Université de Bourgogne
et
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Cyclicité finie des boucles homoclines dans $\mathbb{R}^3$ non dégénérées avec valeurs propres principales réelles en résonance 1:1

par

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présentée par

Louis-Sébastien Guimond

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Thèse acceptée le :

____

À mes grands parents qui m'ont tout donné,

jusqu'à leurs enfants.

SOMMAIRE

Dans cette thèse nous étudions les bifurcations des boucles homoclines des champs de vecteurs dans $\mathbb{R}^3$ qui sont non dégénérées au sens de Deng [**Den93**], twistées et dont les valeurs propres principales sont en résonance 1:1. De tels champs de vecteurs possèdent une 2-variété $\mathcal{M}_\lambda$ invariante dépendant du paramètre et contenant la boucle homocline Γ_0 pour la valeur nulle du paramètre ainsi que toutes les orbites périodiques créées par perturbations de Γ_0 (voir [**Hom96**],[**San96**] ou [**RR96**]). Cette variété est un anneau (cas *non twisté*) ou un ruban de Möbius (cas *twisté*). La dynamique est alors donnée par une application unidimensionnelle $\mathcal{P}_\lambda(t)$ et toutes les orbites périodiques sont de période 1 ou 2. Notre résultat principal est le calcul d'une borne explicite de la cyclicité absolue de ce type de boucle homocline dans le cas twisté, i.e. le nombre d'orbites périodiques générées par perturbation . Pour démontrer ce résultat nous calculons le développement asymptotique d'une fonction $V_\lambda(t)$ liée à $P_\lambda^2(t) - t$, puis en bornons le nombre de zéros.

Dans notre premier article, nous considérons les cas de petites codimensions. Pour calculer la borne, nous projetons la dynamique sur $\mathcal{M}_\lambda$ puis appliquons les techniques exposées par Jebrane et Mourtada [**JM94**] pour l'étude de la boucle en huit dans le plan. Dans le second article, nous étudions le cas général. Dans ce cadre nous ne pouvons projeter la dynamique sur $\mathcal{M}_\lambda$. Les calculs pour obtenir la borne sont alors beaucoup plus techniques et reposent sur une généralisation des techniques exposées dans [**JM94**] ainsi que sur la théorie des fewnomials

de Khovanskiĭ [**Kho91**] permettant de réduire l'étude d'un système d'équations transcendantes à l'étude de systèmes polynomiaux non-dégénérés.

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INTRODUCTION

Je suppose que pour $\mu = 0$, la courbe K soit fermée, mais qu'elle cesse de l'être pour les petites valeurs de μ.

Soit A_0 un point de K. La position de ce point dépendra de μ; pour $\mu = 0$, la courbe K est fermée, de sorte que, après avoir parcouru cette courbe à partir de A_0, on revient au point A_0; si μ est très petit*, il n'en sera plus de même, mais on reviendra passer* très près *de A_0...*

Henri Poincaré, 1899

Voilà un siècle, Poincaré proposait d'étudier la dynamique d'un champ de vecteurs au voisinage d'une orbite périodique à l'aide de l'application premier retour, aussi appelée maintenant *l'application de Poincaré*. Ses travaux lui permirent de constater qu'il y avait deux types de dynamiques, que nous appelons aujourd'hui *chaotiques* et *non chaotiques*. Motivé par l'étude du problème des trois corps, Poincaré constate que l'existence de courbes homoclines peut engendrer une dynamique chaotique. En 1972, Gavrilov et Šil'nikov [**GŠ72**] montrent que les bifurcations homoclines peuvent donner naissance à des *fers-à-cheval* bien que ces bifurcations soient parmis les phénomènes globaux les plus simples.

Le résultat de Gavrilov et Šil'nikov est lié au fait que le point de selle s par lequel passe la boucle homocline peut posséder des valeurs propres dont les parties imaginaires sont grandes. En 1963 Šil'nikov [**Šil63**] obtient que si les valeurs propres principales de s sont réelles de somme non nulle alors, dans un voisinage suffisamment petit de la boucle, de petites perturbations du système engendrent au plus une unique orbite périodique.

Plus récemment, beaucoup ont étudié la famille des champs de vecteurs de $\mathbb{R}^3$ ayant une boucle homocline Γ_0 passant par un point singulier hyperbolique dont les valeurs propres sont réelles. (Sans perte de généralité, on peut supposer qu'il y a une unique valeur propre positive.) La boucle homocline est dite *non dégénérée* au sens de Deng [**Den93**] si: (1) elle rentre en s le long du vecteur propre principal stable; (2) la variété stable et son espace tangent approchent l'origine dans la direction de la variété fortement stable le long de Γ_0.

Si la boucle homocline est non dégénérée et la somme des valeurs propres principales est non nulle, alors elle est de codimension un dans la famille. Il existe trois cas de codimension 2. Dans deux de ces cas la boucle homocline est dégénérée au sens de Deng. Dans le troisième cas, elle est non-dégénérée au sens de Deng et la somme des valeurs propres principales est nulle. Il y a une dizaine d'années, Chow, Deng et Fiedler [**CDF90**] ont étudié ce cas, obtenu les courbes de bifurcations du diagramme de bifurcation et montré que, sous certaines hypothèses, la boucle homocline Γ_0 est de *cyclicité absolue finie* (i.e. donne naissance à un nombre fini d'orbitres périodiques dans tout perturbation). Des études ultérieures ont permis de lever ces hypothèses et d'obtenir le diagramme de bifurcation complet du cas non dégénéré au sens de Deng de codimension 2. Quelques années plus tard Deng [**Den93**] à montré que si Γ_0 est dégénérée, alors la dynamique est chaotique.

Il est maintenant connu que les familles de systèmes d'un multi-paramètre possédant une boucle homocline Γ_0 non dégénérée pour la valeur nulle du paramètre possèdent une 2-variété invariante dépendant du paramètre et contenant Γ_0 pour la valeur nulle du paramètre ainsi que toutes les orbites périodiques créées par perturbations de Γ_0 (voir [**Hom96**], [**San96**] ou [**RR96**]). L'existence de cette variété invariante est importante car elle impose que les orbites périodiques soient de période au plus deux.

Dans le cas où la 2-variété invariante est un anneau orientable (cas *non twisté*), Roussarie et Rousseau [**RR96**] ont montré que pour tout entier naturel k, une boucle homocline non dégénérée de codimension k est de cyclicité absolue finie. De plus ils ont donné une borne explicite (fonction de k) de la cyclicité absolue. Leur approche du problème a ceci de nouveau, dans l'étude des champs de vecteurs dans $\mathbb{R}^3$, qu'ils utilisent le calcul explicite de l'application de premier retour pour obtenir leur résultat, la technique étant utilisée couramment pour les problèmes de cyclicité planaires. En effet, dans le cas non twisté, la dynamique est donnée par les points fixes d'une application unidimensionnelle admettant un développement asymptotique similaire à ceux des applications de retour de certains graphiques planaires (cf. [**RR96**]).

Considérant le cas *twisté*, i.e. pour lequel la variété invariante est un ruban de Möbius, Yanagida [**Yan87**] a montré que les boucles homoclines de codimension supérieure à un peuvent engendrer des orbites périodiques de période deux. En effet, Chow, Deng et Fiedler [**CDF90**] ont démontré l'existence de telles orbites dans le cas de codimension deux. Comme l'existence de la variété invariante impose des orbites périodiques de période inférieure ou égale à deux, il est naturel, dans l'étude de la cyclicité du cas twisté, de considérer une fonction $V_\lambda(t)$ liée à $P_\lambda^2(t) - t$ et d'en borner le nombre de zéros.

Dans cette thèse, nous étudions les bifurcations des boucles homoclines twistées, non dégénérées au sens de Deng et dont les valeurs propres principales sont réelles en résonance 1:1. Notre approche consiste, dans un premier temps, à utiliser des techniques planaires afin d'obtenir le développement asymptotique de $V_\lambda(t)$, puis, subdivisant l'étude en deux cas, à utiliser soit des algorithmes de dérivation-division soit la théorie des fewnomials de Khovanskiĭ afin d'obtenir une borne au nombre de solutions périodiques pouvant être générées par perturbation de Γ_0.

Certaines techniques planaires nous seront fort utiles et ce malgré que toute solution périodique d'un champ planaire soit de période un. En effet, la difficulté de l'étude des orbites de période 2 n'est pas tant leur période mais bien qu'elles passent deux fois au voisinage de la singularité, ceci compliquant alors le calcul du développement asymptotique de $V_\lambda(t)$. Cette difficulté est également présente dans l'étude de certains graphiques planaires, entre autre dans l'étude des "grands" cycles générés par des perturbations de la "boucle en huit" (cf. [**JM94**] and [**KR96**]).

Dans le premier article, nous étudions les cas de petite codimension. Nous pouvons projeter la dynamique sur la 2-variété invariante et ainsi obtenir le développement asymptotique de $V_\lambda(t)$, un développement similaire à celui de l'application premier retour de la boucle planaire en huit. La codimension de la boucle homocline est définie à l'aide de ce développement asymptotique dont le premier terme non nul est intrinsèque. Notre objectif est de borner le nombre de zéros de cette application.

Le problème technique auquel nous devons faire face et qui est aussi présent dans l'étude de la boucle planaire en huit est l'étude des zéros d'une fonction au voisinage de l'origine dont le développement asymptotique possède non pas un mais bien deux types de monômes *généralisés* non analytiques en l'origine (voir chapitre 1). L'approche proposée par Jebrane et Mourtada [**JM94**] est de faire un éclatement des coordonnées. L'éclatement doit être tel que le domaine de définition de la variable d'éclatement soit un intervalle I compact indépendant du paramètre, et qu'en chaque point de cet intervalle, au plus un des deux types de monômes généralisés soit non analytique. La géométrie inhérante au problème suggère l'éclatement.

L'étude de la cyclicité est équivalente à l'étude des zéros de la fonction sur l'intervalle et un argument géométrique nous permet de nous limiter à deux cas: l'étude des zéros de $V_\lambda(t)$ au voisinage de $t = 0$, puis l'étude des zéros de $V_\lambda(t)$ dans un sous intervalle compact I', $I' \subsetneq I$.

L'étude de $V_\lambda(t)$ au voisinage de l'origine consiste à réécrire le développement d'un certaine dérivée $V_\lambda^{(k)}(t)$ de $V_\lambda(t)$ de telle sorte que tous les monômes du développement forment un ensemble de Tchebychev. Nous pouvons alors utiliser l'algorithme de dérivation-division exposé dans [**Rou86**] et obtenir une borne explicite pour le nombre de zéros de la fonction au voisinage de l'origine.

L'étude des zéros de $V_\lambda(t)$ sur un sous intervalle compact I' peut être ramenée à l'étude des zéros d'un polynôme. En effet nous pouvons réécrire le développement asymptotique de $V_\lambda^{(k)}(t)$ comme la perturbation d'un polynôme (non trivial) et pouvons ainsi obtenir une borne en appliquant le théorème de Rolle et un argument de [**JM94**].

Finalement, utilisant un argument de compacité, nous pouvons obtenir une borne du nombre de zéros de $V_\lambda(t)$ sur [0,1]. Notre résultat a comme corollaire de démontrer la complétude du diagramme de bifurcation du cas non dégénéré de codimension deux proposé par Chow, Deng et Fiedler [**CDF90**] (complétude qui a entre autre été montrée dans [**KKO93**]). De plus il peut être généralisé aux boucles homoclines du même type dans $\mathbb{R}^n$.

Dans le second article, nous étudions le cas général et obtenons une borne explicite de la cyclité absolue d'une boucle homocline non dégénérée au sens de Deng de codimension finie arbitraire. Cette borne est fonction de la codimension.

Dans le traitement du cas général, la faible différentiabilité de la variété invariante ne nous permet pas d'y projeter la dynamique et nous devons alors travailler avec une application de Poincaré bidimensionnelle $\mathcal{P}_\lambda(Y,Z)$. Les problèmes techniques qui se posent alors à nous sont non seulement la présence de deux types de monômes généralisés (comme dans le cas de petite codimension), mais de plus le fait que ces monômes sont fonctions des deux variables. Finalement, l'application de Dulac n'est pas inversible en l'origine.

L'étude de la cyclicité est faite en deux temps. En premier lieu nous appliquons le théorème des fonctions implicites afin de ramener l'étude à celle d'une application unidimensionnelle. Pour se faire, premièrement nous cherchons, parmi les changements de paramétrages sur les sections laissant invariante la structure de la forme normale, un paramétrage pour lequel chaque type de monômes généralisés est fonction d'une unique variable. Nous utilisons aussi un éclatement des variables $(Y,Z) = \overline{\Phi}(s,t)$ nous permettant d'étendre l'inverse de l'application de Dulac en l'origine.

La fonction $\delta_\lambda(s,t) = \big(\delta_{1,\lambda}(s,t),\delta_{2,\lambda}(s,t)\big)$ que nous obtenons n'est cependant pas une fonction différentiable de (s,t). Suivant l'idée exposée dans [**Rou97**], nous remarquons que le développement asymptotique de $\delta_{2,\lambda}(s,t)$ correspond au développement asymptotique d'une fonction différentiable $F(s,t,\nu_1,\nu_2,\nu_3)$ où les ν_i sont des monômes généralisés ne dépendant que de la variable t. Nous pouvons alors appliquer le théorème des fonctions implicites pour résoudre s en fonction de t et des trois monômes généralisés, i.e nous pouvons exprimer s comme une fonction $s(t)$ de t. La codimension de la boucle homocline est définie à l'aide du développement asymptotique de $\delta_{1,\lambda}(s(t),t)$ dont le premier terme non nul est intrinsèque.

Comme dans le premier article, la fonction unidimensionnelle $V_\lambda(t) = \delta_{1,\lambda}(s(t),t)$ est définie sur un intervalle compact, l'étude de la cyclicité est équivalente à l'étude des zéros de la fonction sur cet intervalle et un argument géométrique nous permet de limiter notre étude au voisinage de l'origine et a un sous-intervalle compact du premier.

L'étude de $\delta_{1,\lambda}(s(t),t)$ au voisinage de l'origine consiste à réécrire le développement d'une dérivée $\delta_{1,\lambda}^{(k)}(s(t),t)$ de $\delta_{1,\lambda}(s(t),t)$ de telle sorte que tous les monômes du développement forment un ensemble de Tchebychev. Nous pouvons alors utiliser l'algorithme de dérivation-division exposé dans [**RR96**] (ou une légère adaptation) et obtenir une borne explicite pour le nombre de zéros de la fonction au voisinage de l'origine.

Contrairement à l'étude des petites codimensions, l'étude des zéros sur un sous intervalle compact ne peut être ramenée à l'étude des zéros d'un polynôme. Dans le cas général, nous pouvons réécrire le développement de $\delta_{1,\lambda}^{(k)}(s(t),t)$ comme une fonction $\mathcal{G}_\lambda$ dont la partie principale est polynomiale en les trois types de monômes suivant: t, t^μ et $(1-t)^\mu$ avec $\mu(0)$ irrationnel.

Afin d'obtenir une borne explicite pour le nombre de zéros de $\mathcal{G}\big(t,t^\mu,(1-t)^\mu\big)$, nous considérons la partie principale de la fonction comme un système composé d'une fonction polynomiale à trois variables $\mathcal{G}(t,y,z)$ et de deux fonctions transcendantes $y - t^\mu$ et $z - (1-t)^\mu$. Nous appliquons alors la méthode de Khovanskiĭ qui nous permet de réduire l'étude d'un système d'équations transcendantes à l'étude de systèmes polynomiaux non dégénérés. Il est intéressant de noter que notre cas est parmi les cas non triviaux les plus simples de la théorie. Les équations transcendantes du système doivent vérifier certaines propriétées, l'une d'elles étant d'être des solutions séparantes d'une équation polynomiale de Pfaff.

La méthode de Khovanskiĭ, dont nous exposons une partie en appendice du second article, est composée de quatre étapes principales.

i. Nous devons premièrement vérifier que le système initial possède un nombre fini de solutions qui sont dès lors isolées.

ii. Nous déployons ce système transcendant afin d'éliminer toute dégénérescence.

iii. Utilisant le fait que chaque équation transcendante du système définit une solution intégrale d'un système polynomial d'équations de Pfaff, nous plongeons le système dans un système S non dégénéré d'équations polynomiales de Pfaff.

iv. Finalement nous itérons ce processus afin de borner le nombre de zéros du système S par le nombre de racines de systèmes polynomiaux non dégénérés auxquels nous pouvons appliquer le théorème de Bezout.

Nous obtenons ainsi une borne explicite (non optimale) pour la cyclicité des boucles homoclines non dégénérées twistées et dont les valeurs propres principales sont réelles en résonance 1:1. Utilisant un argument de compacité nous pouvons obtenir une borne du nombre de zéros de $\delta_{1,\lambda}(s(t),t)$ sur [0,1].

Chapitre 1

PREMIER ARTICLE: HOMOCLINIC LOOP BIFURCATIONS ON A MÖBIUS BAND

L'article *Homoclinic Loop Bifurcations on a Möbius Band* a été rédigé par Louis-Sébastien Guimond et sera publié dans *Nonlinearity* (1998).

HOMOCLINIC LOOP BIFURCATIONS ON A MÖBIUS BAND.

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Streszczenie. In this paper, we study 1-homoclinic loop bifurcations on a non-orientable 2-manifold: the Möbius band. The technics for studying bifurcating dynamics of the 1-homoclinic loop on this manifold are similar to the ones for a figure eight loop in the plane. We adapt the technics exposed in a paper of Jebrane and Mourtada [JM94] treating the subject: we are able, studying the 2-return map where it exists, to give an explicit bound for the cyclicity of the 1-homoclinic loop for all arbitrary finite codimensions. The key ingredient is a blow-up. A simple corollary is to prove the completeness of the bifurcation diagram given by Chow, Deng and Fiedler in [CDF90].

Introduction

Since M. M. Peixoto [Pei62], it is well known that a planar vector field having a homoclinic loop is structurally unstable. The study of the bifurcations of homoclinic loops requires powerful mathematical tools. In 1986, using expansion with generelized monomials, Roussarie [Rou86] gave an asymptotic expansion of the family of Dulac maps induced by a family of planar vector fields having a hyperbolic saddle point s. He was then able to show that generic planar homoclinic loops had finite cyclicity. (In [Rou86], Roussarie gives an explicit bound for the cyclicity of homoclinic loops, Joyal [Joy88] and Il'yashenko and Yakovenko [IY91] proved that the bound is optimal.)

The bifurcation of homoclinic loops in $\mathbb{R}^3$, as one could expect, is a much more difficult problem. Already in 1972, Gavrilov and Šil'nikov [GŠ72] proved that a homoclinic loop can lead to horseshoes and, in particular, to chaos. However vector fields having a loop through a saddle point, the two principal eigenvalues of which are real, present great similarities with the planar case when generic geometric assumptions are added. Our study is linked to the study of one subfamily of codimension ≥ 2 cases: when the two principal eigenvalues are in 1:1 resonance and no other resonances occur (we say the vector field is strongly 1-resonant). The generic case, when the sum of the principal eigenvalues does not vanish, was first studied by Šil'nikov [Šil63]. In 1987, Yanagida [Yan87] showed that resonant bifurcation

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leads systematically to the birth of periodic curves of period 2. In 1990, Chow, Deng and Fiedler [CDF90] studied the codimension 2 strongly 1-resonant case.

Homoclinic loops of higher codimensions were later studied by Roussarie and Rousseau [RR96]. They noticed that the heuristic argument used on a model in section 2 of [CDF90] could be transformed verbatim into a proof using planar technics: the exact calculation of the transition map in a neighbourhood of the saddle point composed with a C^k-diffeomorphism gives the first return map. A derivation-division algorithm is then used to bound the number of fixed points. The question of the optimality of the bound was not considered. They were able to reduce the problem to the study of homoclinic loops bifurcations on a 2-dimensional manifold (either an open cylinder in the non-twisted case, or a Möbius band in the twisted case), yielding the non-existence of n-orbits for $n > 1$ (non-twisted case) or $n > 2$ (twisted case). They then specialized to the non-twisted case and gave an explicit bound for the cyclicity in all finite codimensions. They did not study the twisted case which present additional difficulties since the second iterate of the Poincaré map must be considered.

The existence of an invariant 2-manifold containing the bifurcating dynamics, which is the key ingredient in the work of Roussarie and Rousseau, was done independently in two other papers, namely the thesis of Homburg [Hom96] and the thesis of Sandstede [San93]. Let us also note that the non-existence of N-homoclinic or N-periodic orbits with $N > 2$ in the non-twisted case and $N > 3$ in the twisted case was also proven by Kisaka, Kokubu, and Oka [KKO93].

In this paper we first prove the finite cyclicity property of finite codimension homoclinic loops on a C^K-Möbius band and give an explicit bound (theorem 7). A simple corollary is to prove the completeness of the bifurcation diagram given by Chow, Deng and Fiedler in [CDF90] of codimension 2 homoclinic loops in $\mathbb{R}^3$ under an additional smoothness condition: the band must be at least C^6. An additional consequence is a result of finite cyclicity for twisted homoclinic loops in $\mathbb{R}^n$ when the codimension is sufficiently small in front of the non-principal eigenvalues.

The paper is divided in three parts as follows.

In the first two parts we look at 1-homoclinic loops on a C^K-Möbius band. The first part contains preliminary results. The second part is devoted to the proof of the finite cyclicity property of a generic loop.

Finally, in the last part we give a result about strongly 1-resonant vector fields in $\mathbb{R}^n$.

Część 1. **Setting up the proof of the C^K-Möbius case.**

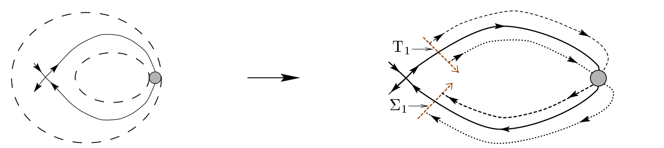
1.1. Notions and visualization

Definition 1. *Let Γ be an orbit of a vector field $X_0(x)$ on a manifold M. We have the following general definitions:*

1. *If the $\alpha-$limit set and $\omega-$limit set of Γ are one and the same saddle point s, then we call $\Gamma \cup \{0\}$ a **homoclinic loop**.*
2. *Let Γ_0 be a homoclinic loop of $X_0(x)$. Fix U a small tubular neighborhood of Γ_0. Assume $\overline{\Gamma} \subseteq U$ with Γ some orbit of $X_\lambda(x)$ intersecting a section of U N times. If Γ is a homoclinic loop then it is called an **N-homoclinic loop**. If Γ is a periodic curve then it is also called an **N-periodic curve**.*
3. *An **N-curve** is either an N-homoclinic loop or an N-periodic curve.*

Remark. As long as U is chosen small enough, the above definitions are independent of the choice of U and are valid for small perturbations $X_\lambda(x)$ of $X_0(x)$.

To help visualize the dynamics on the Möbius band, we use the projection of the band illustrated in figure 1.



(a) Projection of the band. (b) Projection of the vector field.

RYSUNEK 1. Singular projection of the Möbius band on the plane.

More precisely, we will be working in the following framework.

DEFINITION 2 (Framework for the differentiable Möbius band). *Let M^2 be a C^K-smooth Möbius band, and $X_\lambda : M^2 \times \Lambda' \to TM^2$ be a C^K-smooth p-parameter local family of vector fields on M^2 (Λ' is a neighborhood of the origin in $\mathbb{R}^p$). Let $s \in M^2$ be a hyperbolic saddle point with eigenvalues $-\mu_1, \mu_2$ satisfying, for $\lambda = 0$, the resonance relation $0 < \mu_2 = \mu_1$. Let Γ_0 be a homoclinic loop of the vector field X_0 through the saddle s and turning around the Möbius band, and let U be a sufficiently small tubular neighborhood of Γ_0. Let Σ_1 be a transversal of X_λ intersecting the local stable manifold of s and intersecting ∂U in two points, and T_1 a transversal of X_λ intersecting the local unstable manifold of s and intersecting ∂U in two points and this for all $\lambda \in \Lambda'$. We parametrize Σ_1 (resp. T_1) so that the origin corresponds to the intersection point with the invariant stable (resp. unstable) manifold and with orientation as in figure* 1(b).

Take a chart in M^2 around s in which s is the origin. Since $(0,0) \in \mathbb{R}^2 \times \Lambda'$ is hyperbolic, we take a small neighborhood Λ' of $\lambda = 0$ such that the saddle point has eigenvalues $-\mu_1(\lambda) < 0 < \mu_2(\lambda)$, where $\mu_i(0) = \mu_i$.

DEFINITION 3. *The **hyperbolicity ratio** $r(\lambda)$ of the saddle point $(0, \lambda)$ is defined as*

$$r(\lambda) = \frac{\mu_1(\lambda)}{\mu_2(\lambda)}.$$

DEFINITION 4. *Let $\{X_\lambda\}_{\lambda\in\Lambda'}$ be a family of C^K vector fields on M^2. Let Γ be a compact subset of M^2 invariant by X_0. We say that Γ has **finite cyclicity** in the family $\{X_\lambda\}_{\lambda\in\Lambda'}$ if there exists $N \in \mathbb{N}$, $\epsilon > 0$ and a neighborhood Λ_0 of λ_0 in Λ' such that for all $\lambda \in \Lambda_0$, the number $n(\epsilon, \lambda)$ of limit cycles (isolated periodic orbits) γ of X_λ with $\mathrm{dist}_H(\gamma, \Gamma) \leq \epsilon$ is less than N, where dist_H is the Haussdorff distance on compact sets.*

Using the notation of Jebrane and Mourtada, let

$$n(\epsilon, \Lambda_0) = \sup_{\lambda\in\Lambda_0} \{n(\epsilon, \lambda)\}. \tag{1}$$

We can thus define the **cyclicity of Γ** in the family $\{X_\lambda\}_{\lambda\in\Lambda'}$ to be the minimum integer $n(\epsilon,\Lambda_0)$ when ϵ and the diameter of Λ_0 go to 0.

1.2. Geometrical results

The purpose of this section is to give geometric conditions (proofs can be found in [Gui99]) for the existence of periodic solutions which will be of importance to simplify the proof of the finite cyclicity property.

Let $P_\lambda(x)$ be the first return map on Σ_1. Then on the Möbius band M^2:

1. there is at most one 1-periodic curve bifurcating from a homoclinic loop ([RR96]);
2. if there is a 2-curve, then there exists one 1-periodic curve which coexists with the 2-curve: let x_1 and x_2 be the intersection points of the 2-curve with Σ_1, the fixed point of P_λ is located between x_1 and x_2;
3. we only have to look for N-curves with $N=1$ or $N=2$ intersecting some compact interval of Σ_1.

Thus to have limit cycles, there must exists a 1-curve. From 1–3 we have that $Cycl(\Gamma_0)$ is exactly the number of 2-curves plus 1.

Lemma 5. *Let $x_0(\lambda)$ be the fixed point of $P_\lambda(x)$ with $\lambda\in\Lambda$, a small neighborhood of the origin, and $Z_\lambda=\{x\in\Sigma_1 \,|\, P_\lambda^2(x)=x \text{ and } 0<x<x_0(\lambda)\}$. Let*

$$N=\sup_{\lambda\in\Lambda}\Big(Card\big(Z_\lambda\big)\Big). \tag{2}$$

The cyclicity of Γ_0 is equal to $N+1$.

Let $\mathcal{R}_\lambda(y)$ be the regular transition map from T_1 to Σ_1 defined by the flow. In fact, $\mathcal{R}_\lambda(y)$ is C^K since X_λ is of class C^K, and its Taylor expansion is

$$\mathcal{R}_\lambda(y)=d_0(\lambda)+\sum_{i=1}^{K-1}d_i(\lambda)y^i+O_\lambda\left(y^K\right), \tag{3}$$

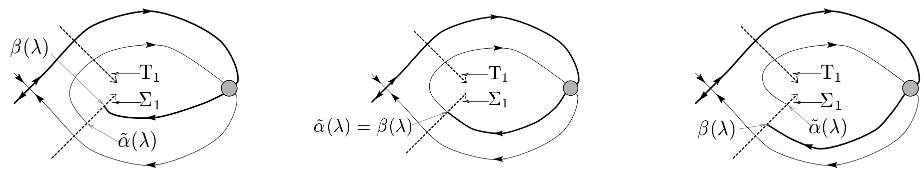
where $d_0(0)=0$ and $d_1(0)<0$ (orientation reversing).

Remark. Since $d_1(0)<0$, there exists a diffeomorphism $H(y)$ such that:

$$H\circ\mathcal{R}_\lambda\circ H^{-1}(y)=d_0(\lambda)+\sum_{i=1}^{[K/2]}\overline{d}_{2i-1}(\lambda)y^i+O_\lambda\left(y^K\right). \tag{4}$$

Proposition 6. *Let the above framework be assumed. Denote by $\beta(\lambda)$ the first intersection of the unstable manifold of the origin with Σ_1 (i.e. $\beta(\lambda)=d_0(\lambda)$), and let $\tilde{\alpha}(\lambda)$ be the second (if it exists) intersection (with reverse time) of the stable manifold of the origin with Σ_1 (cf. figure 2). We have the following results:*

1. *In order to have an N-curve, it is necessary to have $\beta(\lambda)\geq 0$, in which case $\tilde{\alpha}(\lambda)$ exists and is positive. The case $\beta(\lambda)=0$ corresponds to having only a 1-homoclinic curve.*
2. *If $x\in\Sigma_1$ belongs to an N-curve then $x\in[0,\beta(\lambda)]$, the return map $P_\lambda(x)$ is well defined on $[0,\beta(\lambda)]$, and the second iterate of the first return map (call it the 2-return map or the second return map) $P_\lambda^2(x)$ is well defined on $[0,\tilde{\alpha}(\lambda)]$. Moreover, the fixed points of $P_\lambda^2(x)$ (if they exist) are contained in $[0,m(\lambda)]$, where $m(\lambda)\overset{\mathrm{def}}{=}\min\{\beta(\lambda),\tilde{\alpha}(\lambda)\}>0$.*



(a) $\tilde{\alpha}(\lambda) < \beta(\lambda)$ (b) $\tilde{\alpha}(\lambda) = \beta(\lambda)$ (c) $\tilde{\alpha}(\lambda) > \beta(\lambda)$

Rysunek 2. Notation for proposition 6.

Część 2. Proof of the finite cyclicity property.

In this part we give a definition of the codimension for a homoclinic loop on a Möbius band (definition 11). This definition depends only on X_0 and not on the family. We prove the following theorem which is one of our main results:

Theorem 7. *Let Γ_0 as in definition 2. If Γ_0 is of finite codimension and the band is sufficiently smooth (i.e. K sufficiently large), then Γ_0 has finite cyclicity. If Γ_0 is of codimension $3k$ and $K \geq 8k+2$, then $Cycl(\Gamma_0) \leq 3k$. If Γ_0 is of codimension N (where $N = 3k+1$ or $N = 3k+2$) and $K \geq 8k+6$, then $Cycl(\Gamma_0) \leq 3k+2$.*

Corollary 8. *Let Γ_0 be of codimension $3k$ or $3k+2$, then $Cycl(\Gamma_0) \leq$ codimension of Γ_0.*

From corollary 8, we expect the bounds for the cyclicity of Γ_0 to be optimal when the loop is of codimensions $3k$ and $3k+2$. Unfortunately the bounds are not optimal for loops of codimension $3k+1$.

