $a_\gamma$ que les $Q_{\gamma,\delta}$) par :

$$Q_{\gamma,\delta} = q^{a_\gamma - a_\delta} \tag{5.3.8}$$

Les nombres $s_{i,j}$ décrivant l'interaction entre deux lignes dans 5.3.3 peuvent être reliés à des différences de $a_{\alpha_{i,j}}$ (Cf. **[V]** présenté en annexe IX).

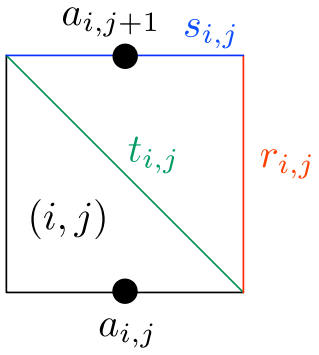


Figure 27 : Présentation des conventions de notation pour les paramètres de Kähler

Traditionnellement, dans le formalisme du vertex topologique, la géométrie d'une variété de Calabi-Yau torique de dimension trois est représentée par un diagramme du type :

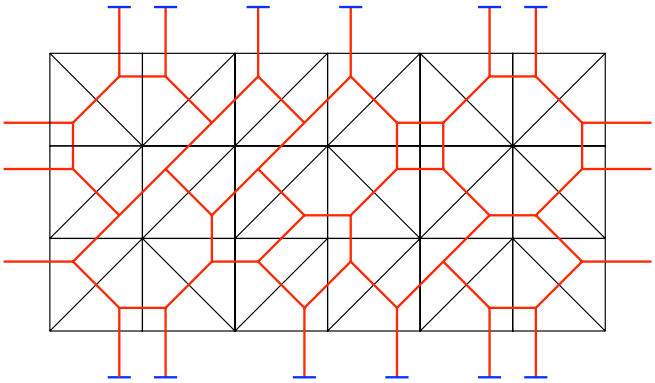

Figure 28 : Exemple de représentations d'une variété de Calabi-Yau torique de dimension 3 par le formalisme du vertex topologique. Le diagramme dual est représenté en rouge. Le choix de l'orientation des diagonales de chaque carré reflète les propriétés géométriques de la variété.

Le passage d'une boîte diagonale haute à une boîte diagonale basse est appelé "transition de flop".

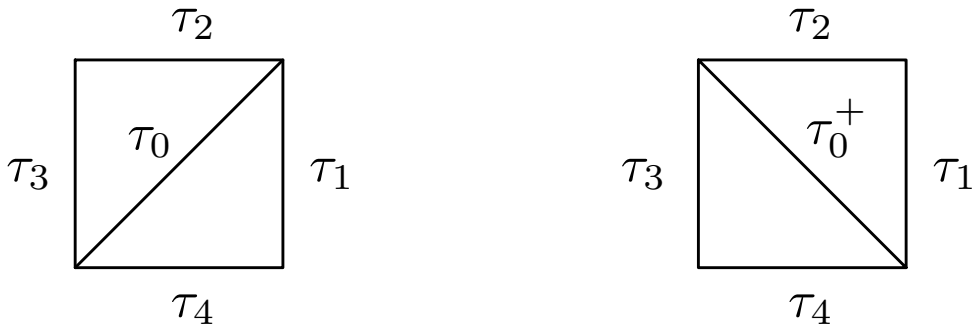


Figure 29 : Illustration d'une transition de flop.

Il est connu que les invariants de Gromov-Witten sont invariants sous les transitions de flop, ce qui signifie que n'importe quelle configuration peut être choisie pour le calcul de ces invariants. En particulier, notre choix fiducial est possible et se décrit par le diagramme :

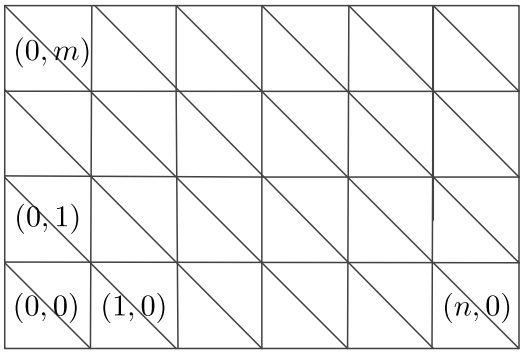


Figure 30 : Représentation de la géométrie fiduciale.

Ainsi, grâce aux résultats précédents, on voit que la fonction de partition des invariants de Gromov-Witten peut être ramenée à une fonction de partition exprimée en termes de partitions planes, qui sont connues pour avoir des liens avec les matrices aléatoires [46, 49, 50, 84, 85]. Notons que si la conjecture s'avère exacte, l'invariance des nombres de Gromov-Witten sous les transitions de flop doit être retrouvée du côté des invariants symplectiques $F^{(g)}$. Dans **[V]** (Cf. annexe IX), il est montré qu'une transition de flop correspond à une transformation symplectique de la courbe spectrale $E(x,y)$ du modèle de matrice décrit dans les prochains paragraphes, qui comme nous l'avons vu précédemment laisse invariants les nombres $F^{(g)}$.

## 5.4 Reformulation en termes de modèles de matrices

Grâce à la formule du vertex topologique et à l'invariance des nombres de Gromov-Witten par transitions de flop, nous avons vu dans le paragraphe précédent qu'il était possible de reformuler la série génératrice des invariants de Gromov-Witten sous la forme d'une fonction de partition impliquant des partitions planes (5.3.3). Dans notre travail **[V]** (Cf. annexe IX), nous avons montré comment il est possible de réécrire cette fonction de partition sous la formule d'un modèle de matrice hermitien. Compte-tenu de la longueur des calculs nécessaires, nous nous contererons ici de mentionner le résultat obtenu, laissant au lecteur le soin de consulter **[V]** en annexe IX de ce mémoire pour les détails de la dérivation. Le modèle de matrice obtenu est :

$$
\begin{aligned}
Z_{\mathrm{MM}}(Q,g_s,\vec{\alpha}_m,\vec{\alpha}_0^{\,T}) \;=\; & \Delta(X(\vec{\alpha}_m))\,\Delta(X(\vec{\alpha}_0))\prod_{i=0}^{m}\int_{H_N(\Gamma_i)} dM_i \prod_{i=1}^{m}\int_{H_N(\mathbb{R}_+)} dR_i \\
& \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,\mathrm{tr}\,V_{\vec{a}_i}(M_i)-V_{\vec{a}_{i-1}}(M_i)} \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,\mathrm{tr}\,V_{\vec{a}_{i-1}}(M_{i-1})-V_{\vec{a}_i}(M_{i-1})} \\
& \prod_{i=1}^{m} e^{\frac{1}{g_s}\mathrm{tr}\,(M_i-M_{i-1})R_i} \prod_{i=1}^{m-1} e^{(S_i+i\pi)\,\mathrm{tr}\ln M_i} \\
& e^{\mathrm{tr}\ln f_0(M_0)}\, e^{\mathrm{tr}\ln f_m(M_m)} \prod_{i=1}^{m-1} e^{\mathrm{tr}\ln f_i(M_i)}. \qquad (5.4.1)
\end{aligned}
$$

où les matrices sont de taille $N=(n+1)\,d$ et où les matrices de début et de fin de chaîne sont données par :

$$
X(\vec{\alpha}_m)=\mathrm{diag}(X(\vec{\alpha}_m)_i)_{i=1,\ldots,N}\,, \qquad X(\vec{\alpha}_m)_{(j-1)d+i}=q^{h_i(\alpha_{j,m})}, \qquad (5.4.2)
$$

$$
X(\vec{\alpha}_0)=\mathrm{diag}(X(\vec{\alpha}_0)_i)_{i=1,\ldots,N}\,, \qquad X(\vec{\alpha}_0)_{(j-1)d+i}=q^{h_i(\alpha_{j,0})}, \qquad (5.4.3)
$$

avec la notation habituelle $\Delta(X)=\prod_{i<j}(X_i-X_j)$ du déterminant de Vandermonde et où enfin les potentiels sont définis par :

$$
V_{\vec{a}_i}(x)=-g_s\sum_{j=1}^{n}\ln g(q^{a_{j,i}}/x) \qquad (5.4.4)
$$

avec $\forall\, i=1,\ldots,m-1$ l'introduction des fonctions $f_i(x)$ par :

$$
f_i(x)=\prod_{j=0}^{n}\frac{g(1)^2\; e^{(\frac{1}{2}+\frac{i\pi}{\ln q})\ln(xq^{1-a_{j,i}})}\; e^{\frac{(\ln(xq^{1-a_{j,i}}))^2}{2g_s}}}{g(xq^{1-a_{j,i}})\,g(q^{a_{j,i}}/x)}. \qquad (5.4.5)
$$

et les cas spéciaux :

$$
f_0(x)=\prod_{j=0}^{n}\frac{g(xq^{1-a_{j,0}-d})}{x^d\,g(xq^{1-a_{j,0}})}\,, \qquad f_{m+1}(x)=\prod_{j=0}^{n}\frac{g(xq^{1-a_{j,m+1}-d})}{x^d\,g(xq^{1-a_{j,m+1}})} \qquad (5.4.6)
$$

Les paramètres de Kähler de la fonction de partition du vertex topologique 5.3.3 s'identifient alors comme $a_{\alpha_{i,j}} = a_{i,j}$ et les couplages $s_{i,j}$ s'identifient avec les nombres $S_i$ par la formule (valable quel que soit le choix de $j$) :

$$S_i = s_{j,i} + \sum_{k \leq j} (a_{k,i-1} - a_{k,i}) + \sum_{k<j} (a_{k,i+1} - a_{k,i}) \tag{5.4.7}$$

Rappelons également que le paramètre $Q$ est contenu dans les $a_j$ par la formule 5.3.8. Le théorème fondamental est alors le suivant (Cf. **[V]**, section 4.4 de l'annexe IX) :

**Théorème 5.1.** *La fonction de partition du vertex topologique 5.3.3 est identique à celle du modele de matrices (à des facteurs de proportionnalité triviaux près) :*

$$Z_{vertex}(\mathscr{M}) = Z_{\mathrm{MM}}(Q, g_s, \vec{\alpha}_m, \vec{\alpha}_0^T) \tag{5.4.8}$$

**Idée de la preuve :**

La preuve de ce théorème est présentée dans mon article **[VI]** réalisé en collaboration avec B. Eynard et A. Kashani Poor et présenté en annexe X. Compte tenu de la longueur importante de la preuve, nous nous contenterons ici de n'en expliciter que les grandes lignes, en laissant au lecteur la possibilité de se référer à l'annexe X pour les détails. L'idée de la preuve est la suivante : Si l'on part du modèle de matrices 5.4.1, on voit que les potentiels $f_i$ ont des pôles simples et que l'intégration correspond donc juste à prendre des résidus en ces pôles. En particulier elle localise les valeurs propres des matrices $M_i$ aux entiers de la forme $q^{h_i}$ et se transforme ainsi en une somme sur des entiers que l'on peut écrire en termes d'une somme sur des partitions. Les intégrales sur les matrices $R_i$, réalisent les transformées de Laplace d'intégrales d'Itzykson-Zuber, c'est à dire des déterminants de Cauchy, i.e. les dénominateurs dans la formule 5.3.4. Les déterminants de Vandermonde proviennent eux de la diagonalisation de l'intégrale sur les matrices $M_i$ et réalisent les numérateurs de la formule 5.3.4. Enfin, les potentiels $f_j$ contenant les fonctions $g$, réalisent les poids de la formule 5.3.5. On retrouve alors la fonction de partition du vertex topologique 5.3.3 établissant ainsi l'égalité.

## 5.5 Analyse du modèle de matrices et conjecture BKMP

D'après le paragraphe précédent, nous avons vu qu'il était possible de reformuler la fonction de partition donnant les invariants de Gromov-Witten d'une variété torique de Calabi-Yau de dimension 3 en un modèle de matrice hermitien donné par 5.4.1. Hélas, le modèle de matrice obtenu est une chaîne de matrices possédant des restrictions de positivité sur les valeurs propres des matrices $R_i$. Cette restriction correspond à la présence de bords dits durs en 0 sur les valeurs propres des matrices $R_i$ et ce type de modèle de matrice (chaîne+ bords durs) n'a pas été étudié en détail pour l'instant. Néanmoins, il n'y a quasiment aucun doute sur la possibilité d'extension des résultats connus sur les chaînes de matrices sans bords durs dans le cas des bords durs moyennant des modifications habituelles liées aux bords durs. En effet, le cas des chaînes de matrices sans bords durs a déjà été traité dans [83]. De même, le cas à une et deux matrices (correspondant à des chaines de longueurs 0 et 1) a été traité avec des bords durs dans [86, 87]. Une fois ce point purement technique établi, un autre problème se présente alors dans la résolution de la conjecture BKMP. En effet, lorsque l'on résout les équations de boucles, la solution $W_1^{(0)}(x)$, ou de manière équivalente la courbe spectrale, n'est pas unique puisqu'il y a autant de solutions que d'extrema du potentiel (il faut alors spécifier l'extremum autour duquel on se situe ou des fractions de remplissage). Or dans le cas de la chaîne de matrices obtenue, le potentiel extrêmement compliqué présente une infinité d'extrema et donc une infinité de solutions. Dans ce cas, il est connu [23] que la véritable solution au problème est alors de rechercher la courbe spectrale qui minimise, parmis les courbes solutions, le premier invariant $F_0$, ce qui en pratique s'avère être extrêmement difficile. En particulier dans notre article **[VI]**, nous montrons seulement que la courbe miroir satisfait effectivement les équations de boucles, mais pas qu'elle minimise globalement $F_0$. Ce point reste donc à éclaircir pour aboutir à une démonstration finale de la conjecture BKMP. Néanmoins dans notre article **[VI]**, nous montrons que la courbe spectrale "minimale", c'est à dire celle de plus petits degré et genre possibles, correspond bien à la courbe miroir recherchée. Malheureusement il n'existe pas à l'heure actuelle de démonstration générale permettant d'affirmer que la courbe spectrale "minimale" cor-

respond toujours à la courbe spectrale minimisant le prépotentiel $F_0$.

# CONCLUSION

Cette thèse a ainsi présenté différentes méthodes utiles pour résoudre les modèles de matrices aléatoires ainsi qu'une application possible en théorie des cordes. A la frontière entre de nombreux domaines et grandes questions des mathématiques actuelles, il apparaît clairement que les matrices aléatoires nous réservent encore bien des surprises. En particulier, la notion d'universalité évoquée pour les cas hermitiens, réels symétriques et quaternioniques self-duaux est une propriété surprenante qui laisse à penser qu'une sorte de théorème central limite encore incompris serait à l'oeuvre dans les modèles de matrices. Si cette propriété surprenante pouvait être généralisée au cas où l'exposant $\beta$ est quelconque, elle ouvrirait sans doute de grandes possibilités tant fondamentales, avec la possibilité de développer la théorie des polynômes orthogonaux, qu'appliquées avec la théorie des cordes topologiques.

L'extension de la notion d'intégrabilité sous-jacente au cas hermitien est également une voie à suivre très prometteuse. En effet, si la méthode, présentée dans cette thèse, des équations de boucles pour le cas où l'exposant $\beta$ est arbitraire aboutissait à des résultats similaires à ceux du cas hermitien, cela ouvrirait de grandes perspectives quant à une meilleure compréhension de l'intégrabilité au sens quantique et des équations différentielles linéaires.

Néanmoins, beaucoup de travail reste encore à accomplir pour parvenir à de tels débouchés, qui hélas, par l'incertitude même de toute recherche, pourraient également s'avérer inacessibles ou chimériques. Espérons donc que l'effervescence de ces dernières années se poursuive et que la théorie des matrices aléatoires continue d'alimenter des domaines variés des mathématiques et de la physique en nous réservant, qui sait, peut être quelques autres grandes découvertes.

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## Annexe I

## Liste des figures

## Annexe II

### Algorithme de Metropolis

L'algorithme de Metropolis-Hastings est un outil puissant en physique statistique et permet de simuler n'importe quel type de distribution de probabilités $f(x_1,\ldots,x_n)$. Un avantage important de l'algorithme est que la connaissance de la fonction de distribution $f(x_1,\ldots,x_n)$ à une constante multiplicative près est suffisante pour implémenter l'algorithme. En particulier, dans notre cas, l'impossibilité de calculer la normalisation $Z_N$ n'est pas un obstacle à l'implémentation concrète de l'algorithme. Le principe de l'algorithme est le suivant : on part d'un état $(x_1,\ldots,x_n)$ quelconque puis on propose un nouvel état $(\tilde{x}_1,\ldots,\tilde{x}_n)$ en fonction de l'état précédent $(x_1,\ldots,x_n)$. Dans notre cas précis, seule une des composantes (tirée aléatoirement) $x_i$ sera changée et sera obtenue par un tirage aléatoire d'une variable normale centrée en $x_i$ et de variance 0.1. On procède alors au calcul du rapport :

$$r = \frac{f(\tilde{x}_1,\ldots,\tilde{x}_n)}{f(x_1,\ldots,x_n)} \tag{II.1}$$

On tire ensuite une variable aléatoire $\alpha$ suivant une loi uniforme sur $[0,1]$. Si $\alpha < r$, on conserve alors le nouvel état $(\tilde{x}_1,\ldots,\tilde{x}_n)$, sinon on conserve l'ancien. On itère enfin ce processus un très grand nombre de fois et l'état final obtenu donne alors un échantillon distribué suivant la loi de probabilité $f(x_1,\ldots,x_n)$.

Dans le cas présenté en 2.4.1, le code en Maple est :

Initialisation des paramètres :

$N := 200$ :

$\varepsilon := 0.5$ :

$c_1 := \cos(\pi * \varepsilon)$ : $c_2 := \cos(2 * \pi * \varepsilon)$ :

$T_c := 1 + 4 * c_1^2$ :

$T := 2 * T_c$ :

$V := z -> 1/T * (z^4/4 - 4 * c_1 * z^3/3 + 2 * c_2 * z^2/2 + 8 * c_1 * z)$ :

Définition de l'état initial :

```
etatini :=Array(1..N) :
for k from 1 to N do etatini[k] :=(-1)^k * 2 * k/N : od :
```

Algorithme de Metropolis-Hastings-Gibbs :

```
Gibbs :=proc(e,NN,sigma)
etatcourant :=e :
for k from 1 to N*NN do
valeurpropreencours :=Generate(integer(range = 1 .. N)) :
oldVP :=etatcourant[valeurpropreencours] :
nouvelleVP :=Sample(RandomVariable(Normal(oldVP,sigma)),1)[1] :
diffpotentiel :=-N/T*V(nouvelleVP)+N/T*V(oldVP) :
for j from 1 to N do
if j<>valeurpropreencours then
diffpotentiel :=diffpotentiel+evalf(2*ln(abs(nouvelleVP-etatcourant[j])))
-evalf(2*ln(abs(oldVP-etatcourant[j]))) :
fi :
od :
ratio :=exp(diffpotentiel) :
alpha :=GenerateFloat() :
if alpha< min(ratio,1) then etatcourant[valeurpropreencours] :=nouvelleVP :
fi :
od :
return(etatcourant) :
end proc :
```

Commande d'exécution :

```
Histogram(Gibbs(etatini,50,0.1), averageshifted = 4) ;
```

## Annexe III

## Quaternions et matrices quaternioniques

Les matrices quaternioniques self-duales, constituent le troisième ensemble standard dans l'étude des matrices aléatoires. Il est connu depuis les années 1950 avec Wigner, et correspond à un ensemble de matrices diagonalisables par un élément du groupe symplectique $Sp(2n)$, généralisant ainsi les matrices hermitiennes et réelles symétriques (respectivement diagonalisables par un élément du groupe unitaire $U(n)$ et orthogonal $O(n)$). Néanmoins, cet ensemble de matrice étant moins connu, nous rappelons dans cet annexe sa définition.

### 1. Les quaternions

Les quaternions constituent une généralisation bien connue des nombres complexes. Ils sont définis par une algèbre complexe (non commutative) de dimension 4 :

$$q = q^{(0)} + q^{(1)}e_1 + q^{(2)}e_2 + q^{(3)}e_3 = q^{(0)} + \vec{q}.\vec{e} \tag{III.1}$$

où les $q^{(i)}$ sont des nombres complexes et les vecteurs de l'algèbre $e_i$ obéissent aux opérations (définissant le produit sur l'algèbre) :

$$\begin{aligned} e_1^2 = e_2^2 = e_3^2 = -1 \\ e_1.e_2 = -e_2.e_1 = e_3 \\ e_2.e_3 = -e_3.e_2 = e_1 \\ e_3.e_1 = -e_1.e_3 = e_2 \end{aligned} \tag{III.2}$$

Les quaternions possèdent une représentation naturelle en termes de matrices $2 \times 2$ :

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad , \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{III.3}$$

avec en plus la matrice unité :

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{III.4}$$

Le conjugué d'un quaternion est alors défini par :

$$\overline{q} = q^{(0)} - \vec{q}.\vec{e} \tag{III.5}$$

qu'il ne faut pas confondre avec son complexe conjugué :

$$q^* = q^{(0)*} - \vec{q}^*.\vec{e} \tag{III.6}$$

Un quaternion vérifiant $q^* = q$ est dit réel, un quaternion avec $q^* = -q$ est dit imaginaire pur et enfin un quaternion avec $\overline{q} = q$ est dit scalaire. Finalement, le conjugué hermitien d'un quaternion est défini comme :

$$q^\dagger = \overline{q}^* \tag{III.7}$$

Un quaternion avec $q^\dagger = q$ est dit hermitien (et il correspond dans sa représentation en termes de matrices à une matrice $2 \times 2$ hermitienne)

### 2. Les matrices "quaternioniques"

Les matrices "quaternioniques" sont des matrices complexes $A$ de taille paire $2n \times 2n$ qui peuvent s'écrire comme une matrice quaternionique $Q$ de taille $n \times n$ grâce à la représentation de dimension deux des quaternions. Par exemple :

$$A = \begin{pmatrix} i & 0 & 0 & i & 1 & 0 \\ 0 & -i & i & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & i \\ -1 & 0 & 0 & 1 & i & 0 \\ 2+3i & 4+5i & 3 & 0 & 3 & 5i \\ -4+5i & 2-3i & 0 & 3 & 5i & 3 \end{pmatrix} \Leftrightarrow Q = \begin{pmatrix} e_1 & e_3 & 1 \\ e_2 & 1 & e_3 \\ 2+3e_1+4e_2+5e_3 & 3 & 3+5e_2 \end{pmatrix} \tag{III.8}$$

Les opérations habituelles sur la matrice $A$ se reflètent sur la matrice $Q$ de la façon suivante :

1. Transposition : $(Q^T)_{i,j} = -e_2.\overline{q}_{j,i}.e_2$
2. Conjugué hermitien : $(Q^\dagger)_{i,j} = q^\dagger_{j,i}$
3. Renversement du temps : $(Q^R)_{i,j} = \overline{q}_{i,j}$

La matrice $Q^R$ est dite duale de $Q$ et une matrice vérifiant $Q^R = Q$ est dite self-duale. Une matrice $A$ de taille $2n \times 2n$ dont la matrice quaternionique $Q$ est self-duale possède $n(2n-1)$ composantes réelles indépendantes.

### 3. Groupe symplectique

Soit la matrice de taille $2n \times 2n$ :

$$Z = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & \dots \\ -1 & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & 1 & \dots & \dots \\ 0 & 0 & -1 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \ddots & \ddots \\ \dots & \dots & \dots & \dots & \ddots & \ddots \end{pmatrix} \tag{III.9}$$

qui est composée de blocs $2 \times 2$ sur la diagonale et nulle ailleurs. Le groupe symplectique $Sp(2n)$ est alors défini comme l'ensemble des matrices $B$ de taille $2n \times 2n$ telles que :

$$Z = BZB^T \tag{III.10}$$

Le groupe symplectique est connu depuis Weyl (1946) et se retrouve très utilisé en physique puisque le formalisme hamiltonien possède intrinsèquement une structure symplectique. La propriété principale dans notre contexte est que l'ensemble des matrices quaternioniques self-duales est invariant sous les transformations $C \mapsto B^R C B$ où $B$ est n'importe quelle matrice symplectique. De plus, n'importe quelle matrice quaternionique self-duale peut être diagonalisée en une matrice quaternionique $D$ réelle et scalaire

par une transformation symplectique :

$$A = BDB^R \tag{III.11}$$

et la mesure de probabilité sur les matrices quaternioniques self-duales est invariante par transformation symplectique.

Toutes ces propriétés permettent de passer du modèle des intégrales de matrices quaternioniques self-duales au problème aux valeurs propres correspondant avec un Jacobien égal au determinant de Vandermonde à la puissance 4. (Cf. [19] pour la démonstration) En particulier, on voit la similitude très forte entre cet ensemble de matrice un peu exotique et les ensembles de matrices hermitiennes et réelles symétriques plus habituels : toutes les matrices de ces ensembles peuvent être diagonalisées par un élément du groupe symplectique, unitaire ou orthogonal et les mesures de probabilité sont invariantes par les transformations correspondantes. Seul le nombre de composantes réelles indépendantes de ces matrices change ce qui aboutit à des exposants du déterminant de Vandermonde différents selon les trois ensembles.

## Annexe IV

## Déterminants de Fredholm

Le déterminant de Fredholm est une notion qui généralise le déterminant d'une matrice dans le cadre d'opérateurs d'un espace de Hilbert qui ne diffèrent de l'identité que par un opérateur vérifiant la propriété dite de trace :

$$\sum_{k=0}^{\infty} \left\langle (A^*A)^{\frac{1}{2}} e_k, e_k \right\rangle < \infty \tag{IV.1}$$

où $<,>$ désigne le produit scalaire de l'espace de Hilbert et $e_k$ est une base orthonormale.

Soit donc $A$ un tel opérateur, on définit alors son déterminant de Fredholm par :

$$\det(Id + A) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \mathrm{Tr}(\lambda^k(A)) \tag{IV.2}$$

où par définition $\forall v_i \in H$ :

$$\lambda^k(A)\ v_1 \wedge v_2 \wedge \cdots \wedge v_k = Av_1 \wedge Av_2 \wedge \cdots \wedge Av_k \tag{IV.3}$$

Une telle définition permet de généraliser les propriétés habituelles du déterminant des matrices. Si $A$ et $B$ sont des opérateurs vérifiant IV.1 et si $T$ est un opérateur inversible alors :

$$\begin{aligned} \det((Id+A)(Id+B)) &= \det(Id+A)\det(I+B) \\ \det(T(Id+A)T^{-1}) &= \det(Id+A) \\ \det(e^A) &= \exp(Tr(A)) \end{aligned} \tag{IV.4}$$

Le cas matriciel est alors un cas particulier des déterminants de Fredholm où $H$ est de dimension finie $m$ :

$$\mathrm{Tr}(\lambda^k(A)) = \sum_{i_1,\ldots,i_n} \frac{(\det(A_{i_p,i_q}))_{p,q=1..n}}{n!} \tag{IV.5}$$

ce qui donne la formule de von Koch (1892) du déterminant :

$$\det(I+zA) = \sum_{n=0}^{m} \frac{z^n}{n!} \sum_{i_1,\ldots,i_n=1}^{m} (\det(A_{i_p,i_q}))_{p,q=1..n} \tag{IV.6}$$

Un deuxième cas, tout spécialement intéressant pour la théorie des matrices aléatoires est celui où l'espace de Hilbert $H$ est l'espace des fonctions de carrés intégrables $\mathscr{L}^2(a,b)$. L'opérateur de trace est alors défini pour un noyau $K(x,y)$ continu sur $(a,b)$ par :

$$\mathrm{Tr}(\lambda^k(K)) = \frac{1}{k!} \int_{(a,b)^k} (\det(K(x_p,x_q)))_{p,q=1..k}\, dx_1 \ldots dx_k \tag{IV.7}$$

ce qui aboutit au déterminant de Fredholm :

$$\det(I+zK) = \sum_{n=0}^{m} \frac{z^n}{n!} \int_{(a,b)^n} (\det(K(x_p,x_q)))_{p,q=1..n}\, dx_1 \ldots dx_n \tag{IV.8}$$

qui peut également se réécrire comme :

$$\det(I-zK) = \exp(-\sum_{k=0}^{\infty} \frac{z^n}{n} \mathrm{Tr} K^n) \tag{IV.9}$$

où la "trace" de $K^n$ est définie comme :

$$\begin{aligned} \mathrm{Tr}(K) &= \int_a^b K(x,x)dx \\ \mathrm{Tr}(K^2) &= \int_a^b \int_a^b K(x,y)K(y,x)\,dx\,dy \\ &\ldots \end{aligned} \tag{IV.10}$$

Notons qu'il est en général rare de pouvoir calculer explicitement les déterminants de Fredholm de façon exacte, même pour des noyaux $K(x,y)$ simples.

## Annexe V

# Double scaling limits of random matrices and minimal $(2m,1)$ models : the merging of two cuts in a degenerate case

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**Abstract** : In this article, we show that the double scaling limit correlation functions of a random matrix model when two cuts merge with degeneracy $2m$ (i.e. when $y \sim x^{2m}$ for arbitrary values of the integer $m$) are the same as the determinantal formulae defined by conformal $(2m,1)$ models. Our approach follows the one developed by Bergère and Eynard in [29] and uses a Lax pair representation of the conformal $(2m,1)$ models (giving Painlevé II integrable hierarchy) as suggested by Bleher and Eynard in [22]. In particular we define Baker-Akhiezer functions associated to the Lax pair to construct a kernel which is then used to compute determinantal formulae giving the correlation functions of the double scaling limit of a matrix model near the merging of two cuts.

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# 1 Introduction

It has been known for a long time that the study of random matrix models in different scaling limits gives rise to a great number of famous integrable equations ; both PDEs of solitonic type (KdV and, more generally, Gelfand-Dikii equations) and ODEs arising from isomonodromic systems (like Painlevé equations). A key idea in these studies is the notion of spectral curves attached to algebraic equations $P(x,y)=0$. The genus of the curve gives the number of intervals on which the eigenvalues of the matrices will accumulate when their size tends to infinity. It is well known that, in the generic case, the curve behaves like $y \sim \sqrt{x-a}$ near a branchpoint $a$ (an extremity of an interval) ; the appropriate double scaling limit gives the celebrated Airy kernel in connection with $(1,2)$ minimal models. But it may happen by taking a fine-tuned limit (see for instance [29]), that the behavior near a branchpoint differs from the generic case and takes the form of $y^p \sim (x-a)^q$. In such a case, it is expected that the double scaling limit is related to the conformal $(p,q)$ minimal model. In their articles [29] and [25], the authors opened the way to rigorous mathematical proofs in order to establish that the correlation functions of the double-scaling limit of a matrix model are the same as the ones defined by determinantal formulae arising from $(p,q)$ models. In their articles, they apply this method to all $(2m+1,2)$ models, i.e. suitable limits of matrix models where the spectral curve behaves like $y^2 \sim x^{2m+1}$ near an endpoint. In this article, we will use the same method for the $(2m,1)$ case which corresponds to a point where two cuts are merging with a degeneracy $2m$. For a generic merging, i.e. $m=1$ it has been proven in [22] that the suitable double scaling limit of the matrix model is connected to the Painlevé II equation. Some similar results have been established with the study of a suitable Riemann-Hilbert problem. For example the case of an even-quartic polynomial has been studied in [90]. It would be also interesting to derive similar results, for these kernels, as the ones proved in [91]. Here, using the approach of [29], we find, as expected, that the correlation functions of the double scaling limit of the merging of two cuts with degeneracy $2m$ are expressed through the Lax system of the Painlevé II hierarchy (see [92] and [93]).

## 2 Double scaling limit in random matrices : the merging of two cuts

### 2.1 Hermitian matrix models and equilibrium density

It is well known in the literature that the study of the Hermitian matrix model with partition function :

$$Z_N = \int_{\mathbb{H}_N} \exp(-N\mathrm{Tr}(V(M)))dM \tag{V.1}$$

with an even polynomial potential

$$V(x) = \sum_{i=1}^{2d} t_i x^i \tag{V.2}$$

can be reduced into an eigenvalue problem : $\lambda = \{\lambda_j, j = 1, ..., N\}$ for the matrix $M$ with distribution :

$$\tilde{Z}_N = \int_{\mathbb{R}^N} \exp\left(2\sum_{1\leq j<k\leq N} \log|\lambda_j - \lambda_k| - N\sum_{i=1}^{N} V(\lambda_j)\right) \tag{V.3}$$

When $N \to \infty$, the distribution of the eigenvalues on the line $d\nu_N(x) = \rho_N(x)dx$ is defined (in the distribution theory sense) by the formula

$$\int_{\mathbb{R}} \phi(x)d\nu_N(x) = \frac{1}{\tilde{Z}_N}\int_{\mathbb{R}^N}\left(\frac{1}{N}\sum_{j=1}^{n}\phi(\lambda_j)\right)\exp\left(2\sum_{1\leq j<k\leq N}\log|\lambda_j-\lambda_k| - 2N\sum_{i=1}^{N}V(\lambda_j)\right). \tag{V.4}$$

For any test function $\phi(x)$ there is a weak limit $d\nu_\infty(x) := \lim_{N\to\infty} d\nu_N(x)$ which is the same as the equilibrium density $d\nu_{eq}(x)$ given by the limit of the empirical density :

$$d\nu_{eq}(x) = \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\delta(x-\lambda_j) \tag{V.5}$$

For details about the existence of the distribution limits ; the equality between the equilibrium density $d\nu_{eq}(x)$ and $d\nu_\infty(x)$ and the following characterizations we refer the reader to [20], [21]. Nowadays, many properties of the equilibrium density are known. For example, we know that the equilibrium density is supported by a finite number of inter-

vals $[a_j, b_j]$, $j = 1, \ldots, q$ and that it is absolutely continuous with respect to the Lebesgue measure :

$$d\nu_{eq}(x) = \rho(x)dx = \frac{1}{2i\pi} h(x) R^{1/2}(x),\ R(x) = \prod_{j=1}^{q} (x - a_j)(x - b_j) \tag{V.6}$$

where $h(x)$ is a polynomial of degree $2d - q - 1$ and $R^{1/2}(x)$ is to be understood as the value on the upper cut of the principal sheet of the complex-valued function $R^{1/2}(z)$ with cuts on $J = \bigcup_{j=1}^{q} [a_j, b_j]$. Eventually, the equilibrium density $d\nu_{eq}(x)$ is completely defined by the knowledge of the extremities $a_j$'s and $b_j$'s and the unknown coefficients of the polynomial $h(x)$. It has been proved that such quantities are uniquely determined by the following set of equations :

1. Connexion between $h(z)$ and the potential $V(z)$ :

$$V'(z) = \underset{z \to \infty}{\text{Pol}} \left( h(z) R^{1/2}(z) \right) \tag{V.7}$$

2. Residue constraint :

$$\underset{z \to \infty}{\text{Res}} \left( h(z) R^{1/2}(z) \right) = -2 \tag{V.8}$$

3. Integrals constraints :

$$\int_{b_j}^{a_j + 1} h(z) R^{1/2}(z) dz = 0\,,\ \forall j \in \{1, \ldots, q-1\} \tag{V.9}$$

Note also that the relation between $h(z)$ and $V(z)$ V.7 can be inverted by :

$$h(z) = \underset{z \to \infty}{\text{Pol}} \left( \frac{V'(z)}{R^{1/2}(z)} \right) \tag{V.10}$$

In theory, the previous set of equation is sufficient to determine the whole solution $d\nu_{eq}(x)$ but, practically, since the equations are highly non-linear, it becomes very hard to compute the unknown coefficients for two or more intervals or for potentials of degree higher than 4. Moreover, in some exceptional situations, the previous set of equations has multiple solutions. In such situations, the good solution is determined by a positivity

condition :

$$h(x) \geq 0,\ \forall x \in J = \bigcup_{j=1}^{q} [a_j, b_j] \tag{V.11}$$

When $\forall x \in \bigcup_{j=1}^{q} ]a_j, b_j[ :\ h(x) > 0$, the potential $V(x)$ and the equilibrium measure $d\nu_{eq}(x)$ are called *regular*. Otherwise the equilibrium density is called *singular* and the corresponding potential is called *critical*, meaning that there is at least one point on $J$ where the equilibrium measure vanishes. For a regular potential, the situation can be summarized with the following picture :

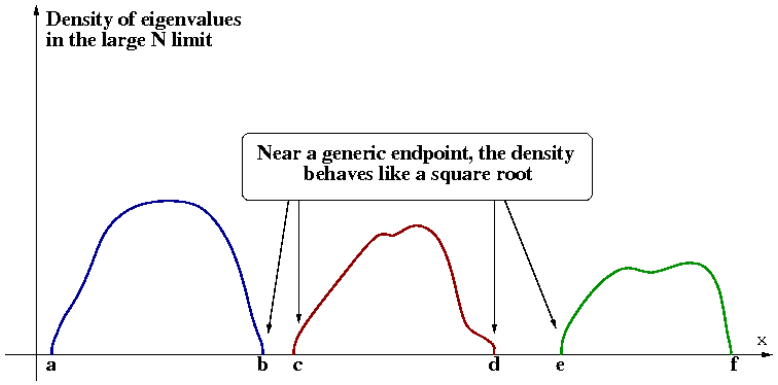


Figure 1 : Example of a typical eigenvalue density for a regular potential. The density is spread here in three intervals

### 2.2 Singular densities for the $(2m, 1)$ case

In order to study what happens at a singular density, one embeds the potential $V(x)$ into a parametric family $V(x,t)$ so that for some $t = t_c$ the problem is at the critical potential : $V(x,t_c) = V_c(x)$. Then the interesting questions are to determine the asymptotics of the eigenvalues correlation functions when $t \to t_c$. Indeed for $t \neq t_c$ the potential is regular and all the previous results stand. Therefore one can define $a_j(t)$, $b_j(t)$ and $h(x,t)$ determining completely the equilibrium density for $t \neq t_c$ and study their limits when $t \to t_c$. In matrix models, it is often interesting to study a modified version of the integral V.1 by introducing a parameter $T$ often referred as "*the temperature*" :

$$Z_N = \int_{\mathbb{H}_N} \exp(-\frac{N}{T}\mathrm{Tr}(V(M)))dM \tag{V.12}$$

It turns out that $T$ can be used as a parameter for the study of singular densities. In order to fit into our previous description, we need to introduce the following notation :

$$V(x,T) = \frac{V(x)}{T} \tag{V.13}$$

In the study of the $(2m,1)$ model, we assume that at $T = T_c$ the potential $V(x,T_c) = V_c(x)$ becomes singular and gives rise to a singular density defined by the following $2m$ singular density :

$$\rho(x,T_c) = \rho_c(x) = \frac{1}{2i\pi}(x-b\varepsilon)^{2m}\sqrt{b^2-x^2} = \frac{1}{2i\pi}h_c(x)\sqrt{b^2-x^2} \tag{V.14}$$

with $\varepsilon \in ]-1,1[$ representing the position of the singular point in the interval $]-b,b[$ supporting the distribution. For $T \neq T_c$, we assume that the density is supported by two intervals $]a_1(T),b_1(T)[$ and $]a_2(T),b_2(T)[$ and define (note the normalization with $\frac{1}{T}$) :

$$\rho(x,T) = \frac{1}{2i\pi T}h(x,T)\sqrt{(x-a_1(T))(x-b_1(T))(x-a_2(T))(x-b_2(T))} \tag{V.15}$$

Note that in order to recover our singular density at $T = T_c$ we must have :

1. $a_1(T) \underset{T\to T_c}{\to} -b$
2. $b_1 \underset{T\to T_c}{\to} b\varepsilon$
3. $a_2(T) \underset{T\to T_c}{\to} b\varepsilon$
4. $b_2(T) \underset{T\to T_c}{\to} b$
5. $h(x,T) \underset{T\to T_c}{\to} h(x)$

The previous assumptions correspond to the merging to two cuts with degeneracy $2m$ (order of the singularity). The most general case would be a singular point $a$ with $\rho_c^q(x) \underset{T\to T_c}{\sim} (x-a)^p$, $(p,q) \in \mathbb{N}^2$, which is expected to correspond to the $(p,q)$ minimal model (for $q > 2$ we are speaking about multi-matrix models). In our case the situation can be summarized with the following pictures :

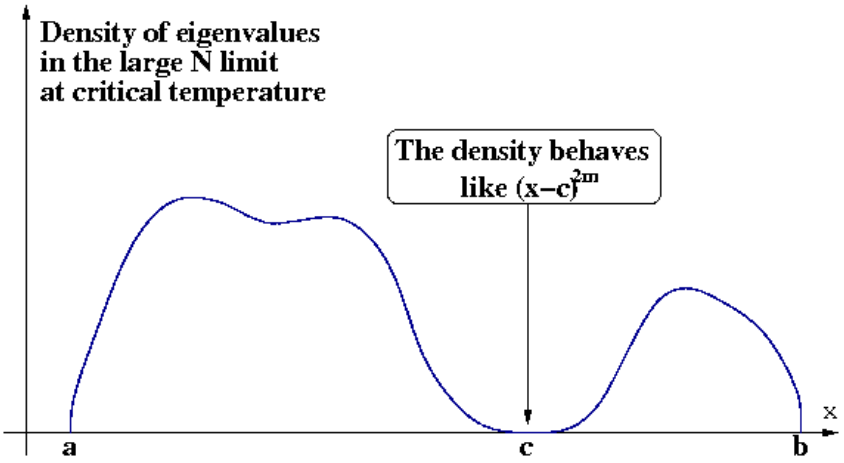


Figure 2 : Example of a critical eigenvalue density for a critical potential. At point $c$, the density is singular and behaves like $x^{2m}$

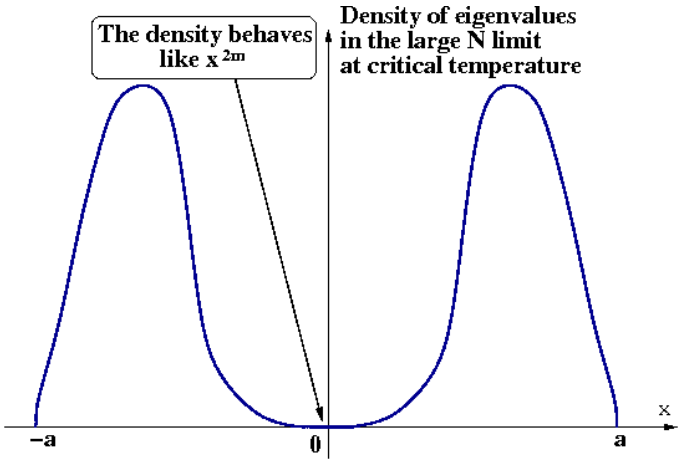


Figure 3 : Example of a critical eigenvalue density for a critical even potential. At the origin, the density is singular and behaves like $x^{2m}$

In [22], the authors studied the case $m = 1$ in details and conjectured some connections with Painlevé II hierarchy for higher $m$.

### 2.3 Double scaling limits in matrix models

In the study of matrix models, one is usually interested in the following functions called *resolvents* :

$$\begin{aligned} \hat{w}_n(x_1,\dots,x_n) &= \left\langle \mathrm{Tr}\left(\frac{1}{x_1-M}\right)\dots\mathrm{Tr}\left(\frac{1}{x_n-M}\right)\right\rangle \\ &= \left\langle \sum_{i_1,\dots,i_n} \mathrm{Tr}\left(\frac{1}{x_1-\lambda_{i_1}}\right)\dots\mathrm{Tr}\left(\frac{1}{x_n-\lambda_{i_n}}\right)\right\rangle \end{aligned} \tag{V.16}$$

and in their cumulants, also known as *correlation functions* :

$$\begin{aligned} \hat{w}_n(x_1,\dots,x_n) &= \left\langle \mathrm{Tr}\left(\frac{1}{x_1-M}\right)\dots\mathrm{Tr}\left(\frac{1}{x_n-M}\right)\right\rangle_c \\ &= \left\langle \sum_{i_1,\dots,i_n} \mathrm{Tr}\left(\frac{1}{x_1-\lambda_{i_1}}\right)\dots\mathrm{Tr}\left(\frac{1}{x_n-\lambda_{i_n}}\right)\right\rangle_c \end{aligned} \tag{V.17}$$

Here, the brackets stand for the integration relatively to the probability measure $Z_N^{-1} d\nu_N(x)$, the $\lambda_i$'s are the eigenvalues of the matrices and the index $c$ stands for the cumulants part (for example : $<AB>_c = <AB> - <A><B>$ and so on). The joint density correlation functions $\rho_n(x_1,\dots,x_n)$ can easily be deduced from the former correlation functions : densities are discontinuities of the resolvents and resolvents are Stieljes transforms of densities. For example :

$$\hat{w}_1(x) = \int \frac{\rho_1(x')}{x-x'}dx' \Longleftrightarrow \rho_1(x) = \frac{1}{2i\pi}\left(\hat{w}_1(x-i0) - \hat{w}_1(x+i0)\right) \tag{V.18}$$

Then we want to use a formal $\frac{1}{T}$ power-series development which unfortunately is not necessarily well-defined for all matrix models. Indeed, if one is interested in *convergent matrix models*, then one must be sure that such a series expansion commutes with integrations. In general, this does not happen and solutions of the convergent matrix model differ from the solutions of the *formal matrix model* (where by definition the development is assumed to exist and to commute with integrations). The explanation of this phenomenon is simple : when we use a series expansion, it automatically ignores the exponentially small factors (one can think, for example, to $\exp(-x^2)$ which has at $x=\infty$

the same asymptotic expansion as the zero function). To sum up, formal matrix models are easier to handle, because by definition the formal expansion exists and we can perform formal operations on it ; but the price to pay is that we only get a part of the convergent solutions (we miss the exponentially decreasing terms). It could appear disappointing to consider just formal matrix models, since they do not carry the whole convergent solutions (and thus leads only to a significative but incomplete part of the convergent solutions), but fortunately differences between formal and convergent matrix models have been well studied, and in [95], [26], the authors show how to reconstruct with theta functions the convergent solutions from the formal ones. From now on, we will place ourselves in the case of formal matrix models, i.e. we assume that there automatically exists an expansion of type :

$$\ln Z_N = \sum_{g=0}^{\infty} \left(\frac{N}{T}\right)^{2-2g} \hat{f}_g \tag{V.19}$$

and

$$\hat{w}_n(x_1,\ldots,x_n) = \sum_{g=0}^{\infty} \left(\frac{N}{T}\right)^{2-2g-n} \hat{w}_n^{(g)}(x_1,\ldots,x_n) \tag{V.20}$$

The numbers $\hat{f}_g$ are called *symplectic or spectral invariants* of the model (invariant relatively to symplectic transformations of the spectral curve). The previous expansion can be understood as a large $N$ expansion and therefore in the limit $N \to \infty$ one expects that the leading value ($g = 0$) corresponds to the "real" large $N$ limit of the model. In fact this intuition is correct and it has been proved that

$$\hat{y}(x) = i\pi\rho_{eq}(x) = \frac{1}{2}V'(x) - \hat{w}_1^{(0)}(x). \tag{V.21}$$

This formula establish a direct link between the equilibrium density and the leading order of the first correlation function. The function $\hat{y}(x)$ (which is up to a trivial rescaling the equilibrium density) is often named the *spectral curve* of the problem. In our case, it

satisfies :

$$\hat{y}^2(x) = \text{Polynomial}(x) = \frac{1}{2T}h^2(x,T)(x-a_1(T))(x-b_1(T))(x-a_2(T))(x-b_2(T)) \tag{V.22}$$

This identity defines the algebraic spectral curve $\hat{y}^2 = P(x)$ where $P$ is a polynomial. We remind the reader that Eynard and Orantin showed in [23] that for any algebraic curve $P(x,y) = 0$ we can associate some symplectic invariants $f_g$ and $w_n^{(g)}(x_1,\ldots,x_n)$. Moreover, when the algebraic curve comes from a matrix model, these invariants are the same as the one we defined earlier in V.16 and V.17.

In our case, the function $\hat{y}(x,T) = \frac{1}{2T}h^2(x,T)(x-a_1(T))(x-b_1(T))(x-a_2(T))(x-b_2(T))$ depends on the temperature $T$ and so are the corresponding invariants $\hat{w}_n(x_1,\ldots,x_n,T)$ and $f_g(T)$. When $T \to T_c$ it is known that $\forall g > 1$, $f_g \to \infty$ and that the correlation functions diverges. This is so because the expansion V.19 reaches its radius of convergence in $T$. In order to recover finite quantities, one has to rescale properly the variables at $T \sim T_c$. In our case we will prove that the good rescaling is given by :

$$x_i = b\varepsilon + (T-T_c)^{\frac{1}{2m}}\xi_i \tag{V.23}$$

so that

$$\hat{y}_{\text{rescaled}}(\xi) = \lim_{T\to T_c} \frac{\hat{y}(b\varepsilon + (T-T_c)^{\frac{1}{2m}}\xi, T)}{T-T_c} \tag{V.24}$$

and

$$\hat{w}^{(g)}_{\text{rescaled},n}(\xi_1,\ldots,\xi_n) = \lim_{T\to T_c} \frac{\hat{w}_n^{(g)}(b\varepsilon + (T-T_c)^{\frac{1}{2m}}\xi_1,\ldots,b\varepsilon + (T-T_c)^{\frac{1}{2m}}\xi_n, T)}{(T-T_c)^n} \tag{V.25}$$

and

$$\hat{f}_{\text{rescaled},g} = \lim_{T\to T_c} (T-T_c)^{-(2-2g)}\hat{f}_g \tag{V.26}$$

are finite quantities and that the new $\hat{w}^{(g)}_{\text{rescaled},n}(\xi_1,\ldots,\xi_n)$ and $\hat{f}_{\text{rescaled},g}$ are the spectral invariants of the rescaled curve $\hat{y}_{\text{rescaled}}(\xi)$. In the general context of matrix model, such a rescaling is called a *double scaling limit* since we have performed a double limit $N \to \infty$

and $T \to T_c$ so that $N(T-T_c)^{2m}$ remains finite :

$$\ln Z_N = \sum_{g=0}^{\infty} \left(\frac{N}{T}\right)^{2-2g} \hat{f}_g \sim \sum_{g=0}^{\infty} \left(\frac{N}{T_c}\right)^{2-2g} (T-T_c)^{(2-2g)} \hat{f}_{\text{rescaled},g} \tag{V.27}$$

From a geometric point of view, this double scaling limit corresponds to a local zoom in the region of the degenerate point $b\varepsilon$. The rate of the zoom depends on both the temperature $T$ and the size of the matrices $N$ so that $N(T-T_c)$ remains finite. It can be illustrated in the following picture :

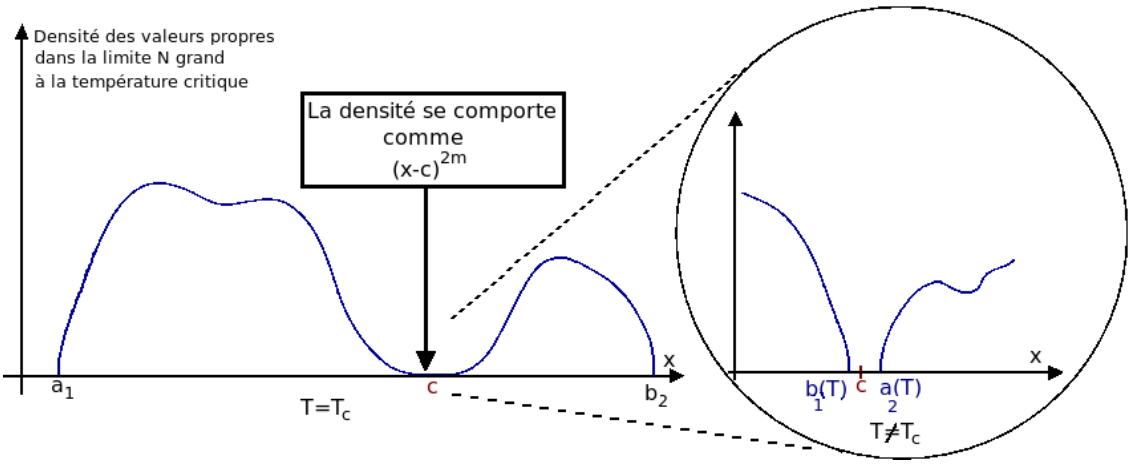


Figure 4 : Example of a critical eigenvalue density near the critical temperature

In the context of matrix models, double scaling limits are often very important because they are expected to give universal (independent of the potential) rescaled spectral curve and correlation functions related to $(p,q)$ minimal models (and thus in our case the $(2m,1)$ minimal model). On the other hand, $(p,q)$ minimal models are studied through string reductions of some well known integrable systems. In the rest of the paper, we will prove that, in the case of the merging of two cuts, the rescaled spectral curve corresponds to the spectral curve of the $(2m,1)$ minimal model. Then, using the method introduced by Bergère and Eynard in [29], we prove that the rescaled correlation functions and the spectral invariants correspond to some "correlation" functions expressed with some determinantal formulae [25] for the $(2m,1)$ minimal model.

### 2.4 The rescaled spectral curve in our $2m$ degenerate matrix model case

In order to get the rescaled spectral curve, we need to perform a few consecutive steps. First we can express explicitly the corresponding critical potential corresponding to $\rho_c(x)$ in V.14 using V.7. The computation is straightforward and uses only the general Taylor expansion of :

$$\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n-2)!}{n!(n-1)!2^{2n-1}} x^n \tag{V.28}$$

It gives :

$$V_c'(x) = \sum_{j=0}^{2m+1} \left( \binom{2m}{j-1} (-b\varepsilon)^{2m+1-j} + \sum_{n=1}^{E(\frac{2m+1-j}{2})} \binom{2m}{2n+j-1} \frac{(-1)^j (2n-2)! \varepsilon^{2(m-n)+1-j} b^{2m+1-j}}{n!(n-1)!2^{2n-1}} \right) x^j \tag{V.29}$$

where $E(\frac{2m+1-j}{2})$ stands for the greatest integer lower or equal to $\frac{2m+1-j}{2}$. The critical temperature is given by :

$$T_c = \frac{b^{2m+2}}{2} \sum_{n=1}^{m+1} \frac{\varepsilon^{2m-2n+2}(2m)!}{n!(2m-2n+2)!(n-1)!2^{2n-1}} \tag{V.30}$$

Then, we need to use some reformulations of conditions V.8 and V.10. Indeed, it is known for a long time (a proof can be found in appendix A of [29] but the results were derived much before) and has been used intensively in [88] that the set of equations V.8 and V.10 leads to the following ordinary differential equations (sometimes called *hodograph equations*) :

$$\begin{aligned}
\frac{d}{dT} a_1(T) &= \frac{4(a_1(T)-x_0(T))}{h(a_1(T),T)(a_1(T)-b_1(T))(a_1(T)-a_2(T))(a_1(T)-b_2(T))} \\
\frac{d}{dT} b_1(T) &= \frac{4(b_1(T)-x_0(T))}{h(b_1(T),T)(b_1(T)-a_1(T))(b_1(T)-a_2(T))(b_1(T)-b_2(T))} \\
\frac{d}{dT} a_2(T) &= \frac{4(a_2(T)-x_0(T))}{h(a_2(T),T)(a_2(T)-b_2(T))(a_2(T)-a_1(T))(a_2(T)-b_1(T))} \\
\frac{d}{dT} b_2(T) &= \frac{4(b_2(T)-x_0(T))}{h(b_2(T),T)(b_2(T)-a_2(T))(b_2(T)-a_1(T))(b_2(T)-b_1(T))}
\end{aligned} \tag{V.31}$$

where the point $x_0(T)$ ($b_1(T) \leq x_0(T) \leq a_2(T)$ ) is determined by :

$$\int_{b_1(T)}^{a_2(T)} \frac{z - x_0(T)}{\sqrt{(b_1(T)-z)(z-a_1(T))(b_2(T)-z)(z-a_2(T))}} dz = 0 \tag{V.32}$$

This set of equations taken at $T = T_c$ for $a_1$ and $b_2$ gives :

$$\begin{array}{rcl} \frac{da_1(T)}{dT}_{|T=T_c} & = & -\frac{2}{(1+\varepsilon)^{2m} b^{2m+1}} \\ \frac{db_2(T)}{dT}_{|T=T_c} & = & \frac{2}{(1-\varepsilon)^{2m} b^{2m+1}} \end{array} \tag{V.33}$$

so that in a neighbourhood of $T_c$ :

$$\boxed{a_1(T) \underset{T=T_c}{\sim} -b - \frac{2}{(1+\varepsilon)^{2m} b^{2m+1}} (T - T_c) + o(T - T_c)} \tag{V.34}$$

$$\boxed{b_2(T) \underset{T=T_c}{\sim} b + \frac{2}{(1-\varepsilon)^{2m} b^{2m+1}} (T - T_c) + o(T - T_c)} \tag{V.35}$$

As mentioned earlier, we expect that the functions $a_j(T)$ and $b_j(T)$ will be analytic functions of $\Delta = (T - T_c)^{\nu}$, where $\nu$ is an exponent that we will determine later. Therefore we introduce the following notations :

$$\begin{array}{rcl} b_1(T) & = & b\varepsilon + \alpha\Delta + \sum_{n=1}^{\infty} b_{1,n}\Delta^n \\ a_2(T) & = & b\varepsilon + \gamma\Delta + \sum_{n=1}^{\infty} a_{2,n}\Delta^n \\ x_0(T) & = & b\varepsilon + X_0\Delta + \sum_{n=1}^{\infty} x_n\Delta^n \\ h(z,T) & = & (z - b\varepsilon)^{2m-1} + P(z)\Delta + \sum_{n=1}^{\infty} h_n(z)\Delta^n \end{array} \tag{V.36}$$

where $P(z)$ and $h_n(z)$ are polynomials of degree at most $2m-2$. In equations V.31 for

$a_2(T)$ and $b_1(T)$ we see that the l.h.s. is of order $(T-T_c)^{\nu-1}$ whereas the r.h.s. is of order $(T-T_c)^{-(2m-1)\nu}$. Hence, to have compatible equations we must have, as announced in the previous subsection, that

$$\boxed{\nu = \frac{1}{2m}} \tag{V.37}$$

The next step is purely technical and consists in proving that $\alpha = -\gamma$. Since it is only a technical point, we postpone this discussion in Appendix V. With the help of this relation we can now determine the rescaled spectral curve.

First remember that for $T = T_c$ , we have V.14 :

$$(z-b\varepsilon)^{2m} = h_c(z) = \underset{z\to\infty}{\text{Pol}}\left(\frac{V_c'(z)}{\sqrt{z^2-b^2}}\right) \tag{V.38}$$

For $T \neq T_c$, reminding that $V(z,T) = \frac{V'(z)}{T}$ and that $\rho(z,T)$ is defined with a factor $\frac{1}{T}$ in V.15 (which will cancel the one of $V(x,T)$) we have :

$$h(z,T) = \underset{z\to\infty}{\text{Pol}}\left(\frac{V'(z)}{\sqrt{(z-a_1)(z-b_2)(z-a_2)(z-b_1)}}\right) \tag{V.39}$$

We now use the fact that up to order $\Delta^{2m-1}$, both $a_1$ and $b_2$ are respectively equal to $-b$ and $b$ (V.34). Therefore we get :

$$h(z,T) = \underset{z\to\infty}{\text{Pol}}\left(\frac{V'(z)}{\sqrt{(z^2-b^2)(z-a_2)(z-b_1)}} + O\left(\Delta^{2m}\right)\right) \tag{V.40}$$

Then, from the definition of $h_c(x)$ we have that :

$$(z-b\varepsilon)^{2m} = \frac{1}{T_c}\underset{z\to\infty}{Pol}\left(\frac{V'(z)}{\sqrt{z^2-b^2}}\right)$$

so that :

$$\frac{V'(z)}{\sqrt{z^2-b^2}} = T_c(z-b\varepsilon)^{2m} + O\left(\frac{1}{z}\right) \tag{V.41}$$

Putting back this identity into V.40 and noticing that $\frac{1}{\sqrt{(z^2-b^2)(z-a_2)(z-b_1)}}$ only gives negative powers of $z$ that will disappear when taking the polynomial part, we find that :

$$\begin{aligned} h(z,T) &= T_c \underset{z\to\infty}{\text{Pol}} \left( \frac{(z-b\varepsilon)^{2m}}{\sqrt{(z-a_2)(z-b_1)}} + O\left(\Delta^{2m}\right) \right) \\ &= T_c \underset{z\to\infty}{\text{Pol}} \left( \frac{(z-b\varepsilon)^{2m-1}}{\sqrt{1+\frac{2b\varepsilon-a_2-b_1}{z-b\varepsilon}+\frac{(b\varepsilon-a_2)(b\varepsilon-b_1)}{(z-b\varepsilon)^2}}} + O\left(\Delta^{2m}\right) \right) \end{aligned} \tag{V.42}$$

We can now insert the Taylor series of the square-root :

$$(1+x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(n!)^2 2^{2n}} x^n \tag{V.43}$$

to get :

$$\begin{aligned} \frac{h(z,T)}{T_c} &= \underset{z\to\infty}{\text{Pol}} \left( (z-b\varepsilon)^{2m-1} \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{(n!)^2 2^{2n}} \left( \frac{2b\varepsilon-a_2-b_1}{z-b\varepsilon} + \frac{(b\varepsilon-a_2)(b\varepsilon-b_1)}{(z-b\varepsilon)^2} \right)^n \right) \right) \\ &\quad + O\left(\Delta^{2m}\right) \\ &= (z-b\varepsilon)^{2m-1} + \\ &\quad \underset{z\to\infty}{\text{Pol}} \Big( \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n (2n)!}{(n!)^2 2^{2n}} \binom{n}{k} (2b\varepsilon-a_2-b_1)^k \left( (b\varepsilon-a_2)(b\varepsilon-b_1) \right)^{n-k} \\ &\quad (z-b\varepsilon)^{2m-1+k-2n} \Big) + O\left(\Delta^{2m}\right) \end{aligned} \tag{V.44}$$

Let's now introduce the following ensemble :

$$I_m = \{(n,k) \in (\mathbb{N}^* \times \mathbb{N}) \,/\, 2n-k \leq 2m-1 \text{ and } k \leq n\} \tag{V.45}$$

Clearly $I_m$ is a finite set and we can rewrite the previous identity as :

$$\begin{aligned} \frac{h(z,T)}{T_c} &= (z-b\varepsilon)^{2m-1} + \\ &\quad \Big( \sum_{(n,k)\in I_m} \frac{(-1)^n (2n)!}{(n!)^2 2^{2n}} \binom{n}{k} (2b\varepsilon-a_2-b_1)^k \left( (b\varepsilon-a_2)(b\varepsilon-b_1) \right)^{n-k} \end{aligned}$$

$$(z-b\varepsilon)^{2m-1+k-2n}\Big) + O\left(\Delta^{2m}\right) \tag{V.46}$$

We can now introduce the series expansion in $\Delta$ :

$$2b\varepsilon - a_2 - b_1 = -(\alpha+\gamma)\Delta + O\left(\Delta^2\right)$$

and

$$(b\varepsilon - a_2)(b\varepsilon - b_1) = \left(\alpha\Delta + \sum_{n=2}^{\infty} a_{2,n}\Delta^n\right)\left(\gamma\Delta + \sum_{n=2}^{\infty} b_{1,n}\Delta^n\right) = \Delta^2\alpha\gamma + O\left(\Delta^3\right)$$

Then we perform the rescaling

$$z = b\varepsilon + \Delta\xi \tag{V.47}$$

We only need to take into account terms with degree strictly less than $\Delta^{2m}$ so that only a few terms remain :

$$\begin{aligned}\frac{h(\xi,\Delta)}{T_c} &= \left(\xi^{2m-1} + \sum_{(n,k)\in I_m} \frac{(-1)^n(2n)!}{(n!)^2 2^{2n}}\binom{n}{k}(-1)^k(\alpha+\gamma)^k(\alpha\gamma)^{n-k}\xi^{2m-1+k-2n}\right)\Delta^{2m-1} \\ &\quad + O\left(\Delta^{2m}\right)\end{aligned} \tag{V.48}$$

so that :

$$h_{\text{rescaled}}(\xi) = T_c\left(\xi^{2m-1} + \sum_{(n,k)\in I_m} \frac{(-1)^n(2n)!}{(n!)^2 2^{2n}}\binom{n}{k}(-1)^k(\alpha+\gamma)^k(\alpha\gamma)^{n-k}\xi^{2m-1+k-2n}\right) \tag{V.49}$$

Eventually we get the rescaled spectral curve by taking into account the trivial term $R^{1/2}(z,T) = \sqrt{(z-a_1(T))(z-a_2(T))(z-b_1(T))(z-b_2(T))}$ with the rescaling V.47 :

$$\begin{aligned}R^{\frac{1}{2}}(b\varepsilon+\xi\Delta,\Delta) &= \sqrt{(b\varepsilon+\xi\Delta-a_1(\Delta))(b\varepsilon+\xi\Delta-a_2(\Delta))(b\varepsilon+\xi\Delta-b_1(\Delta))(b\varepsilon+\xi\Delta-b_2(\Delta))} \\ &= b\sqrt{\varepsilon^2-1}\sqrt{(b\varepsilon+\xi\Delta-a_2(\Delta))(b\varepsilon+\xi\Delta-b_1(\Delta))} + O\left(\Delta^{2m}\right) \\ &= ib\Delta\sqrt{1-\varepsilon^2}\sqrt{(\xi-\alpha)(\xi-\gamma)} + O\left(\Delta^2\right)\end{aligned}$$

(V.50)

so that :

$$\begin{aligned}\rho(b\varepsilon+\xi\Delta,\Delta) &= \frac{b\sqrt{1-\varepsilon^2}}{2\pi}\sqrt{(\xi-\alpha)(\xi-\gamma)} \\ &\left(\xi^{2m-1}+\sum_{(n,k)\in I_m}\frac{(-1)^n(2n)!}{(n!)^2 2^{2n}}\binom{n}{k}(-1)^k(\alpha+\gamma)^k(\alpha\gamma)^{n-k}\xi^{2m-1+k-2n}\right)\Delta^{2m} \\ &+O\left(\Delta^{2m+1}\right) \qquad \text{(V.51)}\end{aligned}$$

giving that :

$$\begin{aligned}\hat{y}_{\text{rescaled}}(\xi) &= b\pi\sqrt{1-\varepsilon^2}\sqrt{(\xi-\alpha)(\gamma-\xi)} \\ &\left(\xi^{2m-1}+\sum_{(n,k)\in I_m}\frac{(-1)^n(2n)!}{(n!)^2 2^{2n}}\binom{n}{k}(-1)^k(\alpha+\gamma)^k(\alpha\gamma)^{n-k}\xi^{2m-1+k-2n}\right) \qquad \text{(V.52)}\end{aligned}$$

In the appendix V, we prove that $\alpha=-\gamma$ so that it eventually leads to :

$$\boxed{\alpha=-\gamma\,,\,:\hat{y}_{\text{rescaled}}(\xi)=b\pi\sqrt{1-\varepsilon^2}\sqrt{(\gamma^2-\xi^2)}\left(\xi^{2m-1}+\sum_{n=1}^{m-1}\frac{(2n)!}{(n!)^2 2^{2n}}\gamma^{2n}\xi^{2m-1-2n}\right)}$$

(V.53)

We can even compute the precise value of $\gamma$. Indeed, using V.45 to compute the leading term of the $\Delta$-expansion of $h(a_2,T)$ and putting it back into V.31 (and using the fact that with the definition of $x_0$ V.32 we have $X_0=0$ when $\alpha+\gamma=0$) we have :

$$\boxed{\alpha=-\gamma \text{ with } \gamma^{2m}=\alpha^{2m}=-\frac{4m}{b^2(1-\varepsilon^2)\left(\sum_{n=0}^{m-1}\frac{(2n)!}{(n!)^2 2^{2n}}\right)}=-\frac{(m!)^2 2^{2m+1}}{b^2(1-\varepsilon^2)(2m)!}}$$

(V.54)

In this case, introducing the new variable $s$ by $\xi = \gamma s$ or equivalently

$$z = b\varepsilon + \gamma \Delta s \tag{V.55}$$

we get :

$$\boxed{\alpha = -\gamma \, , : \hat{y}_{\text{rescaled}}(s) = b\pi\gamma^{2m}\sqrt{1-\varepsilon^2}\sqrt{(1-s^2)}\left(s^{2m-1} + \sum_{n=1}^{m-1} \frac{(2n)!}{(n!)^2 2^{2n}} s^{2m-1-2n}\right)} \tag{V.56}$$

Eventually V.56 shows as expected that when performing a double scaling limit $z = b\varepsilon + \gamma\Delta s$ (with $\gamma$ a complex number given by V.54 whose argument gives oscillations in the (Re($z$),Im($z$)) plane), we recover a universal curve. In the next section, we will see that this rescaled spectral curve V.53 is exactly, (up to the trivial normalization factor $b\sqrt{1-\varepsilon^2}$) the spectral curve arising in the Lax pair representation of the Painlevé II hierarchy with $t_m = 1$, all other $t_j$'s (See next section for a definition) taken to zero and the identification $u_0(t) = \gamma$ (coherently with V.70). Before proceeding in the study of the Lax pair representation, we remind the reader that from general results of Eynard and Orantin [23], the rescaled invariants and correlation functions $\hat{w}^{(g)}_{\text{rescaled},n}$ and $\hat{f}_{\text{rescaled},g}$ are automatically the symplectic invariants and correlation functions of the new rescaled spectral curve $\hat{y}_{\text{rescaled}}(\xi)$ and thus do automatically satisfied the famous loop equations [23].

## 3 Correlation functions and invariants arising in the Lax pair representation of the $(2m,1)$ minimal model

In the previous section, we have found the rescaled spectral curve for a double scaling limit of a $2m$ degenerate merging of two cuts in matrix models. As conjectured in [22], we expect that this universal double scaling limit is connected to the Painlevé II hierarchy. In order to prove this result, we will follow the approach [29] developed and successfully applied for the $(2m+1,2)$ models. It consists in finding a natural spectral

curve $y_{\mathrm{Lax}}(x)$ from a Lax pair representation of the hierarchy and check that it is equal to our rescaled curve defined in the previous section. Then from another work of Bergère and Eynard,[25] we can define from the Lax pair representation some new correlation functions $W_n^{(g)}(x_1,\ldots,x_n)$ and invariants $F_g$ by some determinantal formulae and a suitable kernel. In particular, they proved that these new functions do satisfy the same loop equations as our correlations functions. Eventually, with the study of the pole structure and $W_2^{(0)}$ we will end by proving that our new correlation functions $W_n^{(g)}(x_1,\ldots,x_n)$ and invariants $F_g$ are identical to the rescaled ones defined in the previous section.

### 3.1 A Lax pair representation for the $(2m,1)$ minimal model

In their paper [22], the authors claimed that a good Lax pair representation for the $(2m,1)$ minimal model should be given by a set of two $2\times 2$ matrices $\mathscr{R}(x,t)$ and $\mathscr{D}(x,t)$ satisfying the following Lax pair representation :

$$\begin{aligned}\frac{1}{N}\frac{\partial}{\partial x}\Psi(x,t) &= \mathscr{D}(x,t)\Psi(x,t)\\ \frac{1}{N}\frac{\partial}{\partial t}\Psi(x,t) &= \mathscr{R}(x,t)\Psi(x,t)\end{aligned} \tag{V.57}$$

where $\Psi(x,t)$ is a two by two matrix whose entries will be written as :

$$\Psi(x,t)=\begin{pmatrix}\psi(x,t) & \phi(x,t)\\ \tilde{\psi}(x,t) & \tilde{\psi}(x,t)\end{pmatrix} \tag{V.58}$$

and satisfies the normalization $\det\Psi(x,t)=1$.

The compatibility condition of the Lax pair is then :

$$\left[\frac{1}{N}\frac{\partial}{\partial x}-\mathscr{D}(x,t),\mathscr{R}(x,t)-\frac{1}{N}\frac{\partial}{\partial t}\right]=0 \tag{V.59}$$

In order to specify completely the Lax pair, we need to impose some conditions about

the shape of the matrices $\mathscr{R}(x,t)$ and $\mathscr{D}(x,t)$. In our case we will assume :

$$\mathscr{R}(x,t) = \begin{pmatrix} 0 & x+u(t) \\ -x+u(t) & 0 \end{pmatrix} \tag{V.60}$$

and

$$\mathscr{D}(x,t) = \sum_{k=0}^{m} t_k \mathscr{D}_k(x,t) \tag{V.61}$$

with

$$\mathscr{D}_k(x,t) = \begin{pmatrix} -A_k(x,t) & xB_k(x,t)+C_k(x,t) \\ xB_k(x,t)-C_k(x,t) & A_k(x,t) \end{pmatrix} \tag{V.62}$$

and $A_k$, $B_k$, $C_k$ are polynomials of $x$ of degree respectively $2k-2$, $2k-2$, $2k$. Note that in the literature one can find several different Lax pair corresponding to the same problem. Indeed any conjugation (change of basis) give equivalent matrices that describe the same problem but in different coordinates (see section V). In fact any equivalent Lax pair can be used since the quantities we will define later will be invariant from this choice. In order to have more compact notation, we will use the following convention : a dot will indicate a derivative relatively to $t$ normalized by a coefficient $1/N$, namely :

$$\dot{f}(x,t) \stackrel{\text{def}}{=} \frac{1}{N}\frac{\partial f(x,t)}{\partial t} \tag{V.63}$$

Putting back this specific shape of matrices into the compatibility equation gives the following recursion :

$$\begin{aligned} A_0 &= 0, B_0 = 0, C_0 = 1 \\ C_{k+1} &= x^2 C_k + \check{R}_k(u) \\ B_{k+1} &= x^2 B_k + \hat{R}_k(u) \\ A_{k+1} &= x^2 + \frac{1}{2}\dot{\hat{R}}(u) \end{aligned} \tag{V.64}$$

where $\check{R}_k$ and $\hat{R}_k$ are the modified Gelfand-Dikii polynomials given by the following

recursion :

$$\begin{aligned} \hat{R}_0(u) &= u\,\check{R}_0(u) = \frac{u^2}{2} \\ \hat{R}_{k+1}(u) &= u\check{R}_k(u) - \frac{1}{4}\frac{d^2}{dt^2}\hat{R}_k(u) \\ \frac{d}{dt}\check{R}_k(u) &= u\frac{d}{dt}\hat{R}_k(u) \end{aligned} \tag{V.65}$$

It is then easy to see that the matrices $\mathscr{R}(x,t)$ and $\mathscr{D}(x,t)$ satisfy V.57 if and only if $u(t)$ satisfies the string equation (see details in [22].)

$$\boxed{\sum_{k=0}^{m} t_k \hat{R}_k(u(t)) = -tu(t)} \tag{V.66}$$

which gives an explicit differential equation of order $m$ satisfied by $u(t)$ (since the polynomials $\hat{R}_k$ can be explicitly computed from the recursion V.65). In particular the case $m = 1$ gives Painlevé II equation :

$$\frac{d^2u}{dt^2}(t) = 2u^3(t) + 4(t+t_0)u(t) \tag{V.67}$$

where $t_0$ is a free parameter that can be set to 0 by a time-translation $\tilde{t} = t + t_0$.

Remark : Seculiar equations

As it is always the case for a linear differential equation, we can get a seculiar equation on $\psi(x,t)$ by combining the two components of the differential equation in $t$ given by V.57. In our case, we find that both $\psi(x,t)$ and $\phi(x,t)$ are solution of the seculiar equation :

$$\ddot{\psi}(x,t) - \frac{\dot{u}(t)\dot{\psi}(x,t)}{x+u(t)} = \left(u^2(t) - x^2\right)\psi(x,t) \tag{V.68}$$

which by a simple standard change of variable can be transformed into a Schrodinger-like equation.

### 3.2 Large $N$ development

From the fact that a dot derivative contributes with a factor $\frac{1}{N}$, it is easy to see from the string equation V.66 that $u(t)$ admits a series development at large $N$ :

$$u(t) = \sum_{j=0}^{\infty} \frac{u_j(t)}{N^{2j}} = u_0(t) + \frac{u_1(t)}{N^2} + \dots \tag{V.69}$$

Note : The fact that $u(t)$ admits such a development in $\frac{1}{N^2}$ and not $\frac{1}{N}$ comes from the fact that the modified Gelfand-Dikii polynomials $\hat{R}_k$'s are a sum of terms involving only even numbers of dots-derivatives (i.e. even power of $\frac{1}{N}$).

Putting back this expansion into the string equation V.66 and looking at the power of $N^0$ of the series gives us that $u_0(t)$ must satisfy the following algebraic relation :

$$\boxed{-t = \sum_{j=1}^{m} t_j \frac{(2j)!}{2^{2j}(j!)^2} u_0(t)^{2j}} \tag{V.70}$$

From that result, it is then easy to see that the matrices $\mathscr{R}(x,t)$ and $\mathscr{D}(x,t)$ also admit a large $N$ expansion :

$$\mathscr{R}(x,t) = \begin{pmatrix} 0 & x+u_0(t) \\ -x+u_0(t) & 0 \end{pmatrix} + \frac{1}{N^2} \begin{pmatrix} 0 & u_1(t) \\ u_1(t) & 0 \end{pmatrix} + \dots = \sum_{j=0}^{\infty} \frac{\mathscr{R}_j(x,t)}{N^{2j}} \tag{V.71}$$

and

$$\mathscr{D}(x,t) = \sum_{j=0}^{\infty} \frac{\mathscr{D}_j(x,t)}{N^j} \tag{V.72}$$

where the first matrix can be explicitly computed :

$$\mathscr{D}_0(x,t) = \begin{pmatrix} 0 & t+B_0+C_0 \\ -t+B_0-C_0 & 0 \end{pmatrix} \tag{V.73}$$

with

$$\begin{aligned} B_0 &= \sum_{j=1}^{m} t_j \sum_{k=0}^{j-1} x^{2(j-k)-1} \frac{(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k+1} \\ C_0 &= \sum_{j=1}^{m} t_j \left( \sum_{k=0}^{j} x^{2j} + \sum_{k=0}^{j} x^{2(j-k)} \frac{(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k} \right) \end{aligned} \tag{V.74}$$

It should also be possible to find equations defining recursively the next matrices $\mathscr{R}_j(x,t)$ and $\mathscr{D}_j(x,t)$ by looking at the next orders in the series expansion. But since we will have no use of such results we do not mention them here.

### 3.3 Spectral Curve attached to the Lax pair

By definition, the spectral curve of a differential system like V.57 is given by $\det(yId - \mathscr{D}_0(x,t)) = 0$, that is to say by the large $N$ limit of the eigenvalues of the spectral problem (which we expect to give the large $N$ limit of our matrix model). Note in particular that this definition is independent of a change of basis (conjugation by a matrix). From all the previous results, we can compute this two by two determinant and get :

$$\begin{aligned} y^2 &= \left[ \sum_{j=1}^{m} t_j \sum_{k=0}^{j-1} x^{2(j-k)-1} \frac{(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k+1} + \sum_{j=1}^{m} t_j \left( \sum_{k=0}^{j} x^{2j} + \sum_{k=0}^{j} x^{2(j-k)} \frac{(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k} \right) \right] \\ &\quad \left[ \sum_{j=1}^{m} t_j \sum_{k=0}^{j-1} x^{2(j-k)-1} \frac{(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k+1} - \sum_{j=1}^{m} t_j \left( \sum_{k=0}^{j} x^{2j} + \sum_{k=0}^{j} x^{2(j-k)} \frac{(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k} \right) \right] \\ &= \left[ \sum_{j=1}^{m} t_j \sum_{k=0}^{j-1} x^{2(j-k)-1} \frac{(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k+1} + \sum_{j=1}^{m} t_j \left( \sum_{k=0}^{j-1} x^{2j} + \sum_{k=0}^{j} x^{2(j-k)} \frac{(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k} \right) \right] \\ &\quad \left[ \sum_{j=1}^{m} t_j \sum_{k=0}^{j-1} x^{2(j-k)-1} \frac{(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k+1} - \sum_{j=1}^{m} t_j \left( \sum_{k=0}^{j} x^{2j} + \sum_{k=0}^{j-1} x^{2(j-k)} \frac{(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k} \right) \right] \end{aligned} \tag{V.75}$$

where in the last identity we have use the algebraic equation satisfied by $u_0(t)$ V.70. Then, it is then a straightforward computation to see that the product can be rewritten

as :

$$\boxed{y_{\text{Lax}}^2 = P(x,t) = \left(u_0(t)^2 - x^2\right)\left(\sum_{j=1}^{m} t_j \sum_{k=0}^{j-1} \frac{x^{2(j-k)-1}(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k}\right)^2} \tag{V.76}$$

In particular in the specific case where $\forall j < m : t_j = 0$, and $t_m = 1$, we find that the spectral curve reduces to :

$$\boxed{\forall j < m : t_j = 0, t_m = 1 \Rightarrow y_{\text{Lax}}(x) = \sqrt{u_0(t)^2 - x^2} \sum_{k=0}^{m-1} \frac{x^{2(m-k)-1}(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k}} \tag{V.77}$$

**As expected, with the identification $u_0(t) = \gamma$ we recover exactly the rescaled-spectral curve of our matrix model 2.10.11**.

Note : In V.76 we can see that the only simple zeros of $P(x,t)$ are at $x = \pm u_0(t)$. Moreover since the polynomial $P(x,t)$ is obviously even and that there is no constant term in $x$ in the sum, we get that $P(x,t)$ has a double zero at $x = 0$ and has double roots at some points $\pm\lambda_i$, $i = 1, \ldots, m-1$

### 3.4 Asymptotics of the matrix $\Psi(x,t)$

The next step in the method of [29] is to determine an asymptotic of the functions $\psi(x,t)$ and $\phi(x,t)$. From the Schrodinger-like equation V.68, we have a BKW expansion :

$$\psi(x,t) = g(x,t)e^{Nh(x,t)}\left(1 + \frac{\psi_1(x,t)}{N} + \frac{\psi_2(x,t)}{N^2} + \ldots\right)$$

Putting back into the seculiar equation gives the following result :

$$\begin{aligned}
\psi(x,t) &= \frac{1}{\sqrt{2}}\left(\frac{u_0(t)+x}{u_0(t)-x}\right)^{\frac{1}{4}} e^{N\int^t \sqrt{u_0^2(t')-x^2}dt'}\left(1 + \frac{\psi_1(x,t)}{N} + \ldots\right) \\
\phi(x,t) &= -\frac{1}{\sqrt{2}}\left(\frac{u_0(t)+x}{u_0(t)-x}\right)^{\frac{1}{4}} e^{-N\int^t \sqrt{u_0^2(t')-x^2}dt'}\left(1 + \frac{\phi_1(x,t)}{N} + \ldots\right)
\end{aligned}$$

$$\begin{aligned}
\tilde{\psi}(x,t) &= \frac{1}{\sqrt{2}}\left(\frac{u_0(t)-x}{u_0(t)+x}\right)^{\frac{1}{4}} e^{N\int^t \sqrt{u_0^2(t')-x^2}dt'}\left(1+\frac{\tilde{\psi}_1(x,t)}{N}+\dots\right)\\
\tilde{\phi}(x,t) &= \frac{1}{\sqrt{2}}\left(\frac{u_0(t)-x}{u_0(t)-x}\right)^{\frac{1}{4}} e^{-N\int^t \sqrt{u_0^2(t')-x^2}dt'}\left(1+\frac{\tilde{\phi}_1(x,t)}{N}+\dots\right) \qquad \text{(V.78)}
\end{aligned}$$

One can easily check that at dominant order in $N$ the previous asymptotics gives $\det(\Psi(x,t)) = 1 + O\left(\frac{1}{N}\right)$. The next step is to transform the integration over $t$ in the exponential as a integral over $x$ by using the property of the spectral curve. Indeed, the spectral curve defines a Riemann surface which can be parametrized locally by $x(z,t)$ and $y(z,t)$ where $z$ is a running point on the Riemann surface. Thus, the function $y$ can be seen as both a function of $(z,t)$ or $(x,t)$. In order to avoid confusion here, we will write differently the function when it is seen as a function of $(z,t)$ or as a function of $(x,t)$ (we put a tilda for the function in $(x,t)$ and keep $y$ for the function of $(z,t)$) :

$$\tilde{y}(x,t) = \sqrt{P(x,t)} = y(z(x,t),t) \qquad \text{(V.79)}$$

Then, using standard chain rule derivation, one can compute :

$$\frac{\partial y}{\partial z}\frac{\partial x}{\partial t} - \frac{\partial y}{\partial t}\frac{\partial x}{\partial z} = -\frac{\partial \tilde{y}}{\partial t}\frac{\partial x}{\partial z} \qquad \text{(V.80)}$$

From the expression of the spectral curve V.76 (which gives explicitly $\tilde{y}(x,t)$) one can compute $\frac{\partial \tilde{y}}{\partial t}$ :

$$\begin{aligned}
\frac{\partial \tilde{y}}{\partial t} &= \frac{xu_0(\partial_t u_0)}{\sqrt{u_0^2-x^2}}\left(\sum_{j=1}^m t_j \sum_{k=0}^{j-1} \frac{x^{2(j-1-k)}(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k}\right)\\
&\quad + x(\partial_t u_0)\sqrt{u_0^2-x^2}\left(\sum_{j=1}^m t_j \sum_{k=1}^{j-1} \frac{x^{2(j-1-k)}(2k)!2k}{2^{2k}(k!)^2} u_0(t)^{2k-1}\right)\\
&= \frac{x(\partial_t u_0)}{\sqrt{u_0^2-x^2}}\sum_{j=1}^m t_j \frac{(2j)!(2j)}{2^{2j}(j!)^2} u_0(t)^{2j-1}\\
&= -\frac{x}{\sqrt{u_0^2-x^2}} \qquad \text{(V.81)}
\end{aligned}$$

To get the last identity, we have used the string equation V.70 for $u_0(t)$. Therefore by introducing the parametrization :

$$z^2 = u_0(t)^2 - x^2 \Leftrightarrow x^2 = u_0(t)^2 - z^2 \tag{V.82}$$

one finds that :

$$x'(z,t) = \frac{\partial x}{\partial z} = -\frac{\sqrt{u_0^2 - x^2}}{x} \ , \ \frac{\partial x}{\partial t} = -\frac{u_0(\partial_t u_0)}{x} \tag{V.83}$$

so that eventually :

$$\frac{\partial y}{\partial z}\frac{\partial x}{\partial t} - \frac{\partial y}{\partial t}\frac{\partial x}{\partial z} = -\frac{\partial \tilde{y}}{\partial t}\frac{\partial x}{\partial z} = -\frac{x}{\sqrt{u_0^2 - x^2}}\frac{\sqrt{u_0^2 - x^2}}{x} = -1 \tag{V.84}$$

The last identity can be rewritten as :

$$\boxed{\frac{\partial y}{\partial t}\frac{\partial x}{\partial z} - \frac{\partial y}{\partial z}\frac{\partial x}{\partial t} = 1} \tag{V.85}$$

and interpreted as the remaining of a non-commutative structure of $[P,Q] = \frac{1}{N}$ in the limit $N \to \infty$ which in such situations often transform into a Poisson structure for $y(z,t) \leftrightarrow P$ and $x(z,t) \leftrightarrow Q$ by simply replacing the commutator with a Lie bracket :

$$\{y(z,t), y(z,t)\} = 1 \tag{V.86}$$

With the help of this structure, we can get a reformulation of the integral :

$$\frac{\partial \tilde{y}}{\partial t} = \frac{1}{x'(z)} \tag{V.87}$$

hence :

$$\frac{\partial \int^x \tilde{y} dx}{\partial t} = z \tag{V.88}$$

and

$$\int^t \sqrt{u_0^2(t')-x^2}dt' = \int^t z dt = \int^x \tilde{y} dx \tag{V.89}$$

Eventually we have the following large $N$ developments :

$$\begin{aligned}
\psi(x,t) &= \frac{1}{\sqrt{2}}\left(\frac{u_0(t)+x}{u_0(t)-x}\right)^{\frac{1}{4}} e^{N\int^x \tilde{y}dx}\left(1+\frac{\psi_1(x,t)}{N}+\dots\right) \\
\phi(x,t) &= -\frac{1}{\sqrt{2}}\left(\frac{u_0(t)+x}{u_0(t)-x}\right)^{\frac{1}{4}} e^{-N\int^x \tilde{y}dx}\left(1+\frac{\phi_1(x,t)}{N}+\dots\right) \\
\tilde{\psi}(x,t) &= \frac{1}{\sqrt{2}}\left(\frac{u_0(t)-x}{u_0(t)+x}\right)^{\frac{1}{4}} e^{N\int^x \tilde{y}dx}\left(1+\frac{\tilde{\psi}_1(x,t)}{N}+\dots\right) \\
\tilde{\phi}(x,t) &= \frac{1}{\sqrt{2}}\left(\frac{u_0(t)-x}{u_0(t)-x}\right)^{\frac{1}{4}} e^{-N\int^x \tilde{y}dx}\left(1+\frac{\tilde{\phi}_1(x,t)}{N}+\dots\right)
\end{aligned} \tag{V.90}$$

### 3.5 Kernels and correlation functions in the Lax pair formalism

It was established in [25] that one can define a kernel $K(x_1,x_2)$ and define from it (through determinantal formulae) some functions $W_n(x_1,\dots,x_n)$ that have nice properties. In particular the authors showed in [25] that these functions do satisfy some loop equations and thus are likely to correspond to our matrix model correlation functions. Following [25] we define the kernel by :

$$K(x_1,x_2) = \frac{\psi(x_1)\tilde{\phi}(x_2)-\tilde{\psi}(x_1)\phi(x_2)}{x_1-x_2} \tag{V.91}$$

Then we define the (connected) correlation functions by :

$$W_1(x) = \psi'(x)\tilde{\phi}(x)-\tilde{\psi}'(x)\phi(x) \tag{V.92}$$

$$W_n(x_1,\dots,x_n) = -\frac{\delta_{n,2}}{(x_1-x_2)^2} - (-1)^n \sum_{\sigma=cycles}\prod_{i=1}^{n} K(x_{\sigma(i)},x_{\sigma(i+1)}) \tag{V.93}$$

and eventually we define non-connected functions $W_{n,n-c}$ by determinantal formulae :

$$W_{n,n-c}(x_1,\dots,x_n) = \overset{\prime}{det}(K(x_i,x_j)) \tag{V.94}$$

where the notation $det^{'}$ means that the determinant is computed in the usual way as a sum over permutations $\sigma$ of products $(-1)^{\sigma}\prod_{i=1}^{n}K(x_i,K_{\sigma_i})$, except for terms when $i=\sigma(i)$ and when when $i=\sigma(j)\,,j=\sigma(i)$. In such cases, one must replace $K(x_i,x_i)$ by $W_1(x_i)$ and $K(x_i,x_j)K(x_j,x_i)$ by $-W_2(x_i,x_j)$. For additional details, we invite the reader to look at ([25])

As in our problem we will need the large $N$ developments of these functions, we introduce the notations :

$$\begin{aligned} K(x_1,x_2) &= K_0(x_1,x_2)e^{N\int_{x_2}^{x_1}\tilde{y}dx}\left(1+\sum_{g=1}^{\infty}N^{-g}K^{(g)}(x_1,x_2)\right) \\ W_n(x_1,\ldots,x_n) &= \sum_{g=0}^{\infty}N^{2-2g-n}W_n^{(g)}(x_1,\ldots,x_n) \\ W_{n,n-c}(x_1,\ldots,x_n) &= \sum_{g=0}^{\infty}N^{n-2g}W_{n,n-c}^{(g)}(x_1,\ldots,x_n) \end{aligned} \tag{V.95}$$

Then, we can insert all our previous results concerning the leading terms of the series expansion V.90,V.93 and V.95. It gives :

$$K_0(x_1,x_2)=\frac{1}{2(x_1-x_2)}\left(\left(\frac{u_0+x_1}{u_0-x_1}\right)^{\frac{1}{4}}\left(\frac{u_0-x_2}{u_0+x_2}\right)^{\frac{1}{4}}+\left(\frac{u_0-x_1}{u_0+x_1}\right)^{\frac{1}{4}}\left(\frac{u_0+x_2}{u_0-x_2}\right)^{\frac{1}{4}}\right) \tag{V.96}$$

$$W_1^{(0)}(x)=\tilde{y}(x) \tag{V.97}$$

and

$$W_2^{(0)}(x_1,x_2)=\frac{1}{4(x_1-x_2)^2}\left(-2+\sqrt{\frac{(u_0+x_1)(u_0-x_2)}{(u_0-x_1)(u_0+x_2)}}+\sqrt{\frac{(u_0-x_1)(u_0+x_2)}{(u_0+x_1)(u_0-x_2)}}\right) \tag{V.98}$$

In order to get rid of the square-roots in the expressions above, it is better to introduce a proper parametrization of our spectral curve V.76. Let us define :

$$\boxed{x=\frac{u_0}{2}\left(z+\frac{1}{z}\right)=\frac{u_0(z^2+1)}{2z}\Leftrightarrow z=\frac{1+\sqrt{x^2-u_0^2}}{u_0}} \tag{V.99}$$

In particular, under such a change of variables we obtain several useful identities :

$$\begin{aligned}
\sqrt{\frac{u_0-x}{u_0+x}} &= i\frac{z-1}{z+1} \\
u_0-x &= -\frac{u_0(z-1)^2}{2z} \\
u_0-x &= \frac{u_0(z+1)^2}{2z} \\
\sqrt{u_0^2-x^2} &= \frac{iu_0}{2z}(z+1)(z-1) \\
\frac{dx(z)}{dz} &= \frac{u_0(z^2-1)}{2z^2}
\end{aligned} \tag{V.100}$$

Eventually we can rewrite $W_2^{(0)}$ in terms of the new variable $z$ :

$$W_2^{(0)}(z_1,z_2) = \frac{4z_1^2z_2^2}{u_0^2(z_1^2-1)(z_2^2-1)(z_1z_2-1)^2} \tag{V.101}$$

Although these functions have some interesting features, they still depend on the choice of coordinates on the Riemann surface defined by the spectral curve. Therefore, we introduce similarly to [29] and [23] the corresponding differential forms :

$$\mathscr{W}_n^{(g)}(z_1,\dots,z_n) = W_n^{(g)}(x(z_1),\dots,x(z_n))x'(z_1)\dots x'(z_n) + \delta_{n,2}\delta_{g,0}\frac{x'(z_1)x'(z_2)}{(x(z_1)-x(z_2))^2} \tag{V.102}$$

These differentials are symmetric rational functions of all their variables. Moreover as proved in the crucial theorem V.2 these functions only have poles at $z_i = \pm 1$ (except again $\mathscr{W}_2^{(0)}(z_1,z_2)$ which may have a pole at $x(z_1) = x(z_2)$). Eventually, a direct computation from V.101 gives :

$$\boxed{\mathscr{W}_2^{(0)}(z_1,z_2) = \frac{1}{(z_2-z_1)^2}} \tag{V.103}$$

### 3.6 Loop equations, determinantal formulae, pole structure and unicity

The previous determinantal definitions may seem rather arbitrary, but as we mention before they have the interesting property (proved in [25]) to satisfy the following loop equations.

