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Clones de constantes et de permutations et leur intervalle monoïdal

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SOMMAIRE

En 1941, E. Post présenta le treillis des clones sur deux éléments [**29**]. Depuis, on cherche à connaître les clones plus généralement, ce qui permettrait de mieux comprendre les algèbres universelles. En effet, à chaque algèbre correspond un clone : le clone de ses opérations termes. Malheureusement, le treillis des clones sur un nombre plus grand d'éléments est beaucoup plus compliqué. Même pour trois éléments, il contient déjà $2^{\aleph_0}$ clones et sa structure est très riche et peu connue.

Durant les vingt-cinq dernières années, certains chercheurs ont étudié les intervalles monoïdaux. Ce sont des ensembles de clones ayant en commun le même monoïde d'opérations unaires ; ils forment des intervalles dans le treillis des clones. On connaît maintenant la cardinalité de la plupart des intervalles monoïdaux sur trois éléments et la forme de certains intervalles. Plusieurs des résultats sur trois éléments se généralisent à plus de trois éléments.

Mes recherches de doctorat se sont concentrées sur l'étude des intervalles monoïdaux pour les monoïdes de constantes et de permutations. Ces monoïdes sont particulièrement intéressants puisque le clone des opération termes de plusieurs algèbres connues se retrouvent dans ces intervalles. Ceci m'a mené à la découverte de certains monoïdes affaissants (dont l'intervalle monoïdal ne contient qu'un

clone) et de morceaux de certains intervalles infinis. Au cours de mes travaux, j'ai trouvé certaines relations préservées seulement par des opérations essentiellement unaires.

La présente thèse est par articles. Elle est composée d'une introduction en français et du texte intégral (en anglais) de quatres articles, chacun introduit par un court texte en français. Cette forme de thèse mène inévitablement à certaines répétitions : répétitions de définitions et d'idées de base d'un article à l'autre, répétitions de concepts entre les sections françaises et anglaises. Les articles inclus dans la thèse sont :

– *Collapsing monoids containing permutations and constants*,
– *The monoidal interval for the monoid generated by two constants*,
– *Clones on three elements preserving a binary relation*,
– *The clone of operations preserving a cycle with loops.*

Mots-clés

clone, intervalle monoïdal, monoïde affaissant, opération préservant une relation.

SUMMARY

In 1941, Post presented the lattice of clones on 2 elements [**29**]. Since then, researchers have been trying to understand clones more generally. This would advance our knowledge of universal algebras, since for each algebra there is a corresponding clone : the clone of its term operations. Unfortunately, the lattice of clones on bigger universes is much more complicated. Even for 3 elements, it contains $2^{\aleph_0}$ clones and the structure of the lattice is very rich and generally unknown.

For the last 25 years, certain researchers have been studying monoidal intervals. These are sets of clones sharing the same monoid of unary operations ; they form intervals in the lattice of clones. We now know the cardinality of most of the monoidal intervals on 3 elements and the shape of some of them. Furthermore, several results on 3 elements generalize to more than 3 elements.

My research concentrated on monoidal intervals for monoids of permutations and constants. These are particularly interesting since the clones of term operations of several known algebras are found in these intervals. This led me to discover some collapsing monoids i.e. monoids with singleton monoidal intervals, and parts of certains infinite intervals. In the course of this research, I found certain relations which are preserved only by essentially unary operations.

This thesis is by articles. It is composed of a French introduction followed by the full text (in English) of four articles each introduced by a short explanatory text in French. This type of thesis inevitably leads to repetitions ; notably definitions and basic ideas repeated in each article, as well as key concepts repeated in the French and English sections. The four articles included in the thesis are :

- *Collapsing monoids containing permutations and constants*,
- *The monoidal interval for the monoid generated by two constants*,
- *Clones on three elements preserving a binary relation*,
- *The clone of operations preserving a cycle with loops.*

Keywords

clone, monoidal interval, collapsing monoid, operation preserving a relation.

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INTRODUCTION

0.1. Logique symbolique

On attribue généralement les débuts de la logique symbolique à G. Boole qui publia *The Mathematical Analysis of Logic* en 1847. Les travaux de W.S. Jevons, C.S. Peirce, E. Schroeder et A.Whitehead modifièrent et élaborèrent le système de Boole. En particulier, on proposait d'autres opérations de base et on s'intéressait à la *complétude* de ces nouveaux systèmes de logique ; à savoir si on peut obtenir toutes les phrases logiques possibles à partir des opérations de base. À cette fin, dans les années 1920, E. Post (entre autres) créa un nouvel outil, les tables de vérité qui sont devenues la façon standard de représenter et de calculer les phrases logiques. Dans la même période, J. Łukasiewicz proposa une logique à trois valeurs et E. Post proposa une logique avec un nombre quelconque mais fini de valeurs.

Les recherches sur la complétude et surtout l'identification des phrases logiques à leur table de vérité marquèrent un tournant en logique. On étudiait les opérations elles-mêmes, vues comme des fonctions à plusieurs variables, plutôt que les énoncés logiques et leur interprétation.

Formellement, pour $n \in \mathbb{N}$, une *opération n-aire* sur un ensemble A (nommé *univers*) est une fonction $f : A^n \to A$. On dénote généralement par $\mathcal{O}_A^{(n)}$ l'ensemble des opérations n-aires sur A et on pose $\mathcal{O}_A := \bigcup_{0<n<\omega} \mathcal{O}_A^{(n)}$. Une opération $f \in \mathcal{O}^{(n)}$ *dépend* de sa première variable s'il existe $x, y, x_2 \ldots, x_n \in A$ tels que $f(x, x_2 \ldots, x_n) \neq f(y, x_2, \ldots, x_n)$. Une opération est *essentiellement unaire* si elle dépend d'au plus une variable. On compose ces opérations les unes avec les autres comme suit : soient $f \in \mathcal{O}^{(n)}$ et $g_1, \ldots, g_n \in \mathcal{O}^{(m)}$, leur *composition* est l'opération m-aire $f[g_1, \ldots, g_n]$ définie par

$$f[g_1, \ldots, g_n](x_1, \ldots, x_m) = f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$$

0.2. Clones sur deux éléments : systèmes itérés de Post

En partant de la complétude des logiques symboliques, E. Post s'intéressa aux systèmes non complets. Dans *The two-valued iterative systems of mathematical logic* **[29]**, il décrivit tous les systèmes (ensembles) fermés (pour la composition) de fonctions de vérité. C'est le début de la théorie des clones.

Un *clone* sur A est un sous-ensemble C de $\mathcal{O}_A$ fermé pour la composition et contenant toutes les projections. Nous rappelons que pour $1 \leq i \leq n$, la *i-ème projection* n-aire est $e_i^{(n)} \in \mathcal{O}_A^{(n)}$ définie par $e_i^{(n)}(x_1, \ldots, x_n) = x_i$ pour tous $x_1, \ldots, x_n \in A$. On écrit simplement e pour dénoter l'opération identité $e_1^{(1)}$.

Un clone sur deux éléments $\{0, 1\}$ est donc un système de fonctions de vérité, mais qui en plus contient toutes les projections. Ces clones se retrouvent dans la figure 0.1. Leur description se trouve dans les tableaux 0.1, 0.2 et 0.3 où chaque clone est décrit par un système de générateurs. Pour tout $F \subseteq \mathcal{O}_A$, il existe le

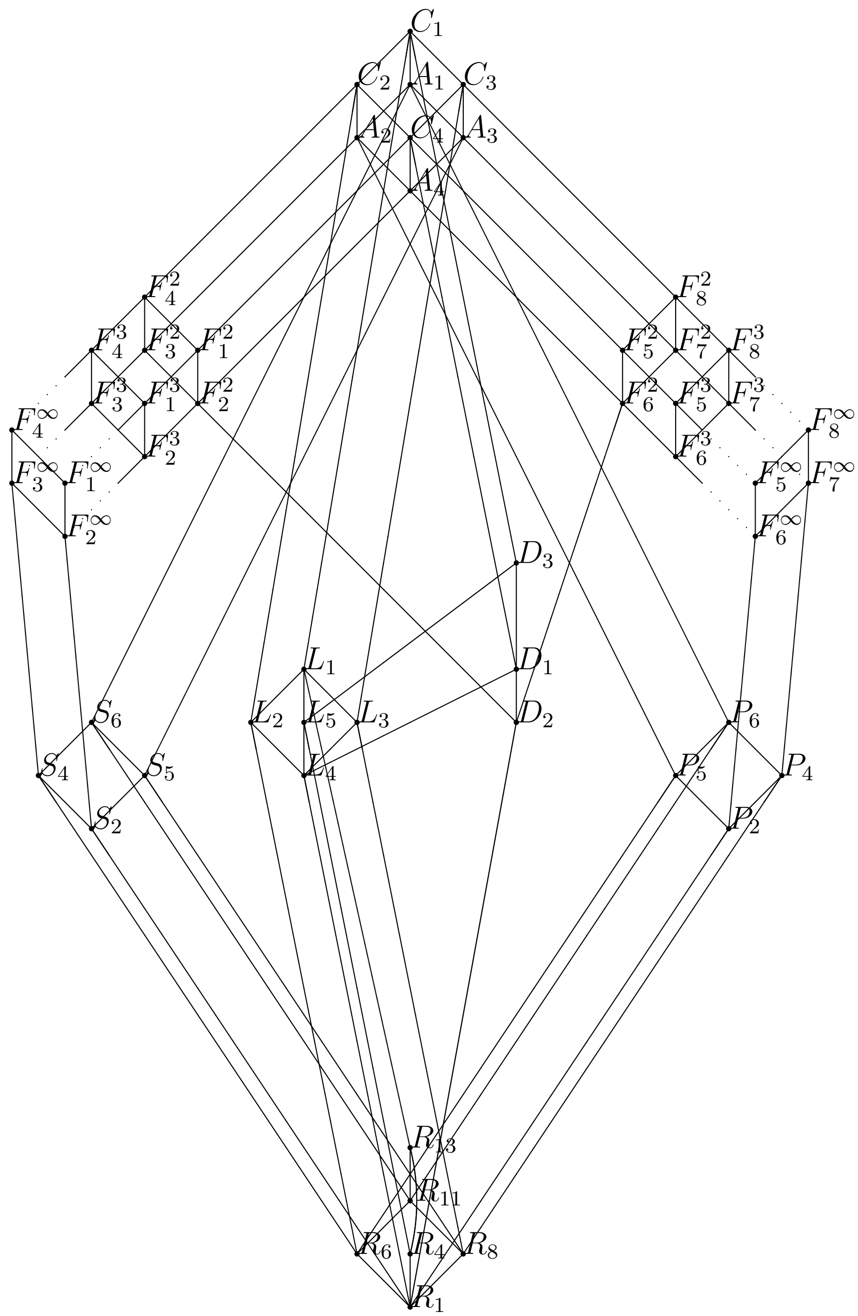


FIG. 0.1. Les clones sur deux éléments

plus petit clone contenant F, le clone *engendré* par F, dénoté par $\langle F \rangle$. On dit alors que les opérations dans F sont des *générateurs* du clone. Voici la liste des opérations utilisées dans les tableaux.

(1) L'identité e et les constantes c_0 et c_1,

(2) Les opérations de logique symbolique : $\vee, \wedge, \neg, \leftrightarrow$, et autres écrites sous forme de phrase logique,

(3) Les additions binaires et ternaires mod 2 : $+$ et $+^{(3)}$,

(4) L'opération majorité, maj, définie par $\text{maj}(x, y, z) = i$ si $x = y = i$ ou $x = z = i$ ou $y = z = i$,

(5) Les opérations n-aires h_n et H_n pour $n = 2, 3, \ldots$, définies par

$$h_n(x_1, \ldots, x_{n+1}) = 0 \text{ ssi } \sum_{i=1}^{n+1} x_i \leq 1$$

$$H_n(x_1, \ldots, x_{n+1}) = 1 \text{ ssi } \sum_{i=1}^{n+1} x_i \geq n$$

	Générateurs		Générateurs		Générateurs
C_1	$\langle \vee, \wedge, \neg \rangle$	D_1	$\langle maj, +^{(3)} \rangle$		
C_2	$\langle \wedge, \leftrightarrow \rangle$	D_2	$\langle maj \rangle$	R_1	$\langle e \rangle = \langle \emptyset \rangle$
C_3	$\langle \vee, + \rangle$	D_3	$\langle maj, \neg \rangle$	R_4	$\langle \neg \rangle$
C_4	$\langle \vee, \wedge, +^{(3)} \rangle$	L_1	$\langle +, \neg \rangle$	R_6	$\langle c_1 \rangle$
A_1	$\langle \vee, \wedge, c_0, c_1 \rangle$	L_2	$\langle \leftrightarrow \rangle$	R_8	$\langle c_0 \rangle$
A_2	$\langle \vee, \wedge, c_1 \rangle$	L_3	$\langle + \rangle$	R_{11}	$\langle c_0, c_1 \rangle$
A_3	$\langle \vee, \wedge, c_0 \rangle$	L_4	$\langle +^{(3)} \rangle$	R_{13}	$\langle c_0, c_1, \neg \rangle$
A_4	$\langle \vee, \wedge \rangle$	L_5	$\langle +^{(3)}, \neg \rangle$		

TAB. 0.1. Description des clones C, A, D, L et R

	Générateurs		Générateurs
S_2	$\langle \vee \rangle$	P_2	$\langle \wedge \rangle$
S_4	$\langle \vee, c_1 \rangle$	P_4	$\langle \wedge, c_0 \rangle$
S_5	$\langle \vee, c_0 \rangle$	P_5	$\langle \wedge, c_1 \rangle$
S_6	$\langle \vee, c_0, c_1 \rangle$	P_6	$\langle \wedge, c_0, c_1 \rangle$

TAB. 0.2. Description des clones S et P

	Générateurs		Générateurs
F_1^2	$\langle maj, a \vee (b \wedge \neg c) \rangle$	F_5^2	$\langle maj, a \wedge (b \vee \neg c) \rangle$
F_1^n	$\langle h_n, a \vee (b \wedge \neg c) \rangle$	F_5^n	$\langle H_n, a \wedge (b \vee \neg c) \rangle$
F_1^∞	$\langle a \vee (b \wedge \neg c) \rangle$	F_5^∞	$\langle a \wedge (b \vee \neg c) \rangle$
F_2^2	$\langle maj, \vee \rangle$	F_6^2	$\langle maj, \wedge \rangle$
F_2^n	$\langle h_n \rangle$	F_6^n	$\langle H_n \rangle$
F_2^∞	$\langle a \vee (b \wedge c) \rangle$	F_6^∞	$\langle a \wedge (b \vee c) \rangle$
F_3^2	$\langle maj, c_1 \rangle$	F_7^2	$\langle maj, c_0 \rangle$
F_3^n	$\langle h_n, c_1 \rangle$	F_7^n	$\langle H_n, c_0 \rangle$
F_3^∞	$\langle a \vee (b \wedge c), c_1 \rangle$	F_7^∞	$\langle a \wedge (b \vee c), c_0 \rangle$
F_4^2	$\langle maj, a \vee \neg b \rangle$	F_8^2	$\langle maj, a \wedge \neg b \rangle$
F_4^n	$\langle h_n, a \vee \neg b \rangle$	F_8^n	$\langle H_n, a \wedge \neg b \rangle$
F_4^∞	$\langle a \vee \neg b \rangle$	F_8^∞	$\langle a \wedge \neg b \rangle$

TAB. 0.3. Description des clones F

0.3. Termes

Au début du vingtième siècle, certaines technologies devinrent tellement compliquées qu'elles durent être étudiées mathématiquement. C'était le cas de l'aiguillage des trains, des circuits électriques, des centrales téléphoniques et plus tard des ordinateurs, entres autres. On partait de composantes simples (types d'aiguillage, portes logiques), et on construisait des systèmes plus compliqués (les chemins de fer de la France, un ordinateur). Le problème était de savoir comment obtenir le résultat voulu à partir des composantes. On s'aperçut qu'on pouvait se représenter les composantes comme des symboles et étudier le problème algébriquement en concaténant ces symboles d'après certaines règles. Dans les années 1930, on fit le lien avec la logique symbolique.

Le problème devenait donc : comment construire une phrase logique, un *terme*, à partir des opérations (composantes) de base. On s'intéressa aussi (surtout en informatique) à quelles opérations possibles on pouvait obtenir à partir des opérations de base. Dans ce cas, la notion d'opération terme est plus utile. Une *opération terme* est une opération obtenue par la composition des opérations de base d'un système donné. L'ensemble de ces opérations termes est donc fermé pour la composition et contient les projections (ceci correspond à pouvoir introduire des variables fictives) ; c'est un clone.

La venue des ordinateurs a donné un nouvel essor à la théorie des clones. De plus, commençant à la fin des années 1950, certains informaticiens s'intéressèrent aux systèmes à plus de deux valeurs. Ils construisirent des portes logiques, et même des ordinateurs basés sur trois valeurs. Ceci poussa les mathématiciens à mieux connaître les clones sur plus de deux éléments.

0.4. Algèbre universelle

En algèbre universelle, on étudie des algèbres en général. Une *algèbre universelle* $\mathcal{A} = \langle A; F \rangle$ est composée d'un ensemble de base ou univers A et d'un ensemble d'opérations de base $F \subseteq \mathcal{O}_A$. L'idée avait déjà été anticipée dans *Universal Algebra* en 1898. A.N. Whitehead y exposa des ressemblances entre les algèbres connues : logique symbolique, arithmétique, algèbre linéaire, etc. Mais l'algèbre universelle ne prit son essor qu'à partir des années 1930 avec les développements en algèbre abstraite initiés par E. Noether. Dans l'étude de l'algèbre universelle, l'approche la plus évidente est de grouper les algèbres en *variétés* ; en classes d'algèbres dont les opérations de base sont de même type et obéissent à certaines règles. Par exemple, on a la variété des groupes, celle des treillis, etc.

Une autre approche consiste à considérer quelles sont toutes les algèbres possibles sur un univers donné A. Pour ce faire, on considère habituellement des algèbres comme étant équivalentes si les opérations de base de l'une peuvent être exprimées comme des opérations termes de l'autre et vice versa. Des algèbres équivalentes sont très semblables ayant les mêmes endomorphismes, congruences, sous-algèbres, etc.

C'est dans cette optique que la théorie des clones devient utile en algèbre universelle. À chaque algèbre $\langle A; F \rangle$, correspond donc le clone de ses opérations terme $\langle F \rangle$, et à chaque clone C, correspond l'algèbre $\langle A; C \rangle$. L'énumération de toutes les algèbres possibles se résume donc à l'énumération de tous les clones. Les algèbres sur un univers de deux éléments correspondent aux clones trouvés par Post. Par exemple L_3 est le clone des opérations termes du groupe $\langle \mathbb{Z}_2; +, -, 0\rangle$, et bien sûr, C_1 correspond à l'algèbre de Boole $\langle \{0,1\}; \wedge, \vee, \neg\rangle$.

Une autre notion qui apparaît régulièrement en algèbre abstraite est celle d'*invariant* ; une relation préservée par les opérations de base de l'algèbre. Rappelons qu'une *relation h-aire* ρ ($h \in \mathbb{N}^+$) est un sous-ensemble de A^h. L'ensemble des relations h-aires sur A est noté $\mathcal{R}_A^{(h)}$, et on pose $\mathcal{R}_A := \bigcup_{0<h<\omega} \mathcal{R}_A^{(h)}$. Maintenant, soient $f \in \mathcal{O}_A^{(n)}$ et $\rho \in \mathcal{R}_A$. L'opération f *préserve* ρ si pour tout $(a_{1,i}, a_{2,i}, \ldots, a_{h,i}) \in \rho$ ($i = 1, \ldots, n$),

$$(f(a_{1,1}, a_{1,2}, \ldots, a_{1,n}), f(a_{2,1}, a_{2,2}, \ldots, a_{2,n}), \ldots, f(a_{h,1}, a_{h,2}, \ldots, a_{h,n})) \in \rho$$

Un important exemple est celui de la *congruence* ; une relation d'équivalence sur A (c'est à dire une relation binaire réflexive, symétrique et transitive) qui est préservée par les opérations de base de l'algèbre. On se demande, par exemple, quelles sont les congruences d'un groupe donné. Plus généralement, on veut connaître les relations qui sont préservées par les opérations de base (et donc les opérations terme) d'une algèbre. L'ensemble de toutes les relations d'arité fini préservées par toutes les opérations dans un sous-ensemble F de $\mathcal{O}_A$, est l'ensemble des *invariants* de F et est dénoté par $\operatorname{Inv} F$.

Un autre exemple est celui des fonctions monotones qui préservent un ordre partiel sur A. On pourrait se demander quelles sont les opérations monotones pour un ordre partiel donné. C'est le problème dual de celui des invariants. Soit $\rho \in \mathcal{R}_A$. L'ensemble des opérations sur A qui préservent ρ est un clone dénoté par $\operatorname{Pol} \rho$. Plus généralement, pour $R \subseteq \mathcal{R}_A$, l'ensemble des opérations sur A qui préservent toutes les relations dans R est un clone dénoté par $\operatorname{Pol} R$. La notation provient du mot *polymorphisme* qui n'est guère utilisé de nos jours.

Pour un ensemble A donné, les fonctions Pol et Inv établissent une correspondance de Galois entre les sous-ensembles de $\mathcal{O}_A$ et ceux de $\mathcal{R}_A$.

Théorème 0.4.1 ([**4**], voir aussi [**36**] page 20-21 et [**6**] page 39)**.** *Soit A un ensemble.*

(1) *Si $F_1 \subseteq F_2 \subseteq \mathcal{O}_A$, alors $\operatorname{Inv} F_1 \supseteq \operatorname{Inv} F_2$, de même, si $S_1 \subseteq S_2 \subseteq \mathcal{R}_A$, alors $\operatorname{Pol} S_1 \supseteq \operatorname{Pol} S_2$.*

(2) *Pour $F \subseteq \mathcal{O}_A$ et $S \subseteq \mathcal{O}_A$, $F \subseteq \operatorname{Pol}\operatorname{Inv} F$ et $S \subseteq \operatorname{Inv}\operatorname{Pol} S$.*

(3) *Pour $F \subseteq \mathcal{O}_A$ et $S \subseteq \mathcal{O}_A$, $\operatorname{Inv} F = \operatorname{Inv}\operatorname{Pol}\operatorname{Inv} F$ et $\operatorname{Pol} S = \operatorname{Pol}\operatorname{Inv}\operatorname{Pol} S$.*

(4) *Si A est fini et $F \subseteq \mathcal{O}_A$, $\operatorname{Pol}\operatorname{Inv} F = \langle F \rangle$ est un clone.*

Cette correspondance de Galois est au centre de l'étude des clones en algèbre universelle.

0.5. Théorie des clones

0.5.1. Clones sur plus de deux éléments

Connaissant essentiellement toutes les algèbres d'ordre 2, c'est à dire les clones sur deux éléments (Fig 0.1), on veut naturellement étendre ces résultats sur k éléments où $k > 2$. Ce n'est pas aussi facile qu'on pourrait le penser. Le treillis des clones sur deux éléments est de cardinalité dénombrable $\aleph_0$, mais le treillis des clones sur trois éléments est de cardinalité $2^{\aleph_0}$ [**17**]. Et c'est la même chose pour les clones sur plus de trois éléments. De plus, la partie infinie du treillis des clones sur deux éléments se limite à huit chaînes reliées entre elles assez simplement.

Mais le treillis des clones sur trois éléments contient des chaînes de cardinalité $2^{\aleph_0}$, et aussi des anti-chaînes de cardinalité $2^{\aleph_0}$. Il est tellement compliqué que V. G. Bodnarchuk, L. Kaluzhnin, V. Kotov et B. A. Romov déclarèrent dans leur article de 1969 que

> Even for $k = 3$, and more so for $k > 3$, the description of all Post algebras (clones) is quite a hopeless task [**4**].

Les mathématiciens ont quand même trouvé des façons d'aborder le problème :

– se concentrer sur trois éléments et espérer que ça se généralise,

– déterminer les clones maximaux et minimaux,

– partitionner les clones en intervalles monoïdaux,

– établir des liens entre les relations et les générateurs.

0.5.2. Généraliser à partir de trois éléments

Une approche qui a fait ses preuves est de commencer par étudier les clones sur trois éléments. On sait que le treillis des clones sur deux éléments représente un cas spécial. On espère que trois éléments suffisent pour nous donner une bonne idée du cas général, et que les théorèmes prouvés sur trois éléments pourront se généraliser à plus de trois éléments assez simplement. Des théorèmes généraux comme celui de I. Rosenberg sur les clones maximaux [**30, 31**], qui est une généralisation d'un théorème de S.V. Jablonskiĭ [**16**] sur trois éléments, entretiennent cet espoir. La présente thèse appuie cette idée : les trois premiers articles qu'elle contient ont commencé par des théorèmes ou des idées sur trois éléments qui ont été généralisés à plus de trois éléments.

Pour mieux comprendre les clones sur trois éléments, certains chercheurs ont établi de grandes listes ordonnées de ces clones. Mentionnons ici les algèbres connues de J. Berman [**2**], les monoïdes de D. Lau [**22, 23**], les clones préservant une relation binaire de A. Fearnley [**8, 11**], et les groupoïdes de J. Berman et S. Burris [**3**]. Ces listes sont des sources d'exemples, elles permettent d'identifier ou au moins de localiser certains nouveaux clones et inspirent de nouveaux résultats plus généraux.

