Université de Montréal

# Approximation des fonctions harmoniques par des séries universelles surconvergentes

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*Cette thèse est dédiée à :*

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# Sommaire

Dans cette thèse, nous étudions l'approximation des fonctions harmoniques par des séries universelles surconvergentes. A quelques différences près, la plupart des résultats établis sont des analogues de ceux obtenus dans le cas de l'approximation des fonctions holomorphes par de telles séries. Dans le cas des fonctions holomorphes, l'approximation est faite pour des fonctions continues sur des compacts et holomorphes à l'intérieur alors que nous approchons des fonctions harmoniques dans un voisinage du compact. Cette différence est due au fait que dans le cas des fonctions holomorphes, on dispose du théorème d'approximation de Mergelyan qui permet une telle approximation, alors que dans le cas des fonctions harmoniques, on utilise uniquement le théorème d'approximation classique de Walsh (analogue harmonique du théorème de Runge).

Le premier article faisant partie de cette thèse a été publié dans la revue Analysis 26. pp 287–293(2006). Dans cet article, nous prouvons pour un ouvert $\Omega \subset \mathbb{R}^N$ l'existence d'une fonction harmonique dont le développement en série de Taylor homogène approche uniformement toutes les fonctions harmoniques hors du domaine de convergence de la série et grâce au théorème de catégorie de Baire, nous obtenons que l'ensemble de telles fonctions est résiduel dans l'espace des fonctions harmoniques dans $\Omega$. Notre approximation est valide sur la frontière du domaine tel qu'obtenu pour la première fois par Nestoridis en 1996 dans le cas des fonctions holomorphes. De plus le centre de notre approximation varie de façon uniforme sur les compacts de $\Omega$.

Le second article de la thèse sera soumis au Journal Canadien de Mathématiques. Dans ce second article, nous prouvons l'existence d'une série formée des translatés de la solution fondamentale de l'équation de Laplace qui est universelle dans l'ensemble des fonctions harmoniques (i) au voisinage d'un compact donné de complémentaire connexe ou (ii) dans un ouvert de $\mathbb{R}^N$ dont le complémentaire dans $\overline{\mathbb{R}}^N$ est connexe. Nous prouvons aussi l'existence de telles séries constituées des dérivées de la solution fondamentale de l'équation de Laplace. Dans les deux cas, les singularités sont connues à l'avance et sont prises hors du domaine de définition de la fonction à approximer. Pour l'approximation par les translatés, nous prouvons qu'il existe de telles séries dont la suite des coefficients est de puissance $p^{\text{ième}}$ ($p > 1$) sommable et qu'on ne peut avoir un tel résultat avec $p = 1$.

**Mots clés** : Séries, Surconvergence, Approximation, fonctions harmoniques, Ensemble résiduel, Solution fondamentale, Ensembles analytiques, sous-ensembles denses.

# summary


In this thesis, we study the approximation of harmonic functions by universal overconvergent series. Most of the results established are analogues of those obtained in the case of approximation of holomorphic functions by such series. In the case of holomorphic functions, the approximation is made for functions which are continuous on a compact set and holomorphic inside this compact set, while our approximation is for functions that are harmonic in a neighborhood of the compact set. This difference is due to the fact that in the case of holomorphic functions, we have at our disposal Mergelyan's approximation theorem, which allows such an approximation, while in the case of harmonic functions, we employ only the classic approximation theorem of Walsh (harmonic analogue of the theorem of Runge).

The fisrt article which is part of this thesis is published in the journal Analysis 26, pp 287–293 (2006). In this article, we prove that for an open subset $\Omega \subset \mathbb{R}^N$, there exists a harmonic function whose homogeneous Taylor expansion approaches uniformly all harmonic functions outside the domain of convergence of the series and thanks to the Baire categoy theorem, we obtain that the class of such functions is residual in the space of harmonic functions in $\Omega$. Our estimate is valid on the border of the domain such as obtained for the first time by Nestoridis in 1996 in the case of holomorphic functions. Furthermore, our approximations are uniform with respect to the variation of the centers of the expansions on the compact subsets of $\Omega$.

The second article will be submitted to the Canadian Journal of Mathematics. In this second article, we prove the existence of a series, whose terms are translates of the fundamental solution of the Laplace operator, which is universal in the space of functions that are harmonic (i) in a neighborhood of a fixed compact set $K$ with connected complement or (ii) in an open subset of $\mathbb{R}^N$ with connected complement in $\overline{\mathbb{R}}^N$. We also prove the existence of such series whose terms are derivatives of a fundamental solution of the Laplace operator. In both cases, the singularities are known in advance and are taken outside the domain of definition of the functions to be approximated. For the case of approximation by translates, we prove that the sequence of coefficients of the universal series can be chosen in $\ell^p (p > 1)$ and that we cannot have such a result with $p = 1$.