**Théorème V.1.** *Loop equations satisfied by the determinantal functions :*

$$P_n(x;x_1,\dots,x_n) = W_{n+2,n-c}(x,x,x_1,\dots,x_n) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{W_n(x,x_1,\dots,x_{j-1},x_{j+1},\dots,x_n) - W_n(x_1,\dots,x_n)}{x-x_j} \tag{V.104}$$

*is a polynomial of the variable x. The previous theorem is equivalently reformulated for the standard connected functions :*

$$P_n(x;x_1,\dots,x_n) = \sum_{h=0}^{g} \sum_{I\subset J} W^{(h)}_{1+|I|}(x,I) W^{(g-h)}_{1+n-|I|}(x,J/I) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{W_n(x,J/\{x_j\}) - W_n(x_1,\dots,x_n)}{x-x_j} \tag{V.105}$$

*is a polynomial of the variable x.*

We emphasize again that loop equations are an essential step because it is well known in the matrix model world [19] that the correlation functions introduced in our first section do satisfy these loop equations. Unfortunately, loop equations generally admit several solutions encoded essentially in the unknown coefficients of the polynomial $P_n$. Therefore we need some additional results to get unicity. The first one deals with the pole structure :

**Théorème V.2.** *Pole Structure :*

*The functions $z \to \psi_k(z,t)$ are rational functions with poles only at $z \in \{\pm i, 0, \infty\}$. The coefficients of these fractions depend on $u_0(t)$ and its derivatives. Hence the determinantal correlation functions $W_n^{(g)}$ are symmetric and rational functions in the variables*

*$z_i$ with poles only at $z_i = \pm 1$.*

Proof : The last part of the theorem is obvious from the definitions as soon as the results regarding the $\psi_k(z,t)$'s are established. This proof is presented in Appendix V and is highly non-trivial. It uses the whole structure of integrability (i.e. the two differential equations V.57) to eliminate other possible poles (at the other zeros of $y_{textLax}(x)$).

With the knowledge of the pole structure of the $W_n^{(g)}$, the fact that they satisfy the loop equations and the knowledge of $W_2^{(0)}$ we have a unicity theorem. In fact under these conditions we can identify our differentials $\mathscr{W}_n^{(g)}$'s with the ones defined by the standard recursion relation introduced by Eynard and Orantin in [23] :

**Théorème V.3.** *The differentials $\mathscr{W}_n^{(g)}$ satisfy the following recursion :*

$$\begin{aligned}\mathscr{W}_{n+1}^{(g)}(z_1,\dots,z_n,z_{n+1}) &= \operatorname*{Res}_{z\to\pm 1} \frac{dz}{2u_0 y(z)(1-\frac{z_{n+1}}{z})(\frac{1}{z}-z_{n+1})}\Big[\mathscr{W}_{n+2}^{(g-1)}(z,\bar{z},z_1,\dots,z_n) \\ &\quad + \sum_{h=0}^{g}{}' \sum_{I\in J} \mathscr{W}_{1+|I|}^{(h)}(z,I)\mathscr{W}_{n+1-|I|}^{(g-h)}(\bar{z},J/I)\Big]\end{aligned} \tag{V.106}$$

*where $J$ is a short-writing for $J=(z_1,\dots,z_n)$ and $\sum_{h=0}^{g}{}' \sum_{I\in J}$ means that we exclude the terms $(h,I)=(0,\emptyset)$ and $(h,I)=(g,J)$ in the sum. The notation $\bar{z}$ stands for the conjugate point of $z$ near the poles where the residue is taken. In our case : $\bar{z}=\frac{1}{z}$*

Note : It is worth noticing that in Eynard and Orantin's notation we have in our case (we omit the dependance in the $t$ parameter) :

$$\begin{aligned}\omega(z) &= y(z)\frac{u_0(z^2-1)}{z^2} \\ y(\bar{z}) &= y(1/z) = -y(z) \\ dE_z(p) &= \frac{1}{2}\int_z^{\frac{1}{z}} \frac{ds}{(s-p)^2} = \frac{1-z^2}{2(z-p)(pz-1)}\end{aligned} \tag{V.107}$$

so that :

$$\frac{dE_z(z_{n+1})}{\omega(z)} = \frac{z^2}{2u_0 y(z)(z-z_{n+1})(1-z_{n+1}z)} = \frac{1}{2u_0 y(z)(1-\frac{z_{n+1}}{z})(\frac{1}{z}-z_{n+1})} \tag{V.108}$$

Proof of V.106 : The unicity proof has been done in various article but for completeness we rederive it here with our notations. First of all Cauchy's theorem states that :

$$\mathscr{W}_{n+1}^{(g)}(z_1,\ldots,z_{n+1}) = \operatorname*{Res}_{z\to z_{n+1}} \frac{dz}{z-z_{n+1}} \mathscr{W}_{n+1}^{(g)}(z_1,\ldots,z_{n+1}) \tag{V.109}$$

We can move the integration contour to enclose all other poles, i.e. only $\pm 1$ in our case :

$$\begin{aligned} \mathscr{W}_{n+1}^{(g)}(z_1,\ldots,z_{n+1}) &= \operatorname*{Res}_{z\to\pm 1} \frac{dz}{z_{n+1}-z} \mathscr{W}_{n+1}^{(g)}(z_1,\ldots,z_{n+1}) \\ &= \operatorname*{Res}_{z\to\pm 1} \frac{x'(z)dz}{z_{n+1}-z} W_{n+1}^{(g)}(x(z_1),\ldots,x(z_{n+1})) \end{aligned} \tag{V.110}$$

Then using the loop equations V.105 and separating the coefficients $W_1^{(0)}$ in the sum gives :

$$\begin{aligned} -2W_1^{(0)}(x)W_{n+1}^{(g)}(x_1,\ldots,x_n,x) &= \sum_{h=0}^{g} {\sum_{I\subset J}}' W_{1+|I|}^{(h)}(x,I)W_{1+n-|I|}^{(g-h)}(x,J/I) \\ &\quad \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{W_n(x,J/\{x_j\}) - W_n(x_j,J/\{x_j\})}{x-x_j} \\ &\quad -P_n^{(g)}(x,x_1,\ldots,x_n) \end{aligned} \tag{V.111}$$

The polynomial $P_n^{(g)}(x,x_1,\ldots,x_n)$ does not contribute to the residue, and after using the relation between $x$ and $z$ we are left with V.106.

## 4 Lax pairs for the (2m,1) minimal model and for the Painlevé II hierarchy

### 4.1 The (2m,1) minimal model and the Flashka-Newell Lax pair

As observed in [22] the string equation V.66 is nothing but the $m^{\text{th}}$ member of the so-called Painlevé II hierarchy. The Painlevé II (PII) hierarchy, a collection of ODEs of order $2m$, arises as a self-similar reduction of the mKdV hierarchy. In the papers [92] and [93] this relationship has been used to construct a Lax pair for the PII hierarchy starting from the relevant Lax pair for the modified KdV hierarchy. We call this PII Lax pair the Flashka-Newell Lax pair since the first member of the hierarchy was find, for the

first time, in [94]. In this subsection we prove that, up to a linear transformation of the wave function and a rescaling of the variables, the Flashka-Newell Lax Pair is equivalent to the $(2m,1)$ minimal model Lax pair. In order to simplify notation we forget, in this section, the rescaling given by $1/N$ over the variables $x$ and $t$. We begin with the case $t_1 = 0 = t_2 = \ldots = t_{m-1}$.

**Proposition V.1.** *Define* $\tilde{\Psi}$ *as a new wave function*

$$\tilde{\Psi} := J\Psi$$

*with*

$$J := \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

*and set* $t_m \longmapsto (4^{m+1}/2)$ *(all other parameters* $t_j$ *equal to* $0$*). Then* $\tilde{\Psi}$ *satisfies the Flashka-Newell Lax pair as written in [92].*

Proof Since $J$ is constant we observe that $\tilde{\Psi}$ solve the Lax system

$$\begin{aligned} \frac{\partial}{\partial x}\tilde{\Psi}(x,t) &= \tilde{\mathscr{D}}_m(x,t)\tilde{\Psi}(x,t) \\ \frac{\partial}{\partial t}\tilde{\Psi}(x,t) &= \tilde{\mathscr{R}}(x,t)\tilde{\Psi}(x,t) \end{aligned} \tag{V.112}$$

with $\tilde{\mathscr{D}}(x,t), \tilde{\mathscr{R}}(x,t)$ obtained through conjugation with $J$ ; i.e.

$$\tilde{\mathscr{R}}(x,t) = J\mathscr{R}(x,t)J^{-1} = \begin{pmatrix} -ix & u \\ u & ix \end{pmatrix}$$

and

$$\tilde{\mathscr{D}}_m(x,t) = \frac{4^{m+1}}{2} J\mathscr{D}_m(x,t)J^{-1} = \frac{4^{m+1}}{2} \begin{pmatrix} -iC_m(x,t) & iA_m(x,t) + xB_m(x,t) \\ -iA_m(x,t) + xB_m(x,t) & iC_m(x,t). \end{pmatrix}$$

These two matrices are exactly the ones appearing in (16a) and (16b) in [92] (modulo the identification $u \longleftrightarrow w, x \longleftrightarrow \lambda, t \longleftrightarrow z$). For the matrix $\tilde{\mathscr{R}}$ this is self-evident. For $\tilde{\mathscr{D}}_m$

we just have to observe that it has the same shape as the matrix written in the right-hand side of (16b) (see eqs (14) ; in particular the polar part in (16b) is zero thanks to (14b)). On the other hand this conditions, plus compatibility condition, determines uniquely $\tilde{\mathscr{D}}_m$.

Of course the result above is extended to the case in which all $t_j$ enter in $\mathscr{D}$ just taking linear combinations of the matrices studied in the previous proposition. This has been done, for the Flashka-Newell pair, in [93] (note, nevertheless, that there the spectral parameter is rotated ; $\lambda \to -i\lambda$). Hence we have the following proposition.

**Proposition V.2.** *Under a rescaling of all time variables $t_j \longrightarrow \frac{4^{j+1}}{2} t_j$ the (2m,1)-minimal model Lax pair is equivalent to the Flashka-Newell Lax pair for the PII hierarchy.*

## 5 Conclusion and outlooks

In section 1, we have established that the double scaling limit of a matrix model with a $2m$-degenerate point can define a universal rescaled spectral curve $\hat{y}_{\text{rescaled}}(x)$. In section 1 we also reminded that the correlation functions and symplectic invariants $\hat{w}_n^{(g)}(x_1,\dots,x_n)$ and $\hat{f}_g$ can also be rescaled in a suitable way in order to give some new functions $\hat{w}^{(g)}_{\text{rescaled,n}}(x_1,\dots,x_n)$ and new symplectic invariants $\hat{f}_{\text{rescaled,g}}$ corresponding respectively to the correlation functions and symplectic invariants of the rescaled curve $\hat{y}_{\text{rescaled}}(x)$. Then, starting from a Lax pair of the Painlevé II hierarchy and using the same method as [29] we have constructed a spectral curve $y_{\text{Lax}}(x)$ which coincides with $\hat{y}_{\text{rescaled}}(x)$ for a natural choice of the flow parameters $t_j$'s. Finally, with the definition of a suitable kernel and determinantal formulae, we have defined in the same way as [29] some functions $W_n^{(g)}$ having interesting properties (loop equations). Studying in details the pole structure and computing $\mathscr{W}_2^{(0)}(z_1,z_2)$, we have eventually shown that the function $W_n^{(g)}$'s are in fact exactly the correlation functions of the curve $y_{\text{Lax}}(x)$. Since the two spectral curves are the same, we have proved the statement :

**Théorème V.4.** *The correlation functions (and spectral curve) of the double scaling limit of a $2m$-degenerate merging of two cuts are the same as the functions $W_n^{(g)}$ (and spectral curve) defined by determinantal formulae of the integrable Painlevé II hierarchy's kernel.*

This result reinforces the links between double scaling limit in matrix models and integrable $(p,q)$ minimal models. With this new result and the one of [29], the two models are shown to be identical for $(p=2m,q=1)$ and $(p=2m+1,q=2)$ $(m\in\mathbb{N}^*)$. However even if this identity is expected to hold for every $(p,q)$, some complete proofs as the one presented here are still missing. Indeed, if our reasoning may seem easy to generalize for arbitrary value of $p$ and $q$, the crucial theorem V.105 establishing that the functions $W_n^{(g)}$ coming from determinantal formulae do satisfy the loop equations (proved in [25]) is only valid for $q\leq 2$ at the moment. Therefore a good approach to the generalization for arbitrary value of $(p,q)$ could be to first extend this theorem for every $(p,q)$ and then to use the method presented here to extend the result.

Another approach could be to use this approach to study other integrable systems whose Lax pairs are known. Indeed, it is possible to perform the same method as the one presented here for any Lax pair. In particular, for every Lax pair, it would be interesting to analyse the associated spectral curve and the corresponding determinantal correlation functions.

**Appendix : Pole structure for $\psi_k(z,t)$**

In order to use the unicity theorem V.105 showing that the $W_n^{(g)}$'s are the expected correlation functions, we need to precise the pole structure of the function $\psi_k(x,t)$'s and $\phi_k(x,t)$'s from which they are defined. In order to determine the functions $\psi_k(x,t)$'s, one can insert the series expansion V.90 into the seculiar equations. Since the case $\psi_k(x,t)$ and $\phi_k(x,t)$'s are similar (they satisfy the same seculiar equation), we will focus only on the $\psi_k(x,t)$'s. The main issue of this appendix is that putting the large $N$ asymptotics of $\psi(x,t)$ V.90 into the seculiar equation a priori gives unwanted poles at the zeros of $y(x)$ for $\psi_k(x,t)$ that we need to rule out. It is the purpose of this appendix to explain how this can be done.

**Study of the differential equation in $t$**

From the fact that $u(t)$ satisfies the string equation we remind the reader that we have V.70 :

$$t = -\sum_{j=1}^{m} t_j \frac{(2j)!}{2^{2j}(j!)^2} u_0(t)^{2j} = P_0(u_0) \tag{V.113}$$

From this, it follows that $\frac{du_0}{dt}$ is :

$$\frac{du_0}{dt} = \frac{1}{P_0'(u_0)} \tag{V.114}$$

Performing more derivations relatively to $t$ can give the derivatives of $u_0(t)$ to any order as a fraction whose denominator is always a power of $P_0'(u_0)$. For example :

$$\begin{aligned} \frac{d^2u_0}{dt^2} &= -\frac{P_0''(u_0)}{(P_0'(u_0))^3} \\ \frac{d^3u_0}{dt^3} &= -\frac{P_0'''(u_0)}{(P_0'(u_0))^4} + 3\frac{(P_0''(u_0))^2}{(P_0'(u_0))^5} \end{aligned} \tag{V.115}$$

and so on.

As a consequence, any power of any derivative of $u_0$ remains a rational function of $u_0$ with poles only at the roots of $P_0'(x)$. For example, expressions like $\frac{du_0}{dt}\frac{d^3u_0}{dt^3} + \left(\frac{du_0}{dt}\right)^2 \frac{d^2u_0}{dt^2}$ will be rational functions of $u_0$ with poles only at the roots of $P_0'(x)$.

Now, putting back the development of $u(t) = u_0(t) + \frac{u_2(t)}{N^2} + \frac{u_3(t)}{N^3} + \dots$ into the full string equation V.66 gives that any subleading order $u_k$ can be expressed as a rational function of $u_0$ with poles only at the roots of $P_0'(x)$.

Eventually, inserting the shape of the function $\psi(x,t)$ into the seculiar equation and evaluating the order $N^{-k}$ gives the following equation $\forall k \geq 2$ :

$$\begin{aligned} \partial_t \psi_{k-1} &= \frac{\partial_{t^2} g}{2gh}\psi_{k-2} - \frac{\partial_t g}{gh}\partial_t \psi_{k-2} - \frac{\partial_{t^2}\psi_{k-2}}{2h} + \frac{1}{2}\left(\frac{\partial_t u}{u+x}\right)_k + \frac{1}{2}\sum_{i=0}^{k-2}\left(\frac{\partial_t u}{u+x}\right)_{k-i}\psi_i \\ &\quad + \frac{\partial_t g}{2gh}\left(\frac{\partial_t u}{u+x}\right)_{k-1} + \frac{1}{2}\sum_{i=0}^{k-2}\left(\frac{\partial_t u}{u+x}\right)_{k-1-i}\left(\psi_i \frac{g_t}{gh} + \frac{\partial_t \psi_i}{h}\right) \end{aligned}$$

$$+\frac{1}{2}\sum_{i=0}^{k-2}\left(u^2\right)_{k-i}\frac{\psi_i}{h} \tag{V.116}$$

where we have written in short :

$$\begin{aligned} \psi_0(x,t) &= 1 \\ h(x,t) &= \sqrt{u_0(t)^2-x^2} \\ g(x,t) &= = \left(\frac{u_0(t)+x}{u_0(t)-x}\right)^{1/4} \end{aligned} \tag{V.117}$$

and the notation $\left(\frac{\partial_t u}{u+x}\right)_k$ stands for the term in $N^{-k}$ in the expansion of $\frac{\partial_t u}{u+x}$. Note in particular that these terms can be expressed as a fraction with poles at $u_0(t)+x=0$ and at $P'(u_0(t))=0$ (the last are independent of $x$). For example the first one is :

$$\partial_t \psi_1(x,t) = \frac{(\partial_t u_0)^2 x^2 (u_0-x)^{\frac{3}{2}}}{4(u_0+x)^{\frac{3}{2}}} + \frac{(u_0+x)^{\frac{1}{2}}}{(u_0-x)^{\frac{1}{2}}} u_2(t) \tag{V.118}$$

where remember that $u_2(t)$ can be expressed as a rational function of $u_0(t)$ whose poles are known are only when $u_0(t)$ is at a root of $P_0'$ (and thus are independent of $x$). From this expression, it is clear that $\psi_1(x,t)$ may only have $x$-dependent singularities at $x=\pm u_0$ and at $x=\infty$.

**Study of the differential equation in $x$**

The technic presented in the previous subsection can be carried out for the differential equation in $x$. Starting with the second equation of the Lax pair V.57 :

$$\frac{1}{N}\frac{\partial}{\partial x}\Psi(x,t) = \begin{pmatrix} -A(x,t) & xB(x,t)+C(x,t) \\ xB(x,t)-C(x,t) & A(x,t) \end{pmatrix} \Psi(x,t) \tag{V.119}$$

we can derive another seculiar equation for both $\psi(x,t)$ and $\phi(x,t)$ :

$$\begin{aligned} 0 &= \frac{1}{N^2}\frac{\partial^2}{\partial x^2}\psi(x,t) - \frac{1}{N^2}\left(\frac{\partial_x(xB+C)}{xB+C}\right)\frac{\partial}{\partial x}\psi(x,t) \\ &\quad + \frac{1}{N}\left(\partial_x A - A\frac{\partial_x(xB+C)}{xB+C}\right)\psi(x,t) - y^2(x,t)\psi(x,t) \end{aligned} \tag{V.120}$$

where we have used that :

$$\det(\Psi) = 1 \Leftrightarrow y^2(x,t) = A(x,t)^2 + x^2 B(x,t)^2 - C(x,t)^2 \tag{V.121}$$

Note in particular in the last identity that the r.h.s. should have a large $N$ development whereas the l.h.s. $y(x)$ given by V.76 does not. Therefore, the l.h.s. must have vanishing subleading orders in $\frac{1}{N^k}, \forall k > 0$.

Moreover, reformulating V.73 give :

$$\begin{aligned} A_0 &= 0 \\ (xB+C)_0 &= y(x,t)\sqrt{\frac{u_0+x}{u_0-x}} \end{aligned} \tag{V.122}$$

where the subscript 0 stands for the first order in the large $N$ expansion. Indeed, it comes from the fact that :

$$\begin{aligned} y^2(x,t) &= (xB+C)_0(xB-C)_0 = P(x,t) = P_1(x,t)P_2(x,t) \\ (xB+C)_0 = P_1(x,t) &= (u_0+x)\left(\sum_{j=1}^{m} t_j \sum_{k=0}^{j-1} \frac{x^{2(j-1-k)}(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k}\right) \\ (xB-C)_0 = P_2(x,t) &= (u_0-x)\left(\sum_{j=1}^{m} t_j \sum_{k=0}^{j-1} \frac{x^{2(j-1-k)}(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k}\right) \end{aligned} \tag{V.123}$$

and eventually :

$$P_2(x,t) = P_1(x,t)\frac{u_0-x}{u_0+x} \tag{V.124}$$

With V.122 it is easy to see that :

$$\left(\frac{\partial_x(xB+C)}{xB+C}\right)_0 = \frac{\partial_x y}{y} + \frac{u_0}{u_0^2 - x^2} \tag{V.125}$$

which will be crucial for the coherence of the computation. Indeed, putting the large $N$ expansion of $\psi(x,t)$ :

$$\psi(x,t) = g(x,t)e^{Nh(x,t)}\left(1 + \frac{\psi_1(x,t)}{N} + \frac{\psi_2(x,t)}{N^2} + \dots\right)$$

into V.120 and comparing the first orders in $\frac{1}{N}$ gives :

$$\begin{aligned} 0 &= g(x,t)y^2(x,t) - g(x,t)y^2(x,t) \\ 0 &= \partial_x(g(x,t)y(x,t)) + y(x,t)\partial_x g(x,t) + g(x,t)y(x,t)\left(\frac{\partial_x(xB+C)}{xB+C}\right)_0 \end{aligned} \tag{V.126}$$

The second equation with the help of V.125 determines $g(x,t)$ coherently with V.117, that is to say :

$$g(x,t) = \left(\frac{u_0(t)+x}{u_0(t)-x}\right)^{1/4}$$

Note now that $\forall k > 0$, the function $\left(\frac{\partial_x(xB+C)}{xB+C}\right)_k$ only has singularities at the singularities of $\frac{1}{xB_0+C_0}$ according to the standard rules of Taylor series for a fraction. The next order, $\frac{1}{N^2}$, gives us the function $\psi_1(x,t)$ (with the notation that a subscript $k$ defines the term in $N^{-k}$ in the expansion at large $N$) :

$$\begin{aligned} \partial_x\psi_1(x,t) \quad = \quad & -\frac{\partial_{x^2}g}{2gy} + \frac{\partial_x g}{2gy}\left(\frac{\partial_x(xB+C)}{xB+C}\right)_0 + \frac{1}{2}\left(\frac{\partial_x(xB+C)}{xB+C}\right)_1 \\ & -\frac{1}{2}\left(\partial_x A - A\frac{\partial_x(xB+C)}{xB+C}\right)_1 \end{aligned} \tag{V.127}$$

From the definition of $g(x,t)$, it is easy to compute :

$$\frac{\partial_x g}{g} = \frac{1}{2}\frac{u_0}{u_0^2 - x^2}$$

l

$$\frac{\partial_{x^2} g}{g} = \frac{u_0 x}{u_0^2 - x^2} + \frac{1}{4}\frac{u_0^2}{(u_0^2-x^2)^2} \tag{V.128}$$

and thus to see that $\partial_x \psi_1(x,t)$ is a function of $x$ that may only have singularities at $x = \pm u_0$, at $x = \infty$ and at the others zeros of $y(x) = 0$. (it is so because $\left(\frac{\partial_x(xB+C)}{xB+C}\right)_1$ have the same singularities as $\frac{1}{xB_0+C_0}$ which by V.122 are only at $x = \pm u_0$, $x = \infty$ and at the zeros of $y(x)$).

It is then possible to extend this result for higher terms in the large $N$ expansion. The power $\frac{1}{N^k}$ gives :

$$\begin{aligned}
\partial_x \psi_{k-1} = & -\frac{\partial_{x^2} g}{2gy}\psi_{k-2} - \frac{\partial_x g}{gy}\partial_x \psi_{k-2} - \frac{1}{y}\partial_{x^2}\psi_{k-2} \\
& + \frac{1}{2}\sum_{i=0}^{k-2}\left(\frac{\partial_x(xB+C)}{xB+C}\right)_{k-1-i}\psi_i + \frac{\partial_x g}{2gy}\sum_{i=0}^{k-2}\left(\frac{\partial_x(xB+C)}{xB+C}\right)_{k-2-i}\psi_i \\
& + \frac{1}{2y}\sum_{i=0}^{k-2}\left(\frac{\partial_x(xB+C)}{xB+C}\right)_{k-2-i}\partial_x\psi_i - \frac{1}{2y}\sum_{i=0}^{k-1-i}\left(\partial_x A - A\frac{\partial_x(xB+C)}{xB+C}\right)_{k-1-i}\psi_i
\end{aligned} \tag{V.129}$$

where we have define $\psi_0 = 1$. The precise form of the relation is mostly irrelevant, but the main fact is that if all the $\psi_i(x,t)$ with $i < k$ are assumed to have singularities only at $x = \pm u_0$, $x = \infty$ and at the other zeros of $y(x) = 0$, then the same is true for $\partial_x \psi_k$ by a simple recursion.

**Pole structure of $\psi_k(x,t)$**

With the help of V.116 and V.129 we are now able to prove that the only singularities of $x \mapsto \psi_k(x,t)$ are at $x = \pm u_0$ and at $x = \infty$.

From V.116 we have shown that $\partial_t \psi_k(x,t)$ can only have singularities at $x = \pm u_0(t)$, at $x = \infty$ and when $u_0(t)$ is at a root of $P_0'$. But from V.129 we have shown that $\partial_x \psi_k(x,t)$ can only have singularities at $x = \pm u_0(t)$, at $x = \infty$ and at the other zeros of $y(x) = 0$ given by $x = \lambda_i(t)$ solution of $\sum_{j=1}^{m} t_j \sum_{k=0}^{j-1} \frac{x^{2(j-k)-1}(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k} = 0$ in V.76. But these poles are incompatible with the former result. Indeed if $\psi_k(x,t)$ had a pole at $x = \lambda_i(t)$, then $\partial_t \psi_k(x,t)$ would also have a pole at $x = \lambda_i(t)$, but we have shown that the only $x$-

dependent singularities of $\partial_t \psi_k(x,t)$ are at $x \pm u_0(t)$ or $x = \infty$ giving rise to a contradiction. **Therefore :** $x \mapsto \psi_k(x,t)$ **has only singularities at** $x = \pm u_0$ **(square-root poles) and** $x = \infty$ **(poles) and in particular has no pole at the other zeros of** $y(x) = 0$. This result is highly non trivial because we need to combine the two differential equations (i.e. the whole integrable structure) to get it. Hence, the structure of integrability seems to play an important underlying role in the pole structure and we can hope that such a result could extend to every integrable system.

**Pole structure in the $z$ variable**

In order to have only poles (and not square root singularities), we want to shift the former result to the $z$ variable defined by :

$$z^2 = \frac{u_0 - x}{u_0 + x} \Leftrightarrow x = u_0 \frac{1 - z^2}{1 + z^2} \tag{V.130}$$

Note that we have the identities :

$$\begin{aligned}
\frac{\partial x}{\partial t} &= (\partial_t u_0) \frac{1 - z^2}{1 + z^2} \\
\frac{\partial x}{\partial z} &= -\frac{4 u_0 z}{(1 + z^2)^2} \\
u_0 + x &= \frac{2 u_0}{1 + z^2} \\
u_0 - x &= \frac{-2 z^2}{1 + z^2} \\
g(z,t) &= \frac{(-u_0)^{\frac{1}{4}}}{z^{\frac{1}{2}}} \\
y(z,t) &= \frac{4 z^2 u_0^2}{(1 + z^2)^2} P_0\left(\left(\frac{1 - z^2}{1 + z^2}\right) u_0\right) \\
\frac{\partial_x g}{g}(z,t) &= \frac{(1 + z^2)^2}{8 u_0^2 z^2} \\
\frac{\partial_{x^2} g}{g}(z,t) &= \frac{(1 + z^2)^2}{4 u_0^2 z^2} + \frac{(1 + z^2)^4}{64 u_0^2 z^4}
\end{aligned} \tag{V.131}$$

Note also that every polynomial in $x$ will give a polynomial in $\frac{1-z^2}{1+z^2}$, that is to say a

rational function in $z$ with poles at $z^2+1=0$.

The rules for derivation gives that :

$$\partial_t \tilde{\psi}_k(z,t) = \partial_t \psi_k(x,t) + \frac{\partial x}{\partial t}\frac{\partial \psi_k(x,t)}{\partial x} \tag{V.132}$$

$$\partial_z \tilde{\psi}_k(z,t) = \frac{\partial x}{\partial z}\partial_x \psi_1(x,t) \tag{V.133}$$

where all these terms are already known from the previous sections. If one uses V.131 and the remark that a polynomial in $x$ will give a rational function in $z$ with poles at $z^2+1=0$ (and remember that functions $A,B,C$ are polynomials in $x$), one can see that the singularities of $\psi_k(x,t)$ at $x=\pm u_0$ (square-root type) and at $x=\infty$ (poles), will transform into poles at $z=0$ ($\Leftrightarrow x=-u_0$), $z=\infty$ ($\Leftrightarrow x=u_0$) and $z=\pm i$ ($\Leftrightarrow x=\infty$).

Hence we have the final result : $\forall k \geq 0$ **: the functions** $z \to \psi_k(z,t)$ **are rational functions with poles only at** $z \in \{\pm i, 0, \infty\}$**. The coefficients of these fractions depend on** $u_0(t)$ **and its derivatives.**

**Appendix : Discussion about** $\alpha = -\gamma$

When computing the rescaled spectral curve in the matrix model double scaling limit, we need to find a relationship between $\alpha$ and $\gamma$ that are given by V.36 :

$$\begin{aligned} b_1(T) &= b\varepsilon + \alpha\Delta + \sum_{n=1}^{\infty} b_{1,n}\Delta^n \\ a_2(T) &= b\varepsilon + \gamma\Delta + \sum_{n=1}^{\infty} a_{2,n}\Delta^n \end{aligned} \tag{V.134}$$

where we remind that $\Delta = (T-T_c)^{\frac{1}{2m}}$. A first argument in favour of the fact that $\alpha=-\gamma$ is the case when $\varepsilon=0$. Indeed, in such a case, the situation is fully symmetric around the singular point 0. Therefore, one expects the two endpoints $b_1(T)$ and $a_2(T)$ to be symmetric around $x=0$ for every value of $T$ around $T_c$. In such a case the identity $\forall T \simeq T_c : a_2(T) = -b_1(T)$ gives $\alpha=-\gamma$. When $\varepsilon \neq 0$, we can carry out a similar

reasoning at first orders in $\Delta$. Indeed, if we center the origin at $b\varepsilon$, then as we observed it several times, the endpoints $a_1$ and $b_2$ can be considered to be respectively $-b$ and $b$ up to order $\Delta^6$. Therefore in the function $R^{\frac{1}{2}}(x)$ they only add a multiplicative trivial factor depending on $\varepsilon$ ($\sqrt{1-\varepsilon^2}$ to be precise) which will not change the symmetry around $b\varepsilon$ of the endpoints $a_2$ and $b_1$ at first orders in $\Delta$.

Eventually, another more explicit approach is to put the developments V.36 into all the equations V.10, V.8, V.9 and V.32 determining $h(z,T)$, $x_0(T)$ and the endpoints $a_1(T), b_1(T), a_2(T)$ and $b_2(T)$. Doing so leads to an algebraic equation of degree $2m$ connecting $\alpha$ and $\gamma$ :

$$Q(\alpha,\eta)=0 \tag{V.135}$$

with $Q$ a symmetric, homogeneous polynomial of degree $2m$. Unfortunately the system does not admit a unique solution as soon as $m>1$. Indeed, although the solution $\alpha=-\gamma$ is always there, when $m>1$ there are also other possibilities such as $\alpha=\lambda\gamma, \lambda\in\mathbb{C}$ and $\gamma$ satisfying an equation of degree $2m$ with complex coefficients. Though it might appear surprising that the set of equations may have several distinct solutions (thus giving several eigenvalues density), one must remember that they are some additional constraints for the solution. Indeed, if one wants to have a density distribution, it means that all quantities involved must at least be real and positive. Therefore only the solution $\alpha=-\gamma$ is possible.

Note : In fact $\alpha$ and $\gamma$ are not necessarily well defined. Indeed, there are only defined up to a multiplicative $(2m)^{\text{th}}$ root of unity since the equation defining them is homogeneous of degree $2m$. This is because the notion of $\Delta=(T-T_c)^{\frac{1}{2m}}$ is also ambiguous, whereas $\Delta^{2m}$, $\alpha^{2m}$ and $\gamma^{2m}$ are well-defined quantities. (which explain why the development in $a_1(T)$ and $b_2(T)$ is well defined). Indeed, if one changes :

$$\forall n\in\{1,\dots,2m-1\}:\ \Delta\to\tilde{\Delta}=\Delta e^{\frac{2in\pi}{2m}}\ ,\ \alpha\to\tilde{\alpha}=\alpha e^{\frac{-2in\pi}{2m}}\ \text{and}\ \gamma\to\tilde{\gamma}=\gamma e^{-\frac{2in\pi}{2m}} \tag{V.136}$$

then V.36 remains unchanged. With the change $\xi\to\tilde{\xi}=\xi e^{-\frac{2in\pi}{2m}}$, the rescaled spectral curve remains unchanged.

## Annexe VI

# The partition function of the two-matrix model as an isomonodromic tau-function

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## Abstract

We consider the Itzykson-Zuber-Eynard-Mehta two-matrix model and prove that the partition function is an isomonodromic tau function in a sense that generalizes Jimbo-Miwa-Ueno's [38]. In order to achieve the generalization we need to define a notion of tau-function for isomonodromic systems where the *ad*–regularity of the leading coefficient is not a necessary requirement.

## 1 Introduction

Random matrices models have been studied for years and have generated important results in many fields of both theoretical physics and mathematics.

The two-matrix model

$$\partial\mu(M_1,M_2) = \mathrm{e}^{-\mathrm{Tr}(V_1(M_1)+V_2(M_2)-M_1M_2)}\partial M_1\partial M_2 \tag{VI.1}$$

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was used to model $2D$ quantum gravity [187] and was investigated from a more mathematical point of view in [32–35, 160, 186, 193]; the *partition function* of the model

$$\mathscr{Z}_N(V_1,V_2) = \int\int \partial\mu(M_1,M_2) \tag{VI.2}$$

has important properties in the large $N$–limit for the enumeration of discrete maps on surfaces [101] of arbitrary genus and it is also known to be a tau-function for the 2–Toda hierarchy. In the case of the Witten conjecture, proved by Kontsevich [115] with the use of matrix integrals not too dissimilar from the above one, the enumerative properties of the tau function imply some nonlinear (hierarchy of) PDEs (the KdV hierarchy for the mentioned example). On a similar level, one expects some hierarchy of PDEs for the case of the two-matrix model and possibly some Painlevé property (namely the absence of movable essential singularities). The Painlevé property is characteristic of tau-functions for isomonodromic families of ODEs that depend on parameters; hence a way of establishing such property for the partition function $\mathscr{Z}_N$ is that of identifying it with an instance of isomonodromic tau function [38, 39].

This is precisely the purpose of this article; we capitalize on previous work that showed how to relate the matrix model to certain biorthogonal polynomials [160, 193] and how these appear in a natural fashion as the solution of certain isomonodromic family [37] .

The paper extends to the case of the two matrix model the work contained in [37, 184, 186]; it uses, however, a different approach, closer to the recent [183].

In [37, 184, 186, 190] the partition function of the one–matrix model (and certain shifted Töplitz determinants) were identified as isomonodromic tau functions by using *spectral residue formulæ* in terms of the spectral curve of the differential equatiоn. Such spectral curve has interesting properties inasmuch as –in the one-matrix case– the spectral invariants can be related to the expectation values of the matrix model. Recently the spectral curve of the two matrix model [32] has been written explicitly in terms of expectation values of the two–matrix model and hence one could use their result and follow a similar path for the proof as the one followed in [186]. Whichever one of the two approaches one chooses, a main obstacle is that the definition of isomonodromic tau function [38, 39] relies on a genericity assumption for the ODE which fails in the case at hand, thus requiring a generalization in the definition.

According to this logic, one of the purposes of this paper is to extend the notion of tau-function introduced by Jimbo-Miwa-Ueno's [38], to the two-matrix Itzykson-Zuber model. This task is accomplished in a rather general framework in Sec. VI.

We then show that the partition function has a very precise relationship with the tau-

function so introduced, allowing us to (essentially) identify it as an isomonodromic tau function (Thm. VI.8).

## 2 A Riemann Hilbert formulation of the two-matrix model

According to the seminal work [160, 193] and following the notations and definitions introduced in [34, 35], we consider paired sequences of monic polynomials $\{\pi_m(x), \sigma_m(y)\}_{m=0\ldots\infty}$ ($m = \deg \pi_m = \deg \sigma_m$), that are biorthogonal in the sense that

$$\iint_{\varkappa} dxdy \pi_m(x)\sigma_n(y)e^{-V_1(x)-V_2(y)+xy} = h_m\delta_{mn}, \quad h_m \neq 0. \tag{VI.3}$$

The functions $V_1(x), V_2(y)$ appearing here are referred to as *potentials*, terminology drawn from random matrix theory, in which such quantities play a fundamental role.

Henceforth, the second potential $V_2(y)$ will be chosen as a polynomial of degree $d_2+1$

$$V_2(y) = \sum_{j=1}^{d_2+1} \frac{v_j}{j} y^j, \quad v_{d_2+1} \neq 0 \tag{VI.4}$$

For the purposes of most of the considerations to follow, the first potential $V_1(x)$ may have very general analyticity properties as long as the manipulations make sense, but for definiteness and clarity we choose it to be polynomial as well.

The symbol $\int\int_{\varkappa}$ stands for any linear combination of integrals of the form

$$\iint_{\varkappa} dxdy := \sum_j \sum_k \varkappa_{jk} \int_{\Gamma_j} dx \int_{\hat{\Gamma}_k} dy, \qquad \varkappa_{ij} \in \mathbb{C} \tag{VI.5}$$

where the contours $\{\hat{\Gamma}_k\}_{k=1\ldots d_2}$ will be chosen as follows. In the $y$–plane, define $d_2+1$ "wedge sectors" $\{\hat{S}_k\}_{k=0\ldots d_2}$ such that $\hat{S}_k$ is bounded by the pairs of rays: $r_k := \{y | \arg y = \theta + \frac{2k\pi}{d_2+1}\}$ and $r_{k-1} := \{y | \arg y = \theta + \frac{2(k-1)\pi}{d_2+1}\}$, where $\theta := \arg v_{d_2+1}$. Then $\hat{\Gamma}_k$ is any smooth oriented contour within the sector $\hat{S}_k$ starting from $\infty$ asymptotic to the ray $r_k$ (or any ray within the sector that is at an angle $< \frac{\pi}{2(2d_2+1)}$ to it, which is equivalent for purposes of integration), and returning to $\infty$ asymptotically along $r_{k-1}$ (or at an angle $< \frac{\pi}{2(2d_2+1)}$ to it). These will be referred to as the "wedge contours". We also define a set of smooth oriented contours $\{\check{\Gamma}_k\}_{k=1,\ldots d_2}$, that have intersection matrix $\check{\Gamma}_j \cap \hat{\Gamma}_k = \delta_{jk}$ with the $\hat{\Gamma}_k$'s, such that $\check{\Gamma}_k$ starts from $\infty$ in sector $\hat{S}_0$, asymptotic to the ray $\check{r}_0 := \{y | \arg(y) = \theta - \frac{\pi}{d_2+1}$ and returns to $\infty$ in sector $\hat{S}_k$ asymptotically along the ray $\check{r}_k := \{y | \arg(y) = \theta + \frac{2(k-\frac{1}{2})}{d_2+1}$. These will be called the "anti-wedge" contours. (See Fig. 1.) The choice of

these contours is determined by the requirement that all moment integrals of the form

$$\int_{\hat{\Gamma}_k} y^j e^{-V_2(y)+xy} dy, \quad \int_{\check{\Gamma}_k} y^k e^{V_2(y)-xy} dy, \quad k = 1, \dots d_2, \quad j \in \mathbb{N} \tag{VI.6}$$

be uniformly convergent in $x \in \mathbb{C}$. In the case when the other potential $V_1(x)$ is also a polynomial, of degree $d_1 + 1$, the contours $\{\Gamma_k\}_{k=1,\dots d_1}$ in the $x$–plane may be defined similarly.

The "partition function" is defined here to be the multiple integral

$$\mathscr{Z}_N := \frac{1}{N!} \iint_{\varkappa^N} \prod_{j=1}^{N} \partial x_j \partial y_j \Delta(X) \Delta(Y) \prod_{j=1}^{N} \mathrm{e}^{-V_1(x_j)-V_2(y_j)+x_j y_j} \tag{VI.7}$$

where $\Delta(X)$ and $\Delta(Y)$ denote the usual Vandermonde determinants and the factor $\frac{1}{N!}$ is chosen for convenience.

Such multiple integral can also be represented as the following determinant

$$\mathscr{Z}_N = \det[\mu_{ij}]_{0 \le i,j \le N-1} \ , \ \ \mu_{ij} := \int_{\varkappa} x^i y^j \mathrm{e}^{-V_1(x)-V_2(y)+xy} \partial x \partial y \tag{VI.8}$$

The denomination of "partition function" comes from the fact [37, 160, 193] that when $\varkappa$ coincides with $\mathbb{R} \times \mathbb{R}$ then $\mathscr{Z}_N$ coincides (up to a normalization for the volume of the unitary group) with the following matrix integral

$$\iint \partial M_1 \partial M_2 \mathrm{e}^{-\mathrm{tr}(V_1(M_1)+V_2(M_2)-M_1 M_2)} \tag{VI.9}$$

extended over the space of Hermitean matrices $M_1, M_2$ of size $N \times N$, namely the normalization factor for the measure $\partial\mu(M_1, M_2)$ introduced in VI.1.

### 2.1 Riemann–Hilbert characterization for the orthogonal polynomials

A Riemann–Hilbert characterization of the biorthogonal polynomials is a crucial step towards implementing a steepest–descent analysis. In our context it is also crucial in order to tie the random matrix side to the theory of isomonodromic deformations.

We first recall the approach given by Kuijlaars and McLaughin (referred to as KM in the rest of the article) in [36], suitably extended and adapted (in a rather trivial way) to the setting and notation of the present work. We quote -paraphrasing and with a minor generalization- their theorem, without proof.

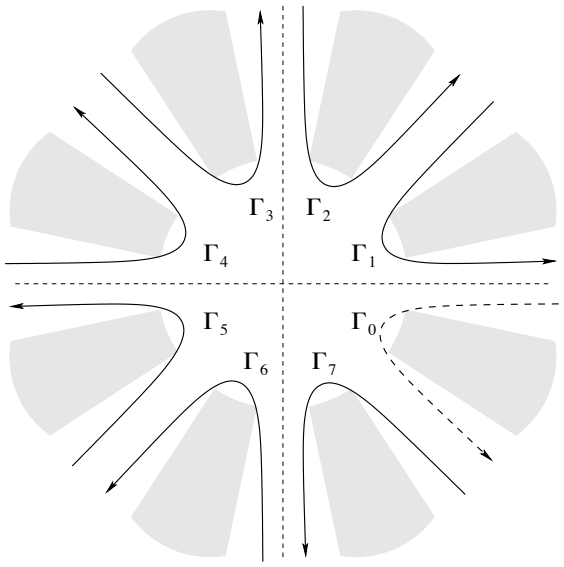


Figure VI.1: Wedge and anti-wedge contours for $V_2(y)$ of degree $D_2+1$

**Theorem VI.1** (Kuijlaars and McLaughin asymptotic)**.** *The monic bi-orthogonal polynomial $\pi_n(x)$ is the $(1,1)$ entry of the solution $\Gamma(x)$ (if it exists) of the following Riemann-Hilbert problem for $\Gamma(x)$.*

1. *The matrix $\Gamma(x)$ is piecewise analytic in $\mathbb{C}\setminus\bigsqcup\Gamma_j$;*

2. *the (non-tangential) boundary values of $\Gamma(x)$ satisfy the relations*

$$\Gamma(x)_+ = \Gamma(x)_- \begin{bmatrix} 1 & w_{j,1} & \dots & w_{j,d_2} \\ & 1 & 0 & 0 \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \quad x \in \Gamma_j \tag{VI.10}$$

$$w_{j,\nu} = w_{j,\nu}(x) := \mathrm{e}^{-V_1(x)} \sum_{k=1}^{d_2} \varkappa_{jk} \int_{\hat{\Gamma}_k} y^{\nu-1} \mathrm{e}^{-V_2(y)+xy} \partial y \tag{VI.11}$$

*3. as $x \to \infty$ we have the following asymptotic expansion*

$$\Gamma(x) \sim \left( I_d + \frac{Y_{N,1}}{x} + \mathscr{O}\left(\frac{1}{x^2}\right)\right) \begin{pmatrix} x^N & 0 & 0 \\ 0 & x^{-m_N-1} Id_{r_N} & 0 \\ 0 & 0 & x^{-m_N} Id_{d_2-r_N} \end{pmatrix} \tag{VI.12}$$

*where we have defined the integers $m_N, r_N$ as follows*

$$N = m_N d_2 + r_N, \quad m_N, r_N \in \mathbb{N}, \quad 0 \leq r_n \leq d_2 - 1 \tag{VI.13}$$

It follows from [36] that the solution $\Gamma_N(x)$ has the following form

$$\Gamma_N(x) := \Gamma(x) := \begin{bmatrix} \pi_N(x) & \mathscr{C}_0(\pi_N) & \dots & \mathscr{C}_{d_2-1}(\pi_N) \\ p_{N-1}(x) & \mathscr{C}_0(p_{N-1}) & \dots & \mathscr{C}_{d_2-1}(p_{N-1}) \\ \vdots & & & \vdots \\ p_{N-d_2}(x) & \mathscr{C}_0(p_{N-d_2}) & \dots & \mathscr{C}_{d_2-1}(p_{N-d_2}) \end{bmatrix}, \tag{VI.14}$$

$$\mathscr{C}_i(f(z)) := \frac{1}{2\pi i} \iint_{\mathscr{H}} \frac{f(x)}{x-z} y^i e^{-V_1(x)-V_2(y)+xy} dy dx \tag{VI.15}$$

where the polynomials denoted above by $p_{N-1}, \dots, p_{N-d_2}$ are some polynomials of degree not exceeding $N-1$, whose detailed properties are largely irrelevant for our discussion; we refer to [36] for these details.

By a left multiplication of this solution by a suitable constant matrix we can see that the matrix

$$\widehat{\Gamma}_N := \begin{bmatrix} \pi_n & \mathscr{C}_0(\pi_n) & \dots & \mathscr{C}_{d_2-1}(\pi_n) \\ \pi_{n-1} & \mathscr{C}_0(\pi_{n-1}) & \dots & \mathscr{C}_{d_2-1}(\pi_{n-1}) \\ \vdots & & & \vdots \\ \pi_{n-d_2} & \mathscr{C}_0(\pi_{n-d_2}) & \dots & \mathscr{C}_{d_2-1}(\pi_{n-d_2}) \end{bmatrix} \tag{VI.16}$$

and $\Gamma_N$ are related as

$$\widehat{\Gamma}_N(x) = U_N \Gamma_N(x) \tag{VI.17}$$

where $U_N$ is a constant matrix (depending on $N$ and on the coefficients of the polynomials but not on $x$). As an immediate consequence, $\widehat{\Gamma}_N$ solves the same RHP as $\Gamma$ with the exception of the normalization at infinity (VI.12).

The present RHP is not immediately suitable to make the connection to the theory

of isomonodromic deformations as described in [38, 39]; we recall that this is the theory that describes the deformations of an ODE in the complex plane which leave the Stokes' matrices (i.e. the so–called *extended monodromy data*) invariant. The solution $\Gamma_N$ (or $\widehat{\Gamma}_N$) does not solve any ODE as formulated, because the jumps on the contours are non constant. If -however- we can relate $\Gamma_N$ with some other RHP with constant jumps, then its solution can be immediately shown to satisfy a polynomial ODE, which allows us to use the machinery of [38, 39]. This is the purpose of the next section.

### 2.2 A RHP with constant jumps

In [37] the biorthogonal polynomials were characterised in terms of an ODE or – which is the same– of a RHP with constant jumps. In order to connect the two formulations we will use some results and we start by defining some auxiliary quantities: for $1 \leq k \leq d_2$, define the $d_2$ sequences of functions $\{\psi_m^{(k)}(x)\}_{m\in\mathbb{N}}$ as follows:

$$\psi_m^{(k)}(x) := \frac{1}{2\pi i}\int_{\check{\Gamma}_k} ds \iint_{\mathcal{H}} dzdw \frac{\pi_m(z)e^{-V_1(z)}}{x-z}\frac{V_2'(s)-V_2'(w)}{s-w}e^{-V_2(w)+V_2(s)+zw-xs}, \quad 1\leq k\leq d_2, \tag{VI.18}$$

and let

$$\psi_m^{(0)}(x) := \pi_m(x)e^{-V_1(x)}. \tag{VI.19}$$

In terms of these define, for $N \geq d_2$, the sequence of $(d_2+1)\times(d_2+1)$ matrix valued functions $\underset{N}{\widehat{\Psi}}(x)$

$$\underset{N}{\widehat{\Psi}}(x) := \begin{bmatrix} \psi_N^{(0)}(x) & \dots & \psi_N^{(d_2)}(x) \\ \vdots & & \vdots \\ \psi_{N-d_2}^{(0)}(x) & \dots & \psi_{N-d_2}^{(d_2)}(x) \end{bmatrix} \tag{VI.20}$$

The following theorem is easily established using the properties of the bilinear concomitant and it is a very special case of the setting of [172] (Cf. Appendix VI for a self-contained re-derivation)

**Theorem VI.2** (Jump discontinuities in $\underset{N}{\widehat{\Psi}}$)**.** *The limits $\underset{N}{\widehat{\Psi}}_\pm$ when approaching the contours $\Gamma_j$ from the left $(+)$ and right$(-)$ are related by the following jump discontinuity conditions*

$$\underset{N}{\widehat{\Psi}}_+(x) = \underset{N}{\widehat{\Psi}}_-(x)\mathbf{H}^{(j)} \tag{VI.21}$$

$$\tag{VI.22}$$

*where*

$$\mathbf{H}^{(j)} := \mathbf{I} - 2\pi i \mathbf{e}_0 \kappa^T$$
$$\hat{\mathbf{H}}^{(j)} = (\mathbf{H}^{(j)})^{-1} = \mathbf{I} + 2\pi i \mathbf{e}_0 \kappa^T \tag{VI.23}$$

$$\mathbf{e}_0 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \kappa := \begin{pmatrix} 0 \\ \varkappa_{j1} \\ \vdots \\ \varkappa_{jd_2} \end{pmatrix} \tag{VI.24}$$

The proof of this theorem is given in Appendix VI. For later convenience we define also

$$\underset{N}{\Psi} := U_N^{-1} \widehat{\underset{N}{\Psi}} \tag{VI.25}$$

The relationship with the matrices $\Gamma_N$, $\widehat{\Gamma}_N$ introduced in the previous section is detailed in the following

**Theorem VI.3** (Factorization theorem)**.** *The following identities hold*

$$\widehat{\underset{N}{\Psi}}(x) = \widehat{\Gamma}_N(x) V(x) W(x) \ , \ \underset{N}{\Psi}(x) = \Gamma_N(x) V(x) W(x) \tag{VI.26}$$

*where*

$$V := \begin{pmatrix} \mathrm{e}^{-V_1(x)} & 0 \\ 0 & V_0, \end{pmatrix} \ , \qquad W(x) := \begin{pmatrix} 1 & 0 \\ 0 & W_0(x) \end{pmatrix} \tag{VI.27}$$

*and $V_0$, $W_0(x)$ are the $d_2 \times d_2$ matrices with elements*

$$(V_0)_{jk} = \begin{bmatrix} v_2 & v_3 & \dots & & v_{d_2+1} \\ v_3 & & & v_{d_2+1} & \\ & & \iddots & & \\ v_{d_2} & v_{d_2+1} & & & \\ v_{d_2+1} & & & & \end{bmatrix} = \tag{VI.28}$$

$$= \begin{matrix} v_{j+k} & \text{if } j+k \le d_2+1 \\ 0 & \text{if } j+k > d_2+1, \end{matrix} \tag{VI.29}$$

$$(W_0(x))_{jk} = \int_{\check{\Gamma}_k} y^{j-1} e^{V_2(y)-xy} dy, \quad 1 \le j,k \le d_2 \tag{VI.30}$$

The proof is a direct verification by multiplication by matrices, noticing that the matrix $V_0$ is nothing but the matrix representation of $\frac{V_2'(y)-V_2'(s)}{y-s}$ as a quadratic form in the bases $1, y, y^2, \dots, y^{d_2-1}$ and $1, s, s^2, \dots, s^{d_2-1}$ (more details are to be found on appendix VI, based on [172]) The RHP for $\underset{N}{\Psi}$ can be read off from that of $\Gamma_N$ and the fact that the jumps are constants. For convenience we collect the information in the following

**Theorem VI.4.** *The matrix $\underset{N}{\Psi}$ is the unique solution of the following RHP:*

1. *Constant Jumps:*

$$\underset{N}{\Psi}_+(x) = \underset{N}{\Psi}_-(x)\mathbf{H}^{(j)} \tag{VI.31}$$

$$\tag{VI.32}$$

2. *Asymptotic at infinity:*

$$\underset{N}{\Psi}(x) \sim \Gamma_N \begin{pmatrix} x^N e^{-V_1(x)} & 0 & 0 \\ 0 & x^{-m_N-1} Id_{r_N} & 0 \\ 0 & 0 & x^{-m_N} Id_{d_2-r_N} \end{pmatrix} \Psi_0(x) \tag{VI.33}$$

*where*

$$\Gamma_N = Id + \frac{Y_{N,1}}{x} + \dots \tag{VI.34}$$

*and where $\Psi_0(x) := V(x)W(x)$ will be referred to as the* bare *solution. Its asymptotic at infinity can be computed by steepest descent, but since it is N–independent,*

*for the sake of brevity, we do not report on it (details are contained in [35]).*

3. $\Psi_N$ *has constant jumps*

4. $\Psi'_N(x)\Psi_N^{-1} = D_N(x)$ *where* $D_N(x)$ *is a polynomial in* $x$

5. $\partial_{u_K}\Psi_N(x)\Psi_N^{-1} = U_{K,N}(x)$ *is polynomial in* $x$.

6. $\partial_{v_J}\Psi_N(x)\Psi_N^{-1} = V_{J,N}(x)$ *is polynomial in* $x$.

7. $\det(\Psi_{N+1}\Psi_N^{-1}) = Cste$

The points (4,5,6,7) in the above theorem can be found in [32, 35]

In the next section we shall define a proper notion of isomonodromic tau function: it should be pointed out that the definition of [38, 39] cannot be applied as such because –as showed in [35]– the ODE that the matrix $\underset{N}{\Psi}$ (or $\underset{N}{\widehat{\Psi}}$) solves, has a highly degenerate leading coefficient at the singularity at infinity.

In the list, the crucial ingredients are the differential equations (in $x$ or relatively to the parameters $u_K$ and $v_J$). First, the fact that $D_N(x)$ is a polynomial comes from explicit computation (See [32] for example). The result concerning the determinant of $R_N(x)$ can also be found in [32] where one has: $\det(\Psi_{N+1}\Psi_N^{-1}) = \det(a_N(x)) = Cste$. The properties concerning the differential equations relatively to parameters can be found in [32] too. Under all these assumptions, we will show that the proof of Jimbo-Miwa-Ueno can be adapted and that we can define a suitable $\tau$-function in the same way Jimbo-Miwa-Ueno did it.

## 3 Definition of the $\tau$-function

In this section, we will place ourselves in a more general context than the one described above; we will show that under few assumptions one can define a good notion of tau-function.

More generally we will denote with $t_a$ the isomonodromic parameters (in our case they are the $u_K$'s and ths $v_J$'s) and a subscript $a$ or $b$ is understood as a derivation relatively to $t_a$ or $t_b$. For a function $f$ of the isomonodromic times we will denote by the usual symbol its differential

$$\partial f = \sum_a \partial_{t_a} f \partial t_a = \sum_a f_a \partial t_a \tag{VI.35}$$

Our setup falls in the following framework that it is useful to ascertain from the specifics of the case at hands. Suppose we are given a matrix

$$\Psi(x) \sim Y(x)\,\Xi(x)\ ,\ \ Y(x) := \left(\mathbf{1} + \frac{Y_1}{x} + \frac{Y_2}{x^2} + \dots\right) x^S \tag{VI.36}$$

where $\Xi(x) = \Xi(x;\mathbf{t})$ is some explicit expression (the "bare" isomonodromic solution) and $S$ is a matrix independent of the isomonodromic times. This implies that if we define the one–form-valued matrix $\mathscr{H}(x;\mathbf{t})$ by

$$\mathscr{H}(x;\mathbf{t}) = \partial\Xi(x;\mathbf{t})\,\Xi(x;\mathbf{t})^{-1} \tag{VI.37}$$

then $\mathscr{H}(x) = \sum \mathscr{H}_a \partial t_a$ (we suppress explicit mention of the $\mathbf{t}$ dependence henceforth) is some solution of the zero-curvature equations:

$$\partial_a \mathscr{H}_b - \partial_b \mathscr{H}_a = [\mathscr{H}_a, \mathscr{H}_b] \tag{VI.38}$$

We will **assume** (which is the case in our setting) that all $\mathscr{H}_a$ are **polynomials** in $x$. We will also use that the dressed deformations $\Omega_a$ given by $\Psi_a = \Omega_a \Psi$ are polynomials. Moreover, according to the asymptotic they are given by:

$$\Omega_a = (Y \mathscr{H}_a Y^{-1})_{pol}. \tag{VI.39}$$

In this very general (and generic) setting we can formulate the definition of a "tau function" as follows

**Definition VI.1.** *The tau-differential is the one-form defined by*

$$\omega := \sum \omega_a \partial t^a := \sum_a \operatorname{res}\operatorname{tr}\left(Y^{-1}Y'\mathscr{H}_a\right)\partial t^a \tag{VI.40}$$

The main point of the matter is that -without any further detail- we can now prove that the tau-differential is in fact closed and hence locally defines a function.

**Theorem VI.5.** *The tau-differential is a closed differential and locally defines a $\tau$–function as*

$$\partial \log \tau = \omega \tag{VI.41}$$

**Proof.** We need to prove the closure of the differential. We first recall the main

relations between the bare and dressed deformations

$$\partial_a Y = \Omega_a Y - Y\mathscr{H}_a \ ; \qquad Y\mathscr{H}_a Y^{-1} = \Omega_a - \mathscr{R}_a \ ; \qquad \mathscr{R}_a := \partial_a Y Y^{-1} \tag{VI.42}$$

We note that -by construction- $\Omega_a = (Y\mathscr{H}_a Y^{-1})_{pol}$ is a polynomial while $\mathscr{R}_a = \mathscr{O}(x^{-1})$ *irrespectively of the form of S*. We compute the cross derivatives directly

$$\begin{aligned}
\partial_a \omega_b &= \mathrm{resTr}\Big( -Y^{-1}(\Omega_a Y - Y\mathscr{H}_a) Y^{-1} Y' \mathscr{H}_b + Y^{-1}(\Omega_a Y - Y\mathscr{H}_a)' \mathscr{H}_b + Y^{-1} Y' \partial_a \mathscr{H}_b \Big) \\
&= \mathrm{resTr}\Big( \mathscr{H}_a Y^{-1} Y' \mathscr{H}_b + Y^{-1}\Omega_a' Y \mathscr{H}_b - Y^{-1} Y' \mathscr{H}_a \mathscr{H}_b - \mathscr{H}_a' \mathscr{H}_b + Y^{-1} Y' \partial_a H_b \Big) \\
&= \mathrm{resTr}\Big( Y^{-1} Y' ([\mathscr{H}_b, \mathscr{H}_a] + \partial_a \mathscr{H}_b) + Y^{-1}\Omega_a' Y \mathscr{H}_b - \overbrace{\mathscr{H}_a' \mathscr{H}_b}^{\text{polynomial}} \Big) \\
&= \mathrm{resTr}\Big( Y^{-1} Y' ([\mathscr{H}_b, \mathscr{H}_a] + \partial_a \mathscr{H}_b) - \Omega_a' \mathscr{R}_b \Big)
\end{aligned} \tag{VI.43}$$

where, in the last step, we have used that $Y\mathscr{H}_b Y^{-1} = \Omega_b - \mathscr{R}_b$ and that the contribution coming from $\Omega_b$ vanishes since it is a polynomial. Rewriting the same with $a \leftrightarrow b$ and subtracting we obtain

$$\begin{aligned}
&\partial_a \omega_b - \partial_b \omega_a = \mathrm{resTr}\Big( 2Y^{-1} Y' [\mathscr{H}_b, \mathscr{H}_a] - \Omega_a' \mathscr{R}_b + \Omega_b' \mathscr{R}_a + Y^{-1} Y' (\partial_a \mathscr{H}_b - \partial_b \mathscr{H}_a) \Big) \\
&= \mathrm{resTr}\Big( Y^{-1} Y' [\mathscr{H}_b, \mathscr{H}_a] - \Omega_a' \mathscr{R}_b + \Omega_b' \mathscr{R}_a + Y^{-1} Y' \big( \overbrace{\partial_a \mathscr{H}_b - \partial_b \mathscr{H}_a + [\mathscr{H}_b, \mathscr{H}_a]}^{=0 \text{ by the ZCC VI.38}} \big) \Big) \\
&= \mathrm{resTr}\Big( Y^{-1} Y' [\mathscr{H}_b, \mathscr{H}_a] - \Omega_a' \mathscr{R}_b + \Omega_b' \mathscr{R}_a \Big)
\end{aligned} \tag{VI.44}$$

Note that, up to this point, we only used the zero curvature equations for the connection $\nabla = \sum(\partial_a - \mathscr{H}_a)\partial t^a$ and the fact that $\mathscr{H}_a$ are polynomials in $x$. We thus need to prove that the last quantity in (VI.44) vanishes: this follows from the following computation, which uses once more the fact that $\mathscr{H}_a$ and $\Omega_a$ are all polynomials. Indeed, we have $\mathrm{resTr}(\mathscr{H}_a' \mathscr{H}_b) = 0$ and hence (using (VI.42))

$$\begin{aligned}
0 = \mathrm{res\,tr}(\mathscr{H}_a' \mathscr{H}_b) &= \mathrm{resTr}\left( \left(Y\mathscr{H}_a Y^{-1}\right)' Y \mathscr{H}_b Y^{-1} \right) - \mathrm{resTr}\left( Y' \mathscr{H}_a \mathscr{H}_b Y^{-1} \right) + \mathrm{resTr}\left( \mathscr{H}_a Y^{-1} Y' \mathscr{H}_b \right) \\
&= \mathrm{resTr}\left( (\Omega_a - \mathscr{R}_a)' (\Omega_b - \mathscr{R}_b) \right) + \mathrm{resTr}\left( Y^{-1} Y' [\mathscr{H}_b, \mathscr{H}_a] \right)
\end{aligned}$$

$$= \mathrm{resTr}\left( \overbrace{\Omega'_a\Omega_b}^{\text{poly}} - \mathscr{R}'_a\Omega_b - \Omega'_a\mathscr{R}_b + \overbrace{\mathscr{R}'_a\mathscr{R}_b}^{=\mathscr{O}(x^{-2})} + Y^{-1}Y'[\mathscr{H}_b,\mathscr{H}_a] \right)$$
$$= \mathrm{resTr}\left( -\mathscr{R}'_a\Omega_b - \Omega'_a\mathscr{R}_b + Y^{-1}Y'[\mathscr{H}_b,\mathscr{H}_a] \right) = 0 \tag{VI.45}$$

Using integration by parts (and cyclicity of the trace) on the first term here above, we obtain precisely the last quantity in (VI.44). The Theorem is proved. **Q.E.D.**

### 3.1 Application to our problem

We now apply the general definition above to our setting, with the identifications $\Psi = \Psi_N$, $Y = \Gamma_N$ (as a formal power series at $\infty$) and $\Xi = \Psi_0$. We will write $Y_N$ instead of $\Gamma_N$ in the expressions below to emphasize that we consider its asymptotic expansion at $\infty$ This reduces the definition of the tau function to the one below

**Definition VI.2.** *The $\tau$-function is defined by the following PDE*

$$d(\log\tau_N) = \operatorname*{Res}_{x\to\infty} \mathrm{Tr}\left(Y_N^{-1}Y_N'\partial(\Psi_0)\Psi_0^{-1}\right) \tag{VI.46}$$

*where $Y_N$ is the formal asymptotic expansion of $\Gamma_N$ at infinity*

$$Y_N = \widetilde{Y}_N \begin{pmatrix} x^N & 0 & 0 \\ 0 & x^{-m_N-1}Id_{r_N} & 0 \\ 0 & 0 & x^{-m_N}Id_{d_2-r_N} \end{pmatrix} \tag{VI.47}$$

**Remark VI.1.** *The matrix S of the previous section in our case becomes:*

$$S = \begin{pmatrix} N & 0 & 0 \\ 0 & (-m_N-1)\,Id_{r_N} & 0 \\ 0 & 0 & -m_N\,Id_{d_2-r_N} \end{pmatrix} \tag{VI.48}$$

*The partial derivatives of $\ln\tau_N$ split into two sets which have different form:*

$$\partial_{u_K}\log\tau_N = -\operatorname*{Res}_{x\to\infty} \mathrm{Tr}\left(Y_N^{-1}Y_N'\frac{x^K}{K}\mathbf{E_{11}}\right) \tag{VI.49}$$

$$\partial_{v_J} \log \tau_N \;=\; \operatorname*{Res}_{x\to\infty} \operatorname{Tr}\left(Y_N^{-1} Y_N' \partial_{v_J}(\Psi_0)\Psi_0^{-1}\right) \tag{VI.50}$$

*where in the last equation the term $\partial_{v_J}(\Psi_0)\Psi_0^{-1}$ has non-zero entries only in the anti-principal minor of size $d_2$.*

One can notice that the situation we are looking at is a generalization of what happen in the one-matrix case. In the 1-matrix model, the matrix $S$ is zero and therefore $Y_N$ are (formal) Laurent series. The matrix $\Psi_0$ matrix is absent in that case since there is only one potential and thus one recovers the usual definition of isomonodromic tau function (see [186]). Note also that in the derivation with respect to $v_J$ we have obtained the second equality using the block diagonal structure of $\Psi_0$ (first row/column does not play a role). It is remarkable that the two systems are completely decoupled, i.e. that in the first one the matrix $\Psi_0$ (containing all the dependance in $V_2$) disappears and that in the second one the matrix $A_0$ (containing the potential $V_1$) also disappears.

### 3.2 Discrete Schlesinger transformation: Tau-function quotient

In this section we investigate the relationship between the tau-function of Def. VI.2 and the partition function $\mathscr{Z}_N$ of the matrix model.

We anticipate that the two object turn out to be the same (up to a nonzero factor that will be explicitly computed, Thm. VI.8): the proof relies on two steps, the first of which we prepare in this section. These are

- proving that they satisy the same recurrence relation
- identifying the initial conditions for the recurrence relation.

We start by investigating the relationship between $\tau_N$ and $\tau_{N+1}$; this analysis is essentially identical to the theory developed in [39] and used in [183], but we report it here for the convenience of the reader.

From the fact that the $\Psi_N$ has constant jumps, we deduce that $\Psi_{N+1}\Psi_N^{-1}$ is an entire function. Moreover asymptotically it looks like:

$$\begin{aligned}\Psi_{N+1}\Psi_N^{-1} \;=\; \tilde{Y}_{N+1} &\begin{pmatrix} x^{N+1}e^{-V_1(x)} & 0 & 0 \\ 0 & x^{-m_{N+1}-1}Id_{r_{N+1}} & 0 \\ 0 & 0 & x^{-m_{N+1}}Id_{d_2-r_{N+1}} \end{pmatrix} \Psi_0(x) \\ &\Psi_0(x)^{-1} \begin{pmatrix} x^{-N}e^{V_1(x)} & 0 & 0 \\ 0 & x^{m_N+1}Id_{r_N} & 0 \\ 0 & 0 & x^{m_N}Id_{d_2-r_N} \end{pmatrix} \tilde{Y}_N \end{aligned} \tag{VI.51}$$

$$\Psi_{N+1}\Psi_N^{-1} = \tilde{Y}_{N+1} \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & Id_{r_N-1} & 0 & 0 \\ 0 & 0 & x^{-1} & 0 \\ 0 & 0 & 0 & Id_{d_2-1-r_N} \end{pmatrix} \tilde{Y}_N \tag{VI.52}$$

Thus, remembering that $\tilde{Y}_N$ is a series $x^{-1}$, Liouville's theorem states that $\Psi_{N+1}\Psi_N^{-1}$ is a polynomial of degree one, and hence, for some constant matrices $R_N^0, R_N^1$ we must have

$$\Psi_{N+1}\Psi_N^{-1} = R_N(x) = R_N^0 + xR_N^1 \tag{VI.53}$$

From the fact that $\det(R_N)$ does not depend on $x$ (last property Thm. VI.3), we know that $R_N^{-1}(x)$ is a polynomial of degree at most one as well (this is easy if one consider the expression of the inverse of a matrix using the co-matrix).

Comparing the asymptotics of $\Psi_{N+1}$ and $R_N(x)\Psi_N$ term-by-term in the expansion in inverse powers of $x$ and after some elementary algebra one obtains ([38] Appendix A):

$$R_N(x) = E_{\alpha_0}x + R_{N,0} \qquad \text{and} \qquad R_N^{-1}(x) = E_1 x + R_{N,0}^{-1} \tag{VI.54}$$

Here we have introduced the notation $\alpha_0 = r_N + 1$ which corresponds to the index of the column where the coefficient $x^{-1}$ is to be found in the asymptotic of $\Psi_{N+1}\Psi_N^{-1}$. This notation is the standard notation used originally by Jimbo-Miwa in a Schlesinger transformation. The matrix $(R_{N,0})_{\alpha,\beta}$ is given by:

$$\begin{array}{lccc} & \beta = \alpha_0 & \beta = 1 & \beta \neq \alpha_0, 1 \\ \alpha = \alpha_0 & \frac{-(Y_{N,2})_{\alpha_0,1} + \sum_{\gamma \neq \alpha_0} (Y_{N,1})_{\alpha_0,\gamma}(Y_{N,1})_{\gamma,1}}{(Y_{N,1})_{\alpha_0,1}} & -(Y_{N,1})_{\alpha_0,1} & -(Y_{N,1})_{\alpha_0,\beta} \\ \alpha = 1 & \frac{1}{(Y_{N,1})_{\alpha_0,1}} & 0 & 0 \\ \alpha \neq \alpha_0, 1 & -\frac{(Y_{N,1})_{\alpha,1}}{(Y_{N,1})_{\alpha_0,1}} & 0 & \delta_{\alpha,\beta} \end{array} \tag{VI.55}$$

and $(R^{-1}_{N,0})_{\alpha,\beta}$ is given by:

$$\begin{array}{ccccc} & \beta=\alpha_0 & \beta=1 & \beta\neq\alpha_0,1 \\ \alpha=\alpha_0 & 0 & (Y_{N,1})_{\alpha_0,1} & 0 \\ \alpha=1 & -\frac{1}{(Y_{N,1})_{\alpha_0,1}} & -\frac{-(Y_{N,2})_{\alpha_0,1}}{(Y_{N,1})_{\alpha_0,1}}+(Y_{N,1})_{1,1} & -\frac{(Y_{N,1})_{\alpha_0,\beta}}{(Y_{N,1})_{\alpha_0,1}} \\ \alpha\neq\alpha_0,1 & 0 & (Y_{N,1})_{\alpha,1} & \delta_{\alpha,\beta} \end{array} \tag{VI.56}$$

While the formulae above might seem complicated, we will use the two important observations:

$$E_{\alpha_0}R^{-1}_{N,0}+R_{N,0}E_1=R^{-1}_{N,0}E_{\alpha_0}+E_1R_{N,0}=0 \tag{VI.57}$$

$$R^{-1}_N(x)R'_N(x)=R^{-1}_{N,0}E_{\alpha_0} \text{ does not depend on } x.$$

The recurrence relation satisfied by the sequence $\{\tau_N\}$ is derived in the next theorem.

**Theorem VI.6.** *Up to multiplication by functions that do not depend on the isomonodromic parameters (i.e. independent of the potentials $V_1, V_2$) the following identity holds*

$$\frac{\tau_{N+1}}{\tau_N}=(Y_1)_{1,\alpha_0} \tag{VI.58}$$

**Proof** The proof follows [39] but we report it here for convenience of the reader. Consider the following identity

$$\Psi_{N+1}=Y_{N+1}\Psi_0=R_NY_N\Psi_0 \tag{VI.59}$$

This implies that

$$Y_{N+1}=R_NY_N \tag{VI.60}$$

Taking the derivative with respect to $x$ gives:

$$Y^{-1}_{N+1}Y'_{N+1}=Y^{-1}_NR^{-1}_NR'_NY_N+Y^{-1}_NY'_N \tag{VI.61}$$

Therefore we have:

$$\partial\log\tau_{N+1}-\partial\log\tau_N \;=\; \operatorname*{Res}_{x\to\infty}\operatorname{Tr}((Y^{-1}_NR^{-1}_NR'_NY_N+Y^{-1}_NY'_N-Y^{-1}_NY_N)\partial(\Psi_0)\Psi^{-1}_0)$$

$$= \operatorname*{Res}_{x\to\infty} \mathrm{Tr}(Y_N^{-1}R_N^{-1}R_N'Y_N\partial(\Psi_0)\Psi_0^{-1}) \tag{VI.62}$$

We now need to "transfer" the exterior derivative from $\Psi_0$ to $Y_N$. This can be done using that $\underset{N}{\Psi} = Y_N\Psi_0$, so that

$$\partial\underset{N}{\Psi} = \partial(Y_N)\Psi_0 + Y_N\partial(\Psi_0)$$

Equivalently:

$$Y_N\partial\Psi_0\Psi_0^{-1}Y_N^{-1} = d(\underset{N}{\Psi})\underset{N}{\Psi}^{-1} - dY_NY_N^{-1} \tag{VI.63}$$

Inserting these identities in the tau quotient we obtain the relation

$$d\log\tau_{N+1} - d\log\tau_N = \operatorname*{Res}_{x\to\infty}\mathrm{Tr}\left(R_N^{-1}R_N'd(\underset{N}{\Psi})\underset{N}{\Psi}^{-1} - R_N^{-1}R_N'dY_NY_N^{-1}\right) \tag{VI.64}$$

The first term is residueless at $\infty$ since $\partial\underset{N}{\Psi}\underset{N}{\Psi}^{-1}$ is polynomial in $x$ and $R_N^{-1}R_N'$ does not depend on $x$. Therefore we are left only with:

$$d\log\tau_{N+1} - d\log\tau_N = -\operatorname*{Res}_{x\to\infty}\mathrm{Tr}(R_N^{-1}R_N'dY_NY_N^{-1}) \tag{VI.65}$$

A direct matrix computation using the explicit form of $R_N$ yields

$$\partial\log\tau_{N+1} - \partial\log\tau_N = \partial\log((Y_{N,1})_{1,\alpha_0}) \tag{VI.66}$$

and hence

$$\frac{\tau_{N+1}}{\tau_N} = (Y_1)_{1,\alpha_0} \tag{VI.67}$$

The last equality is to be understood up to a multiplicative constant not depending on the parameters $u_K$ and $v_J$ in $\tau$. **Q.E.D.**

In order to complete the first step we need to express the entry $(Y_1)_{1,\alpha_0}$ in terms of the ratio of two consecutive partition functions. This is accomplished in the following section.

**Theorem VI.7.** *For the matrix $\Gamma_N$ the asymptotic expansion at infinity (VI.12) is such that*

$$(Y_{N,1})_{1,\alpha_0} = (v_{d_2+1})^S h_N = (v_{d_2+1})^S\frac{\mathscr{Z}_{N+1}}{\mathscr{Z}_N} \tag{VI.68}$$

*where S and* $\alpha_0 \in \{0,1,\ldots,d_2-1\}$ *are defined by the following relation*

$$N = d_2 S + \alpha_0 - 1 \tag{VI.69}$$

**Proof** In order to compute $(Y_{N,1})_{1,\alpha_0}$ it is sufficient to compute the leading term of the expansion at $\infty$ appearing in the first row of the matrix $\Gamma_N$. Recalling the expression (VI.14), we start by the following direct compuation using integration by parts

$$\begin{aligned}\iint_{\kappa} dzdw\,\pi_N(z) z^i w^{k-1} e^{-V_1(z)-V_2(z)+zw} &= \iint_{\kappa} dzdw\,\pi_N(z) e^{-V_1(z)} w^{k-1} e^{-V_2(w)} \frac{d^i}{dw^i}\left(e^{zw}\right) \\ &= (-1)^i \iint_{\kappa} dzdw\,\pi_N(z) e^{-V_1(z)+zw} \frac{d^i}{dw^i}\left(w^{k-1} e^{-V_2(w)}\right) \\ &= \iint_{\kappa} dzdw\,\pi_N(z) q_{d_2 i+k-1}(w) e^{-V_1(z)-V_2(z)+zw} \end{aligned} \tag{VI.70}$$

where $q_{d_2 i+k-1}(w)$ is a polynomial of the indicated degree whose leading coefficient is $v^i_{d_2+1}$. The last RHS is 0 if $d_2 i+k-1 < N$ because of orthogonality. If $d_2 i+k-1 = N$ the integral gives $v^i_{d_2+1} h_N$ by the normality conditions concerning our biorthogonal set. This computation allows us to expand the Cauchy transform of $(\Gamma_N)_{1,\alpha_0}$ near $\infty$ as follows:

$$\begin{aligned}\mathscr{C}(p_N w_{\alpha_0}(x)) &= \frac{1}{2\pi i} \iint_{\kappa} dzdw \frac{\pi_N(z)}{z-w} w^{\alpha_0-1} e^{-V_1(z)-V_2(z)+zw} \\ &= -\sum_{i=0}^{S-1} \frac{1}{2\pi i} \iint_{\kappa} dzdw \pi_N(z) \frac{z^i}{x^{i+1}} w^{\alpha_0-1} e^{-V_1(z)-V_2(z)+zw} \\ &+ \frac{1}{2\pi i} \frac{1}{x^{S+1}} \iint_{\kappa} dzdw \frac{\pi_N(z)}{x-z} z^S w^{\alpha_0-1} e^{-V_1(z)-V_2(z)+zw} + \mathscr{O}(x^{-S-2}) \end{aligned} \tag{VI.71}$$

By orthogonality the first sum vanishes term-by-term and the leading coefficient of the second term is $v^S_{d_2+1} h_N$. **Q.E.D.**

Recalling that the $\tau$-function is only defined up to a multiplicative constant not depending on $N$ nor on the coefficients $u_k$and $v_j$, we have

$$\frac{\tau_{N+1}}{\tau_N} = (v_{d_2+1})^{S_N} \frac{\mathscr{Z}_{N+1}}{\mathscr{Z}_N} \tag{VI.72}$$

where $N = d_2 S_N + \alpha_0 - 1$ Hence for every $n_0$:

$$\tau_N \mathscr{Z}_{n_0} = \mathscr{Z}_N \tau_{n_0} (v_{d_2+1})^{\sum_{j=n_0}^{N-1} S_j} \tag{VI.73}$$

One would like to take $n_0 = 0$ because it enables explicit computations. As we will prove now there is a way of extending naturally all the reasoning down to 0.

The RHP for $\Gamma_N$ (Thm. VI.1) is perfectly well–defined for $N = 0$ and has solution

$$\Gamma_0 = \begin{pmatrix} 1 & \mathscr{C}_0(1) & \mathscr{C}_1(1) & \dots & \mathscr{C}_{d_2-1}(1) \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \ddots & 1 \end{pmatrix}. \tag{VI.74}$$

Consequently we can take

$$\tau_N \mathscr{Z}_0 = (\nu_{d_2+1})^{\Sigma_{j=0}^{N-1} S_j} \mathscr{Z}_N \tau_0 \tag{VI.75}$$

Also note that $\mathscr{Z}_0 \equiv 1$ (by definition).

We can compute $\tau_0$ directly from Def. VI.2 because of the particularly simple and explicit expression of $\Psi_0 = \Gamma_0 \Psi_0$.

$$\partial \ln \tau_0 = \operatorname{resTr} \left( Y_0^{-1} Y_0' \partial \Psi_0 \Psi^{-1} \right) \tag{VI.76}$$

We claim that this expression is identically zero (and hence we can define $\tau_0 \equiv 1$); indeed,

$$Y_0^{-1} Y_0' = \begin{pmatrix} 0 & * & \dots & * \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix} \tag{VI.77}$$

and

$$\partial \Psi_0(x) \Psi_0^{-1}(x) = \begin{pmatrix} \star & 0 & \dots & 0 \\ 0 & \star & \dots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \star & \dots & \star \end{pmatrix} \tag{VI.78}$$

so that the trace of the product is always zero (even before taking the residue). Combining the two results together gives the following theorem:

**Theorem VI.8.** *The isomodromic $\tau$-function and the partition function are related by:*

$$\forall N \in \mathbb{N} : \mathscr{Z}_N = (v_{d_2+1})^{\sum_{j=0}^{N-1} S_j} \tau_N$$

*where we recall that $S_j$ is given by the decomposition of $j+1$ in the Euclidian division by $d_2$: $S_j = E\left[\frac{j+1}{d_2}\right]$. A short computation of the power in $v_{d_2+1}$ gives:*

$$\forall N \in \mathbb{N} : \mathscr{Z}_N = (v_{d_2+1})^{d_2 \frac{\alpha_N(\alpha_N-1)}{2} + \alpha_N(N-\alpha_N d_2)} \tau_N$$

*where $\alpha_N = E\left[\frac{N}{d_2}\right]$*

The presence of the power in $v_{d_2+1}$ is due to a bad normalisation of the partition function itself ($\mathscr{Z}_N$) and can be easily cancelled out by taking $v_{d_2+1} = 1$ from the start (it is just a normalization of the weight function). Moreover it is not surprising because in the work of Bergere and Eynard [182], all results concerning the partition function and its derivatives with respect to parameters have special cases for $u_{d_1+1}$ and $v_{d_2+1}$. It also signals the fact that the RHP is badly defined when $v_{d_2+1} = 0$ because the contour integrals involved diverge and the whole setup breaks down. Indeed if $v_{d_2+1} = 0$ this simply means that $V_2$ is a polynomial of lower degree and thus the RHP that we should set up should be of smaller size from the outset.

## Outlook

In this article, we have restricted ourselves to contours going from infinity to infinity. This allows us to use integration by parts without picking up any boundary term. A natural extension of this work could be to see what happens when contours end in the complex plane, and especially study what happens when the end points moves (models with hard edges). This generalization is important in the computation of the gap probabilities of the Dyson model [194], which correspond to a random matrix model with Gaussian potentials but with the integration restricted to intervals of the real axis.

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## Appendix: Factorization of $\Psi_N$

Starting from the definition of the last $d_2$ columns of $\widehat{\underset{N}{\Psi}}$ (VI.20) we observe that

$$\begin{aligned}\psi_m^{(k)}(x) &:= \frac{1}{2i\pi}\int_{\check\Gamma_k} \mathrm{d}s \iint_{\mathscr{H}} \frac{\pi_m(z)}{x-z}\frac{V_2'(s)-V_2'(w)}{s-w}\mathrm{e}^{-V_1(z)-V_2(w)+V_2(s)+zw-xs}dw dz \qquad \text{(VI.79)}\\ &= \sum_{p,q} v_{q+p}\frac{1}{2i\pi}\iint_{\mathscr{H}}\frac{\pi_m(z)}{x-z}w^{p-1}\mathrm{e}^{-V_1(z)-V_2(w)+zw}\int_{\check\Gamma_k}\mathrm{d}s s^{q-1}\mathrm{e}^{V_2(s)-xs}\\ &= \sum_{p,q}(\widehat\Gamma_N)_{m,p}(V_0)_{p,q}(W_0)_{q,k} = (\widehat\Gamma_N V_0 W_0)_{m,k} \qquad \text{(VI.80)}\end{aligned}$$

This proves Thm. VI.3.

## Appendix: Bilinear concomitant as intersection number

We recall very briefly the result of [172] stating that

$$\left.\frac{V_2'(\partial_x)-V_2'(-\partial_z)}{\partial_x+\partial_z}w(x)f(z)\right|_{z=x} = \int_\Gamma\int_{\check\Gamma}\frac{V_2'(\eta)-V_2'(s)}{\eta-s}\mathrm{e}^{x(\eta-s)-V_2(\eta)+V_2(s)} = 2i\pi\Gamma\#\check\Gamma = \text{constant}\,. \qquad \text{(VI.81)}$$

The last identity is obtained by integration by parts and shows that the bilinear concomitant is just the intersection number of the (homology classes) of the contours $\Gamma,\check\Gamma$. More precisely we get that:

$$\begin{aligned}&\frac{d}{dx}\int_\Gamma\int_{\check\Gamma} ds d\eta\,\frac{V_2'(\eta)-V_2'(s)}{\eta-s}\mathrm{e}^{x(\eta-s)-V_2(\eta)+V_2(s)}\\ =\;& \int_\Gamma\int_{\check\Gamma} ds d\eta\,(V_2'(\eta)-V_2'(s))\mathrm{e}^{x(\eta-s)-V_2(\eta)+V_2(s)}\\ =\;& \int_\Gamma\int_{\check\Gamma} ds d\eta\,\frac{\partial}{\partial\eta}(-e^{-V_2(\eta)})e^{x\eta}e^{-xs+V_2(s)} - \int_\Gamma\int_{\check\Gamma} d\eta ds\,\frac{\partial}{\partial s}(e^{V_2(s)})e^{-xs}e^{x\eta-V_2(\eta)}\\ =\;& x\int_\Gamma\int_{\check\Gamma} ds d\eta\, e^{x\eta-xs-V_2(\eta)+V_2(s)} - x\int_\Gamma\int_{\check\Gamma} ds d\eta e^{x\eta-xs-V_2(\eta)+V_2(s)}\\ =\;& 0 \qquad \text{(VI.82)}\end{aligned}$$

The matrix expression shows that the pairing is indeed a duality since the determinant is nonzero. The undressing matrix $\Psi_0$ (that was originally introduced in Thm. VI.4) is thus

$$\Psi_0 = \left[\begin{array}{c|ccccc} 1 & & & & & \\ \hline & v_2 & v_3 & \cdots & & v_{d_2+1} \\ & v_3 & & & v_{d_2+1} & \\ & & & \iddots & & \\ & v_{d_2} & v_{d_2+1} & & & \\ & v_{d_2+1} & & & & \end{array}\right] \left[\begin{array}{c|ccccc} 1 & & & & \\ \hline & f_1 & f_2 & \cdots & f_{d_2} \\ & f_1' & f_2' & \cdots & f_{d_2}' \\ & \vdots & & & \vdots \\ & f_1^{(d_2-1)} & \cdots & & f_{d_2}^{(d_2-1)} \end{array}\right] \tag{VI.83}$$

where the Wronskian subblock in the second term is constructed by choosing $d_2$ homologically independent contour classes for the integrations $\check{\Gamma}$;

$$f_k(x) := \int_{\check{\Gamma}_k} \mathrm{e}^{-xs+V_2(s)} \mathrm{d}s \ , \ \ k = 1, \ldots, d_2 \ . \tag{VI.84}$$

The dressing matrix $\Psi_0$ exhibits a Stokes' phenomenon (of Airy's type) which is the inevitable drawback of removing the $x$-dependence from the jump matrix. We can now compute the jumps and see that it does not depend on $x$. For the $k$-th column we have:

$$\psi_m^{(k)}(x) := \frac{1}{2\pi i} \int_{\check{\Gamma}_k} ds \iint_{\varkappa} dz dw \frac{\pi_m(z) e^{-V_1(z)}}{x-z} \frac{V_2'(s) - V_2'(w)}{s-w} e^{-V_2(w)+V_2(s)+zw-xs}, \quad 1 \le k \le d_2 \tag{VI.85}$$

gives:

$$\begin{aligned} \psi_m^{(k)}(x)_+ &= \psi_m^{(k)}(x)_- + \psi_m^{(0)}(x) \int\int ds dw \frac{V_2'(s) - V_2'(w)}{s-w} e^{-V_2(w)+V_2(s)+x(w-s)} & \text{(VI.86)} \\ &= \psi_m^{(k)}(x)_- + \psi_m^{(0)}(x) \sum_{j=1}^{d_2} \varkappa_{\ell j} (\Gamma_j^{(y)} \# \check{\Gamma}_k) \ , & \text{(VI.87)} \end{aligned}$$

**Annexe VII**

# Topological expansion of the Bethe ansatz, and non-commutative algebraic geometry

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**Abstract**:
In this article, we define a non-commutative deformation of the "symplectic invariants" (introduced in [23]) of an algebraic hyperelliptical plane curve. The necessary condition for our definition to make sense is a Bethe ansatz. The commutative limit reduces to the symplectic invariants, i.e. algebraic geometry, and thus we define non-commutative deformations of some algebraic geometry quantities. In particular our non-commutative Bergmann kernel satisfies a Rauch variational formula. Those non-commutative invariants are inspired from the large N expansion of formal non-hermitian matrix models. Thus they are expected to be related to the enumeration problem of discrete non-orientable surfaces of arbitrary topologies.

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## 1 Introduction

In [23], the notion of symplectic invariants of a spectral curve was introduced. For any given algebraic plane curve (called spectral curve) of equation:

$$0 = \mathscr{E}(x,y) = \sum_{i,j} \mathscr{E}_{i,j}\, x^i y^j \tag{VII.1}$$

an infinite sequence of numbers

$$F^{(g)}(\mathscr{E}) \qquad , \; g = 0,1,2,\ldots,\infty \tag{VII.2}$$

and an infinite sequence of multilinear meromorphic forms $W_n^{(g)}$ (meromorphic on the algebraic Riemann surface of equation $\mathscr{E}(x,y) = 0$) were defined.