2.1. Generelized monomials and the definition of codimension

We are interested in counting the fixed points of P_λ^2, i.e. the zeroes of $P_\lambda - P_\lambda^{-1}$. One important planar technic is to view the Poincaré map P_λ as the composition of the regular map $\mathcal{R}_\lambda$ (cf. equation (3)) and a transition map D_λ near the saddle point.

The transition map near a saddle point in the plane has been thoroughly studied. Its asymptotic expansion uses generalized monomials which are well ordered and behave adequatly under differentiation (cf. [Rou86], [Mou89], [EM93] and [Rou98]). These generalized monomials have the form $x^i\omega^j(x,\lambda)$ where:

$$\alpha_1(\lambda) = 1 - r(\lambda) \tag{5}$$

$$\omega(x,\lambda) = \begin{cases} \dfrac{x^{-\alpha_1(\lambda)} - 1}{\alpha_1(\lambda)} & \text{if} \quad \alpha_1(\lambda) \neq 0 \\ -\ln(x) & \text{if} \quad \alpha_1(\lambda) = 0 \end{cases} \tag{6}$$

The generalized monomials have the property that for all $k > 0$,

$$\lim_{\alpha_1(\lambda)\to 0} x^k\omega^j(x,\lambda) = -x^k \ln^j(x),$$

and this holds uniformly on $[0, X]$ for any fixed $X > 0$.

Definition 9 ([Mou89]). *Let $K \in \mathbb{N}$, $\psi(x,\lambda)$ a C^K-function on $]0,\epsilon[\times\Lambda_0$ such that $\psi(0,0)=0$, and a positive continuous function $\xi(x,\lambda)$ with $\xi(0,\lambda)=0$. We say that $\psi(x,\lambda)$ is $I_0^K\big(\xi(x,\lambda)\big)$ if for every $n \in \mathbb{N}$ such that $n \le K$, we have*

$$\lim_{x\to 0} \xi^n(x,\lambda)\frac{\partial^n \psi(x,\lambda)}{\partial x^n} = 0$$

uniformly on Λ_0.

The generalized monomials $x^k\omega(x,\lambda)$ are $I_0^K\big(\rho(x,\lambda)\big)$, where $\rho(x,\lambda) = x^{1+\alpha_1(\lambda)}\omega(x,\lambda)$.

Proposition 10 ([Rou86]). *Let the family X_λ as defined in definition 2. Then there exists a decreasing positive sequence of positive numbers $\{\delta_n\}_{n\ge 1}$ and C^∞-functions $\alpha_n(\lambda)$ defined on $\Lambda_n = \big\{\lambda \in \Lambda' \backslash \big|\alpha_1(\lambda)\big| < \delta_n\big\}$ such that for all $K \in \mathbb{N}$, there exists $\epsilon > 0$, a neighborhood Λ_0 of $\lambda = 0$, and transversals Σ_1 and T_1 C^K-parameterized by x and y respectively such that the Dulac map defined from Σ_1 to T_1 can be written in the following form:*

$$D_\lambda(x) = x + \sum_{i=1}^{K} \alpha_i(\lambda)x^i\omega(x,\lambda)\big(1+\psi_i(x,\lambda)\big) + \phi_K(x,\lambda). \tag{7}$$

The function ϕ_K is C^K and K-flat at $x=0$. The functions ψ_i are $I_0^K\big(\rho(x,\lambda)\big)$; more precisely, they are finite linear combinations of monomials $x^n\omega^m(x,\lambda)$ with coefficients being polynomials in $\alpha_l(\lambda)$ where $i \le l \le K$ and $m \in \mathbb{Z}$.

The inverse function $D_\lambda^{-1}(y)$ is now easily computed. Instead of inverting equation (7), it is better to apply proposition 10 to the field with reverse time. Under the hypothesis of proposition 10, the inverse function $D_\lambda^{-1}(y)$ is of the following form:

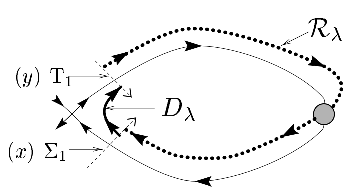
$$D_\lambda^{-1}(y) = y + \sum_{i=1}^{K} \overline{\alpha}_i(\lambda)y^i\overline{\omega}(y,\lambda)\big(1+\overline{\psi}_i(y,\lambda)\big) + \overline{\phi}_K(y,\lambda), \tag{8}$$

with $\overline{\alpha}_1(\lambda) = 1 - 1/r(\lambda)$, $\overline{\alpha}_i(\lambda) = -\alpha_i(\lambda) + p_i(\lambda)$, $p_i(\lambda)$ being polynomials in $\alpha_j(\lambda)$ with $j < i$. Moreover

$$\overline{\omega}(y,\lambda) = \begin{cases} \dfrac{y^{-\overline{\alpha}_1(\lambda)}-1}{\overline{\alpha}_1(\lambda)} & \text{if} \quad \overline{\alpha}_1(\lambda) \ne 0 \\ -\ln(y) & \text{if} \quad \overline{\alpha}_1(\lambda) = 0 \end{cases}, \tag{9}$$

functions $\overline{\psi}_i$ are $I_0^K\big(\bar{\rho}(y,\lambda)\big)$, where $\overline{\rho}(y,\lambda) = y^{1+\overline{\alpha}_1(\lambda)}\overline{\omega}(y,\lambda)$, and $\overline{\phi}_K$ is C^K and K-flat at $y=0$.

Geometrically, the functions $\mathcal{R}_\lambda(y) : \mathrm{T}_1 \to \Sigma_1$ and $D_\lambda(x)$ are given in figure 3.



Rysunek 3. Applications $\mathcal{R}_\lambda(y)$ and $D_\lambda(x)$.

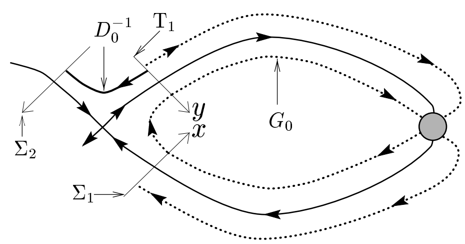
We now want to give a suitable definition of the codimension of a homoclinic loop Γ_0 on M^2. Since 2-curves can occur in the bifurcation of such a loop (cf. [CDF90]), it seems necessary to use the asymptotic expansion of the 2-return map to give this definition. To simplify the computations, we consider a function associated to the 2-return map. We first need to extend equation (8) to the negative values of y. This extension can be found in [RŻ91] (recalled in [Gui99]) and, parameterizing Σ_2 as on figure 4, is of the following form:

$$D_\lambda^{-1}(y) = -\left[y + \sum_{i=1}^{K} \overline{\alpha}_i(\lambda) y^i \overline{\varpi}\big(|y|,\lambda\big)\big(1+\overline{\psi}_i(y,\lambda)\big) + \overline{\phi}_K(y,\lambda)\right]. \tag{10}$$

Now let

$$\mathcal{Z}_\lambda(y) \stackrel{def}{=} D_\lambda^{-1}(y) - G_\lambda(y) \qquad \Big(G_\lambda(y) \stackrel{def}{=} \mathcal{R}_\lambda \circ D_\lambda \circ \mathcal{R}_\lambda(y)\Big). \tag{11}$$

Using equation (10), fonction $\mathcal{Z}_\lambda(y)$ is well defined at least on $\{y \in \mathrm{T}_1 \mid y \leq 0\}$. From proposition 6, when $\beta(\lambda) \geq 0$, there exists $b(\lambda) \geq 0$ with $\beta(\lambda) = 0 \Leftrightarrow b(\lambda) = 0$ such that $\mathcal{Z}_\lambda : \{y \in \mathrm{T}_1 \mid y \leq b(\lambda)\} \to \Sigma_1$. Moreover, the 2-return map is defined only on the interval $[0, b(\lambda)]$ (or a subinterval) on which it has, for all sufficiently small λ, the same number of fixed points as the number of zeroes of function $\mathcal{Z}_\lambda(y)$. Functions $G_0(y)$ and $D_0^{-1}(y)$ are illustrated in figure 4.



Rysunek 4. Application $G_0(y)$ and $D_0^{-1}(y)$.

Since we have the expansion of $\mathcal{R}_0(y)$ on the whole section T_1, from proposition 10 we can get the asymptotic expansion of $G_0(y)$.

As we will show in the next section, in any parameterization, $\mathcal{Z}_\lambda(y)$ at $\lambda = 0$ has the following form:

$$-\mathcal{Z}_0(y) = \sum_{i=0}^{K} \beta_i y^i + \sum_{i=1}^{K} \sigma_i y^i \ln(|y|) + \circ\big(y^K\big), \tag{12}$$

where $\beta_1(0) = 1 + 1/d_1(0)$, $\sigma_i(0) = \alpha_i(0)\big(|d_1(0)|^{1-i} + (-1)^i\big)$, with $\beta_i(0)$ to be defined later.

Definition 11. *Let Γ_0 as given in definition 2, and let the $\beta_i(\lambda)$ and $\sigma_i(\lambda)$ as given in equation (12).*

1. *Γ_0 is **non-degenerate** of finite codimension if one of the $\beta_i(\lambda)$ or $\alpha_i(\lambda)$ does not vanish at $\lambda = 0$.*
2. *If there exists k such that $\alpha_i(0) = 0 = \beta_i(0)$ for $i \leq k$ and $\alpha_{k+1}(0) \neq 0$, then we say that Γ_0 is **logarithmic** of order $k+1$, noted $O_L\big(\Gamma_0\big) = k+1$.*
3. *If there exists k such that $\beta_i(0) = \alpha_i(0) = 0$ for $i < 2k+1$, $\alpha_{2k+1}(0) = 0$, and $\beta_{2k+1}(0) \neq 0$, then we say that Γ_0 is **analytic** of order k, noted $O_A\big(\Gamma_0\big) = k$.*

4. Γ_0 *is of* ***codimension*** $3k$ *if it is logarithmic of order* $2k$*;* ***codimension*** $3k+1$ *if it is logarithmic of order* $2k+1$*;* Γ_0 *is of* ***codimension*** $3k+2$ *if it is analytic of order* $2k+1$.

Proposition 12. *Definitions* 11.2, 11.3 *and* 11.4 *are intrinsic.*

Proof. There is a total order on the monomials appearing in $\mathcal{Z}_0(y)$ $\big($and also in $D_0(x)$ and $D_0^{-1}(y)\big)$,

$$1 \prec y\ln|y| \prec y \prec y^2\ln|y| \prec y^2 \prec \ldots \tag{13}$$

The definitions correspond to the one defined on an expansion such as (13) by means of the lowest order term with non-vanishing coefficient. This is known to be invariant under composition by C^k-diffeomorphisms. □

2.2. Parameterization and the blow-up

The major difficulty that we encounter comes from the terms $\omega\big(\mathcal{R}_\lambda(y)\big)$ in the expansion of function $D_\lambda \circ \mathcal{R}_\lambda(y)$. To simplify $\omega\big(\mathcal{R}_\lambda(y)\big)$, we first need to find a nice parameterization for which $\mathcal{R}_\lambda(y)$ is an affine map. Then we use a property (stated below in lemma 13) of function $\omega(x,\lambda)$ to get a nice expansion of D_λ in that parameterization.

Lemma 13 ([JM94])**.** *Let* $f(x,\lambda)$ *be a* C^K*-function on* $[0,x_0[\times\Lambda$ *such that* $f(0,\lambda)=0$. *Then there exists a* C^K*-function,* $g(x,\lambda)$*, with* $g(0,\lambda)=0$ *and such that for all* $a>0$*, we have*

$$\omega\big(ax(1+f),\lambda\big) = \Big[a+O(\alpha_1(\lambda))\Big]\omega(x,\lambda)+g(x,\lambda)-\ln(a)\Big[1+O\big(\alpha_1(\lambda)\big)\Big]. \tag{14}$$

We have $\beta(\lambda)=\mathcal{R}_\lambda(0)$ and $b(\lambda)=\mathcal{R}_\lambda^{-1}(0)$. Let $x_1\in\Sigma_1$. We note by y its image on T_1 by the diffeomorphism $\mathcal{R}_\lambda^{-1}$, i.e. $y=\mathcal{R}_\lambda^{-1}(x_1)$. We have that

$$y = b(\lambda)+r_\lambda(x_1), \tag{15}$$

where $r_\lambda(0)=0$, and $r_\lambda(x)$ is a smooth diffeomorphism. A priori, $b(\lambda)$ and $\beta(\lambda)$ need not be the same, but since $\mathcal{R}_\lambda\circ\mathcal{R}_\lambda^{-1}(x)$ is smooth, we have the following:

$$\begin{aligned}\partial_x r_\lambda(0) = \partial_x\mathcal{R}_\lambda^{-1}(0) < 0 &\quad (\text{since } \mathcal{R}_\lambda \text{ is orientation reversing})\\ = \frac{1}{\partial_y\mathcal{R}_\lambda(y_1)}\bigg|_{y_1=b(\lambda)} &= \frac{1}{d_1(\lambda)+O\big(b(\lambda)\big)},\end{aligned}$$

for a sufficiently small neighborhood Λ of $\lambda=0$.

Set

$$x = -r_\lambda(x_1) = S_\lambda(x_1) \tag{16}$$

as the new parameterization of Σ_1 (it is smooth). Let us note by $\tilde{\mathcal{R}}_\lambda(y)$ the function $\mathcal{R}_\lambda(y)$ expressed in the new parameterization, i.e. $\tilde{\mathcal{R}}_\lambda(y) = -r_\lambda\big(\mathcal{R}_\lambda(y)\big) = S_\lambda\circ\mathcal{R}_\lambda(y)$, and similarly $\tilde{\mathcal{R}}_\lambda^{-1}(x)=b(\lambda)-x$.

We can suppose that x varies in a domain $[0,x_0]$ which is independent of $\lambda\in\Lambda$. We have that

$$x_1 = r_\lambda^{-1}(-x) = a(\lambda)x+\sum_{i=2}^{K}\eta_i(\lambda)x^i + r_{\lambda,K}(x), \tag{17}$$

where $a^{-1}(\lambda) = -d_1(\lambda) + O\big(b(\lambda)\big)$ and the $\eta_i(\lambda)$ are polynomials in $b(\lambda)$ and $d_j(\lambda)$ with $j \le i$, the function $r_{\lambda,K}$ is C^K and K-flat at $x = 0$.

In parameterization (16), we have that

$$\tilde{R}_\lambda(0) = -r_\lambda\big(\beta(\lambda)\big). \tag{18}$$

Therefore, in the parameterization we have introduced, $\tilde{\mathcal{R}}_\lambda(y)$ is the affine map $\tilde{\mathcal{R}}_\lambda(y) = -r_\lambda\big(\beta(\lambda)\big) - y$ and thus $0 = \tilde{\mathcal{R}}_\lambda \circ \tilde{\mathcal{R}}_\lambda^{-1}(0) = -r_\lambda(\beta(\lambda)) - b(\lambda)$, i.e.

$$\tilde{\mathcal{R}}_\lambda(0) = b(\lambda) = \tilde{\mathcal{R}}_\lambda^{-1}(0) \quad \text{and} \quad \tilde{\mathcal{R}}_\lambda(y) = b(\lambda) - y.$$

Rewriting equations in parameterization (16), the Dulac map has a more complicated expansion than equation (7) since it is composed with a C^k-map.

Lemma 14. *Let the transversal Σ_1. $\tilde{D}_\lambda(x) = D_\lambda \circ S_\lambda^{-1}(x)$ is of the following form:*

$$\tilde{D}_\lambda(x) = \sum_{i=1}^{K} \gamma_i(\lambda) x^i + \sum_{i=1}^{K} \alpha_i(\lambda) a^i(\lambda) x^i \omega(x,\lambda)\big[1 + h_i(x,\lambda)\big] + H_K(x,\lambda), \tag{19}$$

where h_i is a C^K-function in x and verifying $I_0^K\big(\rho(x,\lambda)\big)$. The function $H_K(x,\lambda)$ is K-flat at $x = 0$; $a^{-1}(\lambda) = -d_1(\lambda) + O\big(b(\lambda)\big)$; $\gamma_1(\lambda) = a(\lambda)$ and $\gamma_i(\lambda)$ are polynomials in $\eta_i(\lambda)$, $\alpha_j(\lambda)$ ($j \le i$) and $b(\lambda)$. This result is valid for all K less than the differentiability of X_λ.

Proof. By proposition 10, we have

$$D_\lambda(x_1) = x_1 + \sum_{i=1}^{K} \alpha_i(\lambda) x_1^i \omega(x_1,\lambda)\big(1 + \psi_i(x_1,\lambda)\big) + \tilde{\phi}_K(x_1,\lambda). \tag{20}$$

Using equation (17) and lemma 13, the Dulac map is of the following form in parameterization (16):

$$\tilde{D}_\lambda(x) = \sum_{i=1}^{K} \eta_i(\lambda) x^i + \sum_{i=1}^{K} \alpha_i(\lambda) a^i(\lambda) x^i \big[1 + \check{h}_{1,i}(x,\lambda)\big]\Big[\big(1 + \alpha_1(\lambda)\check{h}_2(x,\lambda)\big)\omega(x,\lambda) + \check{h}_3(x,\lambda)\Big] + \tilde{H}_K(x,\lambda), \tag{21}$$

where $\eta_1(\lambda) = a(\lambda)$, the $\check{h}_{1,i}(x,\lambda)$ are C^K and verify $I_0^K\big(\rho(x,\lambda)\big)$. Setting

$$\check{f}_i(x,\lambda) = \check{h}_2(x,\lambda)\left(1 + \check{h}_{1,i}(x,\lambda)\right) \qquad \check{g}_i(x,\lambda) = \check{h}_3(x,\lambda)\left(1 + \check{h}_{1,i}(x,\lambda)\right),$$

we obtain the following expression for $D_\lambda(x)$:

$$\tilde{D}_\lambda(x) = \sum_{i=1}^{K} \eta_i(\lambda) x^i + \sum_{i=1}^{K} \alpha_i(\lambda) a^i(\lambda) x^i \Big[\big(1 + \alpha_1(\lambda)\check{f}_i(x,\lambda)\big)\omega(x,\lambda) + \check{g}_i(x,\lambda)\Big] + \check{H}_K(x,\lambda), \tag{22}$$

where the $\check{f}_i(x,\lambda)$ are C^K and verify $I_0^K\big(\rho(x,\lambda)\big)$. To get the final form (19), we expand, using Taylor series, the functions $\check{f}_i$ and add the K-first terms to the first term on the right side of equation (22) and the rest to $\check{H}_K(x,\lambda)$. □

Lemma 15. *Let $x \in \Sigma_1$. The function $G_\lambda(y)$ has the following form:*

$$G_\lambda(y) = \beta(\lambda) - b(\lambda) + y + \sum_{i=1}^{K} \beta_i(\lambda)x^i + \sum_{i=1}^{K} \alpha_i(\lambda)d_1(\lambda)a^i(\lambda)x^i\omega(x,\lambda)\big[1 + h_i(x,\lambda)\big] + H_K(x,\lambda), \tag{23}$$

where $x = b(\lambda) - y$, $\beta_1(\lambda) = 1 - a(\lambda)$ and $\beta_i(\lambda)$ are polynomials in $b(\lambda)$, $\eta_j(\lambda)$, $\alpha_j(\lambda)$ and $d_j(\lambda)$ with $j \le i$.

Proof. Since $G_\lambda(y) = R_\lambda \circ \tilde{D}_\lambda \circ \tilde{R}_\lambda(y)$, from equations (3) and (19) we get an expansion of the form of equation (23). The result comes from the fact that

$$\left(\sum_{i=1}^{K} \gamma_i(\lambda)x^i + \sum_{i=1}^{K} \alpha_i(\lambda)a^i(\lambda)x^i\omega(x,\lambda)\big[1 + \check{f}_i(x,\lambda)\big] + \check{F}_K(x,\lambda)\right)^j = \sum_{i=1}^{K} \check{\beta}_i(\lambda)x^i + \sum_{i=1}^{K} \check{\eta}_i(\lambda)a^i(\lambda)x^i\omega(x,\lambda)\big[1 + \check{h}_i(x,\lambda)\big] + \check{H}_K(x,\lambda). \tag{24}$$

□

From lemma 15 and equation (8), we have the following proposition:

Proposition 16. *In the normal form coordinates, the zeroes of $\mathcal{Z}_\lambda(y)$ coincide with the zeroes of the function $\Psi(y,\lambda)$, where*

$$\Psi(y,\lambda) = \sum_{i=0}^{k} \beta_i(\lambda)x^i + \sum_{i=1}^{k} \Big(\alpha_i(\lambda)d_1(\lambda)a^i(\lambda)x^i\omega(x,\lambda)\big[1 + \tilde{f}_i(x,\lambda)\big] + - \overline{\alpha}_i(\lambda)y^i\overline{\omega}(y,\lambda)\big[1 + \overline{\tilde{f}}_i(y,\lambda)\big]\Big) + H_k(x,\lambda) + \overline{\phi}_K(y,\lambda), \tag{25}$$

with $\beta_0(\lambda) = \beta(\lambda) - b(\lambda)$, $x = b(\lambda) - y$. The β_i are the ones given in equation (23), and the $\overline{\alpha}_i(\lambda)$ are the ones given in equation (8). They are C^1 in λ. The $\tilde{f}_i$ and $\overline{\tilde{f}}_i$ are $I_0^K\big(\rho(x,\lambda)\big)$ and $I_0^K\big(\bar{\rho}(y,\lambda)\big)$ respectively.

Remark. The differentiability class K is arbitrarily chosen but sufficiently large to allow all needed differentiations. Let Γ_0 of order k, taking a smaller neighborhood of $\lambda = 0$, we may take $K \ge 4k+2$ if k is odd and $K \ge 4k+6$ if k is even (K is the differentiability class of M^2.). Thus for the codimension two case, $K \ge 6$.

To prove theorem 7, we bound the number of zeroes of $\Psi(y,\lambda)$ on the interval given in proposition 6, i.e. $\big[0, b(\lambda)\big]$ with $b(\lambda) > 0$. (It is of course sufficient to work on $\big[0, m(\lambda)\big]$, but for simplicity reasons, we work on a possibly larger interval.) We will have to work with terms in $\omega\big(b(\lambda) - y\big)$, a function of the two independant variables $b(\lambda)$ and y, both arbitrarily small. To simplify these terms, and then simplify the equation of its derivative, we blow-up the x and y variables in the following way: we set

$$y = b(\lambda)\big(1 - t(\lambda)\big) \quad \big(\text{and } x = t(\lambda)b(\lambda)\big), \tag{26}$$

where, by proposition 6, $t \in [0,1]$.

Proposition 17. *Choosing Λ sufficiently small, there exists $\epsilon > 0$ such that*

$$Cycl(\Gamma_0) \leq \sup_{\lambda\in\Lambda} \Big| \big\{ t \in [0, 1-\epsilon] \mid \Psi\big(b(\lambda)(1-t), \lambda\big) = 0 \big\} \Big|, \tag{27}$$

i.e. we only need to bound uniformly on Λ the number of zeroes of $\Psi\big(b(\lambda)(1-t), \lambda\big)$ on $[0, 1-\epsilon]$ to bound the cyclicity of Γ_0.

Proof. Let $y_0(\lambda) = b(\lambda)\big(1 - t_0(\lambda)\big)$ be the fixed point of $P_\lambda(y)$ on T_1: $t_0(\lambda)$ is a zero of:

$$\big(\tilde{D}_\lambda - \tilde{R}_\lambda^{-1}\big)\big(bt\big) = b(t-1) + abt[(bt)^{-\alpha_1(\lambda)} - 1] + O\big(bt\big) + O\big(bt\omega(bt, \lambda)\big). \tag{28}$$

Moreover

$$\frac{\big(\tilde{D}_\lambda - \tilde{R}_\lambda^{-1}\big)\big(b(\lambda)t\big)\Big|_{t=0}}{b(\lambda)} = -1, \tag{29}$$

thus, for sufficiently small Λ and by continuity of equation (28) with respect to t and λ there exists $\epsilon_1 > 0$ such that $t_0(\lambda) > \epsilon_1$ on Λ.

The result follows from lemma 5 with $\epsilon = 1 - \epsilon_1$. □

2.3. The finite cyclicity property: a proof

2.3.1. **Proof of Theorem 7: case Γ_0 is logarithmic.** This is the case where $d_1(0) = -1$, $\alpha_i(0) = \beta_i(0) = 0$ for $i \leq k-1$ and $\alpha_k(0) \neq 0$.

We first homogenize the principal part of $\Psi\big(b(\lambda)(1-t), \lambda\big)$ with respect to $b(\lambda)$ by means of a blow-up of the coefficients $\alpha_i(\lambda)$ with $i \leq k$. More precisely, following the idea in [JM94], we define the following functions $t_i(\lambda)$ for $i \leq k$.

$$\alpha_i(\lambda) = t_i(\lambda) b^{k-i}(\lambda) \text{ for } 1 \leq i \leq k \tag{30}$$

The $t_i(\lambda)$ are not bounded, but since $\alpha_k(0) \neq 0$, we have that for Λ a neighborhood of $\lambda = 0$ sufficiently small and $\lambda \in \Lambda$, there exists $\delta > 0$ such that

$$L(\lambda) = \left(\sum_{i=1}^{k} t_i^2(\lambda) \right)^{1/2} > \delta.$$

We can then compactify the coefficients space by setting

$$\tau_i \stackrel{\mathrm{d}ef}{=} \frac{t_i}{L},$$

where $\big|\tau_i(\lambda)\big| \leq 1$ for all $i \leq k$, $\lambda \in \Lambda$, and $\sum_{i=1}^{k} \tau_i^2(\lambda) = 1$ for all $\lambda \in \Lambda$, i.e. the new coefficient space is a subset of $\mathbb{S}^k$.

The blow-up destroys the order relation between the coefficients. Therefore it is necessary to divide our study in the following cones in the parameter space. For all $1 \leq j \leq k$, we let

$$E_j \stackrel{\mathrm{d}ef}{=} \left\{ \lambda \in \Lambda \,/\, \big|\tau_j(\lambda)\big| = \sup_{1\leq i\leq k} \big|\tau_i(\lambda)\big| \right\}. \tag{31}$$

Our first step is to compute the $(k+1)^{th}$ derivative of $\Psi(y,\lambda)$ using equation (25). We have that

$$\frac{d\omega}{dx}(x,\lambda) = -x^{-1-\alpha_1(\lambda)} = -x^{-1}\big(\alpha_1(\lambda)\omega(x,\lambda)+1\big) \tag{32}$$

$$\partial_x^j x^i\omega = \begin{cases} *x^{i-j-\alpha_1(\lambda)} & i<j \\ *1+*\omega & i=j \end{cases} \tag{33}$$

$$\frac{\partial^{k+1}x^i\omega\big(1+\tilde{f}_i(x,\lambda)\big)}{\partial x^{k+1}} = x^{i-(k+1)-\alpha_1(\lambda)}\big(A_i(\lambda)+\tilde{F}_i(x,\lambda)\big), \tag{34}$$

where $A_i(0)\neq 0$, $\tilde{f}_i(0,\lambda)=0=\tilde{F}_i(0,\lambda)$, and $*$ are non-vanishing functions of λ. Since we will only be interested in the behavior of functions $\tilde{F}_i(x,\lambda)$ for small x, to simplify the notation we will simply write

$$\frac{\partial^{k+1}x^i\omega\big(1+\tilde{f}_i(x,\lambda)\big)}{\partial x^{k+1}} = x^{i-(k+1)-\alpha_1(\lambda)}\big(A_i(\lambda)+\tilde{f}_i(x,\lambda)\big), \tag{35}$$

noting that now the new $\tilde{f}_i(x,\lambda)$ is only $I_0^{K-(k+1)}\big(\rho(x,\lambda)\big)$. We then have the following expression for the $(k+1)^{\text{th}}$ derivative of equation (25):

$$\begin{aligned}\frac{\partial^{k+1}\Psi(y,\lambda)}{\partial y^{k+1}} = \sum_{i=1}^{k}\Big(&(-1)^k\alpha_i(\lambda)x^{i-(k+1)-\alpha_1(\lambda)}\big[A_i(\lambda)+\tilde{f}_i(x,\lambda)\big] \\ &-\overline{\alpha}_i(\lambda)y^{i-(k+1)-\overline{\alpha}_1(\lambda)}\big[B_i(\lambda)+\overline{\tilde{f}}_i(y,\lambda)\big]\Big)+\tilde{f}_{k+1}(x,\lambda)+\overline{\tilde{f}}_{k+1}(y,\lambda),\end{aligned} \tag{36}$$

where $\tilde{f}_{k+1}(x,\lambda)$ $\big($resp. $\overline{\tilde{f}}_{k+1}(y,\lambda)\big)$ is $C^{K-(k+1)}$ and 1-flat at $x=0$ (resp. $y=0$); $A_i(0)=B_i(0)\neq 0$.

Lemma 18. *Let*

$$\begin{aligned}T_{k+1}(t,\lambda) = \sum_{i=1}^{k}\Big(&\tau_i(\lambda)t^{i-(k+1)-\alpha_1(\lambda)}\big[A_i(\lambda)+f_i(t,\lambda)\big]+ \\ &+(-1)^{k+1}\overline{\tau}_i(\lambda)b^{\alpha_1(\lambda)-\overline{\alpha}_1(\lambda)}(1-t)^{i-(k+1)-\overline{\alpha}_1(\lambda)}\big[B_i(\lambda)+\overline{f}_i(t,\lambda)\big]\Big),\end{aligned} \tag{37}$$

with $\overline{\tau}_i(\lambda)=-\tau_i(\lambda)+\breve{p}_i(\lambda)$, $\breve{p}_i(\lambda)$ are polynomials in $b(\lambda)$ and in $\tau_j(\lambda)$ with $j<i$ such that $\breve{p}_i(0)=0$. The functions $f_i(t,\lambda)$ are $I_0^{K-(k+1)}(t)$ and the functions $\overline{f}_i(t,\lambda)$ are $I_0^{K-(k+1)}(1-t)$.

There exists a function of the form of equation (37) such that the number of zeroes of $\partial_y^{k+1}\Psi(y,\lambda)$ in a small neighborhood of $(0,0)$ in $\mathbb{R}\times\Lambda'$ is the same as the number of zeroes of $T_{k+1}(t,\lambda)$ in $[0,1]\times\Lambda$, where Λ is a small neighborhood of 0 in Λ'.