0.5.3. Clones maximaux et minimaux

Une autre approche consiste à commencer par les *clones maximaux* : clones proprement inclus seulement dans le plus grand clone $\mathcal{O}_A$, et les *clones minimaux* : clones proprement incluant seulement le plus petit clone $\langle e \rangle$ des projections.

Sur deux éléments, les clones maximaux sont C_2, C_3, A_1, L_1, D_3. S.V. Jablonskiĭ a déterminé les clones maximaux sur trois éléments [**16**] et I. Rosenberg a étendu ces résultats à un nombre fini d'éléments [**30, 31**]. En continuant dans la même voie, D. Lau a trouvé les clones sous-maximaux sur trois éléments [**21**]. À partir des clones maximaux, on peut partitionner les clones en classes dépendant de leur inclusion ou non dans chaque clone maximal. M. Miyakawa détermina ces classes sur trois éléments [**26**].

Les clones minimaux sur trois éléments ont été trouvés par B. Csákány [**5**]. Ce résultat a été partiellement étendu à plus de trois éléments.

0.5.4. Intervalles monoïdaux

Les opérations unaires jouent un rôle particulier dans les clones puisqu'elles forment un monoïde avec la composition et l'identité. On peut partitionner les clones d'après leur monoïde d'opérations unaires. C'est l'approche qu'utilisa E. Post dans sa classification des clones sur deux éléments.

Pour ce faire, considérons les sous-monoïdes possibles de $\langle \mathcal{O}_A^{(1)}; \circ, e\rangle$ pour un univers fini A. Pour chaque sous-monoïde M, considérons tous les clones dont les opérations unaires sont exactement M. Ces clones forment un intervalle dans le treillis des clones, c'est l'*intervalle monoïdal* du monoïde M : $Int(M) = \{C \mid C^{(1)} = M\}$. Le clone inférieur de l'intervalle est simplement $\langle M\rangle$ et le clone supérieur est le *stabilisateur* de M, $\mathrm{Sta}(M) = \{f \in \mathcal{O}_A \mid \forall m_1, \ldots, m_n \in M, f[m_1, \ldots, m_n] \in M\}$.

Bien qu'il y ait $2^{\aleph_0}$ clones sur trois (ou plus) éléments, certains intervalles monoïdaux sont finis ou dénombrables. Il y en a même qui n'ont qu'un seul clone, on dit alors que le monoïde est *affaissant*. Á. Szendrei dit au sujet de l'importance des intervalles monoïdaux :

> It is thought that $Lat(A)$ (le treillis des clones sur A) is nice at the top and at the bottom in the sense that the clones belonging to those two parts can be explicitely described, while the middle of the lattice, which contains the families of cardinality $2^{\aleph_0}$, is hopeless. Contrasted to this "horizontal" division, the intervals $Int(M)$ provide a natural "vertical" division of $Lat(A)$ [**36**].

D. Lau a déterminé tous les sous-monoïdes de $\langle \mathcal{O}_{\mathbf{3}}^{(1)}; \circ, e\rangle$ [**22**]. J'ai contribué à cette liste en corrigeant certaines lacunes et en indiquant les monoïdes en double. Une liste corrigée existe en manuscrit [**23**]. Pour presque tous ces monoïdes, on connaît au moins la cardinalité de leur intervalle monoïdal. En particulier, M. Dorman classifia tous les monoïdes affaisants sur trois éléments [**7**].

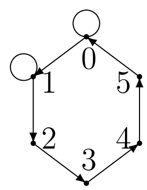
0.5.5. Relations et générateurs

En 1977, I. Rosenberg demanda : “For what clones C does there exist a ρ such that $C = \operatorname{Pol}\rho$.” [**33**]. La raison que ces clones sont intéressants est que tout clone peut être exprimé comme l'intersection de clones de la forme $\operatorname{Pol}\rho$. Ces clones forment donc une sorte de squelette du treillis des clones. Mon mémoire de maîtrise [**8**] traite ce sujet pour trois éléments.

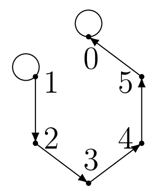
On utilise généralement les relations pour décrire des grands clones, et les opérations pour décrire des petits clones. À titre d'exemple, les clones maximaux sont décrits avec des relations, mais les clones minimaux sont décrits à partir de leur générateur. Il y a des exceptions notables à cette règle générale. Comme nous avons vu plus haut pour $|A| = 2$, le plus grand clone $\mathcal{O}_A$ est engendré par $\{\wedge, \vee, \neg\}$, c'est à dire que $\operatorname{Pol}\emptyset = \langle \wedge, \vee, \neg \rangle$. Le cas dual est plus récent. On dit qu'une relation est *fortement rigide* si elle n'est préservée que par des projections. En 1973, I. Rosenberg trouva une relation fortement rigide ternaire sur deux éléments et une relation fortement rigide binaire pour tout A avec $|A| \geq 3$ [**32**]. Sur trois éléments, $\operatorname{Pol}\{(0,1),(0,2),(1,2),(2,0)\} = \langle e \rangle$

Les clones d'opérations préservant une ou plusieurs relations et qui en plus sont engendrés par des opérations connues explicitement forment des liens dans la correspondance de Galois. Ils sont particulièrement faciles à comparer à d'autres clones étant exprimés sous les deux formes, et ce surtout si les relations et opérations sont simples et peu nombreuses.

Cette thèse présente plusieurs clones qui sont connus à la fois par une relation et par des générateurs. Pour certains, ceci découle trivialement du fait que leur monoïde d'opérations unaires est affaissant, mais d'autres sont plus difficiles. Certains exemples sur six éléments, peuvent être trouvés dans les figures 0.2, 0.3 et 0.4.

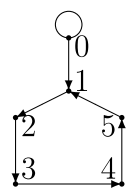
Pol 

$= \langle c_0, c_1 \rangle$

Pol 

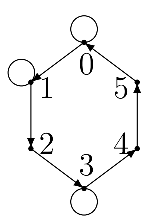
$= \langle c_0, c_1, \max, \min \rangle$ défini pour l'ordre $1 < 2 < 3 < 4 < 5 < 0$

FIG. 0.2. Exemples tirés du chapitre 2

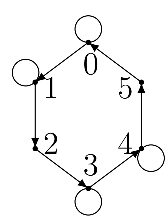
Pol 

$= \langle c_0, \mathrm{maj}_0 \rangle$

FIG. 0.3. Exemple tirés du chapitre 3

Pol 

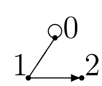
$= \langle c_0, c_1, c_3 \rangle$

Pol 

$= \langle c_0, c_1, (0,3)(1,4)(2,5) \rangle$

FIG. 0.4. Exemples tirés du chapitre 4

Les relations binaires dans ces exemples et partout dans la thèse sont représentées sous forme de graphes. Donc pour $\rho \in A^2$, on écrit $a \to b$ pour représenter $(a,b) \in \rho$. Par exemple, pour $A = \{0,1,2\}$, la relation $\{(0,0),(0,1),(1,0),(1,2)\}$ peut être représentée par le graphe suivant



0.6. Organisation de la thèse

Les chapitres 1, 2, 3 et 4 se composent de quatre articles écrits dans le cadre de mes études de doctorat. Les références de ces articles sont comme suit :

- Anne Fearnley et Ivo Rosenberg. *Collapsing monoids containing permutations and constants.* Algebra Universalis, 50 : 149–156, 2003.
- Anne Fearnley. *The monoidal interval for the monoid generated by two constants.* Soumis pour publication, 2007.
- Anne Fearnley. *Clones on three elements preserving a binary relation.* Algebra Universalis, 56 : 165–177, 2007.
- Anne Fearnley. *Relations made up of a cycle and loops, and their clones.* Soumis pour publication, 2007.

Les contributions du coauteur du premier article sont définies au début du chapitre 1 et les accords et autorisations de publication sont inclus dans l'annexe A.

L'introduction comprend l'historique du sujet, la motivation pour la recherche, les définitions et l'organisation de la thèse. Le chapitre 1 propose certaines classes de monoïdes affaissants ne contenant que des permutations et des constantes. Le chapitre 2 décrit une partie de l'intervalle monoïdal $\mathrm{Int}\langle c_0, c_1\rangle$. On y présente des relations ρ et σ telles que $\mathrm{Pol}\,\rho = \langle c_0, c_1\rangle$ et $\mathrm{Pol}\,\sigma = \langle c_0, c_1, \min, \max\rangle$. Le chapitre 3 trouve une relation ρ telle que $\mathrm{Pol}\,\rho = \langle c_0, \mathrm{maj}_0\rangle$. Le chapitre 4 démontre qu'un grand nombre de clones de constantes et de puissances de cycles sans points fixes préservent des relations binaires assez simples. La conclusion propose des directions possibles de recherche découlant de la thèse. L'annexe A contient l'accord du coauteur de l'article *Collapsing monoids containing permutations and constants* et l'autorisation des éditeurs des articles déjà publiés.

Chapitre 1

COLLAPSING MONOIDS CONTAINING PERMUTATIONS AND CONSTANTS

Auteurs : Anne Fearnley et Ivo G. Rosenberg

1.1. Contribution de l'auteur à l'article

La première version de l'article *Collapsing monoids containing permutations and constants* ne comprenait que les deux premières sections (sections 1.4 et 1.5)et une section qui n'a pas été retenue. Pendant la revision de l'article, I. Rosenberg a généralisé le théorème 1.5.3. Cette généralisation est le théorème 1.6.1 que j'ai inclu dans l'article. Le reste de la section 1.6 contient un corollaire et des travaux additionels que j'ai fait à partir de suggestions de l'éditeur, Á. Szendrei.

1.2. Monoïdes affaissants de constantes et de permutations

En étudiant la littérature sur le sujet des intervalles monoïdaux, j'ai été séduite par le théorème suivant.

Théorème 1.2.1 (P. Pálfy [**27**]). *Soit A un ensemble fini de cardinalité au moins 3. Soit M un monoïde d'opérations sur A contenant toutes les constantes et tel que ses opérations non constantes sont des permutations. Alors $|\operatorname{Int}(M)| \leq 2$, avec égalité si et seulement si M est composé de tous les polynômes unaires d'un espace vectoriel.*

J'ai rapidement trouvé tous les intervalles monoïdaux sur trois éléments déterminés par ce théorème. Certains monoïdes ne contenant que des permutations étaient connus [**34**] et [**28**]. Et deux théorèmes de A. Krokhin indiquaient la cardinalité des intervalles monoïdaux pour les monoïdes ne contenant qu'une constante ou ne contenant que deux constantes [**19**].

J'ai pensé qu'il serait possible de déterminer la cardinalité des intervalles monoïdaux pour tous les monoïdes de permutations et de constantes sur trois éléments. Il ne me restait que les monoïdes $\langle(0,1)\rangle$, $\langle(0,1), c_2\rangle$ et $\langle(0,1), c_0, c_1\rangle$ où $(0,1)$ dénote la transposition de 0 et 1. Je trouvai assez facilement qu'il existait des opérations qui n'étaient pas essentiellement unaires contenues dans certains clones des intervalles $\operatorname{Int}\langle(0,1)\rangle$ et $\operatorname{Int}\langle(0,1), c_2\rangle$. Mais l'intervalle $\operatorname{Int}\langle(0,1), c_0, c_1\rangle$ semblait contenir très peu de clones.

J'ai trouvé en effet que $\langle(0,1), c_0, c_1\rangle$ était affaissant. Mais on pouvait faire mieux. Le théorème suivant (théorème 1.5.3 dans l'article) est un résultat plus général sur un nombre fini d'éléments.

Théorème 1.2.2. *Si $n \geq 4$ et M est un monoïde d'opérations sur $\{0, 1, \ldots, n-1\}$ ne contenant que des permutations et exactement $n-1$ constantes, alors M est affaissant.*

Inspiré par le théorème 1.2.2, I. Rosenberg le généralisa pour qu'il s'applique à plusieurs autres monoïdes, même sur un nombre dénombrable d'élements (théorème 1.6.1 dans l'article). Entre temps, Á. Szendrei prouva que la cardinalité de Int $\langle(0,1), c_2\rangle$ est $2^{\aleph_0}$ [**35**]. Comme on peut voir dans le tableau 1.1, il ne reste que la cardinalité de Int $\langle(0,1)\rangle$ à déterminer, mais au moins on sait que le monoïde n'est pas affaissant.

Depuis, M. Dormán a déterminé tous les monoïdes affaissants sur trois éléments [**7**]. Il s'est servi de cet article entre autres. Il y a 27 monoïdes affaissants sur trois éléments.

1.3. Abstract

In 1941, Post [**29**] presented the complete description of the countably many clones on 2 elements. The structure of the lattice of clones on finitely many (but more than 2) elements is more complex ; in fact the lattice is of cardinality $2^{\aleph_0}$. One approach is to study the monoidal intervals : the set of clones whose unary operations form a given monoid. One surprising fact is that for certain monoids, called collapsing, this interval contains just one clone. This article presents some collapsing monoids containing only constants and permutations.

1.4. Introduction and preliminaries

Let A be a set and k a positive integer. A *k-ary operation* on A is a function $f : A^k \to A$. The set of all k-ary operations on A is denoted by $\mathcal{O}_A^{(k)}$, and $\mathcal{O}_A := \bigcup_{0<k<\omega} \mathcal{O}_A^{(k)}$. For $F \subseteq \mathcal{O}_A$ and $0 < k < \omega$, set $F^{(k)} := F \cap \mathcal{O}_A^{(k)}$.

We denote by c_a the unary *constant function* a defined by $c_a(x) \approx a$ (the symbol '$\approx$' denotes equality for all $x \in A$). For $1 \leq i \leq k$, the k-ary *i-th projection* is $e_i^k(x_1, \ldots, x_k) \approx x_i$.

A set $F \subseteq \mathcal{O}_A$ is *closed under composition* (also called *substitution* or *superposition*), if for all $f \in F^{(k)}$, and $g_1, \ldots, g_k \in F^{(l)}$, the l-ary operation $f[g_1, \ldots, g_k]$ defined by

$$f[g_1, \ldots, g_k](x_1, \ldots, x_l) \approx f(g_1(x_1, \ldots, x_l), \ldots, g_k(x_1, \ldots, x_l))$$

is also in F. A *clone* on A is a subset F of $\mathcal{O}_A$ which contains all the projections and is closed under composition.

It is well known and easy to show that the intersection of an arbitrary set of clones on A is a clone on A. Thus for $F \subseteq \mathcal{O}_A$, we can find the least clone containing F, called the clone *generated* by F and denoted by $\langle F \rangle$. The clones on A, ordered by inclusion, form the complete lattice $\mathcal{L}_A$.

Consider a transformation monoid M of unary operations on A; M contains the identity self-map and is closed under the usual composition $\circ$. We denote by $\operatorname{Int}(M)$ the set of clones C on A such that $C^{(1)} = M$. $\operatorname{Int}(M)$ is an interval in the lattice of clones on A [see **[36]**, page 71] called the *monoidal interval* of M. If $\operatorname{Int}(M)$ contains just one clone, then M is said to be *collapsing*.

1.5. Some collapsing monoids

If A is finite, we may assume that $A = \mathbf{n} := \{0, 1, \dots, n-1\}$. For transformation monoids containing all the constants, we have the following result.

Theorem 1.5.1 (Pálfy [**27**], see also [**36**]). *Let A be a finite set containing at least 3 elements. Let M be a transformation monoid on A containing all the constants and such that its non-constant functions are permutations. Then $|\operatorname{Int}(M)| \leq 2$, and equality holds if and only if M is a monoid of all unary polynomial operations of a vector space.*

Lemma 1.5.2. *Let M be a transformation monoid on a set A. Let $B = \{a \in A : c_a \in M\}$, and let C be a clone on A such that $C^{(1)} = M$.*

(A) For any $f \in C$, we have $f(B, \dots, B) \subseteq B$, therefore f can be restricted to B to get an operation $f|_B$ on B.

(B) In particular, for any $m \in M$, $m(B) \subseteq B$. Moreover, if A is finite, then every permutation in M permutes the elements of B and the elements of $A \setminus B$.

(C) The restriction $M|_B = \{m|_B : m \in M\}$ of M to B is a transformation monoid on B, and the restriction $C|_B = \{f|_B : f \in C\}$ of C to B is a clone such that $(C|_B)^{(1)} = M|_B$.

Proof. Statement (A) is trivially true if $B = \emptyset$. Otherwise, let $f \in C^{(k)}$ and $b_1, \dots, b_k \in B$. Then

$$f(b_1, \dots, b_k) = f(c_{b_1}(b_1), \dots, c_{b_k}(b_1)) = f[c_{b_1}, \dots, c_{b_k}](b_1).$$

Now $f[c_{b_1}, \ldots, c_{b_k}] \in C$ and is unary. Therefore $f[c_{b_1}, \ldots, c_{b_k}] \in M$. Furthermore, it is a constant, thus there exists $b \in B$ such that $f[c_{b_1}, \ldots, c_{b_k}] = c_b$. Therefore $f(b_1, \ldots, b_k) = f[c_{b_1}, \ldots, c_{b_k}](b_1) = c_b(b_1) = b \in B$.

To prove statement (B), let $m \in M$, then by above, $m(B) \subseteq B$. Now let m be a permutation in M and A be finite. Thus $m(B) = B$ and m permutes the elements of B. Therefore, $m(A \setminus B) = A \setminus B$ and m permutes the elements of $A \setminus B$.

Statement (C) is obvious. □

Theorem 1.5.3. *If $n \geq 4$ and M is a transformation monoid on $\mathbf{n}$ containing only permutations and exactly $n-1$ constants, then M is collapsing.*

Proof. Without loss of generality, we can assume that the $n-1$ constants are $c_0, c_1, \ldots, c_{n-2}$. The idea behind the proof is to use Lemma 1.5.2 to reduce the problem to one in $\mathbf{n-1}$, then deduce that the monoid is collapsing with the aid of Theorem 1.5.1.

By Lemma 1.5.2, the permutations of M permute the elements of $\mathbf{n-1}$. Therefore $M^- := M|_{\mathbf{n-1}}$ is a transformation monoid on $\mathbf{n-1}$ satisfying the conditions of Theorem 1.5.1. Let C be a clone in $\mathrm{Int}(M)$, i.e. $C^{(1)} = M$. We must show that all the operations in C are essentially unary. If $f \in C$, then by Lemma 1.5.2, we can restrict f to $\mathbf{n-1}$, and we write $f^- := f|_{\mathbf{n-1}}$. Similarly, $C^- := C|_{\mathbf{n-1}} = \{f^- : f \in C\}$. By Lemma 1.5.2, $(C^-)^{(1)} = M^-$. Therefore, by Theorem 1.5.1, C^- must be a clone of essentially unary operations or be a clone of polynomial operations of a vector space.

Claim 1. *C^- does not contain the clone of polynomial operations of a vector space.*

Proof. Suppose otherwise. Without loss of generality, we may assume that 0 is the zero of the vector space. Since C^- contains the operations $x - y$ and $x + 1$, then C must contain a binary operation f and a unary operation g such that $f^-(x, y) \approx x - y$ and $g^-(x) \approx x + 1$. Thus $f(x, x) = 0$ for all $x \in \mathbf{n-1}$, which implies that the function $f(x, x)$ must be the constant function 0 in C. Hence $f(n-1, n-1) = 0$. The unary operation g^- is a permutation, therefore g cannot be a constant, thus it must be a permutation also, and by Lemma 1.5.2, $g(n-1) = n-1$. Consider the unary operation $h : \mathbf{n} \to \mathbf{n}$ defined by $h(x) \approx f(g(x), x)$. Clearly $h \in C$, therefore $h \in M$. But $h(x) = (x + 1) - x = 1$ if $x \in \mathbf{n-1}$, and $h(n-1) = f(n-1, n-1) = 0$, which is impossible. $\square$

Claim 2. *C is a clone of essentially unary operations.*

Proof. By the previous Claim, C^- is a clone of essentially unary operations. Let $f \in C^{(k)}$, then $f^- \in C^-$ is essentially unary. Thus there exists an $m \in M$ such that f^- is essentially equal to $m^- \in M^-$. Clearly, $m(n-1) = m^-(0)$ if m^- is a constant, and $m(n-1) = n-1$ if m^- is a permutation. We may assume that f^- depends only on its first variable, i.e. that $f^-(x_1, \dots, x_k) \approx m^-(x_1)$.

Let $x_1, \dots, x_k \in \mathbf{n}$. For each $i = 1, \dots, k$, set

$$m_i = \begin{cases} c_{x_i} & \text{if } x_i \neq n-1 \\ id & \text{if } x_i = n-1 \end{cases}$$

Note that $m_i \in M$ for all $i = 1, \dots, k$, therefore $f[m_1, \dots, m_k] \in C$, and thus $f[m_1, \dots, m_k] \in M$. Also, if $y \in \mathbf{n-1}$, then $m_i(y) \in \mathbf{n-1}$, for all $i = 1, \dots, k$. Thus for $y \in \mathbf{n-1}$,

$$f[m_1, \dots, m_k](y) = f(m_1(y), \dots, m_k(y)) = m(m_1(y)).$$

Moreover $f[m_1,\dots,m_k](n-1) = f(m_1(n-1),\dots,m_k(n-1)) = f(x_1,\dots,x_k)$.

We distinguish two cases. If $x_1 \in \mathbf{n-1}$, then $m_1 = c_{x_1}$. For $y \in \mathbf{n-1}$, $f[m_1,\dots,m_k](y) = m(m_1(y)) = m(c_{x_1}(y)) = m(x_1)$. Thus $f[m_1,\dots,m_k] = c_{m(x_1)}$, and $f(x_1,\dots,x_k) = f[m_1,\dots,m_k](n-1) = c_{m(x_1)}(n-1) = m(x_1)$.

Now, if $x_1 = n-1$, then $m_1 = id$. For $y \in \mathbf{n-1}$, $f[m_1,\dots,m_k](y) = m(m_1(y)) = m(id(y)) = m(y)$. Thus $f[m_1,\dots,m_k] = m$, and $f(x_1,\dots,x_k) = f[m_1,\dots,m_k](n-1) = m(n-1) = m(x_1)$. Therefore in both cases, $f(x_1,\dots,x_k) = m(x_1)$. This proves that f is essentially the unary operation m. □

Claim 2 implies that M is collapsing. □

Corollary 1.5.4 (Krokhin [**19**]). *Let $n \geq 4$. Let M be a transformation monoid on* $\mathbf{n}$ *consisting of $n-1$ constants and the identity. Then M is collapsing.*

It is clear that Theorem 1.5.3 need not be true for $n = 3$, because Pálfy's theorem (Theorem 1.5.1) is only valid for 3 or more elements. For example, Corollary 1.5.4 does not hold on 3 elements; the monoid $M = \langle\{c_0, c_1\}\rangle$ is not collapsing, in fact, $|\operatorname{Int}(M)| = 4$ [**19**]. The following result is true for $n \geq 3$.

Corollary 1.5.5. *Let $n \geq 3$. Let M be a transformation monoid on* $\mathbf{n}$ *consisting of the powers of an $(n-1)$-cycle and $n-1$ constants. Then M is collapsing.*

Proof. By Theorem 1.5.3 all that remains is to prove the 3-element case. We must show that $M = \{id, c_0, c_1, \tau\}$ is collapsing where τ is the transposition defined by $\tau(0) = 1$, $\tau(1) = 0$, $\tau(2) = 2$.

The proof is very similar to the proof of Theorem 1.5.3. It is identical up to the reference to Theorem 1.5.1. Here, $\mathbf{n-1} = \mathbf{2}$ and we can use Post's lattice [**29**]. We find that C^- must be one of 3 clones : the clone of essentially unary operations (R_{13}), the clone of polynomial operations of the 2-element vector space (L_1), or the clone of all operations (C_1). The cases $C^- = L_1$ and $C^- = C_1$ lead to a contradiction (see Theorem 1.5.3, Claim 1). Therefore $C^- = R_{13}$, the clone of essentially unary operations, and C is a clone of essentially unary operations (see Theorem 1.5.3, Claim 2). □

We can now state which of the transformation monoids, containing only constants and permutations, are collapsing on the 3-element set **3**. There are 24 submonoids of $S_3 \cup \{c_0, c_1, c_2\}$ in 12 isomorphism classes. Table 1.1 shows one submonoid for each class, the cardinality of its interval (if known), and a reference for the cardinality. The operations used are the constants c_0, c_1, c_2, the transposition τ (transposing 0 and 1), and the cycle σ (adding 1 modulo 3).

1.6. A generalization of Theorem 1.5.3

It is possible to use the ideas in the proof of Theorem 1.5.3 to prove a more general theorem about transformation monoids containing operations other than just constants and permutations and not restricted to finite universes.

Theorem 1.6.1. *Let M be a transformation monoid on a set A such that $B := \{a \in A : c_a \in M\}$ is non-empty and*

(1) $A \setminus B = \{d_1, \dots, d_l\}$ is finite,

(2) every $m \in M$ is completely determined by $m|_B$,

(3) there exist $m_{i,j} \in M$ such that $m_{i,i} = id_A$ and $m_{i,j}(d_i) = d_j$, for all $1 \leq i, j \leq l$ and $i \neq j$,

(4) the monoid $M|_B$ is a collapsing monoid on B.