**Keywords** : series, overconvergence, approximation, residual set, fundamental solution, analytic sets, dense subsets.

# Préliminaires

Dans la théorie de l'approximation, il arrive souvent qu'il existe un objet $u$ qui peut approximer (dans un certain sens) tout objet d'un univers donné. L'objet $u$ est alors appelé objet universel. Le premier cas d'universalité a été obtenu par Fekete [6] en 1914 qui prouva l'existence d'une série de puissance $\sum_{n=1}^{\infty} a_n x^n$ convergente uniquement pour $x = 0$ ayant la propriété que pour toute fonction continue $g$ sur $[-1, 1]$ vérifiant $g(0) = 0$, il existe une suite croissante d'entiers naturels $(n_k)_{k\in\mathbb{N}}$ telle que $\sum_{n=1}^{n_k} a_n x^n \longrightarrow g(x)$ uniformement lorsque $k \to \infty$. En prenant en compte le théorème de Borel, ce résultat s'énonce encore de la façon suivante, il existe **une** fonction $u$ indéfiniment dérivable, avec $u(0) = 0$, telle que la suite des polynômes de Taylor de $u$ approche **toutes** fonctions continues sur $[-1, 1]$ nulle en 0. Ceci a suggéré le terme d'universalité qui fut utilisé pour la première fois par Marcinkiewicz en 1935.

Avec le temps, grand nombre d'objets universels ont été découverts. En 1929 Birkhoff [2] a montré l'existence d'une fonction dont les translatés $f(z+n)$, $(n \geq 1)$ peuvent approcher uniformement sur les compacts de $\mathbb{C}$ toutes les fonctions entières, Seleznev [12] en 1952 a montré l'existence d'une série de Taylor de rayon de convergence zero ayant les propriétés d'universalité dans $\mathbb{C} \setminus \{0\}$. Luh [5] et indépendament, Chui et Parnes [3] ont obtenu ce résultat pour un rayon de convergence quelconque. Plus précisement, ils ont obtenu le résultat suivant : il existe une série de Taylor $\sum_{n=0}^{\infty} a_n z^n$ de rayon de convergence 1, vérifiant pour tout compact $K \subset \{z \in \mathbb{C}, |z| > 1\}$ de complémentaire connexe et pour toute fonction $f$ continue sur $K$ et holomorphe dans $\overset{o}{K}$ il existe une sous suite de la suite $\sum_{k=0}^{n} a_k z^k, n = 0, 1, 2, \cdots$ convergeant vers $f$ uniformement sur $K$. Luh a prouvé que l'ensemble de telles fonctions est dense dans l'espace des fonctions holomorphes dans le domaine. Grosse-Erdmann [4] a prouvé que cet ensemble est en plus résiduel dans cet espace. De telles séries sont dites universellement surconvergentes. Dans un espace métrique $(X, d)$, une série $\sum_{n=0}^{\infty} x_n$, ($(x_n)_n$ est une suite de points de $X$) est dite universelle si la suite des sommes partielles

$\sum_{n=0}^{m} x_n, m = 0, 1, 2, \cdots$ est dense dans $X$. Une série numérique $\sum_{n=0}^{\infty} a_n z^n$ de rayon de convergence $\rho < \infty$ est dite surconvergente lorsqu'il existe une sous suite de la suite des sommes partielles $\sum_{n=0}^{m} a_n z^n, m = 0, 1, 2, \cdots$ qui converge hors du disque de convergence, le premier exemple d'une telle série a été obtenu en 1918 par Jentzsch [7]. D'autre exemples fûrent obtenus par Ostrowski [9, 10, 11] qui donna quelques propriétés de telles séries. Notons que dans les résultats de Chui, Parnes et Luh, l'approximation universelle est valide uniquement sur les compacts ne rencontrant pas la frontière du domaine de définition de la fonction universelle. Leur preuve utilise essentiellement les théorèmes de Mergelyan et de Runge et les méthodes sont constructives. En 1996, Nestoridis [8] a amelioré ces résultats en prouvant que l'approximation est aussi valide sur la frontière, il utilise aussi le théorème de Mergelyan et sa méthode est basée sur le théorème de catégorie de Baire qui simplifie beaucoup la preuve.

Bon nombre des résultats cités plus haut ont été établis pour les fonctions harmoniques. Le premier résultat dans cette direction a été obtenu par Armitage [1] en 2002.