Proof. We first multiply equation (36) by $(-1)^{k+1}b^{\alpha_1(\lambda)+1}(\lambda)/L(\lambda)$ and use the coordinates (26). It is equivalent to the following equation:

$$\begin{aligned}\frac{\partial^{k+1}\overline{\Psi}(t,\lambda)}{\partial t^{k+1}} = \sum_{i=1}^{k}\Big(&\tau_i(\lambda)t^{i-(k+1)-\alpha_1(\lambda)}\big[A_i(\lambda)+f_i(t,\lambda)\big]+ \\ &+(-1)^{k+1}\overline{\tau}_i(\lambda)b^{\alpha_1(\lambda)-\overline{\alpha}_1(\lambda)}(1-t)^{i-(k+1)-\overline{\alpha}_1(\lambda)}\big[B_i(\lambda)+\overline{f}_i(t,\lambda)\big]\Big)+ \\ &+f_{k+1}(t,\lambda)+\overline{f}_{k+1}(t,\lambda).\end{aligned} \tag{38}$$

Looking at $E_j \neq \emptyset$ with $j < k+1$, we can include the function $f_{k+1}(t,\lambda)$ in the term with coefficient $\tau_j(\lambda)$ by letting the following:

$$f_j^{\text{new}}(t,\lambda) = f_j^{\text{old}}(t,\lambda) + \frac{f_{k+1}(t,\lambda)}{\overline{\tau}_j(\lambda)} \cdot t^{-j+(k+1)+\alpha_1(\lambda)}. \tag{39}$$

(To simplify the notation, we again let $f_i = f_i^{\text{new}}$). We do the same with the function $\overline{f}_{k+1}(t,\lambda)$, i.e. we include it in the function $\overline{f}_j(t,\lambda)$. That functions $f_i(t,\lambda)$ are $I_0^{K-(k+1)}(t)$ and functions $\overline{f}_i(t,\lambda)$ are $I_0^{K-(k+1)}(1-t)$ can be shown as follows. In equation (38), the functions $\tilde{f}_i(x,\lambda)$ are $I_0^{K-(k+1)}\big(\rho(x,\lambda)\big)$, the functions $\overline{\tilde{f}}_i(x,\lambda)$ are $I_0^{K-(k+1)}\big(\bar{\rho}(y,\lambda)\big)$, and $\tilde{f}_{k+1}(x,\lambda)$ is of class $C^{K-(k+1)}$ and 1-flat at $x=0$; i.e. we have for all $0 \leq n \leq K-(k+1)$

$$\lim_{x\to 0} \big(x^{1+\alpha_1(\lambda)}\omega\big)^n \frac{\partial^n \tilde{f}_i(x,\lambda)}{\partial x^n} = 0 = \lim_{y\to 0} \big(y^{1+\overline{\alpha}_1(\lambda)}\overline{\omega}\big)^n \frac{\partial^n \overline{\tilde{f}}_i(y,\lambda)}{\partial y^n} \tag{40}$$

uniformly for $\lambda \in \Lambda$. Since $x/x^{1+\alpha_1(\lambda)}\omega(x,\lambda)$ is bounded, we then have the following limits:

$$\lim_{x\to 0} x^n \frac{\partial^n \tilde{f}_i(x,\lambda)}{\partial x^n} = 0 = \lim_{y\to 0} y^n \frac{\partial^n \overline{\tilde{f}}_i(y,\lambda)}{\partial y^n}. \tag{41}$$

Using the variables (26), the following relations are easily obtained for all $0 \leq n \leq K-(k+1)$:

$$\lim_{b(\lambda)\to 0} \frac{\partial^n f_i(t,\lambda)}{\partial t^n} = 0 = \lim_{b(\lambda)\to 0} \frac{\partial^n \overline{f}_i(t,\lambda)}{\partial t^n}, \tag{42}$$

$$\lim_{t\to 0} t^n \frac{\partial^n f_i(t,\lambda)}{\partial t^n} = 0 = \lim_{t\to 1} (1-t)^n \frac{\partial^n \overline{f}_i(t,\lambda)}{\partial t^n} \tag{43}$$

the last limits being uniform in λ. □

From lemma 18, we see that coefficient $b(\lambda)^{\alpha_1(\lambda)-\overline{\alpha}_1(\lambda)}$ will be of some importance. Since both $b(\lambda)$ and $\alpha_1(\lambda)$ tend to zero with the parameter, the quantity $b(\lambda)^{\alpha_1(\lambda)-\overline{\alpha}_1(\lambda)}$ could lead to some serious problems. Let us show:

Lemma 19. *If the 2-return map has a fixed point for small t, then $b(\lambda)^{\alpha_1(\lambda)-\overline{\alpha}_1(\lambda)}$ is bounded from above.*

Proof. Indeed, since $\overline{\alpha}_1(0) = -\alpha_1(0)$, i.e. $\alpha_1(\lambda) - \overline{\alpha}_1(\lambda) \sim 2\alpha_1(\lambda)$, $b(\lambda)^{\alpha_1(\lambda)-\overline{\alpha}_1(\lambda)}$ is not bounded precisely when $\alpha_1(\lambda)$ is negative, in which case $b(\lambda)^{\alpha_1(\lambda)}$ is not bounded. Using proposition 16 we have that

$$\begin{aligned}\Psi(b(1-t),\lambda))|_{t=0} = \beta(\lambda) + b(\lambda)\Big[1 - \Big(b(\lambda)^{-\overline{\alpha}_1(\lambda)} - 1\Big) \times \\ \big(1+\overline{f}_i(b(\lambda),\lambda)\big)\Big] + O\big(b^2(\lambda)[1+\overline{\omega}(b(\lambda),\lambda)]\big).\end{aligned} \tag{44}$$

Assume $b^{\alpha_1(\lambda)}$ is not bounded and can thus be taken as large as necessary. Since $b(\lambda)^{-\alpha_1(\lambda)}$, $\beta(\lambda)$ and $b(\lambda)$ can be taken sufficiently small, we have that for sufficiently small neighborhood Λ of $\lambda = 0$, there exists a constant $M > b^{-1}(\lambda)$ such that for all sufficiently small $t > 0$, $\Psi\big(b(\lambda)(1-t),\lambda\big) < \beta(\lambda) - Mb(\lambda) < 0$. Therefore the 2-return map has no fixed points. □

From proposition 17 and since $y = b(\lambda)(1-t)$, we only have to study the zeroes of equation (37) on the intervals $[0, \epsilon_1]$ and $[\epsilon_2, 1-\epsilon_2]$ with $\epsilon_1 \geq \epsilon_2 > 0$ sufficiently small.

Proposition 20. *There exists $\epsilon_1 > 0$ and a neighborhood $\Lambda_0 \subseteq \Lambda$ of $\lambda = 0$ such that for all $\lambda \in E_j \cap \Lambda_0$, the equation $T_{k+1}(t, \lambda) = 0$ has, on $[0, \epsilon_1]$, at most $(j-1)$ roots.*

Proof. We first multiply equation (37) by $t^{k+\alpha_1(\lambda)}$ to obtain the following equation:

$$\tilde{T}_{k+1}(t, \lambda) = \sum_{i=1}^{k} t^{i-1}\big(\tau_i(\lambda) A_i(\lambda) + h_i(t, \lambda)\big). \tag{45}$$

From equations (42) and (43),

$$\lim_{t\to 0} t^n \frac{\partial^n f(t,\lambda)}{\partial t^n} = 0 = \lim_{b(\lambda)\to 0} \frac{\partial^n \overline{f}(t,\lambda)}{\partial t^n}, \tag{46}$$

$0 \leq n \leq K-(k+1)$. Moreover, since $b(\lambda)^{\alpha_1(\lambda)-\overline{\alpha}_1(\lambda)}$ is bounded on $[0, \epsilon_1]$ for ϵ_1 sufficiently small (by lemma 19), we have that

$$\lim_{t\to 0} t^n \frac{\partial^n h_i(t,\lambda)}{\partial t^n} = 0,$$

i.e. the $h_i(t, \lambda)$ are $I_0^K(t)$. Having such functions allows us to differentiate $j-1$ times and show that, for some small neighborhood Λ_0 of $\lambda = 0$, the number of zeroes of equation (45) in the set $E_j \cap \Lambda_0$ with $t \leq \epsilon_1$ for ϵ_1 sufficiently small is bounded by $j-1$. □

The following lemma states that under some hypothesis, the number of zeroes of equation (37) is bounded by the number of zeroes (counted with multiplicity) of a polynomial on an interval of the form $[t_2, t_3]$ with $t_i > 0$:

Lemma 21 (cf. [JM94]). *Let $0 < t_2 < t_3$. If $T(t, \lambda) \overset{\text{def}}{=} P(t, \lambda) + f(t, \lambda)$ where $P(t, \lambda)$ is some polynomial of degree k with coefficients in λ and $f(t, \lambda)$ such that for all $n \leq k$ we have on $[t_2, t_3]$,*

$$\lim_{\lambda\to 0} \frac{\partial^n f(t,\lambda)}{\partial t^n} = 0, \tag{47}$$

then exists a neighborhood $\Lambda_P \subseteq \Lambda$ of $\lambda = 0$ such that for all $\lambda \in \Lambda_P$, the number of zeroes of $T(t, \lambda)$ on $[t_2, t_3]$ is bounded by the number of zeroes of $P(t, \lambda)$.

Proposition 22. *Let $t_1 \in (0, 1/2)$. There exists a neighborhood $\Lambda_1 \subseteq \Lambda$ of $\lambda = 0$ such that for all $\lambda \in E_j \cap \Lambda_1$, the equation $T_{k+1}(t, \lambda) = 0$ has, on $[t_1, 1-t_1]$, at most $2k-1$ roots.*

Proof. First we write $T_{k+1}(t, \lambda)$ as a polynomial plus some $K-(k+1)$-flat rest $f(t, \lambda)$ and then prove the statement using lemma 21. We will be working with the two functions $F_{k+1}(t, \lambda)$ and $P(t, \lambda)$, with

$$F_{k+1}(t, \lambda) = \left(t^{k+\alpha_1(\lambda)}(1-t)^{k+\overline{\alpha}_1(\lambda)}\right) T_{k+1}(t, \lambda). \tag{48}$$

$P(t,\lambda)$ is the following (non-trivial) polynomial:

$$P(t,\lambda) = \sum_{i=0}^{k-1} \tau_{i+1}\Big(t^i(1-t)^k A_{i+1} + (-1)^k b^{\alpha_1-\overline{\alpha}_1} t^k (1-t)^i B_{i+1}\Big) \tag{49}$$

$$= \sum_{i=0}^{k-1} c_i(\lambda) t^i + o\big(t^{k-1}\big), \tag{50}$$

where if $V_1 = (c_0, c_2 t, \ldots, c_k t^{k-1})$, $V_2 = (\tau_1, \tau_2, \ldots, \tau_k)$ and $\mathcal{M}(t,\lambda)$ is the lower triangular k by k matrix with $m_{ij}(t,\lambda) = *A_j(\lambda)t^{i-1}$ $(i \geq j)$, then $V_1^T = \mathcal{M}(t,\lambda) \cdot V_2^T$. $P(t,\lambda)$ is non-identically zero since $V_2 \neq 0$ and $\mathcal{M}(t,\lambda)$ is inversible for all $(t,\lambda) \in [t_1, 1-t_1] \times \Lambda$.

We then have $F_{k+1}(t,\lambda) = P(t,\lambda) + f(t,\lambda)$, with

$$f(t,\lambda) = \sum_{i=0}^{k-1} \tau_{i+1}\Big(t^i(1-t)^k \big[A_{i+1}\big((1-t)^{\overline{\alpha}_1(\lambda)} - 1\big) + \tag{51}$$

$$+(1-t)^{\overline{\alpha}_1(\lambda)} f_{i+1}(t,\lambda)\big] + (-1)^k b^{\alpha_1-\overline{\alpha}_1} t^k (1-t)^i \big[B_{i+1}\big(t^{\alpha_1(\lambda)}-1\big) + t^{\alpha_1(\lambda)}\overline{f}_{i+1}(t,\lambda)\big]\Big),$$

$0 \leq n \leq K-(k+1)$, $\lim_{\lambda\to 0} \partial_t^n f(t,\lambda) = 0$ uniformly for $t \in [t_1, 1-t_1]$. To prove this limit, which is valid for the $f_i(t,\lambda)$ and the $\overline{f}_i(t,\lambda)$, we only need to show that it is valid for terms of the following form:

$$A_{i+1}\big((1-t)^{-\overline{\alpha}_1(\lambda)} - 1\big) \quad \text{and} \quad B_{i+1}\big(t^{\alpha_1(\lambda)} - 1\big). \tag{52}$$

We now use the fact that $\alpha_1(\lambda)$ and $\overline{\alpha}_1(\lambda)$ converge to zero with λ, and that $t \in [t_1, 1-t_1]$. The case $n = 0$ follows directly from equation (52). For the cases $n > 0$, both terms in equation (52) are of the form $\alpha_1(\lambda)h(t,\lambda)$ and $\overline{\alpha}_1(\lambda)\overline{h}(t,\lambda)$ respectively, where $h(t,\lambda)$ and $\overline{h}(t,\lambda)$ are analytic on $[t_1, 1-t_1]$. We thus obtain the limit.

Let equation (37) and $t_1 > 0$. On the interval $[t_1, 1-t_1]$, the zeroes of $T_{k+1}(t,\lambda)$ are given by the zeroes of $F_{k+1}(t,\lambda)$. The proposition follows from lemma 21 by taking $t_2 = t_1$, $t_3 = 1-t_1$ since $P(t,\lambda)$ is a (non-trivial) polynomial of degree at most $2k+1$ vanishing at 0 and 1. □

From propositions 20 and 22 we have that $T_{k+1}(t,\lambda)$ has, for some small neighborhood $\Lambda_0 \subseteq \Lambda$ of $\lambda = 0$, $3k-1$ roots on $[0, 1-\epsilon]$, for some small ϵ. Consequently, if Γ_0 is logarithmic of order k then, from proposition 17 $Cycl(\Gamma_0) \leq 3k+1$.

Proposition 23. *There exists a neighborhood Λ of $\lambda = 0$ such that for all $\lambda \in E_j \cap \Lambda$, the equation $T_{k+1}(t,\lambda) = 0$ has at most $2k-1$ roots on $[0,1]$ for all $1 \leq j \leq k$.*

Proof. Let $1 \leq j \leq k$. Assume that $T_{k+1}(t,\lambda)$ has d zeroes (d being the maximum) on E_j. Choose a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ converging to 0 such that $T_{k+1}(t,\lambda_n)$ has d zeroes. Of those d zeroes, assume m_0 go to 0 and m_1 go to 1 (the m_i can of course be 0). Let $1-t_1$ be the lower bound of the set of roots that go to 1 and t_2 the upper bound of the set of roots that go to 0. Note $\epsilon_2 = \min\{t_1, t_2\}$, the minimum of the two. We have two cases: when $b(\lambda)^{\alpha_1(\lambda)-\overline{\alpha}_1(\lambda)}$ is bounded, and when it is not.

(i) Suppose $b(\lambda)^{\alpha_1(\lambda)-\overline{\alpha}_1(\lambda)}$ is bounded by some positive constant $M>0$. Taking a converging subsequence, we have that

$$\lim_{n\to\infty} b(\lambda_n)^{\alpha_1(\lambda_n)-\overline{\alpha}_1(\lambda_n)} = v_0 \in [0,M].$$

We then have that $T_{k+1}(t,0)$ is of the following form:

$$T_{k+1}(t,0) = \sum_{i=0}^{k-1} \tau_{i+1}A_{i+1}t^{m_0}(1-t)^{m_1}\Big(t^{i-m_0}(1-t)^{k-m_1} + (-1)^{k+1}v_0 t^{k-m_0}(1-t)^{i-m_1}\Big), \tag{53}$$

The number of roots of $T_{k+1}(t,0)$ with $t\in[\epsilon_2,1-\epsilon_2]$ is thus bounded by the number of roots of the polynomial $P_{k+1}(t)$, where

$$P_{k+1}(t) = \sum_{i=0}^{k-(m_0+1)} \tau_{i+m_0+1}(0)A_{i+m_0+1}(0)t^i(1-t)^{k-m_1} + \sum_{i=0}^{k-(m_1+1)} (-1)^k\tau_{i+m_1+1}(0)A_{i+m_1+1}(0)v_0t^{k-m_0}(1-t)^i. \tag{54}$$

The result follows from the fact that $P(t,0)$ is a (non-trivial) polynomial of degree $2k-(m_0+m_1+1)$ and that in the worst case scenario, $m_0(0)=0=m_1(0)$.

(ii) From equation (50), in the case where $b(\lambda)^{\alpha_1(\lambda)-\overline{\alpha}_1(\lambda)}$ is not bounded, t is a zero of $P(t,\lambda_n)$ on $[\epsilon_2,1-\epsilon_2]$ only if it is a zero of

$$\tilde{P}(t,\lambda_n) = \sum_{i=0}^{k-1} \overline{\tau}_{i+1}(\lambda_n)B_{i+1}(\lambda_n)(1-t)^i, \tag{55}$$

and $\tilde{P}(t,0)$ is of the following form:

$$\tilde{P}(t,0) = \sum_{i=m_1}^{k-1-m_0} \overline{\tau}_{i+1}(\lambda_n)B_{i+1}(\lambda_n)(1-t)^i, \tag{56}$$

i.e. $\tilde{P}(t,0)$ is of degree at most $k-(m_0+m_1+1)$. □

Using proposition 17, part one of theorem 7 is a corollary of proposition 23.

Corollary 24.

1. *If* Γ_0 *is of codimension* $3k$*, i.e* Γ_0 *is logarithmic of order* $2k$*, then* $Cycl(\Gamma_0)\le 3k$.
2. *If* Γ_0 *is of codimension* $3k+1$*, i.e* Γ_0 *is logarithmic of order* $2k+1$*, then* $Cycl(\Gamma_0)\le 3k+2$.

As we announced, we expect the bound to be optimal for loops of codimension $3k$, but for loops of codimension $3k+1$, the bound is not optimal. For the case of codimension 1 ($k=0$), it is shown in [Kuz95] (c.f. theorem 6.4, p. 197) that the homoclinic loop has cyclicity 1 and not 2. The reason of non-optimality comes from our method itself in which we take one to many derivative: in the case of codimension 1, we study the second derivative of $\Psi(y,\lambda)$, whereas the result can be obtained directly by looking at the first derivative which is of the form $t^{1-\alpha_1}\big(1+O(\lambda)\big)$ when $\alpha_1(0)>0$ or $(1-t)t^{1+\alpha_1}\big(1+O(\lambda)\big)$ otherwise.

2.3.2. **Proof of Theorem 7: case Γ_0 is analytic.** This is the case where $\alpha_i(0) = 0 = \beta_i(0)$ for $i \leq k-1$, $\alpha_k(0) = 0$ and $\beta_k(0) \neq 0$ (with k odd).

As for the previous case, we homogenize the principal part of $\Psi\big(b(\lambda)(1-t), \lambda\big)$ with respect to $b(\lambda)$ variables and compactify the space of coefficients $\alpha_i(\lambda)$ with $i \leq k$ and $\beta_k(\lambda)$:

(57) $$\alpha_i(\lambda) = t_i(\lambda) b^{k-i}(\lambda) \text{ for } 1 \leq i \leq k \text{ and } t_{k+1}(\lambda) = \beta_k(\lambda).$$

The $t_i(\lambda)$ are not bounded, but since $\beta_k(0) \neq 0$, we compactify the coefficient space as described in the previous section. Moreover we use the same cones (taking the sup over $1 \leq i \leq k+1$).

Our first step is to compute the k^{th} derivative of $\Psi(y, \lambda)$ using equations (25), (32), (33) and (34). If $k \geq 2$, we obtain the following equation:

(58) $$\begin{aligned} \frac{\partial^k \Psi(y,\lambda)}{\partial y^k} = \sum_{i=1}^{k-1} \Big(& (-1)^{k+1} \alpha_i(\lambda) x^{i-k-\alpha_1(\lambda)} \big[A_i(\lambda) + \tilde{f}i(x,\lambda)\big] \\ & - \overline{\alpha}_i(\lambda) y^{i-k-\overline{\alpha}_1(\lambda)} \big[B_i(\lambda) + \overline{\tilde{f}}_i(y,\lambda)\big]\Big) + (-1)^{k+1}\alpha_k(\lambda)\omega(x,\lambda)\big[A_k(\lambda) \\ + \tilde{f}_k(x,\lambda)\big] - & \overline{\alpha}_k(\lambda)\overline{\omega}(y,\lambda)\big[B_k(\lambda) + \overline{\tilde{f}}_k(y,\lambda)\big] + \beta_k(\lambda)\big[k! + \tilde{F}_{k+1}(x,\lambda) + \overline{\tilde{F}}_{k+1}(y,\lambda)\big], \end{aligned}$$

where functions $\tilde{f}_i(x,\lambda)$ are $I_0^{K-k}\big(\rho(x,\lambda)\big)$ and functions $\overline{\tilde{f}}_i(y,\lambda)$ are $I_0^{K-k}\big(\bar{\rho}(y,\lambda)\big)$, $\tilde{F}(x,\lambda)$ and $\overline{\tilde{F}}_{k+1}(y,\lambda)$ are $k-$flat at $x = 0$ and $y = 0$ respectively, and by hypothesis $\beta_k(0) \neq 0$. (Note that the rest functions have been included in the β_k-term.)

Let, for $k > 1$,

(59) $$\begin{aligned} T_k(t,\lambda) = \sum_{i=1}^{k-1} \Big(& \tau_i(\lambda) t^{i-k-\alpha_1} b^{-\alpha_1}\big[A_i + f_i(t,\lambda)\big] \\ (-1)^k \overline{\tau}_i(\lambda) b^{-\overline{\alpha}_1} & (1-t)^{i-k-\overline{\alpha}_1}\big[B_i + \overline{f}_i(t,\lambda)\big]\Big) + \tau_k(\lambda)\omega(x,\lambda)\big[A_k + f_k(t,\lambda)\big] \\ & (-1)^k \overline{\tau}_k \overline{\omega}(y,\lambda)\big[B_k + \overline{f}_i(t,\lambda)\big] + \tau_{k+1}\big[k! + F_{k+1}(t,\lambda) + \overline{F}_{k+1}(y,\lambda)\big], \end{aligned}$$

where the functions $f_i(t,\lambda)$ are $I_0^{K-k}(t)$ and the functions $\overline{f}_i(t,\lambda)$ are $I_0^{K-k}(1-t)$.

Looking at equation (59), we see why this case is more complicated than equation (37): we have to deal with terms in $\omega(x,\lambda)$ and $\overline{\omega}(y,\lambda)$. Fortunately, we only need to look at $t \in [0, 1-\epsilon]$ for some small ϵ. For values of t in $[\epsilon, 1-\epsilon]$, we divide the proof in two cases: whether all $\tau_i(\lambda)$=0 for $i \leq k$ or not. For t in $[0,\epsilon]$, we use the same differentiation-division algorithm as for equation (37) since $\overline{\omega}(y,\lambda)$ is analytic in t in this interval.

First, we prove that if $t \in [\epsilon, 1-\epsilon]$, then the number of zeroes of $T_k(t,\lambda)$ is bounded. To simplify our study, we will divide the set E_{k+1} in the two following subsets:

$$E^1_{k+1} = \big\{\lambda \in E_{k+1} \mid \tau_i(\lambda) = 0, \quad i \leq k\big\} \qquad \text{and} \qquad E^2_{k+1} = E_{k+1} \backslash E^1_{k+1}.$$

(Note: $E^1_{k+1} = \big\{\lambda \in E_{k+1} \mid \tau_{k+1}(\lambda) = 1\big\}$ and that for $\lambda \in E^2_{k+1}$, there exists $i \leq k$ for which $\tau_i(\lambda) \neq 0$.)

Then, we have the following proposition:

Proposition 25. *Let $\epsilon_2 \in (0, 1/2)$. There exists a neighborhood $\Lambda_{\epsilon_2} \subseteq \Lambda$ of $\lambda = 0$ such that:*

1. *For all $\lambda \in E^1_{k+1} \cap \Lambda_{\epsilon_2}$, the function $T_k(t,\lambda)$ has no zeroes on $[\epsilon_2, 1-\epsilon_2]$.*

2. *For all $\lambda \in E^2_{k+1} \cap \Lambda_{\epsilon_2}$, the function $T_k(t,\lambda)$ has at most $2k$ zeroes on $[\epsilon_2, 1-\epsilon_2]$.*

Proof. Let $\epsilon_2 \in (0,1/2)$ and $t \in [\epsilon_2, 1-\epsilon_2]$, then all $\omega(x,\lambda)$, $\overline{\omega}(y,\lambda)$ and $f_i(t,\lambda)$ are analytic.

First, we easily see that there are no zeroes on E^1_{k+1}.

The study on E^2_{k+1} and on E_j with $j < k+1$ is more complicated. First we divide $T_k(t,\lambda)$ by the unit $\big(k! + F_{k+1}(t,\lambda) + \overline{F}_{k+1}(t,\lambda)\big)$ (but use the same notation for $T_k(t,\lambda)$) and differentiate once again (this kills the term with coefficient $\tau_{k+1}(\lambda)\big)$. Knowing that $\partial/\partial t = b(\lambda)\partial/\partial x$, if $k \geq 2$, we get the following:

$$\frac{\partial T_k(t,\lambda)}{\partial t} = \sum_{i=1}^{k} \Big(\tau_i(\lambda) b^{-\alpha_1(\lambda)} t^{i-(k+1)-\alpha_1(\lambda)} \big[\overline{A}_i(\lambda) + g_i(t,\lambda)\big] + \\ + (-1)^k \overline{\tau}_i(\lambda) b^{-\overline{\alpha}_1(\lambda)} (1-t)^{i-(k+1)-\overline{\alpha}_1(\lambda)} \big[\overline{B}_i(\lambda) + \overline{g}_i(t,\lambda)\big]\Big), \tag{60}$$

with $\overline{A}_i = *A_i$, $\overline{B}_i = *B_i$ where $*$ are non-vanishing functions of λ, and for each λ there exists an $i \leq k$ with $\tau_i(\lambda) \neq 0$. To simplify, we again subdivide the set E^2_{k+1} in the following subsets:

$$E^2_{k+1,j} = \left\{ \lambda \in E^2_{k+1} \;\middle|\; \big|\tau_j(\lambda)\big| = \sup_{i=1,2,\ldots,k} \big|\tau_i(\lambda)\big| > 0 \right\}.$$

Since equation (60) is similar to equation (37), we can use the method exposed in the proof of proposition 22 to bound the zeroes of $\partial_t T_k(t,\lambda)$ on $[\epsilon_2, 1-\epsilon_2]$ for $\lambda \in \Lambda_2 = \Lambda_0$. □

Proposition 26. *There exists $0 < \epsilon_1 < 1$ and a neighborhood $\Lambda'_{\epsilon_1} \subseteq \Lambda$ of $\lambda = 0$ such that for all $\lambda \in \Lambda'_{\epsilon_1} \cap E_j$, the function $T_k(t,\lambda)$ has at most k roots on $(0,\epsilon_1)$.*

Proof. First of all, from lemma 19 we can suppose that both $b^{\alpha_1(\lambda)}(\lambda)$ and $b^{-\overline{\alpha}_1(\lambda)}(\lambda)$ are bounded. In this case, having $T_k(t,\lambda) = 0$ is equivalent to $\overline{T}_k(t,\lambda) = 0$, where if $k \geq 2$:

$$\overline{T}_k(t,\lambda) = \sum_{i=1}^{k-1} t^{i-k-\alpha_1(\lambda)} \big[\tau_i(\lambda)\big(A_i(\lambda) + h_i(t,\lambda)\big)\big] + \\ + b^{\alpha_1(\lambda)} \Big(\tau_k(\lambda)\omega(x,\lambda)\big[A_k(\lambda) + f_k(t,\lambda)\big] + \\ (-1)^k \overline{\tau}_k(\lambda)\overline{\omega}(y,\lambda)\big[B_k(\lambda) + \overline{f}_k(t,\lambda)\big] + \tau_{k+1}(\lambda)\big[k! + F_{k+1}(t,\lambda) + \overline{F}_{k+1}(t,\lambda)\big]\Big). \tag{61}$$

Let $\lambda \in E_j$, we can then include function $\overline{F}_{k+1}(t,\lambda)$ in the term with the factor $\tau_j(\lambda)$.

To get rid of the terms in the summation, we use a differentiation-division algorithm: we differentiate $\overline{T}_k(t,\lambda)$ $(k-1)$-times and before the i^{th} differentiation, we divide by the unit $*(1+h_i)$. Taking a smaller neighborhood of $\lambda = 0$, we have that the number of zeroes of $T_k(t,\lambda)$ is bounded by $(k-1)$ plus the number of zeroes of the following function:

$$\begin{aligned}(62)\quad &\Big(\tau_k(\lambda)\omega(x,\lambda)\big(A_k(\lambda)+f_k(t,\lambda)\big)+\\ &+(-1)^k\overline{\tau}_k(\lambda)\overline{\omega}(y,\lambda)\big(B_k(\lambda)+\overline{f}_k(t,\lambda)\big)\Big)+\tau_{k+1}(\lambda)\big(k!+F_{k+1}(t,\lambda)\big)=0\end{aligned}$$

(we have multiplied by $*b^{-\alpha_1(\lambda)}t^{-\alpha_1(\lambda)}$).

As the function $f_k(t,\lambda)$, the function $\overline{f}_k(t,\lambda)$ now is $I_0^1\big(\rho(tb)\big)$, $\overline{\omega}(y,\lambda)=\overline{\omega}\big(b(1-t),\lambda\big)$ is analytic for small values of t, and $F_{k+1}(t,\lambda)$ is 1-flat at $t=0$. To eliminate the problem with the $\overline{f}_k$, we now divide the equation (62) by the unit $\big(B_k(\lambda)+\overline{f}_k(t,\lambda)\big)$. We get the following equation:

$$\begin{aligned}(63)\quad &\tau_k(\lambda)\omega(x,\lambda)\big(\overline{A}_k(\lambda)+\check{f}_k(t,\lambda)\big)\\ &+(-1)^k\overline{\tau}_k(\lambda)\overline{\omega}(y,\lambda)+\tau_{k+1}(\lambda)\big(C_{k+1}(\lambda)+\check{F}_{k+1}(t,\lambda)\big)=0.\end{aligned}$$

Dividing by $\Big(\omega(x,\lambda)\big(\overline{A}_k(\lambda)+\check{f}_k(t,\lambda)\big)+(-1)^k\tau_k(\lambda)\overline{\omega}(y,\lambda)\Big)$ (which is non-zero since $y=(1-t)b(\lambda)$ with small t) and differentiating one last time with respect to t we obtain an equation equivalent to the simplified equation

$$(64)\quad \big(C_{k+1}+\check{F}_{k+1}(t,\lambda)\big)\big(\overline{A}_k+\check{f}_k(t,\lambda)+O(x^{1+\alpha_1(\lambda)})\big)+O\big(x^{1+\alpha_1(\lambda)}\omega(x,\lambda)\big)+O\big(\check{F}'_{k+1}(t,\lambda)\big).$$

(This result is also true for $k=1$.) We now have the following limits:

$$(65)\qquad \lim_{t\to 0}\check{f}_k(t,\lambda)=0=\lim_{t\to 0}\check{F}_{k+1}(t,\lambda)=\lim_{t\to 0}\check{F}'_{k+1}(t,\lambda)=\lim_{t\to 0}x.$$

Thus choosing ϵ_1 sufficiently small, equation (64) is equivalent to $C_{k+1}(\lambda)\overline{A}_k(\lambda)\neq 0$. We hence have that the number of zeroes of $T_k(t,\lambda)$ in $(0,\epsilon_1)$ is bounded by k. □

From propositions 25 and 26, we have that $T_k(t,\lambda)$ has, for some small neighborhood $\Lambda_0\subseteq\Lambda$ of $\lambda=0$, $3k+2$ roots on $[0,1-\epsilon]$, for some small ϵ. Consequently from proposition 17, if Γ_0 is analytic of order k, then $Cycl(\Gamma_0)\leq 3(k+1)$.