M	$\lvert \mathrm{Int}(M) \rvert$	Reference or proof
$\{id\}$	$2^{\aleph_0}$	Marchenkov [**24**]
$\langle c_0 \rangle$	$2^{\aleph_0}$	Krokhin [**19**]
$\langle c_0, c_1 \rangle$	4	Krokhin [**19**]
$\langle c_0, c_1, c_2 \rangle$	1	Pálfy [**27**]
$\langle \tau \rangle$	> 1	Not weakly transitive [**15**]
$\langle \tau, c_2 \rangle$	$2^{\aleph_0}$	Szendrei [**35**]
$\langle \tau, c_0, c_1 \rangle$	1	Corollary 1.5.5
$\langle \tau, c_0, c_1, c_2 \rangle$	1	Pálfy [**27**]
$\langle \sigma \rangle$	3	Szendrei [**34**]
$\langle \sigma, c_0, c_1, c_2 \rangle$	1	Pálfy [**27**]
S_3	1	Pálfy, Szendrei [**28**]
$S_3 \cup \{c_0, c_1, c_2\}$	2	Pálfy [**27**]

TAB. 1.1. Monoids on 3 elements containing only constants and permutations

Then M is a collapsing monoid on A.

PROOF. The restrictions $m|_B$ and $M|_B$ make sense by Lemma 1.5.2. If $A \setminus B = \emptyset$, then the theorem is trivially true. We assume that $A \setminus B \neq \emptyset$. Let C be a clone in $\mathrm{Int}(M)$, let $f \in C^{(k)}$. By Lemma 1.5.2, we know that $f|_B \in C|_B$. By Condition 4, $M|_B$ is collapsing, therefore $f|_B$ is essentially unary. We may assume that $f|_B$ depends only on its first variable. This implies that there exists $m \in M|_B$ such that $f(b_1, \ldots, b_k) = m(b_1)$ for all $b_1, \ldots, b_k \in B$.

Let m^+ be the unique extension of m to a function in M (Condition 2). Let $a_1, \ldots, a_k \in A$, we must show that $f(a_1, \ldots, a_k) = m^+(a_1)$. For $i = 1, \ldots, k$, define

$$m_i = \begin{cases} c_{a_i} & \text{if } a_i \in B \\ m_{1,s} & \text{if } a_1 \in B \text{ and } a_i = d_s \\ m_{t,s} & \text{if } a_1 = d_t \text{ and } a_i = d_s \end{cases}$$

(where the $m_{i,j}$ are from Condition 3). Since $m_i \in M$ for $i = 1, \ldots, k$ and $f \in C$, then clearly $f[m_1, \ldots, m_k] \in M$. Furthermore, for every $b \in B$, $f[m_1, \ldots, m_k](b) = f(m_1(b), \ldots, m_k(b)) = m(m_1(b))$.

If $a_1 \in B$, then $m_1 = c_{a_1}$, and for $b \in B$, we have $f[m_1, \ldots, m_k](b) = m(m_1(b)) = m(c_{a_1}(b)) = m(a_1)$. Therefore, $f[m_1, \ldots, m_k] = c_{m(a_1)}$ by Condition 2. The definition of m_i and Condition 3 imply, for $i = 1, \ldots, k$, that

$$m_i(d_1) = \begin{cases} c_{a_i}(d_1) = a_i & \text{if } a_i \in B \\ m_{1,s}(d_1) = d_s = a_i & \text{if } a_i = d_s \end{cases}$$

Consequently $f(a_1, \ldots, a_k) = f(m_1(d_1), \ldots, m_k(d_1)) = c_{m(a_1)}(d_1) = m(a_1) = m^+(a_1)$ as required.

Now, if $a_1 \notin B$, then $a_1 = d_t$ for some $1 \leq t \leq l$. For $b \in B$, we have $f[m_1, \ldots, m_k](b) = m(m_1(b)) = m(m_{t,t}(b)) = m(id_B(b)) = m(b)$. Therefore $f[m_1, \ldots, m_k] = m^+$ by Condition 2. Also, the definition of m_i and Condition 3 imply, for $i = 1, \ldots, k$, that

$$m_i(a_1) = m_i(d_t) = \begin{cases} c_{a_i}(d_t) = a_i & \text{if } a_i \in B \\ m_{t,s}(d_t) = d_s = a_i & \text{if } a_i = d_s \end{cases}$$

Consequently, $f(a_1, \ldots, a_k) = f(m_1(a_1), \ldots, m_k(a_1)) = m^+(a_1)$ as required. □

Theorem 1.6.1 helps us to find new collapsing monoids. Since most of the collapsing monoids already known contain only constants and permutations, the following proposition should be useful.

Proposition 1.6.2. *Let M be a transformation monoid on a finite set A such that $B := \{a \in A : c_a \in M\}$ is non-empty. If every $m \in M$ is completely determined by $m|_B$, and the monoid $M|_B$ contains only permutations and constants, then M also contains only permutations and constants. Furthermore the permutations in M permute the elements of B and the elements of $A \setminus B$.*

Proof. The restrictions $m|_B$ and $M|_B$ make sense by Lemma 1.5.2. Choose $m \in M$ not a constant. Then $m|_B$ is not a constant and thus must be a permutation. Let k be the order of $m|_B$, i.e. $(m|_B)^k = id_B$. Clearly id_B must be extended to id_A, which implies that $m^k = id_A$. Now, to prove that m is a permutation, suppose that $m(a_1) = m(a_2)$ for some $a_1, a_2 \in A$. Then $a_1 = m^k(a_1) = m^k(a_2) = a_2$. Therefore m is a permutation as required. The rest is taken from Lemma 1.5.2. □

Proposition 1.6.2 shows that if M is a transformation monoid on a finite set that satisfies Condition 2 of Theorem 1.6.1, then M contains only permutations and constants if and only if $M|_B$ contains only permutations and constants. Note that in Theorem 1.6.1, Condition 2 combined with the assumption that B is non-empty implies that $|B| > 1$. The corollary below restates Theorem 1.6.1 for the special case when M contains only permutations and constants.

Corollary 1.6.3 (A. Szendrei)**.** *Let A be a finite set and B a subset of A with $|B| > 1$. Let $M = \{c_b : b \in B\} \cup G$ where G is a permutation group on A such that $A \setminus B$ is an orbit of G, and the restriction map $G \to G|_B$ is injective. Then M is collapsing if $M|_B$ is.*

There are no collapsing monoids on a 2-element set B. Therefore, in order to get new collapsing monoids from Corollary 1.6.3, we must have $|B| \geq 3$. The case $|A| = |B| + 1$ is settled completely by Theorem 1.5.3. However, if $|A| > |B| + 1$

(and hence $|A| > 4$), then Corollary 1.6.3 yields a large family of new collapsing monoids. Below we give examples of such monoids on 5 and 6 elements.

Example 1. *On 5 elements, let* $A = \{0, 1, 2, 0', 1'\}$ *and* $G = \langle(0, 1)(0', 1')\rangle$. *The monoid* $M = \{c_0, c_1, c_2\} \cup G$ *is collapsing.*

Example 2. *On 6 elements, let* $A = \{0, 1, 2, 0', 1', 2'\}$ *and* $G = \langle(0, 1, 2)(0', 1', 2')\rangle$. *The monoid* $M = \{c_0, c_1, c_2\} \cup G$ *is collapsing.*

1.7. Acknowledgments

The authors are indebted to Ágnes Szendrei for her painstaking editing and excellent suggestions. We especially want to thank her for proposing Lemma 1.5.2, which greatly simplified the proofs of the theorems.

Chapitre 2

THE MONOIDAL INTERVAL FOR THE MONOID GENERATED BY TWO CONSTANTS

Auteur : Anne Fearnley

2.1. L'intervalle du monoïde engendré par deux constantes

A. Krokhin avait énnoncé en 1995 [**19**] que le monoïde composé d'exactement $k-1$ constantes était affaissant sur un univers de k éléments où $k \geq 4$. Il avait aussi trouvé que $Int\langle c_0, c_1\rangle$ avait la forme suivante (figure 2.1) sur trois éléments.

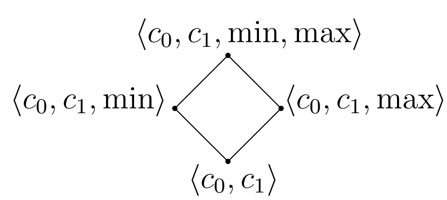


Fig. 2.1. L'intervalle $\mathrm{Int}\langle c_0, c_1\rangle$ sur trois éléments

Comme mentionné plus tôt dans l'introduction, on espère que ce qu'on observe sur trois éléments en théorie des clones se retrouve sur les univers finis de plus de trois éléments. On pourrait s'attendre peut-être à avoir une structure plus complexe (mais quand même semblable) sur plus d'éléments, mais une structure plus simple serait surprenant. Or c'est ce qui arrive dans ces deux théorèmes de A. Krokhin.

J'ai donc pensé à généraliser autrement. Est-ce que la structure de $Int\langle c_0, c_1\rangle$ sur plus de trois éléments pourrait ressembler à la structure de la figure 2.1 ? Cet intervalle a la cardinalité du continu [**19**], mais il se pourrait que la structure apparaisse dans l'intervalle de façon marquante. Dans mon mémoire de maîtrise [**8**], j'avais déjà montré que

$$\mathrm{Pol}\{(0,0),(1,1),(0,1),(1,2),(2,0)\} = \langle c_0, c_1\rangle$$

Semblablement, à partir de mon mémoire et des résultats de Krokhin, il était assez facile de vérifier que

$$\mathrm{Pol}\{(0,0),(1,1),(1,2),(2,0)\} = \langle c_0, c_1, \min, \max\rangle$$

Serait-il possible de généraliser ces résultats à plus de trois éléments ?

J'ai trouvé, en effet, les théorèmes suivants qui sont les théorèmes 2.5.1 et 2.6.1 dans l'article.

Théorème 2.1.1. *Soient* $k \geq 3$ *et* $A = \{0, 1, \ldots, k-1\}$ *et soit*

$$\rho = \{(0,0),(1,1),(0,1),(1,2),(2,3),\ldots,(k-1,0)\}$$

alors $\mathrm{Pol}\,\rho = \langle c_0, c_1\rangle$.

Théorème 2.1.2. *Soient* $k \geq 3$ *et* $A = \{0, 1, \ldots, k-1\}$ *et soit*

$$\sigma = \{(0,0),(1,1),(1,2),(2,3),\ldots,(k-1,0)\}$$

alors $\operatorname{Pol}\sigma = \langle c_0, c_1, \min, \max\rangle$ *où* min *et* max *sont déterminées d'après la chaine* $0 < k-1 < k-2 < \ldots < 2 < 1$.

Ces clones forment le bas et le haut respectivement de losanges semblables à celui de la figure 2.1 dans l'intervalle $Int\langle c_0, c_1\rangle$ sur k éléments. Différents losanges sont obtenus en permutant les éléments autres que 0 et 1. Puisqu'il n'y a pas de telles permutations sur trois éléments, ceci explique en partie la plus grande complexité de l'intervalle sur plus de trois éléments.

Il est rare que ce qui se passe sur deux éléments ressemble à ce qui se passe sur plus de deux éléments en théorie des clones (voir [**1**]). Mais c'est le cas ici puisque l'intervalle $Int\langle c_0, c_1\rangle$ sur deux éléments est

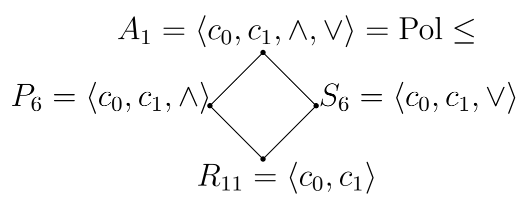


FIG. 2.2. L'intervalle monoïdal $\operatorname{Int}\langle c_0, c_1\rangle$ sur deux éléments

2.2. Abstract

Post (1941) presented the complete description of the countably many clones on 2 elements. The structure of the lattice of clones on finitely many (but more than 2) elements is more complex ; in fact, the lattice is of cardinality $2^{\aleph_0}$. One approach is to study the monoidal intervals : the set of clones whose unary operations form a given monoid. In this article, we study the monoidal interval for the monoid generated by two constants on k elements for k finite. This interval contains the clones of term operations of the bounded lattices of k elements.

2.3. Preliminaries

Let A be a finite set and n a positive integer. An *n-ary operation* on A is a function $f : A^n \to A$. The set of all n-ary operations on A is denoted by $\mathcal{O}_A^{(n)}$, and $\mathcal{O}_A := \bigcup_{0<n<\omega} \mathcal{O}_A^{(n)}$. For $F \subseteq \mathcal{O}_A$, set $F^{(n)} := F \cap \mathcal{O}_A^{(n)}$. For $1 \leq i \leq n$, the n-ary *i-th projection* is defined as $e_i^{(n)}(x_1, \ldots, x_n) = x_i$ for all $x_1, \ldots, x_n$. We write e for the identity operation. For $a \in A$, the n-ary *constant operation a* is defined as $c_a^{(n)}(x_1, \ldots, x_n) = a$ for all $x_1, \ldots, x_n$. We write simply c_a for the unary constant operations $c_a^{(1)}$.

For $f \in \mathcal{O}^{(n)}$, and $g_1, \ldots, g_n \in \mathcal{O}^{(m)}$, we define their composition to be the m-ary operation $f[g_1, \ldots, g_n]$ defined by

$$f[g_1, \ldots, g_n](x_1, \ldots, x_m) = f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$$

A *clone* on A is a subset F of $\mathcal{O}_A$ that contains all projections and is closed under composition. It is well known and easy to prove that the intersection of an arbitrary set of clones on A is a clone on A. Thus for $F \subseteq \mathcal{O}_A$, there exists the least clone containing F, called the clone *generated* by F and denoted by $\langle F \rangle$. Equivalently, $\langle F \rangle$ is the set of term operations of the algebra $\langle A; F \rangle$. The clones on A, ordered by inclusion, form a complete lattice, $\mathcal{L}_A$.

Let h be a positive integer. An *h-ary relation* ρ is a subset of A^h. When dealing with a fixed $\rho \in A^2$, we write $a \to b$ for $(a, b) \in \rho$. The relations may then be drawn as directed graphs. For example for $A = \{0, 1, 2\}$, the relation $\{(0,0), (0,1), (1,0), (1,2)\}$ may be represented as in Figure 2.3

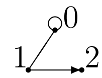


FIG. 2.3. Example of a relation

Let $f \in \mathcal{O}^{(n)}$, and let ρ be an h-ary relation on A. The operation f *preserves* ρ if for all $(a_{1,i}, a_{2,i}, \ldots, a_{h,i}) \in \rho$ $(i = 1, \ldots, n)$,

$$(f(a_{1,1}, a_{1,2}, \ldots, a_{1,n}), f(a_{2,1}, a_{2,2}, \ldots, a_{2,n}), \ldots, f(a_{h,1}, a_{h,2}, \ldots, a_{h,n})) \in \rho$$

The set of operations on A preserving ρ is a clone denoted by $\operatorname{Pol}\rho$.

Consider a transformation monoid M of unary operations on A; M contains the identity self-map e and is closed under the usual composition. Denote by $\operatorname{Int}(M)$ the set of clones C on A such that $C^{(1)} = M$. It is well known that $\operatorname{Int}(M)$ is an interval in the lattice of clones on A, called the *monoidal interval* of M. The smallest clone in $\operatorname{Int}(M)$ is $\langle M \rangle$. The largest clone in $\operatorname{Int}(M)$ is the *stabilizer* of M :

$$\operatorname{Sta}(M) = \{f \in \mathcal{O}_A^{(n)} \mid n > 0 \text{ and } f[m_1, \ldots, m_n] \in M \text{ for all } m_1, \ldots m_n \in M\}$$

$$= \operatorname{Pol}\{(m(a_1), \ldots, m(a_k)) \mid m \in M\}$$

for $A = \{a_1, \ldots, a_k\}$ finite (see [**36**]). If $\operatorname{Int}(M)$ contains just one clone, i.e. $\operatorname{Int}(M) = \{\langle M \rangle\}$, then M is said to be *collapsing*.

2.4. MOTIVATION

In 1995, Krokhin [**19**] showed that the monoidal interval $\operatorname{Int}\langle c_0, c_1, \ldots c_{k-2} \rangle$ on k elements (i.e. for $A = \{0, 1, \ldots, k-1\}$) is collapsing for $k > 3$. For $k = 3$, he showed that the monoidal interval $\operatorname{Int}\langle c_0, c_1 \rangle$ has 4 clones. These correspond to the 3 types of algebras ennumerated by Berman, with free-spectra beginning with 2,3 [**2**]. Its diamond shaped Hasse diagram is in Figure 2.4. Here min and max are defined with respect to the chain $0 < 2 < 1$.

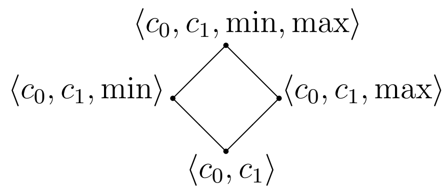


FIG. 2.4. The interval $\mathrm{Int}\langle c_0, c_1\rangle$ on 3 elements

We propose to generalize the interval of Figure 2.4 in a different way, by considering the interval $\mathrm{Int}\langle c_0, c_1\rangle$ for universes of at least 2 elements.

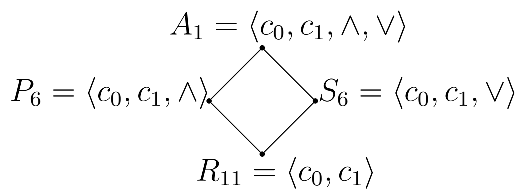


FIG. 2.5. The interval $\mathrm{Int}\langle c_0, c_1\rangle$ on 2 elements

On $\{0, 1\}$, Post [**29**] showed that the interval $\mathrm{Int}\langle c_0, c_1\rangle$ is the one in Figure 2.5 where $\wedge$ and $\vee$ are min and max with respect to the chain $0 < 1$, or equivalently, the conjunction and disjunction from symbolic logic.

In Section 2.7, we show that for any finite bounded distributive lattice $\langle A; \vee, \wedge, 0, 1\rangle$, the interval $[\langle c_0, c_1\rangle, \langle c_0, c_1, \vee, \wedge\rangle]$ in the lattice of clones on A has the same Hasse diagram as in Figure 2.5. Note that for A finite and $|A| > 3$, it is well known that $|\,\mathrm{Int}\langle c_0, c_1\rangle| = 2^{\aleph_0}$ [**19**]. Thus the diamond shaped intervals mentioned above cannot be the whole interval $\mathrm{Int}\langle c_0, c_1\rangle$, but they do appear in the bottom of it.

In [8], it was shown that, for $A = \{0, 1, 2\}$, the clone

$$\mathrm{Pol}\{(0,0),(1,1),(0,1),(1,2),(2,0)\} = \langle c_0, c_1 \rangle$$

which is the smallest clone in the interval. In Section 2.5, we generalize this result to universes of k elements for $k \geq 3$. Note that this is a non-reflexive strongly C-rigid relation [20] (i.e. a relation preserved only by constants and permutations).

Combining results from [19] and [8] for $A = \{0, 1, 2\}$, it is easy to show that

$$\mathrm{Pol}\{(0,0),(1,1),(1,2),(2,0)\} = \langle c_0, c_1, \min, \max \rangle$$

which is the largest clone in the interval. In Section 2.6, we exhibit a relation that is preserved by the clone at the top of each $\min, \max$ diamond for larger universes. This theorem even works for 2 elements; it is the well-known result that $\mathrm{Pol}(\leq) = \langle c_0, c_1, \wedge, \vee \rangle$ (see for example [36]).

2.5. A relation preserved by $\langle c_0, c_1 \rangle$

In this section we exhibit a relation for the smallest clone in the monoidal interval $\mathrm{Int}\langle c_0, c_1 \rangle$. This result is a generalization of a theorem in [8], which dealt with the 3-element case and stated that $\mathrm{Pol}\{(0,0),(1,1),(0,1),(1,2),(2,0)\} = \langle c_0, c_1 \rangle$.

This result ties into research by Länger and Pöschel. A relation ρ is *strongly C-rigid* if every operation on A preserving ρ is a projection or a constant function. Länger and Pöschel [20] presented many reflexive strongly C-rigid relations. The relation in Theorem 2.5.1 is a strongly C-rigid relation that is not reflexive.

Theorem 2.5.1. *Let $k \geq 3$ and $A = \{0, 1, \ldots, k-1\}$. Let*

$$\rho = \{(0,0), (1,1), (0,1), (1,2), (2,3), \ldots, (k-1, 0)\}$$

Then $\operatorname{Pol}\rho = \langle c_0, c_1 \rangle$.

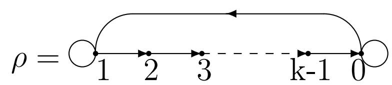


FIG. 2.6. A relation preserved by $\langle c_0, c_1 \rangle$

As an oriented graph, ρ consists of a k-cycle and two loops (Figure 2.6).

Before proving the theorem, let us state a pair of definitions and prove some lemmas. We define two unary operations on A : $x^{\rightarrow}$ defined by $0^{\rightarrow} = 0$, $1^{\rightarrow} = 1$, $j^{\rightarrow} = j+1$ if $1 < j < k-1$ and $(k-1)^{\rightarrow} = 0$, and $x^{\leftarrow}$ defined by $0^{\leftarrow} = 0$, $1^{\leftarrow} = 1$ and $j^{\leftarrow} = j-1$ if $1 < j \leq k-1$. We write $a^{i\leftarrow}$ and $a^{i\rightarrow}$ for the composition i times of the arrow operations. The following proposition follows from the definitions.

Proposition 2.5.2. *(A) $a^{\leftarrow} \to a \to a^{\rightarrow}$ for all $a \in A$.*

(B) $a^{i\leftarrow} \leftarrow a^{(i+1)\leftarrow}$ and $a^{i\rightarrow} \to a^{(i+1)\rightarrow}$ for all $a \in A$ and $i \in \{1, 2, \ldots\}$.

(C) $a^{(k-2)\leftarrow}, a^{(k-2)\rightarrow} \in \{0, 1\}$ for all $a \in A$.

Lemma 2.5.3. *Let $f \in \operatorname{Pol}\rho$ be an n-ary function and let $x_1, \ldots, x_n \in \{0, 1\}$. Then $f(x_1, \ldots, x_n) \in \{0, 1\}$.*

PROOF. For $m = 1, \ldots, n$, $x_m \leftrightarrow x_m$. Thus $f(x_1, \ldots, x_n) \leftrightarrow f(x_1, \ldots, x_n)$. Therefore $f(x_1, \ldots, x_n) \in \{0, 1\}$. □

Lemma 2.5.4. *The unary functions in* $\operatorname{Pol}\rho$ *are exactly c_0, c_1 and e.*

PROOF. Note that $c_0, c_1, e \in \operatorname{Pol}\rho$.

Let $f \in (\operatorname{Pol}\rho)^{(1)}$. By lemma 2.5.3, $f(0), f(1) \in \{0, 1\}$.

Case 1 : If $f(0) = 1$, we have $1 = f(0) \to f(1)$, which along with Lemma 2.5.3, implies that $f(1) = 1$. Now $1 = f(1) \to f(2) \to \ldots \to f(k-1) \to f(0) = 1$; a $(k-1)$-cycle. This implies that $f(2) = \ldots = f(k-1) = 1$. Therefore $f = c_1$.

Case 2 : If $f(1) = 0$, we obtain that $f = c_0$ by a similar reasoning.

Case 3 : If $f(0) = 0$ and $f(1) = 1$, we have $1 = f(1) \to f(2) \to \ldots \to f(k-1) \to f(0) = 0$. This implies that $f(2) = 2, \ldots, f(k-1) = k-1$. Therefore $f = e$. □

Proof. (of Theorem 2.5.1). For every $f \in (\mathrm{Pol}\,\rho)^{(n)}$, we consider the corresponding Boolean function $f|_{\{0,1\}} : \{0,1\}^n \to \{0,1\}$. This is possible because of Lemma 2.5.3. Note that the constants on k elements correspond to the Boolean constants. Now define $(\mathrm{Pol}\,\rho)|_{\{0,1\}} := \{f|_{\{0,1\}} \mid f \in \mathrm{Pol}\,\rho\}$. Clearly, $(\mathrm{Pol}\,\rho)|_{\{0,1\}}$ is a clone on $\{0,1\}$.

Claim 1. $(\mathrm{Pol}\,\rho)|_{\{0,1\}} = \langle c_0, c_1 \rangle$.

Proof. We use Post's classification **[29]**. Since $c_0, c_1 \in (\mathrm{Pol}\,\rho)|_{\{0,1\}}$, clearly, $R_{11} \subseteq (\mathrm{Pol}\,\rho)|_{\{0,1\}}$.

By Lemma 2.5.4, the unary functions in $(\mathrm{Pol}\,\rho)|_{\{0,1\}}$ are exactly c_0, c_1 and e. Thus $\neg \notin (\mathrm{Pol}\,\rho)|_{\{0,1\}}$, which implies that $R_4 \nsubseteq (\mathrm{Pol}\,\rho)|_{\{0,1\}}$

Suppose for the sake of contradiction that $\wedge \in (\mathrm{Pol}\,\rho)|_{\{0,1\}}$. Then there must be some $f \in \mathrm{Pol}\,\rho$ such that $f|_{\{0,1\}} = \wedge$. We thus have

$$1 = f(1,1) \to f(1,2) \to f(1,3) \to \ldots \to f(1,k-1) \to f(1,0) = 0$$

which implies that $f(1,a) = a$ for all $a \in A$.