Their definition was inspired from hermitian matrix models, i.e. in the case where $\mathscr{E} = \mathscr{E}_{\text{M.M.}}$ is the spectral curve ($y(x)$ is the equilibrium density of eigenvalues) of a formal hermitian matrix integral $Z_{\text{M.M.}} = \int dM\, \mathrm{e}^{-N\mathrm{Tr}V(M)}$, the $F^{(g)}$ were such that:

$$\ln Z_{\text{M.M.}} = \sum_{g=0}^{\infty} N^{2-2g} F^{(g)}(\mathscr{E}_{\text{M.M.}}) \tag{VII.3}$$

The $F^{(g)}$'s have many remarkable properties (see [23]), in particular invariance under symplectic deformations of the spectral curve, homogeneity (of degree $2-2g$), holomorphic anomaly equations (modular transformations), stability under singular limits, ... An important property also, is that the following formal series

$$\tau(\mathscr{E}) = \mathrm{e}^{\sum_g N^{2-2g} F^{(g)}(\mathscr{E})} \tag{VII.4}$$

is the "formal" $\tau$ function of an integrable hierarchy.

Although those notions were first developed for matrix models, they extend beyond matrix models, and they make sense for spectral curves which are not matrix models spectral curves. For instance the (non-algebraic) spectral curve $\mathscr{E}_{\text{WP}}(x,y) = (2\pi y)^2 - (\sin(2\pi\sqrt{x}))^2$ is such that $F^{(g)}(\mathscr{E}_{\text{WP}}) = \mathrm{Vol}(\overline{\mathscr{M}}_g)$ is the Weyl-Petersson volume of moduli space of Riemann surfaces of genus $g$ (see [107, 178]). It is conjectured [81] that the $F^{(g)}$'s are deeply related to Gromov-Witten invariants, Hurwitz numbers [154] and topological strings [81]. In particular they are related to the Kodaira-Spencer field theory [103].

There were many attempts to compute also non-hermitian matrix integrals, and an attempt to extend the method of [23] was first made in [98], and here in this paper we deeply improve the result of [98]. The aim of the construction we present here, is to define $F^{(g)}$'s for a "non-commutative spectral curve", i.e. a non commutative polynomial:

$$\mathscr{E}(x,y) = \sum_{i,j} \mathscr{E}_{i,j}\, x^i y^j \qquad , \qquad [y,x] = \hbar \tag{VII.5}$$

For instance we can view $y$ as $y = \hbar\,\partial/\partial x$, and $\mathscr{E}$ is a differential operator, which encodes a linear differential equation.

In this article we choose $\mathscr{E}(x,y)$ of degree 2 in the variable $y$, i.e. the case of a second order linear differential equation, i.e. Schroedinger equation, and we leave to a further work the general case.

Here, in this article, we define some $F^{(g)}(\mathscr{E})$, which reduce to those of [23] in the limit $\hbar \to 0$, and which compute non-hermitian matrix model topological expansions.

For instance consider a formal matrix integral:

$$Z = \int_{E_{2\beta,N}} dM \mathrm{e}^{-N\sqrt{\beta}\mathrm{Tr}V(M)} = \mathrm{e}^{\sum_g N^{2-2g} F^{(g)}} \tag{VII.6}$$

where $E_{2\beta,N}$ is one of the Wigner matrix ensembles [19] of rank $N$: $E_{1,N}$ is the set of real symmetric matrices, $E_{2,N}$ is the set of hermitian matrices, and $E_{4,N}$ is the set of self-dual quaternion matrices (see [19] for a review). We define:

$$\hbar = \frac{1}{N}\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right) \tag{VII.7}$$

Notice that $\hbar = 0$ for hermitian matrices, i.e. the hermitian case is the classical limit $[y,x] = 0$. Notice also that the expected duality $\beta \leftrightarrow 1/\beta$ (cf [97, 180]) corresponds to $\hbar \leftrightarrow -\hbar$, i.e. we expect it to correspond to the duality $x \leftrightarrow y$ (for $\hbar = 0$, the $x \leftrightarrow y$ duality was proved in [23]).

Let us also mention that the topological expansion of non-hermitian matrix integrals is known to be related to the enumeration of unoriented discrete surfaces, and we expect that our $F^{(g)} = \sum_k \hbar^k F^{(g,k)}$ can be interpreted as generating functions of such unoriented surfaces.

So, in this article, we provide a method for computing $F^{(g,k)}$ for any $g$ and $k$ (which is more consise than [98]).

**Outline of the article**

- In section VII, we introduce our recursion kernel $K(x,x')$, and we show that the mere existence of this kernel is equivalent to the Bethe ansatz condition.
- In section VII, we define the $W_n^{(g)}$'s and the $F^{(g)}$'s, and we study their main properties, for instance that $W_n^{(g)}$ is symmetric.
- In section VII, we study the classical limit $\hbar \to 0$, and we show that we recover the algebro-geometric construction of [23].
- This inspires a notion of non-commutative algebraic geometry in section VII.
- In section VII, we study the application to the topological expansion of non-hermitian matrix integrals.
- In section VII, we study the application to the Gaudin model.
- Section VII is the conclusion.
- All the technical proofs are written in appendices for readability.

## 2 Definitions, kernel and Bethe ansatz

Let $V'(x)$ be a rational function (possibly a polynomial), and we call $V(x)$ the **potential**. Let $\alpha_i$ be the poles of $V'(x)$ (one of the poles may be at $\infty$).

For example, the following potential is called **Gaudin potential** (see section VII):

$$V'_{\text{Gaudin}}(x) = x + \sum_{i=1}^{\bar{n}} \frac{S_i}{x-\alpha_i} \tag{VII.8}$$

As another example, we will consider formal matrix models in section VII, for which $V'(x)$ is a polynomial.

However, many other choices can be made.

### 2.1 The problem

Our problem is to find $m$ complex numbers $s_1,\dots,s_m$, as well as two functions $G(x_0,x)$ and $K(x_0,x)$ with the following properties:

1. $G(x_0,x)$ is a rational function of $x$ with poles at $x = s_i$, and a simple pole of residue $+1$ at $x = x_0$, and which behaves as $O(1/x)$ at $x \to \infty$.

2. $G(x_0,x)$ is a rational function of $x_0$ with (possibly multiple) poles at $x_0 = s_i$, and a simple pole at $x_0 = x$, and $G(x_0,x)$ behaves like $O(1/x_0)$ at $x_0 \to \infty$.

3. $B(x_0,x) = -\frac{1}{2}\frac{\partial}{\partial x}G(x_0,x)$ is symmetric: $B(x_0,x) = B(x,x_0)$.

4. $K$ and $G$ are related by the following differential equation:

$$\left(2\hbar\sum_{i=1}^{m}\frac{1}{x-s_i} - V'(x) - \hbar\frac{\partial}{\partial x}\right)K(x_0,x) = G(x_0,x) \tag{VII.9}$$

5. $K(x_0,x)$ is analytical when $x \to s_i$ for all $i = 1,\ldots,m$.

We shall see below that those 5 conditions determine $K$, $G$, and the $s_i$'s. In fact condition 5 is the most important one in this list, it amounts to a **no-monodromy condition**, and we shall see below that it implies that the $s_i$'s must obey the **Bethe-ansatz equation**.

### 2.2 Analytical structure of the kernel $G$

The 4th and 5th conditions imply that $G(x_0,x)$ has at most simple poles at $x = s_i$. Then condition 3 implies that $G(x_0,x)$ has at most double poles at $x_0 = s_i$.

The first 3 conditions imply that there exists a symmetric matrix $A_{i,j}$ such that $G(x_0,x)$ can be written:

$$G(x_0,x) = \frac{1}{x-x_0} + 2\sum_{i,j=1}^{m}\frac{A_{i,j}}{(x-s_i)(x_0-s_j)^2} \tag{VII.10}$$

and therefore:

$$B(x_0,x) = \frac{1}{2}\frac{1}{(x-x_0)^2} + \sum_{i,j=1}^{m}\frac{A_{i,j}}{(x-s_i)^2(x_0-s_j)^2} \tag{VII.11}$$

We will argue in section VII, that $B$ can be viewed as a non=commutative deformation of the algebraic geometry's Bergmann kernel.

### 2.3 Bethe ansatz and monodromies

First, we study the conditions under which the differential equation eq. (VII.9) has no monodromies around $s_i$, in other words the condition under which $K(x_0,x)$ is analytical

when $x \to s_i$, $\forall i$:

$$K(x_0, s_i + \varepsilon) = K(x_0, s_i) + \varepsilon K'(x_0, s_i) + \frac{\varepsilon^2}{2} K''(x_0, s_i) + \frac{\varepsilon^3}{6} K'''(x_0, s_i) + \dots \tag{VII.12}$$

Equating the coefficient of $\varepsilon^{-1}$ in eq. (VII.9), we get:

$$\hbar K(x_0, s_i) = \sum_j \frac{A_{i,j}}{(x_0 - s_j)^2} \tag{VII.13}$$

equating the coefficient of $\varepsilon^0$ in eq. (VII.9), we get:

$$\hbar K'(x_0, s_i) = \frac{-1}{x_0 - s_i} + V'(s_i) K(x_0, s_i) - 2\hbar \sum_{j \neq i} \frac{K(x_0, s_i) - K(x_0, s_j)}{s_i - s_j} \tag{VII.14}$$

and equating the coefficient of $\varepsilon^1$ in eq. (VII.9), we get:

$$\begin{aligned} & 2\hbar \sum_{j \neq i} \frac{K'(x_0, s_i)}{s_i - s_j} - 2\hbar \sum_{j \neq i} \frac{K(x_0, s_i)}{(s_i - s_j)^2} + V''(s_i) K(x_0, s_i) \\ = \quad & V'(s_i) K'(x_0, s_i) - \frac{1}{(s_i - x_0)^2} - 2 \sum_{j \neq i} \sum_k \frac{A_{j,k}}{(s_i - s_j)^2 (x_0 - s_k)^2} \end{aligned} \tag{VII.15}$$

Notice from eq. (VII.13), that $K(x_0, s_i)$ has only double poles in $x_0$, with no residue:

$$\operatorname*{Res}_{x_0 \to s_k} K(x_0, s_i) = 0 \tag{VII.16}$$

Then, taking the residue at $x_0 \to s_k$ in eq. (VII.14), we see that:

$$\hbar \operatorname*{Res}_{x_0 \to s_k} K'(x_0, s_i) = -\delta_{i,k} \tag{VII.17}$$

Then, taking the residue when $x_0 \to s_i$ in eq. (VII.15), implies that the $s_i$'s are Bethe roots, i.e. they must obey the **Bethe equation**:

$$\boxed{\forall i = 1, \dots, m, \qquad 2\hbar \sum_{j \neq i} \frac{1}{s_i - s_j} = V'(s_i)} \tag{VII.18}$$

Then eq. (VII.15) becomes:

$$\frac{1}{(s_i-x_0)^2} = V''(s_i)K(x_0,s_i) + 2\hbar\sum_{j\neq i}\frac{K(x_0,s_i)}{(s_i-s_j)^2} - 2\sum_{j\neq i}\sum_k \frac{A_{j,k}}{(s_i-s_j)^2(x_0-s_k)^2} \quad \text{(VII.19)}$$

i.e. by comparing the coefficient of $1/(x_0-s_k)^2$ on both sides:

$$\boxed{\delta_{i,k} = \frac{1}{\hbar}V''(s_i)A_{i,k} + 2\sum_{j\neq i}\frac{A_{i,k}-A_{j,k}}{(s_i-s_j)^2}} \quad \text{(VII.20)}$$

i.e. $A$ is the inverse of the Hessian matrix $T$:

$$A = T^{-1} \quad , \quad \begin{cases} T_{i,i} = \frac{1}{\hbar}V''(s_i) + 2\sum_{j\neq i}\frac{1}{(s_i-s_j)^2} \\ T_{i,j} = -\frac{2}{(s_i-s_j)^2} \end{cases} \quad \text{(VII.21)}$$

$$T_{i,j} = \frac{1}{\hbar}\,\frac{\partial^2}{\partial s_i \partial s_j}\left(\sum_k V(s_k) - \hbar\sum_{k\neq l}\ln(s_k-s_l)\right) \quad \text{(VII.22)}$$

Therefore the Bethe ansatz equations eq. (VII.18) (as well as eq. (VII.20)) are the necessary conditions for $K(x_0,x)$ to be analytical when $x \to s_i$. Those conditions are necessary, but also sufficient conditions, as one can see by solving explicitely the linear ODE for $K$.

$$K(x_0,x) = \int_c^x dx' G(x_0,x')\, \mathrm{e}^{\frac{1}{\hbar}(V(x')-V(x))} \prod_i \frac{(x-s_i)^2}{(x'-s_i)^2} \quad \text{(VII.23)}$$

**Remark VII.1.** Notice that $K(x_0,x)$ is not analytical everywhere, it has a logarithmic singularity at $x = x_0$, and it has essential singularities at the poles of $V'$.

**Remark VII.2.** Notice that if one solution of the ODE is analytical near all $s_i$'s, then all solutions have that property. Indeed, all the solutions differ by a solution of the homogeneous equation, i.e. by:

$$\prod_i (x-s_i)^2\, \mathrm{e}^{-\frac{1}{\hbar}V(x)} \quad \text{(VII.24)}$$

which is clearly analytical near the $s_i$'s.

So, for the moment, the requirements 1–5 determine $G(x_0,x)$ uniquely, but $K(x_0,x)$ is not unique. Let us choose one possible $K(x_0,x)$, and we prove below in theorem VII.4, that the objects we are going to define, do not depend on the choice of $K$.

**Remark VII.3.** In what follows, it is useful to compute the Taylor expansion of $K$ near a root $s_i$. We write:

$$K(x_0,x) = \sum_{k=0}^{\infty} K_{i,k}(x_0)\ (x-s_i)^k \tag{VII.25}$$

The coefficients $K_{i,k}(x_0)$ are themselves rational fractions of $x_0$, and are computed in appendix VII.

### 2.4 Schroedinger equation

It is well known that the Bethe condition can be rewritten as a Schroedinger equation [89, 96]. We rederive it here for completeness.

Define the wave function:

$$\psi(x) = \prod_{i=1}^{m}(x-s_i)\ \mathrm{e}^{-\frac{1}{2\hbar}V(x)} \qquad , \qquad \omega(x) = \hbar\sum_{i=1}^{m}\frac{1}{x-s_i} \tag{VII.26}$$

$$Y(x) = -2\hbar\frac{\psi'(x)}{\psi(x)} = V'(x) - 2\omega(x) = V'(x) - 2\hbar\sum_i \frac{1}{x-s_i} \tag{VII.27}$$

then compute:

$$\begin{aligned} U(x) &= Y^2 - 2\hbar Y'(x) = 4\hbar^2\frac{\psi''(x)}{\psi(x)} \\ &= V'(x)^2 - 2\hbar V''(x) + 4(\omega(x)^2 - V'(x)\omega(x) + \hbar\omega'(x)) \end{aligned} \tag{VII.28}$$

We have:

$$\begin{aligned} \omega(x)^2 + \hbar\omega'(x) &= \hbar^2\sum_{i,j}\frac{1}{(x-s_i)(x-s_j)} - \hbar^2\sum_i\frac{1}{(x-s_i)^2} \\ &= \hbar^2\sum_{i\neq j}\frac{1}{(x-s_i)(x-s_j)} \end{aligned} \tag{VII.29}$$

which is a rational fraction with only simple poles at the $s_i$'s. The residue at $s_i$ is

$2\hbar^2 \sum_{j\neq i} \frac{1}{s_i - s_j} = \hbar V'(s_i)$, and thus:

$$\omega(x)^2 + \hbar\omega'(x) = \hbar \sum_i \frac{V'(s_i)}{(x - s_i)} \tag{VII.30}$$

which implies:

$$\omega(x)^2 - V'(x)\omega(x) + \hbar\omega'(x) = -\hbar \sum_i \frac{V'(x) - V'(s_i)}{(x - s_i)} \tag{VII.31}$$

and thus:

$$U(x) = V'(x)^2 - 2\hbar V''(x) - 4\hbar \sum_{i=1}^{m} \frac{V'(x) - V'(s_i)}{x - s_i} \tag{VII.32}$$

Therefore $U(x)$ is a rational fraction with poles at the poles of $V'$ (of degree at most those of $V'^2$), in particular it has no poles at the $s_i$'s.

$U$ is the potential for the Schroedinger equation for $\psi$:

$$\boxed{4\hbar^2 \psi'' = U\,\psi} \tag{VII.33}$$

As announced in the introduction, this equation can be encoded in a D-module element:

$$\mathscr{E}(x,y) = y^2 - \frac{1}{4}U(x) \qquad , \qquad y = \hbar \frac{\partial}{\partial x} \qquad , \qquad [y,x] = \hbar \tag{VII.34}$$

i.e.

$$\mathscr{E}(x,y).\psi = 0 \tag{VII.35}$$

Notice that the Schroedinger equation is equivalent to a Ricatti equation for $Y = -2\hbar\psi'/\psi$:

$$\boxed{Y^2 - 2\hbar Y' = U} \tag{VII.36}$$

### 2.5 Classical limit

We shall come back in more detail to the classical limit $\hbar \to 0$ in section VII. However, let us already make a few comments.

• In the classical limit, the Ricatti equation becomes an algebraic equation (hyperelliptical), which we call the (classical) spectral curve:

$$Y_{\rm cl}^2 = U(x) \tag{VII.37}$$

The function $Y_{\rm cl}(x) = \sqrt{U(x)}$ is therefore a multivalued function of $x$, and it should be seen as a meromorphic function on a branched Riemann surface (branching points are the zeroes of $U(x)$). We shall see below that in the limit $\hbar \to 0$, the kernel $B(x_0,x)$ tends towards the Bergmann kernel of that Riemann surface.

In other words the classical limit is expressed in terms of **algebraic geometry**.

In fact, in this article we are going to define non-commutative deformations of certain algebraic geometric objects in section VII.

## 3 Definition of correlators and free energies

In this section, we define the quantum deformations of the symplectic invariants introduced in [23, 105]. The following definitions are inspired from (not hermitian) matrix models. The special case of their application to matrix models will be discussed in section VII.

### 3.1 Definition of correlators

**Definition VII.1.** *We define the following functions $W_n^{(g)}(x_1,\dots,x_n)$ (called n-point correlation function of "genus"[3] g) by the recursion:*

$$W_1^{(0)}(x) = \omega(x) = \hbar \sum_{i=1}^{m} \frac{1}{x - s_i} \qquad , \qquad W_2^{(0)}(x_1,x_2) = B(x_1,x_2) \tag{VII.38}$$

$$W_{n+1}^{(g)}(x_0,J) = \sum_{i=1}^{m} \operatorname*{Res}_{x\to s_i} K(x_0,x) \left( \overline{W}_{n+2}^{(g-1)}(x,x,J) + \sum_{h=0}^{g} {\sum_{I\subset J}}' W_{|I|+1}^{(h)}(x,I) W_{n-|I|+1}^{(g-h)}(x,J/I) \right) \tag{VII.39}$$

[3] here $g$ is any given integer, it has nothing to do with the genus of the spectral curve.

*where $J$ is a collective notation for the variables $J = \{x_1, \ldots, x_n\}$, and where $\sum\sum'$ means that we exclude the terms $(h = 0, I = \emptyset)$ and $(h = g, I = J)$, and where:*

$$\overline{W}_n^{(g)}(x_1, ..., x_n) = W_n^{(g)}(x_1, ..., x_n) - \frac{\delta_{n,2}\delta_{g,0}}{2} \frac{1}{(x_1 - x_2)^2} \tag{VII.40}$$

**Remark VII.4.** This is exactly the same recursion as in [23], the only difference is that the kernel $K$ is not algebraic, but it is solution of the differential equation eq. (VII.9). We shall show in section VII, that in the limit $\hbar \to 0$, it indeed reduces to the definition of [23].

**Remark VII.5.** We say that $W_n^{(g)}$ is the correlation function of genus $g$ with $n$ marked points, and sometimes we say that it has characteristics:

$$\chi = 2 - 2g - n \tag{VII.41}$$

By analogy with algebraic geometry, we say that $W_n^{(g)}$ is stable if $\chi < 0$ and unstable if $\chi \geq 0$. We see that all the stable $W_n^{(g)}$'s have a common recursive definition def.VII.1, whereas the unstable ones appear as exceptions.

**Remark VII.6.** In order for the definition to make sense, we must make sure that the behaviour of each term in the vicinity of $x \to s_i$ is indeed locally meromorphic so that we can compute residues, i.e. there must be no log-singularity near $s_i$. In particular, the requirement of section VII for the kernel $K$ is **necessary**. In other words, a necessary condition for definition eq.VII.39 to make sense, is the **Bethe ansatz** !

### 3.2 Properties of correlators

The main reason of definition. VII.1, is because the $W_n^{(g)}$'s have many beautiful properties, which generalize those of [23].

We shall prove the following properties:

**Theorem VII.1.** *Each $W_n^{(g)}$ is a rational function of all its arguments. It has poles only at the $s_i$'s (except $W_2^{(0)}$, which also has a pole at $x_1 = x_2$). In particular it has no poles at the $\alpha_i$'s. Moreover, it vanishes as $O(1/x_i)$ when $x_i \to \infty$.*

*Proof.* in appendix VII

**Theorem VII.2.** *The $W_n^{(g)}$'s satisfy the loop equation, i.e. Virasoro-like constraints. This means that the quantity:*

$$\begin{aligned} P_{n+1}^{(g)}(x;x_1...,x_n) \quad = \quad & -Y(x)\overline{W}_{n+1}^{(g)}(x,x_1,...,x_n)+\hbar\partial_x\overline{W}_{n+1}^{(g)}(x,x_1...,x_n) \\ & +\sum_{I\subset J}\overline{W}_{|I|+1}^{(h)}(x,x_I)\overline{W}_{n-|I|+1}^{(g-h)}(x,J/I)+\overline{W}_{n+2}^{(g-1)}(x,x,J) \\ & +\sum_j \partial_{x_j}\left(\frac{\overline{W}_n^{(g)}(x,J/\{j\})-\overline{W}_n^{(g)}(x_j,J/\{j\})}{(x-x_j)}\right) \end{aligned} \tag{VII.42}$$

*is a rational fraction of $x$ (possibly a polynomial), with no pole at $x=s_i$. The only possible poles of $P_{n+1}^{(g)}(x;x_1...,x_n)$ are at the poles of $V'(x)$, with degree less than the degree of $V'$.*

*Proof.* in appendix VII

**Theorem VII.3.** *Each $W_n^{(g)}$ is a symmetric function of all its arguments.*

*Proof.* in appendix VII, with the special case of $W_3^{(0)}$ in appendix VII.

**Theorem VII.4.** *The correlation functions $W_n^{(g)}$ are independent of the choice of kernel $K$, provided that $K$ is solution of the equation eq. (VII.9).*

*Proof.* in appendix VII

**Theorem VII.5.** *The 3 point function $W_3^{(0)}$ can also be written:*

$$W_3^{(0)}(x_1,x_2,x_3)=4\sum_i \operatorname*{Res}_{x\to s_i}\frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)} \tag{VII.43}$$

*(In section VII, we interpret this equation as a non-commutative version of Rauch variational formula).*

*Proof.* in appendix VII

**Theorem VII.6.** *Under an infinitesimal variation of the potential $V\to V+\delta V$, we have:*

$$\forall n\geq 0, g\geq 0\,,\quad \delta W_n^{(g)}(x_1,\dots,x_n)=-\sum_i \operatorname*{Res}_{x\to s_i} W_{n+1}^{(g)}(x,x_1,\dots,x_n)\,\delta V(x) \tag{VII.44}$$

*Proof.* in appendix VII

This theorem suggest the definition of the "loop operator":

**Definition VII.2.** *The loop operator $\delta_x$ computes the variation of $W_n^{(g)}$ under a formal variation $\delta_x V(x') = \frac{1}{x-x'}$:*

$$\delta_{x_{n+1}} W_n^{(g)}(x_1,\dots,x_n) = W_{n+1}^{(g)}(x_1,\dots,x_n,x_{n+1}) \tag{VII.45}$$

*The loop operator is a derivation: $\delta_x(uv) = u\delta_x v + v\delta_x u$, and we have $\delta_{x_1}\delta_{x_2} = \delta_{x_2}\delta_{x_1}$, $\delta_{x_1}\partial_{x_2} = \partial_{x_2}\delta_{x_1}$.*

**Theorem VII.7.** *For $n \geq 1$, $W_n^{(g)}$ satify the equation:*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \overline{W}_n^{(g)}(x_1,\dots,x_n) = -\sum_i \operatorname*{Res}_{x_{n+1}\to s_i} V'(x_{n+1})\, \overline{W}_{n+1}^{(g)}(x_1,\dots,x_n,x_{n+1}) \tag{VII.46}$$

*and*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} x_i \overline{W}_n^{(g)}(x_1,\dots,x_n) = -\sum_i \operatorname*{Res}_{x_{n+1}\to s_i} x_{n+1} V'(x_{n+1})\, \overline{W}_{n+1}^{(g)}(x_1,\dots,x_n,x_{n+1}) \tag{VII.47}$$

*Proof.* in appendix VII

**Theorem VII.8.** *For $n \geq 1$, $W_n^{(g)}$ satify the equation:*

$$(2-2g-n-\hbar\frac{\partial}{\partial\hbar})\, \overline{W}_n^{(g)}(x_1,\dots,x_n) = -\sum_i \operatorname*{Res}_{x_{n+1}\to s_i} V(x_{n+1})\, \overline{W}_{n+1}^{(g)}(x_1,\dots,x_n,x_{n+1}) \tag{VII.48}$$

*Proof.* We give a "long" proof in appendix VII.

There is also a short cut:

If one changes $\hbar \to \lambda\hbar$, and $V \to \lambda V$, the $s_i$'s don't change, $B$ and $G$ don't change, and $K$ changes to $\frac{1}{\lambda}K$, thus $W_n^{(g)}$ changes by $\lambda^{2-2g-n}W_n^{(g)}$. The theorem is obtained by computing $\frac{\lambda\partial}{\partial\lambda}\lambda^{2g-2+n}W_n^{(g)} = \sum_k \frac{t_k\partial}{\partial t_k}W_n^{(g)}$, and computing the RHS with theorem VII.6, i.e. $\delta V = V$.

### 3.3 Definition of free energies

So far, we have defined $W_n^{(g)}$ with $n \geq 1$. Now, we define $F^{(g)} = W_0^{(g)}$.

Theorem VII.6, and the symmetry theorem VII.3 imply that:

$$\delta_{x_1} W_1^{(g)}(x_2) = W_2^{(g)}(x_1,x_2) = W_2^{(g)}(x_2,x_1) = \delta_{x_2} W_1^{(g)}(x_1) \tag{VII.49}$$

Thus, the symmetry of $W_2^{(g)}$ implies that there exists a "free energy" $F^{(g)} = W_0^{(g)}$ such that:

$$W_1^{(g)}(x) = \delta_x F^{(g)} \tag{VII.50}$$

which is equivalent to saying that for any variation $\delta V$:

$$\delta F^{(g)} = -\sum_i \operatorname*{Res}_{x \to s_i} W_1^{(g)}(x)\, \delta V(x) \tag{VII.51}$$

Therefore, we know that there must exists some $F^{(g)} = W_0^{(g)}$ which satisfy theorem VII.6 for $n = 0$.

Now, let us give a definition of $F^{(g)}$, inspired from theorem VII.8, and which will be proved to satisfy theorem VII.6 for $n = 0$.

**Definition VII.3.** *We define $F^{(g)} \equiv W_0^{(g)}$ by a solution of the differential equation in $\hbar$:*

$$\forall g \geq 2 \qquad , \qquad (2-2g-\hbar\frac{\partial}{\partial \hbar})F^{(g)} = -\sum_i \operatorname*{Res}_{x \to s_i} W_1^{(g)}(x) V(x) \tag{VII.52}$$

*more precisely:*

$$F^{(g)} = \hbar^{2-2g} \int_0^{\hbar} \frac{d\tilde{\hbar}}{\tilde{\hbar}^{3-2g}} \sum_i \operatorname*{Res}_{x \to s_i} V(x)\ W_1^{(g)}(x)\Big|_{\tilde{\hbar}} \tag{VII.53}$$

*And the unstable cases $2-2g \geq 0$ are defined by:*

$$F^{(0)} = \hbar^2 \sum_{i \neq j} \ln(s_i - s_j) - \hbar \sum_i V(s_i) \tag{VII.54}$$

$$F^{(1)} = \frac{1}{2}\ln\det A + \ln(\Delta(s)^2) + \frac{F^{(0)}}{\hbar^2} \tag{VII.55}$$

*where $\Delta(s) = \prod_{i>j}(s_i - s_j)$ is the Vandermonde determinant of the $s_i$'s.*

**Properties of the $F^{(g)}$'s:**

The definition of the $F^{(g)}$'s, is made so that all the theorems for the $W_n^{(g)}$'s, hold for for $n = 0$ as well. Proofs are given in appendices VII, VII, VII.

Explicit computations of the first few $F^{(g)}$'s are given in section VII and appendix VII.

## 4 Classical limit and WKB expansion

In the $\hbar \to 0$ limit, all quantities can be expanded formally into powers of $\hbar$: Write:

$$W_n^{(g)}(x_1,\dots,x_n) = \sum_k \hbar^k W_n^{(g,k)}(x_1,\dots,x_n) \qquad , \qquad F^{(g)} = \sum_k \hbar^k F^{(g,k)} \tag{VII.56}$$

### 4.1 Classical limit

Here we consider the classical limit $\hbar \to 0$. We noticed in section VII, that in that limit, the Ricatti equation

$$Y^2 - 2\hbar Y' = U = V'^2 - 2\hbar V'' - 4P \tag{VII.57}$$

where $P(x) = \hbar \sum_i \frac{V'(x)-V'(s_i)}{x-s_i}$, becomes an algebraic hyperelliptical equation:

$$Y_{\rm cl}{}^2 = U(x) = V'(x)^2 - 4P(x) \tag{VII.58}$$

i.e.

$$Y(x) \underset{\hbar\to 0}{\sim} Y_{\rm cl}(x) = \sqrt{V'(x)^2 - 4P(x)} \tag{VII.59}$$

$Y_{\rm cl}(x)$ is a multivalued function of $x$, and it should be seen as a meromorphic function on a 2-sheeted Riemann surface, i.e. there is a Riemann surface $\Sigma$ (of equation $0 = \mathscr{E}_{\rm cl}(x,y) = y^2 - 4U(x)$, such that the solutions of $\mathscr{E}_{\rm cl}(x,y) = 0$ are parametrized by two meromorphic functions on $\Sigma$:

$$\mathscr{E}_{\rm cl}(x,y) = 0 \quad \Leftrightarrow \quad \exists z \in \Sigma \begin{cases} x = x(z) \\ y = y(z) \end{cases} \tag{VII.60}$$

The Riemann surface $\Sigma$ has a certain topology[4] characterized by its genus $g$. It has a (non-unique) symplectic basis of $2g$ non-trivial cycles $\mathscr{A}_i \cap \mathscr{B}_j = \delta_{i,j}$.

The meromorphic forms on $\Sigma$ are classified as 1st kind (no pole), 3rd kind (only simple poles), and 2nd kind (multiple poles without residues).

There exists a unique 2nd kind differential $B_{\rm cl}$ on $\Sigma$, called the Bergmann kernel, such that: $B_{\rm cl}(z_1,z_2)$ has a double pole at $z_1 \to z_2$, and no other pole, without residue and

[4]This genus $g$ has nothing to do with the index $g$ of $F^{(g)}$ or $W_n^{(g)}$.

normalized (in any local coordinate $z$) as:

$$B_{\rm cl}(z_1,z_2) \underset{z_2\to z_1}{\sim} \frac{dz_1 dz_2}{(z_1-z_2)^2} + \text{reg} \qquad , \qquad \forall i=1,\dots,g\,,\ \oint_{\mathscr{A}_i} B_{\rm cl} = 0 \tag{VII.61}$$

We define a primitive:

$$G_{\rm cl}(z_0,z) = -2\int^z B_{\rm cl}(z_0,z') \tag{VII.62}$$

which is a 3rd kind differential in the variable $z_0$, it is called $dE_z(z_0)$ in [23].

When $\hbar = 0$, the kernel $K(z_0,z)$ satisfies the equation:

$$K_{\rm cl}(z_0,z) = -\frac{G_{\rm cl}(z_0,z)}{Y_{\rm cl}(z)} = 2\,\frac{\int_c^z B_{\rm cl}(z_0,z')}{Y_{\rm cl}(z)} \tag{VII.63}$$

which coincides with the definition of the recursion kernel in [23].

### 4.2 WKB expansion of the wave function

When $\hbar$ is small but non-zero, we can WKB expand $\psi(x)$, i.e.:

$$\psi(x) \sim \mathrm{e}^{-\frac{1}{2\hbar}\int^x Y_{\rm cl}(x')dx'}\ \frac{1}{\sqrt{Y_{\rm cl}(x)}}\left(1+\sum_k \hbar^k \psi_k(x)\right) \tag{VII.64}$$

i.e.

$$Y \sim Y_{\rm cl} + \sum_{k=1}^{\infty} \hbar^k Y_k \tag{VII.65}$$

The expansion coefficients $Y_k$ can be easily obtained recursively from the Ricatti equation:

$$2Y_{\rm cl}Y_k = 2Y'_{k-1} - \sum_{j=1}^{k-1} Y_j Y_{k-j} \tag{VII.66}$$

For instance:

$$Y_1 = \frac{{Y_{\rm cl}}'}{Y_{\rm cl}} \qquad , \qquad Y_2 = \frac{Y_1'}{Y_{\rm cl}} - \frac{Y_1^2}{2Y_{\rm cl}} = \frac{{Y_{\rm cl}}''}{{Y_{\rm cl}}^2} - \frac{3}{2}\frac{{Y_{\rm cl}}'^2}{{Y_{\rm cl}}^3} \qquad , \qquad \dots \text{etc} \tag{VII.67}$$

### 4.3 $\hbar$ expansion of correlators and energies

The kernel $K(x_0,x)$ can also be expanded:

$$K(x_0,x) = K_{\rm cl}(x_0,x) + \sum_{k=1}^{\infty} \hbar^k K_{(k)}(x_0,x) \tag{VII.68}$$

where $K_{(0)} = K_{\rm cl}$ is the kernel of [23]:

$$K_{\rm cl}(x_0,x) = \frac{dE_{x,o}(x_0)}{Y_{\rm cl}(x)} \tag{VII.69}$$

This implies that the correlators $W_n^{(g)}$ can also be expanded:

$$W_n^{(g)}(x_1,\dots,x_n) = \sum_{k=0}^{\infty} \hbar^k \, W_n^{(g,k)}(x_1,\dots,x_n) \tag{VII.70}$$

where the $W_n^{(g,k)}$ are obtained by the recursion:

$$\begin{aligned} W_{n+1}^{(g,k)}(x_0,J) &= \sum_{l=0}^{k} \sum_i \operatorname*{Res}_{x\to s_i} K_{(k-l)}(x_0,x) \Big[ \overline{W}_{n+2}^{(g-1,l)}(x,x,J) \\ &\quad + \sum_{h=0}^{g} \sum_{j=0}^{l} {\sum_{I\subset J}}' W_{|I|+1}^{(h,j)}(x,I) W_{n-|I|+1}^{(g-h,l-j)}(x,J/I) \Big] \end{aligned} \tag{VII.71}$$

where $J = \{x_1,\dots,x_n\}$.

Therefore, we observe that to leading order in $\hbar$, the $\lim_{\hbar\to 0} W_n^{(g,k)} = W_n^{(g,0)}$ do coincide with the $W_n^{(g)}$ computed with only $K_{\rm cl}$, and thus they coincide with the $W_n^{(g)}$ of [23].

And also, the $\hbar$ expansion must coincide with the diagrammatic rules of [98].

## 5 Non-commutative algebraic geometry

We have seen that in the limit $\hbar \to 0$, the correlation functions and the various functions we are considering, are fundamental objects of algebraic geometry. For instance $B$ is the Bergmann kernel, and $K$ is the recursion kernel of [23], which generates the symplectic invariants $F_g$ and the correlators $W_n^{(g)}$ attached to the spectral curve $Y_{\rm cl}(x)$.

In this paper, when $\hbar \neq 0$, we have defined deformations of those objects, which have

almost the same properties as the classical ones, except that they are no longer algebraic functions.

For instance we have:

- **Spectral curve**

  The algebraic equation of the classical spectral curve is replaced by a linear differential equation:

$$0 = \mathscr{E}(x,y) = \sum_{i,j} \mathscr{E}_{i,j} x^i y^j \quad \rightarrow \quad 0 = \mathscr{E}(x,\hbar\partial)\psi = \sum_{i,j} \mathscr{E}_{i,j} x^i (\hbar\partial)^j \psi \qquad \text{(VII.72)}$$

  In other words the polynomial $\mathscr{E}(x,y)$ is replaced by a non-commutative polynomial with $y = \hbar\partial_x$, i.e. $[y,x] = \hbar$.

  Here, our non-commutative spectral curve is:

$$\mathscr{E}(x,y) = y^2 - U(x) \qquad , \qquad y = \hbar\partial_x \qquad \text{(VII.73)}$$

  Notice that it can be factorized as:

$$\mathscr{E}(x,y) = (y - \frac{Y}{2})(y + \frac{Y}{2}) \qquad \text{(VII.74)}$$

  where $Y(x)$ is solution of $Y^2 - 2\hbar Y' = U$.

- **Bergmann Kernel** $B(x_1,x_2)$

  The non-commutative Bergmann kernel $B(x_1,x_2)$ is closely related to the Inverse of the Hessian $T$, i.e. to $A = T^{-1}$:

$$B(x_1,x_2) = \frac{1}{2(x_1-x_2)^2} + \sum_{i,j} \frac{A_{i,j}}{(x_1-s_i)^2(x_2-s_j)^2} \qquad \text{(VII.75)}$$

  A property of the classical Bergmann kernel $B_{\rm cl}(x_1,x_2)$ is that it computes derivatives, i.e. for any meromorphic function $f(x)$ defined on the spectral curve we have:

$$df(x) = -\operatorname*{Res}_{x_2 \to \text{poles of } f} B_{\rm cl}(x,x_2)\, f(x_2) \qquad \text{(VII.76)}$$

  Here, this property is replaced by: for any function $f(x)$ defined on the non-

commutative spectral curve (i.e. with poles only at the $s_i$'s), we have:

$$f'(x) = -2\sum_i \operatorname*{Res}_{x_2 \to s_i} B(x,x_2)\, f(x_2)\, dx_2 \tag{VII.77}$$

The factor of 2, comes from the fact that the interpretation of $x$, and thus of derivatives with respect to $x$, is slightly different. In the classical case, the differentials are computed in terms of local variables, and $x$ is not a local variable near branch-points. A good local variable near a branchpoint $a$, is $\sqrt{x-a}$. In the non-commutative case, the role of branchpoints seems to be played by the $s_i$'s, and $x$ is a good local variable near $s_i$.

- **Rauch variational formula**: In classical algebraic geometry, on an algebraic curve of equation $\mathscr{E}(x,y) = \sum_{i,j} \mathscr{E}_{i,j} x^i y^j = 0$, the Bergmann kernel depends only on the location of branchpoints $a_i$. The branchpoints are the points where the tangent is vertical, i.e. $dx(a_i) = 0$. Their location is $x_i = x(a_i)$. The Bergmann kernel is only function of the $x_i$'s, and the classical variational Rauch formula reads:

$$\frac{\partial B_{\rm cl}(z_1,z_2)}{\partial x_i} = \operatorname*{Res}_{z \to a_i} \frac{B_{\rm cl}(z,z_1)\, B_{\rm cl}(z,z_2)}{dx(z)} \tag{VII.78}$$

Equivalently, we can parametrize the spectral curve as $x(y)$ instead of $y(x)$, and consider the branchpoints of $y$, i.e. $dy(b_i) = 0$, whose location is $y_i = y(b_i)$, and we have:

$$\frac{\partial B_{\rm cl}(z_1,z_2)}{\partial y_i} = \operatorname*{Res}_{z \to b_i} \frac{B_{\rm cl}(z,z_1)\, B_{\rm cl}(z,z_2)}{dy(z)} \tag{VII.79}$$

Here, in the non-commutative version, theorem VII.5 and theorem VII.6 implies that under a variation of the spectral curve, we have:

$$\delta B(x_1,x_2) = -\frac{1}{2}\sum_i \operatorname*{Res}_{x \to s_i} \frac{B(x,x_1)B(x,x_2)}{Y'(x)}\, \delta Y(x) \tag{VII.80}$$

Consider the branchpoints $b_i$ such that $Y'(b_i) = 0$, and define their location as $Y_i = Y(b_i)$, by moving the integration contours we have:

$$\begin{aligned} \delta B(x_1,x_2) &= \frac{1}{2}\sum_i \operatorname*{Res}_{x \to b_i} \frac{B(x,x_1)B(x,x_2)}{Y'(x)}\, \delta Y(x)\, dx \\ &= \frac{1}{2}\sum_i \delta Y_i \operatorname*{Res}_{x \to b_i} \frac{B(x,x_1)B(x,x_2)}{Y'(x)}\, dx \end{aligned}$$

(VII.81)

i.e.:

$$\frac{\partial B(x_1,x_2)}{\partial Y_i} = \frac{1}{2}\operatorname*{Res}_{x\to b_i} \frac{B(x,x_1)B(x,x_2)}{Y'(x)}\,dx \tag{VII.82}$$

which is thus the quantum version of the Rauch variational formula eq. (VII.79).

Those properties can be seen as the beginning of a dictionary giving the deformations of classical algebraic geometry into non-commutative algebraic geometry.

**Conjecture about the symplectic invariants**

The $F_g$'s of [23] are the symplectic invariants of the classical spectral curve, which means that they are invariant under any cannonical change of the spectral curve which conserves the symplectic form $dx \wedge dy$. For instance they are invariant under $x \to y, y \to -x$.

Here, we conjecture that we may define some non-commutative $F^{(g)}$'s which are invariant under any cannonical transformation which conserves the commutator $[y,x] = \hbar$. This duality should also correspond to the expected duality $\beta \to 1/\beta$ in matrix models, cf [97, 180].

However, to check the validity of this conjecture, one needs to extend our work to differential operators of any order in $y$, and not only order 2. We plan to do this in a forthcoming work.

## 6 Application: non-hermitian Matrix models

The initial motivation for the work of [23], as well as this present work, was initially random matrix models. The classical case corresponds to hermitian matrix models, and here, we show that $\hbar \neq 0$ corresponds in some sense to non-hermitian matrix models [97, 104, 177].

In this section, we show that non-hermitian matrix models satisfy the loop equation eq. (VII.135) of theorem VII.2.

We define the matrix integral over $E_{m,2\beta}$ =set of $m \times m$ matrices of Wigner–type $2\beta$ ($E_{m,1}$ = real symmetric matrices, $E_{m,2}$ = hermitean matrices, $E_{m,4}$ = real quaternion self-dual matrices, see [19]):

$$Z = \int_{E_{m,2\beta}} dM\, \mathrm{e}^{-N\sqrt{\beta}\,\mathrm{Tr}V(M)} \tag{VII.83}$$

where $N$ is some arbitrary constant, not necessarily related to the matrix size $m$.

It is more convenient to rewrite it in terms of eigenvalues of $M$ (see [19]):

$$Z = \int_{\mathscr{C}^m} d\lambda_1 \dots d\lambda_m \prod_{i>j} (\lambda_j - \lambda_i)^{2\beta} \prod_i \mathrm{e}^{-N\sqrt{\beta}V(\lambda_i)} \tag{VII.84}$$

This last expression is well defined for any $\beta$, and not only $1/2, 1, 2$, and for any contour of integration $\mathscr{C}$ on which the integral is convergent.

We also define the correlators:

$$\begin{aligned} \overline{W}_n(x_1,\dots,x_n) &= < \mathrm{Tr}\frac{1}{x_1 - M} \dots \mathrm{Tr}\frac{1}{x_n - M} >_c \\ &= \left(N\sqrt{\beta}\right)^{-n} \frac{\partial}{\partial V(x_1)} \dots \frac{\partial}{\partial V(x_n)} \ln Z \end{aligned} \tag{VII.85}$$

i.e. in terms of eigenvalues:

$$\overline{W}_n(x_1,\dots,x_n) = < \sum_{i_1} \frac{1}{x_1 - \lambda_{i_1}} \dots \sum_{i_n} \frac{1}{x_n - \lambda_{i_n}} >_c \tag{VII.86}$$

In order to match with the notations of section VII, we prefer to shift $\overline{W}_2$ by a second order pole, and we define:

$$W_n(x_1,\dots,x_n) = \overline{W}_n(x_1,\dots,x_n) + \frac{\delta_{n,2}}{2(x_1 - x_2)^2} \tag{VII.87}$$

We are interested in a case where $Z$ has a large $N$ expansion of the form:

$$\ln Z \sim \sum_{g=0}^{\infty} N^{2-2g} F_g \tag{VII.88}$$

and for the correlation functions we assume:

$$W_n(x_1,\dots,x_n) = \frac{1}{\beta^{n/2}} \sum_{g=0}^{\infty} N^{2-2g-n} W_n^{(g)}(x_1,\dots,x_n) \tag{VII.89}$$

### 6.1 Loop equations

The loop equations can be obtained by integration by parts, or equivalently, they follow from the invariance of an integral under a change of variable. By considering the infinitesimal change of variable:

$$\lambda_i \to \lambda_i + \varepsilon \frac{1}{x-\lambda_i} + O(\varepsilon^2) \tag{VII.90}$$

we obtain:

$$\begin{aligned} & N\sqrt{\beta}(V'(x)\,\overline{W}_{n+1}(x,x_1,\dots,x_n) - P_{n+1}(x;x_1,\dots,x_n)) \\ = \;& \beta \sum_{J\subset L} \overline{W}_{1+|J|}(x,J)\,\overline{W}_{1+n-|J|}(x,L/J) \\ & +\beta \overline{W}_{n+2}(x,x,x_1,\dots,x_n) \\ & -(1-\beta)\frac{\partial}{\partial x}\overline{W}_{n+1}(x,x_1,\dots,x_n) \\ & +\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{\overline{W}_n(x,L/\{x_j\}) - \overline{W}_n(x_j,L/\{x_j\})}{x-x_j} \end{aligned} \tag{VII.91}$$

where $P_{n+1}(x;x_1,\dots,x_n))$ is a polynomial in its first variable $x$, of degree $\delta_{n,1} + \deg V - 2$.

If we expand this equation into powers of $N$ using eq. (VII.89), we have $\forall n, g$:

$$\begin{aligned} & V'(x)\,\overline{W}^{(g)}_{n+1}(x,x_1,\dots,x_n) - P^{(g)}_{n+1}(x;x_1,\dots,x_n)) \\ = \;& \sum_{g'=0}^{g} \sum_{J\subset L} \overline{W}^{(g')}_{1+|J|}(x,J)\,\overline{W}^{(g-g')}_{1+n-|J|}(x,L/J) \\ & +\beta \overline{W}^{(g-1)}_{n+2}(x,x,x_1,\dots,x_n) \\ & +\hbar \frac{\partial}{\partial x}\overline{W}^{(g)}_{n+1}(x,x_1,\dots,x_n) \\ & +\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{\overline{W}^{(g)}_n(x,L/\{x_j\}) - \overline{W}^{(g)}_n(x_j,L/\{x_j\})}{x-x_j} \end{aligned} \tag{VII.92}$$

where

$$\hbar = \frac{\sqrt{\beta} - \frac{1}{\sqrt{\beta}}}{N} \tag{VII.93}$$

Those loop equations coincide with the loop equations eq. (VII.42) of theorem VII.2.

Moreover we have:

$$\overline{W}^{(g)}_n = \frac{\partial \overline{W}^{(g)}_{n-1}}{\partial V} \tag{VII.94}$$

and near $x \to \infty$:

$$\sqrt{\beta}\, W_1(x) \sim \frac{m}{x}[N\hbar - \sum_{g=1}^{\infty} (-1)^g \frac{(2g-2)!}{g!(g-1)!}\, (N\hbar)^{1-2g}] \tag{VII.95}$$

i.e.

$$W_1^{(0)}(x) \sim \frac{m\hbar}{x} + O(1/x^2) \qquad , \qquad W_1^{(g)}(x) \sim -\frac{m\hbar}{x}\, \hbar^{-2g}\, \frac{(2g-2)!}{g!(g-1)!} + O(1/x^2) \tag{VII.96}$$

One should notice that the loop equations are independent of the contour $\mathscr{C}$ of integration of eigenvalues. The contour $\mathscr{C}$ is in fact encoded in the polynomial $P_{n+1}(x;x_1,\dots,x_n)$.

### 6.2 Solution of loop equations

To order $g = 0, n = 1$ we have:

$$V'(x)\, W_1^{(0)}(x) - P_1^{(0)}(x) = W_1^{(0)}(x)^2 + \hbar\, \frac{\partial}{\partial x} W_1^{(0)}(x) \tag{VII.97}$$

which is the same as the Ricatti equation eq. (VII.28).

As we said above, the contour $\mathscr{C}$ is in fact encoded in the polynomial $P_1^{(0)}(x)$. From now on, we choose a contour $\mathscr{C}$, i.e. a polynomial $P_1^{(0)}(x)$ such that the solution of the Ricatti equation is rational:

$$W_1^{(0)}(x) = \hbar \sum_{i=1}^{m} \frac{1}{x - s_i} \tag{VII.98}$$

It also has the correct behaviour at $\infty$: $W_1^{(0)}(x) \sim \frac{m\hbar}{x}$. This corresponds to a certain contour $\mathscr{C}$ which we do not determine here.

Since $W_1^{(0)}(x) = \omega(x)$ satisfies the Ricatti equation, i.e. the Bethe ansatz, the kernel $K$ exists, and we can define the functions $K(x_0,x)$, $G(x_0,x)$ and $B(x_0,x)$.

Then, from eq. (VII.94), we see that every $\overline{W}_n^{(g)}$ is going to be a rational fraction of $x$, with poles only at the $s_i$'s. In particular, Cauchy theorem implies:

$$\overline{W}_{n+1}^{(g)}(x_0,x_1,\dots,x_n) = \operatorname*{Res}_{x \to x_0} G(x_0,x)\, \overline{W}_{n+1}^{(g)}(x,x_1,\dots,x_n) \tag{VII.99}$$

and since both $G(x_0,x)$ and $\overline{W}_{n+1}^{(g)}(x,x_1,\dots,x_n)$ are rational fractions, which vanish sufficientely at $\infty$, we may change the integration contour to the other poles of the integrand,

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namely:

$$\begin{aligned} & \overline{W}_{n+1}^{(g)}(x_0,x_1,\ldots,x_n) \\ = \quad & -\sum_i \operatorname*{Res}_{x\to s_i} G(x_0,x)\,\overline{W}_{n+1}^{(g)}(x,x_1,\ldots,x_n) \\ = \quad & -\sum_i \operatorname*{Res}_{x\to s_i} \overline{W}_{n+1}^{(g)}(x,x_1,\ldots,x_n)\,(2\omega(x)-V'(x)-\hbar\partial_x)K(x_0,x) \\ = \quad & -\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\,(2\omega(x)-V'(x)+\hbar\partial_x)\overline{W}_{n+1}^{(g)}(x,x_1,\ldots,x_n) \end{aligned} \tag{VII.100}$$

Now, we insert loop equation eq. (VII.92) in the right hand side, and we notice that the term $P_{n+1}^{(g)}$ and $\frac{\partial}{\partial x_j}\frac{W_n^{(g)}(x_j,L/\{x_j\})}{x-x_j}$ do not have poles at the $s_i$'s, so they don't contribute. We thus get:

$$\begin{aligned} & \overline{W}_{n+1}^{(g)}(x_0,x_1,\ldots,x_n) \\ = \quad & \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\overline{W}_{n+2}^{(g-1)}(x,x,x_1,\ldots,x_n)\right. \\ & \left. +\sum_{g'=0}^{g}\sum_{J\subset L} W_{1+|J|}^{(g')}(x,J)W_{1+n-|J|}^{(g-g')}(x,L/J)\right) \end{aligned} \tag{VII.101}$$

i.e. we find the correlators of def VII.1.

Special care is needed for $W_2^{(0)}$. We have:

$$\begin{aligned} & \overline{W}_2^{(0)}(x_0,x_1,\ldots,x_n) \\ = \quad & -\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\,(2\omega(x)-V'(x)+\hbar\partial_x)\overline{W}_2^{(0)}(x,x_1) \\ = \quad & \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\,\frac{\omega(x)}{(x-x_1)^2} \\ = \quad & \hbar\sum_i \frac{K(x_0,s_i)}{(s_i-x_1)^2} \\ = \quad & \sum_{i,j}\frac{A_{i,j}}{(s_i-x_1)^2(s_j-x_0)^2} \end{aligned} \tag{VII.102}$$

which also agrees with def VII.1.

## 7 Application: Gaudin model

The Gaudin model's Bethe ansatz is obtained for the potential:

$$V'_{\text{Gaudin}}(x) = x + \sum_{i=1}^{\bar{n}} \frac{S_i}{x-\alpha_i} \tag{VII.103}$$

i.e. it corresponds to a Gaussian matrix model with sources:

$$Z = \int_{E_{m,2\beta}} dM\, \mathrm{e}^{-\frac{N\sqrt{\beta}}{2}\mathrm{Tr}M^2} \prod_i \det(\alpha_i - M)^{-NS_i\sqrt{\beta}} \tag{VII.104}$$

with $\hbar = \frac{\sqrt{\beta}-1/\sqrt{\beta}}{N}$.

$Z$ can also be written in eigenvalues:

$$Z = \int d\lambda_1 \dots d\lambda_m \, \frac{\prod_{i=1}^m \mathrm{e}^{-\frac{N\sqrt{\beta}}{2}\lambda_i^2}}{\prod_{i=1}^m \prod_{j=1}^{\bar{n}} (\alpha_j - \lambda_i)^{N\sqrt{\beta}S_j}} \prod_{i>j} (\lambda_i - \lambda_j)^{2\beta} \tag{VII.105}$$

### 7.1 Example

Consider:

$$V'(x) = x - \frac{s^2}{x} \qquad , \qquad V(x) = \frac{x^2}{2} - s^2 \ln x \tag{VII.106}$$

With only 1 root $m = 1$, the solution of the Bethe equation $V'(x) = 0$ is $x = s$.

Thus we have:

$$\omega(x) = \frac{\hbar}{x-s} \tag{VII.107}$$

$$B(x_1,x_2) = \frac{1}{2(x_1-x_2)^2} + \frac{\hbar}{2(x_1-s)^2(x_2-s)^2} \tag{VII.108}$$

We find:

$$W_3^{(0)}(x_1,x_2,x_3) = \frac{\hbar}{2(x_1-s)^2(x_2-s)^2(x_3-s)^2} \left( \frac{1}{x_1-s} + \frac{1}{x_2-s} + \frac{1}{x_3-s} + \frac{1}{2s} \right) \tag{VII.109}$$

$$W_1^{(1)}(x) = \frac{1}{\hbar(x-s)} + \frac{1}{4s(x-s)^2} + \frac{1}{2(x-s)^3} \tag{VII.110}$$

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For the free energies we have:

$$F^{(0)} = \frac{\hbar s^2}{2}(\ln s^2 - 1) \tag{VII.111}$$

$$F^{(1)} = \frac{1}{2}\ln(\frac{\hbar}{2}) + \frac{F^{(0)}}{\hbar^2} \tag{VII.112}$$

$$F^{(2)} = -\frac{1}{12\hbar s^2} - \frac{F^{(0)}}{\hbar^4} \tag{VII.113}$$

$$F^{(3)} = \frac{1}{12\hbar^3 s^2} + \frac{2F^{(0)}}{\hbar^6} \tag{VII.114}$$

and

$$Z = \mathrm{e}^{\Sigma_g N^{2-2g} F^{(g)}} = \mathrm{e}^{-N\sqrt{\beta}V(s)} \frac{1}{\sqrt{2\hbar}} (1 - \frac{1}{12 s^2 N^2 \hbar^2} + \dots) \tag{VII.115}$$

which is indeed the beginning of the saddle point expansion of:

$$Z = \int dx\, \mathrm{e}^{-N\sqrt{\beta}\, V(x)} \tag{VII.116}$$

## 8 Conclusion

In this article, we have defined a special case of non-commutative deformation of the symplectic invariants of [23]. Many of the fundamental properties of [23] are conserved or only slightly modified.

The main difference, is that the recursion kernel, instead of beeing an algebraic function, is given by the solution of a differential equation, otherwise the recursion is the same.

The main drawback of our definition, is that it concerns only a very restrictive subset of possible non-commutative spectral curves. Namely, we considered here only non commutative polynomials $\mathscr{E}(x,y) = \sum_{i,j} \mathscr{E}_{i,j}\, x^i y^j$ with $y = \hbar\partial_x$, of degree 2 in $y$, and such that the differential equation $\mathscr{E}(x,\hbar\partial).\psi = 0$ has a "polynomial" solution of the form $\psi(x) = \prod_{i=1}^{m}(x - s_i)\, \mathrm{e}^{-V(x)/2\hbar}$.

It should be possible to extend our definitions to other "non-polynomial" solutions $\psi$ (with an infinite number of zeroes $m = \infty$ for instance), and/or to higher degrees in $y$. In other words, what we have so far, is only a glimpse on more general structure yet to be discovered.

For example, it is not yet clear how our definitions are related to matrix integrals. We have said that the integration contour for the eigenvalues should be chosen so that the solution of the Schroedinger equation is polynomial of degree $m$, however, it is not known how to find explicitly such integration contours. Conversely, the usual matrix integrals with eigenvalues on the real axis, do probably not correspond to polynomial solutions of the Schroedinger equation. Similarly, it is not clear what the relationship between our definitions and the number of unoriented ribbon graphs is, for the same reason. The solution of the Schroedinger equation for ribbon graphs, should be chosen such that all the $W_n^{(g,k)}$'s are power series in $t$, and it is not known which integration contour it corresponds to, and which solution of the Schroedinger equation it corresponds to.

Therefore it seems necessary to extend our definitions to arbitrary solutions, i.e. to arbitrary integration contours for the matrix integrals. A possibility could be to obtain non-polynomial solutions as limits of polynomial ones.

The extension to higher degree in $y$, can be obtained from multi-matrix integrals, and extension seems rather easy for polynomial solutions again.

Finally, like the symplectic invariants of [23], we expect those "to be defined" non-commutative symplectic invariants, to play a role in several applications to enumerative geometry, and to topological string theory like in [81]. In other words, we expect our $F^{(g)}$'s to be generating functions for intersection numbers in some non-commutative moduli spaces of unoriented Riemann surfaces, whatever it means...

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## Appendix: Expansion of $K$

Since we have to compute residues at the $s_i$'s, we need to compute the Taylor expansion of $K(x_0,x)$ when $x \to s_i$:

$$K(x_0,x) = \sum_k (x-s_i)^k K_{i,k}(x_0) \tag{VII.117}$$

For instance we find:

$$K_{i,0} = \frac{1}{\hbar} \sum_j \frac{A_{i,j}}{(x_0-s_j)^2} \tag{VII.118}$$

$$\hbar K_{i,1}(x_0) = -\frac{1}{(x_0-s_i)} - 2\sum_{a\neq i}\sum_j \frac{A_{a,j}}{(s_a-s_i)\,(x_0-s_j)^2} \tag{VII.119}$$

$$\begin{aligned} \hbar K_{i,3} = & -\hbar\Big(2\sum_{a\neq i}\frac{1}{(s_a-s_i)^2} + \frac{1}{\hbar}V''(s_i)\Big)K_{i,1} \\ & -\hbar\Big(2\sum_{a\neq i}\frac{1}{(s_a-s_i)^3} + \frac{1}{\hbar}\frac{V'''(s_i)}{2}\Big)K_{i,0} \\ & +\frac{1}{(x_0-s_i)^3} + 2\sum_{a\neq i}\sum_j \frac{A_{a,j}}{(s_a-s_i)^3\,(x_0-s_j)^2} \end{aligned} \tag{VII.120}$$

Thanks to property eq. (VII.160), we may assume (but it is not necessary) that:

$$K_{i,2} = 0 \tag{VII.121}$$

Then, we have the recursion for $k \geq 0$:

$$\begin{aligned} & \hbar\Big((1-k)K_{i,k+1} - 2\sum_{a\neq i}\sum_{l=0}^{k}\frac{K_{i,k-l}}{(s_a-s_i)^{l+1}} - \frac{1}{\hbar}\sum_{l=0}^{k}\frac{V^{(l+1)}(s_i)}{l!}K_{i,k-l}\Big) \\ = & -\frac{1}{(x_0-s_i)^{k+1}} - 2\sum_{a\neq i}\sum_j \frac{A_{a,j}}{(s_a-s_i)^{k+1}\,(x_0-s_j)^2} \end{aligned} \tag{VII.122}$$

This proves that each $K_{i,k}(x_0)$ is a rational fraction of $x_0$, with poles at the $s_j$'s.

**Rational fraction of $x_0$**

Thus we write:

$$K_{i,k}(x_0) = \sum_{j,l} \frac{1}{(x_0 - s_j)^{k'}} K_{i,k;j,k'} \tag{VII.123}$$

For instance we have:

$$K_{i,0;j,k'} = \frac{A_{i,j}}{\hbar} \delta_{k',2} \tag{VII.124}$$

$$\hbar K_{i,1;j,k'} = -\delta_{k',1}\delta_{i,j} - 2\delta_{k',2} \sum_{a \neq i} \frac{A_{a,j}}{s_a - s_i} \tag{VII.125}$$

For higher $k$ we have the recursion:

$$\begin{aligned} & \hbar\Big( (1-k) K_{i,k+1;j,k'} - 2\sum_{a\neq i}\sum_{l=1}^{k} \frac{K_{i,k-l;j,k'}}{(s_a - s_i)^{l+1}} - \frac{1}{\hbar}\sum_{l=1}^{k} \frac{V^{(l+1)}(s_i)}{l!} K_{i,k-l;j,k'} \Big) \\ = \quad & -\delta_{i,j}\,\delta_{k',k+1} - 2\delta_{k',2} \sum_{a\neq i} \frac{A_{a,j}}{(s_a - s_i)^{k+1}} \end{aligned} \tag{VII.126}$$

In particular, it shows that if $k' > 2$, then $K_{i,k;i,k'}$ is proportional to $\delta_{i,j}$.

**Generating functions**

We introduce generating functions:

$$R_{i;j,k'}(x) = \sum_i K_{i,k;j,k'} \, (x - s_i)^k \tag{VII.127}$$

We have:

$$\hbar \left( 2\frac{\psi'(x)}{\psi(x)} - \partial_x \right) R_{i;j,k'}(x) = -\delta_{i,j}(x - s_i)^{k'-1} + 2\delta_{k',2} \sum_a \frac{A_{a,j}}{x - s_a} \tag{VII.128}$$

i.e.

$$-\hbar\psi^2(x)\,\partial_x\Big(\frac{R_{i;j,k'}(x)}{\psi^2(x)}\Big) = -\delta_{i,j}(x - s_i)^{k'-1} + \delta_{k',1}c_j + 2\delta_{k',2} \sum_a \frac{A_{a,j}}{x - s_a} \tag{VII.129}$$

In particular with $k' = 1$ we find:

$$R_{i;j,1}(x) = \frac{\delta_{i,j}}{\hbar}\,\psi(x)\phi(x) \tag{VII.130}$$

where

$$\phi(x) = \psi(x)\int^x \frac{dx'}{\psi(x')^2} \qquad , \qquad \phi'(x)\psi(x) - \psi'(x)\phi(x) = 1 \tag{VII.131}$$

**Appendix: Proof of theorem VII.1**

**Theorem VII.1** *Each $W_n^{(g)}$ is a rational function of all its arguments. If $2g+n-2 > 0$, it has poles only at the $s_i$'s. In particular it has no poles at the $\alpha_i$'s, and it vanishes as $O(1/x_i)$ when $x_i \to \infty$.*

**proof:**

It is easy to check that $W_1^{(0)}$, $W_2^{(0)}$ satisfy the theorem.

We will now make a recursion over $-\chi = 2g-2+n$ to prove the result for every $(n,g)$. We write:

$$W_{n+1}^{(g)}(x_0,x_1,\ldots,x_n) = \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\,U_{n+1}^{(g)}(x,x_1,\ldots,x_n) \tag{VII.132}$$

where $J = \{x_1,\ldots,x_n\}$, and

$$U_{n+1}^{(g)}(x,J) = \overline{W}_{n+2}^{(g-1)}(x,x,J) + \sum_{h=0}^{g}\sum_{I\subset J} W_{|I|+1}^{(h)}(x,I)W_{n-|I|+1}^{(g-h)}(x,J/I) \tag{VII.133}$$

First, the recursion hypothesis clearly implies that $U_{n+1}^{(g)}(x,x_1,\ldots,x_n)$ is a rational fraction in all its variables $x,x_1,...x_n$.

Then we Taylor expand $K(x_0,x)$ as in eq. (VII.117) or eq. (VII.123)

$$\begin{aligned} W_{n+1}^{(g)}(x_0,x_1,\ldots,x_n) &= \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\,U_{n+1}^{(g)}(x,x_1,\ldots,x_n) \\ &= \sum_i\sum_k K_{i,k}(x_0)\ \operatorname*{Res}_{x\to s_i}(x-s_i)^k U_{n+1}^{(g)}(x,x_1,\ldots,x_n) \end{aligned} \tag{VII.134}$$

Since $U_{n+1}^{(g)}(x,x_1,\ldots,x_n)$ is a rational fraction of $x$, the sum over $k$ is finite, and therefore,

$W_{n+1}^{(g)}(x_0,x_1,\dots,x_n)$ is a finite sum of rational fractions of $x_0$, with poles at the $s_j$'s, therefore it is a rational fraction of $x_0$ with poles at the $s_j$'s.

It is also clear that $W_{n+1}^{(g)}(x_0,x_1,\dots,x_n)$ is a rational fraction of the other variables $x_1,\dots,x_n$. The poles in those variables are necessarily at the $s_j$'s, because as long as the residues can be computed, $W_{n+1}^{(g)}(x_0,x_1,\dots,x_n)$ is finite. The residue cannot be computed everytime an integration contour gets pinched, and since the integration contours are small circles around the $s_i$'s, the only singularities may occur at the $s_i$'s.

It remains to prove that each $W_n^{(g)}$ behaves like $O(1/x_i)$ at $\infty$. The proof follows the same line: each $K_{i,k}(x_0)$ behaves like $O(1/x_0)$, and by an easy recursion the result holds for all other variables. □

## Appendix: Proof of theorem VII.2

In this subsection we prove theorem VII.2, that all $W_n^{(g)}$'s satisfy the loop equation.

**Theorem VII.2** *The $W_n^{(g)}$'s satisfy the loop equation, i.e. the following quantity $P_{n+1}^{(g)}(x;x_1\dots,x_n)$*

$$\begin{aligned} P_{n+1}^{(g)}(x;x_1\dots,x_n) \quad = \quad & -Y(x)\overline{W}_{n+1}^{(g)}(x,x_1,\dots,x_n)+\hbar\partial_x\overline{W}_{n+1}^{(g)}(x,x_1\dots,x_n) \\ & +\sum_{I\subset J}\overline{W}_{|I|+1}^{(h)}(x,x_I)\overline{W}_{n-|I|+1}^{(g-h)}(x,J/I)+\overline{W}_{n+2}^{(g-1)}(x,x,J) \\ & +\sum_j\partial_{x_j}\left(\frac{\overline{W}_n^{(g)}(x,J/\{j\})-\overline{W}_n^{(g)}(x_j,J/\{j\})}{(x-x_j)}\right) \end{aligned} \tag{VII.135}$$

*is a rational fraction of $x$ (possibly a polynomial), with no pole at $x=s_i$. The only possible poles of $P_{n+1}^{(g)}(x;x_1\dots,x_n)$ are at the poles of $V'(x)$, and their degree is less than the degree of $V'$.*

**proof:**

First, from theorem VII.1, we easily see that $P_{n+1}^{(g)}(x;x_1\dots,x_n)$ is indeed a rational function of $x$. Moreover it clearly has no pole at coinciding points $x=x_j$.

Then we write Cauchy's theorem for $W_{n+1}^{(g)}$:

$$\begin{aligned} W_{n+1}^{(g)}(x_0,\dots,x_n) &= \operatorname*{Res}_{x\to x_0}\frac{1}{x-x_0}W_{n+1}^{(g)}(x,x_1,\dots,x_n) \\ &= \operatorname*{Res}_{x\to x_0}G(x_0,x)\,W_{n+1}^{(g)}(x,x_1,\dots,x_n) \end{aligned} \tag{VII.136}$$

and using again theorem VII.1, i.e. that $W_{n+1}^{(g)}$ has poles only at the $s_i$'s, and that both $W_{n+1}^{(g)}$ and $G(x_0,x)$ behave as $O(1/x)$ for large $x$, we may move the integration contours:

$$W_{n+1}^{(g)}(x_0,...,x_n) = -\sum_i \operatorname*{Res}_{x\to s_i} G(x_0,x)\, W_{n+1}^{(g)}(x,x_1,...,x_n) \tag{VII.137}$$

Then we use the definition of $K$, and integrate by parts:

$$\begin{aligned} W_{n+1}^{(g)}(x_0,...,x_n) &= \sum_i \operatorname*{Res}_{x\to s_i} (Y(x)K(x_0,x)+\hbar K'(x_0,x))W_{n+1}^{(g)}(x,x_1,...,x_n) \\ &= \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\Big(Y(x)W_{n+1}^{(g)}(x,x_1,...,x_n) \\ &- \hbar\partial_x W_{n+1}^{(g)}(x,x_1,...,x_n)\Big) \end{aligned} \tag{VII.138}$$

From the definition we have also

$$\begin{aligned} & W_{n+1}^{(g)}(x_0,...,x_n) \\ = & \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\sum_{h=0}^{g}\sum_{I\subset J} W_{|I|+1}^{(h)}(x,I)W_{n-|I|+1}^{(g-h)}(x,J/I)+\overline{W}_{n+2}^{(g-1)}(x,x,J)\right) \end{aligned} \tag{VII.139}$$

then we shift $W_n^{(g)}$ to $\overline{W}_n^{(g)}$ in the RHS, i.e.:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0,...,x_n) \\ = & \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\Big(\sum_{h=0}^{g}\sum_{I\subset J} \overline{W}_{|I|+1}^{(h)}(x,I)\overline{W}_{n-|I|+1}^{(g-h)}(x,J/I)+\overline{W}_{n+2}^{(g-1)}(x,x,J) \\ & +\sum_{j=1}^{n}\frac{\overline{W}_n^{(g)}(x,J/\{j\})}{(x-x_j)^2}\Big) \\ = & \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\Big(\sum_{h=0}^{g}\sum_{I\subset J} \overline{W}_{|I|+1}^{(h)}(x,I)\overline{W}_{n-|I|+1}^{(g-h)}(x,J/I)+\overline{W}_{n+2}^{(g-1)}(x,x,J) \\ & +\sum_{j=1}^{n}\partial_{x_j}\left(\frac{\overline{W}_n^{(g)}(x,J/\{j\})}{x-x_j}\right)\Big) \\ = & \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\Big(\sum_{h=0}^{g}\sum_{I\subset J} \overline{W}_{|I|+1}^{(h)}(x,I)\overline{W}_{n-|I|+1}^{(g-h)}(x,J/I)+\overline{W}_{n+2}^{(g-1)}(x,x,J) \end{aligned}$$

$$+\sum_{j=1}^{n}\partial_{x_j}\left(\frac{\overline{W}_n^{(g)}(x,J/\{j\})-\overline{W}_n^{(g)}(x_j,J/\{j\})}{x-x_j}\right)\Big) \tag{VII.140}$$

in the last line we have added for free, the term $\overline{W}_n^{(g)}(x_j,J/\{j\})$ because it has no pole at $x=s_i$.

Therefore we have:

$$\begin{aligned}
0 &= \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\Big(-Y(x)W_{n+1}^{(g)}(x,x_1,...,x_n)+\hbar\partial_x W_{n+1}^{(g)}(x,x_1,...,x_n)\\
&\quad+\sum_{h=0}^{g}\sum_{I\subset J}\overline{W}_{|I|+1}^{(h)}(x,I)\overline{W}_{n-|I|+1}^{(g-h)}(x,J/I)+\overline{W}_{n+2}^{(g-1)}(x,x,J)\\
&\quad+\sum_{j=1}^{n}\partial_{x_j}\left(\frac{\overline{W}_n^{(g)}(x,J/\{j\})-\overline{W}_n^{(g)}(x_j,J/\{j\})}{x-x_j}\right)\Big)\\
&= \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)P_{n+1}^{(g)}(x;x_1,...,x_n)\\
&= \sum_i\sum_k K_{i,k}(x_0)\operatorname*{Res}_{x\to s_i}(x-s_i)^k P_{n+1}^{(g)}(x;x_1,...,x_n)
\end{aligned} \tag{VII.141}$$

Notice that this equation holds for any $x_0$. Since $K_{i,k}(x_0)$ is a rational fraction with a pole of degree $k+1$ in $x_0=s_i$, the $K_{i,k}(x_0)$ are linearly independent functions, and thus we must have:

$$\forall k,i \qquad 0=\operatorname*{Res}_{x\to s_i}(x-s_i)^k P_{n+1}^{(g)}(x;x_1,...,x_n) \tag{VII.142}$$

this means that $P_{n+1}^{(g)}$ has no pole at $x=s_i$.

One easily sees that $P_{n+1}^{(g)}(x;x_1,\ldots,x_n)$ is a rational fraction of $x$, and its poles are at most those of $Y(x)$, i.e. at the poles of $V'(x)$. □

## Appendix: Proof of theorem VII.3

**Theorem VII.3** *Each $W_n^{(g)}$ is a symmetric function of all its arguments.*

**proof:**

The special case of $W_3^{(0)}$ is proved in appendix VII above. It is obvious from the definition that $W_{n+1}^{(g)}(x_0,x_1,\ldots,x_n)$ is symmetric in $x_1,x_2,\ldots,x_n$, and therefore we need

to show that (for $n \geq 1$):

$$W_{n+1}^{(g)}(x_0,x_1,J) - W_{n+1}^{(g)}(x_1,x_0,J) = 0 \tag{VII.143}$$

where $J = \{x_2,\dots,x_n\}$. We prove it by recursion on $-\chi = 2g-2+n$.

Assume that every $W_k^{(h)}$ with $2h+k-2 \leq 2g+n$ is symmetric. We have:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0,x_1,J) \\ = \quad & \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\,\Big(W_{n+2}^{(g-1)}(x,x,x_1,J) + 2\,B(x,x_1)W_n^{(g)}(x,J) \\ & + 2\sum_{h=0}^{g}{\sum_{I\in J}}' W_{2+|I|}^{(h)}(x,x_1,I)W_{n-|I|}^{(g-h)}(x,J/I)\Big) \end{aligned} \tag{VII.144}$$

where $\sum'$ means that we exclude the terms $(I=\emptyset, h=0)$ and $(I=J, h=g)$. Notice also that $\overline{W}_{n+2}^{(g-1)} = W_{n+2}^{(g-1)}$ because $n \geq 1$. Then, using the recursion hypothesis, we have:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0,x_1,J) \\ = \quad & 2\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\,B(x,x_1)W_n^{(g)}(x,J) \\ & + \sum_{i,j} \operatorname*{Res}_{x\to s_i}\operatorname*{Res}_{x'\to s_j} K(x_0,x)K(x_1,x')\,\Big(W_{n+3}^{(g-2)}(x,x,x',x',J) \\ & + 2\sum_h {\sum_I}' W_{2+|I|}^{(h)}(x',x,I)W_{1+n-|I|}^{(g-1-h)}(x',x,J/I) \\ & + 2\sum_h {\sum_I}' W_{3+|I|}^{(h)}(x',x,x,I)W_{n-|I|}^{(g-1-h)}(x',J/I) \\ & + 2\sum_h {\sum_{I\in J}}' W_{n-|I|}^{(g-h)}(x,J/I)\Big[W_{3+|I|}^{(h-1)}(x,x',x',I) \\ & + 2\sum_{h'} {\sum_{I'\subset I}}' W_{2+|I'|}^{(h')}(x',x,I')W_{1+|I|-|I'|}^{(h-h')}(x',I/I')\Big]\Big) \end{aligned} \tag{VII.145}$$

Now, if we compute $W_{n+1}^{(g)}(x_1,x_0,J)$, we get the same expression, with the order of integrations exchanged, i.e. we have to integrate $x'$ before integrating $x$. Notice, by moving the integration contours, that:

$$\operatorname*{Res}_{x\to s_i}\operatorname*{Res}_{x'\to s_j} - \operatorname*{Res}_{x'\to s_j}\operatorname*{Res}_{x\to s_i} = -\delta_{i,j}\operatorname*{Res}_{x\to s_i}\operatorname*{Res}_{x'\to x} \tag{VII.146}$$

Moreover, the only terms which have a pole at $x = x'$ are those containing $B(x,x')$. Therefore:

$$\begin{aligned}
& W^{(g)}_{n+1}(x_0,x_1,J) - W^{(g)}_{n+1}(x_1,x_0,J) \\
= \quad & 2\sum_i \operatorname*{Res}_{x\to s_i} \left(K(x_0,x)\,B(x,x_1) - K(x_1,x)\,B(x,x_0)\right) W^{(g)}_n(x,J) \\
& -2\sum_i \operatorname*{Res}_{x\to s_i} \operatorname*{Res}_{x'\to x} K(x_0,x)K(x_1,x')\,B(x,x') \Big( \\
& 2W^{(g-1)}_{1+n}(x',x,J) + 2\sum_h {\sum_{I\in J}}' W^{(g-h)}_{n-|I|}(x,J/I)W^{(h)}_{1+|I|}(x',I)\Big)
\end{aligned} \tag{VII.147}$$

The residue $\operatorname{Res}_{x'\to x}$ can be computed:

$$\begin{aligned}
& W^{(g)}_{n+1}(x_0,x_1,J) - W^{(g)}_{n+1}(x_1,x_0,J) \\
= \quad & 2\sum_i \operatorname*{Res}_{x\to s_i} \left(K(x_0,x)\,B(x,x_1) - K(x_1,x)\,B(x,x_0)\right) W^{(g)}_n(x,J) \\
& -\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\frac{\partial}{\partial x'}\Big(K(x_1,x')\,\Big( \\
& 2W^{(g-1)}_{1+n}(x',x,J) + 2\sum_h {\sum_{I\in J}}' W^{(g-h)}_{n-|I|}(x,J/I)W^{(h)}_{1+|I|}(x',I)\Big)\Big)_{x'=x} \\
= \quad & 2\sum_i \operatorname*{Res}_{x\to s_i} \left(K(x_0,x)\,B(x,x_1) - K(x_1,x)\,B(x,x_0)\right) W^{(g)}_n(x,J) \\
& -\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)K'(x_1,x)\,\Big( \\
& 2W^{(g-1)}_{1+n}(x,x,J) + 2\sum_h {\sum_{I\in J}}' W^{(g-h)}_{n-|I|}(x,J/I)W^{(h)}_{1+|I|}(x,I)\Big) \\
& -\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)K(x_1,x)\frac{\partial}{\partial x'}\Big( \\
& 2W^{(g-1)}_{1+n}(x',x,J) + 2\sum_h {\sum_{I\in J}}' W^{(g-h)}_{n-|I|}(x,J/I)W^{(h)}_{1+|I|}(x',I)\Big)_{x'=x} \\
= \quad & 2\sum_i \operatorname*{Res}_{x\to s_i} \left(K(x_0,x)\,B(x,x_1) - K(x_1,x)\,B(x,x_0)\right) W^{(g)}_n(x,J) \\
& -\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)K'(x_1,x)\,\Big( \\
& 2W^{(g-1)}_{1+n}(x,x,J) + 2\sum_h {\sum_{I\in J}}' W^{(g-h)}_{n-|I|}(x,J/I)W^{(h)}_{1+|I|}(x,I)\Big) \\
& -\frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)K(x_1,x)\frac{\partial}{\partial x}\Big(
\end{aligned}$$

$$2W_{1+n}^{(g-1)}(x,x,J)+2\sum_h{\sum_{I\in J}}' W_{n-|I|}^{(g-h)}(x,J/I)W_{1+|I|}^{(h)}(x,I)\Big) \tag{VII.148}$$

The last term can be integrated by parts, and we get:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0,x_1,J)-W_{n+1}^{(g)}(x_1,x_0,J) \\ = \quad & 2\sum_i \operatorname*{Res}_{x\to s_i}\left(K(x_0,x)\,B(x,x_1)-K(x_1,x)\,B(x,x_0)\right)W_n^{(g)}(x,J) \\ & +\frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i}\Big(K'(x_0,x)K(x_1,x)-K(x_0,x)K'(x_1,x)\Big)\Big( \\ & 2W_{1+n}^{(g-1)}(x,x,J)+2\sum_h{\sum_{I\in J}}' W_{n-|I|}^{(g-h)}(x,J/I)W_{1+|I|}^{(h)}(x,I)\Big) \end{aligned} \tag{VII.149}$$

Then we use theorem VII.2:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0,x_1,J)-W_{n+1}^{(g)}(x_1,x_0,J) \\ = \quad & 2\sum_i \operatorname*{Res}_{x\to s_i}\left(K(x_0,x)\,B(x,x_1)-K(x_1,x)\,B(x,x_0)\right)W_n^{(g)}(x,J) \\ & +\sum_i \operatorname*{Res}_{x\to s_i}\Big(K'(x_0,x)K(x_1,x)-K(x_0,x)K'(x_1,x)\Big)\Big(P_n^{(g)}(x,J) \\ & +(Y(x)-\hbar\partial_x)W_n^{(g)}(x,J)+\sum_j \partial_{x_j}\Big(\frac{W_{n-1}^{(g)}(x_j,J/\{x_j\})}{x-x_j}\Big)\Big) \end{aligned} \tag{VII.150}$$

Since $P_n^{(g)}(x,J)$ and $W_{n-1}^{(g)}(x_j,J/\{x_j\})$ have no poles at the $s_i$'s, we have:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0,x_1,J)-W_{n+1}^{(g)}(x_1,x_0,J) \\ = \quad & 2\sum_i \operatorname*{Res}_{x\to s_i}\left(K(x_0,x)\,B(x,x_1)-K(x_1,x)\,B(x,x_0)\right)W_n^{(g)}(x,J) \\ + \quad & \sum_i \operatorname*{Res}_{x\to s_i}\Big(K'(x_0,x)K(x_1,x)-K(x_0,x)K'(x_1,x)\Big) \\ & (Y(x)-\hbar\partial_x)W_n^{(g)}(x,J) \end{aligned} \tag{VII.151}$$

Notice that:

$$K_0'K_1-K_0K_1'=-\frac{1}{\hbar}\left(G_0K_1-K_0G_1\right) \tag{VII.152}$$

and $B = -\frac{1}{2}G'$, therefore:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0,x_1,J) - W_{n+1}^{(g)}(x_1,x_0,J) \\ = \quad & -\sum_i \operatorname*{Res}_{x\to s_i} \left(K_0 G_1' - K_1 G_0'\right) W_n^{(g)}(x,J) \\ & -\frac{1}{\hbar}\sum_i \operatorname*{Res}_{x\to s_i} \left(G_0 K_1 - K_0 G_1\right) (Y(x) - \hbar\partial_x) W_n^{(g)}(x,J) \end{aligned} \tag{VII.153}$$

we integrate the first line by parts:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0,x_1,J) - W_{n+1}^{(g)}(x_1,x_0,J) \\ = \quad & \sum_i \operatorname*{Res}_{x\to s_i} \left(K_0' G_1 - K_1' G_0\right) W_n^{(g)}(x,J) \\ & +\sum_i \operatorname*{Res}_{x\to s_i} \left(K_0 G_1 - K_1 G_0\right) W_n^{(g)}(x,J)' \\ & -\frac{1}{\hbar}\sum_i \operatorname*{Res}_{x\to s_i} \left(G_0 K_1 - K_0 G_1\right) (Y(x) - \hbar\partial_x) W_n^{(g)}(x,J) \end{aligned} \tag{VII.154}$$

Notice that:

$$K_0' G_1 - G_0 K_1' = -\frac{Y}{\hbar}\left(K_0 G_1 - G_0 K_1\right) \tag{VII.155}$$

So we find

$$W_{n+1}^{(g)}(x_0,x_1,J) - W_{n+1}^{(g)}(x_1,x_0,J) = 0 \tag{VII.156}$$

## Appendix: Proof of theorem VII.4

**Theorem VII.9.** *The correlation functions $W_n^{(g)}$ are independent of the choice of kernel K, provided that K is solution of the equation eq. (VII.9).*

*Proof.* Any two solutions of eq. (VII.9), differ by a homogeneous solution, i.e. by $\psi^2(x)$. Therefore, what we have to prove is that the following quantity vanishes:

$$\sum_i \operatorname*{Res}_{x\to s_i} \psi^2(x) \left[W_{n+2}^{(g-1)}(x,x,J) + \sum_h {\sum_{I\subset J}}' W_{1+|I|}^{(h)}(x,I) W_{1+n-|I|}^{(g-h)}(x,J/I)\right] \tag{VII.157}$$

Using theorem VII.2, we have:

$$\begin{aligned} & \operatorname*{Res}_{x\to s_i} \psi^2(x)\left[W_{n+2}^{(g-1)}(x,x,J)+\sum_h {\sum_{I\subset J}}' W_{1+|I|}^{(h)}(x,I)W_{1+n-|I|}^{(g-h)}(x,J/I)\right] \\ = & \operatorname*{Res}_{x\to s_i} \psi^2(x)\Big(Y(x)W_n^{(g)}(x,J)-\hbar\partial_x W_n^{(g)}(x,J)+P_n^{(g)}(x;J)\Big) \end{aligned} \tag{VII.158}$$

Then we notice that $P_n^{(g)}$ gives no residue, and then we use $Y=-2\hbar\psi'/\psi$, and we integrate by parts:

$$\begin{aligned} = & -\hbar \operatorname*{Res}_{x\to s_i} \psi^2(x)\Big(2\frac{\psi'}{\psi}W_n^{(g)}+\partial_x W_n^{(g)}\Big) \\ = & -\hbar \operatorname*{Res}_{x\to s_i} \partial_x\left(\psi^2\, W_n^{(g)}\right) \\ = & \ 0 \end{aligned} \tag{VII.159}$$

This means that adding to $K(x_0,x)$ a constant times $\psi^2(x)$ doesnot change the $W_n^{(g)}$'s. In fact we may chose a different constant near each $s_i$, or in other words, we may assume that

$$K_{i,2}(x_0)=0 \tag{VII.160}$$

## Appendix: Proof of theorem VII.5

**Theorem VII.1** *The 3 point function $W_3^{(0)}$ is symmetric and we have:*

$$W_3^{(0)}(x_1,x_2,x_3)=4\sum_i \operatorname*{Res}_{x\to s_i} \frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)} \tag{VII.161}$$

**proof:**
The definition of $W_3^{(0)}$ is:

$$\begin{aligned} & W_3^{(0)}(x_0,x_1,x_2) \\ = & 2\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)B(x,x_1)B(x,x_2) \\ = & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} K_0\, G_1'\, G_2' \\ = & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} K_0\left((\hbar K_1''+YK_1'+Y'K_1)(\hbar K_2''+YK_2'+Y'K_2)\right) \end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} K_0\,(\hbar^2 K_1'' K_2'' + \hbar Y(K_1' K_2'' + K_1'' K_2') + \hbar Y'(K_1'' K_2 + K_2'' K_1) \\
& + Y^2 K_1' K_2' + YY'(K_1 K_2' + K_1' K_2) + Y'^2 K_1 K_2\,)
\end{aligned}
\tag{VII.162}
$$

where we have written for short $K_i = K(x_i,x)$, $G_i = G(x_i,x)$, and derivative are w.r.t. $x$.

Since $K(x_i,x)$ has no pole when $x \to s_i$, the first term vanishes. Using the Ricatti equation $Y^2 = 2\hbar Y' + U$ (where $U$ has no pole at $s_i$), we may replace $Y^2$ by $2\hbar Y'$ and $YY'$ by $\hbar Y''$ without changing the residues, i.e.:

$$
\begin{aligned}
& W_3^{(0)}(x_0,x_1,x_2) \\
= \quad & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} K_0\,(\hbar Y(K_1' K_2'' + K_1'' K_2') + \hbar Y'(K_1'' K_2 + K_2'' K_1) \\
& + 2\hbar Y' K_1' K_2' + \hbar Y''(K_1 K_2' + K_1' K_2) + Y'^2 K_1 K_2\,) \\
= \quad & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} K_0\,(\hbar Y(K_1' K_2')' + \hbar Y'(K_1 K_2)'' + \hbar Y''(K_1 K_2)' + Y'^2 K_1 K_2\,) \\
= \quad & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} Y'^2 K_0 K_1 K_2 + \hbar\big(Y'' K_0 (K_1 K_2)' - (YK_0)' K_1' K_2' - (Y' K_0)'(K_1 K_2)'\big) \\
= \quad & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} Y'^2 K_0 K_1 K_2 - \hbar\big((YK_0)' K_1' K_2' + Y' K_0'(K_1 K_2)'\big) \\
= \quad & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} Y'^2 K_0 K_1 K_2 - \hbar Y K_0' K_1' K_2' - \hbar Y'(K_0 K_1' K_2' + K_0' K_1 K_2' + K_0' K_1' K_2)
\end{aligned}
\tag{VII.163}
$$

This expression is clearly symmetric in $x_0, x_1, x_2$ as claimed in theorem VII.3.