Proposition 27. *There exists a neighborhood Λ of $\lambda=0$ such that for all $\lambda\in E_j\cap\Lambda$, the equation $T_k(t,\lambda)=0$ has at most $2k$ roots on $[0,1]$, this for all $1\leq j\leq k$.*

Proof. The proof of this proposition is essentially the same as for proposition 23. However there are some differences: in equation (59), we have to deal with both $b^{-\alpha_1(\lambda)}$ and $b^{-\overline{\alpha}_1(\lambda)}$.

As before we note $\epsilon_2=\min\{t_1,t_2\}$ and let $\big(\lambda_n\big)$ be a converging sequence such that $T_k(t,\lambda_n)$ has d roots.

Multiplying equation (59) by $t^k(1-t)^k$, we get the following equation:

$$(66)\quad S_k(t,\lambda) = \sum_{i=1}^{k-1} \tau_i(\lambda)\Big(t^{i-\alpha_1(\lambda)}(1-t)^k b^{-\alpha_1(\lambda)}\big[A_i(\lambda)+f_i(t,\lambda)\big] + + (-1)^{k+1} b^{-\overline{\alpha}_1(\lambda)} t^k (1-t)^{i-\overline{\alpha}_1(\lambda)}\big[B_i(\lambda)+\overline{f}_i(t,\lambda)\big]\Big) \tau_k(\lambda)t^k(1-t)^k\Big[\Big(\omega(x,\lambda)\big[A_k(\lambda)+f_k(t,\lambda)\big] + + (-1)^{k+1}\overline{\tau}_k(\lambda)\overline{\omega}(y,\lambda)\big[B_k(\lambda)+\overline{f}_i(t,\lambda)\big]\Big) + \tau_{k+1}(\lambda)\big[k! + f_{k+1}(t,\lambda)\big]\Big].$$

(i) Assume $b^{-\alpha_1(\lambda)}$ or $b^{-\overline{\alpha}_1(\lambda)}$ diverges, then using the technics used to prove proposition 23, we show that $S_k(t,\lambda)$ has at most $k-(m_0+m_1+1)$ zeroes on $[\epsilon_2, 1-\epsilon_2]$.

(ii) Assume $b^{-\alpha_1(\lambda)}$ and $b^{-\overline{\alpha}_1(\lambda)}$ both converge. We then have two cases.

1. Assume $\lim_{n\to\infty}\tau_{k+1}(\lambda_n) = 1$. Since $\sigma_k(0)=0$, at the limit we get that $S_k(t,0)$ is equivalent to a polynomial in t of degree 2k and thus has at most 2k roots on the interval.
2. When $\lim\limits_{n\to\infty}\lambda_n \in E^2_{k+1}$, we can then use the same technics as for equation (60). We obtain a polynomial of degree at most $2k-1$ whose number of zeroes is equal to the number of zeroes of $T_k(t,\lambda)$ on $[\epsilon_2, 1-\epsilon_2]$, minus one. Again using the technics used to prove proposition 23, we can show that $S_k(t,\lambda)$ has at most $2k-(m_0+m_1)$ zeroes on $[\epsilon_2, 1-\epsilon_2]$.

□

We get the second part of theorem 7.

Corollary 28. *If Γ_0 of codimension $3k+2$, i.e. analytic of order $2k+1$, then $Cycl(\Gamma_0) \le 3k+2$.*

Note that Jebrane and Mourtada ([JM94]) forgot, in writing the explicit bound, that they where working with the k^{th} derivative and thus their bound is in fact $2(2k+1)$ instead of $3k+2$. This later bound can be improved using a proof like in proposition 27 and would give that $Cycl(\Gamma_0) = 3k$, i.e. the same as in their logarithmic case.

Część 3. Applications: bifurcation of almost planar twisted homoclinic loop of small codimendion in $\mathbb{R}^n$

In [San93], and also in [San96], Sanstede studies homoclinic loops of vector fields in $\mathbb{R}^n$. He shows that if the vector field is sufficiently smooth and satisfies certain global conditions, then there exists a $C^{k,\min_i\{\mu_i\}-k}$ two-dimensional invariant manifold homeomorphic to a Möbius band, where $\mu = (\mu_1, \mu_2, \ldots, \mu_{n-2})$ are the nonprincipal eigenvalues and $k < \min_i\{\mu_i\}$, i.e. the Möbius band is of class at least $C^{[\min_i\{\mu_i\}]}$.

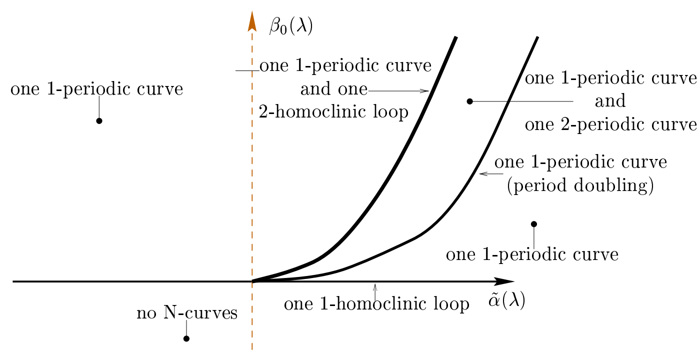
From the remark after proposition 16, we get the following:

Proposition 29. *Let $X_\lambda(x)$ be a strongly 1-resonant vector field in $\mathbb{R}^n$. Let Γ_0 be a homoclinic loop of $X_0(x)$ passing through a hyperbolic saddle at the origin, the saddle having eigenvalues $-1<0<1<\mu_i$, $i=1,\cdots,n-2$. Let the following* *generic geometric assumptions holds:* Γ_0 *is neither of orbit flip nor inclination flip type* [Nau96].

If Γ_0 *is of codimension* $3k$ *and* $\mu_i \geq 8k+2$ *for all* i*, then* $Cycl(\Gamma_0) \leq 3k$*. If* Γ_0 *is of codimension* N *(where* $N = 3k+1$ *or* $3k+2$*) and* $\mu_i \geq 8k+6$ *for all* i*, then* $Cycl(\Gamma_0) \leq 3k+2$*.*

Let us consider the case of codimension 2. In lemma 6.2 of [CDF90], the authors show the existence of a unique C^M-curve of period doubling of the 1-periodic (cf. figure 5), thus the following result is a trivial corollary of proposition 29 and theorem B in [CDF90]:

Corollary 30. *Under the hypothesis of proposition* 29*, if* Γ_0 *is of codimension* 2 *(i.e. of analytic order 1, cf. definition* 11*), then* $Cycl(\Gamma_0) = 2$*, i.e. we have at most one orbit of period 1 and one orbit of period 2 and thus there are no multiple orbits of period 2. The bifurcation diagram is given in figure* 5 *.*



Rysunek 5. Codimension 2 bifurcation diagram. When $\beta_1(0) < -2$, then $\tilde{\alpha}(\lambda) = \alpha_1(\lambda)$, and when $-2 < \beta_1(0) < -1$ then $\tilde{\alpha}(\lambda) = -\alpha_1(\lambda)$.

3.1. Directions for further research

A number of interesting questions remain to be studied. Let us mention some directions for further research.

1. For the non-twisted case it was possible to bound the cyclicity in the high codimension cases even in cases with small values of μ (cf. [RR96]). Indeed the dynamics could be projected on the plane, allowing to by-pass the lack of smoothness of the invariant ring. A similar, although more involved, procedure can be used in the twisted case. The main difference is that we have to work with generalized monomials in two variables: x and y. This result will appear in [GR98].
2. What are exactly the independant coefficients in the development (25), and what is the exact cyclicity?

As a final remark, let us note that theorem 7 can be extended to the study of homoclinic loop bifurcations differentiable surfaces. In fact, on non-orientable surfaces we can have loops for which exists an orientable neighborhood, the bifurcation of which can be treated as in the planar case, and loops for which all sufficiently small neighborhoods are Möbius bands, the bifurcation of which can be treated as we showed in this paper.

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Chapitre 2

SECOND ARTICLE: FINITE CYCLICITY OF FINITE CODIMENSION NONDEGENERATE HOMOCLINIC LOOPS WITH REAL EIGENVALUES IN $\mathbb{R}^3$

L'article *Finite Cyclicity of Finite Codimension Nondegenerate Homoclinic Loops with Real Eigenvalues in* $\mathbb{R}^3$ a été rédigé par Louis-Sébastien Guimond et Christiane Rousseau et a été soumis à *Journal of Dynamics and Differential Equations*.

FINITE CYCLICITY OF FINITE CODIMENSION NONDEGENERATE HOMOCLINIC LOOPS WITH REAL EIGENVALUES IN $\mathbb{R}^3$

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ABSTRACT. In this paper we study homoclinic loops in $\mathbb{R}^3$ which are nondegenerate in the sense of Šil'nikov ([Šil68]) and with real principal eigenvalues in $1:1$ resonance, i.e. loops which have the ***strong inclination property*** and which arise from the equilibrium point along the principal eigenvectors. We are interested here in the higher codimensions. It is known that the dynamics of such systems is given by a 1-dimensional map. Using the ideas exposed in [Gui98], we are able to show that, as for the "nontwisted" loops (cf. [RR96]), this 1-dimensional map admits a nice asymptotic expansion allowing to treat homoclinic loop bifurcations of arbitrarily high codimension and to exhibit an explicit bound for the number of isolated periodic solutions generated under small perturbation. The computations of the bound rely on derivation-division algorithms and Khovanskiĭ's fewnomials theory.

INTRODUCTION

Mathematicians have been interested in homoclinic bifurcations in $\mathbb{R}^3$ for the past thirty years. They are among the simplest "global" bifucations leading to the creation of periodic solutions and can generate complex dynamics: in 1972, Gavrilov and Šil'nikov proved that a homoclinic loop can lead to horseshoes and, in particular, to chaos (cf. [GŠ72]). This is linked to the presence of eigenvalues with large imaginary parts. We consider here the special case of homoclinic loops in $\mathbb{R}^3$, nondegenerate in the sense of Šil'nikov, with real principal eigenvalues in $1:1$ resonance, i.e. loops satisfying the following two properties (if the singular point has two negative eigenvalues): (1) it approaches the equilibrium point (with reverse time) along the principal stable eigenvector; (2) the stable manifold together with its

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tangent space approaches the strong stable manifold along the homoclinic orbit, i.e. in a tubular neighborhood of the invariant 1-manifold the strongly stable manifold is part of the adherence of the stable manifold. In such systems, the bifurcating dynamics is contained in a topological 2-dimensional invariant manifold. ([Hom96], [San93] and [San96], [RR96].) From the view point of chaos, these systems are trivial yielding the hope that the homoclinic loop bifurcates into at most a finite number of periodic orbits. (The invariant 2-manifold is in fact at least C^1, as shown by Sandstede who gave a precise estimate of the class of differentiability.)

The codimension 1 case, when the sum of the principal eigenvalues does not vanish, was studied by Šil'nikov in [Šil63] and [Šil68], and leads under perturbation to a unique periodic orbit. In 1987, Yanagida [Yan87] showed that resonant bifurcation could lead to the birth of periodic curves of period 2. In 1990 Chow, Deng and Fiedler [CDF90] studied the codimension 2 case. Inspired by heuristic arguments on a model (cf. [CDF90, section 2]), their method uses the Šil'nikov's variables and the Lyapunov-Schmidt reduction.

Nondegenerate homoclinic loops (in the sense of Šil'nikov) with the two real principal eigenvalues in $1:1$ resonance were later studied in [RR96] with the following method. By a geometric argument the problem is reduced to the study of homoclinic loop bifurcations on an invariant 2-dimensional manifold (either an oriented annulus, the *nontwisted* case, or a Möbius band, the *twisted* case), yielding the nonexistence of N-orbits for $N > 1$ (nontwisted case) or $N > 2$ (twisted case) (see also [Hom96], [San93] and [KKO93].) The paper then specializes to the nontwisted case: a suitable reduction to normal form allows the exact calculation of the transition map (*Dulac map*) in a neighbourhood of the saddle point and its composition with a C^k–diffeomorphism gives the first return map. The use of a derivation-division algorithm and Khovanskiĭ's fewnomials theory allows to bound the number of fixed points. The method provides a bound for the number of isolated periodic solutions generated under perturbation of the higher codimension homoclinic loops, i.e. what we call *the finite cyclicity property* of the loop and allows to show the finite cyclicity property for all finite codimensions. The optimality of the bound is still an open question which we do not consider here.

The study of the twisted case for small codimension was done in [Gui98] in the case where the Möbius band is sufficiently differentiable. It was done by projecting the dynamics on the band. In the present paper we extend the result to arbitrary finite codimension. Since the Möbius band is not sufficiently differentiable, we do not project the dynamics on the band. We exhibit a bound for the number of isolated periodic solutions generated under perturbation of a twisted homoclinic loop of arbitrary finite codimension (finite cyclicity property).

As we need to study periodic solutions of period 2, it is natural to work with the 2-return map (the second iterate of the Poincaré map) or at least a related function which admits a nice asymptotic expansion. Using the implicit function theorem and geometric constraints, we are able to bring the problem down to the study of a one dimensional map $V_\lambda(t)$ on a compact interval I. We give a definition of the codimension of a homoclinic loop by means of the lowest order term with nonvanishing coefficient of the asymptotic expansion of $V_\lambda(t)$. This definition of the codimension is intrinsic.

To derive the finite cyclicity property we use a blow-up (a method first introduced by Jebrane and Mourtada [JM94]) allowing to divide the discussion in three different cases. The conclusion close to the stable manifold or unstable manifold (end points of I) uses an asymptotic expansion. In the middle region the function $V_\lambda(t)$ is analytic and we use Khovanskiĭ's fewnomials' method of reducing a transcendental system to nondegenerate polynomial ones.

The paper is divided in two parts. The first part contains preliminaries, the definition of codimension and of the function $V_\lambda(t)$. In the second part, we prove the finite cyclicity property of twisted nondegenerate homoclinic loops of finite codimension.

Part 1. The asymptotic expansion of the 1-dimensional map.

1.1. Setting and framework of the problem

Consider a p–parameter family $\mathfrak{X}_\lambda$ of C^∞–vector fields on $\mathbb{R}^3$ which has for $\lambda = 0$ a homoclinic loop Γ_0 through a saddle point at the origin (figure 1). Moreover the origin is a hyperbolic strongly 1-resonant saddle, i.e. the set of eigenvalues of the linearization of $\mathfrak{X}_0$ at the origin of $\mathbb{R}^3$ is $\{\nu_1(0), -\nu_2(0), -\mu(0)\}$ and is such that $0 < \nu_1(0) = \nu_2(0) = 1 < \mu(0)$ and $\mu(0) \notin \mathbb{Q}$ (the only resonances come from $\nu_2(0) = \nu_1(0)$). The resonant monomial u is given by $u = xy$.

Since $(0, 0) \in \mathbb{R}^3 \times \mathbb{R}^p$ is hyperbolic, we take a small neighborhood Λ of $\lambda = 0$ such that the saddle point has eigenvalues $-\nu_1(\lambda) < 0 < \nu_2(\lambda)$.

Definition 1. *The hyperbolic ratio $r(\lambda)$ of the saddle point $(0, \lambda)$ is defined as*

$$r(\lambda) = \frac{\nu_2(\lambda)}{\nu_1(\lambda)}.$$

There exists a C^N–change of coordinates and rescaling of time such that the system defining the family can by written in the neighborhood of the singular point in the following way (cf. theorem 3 in [IY91]):

$$\begin{aligned} \dot{x} &= x \\ \dot{u} &= u\left(\alpha_1(\lambda) + \sum_{i=1}^{K} \alpha_{i+1}(\lambda) u^i\right) \\ \dot{z} &= z\left(-\mu(\lambda) + \sum_{i=1}^{K} \beta_i(\lambda) u^i\right), \end{aligned} \tag{1}$$

where $u = xy$, $\alpha_1(\lambda) = 1 - r(\lambda)$. We can suppose (possibly after scaling) that the normal form is valid in a ball of radius 2.

The first return map (the Poincaré map) is the composition of two maps: a local transition map Δ_λ between two sections to the stable and unstable manifolds, which we call the *Dulac map* as in the planar case and which is defined in a neighborhood U_0 of the singularity and calculated using the normal form coordinates, and a regular map $\mathcal{R}_\lambda$ defined far from the singularity by the flow near Γ_0.

Definition 2. *Let the origin be a saddle point with two negative eigenvalues $-\mu < -\nu_2 < 0$. The homoclinic loop Γ_0 is* ***nondegenerate in the sense of Šil'nikov*** *if it satisfies the following two properties:*

1. *Γ_0 approaches the equilibrium point (with reverse time) along the principal stable eigenvector (i.e. the eigenvector of the eigenvalue $-\nu_2$);*
2. *the stable manifold together with its tangent space approaches the strong stable manifold along the homoclinic orbit, i.e., in a tubular neighborhood of the*

invariant 1-manifold, the strong stable manifold is part of the adherence of the stable manifold.

Let U be a sufficiently small tubular neighborhood of Γ_0. For all $\lambda \in \Lambda \subseteq \mathbb{R}^p$ with Λ a neighborhood of $0 \in \mathbb{R}^p$, let $\Sigma_1 = \{y = 1\}$ be a transversal of $\mathfrak{X}_0$ intersecting the local stable manifold of the origin, and let $\mathrm{T}_1 = \{x = 1\}$ be a transversal of $\mathfrak{X}_0$ intersecting the local unstable manifold of the origin (cf. figure 1). (x, y, z) provides natural parametrizations (x, z) of Σ_1 and (Y_1, Z_1) of T_1 (cf. figure 1).

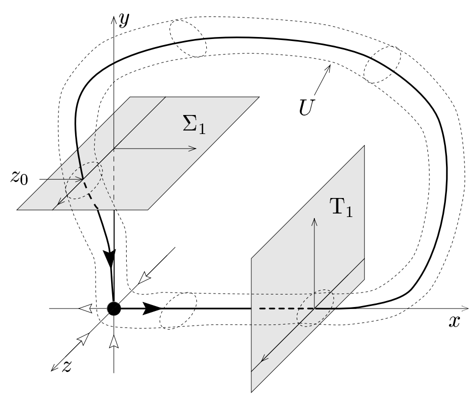


FIGURE 1. The homoclinic loop Γ_0.

The regular transition map $\mathcal{R}_\lambda(Y_1, Z_1)$ from T_1 to Σ_1 is a C^K–orientation preserving diffeomorphism:

$$\mathcal{R}_\lambda(Y_1, Z_1) = \begin{pmatrix} C_0(\lambda) + \sum_{i+j>0}^{K} C_{ij}(\lambda) Y_1^i Z_1^j \\ D_0(\lambda) + \sum_{i+j>0}^{K} D_{ij}(\lambda) Y_1^i Z_1^j \end{pmatrix} + \check{R}_\lambda(Y_1, Z_1), \tag{2}$$

where $C_{10}(0)D_{01}(0) - C_{01}(0)D_{10}(0) > 0$ (orientation preserving), $C_0(0) = 0$ but $D_0(0) = z_0$ need not vanish, and $\check{R}_\lambda(Y_1, Z_1)$ is C^K and K-flat at $Y_1 = 0 = Z_1$.

Lemma 3. *System* (1) *is nondegenerate if* $C_{10}(0) \neq 0$. *The loop is* twisted (*resp.* nontwisted) *if* $C_{10}(0) < 0$ (*resp.* $C_{10}(0) > 0$). *The stable manifold for the twisted case is illustrated in figure* 2.

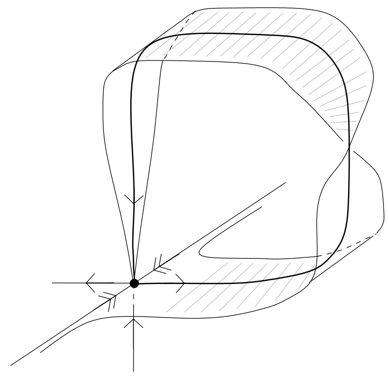


FIGURE 2. The invariant stable manifold in the twisted case.

Definition 4.

1. *Let* $\{\mathfrak{X}_\lambda\}_{\lambda\in\Lambda}$ *be a family of* C^K *vector fields on* $\mathbb{R}^3$ *such as in our framework. We say that* Γ_0 *has* ***finite cyclicity*** *in the family* $\{\mathfrak{X}_\lambda\}_{\lambda\in\Lambda}$ *if there exists* $N \in \mathbb{N}$, $\epsilon > 0$ *and a neighborhood* Λ_0 *of* λ_0 *in* Λ *such that for all* $\lambda \in \Lambda_0$, *the number* $n(\epsilon, \lambda)$ *of isolated periodic orbits* γ *of* $\mathfrak{X}_\lambda$ *with* $\mathrm{dist}_H(\gamma, \Gamma) \le \epsilon$ *is less than* N, *where* dist_H *is the Hausdorff distance on compact sets.*
2. *Let*
$$n(\epsilon, \Lambda_0) = \sup_{\lambda\in\Lambda_0} \{n(\epsilon, \lambda)\}.$$
The ***cyclicity of*** $\mathbf{\Gamma_0}$ *in the family* $\{\mathfrak{X}_\lambda\}_{\lambda\in\Lambda}$ *is the minimum integer* $n(\epsilon, \Lambda_0)$ *when* ϵ *and the diameter of* Λ_0 *go to* 0. *We note it* $Cycl(\Gamma_0, \mathfrak{X}_\lambda)$.
3. *We say that* Γ_0 *has* ***absolute finite cyclicity*** *if there exists a finite upper bound to all* $n(\epsilon, \Lambda_0)$ *in any family* $\{\mathfrak{X}_\lambda\}_{\lambda\in\Lambda}$ *and sufficiently small* ϵ *and we note it* $Cycl(\Gamma_0)$.

1.2. THE RETURN MAP

Let $\mathcal{P}_\lambda = (\mathcal{P}_{1,\lambda}, \mathcal{P}_{2,\lambda})$ be the first return map on T_1, i.e. $\mathcal{P}_\lambda \overset{def}{=} \Delta_\lambda \circ \mathcal{R}_\lambda$.

The transition maps for planar systems have been thoroughly studied. Roussarie for instance uses generalized monomials which are well-ordered and behave adequately under differentiation (cf. [Rou86], [Mou89], and [Rou98]). These monomials have the form $x^i\omega^j(x,\lambda)$ where:

$$\alpha_1(\lambda) = 1 - r(\lambda) \tag{3}$$

$$\omega(x,\lambda) = \begin{cases} \dfrac{x^{-\alpha_1(\lambda)} - 1}{\alpha_1(\lambda)} & \text{if} \quad \alpha_1(\lambda) \neq 0 \\ -\ln(x) & \text{if} \quad \alpha_1(\lambda) = 0 \end{cases} \tag{4}$$

The generalized monomials have the property for all $k > 0$:

$$\lim_{\alpha_1(\lambda)\to 0} x^k\omega^j(x,\lambda) = -x^k \ln^j(x),$$

and this holds uniformly on $[0, X]$ for any fixed $X > 0$.

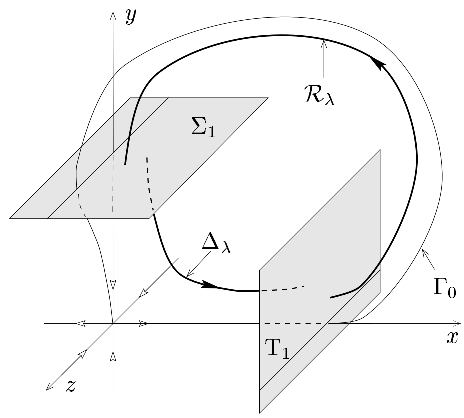


FIGURE 3. The maps $\mathcal{R}_\lambda$ and Δ_λ.

Definition 5.

1. ([Mou89]). *Let* $K \in \mathbb{N}$, $\psi(x,\lambda)$ *a* C^K*–function on* $]0,\epsilon[\times\Lambda_0$ *such that* $\psi(0,0) = 0$*, and a positive continuous function* $\xi(x,\lambda)$ *with* $\xi(0,\lambda) = 0$*. We say that* $\psi(x,\lambda)$ *is* $I_0^K\big(\xi(x,\lambda)\big)$ *if for every* $n \in \mathbb{N}$ *such that* $n \leq K$*, we have*
$$\lim_{x\to 0} \xi^n(x,\lambda)\frac{\partial^n\psi(x,\lambda)}{\partial x^n} = 0$$
uniformly on Λ_0*.*
2. *Let* $\psi(x,\lambda) \in I_0^K\big(\xi(x,\lambda)\big)$*. We say that* $\psi(x,\lambda) \in J_0^K\big(\xi(x,\lambda)\big)$ *if for every* $n \in \mathbb{N}$ *such that* $n \leq K$*, we have*
$$\lim_{\lambda\to 0} \frac{\partial^n\psi(x,\lambda)}{\partial x^n} = 0$$
uniformly on $[0,X]$ *for all fixed* X*.*

The generalized monomials $x^k\omega(x,\lambda)$ are $I_0^K\big(\rho(x,\lambda)\big)$, where $\rho(x,\lambda) = x^{1+\alpha_1(\lambda)} \times \omega(x,\lambda)$.

Lemma 6 ([JM94])**.** *Let* $f(x,\lambda)$ *be a* C^K*–function on* $[0,x_0[\times\Lambda$ *such that* $f(0,\lambda) = 0$*. Then there exists a* C^K*–function,* $g(x,\lambda)$*, with* $g(0,\lambda) = 0$ *and such that for all* $a > 0$*, we have*

$$\omega\big(ax(1+f),\lambda\big) = \Big[1 + O(\alpha_1(\lambda))\Big]\omega(x,\lambda) + g(x,\lambda) - \ln(a)\Big[1 + O\big(\alpha_1(\lambda)\big)\Big]. \tag{5}$$

The Dulac map $\Delta_\lambda = (\Delta_{1,\lambda},\Delta_{2,\lambda})$ from Σ_1 to T_1 has the following form [RR96]:

$$\Delta_\lambda(x,z) = \begin{bmatrix} x + \sum_{i=1}^{K}\alpha_i(\lambda)x^i\omega(x,\lambda)\big(1+\psi_i(x,\lambda)\big) + \phi_{1,K}(x,\lambda) \\ zx^{\mu(\lambda)}\big(1+\varphi_{2,K}(x,\lambda)\big) \end{bmatrix} = \begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix}, \tag{6}$$

where $\psi_i(x,\lambda)$ are $I_0^{K-i}\big(\rho(x,\lambda)\big)$, $\varphi_{2,K}(x,\lambda)$ is $I_0^K\big(\rho(x,\lambda)\big)$, and $\phi_{1,K}(x,\lambda)$ is C^K and K–flat at $x = 0$.

Note that the Dulac map is not one to one for points of the form $(0,z)$. The inverse $\Delta_\lambda^{-1}(Y_1,Z_1)$ of the Dulac map $\Delta_\lambda(x,z)$ is computed by writing equation (6) for the system with reverse time. In fact, for points of T_1 with $Y_1>0$, it is of the form:

$$(7)\quad \Delta_\lambda^{-1}(Y_1,Z_1)=\begin{bmatrix} Y_1+\sum_{i=1}^{K}\overline{\alpha}_i(\lambda)Y_1^i\overline{\omega}(Y_1,\lambda)\big(1+\overline{\psi}_i(Y_1,\lambda)\big)+\overline{\phi}_{1,K}(Y_1,\lambda)\\ Z_1Y_1^{-\mu(\lambda)}\big(1+\overline{\varphi}_{2,K}(Y_1,\lambda)\big)\end{bmatrix} = \begin{pmatrix} x\\ z\end{pmatrix}$$

where functions $\overline{\psi}_i(Y_1,\lambda)$ are $\overline{I}_0^{K-i}\big(\overline{\rho}(Y_1,\lambda)\big)$, $\overline{\varphi}_{2,K}(Y_1,\lambda)$ are $\overline{I}_0^{K}\big(\overline{\rho}(Y_1,\lambda)\big)$, and $\overline{\phi}_{1,K}(Y_1,\lambda)$ is C^K and K–flat at $Y_1=0$. Moreover $\overline{\alpha}_1(\lambda)=-\alpha_1/(1-\alpha_1)$, $\overline{\alpha}_i(\lambda)=-\alpha_i(\lambda)+\overline{p}_i(\lambda)$ where $\overline{p}_i(\lambda)$ is some polynomial in the $\alpha_{i'}$ with $i'<i$. Note that $\Delta_{2,\lambda}^{-1}(Y_1,Z_1)$ is not defined at $Y_1=0$, however $Z_1Y_1^{-\mu(\lambda)}$ is bounded on $\Delta_\lambda(\Sigma_1)$, the region of T_1 where N–curves can appear.

As mentioned in [GR97], a change of coordinates

$$(8)\qquad (x,y,z)\to(\overline{x},\overline{y},\overline{z})$$

tangent to the identity preserving the type of the normal form (1) generates a pair of maps f_λ^1 and f_λ^2 such that

$$(9)\qquad f_\lambda^1\circ\Delta_\lambda\circ f_\lambda^2=\overline{\Delta}_\lambda,$$

where $\overline{\Delta}_\lambda$ is the Dulac map expressed in the $(\overline{x},\overline{y},\overline{z})$ coordinates.

Lemma 7. *In equation (9), the maps f_λ^1 and f_λ^2 have the following form:*

$$(10)\qquad f_\lambda^1(Y,Z)=\left(Y\left(1+\sum_{i=1}^{K}a_i^1(\lambda)Y^i\right),\ Z\left(1+\sum_{i=1}^{K}b_i^1(\lambda)Y^i\right)\right),$$

and

$$(11)\qquad f_\lambda^2(x,z)=\left(x\left(1+\sum_{i=1}^{K}a_i^2(\lambda)x^i\right),\ z\left(1+\sum_{i=1}^{K}b_i^2(\lambda)x^i\right)\right).$$

Proof. Let

$$(12)\qquad f_\lambda^1(Y,Z)=\left(Y\left(1+\sum_{i+j=1}^{K}a_{ij}^1(\lambda)Y^iZ^j\right),\ Z\left(1+\sum_{i+j=1}^{K}b_{ij}^1(\lambda)Y^iZ^j\right)\right)$$

and

$$(13)\qquad f_\lambda^2(x,z)=\left(x\left(1+\sum_{i+j=1}^{K}a_{ij}^2(\lambda)x^iz^j\right),\ z\left(1+\sum_{i+j=1}^{K}b_{ij}^2(\lambda)x^iz^j\right)\right).$$

We want to find coefficients $a_{ij}^\ell(\lambda)$ and $b_{ij}^\ell(\lambda)$ such that the relation $f_\lambda^1\circ\Delta_\lambda\circ f_\lambda^2=\overline{\Delta}_\lambda$ holds. Looking at the coefficients of $x^{j\mu}$ in $f_\lambda^1\circ\Delta_\lambda\circ f_\lambda^2$, one obtains that $a_{ij}^1(\lambda)=a_{ij}^2(\lambda)=0$ and $b_{ij}^1(\lambda)=b_{ij}^2(\lambda)=0$ for $j>0$. □

1.3. Geometric preliminaries

The hypotheses we made on the problem impose important geometric constraints on the bifurcating dynamics.