It follows that $0 = f(0,1) \to f(1,2) = 2$, which is impossible. Therefore $\wedge \notin (\mathrm{Pol}\,\rho)|_{\{0,1\}}$, which implies that $P_2 \nsubseteq (\mathrm{Pol}\,\rho)|_{\{0,1\}}$. Similarly $\vee \notin (\mathrm{Pol}\,\rho)|_{\{0,1\}}$, which implies that $S_2 \nsubseteq (\operatorname{Pol}\rho)|_{\{0,1\}}$. Therefore $(\operatorname{Pol}\rho)|_{\{0,1\}} = R_{11} = \langle c_0, c_1\rangle$ as required. $\square$

Claim 2. $\operatorname{Pol}\rho \subseteq \langle c_0, c_1\rangle$.

Proof. Let $f \in (\operatorname{Pol}\rho)^{(n)}$. By the previous Claim, $f|_{\{0,1\}} \in \langle c_0, c_1\rangle$, thus $f|_{\{0,1\}}$ must be c_0^n or c_1^n or e_m^n for some $m \in \{1, \dots, n\}$.

Case 1 : $f|_{\{0,1\}} = c_0^n$. We will show by induction on i, that $f|_{\{0,\dots,i\}} = c_0^n$ for all $i \in \{1, \dots, k-1\}$. The statement is true for $i = 1$.

Now suppose that $f|_{\{0,\dots,j\}} = c_0^n$ for a certain j with $1 \leq j < k-2$. We must prove that $f|_{\{0,\dots,j+1\}} = c_0^n$. Let $x_1, \dots, x_n \in \{0, \dots, j+1\}$. Note that $x_1^{\leftarrow}, \dots, x_n^{\leftarrow} \in \{0, \dots, j\}$. By Proposition 2.5.2 and the induction hypothesis, $0 = c_0^n(x_1^{\leftarrow}, \dots, x_n^{\leftarrow}) = f(x_1^{\leftarrow}, \dots, x_n^{\leftarrow}) \to f(x_1, \dots, x_n) \to f(x_1^{\rightarrow}, \dots, x_n^{\rightarrow}) \to \dots \to f(x_1^{(k-2)\rightarrow}, \dots, x_n^{(k-2)\rightarrow}) = c_0^n(x_1^{(k-2)\rightarrow}, \dots, x_n^{(k-2)\rightarrow}) = 0$; a $(k-1)$-cycle, so they are all 0. In particular, $f(x_1, \dots, x_n) = 0$, which implies that $f|_{\{0,\dots,j+1\}} = c_0^n$ as required. Therefore, $f = c_0^n \in \langle c_0, c_1\rangle$, by induction.

Case 2 : $f|_{\{0,1\}} = c_1^n$. Then $f = c_1^n \in \langle c_0, c_1\rangle$ in a similar way.

Case 3 : $f|_{\{0,1\}} = e_m^n$ on $\{0,1\}$ for some $m \in \{1, \dots, n\}$. Without loss of generality, $f|_{\{0,1\}} = e_1^n$.

We will show by induction on i, that $f|_{\{0,\dots,i\}} = e_1^n$ for all $i \in \{1, \dots, k-1\}$. The statement is true for $i = 1$.

Now suppose that $f|_{\{0,\dots,j\}} = e_1^n$ for a certain j with $1 \leq j < k-2$. We must prove that $f|_{\{0,\dots,j+1\}} = e_1^n$. Let $x_1, \dots, x_n \in \{0, \dots, j+1\}$. Note that $x_1^{\leftarrow}, \dots, x_n^{\leftarrow} \in \{0, \dots, j\}$. Therefore, $f(x_1, \dots, x_n) \leftarrow f(x_1^{\leftarrow}, \dots, x_n^{\leftarrow}) = x_1^{\leftarrow}$. The only way for $f(x_1, \dots, x_n) \neq x_1$ would be if $x_1 = 0$ and $f(x_1, \dots, x_n) = 1$, or if $x_1 = 1$ and $f(x_1, \dots, x_n) = 2$, or if $x_1 = 2$ and $f(x_1, \dots, x_n) = 1$. By

Proposition 2.5.2,

$$f(x_1,\dots,x_n) \to f(x_1^{\to},\dots,x_n^{\to}) \to \dots \to f(x_1^{(k-2)\to},\dots,x_n^{(k-2)\to}) = x_1^{(k-2)\to} \tag{2.5.1}$$

which is a chain of length $k-2$. If $x_1 = 0$ and $f(x_1,\dots,x_n) = 1$, then (2.5.1) becomes $1 \to \dots \to 0$, which is impossible. If $x_1 = 1$ and $f(x_1,\dots,x_n) = 2$, then (2.5.1) becomes $2 \to \dots \to 1$, which is impossible. If $x_1 = 2$ and $f(x_1,\dots,x_n) = 1$, then (2.5.1) becomes $1 \to \dots \to 2^{(k-2)\to} = 0$, which is impossible. Therefore $f(x_1,\dots,x_n) = x_1$ as required. □

Lemma 2.5.4 implies that $\langle c_0, c_1\rangle \subseteq \operatorname{Pol}\rho$. Using this and Claim 2, we conclude that $\operatorname{Pol}\rho = \langle c_0, c_1\rangle$. □

2.6. Another clone in the interval

Theorem 2.5.1 gave a relation for the bottom of the interval. The following theorem describes a relation for the top of the diamond. Note that for 3 elements, the relation is $\{(0,0),(1,1),(1,2),(2,0)\}$ as expected. Although the theorem is not proved for 2 elements in the same way, we will show that the relation is indeed $\{(0,0),(1,1),(1,0)\}$.

Theorem 2.6.1. *Let $k \geq 3$ and $A = \{0,1,\dots,k-1\}$. Let*

$$\sigma = \{(0,0),(1,1),(1,2),(2,3),\dots,(k-1,0)\}$$

Then $\operatorname{Pol}\sigma = \langle c_0, c_1, \min, \max\rangle$ where $\min$ and $\max$ are defined according to the chain $0 < k-1 < k-2 < \dots < 2 < 1$.

As an oriented graph, σ consists of a chain of length $k-1$ and two loops at its end points (Figure 2.7).

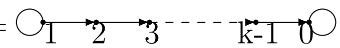
$\sigma =$ 

FIG. 2.7. A relation preserved by $\langle c_0, c_1, \min, \max \rangle$

We will use the operators $x^{\rightarrow}$ and $x^{\leftarrow}$ from Theorem 2.5.1. Note that Proposition 2.5.2 holds for σ.

Lemma 2.6.2 (Post [**29**], see also [**36**])**.** *For $k = 2$, $\operatorname{Pol}\sigma = \langle c_0, c_1, \min, \max \rangle = A_1$, the maximal clone of monotone Boolean functions*

Lemma 2.6.3. *Let $f \in \operatorname{Pol}\sigma$ be an n-ary function and let $x_1, \ldots, x_n \in \{0, 1\}$. Then $f(x_1, \ldots, x_n) \in \{0, 1\}$.*

PROOF. Identical to the proof of Lemma 2.5.3. □

Lemma 2.6.4. *The unary functions in $\operatorname{Pol}\sigma$ are exactly c_0, c_1 and e.*

PROOF. Almost identical to the proof of Lemma 2.5.4. □

PROOF. (of Theorem 2.6.1). The Theorem is true for $k = 2$ by Lemma 2.6.2. From now on, we assume that $k \geq 3$.

Claim 1. *c_0, c_1, $\min$ and $\max$ are in $\operatorname{Pol}\sigma$.*

PROOF. By Lemma 2.6.4, we know that $c_0, c_1 \in \operatorname{Pol}\sigma$.

Let $a_1, a_2, b_1, b_2 \in A$ such that $a_1 \rightarrow a_2$ and $b_1 \rightarrow b_2$. If $a_1 = a_2 = 0$, then $\min(a_1, b_1) = \min(0, b_1) = 0 \rightarrow 0 = \min(0, b_2) = \min(a_2, b_2)$. If $a_1 = a_2 = 1$, then $\min(a_1, b_1) = \min(1, b_1) = b_1 \rightarrow b_2 = \min(1, b_2) = \min(a_2, b_2)$. Similarly, if $b_1 = b_2$, we have that $\min(a_1, b_1) \rightarrow \min(a_2, b_2)$. The last case is if $a_2 = a_1 + 1$, $b_2 = b_1 + 1$ and $a_1, b_1 \neq 0$. Here $\min(a_1, b_1) \rightarrow \min(a_1 + 1, b_1 + 1) = \min(a_2, b_2)$. In all cases $\min(a_1, b_1) \rightarrow \min(a_2, b_2)$.

Similarly, we find that $\max(a_1, b_1) \rightarrow \max(a_2, b_2)$. □

For every $f \in (\operatorname{Pol}\sigma)^{(n)}$, we consider the corresponding Boolean function $f|_{\{0,1\}} : \{0,1\}^n \to \{0,1\}$. This is possible because of Lemma 2.6.3. Note that $\min|_{\{0,1\}} = \wedge$, $\max|_{\{0,1\}} = \vee$ and that the constants become the corresponding Boolean constants. Now define $(\operatorname{Pol}\sigma)|_{\{0,1\}} := \{f|_{\{0,1\}} \mid f \in \operatorname{Pol}\sigma\}$. Clearly, $(\operatorname{Pol}\sigma)|_{\{0,1\}}$ is a clone on $\{0,1\}$.

Claim 2. $(\operatorname{Pol}\sigma)|_{\{0,1\}} = \langle c_0, c_1, \wedge, \vee\rangle$ *the maximal clone of monotone Boolean functions.*

Proof. Since c_0, c_1, $\wedge$ and $\vee$ are in $(\operatorname{Pol}\sigma)|_{\{0,1\}}$, clearly, $A_1 \subseteq (\operatorname{Pol}\sigma)|_{\{0,1\}}$. By **[29]**, $(\operatorname{Pol}\sigma)|_{\{0,1\}}$ must be A_1 or C_1, the clone of all Boolean functions.

By Lemma 2.6.4, the unary functions in $(\operatorname{Pol}\sigma)|_{\{0,1\}}$ are exactly c_0, c_1 and e, thus $\neg \notin (\operatorname{Pol}\sigma)|_{\{0,1\}}$. Therefore $(\operatorname{Pol}\sigma)|_{\{0,1\}} = A_1 = \langle c_0, c_1, \wedge, \vee\rangle$. $\square$

Claim 3. $\operatorname{Pol}\sigma \subseteq \langle c_0, c_1, \min, \max\rangle$.

Proof. Let $f \in (\operatorname{Pol}\sigma)^{(n)}$. By Claim 2, $f|_{\{0,1\}} \in \langle c_0, c_1, \wedge, \vee\rangle$. Therefore, $f|_{\{0,1\}}$ can be written as a term using the functions c_0, c_1, $\wedge$ and $\vee$. Define a new function $g : A^n \to A$ from the term for $f|_{\{0,1\}}$ by replacing all occurrences of the constants by the corresponding constants on A, and by replacing all occurrences of $\wedge$ and $\vee$ by min and max respectively. Clearly, $g \in \operatorname{Pol}\sigma$. We will prove that $g = f$. In fact, we will prove by induction on i, that for all $i = 1, \ldots, k-1$ and for $x_1, \ldots, x_n \in \{0, 1, \ldots, i\}$, we have $f(x_1, \ldots, x_n) \in \{0, 1, \ldots, i\}$ and $g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$,

For $i = 1$, we know by Lemma 2.6.3 that $f(x_1, \ldots, x_n) \in \{0,1\}$ for all $x_1, \ldots, x_n \in \{0,1\}$, and it is trivially true that $g|_{\{0,1\}} = f|_{\{0,1\}}$.

Suppose that for some $j \in \{0, 1, \ldots, k-2\}$ and for all $x_1, \ldots, x_n \in \{0, 1, \ldots, j\}$, we have that $f(x_1, \ldots, x_n) \in \{0, 1, \ldots, j\}$ and $g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$. Let $a_1, \ldots, a_n \in \{0, 1, \ldots, j+1\}$. Then $f(a_1, \ldots, a_n) \leftarrow f(a_1^{\leftarrow}, \ldots, a_n^{\leftarrow}) \in \{0, 1, \ldots, j\}$. Therefore $f(a_1, \ldots, a_n) \in \{0, 1, \ldots, j+1\}$.

Note that by the definition of g, we have $g(a_1, \ldots, a_n) \in \{0, 1, \ldots, j+1\}$. Since $a_1^{\leftarrow}, \ldots, a_n^{\leftarrow} \in \{0, 1, \ldots, j\}$, by the induction hypothesis, $f(a_1^{\leftarrow}, \ldots, a_n^{\leftarrow}) = g(a_1^{\leftarrow}, \ldots, a_n^{\leftarrow})$. We have

$$g(a_1^{\leftarrow}, \ldots, a_n^{\leftarrow}) = f(a_1^{\leftarrow}, \ldots, a_n^{\leftarrow}) \rightarrow f(a_1, \ldots, a_n)$$

$$\text{and } g(a_1^{\leftarrow}, \ldots, a_n^{\leftarrow}) \rightarrow g(a_1, \ldots, a_n)$$

We distinguish 4 cases. CASE 1 : $g(a_1^{\leftarrow}, \ldots, a_n^{\leftarrow}) = 0$. Then $f(a_1, \ldots, a_n) = 0 = g(a_1, \ldots, a_n)$.

CASE 2 : $g(a_1^{\leftarrow}, \ldots, a_n^{\leftarrow}) = 1$. Suppose to the contrary that $f(a_1, \ldots, a_n) \neq g(a_1, \ldots, a_n)$. Then either $f(a_1, \ldots, a_n) = 1$ and $g(a_1, \ldots, a_n) = 2$, or conversely $f(a_1, \ldots, a_n) = 2$ and $g(a_1, \ldots, a_n) = 1$. If $f(a_1, \ldots, a_n) = 1$ and $g(a_1, \ldots, a_n) = 2$, then using Proposition 2.5.2, we have

$$2 = g(a_1, \ldots, a_n) \rightarrow g(a_1^{\rightarrow}, \ldots, a_n^{\rightarrow}) \rightarrow \ldots \rightarrow g(a_1^{(k-2)\rightarrow}, \ldots, a_n^{(k-2)\rightarrow}) \in \{0, 1\}$$

Therefore $g(a_1^{(k-2)\rightarrow}, \ldots, a_n^{(k-2)\rightarrow}) = 0$. Also,

$$1 = f(a_1, \ldots, a_n) \rightarrow f(a_1^{\rightarrow}, \ldots, a_n^{\rightarrow}) \rightarrow \ldots \rightarrow f(a_1^{(k-2)\rightarrow}, \ldots, a_n^{(k-2)\rightarrow}) \in \{0, 1\}$$

is a chain of length $k-2$. Therefore $f(a_1^{(k-2)\rightarrow}, \ldots, a_n^{(k-2)\rightarrow}) = 1$. By the definition of g, $1 = f(a_1^{(k-2)\rightarrow}, \ldots, a_n^{(k-2)\rightarrow}) = g(a_1^{(k-2)\rightarrow}, \ldots, a_n^{(k-2)\rightarrow}) = 0$, a contradiction.

Similarly, it is impossible that $f(a_1, \dots, a_n) = 2$ and $g(a_1, \dots, a_n) = 1$. Therefore $f(a_1, \dots, a_n) = g(a_1, \dots, a_n)$.

Case 3 : $g(\overleftarrow{a_1}, \dots, \overleftarrow{g_n}) = a \in \{2, \dots, k-2\}$. Then $g(a_1, \dots, a_n) = a + 1 = f(a_1, \dots, a_n)$.

Case 4 : $g(\overleftarrow{a_1}, \dots, \overleftarrow{g_n}) = k - 1$. Then $g(a_1, \dots, a_n) = 0 = f(a_1, \dots, a_n)$.

In all cases, $g(a_1, \dots, a_n) = f(a_1, \dots, a_n)$. By induction, $f = g$. Therefore, $f \in \langle c_0, c_1, \min, \max \rangle$. □

By Claim 1 and Claim 3, $\operatorname{Pol}\sigma = \langle c_0, c_1, \max, \min \rangle$. □

When $A = \{0, 1\}$,

$$\begin{aligned} \operatorname{Sta}\langle c_0, c_1 \rangle &= \operatorname{Sta}\{c_0, c_1, e\} \\ &= \operatorname{Pol}\{(0,0), (1,1), (0,1)\} \\ &= \operatorname{Pol}\{(0,0), (1,1), (1,0)\} \\ &= \operatorname{Pol}\sigma \end{aligned}$$

Therefore $\langle c_0, c_1, \max, \min \rangle = \operatorname{Pol}\sigma$ is the largest clone in the interval. This was already known [**29**].

For $A = \{0, 1, 2\}$, we know that $\langle c_0, c_1, \max, \min \rangle$ is the largest clone in the interval $\operatorname{Int}\langle c_0, c_1 \rangle$ [**19**]. Given that $\langle c_0, c_1, \max, \min \rangle = \operatorname{Pol}\sigma$, we can also prove this result easily using the following theorem.

Theorem 2.6.5 (Bodnarčuk, Kalužnin, Kotov, Romov [**4**]). *Let A be a finite set. Let $\rho \subseteq A^h$, and let $\sigma \subseteq A^l$ be a relation without repetitions of coordinates. Then* $\operatorname{Pol}\rho \subseteq \operatorname{Pol}\sigma$ *iff there exist $m \geq l$, $n < m^h$ and an $n \times h$ matrix $X = (x_{i,j})$ with*

$x_{i,j} \in \{1, \ldots, m\}$ such that

$$(a_1, \ldots, a_l) \in \sigma$$

$$\textit{iff there exist } a_{l+1}, \ldots, a_m \textit{ such that}$$

$$\textit{for all } i = 1, \ldots, n, (a_{x_{i,1}}, a_{x_{i,2}}, \ldots, a_{x_{i,h}}) \in \rho$$

Corollary 2.6.6 (See [**19**]). *On $A = \{0, 1, 2\}$, the clone $\langle c_0, c_1, \max, \min\rangle$ is the largest clone in the interval $\mathrm{Int}\langle c_0, c_1\rangle$.*

PROOF. By Theorem 2.6.1, we know that

$$\langle c_0, c_1, \max, \min\rangle = \mathrm{Pol}\{(0,0), (1,1), (1,2), (2,0)\}$$

The largest clone of the interval $\mathrm{Int}\langle c_0, c_1\rangle$ is the stabilizer

$$\mathrm{Sta}\langle c_0, c_1\rangle = \mathrm{Sta}\{c_0, c_1, e\} = \mathrm{Pol}\{(0,0,0), (1,1,1), (0,1,2)\}$$

By Theorem 2.6.5, using the matrix $X = \begin{pmatrix} 3 & 4 & 1 \\ 5 & 3 & 2 \end{pmatrix}$, we obtain

$$\mathrm{Pol}\{(0,0,0), (1,1,1), (0,1,2)\} \subseteq \mathrm{Pol}\{(0,0), (1,1), (1,2), (2,0)\}$$

By Lemma 2.6.4, we know that $\mathrm{Pol}\,\sigma \in \mathrm{Int}\langle c_0, c_1\rangle$, and thus $\mathrm{Pol}\,\sigma = \mathrm{Sta}\langle c_0, c_1\rangle$. This implies that $\langle c_0, c_1, \max, \min\rangle$ is the largest clone in the interval $\mathrm{Int}\langle c_0, c_1\rangle$.

□

For universes of more than 3 elements, it is no longer true that $\mathrm{Sta}\langle c_0, c_1\rangle = \langle c_0, c_1, \max, \min\rangle$. For example, for $A = \{0, 1, 2, 3\}$, the function $\max'$ defined according to the chain $0 < 2 < 3 < 1$ is in $\mathrm{Sta}\langle c_0, c_1\rangle$ but $\max' \notin \langle c_0, c_1, \max, \min\rangle$ where min and max are defined as in Theorem 2.6.1 (according to the chain $0 < 3 < 2 < 1$).

2.7. The structure of the interval

So far, we know what the monoidal interval $\operatorname{Int}\langle c_0, c_1\rangle$ looks like on 2 elements and 3 elements. They can be found in Figures 2.8 and 2.9 (where min and max are defined according to the ordering $0 < 2 < 1$).

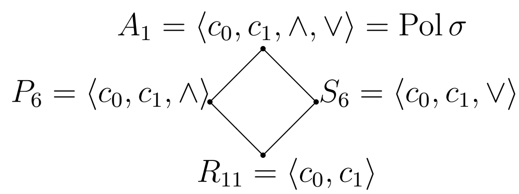


Fig. 2.8. The monoidal interval $\operatorname{Int}\langle c_0, c_1\rangle$ on 2 elements

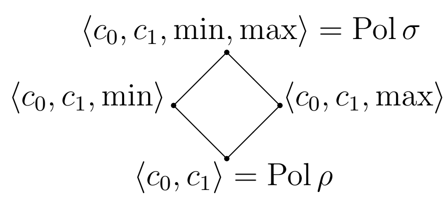


Fig. 2.9. The monoidal interval $\operatorname{Int}\langle c_0, c_1\rangle$ on 3 elements

We want to know what $\operatorname{Int}\langle c_0, c_1\rangle$ looks like in general. This is very difficult since $|\operatorname{Int}\langle c_0, c_1\rangle| = 2^{\aleph_0}$. But, we will present some of the structure near the bottom of the interval.

Proposition 2.7.1. *Let $\langle A; \wedge, \vee, c_0, c_1\rangle$ be a finite lattice with top element 1 and bottom element 0 such that $|A| \geq 2$. Then the clone $\langle c_0, c_1, \wedge, \vee\rangle$ is in the monoidal interval $\operatorname{Int}\langle c_0, c_1\rangle$ for the universe A.*

Proof. Straightforward. □

Theorem 2.7.2. *Let $\langle A; \wedge, \vee, c_0, c_1\rangle$ be a finite distributive lattice with top element 1 and bottom element 0 such that $|A| \geq 2$. Then $[\langle c_0, c_1\rangle, \langle c_0, c_1, \wedge, \vee\rangle]$ is the interval in Figure 2.10.*

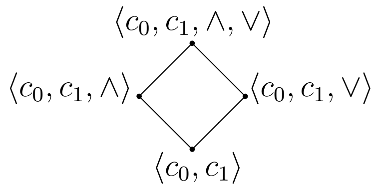


FIG. 2.10. A subinterval of $\mathrm{Int}\langle c_0, c_1\rangle$ for distributive lattices on k elements

Note that for $|A| = 2$, this was proved by Post [**29**]. For $|A| = 3$, it was proved by Krokhin [**19**] since the only possible lattice in this case is $\langle A; \min, \max, c_0, c_1\rangle$, which gives the result stated above.

PROOF. (of Theorem 2.7.2). Let $|A| \geq 2$. Let $f \in \langle c_0, c_1, \wedge, \vee\rangle$ be n-ary and not constant. There exists a non-void and inclusion-free family $\mathcal{F}$ of non-void subsets of $N = \{1, \ldots, n\}$ (i.e. $X \nsubseteq Y$ for all $X, Y \in \mathcal{F}$) such that for all $a_1, \ldots, a_n \in A$

$$f(a_1, \ldots, a_n) = \bigvee_{X \in \mathcal{F}} (\bigwedge_{i \in X} a_i)$$

If $\mathcal{F} = \{\{i\}\}$, then $f = e_i^n$.

Claim 1. *If there exists $X \in \mathcal{F}$ with $|X| \geq 2$, then $\wedge \in \langle c_0, c_1, f\rangle$.*

PROOF. For notational simplicity, let $X = \{1, 2, \ldots, i\}$ where $i \geq 2$. Form $g(x_1, x_2) \approx f(x_1, x_2, c_1, \ldots, c_1, c_0, \ldots, c_0)$ where the first c_0 is in the $(i+1)$-st place. Given that $Y \subset X$ for no $Y \in \mathcal{F}$, clearly $g = \wedge$. $\square$

Claim 2. *If $|\mathcal{F}| \geq 2$, then $\vee \in \langle c_0, c_1, f\rangle$.*

PROOF. Choose distinct $X, Y, \in \mathcal{F}$ with minimum $|X \cup Y|$. For notational simplicity, let $1 \in X \setminus Y$, $2 \in Y \setminus X$ and $X \cup Y = \{1, 2, \ldots, i\}$ where $i \geq 2$.

We claim that every $Z \in \mathcal{F}$ such that $Z \subseteq X \cup Y$ satisfies $1, 2 \in Z$. Indeed suppose to the contrary that $Z \subseteq (X \cup Y) \setminus \{1\}$. Then $1 \notin Z$ and $|Y \cup Z| \leq i - 1$ contrary to the minimality of i. Similarly, $|X \cup Z|$ is contrary to the minimality of

i if $Z \subseteq (X \cup Y) \setminus \{2\}$. Set $h(x_1, x_2) \approx f(x_1, x_2, c_1, \dots c_1, c_0, \dots, c_0)$ where the first c_0 is at the $(i+1)$-st place. Now $h(x_1, x_2) \approx x_1 \vee x_2$ or $h(x_1, x_2) \approx x_1 \vee x_2 \vee (x_1 \wedge x_2)$ depending on if there exists such a Z. In the latter case, the absorption law yeilds $h(x_1, x_2) \approx x_1 \vee x_2$ □

Examining all the possible cases, we obtain exactly Figure 2.10. □

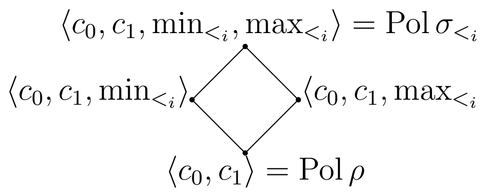


FIG. 2.11. General min-max subintervals

Theorem 2.7.3. *Let* $A = \{0, \dots, k-1\}$ *and* $k \geq 3$. *Let* $\{<_i \mid i \in \{0, \dots, (k-2)!\}\}$ *be all the possible chains of* A *such that 0 is the smallest and 1 the largest elements in the chain. Define*

$$\rho = \{(0,0), (0,1), (1,1), (1,2), (2,3), \dots, (k-1, 0)\}$$

and for each $i \in \{0, \dots, (k-2)!\}$, *define*

$$\sigma_{<_i} = \{(0,0), (1,1), (a_2, 0), (a_3, a_2), \dots, (1, a_{k-1})\}$$

where $0 <_i a_2 <_i a_3 <_i \dots <_i a_{k-1} <_i 1$, *and define* $\max_{<_i}$ *and* $\min_{<_i}$ *according to the ordering* $<_i$. *Then for each* $<_i$, *the interval* $[\operatorname{Pol} \rho, \operatorname{Pol} \sigma_{<_i}]$ *is contained in the monoidal interval* $\operatorname{Int}\langle c_0, c_1 \rangle$ *and has Figure 2.11 as its Hasse diagram.*

PROOF. Follows from Theorems 2.5.1, 2.6.1 and 2.7.2, and Proposition 2.7.1.