Let us give an alternative expression, in the form of the Verlinde or Krichever formula [179]:

$$
W_3^{(0)}(x_1,x_2,x_3) = 4\sum_i \operatorname*{Res}_{x\to s_i} \frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)} \tag{VII.164}
$$

**proof:**

In order to prove formula VII.164, compute:

$$
B(x,x_i) = -\frac{1}{2}G'(x,x_i) = -\frac{1}{2}G_i' = \frac{1}{2}(\hbar K_i'' + YK_i' + Y'K_i) \tag{VII.165}
$$

thus:

$$
\sum_i \operatorname*{Res}_{x\to s_i} \frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)}
$$

$$
\begin{aligned}
&= \frac{1}{8}\sum_i \operatorname*{Res}_{x\to s_i} \frac{1}{Y'(x)}(\hbar K_0''+YK_0'+Y'K_0)(\hbar K_1''+YK_1'+Y'K_1)\\
&\qquad (\hbar K_2''+YK_2'+Y'K_2)\\
&= \frac{1}{8}\sum_i \operatorname*{Res}_{x\to s_i} \frac{\hbar^3}{Y'}K_0''K_1''K_2''+\hbar^2\frac{Y}{Y'}(K_0'K_1''K_2''+K_0''K_1'K_2''+K_0''K_1''K_2')\\
&\quad +\hbar^2(K_0K_1''K_2''+K_0''K_1K_2''+K_0''K_1''K_2)\\
&\quad +\hbar\frac{Y^2}{Y'}(K_0''K_1'K_2'+K_0'K_1''K_2'+K_0'K_1'K_2'')\\
&\quad +\hbar Y(K_0K_1'K_2''+K_0K_1''K_2'+K_0'K_1K_2''+K_0'K_1''K_2+K_0''K_1K_2'+K_0''K_1'K_2)\\
&\quad +\hbar Y'(K_0''K_1K_2+K_0K_1''K_2+K_0K_1K_2'')+\frac{Y^3}{Y'}K_0'K_1'K_2'\\
&\quad +Y^2(K_0K_1'K_2'+K_0'K_1K_2'+K_0'K_1'K_2)\\
&\quad +YY'(K_0'K_1K_2+K_0K_1'K_2+K_0K_1K_2')+Y'^2K_0K_1K_2
\end{aligned}
\tag{VII.166}
$$

Notice that $K_i$ has no pole at the $s_i$'s, and $1/Y'$ has no pole, $Y/Y'$ has no pole, $Y^2/Y'$ has no pole, thus:

$$
\begin{aligned}
&\sum_i \operatorname*{Res}_{x\to s_i} \frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)}\\
&= \frac{1}{8}\sum_i \operatorname*{Res}_{x\to s_i} \hbar Y(K_0K_1'K_2''+K_0K_1''K_2'+K_0'K_1K_2''+K_0'K_1''K_2+K_0''K_1K_2'\\
&\quad +K_0''K_1'K_2)+\hbar Y'(K_0''K_1K_2+K_0K_1''K_2+K_0K_1K_2'')+\frac{Y^3}{Y'}K_0'K_1'K_2'\\
&\quad +Y^2(K_0K_1'K_2'+K_0'K_1K_2'+K_0'K_1'K_2)\\
&\quad +YY'(K_0'K_1K_2+K_0K_1'K_2+K_0K_1K_2')+Y'^2K_0K_1K_2
\end{aligned}
\tag{VII.167}
$$

Notice that $Y^2=2\hbar Y'+U$, thus we may replace $Y^3/Y'$ by $2\hbar Y$, and $Y^2$ by $2\hbar Y'$ and $YY'$ by $\hbar Y''$, thus:

$$
\begin{aligned}
&\sum_i \operatorname*{Res}_{x\to s_i} \frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)}\\
&= \frac{1}{8}\sum_i \operatorname*{Res}_{x\to s_i} \hbar Y(K_0K_1'K_2''+K_0K_1''K_2'+K_0'K_1K_2''+K_0'K_1''K_2+K_0''K_1K_2'\\
&\quad +K_0''K_1'K_2)+\hbar Y'(K_0''K_1K_2+K_0K_1''K_2+K_0K_1K_2'')+2\hbar YK_0'K_1'K_2'\\
&\quad +2\hbar Y'(K_0K_1'K_2'+K_0'K_1K_2'+K_0'K_1'K_2)+\hbar Y''(K_0'K_1K_2+K_0K_1'K_2+K_0K_1K_2')\\
&\quad +Y'^2K_0K_1K_2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{8}\sum_i \operatorname*{Res}_{x\to s_i} \hbar Y(K_0(K_1'K_2')' + K_1(K_0'K_2')' + K_2(K_0'K_1')') \\
&\quad +2\hbar Y K_0'K_1'K_2' + Y'^2 K_0K_1K_2 + \hbar(Y'(K_0'K_1K_2 + K_0K_1'K_2 + K_0K_1K_2'))' \\
&= \frac{1}{8}\sum_i \operatorname*{Res}_{x\to s_i} \hbar Y(K_0(K_1'K_2')' + K_1(K_0'K_2')' + K_2(K_0'K_1')') \\
&\quad +2\hbar Y K_0'K_1'K_2' + Y'^2 K_0K_1K_2 \\
&= -\frac{1}{8}\sum_i \operatorname*{Res}_{x\to s_i} 3\hbar Y K_0'K_1'K_2' + \hbar Y'(K_0K_1'K_2' + K_0'K_1K_2' + K_0'K_1'K_2) \\
&\quad -2\hbar Y K_0'K_1'K_2' - Y'^2 K_0K_1K_2 \\
&= \frac{1}{4}W_3^{(0)}(x_0,x_1,x_2)
\end{aligned}
\tag{VII.168}
$$

**Direct computation**

We write

$$
\begin{aligned}
& W_3^{(0)}(z_1,z_2,z_3) \\
= \quad & 2\sum_i \operatorname*{Res}_{z\to s_i} K(z_1,z)B(z_2,z)B(z_3,z) \\
= \quad & \sum_j\sum_i \frac{A_{i,j}}{(z_2-s_j)^2} \operatorname*{Res}_{z\to s_i} K(z_1,z)\frac{1}{(z-s_i)^2(z_3-z)^2} + \text{sym.} \\
& +2\sum_i\sum_{i'\neq i}\sum_{j,k} \frac{A_{i,j}A_{i',k}}{(z_2-s_j)^2(z_3-s_k)^2} \operatorname*{Res}_{z\to s_i} K(z_1,z)\frac{1}{(z-s_i)^2(z-s_{i'})^2} + \text{sym.} \\
& +2\sum_i\sum_{j,k} \frac{A_{i,j}A_{i,k}}{(z_2-s_j)^2(z_3-s_k)^2} \operatorname*{Res}_{z\to s_i} K(z_1,z)\frac{1}{(z-s_i)^4} \\
= \quad & \sum_j\sum_i \frac{A_{i,j}}{(z_2-s_j)^2}\left(\frac{K_{i,1}(z_1)}{(z_3-s_i)^2} + \frac{2K_{i,0}(z_1)}{(z_3-s_i)^3}\right) + \text{sym.} \\
& +2\sum_i\sum_{i'\neq i}\sum_{j,k} \frac{A_{i,j}A_{i',k}}{(z_2-s_j)^2(z_3-s_k)^2}\left(\frac{K_{i,1}(z_1)}{(s_{i'}-s_i)^2} + \frac{2K_{i,0}(z_1)}{(s_{i'}-s_i)^3}\right) + \text{sym.} \\
& +2\sum_i\sum_{j,k} \frac{A_{i,j}A_{i,k}}{(z_2-s_j)^2(z_3-s_k)^2} K_{i,3}(z_1) \\
= \quad & \sum_j\sum_i \frac{A_{i,j}}{(z_2-s_j)^2}\left(\frac{K_{i,1}(z_1)}{(z_3-s_i)^2} + \frac{2K_{i,0}(z_1)}{(z_3-s_i)^3}\right) + \text{sym.} \\
& +2\sum_i\sum_{i'\neq i}\sum_{j,k} \frac{A_{i,j}A_{i',k}}{(z_2-s_j)^2(z_3-s_k)^2}\left(\frac{K_{i,1}(z_1)}{(s_{i'}-s_i)^2} + \frac{2K_{i,0}(z_1)}{(s_{i'}-s_i)^3}\right) + \text{sym.} \\
& -2\sum_i\sum_{j,k} \frac{A_{i,j}A_{i,k}}{(z_2-s_j)^2(z_3-s_k)^2} T_{i,i}K_{i,1}(z_1)
\end{aligned}
$$

$$
\begin{aligned}
& -2\sum_i\sum_{j,k}\frac{A_{i,j}A_{i,k}}{(z_2-s_j)^2(z_3-s_k)^2}\left(\frac{V'''(s_i)}{2\hbar}+2\sum_{i'\neq i}\frac{1}{(s_{i'}-s_i)^3}\right)K_{i,0}(z_1)\\
& +\frac{2}{\hbar}\sum_i\sum_{j,k}\frac{A_{i,j}A_{i,k}}{(z_2-s_j)^2(z_3-s_k)^2(z_1-s_i)^3}\\
& +\frac{4}{\hbar}\sum_i\sum_{i'\neq i}\sum_l\sum_{j,k}\frac{A_{i,j}A_{i,k}A_{i',l}}{(z_2-s_j)^2(z_3-s_k)^2(s_{i'}-s_i)^3(z_1-s_l)^2}\\
= \quad & \frac{2}{\hbar}\sum_{i,j,k}\frac{A_{i,j}A_{i,k}}{(z_1-s_i)^3(z_2-s_j)^2(z_3-s_k)^2}+\frac{A_{j,i}A_{j,k}}{(z_1-s_i)^2(z_2-s_j)^3(z_3-s_k)^2}\\
& +\frac{A_{k,i}A_{k,j}}{(z_1-s_i)^2(z_2-s_j)^2(z_3-s_k)^3}\\
& +\sum_{i,j,k}\frac{K_{i,1}(z_1)}{(z_2-s_j)^2(z_3-s_k)^2}\Big(A_{j,k}\delta_{i,j}+A_{j,k}\delta_{i,k}-A_{i,j}\sum_{i'}T_{i,i'}A_{i',k}\\
& -A_{i,k}\sum_{i'}T_{i,i'}A_{i',j}\Big)\\
& +2\sum_i\sum_{i'\neq i}\sum_{j,k}\frac{A_{i,j}A_{i',k}}{(z_2-s_j)^2(z_3-s_k)^2}\frac{2K_{i,0}(z_1)}{(s_{i'}-s_i)^3}\ +\text{sym.}\\
& -2\sum_i\sum_{j,k}\frac{A_{i,j}A_{i,k}}{(z_2-s_j)^2(z_3-s_k)^2}\left(\frac{V'''(s_i)}{2\hbar}+2\sum_{i'\neq i}\frac{1}{(s_{i'}-s_i)^3}\right)K_{i,0}(z_1)\\
& +\frac{4}{\hbar}\sum_i\sum_{i'\neq i}\sum_l\sum_{j,k}\frac{A_{i,j}A_{i,k}A_{i',l}}{(z_2-s_j)^2(z_3-s_k)^2(s_{i'}-s_i)^3(z_1-s_l)^2}\\
= \quad & \frac{2}{\hbar}\sum_{l,j,k}\frac{1}{(z_1-s_l)^2(z_2-s_j)^2(z_3-s_k)^2}\sum_i\Big(\frac{\delta_{i,l}A_{i,j}A_{i,k}}{(z_1-s_i)}+\frac{\delta_{i,j}A_{i,l}A_{i,k}}{(z_2-s_i)}\\
& +\frac{\delta_{i,k}A_{i,l}A_{i,j}}{(z_3-s_i)}\Big)\\
& +\frac{4}{\hbar}\sum_{l,j,k}\sum_i\sum_{i'\neq i}\frac{A_{i,j}A_{i,k}A_{i',l}+A_{i,j}A_{i',k}A_{i,l}+A_{i,k}A_{i',j}A_{i,l}-A_{i,j}A_{i,k}A_{i,l}}{(z_1-s_l)^2(z_2-s_j)^2(z_3-s_k)^2(s_{i'}-s_i)^3}\\
& -\frac{1}{\hbar^2}\sum_{l,j,k}\sum_i\frac{A_{i,j}A_{i,k}A_{i,l}V'''(s_i)}{(z_1-s_l)^2(z_2-s_j)^2(z_3-s_k)^2}
\end{aligned}
\tag{VII.169}
$$

Thus we have:

$$
\begin{aligned}
& W_3^{(0)}(z_1,z_2,z_3)\\
= \quad & \frac{2}{\hbar}\sum_{i,j,k,l}\frac{\frac{\delta_{i,l}A_{i,j}A_{i,k}}{(z_1-s_i)}+\frac{\delta_{i,j}A_{i,l}A_{i,k}}{(z_2-s_i)}+\frac{\delta_{i,k}A_{i,l}A_{i,j}}{(z_3-s_i)}}{(z_1-s_l)^2(z_2-s_j)^2(z_3-s_k)^2}
\end{aligned}
$$

$$+\frac{4}{\hbar}\sum_{l,j,k}\sum_{i}\sum_{i'\neq i}\frac{A_{i,j}A_{i,k}A_{i',l}+A_{i,j}A_{i',k}A_{i,l}+A_{i,k}A_{i',j}A_{i,l}-A_{i,j}A_{i,k}A_{i,l}}{(z_1-s_l)^2(z_2-s_j)^2(z_3-s_k)^2(s_{i'}-s_i)^3}$$
$$-\frac{1}{\hbar^2}\sum_{l,j,k}\sum_{i}\frac{A_{i,j}A_{i,k}A_{i,l}V'''(s_i)}{(z_1-s_l)^2(z_2-s_j)^2(z_3-s_k)^2} \quad \text{(VII.170)}$$

## Appendix: Proof of theorem VII.6

**Theorem VII.6** *Under an infinitesimal variation of the potential $V \to V+\delta V$, we have:*

$$\forall n\geq 0, g\geq 0\,, \quad \delta W_n^{(g)}(x_1,\ldots,x_n) = -\sum_i \operatorname*{Res}_{x\to s_i} W_{n+1}^{(g)}(x,x_1,\ldots,x_n)\,\delta V(x) \quad \text{(VII.171)}$$

### Variation of $\omega$

We have:

$$\omega(x) = \hbar\sum_i \frac{1}{x-s_i} \quad \text{(VII.172)}$$

and

$$V'(s_i) = 2\hbar\sum_{j\neq i}\frac{1}{s_i-s_j} \quad \text{(VII.173)}$$

Thus taking a variation we have:

$$\delta V'(s_i)+\delta s_i V''(s_i) = -2\hbar\sum_{j\neq i}\frac{\delta s_i-\delta s_j}{(s_i-s_j)^2} \quad \text{(VII.174)}$$

i.e.

$$\delta V'(s_i) = -\hbar\sum_j T_{i,j}\delta s_j \quad \text{(VII.175)}$$

which implies:

$$\delta s_i = -\frac{1}{\hbar}\sum_j A_{i,j}\delta V'(s_j) \quad \text{(VII.176)}$$

and therefore:

$$\delta\omega(x) = -\sum_{i,j}\frac{A_{i,j}\delta V'(s_j)}{(x-s_i)^2} \quad \text{(VII.177)}$$

which can also be written:

$$\begin{aligned}\delta\omega(x) &= -\sum_k \operatorname*{Res}_{x'\to s_k} \sum_{i,j} \frac{A_{i,j}}{(x-s_i)^2\,(x'-s_j)}\,\delta V'(x') \\ &= -\sum_k \operatorname*{Res}_{x'\to s_k} \sum_{i,j} \frac{A_{i,j}}{(x-s_i)^2\,(x'-s_j)^2}\,\delta V(x') \\ &= -\sum_k \operatorname*{Res}_{x'\to s_k} B(x,x')\,\delta V(x') \end{aligned} \tag{VII.178}$$

and finally we obtain the case $n=1, g=0$ of the theorem:

$$\boxed{\delta\omega(x) = -\sum_k \operatorname*{Res}_{x'\to s_k} B(x,x')\,\delta V(x')} \tag{VII.179}$$

**Variation of $B$**

Consider:

$$\overline{W}_2^{(0)}(x,x') = B(x,x') - \frac{1}{2}\frac{1}{(x-x')^2} = \sum_{i,j} \frac{A_{i,j}}{(x-s_j)^2(x'-s_i)^2} \tag{VII.180}$$

Due to eq. (VII.13) we have:

$$\begin{aligned}\overline{W}_2^{(0)}(x,x') &= \sum_i \frac{\hbar K(x,s_i)}{(x'-s_i)^2} \\ &= \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\,\frac{\omega(z)}{(z-x')^2} \\ &= \frac{\partial}{\partial x'}\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\,\frac{\omega(z)-\omega(x')}{z-x'} \end{aligned} \tag{VII.181}$$

On the other hand, since $\overline{W}_2^{(0)}(x,x')$ has poles only at the $s_i$'s we have:

$$\begin{aligned}\overline{W}_2^{(0)}(x,x') &= \operatorname*{Res}_{z\to x} G(x,z)\,\overline{W}_2^{(0)}(z,x') \\ &= -\sum_i \operatorname*{Res}_{z\to s_i} G(x,z)\,\overline{W}_2^{(0)}(z,x') \\ &= -\sum_i \operatorname*{Res}_{z\to s_i} \left((2\omega(z) - V'(z) + \hbar\partial_z)K(x,z)\right)\,\overline{W}_2^{(0)}(z,x') \end{aligned}$$

$$= \quad -\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \Big((2\omega(z)-V'(z)-\hbar\partial_z)\,\overline{W}_2^{(0)}(z,x')\Big) \tag{VII.182}$$

This implies that $\forall x$:

$$0 = -\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \left((2\omega(z)-V'(z)-\hbar\partial_z)\,\overline{W}_2^{(0)}(z,x') + \frac{\partial}{\partial x'}\frac{\omega(z)-\omega(x')}{z-x'}\right) \tag{VII.183}$$

and therefore, $\overline{W}_2^{(0)}(x,x')$ satisfies the loop equation:

$$(2\omega(x)-V'(x)-\hbar\partial_x)\,\overline{W}_2^{(0)}(x,x') + \frac{\partial}{\partial x'}\frac{\omega(x)-\omega(x')}{x-x'} = -P_2^{(0)}(x,x') \tag{VII.184}$$

where $P_2^{(0)}(x,x')$ has no pole at $x\to s_i$'s.

Then we take the variation:

$$\begin{aligned}(2\omega(x)-V'(x)-\hbar\partial_x)\,\delta\overline{W}_2^{(0)}(x,x') \quad &= \quad -(2\delta\omega(x)-\delta V'(x))\,\overline{W}_2^{(0)}(x,x') \\ &\quad -\frac{\partial}{\partial x'}\frac{\delta\omega(x)-\delta\omega(x')}{x-x'} - \delta P_2^{(0)}(x,x')\end{aligned} \tag{VII.185}$$

$\delta\overline{W}_2^{(0)}(x,x')$ is a rational fraction of $x$, with poles only at the $s_i$'s, and $\delta P_2^{(0)}(x,x')$ has no pole at $x\to s_i$'s. We thus write:

$$\begin{aligned}\delta W_2^{(0)}(x,x') \quad &= \quad \delta\overline{W}_2^{(0)}(x,x') \\ &= \quad \operatorname*{Res}_{z\to x} G(x,z)\,\delta\overline{W}_2^{(0)}(z,x') \\ &= \quad -\sum_i \operatorname*{Res}_{z\to s_i} G(x,z)\,\overline{W}_2^{(0)}(z,x') \\ &= \quad -\sum_i \operatorname*{Res}_{z\to s_i} \big((2\omega(z)-V'(z)+\hbar\partial_z)K(x,z)\big)\ \delta\overline{W}_2^{(0)}(z,x') \\ &= \quad -\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \Big((2\omega(z)-V'(z)-\hbar\partial_z)\,\delta\overline{W}_2^{(0)}(z,x')\Big) \\ &= \quad \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \Big((2\delta\omega(z)-\delta V'(z))\,\overline{W}_2^{(0)}(z,x') \\ &\quad +\frac{\partial}{\partial x'}\frac{\delta\omega(z)-\delta\omega(x')}{z-x'} + \delta P_2^{(0)}(z,x')\Big) \\ &= \quad \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \left((2\delta\omega(z)-\delta V'(z))\,\overline{W}_2^{(0)}(z,x') + \frac{\delta\omega(z)}{(z-x')^2}\right)\end{aligned}$$

$$= \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\,(2\delta\omega(z)-\delta V'(z))\,B(z,x') \tag{VII.186}$$

Then, we use eq. (VII.179), and we get:

$$\begin{aligned}
\delta W_2^{(0)}(x,x') &= -2\sum_i \operatorname*{Res}_{z\to s_i} \sum_k \operatorname*{Res}_{x''\to s_k} K(x,z)\,B(z,x'')\delta V(x'')\,B(z,x') \\
&\quad -\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\,\delta V'(z)\,B(z,x') \\
&= -\sum_i \operatorname*{Res}_{z\to s_i} \sum_k \operatorname*{Res}_{x''\to s_k} K(x,z)\,G(z,x'')\delta V'(x'')\,B(z,x') \\
&\quad -\sum_i \operatorname*{Res}_{z\to s_i} \operatorname*{Res}_{x''\to z} K(x,z)\,G(z,x'')\delta V'(x'')\,B(z,x') \\
&= -\sum_k \operatorname*{Res}_{x''\to s_k} \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\,G(z,x'')\delta V'(x'')\,B(z,x') \\
&= -2\sum_k \operatorname*{Res}_{x''\to s_k} \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\,B(z,x'')\delta V(x'')\,B(z,x')
\end{aligned} \tag{VII.187}$$

We thus obtain the case $n=2, g=0$ of the theorem:

$$\boxed{\delta W_2^{(0)}(x,x') = -\sum_k \operatorname*{Res}_{x''\to s_k} W_3^{(0)}(x,x',x'')\,\delta V(x'')} \tag{VII.188}$$

**Variation of other higher correlators**

We prove by recursion on $2g+n$, that:

$$\boxed{\delta W_{n+1}^{(g)}(x,L) = -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'')\,W_{n+2}^{(g)}(z,L,x'')} \tag{VII.189}$$

where $L=\{x_1,\dots,x_n\}$.

We write:

$$U_{n+1}^{(g)}(z,L) = \overline{W}_{n+2}^{(g-1)}(z,z,L) + \sum_h {\sum_{J\subset L}}' W_{1+|J|}^{(h)}(z,J)\,W_{1+n-|J|}^{(g-h)}(z,L/J) \tag{VII.190}$$

By definition we have:

$$W_{n+1}^{(g)}(x,L) = \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\, U_{n+1}^{(g)}(z,L) \tag{VII.191}$$

From the recursion hypothesis, we have:

$$\begin{aligned}
\delta U_{n+1}^{(g)}(z,L) \quad &= \quad -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'') \Big( W_{n+3}^{(g-1)}(z,z,L,x'') \\
&\quad -2\sum_h {\sum_{J\subset L}}' W_{2+|J|}^{(h)}(z,J,x'')\, W_{1+n-|J|}^{(g-h)}(z,L/J) \Big) \\
&= \quad -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'') \Big( U_{n+2}^{(g)}(z,L,x'') - 2B(z,x'') W_{n+1}^{(g)}(z,L) \Big)
\end{aligned} \tag{VII.192}$$

Thus:

$$\begin{aligned}
&\delta W_{n+1}^{(g)}(x,L) \\
= \quad & \sum_i \operatorname*{Res}_{z\to s_i} \delta K(x,z)\, U_{n+1}^{(g)}(z,L) - \sum_i \operatorname*{Res}_{z\to s_i} K(x,z) \sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'') \Big( \\
& U_{n+2}^{(g)}(z,L,x'') - 2B(z,x'') W_{n+1}^{(g)}(z,L) \Big) \\
= \quad & \sum_i \operatorname*{Res}_{z\to s_i} \delta K(x,z)\, U_{n+1}^{(g)}(z,L) - \sum_k \operatorname*{Res}_{x''\to s_k} \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\, \delta V(x'') \Big( \\
& U_{n+2}^{(g)}(z,L,x'') - 2B(z,x'') W_{n+1}^{(g)}(z,L) \Big) \\
= \quad & \sum_i \operatorname*{Res}_{z\to s_i} \delta K(x,z)\, U_{n+1}^{(g)}(z,L) \\
& +2\sum_k \operatorname*{Res}_{x''\to s_k} \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\, \delta V(x'')\, B(z,x'') W_{n+1}^{(g)}(z,L) \\
& -\sum_k \operatorname*{Res}_{x''\to s_k} \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\, \delta V(x'')\, U_{n+2}^{(g)}(z,L,x'') \\
= \quad & \sum_i \operatorname*{Res}_{z\to s_i} \delta K(x,z)\, U_{n+1}^{(g)}(z,L) \\
& +2\sum_i \operatorname*{Res}_{z\to s_i} \sum_k \operatorname*{Res}_{x''\to s_k} K(x,z)\, \delta V(x'')\, B(z,x'') W_{n+1}^{(g)}(z,L) \\
& +2\sum_i \operatorname*{Res}_{z\to s_i} \operatorname*{Res}_{x''\to z} K(x,z)\, \delta V(x'')\, B(z,x'') W_{n+1}^{(g)}(z,L) \\
& -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'')\, W_{n+2}^{(g)}(z,L,x'')
\end{aligned} \tag{VII.193}$$

We use the loop equation of theorem VII.2, which says that $U^{(g)}_{n+1}(z,L) + (2\omega(z) - V'(z) + \hbar\partial_z)W^{(g)}_{n+1}(z,L)$ has no pole at $z \to s_i$, and thus:

$$\begin{aligned}
& \delta W^{(g)}_{n+1}(x,L) \\
= \quad & -\sum_i \operatorname*{Res}_{z\to s_i} \delta K(x,z)\ (2\omega(z) - V'(z) + \hbar\partial_z)W^{(g)}_{n+1}(z,L) \\
& +2\sum_i \operatorname*{Res}_{z\to s_i} \sum_k \operatorname*{Res}_{x''\to s_k} K(x,z)\ \delta V(x'')\ B(z,x'')W^{(g)}_{n+1}(z,L) \\
& +2\sum_i \operatorname*{Res}_{z\to s_i} \operatorname*{Res}_{x''\to z} K(x,z)\ \delta V(x'')\ B(z,x'')W^{(g)}_{n+1}(z,L) \\
& -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'')\ W^{(g)}_{n+2}(z,L,x'') \\
= \quad & -\sum_i \operatorname*{Res}_{z\to s_i} W^{(g)}_{n+1}(z,L)\ (2\omega(z) - V'(z) - \hbar\partial_z)\delta K(x,z) \\
& +2\sum_i \operatorname*{Res}_{z\to s_i} \sum_k \operatorname*{Res}_{x''\to s_k} K(x,z)\ \delta V(x'')\ B(z,x'')W^{(g)}_{n+1}(z,L) \\
& +2\sum_i \operatorname*{Res}_{z\to s_i} \operatorname*{Res}_{x''\to z} K(x,z)\ \delta V(x'')\ B(z,x'')W^{(g)}_{n+1}(z,L) \\
& -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'')\ W^{(g)}_{n+2}(z,L,x'')
\end{aligned} \tag{VII.194}$$

and we have:

$$(2\omega(z) - V'(z) - \hbar\partial_z)\delta K(x,z) = \delta G(x,z) - (2\delta\omega(z) - \delta V'(z))K(x,z) \tag{VII.195}$$

$$\begin{aligned}
& \delta W^{(g)}_{n+1}(x,L) \\
= \quad & -\sum_i \operatorname*{Res}_{z\to s_i} W^{(g)}_{n+1}(z,L)\ \delta G(x,z) \\
& +\sum_i \operatorname*{Res}_{z\to s_i} W^{(g)}_{n+1}(z,L)\ (2\delta\omega(z) - \delta V'(z))\, K(x,z) \\
& +2\sum_i \operatorname*{Res}_{z\to s_i} \sum_k \operatorname*{Res}_{x''\to s_k} K(x,z)\ \delta V(x'')\ B(z,x'')W^{(g)}_{n+1}(z,L) \\
& +\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \delta V'(z)\ W^{(g)}_{n+1}(z,L) \\
& -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'')\ W^{(g)}_{n+2}(z,L,x'')
\end{aligned} \tag{VII.196}$$

We have:

$$\sum_i \operatorname*{Res}_{z\to s_i} W_{n+1}^{(g)}(z,L)\ \delta G(x,z) = 0 \tag{VII.197}$$

because the integrand is a rational fraction, and we have taken the sum of residues at all poles.

Using eq. (VII.179), we are thus left with:

$$\delta W_{n+1}^{(g)}(x,L) = -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'')\, W_{n+2}^{(g)}(z,L,x'') \tag{VII.198}$$

which proves the recursion hypothesis for $2g+n+1$. QED.

## Appendix: Proof of theorem VII.7

**Theorem VII.7**

For $k = 0, 1$, $W_n^{(g)}$ satify the equation:

$$\begin{aligned} &\Big(-\sum_{i=1}^n x_i^k \frac{\partial}{\partial x_i}\Big) W_n^{(g)}(x_1,\dots,x_n) \\ =\ &\sum_i \operatorname*{Res}_{x_{n+1}\to s_i} x_{n+1}^k V'(x_{n+1})\, W_{n+1}^{(g)}(x_1,\dots,x_n,x_{n+1}) \end{aligned} \tag{VII.199}$$

*Proof.* Since $W_{n+1}^{(g)}$ has poles only at the $s_i$'s we have (with as usual $J = \{x_1,\dots,x_n\}$):

$$\begin{aligned} &\sum_i \operatorname*{Res}_{x\to s_i} x^k V'(x)\, W_{n+1}^{(g)}(J,x) \\ =\ &\sum_i \operatorname*{Res}_{x\to s_i} x^k Y(x)\, W_{n+1}^{(g)}(J,x) \end{aligned} \tag{VII.200}$$

Then using theorem VII.2, we have:

$$\begin{aligned} &\sum_i \operatorname*{Res}_{x\to s_i} x^k V'(x)\, W_{n+1}^{(g)}(J,x) \\ =\ &\sum_i \operatorname*{Res}_{x\to s_i} x^k Y(x)\, W_{n+1}^{(g)}(J,x) \\ =\ &\sum_i \operatorname*{Res}_{x\to s_i} x^k \Big[\hbar\partial_x W_{n+1}^{(g)}(J,x) + U_{n+1}^{(g)}(x,J) - P_{n+1}^{(g)}(x;J) - \sum_{j=1}^n \partial_{x_j} \frac{W_n^{(g)}(J)}{x-x_j}\Big] \end{aligned}$$

$$= \sum_i \operatorname*{Res}_{x\to s_i} x^k \left[\hbar\partial_x W_{n+1}^{(g)}(J,x) + U_{n+1}^{(g)}(x,J)\right] \tag{VII.201}$$

Notice that if $n \geq 1$, $W_{n+1}^{(g)}(J,x)$ behaves like $O(1/x^2)$ at $x \to \infty$, and thus, if $k \leq 1$, $x^k\,\partial_x W_{n+1}^{(g)}(J,x)$ behaves like $O(1/x^2)$. Since we take the residues at all poles, the sum of residues vanish and thus:

$$\begin{aligned} & \sum_i \operatorname*{Res}_{x\to s_i} x^k V'(x)\, W_{n+1}^{(g)}(J,x) \\ = & \sum_i \operatorname*{Res}_{x\to s_i} x^k U_{n+1}^{(g)}(x,J) \end{aligned} \tag{VII.202}$$

Notice that $U_{n+1}^{(g)}(x,J)$ (defined in eq. (VII.190)), behaves at most like $O(1/x^3)$ for large $x$, and thus, if $k \leq 1$, the product $x^k U_{n+1}^{(g)}(x,J)$ is a rational fraction, which behaves like $O(1/x^2)$ for large $x$. Its only poles can be at $x = s_i$ or at $x = x_j$. Therefore the sum of residues at $s_i$'s, can be replaced by the sum of residues at $x_j$'s:

$$\begin{aligned} & \sum_i \operatorname*{Res}_{x\to s_i} x^k V'(x)\, W_{n+1}^{(g)}(J,x) \\ = & -\sum_{j=1}^{n} \operatorname*{Res}_{x\to x_j} x^k U_{n+1}^{(g)}(x,J) \end{aligned} \tag{VII.203}$$

The only terms in $U_{n+1}^{(g)}(x,J)$ which have poles at $x = x_j$, are the terms containing a $B(x,x_j)$, i.e.:

$$\begin{aligned} \sum_i \operatorname*{Res}_{x\to s_i} x^k V'(x)\, W_{n+1}^{(g)}(J,x) &= -2\sum_{j=1}^{n} \operatorname*{Res}_{x\to x_j} x^k B(x,x_j)\, W_n^{(g)}(x, J/\{x_j\}) \\ &= -\sum_{j=1}^{n} \operatorname*{Res}_{x\to x_j} x^k \frac{1}{(x-x_j)^2}\, W_n^{(g)}(x, J/\{x_j\}) \\ &= -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(x_j^k\, W_n^{(g)}(x_1,\ldots,x_n)\right) \end{aligned} \tag{VII.204}$$

## Appendix: Proof of theorem VII.8

**Theorem VII.8:**
*For $n \geq 1$, $W_n^{(g)}$ satify the equation:*

$$(2-2g-n-\hbar\frac{\partial}{\partial\hbar})\,\overline{W}_n^{(g)}(x_1,\ldots,x_n) = -\sum_i \operatorname*{Res}_{x_{n+1}\to s_i} V(x_{n+1})\,\overline{W}_{n+1}^{(g)}(x_1,\ldots,x_n,x_{n+1}) \tag{VII.205}$$

### $\hbar$ derivatives for $w(z)$

We have:

$$V'(s_i) = 2\hbar\sum_{\neq i}\frac{1}{s_i-s_j}$$

Taking the derivative with respect to $\hbar$ gives:

$$\hbar V''(s_i)\partial_\hbar s_i = V'(s_i) - 2\hbar^2\sum_{j\neq i}\frac{\partial_\hbar si - \partial_\hbar s_j}{(s_i-s_j)^2}$$

and so

$$V'(s_i) = \hbar\left(V''(s_i)\partial_\hbar s_i + 2\hbar\sum_{j\neq i}\frac{\partial_\hbar si - \partial_\hbar s_j}{(s_i-s_j)^2}\right)$$

We recognize the general term of the matrix $T$ and find:

$$V'(s_i) = \hbar^2\sum_j T_{i,j}\partial_\hbar s_j$$

Multiplying by the matrix $A$ gives:

$$\boxed{\hbar^2\partial_\hbar s_i = \sum_j A_{i,j}V'(s_j)} \tag{VII.206}$$

We can use this result to compute:

$$\hbar\partial_\hbar\omega(x) \qquad = \qquad \omega(x) + \hbar^2\sum_i\frac{\partial_\hbar si}{(x-s_i)^2}$$

$$
\begin{aligned}
&= \omega(x) + \sum_{i,j} \frac{A_{i,j} V'(s_j)}{(x-s_i)^2} \\
&= \omega(x) + \sum_k \operatorname*{Res}_{x' \to s_k} \sum_{i,j} \frac{A_{i,j} V'(x')}{(x-s_i)^2 (x'-s_j)} \\
&= \omega(x) + \sum_k \operatorname*{Res}_{x' \to s_k} \sum_{i,j} \frac{A_{i,j} V(x')}{(x-s_i)^2 (x'-s_j)^2} \\
&= \omega(x) + \sum_k \operatorname*{Res}_{x' \to s_k} \overline{W}_2^{(0)}(x,x') V(x') \\
&= \omega(x) + \sum_k \operatorname*{Res}_{x' \to s_k} W_2^{(0)}(x,x') V(x')
\end{aligned}
\tag{VII.207}
$$

Thus we have proved the case $n = 1, g = 0$ of the theorem:

$$
\boxed{\hbar \partial_\hbar \omega(x) = \omega(x) + \sum_k \operatorname*{Res}_{x' \to s_k} W_2^{(0)}(x,x') V(x')}
\tag{VII.208}
$$

### $\hbar$ derivatives for $W_2^{(0)}(z)$

We have seen in appendix VII, eq. (VII.184), that $\overline{W}_2^{(0)}(x,x')$ satisfies the loop equation:

$$
(2\omega(x) - V'(x) + \hbar \partial_x)\, \overline{W}_2^{(0)}(x,x') + \frac{\partial}{\partial x'} \frac{\omega(x) - \omega(x')}{x - x'} = -P_2^{(0)}(x,x') \tag{VII.209}
$$

where $P_2^{(0)}(x,x')$ has no pole at $x \to s_i$'s.

Then we take the derivation $\hbar \partial_\hbar$ of this equation:

$$
\begin{aligned}
(2\omega(x) - V'(x) + \hbar \partial_x)\, \hbar \partial_\hbar \overline{W}_2^{(0)}(x,x') + \hbar \partial_x \overline{W}_2^{(0)}(x,x') + 2\hbar \partial_\hbar w(x) \overline{W}_2^{(0)}(x,x') \\
= -\frac{\partial}{\partial x'} \frac{\hbar \partial_\hbar \omega(x) - \hbar \partial_\hbar \omega(x')}{x - x'} - \hbar \partial_\hbar P_2^{(0)}(x,x')
\end{aligned}
\tag{VII.210}
$$

$\hbar \partial_\hbar \overline{W}_2^{(0)}(x,x')$ is a rational fraction of $x$, with poles only at the $s_i$'s, and $\hbar \partial_\hbar P_2^{(0)}(x,x')$ has no pole at $x \to s_i$'s. We thus write:

$$
\hbar \partial_\hbar W_2^{(0)}(x,x')
$$

$$
\begin{aligned}
&= \quad \hbar\partial_\hbar \overline{W}_2^{(0)}(x,x') \\
&= \quad \operatorname*{Res}_{z\to x} G(x,z)\ \hbar\partial_\hbar \overline{W}_2^{(0)}(z,x') \\
&= \quad -\sum_i \operatorname*{Res}_{z\to s_i} G(x,z)\ \hbar\partial_\hbar \overline{W}_2^{(0)}(z,x') \\
&= \quad -\sum_i \operatorname*{Res}_{z\to s_i} \left((2\omega(z)-V'(z)-\hbar\partial_z)K(x,z)\right)\ \hbar\partial_\hbar \overline{W}_2^{(0)}(z,x') \\
&= \quad -\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \left((2\omega(z)-V'(z)+\hbar\partial_z)\ \hbar\partial_\hbar \overline{W}_2^{(0)}(z,x')\right) \\
&= \quad \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \Big((2\hbar\partial_\hbar\omega(z))\ \overline{W}_2^{(0)}(z,x') \\
&\qquad +\frac{\partial}{\partial x'}\frac{\hbar\partial_\hbar\omega(z)+\hbar\partial_\hbar\omega(x')}{z-x'}+\hbar\partial_\hbar P_2^{(0)}(z,x')+\hbar\partial_z\overline{W}_2^{(0)}(z,x')\Big) \\
&= \quad \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \left(2\overline{W}_2^{(0)}(z,x')\ \hbar\partial_\hbar\omega(z)+\frac{\hbar\partial_\hbar\omega(z)}{(z-x')^2}+\hbar\partial_z\overline{W}_2^{(0)}(z,x')\right) \\
&= \quad \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \left(2W_2^{(0)}(z,x')\,\hbar\partial_\hbar\omega(z)+\hbar\partial_z W_2^{(0)}(z,x')\right)
\end{aligned}
\tag{VII.211}
$$

Then, we use eq. (VII.208), and we get:

$$
\begin{aligned}
&\quad \hbar\partial_\hbar W_2^{(0)}(x,x') \\
&= \quad \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \left(2W_2^{(0)}(z,x')w(z)+\hbar\partial_z W_2^{(0)}(z,x')\right) \\
&\qquad +2\sum_{i,k}\operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x''\to s_k} K(x,z)W_2^{(0)}(z,x')W_2^{(0)}(z,x'')V(x'') \\
&= \quad \sum_i \operatorname*{Res}_{z\to s_i} W_2^{(0)}(z,x')\ \Big(2w(z)-\hbar\partial_z\Big)K(x,z) \\
&\qquad +\sum_{i,k}\operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x''\to s_k} K(x,z)W_2^{(0)}(z,x')G(z,x'')V'(x'') \\
&= \quad \sum_i \operatorname*{Res}_{z\to s_i} W_2^{(0)}(z,x')\ (G(x,z)+V'(z)K(x,z)) \\
&\qquad +\sum_{i,k}\operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x''\to s_k} K(x,z)W_2^{(0)}(z,x')G(z,x'')V'(x'') \\
&= \quad \sum_i \operatorname*{Res}_{z\to s_i} W_2^{(0)}(z,x')\ G(x,z) \\
&\qquad +\sum_{i,k}\operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x''\to s_k} K(x,z)W_2^{(0)}(z,x')G(z,x'')V'(x'') \\
&\qquad +\sum_{i}\operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x''\to z} K(x,z)W_2^{(0)}(z,x')G(z,x'')V'(x'') \\
&= \quad \sum_i \operatorname*{Res}_{z\to s_i} W_2^{(0)}(z,x')\ G(x,z)
\end{aligned}
$$

$$
\begin{aligned}
& + \sum_{i,k} \operatorname*{Res}_{x'' \to s_k} \operatorname*{Res}_{z \to s_i} K(x,z) W_2^{(0)}(z,x') G(z,x'') V'(x'') \\
= & \sum_i \operatorname*{Res}_{z \to s_i} W_2^{(0)}(z,x')\, G(x,z) \\
& + 2 \sum_{i,k} \operatorname*{Res}_{x'' \to s_k} \operatorname*{Res}_{z \to s_i} K(x,z) W_2^{(0)}(z,x') B(z,x'') V(x'') \\
= & \sum_i \operatorname*{Res}_{z \to s_i} B(z,x')\, G(x,z) \\
& + \sum_k \operatorname*{Res}_{x'' \to s_k} W_3^{(0)}(x,x',x'') V(x'')
\end{aligned}
\tag{VII.212}
$$

We now use the fact that $G(x,z)$ and $B(z,x')$ are rational fractions whose only poles are $s_i$'s, as well as $z = x$ and $z = x'$, and we write:

$$
\begin{aligned}
& \sum_i \operatorname*{Res}_{z \to s_i} B(z,x')\, G(x,z) \\
= & - \operatorname*{Res}_{z \to x} B(z,x')\, G(x,z) - \operatorname*{Res}_{z \to x'} B(z,x')\, G(x,z) \\
= & - \operatorname*{Res}_{z \to x} B(z,x')\, \frac{1}{z-x} - \frac{1}{2} \operatorname*{Res}_{z \to x'} \frac{1}{(z-x')^2}\, G(x,z) \\
= & - \operatorname*{Res}_{z \to x} B(z,x')\, \frac{1}{z-x} + \operatorname*{Res}_{z \to x'} \frac{1}{z-x'}\, B(x,z) \\
= & -B(x,x') + B(x,x') \\
= & \ 0
\end{aligned}
\tag{VII.213}
$$

So that eventually we have proved the case $n = 2, g = 0$ of the theorem:

$$
\boxed{\hbar \partial_\hbar W_2^{(0)}(x,x') = \sum_k \operatorname*{Res}_{x'' \to s_k} W_3^{(0)}(x,x',x'')\, V(x'')}
\tag{VII.214}
$$

**Recursion for higher correlators**

We proceed by recursion on $2g+n$.

From theorem VII.2, we have that:

$$
\begin{aligned}
& (Y(x) - \hbar \partial_x) \hbar \partial_\hbar W_{n+1}^{(g)}(x,L) \\
= & \ \hbar \partial_\hbar U_{n+1}^{(g)}(x;L) + \hbar \partial_x W_{n+1}^{(g)}(x,L) - W_{n+1}^{(g)}(x,L)\, \hbar \partial_\hbar Y(x)
\end{aligned}
$$

$$-\hbar\partial_\hbar\left(P^{(g)}_{n+1}(x;L)+\sum_{x_j\in L}\frac{\partial}{\partial x_j}\frac{\overline{W}^{(g)}_n(L)}{x-x_j}\right) \tag{VII.215}$$

where the term on the last line has no pole at $x=s_i$. This implies that:

$$\begin{aligned}
&\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left((Y(x)-\hbar\partial_x)\hbar\partial_\hbar W^{(g)}_{n+1}(x,L)\right)\\
=\;&\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\hbar\partial_\hbar U^{(g)}_{n+1}(x;L)+\hbar\partial_x W^{(g)}_{n+1}(x,L)\right.\\
&\left.-W^{(g)}_{n+1}(x,L)\,\hbar\partial_\hbar Y(x)\right)
\end{aligned} \tag{VII.216}$$

We have:

$$\begin{aligned}
&\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left((Y(x)-\hbar\partial_x)\hbar\partial_\hbar W^{(g)}_{n+1}(x,L)\right)\\
=\;&\sum_i \operatorname*{Res}_{x\to s_i} \hbar\partial_\hbar W^{(g)}_{n+1}(x,L)\,(Y(x)+\hbar\partial_x)K(x_0,x)\\
=\;&-\sum_i \operatorname*{Res}_{x\to s_i} \hbar\partial_\hbar W^{(g)}_{n+1}(x,L)\,G(x_0,x)\\
=\;&\operatorname*{Res}_{x\to x_0} \hbar\partial_\hbar W^{(g)}_{n+1}(x,L)\,G(x_0,x)\\
=\;&\hbar\partial_\hbar W^{(g)}_{n+1}(x_0,L)
\end{aligned} \tag{VII.217}$$

and therefore:

$$\begin{aligned}
&\hbar\partial_\hbar W^{(g)}_{n+1}(x_0,L)\\
=\;&\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\hbar\partial_\hbar U^{(g)}_{n+1}(x;L)+\hbar\partial_x W^{(g)}_{n+1}(x,L)-W^{(g)}_{n+1}(x,L)\,\hbar\partial_\hbar Y(x)\right)
\end{aligned} \tag{VII.218}$$

From the recursion hypothesis we have:

$$\begin{aligned}
&\hbar\partial_\hbar U^{(g)}_{n+1}(x;L)\\
=\;&\hbar\partial_\hbar W^{(g-1)}_{n+2}(x,x,L)+\sum_{k=0}^{g}{\sum_{J\subset L}}' W^{(k)}_{1+|J|}(x,J)\hbar\partial_\hbar W^{(g-k)}_{1+n-|J|}(x,L/J)\\
&+\sum_{k=0}^{g}{\sum_{J\subset L}}' W^{(g-k)}_{1+n-|J|}(x,L/J)\hbar\partial_\hbar W^{(k)}_{1+|J|}(x,J)\\
=\;&(2-2(g-1)-(n+2))W^{(g-1)}_{n+2}(x,x,L)+\sum_i \operatorname*{Res}_{x'\to s_i} W^{(g-1)}_{n+3}(x,x,L,x')\,V(x')
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{g}{\sum_{J\subset L}}'(2-2(g-k)-(1+n-|J|))\,W_{1+|J|}^{(k)}(x,J)\,W_{1+n-|J|}^{(g-k)}(x,L/J)\\
&+\sum_{k=0}^{g}{\sum_{J\subset L}}'(2-2k-(1+|J|))\,W_{1+n-|J|}^{(g-k)}(x,L/J)\,W_{1+|J|}^{(k)}(x,J)\\
&+\sum_i \operatorname*{Res}_{x'\to s_i} V(x')\sum_{k=0}^{g}{\sum_{J\subset L}}'W_{2+|J|}^{(k)}(x,J,x')\,W_{1+n-|J|}^{(g-k)}(x,L/J)\\
&+\sum_i \operatorname*{Res}_{x'\to s_i} V(x')\sum_{k=0}^{g}{\sum_{J\subset L}}'W_{1+|J|}^{(k)}(x,J)\,W_{2+n-|J|}^{(g-k)}(x,L/J,x')\\
=\;&(2-2g-n)\,U_{n+1}^{(g)}(x;L)\\
&+\sum_i \operatorname*{Res}_{x'\to s_i} V(x')\,(U_{n+2}^{(g)}(x;x',L)-2B(x,x')W_{n+1}^{(g)}(x,L))
\end{aligned}
\tag{VII.219}
$$

Thus we have:

$$
\begin{aligned}
&\hbar\partial_\hbar W_{n+1}^{(g)}(x_0,L)\\
=\;&(2-2g-n)\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\,U_{n+1}^{(g)}(x;L)\\
&+\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\sum_j \operatorname*{Res}_{x'\to s_j} V(x')\,(U_{n+2}^{(g)}(x;x',L)-2B(x,x')W_{n+1}^{(g)}(x,L))\\
&+\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\hbar\partial_x W_{n+1}^{(g)}(x,L)-W_{n+1}^{(g)}(x,L)\,\hbar\partial_\hbar Y(x)\right)\\
=\;&(2-2g-n)W_{n+1}^{(g)}(x_0,L)\\
&+\sum_j \operatorname*{Res}_{x'\to s_j}\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)V(x')\,(U_{n+2}^{(g)}(x;x',L)-2B(x,x')W_{n+1}^{(g)}(x,L))\\
&+\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\hbar\partial_x W_{n+1}^{(g)}(x,L)-W_{n+1}^{(g)}(x,L)\,\hbar\partial_\hbar Y(x)\right)\\
=\;&(2-2g-n)W_{n+1}^{(g)}(x_0,L)+\sum_j \operatorname*{Res}_{x'\to s_j} V(x')\,W_{n+2}^{(g)}(x_0,x',L)\\
&-2\sum_j \operatorname*{Res}_{x'\to s_j}\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)V(x')\,B(x,x')W_{n+1}^{(g)}(x,L)\\
&+\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\hbar\partial_x W_{n+1}^{(g)}(x,L)-W_{n+1}^{(g)}(x,L)\,\hbar\partial_\hbar Y(x)\right)\\
=\;&(2-2g-n)W_{n+1}^{(g)}(x_0,L)+\sum_j \operatorname*{Res}_{x'\to s_j} V(x')\,W_{n+2}^{(g)}(x_0,x',L)\\
&-2\sum_i \operatorname*{Res}_{x\to s_i}\sum_j \operatorname*{Res}_{x'\to s_j} K(x_0,x)V(x')\,B(x,x')W_{n+1}^{(g)}(x,L)\\
&-2\sum_i \operatorname*{Res}_{x\to s_i}\operatorname*{Res}_{x'\to x} K(x_0,x)V(x')\,B(x,x')W_{n+1}^{(g)}(x,L)\\
&+\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\hbar\partial_x W_{n+1}^{(g)}(x,L)-W_{n+1}^{(g)}(x,L)\,\hbar\partial_\hbar Y(x)\right)
\end{aligned}
\tag{VII.220}
$$

Notice that:

$$\hbar\partial_\hbar Y(x)+2\sum_j \operatorname*{Res}_{x'\to s_j} B(x,x')V(x')+2\operatorname*{Res}_{x'\to x} B(x,x')V(x')=Y(x) \tag{VII.221}$$

therefore:

$$\begin{aligned}
& \hbar\partial_\hbar W_{n+1}^{(g)}(x_0,L) \\
= \;& (2-2g-n)W_{n+1}^{(g)}(x_0,L)+\sum_j \operatorname*{Res}_{x'\to s_j} V(x')\,W_{n+2}^{(g)}(x_0,x',L) \\
& +\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\hbar\partial_x W_{n+1}^{(g)}(x,L)-Y(x)W_{n+1}^{(g)}(x,L)\right) \\
= \;& (2-2g-n)W_{n+1}^{(g)}(x_0,L)+\sum_j \operatorname*{Res}_{x'\to s_j} V(x')\,W_{n+2}^{(g)}(x_0,x',L) \\
& -\sum_i \operatorname*{Res}_{x\to s_i} W_{n+1}^{(g)}(x,L)\,(Y(x)+\hbar\partial_x)K(x_0,x) \\
= \;& (2-2g-n)W_{n+1}^{(g)}(x_0,L)+\sum_j \operatorname*{Res}_{x'\to s_j} V(x')\,W_{n+2}^{(g)}(x_0,x',L) \\
& +\sum_i \operatorname*{Res}_{x\to s_i} W_{n+1}^{(g)}(x,L)\,G(x_0,x) \\
= \;& (2-2g-n)W_{n+1}^{(g)}(x_0,L)+\sum_j \operatorname*{Res}_{x'\to s_j} V(x')\,W_{n+2}^{(g)}(x_0,x',L) \\
& -\operatorname*{Res}_{x\to x_0} W_{n+1}^{(g)}(x,L)\,G(x_0,x) \\
= \;& (2-2g-n)W_{n+1}^{(g)}(x_0,L)+\sum_j \operatorname*{Res}_{x'\to s_j} V(x')\,W_{n+2}^{(g)}(x_0,x',L) \\
& -W_{n+1}^{(g)}(x_0,L) \\
= \;& (2-2g-n-1)W_{n+1}^{(g)}(x_0,L)+\sum_j \operatorname*{Res}_{x'\to s_j} V(x')\,W_{n+2}^{(g)}(x_0,x',L)
\end{aligned} \tag{VII.222}$$

i.e. we have proved the theorem for $2g+n+1$.

## Appendix: Free Energies

Here we consider $g \geq 2$.

The free energies defined in eq. (VII.53), automatically satisfy theorem VII.8, and thus are homogeneous:

$$F^{(g)}(\lambda V,\lambda\hbar)=\lambda^{2-2g}\,F^{(g)}(V,\hbar) \tag{VII.223}$$

Here we show that they satisfy theorem VII.6.

We start from the definition:

$$F^{(g)} = \hbar^{2-2g} \int_0^{\hbar} \frac{d\tilde{\hbar}}{\tilde{\hbar}^{3-2g}} \sum_i \operatorname*{Res}_{x \to s_i} V(x)\ W_1^{(g)}(x) \Big|_{\tilde{\hbar}} \tag{VII.224}$$

and we compute the loop operator applied to $F^{(g)}$:

$$\begin{aligned}
\delta_{x_1} F^{(g)} &= \hbar^{2-2g} \int_0^{\hbar} \frac{d\tilde{\hbar}}{\tilde{\hbar}^{3-2g}} \sum_i \operatorname*{Res}_{x \to s_i} \left( V(x)\ W_2^{(g)}(x,x_1) + \delta_{x_1} V(x)\ W_1^{(g)}(x) \right)_{\tilde{\hbar}} \\
&= \hbar^{2-2g} \int_0^{\hbar} \frac{d\tilde{\hbar}}{\tilde{\hbar}^{3-2g}} \sum_i \operatorname*{Res}_{x \to s_i} \left( V(x)\ W_2^{(g)}(x,x_1) + \frac{W_1^{(g)}(x)}{x - x_1} \right)_{\tilde{\hbar}} \\
&= \hbar^{2-2g} \int_0^{\hbar} \frac{d\tilde{\hbar}}{\tilde{\hbar}^{3-2g}} \left( \left( \sum_i \operatorname*{Res}_{x \to s_i} V(x)\ W_2^{(g)}(x,x_1) \right) - W_1^{(g)}(x_1) \right)_{\tilde{\hbar}} \\
&= \hbar^{2-2g} \int_0^{\hbar} \frac{d\tilde{\hbar}}{\tilde{\hbar}^{3-2g}} \left( \tilde{h}^{2-2g} \frac{d(\tilde{h}^{2g-1} W_1^{(g)}(x_1))}{d\tilde{h}} - W_1^{(g)}(x_1) \right)_{\tilde{\hbar}} \\
&= \hbar^{2-2g} \int_0^{\hbar} \left( \frac{1}{\tilde{h}} d \left( \tilde{h}^{2g-1} W_1^{(g)}(x_1) \right) - \frac{d\tilde{\hbar}}{\tilde{\hbar}^{3-2g}} W_1^{(g)}(x_1) \right)_{\tilde{\hbar}}
\end{aligned} \tag{VII.225}$$

we integrate by parts, and since $2g-2 > 0$, there is no boundary term coming from the bound at 0, and thus:

$$\begin{aligned}
\delta_{x_1} F^{(g)} &= W_1^{(g)}(x_1) + \hbar^{2-2g} \int_0^{\hbar} \left( \tilde{h}^{2g-3} W_1^{(g)}(x_1) - \tilde{\hbar}^{2g-3} W_1^{(g)}(x_1) \right)_{\tilde{\hbar}} d\tilde{h} \\
&= W_1^{(g)}(x_1)
\end{aligned} \tag{VII.226}$$

Therefore we have proved that the loop operator acting on $F^{(g)}$ is indeed $W_1^{(g)}$, i.e. we have proved theorem VII.6.

**Appendix:** $F^{(0)}$

We have defined $F^{(0)}$ as:

$$F^{(0)} = -\hbar \sum_i V(s_i) + \hbar^2 \sum_{i \neq j} \ln(s_i - s_j) \tag{VII.227}$$

• Proof of theorem VII.6 for $F^{(0)}$:
consider a variation $\delta V$, we have:

$$\begin{aligned}\delta F^{(0)} &= -\hbar\sum_i \delta V(s_i) - \hbar\sum_i V'(s_i)\delta s_i + 2\hbar^2\sum_{j\neq i}\frac{\delta s_i}{s_i-s_j}\\ &= -\hbar\sum_i \delta V(s_i)\\ &= -\sum_i \operatorname*{Res}_{x\to s_i}\omega(x)\,\delta V(x)\end{aligned} \tag{VII.228}$$

• Proof of theorem VII.8 for $F^{(0)}$:
we have:

$$\begin{aligned}\hbar\partial_\hbar F^{(0)} &= -\hbar\sum_i V(s_i) + 2\hbar^2\sum_{i\neq j}\ln(s_i-s_j)\\ &\quad -\hbar^2\sum_i\frac{\partial s_i}{\partial\hbar}\left(V'(s_i) - 2\hbar\sum_{j\neq i}\frac{1}{s_i-s_j}\right)\\ &= -\hbar\sum_i V(s_i) + 2\hbar^2\sum_{i\neq j}\ln(s_i-s_j)\\ &= 2F^{(0)} + \hbar\sum_i V(s_i)\\ &= 2F^{(0)} + \sum_i \operatorname*{Res}_{x\to s_i}\omega(x)\,V(x)\end{aligned} \tag{VII.229}$$

Therefore:

$$(2-\hbar\partial_\hbar)F_0 = -\sum_i \operatorname*{Res}_{x\to s_i} V(x)w(x) \tag{VII.230}$$

**Appendix:** $F^{(1)}$

We have defined $F^{(1)}$ as:

$$\begin{aligned}F^{(1)} &= \frac{1}{2}\ln(\det A) + \frac{F^{(0)}}{\hbar^2} + \ln(\Delta(s)^2)\\ &= \frac{1}{2}\ln(\det A) - \frac{1}{\hbar}\sum_i V(s_i) + \sum_{i\neq j}\ln(s_i-s_j) + \sum_{i\neq j}\ln(s_i-s_j)\\ &= \frac{1}{2}\ln(\det A) - \frac{1}{\hbar}\sum_i V(s_i) + 2\sum_{i\neq j}\ln(s_i-s_j)\end{aligned} \tag{VII.231}$$

• Proof of theorem VII.6 for $F^{(1)}$:

Let us start from $W_1^{(1)}$

$$\begin{aligned} W_1^{(1)}(x) &= \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\,\overline{W}_2(z,z) \\ &= \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\left[\frac{A_{i,i}}{(z-s_i)^4} + 2\sum_{j\neq i}\frac{A_{i,j}}{(z-s_i)^2(z-s_j)^2}\right] \\ &= \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\,\frac{A_{i,i}}{(z-s_i)^4} \\ &\quad +2\sum_i\sum_{j\neq i} K'(x,s_i)\,\frac{A_{i,j}}{(s_i-s_j)^2} \\ &\quad -4\sum_i\sum_{j\neq i} K(x,s_i)\,\frac{A_{i,j}}{(s_i-s_j)^3} \end{aligned} \tag{VII.232}$$

We have:

$$\begin{aligned} &\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\,\frac{A_{i,i}}{(z-s_i)^4} \\ =\ & \frac{1}{3}\sum_i \operatorname*{Res}_{z\to s_i} K'(x,z)\,\frac{A_{i,i}}{(z-s_i)^3} \\ =\ & \frac{1}{3}\sum_i \operatorname*{Res}_{z\to s_i} \Big(\frac{2}{z-s_i} + 2\omega_i(z) - \frac{1}{\hbar}V'(z)\Big)K(x,z)\,\frac{A_{i,i}}{(z-s_i)^3} \\ & -\frac{1}{3\hbar}\sum_i \operatorname*{Res}_{z\to s_i} G(x,z)\,\frac{A_{i,i}}{(z-s_i)^3} \end{aligned} \tag{VII.233}$$

Therefore:

$$\begin{aligned} &\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\,\frac{A_{i,i}}{(z-s_i)^4} \\ =\ & \sum_i \operatorname*{Res}_{z\to s_i} \Big(2\omega_i(z) - \frac{1}{\hbar}V'(z)\Big)K(x,z)\,\frac{A_{i,i}}{(z-s_i)^3} \\ & -\frac{1}{\hbar}\sum_i \operatorname*{Res}_{z\to s_i} G(x,z)\,\frac{A_{i,i}}{(z-s_i)^3} \\ =\ & \sum_i \operatorname*{Res}_{z\to s_i} \Big[\frac{2\omega_i(z) - \frac{1}{\hbar}V'(z)}{z-s_i}K(x,z)\Big]\,\frac{A_{i,i}}{(z-s_i)^2} \\ & -\frac{1}{2\hbar}\sum_i \operatorname*{Res}_{z\to s_i} G'(x,z)\,\frac{A_{i,i}}{(z-s_i)^2} \end{aligned}$$

$$
\begin{aligned}
&= \sum_i A_{i,i}\Big[\frac{2\omega_i(z)-\frac{1}{\hbar}V'(z)}{z-s_i}K(x,z)\Big]'_{z=s_i} \\
&\quad +\frac{1}{\hbar}\sum_i \operatorname*{Res}_{z\to s_i} B(x,z)\,\frac{A_{i,i}}{(z-s_i)^2} \\
&= \frac{1}{2}\sum_i (2\omega_i''(s_i)-\frac{1}{\hbar}V'''(s_i))K(x,s_i)A_{i,i} \\
&\quad -\sum_i K'(x,s_i)A_{i,i}T_{i,i} \\
&\quad +\frac{1}{\hbar}\sum_i \operatorname*{Res}_{z\to s_i} B(x,z)\,\frac{A_{i,i}}{(z-s_i)^2}
\end{aligned}
\tag{VII.234}
$$

Notice that:

$$
\begin{aligned}
\operatorname*{Res}_{x\to s} K(x,s_i)\delta V(x) &= \frac{1}{\hbar}\sum_j \operatorname*{Res}_{x\to s}\frac{A_{i,j}\delta V(x)}{(x-s_j)^2} \\
&= \frac{1}{\hbar}\sum_j A_{i,j}\delta V'(s_j) \\
&= -\delta s_i
\end{aligned}
\tag{VII.235}
$$

$$
\operatorname*{Res}_{x\to s} K'(x,s_i)\delta V(x) = -\frac{1}{\hbar}\sum_j \delta_{i,j}\delta V(s_j) - 2\sum_{j\neq i}\frac{\delta s_j}{s_i-s_j}
\tag{VII.236}
$$

$$
\begin{aligned}
\operatorname*{Res}_{x\to s}\operatorname*{Res}_{z\to s_i}\frac{B(x,z)}{(z-s_i)^2}\,\delta V(x) &= \operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x\to s}\frac{B(x,z)}{(z-s_i)^2}\,\delta V(x) \\
&\quad +\operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x\to z}\frac{B(x,z)}{(z-s_i)^2}\,\delta V(x) \\
&= \operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x\to s}\frac{A_{j,l}}{(x-s_l)^2(z-s_j)^2(z-s_i)^2}\,\delta V(x) \\
&\quad +\frac{1}{2}\operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x\to z}\frac{1}{(x-z)^2(z-s_i)^2}\,\delta V(x) \\
&= \hbar\operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x\to s}\frac{K(x,s_j)}{(z-s_j)^2(z-s_i)^2}\,\delta V(x) \\
&\quad +\frac{1}{2}\operatorname*{Res}_{z\to s_i}\frac{1}{(z-s_i)^2}\,\delta V'(z) \\
&= -\hbar\operatorname*{Res}_{z\to s_i}\frac{\delta s_j}{(z-s_j)^2(z-s_i)^2}+\frac{1}{2}\,\delta V''(s_i)
\end{aligned}
$$

$$= \quad 2\hbar \frac{\delta s_j}{(s_i - s_j)^3} + \frac{1}{2}\,\delta V''(s_i) \qquad \text{(VII.237)}$$

That gives:

$$\begin{aligned}
&\operatorname*{Res}_x \operatorname*{Res}_{z\to s_i} \frac{K(x,z)A_{i,i}}{(x-s_i)^4}\,\delta V(x) \\
= \quad & -\frac{1}{2}(2\omega_i''(s_i) - \frac{1}{\hbar}V'''(s_i))\delta s_i A_{i,i} + \frac{1}{\hbar}\sum_j \delta_{i,j}\delta V(s_j)A_{i,i}T_{i,i} \\
& +2\sum_{j\neq i}\frac{\delta s_j}{s_i - s_j}A_{i,i}T_{i,i} + 2\hbar\frac{\delta s_j}{(s_i-s_j)^3}A_{i,i} + \frac{1}{2}\,\delta V''(s_i)A_{i,i} \\
= \quad & \frac{1}{2}\delta(T_{i,i})A_{i,i} + \frac{1}{\hbar}\sum_j \delta_{i,j}\delta V(s_j)A_{i,i}T_{i,i} + 2\sum_{j\neq i}\frac{\delta s_j}{s_i - s_j}A_{i,i}T_{i,i} \qquad \text{(VII.238)}
\end{aligned}$$

and thus:

$$\begin{aligned}
&\operatorname*{Res}_{x\to s} W_1^{(1)}(x)\delta V(x) \\
= \quad & \sum_i \frac{1}{2}\delta(T_{i,i})A_{i,i} + \frac{1}{\hbar}\sum_i\sum_j \delta_{i,j}\delta V(s_j)A_{i,i}T_{i,i} + 2\sum_{j\neq i}\frac{\delta s_j}{s_i - s_j}A_{i,i}T_{i,i} \\
& -2\sum_i\sum_{j\neq i}\frac{\frac{1}{\hbar}\sum_l \delta_{i,l}\delta V(s_l)}{(s_i-s_j)^2}A_{i,j} - 4\sum_i\sum_{j\neq i}\sum_{l\neq i}\frac{\delta s_l}{(s_i-s_l)(s_i-s_j)^2}A_{i,j} \\
& +4\sum_i\sum_{j\neq i}\frac{\delta s_i}{(s_i-s_j)^3}A_{i,j} \\
= \quad & \sum_i \frac{1}{2}\delta(T_{i,i})A_{i,i} + \frac{1}{\hbar}\sum_i\sum_j\sum_l \delta_{i,j}\delta V(s_j)A_{i,l}T_{l,i} + 2\sum_{j\neq i}\frac{\delta s_j}{s_i - s_j}A_{i,i}T_{i,i} \\
& -4\sum_i\sum_{j\neq i}\sum_{l\neq i}\frac{\delta s_l}{(s_i-s_l)(s_i-s_j)^2}A_{i,j} + 4\sum_i\sum_{j\neq i}\frac{\delta s_i}{(s_i-s_j)^3}A_{i,j} \\
= \quad & \frac{1}{2}\mathrm{Tr}A\;\delta T + \frac{1}{\hbar}\sum_i\sum_j\sum_l \delta_{i,j}\delta V(s_j)A_{i,l}T_{l,i} + 2\sum_{j\neq i}\frac{\delta s_j}{s_i - s_j} \\
& +4\sum_{j\neq i}\sum_{l\neq i}\frac{\delta s_j}{(s_i-s_j)(s_i-s_l)^2}A_{i,l} - 4\sum_{i\neq j\neq l}\frac{\delta s_l}{(s_i-s_l)(s_i-s_j)^2}A_{i,j} \\
= \quad & \frac{1}{2}\mathrm{Tr}A\;\delta T + \frac{1}{\hbar}\sum_j \delta V(s_j) - \sum_{j\neq i}\frac{\delta s_i - \delta s_j}{s_i - s_j} \\
= \quad & \frac{1}{2}\delta\mathrm{ln}\mathrm{det}T + \frac{1}{\hbar}\sum_j \delta(V(s_j)) - \frac{1}{\hbar}\sum_j V'(s_j)\delta s_j - \sum_{j\neq i}\frac{\delta s_i - \delta s_j}{s_i - s_j}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\delta \ln\det T + \frac{1}{\hbar}\sum_j \delta(V(s_j)) - 2\sum_j \sum_{i\neq j} \frac{\delta s_j}{s_j - s_i} - \sum_{j\neq i} \frac{\delta s_i - \delta s_j}{s_i - s_j} \\
&= \frac{1}{2}\delta \ln\det T + \frac{1}{\hbar}\sum_j \delta(V(s_j)) - \sum_j \sum_{i\neq j} \frac{\delta s_j - \delta s_i}{s_j - s_i} - \sum_{j\neq i} \frac{\delta s_i - \delta s_j}{s_i - s_j} \\
&= \frac{1}{2}\delta \ln\det T + \frac{1}{\hbar}\sum_j \delta(V(s_j)) - 2\sum_{i\neq j} \frac{\delta s_j - \delta s_i}{s_j - s_i}
\end{aligned}
\tag{VII.239}
$$

That implies:

$$
F_1 = -\frac{1}{2}\ln\det T - \frac{1}{\hbar}\sum_j V(s_j) + 2\sum_{i\neq j} \ln(s_i - s_j) \tag{VII.240}
$$

$$
\boxed{F_1 = \frac{1}{2}\ln\det A - \frac{1}{\hbar}\sum_j V(s_j) + 2\sum_{i\neq j} \ln(s_i - s_j)} \tag{VII.241}
$$

**Appendix: Example $m = 1$**

We choose $s = 0$, and $V'(s) = v_2 s + v_3 s^2 + \sum v_{k+1} s^k$.

We have

$$
\omega(x) = \frac{\hbar}{x} \tag{VII.242}
$$

$$
A = \frac{\hbar}{v_2} \tag{VII.243}
$$

$$
K(x_1, x) = \sum_k K_k(x_1)\, x^k \tag{VII.244}
$$

$$
K_0 = \frac{1}{v_2 x_1^2} \qquad , \qquad K_1 = K_2 = 0 \tag{VII.245}
$$

$$
K_3 = \frac{1}{\hbar x_1^3} - \frac{v_3}{\hbar v_2 x_1^2} \tag{VII.246}
$$

$$
B(x_1, x_2) = \frac{1}{2(x_1 - x_2)^2} + \frac{A}{x_1^2 x_2^2} \tag{VII.247}
$$

$$W_3^{(0)} = \frac{2\hbar}{v_2^2 x_1^2 x_2^2 x_3^2}(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}) - \frac{2\hbar v_3}{v_2^3 x_1^2 x_2^2 x_3^2} \quad \text{(VII.248)}$$

$$\begin{aligned} W_4^{(0)} = & \frac{6\hbar}{v_2^3 x_1^2 x_2^2 x_3^2}(\frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{1}{x_3^2}+\frac{1}{x_4^2}) \\ & + \frac{8\hbar}{v_2^3 x_1^2 x_2^2 x_3^2}(\frac{1}{x_1x_2}+\frac{1}{x_1x_3}+\frac{1}{x_1x_4}+\frac{1}{x_2x_3}+\frac{1}{x_2x_4}+\frac{1}{x_3x_4}) \\ & - \frac{12\hbar v_3}{v_2^4 x_1^2 x_2^2 x_3^2}(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}) + \frac{12\hbar v_3^2}{v_2^5 x_1^2 x_2^2 x_3^2} - \frac{6\hbar v_4}{v_2^4 x_1^2 x_2^2 x_3^2} \end{aligned} \quad \text{(VII.249)}$$

$$W_1^{(1)} = \frac{1}{\hbar x} + \frac{1}{v_2 x^3} - \frac{v_3}{v_2^2 x^2} \quad \text{(VII.250)}$$

$$\begin{aligned} W_2^{(1)} = & \frac{3}{v_2^2 x_1^2 x_2^2}(\frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{2}{3x_1x_2}) + \frac{1}{\hbar v_2 x_1^2 x_2^2} - \frac{4v_3}{v_2^3 x_1^2 x_2^2}(\frac{1}{x_1}+\frac{1}{x_2}) \\ & + \frac{4v_3^2}{v_2^4 x_1^2 x_2^2} - \frac{3v_4}{v_2^3 x_1^2 x_2^2} \end{aligned} \quad \text{(VII.251)}$$

$$\begin{aligned} W_3^{(1)} = & \frac{12}{v_2^3 x_1^2 x_2^2 x_3^2}(\frac{1}{x_1^3}+\frac{1}{x_2^3}+\frac{1}{x_3^3}) \\ & + \frac{12}{v_2^3 x_1^2 x_2^2 x_3^2}(\frac{1}{x_1^2x_2}+\frac{1}{x_2^2x_3}+\frac{1}{x_3^2x_1}+\frac{1}{x_1x_2^2}+\frac{1}{x_2x_3^2}+\frac{1}{x_3x_1^2}) \\ & + \frac{8}{v_2^3 x_1^3 x_2^3 x_3^3} + \frac{2}{\hbar v_2^2 x_1^2 x_2^2 x_3^2}(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}) \\ & - \frac{24v_3}{v_2^4 x_1^2 x_2^2 x_3^2}(\frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{1}{x_3^2}+\frac{1}{x_1x_2}+\frac{1}{x_2x_3}+\frac{1}{x_3x_1}) - \frac{2v_3}{\hbar v_2^3 x_1^2 x_2^2 x_3^2} \\ & + \frac{32v_3^2}{v_2^5 x_1^2 x_2^2 x_3^2}(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}) - \frac{32v_3^3}{v_2^6 x_1^2 x_2^2 x_3^2} - \frac{18v_4}{v_2^4 x_1^2 x_2^2 x_3^2}(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}) \\ & + \frac{42v_3v_4}{v_2^5 x_1^2 x_2^2 x_3^2} - \frac{12v_5}{v_2^4 x_1^2 x_2^2 x_3^2} \end{aligned} \quad \text{(VII.252)}$$

$$W_1^{(2)} = -\frac{1}{\hbar^3 x}+\frac{3}{\hbar v_2^2 x^5}-\frac{5v_3}{\hbar v_2^3 x^4}+\frac{5v_3^2}{\hbar v_2^4 x^3}-\frac{5v_3^3}{\hbar v_2^5 x^2}-\frac{3v_4}{\hbar v_2^3 x^3} +\frac{8v_3 v_4}{\hbar v_2^4 x^2}-\frac{3v_5}{\hbar v_2^3 x^2} \tag{VII.253}$$

$$\begin{aligned} W_2^{(2)} = {} & \frac{15}{\hbar v_2^3 x_1^2 x_2^2}(\frac{1}{x_1^4}+\frac{1}{x_2^4}+\frac{1}{x_1^2 x_2^2})+\frac{12}{\hbar v_2^3 x_1^2 x_2^2}(\frac{1}{x_1^3 x_2}+\frac{1}{x_1 x_2^3})-\frac{1}{\hbar^3 v_2 x_1^2 x_2^2} \\ & -\frac{32v_3}{\hbar v_2^4 x_1^2 x_2^2}(\frac{1}{x_1^3}+\frac{1}{x_2^3})-\frac{30v_3}{\hbar v_2^4 x_1^2 x_2^2}(\frac{1}{x_1 x_2^2}+\frac{1}{x_1^2 x_2})+\frac{45v_3^2}{\hbar v_2^5 x_1^2 x_2^2}(\frac{1}{x_1^2}+\frac{1}{x_2^2}) \\ & +\frac{40v_3^2}{\hbar v_2^5 x_1^3 x_2^3}-\frac{50v_3^3}{\hbar v_2^6 x_1^2 x_2^2}(\frac{1}{x_1}+\frac{1}{x_2})+\frac{50v_3^4}{\hbar v_2^7 x_1^2 x_2^2}-\frac{24v_4}{\hbar v_2^4 x_1^2 x_2^2}(\frac{1}{x_1^2}+\frac{1}{x_2^2}) \\ & -\frac{18v_4}{\hbar v_2^4 x_1^3 x_2^3}+\frac{64v_3 v_4}{\hbar v_2^5 x_1^2 x_2^2}(\frac{1}{x_1}+\frac{1}{x_2})-\frac{109v_3^2 v_4}{\hbar v_2^6 x_1^2 x_2^2}+\frac{24v_4^2}{\hbar v_2^5 x_1^2 x_2^2} \\ & -\frac{18v_5}{\hbar v_2^4 x_1^2 x_2^2}(\frac{1}{x_1}+\frac{1}{x_2})+\frac{50v_3 v_5}{\hbar v_2^5 x_1^2 x_2^2}-\frac{15v_6}{\hbar v_2^4 x_1^2 x_2^2} \end{aligned} \tag{VII.254}$$

$$\begin{aligned} W_1^{(3)} = {} & \frac{2}{\hbar^5 x}+\frac{15}{\hbar^2 v_2^3 x^7}-\frac{3}{\hbar^3 v_2^2 x^5}-\frac{35v_3}{\hbar^2 v_2^4 x^6}+\frac{5v_3}{\hbar^3 v_2^3 x^4}+\frac{50v_3^2}{\hbar^2 v_2^5 x^5}-\frac{5v_3^2}{\hbar^3 v_2^4 x^3} \\ & -\frac{60v_3^3}{\hbar^2 v_2^6 x^4}+\frac{5v_3}{\hbar^3 v_2^5 x^2}+\frac{60v_3^4}{\hbar^2 v_2^7 x^3}-\frac{60v_3^5}{\hbar^2 v_2^8 x^2}-\frac{24v_4}{\hbar^2 v_2^4 x^5}+\frac{3v_4}{\hbar^3 v_2^3 x^3} \\ & +\frac{75v_3 v_4}{\hbar^2 v_2^5 x^4}-\frac{8v_3 v_4}{\hbar^3 v_2^4 x^2}-\frac{125v_3^2 v_4}{\hbar^2 v_2^6 x^3}+\frac{185v_3^3 v_4}{\hbar^2 v_2^7 x^2}+\frac{24v_4^2}{\hbar^2 v_2^5 x^3}-\frac{99v_3 v_4^2}{\hbar^2 v_2^6 x^2} \\ & -\frac{21v_5}{\hbar^2 v_2^4 x^4}+\frac{3v_5}{\hbar^3 v_2^3 x^2}+\frac{56v_3 v_5}{\hbar^2 v_2^5 x^3}-\frac{106v_3^2 v_5}{\hbar^2 v_2^6 x^2}+\frac{45v_4 v_5}{\hbar^2 v_2^5 x^2}-\frac{15v_6}{\hbar^2 v_2^4 x^3} \\ & +\frac{50v_3 v_6}{\hbar^2 v_2^5 x^2}-\frac{15v_7}{\hbar^2 v_2^4 x^2} \end{aligned} \tag{VII.255}$$

The free energies are:

$$F_1 = \frac{1}{2}\ln(v_2/\hbar) \tag{VII.256}$$

$$F_2 = -\frac{5v_3^2}{6\hbar\, v_2^3} + \frac{3v_4}{4\hbar\, v_2^2} \tag{VII.257}$$

$$F_3 = \frac{5v_3^2}{6\hbar^3\, v_2^3} - \frac{5v_3^4}{\hbar^2\, v_2^6} - \frac{3v_4}{4\hbar^3\, v_2^2} + \frac{25v_3^2 v_4}{2\hbar^2\, v_2^5} - \frac{3v_4^2}{\hbar^2\, v_2^4} - \frac{7v_3 v_5}{\hbar^2\, v_2^4} + \frac{5v_6}{2\hbar^2\, v_2^3} \tag{VII.258}$$

**Annexe VIII**

# Topological expansion of the Bethe ansatz and quantum algebraic geometry

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**Abstract**: In this article, we solve the loop equations of the $\beta$-random matrix model, in a way similar to what was found for the case of hermitian matrices $\beta = 1$. For $\beta = 1$, the solution was expressed in terms of algebraic geometry properties of an algebraic spectral curve of equation $y^2 = U(x)$. For arbitrary $\beta$, the spectral curve is no longer algebraic, it is a Schrödinger equation $((\hbar\partial)^2 - U(x)).\psi(x) = 0$ where $\hbar \propto (\sqrt{\beta} - 1/\sqrt{\beta})$. In this article, we find a solution of loop equations, which takes the same form as the topological recursion found for $\beta = 1$. This allows to define natural generalizations of all algebraic geometry properties, like the notions of genus, cycles, forms of 1st, 2nd and 3rd kind, Riemann bilinear identities, and spectral invariants $F_g$, for a quantum spectral curve, i.e. a D-module of the form $y^2 - U(x)$, where $[y,x] = \hbar$. Also, our method allows to enumerate non-oriented discrete surfaces.

## 1 Introduction

### Spectral invariants and algebraic geometry

In [23, 105], was presented the definition of spectral invariants $F_g$ for any algebraic plane curve, i.e. given by a polynomial equation

$$\mathscr{E}(x,y) = \sum_{i,j} \mathscr{E}_{i,j} x^i y^j = 0.$$

Those invariants $F_g(\mathscr{E})$ are defined in terms of algebraic geometry quantities defined on the Riemann surface of equation $\mathscr{E}(x,y) = 0$. Their definition involves residues at branchpoints of some meromorphic forms. Their definition provides a natural basis of meromorphic forms of 1st, 2nd and 3rd kind, and a natural framework for all algebraic geometry notions.

Moreover, the invariants $F_g$ of [23] have many nice properties, for instance their deformations under changes of the complex structure of $\mathscr{E}$ is given by some "special geometry" relations, and provide a natural form-cycle duality. Also, they are invariants under changes of $\mathscr{E}$ which conserve the symplectic form $dx \wedge dy$ in $\mathbb{C} \times \mathbb{C}$, they have nice modular properties, and finally, they define the tau-function of some dispersionfull integrable system associated to $\mathscr{E}$.

Also, those invariants $F_g$ have deep relationships with enumerative geometry, for instance they have been related to the Kodaira-Spencer theory [103], to combinatorics of discrete surfaces (maps), to intersection theory [23, 107], and they are conjectured to be equal to the Gromov-Witten invariants of some toric Calabi-Yau target 3-folds [81].

### Algebraic geometry on "quantum" curves

Here, our goal is to define those notions for a **"quantum curve"**, where $\mathscr{E}(x,y)$ is a non-commutative polynomial of $x$ and $y$:

$$\mathscr{E}(x,y) = \sum_{i,j} \mathscr{E}_{i,j} x^i y^j \qquad , \quad [y,x] = \hbar.$$

The notion of quantum curve has arised in many ways in the litterature [102], and is also called D-modules, i.e. a space of functions quotiented by $\mathrm{Ker}\,\mathscr{E}(x,y)$, where $y = \hbar\partial/\partial x$.

In other words, one has to study functions $\psi(x)$ such that:

$$\mathscr{E}(x, \hbar\partial_x).\psi(x) = 0.$$

In our attempt to define the spectral invariants analogous to those of [23] for such D-modules, we are naturally led to define all analogous properties of algebraic geometry. For instance we define the notions of **branch points**, **sheets**, **genus**, **cycles**, **forms**, **Bergman kernel**, and so on...

Because of non-commutativity, some notions like branch-points, cuts and sheets, become "blurred" or "non-localized", i.e. the branchpoint is no longer a point, but a "region" of the complex plane, and cuts are asymptotic accumulation lines of points.

But, otherwise, it is surprising to find that almost all relationships of classical algebraic geometry, remained unchanged when $\hbar \neq 0$, for instance the Riemann bilinear identity, the Rauch variational formula, and the topological recursion defining the spectral invariants.

Moreover, we shall find, that in order for our quantities to make sense, we must have a "vanishing monodromy" condition, which can be interpreted as a **Bethe ansatz**, and this gives a geometrical interpretation of the Bethe ansatz.

Let us also mention that in a previous article [44], we treated a special case, where the Schrödinger potential $U(x)$ was quantized, and we shall see, under the light of this new work, that it was the case of a degenerate quantum surface, with no branchpoints.

### Hyperelliptical case

Here, for simplicity, we shall restrict ourselves to polynomials of degree 2 in $y$ (called hyperelliptical in algebraic geometry), of the form:

$$\mathscr{E}(x,y) = y^2 - U(x) \qquad , \qquad [y,x] = \hbar$$

i.e. to the Schrödinger equation:

$$\hbar^2 \psi'' = U\psi.$$

We leave the higher degree case for a further work.

### Link with $\beta$ matrix models

The spectral invariants $F_g$ were first introduced for the solution of loop equations arising in the 1-hermitian random matrix model [99, 105]. They were later generalized to other hermitian multi–matrix models [83, 100].

There exist other matrix models, which are defined with non hermitian matrices. In fact it is well known since Wigner [19] that depending on the symmetry of the problem,

it is sometimes interesting to have matrices that are not hermitian. (For example, real-symmetric, unitary, orthogonal or quaternionic, ...). Therefore, it seems reasonable to extend the definition of the spectral invariants for those other models. Those other matrix models are often called $\beta$-ensembles, and they are classified by an exponent $\beta$. The 3 Wigner ensembles (see [19], and we changed $\beta \to \beta/2$) correspond to $\beta = 1$ (hermitian case), $\beta = 1/2$ (real symmetric case), $\beta = 2$ (real self-dual quaternion case), but it is easy to define a $\beta$ one-matrix model for any other value of $\beta$ (see section VIII for more details).

In [98], a first attempt to generalize the solution of [105] to other matrix models was proposed, but it was not as nice as the topological recursion of [105]. In [98], it was assumed that $\beta = O(1)$ when the size $N$ of the random matrix becomes large, and it was found that all spectral invariants were related to a double series expansion of the form:

$$\sum_{g,k} N^{2-2g-k} \, (\sqrt{\beta} - 1/\sqrt{\beta})^k F_{g,k}$$

The coefficients $F_{g,k}$ were computed in [98]. Here, in this article, we shall work at fixed $\hbar = (\sqrt{\beta} - 1/\sqrt{\beta})/N$, instead of fixed $\beta$, i.e. we shall define the resummed $F_g$'s as:

$$F_g(\hbar) = \sum_k \hbar^k F_{g,k}.$$

The $F_{g,k}$'s of [98] can be recovered by computing the semi-classical small $\hbar$ expansion of $F_g(\hbar)$. In this article we shall argue that $F_g(\hbar)$ is the natural generalization of the symplectic invariants of [23] for a "quantum spectral curve" $\mathscr{E}(x,y)$ with $[y,x] = \hbar$.

The tool which we use for studying the $\beta$-matrix model, is the loop equation method. Loop equations are related to the invariance of an integral under change of variable. They can be obtained by integrating by parts. Loop equations for the $\beta$-matrix model have been written many times [104, 106], and here we show how to solve them order by order in $1/N$, at fixed $\hbar$.

The $\beta$-matrix model and its loop equations are explained in section VIII.

## 2 Schrödinger equation and Bethe ansatz

### 2.1 Schrödinger equation, generalities and notation

Let:

$$\hbar^2 \psi''(x) = U(x)\, \psi(x) \tag{VIII.1}$$

be a Schrödinger equation with $U(x)$ a polynomial. Let $U(x)$ be a polynomial of degree $2d$, and define the polynomial "potential" $V(x)$ of degree $d+1$ by its derivative:

$$V'(x) = 2\,(\sqrt{U})_+ = \sum_{k=0}^{d} t_{k+1}\,x^k \tag{VIII.2}$$

where $()_+$ means the polynomial part of the Laurent series at $x \to \infty$. We also define:

$$P(x) = \frac{V'^2(x)}{4} - U(x) - \hbar\frac{V''(x)}{2} \tag{VIII.3}$$

so that $P$ is a polynomial of degree $d-1$.

Eventually, we define:

$$t_0 = \lim_{x\to\infty} \frac{xP(x)}{V'(x)} \tag{VIII.4}$$

**Remark VIII.1.** Just in order to give names to those parameters, let us say that in the language of integrable systems, the coefficients $t_0, t_1, t_2, \ldots, t_{d+1}$ are called the "Casimirs", and the remaining coefficients of $P$ are the "conserved charges". They will play a special role later on in this article. In matrix model language (see section VIII), $t_1, \ldots, t_{d+1}$ are called the times associated to the potential $V(x)$, $t_0$ is often called the temperature, and the remaining coefficients of $P$ are called "filling fractions". In the language of algebraic geometry, parameters $t_k$ with $k \geq 1$ are coupled to 2nd kind meromorphic differential forms, $t_0$ is coupled to 3rd kind, and the remaining coefficients of $P$ are coupled to 1st kind holomorphic differentials, see section VIII about form-cycle duality.

### Stokes Sectors

From the study of the Schrödinger equation we know that the function $\psi(x)$ is subject to the Stokes phenomenon, i.e. although $\psi(x)$ is an entire function, its asymptotics look discontinuous near $\infty$. We therefore need to introduce properly the Stokes sectors by defining the following quantities: Let

$$\theta_0 = \mathrm{Arg}(t_{d+1})$$

be the argument of the leading coefficient of $V(x)$.

We define the Stokes lines going to $\infty$ as:

$$L_k = \left\{ x \,/\, \mathrm{Arg}(x) = -\frac{\theta_0}{d+1} + \pi \frac{k+\frac{1}{2}}{d+1} \right\} \tag{VIII.5}$$

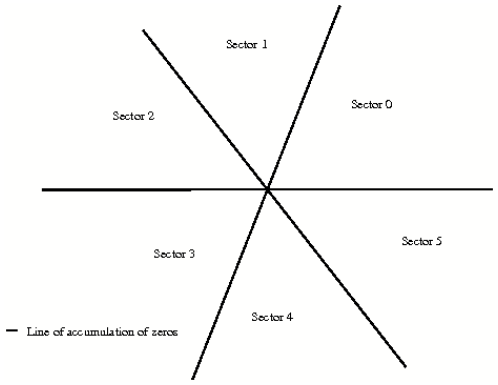
Those are the lines where $\mathrm{Re}V(x)$ vanishes asymptotically.

We define the sectors:

$$S_k = \left\{ \mathrm{Arg}(x) \in ]-\frac{\theta_0}{d+1} + \pi \frac{k-\frac{1}{2}}{d+1}, -\frac{\theta_0}{d+1} + \pi \frac{k+\frac{1}{2}}{d+1}[ \right\} \tag{VIII.6}$$

i.e. $S_k$ is the sector between $L_{k-1}$ and $L_k$.

Notice that in even sectors we have asymptotically $\mathrm{Re}V(x) > 0$ and in odd sectors we have $\mathrm{Re}V(x) < 0$.




Example of sectors for a potential of degree $\deg V = 3$, i.e. $d = 2$. If $\deg V = d+1$ there are $2d+2$ sectors.

**Stokes phenomenon**

Any solution of a linear equation, is analytical where the coefficients of the equation are analytical, and it may possibly have essential singularities where the coefficients are singular. Here, $U(x)$ is an entire function with a singularity (a pole), only at $\infty$, thus, any solution $\psi$ is an entire function with a possible essential singularity at $\infty$. The asymptotics of $\psi$ near $\infty$ are subject to the Stokes phenomenon. This means that, although $\psi$ is analytical in the whole complex plane, its asymptotics at infinity may change from sectors to sectors.

From the study of the Schrödinger equation it is known that in each sector $S_k$, $\psi(x)$

has a large $x$ expansion:

$$\psi(x) \underset{S_k}{\sim} \mathrm{e}^{\pm \frac{1}{2\hbar} V(x)} x^{C_k} \left(A_k + \frac{B_k}{x} + \dots\right)$$

and the sign $\pm$, may jump discontinuously from one sector to another as well as the numbers $A_k, B_k, C_k, \dots$ (and in general, all the coefficients of the series in $\frac{1}{x^j}$ at infinity).

### 2.3 Decreasing solution

Let us consider a specific solution $\psi(x)$ of the Schrödinger equation which is exponentially decreasing in some even sector at infinity. For writing convenience, **we will choose $\psi(z) = \psi_0(z)$ a decreasing solution in sector $S_0$. Without further indication, $\psi(z)$ is now understood to be $\psi_0(z)$ in the rest of the article.** Note that this choice is quite arbitrary at the moment, and one should wonder if the quantities we are about to compute depend on this choice, but we are presently not able to answer this question properly, and leave it for further study.

An important and useful result is the Stokes theorem which claims that if the asymptotics of $\psi(x)$ is exponentially small in some sector, then the same asymptotics holds in the two adjacent sectors (and therefore $\psi(x)$ is exponentially large in those two sectors).

In the general case, (i.e. a generic potential $U(x)$) our solution $\psi(x)$ is decreasing only in sector 0, and is exponentially large in all other sectors. But if the Schrödinger potential $U(x)$ is non-generic (quantized), then there may exist several sectors in which $\psi(x)$ is exponentially small. Due to Stokes theorem, if $\psi$ is exponentially small in some sectors then it must be exponentially large in the adjacent sectors, this implies that there are at most $d+1$ sectors in which $\psi$ is exponentially small.

The case studied in [44] was the most degenerate case, such that $\psi$ is exponentially small in $d+1$ sectors.

### Zeroes of $\psi$

The main difference with our previous article [44] is that we will not restrict ourselves to the case where $\psi(x)$ is a quasi-polynomial which can only be obtained with very non-generic potential $U(x)$. Here $\psi(x)$ is an entire function with an essential singularity at $\infty$, and with isolated zeroes labelled $s_i$:

$$\psi(s_i) = 0 \tag{VIII.7}$$

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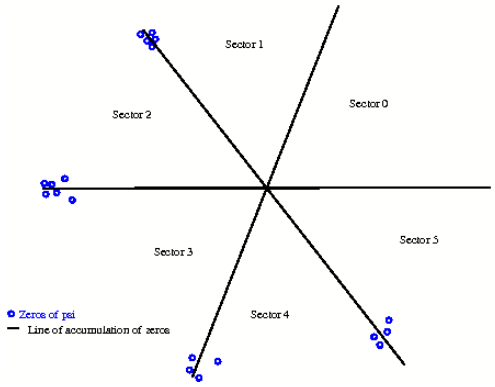
**In particular, the number of zeroes of $\psi$ may be finite or infinite.**

If $\psi(x)$ has an infinite number of zeroes, it is known that the zeroes may only accumulate near $\infty$, and only along the Stokes half-lines $L_j$'s bordering the sectors (see fig.VIII). In fact, there is an accumulation of zeroes along the half–line $L_j$ if and only if $\psi$ is exponentially large on both sides of the half-line.

For example in the case where $\psi(x) = \psi_k(x)$ is a solution that exponentially decreases in sector $k$ then it implies that there is no accumulation of zeroes along the half-lines $L_k$ and $L_{k-1}$.