Definition 8. *Let Γ_0 be a homoclinic loop of $\mathfrak{X}_0(x)$. Fix U a small tubular neighborhood of Γ_0. Assume $\overline{\Gamma} \subseteq U$ with Γ some orbit of $\mathfrak{X}_\lambda(x)$ intersecting a section of U N–times.*

1. *If Γ is an homoclinic loop then it is called an* ***N-homoclinic loop****.*
2. *If Γ is a periodic curve then it is called an* ***N–periodic curve****.*
3. *An* ***N-curve*** *is either an N-homoclinic loop or an N–periodic curve.*

As long as U is chosen small enough, the above definitions are independent of the choice of U.

Facts 9. *In our framework, we have the following facts:*

1. *There exists a $C^{[\mu]}$–Möbius band depending on λ and containing the bifurcating dynamics (cf.* [San93] *and* [San96]*).*
2. *If there is a 2-curve on the Möbius band then there is one and only one 1–periodic curve that coexists with the 2-curve.*
3. *The cyclicity of Γ_0 is bounded by 1 plus the number of 2-curves bifurcating from Γ_0.*
4. *Denote by $\beta(\lambda) = \big(C_0(\lambda), D_0(\lambda)\big)$ the first intersection of W^u with Σ_1 (cf. figure* 4*). A necessary condition for the existence of periodic solutions is $C_0(\lambda) > 0$.*
5. *All fixed points $(Y_1, Z_1) \in \mathrm{T}_1$ of the 2-return map satisfy, for λ sufficiently small, $R_{1,\lambda}(Y_1, Z_1) \in \big[0, C_0(\lambda)\big]$.*

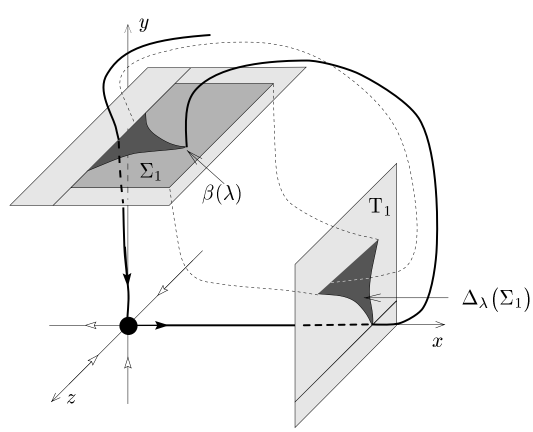


Figure 4. Parts of the bifurcated 1-homoclinic loops in $\mathbb{R}^3$.

1.4. Main result

Definition 10 ([RR96])**.** *The generalized monomials $\{1, x^{i+j\mu}\omega^\ell(x,\lambda) \mid 1 \le i+j \le K, 0 \le \ell \le i,$ and $\ell \le 1$ if $j = 0\}$ are totally ordered with respect to flatness at $x = 0$*

in the following way:

$$x^{i+j\mu}\omega^{\ell}(x,\lambda) \prec x^{i'+j'\mu}\omega^{\ell'}(x,\lambda) \Longleftrightarrow \begin{cases} i+j\mu < i'+j'\mu & \text{or} \\ i+j\mu = i'+j'\mu & \text{and } \ell > \ell'. \end{cases} \tag{14}$$

We will only be working with monomials of the form $(i,0,\ell)$ and $(i,j,0)$.

Definition 11. *Let $k(i_1,i_2,j,\ell)$ denote the number of generalized monomials of the form $(i,0,\ell)$ and $(i,j,0)$ and of order lower than $x^{i_1+ji_2+j\mu}\omega^{\ell}(x,\lambda)$.*

Here are some orders (depending on the value of μ):

$$1 \prec x\omega \prec x \prec x^{\mu} \prec x^2\omega \prec x^2 \prec x^{1+\mu} \prec x^{2\mu} \prec x^3\omega \prec \cdots \quad \text{, if } 1<\mu<1.5 \tag{15}$$

$$1 \prec x\omega \prec x \prec x^{\mu} \prec x^2\omega \prec x^2 \prec x^{1+\mu} \prec x^3\omega \prec x^3 \prec \cdots \quad \text{, if } 1.5<\mu<2 \tag{16}$$

$$1 \prec x\omega \prec x \prec x^2\omega \prec x^2 \prec x^{\mu} \prec x^3\omega \prec x^3 \prec x^{1+\mu} \prec \cdots \quad \text{, if } 2<\mu<3. \tag{17}$$

In equations (15) and (16), $k(2,0,0,1)=4$ and in equation (17), $k(2,0,0,1)=3$.

Definition 12. *Let Γ_0 and equations* (2) *and* (6).

1. Γ_0 *is nondegenerate of* ***finite codimension*** *if it is not degenerate in the sense of Šil'nikov* [Šil68] *and one of the following generic conditions holds:*
   (a) $\alpha_1(0)=0$ *and* $C_{10}(0)\neq -1$*, we say that* Γ_0 *is of type* $(1,0,0,0)$.
   (b) $\exists\, I_1$ *such that* $C_{10}(0)=-1$, $\alpha_i(0)=C_{i0}(0)=0$ *for all* $i<I_1$, $C_{ij}(0)D_{\ell 0}(0)=0$ *for all* $i+j\ell+j\mu<I_1$*, and* $\alpha_{I_1}(0)\neq 0$*, we say that* Γ_0 *is of type* $(I_1,0,0,1)$.
   (c) $\exists\, I_1$ *such that* $C_{10}(0)=-1$, $\alpha_i(0)=C_{i0}(0)=0$ *for all* $i<2I_1+1$, $C_{ij}(0)D_{\ell 0}(0)=0$ *for all* $i+j\ell+j\mu<2I_1+1$, $\alpha_{2I_1+1}(0)=0$*, and* $C_{2I_1+1,0}(0)\neq 0$*, we say that* Γ_0 *is of type* $(2I_1+1,0,0,0)$.
   (d) $\exists\, I_1,I_2,J$*, with* $J>0$*, such that* $C_{10}(0)=-1$, $\alpha_i(0)=C_{i0}(0)=0$ *for all* $i<I_1+I_2J+J\mu$, $C_{ij}(0)D_{\ell 0}(0)=0$ *for all* $i+j\ell+j\mu<I_1+I_2J+J\mu$*, and* $C_{I_1J}(0)D_{I_20}(0)\neq 0$*, we say that* Γ_0 *is of type* $(I_1,I_2,J,0)$.
2. *Let* Γ_0 *be of finite type. If* (I_1,I_2,J,L) *is the type of* Γ_0*, then* Γ_0 *is said to be of* ***codimension*** k *with* $k=k(I_1,I_2,J,L)$.

Proposition 13. *Conditions* 1a-1d *are intrinsic.*

Proof. Using lemma 7 we can simplify the expression of $\mathcal{R}_\lambda$ so that for the first nonvanishing $C_{i'0}$, i' is odd. Moreover, for each $j>0$, the first nonvanishing C_{ij} is intrinsic; the first nonvanishing D_{i0} is intrinsic. Also the first nonvanishing α_i is intrinsic. Indeed, the action of maps as the f_λ^i of lemma 7 allows to simplify the expression of $\mathcal{R}_\lambda$. In the case $C_{10}(0)<0$, we can choose f_λ^i such that:

$$\begin{aligned} &f_\lambda^1 \circ \mathcal{R}_\lambda \circ f_\lambda^{-1}(Y_1,Z_1) \\ &= \begin{pmatrix} C_0(\lambda)+\sum_{i=1}^{K} C_{i,0}(\lambda)Y_1^i + \sum_{\substack{0<i+j<K\\ j>0}} C_{ij}(\lambda)Y_1^iZ_1^j \\ D_0(\lambda)+\sum_{i+j>0}^{K} D_{ij}(\lambda)Y_1^iZ_1^j \end{pmatrix} + \hat{R}_\lambda(Y_1,Z_1), \end{aligned} \tag{18}$$

where, if there exist i' such that $C_{i'0}(0)\neq 0$, the minimum of such i' is odd. □

The finite cyclicity property can be stated in the following way:

Theorem 14. *If Γ_0 is of codimension $k(I_1, I_2, J, L)$, $I = I_1 + JI_2$ and $n = 2(I + [J\mu]) + 1$. Then*

$$Cycl(\Gamma_0) \leq \left[\frac{n(4n^2 + 16n + 37) + 3}{4}\right] = n^3 + 4n^2 + 9n + \left[\frac{n+3}{4}\right].$$

For all values of $\mu > 1$, the condition to have a homoclinic loop of codimension 1 is $\alpha_1 \neq 0$. If $\alpha_1 = 0$ and $C_{10} \neq -1$, then Γ_0 is of codimension 2. For all higher codimensions, the conditions depend on the value of μ. In table 1, we give the conditions for small codimensions.

From now on we will assume that Γ_0 has finite codimension k, i.e. that there exists I_1, I_2, J and L such that Γ_0 is of type (I_1, I_2, J, L).

1.5. New Parametrization on T_1 and the blow-up

It was shown in the study of the C^K–Möbius band (cf. [Gui98]) that the parametrization can play a key role in finding a suitable blow-up that allows us to divide our study in several regions and overcome the difficulties.

Following the ideas introduced by Jebrane and Mourtada in [JM94] and the techniques used in [Gui98], we look for a "good" parametrization of the transversals Σ_1 and T_1.

Let $(Y_1, Z_1) \in \mathrm{T}_1$. We note by (x, z) its image on Σ_1 by the diffeomorphism $\mathcal{R}_\lambda$, i.e. $(x, z) = \mathcal{R}_\lambda(Y_1, Z_1)$. Then from equation (2) we have the following:

$$\mathcal{R}_\lambda(Y_1, Z_1) = \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} C_0(\lambda) \\ D_0(\lambda) \end{pmatrix} + \begin{pmatrix} f_{1,\lambda}(Z_1) \\ f_{2,\lambda}(Z_1) \end{pmatrix} + \begin{pmatrix} r_{1,\lambda}(Y_1, Z_1) \\ r_{2,\lambda}(Y_1, Z_1) \end{pmatrix}, \tag{19}$$

where $f_{i,\lambda}(0) = 0 = r_{i,\lambda}(0, Z)$. Let

$$\Theta(Y_1, Z_1) = \begin{pmatrix} -r_{1,\lambda}(Y_1, Z_1) \\ Z_1 \end{pmatrix}, \tag{20}$$

and set $(Y, Z) = \Theta(Y_1, Z_1)$ as the new parameterization of T_1. Since by hypothesis $C_{10}(0) \neq 0$, we have:

$$Jac_{(Y_1,Z_1)}(Y, Z)(0, 0) = \begin{vmatrix} -C_{10}(\lambda) & 0 \\ 0 & 1 \end{vmatrix} = -C_{10}(\lambda) > 0, \tag{21}$$

for all $\lambda \in \Lambda$. We can thus inverse equation (20). We obtain a solution $(Y_1, Z_1) = \Theta^{-1}(Y, Z)$, where

$$\Theta^{-1}(Y, Z) = \begin{pmatrix} \displaystyle\sum_{\substack{1 \leq i+j \leq K \\ i>0}} \eta_{ij}(\lambda) Y^i Z^j + Y \cdot a_{K,\lambda}(Y, Z), \\ Z \end{pmatrix} \tag{22}$$

in which $a_{K,\lambda}(Y, Z)$ is C^{K-1} and $(K-1)$–flat at $(0, 0)$.

Lemma 15. *The coefficients $\eta_{ij}(\lambda)$ in equation (22) are given below.*

1. $\eta_{10}(\lambda) = \big(-C_{10}(\lambda)\big)^{-1}$;
2. $\eta_{ij}(\lambda) = C_{ij}(\lambda)\eta_{10}^{i+1}(\lambda) + P_{ij}(\lambda)$, *where $P_{ij}(\lambda)$ is a polynomial in $C_{10}^{-1}(\lambda)$ and the $C_{\ell m}(\lambda)$ with $(l, m) \prec (i, j)$.*

Proof. We simply substitute equation (20) in equation (22).

codim / values of μ	3	4	5	6	7	8
$1<\mu<1.5$	$\alpha_1=0$ $C_{10}=-1$ $C_{01}D_0\neq 0$	$\alpha_1=0$ $C_{10}=-1$ $C_{01}D_0=0$ $\alpha_2\neq 0$	$\alpha_1=\alpha_2=0$ $C_{10}=-1$ $C_{01}=0$ $C_{11}D_0\neq 0$ ——— $\alpha_1=\alpha_2=0$ $C_{10}=-1$ $D_0=0$ $C_{01}D_{10}\neq 0$	$\alpha_1=\alpha_2=0$ $C_{10}=-1$ $C_{01}D_0=0$ $C_{01}D_{10}=0$ $C_{11}D_0=0$ $C_{02}D_0\neq 0$	$\alpha_1=\alpha_2=0$ $C_{10}=-1$ $C_{01}D_0=0$ $C_{11}D_0=0$ $C_{01}D_{10}=0$ $C_{02}D_0=0$ $\alpha_3\neq 0$	$\alpha_1=\alpha_2=\alpha_3=0$ $C_{10}=-1$ $C_{01}D_0=0$ $C_{11}D_0=0$ $C_{01}D_{10}=0$ $C_{02}D_0=0$ $C_{30}\neq 0$
$1.5<\mu<2$	$\alpha_1=0$ $C_{10}=-1$ $C_{01}D_0\neq 0$	$\alpha_1=0$ $C_{10}=-1$ $C_{01}D_0=0$ $\alpha_2\neq 0$	$\alpha_1=\alpha_2=0$ $C_{10}=-1$ $C_{01}=0$ $C_{11}D_0\neq 0$ ——— $\alpha_1=\alpha_2=0$ $C_{10}=-1$ $D_0=0$ $C_{01}D_{10}\neq 0$	$\alpha_1=\alpha_2=0$ $C_{10}=-1$ $C_{01}D_0=0$ $C_{11}D_0=0$ $C_{01}D_{10}=0$ $\alpha_3\neq 0$	$\alpha_1=\alpha_2=\alpha_3=0$ $C_{10}=-1$ $C_{01}D_0=0$ $C_{11}D_0=0$ $C_{01}D_{10}=0$ $C_{30}\neq 0$	
$2<\mu<3$	$\alpha_1=0$ $C_{10}=-1$ $\alpha_2\neq 0$	$\alpha_1=\alpha_2=0$ $C_{10}=-1$ $C_{01}D_0\neq 0$	$\alpha_1=\alpha_2=0$ $C_{10}=-1$ $C_{01}D_0=0$ $\alpha_3\neq 0$	$\alpha_1=\alpha_2=\alpha_3=0$ $C_{10}=-1$ $C_{01}D_0=0$ $C_{30}\neq 0$		
$\mu>3$	$\alpha_1=0$ $C_{10}=-1$ $\alpha_2\neq 0$	$\alpha_1=\alpha_2=0$ $C_{10}=-1$ $\alpha_3\neq 0$	$\alpha_1=\alpha_2=\alpha_3=0$ $C_{10}=-1$ $C_{30}\neq 0$			

TABLE 1. Conditions for small codimensions.

1. For the coefficient of Y, we obtain the relation

$$-\eta_{10}(\lambda)C_{10}(\lambda) = 1. \tag{23}$$

2. Using induction, we obtain that the coefficient of $Y_1^I Z_1^J$ is given by the relation

$$(-1)^I C_{10}^I(\lambda)\eta_{IJ}(\lambda) + P'_{ij}(\lambda) - \eta_{10}(\lambda)C_{IJ}(\lambda) = 0. \tag{24}$$

□

Let us note by $\tilde{R}_\lambda(Y,Z) = \mathcal{R}_\lambda \circ \Theta^{-1}(Y,Z)$ the expression of $\mathcal{R}_\lambda$ in the new parameterization (20):

$$\tilde{R}_\lambda(Y,Z) = \begin{pmatrix} C_0(\lambda) \\ D_0(\lambda) \end{pmatrix} + \begin{pmatrix} f_{1,\lambda}(Z) - Y \\ f_{2,\lambda}(Z) + r_{2,\lambda}\big(Y_1(Y,Z), Z\big) \end{pmatrix}, \tag{25}$$

where every function is C^K in (Y,Z,λ).

Consider the displacement map:

$$\tilde{\delta}_\lambda(Y,Z) = \begin{pmatrix} \tilde{\delta}_{1,\lambda}(Y,Z) \\ \tilde{\delta}_{2,\lambda}(Y,Z) \end{pmatrix} = \tilde{G}_\lambda(Y,Z) - \tilde{\Delta}_\lambda^{-1}(Y,Z) \tag{26}$$

$$\begin{pmatrix} \tilde{G}_\lambda(Y,Z) \stackrel{\mathrm{def}}{=} \mathcal{R}_\lambda \circ \Delta_\lambda \circ \mathcal{R}_\lambda \circ \Theta^{-1}(Y,Z) \\ \tilde{\Delta}_\lambda^{-1}(Y,Z) \stackrel{\mathrm{def}}{=} \Delta_\lambda^{-1} \circ \Theta^{-1}(Y,Z) \end{pmatrix}.$$

The map $\tilde{\delta}_\lambda(Y,Z)$ has, for small values of the parameter, the same number of zeroes as the 2-return map.

Let $X = C_0(\lambda) + f_{1,\lambda}(Z) - Y = \tilde{R}_{1,\lambda}(Y,Z)$. Then from equations (6) and (25), we have that:

$$\begin{aligned} &\Delta_\lambda \circ \tilde{R}_\lambda(Y,Z) \\ &= \begin{bmatrix} X + \displaystyle\sum_{i=1}^{K} \alpha_i X^i \omega(X,\lambda)\big(1 + \psi_{i,\lambda}(X)\big) + \phi_{1,K,\lambda}(X) \\ X^\mu \left(\displaystyle\sum_{i+j=0}^{K} D_{ij}(\eta_{10}Y)^i Z^j \big((1 + O(\lambda) + \varphi_{ij,\lambda}(X,Z)\big) + \phi_{2,K,\lambda}(X,Z) \right) \end{bmatrix} \end{aligned} \tag{27}$$

where $D_{00}(\lambda) = D_0(\lambda)$, every coefficient is a function in λ, $\varphi_{ij,\lambda}(X,0)$ is $I_0^K(X)$, $\varphi_{ij,\lambda}(0,Z)$ and $\phi_{i,K,\lambda}(X,Z)$ are C^K and K-flat.

Since the map $(Y,Z) \to (X,Z)$ is a diffeomorphism, we can work with either system of coordinates.

The main step in bounding the number of zeroes of equation (26) is to use an adequate blow-up that will allow us to:

- extend the function $\tilde{\Delta}_\lambda^{-1}(Y,Z)$ to $Y = 0$.
- reduce the dynamics to that of a 1-dimensional map defined on the unit interval via the implicit function theorem;
- divide the study of this dynamics in several regions in order to avoid to have simultaneously X and Y small.

We will blow-up the variables X and Z (this will induce a blow-up of the Y variable).

Before introducing the blow-up, we notice that the system in normal form (1) is invariant under coordinate changes of the form:

$$(x,y,z) = (x,y,Az), \tag{28}$$

so we can assume either $z_0 = 0$ or $z_0 = 1/2$. (We choose $z_0 = 1/2$ instead of $z_0 = 1$ because this allows to work in the region $|z| < B < 1$). Also, from equation (6), we have:

$$\Delta_{2,\lambda}(x,z) = zx^{\mu(\lambda)}\left[1 + \varphi_{2,K,\lambda}(x)\right], \tag{29}$$

where, by fact 9.5, $x \in [0, C_0(\lambda)]$ and, since we are working in a small neighborhood of $(0, z_0) \in \Sigma_1$, $z \in [z_0 - \epsilon_0, z_0 + \epsilon_0]$ with $\epsilon_0 > 0$ as small as we want. We have that:

$$|Z| = |\Delta_{2,\lambda}(x,z)| \leq \big(|z_0| + \epsilon_0\big)x^{\mu(\lambda)}[1 + \epsilon_1] \leq x^{\mu(\lambda)}, \tag{30}$$

where $C_0(\lambda) \leq \epsilon_1$ for all $\lambda \in \Lambda$, i.e. N–curves intersect transversals T_1 and Σ_1 in specific regions which we call *domains of interest*.

This suggests the blow-up $(X, Z) = \Phi(s,t)$, where:

$$\Phi(s,t) = \begin{pmatrix} tC_0(\lambda) \\ sC_0^{\mu(\lambda)}(\lambda) \end{pmatrix} = \begin{pmatrix} X \\ Z \end{pmatrix} \tag{31}$$

where, by fact 9.5, $t \in [0,1]$.

This blow-up has two important consequences:

1. In the blow-up coordinates, the point corresponding to Γ_0 has coordinates $(s,t) = (s_0, 0)$, where $s_0 = z_0\big(-C_{10}(0)\big)^{-\mu}$. Indeed, it is clear, from the construction of the blow-up, that $t = 0$ corresponds to the z–axis on Σ_1. Its inverse image by $\mathcal{R}_\lambda$ yields the upper bound of the coordinate Y of the domain of interest on T_1. This is reflected in the formula (25.1):

$$Y = C_0(\lambda) + f_{1,\lambda}(Z) - X = C_0(\lambda)(1-t) + f_{1,\lambda}(Z). \tag{32}$$

   On the other hand,

$$\begin{aligned} \Delta_{2,\lambda}^{-1}\big(Y_1(Y,Z), Z\big) &= \tilde{\Delta}_{2,\lambda}^{-1}(Y,Z) \\ &= \tilde{\Delta}_{2,\lambda}^{-1}\big(tC_0(\lambda), sC_0^{\mu(\lambda)}(\lambda)\big)\Big|_{|\lambda|=0=t} \\ &= s_0\big(-C_{01}(0)\big)^{\mu} \\ &= \tilde{R}_{2,\lambda}\big(tC_0(\lambda), sC_0^{\mu(\lambda)}(\lambda)\big)\Big|_{|\lambda|=0=t} \\ &= z_0. \end{aligned} \tag{33}$$

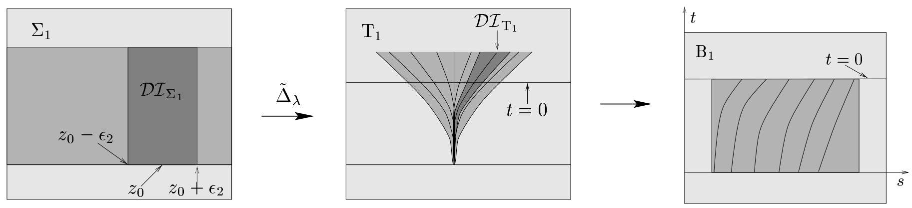
   Equation (33) also implies that $s\big(-C_{10}(0)\big)^{\mu} \in \big[z_0 - \epsilon_0, z_0 + \epsilon_0\big]$.
2. Geometrically, this blow-up acts on T_1 by separating curves with different asymptotic behavior at $Y = 0$ allowing an extension, in the domain of interest on Σ_1, of the diffeomorphism $\tilde{\Delta}_\lambda(x,z)$ to the values with $x = 0$ (cf. figure 5 for the case $z_0 = 1/2$): the domain of interest on Σ_1 ($\mathcal{DI}_{\Sigma_1}$) is illustrated in figure 5(a) ; the image of $\mathcal{DI}_{\Sigma_1}$ by Δ_λ, noted $\mathcal{DI}_{\mathrm{T}_1}$ is illustrated in figure 5(b); finally, we illustrate $\mathcal{DI}_{\mathrm{T}_1}$ in the blown-up coordinates (s,t).

1.6. Dividing the study in two regions

We study the zeroes of equation (26) in the blown-up coordinates for $t \in [0,1]$. It is convenient to divide the study into the following three regions:

$$t \in [0,\epsilon] \quad , \quad t \in [\epsilon, 1-\epsilon] \quad , \text{ and} \quad t \in [1-\epsilon, 1]. \tag{34}$$

In this section we will show that it is sufficient to only consider the values $t \in [0, 1-\epsilon]$, i.e. only the two regions $t \in [0,\epsilon]$ and $t \in [\epsilon, 1-\epsilon]$.



(a) Domain of interest on Σ_1: $\mathcal{DI}_{\Sigma_1}$.

(b) Domain of interest on T_1: $\mathcal{DI}_{\mathrm{T}_1}$.

(c) Blow-up of the domain of interest on T_1.

FIGURE 5. Effect of the blow-up on $\tilde{\Delta}_\lambda(\Sigma_1)$ in the case $z_0 = 1/2$.

Definition 16. *The notation $\mathcal{O}_{A,\lambda}$ is used to denote a function which is at least $J_0^{K-2(I_1+JI_2+[J\mu]+1)}$. (A is a multi-index to number such functions.)*

Let $U(t) = C_0(\lambda)(1-t)$. For all $0 < \epsilon < 1$ we have that in the blown-up coordinates and for $t \in [0, 1-\epsilon]$:

$$Y = C_0(\lambda)(1-t) + f_{1,\lambda}\big(C_0^\mu(\lambda)s\big) = C_0(\lambda)(1-t)\left(1 + \frac{f_{1,\lambda}\big(C_0^\mu(\lambda)s\big)}{C_0(\lambda)(1-t)}\right) = U(t)\big(1 + \mathcal{O}_{1,\lambda}(s,t)\big), \tag{35}$$

and:

$$\begin{aligned} Y_1 &= \eta_{10} U(t)\big(1 + \mathcal{O}_{2,\lambda}(s,t)\big) \\ &= \sum_{i+j=1}^{K} \eta_{ij} C_0^{i+j\mu}(1-t)^i s^j \big(1 + \mathcal{O}_{3,i,j,\lambda}(s,t)\big) + C_0^K \mathcal{O}_{4,K,\lambda}(s,t). \end{aligned} \tag{36}$$

Lemma 17.

$$\begin{aligned} Y_1^i \overline{\omega}(Y_1,\lambda) = \eta_{10}^{i-\overline{\alpha}_1(\lambda)} U^i(t)\overline{\omega}\big(U(t),\lambda\big)\big(1 + \mathcal{O}_{5,i,\lambda}(s,t)\big) \\ + \eta_{10}^i U^i(t)\Big(\overline{\omega}(\eta_{10},\lambda)\big(1 + \mathcal{O}_{6,\lambda}(s,t)\big) + \mathcal{O}_{7,\lambda}(s,t)\Big). \end{aligned} \tag{37}$$

Proof. From equation (36) we have

$$\overline{\omega}(Y_1,\lambda) = \overline{\omega}\big(U(t),\lambda\big)\big(\eta_{10}^{-\overline{\alpha}_1(\lambda)} + \mathcal{O}_{8,\lambda}(s,t)\big) + \overline{\omega}\Big(\eta_{10}\cdot\big(1 + \mathcal{O}_{9,\lambda}(s,t)\big),\lambda\Big). \tag{38}$$

□

Lemma 18.

$$\tilde{\Delta}_\lambda^{-1} \circ \Phi(s,t) = \begin{pmatrix} \sum_{\substack{1\le i+j\le K \\ (i>0)}} \eta_{ij} C_0^{i+j\mu}(1-t)^i s^j \big(\zeta_i(\lambda) + \mathcal{O}_{10,i,j,\lambda}(s,t)\big) \\ \eta_{10}^{-\mu}(1-t)^{-\mu} s\big(1+\mathcal{O}_{11,\lambda}(s,t)\big) \end{pmatrix} + \begin{pmatrix} \sum_{i=1}^{K} \overline{\alpha}_i \eta_{10}^{i-\overline{\alpha}_1} U^i(t)\overline{\omega}\big(U(t),\lambda\big)\big(1+\mathcal{O}_{5,i,\lambda}(s,t)\big) + C_0^K \overline{\phi}_{3,K,\lambda}(s,t) \\ 0 \end{pmatrix} \tag{39}$$

where $\zeta_i(\lambda) = 1 + \eta_{10}^i \overline{\omega}(\eta_{10},\lambda)$ *and* $\overline{\phi}_{3,K,\lambda}$ *is* C^K *and* K*–flat at* 0.

Proof. We substitute equation (37) in the first component of equation (7) and equation (36) in the second. □

Lemma 19. *Let* $\mathcal{X}(t) = C_0(\lambda)t$.

$$\begin{aligned} \tilde{G}_\lambda \circ \Phi(s,t) = &\begin{pmatrix} C_0 \\ D_0 \end{pmatrix} + \begin{pmatrix} C_{10} \\ D_{10} \end{pmatrix} \sum_{i=1}^{K} \alpha_i \mathcal{X}^i(t)\omega\big(\mathcal{X}(t),\lambda\big)\Big(1+\mathcal{O}_{12,i,\lambda}(t)\Big) \\ &+ \sum_{i+j=1}^{K} \begin{pmatrix} C_{ij} \\ D_{ij} \end{pmatrix} \mathcal{X}^{i+j\mu}(t)\Big(D_0 + f_{2,\lambda}(C_0^\mu s) + r_{2,\lambda}\big(\tilde{Y}_1 \circ \Phi(s,t), C_0^\mu s\big)\Big)^j \\ &+ C_0^K \begin{pmatrix} \phi_{3,K,\lambda}(s,t) \\ \phi_{4,K,\lambda}(s,t) \end{pmatrix}, \end{aligned} \tag{40}$$

where $\phi_{i,K,\lambda}$ *are* C^K *and* K*–flat at* 0.

Proof. The result comes from the fact that $C_{10}(\lambda) \neq 0$ and that

$$\alpha_\ell X^\ell \omega(X,\lambda)(1+f) + \alpha_\ell X^{\ell+k}\omega^j(X,\lambda)(1+g) = \alpha_\ell X^\ell \omega(X,\lambda)(1+h) \tag{41}$$

and if $\ell \le k$

$$\alpha_\ell X^\ell \omega(X,\lambda)(1+f) + *\alpha_\ell \alpha_k X^{\ell+k}\omega^j(X,\lambda)(1+g) = \alpha_\ell X^\ell \omega(X,\lambda)(1+h) \tag{42}$$

where if f and g are I_0^K, h is I_0^K. □

Let us note:

$$\delta_{1,\lambda}(s,t) \overset{\mathrm{def}}{=} \tilde{\delta}_{1,\lambda} \circ \Phi(s,t). \tag{43}$$

Proposition 20. *There exists* $\epsilon > 0$ *sufficiently small such that for each intersection point* (s,t) *of a* 2*-periodic orbit with the transversal* T_1 *with* $t \in (1-\epsilon, 1]$*, the* t*-coordinate of the second intersection point necessarily belongs to* $[a, 1-\epsilon]$ *with* $a \in [0,\epsilon]$*. Hence the number of 2–periodic orbits is bounded by the number of fixed points of* P_λ^2 *with* t*-coordinate in* $[0, 1-\epsilon]$.