□

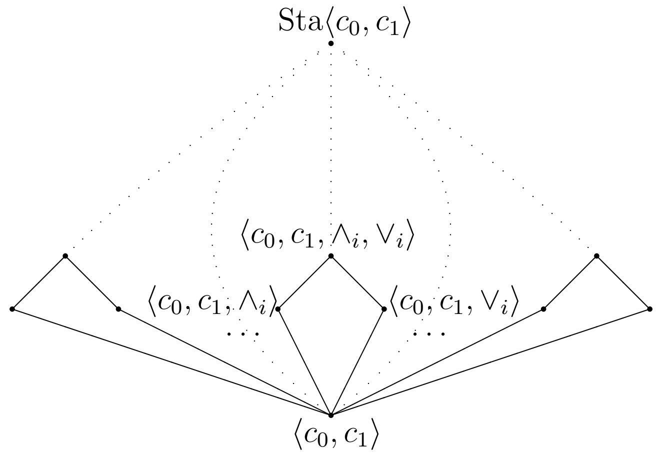


FIG. 2.12. The interval Int$\langle c_0, c_1 \rangle$ for finite universes

Theorem 2.7.4. *Let $A = \{0, \ldots, k-1\}$ and $k \geq 4$. Consider all the possible distributive lattices on A with top element 1 and bottom element 0 :*

$$\{\langle A; \wedge_i, \vee_i, c_0, c_1 \rangle \mid i \in I\}$$

Then the lower part of the monoidal interval Int$\langle c_0, c_1 \rangle$ *contains the diamond shaped intervals $[\langle c_0, c_1 \rangle, \langle c_0, c_1, \wedge_i, \vee_i \rangle]$ as shown in Figure 2.12.*

PROOF. Follows from Proposition 2.7.1 and Theorem 2.7.2. □

Chapitre 3

CLONES ON THREE ELEMENTS PRESERVING A BINARY RELATION

Auteur : Anne Fearnley

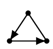
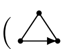
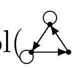
3.1. Contributions nouvelles de l'auteur

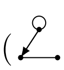
L'article *Clones on three elements preserving a binary relation* annonce les résultats de mon mémoire de maîtrise. Il est inclus dans cette thèse car il contient un nouveau résultat, le théorème 3.6.4 qui a été découvert pendant mon doctorat et qui est représentatif des méthodes que j'ai élaborées dans mes recherches.

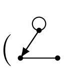
L'article [**10**] présenté dans ce chapitre contient les théorèmes et quelques résultats de mon mémoire de maîtrise [**8**]. La version complète [**11**] étant trop longue, Algebra Universalis a quand même tenu à publier l'article en partie. J'ai l'intention de faire publier la version complète prochainement.

3.2. Clones sur trois éléments qui préservent une opération binaire

La rédaction en vue de publication des résultats de ma maîtrise [**8**], interrompue par la recherche contenue dans [**12**], reprenait de plus belle. Mon mémoire de maîtrise est la liste de tous les clones sur trois éléments qui sont de la forme $\operatorname{Pol}\rho$ pour ρ une relation binaire, et comment ils sont ordonnés par inclusion.

La plupart des inclusions sont prouvées à partir du théorème 3.7.1. Deux petits clones connus complètement génèrent aussi un grand nombre d'inclusions. Ils sont Pol() = Pol() = $\langle e\rangle$ et Pol() = $\langle c_0, c_1\rangle$. Ils sont tous deux généralisés à plus de trois éléments ; le premier dans [**32**] et [**9**], le deuxième dans [**12**] qui est au chapitre 2 de la présente thèse.

Mon mémoire de maîtrise contenait aussi plusieurs petits théorèmes pour prouver l'inclusion d'un clone dans un autre pour laquelle je n'avais pas pu trouver une matrice pour le théorème 3.7.1. Ces petits théorèmes m'agaçaient. Je réussis à trouver quelques matrices pour pour prouver certaines des inclusions. Les cas qui restaient impliquaient tous le clone Pol(). Je prouvai finalement que

Pol() $= \langle c_0, \mathrm{maj}_0\rangle$

où

$$\mathrm{maj}_0(x,y,z) := \begin{cases} i, & \text{if } x=y=i \text{ or } x=z=i \text{ or } y=z=i\,; \\ 0, & \text{sinon.} \end{cases}$$

Inspirée par le théorème 2.5.1, je généralisai le résultat ci-haut de la façon suivante.

Théorème 3.2.1. *Soit* $A = \{0, 1, \ldots, k-1\}$ *où* $k > 2$. *Soit* ρ *une relation binaire sur* A *définie par*

$$\rho = \{(0,0), (0,1), (1,2), \ldots, (k-2, k-1), (k-1, 1)\}$$

(voir la figure 3.1). Alors $\operatorname{Pol}\rho = \langle c_0, \mathrm{maj}_0 \rangle$.

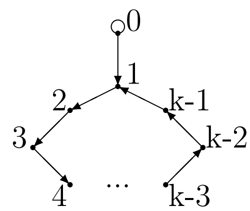


FIG. 3.1. Une relation preservée par $\langle c_0, \mathrm{maj}_0 \rangle$

Bien que ma thèse se concentre surtout sur les clones ne contenant que des constantes et des permutations, les théorèmes 3.2.1 et aussi 2.6.1 montrent que ce genre preuve s'applique plus largement.

3.3. Abstract

We describe the clones on 3 elements that can be expressed as $\operatorname{Pol}\rho$ for ρ a binary relation. We present the poset of these clones ordered by inclusion. This article is a shortened version to give an idea of the whole work.

3.4. Introduction

In 1941, Post presented a complete description of the countably many clones on 2 elements [**29**]. The structure of the lattice of clones on finitely many (but more than 2) elements is more complex ; in fact the lattice is of cardinality $2^{\aleph_0}$ [**17**]. It is hoped that by studying clones on 3 elements, we might get an idea of the general structure of the lattice of clones on any finite set of cardinality greater than 2.

It is also known that every clone C on a set A can be expressed as the clone of those operations preserving a set of relations on A, i.e. $C = \operatorname{Pol} R$ where R is a set of relations on A. This may be rewritten as $C = \bigcap_{\rho \in R} \operatorname{Pol} \rho$. Hence those clones of operations preserving a single relation may be viewed as a sort of skeleton for the whole lattice $\mathcal{L}_A$. If we consider only the clones of operations preserving a single binary relation on 3 elements, we already have 266 clones in 67 equivalence classes. Included among them are 16 of the 18 maximal clones.

The author's Master's thesis was a compilation and an ordering by inclusion of the clones on three elements which preserve one binary relation. This article is a translation and a rewriting of part of that thesis. Errors have been corrected, and several inclusions have been simplified, in part due to Theorem 3.6.4, which is new. Some of the diagrams have been redrawn for clarity and a couple have been added. The Non-Inclusions section of the thesis has been omitted for brevity and because it is straightforward. Due to the great length of the list of all clones on three elements which preserve one binary relation and the size of the diagrams describing all the inclusions between such clones, only a sample is shown in this article. The complete list of relations and all the diagrams can be found in [**11**].

3.5. Definitions

Let A be a finite set and n a positive integer. An *n-ary operation* on A is a function $f : A^n \to A$. The set of all n-ary operations on A is denoted by $\mathcal{O}_A^{(n)}$, and $\mathcal{O}_A := \bigcup_{0<n<\omega} \mathcal{O}_A^{(n)}$. For $1 \le i \le n$, the n-ary *i-th projection* is defined as

$e_i^{(n)}(x_1,\ldots,x_n) = x_i$ for all $x_1,\ldots,x_n$. We write e for the identity operation. For $a \in A$, the n-ary *constant operation* a is defined as $c_a^{(n)}(x_1,\ldots,x_n) = a$ for all $x_1,\ldots,x_n$. We write simply c_a for unary constant operations.

For $f \in \mathcal{O}^{(n)}$, and $g_1,\ldots,g_n \in \mathcal{O}^{(m)}$, we define their composition to be the m-ary operation $f[g_1,\ldots,g_n]$ defined by

$$f[g_1,\ldots,g_n](x_1,\ldots,x_m) = f(g_1(x_1,\ldots,x_m),\ldots,g_n(x_1,\ldots,x_m))$$

A *clone* on A is a subset F of $\mathcal{O}_A$ which contains all the projections and is closed under composition. It is well known and easy to prove that the intersection of an arbitrary set of clones on A is a clone on A. Thus for $F \subseteq \mathcal{O}_A$, there exists the least clone containing F, called the clone *generated* by F and denoted by $\langle F\rangle$. Equivalently, $\langle F\rangle$ is the set of *term operations* of the algebra $\mathbf{A} = \langle A; F\rangle$ usually denoted by $T(\mathbf{A})$. The clones on A, ordered by inclusion, form the complete lattice $\mathcal{L}_A$.

Let h be a positive integer. A *h-ary relation* ρ is a subset of A^h. For $\rho \in A^2$, we write $a \rightarrow b$ for $(a,b) \in \rho$. The relations may then be drawn as directed graphs. For example, for $A = \{0,1,2\}$, the relation $\{(0,0),(0,1),(1,0),(1,2)\}$ may be represented as in Figure 3.2 :

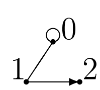


FIG. 3.2. Example of a relation

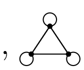
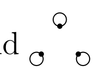
Let $f \in \mathcal{O}^{(n)}$, and let ρ be an h-ary relation on A. The operation f *preserves* ρ if for all $(a_{1,i}, a_{2,i},\ldots,a_{h,i}) \in \rho$ $(i = 1,\ldots,n)$,

$$(f(a_{1,1},a_{1,2},\ldots,a_{1,n}), f(a_{2,1},a_{2,2},\ldots,a_{2,n}),\ldots,f(a_{h,1},a_{h,2},\ldots,a_{h,n})) \in \rho$$

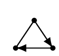
The set of operations on A preserving ρ is a clone denoted by $\operatorname{Pol}\rho$. A relation ρ is *strongly rigid* if it is preserved only by the projections.

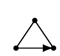
From now on, we will assume that we are working on the set $\mathbf{3} := \{0, 1, 2\}$ and that relations are binary. When we draw them as directed graphs, we will omit the numbers ; they will be assumed to be in the same configuration as in Figure 3.2. Note that the unary relations are equivalent to certain binary relations. For example, the unary relation $\{0, 1\}$ can be represented as $\{(0, 0), (1, 1)\}$, so that we are in fact studying all unary relations as well.

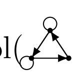
3.6. Description of some clones

It is easy to see that the trivial relations $\emptyset$ and $\mathbf{3}^n$ are preserved by all operations. For binary relations, that means that ,  and  (because it corresponds to the full unary relation) are preserved by all operations.

The following four theorems exhibit relations for which we know all the operations that preserve them.

Theorem 3.6.1 (Rosenberg [**32**])**.** *The relation*  *is strongly rigid.*

Theorem 3.6.2 (Fearnley [**9**])**.** *The relation*  *is strongly rigid.*

Theorem 3.6.3 (Fearnley [**12**])**.** Pol() $= \langle c_0, c_1 \rangle$.

Theorem 3.6.4. *Let $A = \{0, 1, \ldots, k-1\}$ where $k > 2$. Let ρ be the following binary relation on A (see Figure 3.3)*

$$\rho = \{(0, 0), (0, 1), (1, 2), \ldots, (k-2, k-1), (k-1, 1)\}$$

Then $\mathrm{Pol}\,\rho = \langle c_0, \mathrm{maj}_0\rangle$ *where* maj_0 *is the ternary operation defined by*

$$\mathrm{maj}_0(x,y,z) := \begin{cases} i, & \text{if } x=y=i \text{ or } x=z=i \text{ or } y=z=i\,; \\ 0, & \text{otherwise.} \end{cases}$$

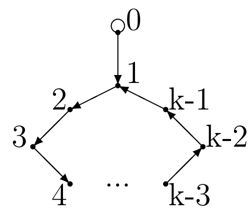


FIG. 3.3. A relation preserved by $\langle c_0, \mathrm{maj}_0\rangle$

Before proving the theorem, let us state a pair of definitions with some of their properties, and prove a lemma. We define two unary operations : $x^{\rightarrow}$ given by $0^{\rightarrow} = 0$, $a^{\rightarrow} = a+1$ if $0 < a < k-1$ and $(k-1)^{\rightarrow} = 1$, and $x^{\leftarrow}$ given by $0^{\leftarrow} = 0$ and $a^{\leftarrow} = a-1$ if $0 < a \leq k-1$. We write $a^{2\rightarrow}$ instead of $a^{\rightarrow\rightarrow}$, and so on. We define $a^{i\leftarrow}$ similarly.

Proposition 3.6.5. *(A)* $a \to a^{\rightarrow}$ *and* $a^{(k-1)\rightarrow} = a$ *for all* $a \in A$.

(B) $a^{\leftarrow} \to a$, $a^{(k-2)\leftarrow} \in \{0,1\}$ *and* $a^{(k-1)\leftarrow} = 0$ *for all* $a \in A$.

(C) Either $\{a, a^{\rightarrow}, a^{2\rightarrow}, \ldots, a^{(k-2)\rightarrow}\} = \{1, 2, \ldots, k-1\}$

or $a = a^{\rightarrow} = a^{2\rightarrow} = \ldots = a^{(k-2)\rightarrow} = 0$.

(D) Let $f \in \mathrm{Pol}\,\rho$ *be an* n*-ary operation. Then*

$$f(x_1^{\rightarrow}, \ldots, x_n^{\rightarrow}) = (f(x_1, \ldots, x_n))^{\rightarrow} \text{ for all } x_1, \ldots, x_n \in A$$

PROOF. Statements (A) and (B) are trivial. Statement (C) is derived from statement (A) since $a \to a^{\rightarrow} \to a^{2\rightarrow} \to \ldots \to a^{(k-1)\rightarrow} = a$.

To prove statement (D), let $x_1, \ldots, x_n \in A$. By (A), we have that

$$f(x_1, \ldots, x_n) \to f(x_1, \ldots, x_n)^{\rightarrow}. \tag{3.6.1}$$

Since $f \in \operatorname{Pol}\rho$, and since by (A), $x_1 \to x_1^{\to}, \ldots, x_n \to x_n^{\to}$, it follows that

$$f(x_1, \ldots, x_n) \to f(x_1^{\to}, \ldots, x_n^{\to}). \tag{3.6.2}$$

If $f(x_1, \ldots, x_n) \neq 0$, then $f(x_1^{\to}, \ldots, x_n^{\to})$ is uniquely determined, so by equations (3.6.1) and (3.6.2), we have $f(x_1^{\to}, \ldots, x_n^{\to}) = (f(x_1, \ldots, x_n))^{\to}$ as required. If $f(x_1, \ldots, x_n) = 0$ and $f(x_1^{\to}, \ldots, x_n^{\to}) = 0$, then the equality is fulfilled. However, if $f(x_1, \ldots, x_n) = 0$ and $f(x_1^{\to}, \ldots, x_n^{\to}) = 1$, then from (3.6.2), it follows that $1 = f(x_1^{\to}, \ldots, x_n^{\to}) \to f(x_1^{2\to}, \ldots, x_n^{2\to}) \to \ldots \to f(x_1^{(k-1)\to}, \ldots, x_n^{(k-1)\to}) = f(x_1, \ldots, x_n) = 0$, which is a contradiction. □

Lemma 3.6.6. *Let $f \in \operatorname{Pol}\rho$ be an n-ary operation, and let $x_1, \ldots, x_n \in \{0, 1\}$. Then $f(0, \ldots, 0) = 0$ and $f(x_1, \ldots, x_n) \in \{0, 1\}$.*

Proof. Since $0 \leftrightarrow 0$, we have $f(0, \ldots, 0) \leftrightarrow f(0, \ldots, 0)$. Thus $f(0, \ldots, 0) = 0$. Now $0 \to x_i$ for $i = 1, \ldots, n$, thus $0 = f(0, \ldots, 0) \to f(x_1, \ldots, x_n)$. Therefore $f(x_1, \ldots, x_n) \in \{0, 1\}$. □

Proof. (of Theorem 3.6.4). We begin by showing that $\langle c_0, \mathrm{maj}_0 \rangle \subseteq \operatorname{Pol}\rho$. Then, for an operation $f \in \operatorname{Pol}\rho$, we consider its diagonal, which, being unary, must be a constant or the identity. We show that a constant diagonal implies that the original operation is a constant. When the diagonal is the identity, we consider the restriction of f to $\{0, 1\}$. The Boolean clones are all known [**29**]. We find that $f|_{\{0,1\}} \subseteq \langle c_0, \mathrm{maj} \rangle$ on $\{0, 1\}$. Finally, we show that what happens on $\{0, 1\}$ determines what happens on the whole set. This completes the proof.

Claim 1. *c_0 and maj_0 are in $\operatorname{Pol}\rho$.*

Proof. For c_0, note that if $a \to b$ then $c_0(a) = 0 \to 0 = c_0(b)$. Therefore $c_0 \in \operatorname{Pol}\rho$.

For maj_0, let $a_1, a_2, b_1, b_2, c_1, c_2 \in A$ such that $a_1 \to a_2$, $b_1 \to b_2$ and $c_1 \to c_2$. If $a_1 = b_1 \neq 0$, then $a_2 = a_1^{\rightarrow} = b_2$. Thus $\text{maj}_0(a_1, b_1, c_1) = \text{maj}_0(a_1, a_1, c_1) = a_1 \to a_1^{\rightarrow} = a_2 = \text{maj}_0(a_2, a_2, c_2) = \text{maj}_0(a_2, b_2, c_2)$ as required. The cases where $a_1 = c_1 \neq 0$ and $b_1 = c_1 \neq 0$ are similar. In all other cases, $\text{maj}_0(a_1, b_1, c_1) = 0$, and the only way we could fail to have $\text{maj}_0(a_1, b_1, c_1) \to \text{maj}_0(a_2, b_2, c_2)$ would be if $\text{maj}_0(a_2, b_2, c_2) \notin \{0, 1\}$. For that to happen, we would need to have $a_2 = b_2 \notin \{0, 1\}$ or $a_2 = c_2 \notin \{0, 1\}$ or $b_2 = c_2 \notin \{0, 1\}$, which have already been covered in a previous case. Therefore $\text{maj}_0 \in \operatorname{Pol} \rho$. $\square$

Let $f \in \operatorname{Pol} \rho$ be an n-ary operation. Define its diagonal operation $d : A \to A$ by $d(x) := f(x, \dots, x)$ for all $x \in A$.

Claim 2. *Either $d = c_0$ or $d = e$.*

Proof. By Lemma 3.6.6, $d(0) = f(0, \dots, 0) = 0$ and $d(1) \in \{0, 1\}$. If $d(1) = 0$, we have $d(2) \to \dots \to d(k-1) \to d(1) = 0$. Therefore $d(2) = \dots = d(k-1) = 0$, which implies that $d = c_0$. If $d(1) = 1$, we have $1 = d(1) \to d(2) \to \dots \to d(k-1) \to d(1) = 1$. Therefore $d(a) = a$ for all $a \in A$, which implies that $d = e$. $\square$

Claim 3. *If $d = c_0$ then $f = c_0^{(n)}$.*

Proof. Let $x_1, \dots, x_n \in \{0, 1\}$, then $x_i^{(k-2)\rightarrow} \in \{k-1, 0\}$ and hence $x_i^{(k-2)\rightarrow} \to 1$ for all $i = 1, \dots, n$. By Propositions 3.6.5(A) and (D), we obtain $f(x_1, \dots, x_n) \to f(x_1^{\rightarrow}, \dots, x_n^{\rightarrow}) \to \dots \to f(x_1^{(k-2)\rightarrow}, \dots, x_n^{(k-2)\rightarrow}) \to f(1, \dots, 1) = d(1) = 0$. Therefore $f(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in \{0, 1\}$. Now let $x_1, \dots, x_n \in A$. By Propositions 3.6.5(A), 3.6.5(D) and the result above, we have

$$f(x_1, \dots, x_n) \leftarrow f(x_1^{\leftarrow}, \dots, x_n^{\leftarrow}) \leftarrow \dots \leftarrow f(x_1^{(k-2)\leftarrow}, \dots, x_n^{(k-2)\leftarrow}) = 0.$$

Thus $f(x_1,\dots,x_n)\in\{0,\dots,k-2\}$. Similarly,

$$(f(x_1,\dots,x_n))^{\rightarrow} = f(x_1^{\rightarrow},\dots,x_n^{\rightarrow}) \in \{0,\dots,k-2\}.$$

In the same way, we find that

$$(f(x_1,\dots,x_n))^{2\rightarrow},\dots,(f(x_1,\dots,x_n))^{(k-2)\rightarrow} \in \{0,\dots,k-2\}.$$

By Proposition 3.6.5(C), this implies that $f(x_1,\dots,x_n)=0$. $\square$

For every $f : A^n \to A$ in $\operatorname{Pol}\rho$, we consider the corresponding Boolean operation $f|_{\{0,1\}} : \{0,1\}^n \to \{0,1\}$. This is possible because of Lemma 3.6.6. Note that $\mathrm{maj}_0|_{\{0,1\}} = \mathrm{maj}$ (the usual Boolean majority operation) and that c_0 become the corresponding Boolean constant c_0. Now define $(\operatorname{Pol}\rho)|_{\{0,1\}} := \{f|_{\{0,1\}} \mid f \in \operatorname{Pol}\rho\}$. Clearly, $(\operatorname{Pol}\rho)|_{\{0,1\}}$ is a clone on $\{0,1\}$.

Claim 4. $(\operatorname{Pol}\rho)|_{\{0,1\}} = \langle c_0, \mathrm{maj}\rangle$.

Proof. Using Post's classification **[29]**, we can find what $(\operatorname{Pol}\rho)|_{\{0,1\}}$ is. By Claim 2, we need only consider the clones on $\{0,1\}$ which contain c_0, but not c_1. These clones are shown in Figure 3.4, along with the clones F_5^∞, D_2 and S_2, which are refered to in this proof, and a few other clones needed to situate them.

Since c_0 and maj are in $(\operatorname{Pol}\rho)|_{\{0,1\}}$, that means that $\langle c_0\rangle = R_8$ and $\langle \mathrm{maj}\rangle = D_2$ are included in $(\operatorname{Pol}\rho)|_{\{0,1\}}$, hence $F_7^2 = \langle c_0, \mathrm{maj}\rangle \subseteq (\operatorname{Pol}\rho)|_{\{0,1\}}$.

But the operation $g(x,y,z) := x\wedge(y\vee\neg z) \notin (\operatorname{Pol}\rho)|_{\{0,1\}}$ since otherwise, there would be an operation $f\in\operatorname{Pol}\rho$ such that $f|_{\{0,1\}}(x,y,z) = x\wedge(y\vee\neg z)$. Such an operation would have the property that $1 = f(1,0,0)\to f(2,0,0)\to\dots\to f(k-1,0,0)\to f(1,0,1)=0$, which is impossible. Therefore $\langle g\rangle = F_5^\infty \nsubseteq (\operatorname{Pol}\rho)|_{\{0,1\}}$.