If $U(x)$ is generic, then $\psi$ has an infinite number of zeroes, and the zeroes accumulate at $\infty$ along all half-lines $L_j$ with $j \neq 1, 2d+1$ (because remember that $\psi$ is implicitely assumed to be $\psi_0$ which decreases in sector 0), i.e. there are generically $2d$ half-lines of zeroes. The situation is illustrated in fig VIII.




The zeroes of $\psi$ accumulate near $\infty$ along the half-lines bordering sectors where $\psi_0$ is exponentially large on both sides. In particular, there is no accumulation of zeroes along $L_0$ and $L_{2d+1}$.

If $U(x)$ is non-generic (quantized), then there are additional sectors in which $\psi$ is exponentially small, and thus there can be no zeroes accumulating along the two half-lines bordering these sectors. Remember that from Stokes theorem, each time we have a new sector in which $\psi$ is decreasing, we have two half-lines less of zeroes. Therefore, the number of half-lines of zeroes is always even, and we call it:

**Definition VIII.1.** *The genus g of the Schrödinger equation is defined by:*

$$2\mathfrak{g} + 2 = \#\ \textit{half-lines of zeroes} \tag{VIII.8}$$

*And if $\psi$ has a finite number of zeroes (i.e. there is no half-line of zeroes), we define $\mathfrak{g} = -1$. We have*

$$-1 \leq g \leq d-1 \tag{VIII.9}$$

Note also that the definition of $g$ a priori depends on the choice of the solution $\psi = \psi_0$ since two different solutions of the same Schrödinger equation may have different numbers of semi-lines of zero accumulation.

An exception, is in the special cases $g = -1$ where it is easy to see that every choice of $\psi = \psi_{2k}$ would give the same value of $g$.

Indeed, consider $g_{2k}$ and $g_{2k'}$ be the genus defined from the solutions $\psi_{2k}$ exponentially small in sector $S_{2k}$ and $\psi_{2k'}$ exponentially small in sector $S_{2k'}$:

if $g_{2k} = -1$, this means that $\psi_{2k}$ is exponentially small in all even sectors, in particular in sector $S_{2k'}$, and therefore $\psi_{2k} \propto \psi_{2k'}$, and therefore $g_{2k'} = -1$.

**Case** $g = -1$

The case $g = -1$ was studied in [44]. This is the case where $\psi$ has only a finite number of zeroes, it is a quasipolynomial:

$$\psi(x)\, \mathrm{e}^{\frac{\hbar}{2} V(x)} = \text{polynomial}. \tag{VIII.10}$$

Notice that in order to diminish $g$ by 1, we need to quantize one parameter of $U$, and therefore to reach $g$, we need to quantize $d - 1 - g$ parameters. In particular, to reach $g = -1$, we need to quantize $d$ parameters, i.e. $P$ is completely fixed in terms of $V'$, and in particular, $t_0$ is quantized.

In the applications to random matrices, $t_0$ is usually a free parameter (called the temperature) and is never considered quantized, and therefore the case $g = -1$ is never obtained in random matrices.

Another way to say that, is that the case $g = -1$ has no $\hbar \to 0$ classical limit, and therefore in classical geometry we always have $g \geq 0$.

### 2.4 Resolvent

The first ingredient of our strategy is to define a resolvent similar to the one in matrix models.

**Definition VIII.2.** *We define the resolvent for a generic solution $\psi$ by:*

$$\omega(x) = \hbar \frac{\psi'(x)}{\psi(x)} + \frac{V'(x)}{2} \tag{VIII.11}$$

It is clear that this function is analytical except at the zeros of $\psi(x)$ where it has **simple** poles with residue $\hbar$:

$$\omega(x) \underset{x\to s_i}{\sim} \frac{\hbar}{x-s_i} + \text{reg.} \tag{VIII.12}$$

It also has a possible essential singularity at infinity with the same location of discontinuities as $\psi(x)$. Eventually, note again that the definition of $\omega(x)$ depends on the choice of $\psi(x)$.

### 2.5 Sheets

In sector $S_k$ we have the asymptotic:

$$\psi(x) \underset{S_k}{\sim} e^{\frac{\eta_k}{2\hbar}V(x)} x^{-\frac{\eta_k t_0}{\hbar} - d\frac{1+\eta_k}{2}} \left(A_k + \frac{B_k}{x} + \dots\right) \tag{VIII.13}$$

where $\eta_k = \pm 1$. That translates for the resolvent to:

$$\omega(x) \underset{x\to\infty_k}{\sim} \frac{1+\eta_k}{2}\left(V'(x) - \hbar\frac{d}{x}\right) - \frac{\eta_k t_0}{x} + O(1/x^2), \tag{VIII.14}$$

Therefore it depends if the solution $\psi$ is exponentially big or small in sector $k$ (and of course on the parity of $k$). For a generic $g = d-1$ solution which is exponentially big in every sector except $S_0$ (and thus has an alternating sign in the exponential) then $\eta_k = (-1)^k$ (except $\eta_0 = -1$).

**Definition VIII.3.** *We call* **"physical sheet"**, *the union of sectors where $\eta_k = -1$, in those sectors we have:*

$$\omega(x) \sim \frac{t_0}{x} + O(1/x^2) \tag{VIII.15}$$

*Notice that the sectors $S_0, S_1$ and $S_{2d+1}$ are always in the physical sheet.*

*And we call* **"second sheet"**, *the union of sectors where $\eta_k = +1$, in those sectors we have:*

$$\omega(x) \sim V'(x) + O(1/x) \tag{VIII.16}$$

This definition comes from the analogy with the resolvent in matrix model (see section VIII for details).

For a generic potential $U(x)$, all odd sectors are in the physical sheet, and all even sectors except $S_0$ are in the second sheet.

Notice that if $g = -1$, there is only the physical sheet, i.e. there is no second sheet.

### 2.6 The Bethe ansatz

In the polynomial case studied before [44], a key ingredient for establishing results was the Bethe ansatz. This ansatz basically deals with the behaviour of $\omega(x)$ around zeroes of $\psi$. The zeroes of $\psi$ are called "Bethe roots".

The Bethe ansatz can be formulated in many ways. One way to formulate it, is to say that $1/\psi^2$ has no residue at the $s_i$'s:

$$\operatorname*{Res}_{s_i} \frac{1}{\psi^2(x)} = 0 \tag{VIII.17}$$

in this way, it will play a key role in defining contour integrals, because all integrals of the type $\int dx/\psi^2(x)$ are insensitive to the exact location integration path with respect to the $s_i$'s, i.e. such integrals will depend only on the homotopy classes of paths.

Equation (VIII.17) can also be formulated, in a form very similar to the Bethe ansatz in the Gaudin model [89, 110] as follows:

**Theorem VIII.1.** *The roots $s_i$ of $\psi$ satisfy the Bethe ansatz:*

$$\forall i, \qquad V'(s_i) = 2 \lim_{x \to s_i} \left( \omega(x) - \frac{\hbar}{x - s_i} \right). \tag{VIII.18}$$

It is a regularized version of the Bethe equation for Gaudin model:

$$\forall i, \qquad V'(s_i) \text{ ``} = \text{ ''} 2\hbar \sum_{j \neq i} \frac{1}{s_i - s_j}$$

when the number of zeros is infinite and the sum is ill-defined.

*Proof.* This theorem is a classical result and is easy, it just consists in rewriting the Schrödinger equation as a Ricatti equation. We proceed the same way as in [44] and compute:

$$\begin{aligned}
& V'(x)\omega(x) - \omega^2(x) - \hbar\omega'(x) \\
= \; & V'(x)(\hbar\frac{\psi'(x)}{\psi(x)} + \frac{V'(x)}{2}) - \left( \frac{V'(x)^2}{4} + \hbar V'(x)\frac{\psi'(x)}{\psi(x)} + \hbar^2 \frac{\psi'(x)^2}{\psi^2(x)} \right) \\
& -\hbar \left( \hbar\frac{\psi''(x)}{\psi(x)} - \hbar\frac{\psi'^2(x)}{\psi^2(x)} + \frac{V''(x)}{2} \right) \\
= \; & \frac{V'(x)^2}{4} - \hbar^2 \frac{\psi''(x)}{\psi(x)} - \hbar\frac{V''(x)}{2}
\end{aligned}$$

cliv

$$\begin{aligned} &= \frac{V'(x)^2}{4} - U(x) - \hbar\frac{V''(x)}{2} \\ &= P(x) \end{aligned} \tag{VIII.19}$$

which is a polynomial in $x$, of degree $d-1$.

From its definition, it is clear that $\omega^2+\hbar\omega'$ has no double pole at the $s_i$'s, but it could have simple poles. Consider now a zero $s_i$ of $\psi$, and define:

$$\bar{\omega}_i(x) = \omega(x) - \frac{\hbar}{x-s_i}$$

Then, $\bar{\omega}_i(x)$ is regular at $x=s_i$, and we may compute $\bar{\omega}_i(s_i)$. Compute:

$$\begin{aligned} \operatorname*{Res}_{x\to s_i} \omega^2(x)+\hbar\omega'(x) &= \operatorname*{Res}_{x\to s_i} \bar{\omega}_i^2(x) + 2\hbar\frac{\bar{\omega}_i(x)}{x-s_i} + \frac{\hbar^2}{(x-s_i)^2} + \hbar\bar{\omega}_i'(x) - \frac{\hbar^2}{(x-s_i)^2} \\ &= \operatorname*{Res}_{x\to s_i} 2\hbar\frac{\bar{\omega}_i(x)}{x-s_i} \\ &= 2\hbar\,\bar{\omega}_i(s_i) \end{aligned} \tag{VIII.20}$$

On the other hand we have, from eq. (VIII.19) we have:

$$\begin{aligned} \operatorname*{Res}_{x\to s_i} \omega^2(x)+\hbar\omega'(x) &= \operatorname*{Res}_{x\to s_i} V'(x)\omega(x) - P(x) \\ &= \operatorname*{Res}_{x\to s_i} V'(x)\omega(x) \\ &= \hbar V'(s_i) \end{aligned} \tag{VIII.21}$$

Therefore we find :

$$\forall i, \qquad V'(s_i) = 2\,\bar{\omega}_i(s_i).$$

This equation is the Bethe equation for the roots $s_i$'s. Note that the potential $V'(x)$ is completely determined by the data of the potential $U(x)$ and does not depend on $\psi$. In particular, in the case where there are only a finite number of $s_i$'s we recognize the Bethe equation for Gaudin model [44]:

$$\forall i, \qquad V'(s_i) = 2\hbar\sum_{j\neq i}\frac{1}{s_i-s_j}$$

which were completely defining the $s_i$'s.

## 3 Towards a "Quantum Riemann Surface"

From the definition of our non-commutative spectral curve (i.e the Schrödinger equation), it is tempting to generalize the classical notions kwown in algebraic geometry and Riemann surfaces to our "quantum" case ("quantum" is not to be understood as "quantized" but as "non-commutative" $[y,x]=\hbar$). For a Riemann surface, the central notions are those of cuts, sheets, genus, cycles and meromorphic differentials forms of 1st, 2nd and 3rd kind. In our context, the picture needs a proper adaptation in order to recover the terminology of Riemann surfaces and algebraic geometry.

In this section we will define the notions of genus, $\mathscr{A}$-cycles, $\mathscr{B}$-cycles and the first kind differentials dual to them. Here, let us assume that $g \geq 0$.

### 3.1 Cuts

First, we like to think of the 2 sheets, as the sectors which correspond to the 2 possible behaviors of the resolvent at $\infty$: $\omega(x) \sim t_0/x$ (physical sheet) or $\omega(x) \sim V'(x)$ (second sheet).

Then, we consider the cuts as sets of roots $s_i$'s. In some sense, each pair of half lines of accumulation of zeroes can be thought of as a cut.

**Definition VIII.4.** *We define cuts as pairs of half-lines of zeroes.*
*There is some arbitrariness in grouping the half-lines of zeroes by pairs.*

There is $g+1$ cuts, like in classical algebraic geometry, and notice that the case $g=-1$ which has no classical counterpart, has no cuts.

Notice that, contrarily to classical geometry, where the endpoints of the cuts are zeroes of $U(x)$, here the endpoints are somehow blurred, we may move a finite number of $s_i$'s from one cut to another.

### 3.2 Cycles

In standard algebraic geometry, the non-contractible $\mathscr{A}$-cycles are often thought of as surrounding cuts in the physical sheet, and their dual $\mathscr{B}$-cycles are going through the cuts, from one sheet to the other, see fig VIII.

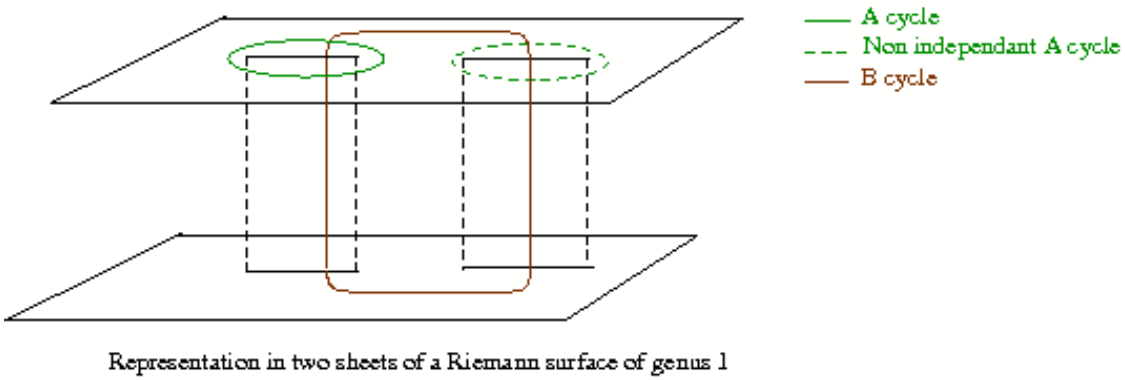



Figure 6: Representation in two sheets of a Riemann surface of genus 1.

### A-Cycles

Consider the complex plane from which we remove the second sheet (sectors where $\omega(x) \sim V'(x)$). It is clear that it contains $g+1$ sectors near $\infty$, and there are $g$ homologically linearly independent contours which link them.

**Definition VIII.5.** *We define $\mathscr{A}$-cycles $\mathscr{A}_1, \ldots, \mathscr{A}_g$ as $g$ linearly independent non-contractible contours going from $\infty$ to $\infty$ in the physical sheet.*

*A choice of $\mathscr{A}$-cycles is not unique.*

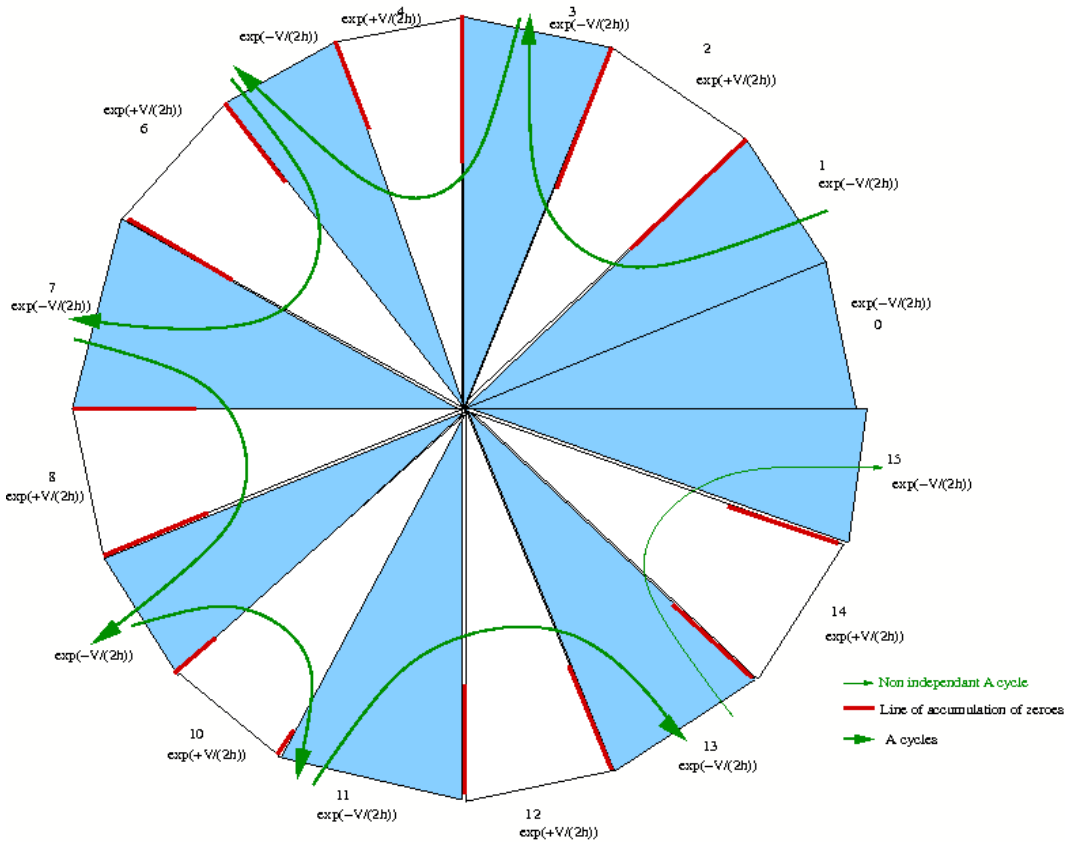
Remark that this notion really makes sense only for $g \geq 1$.

Notice that each time $\psi(x) \sim \mathrm{e}^{-V(x)/2\hbar}$ in an even sector, it means it is exponentially small and thus it also behaves like $\mathrm{e}^{-V(x)/2\hbar}$ in the neighboring odd sectors. That means that we can always choose $\mathscr{A}$-cycles going from odd sector to odd sector.

Since the first sheet and second sheet are separated by half-lines of accumulations of zeroes, every $\mathscr{A}$-cycle surrounds an even number of such half-lines of accumulations of zeroes, i.e. surrounds the cuts in the physical sheet. Like in standard algebraic geometry, the cuts are identified as pairs of half-lines of zeroes accumulations and the $\mathscr{A}$ cycles are going enclosing these cuts.

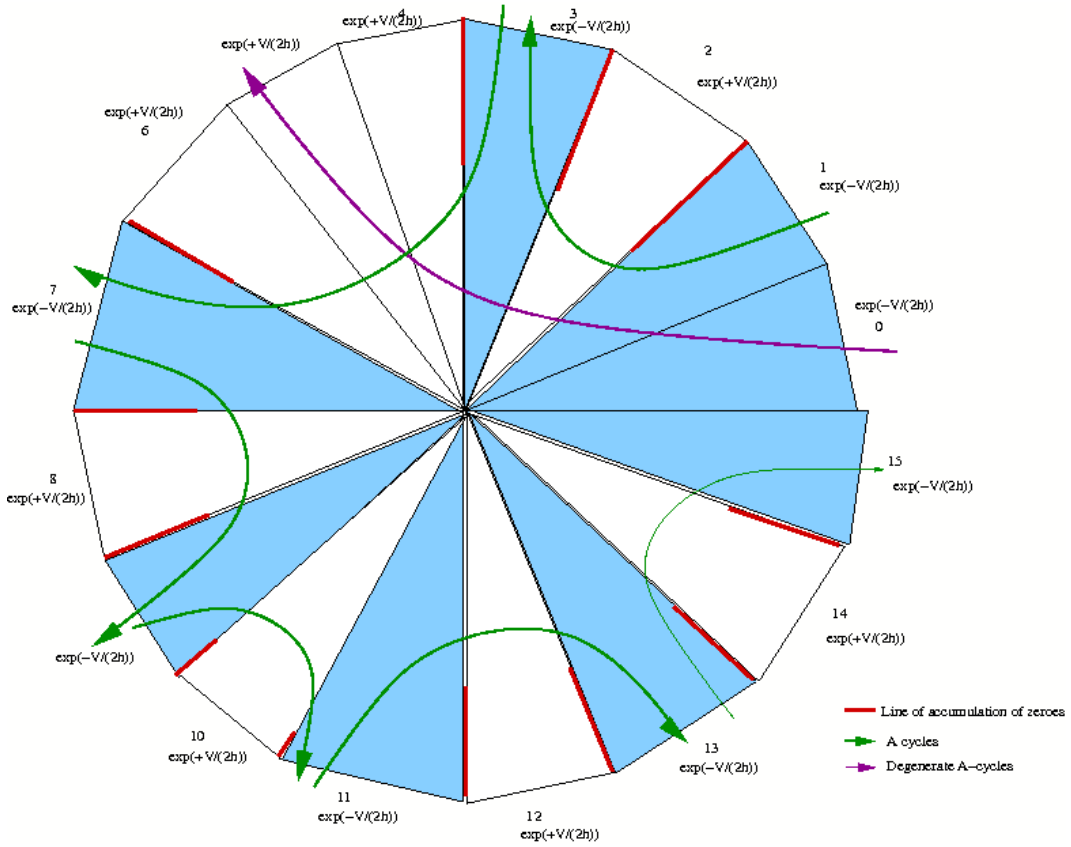
### Examples

In the generic case $g = d-1$, we can define $d$ $\mathscr{A}$-cycles but only $d-1$ are linearly independent. See picture where $d = 7$:

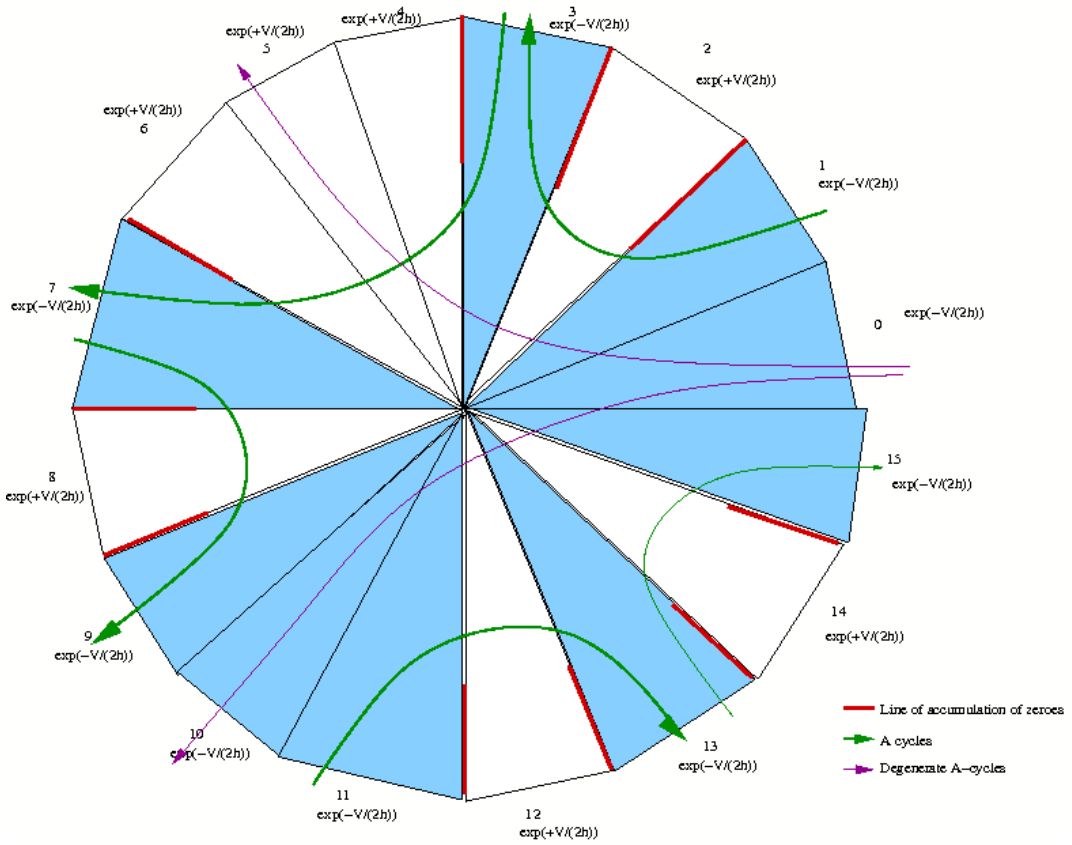



We clearly see that the dashed contour is not linearly independent with the others since the global sum of the contours (dashed included) is contractible in the physical sheet.

For a non-generic case, there are sectors at infinity where $\psi$ is exponentially small. In these cases, the definition of the contours need some adaptations because these sectors correspond to "degenerate" cuts. Here are a few examples of how to deal with these cases. Basically, each time there are two sectors where $\psi$ is small we can replace one of the standard $\mathscr{A}$ cycle, by a $\hat{\mathscr{A}}$ cycle (sometimes called also "degenerate" $\mathscr{A}$ cycles) that connect them. Here are some examples of the contours in more and more peculiar situations for $d = 7$:




From then it is easy to generalize into more complicated frames:




It is then easy to generalize the method in more sophiticated situations.

In the extreme case where $\psi$ is exponentially small in all even sectors, there are only $d$ independant "degenerate" $\hat{\mathscr{A}}$ cycles and no $\mathscr{A}$ cycles, the genus is $g = -1$. This is the polynomial case studied in [44] where there are no $\mathscr{A}$ cycles.

From the definitions, it is easy to see that the genus $g$ defined above corresponds to the number of independant $\mathscr{A}$ cycles (we exclude the $\hat{\mathscr{A}}$ cycles). It is also obvious that the sum of independant $\mathscr{A}$ and $\hat{\mathscr{A}}$ cycles always equals $d - 1$.

### B-Cycles

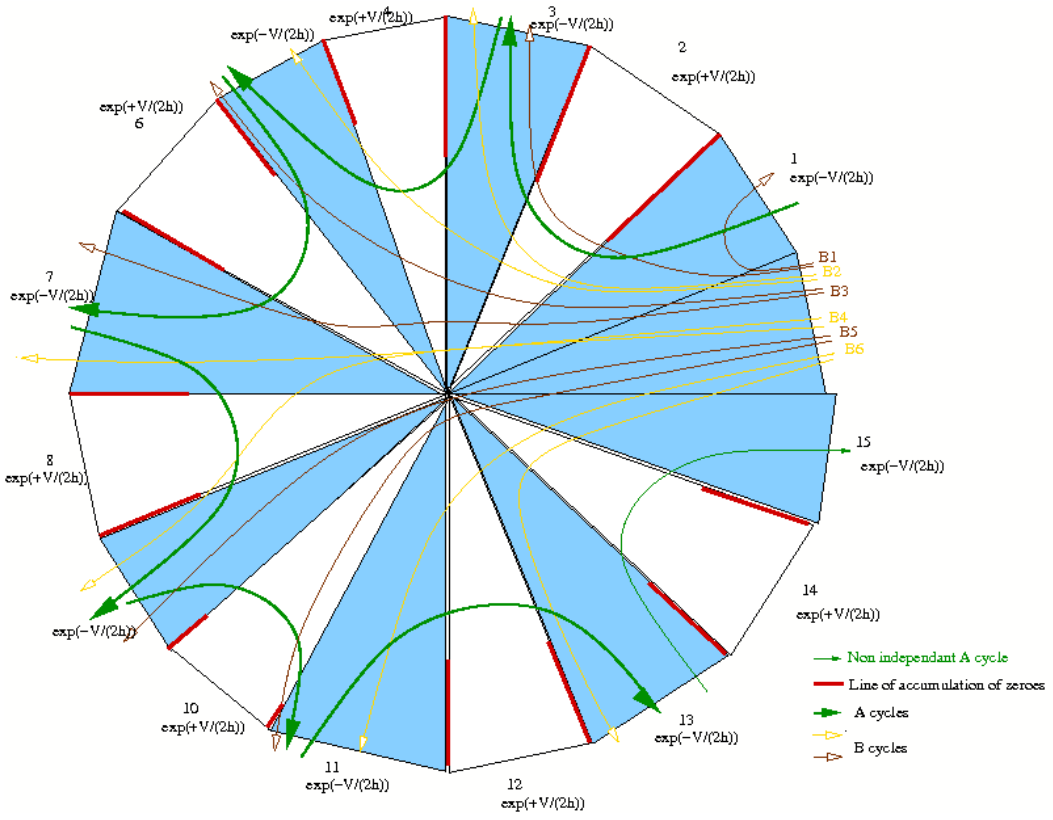
As in classical algebraic geometry, it is standard to define the $\mathscr{B}$ cycles with an origin lying in the non-independant cut. Moreover, although it would be possible to define $\hat{\mathscr{B}}$ cycles attached to the $\hat{\mathscr{A}}$ cycles, we prefer limiting ourselves to the definition of $\mathscr{B}$ cycles attached only to the $\mathscr{A}$ cycles. Basically, they start from the non-independant cut, goes through their corresponding $\mathscr{A}$ cycle and end at infinity in the same sector as their corresponding $\mathscr{A}$ cycle. As there are two sectors in which their corresponding $\mathscr{A}$ cycle ends, we double them so that one goes into one sector and the other one in the second sector. We also choose the whole so that they intersect only with their corresponding

$\mathscr{A}$-cycles:

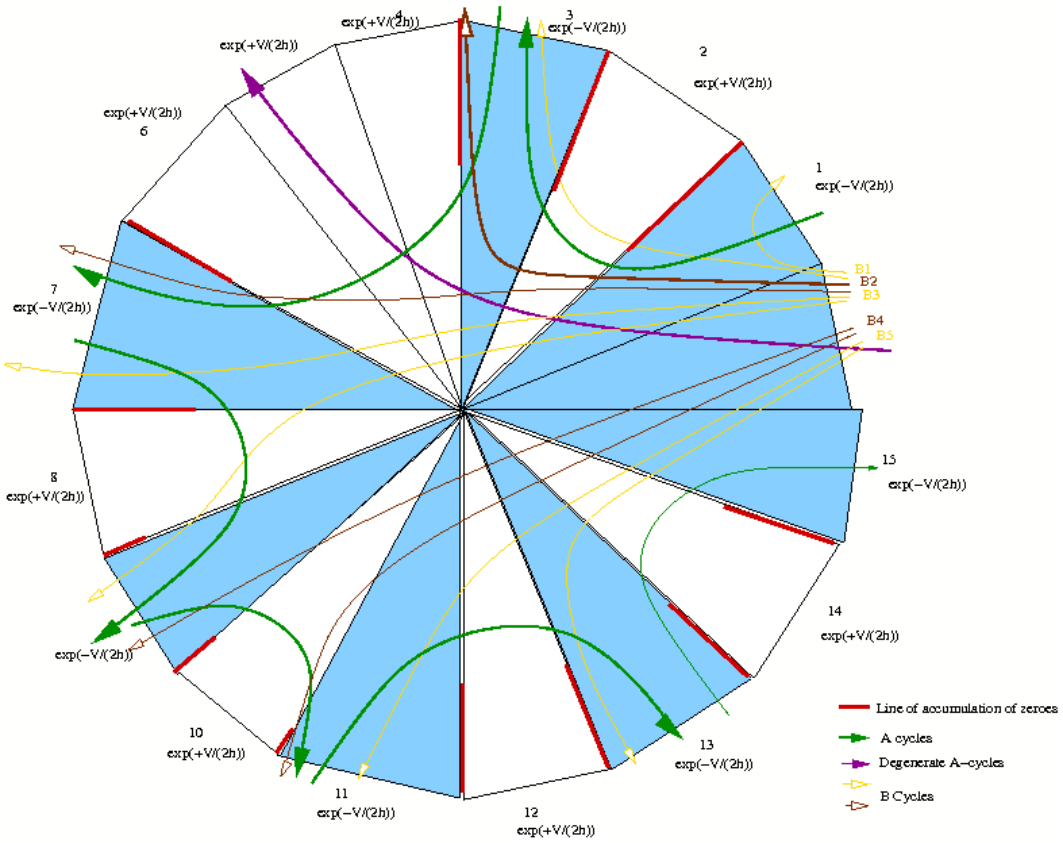
$$\mathscr{A}_\alpha \cap \mathscr{B}_\beta = 2\delta_{\alpha,\beta} \qquad \text{(VIII.22)}$$

This definition is easier understandable with the following pictures:

Generic case:




And in a degenerate case:




### 3.3 First kind functions

After defining the cycles, another important step is to define the equivalent of the first, second and third kind differentials. In this section, we propose a definition of the first kind differentials.

Let $h_k$, $k = 1, \dots, d-1$, be a basis (arbitrary for the moment, but we will choose it orthonormal later on), of the complex vector space of polynomials of degree $\leq d-2$. To have more convenient notation, we will label the $\hat{\mathscr{A}}$-cycles as $\mathscr{A}_\alpha$ , $g+1 \leq \alpha \leq d-1$ and the standard $\mathscr{A}$ are labelled $\mathscr{A}_\alpha$, $1 \leq \alpha \leq g$.

Consider the following functions:

$$v_k(x) = \frac{1}{\hbar\, \psi^2(x)} \int_{\infty_0}^{x} h_k(x')\, \psi^2(x')\, dx' \qquad , \qquad \deg h_k \leq d-2. \tag{VIII.23}$$

Notice that, thanks to the Bethe ansatz, $v_k(x)$ has double poles with vanishing residues at the $s_j$'s (the zeroes of $\psi$), and behaves like $O(1/x^2)$ in sector $S_0$ and in sectors where $\psi$ is exponentially large. (because the polynomial is of degree less than $d-2$). Therefore,

the following integrals are well defined:

$$I_{k,\alpha} = \oint_{\mathscr{A}_\alpha} v_k(x)\,dx \qquad , \qquad \alpha = 1,\dots,g, k = 1,\dots,d-1. \tag{VIII.24}$$

For the degenerate contours $\hat{\mathscr{A}_\alpha}$, we cannot take the integral since it would not converge. We define instead:

$$I_{k,\alpha} = \int_{\hat{\mathscr{A}_\alpha}} h_k(x)\,\psi^2(x)\,dx \qquad , \qquad \alpha = g+1,\dots,d-1, k = 1,\dots,d-1. \tag{VIII.25}$$

The matrix $I_{k,\alpha}$ with $k,\alpha = 1,\dots,d-1$ is a square matrix, which gives a pairing between the set of paths $\{\ \mathscr{A}_\alpha, \hat{\mathscr{A}_\alpha}\ \}$ and the space of polynomials of degree at most $d-2$. Let us choose a basis $h_k$, dual to the $\mathscr{A}$-cycles, i.e.:

$$I_{k,\alpha} = \delta_{k,\alpha}. \tag{VIII.26}$$

Choosing this set of polynomials gives then the following relations:

$$\forall\, i = 1,\dots,g, j = 1,\dots,d-1\,, \qquad \oint_{\mathscr{A}_i} v_j(x)\,dx = \delta_{i,j} \tag{VIII.27}$$

$$\forall i = g+1,\dots,d-1,\ j = 1,\dots,d-1 : \int_{\hat{\mathscr{A}_i}} h_j(x)\,\psi^2(x)\,dx = \delta_{j,i} \tag{VIII.28}$$

Moreover, from the definitions, we get an asymptotic expression of $v_k(x)$ at infinity:

**Theorem VIII.2.** *The functions $v_k(x)$ with $k \le g$ are such that:*

$$k = 1,\dots,g, \qquad v_k(x) = O(x^{-2}) \tag{VIII.29}$$

*in all sectors at infinity.*

*And the functions $v_k(x)$ with $g+1 \le k \le d-1$ are such that:*

$$k = g+1,\dots,d-1, \qquad v_k(x) = O(x^{-2}) \tag{VIII.30}$$

*in all sectors except in the sector where $\hat{\mathscr{A}_k}$ ends, where we have:*

$$v_k(x) = \frac{1}{\hbar\psi(x)^2} + O(1/x^2). \tag{VIII.31}$$

*Proof.* In sector $\infty_0$, we clearly have $v_k(x) \sim O(x^{\deg h_k - d}) = O(x^{-2})$. And in a sector $S_i$ where $\psi$ is exponentially small we have:

$$v_k(x) = \frac{1}{\hbar \psi^2(x)} \left[ \int_{\infty_i}^{x} h_k(x')\, \psi^2(x')\, dx' + \int_{\infty_0}^{\infty_i} h_k(x')\, \psi^2(x')\, dx' \right], \tag{VIII.32}$$

and due to our choice of basis eq. (VIII.26), we have

$$v_k(x) = \frac{\delta_{i,i_k}}{\hbar \psi(x)^2} + \frac{1}{\hbar \psi^2(x)} \int_{\infty_i}^{x} h_k(x')\, \psi^2(x')\, dx' = \frac{\delta_{i,i_k}}{\hbar \psi(x)^2} + O(1/x^2), \tag{VIII.33}$$

in sector $S_i$.

We claim that the function $v_k(x)\, k = 1, \ldots, g$ are the generalization of holomorphic forms (1st kind differentials).

**Remark VIII.2.** Classical limit.

The small $\hbar$ BKW expansion $\psi \sim \mathrm{e}^{\pm \frac{1}{2\hbar} \int \sqrt{U}}$ gives:

$$v_k(x) \sim \frac{\pm h_k(x)}{\sqrt{U(x)}} \tag{VIII.34}$$

and $v_k(x)dx$ are indeed the holomorphic forms on the algebraic curve $y^2 = U(x)$.

### 3.4 Riemann matrix of periods

An interesting quantity in standard algebraic geometry is the Riemann matrix of periods which is the integrations of the holomorphic differentials over $\mathscr{B}$-cycles. Now that we have defined properly the cycles, we can define a similar "quantum" Riemann period matrix $\tau_{i,j}$, $i, j = 1, \ldots, g$ by:

$$\tau_{i,j} \overset{\text{def}}{=} \oint_{\mathscr{B}_i} v_j(x)\, dx. \tag{VIII.35}$$

Note that this definition makes sense since $v_j(x)$ $(j = 1, \ldots, g)$ behaves as $O(1/x^2)$ in the sectors where the $\mathscr{B}$-cycles go. Also, thanks to the Bethe ansatz, $v_j$ has no residue at the roots $s_i$'s, therefore those integrals depend only on the homology class of $\mathscr{B}$-cycles, and not on a representent.

Like for the classical Riemann matrix of periods we have the following property:

**Theorem VIII.3.** *The period matrix $\tau$ is symmetric: $\tau_{i,j} = \tau_{j,i}$.*

*Proof.* We anticipate on results which shall be proved later, but which don't depend on this theorem. The proof comes directly from theorem VIII.14 below, since:

$$\oint_{\mathscr{B}_\beta} dx \oint_{\mathscr{B}_\alpha} B(x,z)dz = 2i\pi \oint_{\mathscr{B}_\beta} dx v_\alpha(x) = 2i\pi\tau_{\beta,\alpha}$$

and from the symmetry theorem VIII.15 for the Bergman kernel $B(x,z) = B(z,x)$:

$$\oint_{\mathscr{B}_\beta} dx \oint_{\mathscr{B}_\alpha} B(x,z)dz = \oint_{\mathscr{B}_\alpha} dz \oint_{\mathscr{B}_\beta} dx B(x,z) = 2i\pi \oint_{\mathscr{B}_\alpha} dz v_\beta(z) = 2i\pi\tau_{\alpha,\beta}.$$

### 3.5 Filling fractions

In random matrices, the notion of filling fractions, is just the $\mathscr{A}$-cycle integrals of the resolvent. Here, we easily generalize it by the definition:

**Definition VIII.6.** *The filling fractions $\varepsilon_1,\dots,\varepsilon_d$ are defined as follows:*

$$\alpha = 1,\dots,g, \qquad \varepsilon_\alpha = \frac{1}{2i\pi}\oint_{\mathscr{A}_\alpha}\left(\omega(x) - \frac{t_0}{x}\right) + \frac{t_0 n_\alpha}{(d+1)} \tag{VIII.36}$$

*where the integer $n_\alpha$ is half the number of Stokes half-lines surrounded by the cycle $\mathscr{A}_\alpha$. In other words, $\frac{2n_\alpha}{2d+2}$ corresponds to the angular fraction of the complex plane defined by the cycle $\mathscr{A}_\alpha$.*

*For $\alpha = g+1,\dots,d-1$ we define*

$$\alpha = g+1,\dots,d-1, \qquad \varepsilon_\alpha = 0 \tag{VIII.37}$$

*And for $\alpha = d$, we choose a non-independent $\mathscr{A}$-cycle $\mathscr{A}_d$, which surrounds all the $s_i$'s which are not surrounded by $\mathscr{A}_1,\dots,\mathscr{A}_g$, and define:*

$$\varepsilon_d = \frac{1}{2i\pi}\oint_{\mathscr{A}_d}\left(\omega(x) - \frac{t_0}{x}\right) + \frac{t_0 n_d}{(d+1)} \tag{VIII.38}$$

Note that this definition makes sense because all the cycles $\mathscr{A}_\alpha$ go from an infinity where $\omega(x) - \frac{t_0}{x} \sim O\left(\frac{1}{x^2}\right)$. Note also that this definition depends on the exact locus of the contour $\mathscr{A}_\alpha$ and not only on its homotopy class, since $\omega(x)$ has simple poles at the $s_i$'s with residue $\hbar$. If we deform the contour $\mathscr{A}_\alpha$, the filling fractions can change by

some integer times $\hbar$.

In other words, the filling fractions are "blurred" when $\hbar \neq 0$, they are defined modulo an integer times $\hbar$. In the classical limit $\hbar \to 0$, they become deterministic.

We have:

**Theorem VIII.4.**

$$\sum_{\alpha=1}^{d} \varepsilon_\alpha = t_0 \tag{VIII.39}$$

*Proof.* When we perform the sum over the contours $\mathscr{A}_\alpha$, the contour $\mathscr{A}_d$ was defined as the "complementary" of the others, i.e. so that the sum is contractible. Since the function $x \to \omega(x) - t_0/x$ is integrable at infinity, we find that its global integral is null. With the same argument, it is easy to see that $\sum_{\alpha=1}^{d} n_\alpha = (d+1)$ because we take all Stokes lines once and only once. Therefore we get:

$$\sum_{\alpha=1}^{d} \varepsilon_\alpha = 0 + \frac{t_0}{d+1} \sum_{\alpha=1}^{d} n_\alpha = t_0.$$

Note that it also tells us that only $d-1$ of the epsilon's are independant.

**Remark VIII.3.** In the case $g = -1$, the only filling fraction is $\varepsilon_d = t_0$, and it is also the sum of residues of $\omega$ at the $s_i$'s:

$$\varepsilon_d = t_0 = \sum_i \operatorname*{Res}_{s_i} \omega = \hbar \# \{s_i\}$$

This shows again, that $g = -1$ corresponds to a case where $t_0$ is quantized, namely $t_0$ is an integer times $\hbar$:

$$t_0/\hbar \in \mathbb{N}.$$

## 4 Kernels

One of the key geometric objects in [44] and in [23], is the "recursion kernel" $K(x,z)$. It was used in the context of matrix models, to find a solution of loop equations. Here, it will also allow us to define the 3rd and 2nd kind differentials.

### 4.1 The recursion kernel $K$

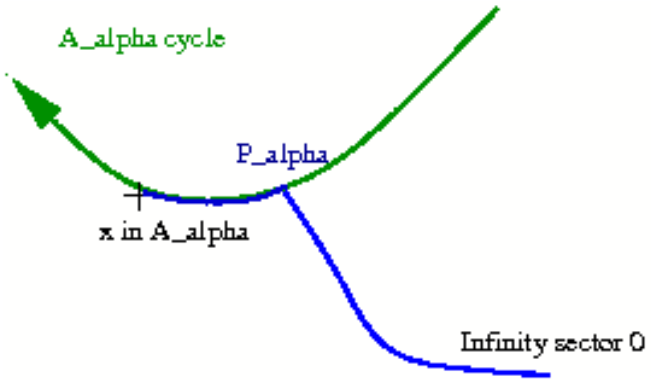
First we define:

$$\hat{K}(x,z) = \frac{1}{\hbar} \frac{1}{\psi^2(x)} \int_{\infty_0}^{x} \psi^2(x') \frac{dx'}{x'-z} \tag{VIII.40}$$

and for each $\alpha = 1, \ldots, g$, we choose a point $P_\alpha \in \mathscr{A}_\alpha$ and we define:

$$\hbar C_\alpha(z) = \oint_{\mathscr{A}_\alpha} \frac{dx''}{\psi^2(x'')} \int_{\infty_0}^{P_\alpha} \psi^2(x') \frac{dx'}{x'-z} + \oint_{\mathscr{A}_\alpha} \frac{dx''}{\psi^2(x'')} \int_{P_\alpha}^{x''} \psi^2(x') \frac{dx'}{x'-z} \quad \text{(VIII.41)}$$

where in the last integral, the integration contour between $P_\alpha$ and $x''$, is along $\mathscr{A}_\alpha$. This is described in fig.VIII.




Picture of the path of integration used for the definition of the kernel $K(x,z)$.

For each $\alpha = g+1, \ldots, d-1$, we define:

$$C_\alpha(z) = \int_{\hat{\mathscr{A}}_\alpha} \psi^2(x') \frac{dx'}{x'-z}. \quad \text{(VIII.42)}$$

We now need to describe the domain of definition of these functions.

First, one can see that for a fixed $x$, these functions are defined for $z$ outside of some "cuts" (see figure VIII)

• Choose a path between $\infty_0$ and $x$, then $\hat{K}(x,z)$ is defined for $z$ outside of this path. Across the path $]\infty_0, x]$, $\hat{K}(x,z)$ has a discontinuity:

$$\delta\hat{K}(x,z) = \frac{2i\pi}{\hbar} \frac{\psi^2(z)}{\psi^2(x)} \quad \text{(VIII.43)}$$

• For each $\alpha = 1, \ldots g$, choose a path between $\infty_0$ and $P_\alpha$, then $C_\alpha(z)$ is defined for $z$ outside of this path, and outside $\mathscr{A}_\alpha$. Across the path $]\infty_0, P_\alpha]$, $C_\alpha(z)$ has a discontinuity:

$$\delta C_\alpha(z) = \frac{2i\pi}{\hbar} \psi^2(z) \oint_{\mathscr{A}_\alpha} \frac{dx''}{\psi^2(x'')} \quad \text{(VIII.44)}$$

and across the path $\mathscr{A}_\alpha$, $C_\alpha(z)$ has a discontinuity:

$$\delta C_\alpha(z) = \frac{2i\pi\ \psi^2(z)}{\hbar} \int_{P_\alpha}^{z} \frac{dx''}{\psi^2(x'')} \tag{VIII.45}$$

• For each $\alpha = g+1,\dots,d-1$, $C_\alpha(z)$ is defined for $z$ outside of the path $\hat{\mathscr{A}}_\alpha$. Across the path $\hat{\mathscr{A}}_\alpha$, $C_\alpha(z)$ has a discontinuity:

$$\delta C_\alpha(z) = 2i\pi\ \psi^2(z) \tag{VIII.46}$$

From these remarks, we now define the recursion kernel $K(x,z)$ by:

**Definition VIII.7.** *Definition of the recursion kernel:*

$$K(x,z) = \hat{K}(x,z) - \sum_{\alpha=1}^{d-1} v_\alpha(x)\, C_\alpha(z) \tag{VIII.47}$$

*it is defined for $z$ outside the cuts mentionned above.*

For a fixed $z$, the analytical properties in $x$ of $K(x,z)$ are the same as those of $\hat{K}(x,z)$ since all $v_\alpha(x)$ are analytic. For a fixed $z$, the primitive of $\psi^2(x')\frac{dx'}{x'-z}$ can be defined locally but not globaly on the complex plane. In fact there is a logarithmic cut to be arbitrarily chosen on $]\infty_0, z]$. Anywhere out of this cut the function $x \to K(x,z)$ is analytic.

### Properties of kernel $K$

The definition of the kernel $K(x,z)$ might seem arbitrary at first glance. But in fact, the main reason for the introduction of such kernel is that it has many interesting properties:

It is clear from our definitions that:

**Theorem VIII.5.** *For a given $z$, the kernel $K$ behaves like:*

$$K(x,z) \sim O(x^{-2}) \tag{VIII.48}$$

*when $x \to \infty$ in all sectors.*

*Proof.* The result is obvious for sector $S_0$ and for sectors where $\psi$ is exponentially big. When it is not, the fact that we substract $C_\alpha, \alpha = g+1,\dots,d-1$ gives the result.

**Theorem VIII.6.** *We have in all sectors at infinity :*

$$K(x,z) \underset{z\to\infty}{\sim} O(z^{-d}). \tag{VIII.49}$$

*More precisely we have:*

$$K(x,z) \sim -\sum_{k=d-1}^{\infty} \frac{K_k(x)}{z^{k+1}} \tag{VIII.50}$$

*with*

$$\hat{K}_k(x) = \frac{1}{\hbar\psi^2(x)} \int_{\infty_0}^{x} x'^k\, \psi^2(x')\, dx', \tag{VIII.51}$$

*and*

$$K_k(x) = \hat{K}_k(x) - \sum_{\alpha=1}^{g} v_\alpha(x) \oint_{\mathscr{A}_\alpha} \hat{K}_k(x')\, dx' - \sum_{\alpha=g+1}^{d-1} v_\alpha(x) \oint_{\mathscr{A}_\alpha} \psi^2(x')\, x'^k\, dx'. \tag{VIII.52}$$

*Proof.* It is clear that

$$\hat{K}(x,z) \sim -\sum_{k=0}^{\infty} \frac{\hat{K}_k(x)}{z^{k+1}} \tag{VIII.53}$$

where

$$\hat{K}_k(x) = \frac{1}{\hbar\psi^2(x)} \int_{\infty_0}^{x} x'^k\, \psi^2(x')\, dx', \tag{VIII.54}$$

and therefore

$$K_k(x) = \hat{K}_k(x) - \sum_{\alpha=1}^{g} v_\alpha(x) \oint_{\mathscr{A}_\alpha} \hat{K}_k(x')\, dx' - \sum_{\alpha=g+1}^{d-1} v_\alpha(x) \oint_{\mathscr{A}_\alpha} \psi^2(x')\, x'^k\, dx' \tag{VIII.55}$$

Now, if $k \leq d-2$, notice that $x'^k$ is a polynomial of degree $\leq d-2$, and it is thus a linear combinations of $h_\alpha(x)$'s:

$$x'^k = \sum_{\beta=1}^{d-1} b_{k,\beta}\, h_\beta(x') \tag{VIII.56}$$

This implies:

$$\hat{K}_k(x) = \sum_{\beta=1}^{d-1} b_{k,\beta}\, v_\beta(x) \tag{VIII.57}$$

Taking now the integral over an $\mathscr{A}$ cycle and using the normalization choice of $h_k(x)$

gives: If $\alpha \leq g$

$$\oint_{\mathscr{A}_\alpha} \hat{K}_k(x')\,dx' = b_{k,\alpha} \tag{VIII.58}$$

and if $\alpha > g$

$$\oint_{\mathscr{A}_\alpha} \psi^2(x')\,x'^k\,dx' = b_{k,\alpha} \tag{VIII.59}$$

This implies that $K_k(x) = 0$ if $k \leq d-2$, and therefore

$$K(x,z) = O(z^{-d}). \tag{VIII.60}$$

**Theorem VIII.7.** *Let $\alpha = 1,\dots,g$, and $z$ on the side of $\mathscr{A}_\alpha$ which does not contain $\infty_0$, then:*

$$\oint_{\mathscr{A}_\alpha} K(x,z)\,dx = 0 \tag{VIII.61}$$

*Proof.* Notice that if $z$ is on that side of $\mathscr{A}_\alpha$, we have $C_\alpha(z) = \oint_{\mathscr{A}_\alpha} \hat{K}(x,z)\,dx$, and therefore $\oint_{\mathscr{A}_\alpha} K(x,z)\,dx = 0$. In fact one can see that the addition of the part with the $C_\alpha(z)$ was just put there to cancel out the $\mathscr{A}$-cycle integrals.

### 4.2 Third kind differential: kernel $G(x,z)$

The second important kernel to define is the equivalent of the third kind differential. In [44] this kernel was computed from $K$ by derivation, and we use the same definition.

**Definition VIII.8.** *We define the kernel $G(x,z)$ by:*

$$G(x,z) = -\hbar\,\psi^2(z)\,\partial_z \frac{K(x,z)}{\psi^2(z)} = 2\hbar \frac{\psi'(z)}{\psi(z)} K(x,z) - \hbar \partial_z K(x,z) \tag{VIII.62}$$

From an easy integration by parts we find:

$$\begin{aligned} G(x,z) \quad = \quad & -\frac{1}{x-z} + \frac{2}{\psi^2(x)} \int_{\infty_0}^{x} \frac{dx'}{x'-z}\,\psi^2(x') \left( \frac{\psi'(x')}{\psi(x')} - \frac{\psi'(z)}{\psi(z)} \right) \\ & -\hbar \sum_\alpha v_\alpha(x)\,\psi^2(z) \partial_z \frac{C_\alpha(z)}{\psi^2(z)} \end{aligned} \tag{VIII.63}$$

which shows that near $x = z$ we have $G(x,z) \sim \frac{1}{z-x}$, i.e. there is a simple pole of residue 1 at $z = x$. Note in particular that $\frac{1}{x'-z}\left( \frac{\psi'(x')}{\psi(x')} - \frac{\psi'(z)}{\psi(z)} \right)$ has no singularity at $x' = z$ and therefore for a fixed $z$, there is no more any logarithmic cut $]\infty, z]$ as we had for $K(x,z)$.

Note again that a priori, this function of $z$ has the same lines of discontinuity as the kernel $K(x,z)$. But notice that the definition of $G$ ensures that all discontinuities of $K$ which are proportional to $\psi^2(z)$ cancel.

**Theorem VIII.8.** *$G(x,z)$ is an analytical function of $x$, with a simple pole at $x=z$ with residue $-1$, and double poles at the $s_j$'s (zeros of $\psi(x)$) with vanishing residue, and possibly an essential singularity around $\infty$.*

*$G(x,z)$ is an analytical function of $z$, with a simple pole at $z=x$ with residue $+1$, simple poles at $z=s_j$, and with a discontinuity across $\mathscr{A}_\alpha$-cycles with $\alpha=1,\dots,g$ (and thus no discontinuity on $\hat{\mathscr{A}_\alpha}$):*

$$\delta G(x,z) = -2i\pi\, v_\alpha(x) \tag{VIII.64}$$

*Proof.* $K(x,z)$ is discontinuous when $z$ crosses either $]\infty_0,x]$, $]\infty_0,P_\alpha]$ or $\mathscr{A}_\alpha$. However, the discontinuity of $K(x,z)$ across $]\infty_0,x]$, $]\infty_0,P_\alpha]$, and $\hat{\mathscr{A}_\alpha}$ is proportional to $\psi^2(z)$, and this means by derivation that $G(x,z)$ is not discontinuous there. Across $\mathscr{A}_\alpha$ with $\alpha\le g$, the discontinuity of $K(x,z)$ is given by eq. (VIII.45), and thus, the discontinuity of $G(x,z)$ is $\delta G(x,z) = -2i\pi\, v_\alpha(x)$.

Since $K(x,z)$ is regular when $z=s_j$, then it is clear that $G(x,z)$ has simple poles at $z=s_j$, with residue $-2\hbar K(x,s_j)$.

In the variable $x$, it is clear from the definition and from the Bethe ansatz VIII.18, that $K(x,z)$ has double poles at $x=s_j$ without residue, and this properties follows for $G(x,z)$.

**Theorem VIII.9.**

$$G(x,z) = O(1/x^2) \tag{VIII.65}$$

*when $x\to\infty$ in all sectors.*

*And at large $z$ in sector $S_k$:*

$$\lim_{z\to\infty_k} G(x,z) = G(x,\infty_k) = \eta_k\, t_{d+1}\, K_{d-1}(x) \tag{VIII.66}$$

*where $\eta_k=\pm1$ is such that $\psi\sim \mathrm{e}^{\eta_k V/2\hbar}$ in sector $S_k$.*

*Proof.* The large $x$ behavior follows from theorem VIII.5. The large $z$ behavior is given by theorem VIII.6, i.e. $G(x,z)\sim \eta_k V'(z)K(x,z)\sim \eta_k t_{d+1}K_{d-1}(x)$. The sign depends on the behavior of the solution in this sector. .

**Theorem VIII.10.** *Let $\alpha = 1,\dots,g$, and $z$ on the side of $\mathscr{A}_\alpha$ which does not contain $\infty_0$, then:*

$$\oint_{\mathscr{A}_\alpha} G(x,z)\,dx = 0 \tag{VIII.67}$$

*Proof.* Immediate from theorem VIII.7

**Semi-classical limit**

We claim that this kernel is the quantum version of the third kind differential. Indeed, in classical algebraic geometry a third kind differential is characterized by analyticity except a simple pole with non vanishing residue and a proper normalization on $\mathscr{A}$-cycles. Here, apart from the discontinuity along the $\mathscr{A}$-cycles which is expected since these contours represent the "quantum cuts", we have analyticity (apart from the $s_i$'s which also define the cuts), a simple pole with residue and a good normalization on $\mathscr{A}$-cycles.

In the BKW semiclassical expansion we have $\psi \sim \mathrm{e}^{\frac{\pm 1}{\hbar}\int\sqrt{U}}$ and thus

$$\hat{K}(x,z) \sim \frac{2}{x-z}\frac{1}{\sqrt{U(x)}} \tag{VIII.68}$$

and

$$K(x,z) \sim \frac{1}{x-z}\frac{1}{2\sqrt{U(x)}} - \sum_\alpha v_\alpha(x)\,C_\alpha(z) \tag{VIII.69}$$

and

$$G(x,z) \sim 2\sqrt{U(z)}K(x,z) \sim \frac{1}{x-z}\frac{\sqrt{U(z)}}{\sqrt{U(x)}} - 2\sum_\alpha v_\alpha(x)\,C_\alpha(z)\sqrt{U(z)} \tag{VIII.70}$$

The form $G(x,z)dx$ has thus a simple pole at $x=z$, in the physical sheet with residue $+1$ and in the other sheet with residue $-1$, and it is normalized on $\mathscr{A}$-cycles $\oint_{\mathscr{A}_i} G(x,z)dx = 0$. This is indeed the usual 3rd kind differential in classical algebraic geometry.

### 4.3 The Bergman kernel $B(x,z)$

In classical algebraic geometry, the Bergman kernel is the fundamental second kind differential, it is the derivative of the 3rd kind differential, and it is another major tool in classical algebraic geometry. Following the same definition as in [44], we define:

$$B(x,z) = -\frac{1}{2}\,\partial_z G(x,z). \tag{VIII.71}$$

The kernel $B$ is going to be called the "quantum" Bergman kernel.

**Theorem VIII.11.** *$B(x,z)$ is an analytical function of $x$, with a double pole at $x=z$ with no residue, and double poles at the $s_j$'s with vanishing residues, and possibly an essential singularity around $\infty$.*

*$B(x,z)$ is an analytical function of $z$, with a double pole at $z=x$ with no residue, and double poles at the $s_j$'s with vanishing residues, and possibly an essential singularity around $\infty$.* ***In particular it has no discontinuity along the $\mathscr{A}$ cycles, it is defined analytically in the whole complex plane except at those double poles.***

*Proof.* Those properties follow easily from those of $G(x,z)$ of theorem VIII.8. In particular, it is important to notice that the only discontinuity of $G(x,z)$ is along the $\mathscr{A}$-cycles, and is independent of $z$, therefore $B(x,z)$ has no discontinuity there.

**Properties of the Bergman kernel**

**Theorem VIII.12.**

$$B(x,z) = O(1/x^2) \tag{VIII.72}$$

*when $x \to \infty$ in all sectors.*

*And*

$$B(x,z) = O(1/z^2) \tag{VIII.73}$$

*when $z \to \infty$ in all sectors.*

*Proof.* Follows from the large $x$ and $z$ behaviors of $G(x,z)$.

**Theorem VIII.13.** *$B$ satisfies the loop equations:*

$$(2\frac{\psi'(x)}{\psi(x)} + \partial_x)\left(B(x,z) - \frac{1}{2(x-z)^2}\right) + \partial_z \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z} = P_2^{(0)}(x,z) \tag{VIII.74}$$

*where $P_2^{(0)}(x,z)$ is a polynomial in $x$ of degree at most $d-2$. And*

$$(2\frac{\psi'(z)}{\psi(z)} + \partial_z)\left(B(x,z) - \frac{1}{2(x-z)^2}\right) + \partial_x \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z} = \tilde{P}_2^{(0)}(z,x) \tag{VIII.75}$$

*where $\tilde{P}_2^{(0)}(z,x)$ is a polynomial in $z$ of degree at most $d-2$.*

*Proof.* This theorem is crucial for all what follows, and its proof is rather non-trivial. Since it is very long and technical, we present the proof in appendix VIII. Those equations are indeed the loop equations for the 2-point function in the $\beta$ matrix model, see section VIII.

**Theorem VIII.14.** *We have for every* $\alpha = 1, \dots, g$*:*

$$\oint_{\mathscr{A}_\alpha} B(x,z)\,dx = 0 \qquad , \qquad \oint_{\mathscr{A}_\alpha} B(x,z)\,dz = 0 \tag{VIII.76}$$

*and*

$$\oint_{\mathscr{B}_\alpha} B(x,z)\,dz = 2i\pi v_\alpha(x) \tag{VIII.77}$$

*Proof.* The vanishing of $\mathscr{A}$-cycle integrals in the $x$ variable is by construction and can be seen as the consequence of the same result known for $G(x,z)$ on one side of $\mathscr{A}$ and the fact that $B(x,z)$ has no discontinuity along the $\mathscr{A}$-cycles. (Therefore, the nullity extend on both sides which no longer need to be treated separately).

For the $z$ variable, notice that if $\mathscr{A}_\alpha = ]\infty_i, \infty_j[$ goes from $\infty_i$ to $\infty_j$, where both $\infty_i$ and $\infty_j$ are in the physical sheet, we have:

$$\oint_{\mathscr{A}_\alpha} B(x,z)dz = \int_{\infty_i}^{\infty_j} B(x,z)dz = -\frac{1}{2}\left(G(x,\infty_j) - G(x,\infty_i)\right) \tag{VIII.78}$$

and from theorem VIII.9 $G(x,\infty_i) = \eta_i t_{d+1} K_{d-1}(x)$, we get:

$$\oint_{\mathscr{A}_\alpha} B(x,z)dz = \int_{\infty_i}^{\infty_j} B(x,z)dz = \frac{\eta_i - \eta_j}{2}\, t_{d+1}\, K_{d-1}(x) \tag{VIII.79}$$

and since $\infty_i$ and $\infty_j$ are both in the physical sheet we have $\eta_i = \eta_j = -1$, and therefore

$$\oint_{\mathscr{A}_\alpha} B(x,z)dz = 0. \tag{VIII.80}$$

And similarly, when performing the integral over $\mathscr{B}_\alpha$, the contribution from infinities cancels out since the contour goes in the same sheet. But since $\mathscr{B}_\alpha$ intersects its corresponding $\mathscr{A}_\alpha$ (and only this one) where the primitive $-\frac{1}{2}G(x,z)$ is discontinous, the result is the jump of $G(x,z)$ along this $\mathscr{A}_\alpha$, that is to say $i\pi v_\alpha(x)$. Eventually, since $\mathscr{B}_\alpha$ and $\mathscr{A}_\alpha$ intersect twice, we find eq. (VIII.14).

One of our key theorems is:

**Theorem VIII.15.** *$B(x,z)$ is symmetric*

$$B(x,z) = B(z,x) \tag{VIII.81}$$

*Proof.* The proof relies essentially on the fact that $B(x,z)$ satisfies the loop equation in the two variables. We have:

$$\begin{aligned} & (2\frac{\psi'(z)}{\psi(z)} + \partial_z)\,(2\frac{\psi'(x)}{\psi(x)} + \partial_x)\,(B(x,z) - \frac{1}{2(x-z)^2}) \\ = \quad & (2\frac{\psi'(z)}{\psi(z)} + \partial_z)\left(P_2^{(0)}(x,z) - \partial_z \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z}\right) \\ = \quad & (2\frac{\psi'(x)}{\psi(x)} + \partial_x)\left(\tilde{P}_2^{(0)}(z,x) - \partial_x \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z}\right) \end{aligned} \tag{VIII.82}$$

This implies:

$$\begin{aligned} & (2\frac{\psi'(z)}{\psi(z)} + \partial_z)\,P_2^{(0)}(x,z) - (2\frac{\psi'(x)}{\psi(x)} + \partial_x)\,\tilde{P}_2^{(0)}(z,x) \\ = \quad & (2\frac{\psi'(z)}{\psi(z)} + \partial_z)\partial_z \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z} \\ & -(2\frac{\psi'(x)}{\psi(x)} + \partial_x)\partial_x \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z} \\ = \quad & 2\frac{U(x)-U(z)}{(x-z)^2} - \frac{U'(x)+U'(z)}{x-z} \end{aligned} \tag{VIII.83}$$

and therefore:

$$\begin{aligned} & (x-z)^2\,(2\frac{\psi'(z)}{\psi(z)} + \partial_z)\,P_2^{(0)}(x,z) + 2U(z) + (x-z)U'(z) \\ = \quad & (x-z)^2\,(2\frac{\psi'(x)}{\psi(x)} + \partial_x)\,\tilde{P}_2^{(0)}(z,x) + 2U(x) + (z-x)U'(x) \\ \stackrel{\text{def}}{=} \quad & R(x,z) \end{aligned} \tag{VIII.84}$$

Here, the first line is a polynomial in $x$, whereas the second line is also a polynomial in $z$. Therefore, $R(x,z)$ is a polynomial in both variables, of degree at most $d$ in each variable. Moreover, we must have:

$$R(x,x) = 2U(x) \tag{VIII.85}$$

Therefore we must have:

$$R(x,z) = \frac{1}{\hbar^2}\left(\frac{1}{2}V'(x)V'(z) - \hbar\,\frac{V'(x)-V'(z)}{x-z} - P(x) - P(z)\right) + (x-z)^2\tilde{R}(x,z) \tag{VIII.86}$$

where $\tilde{R}(x,z)$ is a polynomial of both variables of degree at most $d-2$ in each variable.

Putting this back into VIII.84 and using the symmetry $x \leftrightarrow z$ it implies that:

$$(2\frac{\psi'(z)}{\psi(z)} + \partial_z)\,(P_2^{(0)}(x,z) - \tilde{P}_2^{(0)}(x,z)) = \tilde{R}(x,z) - \tilde{R}(z,x) \tag{VIII.87}$$

Then, we can decompose the r.h.s into the basis $h_\alpha(x)h_\beta(z)$ introduced in VIII.27:

$$\tilde{R}(x,z) - \tilde{R}(z,x) = \sum_{\alpha,\beta=1}^{d-1} (\tilde{R}_{\alpha,\beta} - \tilde{R}_{\beta,\alpha}) h_\alpha(x) h_\beta(z) \tag{VIII.88}$$

Integrating the differential equation eq. (VIII.87) then gives:

$$P_2^{(0)}(x,z) - \tilde{P}_2^{(0)}(x,z) = \sum_{\alpha,\beta=1}^{d-1} (\tilde{R}_{\alpha,\beta} - \tilde{R}_{\beta,\alpha}) h_\alpha(x)\, v_\beta(z) + A_1(x) \tag{VIII.89}$$

where $A_1(x)$ is some integration constant.

Then using the loop equations VIII.13 we find by substraction that:

$$\left(2\frac{\psi'(y)}{\psi(y)} + \partial_y\right)(B(y,z) - B(z,y)) = P_2^{(0)}(y,z) - \tilde{P}_2^{(0)}(y,z) \tag{VIII.90}$$

and again, integrating this differential equation we find:

$$B(x,z) - B(z,x) = \sum_{\alpha,\beta=1}^{d-1} (\tilde{R}_{\alpha,\beta} - \tilde{R}_{\beta,\alpha}) v_\alpha(x)\, v_\beta(z) + A(x) + \tilde{A}(z) \tag{VIII.91}$$

where $(2\psi'/\psi + \partial)A = A_1$, and $\tilde{A}(z)$ is some other integration constant.

The large $x$ and large $z$ behavior of $B$ imply that $A(x) = \tilde{A}(z) = 0$. We thus get:

$$B(x,z) - B(z,x) = \sum_{\alpha,\beta} (\tilde{R}_{\alpha,\beta} - \tilde{R}_{\beta,\alpha})\, \tilde{v}_\alpha(x) \tilde{v}_\beta(z) \tag{VIII.92}$$

Then, using theorem VIII.14

$$\oint_{\mathscr{A}_\alpha} B(x,z)dx = 0 = \oint_{\mathscr{A}_\beta} B(x,z)dz \tag{VIII.93}$$

We find:

$$\forall \alpha,\beta, \qquad \tilde{R}_{\alpha,\beta} = \tilde{R}_{\beta,\alpha} \tag{VIII.94}$$

that is to say by VIII.92 that the Bergman kernel is symmetric.

We claim that all these properties are essential to name this function a "quantum Bergman kernel". Indeed, the symmetry is absolutely necessary and is completely non-trivial. The fact that $B(x,z)$ has no discontinuity is also essential since in standard algebraic geometry, it is defined everywhere on the Riemann surface. Using all these kernels and their properties, we can then generalize easily the recursion of [44, 105] defining the correlation functions.

## 4.4 Meromorphic forms and properties

### Definition of meromorphic forms

**Definition VIII.9.** *A meromorphic form $\mathscr{R}(x)$ is defined as:*

$$\mathscr{R}(x) = \frac{1}{\hbar\psi^2(x)} \int_{\infty_0}^{x} r(x')\,\psi^2(x')\,dx' \tag{VIII.95}$$

*where $r(x)$ is a rational function of x, which behaves at most like $O(x^{d-2})$ at large x, and whose poles $r_i$ are such that:*

$$\operatorname*{Res}_{x\to r_i} \psi^2(x)\,r(x) = 0 \tag{VIII.96}$$

*and for all degenerate $\hat{\mathscr{A}}_\alpha$ cycles*

$$\int_{\hat{\mathscr{A}}_\alpha} \psi^2(x')r(x')\,dx' = 0. \tag{VIII.97}$$

It is easy to see, that with this definition, the holomorphic forms $v_\alpha(x)$, the kernels $G(x,z)$ and $B(x,z)$ are meromorphic forms of $x$.

**Analiticity properties**

A meromorphic forms $\mathscr{R}(x)$, has poles at $x = r_i$ the poles of $r(x)$, with degree 1 less than that of $r$, it behaves like $O(x^{-2})$ in all sectors of the physical sheet. From the Bethe ansatz, it has double poles at the $s_i$'s, with vanishing residues.

In particular, it has an accumulation of poles along the half-lines $L_i$ of accumulations of zeroes of $\psi$.

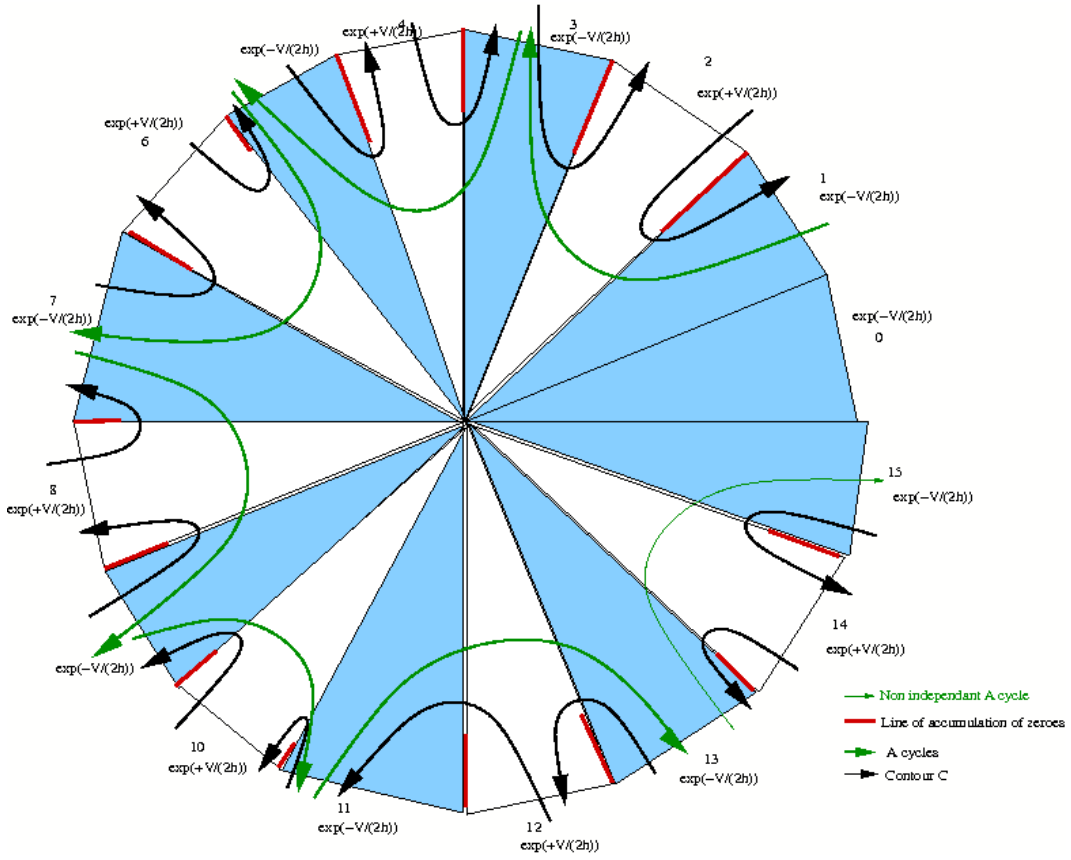
Also, notice that the following integrals are well defined, and independent of homotopic deformations of $\mathscr{A}_\alpha$ (in particular independent of where are the $s_i$'s):

$$\oint_{\mathscr{A}_\alpha} \mathscr{R}(x)dx. \tag{VIII.98}$$

**The integration contours around branch-points**

Let us choose some contour $\mathscr{C}_i$, such that each $\mathscr{C}_i$ surrounds (in the trigonometric direction) a half-line $L_i$ of accumulation of zeroes. In other words it surrounds a "branch point". Let us also assume that $\sum_i \mathscr{C}_i$ surrounds all roots of $\psi$, i.e. each root of $\psi$ is enclosed in one $\mathscr{C}_i$. We also assume that contours $\mathscr{C}_i$ and $\mathscr{A}_\alpha$ do not intersect (they have vanishing intersection numbers):

$$\forall i = 1,\dots,2g+2, \quad ,\forall \alpha = 1,\dots,d-1, \qquad \mathscr{C}_i \cap \mathscr{A}_\alpha = 0 \tag{VIII.99}$$




**Riemann bilinear identity**

For the Riemann bilinear identity, we need the following useful lemma, which we shall use very often in this article:

**Lemma VIII.1.** *For every analytical function $f(x)$ which behaves at infinity at most like $f(x) = O\left(x^{d-2}\right)$ in all directions, and such that it has no singularities inside every contour $\mathscr{C}_i$ (and thus must be regular at the root $s_j$'s) we have, for $x_0$ outside of all $\mathscr{A}$-cycles (i.e. on the same side as $\infty_0$) :*

$$\forall i, \qquad \frac{1}{2i\pi}\oint_{\mathscr{C}_i} dx\, K(x_0,x)\, f(x) = 0$$

*Proof.* Clearly, the contours $\mathscr{C}_i$ enclose no singularity of $K(x_0,x)f(x)$ and can be contracted to 0.

Then we can write the bilinear Riemann identity:

**Theorem VIII.16. Riemann bilinear identity**

*Consider a meromorphic form $\mathscr{R}(x)$, with poles $r_i$.*
*Then we have for $x$ outside of all $\mathscr{A}$-cycles (i.e. on the same side as $\infty_0$):*

$$\mathscr{R}(x) = -\sum_i \operatorname*{Res}_{r_i} G(x,z)\mathscr{R}(z)dz + \sum_{\alpha=1}^{g} v_\alpha(x) \oint_{\mathscr{A}_\alpha} \mathscr{R}(z)\,dz. \tag{VIII.100}$$

*Proof.* Since $G(x,z) = 1/(z-x) + \ldots$, we write Cauchy formula:

$$\mathscr{R}(x) = \operatorname*{Res}_{z\to x} G(x,z)\,\mathscr{R}(z)\,dz \tag{VIII.101}$$

and we deform the contour of integration from a small circle around $x$, to contours enclosing all other singularities, i.e. the $r_i$'s and the $s_i$'s. By doing so, $G(x,z)$ has to cross the $\mathscr{A}$-cycles, and picks a discontinuity equal to $2i\pi\, v_\alpha(x)$ i.e. independent of $z$, so the contour integral of the product factorizes for each $\mathscr{A}_\alpha$. We thus arrive to:

$$\begin{aligned}\mathscr{R}(x) &= -\sum_i \operatorname*{Res}_{r_i} G(x,z)\mathscr{R}(z)dz - \sum_i \frac{1}{2i\pi}\oint_{\mathscr{C}_i} G(x,z)\mathscr{R}(z)dz \\ &\quad + \sum_{\alpha=1}^{g} v_\alpha(x) \oint_{\mathscr{A}_\alpha} \mathscr{R}(z)\,dz.\end{aligned} \tag{VIII.102}$$

Then, we need to compute

$$\oint_{\mathscr{C}_i} G(x,z)\mathscr{R}(z)dz.$$

Write that $G(x,z) = \psi^2(z)\,\partial_z K(x,z)/\psi^2(z)$, and integrate by parts:

$$\oint_{\mathscr{C}_i} G(x,z)\mathscr{R}(z)dz = -\oint_{\mathscr{C}_i} K(x,z)r(z)dz$$

and using lemma VIII.1, we see that this vanishes.

## 5 Definition of correlators and free energies

In this section, we define the quantum deformations of the correlation functions introduced in [23, 105]. Although the following definitions are inspired from (non hermitian) matrix models (see section VIII), they are valid in the present framework of an arbitrary Schrödinger equation, not necessarily linked to a matrix model. The special case of their application to matrix models will be discussed in section VIII.

### 5.1 Definition of correlators

**Definition VIII.10.** *We define the following functions $W_n^{(g)}(x_1,\dots,x_n)$ called $n$-**point correlation function of "genus"** $g$ by the recursion*[1]*:*

$$W_1^{(0)}(x) = \omega(x) \qquad , \qquad W_2^{(0)}(x_1,x_2) = B(x_1,x_2) \tag{VIII.103}$$

$$\begin{aligned} W_{n+1}^{(g)}(x_0,J) \quad = \quad & \frac{1}{2i\pi}\sum_{i=1}^{2g+2}\oint_{\mathscr{C}_i} dx\, K(x_0,x)\Big(\overline{W}_{n+2}^{(g-1)}(x,x,J) \\ & + \sum_{h=0}^{g}{\sum_{I\subset J}}' W_{|I|+1}^{(h)}(x,x_I)W_{n-|I|+1}^{(g-h)}(x,J/I)\Big) \end{aligned} \tag{VIII.104}$$

*where $J$ is a collective notation for the variables $J=\{x_1,\dots,x_n\}$, and where $\sum\sum'$ means that we exclude the terms $(h=0,I=\emptyset)$ and $(h=g,I=J)$, and where:*

$$\overline{W}_n^{(g)}(x_1,...,x_n) = W_n^{(g)}(x_1,...,x_n) - \frac{\delta_{n,2}\delta_{g,0}}{2}\frac{1}{(x_1-x_2)^2} \tag{VIII.105}$$

*Here $x_0$ and all the $x_i'$s are outside of the $\mathscr{A}$-cycles, i.e. on the same side as $\infty_0$. The contour $\mathscr{C}_i$ (defined in section VIII) is a contour which surrounds the branchpoint $L_i$, i.e. a half-line of accumulation of zeroes, and chosen such that every $s_j$ is surrounded by exactly one $\mathscr{C}_i$, and such that $\mathscr{C}_i$ doesn't intersect any $\mathscr{A}$-cycle. Very often we shall write*

$$\mathscr{C} = \sum_{i=1}^{2g+2}\mathscr{C}_i. \tag{VIII.106}$$

Appart from the precise definition of the kernel $K$, this definition is exactly the same topological recursion as in [23], a sum of residues around all branchpoints of the same expression. In other words, the topological recursion is independent of $\hbar$.

To shorten equation we will introduce the notation:

$$U_n^{(g)}(x,J) \quad = \quad \overline{W}_{n+2}^{(g-1)}(x,x,J) + \sum_{I\subset J}\overline{W}_{|I|+1}^{(h)}(x,x_I)\overline{W}_{n-|I|+1}^{(g-h)}(x,J/I)$$

[1] here $g$ is any given integer, it has nothing to do with the genus $g$ of the spectral curve.

$$+\sum_j \partial_{x_j}\left(\frac{\overline{W}_n^{(g)}(x,J/\{j\})-\overline{W}_n^{(g)}(x_j,J/\{j\})}{(x-x_j)}\right) \qquad \text{(VIII.107)}$$

To get:

**Theorem VIII.17.**

$$W_{n+1}^{(g)}(x_0,J)=\frac{1}{2i\pi}\oint_{\mathscr{C}} dx\,K(x_0,x)\,U_n^{(g)}(x,J) \qquad \text{(VIII.108)}$$

*Proof.* The only difference with the definition, is when we face a term like $B(x,x_j)W_n^{(g)}(x,J/\{j\})$. (note that there are twice this term). It can be split into two terms: $\overline{B}(x,x_j)W_n^{(g)}(x,J/\{j\})$ and $\frac{1}{(x-x_j)^2}W_n^{(g)}(x,J/\{j\})$. The second term compensate exactly the $\partial_{x_j}\frac{\overline{W}_n^{(g)}(x,J/\{j\})}{(x-x_j)}$. Thus, the only difference between the two definitions is the term: $\frac{1}{2i\pi}\oint_{\mathscr{C}} dx\,K(x_0,x)\sum_j \partial_{x_j}\frac{\overline{W}_n^{(g)}(x_j,J/\{j\})}{(x-x_j)}$. Therefore the definitions are only the same if these terms are null. This is the case because of Lemma VIII.1.

### 5.2 Properties of correlators

The main reason of definition. VIII.10, is because the $W_n^{(g)}$'s have many beautiful properties, which generalize those of [23], and in particular they provide a solution of loop equations. We shall prove the following properties:

**Theorem VIII.18.** *Each $W_n^{(g)}(x_1,\dots,x_n)$ with $2-2g-n<0$, is an analytical functions of all its arguments, with poles only when $x_i \to s_j$. Moreover, it vanishes at least as $O\left(1/x_i^2\right)$ when $x_i \to \infty$ in all sectors. It has no discontinuity across $\mathscr{A}$-cycles.*

*Proof.* in appendix VIII

**Theorem VIII.19.** *For all $(n,g)\neq(0,0)$ we have*

$$\forall \alpha \leq g : \oint_{\mathscr{A}_\alpha} W_{n+1}^{(g)}(x_0,x_1,...,x_n)dx_1 = 0 \qquad \text{(VIII.109)}$$

$$\forall \alpha \leq g : \oint_{\mathscr{A}_\alpha} W_{n+1}^{(g)}(x_0,x_1,...,x_n)dx_0 = 0 \qquad \text{(VIII.110)}$$

*Proof.* We clearly have these properties for $W_2^{(0)}(x_0,x_1)$. By an easy recursion, the first property holds for $x_1,\dots,x_n$. The case of the variable $x_0$ is special and requires explanation. Indeed for fixed values of $x_1,...,x_n$, the dependance in $x_0$ comes from $K(x_0,x)$. The

theorem then comes from a permutation of integrals. Indeed, since the contour $\mathscr{C}$ never crosses any $\mathscr{A}$-cycles by prescription then we can permute the integrals in $x$ and $x_0$. The nullity of the integral for $K(x_0,x)$ in VIII.7 then gives the result.

**Theorem VIII.20.** *For $2-2g-n<0$, the $W_n^{(g)}$'s satisfy the loop equation, i.e. Virasoro-like constraints. This means that the quantity:*

$$\begin{aligned} P_{n+1}^{(g)}(x;x_1\ldots,x_n) \qquad = \qquad & 2\hbar\frac{\psi'(x)}{\psi(x)}\overline{W}_{n+1}^{(g)}(x,x_1,\ldots,x_n)+\hbar\partial_x\overline{W}_{n+1}^{(g)}(x,x_1\ldots,x_n) \\ & +\sum_{I\subset J}\overline{W}_{|I|+1}^{(h)}(x,x_I)\overline{W}_{n-|I|+1}^{(g-h)}(x,J/I)+\overline{W}_{n+2}^{(g-1)}(x,x,J) \\ & +\sum_j\partial_{x_j}\left(\frac{\overline{W}_n^{(g)}(x,J/\{j\})-\overline{W}_n^{(g)}(x_j,J/\{j\})}{(x-x_j)}\right) \end{aligned} \tag{VIII.111}$$

*is a polynomial in the variable x, of degree at most $d-2$.*

*Proof.* in appendix VIII

**Theorem VIII.21.** *Each $W_n^{(g)}$ is a symmetric function of all its arguments.*

*Proof.* in appendix VIII, with the special case of $W_3^{(0)}$ in appendix VIII.

**Theorem VIII.22.** *The 3 point function $W_3^{(0)}$ can also be written:*

$$W_3^{(0)}(x_1,x_2,x_3)=\frac{4}{2i\pi}\sum_i\oint_{\mathscr{C}_i}\frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)}\,dx \tag{VIII.112}$$

*(this can be seen as a quantum version of Rauch variational formula)*

*Proof.* in appendix VIII

**Theorem VIII.23.** *For $2-2g-n<0$, $W_n^{(g)}(x_1,\ldots,x_n)$ is homogeneous of degree $2-2g-n$:*

$$\left(\hbar\frac{\partial}{\partial\hbar}+\sum_{j=0}^{d+1}t_j\frac{\partial}{\partial t_j}+\sum_{i=1}^{g}\varepsilon_i\frac{\partial}{\partial\varepsilon_i}\right)W_n^{(g)}(x_1,\ldots,x_n)=(2-2g-n)\,W_n^{(g)}(x_1,\ldots,x_n) \tag{VIII.113}$$

*Proof.* Under a change $t_k\to\lambda t_k$, $\hbar\to\lambda\hbar$, $\varepsilon_i\to\lambda\varepsilon_i$, the Schrödinger equation remains unchanged, and thus $\psi$ is unchanged. The kernel $K$ is changed to $K/\lambda$ and nothing else is changed. By recursion, $W_n^{(g)}$ is changed by $\lambda^{2-2g-n}$.

## 6 Deformations

In this section, we will consider the variations of correlators $W_n^{(g)}$ under infinitesimal variations of the Schrödinger potential $U(x)$ or $\hbar$. Infinitesimal variations of the resolvent $\omega(x)$ can be decomposed on the basis of "meromorphic forms", and forms can be put in duality with cycles. The duality kernel pairing is the Bergman kernel. We will find in this section, that the classical $\hbar = 0$ formulae remain valid for $\hbar \neq 0$, and generalize the corresponding form-cycle duality in special geometry.

### 6.1 Variation of the resolvent

Let's consider an infinitesimal polynomial variation:

$$U \to U + \delta U \qquad , \qquad \hbar \to \hbar + \delta\hbar$$

where $\delta U$ is a polynomial of degree: $\deg \delta U \leq 2d$. Since we have written $U = V'^2/4 - \hbar V''/2 - P$, we have:

$$\delta U = \frac{V'}{2}\delta V' - \frac{\hbar}{2}\delta V'' - \frac{\delta\hbar}{2}\delta V'' - \delta P \tag{VIII.114}$$

with

$$\delta V'(x) = \sum_{k=1}^{d+1} \delta t_k x^{k-1}, \tag{VIII.115}$$

and $\delta P$ is of degree at most $d-1$:

$$\delta P = (t_{d+1}\delta t_0 + t_0 \delta t_{d+1})x^{d-1} + \text{lower degree}. \tag{VIII.116}$$

Let us compute $\delta\psi$, or more precisely $f = \delta \ln\psi = \delta\psi/\psi$, let us write it:

$$\delta\psi(x) = f(x)\psi(x) \tag{VIII.117}$$

The Schrödinger equation $\hbar^2\psi'' = U\psi$ implies:

$$\hbar^2(f\psi)'' - Uf\psi = \delta U\psi - \delta\hbar^2\psi'' \tag{VIII.118}$$

i.e.

$$\hbar^2(f''\psi + 2f'\psi') = (\delta U - 2\frac{\delta\hbar}{\hbar}U)\,\psi \tag{VIII.119}$$

Multiplying by $\psi$ we get:

$$\hbar^2(f'\psi^2)' = (\delta U - 2\frac{\delta\hbar}{\hbar}U)\,\psi^2 \tag{VIII.120}$$

i.e.:

$$\delta(\psi'/\psi) = f'(x) = \frac{1}{\hbar^2\,\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\,(\delta U(x') - 2\frac{\delta\hbar}{\hbar}U(x'))\,dx'. \tag{VIII.121}$$

therefore, since $\omega = V'/2 + \hbar\psi'/\psi$:

$$\boxed{\delta\omega(x) = \frac{\delta V'(x)}{2} + \delta\hbar\,\frac{\psi'(x)}{\psi(x)} + \frac{1}{\hbar\,\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\,(\delta U(x') - 2\frac{\delta\hbar}{\hbar}U(x'))\,dx'.} \tag{VIII.122}$$

If we write:

$$\delta U = \frac{V'}{2}\delta V' - \frac{\hbar}{2}\delta V'' - \frac{\delta\hbar}{2}V'' - \delta P \tag{VIII.123}$$

where $\delta P$ is of degree at most $d-1$, and $V'/2 = \omega - \hbar\psi'/\psi$, we have by integration by parts:

$$\begin{aligned}\delta\omega(x) \;=\; & \frac{1}{\hbar\,\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\Big(\omega(x')\delta V'(x') - \delta P(x') \\ & - \delta\hbar\,(\omega'(x') - \frac{1}{2}V''(x'))\Big)\,dx'.\end{aligned} \tag{VIII.124}$$

### 6.2 Decomposition of variations

$U(x)$ is a polynomial of degree $2d$, it has $2d+1$ independent coefficients. If we assume that we have a solution of genus $g < d-1$, this means that $U$ is non generic, and satisfies $d-1-g$ constraints. In the space of all possible $U$'s, we shall consider the submanifold corresponding to $U$ of genus $g$, which is a submanifold of dimension

$$\dim = d + 2 + g \tag{VIII.125}$$

and we shall consider variations of $U$ within that submanifold. Variations transverse to the genus $g$ submanifold, are variations of higher genus and should be computed within a higher genus submanifold.

Instead of the $d+2+g$ independent coefficients of the polynomial $U$, it is more convenient to choose a system of "flat" coordinates in our genus $g$ submanifold, given by:

$$t_0, t_1, \dots, t_{d+1},\ \varepsilon_1, \dots, \varepsilon_g. \tag{VIII.126}$$

We have indeed $d+2+g$ coordinates.