Proof. We are looking for orbits of period 2. Any such orbit generates two fixed points of the 2-return map. Also, when there exists an orbit of period 2, the orbit of period 1 exists.

Let M^2 be the 2-dimensional invariant manifold containing all the bifurcating dynamics, $t_1(\lambda)$ and $t_2(\lambda)$ be the t-coordinates of the intersection points of the orbit of period 2 with T_1, and $t_0(\lambda)$ be the one with the orbit of period 1. It was shown in [RR96] that the intersection of M^2 with Σ_1 (and thus with T_1) is a graph, thus $t_1(\lambda) < t_0(\lambda) < t_2(\lambda)$.

To show the lemma, it is then sufficient to show that, whenever an orbit of period 2 exists, then $t_1(\lambda)$ and $t_2(\lambda)$ can not be both close to 1 for $\lambda \in \Lambda$.

Let us first look at equation (26). We have that:

$$\frac{\delta_{1,\lambda}(s,t)}{C_0(\lambda)} = 1 + C_{10}(\lambda)C_0^{-\alpha_1(\lambda)}(\lambda)t^{1-\alpha_1(\lambda)} - \eta_{10}(\lambda)C_0^{-\overline{\alpha}_1(\lambda)}(\lambda)(1-t)^{1-\overline{\alpha}_1(\lambda)} + O(\lambda). \tag{44}$$

Let $\mathcal{L}_{12}$ be the straight line in T_1 passing through (t_1, s_1) and (t_2, s_2). $\mathcal{L}_{12}$ can be parametrized by t. The first derivative of the restriction of $\delta_{1,\lambda}(s,t)$ to $\mathcal{L}_{12}$ is of the form:

$$* C_0^{-\alpha_1(\lambda)}(\lambda)t^{-\alpha_1(\lambda)} + * C_0^{-\overline{\alpha}_1(\lambda)}(\lambda)(1-t)^{-\overline{\alpha}_1(\lambda)} + O(\lambda). \tag{45}$$

Since $\delta_{1,\lambda}(s,t)$ has at least two zeroes in $\mathcal{L}_{12}$, equation (45) has at least one zero t_3. Thus for any $\epsilon_1 > 0$ sufficiently small, both $C_0^{-\alpha_1(\lambda)}$ and $C_0^{-\overline{\alpha}_1(\lambda)}$ must be bounded, i.e. we are interested in the region Λ_1 of the parameter space Λ where there exists $m, M > 0$ such that:

$$0 < m < C_0^{-\alpha_1(\lambda)}(\lambda) < M. \tag{46}$$

Indeed, when $t_3 \in [\epsilon_1, 1-\epsilon_1]$, condition (46) follows directly from the vanishing of (45). If $t_3 \in [0, \epsilon_1)$ or $t_3 \in (1-\epsilon_1, 1]$, then we need to make the discussion in the two cases $\alpha_1 < 0$ or $\alpha_1 > 0$. In the case $\alpha_1 > 0$ and for sufficiently small λ, $C_0^{-\alpha_1(\lambda)}(\lambda) \gg 0$ and $C_0^{-\overline{\alpha}_1(\lambda)}(\lambda)$ is small. Moreover, we have that for all $t \in (0,1)$:

$$0 \le (1-t)^{-\overline{\alpha}_1(\lambda)} \le 1; \tag{47}$$

$$t^{-\alpha_1(\lambda)} \ge 1. \tag{48}$$

From equation (48), $C_0^{-\alpha_1(\lambda)}(\lambda)t^{-\alpha_1(\lambda)} \gg 0$. The vanishing of equation (45) at t_3 excludes t_3 small and t_3 large.

In the case $\alpha_1 < 0$ we use the same argument as in the case $\alpha_1 > 0$ where we interchange t and $(1-t)$, and also $\alpha_1(\lambda)$ and $\overline{\alpha}_1(\lambda)$. □

Part 2. The finite Cyclicity Property

2.1. Solving for s in the region $t \in [0, 1-\epsilon]$

We can use the implicit function theorem to solve $\delta_{2,\lambda}(s,t) = \tilde{\delta}_{2,\lambda} \circ \Phi(s,t) = 0$ yielding s as a function of t.

Let us introduce the two following variables:

$$\nu_1 = \mathcal{X}(t)\omega\big(\mathcal{X}(t), \lambda\big) \qquad \text{and} \qquad \nu_2 = U(t)\varpi\big(U(t), \lambda\big). \tag{49}$$

Since $Y = U(t) + f_{1,\lambda}\big(C_0^{\mu}s\big)$ and using lemma 17, we can consider the principal part of function $\delta_{2,\lambda}(s,t)$ as a C^K function of the variables s, t, t^{μ}, ν_1 and ν_2, and the higher order terms as a C^K function of s and t. We use the notation:

$$F_\lambda\left(s, t, t^{\mu}, \nu_1, \nu_2\right) = \delta_{2,\lambda}(s,t), \tag{50}$$

i.e. F_λ is C^K in its variables. For all points of the curve

$$s_1(t) = D_0\eta_{10}^{\mu}(0)(1-t)^{\mu}, \tag{51}$$

we have:

$$\begin{cases} F_0\left(s_1(t),t,0,0\right)=0 \\ \partial_s F_0\left(s_1(t),t,0,0\right) = -\big(-C_{10}^{-1}(0)(1-t)\big)^{-\mu} < 0. \end{cases} \tag{52}$$

We can apply the implicit function theorem to equation (50) to solve for s around any solution of equation (51) in a small neighborhood of $\lambda = 0$. Moreover, for a sufficiently small neigbourhood Λ' of $\lambda = 0$ we can write s explicitly in terms of (t, t^μ, ν_1, ν_2) which are functions of t only. From lemmae 18 and 19, equation $F_\lambda = 0$ is equivalent to (after substitution of the ν_i using equation (49)):

$$\begin{aligned} &s\cdot\big(1+\mathcal{O}_{13,\lambda}(s,t)\big) = \\ &\eta_{10}^{\mu}(1-t)^{\mu}\left[\sum_{i=0}^{K} D_{i0}\left(C_0t+\sum_{j=1}^{K}\alpha_j(C_0t)^j\omega(C_0t,\lambda)\big(1+\mathcal{O}_{14,j}(t)\big)\right)^i + D_{01}(C_0t)^{\mu}\right.\\ &\qquad\left.\times\left(\sum_{i=0}^{K} D_{i0}\eta_{10}^{i}\big(C_0(1-t)\big)^i\big(1+\mathcal{O}_{15,i,\lambda}(s,t)\big) + D_{01}C_0^{\mu}s\big(1+\mathcal{O}_{16,\lambda}(s,t)\big)\right)\right]. \end{aligned} \tag{53}$$

Lemma 21. *The zeroes of $\delta_{2,\lambda}(s,t)$ in the neighbourhood of a solution of equation* (51) *are of the form $\big(s(t),t\big)$ where:*

$$\begin{aligned} s(t) = {}&\eta_{10}^{\mu}(1-t)^{\mu} \\ &\times\sum_{i=0}^{K} D_{i0}\left[\Big(C_0t\big(1+\mathcal{O}_{17}(t)\big) \quad + \alpha_1(C_0t)\omega(C_0t,\lambda)\big(1+\mathcal{O}_{18}(t)\big)\Big)^i\right.\\ &\qquad\qquad\left. + D_{01}\eta_{10}^{i}C_0^{i+\mu}t^{\mu}(1-t)^i\big(1+\mathcal{O}_{19,i}(t)\big)\right] \end{aligned} \tag{54}$$

Proof. Equation (54) is obtained directly from equation (53) using the fact that since $D_{10}\neq 0$ or $D_{01}\neq 0$ (because $\mathcal{R}_\lambda$ is a diffeomorphism) we can group all terms either in a term with a coefficient $D_{i0}\neq 0$ or with coefficient D_{01}. □

We use the notation:

$$V_\lambda(t) = \delta_{1,\lambda}\big(s(t),t\big). \tag{55}$$

Proposition 22. *The fixed points of the 2–return map are in one to one correspondance with the zeroes of the map $V_\lambda(t)$, where:*

$$V_\lambda(t) = c + \sum_{i=1}^{K} C_0^i \Bigg(\alpha_i t^i \omega(C_0 t, \lambda)\big(1 + \mathcal{O}_{20,i}(t)\big) - \overline{\alpha}_i (1-t)^i \overline{\omega}(C_0(1-t), \lambda)\big(1 + \mathcal{O}_{21,i}(t)\big)\Bigg) + \sum_{i+j=1}^{K} C_0^{i+j\mu} \sum_{\substack{||M||=j\\ M=(m_\ell)\\ 0\le \ell \le k}} \binom{j}{M} \prod_{\ell=0}^{K} \big(D_{\ell 0} C_0^\ell\big)^{m_\ell} \Bigg(C_{ij} t^{i+j\mu} (1-t)^{\sum \ell m_\ell} \big(1 + \mathcal{O}_{22,M,i,j}(t)\big) - (1-\delta_{i0}) \eta_{ij} (1-t)^{i+j\mu} t^{\sum \ell m_\ell} \big(1 + \mathcal{O}_{23,M,i,j}(t)\big)\Bigg) \tag{56}$$

where $c = c(\lambda)$ is some constant, δ_{i0} is the Kronecker delta, and $\binom{j}{M}$ is the multinomial coefficient:

$$\binom{j}{M} = \frac{j!}{m_1! m_2! \cdots m_K!}. \tag{57}$$

Remark: Note that Γ_0 is of finite codimension if and only if at least one of the coefficients in $V_\lambda(t)$ is nonvanishing, up to an adequate power of C_0.

Proof of proposition 22. We need to apply lemmae 18 and 19 in which we replace s by its value $s(t)$ given in lemma 21. To substitute it in equations (39) and (40) we first need to calculate s^j and $\big(D_0 + f_2, \lambda(C_0^\mu s) + \tilde{r}_{2,\lambda} \circ \Phi(s,t)\big)^j$.

$$s^j(t) = \eta_{10}^{j\mu} (1-t)^{j\mu} \Bigg(\sum_{\substack{||M||=j\\ M=(m_\ell)}} \binom{j}{M} \prod_{\ell=0}^{K} \big(D_{\ell 0} C_0^\ell t^\ell\big)^{m_\ell} \big(1 + \mathcal{O}_{24,M}(t)\big) + \alpha_1(C_0 t)\omega(C_0 t, \lambda) \cdot F_{1,j,\lambda}(t)\Bigg). \tag{58}$$

Also we have:

$$
\begin{aligned}
(59)\quad & D_0 + f_{2,\lambda}(C_0^\mu s) + \tilde{r}_{2,\lambda}\circ\Phi(s,t)\\
&= D_0\big(1+\mathcal{O}_{25}(t)\big) + D_{01}C_0^\mu s\big(1+\mathcal{O}_{26}(t)\big)\\
&\quad + \sum_{i=1}^{K} D_{i0}\eta_{10}^i C_0^i(1-t)^i\big(1+\mathcal{O}_{27,i}(t)\big)\\
&= D_0\big(1+\mathcal{O}_{28}(t)\big) + \sum_{i=1}^{K} D_{i0}\eta_{10}^i C_0^i(1-t)^i\big(1+\mathcal{O}_{29,i}(t)\big)\\
&\quad + D_{01}C_0^\mu\eta_{10}^\mu(1-t)^\mu\sum_{i=1}^{K} D_{i0}\Bigg(\Big(C_0t\big(1+\mathcal{O}_{30,i}(t)\big)\\
&\quad + \alpha_1C_0t\omega(C_0t,\lambda)\big(1+\mathcal{O}_{31,i}(t)\big)\Big)^i + D_{01}C_0^{i+\mu}t^\mu(1-t)^i\big(1+\mathcal{O}_{32,i}(t)\big)\Bigg)\\
&= D_0\big(1+\mathcal{O}_{28,i}(t)\big) + \sum_{i=1}^{K} D_{i0}C_0^i(1-t)^i\Big(\eta_{10}^i\big(1+\mathcal{O}_{29,i}(t)\big) + *C_0^\mu t^\mu\mathcal{O}_{33,i}(t)\Big)\\
&\quad + D_{01}C_0^\mu\eta_{10}^\mu(1-t)^\mu\sum_{i=1}^{K} D_{i0}\Big(C_0t\big(1+\mathcal{O}_{30,i}(t)\big) + \alpha_1C_0t\omega(C_0t,\lambda)\big(1+\mathcal{O}_{31,i}(t)\big)\Big)^i\\
&= D_0\big(1+\mathcal{O}_{28}(t)\big) + \sum_{i=1}^{K} D_{i0}\eta_{10}^i C_0^i(1-t)^i\big(1+\mathcal{O}_{34,i}(t)\big)\\
&\quad + D_{01}C_0^\mu\eta_{10}^\mu(1-t)^\mu\sum_{i=1}^{K} D_{i0}\alpha_1C_0t\omega(C_0t,\lambda)\big(1+\mathcal{O}_{31,i}(t)\big)F_{2,i,\lambda}(t),
\end{aligned}
$$

where $F_{2,1,\lambda}(t)\equiv 1$ and for $i>1$, the $F_{2,i,\lambda}(t)$ are $I_0^K(t)$. Therefore:

$$
\begin{aligned}
(60)\quad & \big(D_0 + f_{2,\lambda}(C_0^\mu s) + \tilde{r}_{2,\lambda}\circ\Phi(s,t)\big)^j\\
&= \sum_{\substack{||M||=j\\ M=(m_\ell)}} \binom{j}{M}\prod_{\ell=0}^{K}\big(D_{\ell0}\eta_{10}^\ell C_0^\ell(1-t)^\ell\big)^{m_\ell}\big(1+\mathcal{O}_{35,M}(t)\big)\\
&\qquad + \alpha_1C_0^{1+\mu}t\omega(C_0t,\lambda)F_{2,\lambda}(t),
\end{aligned}
$$

where, for terms in ω, all the D_{i0} are included in $F_{2,\lambda}(t)$ which is $I_0^K(t)$. The result follows from lemmae 18 and 19. We have used the hypothesis that Γ_0 is of finite codimension to get rid of the higher order terms in the expansion. Indeed there exists at least one nonvanishing term of the expansion in which we can include the higher order terms. □

Corollary 23. *For codimensions* 1 *and* 2*,* $V_\lambda(t)$ *is of the same form as studied in* [Gui98]*, i.e.*

$$
\begin{aligned}
(61)\quad V_\lambda(t) &= c + C_0\Big(\alpha_1t\omega(C_0t,\lambda)\big(1+\mathcal{O}_{20,1}(t)\big)\\
&\quad - \overline{\alpha}_1(1-t)\overline{\omega}(C_0(1-t),\lambda))\big(1+\mathcal{O}_{21,1}(t)\big)\Big) + C_0t(C_{10}+\eta_{10})\big(1+\mathcal{O}_{36}(t)\big).
\end{aligned}
$$

We will limit our study to codimensions $k > 2$ $\big($i.e. $\alpha_1(0) = 0$ and $C_{10}(0) = -1 = -\eta_{10}(0)\big)$.

2.2. The differentiability properties of the generalized monomials

In the region $t \in [0, \epsilon]$, we use a derivation-division algorithm on $V_\lambda(t)$ which is a generalization of Rolle's theorem. Each derivation must kill one term. In between the derivations we multiply the function by functions which are positive for t in the whole region $(0, 1-\epsilon)$. The details of the algorithm are long to write and lead to an explicit bound which is a function of $\mu(\lambda)$.

We recall the nice differential properties of the generalized monomials (which can be found in [RR96] for instance).

1. Everywhere in the sequel, $*$ denotes a nonvanishing constant (which may be a differentiable function of λ).
2.

$$\frac{d\omega(x,\lambda)}{dx} = -x^{-1-\alpha_1(\lambda)} = x^{-1}\big(\alpha_1(\lambda)\omega(x,\lambda)+1\big). \tag{62}$$

3. The derivative of a monomial $g = x^\beta \omega^\ell(x,\lambda)$ is:

$$\frac{dg}{dx} = *x^{\beta-1}\omega^\ell(x,\lambda)[1+g_1(x,\lambda)], \tag{63}$$

where $g_1(x,\lambda)$ is $I_0^K(x)$.

4. More generally, if i and ℓ are integers such that $\ell \le i \le h$, then:

$$\frac{d^h\big(x^i\omega^\ell(x,\lambda)\big)}{dx^h} = \begin{cases} x^{i-h-\alpha_1(\lambda)}\displaystyle\sum_{j=0}^{\ell-1} *\omega^j(x,\lambda) & \text{if } i < h \\ \displaystyle\sum_{j=0}^{\ell} *\omega^j(x,\lambda) & \text{if } i = h. \end{cases} \tag{64}$$

5. If $h < \beta$:

$$\frac{d^h\big(x^\beta\omega^\ell(x,\lambda)\big)}{dx^h} = *x^{\beta-h}\omega^\ell(x,\lambda)[1+f_{\beta h\ell}(x,\lambda)], \tag{65}$$

where $f_{\beta h \ell}(x,\lambda)$ is $I_0^K(x)$.

The n–th derivative of a generalized monomial $f_1 = x^{i+j\mu}\omega^\ell(x,\lambda)$ is thus given by:

$$\frac{\partial^n f_1}{\partial x^n} = \begin{cases} *x^{i-n+j\mu(\lambda)}\omega^\ell\big(x,\lambda\big)[1+f_{ij\ell n}(x,\lambda)] & \text{if } j \ge 1 \text{ or } j = 0 \text{ and } n < i \\ \displaystyle\sum_{k=0}^{\ell} *\omega^k\big(x,\lambda\big) & \text{if } j = 0 \text{ and } n = i \\ *x^{i-n-\alpha_1(\lambda)}\omega^{\ell-1}\big(x,\lambda\big)[1+f_{ij\ell n}(x,\lambda)] & \text{if } j = 0 \text{ and } n > i, \end{cases} \tag{66}$$

where $f_{ijln}(x,\lambda)$ are $I_0^{K-n}(x)$.

Lemma 24.

1. *Let* $f_i(X,\lambda)$ *be* $I_0^{K-n}\big(\rho(X,\lambda)\big)$, *and let* $F_i(t) \stackrel{\text{def}}{=} f_i\big(tC_0(\lambda)\big)$. *Then* $F_i(t)$ *is* $J_0^{K-n}(t)$.
2. *Let* $\overline{f}_i(Y,\lambda)$ *be* $I_0^{K-n}\big(\overline{\rho}(Y,\lambda)\big)$, *and let* $\overline{F}_i(t) \stackrel{def}{=} \overline{f}_i\big(C_0(\lambda)(1-t+\tilde{\overline{f}}_{1,\lambda}(t))\big)$. *Then on* $[0,1-\epsilon]$, $\overline{F}_i(t,\lambda)$ *is analytic in* t *and* $\lim_{\lambda\to 0}\overline{F}_i(t,\lambda) = 0$ *uniformly.*

Proof. We have that for all $0 \le n \le K-(k+1)$:

$$\lim_{X\to 0}\left(X^{1+\alpha_1(\lambda)}\omega\right)^n \frac{\partial^n f_i(X,\lambda)}{\partial X^n} = 0 = \lim_{Y\to 0}\left(Y^{1+\overline{\alpha}_1(\lambda)}\overline{\omega}\right)^n \frac{\partial^n \overline{f}_i(Y,\lambda)}{\partial Y^n}, \tag{67}$$

uniformly for $\lambda \in \Lambda$. Since $\dfrac{X}{X^{1+\alpha_1(\lambda)}\omega(X,\lambda)}$ is bounded, we then have the following limit:

$$\lim_{X\to 0} X^n \frac{\partial^n f_i(X,\lambda)}{\partial X^n} = 0 = \lim_{Y\to 0} Y^n \frac{\partial^n \overline{f}_i(Y,\lambda)}{\partial Y^n}. \tag{68}$$

We easily obtain that for all $0 \le n \le K-(k+1)$:

$$\lim_{C_0(\lambda)\to 0} \frac{\partial^n \overline{F}_i(t,\lambda)}{\partial t^n} = \lim_{C_0(\lambda)\to 0} \frac{\partial^n F_i(t,\lambda)}{\partial t^n} = 0 = \lim_{(t,\lambda)\to(0,0)} t^n \frac{\partial^n F_i(t,\lambda)}{\partial t^n}, \tag{69}$$

the first limits being uniform in λ. □

2.3. Algorithm for $t \in [0,\epsilon]$ with Γ_0 of codimension k

In this section, the notation $\mathcal{O}_\lambda(t)$ is used to note a function such that if we note $\mathcal{O}_\lambda(0) = f(\lambda)$, then $f(\lambda) = O(\lambda)$ and $\mathcal{O}_\lambda(t) - f(\lambda)$ is at least $J_0^{K-2(I_1+JI_2+[J\mu]+1)}(t)$. Thus

$$\lim_{(t,\lambda)\to(0,0)} \partial_t^j \mathcal{O}_\lambda(t) = 0, \tag{70}$$

for all $0 \le j \le K - 2(I_1 + JI_2 + [J\mu] + 1)$.

2.3.1. **Case 1: $\boldsymbol{\Gamma_0}$ of type $\boldsymbol{(I_1, I_2, J, L)}$ with $\boldsymbol{(J,L) \ne (0,0)}$.** This is the case where $\alpha_{I_1}(0) \ne 0$ or $C_{I_1 J} D_{I_2 0} \ne 0$. Let $I = I_1 + JI_2$ and

$$I_3 = \begin{cases} I_2 & \text{if } I_2 \ne 0 \\ I = I_1 & \text{otherwise.} \end{cases} \tag{71}$$

The introduction of I_3 is motivated by the fact that when $D_{I_2 0} \ne 0$, then terms D_{i0} with $i > I_2$ can be grouped with the $D_{I_2 0}$ term.

Lemma 25. *For $t \in (0,\epsilon]$, the vanishing of the $(I+[J\mu]+1)^{th}$ derivative of equation (56) is equivalent to the vanishing of:*

$$\begin{aligned}\overline{T}_{I+[J\mu]+1,\lambda}(t) = &\sum_{i=1}^{I+[J\mu]} *C_0^i \alpha_i t^i \bigl(1 + \mathcal{O}_{37,i,\lambda}(t)\bigr) \\ &+ \sum_{\substack{1\le i+j \le I+J\mu \\ j\ne 0}} C_0^{i+j\mu} p_{ij}(\lambda) t^{i+j\mu+\alpha_1(\lambda)} \bigl(1 + \mathcal{O}_{38,i,j,\lambda}(t)\bigr),\end{aligned} \tag{72}$$

where

$$p_{ij}(\lambda) \overset{\mathrm{def}}{=} \sum_{\substack{||M||=j \\ M=(m_\ell) \\ 0\le \ell \le I_3}} \sum_{\substack{i \le i_1 + \sum m_\ell \ell \le I + [(J-j)\mu] \\ 0 \le i_1 \le i}} *C_0^{i_1 + \sum m_\ell \ell - i} C_{i_1 j} \left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{m_\ell}\right), \tag{73}$$

and $*$ are nonvanishing functions of λ.

Proof. For $t \in (0, \epsilon]$, $V_\lambda(t)$ (equation (56)) is of the following form:

$$V_\lambda(t) = c + \sum_{i=1}^{I+[J\mu]} C_0^i \Big(\alpha_i t^i \omega(C_0 t, \lambda)\big(1 + \mathcal{O}_{38,i,\lambda}(t)\big) - \overline{\alpha}_i (1-t)^i \overline{\omega}(C_0(1-t), \lambda))\big(1 + \mathcal{O}_{21,i,\lambda}(t)\big)\Big) + \mathcal{V}_\lambda(t), \tag{74}$$

where, using the relation $(1-t)^A = \sum_{i'=0}^{A} * t^{i'}$, the rest function $\mathcal{V}_\lambda(t)$ is of the form:

$$\begin{aligned}\mathcal{V}_\lambda(t) = &\sum_{i+j\mu=1}^{I+J\mu} C_0^{i+j\mu} \sum_{\substack{||M||=j\\ M=(m_\ell)\\ 0\le \ell \le k}} \binom{j}{M} \prod_{\ell=0}^{I_3} \big(D_{\ell 0} C_0^\ell\big)^{m_\ell} \Big(C_{ij} t^{i+j\mu} (1-t)^{\sum \ell m_\ell} \\ &\times \big(1 + \mathcal{O}_{30,M,i,j}(t)\big) - (1-\delta_{i0})\eta_{ij}(1-t)^{i+j\mu} t^{\sum \ell m_\ell}\big(1 + \mathcal{O}_{31,M,i,j}(t)\big)\Big) \\ = &\sum_{i+j\mu=1}^{I+J\mu} C_0^{i+j\mu} t^i \Bigg[\sum_{\substack{||M||=j\\ M=(m_\ell)\\ 0\le \ell \le I_3}} \sum_{\substack{i \le i_1 + \sum m_\ell \ell \le I + [(J-j)\mu]\\ 0 \le i_1 \le i}} * C_0^{i_1 + \sum m_\ell \ell - i} C_{i_1 j} \left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{m_\ell}\right) t^{j\mu} \\ &\times \big(1 + \mathcal{O}_{39,M,i_1,j,\lambda}(t)\big) - \sum_{\substack{||N||=j\\ N=(n_\ell)\\ 0\le \ell\le I_3\\ 0\le \sum n_\ell \ell \le i}} \sum_{i \le i_2 + \sum n_\ell \ell \le I + [(J-j)\mu]} *(1-\delta_{i_2 0}) \\ &\times C_0^{i_2 + \sum n_\ell \ell - i} \eta_{i_2 j} \left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{n_\ell}\right) (1-t)^{j\mu}\big(1 + \mathcal{O}_{40,N,i_2,j,\lambda}(t)\big)\Bigg] + C_0^{I+J\mu}\mathcal{O}_{41,k,\lambda}(t).\end{aligned} \tag{75}$$

The rest function $C_0^{I+J\mu}\mathcal{O}_{41,k,\lambda}(t)$ can be included in the term with coefficient $*C_0^I \alpha_I$ or $*C_0^{I+J\mu} C_{I_1 J} D_{I_2 0}^J$. The $(I + [J\mu] + 1)^{\text{th}}$ derivative of equation $V_\lambda(t)$ is then of the form:

$$\sum_{i=1}^{I+[J\mu]} * C_0^{i-\alpha_1(\lambda)} \alpha_i \Big(t^{i-(I+[J\mu]+1+\alpha_1(\lambda))}\big(1 + \mathcal{O}_{42,i,\lambda}(t)\big) + *\big(1 + \mathcal{O}_{43,i,\lambda}(t)\big)\Big) + \mathcal{V}_{I+[J\mu]+1,\lambda}(t), \tag{76}$$

where $\mathcal{V}_{I+[J\mu]+1,\lambda}(t)$ is of the following form:

$$\begin{aligned}&\mathcal{V}_{I+[J\mu]+1,\lambda}(t) \\ &= \sum_{\substack{1 \le i+j\mu \le I+J\mu\\ j\neq 0}} C_0^{i+j\mu} \Bigg[\sum_{\substack{||M||=j\\ M=(m_\ell)\\ 0\le \ell\le I_3}} \sum_{\substack{i\le i_1 + \sum m_\ell \ell \le I + [(J-j)\mu]\\ 0\le i_1 \le i}} * C_0^{i_1 + \sum m_\ell \ell - i} C_{i_1 j} \\ &\qquad \times \left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{m_\ell}\right) t^{i-(I+[J\mu]+1)+j\mu}\big(1 + \mathcal{O}_{44,M,i_1,j,\lambda}(t)\big)\end{aligned} \tag{77}$$

$$- \sum_{\substack{||N||=j\\ N=(n_\ell)\\ 0\le \ell\le I_3\\ 0\le \sum n_\ell \ell \le i}} \sum_{i\le i_2+\sum n_\ell \ell \le I+[(J-j)\mu]} *C_0^{i_2+\sum n_\ell \ell - i}(1-\delta_{i_2 0})\eta_{i_2 j}\left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{n_\ell}\right) \times \big(1+\mathcal{O}_{45,N,i,j,\lambda}(t)\big)\Bigg).$$

Indeed for all $i + j\mu < I + [J\mu] + 1$:

$$\frac{d^{I+[J\mu]+1} t^i (1-t)^{j\mu}}{dt^{I+[J\mu]+1}} = \big(1+\mathcal{O}_{46,i,j,\lambda}(t)\big). \tag{78}$$

We multiply equation (76) by $t^{I+[J\mu]+1+\alpha_1(\lambda)}$ and in the first summation we include $C_0^{-\alpha_1(\lambda)}$ in $*$ using equation (46). We can then factorise t^i in the term with coefficient $*C_0^i\alpha_i$:

$$\Big(t^i\big(1+\mathcal{O}_{42,i,\lambda}(t)\big) + *t^{I+[J\mu]+1+\alpha_1(\lambda)}\big(1+\mathcal{O}_{43,i,\lambda}(t)\big)\Big) = t^i\big(1+\mathcal{O}_{37,i,\lambda}(t)\big). \tag{79}$$

Moreover, if $j \neq 0$,

$$\eta_{i_2 j}(\lambda) = C_{i_2 j}(\lambda) + \sum_{0<j'<j,\, i'<i_2} C_{i'j'}(\lambda)\cdot O(\lambda). \tag{80}$$

Hence all terms in the second summation of $\mathcal{V}_{I+[J\mu]+1,\lambda}(t)$ $\big($equation (77)$\big)$ have the form $*C_{i'j'}\left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{m_\ell}\right)$ multiplied by at least the same power of C_0 and a greater power of t than the corresponding term in the first summation. Thus:

$$\mathcal{V}_{I+[J\mu]+1,\lambda}(t) = \sum_{i+j\mu=1}^{I+J\mu} C_0^{i+j\mu} p_{ij}(\lambda) t^{i+j\mu-\alpha_1(\lambda)}\big(1+\mathcal{O}_{38,i,j,\lambda}(t)\big). \tag{81}$$

Indeed, fix (i, i_2, j, N) with:

$$\begin{cases} ||N|| = j \\ 0 \le \sum_{\ell=0}^{I_3} n_\ell \ell \le i \\ i \le i_2 + \sum_{\ell=0}^{I_3} n_\ell \ell \le I + [(J-j)\mu], \end{cases} \tag{82}$$

then from equation (80):

$$\eta_{i_2 j}\left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{n_\ell}\right) t^{I+[J\mu]+1} = \left(C_{i_2 j}(\lambda) + \sum_{0<j'<j,\, i'<i_2} C_{i'j'}(\lambda)\cdot O(\lambda)\right)\left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{n_\ell}\right) t^{I+[J\mu]+1}. \tag{83}$$

Moreover, if $j > j'$ and $N = N' + N''$ with any $N' = (n'_\ell)$ such that $||N'|| = j'$, then:

$$\prod_{\ell=0}^{I_3} D_{\ell 0}^{n_\ell} = \prod_{\ell=0}^{I_3} D_{\ell 0}^{n'_\ell} \times \prod_{\ell=0}^{I_3} D_{\ell 0}^{n''_\ell}. \tag{84}$$

Let equation (83) and for each $j' < j$ choose such a N'. Then:

$$\eta_{i_2 j}\left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{n_\ell}\right) t^{I+[J\mu]+1} = \left(C_{i_2 j}(\lambda)\left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{n_\ell}\right) + \sum_{\substack{1<i'+j'\mu<i_2+j\mu \\ (j'\neq 0)}} C_{i' j'}(\lambda)\left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{n'_\ell}\right)\cdot O(\lambda)\right) t^{I+[J\mu]+1}, \tag{85}$$

i.e. the term in equation (85) with coefficient $C_{i'j'}(\lambda)$ can be included in the rest function of the corresponding term in the first summation of equation (76) with $M = N'$. □

Let the following homogenization of the coefficients.

$$\begin{cases} *\alpha_i(\lambda) = C_0^{(I-i)+J\mu}\overline{\tau}_i(\lambda) \\ *p_{ij}(\lambda) = *C_0(\lambda)^{(I-i)+(J-j)\mu}\overline{\rho}_{ij}(\lambda) \quad (j\neq 0). \end{cases} \tag{86}$$

Coefficients $\overline{\tau}_i(\lambda)$ and $\overline{\rho}_{ij}(\lambda)$ may not be bounded at 0. To eliminate this problem, let:

$$L(\lambda) = \left(\sum_{\substack{(i,j)\leq(I,J) \\ j\neq 0}} \left(\overline{\tau}_i^2(\lambda) + \overline{\rho}_{ij}^2(\lambda)\right)\right)^{1/2} > \delta > 0, \tag{87}$$

where the first inequality comes from the finite codimension hypothesis. Indeed:

$$\begin{array}{ll} \overline{\tau}_I(0) = \alpha_I(0) \neq 0 & \text{if } L = 1 \\ \overline{\rho}_{IJ}(0) = p_{IJ}(0) = C_{I_1 J}(0) D_{I_2 0}^{J}(0) \neq 0 & \text{if } L = 0 \neq J. \end{array} \tag{88}$$

Let:

$$\begin{cases} \tau_i(\lambda) = \dfrac{\overline{\tau}_i(\lambda)}{L(\lambda)} \\ \rho_{ij}(\lambda) = \dfrac{\overline{\rho}_{ij}(\lambda)}{L(\lambda)}. \end{cases} \tag{89}$$

Remark 26. There are $N = k(I_1, I_2, J, L) - (I + [J\mu])$ equations in system (89). Hence, even if $L(\lambda)$ is not bounded at $\lambda = 0$, at least one of the inequalities $\tau_i(\lambda) \geq 1/N$ or $\rho_{ij}(\lambda) \geq 1/N$ is satisfied.