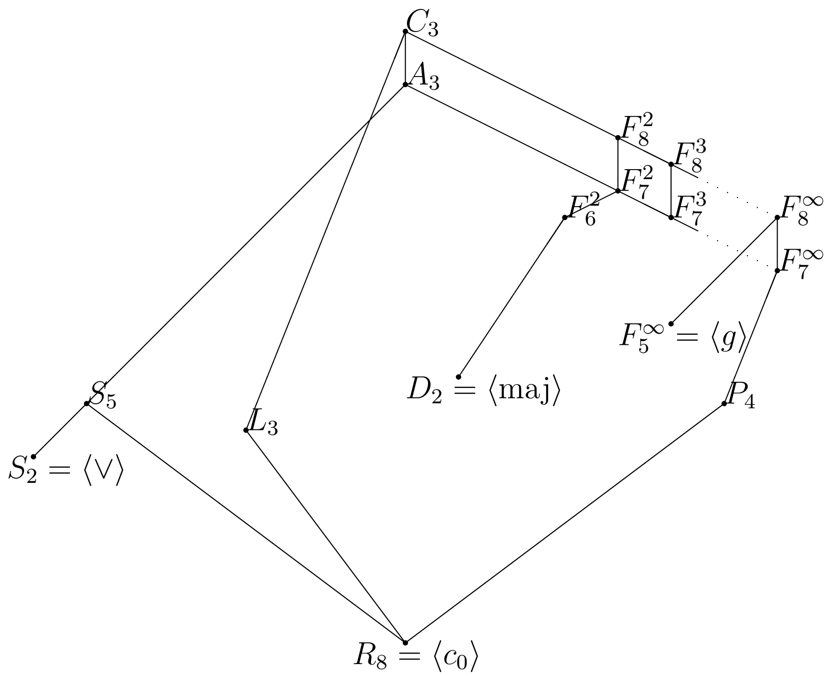


FIG. 3.4. A part of Post's lattice

The operation $x \vee y \notin (\operatorname{Pol}\rho)|_{\{0,1\}}$, since otherwise, there would be a operation $f \in \operatorname{Pol}\rho$ such that $f|_{\{0,1\}}(x,y) = x \vee y$. Such an operation would have the property that

$$1 = f(0,1) \to f(1,2) \to \ldots \to f(k-3,k-2) \to f(k-2,k-1) \to f(k-1,1)$$

which implies that $f(k-1,1) = 1$, and

$$1 = f(1,0) \to f(2,0) \to \ldots \to f(k-2,0) \to f(k-1,1) = 1$$

which is impossible because it is only a $(k-2)$-cycle. Therefore $\langle \vee \rangle = S_2 \not\subseteq (\operatorname{Pol}\rho)|_{\{0,1\}}$. Hence $(\operatorname{Pol}\rho)|_{\{0,1\}} = F_7^2 = \langle c_0, \mathrm{maj} \rangle$. $\square$

Claim 5. *Let $f, g \in \operatorname{Pol}\rho$ such that $f|_{\{0,1\}} = g|_{\{0,1\}}$. Then $f = g$.*

Proof. Let f, g be n-ary and let $x_1, \ldots, x_n \in A$. By Proposition 3.6.5, we have for all $i \in \{0, \ldots, k-2\}$,

$$(g(x_1, \ldots, x_n))^{i\rightarrow} = g(x_1^{i\rightarrow}, \ldots, x_n^{i\rightarrow}) \leftarrow g((x_1^{i\rightarrow})^{\leftarrow}, \ldots, (x_n^{i\rightarrow})^{\leftarrow}) \leftarrow \ldots$$

$$\leftarrow g((x_1^{i\rightarrow})^{(k-2)\leftarrow}, \ldots, (x_n^{i\rightarrow})^{(k-2)\leftarrow}) = f((x_1^{i\rightarrow})^{(k-2)\leftarrow}, \ldots, (x_n^{i\rightarrow})^{(k-2)\leftarrow})$$

$$\rightarrow \ldots \rightarrow f((x_1^{i\rightarrow})^{\leftarrow}, \ldots, (x_n^{i\rightarrow})^{\leftarrow}) \rightarrow f(x_1^{i\rightarrow}, \ldots, x_n^{i\rightarrow}) = (f(x_1, \ldots, x_n))^{i\rightarrow}$$

Furthermore, by Lemma 3.6.6, $f((x_1^{i\rightarrow})^{(k-2)\leftarrow}, \ldots, (x_n^{i\rightarrow})^{(k-2)\leftarrow}) \in \{0, 1\}$. If there exists an i such that $f((x_1^{i\rightarrow})^{(k-2)\leftarrow}, \ldots, (x_n^{i\rightarrow})^{(k-2)\leftarrow}) = 1$, then for that i we would have

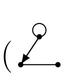
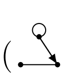
$$(g(x_1, \ldots, x_n))^{i\rightarrow} = k - 1 = (f(x_1, \ldots, x_n))^{i\rightarrow}$$

which implies that

$$g(x_1, \ldots, x_n) = (k-1)^{((k-1)-i)\rightarrow} = f(x_1, \ldots, x_n)$$

as required. Now suppose that $f((x_1^{i\rightarrow})^{(k-2)\leftarrow}, \ldots, (x_n^{i\rightarrow})^{(k-2)\leftarrow}) = 0$ for all $i \in \{0, \ldots, k-2\}$. In that case, $(f(x_1, \ldots, x_n))^{i\rightarrow}, (g(x_1, \ldots, x_n))^{i\rightarrow} \in \{0, \ldots, k-2\}$. Therefore, by Proposition 3.6.5, $g(x_1, \ldots, x_n) = 0 = f(x_1, \ldots, x_n)$ as required. $\square$

Thus $f \in \operatorname{Pol} \rho$ is entirely determined by $f|_{\{0,1\}}$. By Claim 4, $f|_{\{0,1\}}$ can be written as a term made up of c_0 and maj on $\{0, 1\}$. If we replace all occurrences of c_0 and maj in the term by the corresponding operations on A, we must obtain f. Therefore f can be written as a term made up of maj_0 and c_0 on A. In other words, $f \in \langle c_0, \text{maj}_0 \rangle$. Therefore, by Claim 1, $\operatorname{Pol} \rho = \langle c_0, \text{maj}_0 \rangle$. $\square$

Corollary 3.6.7. *For* $A = \{0, 1, 2\}$, Pol() = Pol() $= \langle c_0, \text{maj}_0 \rangle$.

3.7. Inclusions

For the relations not covered in Section 3.6, we must find out which clones are included in which others. The following theorem provides a way of finding inclusions directly from the relations. Note that a relation $\rho \subseteq A^h$ is *without repetitions* if $\rho \nsubseteq \{(a_1,\dots,a_h) \in A^h \mid a_i = a_j \text{ if } (i,j) \in \epsilon\}$ for any equivalence $\epsilon \in \{1,\dots,h\}^2$ other than the trivial equivalence $\{(1,1),(2,2),\dots,(h,h)\}$. For binary relations on 3 elements, this means that $\rho \nsubseteq \{(0,0),(1,1),(2,2)\}$. Also note that if $\rho \subseteq \{(0,0),(1,1),(2,2)\}$, then $\operatorname{Pol}\rho = \operatorname{Pol}\rho^{(1)}$ where $\rho^{(1)} = \{a \mid (a,a) \in \rho\}$, which is without repetitions.

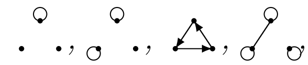
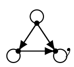
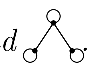
Theorem 3.7.1 (Bodnarčuk, Kalužnin, Kotov, Romov [**4**])**.** *Let A be a finite set. Let $\rho \subseteq A^h$, and let $\sigma \subseteq A^l$ be a relation without repetitions. Then $\operatorname{Pol}\rho \subseteq \operatorname{Pol}\sigma$ iff there exist $m \geq l$, $n < m^h$ and an $n \times h$ matrix $X = (x_{ij})$ with $x_{ij} \in \{1,\dots,m\}$ such that $(a_1,\dots,a_l) \in \sigma$ iff there exist $a_{l+1},\dots,a_m$ such that for all $i = 1,\dots,n$, $(a_{x_{i,1}}, a_{x_{i,2}},\dots,a_{x_{i,h}}) \in \rho$*

We can greatly decrease the number of relations we need to examine by considering the following results.

Proposition 3.7.2. *Let π be a permutation of A, ρ an h-ary relation on A and f an n-ary operation on A. Set $\pi(\rho) := \{(\pi(a_1),\dots,\pi(a_h)) \mid (a_1,\dots,a_h) \in \rho\}$, and $f_\pi : A^n \to A$ defined by $f_\pi(x_1,\dots,x_n) = \pi(f(\pi^{-1}(x_1),\dots,\pi^{-1}(x_n)))$. Then $\operatorname{Pol}(\pi(\rho)) = (\operatorname{Pol}\rho)_\pi := \{f_\pi \mid f \in \operatorname{Pol}\rho\}$.*

Corollary 3.7.3. *Let ρ be a binary relation on A. Set $\rho' = \{(b,a) \mid (a,b) \in \rho\}$. Then $\operatorname{Pol}\rho = \operatorname{Pol}\rho'$.*

Proof. Use Theorem 3.7.1 with the matrix $\begin{pmatrix} 1 & 0 \end{pmatrix}$. □

Theorem 3.7.4 (Jablonskij [**16**] and Rosenberg [**30**] and [**31**])**.** *There are 18 maximal clones in the lattice of clones on 3 elements. The 16 that are preserved by binary relations, are of 6 types. These relations are :*   *and* 

Theorem 3.7.1 gives a necessary and sufficient condition for inclusion. In practise, proving that there is no inclusion, using the theorem, is difficult. To show that the clone $\mathrm{Pol}\,\rho$ is not included in the clone $\mathrm{Pol}\,\sigma$, we only need to find an operation f which preserves ρ but not σ. We have found such a function for each pair of relations σ, ρ for which $\mathrm{Pol}\,\rho \nsubseteq \mathrm{Pol}\,\sigma$. Finding such operations is straightforward : the operations used are at most 4-ary, and almost all are at most binary. The tables of operations will not be included in this paper, but they can be found in [**8**].

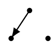
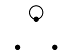
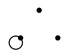
The complete version of this paper contains a list of all the binary relations and what relations they are equivalent to and why. For each relation ρ, up to equivalence, we indicate which relations σ are such that $\mathrm{Pol}\,\rho \subseteq \mathrm{Pol}\,\sigma$ minimally (within the scope of this study), along with the matrix or theorem that proves it. Here, we present the relations of cardinality 0, 1 and 2 to give an idea of what is known. The complete list of relations can be found in [**11**](Section 5, pp.7-19).

Cardinality 0

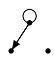
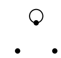
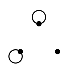
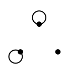
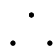
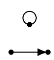
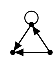
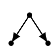
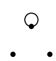
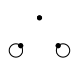
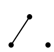
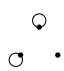
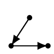
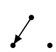
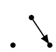
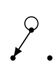
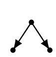
 Greatest clone (Section 3.6)

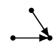
Cardinality 1

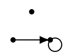
 < Maximal clone

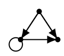
	$<$		$\approx \{0\}$	$\begin{pmatrix} 1 & 2 \end{pmatrix}$
	$<$		$\approx \{1\}$	$\begin{pmatrix} 2 & 1 \end{pmatrix}$

Cardinality 2

	$<$		$\approx \{0\}$	$\begin{pmatrix} 1 & 1 \end{pmatrix}$
	$<$		$\approx \{0, 1\}$	$\begin{pmatrix} 2 & 1 \end{pmatrix}$
	$<$			Maximal clone
	$<$			$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$
	$<$		$\approx \{0\}$	$\begin{pmatrix} 1 & 2 \end{pmatrix}$
	$<$		$\approx \{1, 2\}$	$\begin{pmatrix} 2 & 1 \end{pmatrix}$
	$<$		$\approx \{0, 1\}$	$\begin{pmatrix} 1 & 2 \end{pmatrix}$
	$<$			$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$
	$<$			$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$
	$<$			$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$
	$<$			$\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \end{pmatrix}$
	$<$			$\begin{pmatrix} 1 & 3 \\ 3 & 4 \\ 5 & 2 \end{pmatrix}$

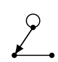
$<$  $\begin{pmatrix} 1 & 3 \\ 4 & 2 \\ 5 & 4 \end{pmatrix}$

$<$  $\begin{pmatrix} 3 & 1 \\ 4 & 2 \\ 5 & 4 \end{pmatrix}$

$<$  $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

3.8. Inclusion diagram

The whole inclusion diagram for clones preserving one binary relation can be found in the complete version of this paper on the author's website [**11**](Section 7, pp. 29-45). It is separated into sub-diagrams which are either one or more intervals, or a set of clones satisfying a certain property. Only the first sub-diagram is included here. All the diagrams are Hasse diagrams. That is $A \subseteq B$ is indicated by linking A and B by a line such that A is lower than B on the page. Throughout, we write simply the relation to represent the clone preserving that relation.

Figures 3.5 and 3.6 show the interval from  to  in two parts. These are the clones containing the constant c_0 but no other constants. The greatest and smallest clones are also indicated.

3.9. Conclusion

I have found the clones on 3 elements preserving one binary relation to be a useful framework and source of examples for understanding clones in general. An obvious way to extend this research would be to consider relations of greater arity. One could also consider binary relations on 4 or more elements to look for

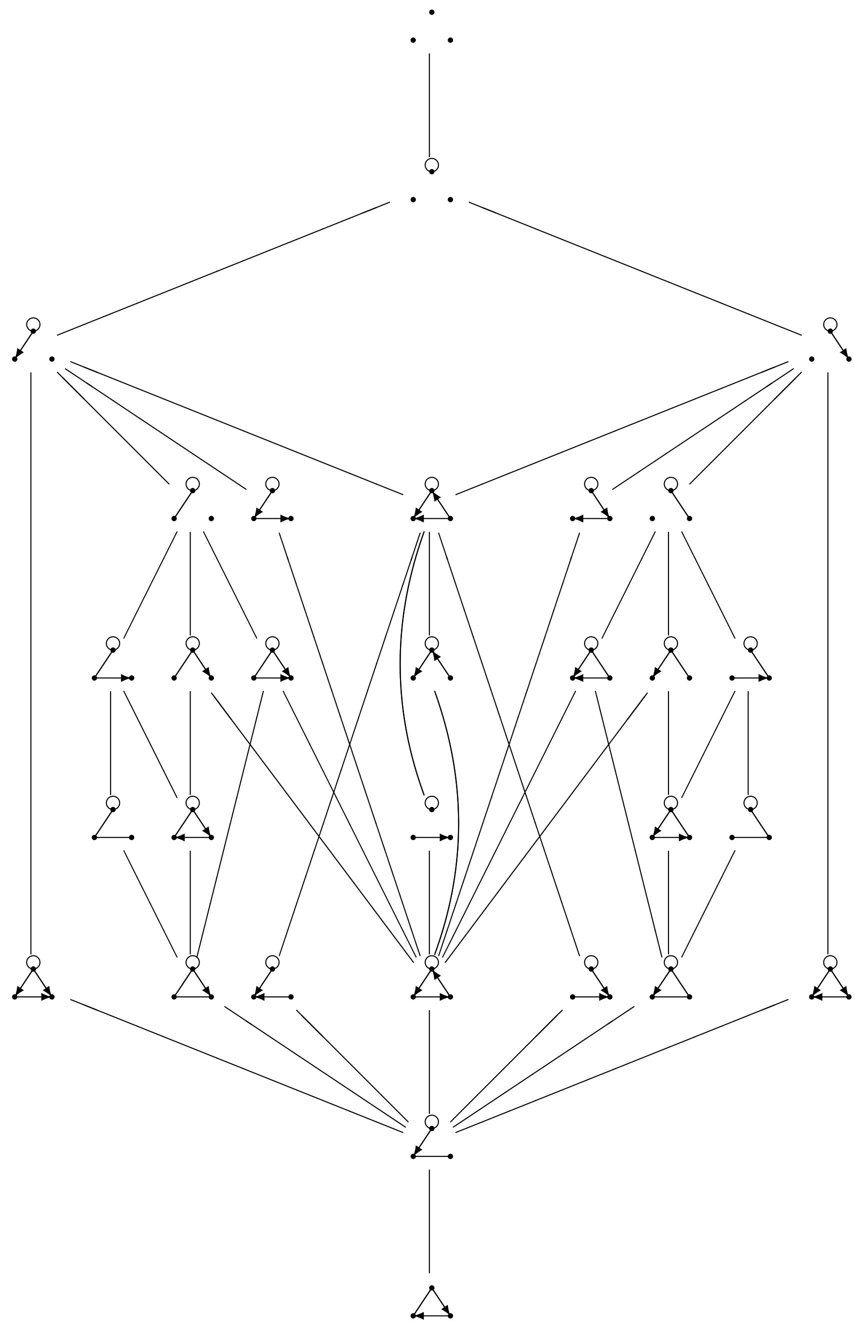


Fig. 3.5. Interval of clones containing the constant c_0, but no other constants. Part 1.

any generalizations. This approach has yielded some results **[9]**, **[12]**. I am also interested in linking up the clones in this study with other lists of clones on 3 elements.

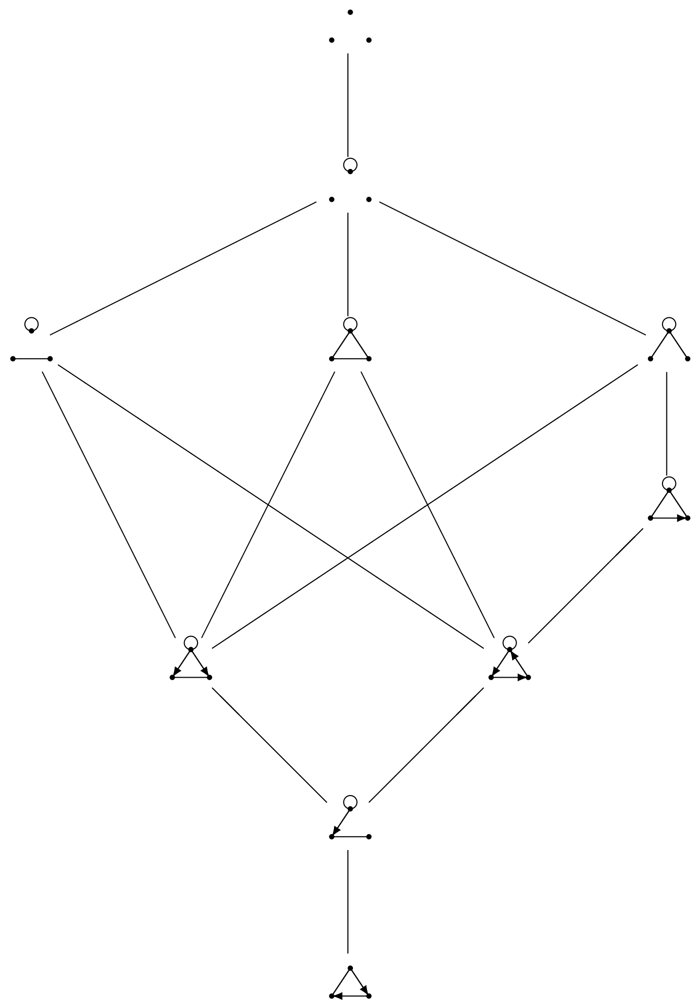


FIG. 3.6. Interval of clones containing the constant c_0, but no other constants. Part 2.

Chapitre 4

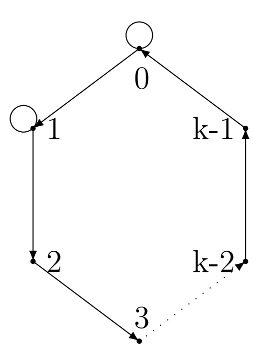
THE CLONE OF OPERATIONS PRESERVING A CYCLE WITH LOOPS

Auteur : Anne Fearnley

4.1. Le clone d'opérations préservant un cycle et des boucles

Après la rédaction de l'article *The monoidal interval for the monoid generated by two constants* (chapitre 2), je sentais qu'il y avait plus à dire au sujet des relations formées d'un k-cycle et de boucles sur k éléments. J'étais intriguée par les ennoncés suivants pour $k \geq 4$:

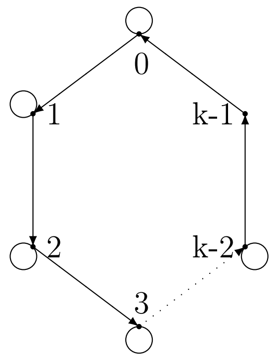
Pol



$= \langle c_0, c_1 \rangle$

qui est le théorème 2.5.1 de la présente thèse et

Pol



$= \langle c_0, c_1, \dots c_{k-2} \rangle$

qui est une conséquence facile du théorème 4.1.1. Puisque les seules opérations unaires préservant cette relation sont les constantes $c_0, \dots, c_{k-2}$, le théorème 4.1.1, garantie que les seules opérations (de n'importe quelle arité) préservant la relation sont engendrées par les constantes $c_0, \dots, c_{k-2}$.

Théorème 4.1.1 (A. Krokhin [**19**])**.** *Soit* $A = \{0, 1, \dots, k-1\}$ *tel que* $k > 3$*. Le monoide* $\langle c_0, c_1, \dots c_{k-2} \rangle$ *est affaissant sur* A*.*

En même temps, I. Rosenberg me demanda si l'on pouvait étendre le théorème 2.5.1 aux relations formées d'un k-cycle et de deux boucles pas nécessairement voisines.

Ces idées portèrent fruit au delà de nos espérances. Dans l'article, je considère toutes les relations binaires sur k éléments qui sont formées d'un k-cycle et de boucles quand elles sont écrites sous forme de graphe. Je montre que si $k \geq 3$ et que la relation a au moins deux boucles, alors la relation n'est préservée que par des opérations essentiellement unaires. Dans tous les autres cas, la relation est préservée par des opérations dépendant de plusieurs variables.

4.2. Abstract

We consider all the binary relations on k elements which, when viewed as directed graphs, consist of a k-cycle and some loops. If $k \geq 3$ and the relation has at least 2 loops, we show that it is only preserved by essentially unary operations. In all other cases, the relation is preserved by operations that depend on a greater number of variables.

4.3. Preliminaries

Let A be a finite set and n a positive integer. An *n-ary operation* on A is a function $f : A^n \to A$. The set of all n-ary operations on A is denoted by $\mathcal{O}_A^{(n)}$, and $\mathcal{O}_A := \bigcup_{0<n<\omega} \mathcal{O}_A^{(n)}$. For $F \subseteq \mathcal{O}_A$, set $F^{(n)} := F \cap \mathcal{O}_A^{(n)}$.

For $1 \leq i \leq n$, the n-ary *i-th projection* is defined as $e_i^{(n)}(x_1, \ldots, x_n) = x_i$ for all $x_1, \ldots, x_n$. We write e for the identity operation. For $a \in A$, the n-ary *constant operation a* is $c_a^{(n)}(x_1, \ldots, x_n) = a$ for all $x_1, \ldots, x_n$. We write simply c_a for the unary constant operations $c_a^{(1)}$. An operation $f \in \mathcal{O}^{(n)}$ *depends* on its first variable if there exist $x, y, x_2 \ldots, x_n \in A$ such that $f(x, x_2 \ldots, x_n) \neq f(y, x_2, \ldots, x_n)$. An operation is *essentially unary* if it depends on at most one of its variables.

For $f \in \mathcal{O}^{(n)}$, and $g_1, \ldots, g_n \in \mathcal{O}^{(m)}$, we define their composition to be the m-ary operation $f[g_1, \ldots, g_n]$ defined by

$$f[g_1, \ldots, g_n](x_1, \ldots, x_m) = f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$$

A *clone* on A is a subset F of $\mathcal{O}_A$ that contains all projections and is closed under composition. It is well known and easy to prove that the intersection of an arbitrary set of clones on A is a clone on A. Thus for $F \subseteq \mathcal{O}_A$, there exists the least clone containing F, called the clone *generated* by F and denoted by $\langle F\rangle$. Equivalently, $\langle F\rangle$ is the set of term operations of the algebra $\langle A;F\rangle$. The clones on A, ordered by inclusion, form a complete lattice, $\mathcal{L}_A$.

Let h be a positive integer. An *h-ary relation* ρ is a subset of A^h. When dealing with a fixed $\rho \in A^2$, we write $a \to b$ for $(a,b) \in \rho$. The relations may then be drawn as directed graphs. For example for $A = \{0,1,2\}$, the relation $\{(0,0),(0,1),(1,0),(1,2)\}$ may be represented as in Figure 4.1

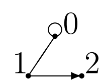


FIG. 4.1. Example of a relation

Let $f \in \mathcal{O}^{(n)}$, and let ρ be an h-ary relation on A. The operation f *preserves* ρ if for all $(a_{1,i}, a_{2,i}, \dots, a_{h,i}) \in \rho$ $(i = 1, \dots, n)$,

$$(f(a_{1,1}, a_{1,2}, \dots, a_{1,n}), f(a_{2,1}, a_{2,2}, \dots, a_{2,n}), \dots, f(a_{h,1}, a_{h,2}, \dots, a_{h,n})) \in \rho$$

The set of operations on A preserving ρ is a clone denoted by $\operatorname{Pol}\rho$.

Consider a transformation monoid M of unary operations on A; M contains the identity self-map e and is closed under the usual composition. Denote by $\operatorname{Int}(M)$ the set of clones C on A such that $C^{(1)} = M$. It is well known that $\operatorname{Int}(M)$ is an interval in the lattice of clones on A, (see [**36**]) called the *monoidal interval* of M. If $\operatorname{Int}(M)$ contains just one clone, then M is said to be *collapsing*.

4.4. Some collapsing monoids

As we will see in Section 4.5, the unary operations preserving a k-cycle and at least 2 loops are some constants and some special permutations. It is interesting to find out which monoids of that type are collapsing. We begin with the well known theorem by Pálfy.

Theorem 4.4.1 (Pálfy [**27**], see also [**36**]). *Let A be a finite set containing at least 3 elements. Let M be a transformation monoid on A containing all the constants and such that its non-constant operations are permutations. Then $|\operatorname{Int}(M)| \leq 2$, and equality holds if and only if M is a monoid of all unary polynomial operations of a vector space.*

Corollary 4.4.2. *Let A be a finite set containing at least 3 elements. Let M be a transformation monoid on A. If M consists exactly of all the constants and the identity, then M is collapsing.*

Theorem 4.4.3 (Krokhin [**19**]). *Let $A = \{0, 1, \ldots, k-1\}$ where $k \geq 4$. The monoid $\langle c_0, c_1, \ldots c_{k-2} \rangle$ is collapsing on A.*

The only collapsing monoids containing only constants and the identity are given by Corollary 4.4.2 and Theorem 4.4.3. In fact all other monoids of that type give rise to monoidal intervals of cardinality $2^{\aleph_0}$ [**17**], [**19**], [**28**].