Let us write the variations as:

$$\delta U = \sum_{k=0}^{d+1} U_{t_k}\, \delta t_k + \sum_{i=1}^{g} U_{\varepsilon_i} \delta \varepsilon_i + U_\hbar \delta \hbar. \tag{VIII.127}$$

**Variations relatively to the filling fractions**

For the filling fraction $\delta\varepsilon_\alpha$ we have $\delta V' = 0$ and thus:

$$\delta U(x) = -\delta P(x) \tag{VIII.128}$$

where $\deg \delta P \leq d-2$, so we decompose it on the basis of $h_\alpha$'s:

$$\delta P(x) = \sum_{\alpha'} c_{\alpha'}\, h_{\alpha'}. \tag{VIII.129}$$

and therefore, from eq. (VIII.122):

$$\delta\omega(x) = -\sum_{\alpha'} c_{\alpha'}\, v_{\alpha'}(x). \tag{VIII.130}$$

Since $2i\pi\varepsilon_{\alpha'} = \oint_{\mathscr{A}_{\alpha'}} \omega$, we have:

$$2i\pi\, \delta_{\alpha,\alpha'} = \oint_{\mathscr{A}_{\alpha'}} \delta\omega = -\sum_{\alpha''} \oint_{\mathscr{A}_{\alpha'}} c_{\alpha''}\, v_{\alpha''} = -c_{\alpha'} \tag{VIII.131}$$

This implies:

$$U_{\varepsilon_\alpha}(x) = 2i\pi h_\alpha(x) \tag{VIII.132}$$

and

$$\boxed{\delta_{\varepsilon_i}\omega(x) = 2i\pi\, v_\alpha(x) = \oint_{\mathscr{B}_\alpha} B(x,z)\,dz.} \tag{VIII.133}$$

We shall say that the flat coordinate $\varepsilon_\alpha$ is dual to the holomorphic form $v_\alpha$, which is itself dual to the cycle $\mathscr{B}_\alpha$:

$$\varepsilon_\alpha \text{"} = \text{"} \frac{1}{2i\pi} \oint_{\mathscr{A}_\alpha} \omega \qquad , \qquad \delta_{\varepsilon_\alpha}\omega = 2i\pi\, v_\alpha = \oint_{\mathscr{B}_\alpha} B. \tag{VIII.134}$$

**Variations relatively to $t_0$**

We have:

$$\delta U(x) = -\delta P(x) = -t_{d+1}\, x^{d-1} + Q(x) \tag{VIII.135}$$

where $\deg Q \leq d-2$. Using eq. (VIII.122) we get:

$$\delta\omega(x) = \frac{1}{\psi^2(x)} \int_{\infty_0}^{x} \left(-t_{d+1} x'^{d-1} + Q(x')\right) \psi^2(x')\,dx' \tag{VIII.136}$$

and the polynomial $Q$ is chosen such that $\oint_{\mathscr{A}_i} \delta\omega = 0$ so that when decomposing $Q(x)$ on the basis $v_\alpha(x)$ and performing integrals over $\mathscr{A}$-cycles one finds the coefficients of the decomposition as integrals. Therefore we have:

$$\begin{aligned} \delta\omega(x) \qquad &= \qquad -t_{d+1}\, K_{d-1}(x) \\ &= \qquad -t_{d+1} \left( \hat{K}_{d-1}(x) - \sum_{\alpha=1}^{g} v_\alpha(x) \oint_{\mathscr{A}_\alpha} \hat{K}_{d-1}(x')\,dx' - \sum_{\alpha=g+1}^{d-1} v_\alpha(x) \oint_{\mathscr{A}_\alpha} \psi^2(x')\, x'^{d-1}\,dx' \right) \end{aligned} \tag{VIII.137}$$

where

$$\hat{K}_k(x) = \frac{1}{\psi^2(x)} \int_{\infty_0}^{x} x'^k\, \psi^2(x')\,dx', \tag{VIII.138}$$

and $K_k(x)$ is the $k^{\text{th}}$ term in the large $z$ expansion of $K(x,z) = -\sum_{k=0}^{\infty} \frac{K_k(x,z)}{z^{k+1}}$ computed in theorem VIII.6. From theorem VIII.9 we have $G(x,\infty_i) = \eta_i t_{d+1} K_{d-1}(x)$. This shows that

$$\boxed{\delta_{t_0}\omega(x) = G(x,\infty_0) = \frac{1}{2}\left(G(x,\infty_0) - G(x,\infty_-)\right) = \int_{\infty_0}^{\infty_-} B(x,z)\,dz} \tag{VIII.139}$$

where $\infty_0$ is in the physical sheet, and $\infty_-$ is any infinity chosen in the second sheet.

We shall say that the flat coordinate $t_0$ is dual to the 3rd kind meromorphic form $-2G(x,\infty_0)$, which is itself dual to the chain $[\infty_0,\infty_-]$:

$$\boxed{t_0 = \operatorname*{Res}_{\infty_0} \omega \qquad , \qquad \delta_{t_0}\omega = -2G(x,\infty_0) = \int_{\infty_0}^{\infty_-} B(x,z)\,dz} \tag{VIII.140}$$

where Res means the coefficient of $1/z$ in the given sector.

**Variation relatively to $t_k, k = 1 \ldots d$**

For $k = 1,\ldots,d$ we have:

$$U_{t_k}(x) = \frac{V'(x)}{2}\,x^{k-1} - Q(x) \qquad , \qquad \deg Q \le d-2 \tag{VIII.141}$$

and $Q$ is chosen such that $\oint_{\mathscr{A}_i} \delta\omega = 0$. Using eq. (VIII.122) we write:

$$\delta\omega(x) = \delta\hat{\omega}(x) - \sum_\alpha v_\alpha(x) \oint_{\mathscr{A}_\alpha} \delta\hat{\omega}(x')\,dx' \tag{VIII.142}$$

where

$$\delta\hat{\omega}(x) = \frac{1}{\psi^2(x)} \int_{\infty_0}^{x} \frac{V'(x')}{2}\,x'^{k-1}\,\psi^2(x')\,dx' \tag{VIII.143}$$

Since $V'(x') = \sum_j t_{j+1} x'^j$, we have:

$$2\delta\omega(x) = \sum_{j=0}^{d} t_{j+1} K_{k+j-1} \tag{VIII.144}$$

Let us compare it with the large $z$ behaviour of $G(x,z)$ in the physical sheet. We have:

$$G(x,z) = V'(z)\,K(x,z) + O(z^{-d-1}) \tag{VIII.145}$$

which means that the large $z$ expansion of $G(x,z) = \sum_k G_k(x)\,z^{-k}$ is given for $k = 1,\ldots,d$ by:

$$G_k(x) = -\sum_{j=0}^{d} t_{j+1} K_{k+j-1} \tag{VIII.146}$$

and therefore

$$\delta\omega(x) = -\frac{1}{2}G_k(x) \tag{VIII.147}$$

If we write the large $z$ expansion of $B(x,z)$ in the physical sheet, we have

$$B(x,z) = \sum_k B_k(x)\, z^{-k-1} = -\frac{1}{2}\sum_k k\, G_k(x,z) z^{-k-1} \tag{VIII.148}$$

and thus

$$\boxed{\delta_{t_k}\omega(x) = \frac{1}{k}B_k(x) = \operatorname*{Res}_{\infty_0} \frac{z^k}{k} B(x,z)\, dz} \tag{VIII.149}$$

We shall say that the flat coordinate $t_k$ is dual to the 2nd kind meromorphic form $\frac{1}{k}B_k(x)$, which is itself dual to a residue of $B$.

**Variations relatively to $t_{d+1}$**

When $k = d+1$, we have a few additional terms of degree $> d-2$:

$$U_{t_{d+1}}(x) = \frac{V'(x)}{2}x^d - \frac{d\hbar}{2}x^{d-1} - t_0 x^{d-1} - Q(x) \qquad , \qquad \deg Q \leq d-2 \tag{VIII.150}$$

and $Q$ is chosen such that $\oint_{\mathscr{A}_i} \delta\omega = 0$. Using eq. (VIII.122) we write:

$$\delta\omega(x) = \delta\hat{\omega}(x) - \sum_\alpha v_\alpha(x) \oint_{\mathscr{A}_\alpha} \delta\hat{\omega}(x')\, dx' \tag{VIII.151}$$

where

$$\delta\hat{\omega}(x) = \frac{1}{\psi^2(x)} \int_{\infty_0}^x \left( \frac{V'(x')}{2}x'^d - \frac{d\hbar}{2}x'^{d-1} - t_0 x'^{d-1} \right) \psi^2(x')\, dx' \tag{VIII.152}$$

In other words we have:

$$2\delta\omega(x) = \sum_{j=0}^{d} t_{j+1} K_{d+j} - d\hbar K_{d-1} - 2t_0 K_{d-1} \tag{VIII.153}$$

Let us compare it with the large $z$ behaviour of $G(x,z)$. We have:

$$G(x,z) = (V'(z) - \frac{2t_0}{z})\, K(x,z) - \hbar \partial_z K(x,z) + O(z^{-d-2}) \tag{VIII.154}$$

which means that the large $z$ expansion of $G(x,z) = \sum_k G_k(x)\, z^{-k}$ is given for $k = d+1$ by:

$$G_{d+1}(x) = -\sum_{j=0}^{d} t_{j+1} K_{d+j} + \hbar\, d\, K_{d-1} + 2t_0 K_{d-1} \tag{VIII.155}$$

and therefore

$$\delta\omega(x) = -\frac{1}{2} G_{d+1}(x) \tag{VIII.156}$$

If we write the large $z$ expansion of $B(x,z)$, we have

$$B(x,z) = \sum_k B_k(x)\, z^{-k-1} = -\frac{k}{2} \sum_k G_k(x,z) z^{-k-1} \tag{VIII.157}$$

and thus

$$\boxed{\delta_{t_{d+1}} \omega(x) = \frac{1}{d+1} B_{d+1}(x) = \operatorname*{Res}_{\infty_0} \frac{z^d}{d} B(x,z)\, dz} \tag{VIII.158}$$

We shall say that the flat coordinate $t_{d+1}$ is dual to the 2nd kind meromorphic form $\frac{1}{d+1} B_{d+1}(x)$, which is itself dual to a residue of $B$.

### 6.3 Variation relatively to $\hbar$

We have:

$$\delta_\hbar \omega(x) = -\frac{1}{\hbar\, \psi^2(x)} \int_{\infty_0}^{x} \psi^2(x') \left( \omega'(x') - \frac{1}{2} V''(x') - \delta_\hbar P(x') \right) dx' \tag{VIII.159}$$

where $\delta_\hbar P$ is a polynomial of degree $\leq d-2$ chosen so that $\oint_{\mathscr{A}_i} \delta\omega = 0$. For the moment, we have not found a good way of writing this expression as an integral with $B$, and we leave that question for a future work.

cxc

### 6.4 Form-cycle duality

Notice that in all cases, except $\delta_\hbar$, there exist a cycle $\delta\omega^*$ and a function $\Lambda^*_{\delta\omega}$ such that:

$$\delta\omega(x) = \int_{\delta\omega^*} B(x,z)\,\Lambda^*_{\delta\omega}(z)\,dz. \tag{VIII.160}$$

We will use this generic notation later on in order to avoid specifying the 3 different cases.

Under a suitable reparametrization $z \to z'$ such that $dz' = \Lambda^*_{\delta\omega}(z)\,dz$, we say that $\delta\omega^*$ in the variable $z'$ is the cycle dual to the "meromorphic form" $\delta\omega$.

### 6.5 Variation of higher correlators

The following theorem allows to compute the infinitesimal variation of any $W_n^{(g)}$ under a variation of the Schrödinger equation. It tells about the "complex structure deformation" of our quantum Riemann surface. It can be regarded as special geometry relations.

**Theorem VIII.24.** *Under an infinitesimal deformation $U \to U + \delta U$, we have:*

$$\delta W_n^{(g)}(x_1,\dots,x_n) = \int_{\delta\omega^*} W_{n+1}^{(g)}(x_1,\dots,x_n,x')\,\Lambda^*(x')\,dx' \tag{VIII.161}$$

*where $(\delta\omega^*, \Lambda^*_{\delta\omega})$ is the dual cycle to the deformation of the resolvent $\omega \to \omega + \delta\omega$.*

*Proof.* The loop equation for $W_n^{(g)}(x,J)$ is:

$$(2\omega(x) - V'(x) + \hbar\partial_x)\,W_n^{(g)}(x,J) + U_n^{(g)}(x,x;J) = P_n^{(g)}(x,J) \tag{VIII.162}$$

taking a variation $\delta$ we have:

$$\begin{aligned}&(2\omega(x) - V'(x) + \hbar\partial_x)\,\delta W_n^{(g)}(x,J) + (2\delta\omega(x) - \delta V'(x))W_n^{(g)}(x,J) + \delta U_n^{(g)}(x,x;J)\\ &= \delta P_n^{(g)}(x,J)\end{aligned} \tag{VIII.163}$$

notice that $\delta P_n^{(g)}(x,J)$ is a polynomial in $x$, of degree at most $d-2$.

On the other hand, consider the loop equation for $W_{n+1}^{(g)}(x,J,x')$ and multiply it by $\Lambda^*(x')$ and integrate $x'$ along $\omega^*$, one gets:

$$(2\omega(x) - V'(x) + \hbar\partial_x)\int_{\omega^*} W_{n+1}^{(g)}(x,J,x')\Lambda^*(x')dx' + \int_{\omega^*} \delta U_{n+1}^{(g)}(x,x;J,x')\Lambda^*(x')dx'$$

$$= \int_{\omega^*} P_{n+1}^{(g)}(x,J,x')\Lambda^*(x')dx' \tag{VIII.164}$$

That gives by recursion hypothesis for the computation of $\int_{\omega^*} \delta U_{n+1}^{(g)}(x,x;J,x')\Lambda^*(x')dx'$ and using VIII.122:

$$\begin{aligned} & (2\omega(x) - V'(x) + \hbar\partial_x)\left(\int_{\omega^*} W_{n+1}^{(g)}(x,J,x')\Lambda^*(x')dx' - \delta W_n^{(g)}(x,J)\right) \\ = \;& \delta P_n^{(g)}(x,J) - \int_{\omega^*} P_{n+1}^{(g)}(x,J,x')\Lambda^*(x')dx' \\ = \;& \sum_i \alpha_i(J)\, h_i(x) \end{aligned} \tag{VIII.165}$$

where the right hand side is a polynomial of degee at most $d-2$ in $x$, which can be decomposed on the basis $h_i(x)$.

Solving the differential equation gives:

$$\int_{\omega^*} W_{n+1}^{(g)}(x,J,x')\Lambda^*(x')dx' - \delta W_n^{(g)}(x,J) = \sum_i \alpha_i(J)\, v_i(x) \tag{VIII.166}$$

but since $W_n^{(g)}$ and $W_{n+1}^{(g)}$ are normalized on $\mathscr{A}$-cycles, this implies $\alpha_i = 0$, i.e.:

$$\int_{\omega^*} W_{n+1}^{(g)}(x,J,x')\Lambda^*(x')dx' = \delta W_n^{(g)}(x,J) \tag{VIII.167}$$

## 7 Free energies

We use the variations and theorem VIII.23 to define the $F_g$'s.

Theorem VIII.23 gives:

$$(2-2g-n-\hbar\,\partial_\hbar)\, W_n^{(g)} = \left(t_0\,\partial_{t_0} + \sum_{k=1}^{d+1} t_k\,\partial_{t_k} + \sum_{i=1}^{g} \varepsilon_i \partial_{\varepsilon_i}\right) W_n^{(g)} \tag{VIII.168}$$

And in the previous section, we have seen how to write the derivatives of $W_n^{(g)}$ as integrals of $W_{n+1}^{(g)}$, that gives:

$$(2-2g-n-\hbar\,\partial_\hbar)\, W_n^{(g)} = \hat{H}.W_{n+1}^{(g)} \tag{VIII.169}$$

where $\hat{H}$ is the linear operator acting as follows:

$$\hat{H}.f(x) = t_0 \int_{\infty_0}^{\infty_-} f + \sum_{j=1}^{d+1} \operatorname*{Res}_{\infty_0} \frac{t_j x^j}{j} f + \sum_{i=1}^{g} \varepsilon_i \oint_{\mathscr{B}_i} f. \tag{VIII.170}$$

Those equations allow to define $W_0^{(g)} = F_g$ for $n = 0$ and $g \geq 2$ as:

**Definition VIII.11.** *We define $F_g$ for $g \geq 2$ such that:*

$$(2 - 2g - \hbar\,\partial_\hbar)\, F_g = \hat{H}.W_1^{(g)} \tag{VIII.171}$$

It would remain to find the correct definitions of $F_0$ (called the prepotential) and $F_1$. $F_0$ and $F_1$ should be such that under every deformation $\delta = \partial_{t_k}, \partial_{t_0}, \partial_{\varepsilon_i}$ we should have

$$\delta F_g = H_\delta\, W_1^{(g)}. \tag{VIII.172}$$

For example $\partial F_0/\partial t_k = \operatorname{Res} x^k \omega(x)/k$ i.e. the coefficient of the term $1/x^{k-1}$ in the expansion of $\omega(x)$ near $\infty_0$.

We leave the definitions of $F_0$ and $F_1$ for a future work.

## 8 Classical and quantum geometry: summary

Let us summarize the comparison between classical algebraic geometry, and its quantum counterpart introduced here.

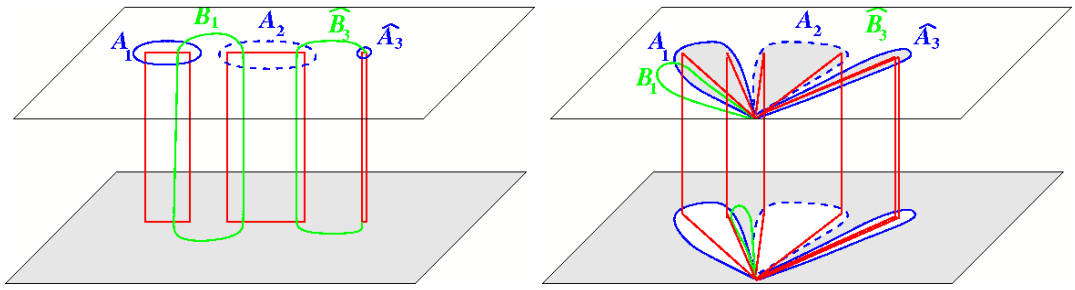



Figure VIII.1: Classical case $\hbar = 0$ of a two sheeted Riemann surface. The branchpoints are paired (in an arbitrary way) to form cuts, and the two sheets are glued along the cuts. Another possibility, is to draw a cut from each branchpoint to $\infty$. The $\mathscr{A}$-cycles surround pairs of branchpoints in the physical sheet. There are also some degenerate branchpoints, which correspond to cuts of vanishing length.

**Summary**

| | **classical** $\hbar = 0$ | **quantum** |
|---|---|---|
| plane curve: | $E(x,y) = \sum_{i,j} E_{i,j} x^i y^j$ | $E(x,y) = \sum_{i,j} E_{i,j} x^i y^j$, $\quad [y,x] = \hbar$ |
| | $E(x,y) = 0$ | $E(x,\hbar\partial_x)\psi = 0$ |
| hyperelliptical plane curve: | $y^2 = U(x)$ | $y^2 - U(x)$, $\quad [y,x] = \hbar$, |
| | $\deg U = 2d$ | $\hbar^2\psi'' = U\,\psi$ |
| Potential: | | $V'(x) = 2(\sqrt{U(x)})_+$ |
| 2 sheets: | $y \sim_\infty \pm\frac{1}{2}V'(x)$ | $\hbar\psi'/\psi \sim_\infty \pm\frac{1}{2}V'(x) \sim_{\infty_k} \frac{\eta_k}{2}V'(x)$ |
| | | choice $\psi = \psi_0 \searrow$ in sector $\infty_0$, $\eta_0 = -1$ |
| resolvent: | $\omega(x) = V'(x)/2 + y.$ | $\omega(x) = V'(x)/2 + \hbar\frac{\psi'}{\psi}.$ |
| physical sheet: | $y \sim_\infty -\frac{1}{2}V'(x)$, $\omega \sim t_0/x$ | $\hbar\psi'/\psi \sim_\infty -\frac{1}{2}V'(x)$, $\omega \sim t_0/x$ |
| | | sectors where $\psi \sim e^{-\frac{V}{2\hbar}}$ |
| branchpoints: | simple zeroes of $U(x)$ | half-lines of accumulations |
| | $U(a_i) = 0$, $U'(a_i) \neq 0$ | of zeroes of $\psi$ |
| | $i = 1,\ldots,2g+2$ | $L_i$, $i = 1,\ldots,2g+2$ |
| genus $g$: | | $2g+2 = \#$ branch points |
| | | $-1 \leq g \leq d-1$ |
| double points: | double zeroes of $U(x)$ | half-lines without accumulations |
| | $U(\hat{a}_i) = 0$, $U'(\hat{a}_i) = 0$ | of zeroes of $\psi$ |
| genus $g = -1$ | degenerate surface | $\psi\, e^{V/2\hbar}$ =polynomial |
| $\mathscr{A}_\alpha$-cycles | surround pairs of | surround pairs of half-lines |
| $\alpha = 1,\ldots,g$ | branchpoints | of accumulating zeroes |

| **Summary** | | |
|---|---|---|
| | classical $\hbar = 0$ | quantum |
| 3rd kind form | $G(x,z) \sim_{x\to z} 1/(z-x)$ | |
| | $G(x,z) = (2\omega(z) - V'(z) - \hbar\partial_z)K(x,z)$ | |
| Recursion kernel | $K(x,z)$ | |
| | $K(x,z) = \hat{K}(x,z) - \sum_\alpha v_\alpha(x) C_\alpha(z)$ | |
| | $C_\alpha(z) = \oint_{\mathscr{A}_\alpha} \hat{K}(x',z)dx'$ | |
| | $\hat{K}(x,z) = \frac{1}{z-x}\frac{1}{2\sqrt{U(x)}}$ | $\hat{K}(x,z) = \frac{1}{\hbar\psi^2(x)}\int_{\infty_0}^x \frac{\psi^2(x')dx'}{x'-z}$ |
| Bergman kernel | $B(x,z) = -\frac{1}{2}\partial_z G(x,z)$ | |
| 2nd kind | $B(x,z) \sim 1/2(x-z)^2$ | |
| Symmetry: | $B(x,z) = B(z,x)$ | |
| | $\oint_{\mathscr{A}_\alpha} B(x,z)dx = 0$ | |
| | $\oint_{\mathscr{B}_\alpha} B(x,z)dx = 2i\pi\, v_\alpha(z)$ | |
| Meromorphic | $\mathscr{R}(x)dx = \frac{r(x)dx}{2\sqrt{U(x)}}$ | $\mathscr{R}(x) = \frac{1}{\hbar\psi^2(x)}\int_{\infty_0}^x r(x')\,\psi^2(x')\,dx'$ |
| forms | $r(x)$ =rational with poles $r_i$, $r(x) = O(x^{d-2})$ | |
| | | $\mathrm{Res}_{r_i} r(x')\psi^2(x') = 0$ |
| Higher correlators | $W_{n+1}^{(g)}(x,J) = \sum_i \frac{1}{2i\pi}\oint_{\mathscr{C}_i} K(x,z)\,dz \Big( W_{n+2}^{(g-1)}(z,z,J)$ | |
| | $+ \sum'_{h+h'=g, I\uplus I'=J} W_{1+\lvert I\rvert}^{(h)}(z,I) W_{1+\lvert I'\rvert}^{(h')}(z,I') \Big)$ | |
| | where $\mathscr{C}_i$ surrounds the branchpoint $L_i$ | |
| Symmetry | $W_n^{(g)}(x_1,x_2,\dots,x_n) = W_n^{(g)}(x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(n)}), \qquad \sigma \in S_n$ | |
| Variations and dual cycle | $U(x) \to U(x) + \delta U(x)$ | |
| | $\delta U^*$: $\delta\omega(x) = \int_{\delta U^*} B(x,x')\,\Lambda_{\delta U}(x')\,dx'$ | |
| $\delta V' = \sum \delta t_k x^{k-1}$ | $\delta_{t_k}\omega(x) = \mathrm{Res}_\infty B(x,x')\frac{x'^k}{k}dx'$ | |
| variation $\delta t_0$ | $\delta_{t_0}\omega(x) = \int_{\infty_0}^{\infty_-} B(x,x')\,dx'$ | |
| variation $\delta\varepsilon_i$ | $\delta_{\varepsilon_i}\omega(x) = \oint_{\mathscr{B}_i} B(x,x')\,dx'$ | |
| Variations of higher correlators | $\delta W_n^{(g)}(x_1,\dots,x_n) = \int_{\delta U^*} W_{n+1}^{(g)}(x_1,\dots,x_n,x')\,\Lambda_{\delta U}(x')\,dx'$ | |
| Rauch formula | $W_3(x_1,x_2,x_3) = \oint_{\mathscr{C}} \frac{B(x_1,z)B(x_2,z)B(x_3,z)}{4Y'(z)}\,dz$ | |

## 9 Application: Matrix models

The reason why we introduced those $W_n^{(g)}$'s is because they satisfy the loop equations for $\beta$-random matrix ensembles.

Consider a (possibly formal) matrix integral:

$$Z = \int_{E_{N,\beta}} dM \, \mathrm{e}^{-\frac{N\sqrt{\beta}}{t_0} \mathrm{tr} V(M)} \tag{VIII.173}$$

where $V(x)$ is some polynomial, and where $E_{N,1} = H_N$ is the set of hermitian matrices of size $N$, $E_{N,1/2}$ is the set of real symmetric matrices of size $N$ and $E_{N,2}$ is the set of quaternion self dual matrices of size $N$ (see [19]).

Alternatively, we can integrate over the angular part and get an integral over eigenvalues only [19]:

$$Z = \int d\lambda_1 \dots d\lambda_N \, \Delta(\lambda)^{2\beta} \prod_{i=1}^{N} \mathrm{e}^{-\frac{N\sqrt{\beta}}{t_0} V(\lambda_i)} \tag{VIII.174}$$

where $\Delta(\lambda) = \prod_{i<j}(\lambda_j - \lambda_i)$ is the Vandermonde determinant.

This allows to generalize the matrix model to arbitrary values of $\beta$. In particular, we shall choose $\beta$ of the form:

$$\sqrt{\beta} = \frac{\hbar N}{2t_0} \left( 1 \pm \sqrt{1 + \frac{4t_0^2}{\hbar^2 N^2}} \right) \tag{VIII.175}$$

i.e.

$$\hbar = \frac{t_0}{N} \left( \sqrt{\beta} - \frac{1}{\sqrt{\beta}} \right). \tag{VIII.176}$$

Notice that $\hbar = 0$ correspond to the hermitian case $\beta = 1$, and $\hbar \to -\hbar$ corresponds to $\beta \to 1/\beta$.

### 9.1 Correlators and loop equations

Then we define the correlators:

$$W_k(x_1, \dots, x_k) = \beta^{k/2} \left\langle \sum_{i_1,\dots,i_k} \frac{1}{x_1 - \lambda_{i_1}} \dots \frac{1}{x_k - \lambda_{i_k}} \right\rangle_c \tag{VIII.177}$$

and

$$W_0 = F = \ln Z. \tag{VIII.178}$$

And we assume (this is automatically true if we are considering formal matrix integrals), that there is a large $N$ expansion of the type (where we assume $\hbar = O(1)$):

$$W_k(x_1,\dots,x_k) = \sum_{g=0}^{\infty} (N/t_0)^{2-2g-k} W_k^{(g)}(x_1,\dots,x_k) \tag{VIII.179}$$

$$W_0 = F = \sum_g (N/t_0)^{2-2g} W_0^{(g)} = \sum_g (N/t_0)^{2-2g} F_g. \tag{VIII.180}$$

The loop equations are obtained by integration by parts, for example:

$$0 = \sum_i \int d\lambda_1 \dots d\lambda_N \frac{\partial}{\partial \lambda_i} \left( \frac{1}{x-\lambda_i} \Delta(\lambda)^{2\beta} \prod_j \mathrm{e}^{-\frac{N\sqrt{\beta}}{t_0} V(\lambda_j)} \right) \tag{VIII.181}$$

gives:

$$\begin{aligned}
0 &= \sum_i \left\langle \frac{1}{(x-\lambda_i)^2} + 2\beta \sum_{j\neq i} \frac{1}{x-\lambda_i}\frac{1}{\lambda_i-\lambda_j} - \frac{N\sqrt{\beta}}{t_0}\frac{V'(\lambda_i)}{x-\lambda_i} \right\rangle \\
&= \sum_i \left\langle \frac{1}{(x-\lambda_i)^2} + \beta \sum_{j\neq i} \frac{1}{x-\lambda_i}\frac{1}{x-\lambda_j} - \frac{N\sqrt{\beta}}{t_0}\frac{V'(\lambda_i)}{x-\lambda_i} \right\rangle \\
&= \sum_i \left\langle \frac{1-\beta}{(x-\lambda_i)^2} + \beta \sum_{j} \frac{1}{x-\lambda_i}\frac{1}{x-\lambda_j} - \frac{N\sqrt{\beta}}{t_0}\frac{V'(\lambda_i)}{x-\lambda_i} \right\rangle \\
&= (\beta-1)\frac{1}{\sqrt{\beta}} W_1'(x) + \beta\left(\frac{1}{\beta} W_1^2(x) + \frac{1}{\beta} W_2(x,x)\right) \\
&\qquad - \frac{N\sqrt{\beta}}{t_0}\left( \frac{1}{\sqrt{\beta}} V'(x) W_1(x) - \sum_i \left\langle \frac{V'(x)-V'(\lambda_i)}{x-\lambda_i} \right\rangle \right)
\end{aligned} \tag{VIII.182}$$

We define the polynomial

$$P_1(x) = \sqrt{\beta} \sum_i \left\langle \frac{V'(x)-V'(\lambda_i)}{x-\lambda_i} \right\rangle = (V' W_1)_+. \tag{VIII.183}$$

We thus have:

$$W_1^2(x)+\hbar\frac{N}{t_0}W_1'(x)+W_2(x,x)=\frac{N}{t_0}\left(V'(x)W_1(x)-P_1(x)\right) \tag{VIII.184}$$

Using the expansion eq. (VIII.179), that gives the Ricatti equation

$$W_1^{(0)}(x)^2+\hbar\,\partial_x W_1^{(0)}(x)=V'(x)W_1^{(0)}(x)-P_1^{(0)}(x) \tag{VIII.185}$$

which is satisfied by $\omega(x)$:

$$W_1^{(0)}(x)=\omega(x). \tag{VIII.186}$$

generalizing to the integration by parts of

$$0=\sum_i\int d\lambda_1\dots d\lambda_N\frac{\partial}{\partial\lambda_i}\Big(\frac{1}{x-\lambda_i}\sum_{i_1,\dots,i_k}\prod_{j=1}^k\frac{1}{x_j-\lambda_{i_j}}\quad\Delta(\lambda)^{2\beta}\prod_j \mathrm{e}^{-\frac{N\sqrt{\beta}}{t_0}V(\lambda_j)}\Big) \tag{VIII.187}$$

and using the expansion eq. (VIII.179) to higher orders in $t_0/N$, one gets the loop equations of theorem VIII.20, where

$$\begin{aligned}P_{k+1}(x;x_1,\dots,x_k) &= \sum_g (N/t_0)^{2-2g-k}P_{k+1}^{(g)}(x;x_1,\dots,x_k)\\ &= \beta^{k/2}\left\langle\sum_i\frac{V'(x)-V'(\lambda_i)}{x-\lambda_i}\sum_{i_1,\dots,i_k}\prod_{j=1}^k\frac{1}{x_j-\lambda_{i_j}}\right\rangle_c\end{aligned} \tag{VIII.188}$$

In other words, **the correlation functions of $\beta$ matrix models, obey the topological recursion of def. VIII.10**.

**Remark:**

In [44], a solution of loop equations for the $\beta$-matrix ensemble was proposed, but that solution was such that $U(x)$ was non-generic, corresponding to $g=-1$, and that $\psi(x)$ had only a finite number of zeroes. This case implied that $t_0$ was quantized. Generic matrix models cannot correspond to that situation.

That solution was thus not very useful for actual matrix models. Here instead, we have the solution for every $U(x)$, i.e. every contour of integration for the $\lambda_i$'s, and therefore we have the solution of loop equations for the actual matrix model.

### 9.2 Example: real eigenvalues

Very often, we are interested in a matrix model with real potential $V(x)$ of even degree (i.e. $d$ is odd) and such that the eigenvalues are integrated along the real axis. The resolvent $\omega(x)$ is the Stieljes transform of the density of eigenvalues:

$$\omega(x) = \frac{t_0\sqrt{\beta}}{N}\int_{\mathbb{R}} \frac{\rho(x')\,dx'}{x-x'} \qquad , \qquad \rho(x) = \left\langle \sum_i \delta(x-\lambda_i) \right\rangle \tag{VIII.189}$$

Let us consider that it is defined by this integral in the upper half-plane for $x \in \mathbb{H}_+$, and it is extended to the lower half-plane by analytical continuation.

By definition, $\omega(x)$ is regular in the upper half-plane, therefore we look for a $\psi(x)$ which has no zero in the upper half-plane, i.e. no zero on the half-lines $L_0, L_1, \dots, L_{d-1}$. I.e. it has at most $d+1$ half-lines of zeroes , and thus:

$$g \leq \frac{d-1}{2}.$$

## 10 Non-oriented Ribbon graphs

Consider the set of all closed connected ribbon graphs obtained by gluing the pieces represented in fig. VIII.2. Closed means every half-edge is glued to another half-edge. Connected means every vertex is connected to any other vertex. See for example fig. VIII.3.

We define the genus of such a ribbon graph $\mathscr{G}$ as follows. We replace every twisted edge of $\mathscr{G}$ by a non-twisted one, we thus obtain another ribbon graph $\mathscr{G}'$, which is oriented. We define the genus of $\mathscr{G}$ equal to that of $\mathscr{G}'$:

$$g(\mathscr{G}) = g(\mathscr{G}').$$

The genus of $\mathscr{G}'$ is computed as usual for oriented ribbon graphs, from the Euler characteristics of $\mathscr{G}'$:

$$\chi(\mathscr{G}') = 2-2g = \#\{\text{vertices}(\mathscr{G})\} - \#\{\text{edges}(\mathscr{G})\} + \#\{\text{single lines}(\mathscr{G}')\}$$

where single lines are the lines bordering each side of the ribbon edges. One should follow single lines and see how many connected single lines a graph contains. Obviously $\mathscr{G}$ and $\mathscr{G}'$ have the same number of fat vertices and fat edges (each edge containing two

single lines), but they may have different number of single lines.

This defines what we call the genus $g$ of a ribbon graph.

For a given Ribbon graph $\mathscr{G}$ we call:

- $n_i(\mathscr{G}) =$ #unmarked vertices of degree $i$, for $3 \leq i \leq d+1$,
- $l_i(\mathscr{G}) =$size of the $i^{\text{th}}$ marked vertex, we have $l_i(\mathscr{G}) \geq 1$.
- $e(\mathscr{G}) =$ #edges,
- $q(\mathscr{G}) =$ #twisted edges,
- $v(\mathscr{G}) =$ #connected single lines,
- $g(\mathscr{G}) =$genus,
- $\#Aut(\mathscr{G}) =$symmetry factor of $\mathscr{G}$.

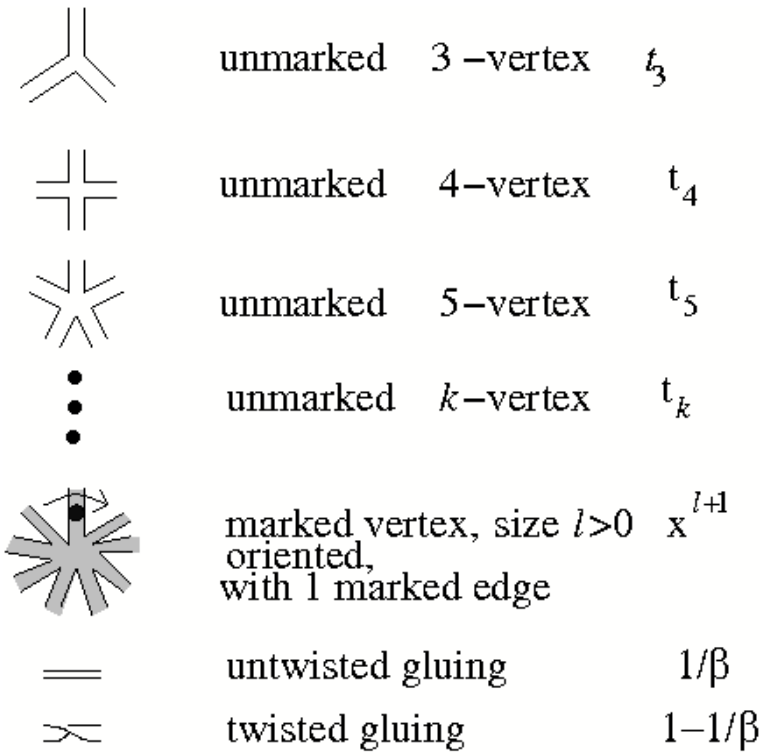



Figure VIII.2: Consider the set of ribbon graphs obtained by gluing those vertices. Marked vertices are of degree $l \geq 1$, they are oriented and have one marked half-edge. Unmarked vertices are unoriented, and are of degree $\geq 3$. Vertices are glued together by their half-edges, either twisted (with weight $1/\beta$) or untwisted (with weight $1-1/\beta$).

**Definition VIII.12.** *Let $\mathbb{M}_k^{(g)}(v')$, be the set of ribbon graphs $\mathscr{G}$ with k marked vertices,*

cc

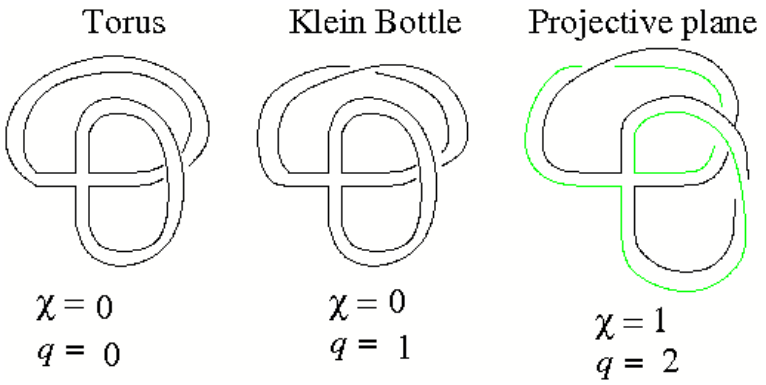



Figure VIII.3: Examples of ribbon graphs of genus $g = 1$.

*$q$ twisted edges, and of genus $g$, and such that $\mathscr{G}'$ has $v(\mathscr{G}') = v'$ connected single-lines,*
.

**Proposition VIII.1.** $\mathbb{M}_k^{(g)}(v')$ *is a finite set.*

*Proof.* The number of vertices of $\mathscr{G}'$ is:

$$\#\{\text{vertices}\} = k + \sum_{i\geq 3} n_i$$

The number of edges is twice the number of half-edges, i.e.

$$2\#\{\text{edges}\} = \sum_{i\geq 3} i n_i + \sum_{i=1}^{k} l_i$$

That gives:

$$2 - 2g = \#\{\text{vertices}\} - \#\{\text{edges}\} + v' = k - \frac{1}{2}\sum_{i\geq 3}(i-2)n_i - \frac{1}{2}\sum_{i=1}^{k} l_i + v$$

i.e.

$$k + v' + 2g - 2 = \frac{1}{2}\sum_{i\geq 3}(i-2)n_i + \frac{1}{2}\sum_{i=1}^{k} l_i$$

Since the left hand side is fixed, we see that the number and size of vertices are bounded, so that there is only a finite number of possible oriented ribbon graphs $\mathscr{G}'$. Since $\mathscr{G}'$ has a

bounded number of edges, there is only a finite number of possibilities of twisting them, i.e. there are also only a finite number of graphs $\mathscr{G}$.

### 10.1 Generating functions

In order to enumerate the sets $\mathbb{M}_k^{(g)}(v')$, we define the following generating functions:

**Definition VIII.13.** *We define:*

$$\begin{aligned} & W_k^{(g)}(x_1,\dots,x_k;t_3,\dots,t_{d+1},\beta;t_0) \\ = \;& \beta^{-k/2}\sum_{v'\geq 1} t_0^{v'} \sum_{\mathscr{G}\in\mathbb{M}_k^{(g)}(v')} \frac{1}{\#\mathrm{Aut}(\mathscr{G})} \frac{t_3^{n_3(\mathscr{G})} t_4^{n_4(\mathscr{G})} \dots t_{d+1}^{n_{d+1}(\mathscr{G})}}{x_1^{l_1(\mathscr{G})} x_2^{l_2(\mathscr{G})} \dots x_k^{l_k(\mathscr{G})}} \beta^{-e(\mathscr{G})} (\beta-1)^{q(\mathscr{G})} \\ & +\delta_{k,1}\delta_{g,0}\delta_{q,0}\frac{t_0}{x_1} + \delta_{k,2}\delta_{g,0}\delta_{q,0}\frac{1}{2(x_1-x_2)^2}. \end{aligned} \tag{VIII.190}$$

*It is a formal series in powers of $t_0$.*

*Most often, for readability, we shall write only the dependence in the $x_i$'s:*

$$W_k^{(g)}(x_1,\dots,x_k;t_3,\dots,t_{d+1},\beta;t_0) \equiv W_k^{(g)}(x_1,\dots,x_k).$$

*Also, for $k=0$ we write*

$$W_0^{(g)} = F_g.$$

### 10.2 Tutte's recursive equations

Tutte's equation is a recursion on the number of edges to construct the ribbon graphs. It consists in finding a bijection between ribbon graphs of various ensembles, by recursion on the number of edges. Let $\mathbb{M}_{l_1,\dots,l_k}^{(g)}$ be the set of ribbon graphs of genus $g$, and with $k$ marked vertices of size $l_1,\dots,l_k$.

Consider a ribbon graph $\mathscr{G}\in\mathbb{M}_{l_0+1,L}^{(g)}$ where $L=\{l_1,\dots,l_k\}$, with marked vertices of degrees $l_0+1, L$.

Consider the marked edge of marked face 0. It is either twisted or untwisted. Several mutually exclusive situations may occur (see fig VIII.4):

• on the other side of the marked edge, there is an unmarked vertex of size $j+1$ with $j\geq 2$. We then shrink the marked edge to concatenate the two vertices into one marked vertex of degree $l_0+j$. The orientation is inherited from the initial marked vertex, and the marked edge is chosen as the first edge to the left of the shrinked edge. It is clear

that we don't change the number of single lines in $\mathscr{G}$ or $\mathscr{G}'$. We decrease the number of vertices and edges by 1, so we don't change the genus. We thus get a ribbon graph in $M^{(g)}_{l_0+j,L}$, and this is weighted with weight $t_{j+1}\,(1/\beta+(1-1/\beta))=t_{j+1}$.

• on the other side of the marked edge, there is the marked vertex $i\neq 0$, of size $l_i\geq 1$. We then shrink the marked edge to concatenate the two vertices into one marked vertex of degree $l_0+l_i-1$. The orientation is inherited from the initial marked vertex, and the marked edge is chosen as the first edge to the left of the shrinked edge. It is also clear that we don't change the genus. Since we forget the marking of the other face, we shall get a symmetry factor $l_i$, corresponding to the $l_i$ places where we glue to the $i^{\text{th}}$ marked vertex. We thus get a ribbon graph in $M^{(g)}_{l_0+l_i-1,L/\{l_i\}}$, and this is weighted with weight $l_i$.

• on the other side of the marked edge, there is the same marked vertex 0. Again we shall shrink the marked edge, i.e. shrink the 2 single lines. Several sub-situations may occur:

$*$ if the edge is untwisted, shrinking the 2 single lines splits the marked vertex of size $l_0+1$ into two vertices of size $l'$ and $l_0-l'-1$. They inherit their orientation and marked edge from the initial marked vertex. We have increased the number of marked vertices by 1. The two new vertices are either connected together, or not.

$**$ If they are not connected, this means that the number of other marked vertices and the genus simply add up. We thus get two ribbon graphs in $M^{(g')}_{l',L'}\times M^{(g-g')}_{l_0-l'-1,L/L'}$, and this is weighted with weight $1/\beta$.

$**$ If they are connected, we see that we get a new ribbon graph, with one more vertex, 1 less edge, and we have not changed the connectivity of single lines. The genus has thus decreased by 1. We thus get a ribbon graph in $M^{(g-1)}_{l',l_0-l'-1,L}$, and this is weighted with weight $1/\beta$.

$*$ if the edge is twisted, shrinking the 2 single lines doesn't split the marked vertex. Instead we get a new vertex of size $l_0-1$. We assign to it the orientation of the half-vertex situated left of the marked edge, and we mark the edge left of the initial one. We have decreased $q$ by 1, and the genus is unchanged. We thus get a ribbon graph in $M^{(g)}_{l_0-1,L}$, and this is weighted with weight $(1-1/\beta)\,l_0$ (indeed, there are $l_0$ places where we can glue the marked edge).

For the generating function, those bijections read:

$$\begin{aligned} x\,W^{(g)}_{k+1}(x,X_L) &= \sum_{j=2}^{d} t_{j+1}\,x^j\,W^{(g)}_{k+1}(x,X_L) \\ &\quad +\frac{1}{\sqrt{\beta}}\sum_{i=1}^{k}\partial_{x_i}\,\frac{W^{(g)}_k(x,X_{L/\{x_i\}})-W^{(g)}_k(x_i,X_{L/\{x_i\}})}{x-x_i} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\sqrt{\beta}} \sum_{g',L'\subset L} W_{1+\#L'}^{(g')}(x,X_{L'}) W_{1+k-\#L'}^{(g-g')}(x,X_{L/L'}) \\
&+\frac{1}{\sqrt{\beta}} W_{k+2}^{(g-1)}(x,x,X_L) \\
&+(1-\frac{1}{\beta})\,\partial_x W_{k+1}^{(g)}(x,X_L) \\
&+\frac{1}{\sqrt{\beta}} P_{k+1}^{(g)}(x,X_L)
\end{aligned}
\tag{VIII.191}
$$

we define

$$
V'(x) = \sqrt{\beta}\left(x - \sum_{j=2}^{d} t_{j+1} x^j\right) \tag{VIII.192}
$$

and the last term $P_{k+1}^{(g)}(x;X_L)$ accounts for all the boundary terms, and it is necessarily equal to:

$$
P_{k+1}^{(g)}(x;X_L) = \left(V'(x)\, W_{k+1}^{(g)}(x;X_L)\right)_+ . \tag{VIII.193}
$$

This can be rewritten:

$$
\begin{aligned}
V'(x)\, W_{k+1}^{(g)}(x,X_L) \;=\; & \sum_{i=1}^{k} \partial_{x_i} \frac{W_k^{(g)}(x,X_{L/\{x_i\}}) - W_k^{(g)}(x_i,X_{L/\{x_i\}})}{x-x_i} \\
& + \sum_{g',L'} W_{1+\#L'}^{(g')}(x,X_{L'}) W_{1+k-\#L'}^{(g-g')}(x,X_{L/L'}) \\
& + W_{k+2}^{(g-1)}(x,x,X_L) \\
& + \hbar\,\partial_x W_{k+1}^{(g)}(x,X_L) \\
& + P_{k+1}^{(g)}(x,X_L)
\end{aligned}
\tag{VIII.194}
$$

where

$$
\hbar = \sqrt{\beta} - 1/\sqrt{\beta}.
$$

In other words, the $W_n^{(g)}$'s defined in section VIII provide a solution to Tutte's equations. They are the generating functions counting our non-oriented ribbon graphs. One just needs to find the polynomial $P_1^{(0)}(x)$, i.e. $U(x)$, and the choice of $\psi$ which is such that $W_1^{(0)}$ is a formal power series in $t_0$.

## 11 Conclusion

In this article, we have defined some "quantum" versions of quantities known in algebraic geometry and applied them to the resolution of the loop equations in the arbitrary $\beta$-random matrix model case, and in particular the enumeration of some non-orientable ribbon graphs.

Our formalism recovers standard algebraic geometry and the invariants of [44] in the classical limit $\hbar \to 0$.

Instead of an albebraic equation, we have to deal with a differential equation, which we interpreted as a "quantum spectral curve", and we were able to generalize the basic notions arising in classical algebraic geometry, like genus, sheets, branchpoints, meromorphic forms, of 1st kind, 2nd kind, 3rd kind, matrix of periods,...

It is surprising to see that the notion of branchpoints become "blurred", a branchpoint is no longer a point, but an asymptotic accumulation line. Also, there are two sheets, corresponding of the two possible large $x$ asymptotic behaviors of $\psi(x) \sim \exp \pm V/2\hbar$, but in fact any solution is a linear combination of these two, so that we could say that we are always in a "linear superposition" of two states like in quantum mechanics.

Another surprising thing, is that, in order for any cohomology theory to make sense, we need the cycle integrals of any forms to depend only on the homology class of the cycles, i.e. we need all forms to have vanishing residues at the $s_i$'s. This "no-monodromy" condition is equivalent to a Bethe ansatz satisfied by the $s_i$'s, like in the Gaudin model [96]. This provides a geometric interpretation of the Bethe ansatz, as the condition for cohomology to make sense.

However, we still lack of a complete understanding of the situation, since most of our results explicitely depend on an initial sector $S_0$ which we choose, whereas in algebraic geometry most of them only depend on the spectral curve and not on its parametrization. For instance the genus itself depends on a choice of sector. In some sense, the genus is no longer deterministic.

Moreover, we still lack the proper definition of the spectral invariants $F_g$, indeed we have defined the $F_g$'s only through solving a differential equation with respect to $\hbar$, which is not as explicit as [23] or [98]. Out of the $F_g$'s, we could expect the possibility to make the link with integrable systems and define a "quantum Tau-function", like in [23].

Also, we restricted ourselves to the case of hyperelliptic curves, i.e. second order differential equations, or also a 1-matrix model. In a forthcoming paper, we shall generalize all this construction to arbitrary linear differential equations of any order, and generalize to a 2-matrix model. This work is underway, almost finished and the article is being written at this time. As for the hyperelliptical case, the notions of genus, sheets,

branchpoints, forms, $W_n^{(g)}$'s ... can be defined. Again there is a Bethe ansatz ensuring a no-monodromy condition so that all cycle integrals depend only on the homology class of cycles. So, there is no qualitative change, the difference is only technical, because the hyperelliptical case has big simplifications due to the involutive symmetry. The difference between the hyperelliptical case and the general case is comparable to the difference between [105] and [100], i.e. the definition of the kernel $K$ is really more complicated, and there is a rather "big" technical step.

Then is would be interesting to see if the $F_g$'s have some sort of symplectic invariance, or more precisely some "canonical invariance", i.e. are unchanged under any change $(x,y) \to (\tilde{x},\tilde{y})$ such that $[\tilde{y},\tilde{x}] = [y,x] = \hbar$.

Finally, let us mention that we have developped a new geometrical approach to the study of D-modules, and it would be interesting to see how to relate it to more standard approaches, and also to the resurgence theory for studying the Schrödinger equation.

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**Appendix 1: Proof of the loop equation for $B(x,z)$**

Let's first proove the first loop equation for $B(x,z)$: Let's define:

$$\hat{B}(x,z) = \frac{1}{2}\partial_z\left(2\frac{\psi'(z)}{\psi(z)} - \partial_z\right)\hat{K}(x,z) \tag{VIII.195}$$

i.e. we have:

$$B(x,z) = \hat{B}(x,z) - \sum_{\alpha=1}^{d-1} v_\alpha(x) \oint_{\mathscr{A}_\alpha} \hat{B}(x'',z)dx'' \tag{VIII.196}$$

Since $(2\frac{\psi'(x)}{\psi(x)} + \partial_x)\, v_\alpha(x) = h_\alpha(x)$ is a polynomial of degree $\leq d-2$, it suffices to prove eq. (VIII.74) for $\hat{B}(x,z)$.

Let us compute:

$$\begin{aligned}(2\frac{\psi'(x)}{\psi(x)}+\partial_x)\,\hat{B}(x,z) &= \frac{1}{2}\,\partial_z\,(2\frac{\psi'(z)}{\psi(z)}-\partial_z)\,\frac{1}{x-z}\\ &= \frac{1}{2}\,\partial_z\,(2\frac{\psi'(z)}{\psi(z)(x-z)}-\frac{1}{(x-z)^2})\\ &= -\frac{1}{(x-z)^3}+\partial_z\,\frac{\psi'(z)}{\psi(z)(x-z)}\end{aligned} \tag{VIII.197}$$

and therefore:

$$(2\frac{\psi'(x)}{\psi(x)}+\partial_x)\,\left(\hat{B}(x,z)-\frac{1}{2(x-z)^2}\right)+\partial_z\,\frac{\frac{\psi'(x)}{\psi(x)}-\frac{\psi'(z)}{\psi(z)}}{x-z}=0 \tag{VIII.198}$$

This proves eq. (VIII.74), with:

$$P_2^{(0)}(x,z)=-\sum_{\alpha=1}^{g}h_\alpha(x)\oint_{\mathscr{A}_\alpha}\hat{B}(x'',z)dx''-\sum_{\alpha=g+1}^{d-1}h_\alpha(x)\oint_{\mathscr{A}_\alpha}dx''\,\partial_{x''}\,\psi^2(x'')\hat{B}(x'',z). \tag{VIII.199}$$

Let's now proove the second loop equation for $B(x,z)$: Similarly, let us compute $(2\frac{\psi'(z)}{\psi(z)}+\partial_z)\,\hat{B}(x,z)$:

$$(2\frac{\psi'(z)}{\psi(z)}+\partial_z)\,\hat{B}(x,z)=\frac{1}{2}(2\frac{\psi'(z)}{\psi(z)}+\partial_z)\,\partial_z\,(2\frac{\psi'(z)}{\psi(z)}-\partial_z)\,\hat{K}(x,z) \tag{VIII.200}$$

Notice that the operator $\hat{U}(z)=\frac{1}{2}(2\frac{\psi'(z)}{\psi(z)}+\partial_z)\,\partial_z\,(2\frac{\psi'(z)}{\psi(z)}-\partial_z)$, is equal to:

$$\hat{U}(z)=-\frac{1}{2}\,\partial_z^3+2U(z)\partial_z+U'(z) \tag{VIII.201}$$

which is also known in the litterature as the Gelfand-Dikii operator [101] (The Gelfand-Dikii differential polynomials $R_k(U)$ are computed recursively by $R_0=1$ and $\partial_z R_{k+1}=\hat{U}\,.R_k$), which plays a key role in the KdV hierarchy.

However, independently of any relationship with KdV, we get:

$$(2\frac{\psi'(z)}{\psi(z)}+\partial_z)\,\hat{B}(x,z)$$

$$
\begin{aligned}
&= \frac{1}{\psi^2(x)} \int_{\infty_0}^{x} \psi^2(x')\,dx'\,\hat{U}(x').\frac{1}{x'-z} \\
&= \frac{1}{\psi^2(x)} \int_{\infty_0}^{x} \psi^2(x')\,dx'\,\Big(-\frac{3}{(x'-z)^4}+\frac{2U(z)}{(x'-z)^2}+\frac{U'(z)}{x'-z}\Big)
\end{aligned}
\tag{VIII.202}
$$

We integrate the first term by parts three times, and we write $Y=\psi'/\psi$ (we have $Y'+Y^2=U$):

$$
\begin{aligned}
&(2\frac{\psi'(z)}{\psi(z)}+\partial_z)\,\hat{B}(x,z) \\
= \;& \frac{1}{(x-z)^3}-\frac{2}{\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\,dx'\,\frac{Y(x')}{(x'-z)^3} \\
&+\frac{1}{\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\,dx'\,\Big(\frac{2U(z)}{(x'-z)^2}+\frac{U'(z)}{x'-z}\Big) \\
= \;& \frac{1}{(x-z)^3}+\frac{Y(x)}{(x-z)^2}-\frac{1}{\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\,dx'\,\frac{Y'(x')+2Y^2(x')}{(x'-z)^2} \\
&+\frac{1}{\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\,dx'\,\Big(\frac{2U(z)}{(x'-z)^2}+\frac{U'(z)}{x'-z}\Big) \\
= \;& \frac{1}{(x-z)^3}+\frac{Y(x)}{(x-z)^2}+\frac{1}{\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\,dx'\,\frac{Y'(x')}{(x'-z)^2} \\
&+\frac{1}{\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\,dx'\,\Big(\frac{2(U(z)-U(x'))}{(x'-z)^2}+\frac{U'(z)}{x'-z}\Big) \\
= \;& \frac{1}{(x-z)^3}+\frac{Y(x)}{(x-z)^2}-\frac{Y'(x)}{x-z}+\frac{1}{\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\,dx'\,\frac{Y''(x')+2Y(x')Y'(x')}{(x'-z)^2} \\
&+\frac{1}{\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\,dx'\,\Big(\frac{2(U(z)-U(x'))}{(x'-z)^2}+\frac{U'(z)}{x'-z}\Big) \\
= \;& \frac{1}{(x-z)^3}-\frac{\partial}{\partial x}\frac{Y(x)}{x-z}+\frac{1}{\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\,dx'\,\Big(\frac{2(U(z)-U(x'))}{(x'-z)^2}+\frac{U'(z)+U(x')}{x'-z}\Big)
\end{aligned}
\tag{VIII.203}
$$

This implies that:

$$
\begin{aligned}
&(2\frac{\psi'(z)}{\psi(z)}+\partial_z)\Big(\hat{B}(x,z)-\frac{1}{2(x-z)^2}\Big)+\frac{\partial}{\partial x}\frac{Y(x)-Y(z)}{x-z} \\
=\;& \frac{1}{\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\,dx'\,\Big(\frac{2(U(z)-U(x'))}{(x'-z)^2}+\frac{U'(z)+U(x')}{x'-z}\Big) \\
=\;& Q(z,x)
\end{aligned}
\tag{VIII.204}
$$

which is clearly a polynomial in $z$. Taking integrals over $x$ along $\mathscr{A}_\alpha$ does not change its

structure in $z$, and therefore:

$$\begin{aligned} & (2\frac{\psi'(z)}{\psi(z)}+\partial_z)\left(B(x,z)-\frac{1}{2(x-z)^2}\right)+\frac{\partial}{\partial x}\frac{Y(x)-Y(z)}{x-z} \\ = & \frac{1}{\psi^2(x)}\int_{\infty_0}^{x}\psi^2(x')\,dx'\left(\frac{2(U(z)-U(x'))}{(x'-z)^2}+\frac{U'(z)+U(x')}{x'-z}\right) \\ = & \tilde{P}_2^{(0)}(z,x) \end{aligned} \tag{VIII.205}$$

is of the required form.

By looking at the behavior of the various terms in the LHS of eq. (VIII.75) when $z\to\infty$, we find that $\tilde{P}_2^{(0)}(z,x)$ is a polynomial of degree at most $d-2$ in $z$.

## Appendix 2: Proof of theorem VIII.18

**Theorem VIII.18** *Each $W_n^{(g)}(x_1,\dots,x_n)$ with $2-2g-n<0$, is an analytical functions of all its arguments, with poles only when $x_i\to s_j$. Moreover, it vanishes at least as $O\left(1/x_i^2\right)$ when $x_i\to\infty$ in all sectors. It has no discontinuity across $\mathscr{A}$-cycles.*

**proof:**

We proceed by recursion on $2g+n$. The theorem is true for $W_2^{(0)}$. Assume it is true up to $2g+n$, we shall prove it for $W_{n+1}^{(g)}(x_0,x_1,\dots,x_n)$.

The integrand $U_n^{(g)}$ of theorem VIII.17 is singular only at $x=s_j$'s. As long as $x_0$ is away from the $s_j$'s, we can continuously deform the $\mathscr{A}$-cycles and the contour $\mathscr{C}$ in order to have $x_0$ outside of the $\mathscr{A}$-cycles, and the integral can be evaluated and is analytical in $x_0$. When $x_0$ approaches $s_i$, we define $\hat{\mathscr{C}}_i$, a contour which surrounds all roots except $s_i$, i.e:

$$\oint_{\mathscr{C}} = \oint_{\hat{\mathscr{C}}_i} + 2i\pi \operatorname*{Res}_{s_i} \tag{VIII.206}$$

The integral over $\hat{\mathscr{C}}_i$ can be evaluated and is convergent, thus it is analytical in $x_0$.

From the recursion hypothesis, all terms in the integrand are meromorphic in the vicinity of $s_i$, and thus the residue at $s_i$ can be computed by taking a finite Taylor expansion of $K(x_0,x)=\sum_k (x-s_i)^k K_{i,k}(x_0)$ in the vicinity of $x\to s_i$. The result is a finite sum of terms of the type $K_{i,k}(x_0)$. It is easy to see from the definition of $K$, that each $K_{i,k}(x_0)$ has only poles at $x_0=s_i$. Thus we have proved that $W_{n+1}^{(g)}$ has poles at the $s_i$'s in its first variable.

In the other variables, the result comes from an obvious recursion.

□

## Appendix 3: Proof of theorem VIII.20

In this subsection we prove theorem VIII.20, that all $W_n^{(g)}$'s satisfy the loop equation.

$$\begin{aligned} P_{n+1}^{(g)}(x,x_1...,x_n) \quad = \quad & 2\hbar\frac{\psi'(x)}{\psi(x)}\overline{W}_{n+1}^{(g)}(x,x_1...,x_n)+\hbar\partial_x\overline{W}_{n+1}^{(g)}(x,x_1,...,x_n) \\ & +\sum_{I\subset J}\overline{W}_{|I|+1}^{(h)}(x,x_I)\overline{W}_{n-|I|+1}^{(g-h)}(x,J/I)+\overline{W}_{n+2}^{(g-1)}(x,x,J) \\ & +\sum_j\partial_{x_j}\left(\frac{\overline{W}_n^{(g)}(x,J/\{j\})-\overline{W}_n^{(g)}(x_j,J/\{j\})}{(x-x_j)}\right) \end{aligned} \tag{VIII.207}$$

is a polynomial in $x$ of degree at most $d-2$.

**proof:**

From the definition we have:

$$\begin{aligned} W_{n+1}^{(g)}(x,J) &= \frac{1}{2i\pi}\oint_{\mathscr{C}}dz\,K(x,z)\,U_n^{(g)}(z,J) \\ &= \frac{1}{2i\pi}\oint_{\mathscr{C}}dz\,\hat{K}(x,z)\,U_n^{(g)}(z,J) \\ &\quad -\sum_\alpha\frac{v_\alpha(x)}{2i\pi}\oint_{\mathscr{C}}dz\,C_\alpha(z)\,U_n^{(g)}(z,J) \end{aligned} \tag{VIII.208}$$

Then, notice that $\hat{K}(x,z)$ has a logarithmic cut along $]\infty_0,x]$, and the discontinuity across that cut is:

$$\delta\hat{K}(x,z)=\frac{2i\pi}{\hbar}\frac{\psi^2(z)}{\psi^2(x)} \tag{VIII.209}$$

$U_n^{(g)}$ has no singularity outside of $\mathscr{C}$, and thus we can deform the contour into a contour enclosing only the logarithmic cut of $\hat{K}(x,z)$, and therefore:

$$\frac{1}{2i\pi}\oint_{\mathscr{C}}dz\,\hat{K}(x,z)\,U_n^{(g)}(z,J)=-\frac{1}{\hbar}\int_{\infty_0}^{x}dz\,\frac{\psi^2(z)}{\psi^2(x)}\,U_n^{(g)}(z,J) \tag{VIII.210}$$

We then apply the operator: $2\frac{\psi'(x)}{\psi(x)}+\partial_x$, that gives:

$$(2\hbar\frac{\psi'(x)}{\psi(x)}+\hbar\partial_x)\,\frac{1}{2i\pi}\oint_{\mathscr{C}}dz\,\hat{K}(x,z)\,U_n^{(g)}(z,J)=-U_n^{(g)}(x,J) \tag{VIII.211}$$

and therefore:

$$\begin{aligned} P_{n+1}^{(g)}(x,J) &= (2\hbar\frac{\psi'(x)}{\psi(x)}+\hbar\partial_x)W_{n+1}^{(g)}(x,J)+U_n^{(g)}(x,J) \\ &= -(2\hbar\frac{\psi'(x)}{\psi(x)}+\hbar\partial_x)\sum_\alpha \frac{v_\alpha(x)}{2i\pi}\oint_{\mathscr{C}} dz\, C_\alpha(z)\, U_n^{(g)}(z,J) \\ &= -\sum_\alpha h_\alpha(x)\oint_{\mathscr{C}} dz\, C_\alpha(z)\, U_n^{(g)}(z,J) \end{aligned} \tag{VIII.212}$$

which is indeed a polynomial of $x$ of degree at most $d-2$.

□

## Appendix 4: Proof of theorem VIII.22

**Theorem VIII.22** *The 3 point function $W_3^{(0)}$ is symmetric and we have:*

$$W_3^{(0)}(x_1,x_2,x_3) = \frac{4}{2i\pi}\oint_{\mathscr{C}} dx\, \frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)} \tag{VIII.213}$$

*where $Y(x) = -2\hbar\frac{\psi'(x)}{\psi(x)}$*

**proof:**
The definition of $W_3^{(0)}$ is:

$$\begin{aligned} & W_3^{(0)}(x_0,x_1,x_2) \\ = \quad & \frac{1}{i\pi}\oint_{\mathscr{C}} dx\, K(x_0,x)B(x,x_1)B(x,x_2) \\ = \quad & \frac{1}{4i\pi}\oint_{\mathscr{C}} dx\, K_0\, G_1^{'}\, G_2' \\ = \quad & \frac{1}{4i\pi}\oint_{\mathscr{C}} dx\, K_0\left((\hbar K_1''+YK_1'+Y'K_1)(\hbar K_2''+YK_2'+Y'K_2)\right) \\ = \quad & \frac{1}{4i\pi}\oint_{\mathscr{C}} dx\, K_0\,(\hbar^2K_1''K_2''+\hbar Y(K_1'K_2''+K_1''K_2')+\hbar Y'(K_1''K_2+K_2''K_1) \\ & +Y^2K_1'K_2'+YY'(K_1K_2'+K_1'K_2)+Y'^2K_1K_2\,) \end{aligned} \tag{VIII.214}$$

where we have written for short $K_p = K(x_p,x)$, $G_p = G(x_p,x)$, and derivative are w.r.t. $x$. Note also that introducing $K_1$ and $K_2$ makes appear some additional and arbitrary logarithmic cuts from $x_1$ to $\infty_0$ and from $x_2$ to $\infty_0$. But these cuts can be chosen arbitrarily since from the definition of $W_3^{(0)}(x_0,x_1,x_2)$ it should not depend on that. Remember also

that to use this definition of $W_3^{(0)}$ we need to assume that $x_1$ and $x_2$ are not circled by the contour $\mathscr{C}$. Therefore we can choose the logarithmic cut of $K_1$ and $K_2$ inside the contour $\mathscr{C}$ like we have done it for $x_0$. We now see that for example $K_0K_1K_2$ has no singularity outside $\mathscr{C}$ and thus will not contribute because of theorem VIII.1. Many other manipulations involving globally defined functions with no singularities outside $\mathscr{C}$ can be done.

For example, using the Ricatti equation $Y_i^2 = 2\hbar Y_i' + 4U$, we may replace $Y_i^2$ by $2\hbar Y_i'$ and $Y_iY_i'$ by $\hbar Y_i''$.

$$\begin{aligned}
& W_3^{(0)}(x_0,x_1,x_2) \\
= \quad & \frac{1}{4i\pi}\oint_{\mathscr{C}} dx\, K_0\,(\hbar Y(K_1'K_2''+K_1''K_2')+\hbar Y'(K_1''K_2+K_2''K_1) \\
& +2\hbar Y'K_1'K_2'+\hbar Y''(K_1K_2'+K_1'K_2)+Y'^2K_1K_2\,) \\
= \quad & \frac{1}{4i\pi}\oint_{\mathscr{C}} dx\, K_0\,(\hbar Y(K_1'K_2')'+\hbar Y'(K_1K_2)''+\hbar Y''(K_1K_2)'+Y'^2K_1K_2\,) \\
= \quad & \frac{1}{4i\pi}\oint_{\mathscr{C}} dx\, Y'^2K_0K_1K_2+\hbar\big(Y''K_0(K_1K_2)'-(YK_0)'K_1'K_2'-(Y'K_0)'(K_1K_2)'\big) \\
= \quad & \frac{1}{2}\frac{1}{4i\pi}\oint_{\mathscr{C}} dx\, Y'^2K_0K_1K_2-\hbar\big((YK_0)'K_1'K_2'+Y'K_0'(K_1K_2)'\big) \\
= \quad & \frac{1}{4i\pi}\oint_{\mathscr{C}} dx\, Y'^2K_0K_1K_2-\hbar YK_0'K_1'K_2'-\hbar Y'(K_0K_1'K_2'+K_0'K_1K_2'+K_0'K_1'K_2)
\end{aligned} \tag{VIII.215}$$

This expression is clearly symmetric in $x_0,x_1,x_2$ as claimed in theorem VIII.21.

Let us give an alternative expression, in the form of the Verlinde or Krichever formula.

$$W_3^{(0)}(x_0,x_1,x_2) = \frac{2}{i\pi}\oint_{\mathscr{C}} dx\,\frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)} \tag{VIII.216}$$

*Proof.* In order to prove formula VIII.216, compute:

$$B(x,x_i) = -\frac{1}{2}G'(x,x_i) = -\frac{1}{2}G_i' = \frac{1}{2}(\hbar K_i''+YK_i'+Y'K_i) \tag{VIII.217}$$

thus:

$$\begin{aligned}
& \frac{1}{2i\pi}\oint_{\mathscr{C}} dx\,\frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)} \\
= \quad & \frac{1}{16i\pi}\oint_{\mathscr{C}} dx\,\frac{1}{Y'(x)}(\hbar K_0''+YK_0'+Y'K_0)(\hbar K_1''+YK_1'+Y'K_1)(\hbar K_2''+YK_2'+Y'K_2) \\
= \quad & \frac{1}{16i\pi}\oint_{\mathscr{C}} dx\,\frac{\hbar^3}{Y'}K_0''K_1''K_2''+\hbar^2\frac{Y}{Y'}(K_0'K_1''K_2''+K_0''K_1'K_2''+K_0''K_1''K_2')
\end{aligned}$$

$$
\begin{aligned}
&+\hbar^2(K_0K_1''K_2''+K_0''K_1K_2''+K_0''K_1''K_2)+\hbar\frac{Y^2}{Y'}(K_0''K_1'K_2'+K_0'K_1''K_2'+K_0'K_1'K_2'')\\
&+\hbar Y(K_0K_1'K_2''+K_0K_1''K_2'+K_0'K_1K_2''+K_0'K_1''K_2+K_0''K_1K_2'+K_0''K_1'K_2)\\
&+\hbar Y'(K_0''K_1K_2+K_0K_1''K_2+K_0K_1K_2'')+\frac{Y^3}{Y'}K_0'K_1'K_2'\\
&+Y^2(K_0K_1'K_2'+K_0'K_1K_2'+K_0'K_1'K_2)+YY'(K_0'K_1K_2+K_0K_1'K_2+K_0K_1K_2')\\
&+Y'^2K_0K_1K_2
\end{aligned}
\tag{VIII.218}
$$

$$
\begin{aligned}
&\frac{1}{2i\pi}\oint_{\mathscr{C}}dx\,\frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)}\\
=\quad&\frac{1}{16i\pi}\oint_{\mathscr{C}}dx\,\hbar Y(K_0K_1'K_2''+K_0K_1''K_2'+K_0'K_1K_2''+K_0'K_1''K_2+K_0''K_1K_2'+K_0''K_1'K_2)\\
&+\hbar Y'(K_0''K_1K_2+K_0K_1''K_2+K_0K_1K_2'')+\frac{Y^3}{Y'}K_0'K_1'K_2'\\
&+Y^2(K_0K_1'K_2'+K_0'K_1K_2'+K_0'K_1'K_2)+YY'(K_0'K_1K_2+K_0K_1'K_2+K_0K_1K_2')\\
&+Y'^2K_0K_1K_2
\end{aligned}
\tag{VIII.219}
$$

Notice that $Y^2=2\hbar Y'+4U$, thus we may replace $Y^3/Y'$ by $2\hbar Y$, and $Y^2$ by $2\hbar Y'$ and $YY'$ by $\hbar Y''$, for the same reasons as before. Thus:

$$
\begin{aligned}
&\frac{1}{2i\pi}\oint_{\mathscr{C}}dx\,\frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)}\\
=\quad&\frac{1}{16i\pi}\oint_{\mathscr{C}}dx\,\hbar Y(K_0K_1'K_2''+K_0K_1''K_2'+K_0'K_1K_2''+K_0'K_1''K_2+K_0''K_1K_2'+K_0''K_1'K_2)\\
&+\hbar Y'(K_0''K_1K_2+K_0K_1''K_2+K_0K_1K_2'')+2\hbar YK_0'K_1'K_2'\\
&+2\hbar Y'(K_0K_1'K_2'+K_0'K_1K_2'+K_0'K_1'K_2)+\hbar Y''(K_0'K_1K_2+K_0K_1'K_2+K_0K_1K_2')\\
&+Y'^2K_0K_1K_2\\
=\quad&\frac{1}{16i\pi}\oint_{\mathscr{C}}dx\,\hbar Y(K_0(K_1'K_2')'+K_1(K_0'K_2')'+K_2(K_0'K_1')')\\
&+2\hbar YK_0'K_1'K_2'+Y'^2K_0K_1K_2+\hbar(Y'(K_0'K_1K_2+K_0K_1'K_2+K_0K_1K_2'))'\\
=\quad&\frac{1}{16i\pi}\oint_{\mathscr{C}}dx\,\hbar Y(K_0(K_1'K_2')'+K_1(K_0'K_2')'+K_2(K_0'K_1')')\\
&+2\hbar YK_0'K_1'K_2'+Y'^2K_0K_1K_2\\
=\quad&-\frac{1}{16i\pi}\oint_{\mathscr{C}}dx\,3\hbar YK_0'K_1'K_2'+\hbar Y'(K_0K_1'K_2'+K_0'K_1K_2'+K_0'K_1'K_2)\\
&-2\hbar YK_0'K_1'K_2'-Y'^2K_0K_1K_2\\
=\quad&\frac{1}{8i\pi}\oint_{\mathscr{C}}dx\,W_3^{(0)}(x_0,x_1,x_2)
\end{aligned}
\tag{VIII.220}
$$

## Appendix 5: Proof of theorem VIII.21

**Theorem VIII.21** *Each $W_n^{(g)}$ is a symmetric function of all its arguments.*

**proof:**

The special case of $W_3^{(0)}$ is proved in appendix VIII.213 above.

It is obvious from the definition that $W_{n+1}^{(g)}(x_0,x_1,\ldots,x_n)$ is symmetric in $x_1,x_2,\ldots,x_n$, and therefore we need to show that (for $n \geq 1$):

$$W_{n+1}^{(g)}(x_0,x_1,J) - W_{n+1}^{(g)}(x_1,x_0,J) = 0 \tag{VIII.221}$$

where $J = \{x_2,\ldots,x_n\}$. We prove it by recursion on $-\chi = 2g-2+n$.

Assume that every $W_k^{(h)}$ with $2h+k-2 \leq 2g+n$ is symmetric. We have:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0,x_1,J) \\ = \quad & \frac{1}{2\pi i}\oint_{\mathscr{C}} dx\, K(x_0,x)\,\Big(W_{n+2}^{(g-1)}(x,x,x_1,J) + 2\,B(x,x_1)W_n^{(g)}(x,J) \\ & +2\sum_{h=0}^{g}{\sum_{I\in J}}' W_{2+|I|}^{(h)}(x,x_1,I)W_{n-|I|}^{(g-h)}(x,J/I)\Big) \end{aligned} \tag{VIII.222}$$

where $\sum'$ means that we exclude the terms $(I=\emptyset, h=0)$ and $(I=J, h=g)$. Notice also that $\overline{W}_{n+2}^{(g-1)} = W_{n+2}^{(g-1)}$ because $n \geq 1$. Then, using the recursion hypothesis, we have:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0,x_1,J) \\ = \quad & 2\oint_{\mathscr{C}} dx\, K(x_0,x)\,B(x,x_1)W_n^{(g)}(x,J) \\ & +\oint_{\mathscr{C}} dx \oint_{\mathscr{C}} dx'\, K(x_0,x)K(x_1,x')\,\Big(W_{n+3}^{(g-2)}(x,x,x',x',J) \\ & +2\sum_h{\sum_I}' W_{2+|I|}^{(h)}(x',x,I)W_{1+n-|I|}^{(g-1-h)}(x',x,J/I) \\ & +2\sum_h{\sum_I}' W_{3+|I|}^{(h)}(x',x,x,I)W_{n-|I|}^{(g-1-h)}(x',J/I) \\ & +2\sum_h{\sum_{I\in J}}' W_{n-|I|}^{(g-h)}(x,J/I)\Big[W_{3+|I|}^{(h-1)}(x,x',x',I) \end{aligned}$$

$$+2\sum_{h'}\sum_{I'\subset I}{}' W^{(h')}_{2+|I'|}(x',x,I')W^{(h-h')}_{1+|I|-|I'|}(x',I/I')\Big]\Big) \tag{VIII.223}$$

Now, if we compute $W^{(g)}_{n+1}(x_1,x_0,J)$, we get the same expression, with the order of integrations exchanged, i.e. we have to integrate $x'$ before integrating $x$. Notice, by moving the integration contours, that:

$$\oint_{\mathscr{C}} dx \oint_{\mathscr{C}} dx' - \oint_{\mathscr{C}} dx' \oint_{\mathscr{C}} dx = -\oint_{\mathscr{C}} dx \frac{1}{2\pi i}\operatorname*{Res}_{x'\to x} \tag{VIII.224}$$

Moreover, the only terms which have a pole at $x=x'$ are those containing $B(x,x')$. Therefore:

$$\begin{aligned} & W^{(g)}_{n+1}(x_0,x_1,J) - W^{(g)}_{n+1}(x_1,x_0,J) \\ = \quad & 2\oint_{\mathscr{C}} dx\ \left(K(x_0,x)\,B(x,x_1) - K(x_1,x)\,B(x,x_0)\right) W^{(g)}_n(x,J) \\ & -2\oint_{\mathscr{C}} dx\ \frac{1}{2i\pi}\operatorname*{Res}_{x'\to x} K(x_0,x)K(x_1,x')\,B(x,x')\ \Big( \\ & 2W^{(g-1)}_{1+n}(x',x,J) + 2\sum_{h}\sum_{I\in J}{}' W^{(g-h)}_{n-|I|}(x,J/I)W^{(h)}_{1+|I|}(x',I)\Big) \end{aligned} \tag{VIII.225}$$

The residue $\operatorname{Res}_{x'\to x}$ can be computed:

$$\begin{aligned} & W^{(g)}_{n+1}(x_0,x_1,J) - W^{(g)}_{n+1}(x_1,x_0,J) \\ = \quad & 2\oint_{\mathscr{C}} dx\ \left(K(x_0,x)\,B(x,x_1) - K(x_1,x)\,B(x,x_0)\right) W^{(g)}_n(x,J) \\ & -\oint_{\mathscr{C}} dx\ K(x_0,x)\frac{\partial}{\partial x'}\Big(K(x_1,x')\ \Big( \\ & 2W^{(g-1)}_{1+n}(x',x,J) + 2\sum_{h}\sum_{I\in J}{}' W^{(g-h)}_{n-|I|}(x,J/I)W^{(h)}_{1+|I|}(x',I)\Big)\Big)_{x'=x} \\ = \quad & 2\oint_{\mathscr{C}} dx\ \left(K(x_0,x)\,B(x,x_1) - K(x_1,x)\,B(x,x_0)\right) W^{(g)}_n(x,J) \\ & -\oint_{\mathscr{C}} dx\, K(x_0,x)K'(x_1,x)\ \Big( \\ & 2W^{(g-1)}_{1+n}(x,x,J) + 2\sum_{h}\sum_{I\in J}{}' W^{(g-h)}_{n-|I|}(x,J/I)W^{(h)}_{1+|I|}(x,I)\Big) \\ & -\oint_{\mathscr{C}} dx\, K(x_0,x)K(x_1,x')\frac{\partial}{\partial x'}\Big( \end{aligned}$$

$$
\begin{aligned}
& 2W_{1+n}^{(g-1)}(x',x,J) + 2\sum_h {\sum_{I\in J}}' W_{n-|I|}^{(g-h)}(x,J/I) W_{1+|I|}^{(h)}(x',I) \Big)_{x'=x} \\
= \quad & 2\oint_{\mathscr{C}} dx\ \left(K(x_0,x)\,B(x,x_1) - K(x_1,x)\,B(x,x_0)\right) W_n^{(g)}(x,J) \\
& -\oint_{\mathscr{C}} dx\, K(x_0,x)K'(x_1,x)\ \Big( \\
& 2W_{1+n}^{(g-1)}(x,x,J) + 2\sum_h {\sum_{I\in J}}' W_{n-|I|}^{(g-h)}(x,J/I) W_{1+|I|}^{(h)}(x,I)\Big) \\
& -\frac{1}{2}\oint_{\mathscr{C}} dx\, K(x_0,x)K(x_1,x)\frac{\partial}{\partial x}\Big( \\
& 2W_{1+n}^{(g-1)}(x,x,J) + 2\sum_h {\sum_{I\in J}}' W_{n-|I|}^{(g-h)}(x,J/I) W_{1+|I|}^{(h)}(x,I) \Big)
\end{aligned}
$$

(VIII.226)

The last term can be integrated by parts, and we get:

$$
\begin{aligned}
& W_{n+1}^{(g)}(x_0,x_1,J) - W_{n+1}^{(g)}(x_1,x_0,J) \\
= \quad & 2\oint_{\mathscr{C}} dx\ \left(K(x_0,x)\,B(x,x_1) - K(x_1,x)\,B(x,x_0)\right) W_n^{(g)}(x,J) \\
& +\frac{1}{2}\oint_{\mathscr{C}} dx\ \Big(K'(x_0,x)K(x_1,x) - K(x_0,x)K'(x_1,x)\Big)\ \Big( \\
& 2W_{1+n}^{(g-1)}(x,x,J) + 2\sum_h {\sum_{I\in J}}' W_{n-|I|}^{(g-h)}(x,J/I) W_{1+|I|}^{(h)}(x,I)\Big)
\end{aligned}
$$

(VIII.227)

Then we use theorem VIII.20:

$$
\begin{aligned}
& W_{n+1}^{(g)}(x_0,x_1,J) - W_{n+1}^{(g)}(x_1,x_0,J) \\
= \quad & 2\oint_{\mathscr{C}} dx\ \left(K(x_0,x)\,B(x,x_1) - K(x_1,x)\,B(x,x_0)\right) W_n^{(g)}(x,J) \\
& +\oint_{\mathscr{C}} dx\ \Big(K'(x_0,x)K(x_1,x) - K(x_0,x)K'(x_1,x)\Big)\ \Big(P_n^{(g)}(x,J) \\
& +(Y(x) - \hbar\partial_x) W_n^{(g)}(x,J) + \sum_j \partial_{x_j}\Big(\frac{W_{n-1}^{(g)}(x_j,J/\{x_j\})}{x-x_j}\Big)\Big)
\end{aligned}
$$

(VIII.228)

Since $P_n^{(g)}(x,J)$ and $W_{n-1}^{(g)}(x_j,J/\{x_j\})$ are entire functions of $x$, we can use the usual thorem VIII.1 to say that they do not contribute. (Note again that we choose the logarithmic cut of $K_1$ inside the contour $\mathscr{C}$, and that we can do that because the contour $\mathscr{C}$

contains $x_1$.)

$$\begin{aligned} & W^{(g)}_{n+1}(x_0,x_1,J) - W^{(g)}_{n+1}(x_1,x_0,J) \\ = \quad & 2\oint_{\mathscr{C}} dx\ \left(K(x_0,x)\,B(x,x_1) - K(x_1,x)\,B(x,x_0)\right) W^{(g)}_n(x,J) \\ + \quad & 2\oint_{\mathscr{C}} dx\ \Big(K'(x_0,x)K(x_1,x) - K(x_0,x)K'(x_1,x)\Big) \\ & (Y(x)-\hbar\partial_x)W^{(g)}_n(x,J) \end{aligned} \tag{VIII.229}$$

Notice that:

$$K'_0K_1 - K_0K'_1 = -\frac{1}{\hbar}\left(G_0K_1 - K_0G_1\right) \tag{VIII.230}$$

and $B = -\frac{1}{2}G'$, therefore:

$$\begin{aligned} & W^{(g)}_{n+1}(x_0,x_1,J) - W^{(g)}_{n+1}(x_1,x_0,J) \\ = \quad & -2\oint_{\mathscr{C}} dx\ \left(K_0G'_1 - K_1G'_0\right) W^{(g)}_n(x,J) \\ & -\frac{1}{\hbar}2\oint_{\mathscr{C}} dx\ \Big(G_0K_1 - K_0G_1\Big)\ (Y(x)-\hbar\partial_x)W^{(g)}_n(x,J) \end{aligned} \tag{VIII.231}$$

we integrate the first line by parts:

$$\begin{aligned} & W^{(g)}_{n+1}(x_0,x_1,J) - W^{(g)}_{n+1}(x_1,x_0,J) \\ = \quad & \oint_{\mathscr{C}} dx\ \left(K'_0G_1 - K'_1G_0\right) W^{(g)}_n(x,J) \\ & + \oint_{\mathscr{C}} dx\ \left(K_0G_1 - K_1G_0\right) W^{(g)}_n(x,J)' \\ & -\frac{1}{\hbar}\oint_{\mathscr{C}} dx\ \Big(G_0K_1 - K_0G_1\Big)\ (Y(x)-\hbar\partial_x)W^{(g)}_n(x,J) \end{aligned} \tag{VIII.232}$$

Notice that:

$$K'_0G_1 - G_0K'_1 = -\frac{Y}{\hbar}\left(K_0G_1 - G_0K_1\right) \tag{VIII.233}$$

So we find

$$W^{(g)}_{n+1}(x_0,x_1,J) - W^{(g)}_{n+1}(x_1,x_0,J) = 0 \tag{VIII.234}$$

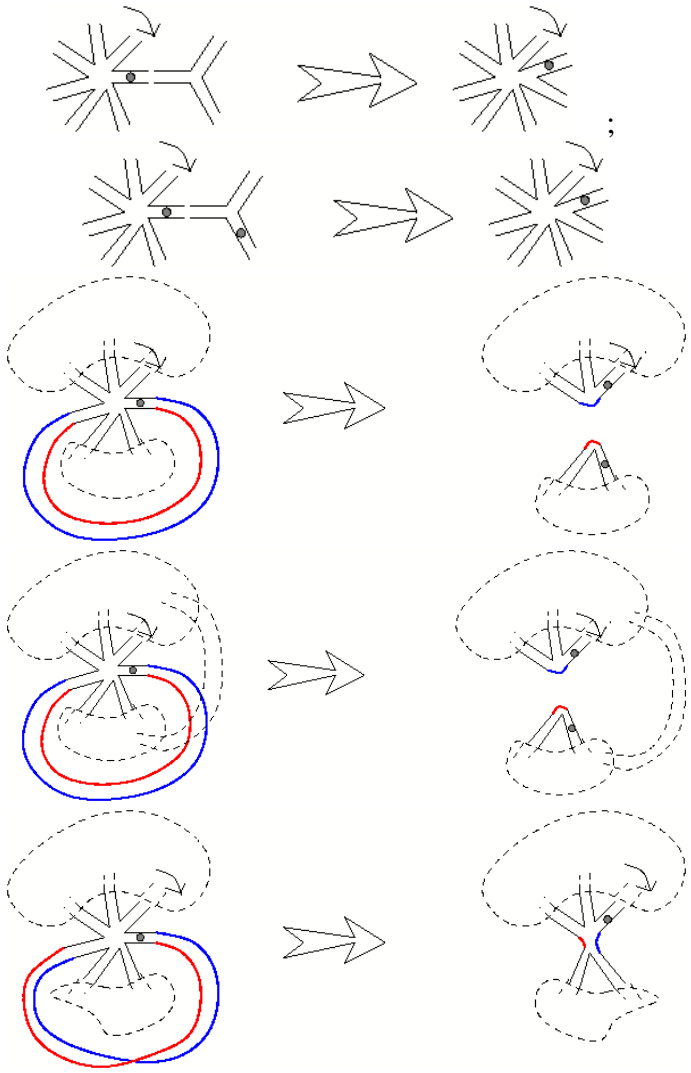

Figure VIII.4: When we shrink the single lines of a marked edge, several possibilities may occur: 1) the other side is an umarked vertex of order $j+1$, we get a new vertex of order $l_0+j$; 2) the other side is a marked vertex of order $l_i$, we get a new vertex of order $l_0+l_i-1$; 3)4) the other side is the same vertex and the edge is untwisted. Then shrinking the edge splits the vertex into two vertices, this may disconnect the graph or not; 5) the other side is the same vertex and the edge is twisted. Then shrinking the edge doesn't disconnect the vertex.

**Annexe IX**

# A matrix model for the topological string I : Deriving the matrix model

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We construct a matrix model that reproduces the topological string partition function on arbitrary toric Calabi-Yau 3-folds. This demonstrates, in accord with the BKMP "remodeling the B-model" conjecture, that Gromov-Witten invariants of any toric Calabi-Yau 3-fold can be computed in terms of the spectral invariants of a spectral curve. Moreover, it proves that the generating function of Gromov-Witten invariants is a tau function for an integrable hierarchy. In a follow-up paper, we will explicitly construct the spectral curve of our matrix model and argue that it equals the mirror curve of the toric Calabi-Yau manifold.

## 1 Introduction

In the topological string A-model, the object of study is the moduli space of maps from a Riemann surface $\Sigma_g$ of genus $g$ to a given Calabi-Yau target space $\mathfrak{X}$. Its partition function is the generating function of Gromov-Witten invariants of $\mathfrak{X}$, which roughly speaking count these maps.

In recent years, deep connections have been unrooted between the topological string on various geometries and random matrix models. A classic result in the field is that intersection numbers, which are related to the Gromov-Witten theory of a point, are computed by the Kontsevich matrix integral [115], see also [116]. In the Dijkgraaf-Vafa conjecture [117] such a connection is obtained between the topological B-model on certain non-compact Calabi-Yau manifolds and a 1-matrix model. A novel type of matrix model [118] inspired by Chern-Simons theory is associated to the topological string in [119], yielding matrix model descriptions of target spaces obtained from the cotangent space of lens spaces via geometric transition. This work is extended to chains of lens spaces and their duals in [120].

In the 20 years that have passed since topological string theory was formulated [78, 79], various techniques have been developed for computing the corresponding partition function. The topological vertex method [76] solves this problem completely for toric Calabi-Yau 3-folds at large radius, furnishing the answer as a combinatorial sum over partitions. On geometries with unit first Betti number (the conifold and $\mathscr{O}(-2) \to \mathbb{CP}^1 \times \mathbb{C}$), this formalism yields the partition function as a sum over a single partition with Plancherel measure. In [85], such a sum was rewritten as a 1-matrix integral. More complicated examples, such as the topological string on geometries underlying Seiberg-Witten $SU(n)$ theory, can be written as sums over multiple partitions [121–123]. 1-matrix integrals that reproduce the corresponding partition functions were formulated in [124]. Multi-matrix integrals have arisen in rewriting the framed vertex as a chain of matrices integral [84]. Its Hurwitz-numbers limit (infinite framing of the framed vertex geometry) was shown to be reproduced by a 1-matrix model with an external field in [57, 125].

Here, generalizing the method of [85], we are able to formulate a matrix model which reproduces the topological string partition function on a certain fiducial geometry, which we introduce in the next section. Flop transitions and limits in the Kähler cone relate the fiducial geometry to an arbitrary toric Calabi-Yau manifold. As we can follow the effect of both of these operations on the topological string partition function, our matrix model provides a description for the topological string on an arbitrary toric Calabi-Yau manifold.

By providing a matrix model realization, we are able to transcribe deep structural

insights into matrix models to the topological string setting. E.g., our matrix model involves a chain of matrices, and chain of matrices integrals are always tau functions for an integrable system. Our matrix model realization hence proves integrability of the generating function of Gromov-Witten invariants. Moreover, matrix models satisfy loop equations, which are known to be equivalent to W-algebra constraints. A general formal solution to these equations was found in [126], centered around the introduction of an auxiliary Riemann surface, referred to as the spectral curve of the system. The partition and correlation functions of the matrix model are identified with so-called symplectic invariants of this curve [23]. The BKMP conjecture [81], building on work of [127], identifies the spectral invariants of the mirror curve to a toric Calabi-Yau manifold with the topological string partition function with the Calabi-Yau manifold as target space. In a forthcoming publication [69], we will compute the spectral curve of our matrix model explicitly, thus establishing the validity of this conjecture.

Finally, we would like to emphasize that many different matrix models can yield the same partition function (justifying the choice of indefinite article in the title of this paper). An interesting open problem consists in identifying invariants of such equivalent matrix models. A promising candidate for such an invariant is the symplectic class of the matrix model spectral curve.

The outline of this paper is as follows. In section IX, after a very brief review of toric geometry basics, we introduce the fiducial geometry and the notation that we will use in discussing it throughout the paper. We also review the transformation properties of the topological string partition function under flop transitions, which will relate the fiducial to an arbitrary toric geometry, in this section. We recall the topological vertex formalism and its application to geometries on a strip [128] in section IX. Section IX contains our main result: we introduce a chain of matrices matrix model and demonstrate that it reproduces the topological string partition function on the fiducial geometry. By the argument above, we thus obtain a matrix model description for the topological string on an arbitrary toric Calabi-Yau manifold, in the large radius limit. We discuss implications of this result in section IX, and point towards avenues for future work in section IX.