We divide $\overline{T}_{I+[J\mu]+1,\lambda}(t)$ (equation (72)) by $C_0^{I+J\mu}L(\lambda)$:

$$T_{I+[J\mu]+1,\lambda}(t) = \sum_{i=1}^{I+[J\mu]} \tau_i t^i\left(1 + \mathcal{O}_{47,i,\lambda}(t)\right) + \sum_{\substack{1<i+j\mu\leq I+J\mu \\ j\neq 0}} \rho_{ij} t^{i+j\mu+\alpha_1}\left(1 + \mathcal{O}_{48,i,j,\lambda}(t)\right). \tag{90}$$

Proposition 27. *For sufficiently small $\lambda \in \Lambda$, $V_\lambda(t)$ has at most $k(I_1, I_2, J, L) + 1$ zeroes in $[0, \epsilon]$.*

Proof. All terms corresponding to polynomial terms in equation (56) have been killed by derivation, thus there are at most $k(I_1, I_2, J, L) - (I + [J\mu])$ terms in equation (90). Moreover, monomials t^i and $t^{i+j\mu+\alpha_1(\lambda)}$ with $j \neq 0$ are well ordered and form a Chebyshev system (cf. [RR96]). Using a derivation-division algorithm in each cone where either τ_i or ρ_{ij} is the largest coefficient, we thus obtain that for sufficiently small $\lambda \in \Lambda$, $T_{I+[J\mu]+1,\lambda}(t)$ has at most $k(I_1, I_2, J, L) - (I + [J\mu])$ zeroes in $[0, \epsilon]$. The result follows from Rolle's theorem. □

Remark 28. As stated in the previous proof, monomials t^i and $t^{i+j\mu+\alpha_1(\lambda)}$ with $j \neq 0$ are well ordered and form a Chebyshev system. If a function has an expansion in these monomials and if at least one of the coefficients is nonvanishing, then a derivation-division algorithm yields that the number of its small zeroes is at most the order of the nonvanishing coefficient minus one.

2.3.2. **Case 2: Γ_0 of type $(2I_1 + 1, 0, 0, 0)$.** When Γ_0 is of type $(I, 0, 0, 0)$, with $I = 2I_1 + 1$, we must be careful in the algorithm not to kill the leading term t^I with coefficient $*C_0^I C_{I0}$. Indeed, following the proof of lemma 25, the I^{th} derivative of equation (56) is of the form

$$\begin{aligned}(91)\quad \overline{\overline{\overline{T}}}_{I,\lambda}(t) &= \sum_{i=1}^{I-1} *C_0^i \alpha_i t^{i-I-\alpha_1(\lambda)}\big(1 + \mathcal{O}_{49,i,\lambda}(t)\big) \\ &+ *C_0^I \alpha_I \Big(\omega\big(C_0 t, \lambda\big)\big(1 + \mathcal{O}_{50,I_1,\lambda}(t)\big) + *\overline{\omega}\big(C_0(1-t), \lambda\big)\big(1 + \mathcal{O}_{51,I_1,\lambda}(t)\big)\Big) \\ &+ \left(\sum_{\substack{1 \le i+j\mu \le I \\ j \neq 0}} *C_0^{i+j\mu} p_{ij}(\lambda) t^{i-I+j\mu}\big(1 + \mathcal{O}_{52,M,i,j}(t)\big)\right) + *C_0^I p_{I0}\big(1 + \mathcal{O}_{53,I,\lambda}(t)\big).\end{aligned}$$

where, up to multiplication by a nonvanishing function of λ, the $p_{ij}(\lambda)$ are the ones given in equation (73), and

$$(92)\qquad p_{I0}(\lambda) = *\big(C_{I0}(\lambda) + (-1)^{I+1}\eta_{I0}(\lambda)\big).$$

Let the homogenization given in equation (86). We subdivide the parameter space in the following cones:

$$(93)\qquad E_\ell(\Lambda_1) \stackrel{\text{def}}{=} \big\{\lambda \in \Lambda_1 \bigm| |\tau_\ell|(\lambda) = \max_{\substack{k \le I \\ (m,n) \le (I,0)}} \big(|\tau_k(\lambda)|, |\rho_{mn}(\lambda)|\big)\big\}$$

$$(94)\qquad E_{ij}(\Lambda_1) \stackrel{\text{def}}{=} \big\{\lambda \in \Lambda_1 \bigm| |\rho_{ij}(\lambda)| = \max_{\substack{k \le I \\ (m,n) \le (I,0)}} \big(|\tau_k(\lambda)|, |\rho_{mn}(\lambda)|\big)\big\},$$

with $0 \in \Lambda_1 \subseteq \Lambda$.

The only cone which requires a discussion different from proposition 27 is the cone $E_{I0}(\Lambda_1)$. We need to subdivide the cone $E_{I0}(\Lambda_1)$ in the following ones:

$$(95)\qquad E_{I0}^1(\Lambda_1) \stackrel{\text{def}}{=} \big\{\lambda \in E_{I0}(\Lambda_1) \bigm| |\tau_I| \le |\tau_1|\big\}$$

$$(96)\qquad E_{I0}^2(\Lambda_1) = \Lambda_1 \backslash E_{I0}^1(\Lambda_1).$$

Since $\big(\tau_i(\lambda), \rho_{ij}(\lambda)\big) \in \mathbb{S}^k$,

$$\Lambda_1 = \left(\bigcup_{\ell=0}^{I} E_\ell(\Lambda_1)\right) \bigcup \left(\bigcup_{\substack{0<i+j\mu<I \\ (j\neq 0)}} E_{ij}(\Lambda_1)\right) \bigcup E_{I0}(\Lambda_1). \tag{97}$$

Notice that if:

$$E'_\ell(\Lambda_1) \overset{\mathrm{def}}{=} \Big\{\lambda \in E^2_{I0}(\Lambda_1) \;\Big|\; |\tau_\ell(\lambda)| = \max_{\substack{k\leq I \\ (m,n)<(I,0)}} \big(|\tau_k(\lambda)|, |\rho_{mn}(\lambda)|\big)\Big\} \tag{98}$$

$$E'_{ij}(\Lambda_1) \overset{\mathrm{def}}{=} \Big\{\lambda \in E^2_{I0}(\Lambda_1) \;\Big|\; |\rho_{ij}(\lambda)| = \max_{\substack{k\leq I \\ (m,n)<(I,0)}} \big(|\tau_k(\lambda)|, |\rho_{mn}(\lambda)|\big)|\Big\}, \tag{99}$$

then

$$E^2_{I0}(\Lambda_1) = \left(\bigcup_{\ell=0}^{I} E'_\ell(\Lambda_1)\right) \bigcup \left(\bigcup_{\substack{0<i+j\mu<I \\ (j\neq 0)}} E'_{ij}(\Lambda_1)\right). \tag{100}$$

Proposition 29. *For sufficiently small $\lambda \in \Lambda$, $V_\lambda(t)$ has at most $k(I,0,0,0)$ zeroes in $[0,\epsilon]$.*

Proof: We first divide equation (91) by $C_0^I L(\lambda)$ and note by $\tilde{T}_{I,\lambda}(t)$ the resulting equation.

1. Let $\lambda \in E^1_{I0}(\Lambda_)$. In $\tilde{T}_{I,\lambda}(t)$, we group the terms with coefficient in τ_I with the terms with coefficient in τ_1 ($|\tau_I/\tau_1| < 1$ if $\tau_1 \neq 0$ or both terms vanish). The monomials in $\tilde{T}_{I,\lambda}$ are then well ordered and form a Chebyshev system, the result follows from remark 28.
2. Let $\lambda \in E^2_{I0}(\Lambda_1)$. We first divide $\tilde{T}_{I,\lambda}(t)$ by $\big(1+\mathcal{O}_{53,I,\lambda}(t)\big)$ and then differentiate once with respect to t. We obtain a function whose vanishing is equivalent to the vanishing of $T_{I+1,\lambda}(t)$, see equation (90). The result follows from proposition 27 and equation (100). □

2.4. Algorithm for $t \in [\epsilon, 1-\epsilon]$ with Γ_0 of codimension k

2.4.1. Case 1: $\boldsymbol{\Gamma_0}$ of type $\boldsymbol{(I_1, I_2, J, L)}$ with $\boldsymbol{(J, L) \neq (0,0)}$.

Let $I = I_1 + JI_2$ and, as in the previous section,

$$I_3 = \begin{cases} I_2 & \text{if } I_2 \neq 0 \\ I = I_1 & \text{otherwise.} \end{cases} \tag{101}$$

Lemma 30. *For $t \in [\epsilon, 1-\epsilon]$ the vanishing of the $(I+[J\mu]+1)^{th}$ derivative of equation (56) is equivalent to the vanishing of:*

$$\begin{aligned}\overline{\overline{\overline{T}}}_{I+[J\mu]+1,\lambda}(t) = &\sum_{i=1}^{I+[J\mu]} *C_0^i\alpha_i\Big(t^i(1-t)^{I+[J\mu]+1}\big(1+\mathcal{O}_{54,i,\lambda}(t)\big) \\ &+ (-1)^{I+[J\mu]}t^{I+[J\mu]+1}(1-t)^i\big(1+\mathcal{O}_{55,i,\lambda}(t)\big)\Big) + \sum_{\substack{1\leq i+j\mu\leq I+J\mu \\ j\neq 0}} C_0^{i+j\mu}t^i \\ \times \Big(&\overline{q}_{ij}t^{j\mu}(1-t)^{I+[J\mu]+1}\big(1+\mathcal{O}_{56,i,j,\lambda}(t)\big) - \overline{\overline{q}}_{ij}t^{I+[J\mu]+1}(1-t)^{j\mu}\big(1+\mathcal{O}_{57,i,j,\lambda}(t)\big)\Big),\end{aligned} \tag{102}$$

where:

$$(103)\quad \overline{q}_{ij}(\lambda) \stackrel{\mathrm{def}}{=} \sum_{\substack{||M||=j\\ M=(m_\ell)\\ 0\le \ell\le I_3}} \sum_{\substack{0\le i_1\le i\\ i\le i_1+\sum m_\ell \ell\le I+(J-j)\mu}} *C_0^{i_1+\sum m_\ell \ell - i}\left(\prod_{\ell=0}^{I_3}\left(D_{\ell 0}(\lambda)\right)^{m_\ell}\right) C_{i_1 j}(\lambda),$$

and:

$$(104)\quad \overline{\check{q}}_{ij}(\lambda) \stackrel{\mathrm{def}}{=} \sum_{\substack{||N||=j\\ N=(n_\ell)\\ 0\le \ell\le I_3}} \sum_{\substack{0\le \sum n_\ell \ell\le i\\ i\le i_2+\sum n_\ell \ell\le I+(J-j)\mu}} *C_0^{i_2+\sum n_\ell \ell - i}\left(\prod_{\ell=0}^{I_3}\left(D_{\ell 0}(\lambda)\right)^{n_\ell}\right)(1-\delta_{i_2 0})\eta_{i_2 j}(\lambda).$$

Remark: All coefficients $\overline{q}_{ij}(\lambda)$ in the summation are equal, up to multiplication by a nonvanishing function of λ, to the corresponding coefficient $p_{ij}(\lambda)$ defined in equation (73).

Proof of lemma 30. The $(I+[J\mu]+1)^{\mathrm{th}}$ derivative of equation (56) has the form:

$$(105)\quad \sum_{i=1}^{I+[J\mu]} *C_0^{i-\alpha_1(\lambda)}\Big(\alpha_i t^{i-(I+[J\mu]+1+\alpha_1(\lambda))}\big(1+\mathcal{O}_{58,i,\lambda}(t)\big) + (-1)^{I+[J\mu]+1}\overline{\alpha}_i(1-t)^{i-(I+[J\mu]+1+\overline{\alpha}_1(\lambda))}\big(1+\mathcal{O}_{59,i,\lambda}(t)\big)\Big) + \overline{\mathcal{V}}_{I+[J\mu]+1,\lambda}(t),$$

where

$$(106)\quad \overline{\mathcal{V}}_{I+[J\mu]+1,\lambda}(t) = \sum_{\substack{1\le i+j\mu\le I+J\mu\\ j\neq 0\\ ||M||=j\\ A+B=I+[J\mu]+1\\ 0\le B\le \sum m_\ell \ell\\ i_1+\sum m_\ell \ell = i}} C_0^{i+j\mu}\Big(p_{i_1,j,A,B,M}t^{i_1+j\mu-A}(1-t)^{\sum m_\ell \ell - B}\big(1+\mathcal{O}_{60,M,i_1,j,A,B,\lambda}(t)\big) - \check{p}_{i_1,j,A,B,M}t^{\sum m_\ell \ell - B}(1-t)^{i_1+j\mu-A}\big(1+\mathcal{O}_{61,M,i_1,j,A,B,\lambda}(t)\big)\Big),$$

with

$$(107)\qquad p_{i_1,j,A,B,M}(\lambda) \stackrel{\mathrm{def}}{=} f_{1,i_1,j,A,B,M}(\lambda)\left(\prod_{l=0}^{I_3}\left(D_{l0}(\lambda)\right)^{m_l}\right) C_{i_1 j}(\lambda),$$

and

$$(108)\qquad \check{p}_{i_1,j,A,B,M}(\lambda) \stackrel{\mathrm{def}}{=} f_{2,i_1,j,A,B,M}(\lambda)\left(\prod_{l=0}^{I_3}\left(D_{l0}(\lambda)\right)^{m_l}\right)(1-\delta_{i_1 0})\eta_{i_1 j}(\lambda),$$

where coefficients $f_{i,i_1,j,A,B,M}(\lambda)$ are nonvanishing functions appearing as a result of the derivations which, in the sequel, we simply write as $*$.

We multiply equation (105) by $t^{I+[J\mu]+1}(1-t)^{I+[J\mu]+1}$ and, in the first summation, we include $C_0^{-\alpha_1(\lambda)}$ in $*$ using equation (46). We can then factorise t^i in the term with coefficient $*C_0^i\alpha_i$. Indeed, using the identity $t^{i-\alpha_1(\lambda)} = t^i\big(1+(t^{-\alpha_1(\lambda)}-1)\big) = t^i\big(1+\mathcal{O}_{62,i,\lambda}(t)\big)$:

$$\begin{aligned}(109)\quad & t^{i-\alpha_1(\lambda)}(1-t)^{I+[J\mu]+1}\big(1+\mathcal{O}_{58,i,\lambda}(t)\big) \\ &\qquad + (-1)^{I+[J\mu]}t^{I+[J\mu]+1}(1-t)^{i-\overline{\alpha}_1(\lambda)}\big(1+\mathcal{O}_{59,i,\lambda}(t)\big) \\ &= t^i(1-t)^{I+[J\mu]+1}\big(1+\mathcal{O}_{54,i,\lambda}(t)\big) + (-1)^{I+[J\mu]}t^{I+[J\mu]+1}(1-t)^i\big(1+\mathcal{O}_{55,i,\lambda}(t)\big).\end{aligned}$$

From equation (105), we then obtain a function of the form:

$$\begin{aligned}(110)\quad & \sum_{i=1}^{I+[J\mu]} *C_0^i\alpha_i\Big(t^i(1-t)^{I+[J\mu]+1}\big(1+\mathcal{O}_{54,i,\lambda}(t)\big) \\ &\qquad + (-1)^{I+[J\mu]+1}t^{I+[J\mu]+1}(1-t)^i\big(1+\mathcal{O}_{55,i,\lambda}(t)\big)\Big) + \overline{\overline{\mathcal{V}}}_{I+[J\mu]+1,\lambda}(t),\end{aligned}$$

where:

$$\begin{aligned}(111)\quad \overline{\overline{\mathcal{V}}}_{I+[J\mu]+1,\lambda}(t) &= \sum_{\substack{1\le i+j\mu\le I+J\mu \\ j\neq 0}} C_0^{i+j\mu} \sum_{\substack{||M||=j \\ i_1+\sum m_\ell \ell=i}} \sum_{\substack{A+B=I+[J\mu]+1 \\ 0\le B\le \sum m_\ell \ell}} \Bigg[(1-t)^{I+[J\mu]+1} \\ &\times \Big(p_{i_1,j,A,B,M}t^{i_1+I+[J\mu]+1+j\mu-A}(1-t)^{\sum m_\ell \ell-B}\big(1+\mathcal{O}_{60,M,i_1,j,A,B,\lambda}(t)\big)\Big) \\ -t^{I+[J\mu]+1}&\Big(\check{p}_{i_1,j,A,B,M}t^{\sum m_\ell\ell-B}(1-t)^{i_1+I+[J\mu]+1+j\mu-A}\big(1+\mathcal{O}_{61,M,i_1,j,A,B,\lambda}(t)\big)\Big)\Bigg].\end{aligned}$$

Consider, for fixed (i,j,M), the polynomial:

$$\begin{aligned}(112)\quad & \sum_{i\le i_1+\sum m_\ell\ell\le I+(J-j)\mu} \sum_{\substack{A+B=I+[J\mu]+1 \\ 0\le B\le \sum m_\ell\ell}} C_0^{i_1+\sum m_\ell\ell-i} p_{i_1,j,A,B,M}t^{i_1+I+[J\mu]+1-A} \\ &\qquad\qquad \times (1-t)^{\sum m_\ell\ell-B},\end{aligned}$$

and, for fixed (i,j,N), the polynomial:

$$\begin{aligned}(113)\quad & \sum_{i\le i_2+\sum n_\ell\ell\le I+(J-j)\mu} \sum_{\substack{A+B=I+[J\mu]+1 \\ 0\le B\le \sum n_\ell\ell}} C_0^{i_2+\sum n_\ell\ell-i}\check{p}_{i_1,j,A,B,N} \\ &\qquad\qquad \times t^{\sum n_\ell\ell-B}(1-t)^{i_2+I+[J\mu]+1-A}.\end{aligned}$$

Let $\overline{p}_{i,j,M}(\lambda)$ $\big($resp. $\overline{\check{p}}_{i,j,N}(\lambda)\big)$ be the coefficient of the monomial t^i in equation (112) $\big($resp. (113)$\big)$ after expansion of terms of the form $(1-t)^a$. Then from equation (111):

$$\begin{aligned}(114)\quad \overline{\overline{\mathcal{V}}}_{I+[J\mu]+1,\lambda}(t) &= \sum_{\substack{1\le i+j\mu\le I+J\mu \\ j\neq 0}} C_0^{i+j\mu}\Bigg[t^{j\mu}(1-t)^{I+[J\mu]+1} \\ &\times\Bigg(\sum_{\substack{||M||=j \\ i_1+\sum m_\ell\ell=i}} \sum_{\substack{A+B=I+[J\mu]+1 \\ 0\le B\le\sum m_\ell\ell}} p_{i_1,j,A,B,M}t^{i_1+I+[J\mu]+1-A}(1-t)^{\sum m_\ell\ell-B}\end{aligned}$$

$$\times\left(1+\mathcal{O}_{60,M,i_1,j,A,B,\lambda}(t)\right)\Bigg)-t^{I+[J\mu]+1}(1-t)^{j\mu}$$

$$\times\Bigg(\sum_{\substack{||N||=j\\ i_2+\sum n_\ell\ell=i}}\ \sum_{\substack{A+B=I+[J\mu]+1\\ 0\le B\le\sum n_\ell\ell}}\check{p}_{i_2,j,A,B,N}t^{\sum n_\ell\ell-B}(1-t)^{i_2+I+[J\mu]+1-A}$$

$$\times\left(1+\mathcal{O}_{61,N,i_2,j,A,B,\lambda}(t)\right)\Bigg)\Bigg]$$

$$=\sum_{\substack{1\le i+j\mu\le I+J\mu\\ j\neq 0}}C_0^{i+j\mu}t^i\Bigg[t^{j\mu}(1-t)^{I+[J\mu]+1}\Bigg(\sum_{\substack{||M||=j\\ M=(m_\ell)\\ 0\le\ell\le I_3}}\overline{p}_{i,j,M}\left(1+\mathcal{O}_{63,M,i,j,\lambda}(t)\right)\Bigg)$$

$$-t^{I+[J\mu]+1}(1-t)^{j\mu}\Bigg(\sum_{\substack{||N||=j\\ N=(n_\ell)\\ 0\le\ell\le I_3}}\overline{\check{p}}_{i,j,N}\left(1+\mathcal{O}_{64,i,j,\lambda}(t)\right)\Bigg)\Bigg].\qquad\square$$

Let the following homogenization of the coefficients:

$$\begin{cases} *\alpha_i(\lambda)=C_0^{(I-i)+J\mu}\overline{\tau}_i(\lambda)\\ *\overline{q}_{ij}(\lambda)=C_0(\lambda)^{I-i+(J-j)\mu}\overline{\rho}_{ij}(\lambda),\\ *\overline{\check{q}}_{ij}(\lambda)=C_0(\lambda)^{I-i+(J-j)\mu}\overline{\check{\rho}}_{ij}(\lambda).\end{cases}\tag{115}$$

Once again coefficients $\overline{\tau}_i(\lambda)$, $\overline{\rho}_{ij}(\lambda)$ and $\overline{\check{\rho}}_{ij}(\lambda)$ may not be bounded at 0. However, since

$$\overline{\rho}_{I,J}(\lambda)=C_{I_1J}(\lambda)\Bigg(D^J_{I_2,0}(\lambda)+\sum_{i_2=0}^{I_2}D_{i_2,0}h_{I_1,i_2,J}(\lambda)\Bigg)+\sum_{i_1+ji_2+j\mu=1}^{I_1+JI_2+J\mu}C_{i_1,j}D_{i_2,0}h_{i_1,i_2,j}(\lambda)\tag{116}$$

where $h_{i_1,i_2,j}(\lambda)$ are polynomials in $C_{ij}(\lambda)$ and $D_{\ell 0}(\lambda)$, either $\overline{\tau}_I(0)=\alpha_I(0)\neq 0$ or $\overline{\rho}_{I,J}(0)=C_{I_1J}(0)D^J_{I_2,0}(0)\neq 0$. We can thus compactify the coefficients space as we did in section 2.3.1.

Lemma 31. *For $t\in[\epsilon,1-\epsilon]$ the vanishing of the $(I+[J\mu]+1)^{th}$ derivative of equation (56) is equivalent to the vanishing of $\mathcal{G}_{I+[J\mu]+1,\lambda}\left(t,(1-t)^\mu,t^\mu\right)$, where:*

$$\mathcal{G}_{I+[J\mu]+1,\lambda}(t,y,z)=\sum_{i+(j+l)\mu=1}^{2I+[J\mu]+J\mu+1}\xi_{ijl}t^iy^jz^l+\mathcal{O}_{65,\lambda,k}(t,y,z),\tag{117}$$

with $\xi_{ijl}(\lambda)$ polynomials in $\tau_{i'}(\lambda)$, $\rho_{i'j'}(\lambda)$ and $\check{\rho}_{i'j'}(\lambda)$.

Proof. We divide $\overline{\overline{\overline{T}}}_{I+[J\mu]+1,\lambda}(t)$ (equation (102)) by $C_0^{I+J\mu}L(\lambda)$ and obtain:

$$\tilde{\tilde{\tilde{T}}}_{I+[J\mu]+1,\lambda}(t)=\sum_{i=1}^{I+[J\mu]}\tau_i\left(t^i(1-t)^{I+[J\mu]+1}+(-1)^{I+[J\mu]}(1-t)^it^{I+[J\mu]+1}\right)+\sum_{\substack{1\le i+j\mu\le I+J\mu\\ j\neq 0}}t^i\left(\rho_{ij}t^{j\mu}(1-t)^{I+[J\mu]+1}-\check{\rho}_{ij}t^{I+[J\mu]+1}(1-t)^{j\mu}\right)+\mathcal{O}_{66,k}(t).\tag{118}$$

The result follows by setting $\xi_{ijl}(\lambda)$ such that $\mathcal{G}\big(t,(1-t)^\mu,t^\mu\big) = \tilde{\tilde{\tilde{T}}}_{I+[J\mu]+1,\lambda}(t)$. □

Proposition 32. *Let $n = 2(I+[J\mu])+1$. For sufficiently small $\lambda \in \Lambda$, $V_\lambda(t)$ has at most $\frac{1}{2}\big(n(4n^2+16n+37)+1\big)$ zeroes in $[\epsilon, 1-\epsilon]$.*

To prove this proposition, we will need the following lemma:

Lemma 33. *Let $0 < t_2 < t_3$. If $T(t,\lambda) \stackrel{def}{=} P(t,\lambda)+f(t,\lambda)$ where $P(t,\lambda)$ and $f(t,\lambda)$ are some analytic fonctions depending on λ and $f(t,\lambda)$ such that for all $n \le k$ we have on $[t_2,t_3]$,*

$$\lim_{\lambda\to 0} \frac{\partial^n f(t,\lambda)}{\partial t^n} = 0. \tag{119}$$

We suppose $P(t,0) \not\equiv 0$. Let N be a bound for the number of zeroes of $P(t,\lambda)$ on $[t_2,t_3]$, for λ in a neighborhood of Λ_0. Then there exists a neighborhood $\Lambda_P \subseteq \Lambda_0$ of $\lambda = 0$ such that $T(t,\lambda)$ has at most N zeroes on $[t_2,t_3]$.

Proof. This result is stated in [JM94] for P a polynomial. The proof is similar.

Let $N \in \mathbb{N}$ such that for all sufficiently small λ, $P(t,\lambda)$ has at most N zeroes counted with multiplicities on $[t_2,t_3]$. Moreover, assume there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to 0 and such that $T(t,\lambda_n)$ has at least M zeroes counted with multiplicities in $[t_2,t_3]$:

$$t_n^{(1)} \le t_n^{(2)} \le \cdots \le t_n^{(M)}. \tag{120}$$

We can take a subsequence $(\lambda_{n_k})_{n_k\in\mathbb{N}}$ such that the $t_{n_k}^{(i)}$ converge on $[t_2,t_3]$ to $t^{(i)}$ with:

$$t^{(1)} \le t^{(2)} \le \cdots \le t^{(M)}. \tag{121}$$

Since $\lim_{\lambda\to 0} f(t,\lambda) = 0$ uniformly on $[t_2,t_3]$, we have that $P\big(t^{(i)},0\big) = 0$ for all $1 \le i \le M$.

We now show that the zeroes $t^{(i)}$ of $P(t,0)$ are counted with multiplicities: this is done using Rolle's theorem. Let $t^{(j)} = t^{(j+1)} = \cdots = t^{(j+s)}$. Using Rolle's theorem for the derivatives of $T(t,\lambda_{n_k})$, we can find convergent sequences $(t_{n_k,\ell})$ with $\lim_{n_k\to\infty} t_{n_k,\ell} = t^{(j)}$ such that $t_{n_k}^{(j)} \le t_{n_k,\ell} \le t_{n_k}^{(j+s)}$ and

$$\frac{\partial^\ell T}{\partial t^\ell}\big(t_{n_k,\ell},\lambda_{n_k}\big) = 0 \qquad \text{for } \ell \le j-1. \tag{122}$$

Since:

$$\lim_{\lambda\to 0} \frac{\partial^n f(t,\lambda)}{\partial t^n} = 0, \tag{123}$$

uniformly in $t \in [t_2,t_3]$, we have:

$$\frac{\partial^\ell P}{\partial t^\ell}\big(t^{(j)},0\big) = 0 \qquad \text{for } \ell \le j-1. \tag{124}$$

Therefore $T(t,\lambda)$ has at most N zeroes counted with multiplicities, i.e. $M \le N$. □

Proposition 32 can be proven using lemma 33 and the following theorem:

Theorem 34. *Let $f_1(t,t^\mu,(1-t)^\mu)$ be a polynomial of degree at most n in the variables t, t^μ and $(1-t)^\mu$, $f \not\equiv 0$, and let $P_n(t) = f_1(t,t^\mu,(1-t)^\mu)$. Then for* *all $\epsilon > 0$ and μ irrational, the number of zeroes of $P_n(t)$ on $[\epsilon, 1-\epsilon]$ counted with multiplicity is bounded from above. Moreover*

$$\#_0\big(P_n(t)\big) \stackrel{def}{=} \#\big\{t \in [\epsilon, 1-\epsilon] \mid P_n(t) = 0\big\} \le n(2n^2 + 8n + 18), \tag{125}$$

where the solutions t are counted with multiplicities.