We will now look at monoids on $A = \{0, 1, \ldots, k-1\}$ containing constants and those special permutations which we will call shifts. A *shift* is a unary operation that is a power of the permutation $s = (0, 1, \ldots, k-1)$. Note that $s^a(x) = a \dotplus x$ for all $x \in A$ where $\dotplus$ denotes addition mod k.

Theorem 4.4.4. *Let $A = \{0, \ldots, k-1\}$ where $k \geq 3$. Consider the monoid $M = \langle c_0, c_1, \ldots, c_{k-1}, s^j \rangle$ on A where $j \in \mathbb{N}$. M is collapsing.*

Proof. Note that the case $j = 0$ is simply Corollary 4.4.2. By Theorem 4.4.1, either M is collapsing, or there is a clone whose monoid of unary operations is exactly M and contains a binary operation (corresponding to the addition in the vector space) depending on both its variables.

Let C be a clone on A such that $C^{(1)} = M$. Let $f \in C$ be a binary operation. Let us define some unary operations associated with the table of f : the diagonal $d(x) = f(x,x)$, the horizontals $h_i(x) = f(i,x) = f(c_i(x),x)$ and the verticals $v_i(x) = f(x,i) = f(x,c_i(x))$ for all $i, x \in A$. Note that $d, h_i, v_i \in C$ for every $i \in A$; hence $d, h_i, v_i \in M$ for every $i \in A$. Note also that $h_x(y) = f(x,y) = v_y(x)$ for all $x, y \in A$.

Since $d \in M$ and $d(0) = f(0,0)$, it is clear that d can only be either $c_{f(0,0)}$ or $s^{f(0,0)}$. Note that $s^{f(0,0)}$ might not even be in M. This is not a problem in what follows since we need only omit all reference to the function $s^{f(0,0)}$.

Claim 1. *If $d = c_{f(0,0)}$, then $f = c^{(2)}_{f(0,0)}$.*

PROOF. Since $h_0(0) = f(0,0)$, thus $h_0 = c_{f(0,0)}$ or $h_0 = s^{f(0,0)}$. Suppose that $h_0 = s^{f(0,0)}$, then $v_1(0) = h_0(1) = s^{f(0,0)}(1) = 1 \dotplus f(0,0)$ and $v_1(1) = d(1) = c_{f(0,0)}(1) = f(0,0)$. This is impossible since $v_1 \in M$. Therefore $h_0 = c_{f(0,0)}$. Similarly, $v_0 = c_{f(0,0)}$. Now $h_1(0) = v_0(1) = c_{f(0,0)}(1) = f(0,0)$ and $h_1(1) = d(1) = c_{f(0,0)}(1) = f(0,0)$. Therefore $h_1 = c_{f(0,0)}$.

Now, let $i \in A$, we have $v_i(0) = h_0(i) = c_{f(0,0)}(i) = f(0,0)$ and $v_i(1) = h_1(i) = c_{f(0,0)}(i) = f(0,0)$. Therefore $v_i = c_{f(0,0)}$ for all $i \in A$. Finally, let $x, y \in A$, then $f(x,y) = v_y(x) = c_{f(0,0)}(x) = f(0,0)$. Therefore $f = c^{(2)}_{f(0,0)}$ as required. □

Claim 2. *If $d = s^{f(0,0)}$, then either $f = s^{f(0,0)} \circ e^{(2)}_1$ or $f = s^{f(0,0)} \circ e^{(2)}_2$.*

PROOF. Since $h_0(0) = f(0,0)$, thus $h_0 = c_{f(0,0)}$ or $h_0 = s^{f(0,0)}$.

CASE 1 : $h_0 = c_{f(0,0)}$. Let us consider v_1 and v_2 : we have $v_1(0) = h_0(1) = c_{f(0,0)}(1) = f(0,0)$ and $v_1(1) = d(1) = s^{f(0,0)}(1) = 1 \dotplus f(0,0)$. Therefore $v_1 =$

$s^{f(0,0)}$. Similarly, $v_2(0) = f(0,0)$ and $v_2(2) = 2 \dotplus f(0,0)$, which imply that $v_2 = s^{f(0,0)}$. Now let us consider all the horizontals : for $i \in A$, we have $h_i(1) = v_1(i) = s^{f(0,0)}(i) = i \dotplus f(0,0)$ and $h_i(2) = v_2(i) = s^{f(0,0)}(i) = i \dotplus f(0,0)$. Therefore $h_i = c_{i \dotplus f(0,0)}$ for all $i \in A$. Let $x, y \in A$, we have $f(x,y) = h_x(y) = c_{x \dotplus f(0,0)}(y) = x \dotplus f(0,0) = s^{f(0,0)}(x)$. Therefore $f = s^{f(0,0)} \circ e_1^{(2)}$ as required.

CASE 2 : $h_0 = s^{f(0,0)}$. Let us consider v_1 and v_2 : we have $v_1(0) = h_0(1) = s^{f(0,0)}(1) = 1 \dotplus f(0,0)$ and $v_1(1) = d(1) = s^{f(0,0)}(1) = 1 \dotplus f(0,0)$. Therefore $v_1 = c_{(1 \dotplus f(0,0))}$. Similarly, $v_2(0) = 2 \dotplus f(0,0)$ and $v_2(2) = 2 \dotplus f(0,0)$, which imply that $v_2 = c_{(2 \dotplus f(0,0))}$. Now let us consider all the horizontals : for $i \in A$, we have $h_i(1) = v_1(i) = c_{(1 \dotplus f(0,0))}(i) = 1 \dotplus f(0,0)$ and $h_i(2) = v_2(i) = c_{(2 \dotplus f(0,0))}(i) = 2 \dotplus f(0,0)$. Therefore $h_i = s^{f(0,0)}$ for all $i \in A$. Now, let $x, y \in A$, we have $f(x,y) = h_x(y) = s^{f(0,0)}(y) = y \dotplus f(0,0) = s^{f(0,0)}(y)$. Therefore $f = s^{f(0,0)} \circ e_2^{(2)}$ as required □

By the Claims, f is essentially unary. Therefore M must be collapsing. □

Corollary 4.4.5. *Let $A = \{0, \ldots, k-1\}$ where $k \geq 3$. Let ρ be the binary relation on A defined by $\rho = \{(0,1), (1,2), \ldots, (k-1,0), (0,0), \ldots, (k-1,k-1)\}$ Then* $\operatorname{Pol}\rho = \langle c_0, s \rangle$.

PROOF. By Theorem 4.4.4, $\langle c_0, s \rangle = \langle c_0, c_1, \ldots c_{k-1}, s \rangle$ is collapsing. Thus it suffices to show that $(\operatorname{Pol}\rho)^{(1)} = \langle c_0, s \rangle$. Clearly $c_0, s \in (\operatorname{Pol}\rho)^{(1)}$. Let $f \in (\operatorname{Pol}\rho)^{(1)}$. If $f(0) = f(1)$, we have

$$f(0) = f(1) \to f(2) \to \ldots \to f(k-1) \to f(0)$$

a $(k-1)$-cycle, hence they are all equal. Therefore $f = c_{f(0)} = s^{f(0)} \circ c_0 \in \langle c_0, s\rangle$. Now suppose that $f(0) \neq f(1)$. Since $f(0) \to f(1)$, then $f(1) = f(0) \dotplus 1$, and

$$f(0) \dotplus 1 = f(1) \to f(2) \to \ldots \to f(k-1) \to f(0)$$

which implies that $f(x) = f(0) \dotplus x$ for every $x \in A$. Therefore $f = s^{f(0)} \in \langle c_0, s\rangle$. In both cases $f \in \langle c_0, s\rangle$ as required. □

Using the following theorem and Theorem 4.4.4 we can find a result analoguous to Theorem 4.4.3 for constants and shifts.

Theorem 4.4.6 (Á. Szendrei, see [**14**])**.** *Let A be a finite set and L a subset of A with $|L| > 1$. Let $M = \{c_a \mid a \in L\} \cup G$ where G is a permutation group on A such that $A \setminus L$ is an orbit under the action of G, and the restriction map $G \to G|_L$ is injective. Then M is collapsing if $M|_L$ is.*

Corollary 4.4.7. *Let $A = \{0, \ldots, k-1\}$ where $k \geq 3$. Let M be a monoid on A containing only constants (at least 3) and shifts. Let $L = \{a \in A \mid c_a \in M\}$. If $A \setminus L$ is an orbit under the action of the permutations in M, then M is collapsing.*

Proof. The permutations of M must form a group G since A is finite; we may write $M = \{c_a \mid a \in L\} \cup G$ as in Theorem 4.4.6. Take the smallest power t such that $s^t \in G$ and set $r = s^t$. It is well known from Group Theory, that $G = \langle r\rangle$. It is easy to verify that the restriction map $G \to G|_L$ is injective. By Theorem 4.4.6, M is collapsing as long as $M|_L$ is. But $M|_L$ is indeed collapsing by Theorem 4.4.4 implying the desired result. □

4.5. The relations made up of a cycle with loops

We are interested in studying binary relations on $A = \{0, \dots, k-1\}$ which, when represented as graphs, are made up of a k-cycle and a certain number of loops. As usual the case $A = \{0, 1\}$ is exceptional. A relation with no loops or with only one loop is also different from the rest. These cases will be discussed in Section 4.6.

We already know what happens when we have a k-cycle and 2 loops that follow one another on the cycle.

Theorem 4.5.1 (Fearnley [**12**]). *Let* $k \geq 3$ *and* $A = \{0, 1, \dots, k-1\}$. *Let*

$$\rho = \{(0,0), (1,1), (0,1), (1,2), (2,3), \dots, (k-1, 0)\}$$

Then $\operatorname{Pol} \rho = \langle c_0, c_1 \rangle$.

We will extend this result to all relations made up of a k-cycle and at least 2 loops placed anywhere on the cycle. In Theorem 4.5.3, we show that these relations are preserved only by constants, and operations that are essentially shifts (powers of the permutation $s = (0, 1, \dots, k-1)$).

The basic idea is to consider what happens on the set $L = \{a_1, \dots, a_l\}$ of elements that have loops. Without loss of generality, we may assume throughout that $0 \in L$. As in [**12**], we define two unary operations on $A : x^{\rightarrow}$ defined by $a^{\rightarrow} = a$ if $a \in L$ and $a^{\rightarrow} = a \dotplus 1$ if $a \notin L$, and $x^{\leftarrow}$ defined by $a^{\leftarrow} = a$ if $a \in L$ and $a^{\leftarrow} = a \dotplus (k-1)$ if $a \notin L$ where $\dotplus$ is the addition in $\mathbb{Z}_k$. We write $a^{2\rightarrow}$ instead of $a^{\rightarrow\rightarrow}$, and so on. We define $a^{i\leftarrow}$ similarly. The following proposition follows from the definitions.

Proposition 4.5.2. *(A)* $a^{\leftarrow} \rightarrow a \rightarrow a^{\rightarrow}$ *for all* $a \in A$.

(B) $a^{i\leftarrow} \leftarrow a^{(i+1)\leftarrow}$ *and* $a^{i\rightarrow} \rightarrow a^{(i+1)\rightarrow}$ *for all* $a \in A$ *and* $i \in \{1, 2, \ldots\}$.

(C) $a^{(k-l)\leftarrow}, a^{(k-l)\rightarrow} \in L$ *for all* $a \in A$.

Theorem 4.5.3. *Let* $A = \{0, \ldots, k-1\}$ *where* $k \geq 3$. *Let* ρ *be the binary relation on* A *defined by* $\rho = \{(0,1), (1,2), \ldots, (k-1, 0), (a_1, a_1), \ldots, (a_l, a_l)\}$ *where* $2 \leq l \leq k$ *and* $a_1, \ldots, a_l \in A$. *Then* $\operatorname{Pol}\rho = \langle c_{a_1}, \ldots, c_{a_l}, s^j \rangle$ *for some* $j \in \{0, 1, 2, \ldots\}$.

As stated above, we will assume that $0 \in L$ throughout the proofs of the lemmas and the theorem. Theorem 4.5.3 naturally separates into 2 cases depending on whether the relation has rotational symmetry. The case without rotational symmetry is generally easier.

Lemma 4.5.4. *There exists a permutation,* $r = s^t$, *such that* $\operatorname{Pol}\rho \cap S_k = \langle r \rangle$.

Proof. If $\operatorname{Pol}\rho \cap S_k = \{e\}$ (i.e. if ρ has no rotational symmetry), then set $j = 0$ and $r = e$. Otherwise, let $p \in \operatorname{Pol}\rho \cap S_k$ such that $p \neq e$. Since $p \in \operatorname{Pol}\rho$, we have that $p(0) \rightarrow p(1)$, and since $p \in S_k$, we know that $p(0) \neq p(1)$. Therefore $p(1) = p(0) \dotplus 1$. Similarly $p(2) = p(0) \dotplus 2$. In general $p(x) = p(0) \dotplus x$ for all $x \in A$. Therefore $p = s^{p(0)}$. As in Corollary 4.4.7, take the smallest power t such that $s^t \in \operatorname{Pol}\rho$ and set $r = s^t$. Thus $\operatorname{Pol}\rho \cap S_k = \langle r \rangle$ as required. □

Lemma 4.5.5 (Fearnley and Rosenberg [**14**])**.** *Let* M *be a transformation monoid on a set* A. *Let* $L = \{a \in A \mid c_a \in M\}$, *and let* C *be a clone on* A *such that* $C^{(1)} = M$.

(A) For any $f \in C$, *we have* $f(L, \ldots, L) \subseteq L$, *therefore* f *can be restricted to* L *to get an operation* $f|_L$ *on* L.

(B) *In particular, for any $m \in M$, $m(L) \subseteq L$. Moreover, if A is finite, then every permutation in M permutes the elements of L and the elements of $A \setminus L$.*

(C) *The restriction $M|_L = \{m|_L \mid m \in M\}$ of M to L is a transformation monoid on L, and the restriction $C|_L = \{f|_L \mid f \in C\}$ of C to L is a clone such that $(C|_L)^{(1)} = M|_L$.*

Lemma 4.5.6. *For $x \in A$, denote by $r^*(x)$ the orbit of x under the action of $\langle r \rangle$. Let $R = \{0, \ldots, r(0) \dotplus (k-1)\}$*

(A) *For each $x \in A$, $r^*(x) \subseteq L$ or $r^*(x) \subseteq A \setminus L$,*

(B) *For each $x \in A$, there exists a unique $x^- \in R$ such that $x \in r^*(x^-)$, i.e. such that $x = r^i(x^-)$ for some integer i.*

(C) *$r^*(R) = A$.*

(D) *$x^- = y^-$ if and only if $x \in r^*(y)$.*

(E) *For any $x \in A$, $(x^{\rightarrow})^- = ((x^-)^{\rightarrow})^-$ and $(x^{\leftarrow})^- = (x^-)^{\leftarrow}$.*

(F) *For any $x \in A$, $(x^{n\rightarrow})^- = ((x^-)^{n\rightarrow})^-$ and $(x^{n\leftarrow})^- = (x^-)^{n\leftarrow}$.*

Proof. Note that in the case without rotational symmetry, this lemma becomes trivial since $r^*(x) = \{x\}$ for all $x \in A$, $R = A$, and $x^- = x$ for all $x \in A$. In all other cases, $R = \{0, \ldots, r(0) - 1\}$

Statement (A) is true by Lemma 4.5.5 (B). Statements (B), (C) and (D) are obvious. To prove statement (E), let $x \in A$. There exists an integer i such that $x = r^i(x^-) = x^- \dotplus it$, since $r = s^t$. If $x \in L$, then $x^- \in L$, and we have $(x^{\rightarrow})^- = x^- = (x^-)^{\rightarrow} = ((x^-)^{\rightarrow})^-$. If $x \notin L$, then $x^- \notin L$. Now, $((x^-)^{\rightarrow})^- = (x^- \dotplus 1)^-$, and $(x^{\rightarrow})^- = (x \dotplus 1)^- = ((x^- \dotplus it) \dotplus 1)^- = ((x^- \dotplus 1) \dotplus it)^- = (r^i(x^- \dotplus 1))^- = (x^- \dotplus 1)^-$.

The second part is similar, it is simpler to state because $0 \in L$. Statement (F) is derived from Statement (E) by a simple induction. □

Lemma 4.5.7. $(\operatorname{Pol}\rho)^{(1)} = \langle r, c_{a_1}, \ldots, c_{a_l}\rangle$ *as a monoid.*

PROOF. Remembering that r is defined to be e if ρ has no rotational symmetry (Lemma 4.5.4), it is clear that $r, c_{a_1}, \ldots, c_{a_l} \in (\operatorname{Pol}\rho)^{(1)}$. Let $f \in (\operatorname{Pol}\rho)^{(1)}$. Since $0 \in L$, $0 \leftrightarrow 0$, thus $f(0) \leftrightarrow f(0)$ and therefore $f(0) \in L$. We have $f(0) \rightarrow f(1)$, therefore $f(1) \in \{f(0), f(0) \dot{+} 1\}$.

If $f(1) = f(0)$, then we have

$$f(0) = f(1) \rightarrow f(2) \rightarrow \ldots \rightarrow f(k-1) \rightarrow f(0)$$

a $k-1$-cycle, thus $f(0) = f(1) = \ldots = f(k-1)$. Therefore $f = c_{f(0)}$.

Now, if $f(1) = f(0) \dot{+} 1$, then we have

$$f(0) \dot{+} 1 = f(1) \rightarrow f(2) \rightarrow \ldots \rightarrow f(k-1) \rightarrow f(0)$$

which implies that $f(x) = f(0) \dot{+} x$ for all $x \in A$. Therefore f is the permutation $s^{f(0)} \in \operatorname{Pol}\rho \cap S_k$. By Lemma 4.5.4, $f \in \langle r\rangle$.

In both cases, $f \in \langle r, c_{a_1}, \ldots, c_{a_l}\rangle$. □

PROOF. (of Theorem 4.5.3). By Lemma 4.5.5 (C), $(\operatorname{Pol}\rho)|_L$ is a clone on L, and by Lemma 4.5.7, its unary operations are exactly the identity, all the constants and the powers of $r|_L$.

Claim 1. *If ρ has only 2 loops and no rotational symmetry, then* $(\operatorname{Pol}\rho)|_L = \langle c_{a_1}, c_{a_2}\rangle$.

PROOF. Since $0 \in L$, let us write $L = \{0, a\}$. We need only study the case $2 \leq a \leq k-2$, since Theorem 4.5.1 shows the Claim to be true when $a = 1$ or $a = k-1$ (by renumbering).

We use Post's classification **[29]** making the 0 correspond to the 'false' of Post and a to 'true'. By Lemmas 4.5.5 and 4.5.7, the unary operations in $(\mathrm{Pol}\,\rho)|_L$ are exactly c_0, c_a and e. The monoidal interval on 2-elements for the monoid $\{c_0, c_a, e\}$ is represented in Figure 4.2

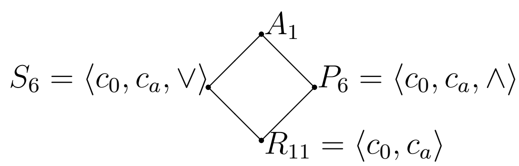


FIG. 4.2. The monoidal interval $Int\{c_0, c_a, e\}$ on {0, a}

Now suppose for the sake of contradiction that $\wedge \in (\mathrm{Pol}\,\rho)|_L$. Then there must be some $f \in \mathrm{Pol}\,\rho$ such that $f|_L = \wedge$; i.e. $f(0,0) = f(a,0) = f(0,a) = 0$ and $f(a,a) = a$. We thus have

$$a = f(a,a) \to f(a, a \dotplus 1) \to \ldots \to f(a,0) = 0 \to f(a,1) \to \ldots \to f(a,a) = a$$

which implies that $f(a,x) = x$ for all $x \in A$. Also,

$$0 = f(0,0) \to f(0,1) \to \ldots \to f(0,a) = 0$$

an a-cycle. Since $a < k$, all the elements of the cycle must be equal to 0. In particular, $f(0,1) = 0$. Consider the following chain :

$$0 = f(0,1) \to f(1,2) \to \ldots \to f(a, a \dotplus 1) = a \dotplus 1$$

This is impossible since $a \neq k-1$. Hence $\wedge \notin (\mathrm{Pol}\,\rho)|_L$, which implies that $P_6 \nsubseteq (\mathrm{Pol}\,\rho)|_L$. Similarly $\vee \notin (\mathrm{Pol}\,\rho)|_L$, which implies that $S_6 \nsubseteq (\mathrm{Pol}\,\rho)|_L$.

Therefore $(\operatorname{Pol}\rho)|_L = R_{11} = \langle c_0, c_a\rangle$ as required. $\square$

Claim 2. *If ρ has only 2 loops and exhibits rotational symmetry, then* $(\operatorname{Pol}\rho)|_L = \langle c_0, c_{k/2}, (0, k/2)\rangle$.

PROOF. The only way that ρ can have rotational symmetry is if k is divisible by 2 and $L = \{0, k/2\}$. By Lemmas 4.5.5 and 4.5.7, $((\operatorname{Pol}\rho)|_L)^{(1)} = \{e, c_0, c_{k/2}, (0, k/2)\}$. Note that since $k \geq 3$, we know that $k/2 < k - 1$.

We use Post's classification **[29]** making the 0 correspond to the 'false' of Post and $k/2$ to 'true'. Clearly, the permutation $(0, k/2)$ corresponds to negation. The monoidal interval on 2-elements for the monoid $\{c_0, c_{k/2}, \neg, e\}$ is represented in Figure 4.3

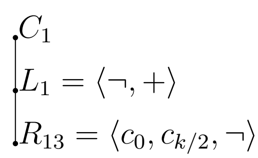


FIG. 4.3. The monoidal interval $Int\{c_0, c_{k/2}, \neg, e\}$ on {0, k/2}

Now suppose for the sake of contradiction that $+ \in (\operatorname{Pol}\rho)|_L$ where $+$ is addition mod 2. Then there must be some $f \in \operatorname{Pol}\rho$ such that $f|_L = +$; i.e. $f(0,0) = f(k/2, k/2) = 0$ and $f(0, k/2) = f(k/2, 0) = k/2$. We thus have

$$0 = f(0,0) \to f(0,1) \to \ldots \to f(0, k/2) = k/2$$

which implies that $f(0,1) = 1$. Similarly,

$$k/2 = f(0, k/2) \to f(1, k/2) \to \ldots \to f(k/2 - 1, k/2) \to f(k/2, k/2) = 0$$

which implies that $f(k/2 - 1, k/2) = k - 1$. Therefore

$$1 = f(0,1) \to f(1,2) \to \ldots \to f(k/2 - 1, k/2) = k - 1 \neq k/2$$

which is impossible.

Hence $+ \notin (\mathrm{Pol}\,\rho)|_L$, which implies that $L_1 \nsubseteq (\mathrm{Pol}\,\rho)|_L$. Therefore $(\mathrm{Pol}\,\rho)|_L = R_{13} = \langle c_0, c_{k/2}, (0, k/2)\rangle$ as required. □

Claim 3. *If ρ has more than 2 loops, then* $(\mathrm{Pol}\,\rho)|_L = \langle c_{a_1}, \dots, c_{a_l}, r|_L\rangle$.

Proof. Case 1 : If ρ has no rotational symmetry, then the unary operations of $(\mathrm{Pol}\,\rho)|_L$ are $e, c_{a_1}, \dots, c_{a_l}$, i.e. all the constants and the identity on L. By Corollary 4.4.2, $(\mathrm{Pol}\,\rho)|_L$ is essentially unary as a clone on L. Therefore $(\mathrm{Pol}\,\rho)|_L = \langle c_{a_1}, \dots, c_{a_l}\rangle$.

Case 2 : If ρ has rotational symmetry, then by Lemma 4.5.7, the unary operations in $\mathrm{Pol}\,\rho$ are $e, c_{a_1}, \dots, c_{a_l}$ and the powers of r. By Lemma 4.5.5 (C), the unary operations of $(\mathrm{Pol}\,\rho)|_L$ are all the constants and the powers of $r|_L$. Note that $r|_L$ is a power of the permutation $(a_1, \dots, a_l)$. By Theorem 4.4.4 the monoid generated by all the constants and $r|_L$ is collapsing as a monoid on L. Therefore $(\mathrm{Pol}\,\rho)|_L = \langle c_{a_1}, \dots, c_{a_l}, r|_L\rangle$ as required.

In both cases, $(\mathrm{Pol}\,\rho)|_L = \langle c_{a_1}, \dots, c_{a_l}, r|_L\rangle$, since we had defined in Lemma 4.5.4 that $r = e$ whenever ρ was without rotational symmetry. □

Considering the relation ρ as a directed graph, each point in the graph must be either on a loop or in a *loop-less chain* : $]a, a'[$ defined as the segment $\{x_1, \dots, x_m\} \subseteq A$ such that $a, a' \in L$, $x_1, \dots, x_m \notin L$ and $a \to x_1 \to \dots \to x_m \to a'$.