## 2 The fiducial geometry and flop transitions

Toric geometries present a rich class of very computable examples for many questions in algebraic geometry. The topological vertex formalism provides an algorithm for computing the generating function for Gromov-Witten invariants on toric 3 dimensional Calabi-Yau manifolds. These are necessarily non-compact and have rigid complex structure.

The geometry of toric manifolds of complex dimension $d$ can be encoded in terms of a $d$ dimensional fan $\Sigma$, consisting of cones of dimensions 0 to $d$. We denote the set of all $n$ dimensional cones as $\Sigma(n)$. Each such $n$-cone represents the closure of a $(C^*)^{d-n}$ orbit. In particular, 1-cones correspond to hypersurfaces, and for $d = 3$, our case of interest, 2-cones correspond to curves.

The fan for the class of geometries we are interested in is constructed by triangulating a finite connected region of the $\mathbb{Z}^2$ lattice containing the origin, embedding this lattice in $\mathbb{Z}^3$ within the $(x,y)$ plane at $z = 1$, and defining the cones of the fan via half-lines emanating at the origin and passing through the vertices of this triangulation.[1]

We can associate a dual diagram to such toric fans, a so-called web diagram, spanned by lines orthogonal to the projection of 2-cones onto the $\mathbb{Z}^2$ lattice. In the web diagram, the relation between the dimension of the components of the diagram and the submanifold of the toric geometry they represent coincide: 3-cones (points) correspond to vertices, and 2-cones (curves) to lines, see figure IX.1.

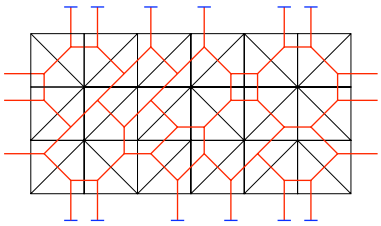

Figure IX.1: Example of a box triangulation, corresponding to a 3 dimensional toric fan. The diagram in red is the dual web diagram. Vertices of the triangulations (faces of the web diagram) correspond to 1-cones, edges correspond to 2-cones, and faces (vertices of the dual) correspond to 3-cones.

## 2.1 The fiducial geometry

The geometry $\mathfrak{X}_0$ we will take as the starting point of our considerations is depicted in figure IX.2.

Since the torically invariant curves play a central role in our considerations, we introduce a labeling scheme for these in figure IX.3: $(i, j)$ enumerates the boxes as in figure IX.2, and we will explain the $a$-parameters further below.

In the following, we will, when convenient, use the same notation for a torically invariant curve $\Sigma$, its homology class $[\Sigma] \in H_2(\mathfrak{X}_0, \mathbb{Z})$, and its volume or associated Kähler parameter $\int_\Sigma J$, given a Kähler form $J$ on $\mathfrak{X}_0$. The classes of the curves $r_{i,j}, s_{i,j}, t_{i,j}$ introduced in figure IX.3 are not independent. To determine the relations among these, we

[1] The canonical class of a toric manifold is given by the sum over all torically invariant divisors. The construction sketched above guarantees that this sum is principal, hence the canonical class trivial: the monomial associated to the 1-cone $(0,0,1)$ generates the class in question. See e.g. [129].

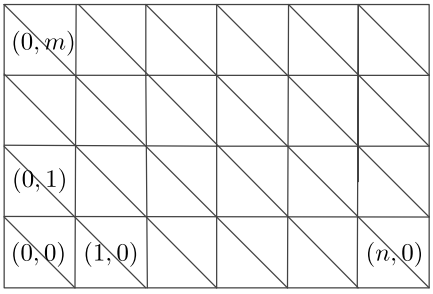


Figure IX.2: Fiducial geometry $\mathfrak{X}_0$ with boxes numbered.

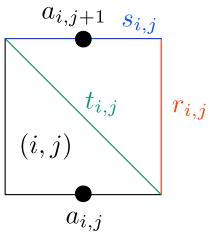


Figure IX.3: Labeling curve classes, and introducing $a$-parameters.

follow [130, page 39, 40]. Consider the integer lattice $\Lambda$ spanned by formal generators $e_\rho$, with $\rho \in \Sigma(1)$ 1-cones of the toric fan,

$$\Lambda = \{ \sum_{\rho \in \Sigma(1)} \lambda_\rho e_\rho | \lambda_\rho \in \mathbb{Z} \} . \tag{IX.1}$$

Each torically invariant curve, corresponding to a 2-cone of the fan, maps to a relation between 1-cones, and thus to an element of the lattice $\Lambda$, as follows: a 2-cone $\sigma$ is spanned by two integral generators $v_1$ and $v_2$, and it is contained in precisely two 3-cones, which are each spanned by $v_1, v_2$ and one additional generator $v_3$, $v_4$ respectively. These vectors satisfy the relation $\sum_{i=1}^4 \lambda_i v_i = 0$, where the $\lambda_i$ can be chosen as relatively prime integers, and as $v_3$ and $v_4$ lie on opposite sides of $\sigma$, we can assume that $\lambda_3, \lambda_4 > 0$. [130] shows that on a smooth variety, the sublattice $\Lambda_h$ generated by the elements $\sum_{i=1}^4 \lambda_i e_i$ of $\Lambda$ is isomorphic to $H_2(\mathfrak{X}_0, \mathbb{Z})$. We call this isomorphism $\lambda$,

$$\lambda : H_2(\mathfrak{X}_0, \mathbb{Z}) \to \Lambda_h . \tag{IX.2}$$

Figure IX.4 exemplifies this map.

It allows us to easily work out the relation between the various curve classes. Con-

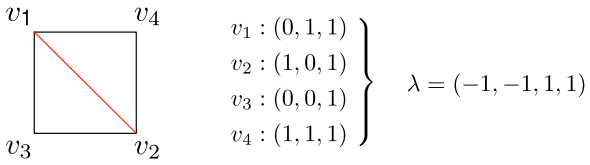


Figure IX.4: The 2-cone $\sigma$ corresponds to the relation $\vec{\lambda}$ among 1-cones.

sider figure IX.5.

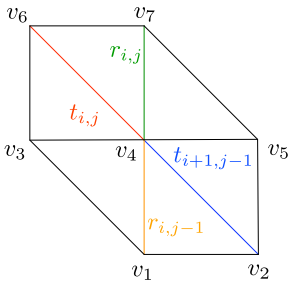


Figure IX.5: Determining the relation between curve classes.

The images of the curve classes depicted there under $\lambda$ are,

$$\begin{aligned}
\lambda(r_{i,j}) &= e_5 + e_6 - e_4 - e_7\,, \\
\lambda(r_{i,j-1}) &= e_2 + e_3 - e_1 - e_4\,, \\
\lambda(t_{i,j}) &= e_3 + e_7 - e_4 - e_6\,, \\
\lambda(t_{i+1,j-1}) &= e_1 + e_5 - e_2 - e_4\,.
\end{aligned}$$

We read off the relation

$$t_{i,j} + r_{i,j} = t_{i+1,j-1} + r_{i,j-1}\,. \tag{IX.3}$$

By symmetry, we also have

$$t_{i,j} + s_{i,j-1} = t_{i+1,j-1} + s_{i+1,j-1}\,. \tag{IX.4}$$

A moment's thought convinces us that this constitutes a complete basis for the space of relations. We can solve these in terms of the classes of the curves $r_i, s_i, t_{i,j}$, $i,j = 0,1,\ldots$ depicted in figure IX.6, which hence generate $H_2(\mathfrak{X}_0,\mathbb{Z})$. The explicit relations are

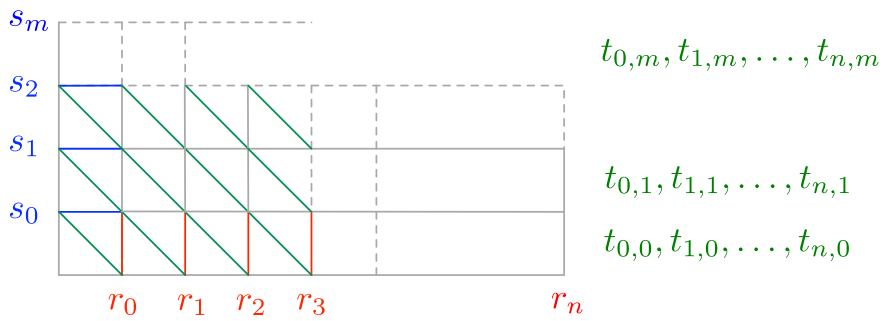


Figure IX.6: Fiducial geometry with choice of basis of $H_2(\mathfrak{X}_0, \mathbb{Z})$.

$$\begin{aligned} r_{i,j} &= r_i + \sum_{k=1}^{j} (t_{i+1,k-1} - t_{i,k}) \,, \\ s_{i,j} &= s_j + \sum_{k=1}^{i} (t_{k-1,j+1} - t_{k,j}) \,. \end{aligned}$$

Our computation for the partition function on $\mathfrak{X}_0$ will proceed by first considering the horizontal strips in the toric fan describing the geometry, as depicted in figure IX.2, individually, and then applying a gluing algorithm to obtain the final result.

For each strip, we find it convenient to write the curve class $w_{IJ} \in H_2(\mathfrak{X}_0, \mathbb{Z})$ of the curve extending between two 3-cones which we label by $I$ and $J$ (recall that 3-cones correspond to vertices in the dual web diagram), with $J$ to the right of $I$, as the difference between two parameters $a_I$ and $a_J$ associated to each 3-cone,

$$w_{IJ} = a_I - a_J \,. \tag{IX.5}$$

We call these parameters, somewhat prosaically, $a$-parameters. It is possible to label the curve classes in this way due to their additivity along a strip. In terms of the notation introduced in figure IX.3, we obtain

$$t_{i,j} = a_{i,j} - a_{i,j+1} \quad , \qquad r_{i,j} = a_{i,j+1} - a_{i+1,j} \,. \tag{IX.6}$$

By invoking the relation (IX.3), we easily verify that upon gluing two strips, the curve class of a curve extending between two 3-cones $I$ and $J$ on the lower strip is equal to the class of the curve between the 3-cones $I'$ and $J'$ on the upper strip, where the cones

$I$ and $I'$ are glued together, as are the cones $J$ and $J'$,

$$w_{IJ} = w_{I'J'}\,. \tag{IX.7}$$

This allows us to identify the parameters $a_I = a_{I'}$ and $a_J = a_{J'}$ associated to 3-cones glued together across strips.

Note that the basic curve classes $s_i$ are not captured by the parameters $a_{i,j}$.

### 2.2 Flop invariance of toric Gromov-Witten invariants

Under the proper identification of curve classes, Gromov-Witten invariants (at least on toric manifolds) are invariant under flops. Assume $\mathfrak{X}$ and $\mathfrak{X}^+$ are related via a flop transition, $\phi : \mathfrak{X} \to \mathfrak{X}^+$. In a neighborhood of the flopped $(-1,-1)$ curve, the respective toric diagrams are depicted in figure IX.7.

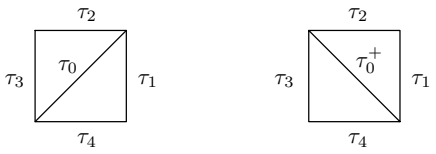


Figure IX.7: $\mathfrak{X}$ and $\mathfrak{X}^+$ in the vicinity of the (-1,-1) curve.

The 1-cones of $\Sigma_{\mathfrak{X}}$, corresponding to the toric invariant divisors of $\mathfrak{X}$, are not affected by the flop, hence can be canonically identified with those of $\mathfrak{X}^+$. The 2-cones $\tau_i$ in these diagrams correspond to toric invariant 2-cycles $C_i, C_i^+$ in the geometry. The curve classes of $\mathfrak{X}$ push forward to classes in $\mathfrak{X}^+$ via

$$\phi_*([C_0]) = -[C_0^+]\,, \quad \phi_*([C_i]) = [C_i^+] + [C_0^+]\,. \tag{IX.8}$$

All other curve classes of $\mathfrak{X}$ are mapped to their canonical counterparts in $\mathfrak{X}^+$. Under appropriate analytic continuation and up to a phase factor (hence the $\propto$ in the following formula), the following identity then holds [77, 128, 131],

$$Z_{GW}(\mathfrak{X}, Q_0, Q_1, \ldots, Q_4, \vec{Q}) \propto Z_{GW}(\mathfrak{X}^+, 1/Q_0, Q_0Q_1, \ldots, Q_0Q_4, \vec{Q})\,, \tag{IX.9}$$

i.e.

$$GW_g(\mathfrak{X}, Q_0, Q_1, \ldots, Q_4, \vec{Q}) = GW_g(\mathfrak{X}^+, 1/Q_0, Q_0Q_1, \ldots, Q_0Q_4, \vec{Q})\,.$$

Any toric Calabi-Yau manifold $\mathfrak{X}$ with Kähler moduli $\vec{Q}$ can be obtained from a sufficiently large fiducial geometry $(\mathfrak{X}_0, \vec{Q}_0)$ upon performing a series of flop transitions and taking unwanted Kähler moduli of $\mathfrak{X}_0$ to $\infty$. Once we obtain a matrix model reproducing the topological string partition function on the fiducial geometry, extending the result to arbitrary toric Calabi-Yau 3-folds will therefore be immediate.

As an example, we show how to obtain the $\mathbb{P}^2$ geometry from the fiducial geometry with $2 \times 2$ boxes in figure IX.8.

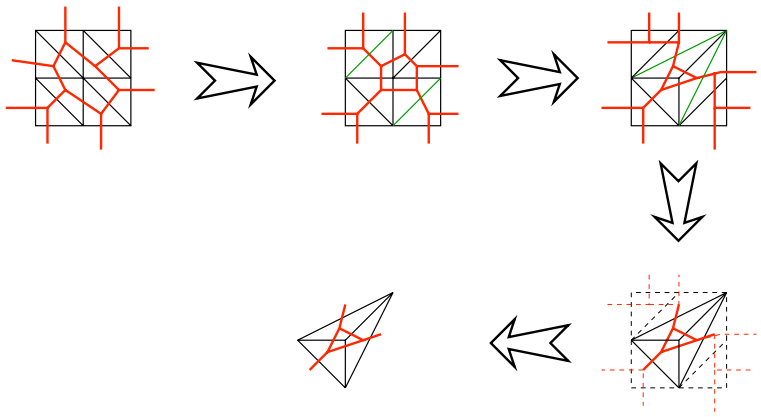

Figure IX.8: We obtain local $\mathbb{P}^2$ from the fiducial geometry with $2 \times 2$ boxes by performing five flops and then sending the Kähler parameters of the unwanted edges to $\infty$.

## 3 The partition function via the topological vertex

### 3.1 Gromov-Witten invariants

Gromov-Witten invariants $\mathcal{N}_{g,D}(\mathfrak{X})$ roughly speaking count the number of maps from a Riemann surface of genus $g$ into the target space $\mathfrak{X}$, with image in a given homology class $D = (D_1, \ldots, D_k) \in H_2(\mathfrak{X}, \mathbb{Z})$. They can be assembled into a generating series

$$GW_g(\mathfrak{X}, Q) = \sum_D \mathcal{N}_{g,D}(\mathfrak{X})\, Q^D. \tag{IX.10}$$

Each $GW_g(\mathfrak{X}, Q)$ is a formal series in powers $Q^D = \prod_i Q_i^{D_i}$ of the parameters $Q = (Q_1, Q_2, \ldots, Q_k)$, the exponentials of the Kähler parameters.

We can introduce a generating function for Gromov-Witten invariants of all genera

by introducing a formal parameter $g_s$ (the string coupling constant) and writing

$$GW(\mathfrak{X},Q,g_s)=\sum_{g=0}^{\infty} g_s^{2g-2}\,GW_g(\mathfrak{X},Q)\,. \tag{IX.11}$$

It is in fact more convenient to introduce disconnected Gromov-Witten invariants $\mathscr{N}^*_{\chi,D}(\mathfrak{X})$, for possibly disconnected surfaces, of total Euler characteristics $\chi$, and to define

$$Z_{GW}(\mathfrak{X},Q,g_s)=e^{GW(\mathfrak{X},Q,g_s)}=\sum_D Q^D\sum_\chi g_s^{-\chi}\,\mathscr{N}^*_{\chi,D}(\mathfrak{X}). \tag{IX.12}$$

For toric Calabi-Yau manifolds, an explicit algorithm was presented in [76] for computing $Z_{GW}$ via the so-called topological vertex formalism, proved in [132, 133].

### 3.2 The topological vertex

In the topological vertex formalism, each vertex of the web diagram contributes a factor $C_q(\alpha,\beta,\gamma)$ to the generating function of GW-invariants, where the $\alpha,\beta,\gamma$ are Young tableaux associated to each leg of the vertex, and $C_q(\alpha,\beta,\gamma)$ is a formal power series in the variable $q$, where

$$q=e^{-g_s}.$$

Topological vertices are glued along edges (with possible framing factors, see [76]) carrying the same Young tableaux $\alpha$ by performing a sum over $\alpha$, weighted by $Q^{|\alpha|}$, with $Q$ encoding the curve class of this connecting line,

$$Z_{\text{vertex}}(\mathfrak{X},Q,q)=\sum_{\text{Young tableaux }\alpha_e}\ \prod_{\text{edges }e} Q_e^{|\alpha_e|}\ \prod_{\text{vertices }v=(e_1,e_2,e_3)} C_q(\alpha_{e_1},\alpha_{e_2},\alpha_{e_3})\,. \tag{IX.13}$$

Note that in practical computations, the sum over representations can ordinarily not be performed analytically. A cutoff on the sum corresponds to a cutoff on the degree of the maps being counted.

The equality

$$Z_{GW}(\mathfrak{X},Q,g_s)=Z_{\text{vertex}}(\mathfrak{X},Q,q) \tag{IX.14}$$

holds at the level of formal power series in the $Q$'s, referred to as the large radius expansion. It was proved in [133] that the log of the right hand side indeed has a power series expansion in powers of $g_s$.

### 3.3 Notations for partitions and q-numbers

Before going further in the description of the topological vertex formula, we pause to fix some notations and introduce special functions that we will need in the following.

#### Representations and partitions

Representations of the symmetric group are labelled by Young tableaux, or Ferrer diagrams. For a representation $\gamma$, we introduce the following notation:

- $\gamma_i$: number of boxes in the $i$-th row of the Young tableau associated to the representation $\gamma$, $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_d \geq 0$.
- The weight $|\gamma| = \sum_i \gamma_i$: the total number of boxes in the corresponding Young tableau.
- The length $l(\gamma)$: the number of non-vanishing rows in the Young tableau, i.e. $\gamma_i = 0$ iff $i > l(\gamma)$.
- The Casimir $\kappa(\gamma) = \sum_i \gamma_i(\gamma_i - 2i + 1)$.
- $\gamma^T$ denotes the conjugate representation, which is obtained by exchanging the rows and columns of the associated Young tableau. We have $|\gamma^T| = |\gamma|$, $l(\gamma^T) = \gamma_1$, and $\kappa(\gamma^T) = -\kappa(\gamma)$.

An integer $d > 0$ will denote a cut-off on the length of representations summed over,

$$l(\gamma) \leq d.$$

Most expressions we are going to write will in fact be independent of $d$, and we shall argue in [69], following the same logic as in [85] based on the arctic circle property [134], that our results depend on $d$ only non-perturbatively.

To each representation $\gamma$, we shall associate a parameter $a$ as introduced in (IX.5).

Instead of dealing with a partition $\gamma$, characterized by the condition $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_d \geq 0$, it will prove convenient to define the quantities

$$h_i(\gamma) = \gamma_i - i + d + a\,, \tag{IX.15}$$

which satisfy instead

$$h_1 > h_2 > h_3 > \cdots > h_d \geq a\,. \tag{IX.16}$$

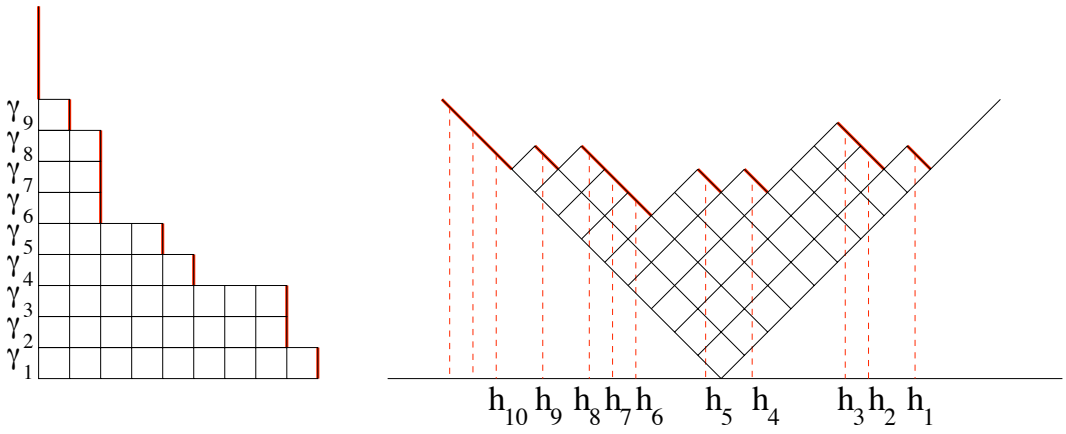


Figure IX.9: Relation between a partition $\gamma$ and $h(\gamma)$.

The relation between $\gamma$ and $h(\gamma)$, for the off-set $a = 0$, is depicted in figure IX.9.

We finally introduce the functions

$$x_i(\gamma) = q^{h_i(\gamma)} \,. \tag{IX.17}$$

In terms of the $h_i(\gamma)$, we have

$$\kappa(\gamma) = \sum_i h_i^2 - (2d + 2a - 1) \sum_i h_i + d\, C_{d,a} \,,$$

where $C_{d,a} = \frac{1}{3}(d-1)(2d-1) + a(a+2d-1)$.

### q-numbers

We choose a string coupling constant $g_s$ such that the quantum parameter $q = e^{-g_s}$ satisfies $|q| < 1$. A $q$-number $[x]$ is defined as

$$[x] = q^{-\frac{x}{2}} - q^{\frac{x}{2}} = 2 \sinh \frac{x g_s}{2} \,. \tag{IX.18}$$

$q$-numbers are a natural deformation away from the integers; in the limit $q \to 1$, $\frac{1}{g_s}[x] \to x$.

We also define the $q$-product

$$g(x) = \prod_{n=1}^{\infty} (1 - \frac{1}{x} q^n) \,.$$

The function $g(x)$ is related to the quantum Pochhammer symbol, $g(x) = [q/x;q]_\infty$, and to the $q$-deformed gamma function via $\Gamma_q(x) = (1-q)^{1-x} g(1)/g(q^{1-x})$. $g(x)$ satisfies the functional relation

$$g(qx) = (1 - \frac{1}{x})\, g(x)\,.$$

For $\Gamma_q$, this implies $\Gamma_q(x+1) = \frac{1-q^x}{1-q}\,\Gamma_q(x)$, the quantum deformation of the functional equation $\Gamma(x+1) = x\Gamma(x)$ of the gamma function, which is recovered in the classical limit $q \to 1$. The central property of $g(x)$ for our purposes is that it vanishes on integer powers of $q$,

$$g(q^n) = 0 \quad \text{if } n \in \mathbb{N}^*. \tag{IX.19}$$

Moreover, it has the following small $\ln q$ behavior,

$$\ln g(x) = \frac{1}{\ln q} \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} (\ln q)^n \, \mathrm{Li}_{2-\mathrm{n}}(1/\mathrm{x})\,, \tag{IX.20}$$

where $\mathrm{Li}_\mathrm{n}(\mathrm{x}) = \sum_{\mathrm{k}=1}^{\infty} \frac{\mathrm{x}^\mathrm{k}}{\mathrm{k}^\mathrm{n}}$ is the polylogarithm, and $B_n$ are the Bernouilli numbers

$$B_0 = 1\,, \quad B_1 = -\frac{1}{2}\,, \quad B_2 = \frac{1}{6}\,, \quad \dots$$

$B_{2k+1} = 0$ if $k \geq 1$ (see the appendix).

We shall also need the following function $f(x)$,

$$\begin{aligned} \frac{1}{f(x)} &= \frac{g(x)\, g(q/x)}{g(1)^2\, \sqrt{x}}\, e^{\frac{(\ln x)^2}{2\ln q}}\, e^{\frac{-i\pi \ln x}{\ln q}} \\ &= \frac{-\ln q}{\theta'(\frac{1}{2} - \frac{i\pi}{\ln q}, -\frac{2i\pi}{\ln q})}\, \theta\left(\frac{\ln x}{\ln q} + \frac{1}{2} - \frac{i\pi}{\ln q}, \frac{-2i\pi}{\ln q}\right), \end{aligned}$$

where $\theta$ is the Riemann theta-function for the torus of modulus $-2i\pi/\ln q$. This relationship is the quantum deformation of the classical gamma function identity

$$e^{-i\pi x}/\Gamma(1-x)\Gamma(x) = \sin(\pi x)/\pi\,.$$

### 3.4 The partition function via the vertex

We begin by considering a single horizontal strip of the fiducial geometry, as depicted in figure IX.10.

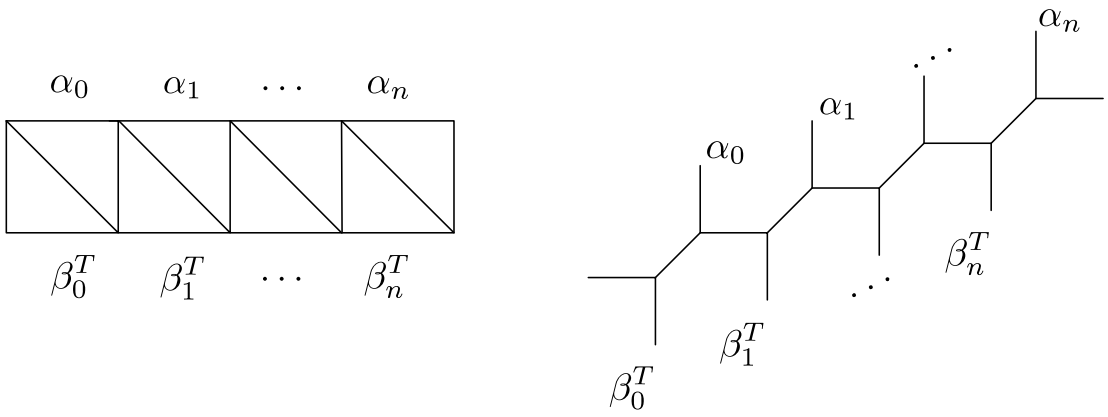


Figure IX.10: A horizontal strip of the fiducial geometry and its corresponding web diagram.

Of the three legs of the vertex, two point in the direction of the strip and connect the vertex to its neighbors. One leg points out of the strip, either above or below. This leg carries a free representation, $\alpha_i$ or $\beta_i^T$ in the notation of figure IX.10. The partition function will hence depend on representations, one per vertex (i.e. face of the triangulation).

A note on notation: since each 3-cone carries a representation (which up to the final paragraph of this subsection is held fixed) and an a-parameter (see figure IX.3), we will identify the a-parameters by the corresponding representations when convenient.

Using the topological vertex, it was shown in [128] that the A-model topological string partition function of the strip is given by a product of terms, with the individual factors depending on the external representations and all possible pairings of these. Applied to the fiducial strip, the results there specialize to

$$Z_{\text{strip}}(\alpha_0;\beta^T) \;=\; \prod_{i=0}^{n} \frac{[\alpha_i][\beta_i^T]}{[\beta_i,\alpha_i^T]_{Q_{\beta_i,\alpha_i}}} \; \frac{\prod_{i<j}[\alpha_i,\alpha_j^T]_{Q_{\alpha_i,\alpha_j}} \prod_{i<j}[\beta_i,\beta_j^T]_{Q_{\beta_i,\beta_j}}}{\prod_{i<j}[\alpha_i,\beta_j^T]_{Q_{\alpha_i,\beta_j}}[\beta_i,\alpha_j^T]_{Q_{\beta_i,\alpha_j}}} \tag{IX.21}$$

We explain each factor in turn.

• Each vertex $\gamma=\alpha_i$ or $\gamma=\beta_i^T$ contributes a representation dependent factor to the partition function, which we have denoted by $[\gamma]$. It is the $n\to\infty$ limit of the Schur

polynomial evaluated for $x_i = q^{\frac{1}{2}-i}$, $i = 1,\dots,n$, given explicitly by

$$\begin{aligned}
[\gamma] &= (-1)^d q^{\frac{1}{4}\kappa(\gamma)} \prod_{1\le i<j\le d} \frac{[\gamma_i-\gamma_j+j-i]}{[j-i]} \prod_{i=1}^{d}\prod_{j=1}^{\gamma_i} \frac{1}{[d+j-i]} \\
&= \prod_{1\le i<j\le d} (q^{h_j}-q^{h_i}) \prod_{i=1}^{d} \left( \frac{g(q^{a_\gamma-h_i})}{g(1)}\, q^{\frac{1}{2}h_i^2-(a_\gamma+d-1)h_i+\frac{a\gamma(a\gamma+d-1)}{2}+\frac{(d-1)(2d-1)}{12}} \right) \\
&= \Delta(X(\gamma))\, e^{-\frac{1}{g_s}\mathrm{tr}U(X(\gamma),a_\gamma)}\, e^{-\frac{1}{g_s}\mathrm{tr}U_1(X(\gamma),a_\gamma)}.
\end{aligned}$$

We recall that $h_i(\gamma) = \gamma_i - i + d + a_\gamma$, and we have defined $x_i = q^{h_i}$ and the diagonal matrix $X(\gamma) = \mathrm{diag}(q^{h_1}, q^{h_2}, \dots, q^{h_d})$. Furthermore, $\Delta(X)$ denotes the Vandermonde determinant of the matrix $X$,

$$\Delta(X) = \prod_{1\le i<j\le d} (x_j - x_i)\,, \tag{IX.22}$$

and we have written

$$U(X,a) = -g_s \ln\left( \frac{g(\frac{q^a}{X})}{g(1)} \right), \tag{IX.23}$$

$$U_1(X,a) = \frac{(\ln X)^2}{2} - (a+d-1)\ln X \ln q + C(a,d)\,, \tag{IX.24}$$

where $C(a,d) = \frac{a(a+d-1)}{2} + \frac{(d-1)(2d-1)}{12}$.

We have

$$[\gamma] = q^{\frac{\kappa(\gamma)}{2}}[\gamma^T]\,, \quad \kappa(\gamma^T) = -\kappa(\gamma)\,,$$

and thus

$$[\gamma^T] = \Delta(X(\gamma))\, e^{-\frac{1}{g_s}\mathrm{tr}U(X(\gamma),a_\gamma)}\, e^{-\frac{1}{g_s}\mathrm{tr}\tilde{U}_1(X(\gamma),a_\gamma)}\,,$$

where

$$\tilde{U}_1(X,a) = \frac{1}{2}\ln X \ln q + \tilde{C}(a,d). \tag{IX.25}$$

$\tilde{C}_{a,d}$ is another constant which depends only on $a$ and $d$ and which will play no role for our purposes.

• In addition, each pair of representations contributes a factor, reflecting the contribution of the curve extended between the respective vertices. In the nomenclature of [128], the representations $\alpha_i$ are all of same type, and of opposite type relative to the $\beta_i$. If we take $i < j$, representations of same type (corresponding to (-2,0) curves) contribute

a factor of

$$[\alpha_i, \alpha_j^T] \quad \text{or} \quad [\beta_i^T, \beta_j]\,,$$

whereas representations of different type (corresponding to (-1,-1) curves) contribute a factor of

$$\frac{1}{[\alpha_i, \beta_j]} \quad \text{or} \quad \frac{1}{[\beta_i^T, \alpha_j^T]}\,.$$

The pairing is given by [122, 128, 135, 136]

$$\begin{aligned}
[\gamma, \delta^T] &= Q_{\gamma,\delta}^{-\frac{|\gamma|+|\delta|}{2}} q^{-\frac{\kappa(\gamma)-\kappa(\delta)}{4}} \prod_{i=1}^{d}\prod_{j=1}^{d} \frac{[h_i(\gamma)-h_j(\delta)]}{[a_\gamma - a_\delta + j - i]} \\
&\quad \times \prod_{i=1}^{d}\prod_{j=1}^{\gamma_i} \frac{1}{[a_\gamma - a_\delta + j - i + d]} \prod_{i=1}^{d}\prod_{j=1}^{\delta_i} \frac{1}{[a_\gamma - a_\delta - j + i - d]} \prod_{k=0}^{\infty} g(Q_{\gamma,\delta}^{-1} q^{-k}) \\
&= (-1)^{\frac{d(d-1)}{2}} \prod_{i=1}^{d} \frac{q^{\frac{1}{2}(h_i(\delta)^2 - h_i(\delta)(2a_\gamma + 2d - 1) - a_\delta^2 + 2a_\gamma a_\delta + (d-2i)a_\gamma + (2i-d-1)a_\delta)}}{[a_\gamma - a_\delta]^d} \prod_{i=1}^{d} (-1)^{\delta_i} \\
&\quad \prod_{i,j=1}^{d} (q^{h_j(\delta)} - q^{h_i(\gamma)}) \prod_{i=1}^{d} \frac{g(q^{a_\gamma - h_i(\delta)})}{g(q^{a_\gamma - a_\delta})} \frac{g(q^{a_\delta - h_i(\gamma)})}{g(q^{a_\delta - a_\gamma})} \\
&\propto \Delta(X(\gamma), X(\delta))\, e^{-\frac{1}{g_s}\mathrm{Tr}U(X(\gamma), a_\delta)} e^{-\frac{1}{g_s}\mathrm{Tr}U(X(\delta), a_\gamma)}\, e^{-\frac{1}{g_s}\left(\mathrm{Tr}U_2(X(\gamma), a_\delta) + \mathrm{Tr}\tilde{U}_2(X(\delta), a_\gamma)\right)}\,,
\end{aligned} \tag{IX.26}$$

where the square brackets on the RHS denote $q$-numbers as defined in (IX.18), the symbol $\Delta(X(\gamma), X(\delta))$ signifies

$$\Delta(X(\gamma), X(\delta)) = \prod_{i,j} (X_i(\delta) - X_j(\gamma)) = \prod_{i,j} (q^{h_i(\delta)} - q^{h_j(\gamma)})\,, \tag{IX.27}$$

and

$$U_2(X, a) = 0\,,$$

$$\tilde{U}_2(X, a) = \frac{(\ln X)^2}{2} - (a + d - \frac{1}{2}) \ln X \ln q + i\pi \ln X.$$

The parameter $Q_{\gamma,\delta}$ reflects, given a choice of Kähler class $J$ of the metric on $\mathfrak{X}_0$, the

curve class of the curve $\mathscr{C}$ extended between the vertices labeled by $\gamma$ and $\delta$ via

$$w_{\gamma,\delta} = \int_{\mathscr{C}} J\,, \qquad Q_{\gamma,\delta} = q^{w_{\gamma,\delta}}\,. \tag{IX.28}$$

By the definition of the a-parameters,

$$w_{\gamma,\delta} = a_\gamma - a_\delta. \tag{IX.29}$$

Substituting these expressions into (IX.21), we obtain

$$\begin{aligned} &Z_{\rm strip}(\alpha_0,\ldots,\alpha_n;\beta_0^T,\ldots,\beta_n^T) = \\ &= \frac{\prod_i \Delta(X(\alpha_i))\prod_{i<j}\Delta(X(\alpha_i),X(\alpha_j))\ \prod_i \Delta(X(\beta_i))\prod_{i<j}\Delta(X(\beta_i),X(\beta_j))}{\prod_{i,j}\Delta(X(\alpha_i),X(\beta_j))} \\ &\times \prod_i e^{-\frac{1}{g_s}\,{\rm tr}(V_{\vec a}(X(\alpha_i))-V_{\vec b}(X(\alpha_i)))}\ \prod_i e^{-\frac{1}{g_s}\,{\rm tr}V_i(X(\alpha_i))} \\ &\times \prod_i e^{\frac{1}{g_s}\,{\rm tr}(V_{\vec a}(X(\beta_i))-V_{\vec b}(X(\beta_i)))}\ \prod_i e^{-\frac{1}{g_s}\,{\rm tr}\tilde V_i(X(\beta_i))}\,, \end{aligned} \tag{IX.30}$$

where we have denoted by $\vec a = (a_0,a_1,\ldots,a_n)$ (resp. $\vec b = (b_0,b_1,\ldots,b_n)$) the a-parameters of representations on the upper side (resp. lower side) of the strip, and defined

$$V_{\vec a}(X) = -g_s\sum_{j=0}^{n}\ln(g(q^{a_j}/X))\,, \tag{IX.31}$$

and

$$V_i(X) = \ln X\ln q\,\left(\frac{1}{2} - \sum_{j\le i}(a_j-b_j)\right) + i\pi\ln X\,, \tag{IX.32}$$

$$\tilde V_i(X) = \ln X\ln q\,\left(\frac{1}{2} - \sum_{j<i}(b_j-a_j)\right). \tag{IX.33}$$

### 3.5 Gluing strips

To obtain the partition function for the full multistrip fiducial geometry $\mathfrak{X}_0$, we must glue these strips along the curves labelled $s_{i,j}$ in figure IX.3.

Denoting the representations $\alpha_{j,i}$ on line $i$ collectively by

$$\vec{\alpha}_i = (\alpha_{0,i}, \alpha_{1,i}, \ldots, \alpha_{n,i}), \tag{IX.34}$$

this yields

$$Z_{\text{vertex}}(\mathfrak{X}_0) = Z_{(n,m)}(\vec{\alpha}_{m+1}, \vec{\alpha}_0^T) = \sum_{\alpha_{j,i},\, j=0,\ldots,n;\, i=1,\ldots,m} \prod_{i=1}^{m+1} Z_{\text{strip}}(\vec{\alpha}_i, \vec{\alpha}_{i-1}^T) \prod_{j=0}^{n} \prod_{i=1}^{m} q^{s_{j,i}|\alpha_{j,i}|}. \tag{IX.35}$$

Our goal now is to find a matrix integral which evaluates to this sum.

## 4 The matrix model

### 4.1 Definition

Consider the fiducial geometry $\mathfrak{X}_0$ of size $(n+1)\times(m+1)$, with Kähler parameters $t_{i,j} = a_{i,j} - a_{i,j+1}$, $r_{i,j} = a_{i,j+1} - a_{i+1,j}$, and $s_{i,j}$, as depicted in figures IX.3 and IX.6. We write

$$\vec{a}_i = (a_{0,i}, a_{1,i}, \ldots, a_{n,i}). \tag{IX.36}$$

Assume that the external representations are fixed to $\vec{\alpha}_{m+1} = (\alpha_{0,m+1}, \alpha_{1,m+1}, \ldots, \alpha_{n,m+1})$ on the upper line, and $\vec{\alpha}_0 = (\alpha_{0,0}, \alpha_{1,0}, \ldots, \alpha_{n,0})$ on the lower line (for most applications, one prefers to choose these to be trivial).

We now define the following matrix integral $\mathscr{Z}_{\text{MM}}$ (MM for Matrix Model),

$$\begin{aligned}
\mathscr{Z}_{\text{MM}}(Q, g_s, \vec{\alpha}_{m+1}, \vec{\alpha}_0^T) \;=\; & \Delta(X(\vec{\alpha}_{m+1}))\, \Delta(X(\vec{\alpha}_0)) \prod_{i=0}^{m+1} \int_{H_N(\Gamma_i)} dM_i \prod_{i=1}^{m+1} \int_{H_N(\mathbb{R}_+)} dR_i \\
& \prod_{i=1}^{m} e^{\frac{-1}{g_s}\operatorname{tr}\left[V_{\vec{a}_i}(M_i) - V_{\vec{a}_{i-1}}(M_i)\right]} \prod_{i=1}^{m} e^{\frac{-1}{g_s}\operatorname{tr}\left[V_{\vec{a}_{i-1}}(M_{i-1}) - V_{\vec{a}_i}(M_{i-1})\right]} \\
& \prod_{i=1}^{m+1} e^{\frac{1}{g_s}\operatorname{tr}(M_i - M_{i-1})R_i} \prod_{i=1}^{m} e^{(S_i + \frac{i\pi}{g_s})\operatorname{tr}\ln M_i} \\
& e^{\operatorname{trln} f_0(M_0)}\, e^{\operatorname{trln} f_{m+1}(M_{m+1})} \prod_{i=1}^{m} e^{\operatorname{trln} f_i(M_i)}.
\end{aligned} \tag{IX.37}$$

All matrices are taken of size

$$N = (n+1)\, d, \tag{IX.38}$$

where $d$ is the cut-off discussed in section IX. We have introduced the notation

$$X(\vec{\alpha}_{m+1}) = \mathrm{diag}(X(\vec{\alpha}_{m+1})_i)_{i=1,\dots,N}\,, \qquad X(\vec{\alpha}_{m+1})_{jd+k} = q^{h_k(\alpha_{j,m+1})}, \tag{IX.39}$$

$$X(\vec{\alpha}_0) = \mathrm{diag}(X(\vec{\alpha}_0)_i)_{i=1,\dots,N}\,, \qquad X(\vec{\alpha}_0)_{jd+k} = q^{h_k(\alpha_{j,0})}, \tag{IX.40}$$

for $k = 1,\dots,d$, $j = 0,\dots,n$. $\Delta(X) = \prod_{i<j}(X_i - X_j)$ is the Vandermonde determinant. $V_{\vec{d}_i}(x)$ was introduced in (IX.31). For $i = 1,\dots,m$, we have defined

$$f_i(x) = \prod_{j=0}^{n} \frac{g(1)^2\, e^{(\frac{1}{2}+\frac{i\pi}{\ln q})\ln(xq^{1-a_{j,i}})}\, e^{\frac{(\ln(xq^{1-a_{j,i}}))^2}{2g_s}}}{g(xq^{1-a_{j,i}})\, g(q^{a_{j,i}}/x)}\,. \tag{IX.41}$$

The denominator of these functions induces simple poles at $x = q^{a_{j,i}+l}$ for $j = 0,\dots,n$ and $l \in \mathbb{Z}$. The numerator is chosen such that they satisfy the relation $f_i(qx) = f_i(x)$. This enforces a simple $l$ dependence of the residues taken at $x = q^{a_{j,i}+l}$, given by a prefactor $q^l$ – a fact which will be important in the following. These residues are in fact given by

$$\operatorname*{Res}_{q^{a_{j,i}+l}} f_i(x) = q^{a_{j,i}+l}\, \hat{f}_{j,i} = -q^{a_{j,i}+l} \prod_{k\neq j} \frac{g(1)^2\, e^{(\frac{1}{2}+\frac{i\pi}{\ln q})(1+a_{j,i}-a_{k,i})\ln q}\, e^{\frac{(\ln(q^{1+a_{j,i}-a_{k,i}}))^2}{2g_s}}}{g(q^{a_{j,i}-a_{k,i}})\,(1-q^{a_{k,i}-a_{j,i}})\, g(q^{a_{k,i}-a_{j,i}})}\,, \tag{IX.42}$$

where $\hat{f}_{j,i}$ is independent of the integer $l$.

The parameters $S_i$ are defined by

$$S_i = s_{0,i-1} + t_{0,i-1} = s_{j,i-1} - \sum_{k<j} t_{k,i} + \sum_{k\leq j} t_{k,i-1}\,. \tag{IX.43}$$

The final equality holds for arbitrary $j$, and can be verified upon invoking (IX.7) repeatedly.

For $i = 0$ and $i = m+1$, we define

$$f_0(x) = \frac{1}{\prod_{j=0}^{n}\prod_{i=1}^{d}(x - q^{h_i(\alpha_{j,0})})}\,, \tag{IX.44}$$

$$f_{m+1}(x) = \frac{1}{\prod_{j=0}^{n}\prod_{i=1}^{d}(x - q^{h_i(\alpha_{j,m+1})})}\,. \tag{IX.45}$$

Notice that if the representations $\vec{\alpha}_0$ or $\vec{\alpha}_{m+1}$ are trivial, i.e. $h_i(\alpha_{j,0}) = d - i + a_{j,0}$ or

$h_i(\alpha_{j,m+1}) = d - i + a_{j,m+1}$, we have

$$f_0(x) = \prod_{j=0}^{n} \frac{g(xq^{1-a_{j,0}-d})}{x^d\, g(xq^{1-a_{j,0}})}\,, \qquad f_{m+1}(x) = \prod_{j=0}^{n} \frac{g(xq^{1-a_{j,m+1}-d})}{x^d\, g(xq^{1-a_{j,m+1}})} \tag{IX.46}$$

respectively. The functions $f_0$ and $f_{m+1}$ have simple poles at $x = q^{h_l(\alpha_{j,0})}$ (resp. $x = q^{h_l(\alpha_{j,m+1})}$) for $l = 1, \dots, d$, with residue

$$\hat{f}_{j,0;l} = \operatorname*{Res}_{q^{h_l(\alpha_{j,0})}} f_0(x) = \frac{1}{\prod_{j' \neq j} \prod_{i=1}^{d} (q^{h_l(\alpha_{j,0})} - q^{h_i(\alpha_{j',0})})} \frac{1}{\prod_{i \neq l} (q^{h_l(\alpha_{j,0})} - q^{h_i(\alpha_{j,0})})}\,, \tag{IX.47}$$

$$\hat{f}_{j,m+1;l} = \operatorname*{Res}_{q^{h_l(\alpha_{j,m+1})}} f_{m+1}(x) = \frac{1}{\prod_{j' \neq j} \prod_{i=1}^{d} (q^{h_l(\alpha_{j,m+1})} - q^{h_i(\alpha_{j',m+1})})} \frac{1}{\prod_{i \neq l} (q^{h_l(\alpha_{j,m+1})} - q^{h_i(\alpha_{j,m+1})})}\,. \tag{IX.48}$$

The $l$ dependence here is more intricate than above, but this will not play any role since the partitions $\alpha_{j,0}$ and $\alpha_{j,m+1}$ are kept fixed, not summed upon.

The integration domains for the matrices $R_i$ are $H_N(\mathbb{R}_+^N)$, i.e. the set of hermitian matrices having only positive eigenvalues. For the matrices $M_i, i = 1, \dots, m$, the integration domains are $H_N(\Gamma_i)$, where

$$\Gamma_i = \prod_{j=0}^{n} (\gamma_{j,i})^d\,. \tag{IX.49}$$

$\gamma_{j,i}$ is defined as a contour which encloses all points of the form $q^{a_{j,i}+\mathbb{N}}$, and does not intersect any contours $\gamma_{k,l}$, $(j,i) \neq (k,l)$. For this to be possible, we must require that the differences $a_{j,i} - a_{j',i'}$ be non-integer. The normalized logarithms of two such contours are depicted in figure IX.11.

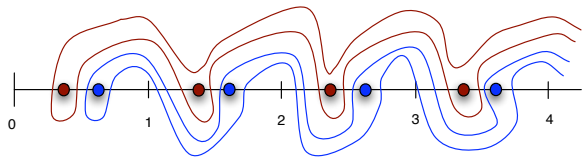


Figure IX.11: Two contours surrounding points $a + \mathbb{N}$ and $b + \mathbb{N}$, such that $a - b \notin \mathbb{Z}$.

We have defined

$$H_N(\Gamma_i) = \{M = U\,\Lambda\,U^\dagger\,, \quad U \in U(N)\,,\ \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N) \in \Gamma_i\}\,, \tag{IX.50}$$

i.e. $H_N(\Gamma_i)$ is the set of normal matrices with eigenvalues on $\Gamma_i$. By definition, the measure on $H_N(\Gamma_i)$ is (see [19])

$$dM = \frac{1}{N!}\,\Delta(\Lambda)^2\,dU\,d\Lambda\,, \tag{IX.51}$$

where $dU$ is the Haar measure on $U(N)$, (normalized not to 1, but to a value depending only on $N$, such that the Itzykson-Zuber integral evaluates as given in (IX.59) with prefactor 1), and $d\Lambda$ is the product of the measures for each eigenvalue along its integration path.

The integration domains for the matrices $M_0$, $M_{m+1}$ are $H_N(\Gamma_0)$, $H_N(\Gamma_{m+1})$ respectively, where

$$\Gamma_0 = (\sum_{j=0}^{n} \gamma_{j,0})^N\,, \qquad \Gamma_{m+1} = (\sum_{j=0}^{n} \gamma_{j,m+1})^N\,. \tag{IX.52}$$

The goal of the rest of this section is to prove that the matrix integral (IX.37) reproduces the topological string partition function for target space the fiducial geometry $\mathfrak{X}_0$.

### 4.2 Diagonalization

Let us first diagonalize all matrices. We write

$$M_i = U_i X_i U_i^\dagger\,, \tag{IX.53}$$

$$R_i = \tilde{U}_i Y_i \tilde{U}_i^\dagger\,, \tag{IX.54}$$

where $U_i$ and $\tilde{U}_i$ are unitary matrices.

By the definition (IX.51), the measures $dM_i$ and $dR_i$ are given by

$$dM_i = \frac{1}{N!}\,\Delta(X_i)^2\,dU_i\,dX_i \quad , \quad dR_i = \frac{1}{N!}\,\Delta(Y_i)^2\,d\tilde{U}_i\,dY_i\,. \tag{IX.55}$$

The matrix integral thus becomes

$$\begin{aligned}\mathscr{Z}_{\rm MM}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^T) \;=\; & \frac{\Delta(X(\vec{\alpha}_{m+1}))\,\Delta(X(\vec{\alpha}_0))}{(N!)^{2m+3}} \prod_{i=0}^{m+1}\int_{\Gamma_i} dX_i\,\Delta(X_i)^2 \prod_{i=1}^{m+1}\int_{\mathbb{R}_+^N} dY_i\,\Delta(Y_i)^2 \\ & \prod_{i=0}^{m+1} dU_i \prod_{i=1}^{m+1} d\tilde{U}_i\end{aligned}$$

$$\prod_{i=1}^{m} e^{\frac{-1}{g_s}\,\mathrm{tr}\left[V_{\vec{a}_i}(X_i)-V_{\vec{a}_{i-1}}(X_i)\right]}\ \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,\mathrm{tr}\left[V_{\vec{a}_{i-1}}(X_{i-1})-V_{\vec{a}_i}(X_{i-1})\right]}$$
$$\prod_{i=1}^{m+1} e^{\frac{1}{g_s}\mathrm{tr}X_i U_i^\dagger \tilde{U}_i Y_i \tilde{U}_i^\dagger U_i}\ e^{\frac{-1}{g_s}\mathrm{tr}X_{i-1}U_{i-1}^\dagger \tilde{U}_i Y_i \tilde{U}_i^\dagger U_{i-1}}\ \prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\,\mathrm{tr}\ln X_i}$$
$$e^{\mathrm{tr}\ln f_0(X_0)}\ e^{\mathrm{tr}\ln f_{m+1}(X_{m+1})}\ \prod_{i=1}^{m} e^{\mathrm{tr}\ln f_i(X_i)}\,.$$

Next, we introduce the matrices $\hat{U}_i$, $\check{U}_i$, for $i=1,\ldots,m+1$, via

$$\hat{U}_i = U_i^\dagger \tilde{U}_i \quad , \quad \check{U}_i = \tilde{U}_i^\dagger U_{i-1}\,. \tag{IX.56}$$

We can express $U_0,\ldots,U_{m+1}$, and $\tilde{U}_1,\ldots,\tilde{U}_{m+1}$, in terms of these matrices and $U_{m+1}$,

$$U_i = U_{m+1}\,\hat{U}_{m+1}\check{U}_{m+1}\hat{U}_m\check{U}_m\ldots\hat{U}_{i+1}\check{U}_{i+1}\,, \tag{IX.57}$$

$$\tilde{U}_i = U_{m+1}\,\hat{U}_{m+1}\check{U}_{m+1}\hat{U}_m\check{U}_m\ldots\hat{U}_{i+1}\check{U}_{i+1}\hat{U}_i\,. \tag{IX.58}$$

With this change of variables, we arrive at

$$\begin{aligned}\mathscr{Z}_{\mathrm{MM}}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^T) \;=\; & \frac{\Delta(X(\vec{\alpha}_{m+1}))\,\Delta(X(\vec{\alpha}_0))}{(N!)^{2m+3}}\prod_{i=0}^{m+1}\int_{\Gamma_i} dX_i\,\Delta(X_i)^2\prod_{i=1}^{m+1}\int_{\mathbb{R}_+^N} dY_i\,\Delta(Y_i)^2\\
& \int dU_{m+1}\prod_{i=1}^{m+1} d\hat{U}_i\prod_{i=1}^{m+1} d\check{U}_i\\
& \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,\mathrm{tr}\left[V_{\vec{a}_i}(X_i)-V_{\vec{a}_{i-1}}(X_i)\right]}\ \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,\mathrm{tr}\left[V_{\vec{a}_{i-1}}(X_{i-1})-V_{\vec{a}_i}(X_{i-1})\right]}\\
& \prod_{i=1}^{m+1} e^{\frac{1}{g_s}\mathrm{tr}X_i \hat{U}_i Y_i \hat{U}_i^\dagger}\ e^{\frac{-1}{g_s}\mathrm{tr}X_{i-1}\check{U}_i^\dagger Y_i \check{U}_i}\ \prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\,\mathrm{tr}\ln X_i}\\
& e^{\mathrm{tr}\ln f_0(X_0)}\ e^{\mathrm{tr}\ln f_{m+1}(X_{m+1})}\ \prod_{i=1}^{m} e^{\mathrm{tr}\ln f_i(X_i)}\,.\end{aligned}$$

Notice that the integral over $U_{m+1}$ decouples, and $\int dU_{m+1} = \mathrm{Vol}(U(N))$.

### 4.3 Itzykson-Zuber integral and Cauchy determinants

The $\hat{U}_i$ and $\check{U}_i$ appear in the form of Itzykson-Zuber integrals [140],

$$I(X,Y) = \int dU\, e^{\mathrm{tr}XUYU^\dagger} = \frac{\det_{p,q}(e^{x_p y_q})}{\Delta(X)\,\Delta(Y)}\,, \tag{IX.59}$$

where $x_p$ and $y_q$ are the eigenvalues of $X$ and $Y$. We thus have

$$\begin{aligned}
\mathscr{Z}_{\rm MM}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^T) \;\propto\; & \frac{\Delta(X(\vec{\alpha}_{m+1}))\,\Delta(X(\vec{\alpha}_0))}{(N!)^{2m+3}} \prod_{i=0}^{m+1}\int_{\Gamma_i} dX_i\,\Delta(X_i)^2 \prod_{i=1}^{m+1}\int_{\mathbb{R}_+^N} dY_i\,\Delta(Y_i)^2 \\
& \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_i}(X_i)-V_{\vec{a}_{i-1}}(X_i)\right]} \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_{i-1}}(X_{i-1})-V_{\vec{a}_i}(X_{i-1})\right]} \\
& \prod_{i=1}^{m+1} I(\frac{1}{g_s}X_i,Y_i)\; I(-\frac{1}{g_s}X_{i-1},Y_i) \prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\,{\rm tr}\ln X_i} \\
& e^{{\rm tr}\ln f_0(X_0)}\; e^{{\rm tr}\ln f_{m+1}(X_{m+1})} \prod_{i=1}^{m} e^{{\rm tr}\ln f_i(X_i)} \\
\propto\; & \frac{\Delta(X(\vec{\alpha}_{m+1}))\,\Delta(X(\vec{\alpha}_0))}{(N!)^{2m+3}} \prod_{i=0}^{m+1}\int_{\Gamma_i} dX_i \prod_{i=1}^{m+1}\int_{\mathbb{R}_+^N} dY_i \\
& \Delta(X_0)\,\Delta(X_{m+1}) \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_i}(X_i)-V_{\vec{a}_{i-1}}(X_i)\right]} \\
& \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_{i-1}}(X_{i-1})-V_{\vec{a}_i}(X_{i-1})\right]} \prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\,{\rm tr}\ln X_i} \\
& \prod_{i=1}^{m+1} \det_{p,q}\!\left(e^{\frac{1}{g_s}(X_i)_p\,(Y_i)_q}\right) \det_{p,q}\!\left(e^{\frac{-1}{g_s}(X_{i-1})_p\,(Y_i)_q}\right) \\
& e^{{\rm tr}\ln f_0(X_0)}\; e^{{\rm tr}\ln f_{m+1}(X_{m+1})} \prod_{i=1}^{m} e^{{\rm tr}\ln f_i(X_i)}\,,
\end{aligned}$$

where we have dropped an overall sign, powers of $g_s$, and the group volume ${\rm Vol}(U(N))$ which are constant prefactors of no interest to us.

Next, we perform the integrals over $Y_i$ along $\mathbb{R}_+^N$.

$$\begin{aligned}
& \int_{\mathbb{R}_+^N} dY \det_{p,q}\!\left(e^{\frac{1}{g_s}(X_i)_p\,(Y)_q}\right) \det_{p,q}\!\left(e^{\frac{-1}{g_s}(X_{i-1})_p\,(Y)_q}\right) \\
=\; & \sum_{\sigma}\sum_{\tilde{\sigma}} (-1)^{\sigma}(-1)^{\tilde{\sigma}} \prod_{p=1}^{N}\int_0^{\infty} dy_p\, e^{\frac{y_p}{g_s}((X_i)_{\sigma(p)}-(X_{i-1})_{\tilde{\sigma}(p)})} \\
=\; & \sum_{\sigma}\sum_{\tilde{\sigma}} (-1)^{\sigma}(-1)^{\tilde{\sigma}} \prod_{p=1}^{N} \frac{g_s}{(X_{i-1})_{\tilde{\sigma}(p)}-(X_i)_{\sigma(p)}} \\
=\; & N!\,g_s^N \det_{p,q}\left(\frac{1}{(X_{i-1})_p-(X_i)_q}\right).
\end{aligned}$$

Note that the integral is only convergent for $(X_i)_{\sigma(p)}-(X_{i-1})_{\tilde{\sigma}(p)}<0$. For $X_i$ that violate this inequality, we will define the integral via its analytic continuation given in the third line.

An application of the Cauchy determinant formula,

$$\det\left(\frac{1}{x_i+y_j}\right)_{1\le i<j\le n}=\frac{\prod_{1\le i<j\le n}(x_j-x_i)(y_j-y_i)}{\prod_{i,j=1}^{n}(x_i+y_j)}\,, \tag{IX.60}$$

yields

$$\int_{\mathbb{R}_+^N} dY \det_{p,q}(e^{\frac{1}{g_s}(X_i)_p\,(Y)_q})\ \det_{p,q}(e^{\frac{-1}{g_s}(X_{i-1})_p\,(Y)_q}) = (-1)^{\binom{N}{2}} N!\, g_s^N\ \frac{\Delta(X_i)\,\Delta(X_{i-1})}{\Delta(X_{i-1},X_i)}\,, \tag{IX.61}$$

where the notation $\Delta(X_{i-1},X_i)$ was introduced in (IX.27). Evaluating the $Y_i$ integrals thus, and continuing to drop overall signs and powers of $g_s$, our matrix integral becomes

$$\begin{aligned}\mathscr{Z}_{\rm MM}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^{\,T}) \;\propto\;& \frac{\Delta(X(\vec{\alpha}_{m+1}))\,\Delta(X(\vec{\alpha}_0))}{(N!)^{m+3}}\ \prod_{i=0}^{m+1}\int_{\Gamma_i} dX_i\,\Delta(X_i)^2\\ & \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,\mathrm{tr}\left[V_{\vec{a}_i}(X_i)-V_{\vec{a}_{i-1}}(X_i)\right]}\ \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,\mathrm{tr}\left[V_{\vec{a}_{i-1}}(X_{i-1})-V_{\vec{a}_i}(X_{i-1})\right]}\\ & \prod_{i=1}^{m+1}\frac{1}{\Delta(X_{i-1},X_i)}\ \prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\,\mathrm{tr}\ln X_i}\\ & e^{\mathrm{tr}\ln f_0(X_0)}\ e^{\mathrm{tr}\ln f_{m+1}(X_{m+1})}\ \prod_{i=1}^{m} e^{\mathrm{tr}\ln f_i(X_i)}\,.\end{aligned}$$

### 4.4 Recovering the sum over partitions

Following the steps introduced in [124] in reverse, we next decompose the diagonal matrix $X_i$ into blocks,

$$X_i=\mathrm{diag}\,(X_{0,i},X_{1,i},\ldots,X_{n,i})\,,$$

where each matrix $X_{j,i}$ is a $d\times d$ diagonal matrix whose eigenvalues are integrated on the contours $\gamma_{j,i}$ surrounding points of the form $q^{a_{j,i}+\mathbb{N}}$. We arrive at

$$\begin{aligned}\mathscr{Z}_{\rm MM}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^{\,T}) \;\propto\;& \frac{\Delta(X(\vec{\alpha}_{m+1}))\,\Delta(X(\vec{\alpha}_0))}{(N!)^{m+3}}\ \prod_{i=0}^{m+1}\prod_{j=0}^{n}\int_{(\gamma_{j,i})^d} dX_{j,i}\\ & \Delta(X_0)\,\Delta(X_{m+1})\ \prod_{i=1}^{m+1}\frac{\Delta(X_{i-1})\Delta(X_i)}{\Delta(X_{i-1},X_i)}\\ & \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,\mathrm{tr}\left[V_{\vec{a}_i}(X_i)-V_{\vec{a}_{i-1}}(X_i)\right]}\ \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,\mathrm{tr}\left[V_{\vec{a}_{i-1}}(X_{i-1})-V_{\vec{a}_i}(X_{i-1})\right]}\end{aligned}$$

$$e^{\mathrm{trln} f_0(X_0)}\, e^{\mathrm{trln} f_{m+1}(X_{m+1})} \prod_{i=1}^{m} e^{\mathrm{trln} f_i(X_i)} \prod_{i=1}^{m} e^{(S_i + \frac{i\pi}{g_s})\,\mathrm{tr}\ln X_i},$$

with

$$\frac{\Delta(X_{i-1})\Delta(X_i)}{\Delta(X_{i-1}, X_i)} = \frac{\prod_j \Delta(X_{j,i-1}) \prod_j \Delta(X_{j,i}) \prod_{j<l} \Delta(X_{j,i-1}, X_{l,i-1}) \prod_{j<l} \Delta(X_{j,i}, X_{l,i})}{\prod_{j,l} \Delta(X_{j,i-1}, X_{l,i})}. \tag{IX.62}$$

Our next step is to evaluate the $dX_{j,i}$ integrals via Cauchy's residue theorem. The poles of the integrands lie at the poles of $f_i$, and the zeros of $\Delta(X_{i-1}, X_i)$. However, we have been careful to define our contours $\gamma_{j,i}$ in a way that only the poles of $f_i$ contribute. These lie at the points $q^{a_{j,i}+\mathbb{N}}$. Hence, the integrals evaluate to a sum of residues over the points

$$(X_{j,i})_l = q^{a_{j,i}+(h_{j,i})_l}, \tag{IX.63}$$

where each $(h_{j,i})_l$ is a positive integer.

Since the integrand contains a Vandermonde of the eigenvalues of $X_{j,i}$, the residues vanish whenever two eigenvalues are at the same pole of $f_i$, i.e. if two $(h_{j,i})_l$ coincide. Moreover, since the integrand is symmetric in the eigenvalues, upon multiplication by $N!$, we can assume that the $(h_{j,i})_l$ are ordered,

$$(h_{j,i})_1 > (h_{j,i})_2 > (h_{j,i})_3 > \cdots > (h_{j,i})_d \geq 0. \tag{IX.64}$$

The $(h_{j,i})_l$ hence encode a partition $\alpha_{j,i}$ via $(h_{j,i})_l = (\alpha_{j,i})_l - i + d$, and we have reduced our integrals to a sum over partitions. In terms of the function $h_l(\alpha)$ introduced in (IX.15),

$$(X_{j,i})_l = q^{h_l(\alpha_{j,i})}, \quad h_l(\alpha_{j,i}) = (h_{j,i})_l + a_{j,i}, \tag{IX.65}$$

$$h_1(\alpha_{j,i}) > h_2(\alpha_{j,i}) > \cdots > h_d(\alpha_{j,i}) \geq a_{j,i}. \tag{IX.66}$$

Notice that unlike $f_i$, $i = 1, \ldots, m$, $f_0$ and $f_{m+1}$ only have a finite number of $N = (n+1)d$ poles. Since the $(h_{j,0})_l$, $(h_{j,m+1})_l$ respectively can be chosen pairwise distinct and ordered, $f_0$ and $f_{m+1}$ act as delta functions in the integrals over the $N \times N$ matrices $X_0$ and $X_{m+1}$, and fix these to the prescribed values $X(\vec{\alpha}_0)$ and $X(\vec{\alpha}_{m+1})$ respectively.

Performing the integrals hence yields

$$\begin{aligned}\mathscr{Z}_{\rm MM}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^T) \;\; &\propto \;\; \Delta(X(\vec{\alpha}_{m+1}))^2\Delta(X(\vec{\alpha}_0))^2 \\ &\sum_{\{\alpha_{j,i}|j=0,\ldots,n;i=1,\ldots,m+1\}} \prod_{i=1}^{m+1}\frac{\Delta(X(\vec{\alpha}_{i-1}))\Delta(X(\vec{\alpha}_i))}{\Delta(X(\vec{\alpha}_{i-1}),X(\vec{\alpha}_i))} \\ &\prod_{i=1}^m e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_i}(X(\vec{\alpha}_i))-V_{\vec{a}_{i-1}}(X(\vec{\alpha}_i))\right]}\;\prod_{i=1}^m e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_{i-1}}(X(\vec{\alpha}_{i-1}))-V_{\vec{a}_i}(X(\vec{\alpha}_{i-1}))\right]} \\ &\prod_{i=1}^m e^{(S_i+\frac{i\pi}{g_s})\,{\rm tr}\ln X(\vec{\alpha}_i)}\prod_{i=0}^{m+1}\prod_{j=0}^n\prod_{l=1}^d\left(\operatorname*{Res}_{q^{h_l(\alpha_{j,i})}} f_i\right).\end{aligned}$$

Notice that

$$\prod_j\prod_l \operatorname*{Res}_{q^{h_l(\alpha_{j,0})}} f_0 = \frac{1}{\Delta(X(\vec{\alpha}_0))^2}\,, \tag{IX.67}$$

$$\prod_j\prod_l \operatorname*{Res}_{q^{h_l(\alpha_{j,m+1})}} f_{m+1} = \frac{1}{\Delta(X(\vec{\alpha}_{m+1}))^2}\,. \tag{IX.68}$$

Furthermore,

$$\operatorname*{Res}_{q^{h_l(\alpha_{j,i})}} f_i = q^{h_l(\alpha_{j,i})}\;\hat{f}_{j,i}\,, \tag{IX.69}$$

where $\hat{f}_{j,i}$ computed in (IX.42) is independent of $h_l(\alpha_{j,i})$. We thus have

$$e^{S_i\,{\rm trln}X(\vec{\alpha}_i)}\prod_{j=0}^n\prod_{l=1}^d\left(\operatorname*{Res}_{q^{h_l(\alpha_{j,i})}} f_i\right) = e^{(S_i+1){\rm trln}X(\vec{\alpha}_i)}\;\prod_{j=0}^n(\hat{f}_{j,i})^d\,. \tag{IX.70}$$

Upon substituting the expression (IX.43) for $S_i$, we finally arrive at

$$\begin{aligned}&\mathscr{Z}_{\rm MM}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^T) \\ \propto \quad &\prod_{i=1}^m\prod_{j=0}^n(\hat{f}_{j,i})^d\sum_{\{\alpha_{j,i}|j=0,\ldots,n;i=1,\ldots,m+1\}} \\ &\prod_{i=1}^{m+1}\frac{\prod_j\Delta(X(\alpha_{j,i-1}))\prod_j\Delta(X(\alpha_{j,i}))\prod_{j<l}\Delta(X(\alpha_{j,i-1}),X(\alpha_{l,i-1}))\;\prod_{j<l}\Delta(X(\alpha_{j,i}),X(\alpha_{l,i}))}{\prod_{j,l}\Delta(X(\alpha_{j,i-1}),X(\alpha_{l,i}))} \\ &\prod_{i=1}^m e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_i}(X(\vec{\alpha}_i))-V_{\vec{a}_{i-1}}(X(\vec{\alpha}_i))\right]}\;\prod_{i=1}^m\prod_{k=0}^n e^{(\frac{1}{2}-\sum_{j\le k}(a_{j,i}-a_{j,i-1})-\frac{i\pi}{g_s})\,{\rm trln}X(\alpha_{k,i})} \\ &\prod_{i=1}^m e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_{i-1}}(X(\vec{\alpha}_{i-1}))-V_{\vec{a}_i}(X(\vec{\alpha}_{i-1}))\right]}\;\prod_{i=1}^m\prod_{k=0}^n e^{(\frac{1}{2}-\sum_{j<k}(a_{j,i}-a_{j,i+1}))\,{\rm trln}X(\alpha_{k,i})}\end{aligned}$$

$$\prod_{i=1}^{m}\prod_{j=0}^{n} e^{s_{j,i}\mathrm{trln}X(\alpha_{j,i})}\,. \tag{IX.71}$$

Comparing to (IX.30) and (IX.35), we conclude

$$\mathscr{Z}_{\mathrm{MM}}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^T) \propto \sum_{\alpha_{j,i},j=0,\ldots,n;i=1,\ldots,m} \prod_{i=1}^{m+1} Z_{\mathrm{strip}}(\vec{\alpha}_i,\vec{\alpha}_{i-1}^T) \prod_{j=0}^{n}\prod_{i=1}^{m} q^{s_{j,i}|\alpha_{j,i}|}\,,$$

i.e.

$$\mathscr{Z}_{\mathrm{MM}}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^T) \propto Z_{\mathrm{vertex}}(\mathfrak{X}_0) = e^{\sum_g g_s^{2g-2} GW_g(\mathfrak{X}_0)}\,. \tag{IX.72}$$

Up to a trivial proportionality constant, we have thus succeeded in rewriting the topological string partition function on the fiducial geometry $\mathfrak{X}_0$ as a *chain of matrices* matrix integral. By our reasoning in section IX, this result extends immediately to arbitrary toric Calabi-Yau 3-folds as follows. We have argued that any such 3-fold can be obtained from a sufficiently large choice of fiducial geometry via flops and limits. The respective partition functions are related via (IX.9). Upon the appropriate variable identification, we hence arrive at a matrix model representation of the topological string on an arbitrary toric Calabi-Yau 3-fold.

## 5 Implications of our result

We have rewritten the topological string partition function as a matrix integral. This allows us to bring the rich theory underlying the structure of matrix models to bear on the study of topological string.

The type of matrix integral we have found to underlie the topological string on toric Calabi-Yau 3-folds is a so-called chain of matrices. This class of models has been studied extensively [19, 141], and many structural results pertaining to it are known.

### 5.1 Loop equations and Virasoro constraints

The loop equations of matrix models provide a set of relations among correlation functions. They are Schwinger-Dyson equations; they follow from the invariance of the matrix integral under a change of integration variables, or by an integration by parts argument.

Loop equations for a general chain of matrices have been much studied in the literature, in particular in [83, 101, 142, 143]. They can be viewed as W-algebra constraints

(a generalization of Virasoro constraints) [144]. Having expressed the topological string partition function as a matrix integral, we can hence conclude that Gromov-Witten invariants satisfy W-algebra constraints.

Moreover, a general formal solution of the loop equations for a chain of matrices matrix model was found in [83], and expressed in terms of so-called symplectic invariants $F_g$ of a spectral curve. The spectral curve for a matrix integral is related to the expectation value of the resolvent of the first matrix in the chain,

$$W(x) = \left\langle \mathrm{tr} \frac{1}{x - M_0} \right\rangle^{(0)} . \tag{IX.73}$$

The superscript $^{(0)}$ indicates that the expectation value is evaluated to planar order in a Feynman graph expansion. The symplectic invariants $F_g(\mathscr{C})$ of an arbitrary spectral curve $\mathscr{C}$ were defined in [23]. [83] proved that for any chain of matrices integral $Z$, one has

$$\ln Z = \sum_g F_g(\mathscr{C}) \tag{IX.74}$$

with $\mathscr{C}$ the spectral curve associated to the matrix integral.

Calculating the spectral curve of a chain of matrices matrix model with complicated potentials poses some technical challenges. We will present the spectral curve for our matrix model (IX.37) in a forthcoming publication [69].

### 5.2 Mirror symmetry and the BKMP conjecture

The mirror $\hat{\mathfrak{X}}$ of a toric Calabi-Yau 3-fold $\mathfrak{X}$ is a conic bundle over $\mathbb{C}^* \times \mathbb{C}^*$. The fiber is singular over a curve, which we will refer to as the mirror curve $\mathscr{S}_{\hat{\mathfrak{X}}}$ of $\hat{\mathfrak{X}}$. It is a plane curve described by an equation

$$\mathscr{S}_{\hat{\mathfrak{X}}} \qquad : \qquad H(e^x, e^y) = 0 , \tag{IX.75}$$

where $H$ is a polynomial whose coefficients follow from the toric data of $\mathfrak{X}$ and the Kähler parameters of the geometry.

Mirror symmetry is the statement that the topological A-model partition function with target space $\mathfrak{X}$ is equal to the topological B-model partition function with target space $\hat{\mathfrak{X}}$.

Extending work of Mariño [127] proposing a relation between the formalism of [23] and open and closed topological string amplitudes, Bouchard, Klemm, Mariño and Pas-

quetti (BKMP) conjecture in [81] that

$$GW_g(\mathfrak{X}) \stackrel{?}{=} F_g(\mathscr{S}_{\hat{\mathfrak{X}}})\,. \tag{IX.76}$$

Here, the $F_g$'s are the symplectic invariants introduced in [23]. The main interest of this conjecture is that it provides a systematic method for computing the topological string partition function, genus by genus, away from the large radius limit, and without having to solve differential equations.

This conjecture was motivated by the fact that symplectic invariants have many intriguing properties reminiscent of the topological string free energies. They are invariant under transformations $\mathscr{S} \to \tilde{\mathscr{S}}$ which conserve the symplectic form $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$, whence their name [23]. They satisfy holomorphic anomaly equations [145], they have an integrable structure similar to Givental's formulae [146–150], they satisfy some special geometry relations, WDVV relations [151], and they give the Witten-Kontsevich theory as a special case [23, 107].

BKMP succesfully checked their claim for various examples to low genus.

The conjecture was proved for arbitrary genus in [85] for $\mathfrak{X}$ a Hirzebruch rank 2 bundle over $\mathbb{P}^1$ (this includes the conifold). Marshakov and Nekrasov [123] proved $F_0 = GW_0$ for the family of $SU(n)$ Seiberg-Witten models. Klemm and Sulkowski [124], generalizing [85] to Nekrasov's sums over partitions for $SU(n)$ Seiberg-Witten gauge theories, proved the relation for $F_0$, building on work in [152]. In fact, it appears straightforward to extend their computation to arbitrary genus $F_g$. In [153], Sulkowski provided a matrix model realization of $SU(n)$ gauge theory with a massive adjoint hypermultiplet, again using a generalization of [85] for more general sums over partitions. Bouchard and Mariño [58] noticed that an infinite framing limit of the BKMP conjecture for the framed vertex $\mathfrak{X} = \mathbb{C}^3$ implies another conjecture for the computation of Hurwitz numbers, namely that the Hurwitz numbers of genus $g$ are the symplectic invariants of genus $g$ for the Lambert spectral curve $e^x = y e^{-y}$. That conjecture was proved recently by another generalization of [85] using a matrix model for summing over partitions [57], and also by a direct cut and join combinatorial method [155]. The BKMP conjecture was also proved for the framed vertex $\mathfrak{X} = \mathbb{C}^3$ in [80, 82], using the ELSV formula and a cut and join combinatorial approach.

Since we have demonstrated that the topological string partition function is reproduced by a matrix model, we can conclude that the Gromov-Witten invariants coincide

with the symplectic invariants

$$\sum_g g_s^{2g-2} GW_g = \sum_G F_g(\mathscr{C}), \tag{IX.77}$$

with $\mathscr{C}$ the spectral curve of our matrix model. We will compute $\mathscr{C}$ explicitly in a forthcoming work [69], and demonstrate that it indeed coincides, up to symplectic transformations, with the mirror curve $\mathscr{S}_{\hat{\mathfrak{x}}}$, thus proving the BKMP conjecture for arbitrary toric Calabi-Yau 3-folds, in the large radius limit.

### 5.3 Simplifying the matrix model

The matrix models associated to the conifold or to geometries underlying Seiberg-Witten theory have a remarkable property: the spectral curve is the same (perturbatively and up to symplectic transformations) as the one of a simpler matrix model with all $g$-functions replaced by only the leading term in their small $\ln q$ expansion. We will demonstrate in a forthcoming work [69] that this property also holds for our matrix integral (IX.37). We can hence simplify the potentials of our matrix model, arriving at

$$\begin{aligned}\mathscr{Z}_{\rm simp}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^T) &= \Delta(X(\vec{\alpha}_{m+1}))\,\Delta(X(\vec{\alpha}_0))\prod_{i=0}^{m+1}\int_{H_{\bar{n}_i}(\Gamma_i)} dM_i \prod_{i=1}^{m+1}\int_{H_{\bar{n}}(\mathbb{R}_+)} dR_i \\ &\prod_{i=1}^{m} e^{\frac{1}{g_s}\,{\rm tr}\sum_{j=0}^{n}(Li_2(q^{a_{j,i}}/M_i)-Li_2(q^{a_{j,i-1}}/M_i))} \\ &\prod_{i=0}^{m-1} e^{\frac{1}{g_s}\,{\rm tr}\sum_{j=0}^{n}(Li_2(q^{a_{j,i}}/M_i)-Li_2(q^{a_{j,i+1}}/M_i))} \\ &\prod_{i=1}^{m+1} e^{\frac{1}{g_s}{\rm tr}(M_i-M_{i-1})R_i}\ \prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\,{\rm tr}\ln M_i},\end{aligned}$$

where the matrix $M_i$ is of size $\bar{n}_i = \sum_j \bar{n}_{j,i}$.

#### Classical limit

In the classical limit, the dilogarithm $\mathrm{Li}_2$ becomes the function $x\ln x$, and we have

$$\begin{aligned}\mathscr{Z}_{\rm eff.\,cl}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^T) &= \Delta(X(\vec{\alpha}_{m+1}))\,\Delta(X(\vec{\alpha}_0))\prod_{i=0}^{m+1}\int_{H_{\bar{n}_i}(\Gamma_i)} dM_i \prod_{i=1}^{m+1}\int_{H_{\bar{n}}(\mathbb{R}_+)} dR_i \\ &\prod_{i=1}^{m} e^{\frac{1}{g_s}\,{\rm tr}\sum_{j=0}^{n}(M_i-a_{j,i})\ln(a_{j,i}-M_i)-(M_i-a_{j,i-1})\ln(a_{j,i-1}-M_i)}\end{aligned}$$

$$\prod_{i=0}^{m-1} e^{\frac{1}{g_s}\operatorname{tr}\sum_{j=0}^{n}(M_i-a_{j,i})\ln(a_{j,i}-M_i)-(M_i-a_{j,i+1})\ln(a_{j,i+1}-M_i)}$$
$$\prod_{i=1}^{m+1} e^{\frac{1}{g_s}\operatorname{tr}(M_i-M_{i-1})R_i} \prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\operatorname{tr}\ln M_i}.$$

This model shares features with the Eguchi-Yang matrix model [156], see also [123].

## 6 Conclusion

We have rewritten the topological vertex formula for the partition function of the topological A-model as a matrix integral.

Having expressed the topological string in terms of a matrix model, we can bring the immense matrix model toolkit which has been developed since the introduction of random matrices by Wigner in 1951 to bear on questions concerning the topological string and Gromov-Witten invariants. We already started down this path in section IX above. Going further, we can apply the method of bi-orthogonal polynomials [19] to our matrix model to unearth the integrable system structure (Miwa-Jimbo [38, 157]) underlying the topological string, at least in the case of toric targets, together with its Lax pair, its Hirota equations (which arise as orthogonality relations), etc. In a related vein, free fermions [158, 159] arise in the theory of matrix models when invoking determinantal formulae to express the matrix model measure [160]. It will be very interesting to explore how this is related to the occurrence of free fermions in topological string theory, as studied in [102, 161–163]. More generally, one should study what can be learned about the non-perturbative topological string from its perturbative reformulation as a matrix model, as in the works [71, 164, 166, 167]. A recurrent such question, which could be addressed in the matrix model framework (in fact, it was already latently present in the calculations in this work), is that of the quantization of Kähler parameters.

On a different note, notice that the matrix model derived in this article, with a potential which is a sum of logs of $q$-deformed $\Gamma$ functions, looks very similar to the matrix model counting plane partitions introduced in [84]. This is a hint that it could be possible to recover the topological vertex formula, corresponding to the topological string with target $\mathbb{C}^3$ and appropriate boundary conditions, directly from the matrix model approach. Either along these lines or the lines pursued in this paper, it would be interesting to derive a matrix model related to the Nekrasov deformation [121, 168] of the topological string.

A completely open question is whether the close relation between topological strings and matrix models persists beyond toric target spaces, and more ambitiously yet, whether

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there exists a general notion of geometry underlying matrix models.

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**Appendix: q-product**

The $g$-function, which plays a central role in the definition of our matrix model, is defined as an infinite product,

$$g(x) = \prod_{n=1}^{\infty} (1 - \frac{1}{x} q^n) \,. \tag{IX.78}$$

It is the quantum Pochhammer symbol $g(x) = [q/x;q]_\infty$, and it is related to the $q$-deformed gamma function via $\Gamma_q(x) = (1-q)^{1-x} g(1)/g(q^{1-x})$.

The RHS is convergent for $|q| < 1$ and arbitrary complex $x \neq 0$. $g(x)$ satisfies the functional equation

$$g(qx) = (1 - \frac{1}{x})\, g(x) \,. \tag{IX.79}$$

For $n \in \mathbb{N}$, we have

$$g(q^n) = 0 \tag{IX.80}$$

and

$$g'(q^n) = (-1)^{n-1} g(1)\, q^{-\frac{n(n+1)}{4}} \prod_{m=1}^{n-1} [m] = g(1)\, q^{-\frac{n(n+1)}{2}}\, [n-1]! = (-1)^{n-1} q^{-\frac{n(n+1)}{2}} \frac{g(1)^2}{g(q^{1-n})} \,. \tag{IX.81}$$

Via the triple product representation of the theta function,

$$\theta(z;\tau) = \prod_{m=1}^{\infty}(1-e^{2\pi i m\tau})(1+e^{(2m-1)\pi i\tau+2\pi iz})(1+e^{(2m-1)\pi i\tau-2\pi iz}), \tag{IX.82}$$

we obtain the identity

$$\theta\left(\frac{1}{2}+\frac{1}{4\pi i}\ln\frac{q}{x^2};\ \frac{\ln q}{2\pi i}\right) = g(x)g(\frac{q}{x})g(1). \tag{IX.83}$$

We have

$$\frac{g(x)\,g(q/x)}{g(1)^2\,\sqrt{x}}\,e^{\frac{(\ln x)^2}{2\ln q}}\,e^{\frac{-i\pi\ln x}{\ln q}} = \frac{-\ln q}{\theta'(\frac{1}{2}-\frac{i\pi}{\ln q},-\frac{2i\pi}{\ln q})}\;\theta\left(\frac{\ln x}{\ln q}+\frac{1}{2}-\frac{i\pi}{\ln q},\frac{-2i\pi}{\ln q}\right) \tag{IX.84}$$

where $\theta$ is the Riemann theta-function for the torus of modulus $-2i\pi/\ln q$.

At small $\ln q$, the following expansion is valid,

$$\ln g(x) = \frac{1}{\ln q}\sum_{n=0}^{\infty}\frac{(-1)^n B_n}{n!}(\ln q)^n\,\mathrm{Li}_{2-\mathrm{n}}(1/\mathrm{x}), \tag{IX.85}$$

where we have used the definition of the Bernoulli numbers $B_n$ as the coefficients in the expansion of $t/(e^t-1)$,

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty}B_n\frac{t^n}{n!}. \tag{IX.86}$$

$\mathrm{Li}_{\mathrm{n}}$ is the polylogarithm function, defined as

$$\mathrm{Li}_{\mathrm{n}}(\mathrm{x}) = \sum_{\mathrm{k}=1}^{\infty}\frac{\mathrm{x}^{\mathrm{k}}}{\mathrm{k}^{\mathrm{n}}}. \tag{IX.87}$$

This is a generalization of the logarithm function, recovered at $n=1$,

$$\mathrm{Li}_1(\mathrm{x}) = -\ln(1-\mathrm{x}). \tag{IX.88}$$

It satisfies the functional relation

$$\mathrm{Li}'_{\mathrm{n}}(\mathrm{x}) = \frac{1}{\mathrm{x}}\mathrm{Li}_{\mathrm{n}-1}(\mathrm{x}). \tag{IX.89}$$

Note in particular that this implies that $\mathrm{Li_n}$ is an algebraic function of $x$ for $n \leq 0$. E.g.,

$$\mathrm{Li_0(x)} = \frac{\mathrm{x}}{1-\mathrm{x}}\,. \tag{IX.90}$$

We also define the function

$$\psi_q(x) = x\frac{g'(x)}{g(x)}\,. \tag{IX.91}$$

Using the functional equation (IX.89) of the polylogarithm, we find its small $\ln(q)$ expansion

$$\begin{aligned}\psi_q(x) &= -\frac{1}{\ln q}\sum_{n=0}^{\infty}\frac{(-1)^n B_n}{n!}(\ln q)^n\,\mathrm{Li_{1-n}(1/x)} \\ &= \frac{1}{\ln q}\left[\ln(1-\frac{1}{x}) - \frac{\ln q}{2(x-1)} - \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}\,(\ln q)^{2n}\,\mathrm{Li_{1-2n}(x)}\right].\end{aligned}$$

For the second equality, we have used $B_0 = 1, B_1 = -\frac{1}{2}$, and $B_{2n+1} = 0$ for $n > 1$.

We have near $x \to \infty$

$$\psi_q(x) \sim \frac{q}{1-q}\,\frac{1}{x} + O(x^{-2}) \tag{IX.92}$$

and near $x \to 0$:

$$\psi(x) \sim \frac{1}{2} + \frac{i\pi + \ln x}{g_s} + O(x). \tag{IX.93}$$

**Annexe X**

# A matrix model for the topological string II : The spectral curve and mirror geometry

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In a previous paper, we presented a matrix model reproducing the topological string partition function on an arbitrary given toric Calabi-Yau manifold. Here, we study the spectral curve of our matrix model and thus derive, upon imposing certain minimality assumptions on the spectral curve, the large volume limit of the BKMP "remodeling the B-model" conjecture, the claim that Gromov-Witten invariants of any toric Calabi-Yau 3-fold coincide with the spectral invariants of its mirror curve.

## 1 Introduction

In a previous paper [68], we presented a matrix model that computes the topological string partition function at large radius on an arbitrary toric Calabi-Yau manifold $\mathfrak{X}$. The goal of this paper is to determine the corresponding spectral curve $\mathscr{S}$.

That the partition function of a matrix model can be recovered to all genus from its spectral curve was first demonstrated in [105]. [23] pushed this formalism further, showing that symplectic invariants $F_g(\mathscr{S})$ can be defined for any analytic affine curve $\mathscr{S}$, with no reference to an underlying matrix model. These invariants coincide with the partition function of a matrix model when $\mathscr{S}$ is chosen as the associated spectral curve. The symplectic invariants $F_g$ satisfy many properties reminiscent of the topological string partition function [107, 145, 150, 151], motivating Bouchard, Klemm, Mariño, and Pasquetti (BKMP) [81], building on work of Mariño [127], to conjecture that $F_g(\mathscr{S})$ in fact coincides with the topological string partition function on the toric Calabi-Yau manifold with mirror curve $\mathscr{S}$. BKMP successfully checked their claim for various examples, at least to low genus. The conjecture was subsequently proved in numerous special cases [80, 82, 85, 123, 124, 153].

Bouchard and Mariño [154] noticed that an infinite framing limit of the BKMP conjecture for the framed vertex, $\mathfrak{X} = \mathbb{C}^3$, implies a conjecture for the computation of Hurwitz numbers, namely that the Hurwitz numbers of genus $g$ are the symplectic invariants of genus $g$ for the Lambert spectral curve $e^x = y e^{-y}$. This conjecture was proved recently by a generalization of [85] using a matrix model for summing over partitions [57], and also by a direct combinatorial method [155]. Matrix models and the BKMP conjecture related to toric Calabi-Yau geometries arising from the triangulation of a strip were recently studied in [169].

In this paper, we derive the large radius limit of the BKMP conjecture for *arbitrary* toric Calabi-Yau manifolds, but with one caveat: to determine the spectral curve of our matrix model, we must make several minimality assumptions along the way. To elevate our results to a rigorous proof of the BKMP conjecture, one needs to establish a uniqueness result underlying our prescription for finding the spectral curve to justify these minimal choices. Such a uniqueness result does not exist to date.

Recall that in [68], we first compute the topological string partition function on a toric Calabi-Yau geometry $\mathfrak{X}_0$ which we refer to as fiducial. We then present a matrix model which reproduces this partition function. Flops and limits in the Kähler cone relate $\mathfrak{X}_0$ to an arbitrary toric Calabi-Yau 3-fold. As we can follow the action of these operations on the partition function, we thus arrive at a matrix model for the topological string on any toric Calabi-Yau 3-folds. Here, we follow the analogous strategy, by first computing the spectral curve of the matrix model associated to $\mathfrak{X}_0$, and then studying

the action of flops and limits on this curve.

The plan of the paper is as follows. In section X, we introduce the fiducial geometry $\mathfrak{X}_0$ and its mirror. The matrix model reproducing the partition function on $\mathfrak{X}_0$, as derived in [68], is a chain of matrices matrix model. It is summarized in section X and appendix X. We review general aspects of this class of matrix models and their solutions in section X. In section X, we determine a spectral curve which satisfies all specifications outlined in section X, and demonstrate that it coincides, up to symplectic transformations, with the B-model mirror of the fiducial geometry. While in our experience with simpler models, the conditions of section X on the spectral curve specify it uniquely, we lack a proof of this uniqueness property. We thus provide additional consistency arguments for our proposal for the spectral curve in section X. Flops and limits in the Kähler cone relate the fiducial to an arbitrary toric Calabi-Yau manifold. Following the action of these operations on both sides of the conjecture in section X completes the argument yielding the BKMP conjecture for arbitrary toric Calabi-Yau manifolds in the large radius limit. We conclude by discussing possible avenues along this work can be extended.

## 2 The fiducial geometry and its mirror

### 2.1 The fiducial geometry

In [68], we derived a matrix model reproducing the topological string partition function on the toric Calabi-Yau geometry $\mathfrak{X}_0$ whose toric fan is depicted in figure X.1. We refer to $\mathfrak{X}_0$ as our fiducial geometry; we will obtain the partition function on an arbitrary toric Calabi-Yau manifolds by considering flops and limits of $\mathfrak{X}_0$.

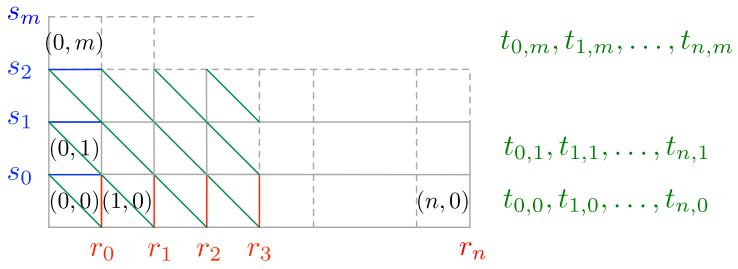


Figure X.1: Fiducial geometry $\mathfrak{X}_0$ with boxes numbered and choice of basis of $H_2(\mathfrak{X}_0, \mathbb{Z})$.

We have indicated a basis of $H_2(\mathfrak{X}_0, \mathbb{Z})$ in figure X.1. Applying the labeling scheme introduced in figure X.2, the curve classes of our geometry are expressed in this basis as

follows,

$$\begin{aligned} r_{i,j} &= r_i + \sum_{k=1}^{j} (t_{i+1,k-1} - t_{i,k}) \\ s_{i,j} &= s_j + \sum_{k=1}^{i} (t_{k-1,j+1} - t_{k,j}) \,. \end{aligned}$$

It proves convenient to express these classes as differences of what we will refer to as $a$-parameters [68], defined via

$$t_{i,j} = a_{i,j} - a_{i,j+1} \quad , \qquad r_{i,j} = a_{i,j+1} - a_{i+1,j} \,. \tag{X.1}$$

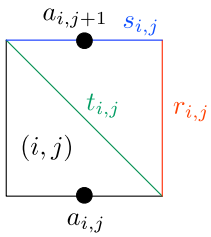


Figure X.2: Labeling curve classes, and introducing $a$-parameters.