Proof. The proof is delayed until section 2.6. □

Proof of proposition 32. As shown in the proof of lemma 31, the vanishing of the $(I+[J\mu]+1)^{\text{th}}$ derivative of equation (56) is equivalent to the vanishing of $\mathcal{G}_{I+[J\mu]+1,\lambda}\big(t, t^\mu, (1-t)^\mu\big)$ which is of the form stated in lemma 33: let

$$P_\lambda(t, y, z) = \sum_{i+(j+l)\mu=1}^{2I+[J\mu]+J\mu+1} \xi_{ijl} t^i y^j z^l, \tag{126}$$

and

$$f_\lambda(t, y, z) = \mathcal{O}_{65,\lambda,k}(t, y, z), \tag{127}$$

then from equation (118):

$$\mathcal{G}_{I+[J\mu]+1,\lambda}\big(t, t^\mu, (1-t)^\mu\big) = P_\lambda\big(t, t^\mu, (1-t)^\mu\big) + f_\lambda\big(t, t^\mu, (1-t)^\mu\big). \tag{128}$$

To conclude, we use theorem 34. To apply the theorem we must simply show that $P_\lambda\big(t, t^\mu, (1-t)^\mu\big)$ is not trivial. We compactify the coefficient space as we did in section 2.3.2.

1. Let $\lambda \in E_{i_1}(\Lambda_1)$, then:

$$\mathcal{G}_\lambda(t, y, z) = t(1-t)P_{1,\lambda}(t) + \sum_{\substack{1 \le i+(j+l)\mu \le 2I+[J\mu]+J\mu \\ j+l>0}} \xi_{ijl} t^i y^j z^l + \mathcal{O}_{67,k}(t, x, z), \tag{129}$$

   where $P_{1,\lambda}(t)$ is the following (nontrivial) polynomial:

$$\begin{aligned} P_{1,\lambda}(t) &= \sum_{i=0}^{I+[J\mu]-1} \tau_{i+1}\big(t^i(1-t)^{I+[J\mu]} + (-1)^{I+[J\mu]}(1-t)^i t^{I+[J\mu]}\big) \\ &= \sum_{i=0}^{I+[J\mu]-1} c_i(\lambda) t^i + o(t^{I+[J\mu]-1}), \end{aligned} \tag{130}$$

   where the $c_i(\lambda)$ are obtained by expanding all terms $(1-t)^A$. $P_{1,\lambda}(t)$ is nontrivial. Indeed, let $V_1(t) = (c_0, c_1 t, \ldots, c_{I+[J\mu]-1} t^{I+[J\mu]-1})$, $V_2 = (\tau_1, \tau_2, \ldots, \tau_{I+[J\mu]})$ and let $\mathcal{M}_\lambda(t)$ be the lower triangular $(I + [J\mu] \times I + [J\mu])$ matrix with $m_{ij,\lambda}(t) = *t^{i-1}$ such that $V_1^T(t) = \mathcal{M}_\lambda(t) \cdot V_2^T$. Then $P_{1,\lambda}(t)$ is not identically zero since $V_2 \ne 0$ ($\tau_{i_1} \ne 0$) and $\mathcal{M}_\lambda(t)$ is inversible for all $(t, \lambda) \in [\epsilon, 1-\epsilon] \times \Lambda_1$.
2. Let $\lambda \in E_{i_1 j_1}(\Lambda_1)$, then

$$\mathcal{G}_\lambda(t, y, z) = (1-t)^{I+[J\mu]} z^{j_1} P_{2,\lambda}(t) + \sum_{j \ne j_1} \xi_{ijl} t^i y^j z^l + \mathcal{O}_{68,k}(t, x, z), \tag{131}$$

   where $P_{2,\lambda}(t, z)$ is the following (nontrivial) polynomial:

$$P_{2,\lambda}(t) = \sum_{i=0}^{I+[(J-j)\mu]} \rho_{ij_1} t^i. \tag{132}$$

We have thus shown that for sufficiently small $\lambda \in \Lambda$, $\tilde{\tilde{\check{T}}}_{I+[J\mu]+1,\lambda}(t)$ has at most $n(2n^2+8n+18)$ zeroes and the result follows yielding at most $n(2n^2+8n+18)+\frac{1}{2}(n+1)$ zeroes for $V_\lambda(t)$. □

2.4.2. **Case 2: Γ_0 be of type $(2I+1,0,0,0)$.**

Proposition 35. *Let $n = 4I+3$. For sufficiently small $\lambda \in \Lambda$, $V_\lambda(t)$ has at most $\frac{1}{2}\big(n(4n^2+18n+37)+1\big)$ zeroes in $[\epsilon, 1-\epsilon]$.*

Proof. We proceed as in the proof of proposition 29, but we subdivide the cone $E_{I0}(\Lambda_1)$ in a different way.

We have that the vanishing of the I^{th} derivative of equation (56) is equivalent to the vanishing of:

$$\begin{aligned}(133)\quad \check{T}_{I,\lambda}(t) = \sum_{i=1}^{I-1} \tau_i \Big(t^{i-I-\alpha_1}\big(1+\mathcal{O}_{69,i,\lambda}(t)\big) + *(1-t)^{i-I-\overline{\alpha}_1}\big(1+\mathcal{O}_{70,i,\lambda}(t)\big)\Big)\\ + \tau_I\Big(\omega\big(C_0 t,\lambda\big)\big(1+\mathcal{O}_{71,I,\lambda}(t)\big) + *\overline{\omega}\big(C_0(1-t),\lambda\big)\big(1+\mathcal{O}_{72,I,\lambda}(t)\big)\Big)\\ + \sum_{\substack{1\le i+j\mu\le I+J\mu\\ j\neq 0}} \Big(\rho_{ij} t^{i+j\mu-I}\big(1+\mathcal{O}_{73,i,j,\lambda}(t)\big) - *\check{\rho}_{ij} t^i (1-t)^{j\mu-I}\big(1+\mathcal{O}_{74,i,j,\lambda}(t)\big)\Big)\\ + \rho_{I0}\big(1+\mathcal{O}_{75,I,\lambda}(t)\big).\end{aligned}$$

Let:

$$(134)\qquad E_{I0}^{1''}(\Lambda_1) \overset{def}{=} \big\{\lambda\in\Lambda_1 \;\big|\; |\tau_I| \le |\lambda|\cdot|\tau_1|\big\}$$

$$(135)\qquad E_{I0}^{2''}(\Lambda_1) = \Lambda_1 \backslash E_{I0}^{1''}(\Lambda_1).$$

As before, if:

(136)
$$E_\ell'''(\Lambda_1) \overset{def}{=} \big\{\lambda \in E_{I0}^{2''}(\Lambda_1) \;\big|\; |\tau_\ell(\lambda)| = \max_{\substack{k<I\\ (m,n)<(I,0)}} \big(|\tau_k(\lambda)|, |\rho_{mn}(\lambda)|, |\tau_I(\lambda)|\big)\big\}$$

(137)
$$E_{ij}'''(\Lambda_1) \overset{def}{=} \big\{\lambda \in E_{I0}^{2''}(\Lambda_1) \;\big|\; |\rho_{ij}(\lambda)| = \max_{\substack{k<I\\ (m,n)<(I,0)}} \big(|\tau_k(\lambda)|, |\rho_{mn}(\lambda)|, |\tau_I(\lambda)|\big)\big\},$$

then

$$(138)\qquad E_{I0}^{2''}(\Lambda_1) = \left(\bigcup_{\ell=0}^{I} E_\ell'''(\Lambda_1)\right) \bigcup \left(\bigcup_{\substack{0<i+j\mu<I\\ (j\neq 0)}} E_{ij}'''(\Lambda_1)\right).$$

1. Let $\lambda \in E_{I0}^{1''}(\Lambda_1)$. In $\check{T}_{I,\lambda}(t)$ we group the term with coefficient in τ_I with the term with coefficient in τ_1 ($|\tau_I/\tau_1| \le |\lambda|$ if $\tau_i \neq 0$ or both terms vanish). We obtain a function of the form of $\tilde{\tilde{\check{T}}}_{I,\lambda}$, see equation (118). Note that in proposition 29, the term with coefficient in τ_I is added as $O(t)$ whereas here it is added as $O(\lambda)$.
2. Let $\lambda \in E_{I0}^{2''}(\Lambda_1)$. We divide $\check{T}_{I,\lambda}(t)$ by $\big(1+\mathcal{O}_{75,I,\lambda}(t)\big)$ which is nonzero on $[\epsilon, 1-\epsilon]$ for λ in a sufficiently small neighborhood and differentiate once more with respect to t. We obtain a function of the form of $\tilde{\tilde{\check{T}}}_{I+1,\lambda}$ (but in which all coefficients may be small), see equation (118).

The result follows using the same argumentation as in the proof of proposition 32. □

2.5. General conclusion for $t \in [0,1]$ with Γ_0 of codimension k

Proposition 36. *Let Γ_0 be of type (I_1, I_2, J, L), $I = I_1 + JI_2$, and $n = 2(I + [J\mu]) + 1$. There exists a neighborhood Λ_0 of $\lambda = 0$ such $V_\lambda(t)$ has at most $N = \frac{1}{2}\big(n(4n^2 + 16n + 37) + 1\big)$ roots on $[0,1]$.*

Proof. As we saw in the previous sections, we can divide the coefficient space in several cones noted $E_\ell(\Lambda_1)$ and $E_{ij}(\Lambda_1)$. We prove the result on each cone.

Let us restrict the parameter space to any of the cones. Moreover, assume the n^{th}-derivative $V_\lambda^{(n)}(t)$ of $V_\lambda(t)$ has a maximum of d zeroes on this cone. Choose a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ converging to 0 such that $V_{\lambda_i}^{(n)}(t)$ has d zeroes. Of those d zeroes, assume m_0 go to 0 and m_1 go to 1 (the m_ℓ can of course be 0). Let $1 - t_1$ be the lower bound of the set of roots that go to 1 and t_2 the upper bound of the set of roots that go to 0. Note $\epsilon_2 = \min\{t_1, t_2\}$, the minimum of the two.

We then have that $V_0^{(n)}(t)$ is of the following form:

$$
\begin{aligned}
(139)\quad V_0^{(n)}(t) = &\sum_{i=1}^{I+[J\mu]} \tau_i \big(t^i(1-t)^{I+[J\mu]+1} + (-1)^{I+[J\mu]}(1-t)^i t^{I+[J\mu]+1}\big) \\
&+ \sum_{\substack{1\le i+j\mu \le I+J\mu \\ j\neq 0}} t^i\Big(\rho_{ij} t^{j\mu}(1-t)^{I+[J\mu]+1} - \check{\rho}_{ij} t^{I+[J\mu]+1}(1-t)^{j\mu}\Big) + V_{k,0}^{(n)}(t),
\end{aligned}
$$

in which we can factorize $t^{m_0}(1-t)^{m_1}$ and where:

$$
V_{k,0}^{(n)}(t) = \begin{cases} \tau_{I+[J\mu]}(1-t)^{I+[J\mu]} t^{I+[J\mu]}\Big((1-t) + (-1)^{I+[J\mu]} t\Big) & \begin{cases}\text{if } (J,L) \neq (0,0) \\ \text{or if the cone is} \\ \qquad E_{I0}^2(\Lambda_1);\end{cases} \\ \rho_{I0}(1-t)^I t^I & \text{otherwise.}\end{cases}
$$

Let $\tilde{\mathcal{G}}_\lambda\big(t, (1-t)^\mu, t^\mu\big) = V_0^{(n)}(t)$. As in the proof of proposition 32, the result follows from Khovanskiĭ's fewnomials theory if equation (139) is nontrivial, which was proven either in proposition 32 or in proposition 35, since $\tilde{\mathcal{G}}_\lambda(t, y, z)$ is of degree at most $2(I + [J\mu]) + 1 - m_0 - m_1$. □

Corollary 37. *Let Γ_0 be of type (I_1, I_2, J, L), $I = I_1 + JI_2$, and $n = 2(I + [J\mu]) + 1$. Then $Cycl(\Gamma_0) \le \frac{1}{4}\big(n(4n^2 + 10n + 37) + 3\big)$.*

Proof. The result follows from proposition 36 and facts 9. □

2.6. Khovanskiĭ's fewnomial theory and proof of theorem 34.

In this section we prove theorem 34. The result is obtained using Khovanskiĭ's method of reducing a transcendental system to nondegenerate polynomial ones; our setting is one of the simplest nontrivial cases of the theory. The theory in its full generality can be found in [Kho91]. In their article [IY95], Il'yashenko and Yakovenko used the theory to bound the cyclicity of elementary polycycles on $\mathbb{R}^2$ in generic families. Section 2 of their paper is certainly a good introduction to the subject. We illustrate the theory for the simplest case, when $P_n(t)$ has nondegenerate zeroes.

2.6.1. **The zeroes of $P_n(t)$ are solutions of a system of transcendental equations on $\mathbb{R}^3$.** We first transform the problem of bounding the number of zeroes of $P_n(t)$ to bounding the number of solutions of a transcendental system on $\mathbb{R}^3$.

Define the following two functions on $\mathbb{R}^3$:

$$\begin{cases} f_{2,A}(t,y,z) \stackrel{\mathrm{def}}{=} y - At^{\mu} \\ f_{3,B}(t,y,z) \stackrel{\mathrm{def}}{=} z - B(1-t)^{\mu}, \end{cases} \tag{140}$$

where $(A,B) \in {\mathbb{R}^+}^2$ and consider the system of transcendental equations:

$$\mathsf{S}_0 = \begin{cases} f_1(t,y,z) = 0 \\ f_{2,1}(t,y,z) = 0 \\ f_{3,1}(t,y,z) = 0, \end{cases} \tag{141}$$

defined on $\mathcal{D}_\epsilon$, where:

$$\mathcal{D}_\epsilon \stackrel{\mathrm{def}}{=} [\epsilon, 1-\epsilon] \times \left[\epsilon^{\mu}, (1-\epsilon)^{\mu}\right]^2 \subseteq \mathbb{R}^3. \tag{142}$$

Lemma 38. *Solving $P_n(t) = 0$ on $[\epsilon, 1-\epsilon]$ is equivalent to solving system S_0 on $\mathcal{D}_\epsilon$.*

We use Khovanskiĭ's method to compute an explicit upper bound for the number of isolated zeroes of the transcendental system S_0, system (141). The method consists in transforming the transcendental problem in algebraic ones, allowing to use Bezout's theorem. This is done in four main steps:

1. We verify that the system has a finite number of solutions which are then isolated.
2. We unfold the transcendental system in a family of system where all degeneracies have been eliminated in the generic systems.
3. Using the fact that manifolds $\{f_{2,A} = 0\}$ and $\{f_{3,B} = 0\}$ are integral separating solutions of polynomial Pfaff equations (to be defined below), we embed the system in a nondegenerate system S of Pfaff forms and polynomials. Indeed the transcendental functions $f_{2,1}$ and $f_{3,1}$ in S_0 are separating solutions of polynomial Pfaff 1-forms. For instance the function $f_{2,1}$ is an integral solution of the polynomial Pfaff 1-form
$$w_2 \stackrel{\mathrm{def}}{=} t dy - \mu y dt, \tag{143}$$
and the function $f_{3,1}$ is an integral solution of the polynomial Pfaff 1-form
$$w_3 \stackrel{\mathrm{def}}{=} (1-t) dz + \mu z dt. \tag{144}$$
The two solutions in $\mathcal{D}_\epsilon$ are given in figure 6.
4. Finally we iterate Khovanskiĭ's reduction method to bound the number of zeroes of S by the sum of the number of zeroes of polynomial systems having nondegenerate roots and to which we can apply Bezout's theorem.

2.6.2. **The smooth manifold with boudary $M_\epsilon \supseteq \mathcal{D}_\epsilon$.** The theory applies to systems defined on smooth manifolds. We must thus define a smooth manifold with boundary M_ϵ which contains $\mathcal{D}_\epsilon$ and on which the system (140) is smooth. Let

$$M_\epsilon \stackrel{\mathrm{def}}{=} \left\{(t,y,z) \in \mathbb{R}^3 | F(t,y,z) \stackrel{\mathrm{def}}{=} t(1-t)y(1-y)z(1-z) \geq \epsilon^{3\mu}(1-\epsilon)^3\right\}, \tag{145}$$

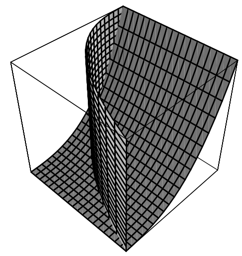


FIGURE 6. The manifolds $\{f_{2,1}=0\}$ and $\{f_{3,1}=0\}$ in $\mathcal{D}_\epsilon$.

and note its interior M_ϵ^0. We then have that $M_\epsilon \supseteq \mathcal{D}_\epsilon$. We can also choose ϵ small and such that the algebraic surface $f_1 = 0$ is in general position with respect to the boundary $F = \epsilon^{3\mu}(1-\epsilon)^3$.

2.6.3. **Bounding the number of solutions of S_0.**

Lemma 39. *For all $(A,B) \in {\mathbb{R}^+}^2$, the system*

$$\mathcal{S}_{0,A,B} = \begin{cases} f_1(t,y,z) = 0 \\ f_{2,A}(t,y,z) = 0 \\ f_{3,B}(t,y,z) = 0 \end{cases} \tag{146}$$

has a finite number of solutions.

Proof. By hypothesis the polynomial $f_1(t,y,z)$ has at least one nonzero coefficient.

1. If f_1 is a polynomial in only one of the variables t, y, or z, the result follows (e.g. from Rolle's theorem).
2. If f_1 is a polynomial in at least the y and t variables, we can write f_1 as a function of y of the following form in the neighbourhood of 0:

$$f_1(t,y,z) = \sum_{i\geq 0} \tilde{f}_i(t,z)y^i. \tag{147}$$

   Zeroes of f_1 are thus solutions of the following equation:

$$-\tilde{f}_0(t,z) = \sum_{i\geq 1} \tilde{f}_i(t,z)y^i. \tag{148}$$

   There exists $i \in \mathbb{N}^+$ such that after expanding $z = B(1-t)^\mu$ (if it occurs in the $\tilde{f}_i$) as a function of t in the neighbourhood of 0 and substituting $y = At^\mu$, equation (148) can be written in the following form:

$$a_{k_1}t^{k_1}\big(1+O(t)\big) = b_{k_2}t^{k_2+i\mu}\big(1+O(t)\big), \tag{149}$$

   where $b_{k_2} \neq 0$. Let $k_3 = \min\{k_1, k_2+i\mu\}$, and let c_{k_3} be the nonzero coefficient corresponding to k_3. Dividing equation (149) by t^{k_3} and taking $t=0$, we get that there exists $\epsilon > 0$ sufficiently small such that the system has no zeroes for $t \in (0,\epsilon)$. From the analycity of the functions on $(0,1)^3$, we have that on any M_ϵ with $\epsilon > 0$ the system has a finite number of solutions.
3. If f_1 is a polynomial in only the z and t variables, we use the same argument as in the previous case where we interchange y and z, and expand around $z = 0$ and $t = 1$.

□

2.6.4. **Khovanskiĭ's reduction procedure.** In this section, we will only consider the case where $f_1 = 0$ is a nondegenerate algebraic surface (a regular surface), i.e. $f_1 = 0$ has no singular points in M_ϵ, and $f_1 = f_{3,1} = 0$ is a nondegenerate curve in $f_{3,1} = 0$. This simple case illustrates all important geometric ideas of the method. The result is true for a general algebraic surface $f_1 = 0$, but the generalization of the method is much more technical since we need to control all possible pathologies (cf. [Kho91], chapter 3).

Definition 40. *A **contact point** of a curve and a vector field in the plane is a point of the curve in which the tangent vector to the curve and the vector of the vector field are collinear.*

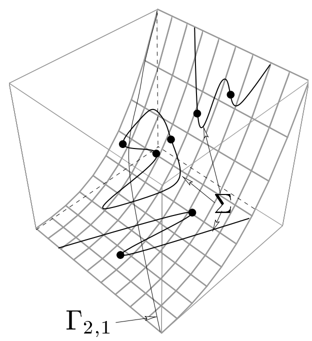


FIGURE 7. Example of contact points on $\{f_{3,1} = 0\}$: $\Sigma = \{f_1 = f_{3,1} = 0\}$ and $\Gamma_{2,1} = \{f_{2,1} = f_{3,1} = 0\}$

It is easily seen that between two points of intersection of a connected component of $f_1 = f_{3,1} = 0$ with $f_{2,1} = 0$ there exists a contact point of $f_1 = f_{3,1} = 0$ and w_2 (figure 7) . Hence

$$\#_0\big(P_n(t)\big) \leq \text{number of contact points of } f_1 = f_{2,1} = 0 \text{ and } w_2 = 0 \\ + \text{number of noncompact connected components of } f_1 = f_{3,1} = 0. \tag{150}$$

Define the following map $*$ mapping 3-forms to functions.

Definition 41. *Let $\alpha = f dx \wedge dy \wedge dz$ be a 3-form on M_ϵ. Then*

$$*(\alpha) \stackrel{def}{=} f. \tag{151}$$

The equation of the contact points on $f_1 = f_{3,1} = 0$ is given by:

$$f_1 = f_{3,1} = 0 \tag{152}$$

$$W_1 \stackrel{def}{=} *(w_3 \wedge w_2 \wedge df_1) \qquad (\deg W_1 = n+1), \tag{153}$$

which we can again consider as a Pfaffian system:

$$\mathsf{S}_1 = \begin{cases} f_1 = 0 \\ W_1 = 0 \\ w_3 = 0. \end{cases} \tag{154}$$

Each noncompact connected component of $f_1 = f_{3,1} = 0$ intersects ∂M_ϵ in at least two points. Hence the number of noncompact components is bounded by:

$$\frac{1}{2}\#\{f_1 = f_{3,1} = F = 0\}, \tag{155}$$

where $\partial M_\epsilon = \{F = 0\}$. We can also consider (155) as a Pfaffian system:

$$\mathsf{S}_2 = \begin{cases} f_1 = 0 \\ F = 0 \\ w_3 = 0. \end{cases} \tag{156}$$

The elimination of w_3 in systems (154) and (156) is similar although it is simpler in system (156). We now consider the curve $f_1 = W_1 = 0$ which for the moment we suppose regular.

Between two intersection points of $f_1 = W_1 = 0$ with $f_{3,1} = 0$ there is at least one contact point with w_3. Hence:

$$\begin{aligned} \#_0\{f_1 = W_1 = f_{3,1} = 0\} &\leq \#\big\{f_1 = W_1 = *(df_1 \wedge dW_1 \wedge w_3)\big\} \\ &\quad + \frac{1}{2}\#\{f_1 = W_1 = F = 0\} \\ &= 2n^2(n+1) + 3n(n+1). \end{aligned} \tag{157}$$

Let

$$W_2 \stackrel{\mathrm{def}}{=} *(df_1 \wedge dW_1 \wedge w_3) \qquad (\deg = 2n) \tag{158}$$

$$W_3 \stackrel{\mathrm{def}}{=} *(df_1 \wedge dF \wedge w_3) \qquad (\deg = n+5). \tag{159}$$

In the case of system (156), $F = 0$ is a compact manifold without boundary. Hence:

$$\#_0\{f_1 = f_{3,1} = F = 0\} \leq \#\big\{f_1 = F = W_3\big\} = 6n(n+5). \tag{160}$$

Therefore:

$$\begin{aligned} \#_0\big(P_n(t)\big) &\leq \#\big\{f_1 = W_1 = W_2\big\} \\ &\quad + \frac{1}{2}\Big(\#\{f_1 = W_1 = F = 0\} + \#\big\{f_1 = F = W_3\big\}\Big), \end{aligned} \tag{161}$$

i.e.

$$\#_0\big(P_n(t)\big) \leq 2n^2(n+1) + 3n(n+1) + 3n(n+5) = n(2n^2 + 8n + 18). \tag{162}$$

2.6.5. **The case of degenerate systems.** As we have seen, the case of degenerate systems can be of different nature:

1. the intersections are not transversal: the remedy is to count points with multiplicity;
2. the surface $f_1 = 0$ is not regular;
3. the intersection of the surface $f_1 = 0$ with $f_{3,1} = 0$ is not a regular curve;
4. the curve $f_1 = W_1 = 0$ is not regular.

The solution exhibited by Khovanskiĭ is to introduce an unfolding of the Pfaffian system:

$$\mathsf{S}_{\lambda,6} = \begin{cases} f_{1,\lambda} \overset{\mathrm{def}}{=} f_1(t) + \displaystyle\sum_{i+j\mu=0}^{k} a_{ijl} t^i y^j z^l = 0 \\ w_{2,\lambda} \overset{\mathrm{def}}{=} w_2 + \displaystyle\sum_{i=1}^{3} (\xi_{2i0} + \xi_{2i1} t + \xi_{2i2} y + \xi_{2i3} z) dx_i = 0 \\ w_{3,\lambda} \overset{\mathrm{def}}{=} w_3 + \displaystyle\sum_{i=1}^{3} (\xi_{3i0} + \xi_{3i1} t + \xi_{3i2} y + \xi_{3i3} z) dx_i = 0 \end{cases} \tag{163}$$

with $x_1 = t$, $x_2 = y$, $x_3 = z$ and $\lambda = (a_{ijl}, \xi_{2ij}, \xi_{3ij})$.

We repeat the previous argument (section 2.6.4) for all systems $\mathsf{S}_{\lambda,6}$ where λ is a regular value of the parameter (a value for which none of the previous pathologies occur) in a small neighborhood Λ of 0. Let $\Lambda_0 \subseteq \Lambda$ be the set of regular values of the parameter and $B(\lambda)$ the bound obtained by the method. (This set Λ_0 is of full mesure, cf. [Kho91, prop. 3, section 3.9].) Then

$$\#_0\big(P_n(t)\big) \leq \max_{\lambda \in \Lambda_0} B(\lambda). \tag{164}$$

This ends the proof of theorem 34.

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CONCLUSION

"My dear Watson, try a little analysis yourself," said he, with a touch of impatience. "You know my methods. Apply them, and it will be instructive to compare results."

– Conan Doyle, *The Sign of the Four (1890)*

En guise de conclusion nous mentionnons quelques-unes des questions intéressantes soulevées par nos travaux et auxquelles nous aimerions nous attaquer.

1. Afin d'améliorer l'énoncé du théorème 12 du second article, nous voudrions montrer le résultat suivant:

   Si aucune des conditions 10.1.a–10.1.d n'est vérifiée, alors pour tout N et tout voisinage U_0 de Γ_0, il existe une perturbation $\mathfrak{X}_\lambda$ de $\mathfrak{X}_0$ possédant au moins N orbites périodiques dans U_0.

2. Les bornes de la cyclicité absolue que nous avons obtenues ne sont pas optimales. La question suivante se pose alors d'elle-même: Quels sont les coefficients essentiels du développement des équations (25) (premier article) et (60) (second article), et quelle est la cyclicité absolue exacte d'une boucle homocline de codimension k du type étudié?

3. Quels sont les diagrammes de bifurcation pour les cas de petite codimension (supérieure à 2)?

4. Dans nos travaux, nous avons toujours pris comme hypothèse qu'il y avait une unique résonance. Notre résultat est-il valide en la présence d'une résonance suplémentaire $\mu \in \mathbb{Q}$? Très certainement la réponse est affirmative dans le cas où les conditions de généricité viennent de termes d'ordre inférieur aux nouveaux monômes résonants. Quels sont les nouveaux phénomènes géométriques apparaissant lorsque les conditions de généricité concernent des termes d'ordre supérieur ou égal à celui des premiers monômes résonants?

5. Soit un système dans $\mathbb{R}^n$ de même type que ceux de notre étude et dont le point de selle a pour valeurs propres $\nu_1(0) = -\nu_2(0) = 1$ et, pour $3 \leq i \leq n$, $|\mu_i(0)| > 1$. Pour un tel système, Sandstede [**San96**] a montré l'existence de la 2-variété invariante, il nous semble donc qu'une légère adaptation de notre argumentation permet de généraliser à $\mathbb{R}^n$ le théorème 12 du second article: en l'occurrence de considérer l'éclatement

$$\begin{aligned}\Phi(s_1,s_2,\ldots,s_{n-2},t) &= \big(C_0(\lambda)s_1,C_0(\lambda)s_2,\ldots,C_0(\lambda)s_{n-2},C_0(\lambda)t\big)\\ &= (Z_1,Z_2,\ldots,Z_{n-2},X),\end{aligned}$$

et d'utiliser le théorème des fonctions implicites pour résoudre, en fonction de t, $(s_1,s_2,\ldots,s_{n-2})$. L'équation unidimensionnelle serait alors de la même forme que dans le second article après substitution de $j\mu$ par $\sum_{j=3}^{n} a_j\mu_j$.

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Cyclicité finie des boucles homoclines dans $\mathbb{R}^3$ non dégénérées avec valeurs propres principales réelles en résonance 1:1

par Louis-Sébastien Guimond

Résumé

Dans cette thèse nous étudions les bifurcations des boucles homoclines des champs de vecterus dans $\mathbb{R}^3$ qui sont non dégénérées au sens de Deng, twistées et dont les valeurs propres principales sont en résonance 1:1. De tels champs de vecterus possèdent une 2-variété $\mathcal{M}_\lambda$ invariante dépendant du paramètre et contenant la boucle homocline Γ_0 pour la valeur nulle du paramètre ainsi que toutes les orbites périodiques créées par perturbations de Γ_0. Cette variété est un anneau (cas *non twisté*) ou un ruban de Möbius (cas *twisté*). La dynamique est alors donnée par une application unidimensionnelle $\mathcal{P}_\lambda(t)$ et toutes les orbites périodiques sont de périodes 1 ou 2. Notre résultat principal est le calcul d'une borne explicite de la cyclicité absolue de ce type de boucle homocline dans le cas twisté, i.e. le nombre d'orbites périodiques générées par perturbation . Pour démontrer ce résultat nous calculons le développement asymptotique d'une fonction $V_\lambda(t)$ liée à $P_\lambda^2(t) - t$, puis en bornons le nombre de zéros.

Dans notre premier article, nous considérons les cas de petites codimensions. Pour calculer la borne, nous projetons la dynamique sur $\mathcal{M}_\lambda$ puis appliquons les techniques exposées par Jebrane et Mourtada pour l'étude de la boucle en huit dans le plan. Dans le second article, nous étudions le cas général. Dans ce cadre nous ne pouvons projeter la dynamique sur $\mathcal{M}_\lambda$. Les calculs pour obtenir la borne sont alors beaucoup plus techniques et reposent sur une généralisation des techniques exposées par Jebrane et Mourtada ainsi que sur la théorie des fewnomials de Khovanskiĭ permettant de réduire l'étude d'un système d'équations transcendantes à l'étude de systèmes polynomiaux non-dégénérés.

Discipline: mathématiques

Mots clefs: champs de vecteurs; application premier retour; ruban de Möbius; boucle homocline twistée; monômes généralisés; éclatement; algorithme derivation-division; fewnomials.

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