If ρ had no rotational symmetry, we could index the elements that do not have loops, then prove the rest of the theorem by induction, adding those elements one by one. The possibility that ρ has rotational symmetry can be handled by using orbits instead of just elements. With that in mind, let us index the elements of $R =$

$\{0, \ldots, r(0)-1\}$ that do not have loops by setting $R \setminus (L \cap R) = \{b_1, b_2, \ldots, b_m\}$ where $b_1 < b_2 < \ldots < b_m$. Note that $r^*(b_i) \cap r^*(b_j) = \emptyset$ whenever $i \neq j$ and $L \cup r^*(b_1) \cup r^*(b_2) \cup \ldots \cup r^*(b_m) = A$. The rest of the proof is basically an induction obtained by adding the orbits $r^*(b_i)$ one by one. We use the following fact to establish the induction.

Claim 4. *Let $x \in L \cup r^*(b_1) \cup \ldots \cup r^*(b_{j+1})$ where $j < m$. Then $x^{\leftarrow} \in L \cup r^*(b_1) \cup \ldots \cup r^*(b_j)$. Furthermore, let b_{j+1} be in the loop-less chain $]b'-1, a'[$. Then $x^{(a' \dot{+} (k-b'))\rightarrow} \in L \cup r^*(b_1) \cup \ldots \cup r^*(b_j)$.*

Proof. Let $x \in L \cup r^*(b_1) \cup \ldots \cup r^*(b_{j+1})$. If $x \in L$, then $x^{\leftarrow} \in L$. If $x \in r^*(b_i)$ for some $1 \leq i \leq j+1$, then $x^- = b_i$. By Lemma 4.5.6 (E), $(x^{\leftarrow})^- = (x^-)^{\leftarrow} = b_i^{\leftarrow} \in L \cup \{b_1, \ldots, b_j\}$. Therefore $x^{\leftarrow} \in L \cup r^*(b_1) \cup \ldots \cup r^*(b_j)$ as required.

For the second part, note that $(a' \dot{+} (k-b'))$ is really just the length of the loop-less chain, written in such a way so as to work even with the 0.

Case 1 : If $x \in L$, then $x^{(a' \dot{+} (k-b'))\rightarrow} = x \in L$.

Case 2 : If $x \notin L$ and $x^- \in\,]b'-1, a'[$, then by Lemma 4.5.6 (F),

$$(x^{(a' \dot{+} (k-b'))\rightarrow})^- = ((x^-)^{(a' \dot{+} (k-b'))\rightarrow})^- = (a')^-$$

But $a' \in L$, so by Lemma 4.5.6 (A), $(a')^- \in L$, and hence $x^{(a' \dot{+} (k-b'))\rightarrow} \in L$.

Case 3 : If $x \notin L$ and $x^- \notin\,]b'-1, a'[$, then $x^- < b'-1$. This implies that $(x^-)^{(a' \dot{+} (k-b'))\rightarrow} \leq b'-1 < b_{j+1}$, which means that $(x^-)^{(a' \dot{+} (k-b'))\rightarrow} \in R$. Therefore by Lemma 4.5.6 (F),

$$(x^-)^{(a' \dot{+} (k-b'))\rightarrow} = ((x^-)^{(a' \dot{+} (k-b'))\rightarrow})^- = (x^{(a' \dot{+} (k-b'))\rightarrow})^-$$

impliying that $(x^{(a'\dot{+}(k-b'))\rightarrow})^{-} < b_{j+1}$. Therefore $(x^{(a'\dot{+}(k-b'))\rightarrow})^{-} \in L\cup\{b_1,\ldots,b_j\}$, hence $x^{(a'\dot{+}(k-b'))\rightarrow} \in L \cup r^*(b_1) \cup \ldots \cup r^*(b_j)$.

In all cases $x^{(a'\dot{+}(k-b'))\rightarrow} \in L \cup r^*(b_1) \cup \ldots \cup r^*(b_j)$ as required. □

Claim 5. *Let $f \in \operatorname{Pol}\rho$. If $f|_L$ is a constant, then f is a constant.*

PROOF. Set $f|_L = c^{(n)}_{f(0,\ldots,0)}$. Let us show by induction on j that

$$f|_{L\cup r^*(b_1)\cup\ldots\cup r^*(b_j)} = c^{(n)}_{f(0,\ldots,0)}$$

The base case $j = 0$ is simply the statement that $f|_L = c^{(n)}_{f(0,\ldots,0)}$. Now suppose that $f|_{L\cup r^*(b_1)\cup\ldots\cup r^*(b_j)} = c^{(n)}_{f(0,\ldots,0)}$ for some $0 \le j < m$. Now let $x_1,\ldots,x_n \in L \cup r^*(b_1) \cup \ldots \cup r^*(b_{j+1})$. Notice that by Claim 4, $x_i^{\leftarrow} \in L \cup r^*(b_1) \cup \ldots \cup r^*(b_j)$ for each $i \in \{1,\ldots,n\}$. Thus by Proposition 4.5.2 (C), we have

$$\begin{aligned} f(0,\ldots,0) = c^{(n)}_{f(0,\ldots,0)}(x_1^{\leftarrow},\ldots,x_n^{\leftarrow}) &= f|_{L\cup r^*(b_1)\cup\ldots\cup r^*(b_j)}(x_1^{\leftarrow},\ldots,x_n^{\leftarrow}) \\ &= f(x_1^{\leftarrow},\ldots,x_n^{\leftarrow}) \\ &\rightarrow f(x_1,\ldots,x_n) \\ &\rightarrow f(x_1^{\rightarrow},\ldots,x_n^{\rightarrow}) \\ &\rightarrow \ldots \\ &\rightarrow f(x_1^{(k-l)\rightarrow},\ldots,x_n^{(k-l)\rightarrow}) \\ &= f|_L(x_1^{(k-l)\rightarrow},\ldots,x_n^{(k-l)\rightarrow}) \\ &= c^{(n)}_{f(0,\ldots,0)}(x_1^{(k-l)\rightarrow},\ldots,x_n^{(k-l)\rightarrow}) = f(0,\ldots,0) \end{aligned}$$

a $(k-l+1)$-cycle. Since $l>1$, they must all be equal to $f(0,\ldots,0)$. In particular, $f(x_1,\ldots,x_n)=f(0,\ldots,0)\in L$. Hence $f|_{L\cup r^*(b_1)\cup\ldots\cup r^*(b_{j+1})}$ is well defined and $f|_{L\cup r^*(b_1)\cup\ldots\cup r^*(b_{j+1})}=c^{(n)}_{f(0,\ldots,0)}$. By induction, we conclude that $f=c^{(n)}_{f(0,\ldots,0)}$. □

Claim 6. *Let $f\in\operatorname{Pol}\rho$. If $f|_L$ is essentially a power of $r|_L$, then f is essentially a power of r.*

Proof. This proof may be easier to understand if you consider first the case where ρ has no rotational symmetry and $f|_L$ is the first projection e_1. In that case $f|_L$ is essentially $e=s^0=s^{f(0,\ldots,0)}$.

When we say $f|_L$ is essentially a power of $r|_L$, we mean that $f|_L$ is essentially unary and equals a power of $r|_L$ in the only variable it depends on. Without loss of generality, we may assume that $f|_L$ depends only on its first variable; in other words $f|_L=(r|_L)^i\circ e_1^{(n)}$ for some integer i. In fact, $r^i=s^{f(0,\ldots,0)}$, so $(r|_L)^i=r^i|_L=(s^{f(0,\ldots,0)})|_L$. Note that these operations are well defined by Lemma 4.5.5.

Let us show by induction on j that

$$f|_{L\cup r^*(b_1)\cup\ldots\cup r^*(b_j)}=s^{f(0,\ldots,0)}|_{L\cup r^*(b_1)\cup\ldots\cup r^*(b_j)}\circ e_1^{(n)}$$

The base case $j=0$ is simply the statement that $f|_L=s^{f(0,\ldots,0)}|_L\circ e_1^{(n)}$. Now, suppose that $f|_{L\cup r^*(b_1)\cup\ldots\cup r^*(b_j)}=s^{f(0,\ldots,0)}|_{L\cup r*(b_1)\cup\ldots\cup r^*(b_j)}\circ e_1^{(n)}$ for some $0\le j<m$. Let $x_1,\ldots,x_n\in L\cup r^*(b_1)\cup\ldots\cup r^*(b_{j+1})$. Notice that by Claim 4, $x_i^{\leftarrow}\in L\cup r^*(b_1)\cup\ldots\cup r^*(b_j)$ for each $i\in\{1,\ldots,n\}$. Thus by Proposition 4.5.2 (C), we

have

$$
\begin{aligned}
x_1^{\leftarrow} \dotplus f(0,\ldots,0) &= s^{f(0,\ldots,0)}(x_1^{\leftarrow}) \qquad (4.5.1)\\
&= f|_{L\cup r*(b_1)\cup\ldots\cup r^*(b_j)}(x_1^{\leftarrow},\ldots,x_n^{\leftarrow})\\
&= f(x_1^{\leftarrow},\ldots,x_n^{\leftarrow})\\
&\to f(x_1,\ldots,x_n)\\
&\to f(x_1^{\rightarrow},\ldots,x_n^{\rightarrow})\\
&\to \ldots\\
&\to f(x_1^{(k-l)\rightarrow},\ldots,x_n^{(k-l)\rightarrow})\\
&= f|_L(x_1^{(k-l)\rightarrow},\ldots,x_n^{(k-l)\rightarrow})\\
&= s^{f(0,\ldots,0)}(x_1^{(k-l)\rightarrow})\\
&= x_1^{(k-l)\rightarrow} \dotplus f(0,\ldots,0)
\end{aligned}
$$

Case 1 : $x_1 \in L$. In this case, $x_1^{\leftarrow} = x_1 = x_1^{\rightarrow}$, and (4.5.1) becomes

$$
\begin{aligned}
x_1 \dotplus f(0,\ldots,0) &= x_1^{\leftarrow} \dotplus f(0,\ldots,0)\\
&\to f(x_1,\ldots,x_n)\\
&\to \ldots\\
&\to x_1^{(k-l)\rightarrow} \dotplus f(0,\ldots,0)\\
&= x_1 \dotplus f(0,\ldots,0)
\end{aligned}
$$

a $(k-l+1)$-cycle. Since $l > 1$, they must all be equal. In particular, $f(x_1,\ldots,x_n) = x_1 \dotplus f(0,\ldots,0) = s^{f(0,\ldots,0)}(x_1)$ as required.

CASE 2 : $x_1, x_1 - 1 \notin L$. Note that since $0 \in L$, $x_1 \neq 0$, then $x_1^{\leftarrow} = x_1 \dot{+} (k-1) = x_1 - 1$. Equation (4.5.1) becomes

$$(x_1 - 1) \dot{+} f(0, \ldots, 0) = x_1^{\leftarrow} \dot{+} f(0, \ldots, 0) \to f(x_1, \ldots, x_n)$$

But $x^{\leftarrow} \notin L$, which implies that $s^{f(0,\ldots,0)}(x_1^{\leftarrow}) = x_1^{\leftarrow} \dot{+} f(0, \ldots, 0) \notin L$. Therefore $f(x_1, \ldots, x_n) = x_1 - 1 \dot{+} f(0, \ldots, 0) \dot{+} 1 = x_1 \dot{+} f(0, \ldots, 0) = s^{f(0,\ldots,0)}(x_1)$ as required.

CASE 3 : x_1 is in the loop-less chain $]x_1 - 1, a[$, and b_{j+1} is in the loop-less chain $]b' - 1, a'[$ (note that the two loopless chains could be the same). By Claim 4, $x_i^{(a' \dot{+} (k-b'))\to} \in L \cup r^*(b_1) \cup \ldots \cup r^*(b_j)$ for $i = 1, \ldots n$. We get from (4.5.1) :

$$\begin{aligned} (x_1 - 1) \dot{+} f(0, \ldots, 0) &\to f(x_1, \ldots, x_n) \\ &\to \ldots \\ &\to f(x_1^{(a' \dot{+} (k-b'))\to}, \ldots, x_n^{(a' \dot{+} (k-b'))\to}) \\ &= f|_{L \cup r^*(b_1) \cup \ldots \cup r^*(b_j)}(x_1^{(a' \dot{+} (k-b'))\to}, \ldots, x_n^{(a' \dot{+} (k-b'))\to}) \\ &= s^{f(0,\ldots,0)}(x_1^{(a' \dot{+} (k-b'))\to}) \\ &= x_1^{(a' \dot{+} (k-b'))\to} \dot{+} f(0, \ldots, 0) \end{aligned} \tag{4.5.2}$$

If the chain $]b' - 1, a'[$ is no longer than the chain $]x_1 - 1, a[$, then $x_1^{(a' \dot{+} (k-b'))\to} = x_1 \dot{+} a' \dot{+} (k - b')$ and (4.5.2) implies that

$$f(x_1, \ldots, x_n) = (x_1 - 1) \dot{+} f(0, \ldots, 0) \dot{+} 1 = x_1 \dot{+} f(0, \ldots, 0) = s^{f(0,\ldots,0)}(x_1)$$

as required.

Suppose now that the chain $]b'-1,a'[$ is longer than the chain $]x_1-1,a[$. From (4.5.1), we have $f(x_1,\dots,x_n) \leftarrow (x_1-1)\dot{+}f(0,\dots,0) = s^{f(0,\dots,0)}(x_1-1) \in L$ by Lemma 4.5.5 (B) since $x_1-1 \in L$. Therefore

$$\begin{aligned} f(x_1,\dots,x_n) &\in \{(x_1-1)\dot{+}f(0,\dots,0),(x_1-1)\dot{+}f(0,\dots,0)\dot{+}1\} \\ &\in \{(x_1-1)\dot{+}f(0,\dots,0),x_1\dot{+}f(0,\dots,0)\} \\ &\in \{s^{f(0,\dots,0)}(x_1-1),s^{f(0,\dots,0)}(x_1)\} \end{aligned}$$

If $f(x_1,\dots,x_n) = s^{f(0,\dots,0)}(x_1)$, we are done. So suppose that

$$f(x_1,\dots,x_n) = s^{f(0,\dots,0)}(x_1-1) = (x_1-1)\dot{+}f(0,\dots,0)$$

We have :

$$\begin{aligned} (x_1-1)\dot{+}f(0,\dots,0) &= f(x_1,x_2,\dots,x_n) \\ &\to f(x_1\dot{+}1,x_2^{\rightarrow},\dots,x_n^{\rightarrow}) \\ &\to \dots \\ &\to f(x_1\dot{+}(k-1),x_2^{(k-1)\rightarrow},\dots,x_n^{(k-1)\rightarrow}) \\ &= f(x_1-1,x_2^{(k-1)\rightarrow},\dots,x_n^{(k-1)\rightarrow}) \\ &= f|_L(x_1-1,x_2^{(k-1)\rightarrow},\dots,x_n^{(k-1)\rightarrow}) \\ &= s^{f(0,\dots,0)}(x_1-1) \\ &= (x_1-1)\dot{+}f(0,\dots,0) \end{aligned}$$

a $(k-1)$-cycle, so they are all equal to $(x_1-1) \dotplus f(0,\dots,0)$. In particular,

$$(x_1-1) \dotplus f(0,\dots,0) = f(x_1 \dotplus (a-x_1), x_2^{(a-x_1)\rightarrow}, \dots, x_n^{(a-x_1)\rightarrow}) \tag{4.5.3}$$
$$= f(a, x_2^{(a-x_1)\rightarrow}, \dots, x_n^{(a-x_1)\rightarrow})$$

Since $a \in L$, by modifying (4.5.2) a bit, we get :

$$a \dotplus f(0,\dots,0) = f(a, x_2^{\leftarrow}, \dots, x_n^{\leftarrow})$$
$$\rightarrow f(a, x_2, \dots, x_n)$$
$$\rightarrow \dots$$
$$\rightarrow f(a, x_2^{(a' \dotplus (k-b'))\rightarrow}, \dots, x_n^{(a' \dotplus (k-b'))\rightarrow})$$
$$= a \dotplus f(0,\dots,0)$$

a cycle of length $(a' \dotplus (k-b'))+1 < k$. Therefore every element is equal to $a \dotplus f(0,\dots,0)$. Since the chain $]b'-1, a'[$ is longer than the chain $]x_1-1, a[$, we get in particular :

$$a \dotplus f(0,\dots,0) = f(a, x_2^{(a-x_1)\rightarrow}, \dots, x_n^{(a-x_1)\rightarrow})$$

Hence, by (4.5.3), $x_1 - 1 = a$, which is impossible since $a > x_1$.

Therefore, in all four cases,

$$f(x_1,\dots,x_n) = s^{f(0,\dots,0)}(x_1) = r^i(x_1) \in L \cup r^*(b_1) \cup \dots \cup r^*(b_{j+1})$$

Hence $f|_{L\cup r^*(b_1)\cup\dots\cup r^*(b_{j+1})}$ is well defined and equals $s^{f(0,\dots,0)} \circ e_1^{(n)}$. By induction, we conclude that $f = s^{f(0,\dots,0)} \circ e_1^{(n)}$. □

Claim 7. $\operatorname{Pol}\rho \subseteq \langle c_{a_1}, \dots, c_{a_l}, r\rangle$.

Proof. Let $f \in \operatorname{Pol}\rho$ be an n-ary operation. If ρ has no rotational symmetry, then by Claims 1 and 3, $f|_L \in \{c_a^{(n)} \mid a \in L\, n \in \mathbb{N}\} \cup \{e_m^{(n)} \mid m \in \{1, \dots, n\}, n \in \mathbb{N}\}$. This implies, by Claims 5 and 6, that $f \in \langle c_{a_1}, \dots, c_{a_l} \rangle$ as required.

Now, if ρ has rotational symmetry, then by Claims 2 and 3, $f|_L \in \{c_a^{(n)} \mid a \in L, n \in \mathbb{N}\} \cup \{(r|_L)^i \circ e_m^{(n)} \mid m \in \{1, \dots, n\}, i, n \in \mathbb{N}\}$. This implies, by Claims 5 and 6, that $f \in \langle c_{a_1}, \dots, c_{a_l}, r \rangle$ as required.

In both cases, $f \in \langle c_{a_1}, \dots, c_{a_l}, r \rangle$, since we had defined in Lemma 4.5.4 that $r = e$ whenever ρ was without rotational symmetry. □

Lemma 4.5.7 implies that $\langle c_{a_1}, \dots, c_{a_l}, r \rangle \subseteq \operatorname{Pol}\rho$. Using this and Claim 7, we conclude that $\operatorname{Pol}\rho = \langle c_{a_1}, \dots, c_{a_l}, r \rangle$. □

4.6. Special cases

Theorem 4.5.3 states that on a universe A of at least 3 elements, a relation made up of a $|A|$-cycle and at least 2 loops is preserved only by essentially unary operations. What can we say about a cycle and loops on fewer than 3 elements or with fewer than 2 loops?

For $A = \{0, 1\}$, the relations made up of a 2-cycle and some loops are $\rho_0 = \{(0, 1), (1, 0)\}$, $\rho_1 = \{(0, 1), (1, 0), (0, 0)\}$ (and the dual $\{(0, 1), (1, 0), (1, 1)\}$), and $\rho_2 = \{(0, 1), (1, 0), (0, 0), (1, 1)\}$. Using Post's classification [**29**], it is easy to check that $\operatorname{Pol}\rho_0 = D_3$, $\operatorname{Pol}\rho_1 = F_8^2$ and $\operatorname{Pol}\rho_2 = \operatorname{Pol} A^2 = C_1$, which are all clones containing operations other than the essentially unary ones.

For $A = \{0, \dots, k - 1\}$ where $k \geq 3$, let us consider the relation

$$\rho = \{(0, 1), (1, 2), \dots, (k - 1, 0)\}$$

made up of only a k-cycle. Let $f : A^2 \to A$ be defined as

$$f(x,y) = \begin{cases} x & \text{if } x \neq y\,; \\ x \dotplus 1 & \text{if } x = y. \end{cases}$$

It is easy to check that $f \in \operatorname{Pol}\rho$ and that it is not essentially unary. For more information about this clone and the monoidal interval it is in, please refer to [**34**]

Now, let us consider the relation $\sigma = \{(0,0),(0,1),(1,2),\ldots,(k-1,0)\}$ on the same A, made up of a k-cycle and one loop. In that case, $eq_0 \in \operatorname{Pol}\sigma$ where eq_0 is defined by $eq_0(x,y) = x$ if $x = y$ and $eq_0(x,y) = 0$ otherwise. It is obvious that eq_0 is not essentially unary.

Therefore, Theorem 4.5.3 states all the cases where a relation consisting of an $|A|$-cycle and some loops is preserved only by essentially unary operations.

4.7. Ideas for future research

While working on this paper, I was struck by the similarities between monoids generated by only constants and those generated by constants and a power of a cyclic fixed point free permutation. This is already alluded to by Pálfy (Theorem 4.4.1) in a more general form. The similarity is most obvious when comparing Corollary 4.4.2 and Theorem 4.4.4. For universes of at least 4 elements, Theorem 4.4.3 and Corollary 4.4.7 are another pair of similar results. Theorem 4.4.3 deals with monoids that have constants on all but one element of the universe, and Corollary 4.4.7 deals with monoids that have certain fixed point free permutations and constants associated to every element except for one orbit under the action of those permutations.

I used this idea in Theorem 4.5.3 ; I had proved it first in the case without rotational symmetries, which is much simpler. Inspired by the link between Theorem 4.4.3 and Corollary 4.4.7, I was able to transform the proof of the case without rotational symmetry to the one with rotational symmetry (rotational symmetry implying the existence of a cyclic fixed point free permutation). The induction in the proof for the case with rotational symmetry works by adding orbits of non-constant elements one by one, similarly to the original induction on the non-constant elements themselves in the case without rotational symmetry.

All this makes me wonder if it is possible to extend this idea further. For example, could we use what we know about the size and structure of the monoidal intervals for monoids containing only constants (see for example [**19**]), to learn more about the monoidal intervals generated by a power of a cyclic fixed point free permutation and some constants ?

We might even be able to go further. Theorem 4.4.1 and Theorem 4.4.6 deal more generally with certain permutation groups. Could we generalize results about monoids of constants to those made up of some constants and certain permutations ?

Chapitre 5

CONCLUSION

Mes recherches de doctorat suggèrent plusieurs directions possibles pour des recherches futures. Il serait intéressant de connaître la cardinalité de tous les intervalles monoïdaux sur trois éléments. Cette tâche semble maintenant réalisable étant donné tout ce qui a été découvert dans les vingt-cinq dernières années. Il serait aussi intéressant de mieux connaître la structure de ces intervalles, incluant une idée de la structure des intervalles infinis.

On pourrait peut-être généraliser ces informations à plus de trois éléments. Les intervalles pour les monoïdes de constantes et de permutations découlants des chapitres 2 et 4 ([**12, 13**]) me semblent particulièrement intéressants. Dans le chapitre 2, on trouve une ébauche du bas de l'intervalle $\mathrm{Int}\langle c_0, c_1\rangle$ et il me semble qu'on pourrait développer cette idée pour en apprendre plus sur la structure de l'intervalle au complet.

Pour mieux comprendre le treillis des clones sur trois éléments, il faudrait faire des liens entre tout ce qui est connu. En particulier, il faudrait placer les clones des grandes listes (algèbres de J. Berman [**2**], clones d'opérations préservant une relation binaire de A. Fearnley [**8**], groupoïdes de J. Berman et S. Burris [**3**]) dans les partitions connues (intervalles monoïdaux et partition de Miyakawa [**26**]). En plus de rassembler les informations, ce travail pourrait nous guider vers des régions du treillis des clones sur k éléments qui pourraient être particulièrement intéressantes et proposer des résultats généraux.

Dans cette optique, ces clones qui sont connus, et par leurs générateurs et comme préservant certaines relations, peuvent jouer un rôle important. Ceci est surtout vrai quand les générateurs et les relations sont simples et peu nombreux. De tels clones sont faciles à comparer aux autres (pour l'inclusion) et à placer dans les partitions. Ils servent en quelque sorte d'ancres dans le treillis. On en trouve dans les chapitres 2, 3 et 4, et j'espère en trouver d'autres au cours de mes recherches futures.

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Annexe A

ACCORD DES COAUTEURS

ET PERMISSIONS DES ÉDITEURS

A.1. Article *Collapsing monoids*

A.1.1. Identification de l'étudiant

Nom de l'étudiant : Anne Fearnley

Code permanent : FEAA09616309

A.1.2. Description de l'article

Anne Fearnley et Ivo Rosenberg. *Collapsing monoids containing permutations and constants.* Algebra Universalis, 50 : 149–156, 2003.

A.1.3. Déclaration des coauteurs autres que l'étudiant

À titre de coauteur de l'article identifié ci-haut, je suis d'accord pour que ANNE FEARNLEY inclut cet article dans sa thèse de doctorat qui a pour titre : Clones de constantes et de permutations et leur intervalle monoïdal.

IVO ROSENBERG ____

Coauteur Signature Date

A.1.4. Permission de l'éditeur

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>Date: Sat, 17 Jun 2006 10:34:57 -0400
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>To: hempfling@birkhauser.net
>Subject: Permission to include articles in a thesis

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>Anne Fearnley
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>Département de Mathématiques et statistique
>Université de Montréal
>CP 6128
>succ. Centre-Ville Montréal, Qc
>H3C 3J7
>CANADA

A.2. ARTICLE *Clones preserving a binary relation*

A.2.1. Identification de l'étudiant

Nom de l'étudiant : Anne Fearnley

Code permanent : FEAA09616309

A.2.2. Description de l'article

Anne Fearnley. *Clones on three elements preserving a binary relation.* Algebra Universalis, 56 : 165–177, 2007.

A.2.3. Permission de l'éditeur

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