## 2.2 The mirror of the fiducial geometry

The Hori-Vafa prescription [170] allows us to assign a mirror curve to a toric Calabi-Yau manifold. Each torically invariant divisor, corresponding to a 1-cone $\rho \in \Sigma(1)$, is mapped to a $\mathbb{C}^*$ variable $e^{-Y_\rho}$. These are constrained by the equation

$$\sum_{\rho \in \Sigma(1)} e^{-Y_\rho} = 0 \,. \tag{X.2}$$

Relations between the 1-cones, as captured by the lattice $\Lambda_h$ introduced in section (2.1) of [68], map to relations between these variables: for $\sigma \in \Sigma(2)$,

$$\sum_{\rho \in \Sigma(1)} \lambda_\rho(\sigma) Y_\rho = W_\sigma \,. \tag{X.3}$$

The $W_\sigma$ are complex structure parameters of the mirror geometry, related to the Kähler parameters $w_\sigma = r_{i,j}, s_{i,j}, \ldots$ introduced in the previous subsection via the mirror map, as we will explain in the next subsection.

The Hori-Vafa prescription gives rise to the following mirror curve $\mathscr{C}_{\mathfrak{X}_0}$ of our fiducial geometry $\mathfrak{X}_0$,

$$\sum_{i=0}^{n+1}\sum_{j=0}^{m+1} x_{i,j} = 0\,. \tag{X.4}$$

We have here labeled the 1-cones by coordinates $(i,j)$, beginning with $(0,0)$ for the cone $(0,0,1)$ in the bottom left corner of box $(0,0)$ as labeled in figure X.1, and introduced the notation

$$x_{i,j} = e^{-Y_{i,j}}\,. \tag{X.5}$$

Eliminating dependent variables by invoking (X.3) yields an equation of the form

$$\sum_{i=0}^{n+1}\sum_{j=0}^{m+1} c_{i,j} z_{i,j} = 0\,. \tag{X.6}$$

Here,

$$z_{i,j} = x_0^{1-i-j} x_1^i x_2^j\,,$$

where we have defined

$$x_0 = x_{0,0}\,,\; x_1 = x_{1,0}\,,\; x_2 = x_{0,1}\,. \tag{X.7}$$

$(x_0 : x_1 : x_2)$ define homogeneous coordinates on $\mathbb{CP}^2$. The form of the equation is independent of the choice of triangulation of the toric diagram. What does depend on this choice are the coefficients $c_{i,j}$. It is not hard to write these down for the fiducial geometry $\mathfrak{X}_0$ with the choice of basis for $H_2(\mathfrak{X}_0,\mathbb{Z})$ indicated in figure X.1. Explicitly, the relations between the coordinates of the mirror curve (X.4) are

$$x_{i,0} = \frac{x_{i-1,0}x_{i-1,1}}{x_{i-2,1}} e^{R_{i-2}}\,, \quad x_{0,j} = \frac{x_{0,j-1}x_{1,j-1}}{x_{1,j-2}} e^{S_{j-2}}\,, \quad x_{i,j} = \frac{x_{i-1,j}x_{i,j-1}}{x_{i-1,j-1}} e^{T_{i-1,j-1}}\,. \tag{X.8}$$

Solving in terms of $x_0, x_1, x_2$ yields the coefficients $c_{0,0} = c_{0,1} = c_{1,0} = 1$,

$$c_{i,0} = \exp\left[\sum_{k=1}^{i-1}(i-k)(R_{k-1} + T_{k-1,0})\right]\,,$$
$$c_{0,j} = \exp\left[\sum_{l=1}^{j-1}(j-l)(S_{l-1} + T_{0,l-1})\right]\,,$$

and for $i, j > 0$

$$c_{i,j} = \exp\left[(i+j-1)T_{0,0} + \sum_{k=1}^{i-1}(i-k)(R_{k-1}+T_{k,0}) + \sum_{l=1}^{j-1}(j-l)(S_{l-1}+T_{0,l}) + \sum_{k=1}^{i-1}\sum_{l=1}^{j-1}T_{k,l}\right].$$

Note that the number of coefficients $c_{i,j}$, up to an overall rescaling, is equal to the number of independent curve classes $r_i$, $s_j$, $t_{i,j}$.

In [171], the thickening prescription was put forth for determining the genus and number of punctures of the mirror curve: one is to thicken the web diagram of the original geometry to obtain the Riemann surface of the mirror geometry. The procedure is illustrated in figure X.3. We will now verify this procedure by studying the curve (X.6) explicitly.

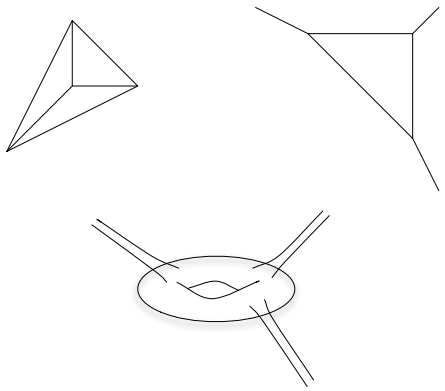

Figure X.3: Example of the thickening prescription: depicted are the fan for $\mathscr{O}(-3) \to \mathbb{P}^2$, the corresponding web diagram, and the mirror curve obtained via the thickening prescription.

Let's consider the curve (X.6) for a single strip (i.e. $n = 0$) of length $m+1$,

$$x_0^{m+2} + x_0^{m+1}x_1 + x_0^{m+1}x_2 + c_{1,1}x_0^m x_1 x_2 + c_{2,0}x_0^m x_1^2 + c_{2,1}x_0^{m-1}x_1^2 x_2 + c_{3,0}x_0^{m-1}x_1^3 + \ldots \\ + c_{m+1,0}x_0 x_1^{m+1} + c_{m+1,1}x_1^{m+1}x_2 = 0\,. \quad \text{(X.9)}$$

Note that the equation is of degree $m+2$, but the point $(0:0:1)$ is an $m+1$-tuple point. By choosing the coefficients to be generic, we can arrange for this singular point to be ordinary. The genus formula then yields

$$\mathfrak{g} = \frac{(d-1)(d-2)}{2} - \frac{m(m+1)}{2} = 0\,.$$

In terms of the physical variables $Y_i$, any point on the curve with a vanishing homogeneous coordinate corresponds to a puncture. The punctures on the curve (X.9) thus lie at

$$\begin{aligned}(0:0:1) &: \quad m+1\\ (0:1:0) &: \quad 1\\ (1:x_1^i:0) &: \quad m+1\\ (1:0:-1) &: \quad 1\,,\end{aligned}$$

where $x_1^i$, $i=1,\ldots,m+1$, are the solutions of the equation

$$1+x_1+\sum_{j=1}^{m} d_i x_1^{i+1} \;=\; 0\,.$$

Note that we reproduce the $2m+4$ punctures expected from the thickening prescription of the toric diagram.

For the general case parametrized by $(m,n)$, the degree of the curve is $d=m+n+2$, and we have an ordinary $m+1$-tuple point at $(0:0:1)$ and an ordinary $n+1$-tuple point at $(0:1:0)$. The genus formula now yields

$$g=\frac{(m+n)(m+n+1)}{2}-\frac{m(m+1)}{2}-\frac{n(n+1)}{2}=mn\,.$$

The punctures lie at

$$\begin{aligned}(0:0:1) &: \quad m+1\\ (0:1:0) &: \quad n+1\\ (1:x_1^i:0) &: \quad m+1\\ (1:0:x_2^j) &: \quad n+1\,,\end{aligned}$$

with $x_1^i$ the roots of $\sum_{i=0}^{m+1} c_{i,0}x_1^i=0$ and $x_2^j$ the roots of $\sum_{j=0}^{n+1} c_{0,j}x_2^j=0$. Again, we see that we reproduce the thickening prescription.

### 2.3 The mirror map

Above, we have distinguished between Kähler ($A$-model) parameters $w_\sigma$ and complex structure ($B$-model) parameters $W_\sigma$. At large radius/complex structure, these are identified between mirror pairs, but this identification is corrected by the so-called mirror map,[1]

$$W_\sigma = w_\sigma + \mathcal{O}(e^{-w_\sigma}) \,. \tag{X.10}$$

The exponentials of the parameters $W_\sigma$ appear as coefficients in the equation defining the mirror curve. They are global coordinates on the complex structure moduli space of the mirror curve. To compare expressions obtained in the A-model to those obtained in the B-model, all expressions are conventionally expressed in terms of flat coordinates $w_\sigma$. On the A-model side, these coordinates enter (in exponentiated form denoted generically as $Q_{\alpha,\beta}$ below) in the definition of the topological vertex. On the B-model side, they arise as the appropriate periods of a meromorphic one-form $\lambda$, defined in terms of the affine variables $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$ in the patch $x_0 \neq 0$ of the curve (X.6) as

$$\lambda = \log y \frac{dx}{x} \,. \tag{X.11}$$

By calculating these periods as a function of the coefficients defining the mirror curve, we obtain the mirror map (X.10).

The coordinates $w_\sigma$ are not globally defined functions on the complex structure moduli space. In the slightly clearer compact setting, this is due to the fact that the symplectic basis $\{\alpha_A, \beta^A\}$ of $H^3(\mathfrak{X}, \mathbb{Z})$ in which we expand $\Omega$ (the compact analogue of the meromorphic 1-form $\lambda$ introduced above) such that the coefficients of $\alpha_A$ furnish our (local) coordinate system of the complex structure moduli space, undergo monodromy when transported around a singularity in moduli space.[2] A good choice of coordinates in the vicinity of a singular divisor $D$ hence involves a choice of basis forms that are invariant

[1] One could take exception to this nomenclature, arguing that the parameters $W_\sigma$ are the geometric parameters on both sides of the mirror, and refer to the $w_\sigma$ as the instanton or quantum corrected parameters. In such conventions, the curve classes in the various toric diagrams should be labeled by upper case letters.

[2] Note that the symplectic basis makes no reference to complex structure, one might hence be led to believe that a global choice (i.e. one valid for any choice of complex structure) should be possible. This is not so. We consider the family $\pi : \mathscr{X} \to \mathscr{S}$, with $\mathscr{S}$ the complex structure moduli space. The fiber over each point $w \in \mathscr{S}$, $\pi^{-1} w = X_w$, is the Calabi-Yau manifold with the respective complex structure. $H^n(X_w, \mathbb{C})$ fit together to form a vector bundle $\mathscr{F}_0$ over $\mathscr{S}$, with a canonical flat connection, the Gauss-Manin connection. Using this connection, we can parallel transport a symplectic basis of $H^3(X_w, \mathbb{C})$ along a curve in $\mathscr{S}$. As $\mathscr{S}$ is not generically simply connected (due to the existence of degeneration points of the geometry), this transport may exhibit monodromy. Note that $\Omega$ can be defined as the section of a sheaf in the Hodge filtration of $H^3$ which extends to the singular divisor, hence is single valued. The monodromy in our choice of flat coordinates is therefore entirely due to the choice of symplectic basis.

under monodromy around that divisor.

## 3 Our matrix model

We derived a chain of matrices matrix model that reproduces the topological string partition function on $\mathfrak{X}_0$ in [68]. For $\mathfrak{X}_0$ of size $(n+1)\times(m+1)$, as depicted in figure X.1, it is given by

$$\begin{aligned} Z_{\rm MM}(\vec{Q},g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^T) &= \Delta(X(\vec{\alpha}_{m+1}))\,\Delta(X(\vec{\alpha}_0))\prod_{i=0}^{m+1}\int_{H_N(\Gamma_i)}dM_i\prod_{i=1}^{m+1}\int_{H_N(\mathbb{R}_+)}dR_i \\ &\prod_{i=1}^{m}e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_i}(M_i)-V_{\vec{a}_{i-1}}(M_i)\right]}\ \prod_{i=1}^{m}e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_{i-1}}(M_{i-1})-V_{\vec{a}_i}(M_{i-1})\right]} \\ &\prod_{i=1}^{m+1}e^{\frac{1}{g_s}{\rm tr}(M_i-M_{i-1})R_i}\ \prod_{i=1}^{m}e^{(S_i+\frac{i\pi}{g_s}){\rm tr}\ln M_i} \\ &e^{{\rm trln}f_0(M_0)}\ e^{{\rm trln}f_{m+1}(M_{m+1})}\prod_{i=1}^{m}e^{{\rm trln}f_i(M_i)}\,. \end{aligned} \tag{X.12}$$

We give the explicit expressions for the various functions entering in this definition in appendix X. Here, we briefly explain some of its general features.

The matrix model (X.12) is designed to reproduce the topological string partition function on the toric Calabi-Yau manifold $\mathfrak{X}_0$ as computed using the topological vertex [76]. Recall that in this formalism, the dual web diagram to the toric diagram underlying the geometry is decomposed into trivalent vertices. Each such vertex contributes a factor $C(\alpha_i,\alpha_j,\alpha_k)$ [76], where $\alpha_i$ denote Young tableaux (partitions) of arbitrary size, one associated to each leg of the vertex. Legs of different vertices are glued by matching these Young tableaux and summing over them with appropriate weight.

Aside from the coupling constant $g_s$ and Kähler parameters of the geometry, denoted collectively as $\vec{Q}$, the matrix model (X.12) depends on partitions $\vec{\alpha}_0$, $\vec{\alpha}_{m+1}$ associated to the outer legs of the web diagram, which we choose to be trivial in this paper. The two classes of integrals $dR_i$ and $dM_i$ correspond to the two steps in which the topological string partition function on the fiducial geometry $\mathfrak{X}_0$ can be evaluated: First, the geometry can be decomposed into $m+1$ horizontal strips, with partitions $\alpha_{j,i+1}$ and $\alpha_{j,i}$ associated to the upper and lower outer legs of the associated strip web diagram. $j=0,\ldots,n$ counts the boxes in figure X.1 in the horizontal direction, $i=0,\ldots,m+1$ is essentially the strip index. Each such strip has a $dR_i$ integration associated to it. The partition function on such strips was calculated in [128]. Following [124], we introduce two matrices $M_i$, $M_{i+1}$ per strip. Their eigenvalues encode the partitions $\alpha_{j,i}$ and $\alpha_{j,i+1}$

for all $j$. To work with finite size matrices, we introduce a cut-off $d$ on the number of rows of the Young tableaux we sum over. As we argue in section X, our matrix model depends on $d$ only non-perturbatively. The strip partition function is essentially given by the Cauchy determinant of the two matrices $M_i$, $M_{i+1}$ [68], and the $dR_i$ integrals are the associated Laplace transforms. Gluing the strips together involves summing over the partitions $\alpha_{j,i}$. This step is implemented by the $dM_i$ integrations. To obtain a discrete sum over partitions from integration, we introduce functions $f_i(M_i)$ with integrally spaced poles. Integrating $M_i$ along appropriate contours then yields the sum over partitions as a sum over residues, the potentials $V_{\vec{a}_i}$ chosen to provide the proper weight per partition.

## 4 Generalities on solving matrix models

### 4.1 Introduction to the topological expansion of chain of matrices

Chain of matrices matrix models have been extensively studied (see Mehta's book [19] and the review article [101]), and the computation of their topological expansion was performed recently in [83, 143].

The solution provided in [83] is based on the computation of the spectral curve $\mathscr{S}_{\rm MM}$ of the matrix model. In [83, 143], only the case of potentials whose derivatives are rational functions is considered, and similarly to the one matrix model, the planar[3] expectation value of the resolvent of the first matrix of the chain is shown to satisfy an algebraic equation. The spectral curve is defined to be the solution locus of this equation. A general recipe is provided in [83, 143] to obtain the spectral curve from algebraic equations and analyticity properties related to rational potentials and integration contours. Here, our potentials contain logs of $g$-functions. As they are not rational, we will have to present a slight extension of the recipe of [83] in section X. This extension from rational potentials to analytical potentials, although not published, is straightforward, and the derivation of these results will appear soon. In some sense, the derivative of $\ln g(x)$ can be viewed as a rational function with an infinite number of simple poles, i.e. as a limit of a rational function. More precisely, as an expansion in powers of $q$, to each order, it is a rational function. Since the spectral curve can be described by local properties,

[3]For matrix models with $N$-independent polynomial potentials whose $g_s$ dependence is given by an overall prefactor, the planar limit coincides with the large $N$ limit, but this correspondence can fail if the potential or the integration contours have a non-trivial $N$ or $g_s$ dependence. The planar limit is defined by keeping only planar graphs in the Feynman graph perturbative expansion around an extremum of the potential. However, it is helpful to have in mind the intuitive picture that the planar limit is similar to a large $N$ limit.

independent of the number of poles, one can take the limit of the recipe of [83, 143]. This is what we shall do in section X below.

Having found the spectral curve $\mathscr{S}_{\rm MM}$ of the matrix model, we will compute its symplectic invariants

$$F_g(\mathscr{S}_{\rm MM}), \quad g = 0,1,2,3,\dots$$

Symplectic invariants $F_g(\mathscr{S})$ can be computed for any analytical plane curve $\mathscr{S}$, and thus in particular for $\mathscr{S} = \mathscr{S}_{\rm MM}$. For a general $\mathscr{S}$ they were first introduced in [23], as a generalization of the solution of matrix models loop equations of [105]. Their definition is algebraic and involves computation of residues at branch points of $\mathscr{S}$. We recall the definition below in section X.

### 3.2 Definition of the general chain of matrices

We consider chain of matrices matrix models of the form

$$Z = \int_{\mathscr{E}} dM_1 \dots dM_L \, e^{-\frac{1}{g_s}\mathrm{Tr}\sum_{i=1}^{L} V_i(M_i)} \, e^{\frac{1}{g_s}\mathrm{Tr}\sum_{i=1}^{L-1} c_i M_i M_{i+1}} . \tag{X.13}$$

Note that aside from the potentials $V_i(M_i)$, the only interactions are between nearest neighbors, whence the name "chain of matrices." Chain of matrices matrix models can be solved when the interaction terms between different matrices are of the form $\mathrm{Tr} M_i M_{i+1}$, as is the case here.

$\mathscr{E}$ can be any ensemble of $L$ normal matrices of size $N \times N$, i.e. a submanifold of $\mathbb{C}^{LN^2}$ of real dimension $LN^2$, such that the integral is convergent. $\mathscr{E}$ can be many things; for a chain of matrices model, it is characterized by the contours on which eigenvalues of the various normal matrices are integrated (see [172] for the 2-matrix model case). For (X.13) to have a topological expansion, $\mathscr{E}$ must be a so-called steepest descent ensemble (see [58], section 5.5). For a generic ensemble $\mathscr{E}$ which would not be steepest descent, $\ln Z$ would be an oscillating function of $1/g_s$, and no small $g_s$ expansion would exist, see [95].

The matrix model introduced in [68] and reproduced in section X was defined to reproduce the topological string partition function, which is defined as a formal series in $g_s$, and therefore has a topological expansion by construction.

An ensemble $\mathscr{E}$ is characterized by filling fractions $n_{j,i}$,

$$\mathscr{E} = \prod_{i=1}^{L} \mathscr{E}_i \quad , \quad \mathscr{E}_i = H_N(\gamma_{1,i}^{n_{1,i}} \times \gamma_{2,i}^{n_{2,i}} \times \dots \times \gamma_{k_i,i}^{n_{k_i,i}}) , \tag{X.14}$$

where $H_N(\gamma_1^{n_1} \times \cdots \times \gamma_k^{n_k})$ is the set of normal matrices with $n_1$ eigenvalues on path $\gamma_1$, $n_2$ eigenvalues on path $\gamma_2$, ..., $n_k$ eigenvalues on path $\gamma_k$.

As the filling fractions $n_{j,i}$ must satisfy the relation

$$\sum_{j=1}^{k_i} n_{j,i} = N \tag{X.15}$$

for all $i$, only $\sum_i (k_i - 1)$ of them are independent.

We also allow some paths $\gamma_{j,i}$ to have endpoints where $e^{-\mathrm{Tr}\sum_{i=1}^{L}(V_i(M_i)-M_iM_{i+1})} \neq 0$ – indeed, in our matrix model, the matrices $R_i$ are integrated on $H(\mathbb{R}_+^N)$.

**The resolvent**

The spectral curve encodes all $W_i(x)$, the planar limits (see footnote 3) of the resolvents of the matrices $M_i$,

$$W_i(x) = g_s \left\langle \mathrm{tr} \frac{1}{x - M_i} \right\rangle_{planar}, \tag{X.16}$$

see equation (X.25) below. The respective $W_i$ can be expressed as the Stieljes transform

$$W_i(x) = \int \frac{\rho_i(x')dx'}{x - x'} \tag{X.17}$$

of the planar expectation value of the eigenvalue density $\rho_i(x)$ of the matrix $M_i$,

$$\rho_i(x) = g_s \left\langle \mathrm{tr}\delta(x - M_i) \right\rangle_{planar}. \tag{X.18}$$

By general properties of Stieljes transforms, singularities of $W_i(x)$ coincide with the support of the distribution $\rho_i(x)dx$:

- Simple poles of $W_i(x)$ correspond to delta distributions i.e. isolated eigenvalues.
- Multiple poles correspond to higher derivatives of delta distributions.
- Cuts correspond to finite densities, the density being the discontinuity of $W_i(x)$ along the cut,

$$\rho_i(x) = \frac{1}{2i\pi} \left( W_i(x - i0) - W_i(x + i0) \right). \tag{X.19}$$

In particular, cuts emerging from algebraic singularities (generically square root singularities) correspond to densities vanishing algebraically (generically as square roots) at the endpoints of the cut. Cuts emerging from logarithmic singularities correspond to constant densities.

**The spectral curve of the general chain of matrices**

When all $V_i'$ are rational, the spectral curve was found in [83, 143], and it is algebraic. We present here a generalization of this result to more general potentials.

The spectral curve can be obtained by the following procedure:

1. Consider a compact Riemann surface $\mathscr{C}$ of genus

$$\mathfrak{g} = \sum_{i=1}^{L} (k_i - 1)\,, \tag{X.20}$$

where $k_i$ denotes the number of cuts of the $i$-th matrix, as implicitly defined in (X.14).

2. Look for $L+2$ functions on $\mathscr{C}$,

$$x_0(z), x_1(z), x_2(z), \ldots, x_L(z), x_{L+1}(z) : \mathscr{C} \to \mathbb{CP}^1\,. \tag{X.21}$$

The $x_i$ are to be holomorphic away from points $z \in \mathscr{C}$ at which $V_{i-1}'(x_{i-1})$ or $V_{i+1}'(x_{i+1})$ become singular, and satisfy the functional relations

$$c_{i-1}x_{i-1}(z) + c_i x_{i+1}(z) = V_i'(x_i(z))\,. \tag{X.22}$$

Recall that the $c_i$ are the coefficients of the interaction potentials in (X.13). We have set $c_0 = c_L = 1$.

For each $i = 1, \ldots, L$, the Riemann surface $\mathscr{C}$ can be realized as a branched covering of $\mathbb{CP}^1$ by the projection $x_i : \mathscr{C} \to \mathbb{CP}^1$. A choice of branched covering is not unique: the choice consists in the set of cuts connecting branch points (recall that these are points at which $dx_i(z) = 0$). We will determine an appropriate covering below in step 4.

3. If some path $\gamma_{j,i}$ has an endpoint $a$ (called "hard edge" in the matrix model literature, see [86]), then choose a pre-image $a_i \in x_i^{-1}(a)$ and require

$$dx_i(a_i) = 0 \quad \text{and} \quad x_{i-1}(z) \text{ has a simple pole at } z = a_i. \tag{X.23}$$

The topological recursion is proved in [83] without hard edges, but it is not difficult to see, by mixing the results of [86], [173] and [83], that the topological recursion continues to hold in the presence of hard edges. The proof will appear in a forthcoming publication. Here, we shall assume that it holds.

4. Choose some contours $\widehat{\mathscr{A}_{j,i}}$, $j = 1, \dots, k_i$ in $\mathbb{CP}^1$, such that each $\widehat{\mathscr{A}_{j,i}}$ surrounds all points of the contour $\gamma_{j,i}$ (related to the matrix ensemble $\mathscr{E}_i$ defined in (X.14)) in the clockwise direction and no other contour $\gamma_{j',i}$. For $x \in \mathbb{CP}^1$ not enclosed in the contours $\widehat{\mathscr{A}_{j,i}}$, $j = 1, \dots, k_i$, and given a connected component $\mathscr{A}_{j,i}$ of the pre-image of the contour $\widehat{\mathscr{A}_{j,i}}$ under $x_i$,

$$\mathscr{A}_{j,i} \subset x_i^{-1}(\widehat{\mathscr{A}_{j,i}}), \tag{X.24}$$

define the function

$$W_i(x) = \frac{c_{i-1}}{2i\pi} \sum_{j=1}^{k_i} \oint_{\mathscr{A}_{j,i}} \frac{x_{i-1}(z)\, dx_i(z)}{x - x_i(z)}. \tag{X.25}$$

Generalizing [143] to non-polynomial potentials, we claim that a choice of $\mathscr{A}_{j,i}$ exists such that $W_i(x)$ is the planar limit of the resolvent of the matrix $M_i$. In the following, it is this choice that will be referred to as $\mathscr{A}_{j,i}$.

Notice that not all $\mathscr{A}_{j,i}$ will be homologically independent on $\mathscr{C}$. We require that we have $\mathfrak{g} = \sum_{i=1}^{L}(k_i - 1)$ homologically independent $\mathscr{A}_{j,i}$'s, which coincides with the genus of $\mathscr{C}$. As a condition on the choice of branched covering, we impose that $\mathscr{A}_{j,i}$ and $a_i$ lie on the same sheet of $x_i$. This condition, in our experience, uniquely fixes this choice. We will assume that this is the case. We refer to the sheet of $x_i$ containing $\mathscr{A}_{j,i}$ and $a_i$ as the physical sheet for $x_i$.

5. In accord with (X.19), we consider the discontinuity of $W_i(x)$ along the $j$-th cut. It

is given by

$$\begin{aligned} \underset{j}{\text{Disc}}\ W_i(x) &= \frac{1}{2\pi i}\left(W_i(x_+) - W_i(x_-)\right) \\ &= \frac{1}{2\pi i} c_{i-1} \underset{j}{\text{Disc}}\ x_{i-1}\,, \end{aligned} \tag{X.26}$$

as we explain in figure X.5.

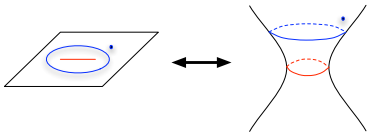

Figure X.4: The integration contour $\widehat{\mathscr{A}_{j,i}}$ on the $x$-plane, and its image $\mathscr{A}_{j,i}$ on $\mathscr{C}$.

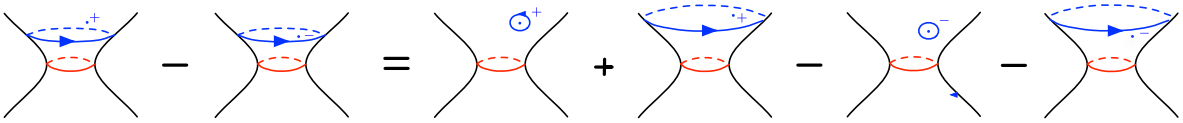

Figure X.5: The preimage of the points $x_+$ and $x_-$ of (X.26) are depicted as dots in the above diagram, $\widehat{\mathscr{A}_{j,i}}$ is given by the blue contour, and the preimage of the cut is drawn in red. To take the limit $x_+ \to x_-$, one must first shift the contours. The second and fourth term on the RHS of the above diagrammatic equation then cancel, yielding the RHS of (X.26).

The definition (X.14) of the matrix ensemble $\mathscr{E}_i$ is the condition that there are $n_{j,i}$ eigenvalues of $M_i$ on the contour $\gamma_{j,i}$, hence corresponds to imposing the filling fraction conditions

$$\frac{1}{2\pi i}\oint_{\mathscr{A}_{j,i}} c_{i-1}x_{i-1}dx_i = g_s\, n_{j,i} \tag{X.27}$$

for $i = 1,\dots,L$, $j = 1,\dots,k_i$.

In our experience, the conditions enumerated above have a unique solution and define a unique spectral curve. As emphasized in the introduction, a formal uniqueness proof is however still lacking.

The spectral curve is defined as the data of the Riemann surface $\mathscr{C}$, and the two functions $x_1(z)$ and $x_2(z)$,

$$\boxed{\mathscr{S}_{MM} = (\mathscr{C}, x_1, x_2).} \tag{X.28}$$

### 3.3 Symplectic invariants of a spectral curve

Once we have found the spectral curve $\mathscr{S}_{MM}$ of our matrix model, we can compute the coefficients $F_g$ in the topological expansion of its partition function,

$$\ln Z = \sum_{g=0}^{\infty} g_s^{2g-2} F_g \,, \tag{X.29}$$

by computing the symplectic invariants of this curve,

$$F_g = F_g(\mathscr{S}_{MM}) \,, \tag{X.30}$$

following [83].

Let us recall the definition of these invariants for an arbitrary spectral curve $\mathscr{S}$.

Let $\mathscr{S} = (\mathscr{C}, x, y)$ be a spectral curve, comprised of the data of a Riemann surface $\mathscr{C}$ and two functions $x(z), y(z) : \mathscr{C} \to \mathbb{C}$, meromorphic on $\mathscr{C}$ away from a finite set of points (we wish to allow logarithms).[4] We will assume that $dx$ is a meromorphic form on all of $\mathscr{C}$.

#### Branchpoints

Let $a_i$ be the branch points of the function $x$,

$$dx(a_i) = 0. \tag{X.31}$$

We assume that all branch points are simple, i.e. that $dx$ has a simple zero at $a_i$. This implies that in the vicinity of $a_i$, the map $x$ is $2:1$. We introduce the notation $\bar{z} \neq z$ such that

$$x(\bar{z}) = x(z). \tag{X.32}$$

$\bar{z}$ is called the conjugate point to $z$, and it is defined only in the vicinity of branch points, as depicted in figure X.6.

We also require that the branch points of $x$ and $y$ do not coincide, such that $dy(a_i) \neq 0$ and $y(z)$ therefore has a square-root branchcut as a function of $x$ at $x(a_i)$. If $y$ is finite at

[4] In fact, the most general setting in which this formalism is valid has not yet been established. We will state it within the generality we need here i.e. we assume that $dx$ is meromorphic forms on $\mathscr{C}$ (this allows $x$ and $y$ to have logarithms).

$a_i$, its local behavior is hence given by

$$y(z) \sim y(a_i) + C_i \sqrt{x(z) - x(a_i)}\,. \tag{X.33}$$

If $a_i$ corresponds to a hard edge, we require $y$ to have a pole here. Its local behavior is hence given by

$$y(z) \sim \frac{C_i}{\sqrt{x(z) - x(a_i)}}\,. \tag{X.34}$$

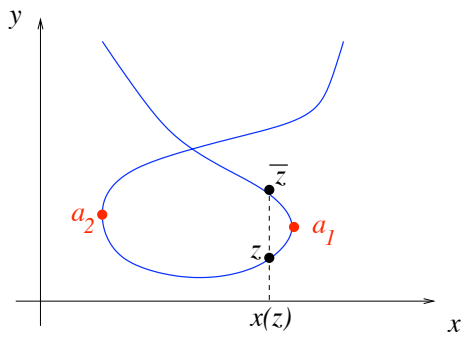


Figure X.6: At a regular branch point $a \in \mathscr{C}$ of $x$, $y$ as a function of $x$ has a branchcut $y \sim y(a) + C\sqrt{x - x(a)}$. If $z$ is a point on one branch near $a$, we call $\bar{z}$ the conjugate point on the other branch; it has the same $x$ projection, $x(\bar{z}) = x(z)$. Notice that $\bar{z}$ is defined only locally near branch points. If we follow $z$ from $a_1$ to $a_2$, $\bar{z}$ may have to jump from one branch to another.

### Bergman kernel

On a curve $\mathscr{C}$, there exists a unique symmetric 2-form $B(z_1, z_2)$ with a double pole on the diagonal $z_1 = z_2$ and no other poles, with the following normalization on $\mathscr{A}$-cycles,

$$\oint_{z_2 \in \mathscr{A}_{j,i}} B(z_1, z_2) = 0. \tag{X.35}$$

In any local coordinate near $z_1 = z_2$, one has

$$B(z_1, z_2) \sim \frac{dz_1\, dz_2}{(z_1 - z_2)^2} + \text{regular}\,. \tag{X.36}$$

$B$ is called the Bergman kernel of $\mathscr{C}$, or the fundamental 2-form of the second kind [109].

**Recursion kernel**

We now define the recursion kernel $K$ as

$$K(z_0,z)=\frac{\int_{\bar z}^{z}B(z_0,z')}{2(y(\bar z)-y(z))\,dx(z)}\,. \tag{X.37}$$

This kernel is a globally defined 1-form in the variable $z_0\in\mathscr{C}$. In the variable $z$, it is the inverse of a 1-form (that means we have to multiply it with a quadratic differential before computing any integral with it); it is defined only locally near branch points of $x$, such that $K(z_0,\bar z)=K(z_0,z)$. At the branch points, it has simple poles,

$$K(z_0,z)\sim-\frac{B(z_0,z)}{2\,dx(z)\,dy(z)}+\text{regular}. \tag{X.38}$$

**Topological recursion**

Correlation forms $W_n^{(g)}(z_1,\dots,z_n)$ (not to be confused with the resolvents $W_i(z)$ introduced above) are symmetric $n$-forms defined by

$$W_1^{(0)}(z)=-y(z)dx(z)\,, \tag{X.39}$$

$$W_2^{(0)}(z_1,z_2)=B(z_1,z_2)\,, \tag{X.40}$$

and then by recursion (we write collectively $J=\{z_1,\dots,z_n\}$),

$$\begin{aligned}W_{n+1}^{(g)}(z_0,J) &= \sum_i \operatorname*{Res}_{z\to a_i}K(z_0,z)\Big[W_{n+2}^{(g-1)}(z,\bar z,J)\\ &\quad+\sum_{h=0}^{g}{\sum_{I\subset J}}'W_{1+|I|}^{(h)}(z,I)\,W_{1+n-|I|}^{(g-h)}(\bar z,J\setminus I)\Big]\end{aligned}$$

where $\sum_I'$ is the sum over all subsets of $J$, restricted to $(h,I)\neq(0,\emptyset)$ and $(h,I)\neq(g,J)$.

Although it is not obvious from the definition, the forms $W_n^{(g)}$ are symmetric. For $2-2g-n<0$, they are meromorphic $n$-forms with poles only at branch points. These poles are of degree at most $6g-4+2n$, and have vanishing residues.

For the one matrix model, the $W_n^{(g)}$ coincide with the $n$-point function of the trace of the resolvent at order $g$ in the topological expansion.

**Symplectic invariants**

Finally, for $g \geq 2$, we define the symplectic invariants $F_g$ (also denoted $W_0(g)$ in [23]) by

$$F_g(\mathscr{S}) = \frac{1}{2-2g} \sum_i \operatorname*{Res}_{z \to a_i} \Phi(z) W_1^{(g)}(z), \tag{X.41}$$

where $\Phi$ is any function defined locally near branch points of $x$ such that $d\Phi = ydx$.

The definitions of $F_1$ and $F_0$ are more involved and we refer the reader to [23]. $F_0$ is called the prepotential, and $F_1$ is closely related to the determinant of the Laplacian on $\mathscr{C}$ with metrics $|ydx|^2$, see [174, 175].

The $F_g(\mathscr{S})$'s depend only on the orbit of $\mathscr{S}$ under the group of transformations generated by

$\mathfrak{R}$ : $\mathscr{S} \mapsto \tilde{\mathscr{S}} = (\mathscr{C}, x, y + R(x))$ where $R(x)$ is any rational function of $x$,

$\mathfrak{F}$ : $\mathscr{S} \mapsto \tilde{\mathscr{S}} = (\mathscr{C}, f(x), y/f'(x))$ where $f(x)$ is an analytical function of $x$, with $f'$ rational, such that $df = f'dx$ has the same number of zeroes as $dx$,

$\mathfrak{S}$ : $\mathscr{S} \mapsto \tilde{\mathscr{S}} = (\mathscr{C}, y, -x)$.

These transformations are symplectic, i.e. they leave $dx \wedge dy$ invariant.

The symplectic invariants are homogeneous of degree $2-2g$,

$$F_g(\mathscr{C}, x, \lambda y) = \lambda^{2-2g} F_g(\mathscr{C}, x, y). \tag{X.42}$$

In particular, they are invariant under the parity transformation $F_g(\mathscr{C}, x, -y) = F_g(\mathscr{C}, x, y)$.

## 5 The spectral curve for the topological string's matrix model

Applying the procedure outlined in section X to our matrix model, we will determine a spectral curve $\mathscr{S}_{MM}(\mathfrak{X}_0)$ in this section. [83] demonstrated that for a chain of matrices, we have

$$\ln Z = \sum_g g_s^{2g-2} F_g(\mathscr{S}_{\rm MM}), \tag{X.43}$$

with $F_g$ the symplectic invariants of [23]. In our case, since we have engineered our matrix model to yield[5] $GW_g(\mathfrak{X}_0)$ as its partition function, re-computing the partition

[5] As we have here reserved the notation $F_g$ for the symplectic invariants of our matrix model, we refer to the topological string free energies as $GW_g$.

function via the methods of [83] will yield

$$GW_g(\mathfrak{X}_0) = F_g(\mathscr{S}_{\rm MM})\,. \tag{X.44}$$

This relation is already quite interesting, as it allows for explicit computation of the Gromov-Witten invariants. Our goal however will be to go further. We will argue that $\mathscr{S}_{\rm MM}$ is symplectically equivalent to the mirror spectral curve $\mathscr{S}_{\hat{\mathfrak{X}}_0}$ of section X,

$$\mathscr{S}_{\rm MM} \sim \mathscr{S}_{\mathfrak{X}_0}\,. \tag{X.45}$$

Since the $F_g$'s are symplectic invariants, this will imply the BKMP conjecture for $\mathfrak{X}_0$, i.e.

$$GW_g(\mathfrak{X}_0) = F_g(\mathscr{S}_{\mathfrak{X}_0}). \tag{X.46}$$

### 5.1 Applying the chain of matrices rules

We now apply the rules of section X to the chain of matrices model introduced in section X.

- Recall that the integration ensembles for the matrices $M_0$ and $M_{m+1}$ are such that for each matrix, all eigenvalues are integrated on the same contour (X.113). Hence, $k_0 = k_{m+1} = 1$, and the corresponding filling fractions are equal to $N$. For $i = 1,\dots,m$, the matrix $M_i$ is integrated on $H(\gamma_{0,i}^d \times \gamma_{2,i}^d \times \cdots \times \gamma_{n,i}^d)$, where $\gamma_{j,i}$ is a contour which surrounds all points of the form $q^{a_{j,i}+\mathbb{N}}$. There are thus $k_i = n+1$ filling fractions, each equal to $d$. The matrices $R_i$ are integrated on $H(\mathbb{R}_+^N)$. We denote the number of their cuts by $\tilde{k}_i$. Hence, $\tilde{k}_i = 1$, with the respective filling fraction equal to $N$.

  According to condition 1 of section X, the genus of the spectral curve $\mathscr{C}$ is thus given by

$$\mathfrak{g} = \sum_{i=0}^{m+1}(k_i - 1) + \sum_{i=1}^{m+1}(\tilde{k}_i - 1) = nm\,.$$

- Following condition 2 of section X, we introduce functions $x_i(z)$, $i = 0,\dots,m+1$, associated to the matrices $M_i$, and functions $y_i(z)$, $i = 1,\dots,m+1$, associated to the matrices $R_i$, as well as two additional functions $y_0(z)$ and $y_{m+2}(z)$ at the ends of the chain.

  They must satisfy the following requirements:

– Since there is no potential for the matrices $R_i$, equation (X.22) implies that we have, for $i = 1, \ldots, m+1$,

$$x_i(z) - x_{i-1}(z) = 0\,. \tag{X.47}$$

We can hence suppress the index $i$ on these functions, $x(z) = x_i(z)$.

– For $i = 1, \ldots, m$, equation (X.22) gives

$$y_i(z) - y_{i+1}(z) = 2V'_{\vec{a}_i}(x(z)) - V'_{\vec{a}_{i+1}}(x(z)) - V'_{\vec{a}_{i-1}}(x(z)) - g_s \frac{f_i'(x(z))}{f_i(x(z))} - \frac{g_s S_i + i\pi}{x(z)} \tag{X.48}$$

and

$$y_0(z) - y_1(z) = V'_{\vec{a}_0}(x(z)) - V'_{\vec{a}_1}(x(z)) - g_s \frac{f_0'(x(z))}{f_0(x(z))}\,, \tag{X.49}$$

$$y_{m+1}(z) - y_{m+2}(z) = V'_{\vec{a}_{m+1}}(x(z)) - V'_{\vec{a}_m}(x(z)) - g_s \frac{f_{m+1}'(x(z))}{f_{m+1}(x(z))}\,. \tag{X.50}$$

More explicitly, in terms of the function

$$\psi_q(x) = x g'(x)/g(x)\,,$$

whose small $g_s$ expansion

$$\begin{aligned} \psi_q(x) &= -\frac{1}{\ln q} \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} (\ln q)^n \, \mathrm{Li}_{1-\mathrm{n}}(1/\mathrm{x}) \\ &= \frac{1}{\ln q} \Big[ \ln(1 - \frac{1}{x}) - \frac{\ln q}{2(x-1)} - \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} (\ln q)^{2n} \mathrm{Li}_{1-2\mathrm{n}}(\mathrm{x}) \Big]\,. \end{aligned}$$

we worked out in appendix A of [68], we obtain

$$\begin{aligned} & x(z)(y_{i+1}(z) - y_i(z)) \\ &= i\pi + g_s S_i - g_s \sum_j (2\psi_q(q^{a_{j,i}}/x(z)) - \psi_q(q^{a_{j,i+1}}/x(z)) - \psi_q(q^{a_{j,i-1}}/x(z))) \\ &\quad + g_s \frac{x(z) f_i'(x(z))}{f_i(x(z))}\,, \end{aligned} \tag{X.51}$$

as well as

$$\begin{aligned} x(z)(y_1(z)-y_0(z)) &= -g_s\sum_j \psi_q(q^{a_{j,0}}/x(z)) + g_s\sum_j \psi_q(q^{a_{j,1}}/x(z)) \\ &\quad -g_s\sum_j\sum_{k=0}^{d-1}\frac{x(z)}{x(z)-q^{a_{j,0}+k}}\,, \end{aligned}$$

$$\begin{aligned} x(z)(y_{m+2}(z)-y_{m+1}(z)) &= -g_s\sum_j \psi_q(q^{a_{j,m+1}}/x(z)) + g_s\sum_j \psi_q(q^{a_{j,m}}/x(z)) \\ &\quad -g_s\sum_j\sum_{k=0}^{d-1}\frac{x(z)}{x(z)-q^{a_{j,m+1}+k}} \end{aligned}$$

Note that we have explicitly used the fact that the partitions $\alpha_{j,m+1}$ and $\alpha_{j,0}$ are chosen to be trivial.

– Since the integral over $R_i$ is over $H_N(\mathbb{R}_+)$, i.e. its eigenvalues are integrated on $\mathbb{R}_+$, the integration contour has an endpoint (hard edge) at $y_i = 0$. Condition 3 hence requires that at a pre-image $y_i^{-1}(0)$, which we will refer to as $\infty_i$, the following holds

$$y_i(\infty_i) = 0\,, \quad dy_i(\infty_i) = 0\,, \quad x(z) \text{ has a simple pole at } z = \infty_i\,. \tag{X.52}$$

Furthermore, introducing a local parameter $z$ in the neighborhood of $\infty_i$, the above translates into

$$y_i(z) \sim z^2\,, \quad x(z) \sim 1/z\,. \tag{X.53}$$

Hence, $\forall\, i = 1,\ldots,m+1$,

$$y_i \sim \mathcal{O}(1/x^2)\,. \tag{X.54}$$

- The relations (X.51) imply that near $\infty_i$, we have

$$x(y_{j+1}-y_j) \underset{z\to\infty_i}{\sim} i\pi + g_s S_j + g_s\sum_{l=0}^{n}(2a_{l,j}-a_{l,j+1}-a_{l,j-1}) + O(1/x)\,. \tag{X.55}$$

In particular, it follows that $\infty_j \neq \infty_i$. Thus, all points $\{\infty_1,\ldots,\infty_{m+1}\} \subset x^{-1}(\infty)$ are distinct , i.e. condition 3 requires that $x^{-1}(\infty)$ have at least $m+1$ points.

We will make the minimal assumption that $x^{-1}(\infty)$ has exactly $m+1$ elements

that are simple poles of $x$, and that $x$ has no further singularities, i.e. that $x$ is a meromorphic function of degree $m+1$ on $\mathscr{C}$.

- By condition 5, since for $i=0,\dots,m+2$ there are $d$ eigenvalues of $M_i$ of the form $q^{a_{j,i}+\mathbb{N}}$ surrounded by the path $\widehat{\mathscr{A}_{j,i}}$, we have the $(m+2)\times(n+1)$ filling fraction conditions

$$\frac{1}{2i\pi}\oint_{\mathscr{A}_{j,i}} y_i\,dx = d\,g_s \qquad \text{for } i=0,\dots,m+1,\ \ j=0,\dots,n\,. \tag{X.56}$$

$x$ hence defines an $m+1$ sheeted cover of $\mathbb{CP}^1$. Considering the function $\ln x$ instead, with singularities at $x=0$ and $x=\infty$, each sheet of this cover is mapped to a cylinder. We have depicted this covering in figure X.7, and indicated the singularities of $y_i$ on each sheet: algebraic cuts are represented by vertical cylinders, and poles and logarithmic cuts by grey strips.

In sheet $i$ we have represented some contours $\mathscr{A}_{j,i}$ whose image under the projection $x:\mathscr{C}\to\mathbb{CP}^1$ surrounds all points of type $q^{a_{j,i}+\mathbb{N}}$.

For $i=1,\dots,m$, the resolvent $W_i(x)$ of the $i^{\text{th}}$ matrix $M_i$ is computed as a contour integral around the sum over $j$ of cycles $\mathscr{A}_{j,i}$ on sheet $i$,

$$W_i(x)=\sum_{j=0}^{n}\frac{1}{2i\pi}\oint_{\mathscr{A}_{j,i}}\frac{y_i(z')dx(z')}{x-x(z')}\,. \tag{X.57}$$

Also, as argued in [68], the potentials of $M_0$ and $M_{m+1}$ are such that in fact the matrices $M_0$ and $M_{m+1}$ are frozen, and thus their resolvents contain only poles. In terms of the functions $y_0$ and $y_{m+1}$, we conclude that the singularities of $y_0$ in $\mathscr{A}_{j,0}$ in sheet 1 and the singularities of $y_{m+1}$ in $\mathscr{A}_{j,m+1}$ in sheet $m+1$ can be only poles, not cuts.

Since condition 1 requires that the genus be $\mathfrak{g}=nm$, we see that there can be no other cuts than the ones already discussed – the genus would be higher, otherwise.

### 5.2 Symplectic change of functions

The spectral curve of the matrix model is $\mathscr{S}_{MM}=(\mathscr{C},x,y_0)$, and our goal is to relate it to the mirror curve described in section X. The mirror curve is described via the algebraic equation (X.6) in the two functions $x_1,x_2:\mathscr{C}_{mirror}\to\mathbb{CP}^1$ (in the patch $x_0=1$). We wish to obtain a similar algebraic description of $\mathscr{C}$. Due to log singularities in $y_0$, to be traced to the small $g_s$ behavior of $\psi_q(x)$, an algebraic equation in the variables $(x,y_0)$ cannot exist (recall that $x$ is meromorphic). In this section, we shall, via a series of symplectic

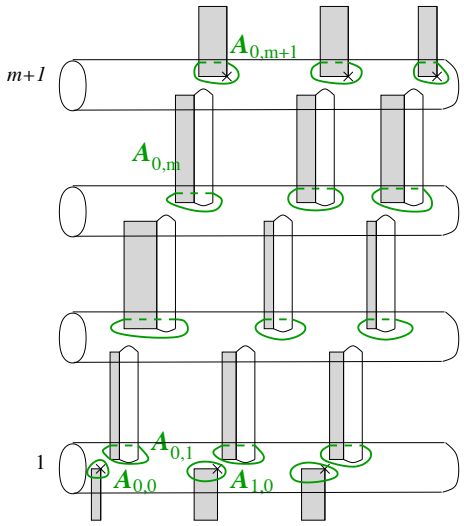


Figure X.7: The spectral curve of our matrix model can be represented as follows. The cover of $\mathbb{CP}^1$ provided by $x$ has $m+1$ sheets. Instead of the projective plane of $x$, we represent the sheets of $\ln x$, which are cylinders. Cycles $\mathscr{A}_{j,i}$ appear in sheets $i-1$ and $i$. They enclose singularities of the resolvent $W_i$. Algebraic cuts are represented as vertical cylinders, and poles and log singularities are represented as grey strips. There is only one cycle $\mathscr{A}_{j,0}$ (which is in sheet 0) and one $\mathscr{A}_{j,m+1}$ (in sheet $m$), and they enclose only poles or log singularities of $y_0$ resp. $y_{m+1}$.

transformations on the $y_i$ of the type enumerated in section X, arrive at functions $Y_i$ that are meromorphic on $\mathscr{C}$, and hence each present a viable candidate to pair with $x$ to yield an algebraic equation for $\mathscr{C}$.

Essentially, we wish to introduce the exponentials of $y_i$. While this will eliminate the log singularities, poles in $y_i$ would be elevated to essential singularities. We hence first turn to the question of eliminating these poles.

### The arctic circle property

On the physical sheet, the interpretation of a pole of $y_i$ is as an eigenvalue of the matrix $M_i$ with delta function support. Such a so-called frozen eigenvalue can arise in the following way:

The sum over all partitions is dominated by partitions close to a typical equilibrium partition, i.e. a saddle point. The typical partition has a certain typical length referred to as its equilibrium length $\bar{n}$. All partitions with a length very different from the equilibrium length contribute only in an exponentially small way (and thus non-perturbatively) to the full partition function. Introducing a cutoff on the length of partitions which is larger than the equilibrium length hence does not change the perturbative part of the partition function. Now recall that when we defined the $h_i(\gamma)$ of a representation $\gamma$ in

appendix X, we introduced an arbitrary maximal length $d$ such that $l(\gamma) \leq d$ and set

$$h_i(\gamma) = a_\gamma + d - i + \gamma_i \,. \tag{X.58}$$

Setting $\gamma_i = 0$ for $d \geq i > \bar{n}$ yields $h_i$ that do not depend on the integration variables, hence are frozen at fixed values. This behavior is referred to as the arctic circle property [134], as all eigenvalues beyond the arctic circle situated at equilibrium length $\bar{n}$ are frozen.

Returning to our matrix model, the eigenvalues of the matrices $M_i$ are given by $q^{(h_{j,i})_l}$. For $d \geq l > n_{j,i}$, they are frozen, and thus contribute poles to $y_i$ by (X.25) (recall that poles of the resolvent correspond to eigenvalues with delta function support) in the physical sheet. We will assume that these are the only poles in the physical sheet and we subtract them to obtain new functions $\tilde{y}_i$,

$$\tilde{y}_0(z) = x(z) y_0(z) - \sum_j \sum_{k=0}^{d-1} \frac{g_s x(z)}{x(z) - q^{a_{j,0}+k}} \,,$$

$$\tilde{y}_{m+2}(z) = x(z) y_{m+2}(z) + \sum_j \sum_{0}^{d-1} \frac{g_s x(z)}{x(z) - q^{a_{j,m+1}+k}}$$

and for $i = 1, \ldots, m+1$,

$$\tilde{y}_i(z) = x(z) y_i(z) - \sum_j \sum_{k=0}^{d-n_{j,i}-1} \frac{g_s x(z)}{x(z) - q^{a_{j,i}+k}} + \sum_j \sum_{k=0}^{d-n_{j,i-1}-1} \frac{g_s x(z)}{x(z) - q^{a_{j,i-1}+k}} \,.$$

We have set

$$n_{j,0} = 0 \,, \quad n_{j,m+1} = 0 \,. \tag{X.59}$$

Notice that at large $x(z)$ in sheet $i$ we have

$$\tilde{y}_0 \sim O(1/x(z)) \,, \qquad \tilde{y}_{m+2} \sim O(1/x(z)) \tag{X.60}$$

and for $i = 1, \ldots, m+1$

$$\tilde{y}_i \sim g_s \sum_j (n_{j,i} - n_{j,i-1}) + O(1/x(z)). \tag{X.61}$$

As a general property of $\psi_q$, we have for any integer $n_{j,i} \leq d$

$$\psi_q(q^{a_{j,i}}/x) = \psi_q(q^{a_{j,i}+d-n_{j,i}}/x) + \sum_{k=0}^{d-n_{j,i}-1} \frac{x}{x-q^{a_{j,i}+k}}\,. \tag{X.62}$$

Hence, the loop equations for the new functions $\tilde{y}_i$ read

$$\begin{aligned}
\tilde{y}_{i+1}(z) - \tilde{y}_i(z) &= i\pi + g_s S_i \\
&\quad + g_s \sum_j (2\psi_q(q^{a_{j,i}+d-n_{j,i}}/x(z)) - \psi_q(q^{a_{j,i+1}+d-n_{j,i+1}}/x(z)) \\
&\quad - \psi_q(q^{a_{j,i-1}+d-n_{j,i-1}}/x(z))) + g_s \frac{x(z) f_i'(x(z))}{f_i(x(z))}\,, \\
\tilde{y}_1(z) - \tilde{y}_0(z) &= g_s \sum_j \psi_q(q^{a_{j,0}+d}/x(z)) - g_s \sum_j \psi_q(q^{a_{j,1}+d-n_{j,1}}/x(z))\,, \\
\tilde{y}_{m+2}(z) - \tilde{y}_{m+1}(z) &= g_s \sum_j \psi_q(q^{a_{j,m+1}+d}/x(z)) - g_s \sum_j \psi_q(q^{a_{j,m}+d-n_{j,m}}/x(z))\,.
\end{aligned}$$

The $n_{j,i}$ in the above definitions are defined as the equilibrium lengths, i.e. by the property that the functions $\tilde{y}_i$ have no poles on their physical sheet. That such a choice of $n_{j,i}$ exists is suggested by the arctic circle property.

Note that the $n_{j,i}$ can also be specified by the fact that $q^{a_{j,i}+d-n_{j,i}}$ be the beginning of the cut encircled by $\gamma_{j,i}$. As we have identified the discontinuities of $y_i$ to lie across branchcuts of $x$, this implies that $x$ has ramification points at the element of $x^{-1}(q^{a_{j,i}+d-n_{j,i}})$ lying on the physical sheets of $y_i$.

Note that the arctic circle property also implies the perturbative independence of our expressions from the arbitrary cut-off $d$. Changing $d$ to $d+d'$ merely introduces $d'$ new frozen eigenvalues $h_i$. This independence from $d$ is important in establishing the equality between the topological string partition function and our matrix integral (X.12), as the topological vertex formulae in fact are formulated in the limit $d \to \infty$.

**Obtaining globally meromorphic functions**

We have arrived at functions $\tilde{y}_i$ that have no poles on their physical sheet, and are thus safely exponentiated there. We wish now to use the loop equations to obtain functions which are globally well-behaved.

To this end, we note that since the Gromov-Witten invariants are defined as a formal power series in $g_s$, we can compute the spectral curve order by order in $g_s$, invoking the

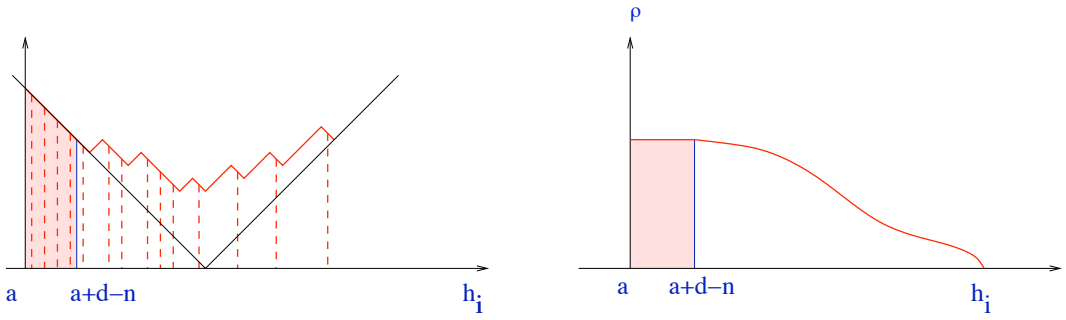


Figure X.8: We shift the cut-off $d$ on the representation lengths, $d \to n_{j,i}$, with $n_{j,i}$ chosen such that frozen eigenvalues in the expected distribution of the $h_i$ are suppressed. In the limit of vanishing spacing ($g_s \to 0$), the equidistant frozen eigenvalues give rise to a constant eigenvalue density region.

following small $\ln q$ expansion [68]:

$$\begin{aligned}\psi_q(q^{a_{j,i}+d-n_{j,i}}/x) \;\sim\; & -\frac{1}{g_s}\ln(1-\frac{x}{q^{a_{j,i}+d-n_{j,i}}})+\frac{x}{2(x-q^{a_{j,i}+d-n_{j,i}})} \\ & +\frac{1}{g_s}\sum_{n=1}^{\infty}\frac{B_{2n}\,g_s^{2n}}{(2n)!}\,\mathrm{Li}_{1-2n}(\mathrm{q}^{\mathrm{d}-\mathrm{n_{j,i}}+\mathrm{a_{j,i}}}/\mathrm{x})\,.\end{aligned}$$

The functions $f_i'/f_i$ are completely non-perturbative; one can easily check with the above expansion that they can be replaced by 0 to every order in $g_s$.

Introducing new functions $X(z)$ and $Y_i$ by the formulae

$$x(z) = q^d\,X(z)\,, \tag{X.63}$$

$$\tilde{y}_0(z) = \ln Y_0(z)\,, \tag{X.64}$$

$$\tilde{y}_{m+2}(z) = \ln Y_{m+2}(z)\,, \tag{X.65}$$

and for $i = 1,\ldots,m+1$

$$\begin{aligned}\tilde{y}_i(z) \;=\; & \ln Y_i(z)+\sum_j\frac{X(z)g_s}{2(X(z)-q^{a_{j,i}-n_{j,i}})}+\frac{1}{g_s}\sum_j\sum_{n=1}^{\infty}\frac{B_{2n}\,g_s^{2n}}{(2n)!}\,\mathrm{Li}_{1-2n}(\mathrm{q}^{\mathrm{a_{j,i}}-\mathrm{n_{j,i}}}/\mathrm{X(z)}) \\ & -\sum_j\frac{X(z)g_s}{2(X(z)-q^{a_{j,i-1}-n_{j,i-1}})}-\frac{1}{g_s}\sum_j\sum_{n=1}^{\infty}\frac{B_{2n}\,g_s^{2n}}{(2n)!}\,\mathrm{Li}_{1-2n}(\mathrm{q}^{\mathrm{a_{j,i-1}}-\mathrm{n_{j,i-1}}}/\mathrm{X(z)})\end{aligned}$$

yields loop equations that are algebraic on their right hand side,

$$\begin{aligned}
\frac{Y_i}{Y_{i+1}} &= -e^{-g_s S_i}\prod_j \frac{(X-q^{a_{j,i+1}-n_{j,i+1}})(X-q^{a_{j,i-1}-n_{j,i-1}})}{(X-q^{a_{j,i}-n_{j,i}})^2} \\
&\qquad \prod_j q^{2(a_{j,i}-n_{j,i})-(a_{j,i+1}-n_{j,i+1})-(a_{j,i-1}-n_{j,i-1})}\,, \\
\frac{Y_0}{Y_1} &= \prod_j \frac{(X-q^{a_{j,1}-n_{j,1}})}{(X-q^{a_{j,0}})}\prod_j q^{a_{j,0}-(a_{j,1}-n_{j,1})}\,, \\
\frac{Y_{m+1}}{Y_{m+2}} &= \prod_j \frac{(X-q^{a_{j,m}-n_{j,m}})}{(X-q^{a_{j,m+1}})}\prod_j q^{a_{j,m+1}-(a_{j,m}-n_{j,m})}\,,
\end{aligned} \tag{X.66}$$

i.e.

$$\frac{Y_i}{Y_0} = e^{g_s(S_1+\cdots+S_{i-1})}\prod_j q^{(a_{j,i}-n_{j,i})-(a_{j,i-1}-n_{j,i-1})}\prod_j \frac{X-q^{a_{j,i-1}-n_{j,i-1}}}{X-q^{a_{j,i}-n_{j,i}}}\,.$$

Since we have argued that the $Y_i$ are holomorphic on their physical sheet, and the ratio $Y_i/Y_{i+1}$ is purely algebraic, we conclude that the $Y_i$ are meromorphic functions on all of $\mathscr{C}$. This was the goal we had set out to achieve.

Note that the above changes of variables have modified the asymptotics at infinity and the integrals over the $\mathscr{A}$-cycles. More precisely, we have

$$\forall i \in [1,m+1] : \ln Y_i \underset{\infty_i}{\sim} \tilde{y}_i \underset{\infty_i}{\sim} g_s \sum_j (n_{j,i}-n_{j,i-1}) + O(\frac{1}{x})\,, \tag{X.67}$$

$$\ln Y_0 = \tilde{y}_0 \underset{\infty}{\sim} O(\frac{1}{x})\,, \tag{X.68}$$

$$\ln Y_{m+2} = \tilde{y}_{m+2} \underset{\infty}{\sim} O(\frac{1}{x})\,. \tag{X.69}$$

The filling fraction equation reads

$$\frac{1}{2i\pi}\int_{\mathscr{A}_{j,i}} \frac{\tilde{y}_i(z)}{x(z)} dx(z) = g_s(d-(d-n_{j,i})) = g_s n_{j,i}\,. \tag{X.70}$$

In terms of $Y_0$, these conditions can be rewritten as

$$\ln Y_0 \underset{\infty_i}{\sim} -g_s(S_1+\cdots+S_{i-1}) + g_s\sum_{j=0}^{n}(a_{j,i}-a_{j,i-1}) + O(\frac{1}{x}) \tag{X.71}$$

and

$$\begin{aligned}
\frac{1}{2i\pi}\int_{\mathscr{A}_{j,i}} \ln Y_0 \,\frac{dX}{X} &= \frac{1}{2i\pi}\int_{\mathscr{A}_{j,i}} \ln Y_i \,\frac{dX}{X} + \frac{1}{2i\pi}\int_{\mathscr{A}_{j,i}} \ln X \; d\ln\left(\frac{Y_i}{Y_0}\right) \\
&= \frac{1}{2i\pi}\int_{\mathscr{A}_{j,i}} \ln Y_i \,\frac{dX}{X} + \frac{1}{2i\pi}\int_{\mathscr{A}_{j,i}} \ln X \; d\left(\sum_{k=1}^{i} \ln\frac{Y_k}{Y_{k-1}}\right) \\
&= \frac{1}{2i\pi}\int_{\mathscr{A}_{j,i}} \ln Y_i \,\frac{dX}{X} - \frac{1}{2i\pi}\int_{\mathscr{A}_{j,i}} \ln X \; \left(\sum_l \frac{dX}{X - q^{a_{l,i}-n_{l,i}}} - \frac{dX}{X - q^{a_{l,i-1}-n_{l,i-1}}}\right) \\
&= \frac{1}{2i\pi}\int_{\mathscr{A}_{j,i}} \ln Y_i \,\frac{dX}{X} + g_s(a_{j,i} - n_{j,i}) \\
&= g_s n_{j,i} + g_s\,(a_{j,i} - n_{j,i}) \\
&= g_s a_{j,i}\,.
\end{aligned}$$

### 5.3 Recovering the mirror curve

We have argued above that $X$ and $Y_i$, and hence in particular $Y_0$, are meromorphic functions on $\mathscr{C}$. There must hence exist a polynomial $H(X,Y)$ such that (see e.g. Theorem 5.8.1 in [176])

$$H(X,Y_0) = 0\,. \tag{X.72}$$

The facts that $X$ provides an $m+1$ sheeted cover of $\mathbb{CP}^1$ and that $Y_0$ may have $n+1$ poles in its physical sheet imply that the polynomial $H$ has degrees at least $(n+1,m+1)$. As above, we shall choose the minimal hypothesis that it has exactly these degrees. Thus,

$$H(X,Y) = \sum_{i=0}^{m+1}\sum_{j=0}^{n+1} H_{i,j} X^j Y^i. \tag{X.73}$$

As we saw in section X, projectivizing a generic polynomial of these degrees (yielding a homogeneous polynomial of degree $m+n+2$) indeed gives rise to a curve of genus $\mathfrak{g} = nm$.

We now need to determine the $(n+2)(m+2)-1$ unknown coefficients of $H$ ($H$ is defined up to a global multiplicative constant).

The cycle integrals

$$\oint_{\mathscr{A}_{j,i}} \ln Y_0 \,\frac{dX}{X} = 2i\pi g_s\, a_{j,i} \tag{X.74}$$

provide $(n+1)m$ constraints on the coefficients of $H$. We also have $m+1$ constraints for

the behavior at $\infty_i$, $i = 1,\ldots,m+1$,

$$\operatorname*{Res}_{\infty_i} \ln Y_0 \, \frac{dX}{X} = g_s(S_1 + \cdots + S_{i-1}) - g_s \sum_{j=0}^{n} (a_{j,i} - a_{j,i-1}) \,. \tag{X.75}$$

Finally, requiring that $Y_0$ has poles at $q^{a_{j,0}}$ and $Y_{m+2}$ has zeroes at $q^{a_{j,m+1}}$ gives another $2(n+1)$ constraints, which we may write as

$$\operatorname*{Res}_{q^{a_{j,0}}} \ln X \, \frac{dY_0}{Y_0} = g_s a_{j,0} \,, \tag{X.76}$$

$$\operatorname*{Res}_{q^{a_{j,m+1}}} \ln X \, \frac{dY_{m+2}}{Y_{m+2}} = g_s a_{j,m+1} \,. \tag{X.77}$$

This gives enough equations to completely determine $H$. Knowing $H$, we know the location of branch points as functions of $a_{j,i}$'s and $S_i$'s, and can hence determine the $n_{j,i}$ by requiring that $q^{a_{j,i}-n_{j,i}}$ be a branch point.

Notice that we can choose to express the period integrals in any linear combination of $\mathscr{A}$-cycles. In particular,

$$\oint_{\mathscr{A}_{j,i+1}-\mathscr{A}_{j,i}} \ln Y_0 \, \frac{dX}{X} = 2i\pi g_s \, (a_{j,i+1} - a_{j,i}) = 2i\pi \, g_s t_{j,i} \,, \tag{X.78}$$

$$\oint_{\mathscr{A}_{j,i+1}-\mathscr{A}_{j+1,i}} \ln Y_0 \, \frac{dX}{X} = 2i\pi g_s \, (a_{j,i+1} - a_{j+1,i}) = 2i\pi \, g_s r_{j,i} \,. \tag{X.79}$$

Similarly, we may also take linear combinations of $\mathscr{A}$-cycles together with circles surrounding the poles or zeroes of $x$ in order to get the $s_{j,i}$ classes. We hence conclude that the periods of the curve $H(X,Y_0) = 0$ yield the quantum corrected Kähler parameters of the fiducial toric geometry $\mathfrak{X}_0$, allowing us to identify it with the corresponding mirror curve.

### 5.4 Topological expansion and symplectic invariants

Following [83], we obtained

$$\mathscr{S}_{MM} = (\mathscr{C}, x, y_0) \tag{X.80}$$

as the spectral curve of our matrix model at the end of section X.

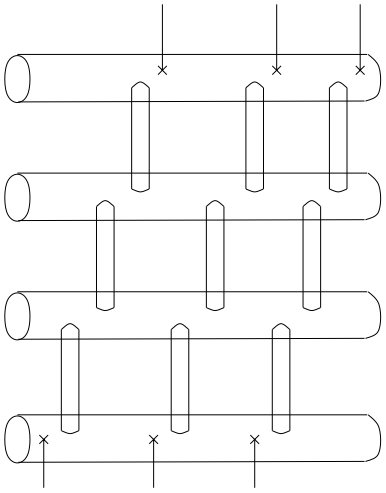

Figure X.9: The spectral curve $(X, \frac{1}{X}\ln(Y_0))$ has the following structure: $X(z)$ is a meromorphic function of degree $m+1$ on a curve of genus $\mathfrak{g} = nm$. Therefore it has $m+1$ poles and $m+1$ zeroes. It provides a branched covering of $\mathbb{CP}^1$. We prefer to represent $\ln X$ instead of $X$, and thus we have $m+1$ copies of the $\ln X$-cylinder. In each sheet there is one zero and one pole of $X$. $Y_0$ is a meromorphic function of degree $n+1$, so that it has $n+1$ zeroes in sheet 0, and $n+1$ poles in sheet $m+1$. We recognize the mirror curve $\mathscr{S}_{\hat{\mathfrak{X}}_0}$, which is a thickening of the toric web diagram.

As reviewed in section X, we can compute the corresponding symplectic invariants $F_g(\check{\mathscr{S}}_{MM})$, which assemble to yield the matrix model partition function [83],

$$\ln Z = \sum_g g_s^{2g-2} F_g(\mathscr{S}_{MM}) \,. \tag{X.81}$$

The symplectic transformation $\mathfrak{R}$ of section X maps $(\mathscr{C}, x, y_0)$ to $(\mathscr{C}, x, \frac{1}{x}\ln Y_0)$ order by order in $g_s$. $\mathfrak{F}$ maps this to $(\mathscr{C}, X, \frac{1}{X}\ln Y_0)$, and a second application of $\mathfrak{F}$ yields

$$\hat{\mathscr{S}}_{MM} = (\mathscr{C}, \ln X, \ln Y_0) \,. \tag{X.82}$$

By the symplectic invariance of the $F_g$, we therefore have, order by order in powers of $g_s$,

$$F_g(\mathscr{S}_{MM}) = F_g(\hat{\mathscr{S}}_{MM}) \,. \tag{X.83}$$

Since our matrix model was engineered to reproduce the Gromov-Witten invariants of $\mathfrak{X}_0$, we have arrived at

$$GW_g(\mathfrak{X}_0) = F_g(\mathscr{C}, \ln X, \ln Y_0) \,, \tag{X.84}$$

with $X$ and $Y_0$ obeying the algebraic equation

$$H_0(X,Y_0)=0 \tag{X.85}$$

which coincides with the equation (X.6) describing the mirror curve of $\mathfrak{X}_0$.

Given our minimality assumptions on the spectral curve, we have thus derived the BKMP conjecture for the fiducial geometry $\mathfrak{X}_0$.

### 5.5 The small $q$ limit and the thickening prescription

The above derivation of the spectral curve for the matrix model is not fully rigorous, as we have relied on making minimal assumptions along the way. Although the spectral curve we have found here satisfies all the constraints of section X, to prove that it is the spectral curve of our matrix model requires a uniqueness result which we currently do not have.

In this section, we provide a heuristic argument that the qualitative behavior of the spectral curve and the mirror curve coincide at small $q$.

At small $q$, only very small partitions contribute to the matrix integral. Almost all eigenvalues of $M_i$ are frozen to the values $q^{a_{j,i}+d-l}$. By the arguments in section X, the resolvent $W_i(x)$ hence behaves at small $q$ as

$$W_i(x)\sim\sum_{j=0}^{n}\sum_{l=1}^{d}\frac{1}{x-q^{a_{j,i}-l+d}}+\text{small cut near}\,q^{a_{j,i}+d}\,. \tag{X.86}$$

Pictorially, the size of the cuts is shrinking in this limit, replacing the spectral curve by its skeleton, see figure X.10.

On the other hand, the mirror curve is a priori a tree level quantity, hence does not depend on $q=e^{-g_s}$. However, recall that we have defined the Kähler parameters $Q$ associated to a curve $\mathscr{C}$ as

$$Q=q^{\int_{\mathscr{C}}J}\,. \tag{X.87}$$

The large $q$ limit hence corresponds to the large curve class limit, i.e. the distance between the vertices of the pairs of pants out of which the mirror curve is constructed is taken to infinity. Just as the spectral curve, the mirror curve thus collapses to its skeleton in the $q\to 0$ limit.

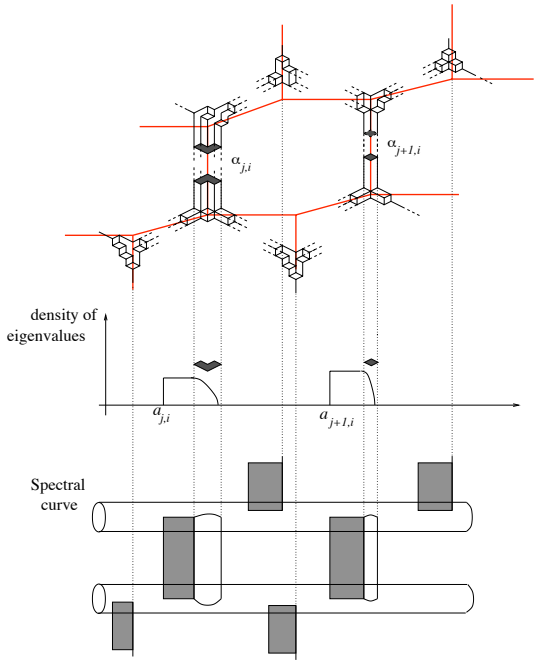


Figure X.10: In the small $q$ limit, only very small partitions contribute to the matrix integral, therefore the density of eigenvalues of $M_i$ tends to the flat density (a Dirac comb of equidistant delta functions), the non-flat part, which reflects the cuts of the spectral cut, shrinks to zero.

## 6 The general BKMP conjecture

So far, we have obtained the BKMP conjecture only for the fiducial geometry $\mathfrak{X}_0$. Studying the behavior of the partition function under flop transitions will allow us to extend our argument to arbitrary toric geometries.

### 6.1 Flop invariance of toric Gromov-Witten invariants

Under the proper identification of curve classes, Gromov-Witten invariants (at least on toric manifolds) are invariant under flops. Assume the toric Calabi-Yau manifolds $\mathfrak{X}$ and $\mathfrak{X}^+$ are related via a flop transition, $\phi : \mathfrak{X} \to \mathfrak{X}^+$. In a neighborhood of the flopped $(-1,-1)$ curve, the respective toric diagrams are depicted in figure X.11.

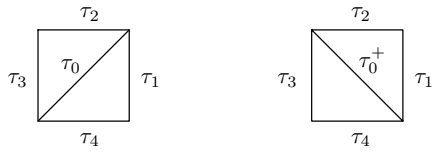


Figure X.11: $\mathfrak{X}$ and $\mathfrak{X}^+$ in the vicinity of the (-1,-1) curve.

The 1-cones of $\Sigma_{\mathfrak{X}}$, corresponding to the toric invariant divisors of $\mathfrak{X}$, are not affected by the flop, hence can be canonically identified with those of $\mathfrak{X}^+$. The 2-cones $\tau_i$ in these diagrams correspond to toric invariant 2-cycles $C_i, C_i^+$ in the geometry. The curve classes of $\mathfrak{X}$ push forward to classes in $\mathfrak{X}^+$ via

$$\phi_*([C_0]) = -[C_0^+]\,, \quad \phi_*([C_i]) = [C_i^+] + [C_0^+]\,. \tag{X.88}$$

All other curve classes $\vec{C}$ of $\mathfrak{X}$ are mapped to their canonical counterparts in $\mathfrak{X}^+$. Under appropriate analytic continuation and up to a phase factor (hence the $\propto$ in the following formula), the following identity then holds [77, 128, 131],

$$Z_{GW}(\mathfrak{X}, Q_0, Q_1, \ldots, Q_4, \vec{Q}) \propto Z_{GW}(\mathfrak{X}^+, 1/Q_0, Q_0Q_1, \ldots, Q_0Q_4, \vec{Q})\,,$$

i.e.

$$GW_g(\mathfrak{X}, Q_0, Q_1, \ldots, Q_4, \vec{Q}) = GW_g(\mathfrak{X}^+, 1/Q_0, Q_0Q_1, \ldots, Q_0Q_4, \vec{Q})\,.$$

### 6.2 Proof of flop invariance via mirror symmetry

Flop invariance of Gromov-Witten invariants upon the identification (X.88) is immediate upon invoking mirror symmetry, as (X.88) maps the mirror curve of $\mathfrak{X}$ to that of $\mathfrak{X}^+$. The proof is a simple computation.

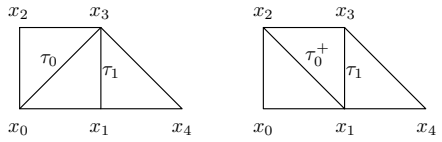


Figure X.12: $\mathfrak{X}$ and $\mathfrak{X}^+$ in the vicinity of the (-1,-1) curve.

Let us introduce the notation $t_0, t_1, t_0^+, t_1^+$ for the Kähler volume of the curve classes $C_i, C_i^+$ corresponding to the respective 2-cones. In terms of these, we obtain for the mirror curve of $\mathfrak{X}$

$$x_0 + x_1 + x_2 + \frac{x_1x_2}{x_0}e^{T_0} + \frac{x_1^2}{x_0}e^{-T_1} = 0\,, \tag{X.89}$$

while the mirror curve of $\mathfrak{X}^+$ is given by

$$x_0 + x_1 + x_2 + \frac{x_1x_2}{x_0}e^{-T_0^+} + \frac{x_1x_3}{x_2}e^{-T_1^+} = 0\,. \tag{X.90}$$

Upon invoking $x_3 = \frac{x_1 x_2}{x_0} e^{-T_0^+}$, we easily verify that the identification (X.88) maps these curves and their associated meromorphic 1-forms $\lambda$ into each other.

### 6.3 The BKMP conjecture

Any toric Calabi-Yau manifold $\mathfrak{X}$ with Kähler moduli $\vec{Q}$ can be obtained from a sufficiently large fiducial geometry $(\mathfrak{X}_0, \vec{Q}_0)$ upon performing a series of flop transitions and taking unwanted Kähler moduli of $\mathfrak{X}_0$ to $\infty$, see figure X.13 for an example.

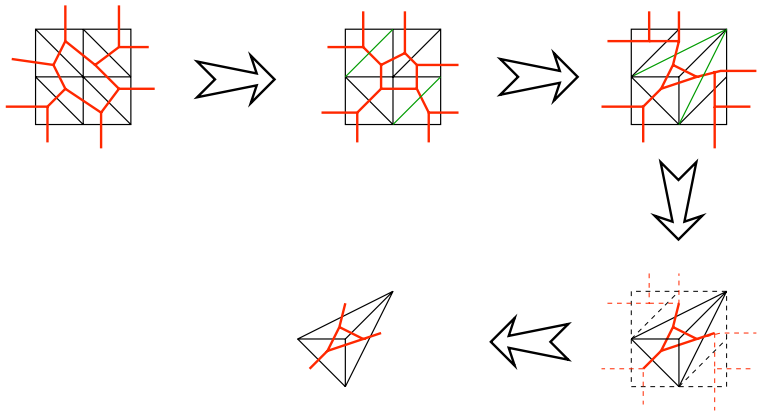

Figure X.13: Example: We obtain local $\mathbb{P}^2$ from the fiducial geometry with $2 \times 2$ boxes by performing five flops and then sending the Kähler parameters of the unwanted edges to $\infty$.

The Kähler moduli of $\mathfrak{X}$ are related to those of $\mathfrak{X}_0$ by some relation $\vec{Q} = f(\vec{Q}_0)$. We have just argued that the mirror curves of $\mathfrak{X}_0$ and $\mathfrak{X}$ are equal upon this identification,

$$\mathscr{S}_{\mathfrak{X},\vec{Q}} = \mathscr{S}_{\mathfrak{X}_0,\vec{Q}_0}\,, \tag{X.91}$$

as are the respective Gromov-Witten invariants,

$$GW_g(\mathfrak{X},\vec{Q}) = GW_g(\mathfrak{X}_0,\vec{Q}_0)\,. \tag{X.92}$$

Given the BKMP conjecture for the fiducial geometry,

$$GW_g(\mathfrak{X}_0,\vec{Q}_0) = F_g(\mathscr{S}_{\mathfrak{X}_0,\vec{Q}_0})\,, \tag{X.93}$$

its validity thus follows for any toric Calabi-Yau manifold:

$$\boxed{GW_g(\mathfrak{X},\vec{Q}) = F_g(\mathscr{S}_{\mathfrak{X},\vec{Q}})} \tag{X.94}$$

## 7 Conclusion

Taking our matrix model from [68] as a starting point and imposing certain minimality conditions on the spectral curve, we have thus derived the BKMP conjecture, for closed topological strings, for all toric Calabi-Yau manifolds in the large radius limit. As we have emphasized throughout, elevating our procedure to a formal proof of the conjecture requires a more rigorous derivation of the spectral curve of our matrix model.

It should also be possible to extend our argument to open Gromov-Witten invariants by invoking loop operators, which relate closed to open invariants. In [23], such an operator was defined in the matrix model context. An analogous operator should also exist in the theory of Gromov-Witten invariants. Establishing the equivalence of these two loop operators would allow us to conclude that the $W_n^{(g)}$'s of the spectral curve $\mathscr{S}_{\mathfrak{X}}$ are the open Gromov-Witten invariants of $\mathfrak{X}$.

Finally, our treatment of the BKMP conjecture took place at large radius. One should study the behavior of the matrix model as one moves away from large radius e.g. to orbifold points, and see whether the phase transitions of the topological string are captured accurately by the matrix model. Of course, the main tool on the topological string side employed in this work, the topological vertex, is no longer applicable in these regions of moduli space.

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## Appendix: The matrix model

In this appendix, which is mainly a reprint of section 4 of [68], we present the matrix model which reproduces the topological string partition function on the fiducial geometry $\mathfrak{X}_0$, and whose spectral curve we derive in the text.

SConsider the fiducial geometry $\mathfrak{X}_0$ of size $(n+1)\times(m+1)$, with Kähler parameters $t_{i,j} = a_{i,j} - a_{i,j+1}$, $r_{i,j} = a_{i,j+1} - a_{i+1,j}$, and $s_{i,j}$, as depicted in figure X.1. We write

$$\vec{a}_i = (a_{0,i}, a_{1,i}, \ldots, a_{n,i}). \tag{X.95}$$

Assume that the external representations are fixed to $\vec{\alpha}_{m+1} = (\alpha_{0,m+1}, \alpha_{1,m+1}, \ldots, \alpha_{n,m+1})$ on the upper line, and $\vec{\alpha}_0 = (\alpha_{0,0}, \alpha_{1,0}, \ldots, \alpha_{n,0})$ on the lower line. For the most part, we will choose these to be trivial.

We now define the following matrix integral $\mathscr{Z}_{\rm MM}$ (MM for Matrix Model),

$$\begin{aligned}
\mathscr{Z}_{\rm MM}(Q, g_s, \vec{\alpha}_{m+1}, \vec{\alpha}_0^T) \;=\; & \Delta(X(\vec{\alpha}_{m+1}))\,\Delta(X(\vec{\alpha}_0)) \prod_{i=0}^{m+1} \int_{H_N(\Gamma_i)} dM_i \prod_{i=1}^{m+1} \int_{H_N(\mathbb{R}_+)} dR_i \\
& \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,\mathrm{tr}\left[V_{\vec{a}_i}(M_i) - V_{\vec{a}_{i-1}}(M_i)\right]} \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,\mathrm{tr}\left[V_{\vec{a}_{i-1}}(M_{i-1}) - V_{\vec{a}_i}(M_{i-1})\right]} \\
& \prod_{i=1}^{m+1} e^{\frac{1}{g_s}\mathrm{tr}(M_i - M_{i-1})R_i} \prod_{i=1}^{m} e^{(S_i + \frac{i\pi}{g_s})\,\mathrm{tr}\ln M_i} \\
& e^{\mathrm{trln} f_0(M_0)}\; e^{\mathrm{trln} f_{m+1}(M_{m+1})} \prod_{i=1}^{m} e^{\mathrm{trln} f_i(M_i)}\,.
\end{aligned} \tag{X.96}$$

All matrices are taken of size

$$N = (n+1)\,d\,. \tag{X.97}$$

$d$ denotes a cut-off on the size of the matrices, on which, as discussed in section X, the partition function depends only non-perturbatively. We have introduced the notation

$$X(\vec{\alpha}_{m+1}) = \mathrm{diag}(X(\vec{\alpha}_{m+1})_i)_{i=1,\ldots,N}\,, \qquad X(\vec{\alpha}_{m+1})_{jd+k} = q^{h_k(\alpha_{j,m+1})}, \tag{X.98}$$

$$X(\vec{\alpha}_0) = \mathrm{diag}(X(\vec{\alpha}_0)_i)_{i=1,\ldots,N}\,, \qquad X(\vec{\alpha}_0)_{jd+k} = q^{h_k(\alpha_{j,0})}, \tag{X.99}$$

for $k = 1, \ldots, d$, $j = 0, \ldots, n$, where

$$h_i(\gamma) = \gamma_i - i + d + a\,. \tag{X.100}$$

$\Delta(X) = \prod_{i<j}(X_i - X_j)$ is the Vandermonde determinant. The potentials $V_{\vec{a}_i}(x)$ are given by

$$V_{\vec{a}}(X) = -g_s \sum_{j=0}^{n} \ln(g(q^{a_j}/X)) \tag{X.101}$$

in terms of the $q$-product

$$g(x) = \prod_{n=1}^{\infty}(1 - \frac{1}{x}q^n)\,.$$

For $i = 1, \ldots, m$, we have defined

$$f_i(x) = \prod_{j=0}^{n} \frac{g(1)^2\, e^{(\frac{1}{2}+\frac{i\pi}{\ln q})\ln(xq^{1-a_{j,i}})}\, e^{\frac{(\ln(xq^{1-a_{j,i}}))^2}{2g_s}}}{g(xq^{1-a_{j,i}})\, g(q^{a_{j,i}}/x)}\,. \tag{X.102}$$

The denominator of these functions induces simple poles at $x = q^{a_{j,i}+l}$ for $j = 0, \ldots, n$ and $l \in \mathbb{Z}$. The numerator is chosen such that they satisfy the relation $f_i(qx) = f_i(x)$. This enforces a simple $l$-dependence of the residues taken at $x = q^{a_{j,i}+l}$, given by a prefactor $q^l$ – a fact which will be important in the following. These residues are in fact given by

$$\operatorname*{Res}_{q^{a_{j,i}+l}} f_i(x) = q^{a_{j,i}+l}\, \hat{f}_{j,i} = -\, q^{a_{j,i}+l} \prod_{k\neq j} \frac{g(1)^2\, e^{(\frac{1}{2}+\frac{i\pi}{\ln q})(1+a_{j,i}-a_{k,i})\ln q}\, e^{\frac{(\ln(q^{1+a_{j,i}-a_{k,i}}))^2}{2g_s}}}{g(q^{a_{j,i}-a_{k,i}})\,(1-q^{a_{k,i}-a_{j,i}})g(q^{a_{k,i}-a_{j,i}})}\,, \tag{X.103}$$

where $\hat{f}_{j,i}$ is independent of the integer $l$.

The parameters $S_i$ are defined by

$$S_i = s_{0,i-1} + t_{0,i-1} = s_{j,i-1} - \sum_{k<j} t_{k,i} + \sum_{k\leq j} t_{k,i-1}\,. \tag{X.104}$$

The final equality holds for arbitrary $j$ [68].

For $i = 0$ and $i = m+1$, we define

$$f_0(x) = \frac{1}{\prod_{j=0}^{n}\prod_{i=1}^{d}(x - q^{h_i(\alpha_{j,0})})}\,, \tag{X.105}$$

$$f_{m+1}(x) = \frac{1}{\prod_{j=0}^{n}\prod_{i=1}^{d}(x - q^{h_i(\alpha_{j,m+1})})}\,. \tag{X.106}$$

Notice that if the representations $\vec{\alpha}_0$ or $\vec{\alpha}_{m+1}$ are trivial, i.e. $h_i(\alpha_{j,0}) = d - i + a_{j,0}$ or $h_i(\alpha_{j,m+1}) = d - i + a_{j,m+1}$, we have

$$f_0(x) = \prod_{j=0}^{n} \frac{g(xq^{1-a_{j,0}-d})}{x^d\, g(xq^{1-a_{j,0}})}, \qquad f_{m+1}(x) = \prod_{j=0}^{n} \frac{g(xq^{1-a_{j,m+1}-d})}{x^d\, g(xq^{1-a_{j,m+1}})} \tag{X.107}$$

respectively. The functions $f_0$ and $f_{m+1}$ have simple poles at $x = q^{h_l(\alpha_{j,0})}$ (resp. $x = q^{h_l(\alpha_{j,m+1})}$) for $l = 1, \dots, d$, with residue

$$\hat{f}_{j,0;l} = \operatorname*{Res}_{q^{h_l(\alpha_{j,0})}} f_0(x) = \frac{1}{\prod_{j' \neq j} \prod_{i=1}^{d} (q^{h_l(\alpha_{j,0})} - q^{h_i(\alpha_{j',0})})} \frac{1}{\prod_{i \neq l} (q^{h_l(\alpha_{j,0})} - q^{h_i(\alpha_{j,0})})}, \tag{X.108}$$

$$\hat{f}_{j,m+1;l} = \operatorname*{Res}_{q^{h_l(\alpha_{j,m+1})}} f_{m+1}(x) = \frac{1}{\prod_{j' \neq j} \prod_{i=1}^{d} (q^{h_l(\alpha_{j,m+1})} - q^{h_i(\alpha_{j',m+1})})} \frac{1}{\prod_{i \neq l} (q^{h_l(\alpha_{j,m+1})} - q^{h_i(\alpha_{j,m+1})})}. \tag{X.109}$$

The $l$ dependence here is more intricate than above, but this will not play any role since the partitions $\alpha_{j,0}$ and $\alpha_{j,m+1}$ are kept fixed, and not summed upon.

The integration domains for the matrices $R_i$ are $H_N(\mathbb{R}_+^N)$, i.e. the set of hermitian matrices having only positive eigenvalues. For the matrices $M_i, i = 1, \dots, m$, the integration domains are $H_N(\Gamma_i)$, where

$$\Gamma_i = \prod_{j=0}^{n} (\gamma_{j,i})^d\,. \tag{X.110}$$

$\gamma_{j,i}$ is defined as a contour which encloses all points of the form $q^{a_{j,i}+\mathbb{N}}$, and does not intersect any contours $\gamma_{k,l}$, $(j,i) \neq (k,l)$. For this to be possible, we must require that the differences $a_{j,i} - a_{j',i'}$ be non-integer. The normalized logarithms of two such contours are depicted in figure X.14.

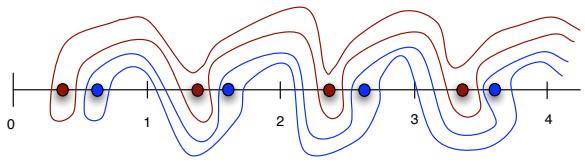


Figure X.14: Two contours surrounding points $a + \mathbb{N}$ and $b + \mathbb{N}$, such that $a - b \notin \mathbb{Z}$.

We have defined

$$H_N(\Gamma_i) = \{M = U \Lambda U^\dagger, \quad U \in U(N),\ \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N) \in \Gamma_i\}, \tag{X.111}$$

i.e. $H_N(\Gamma_i)$ is the set of normal matrices with eigenvalues on $\Gamma_i$. By definition, the measure on $H_N(\Gamma_i)$ is (see [19])

$$dM = \frac{1}{N!}\,\Delta(\Lambda)^2\,dU\,d\Lambda\,, \tag{X.112}$$

where $dU$ is the Haar measure on $U(N)$, and $d\Lambda$ is the product of the measures for each eigenvalue along its integration path.

The integration domains for the matrices $M_0$, $M_{m+1}$ are $H_N(\Gamma_0)$, $H_N(\Gamma_{m+1})$ respectively, where

$$\Gamma_0 = (\sum_{j=0}^{n} \gamma_{j,0})^N\,, \qquad \Gamma_{m+1} = (\sum_{j=0}^{n} \gamma_{j,m+1})^N\,. \tag{X.113}$$