Nous rappelons qu'une fonction $f$ de classe $C^2$ définie dans un ouvert $\Omega \subset \mathbb{R}^N$ à valeur dans $\mathbb{R}$ est dite harmonique si elle vérifie l'équation aux dérivées partielles $\triangle f = 0$, appelée équation de Laplace, où $\triangle = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2}$. Les fonctions harmoniques jouent un rôle crucial dans plusieurs branches des mathématiques, de physique et de l'ingénierie. Ces fonctions (comme toutes les fonctions analytiques) sont développables en série de Taylor en tout point du domaine de définition, mais contrairement au cas des fonctions analytiques d'une variable où on peut trouver le domaine de convergence à l'aide des théorèmes d'Abel, Cauchy et Cauchy-Hadamard, le domaine de convergence de la série de Taylor d'une fonction analytique de plusieurs variables n'a pas une forme géométrique simple. Si $f$ est harmonique dans une boule $B$ de centre $a$, la série de Taylor de $f$ en $a$ ne converge pas nécessairement vers $f$ dans $B$. Toutefois, si nous regroupons les termes de même degré, alors la nouvelle série obtenue converge vers $f$ dans $B$. Nous travaillons avec de telles séries que nous appelons développements homogènes de $f$. Les preuves de l'existence des séries universelles (et aussi des fonctions universelles) se divisent en deux groupes : 1) celles par construction (elles sont plus longues), 2) celles qui utilisent le théorème de catégorie de Baire. Nous utilisons la deuxième méthode pour nos preuves.

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# Chapitre 1

# Universal Overconvergence of Homogeneous Expansions of Harmonic Functions

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## 1.1 abstract

On certain domains of Euclidean space, we establish the existence of harmonic functions, which are universal in the sense that their Taylor polynomials approximate all plausibly approximable functions in the complement of the domain.

## 1.2

Let $0 \leq r < +\infty$. There exists a power series $\sum a_n z^n$, with radius of convergence $r$ having the remarkable property that, for each compact set $K$ in the complement of the disc $\Omega_r = \{z : |z| \leq r\}$, for which the complement of $K$ is connected and for each function $f$ holomorphic on $K$, there exists a subsequence of the partial sums of $\sum a_n z^n$ which converges uniformly to $f$ on $K$. This result is due to Seleznev [11] for $r = 0$ and, for the general case, to Luh [6] and Chui and Parnes [3]. Such a power series is said to be universally overconvergent. Armitage [1] proved analogous results on universal overconvergence for functions, which are harmonic in a domain $\Omega$ of $\mathbb{R}^N$ and extend continuously to the boundary of $\Omega$. In the present paper, we consider universal harmonic functions in more general domains. However we do not consider the boundary behaviour of such functions. We hope to do so in the near future. For a general survey of various kinds of universality, see [4].

Let $E$ be a subset of $\mathbb{R}^N$, we denote by $\mathcal{H}(E)$ the space of functions $u$, each of which is harmonic on some neighourhood (depending on $u$) of $E$. If $u$ is harmonic in a ball $B$ centered at a point $y$, then the Taylor series for $u$ about $y$ need not converge in the ball $B$, however, if the terms of the same degree are grouped, then this grouped Taylor series indeed converges to $u$ in the ball $B$. The terms are homogeneous polynomials which are also harmonic. We call this grouped Taylor series the homogeneous expansion of $u$ with center $y$. For an open subset $\Omega$ of $\mathbb{R}^N$, a harmonic function $u \in \mathcal{H}(\Omega)$ and $y \in \Omega$, let $S_n(u, y)(x)$ denote the $n$th partial sum of the homogeneous expansion of $u$ with center $y$. A harmonic function $v \in \mathcal{H}(\Omega)$ is said to be *universal* with respect to $\mathcal{H}(\Omega)$ if, for each compact set $L \subset \Omega$ and each compact set $K \subset \mathbb{R}^N$ disjoint from $\Omega$ and with $\mathbb{R}^N \setminus K$ connected and for each $u \in \mathcal{H}(K)$, there is an increasing sequence of natural numbers $\{n_k\}$, such that

$$\max_{x \in K, y \in L} |S_{n_k}(v, y)(x) - u(x)| \to 0 \qquad (k \to \infty).$$

We denote by $\mathcal{U}(\Omega)$ the family of universal functions with respect to $\mathcal{H}(\Omega)$.

We endow the space $\mathcal{H}(\Omega)$ with the topology of uniform convergence on compact subsets. The space $\mathcal{H}(\Omega)$ is a Fréchet space and so is of second Baire category. A subset of $\mathcal{H}(\Omega)$ is said to be *residual* if its complement in $\mathcal{H}(\Omega)$ is of first category. Let $\overline{\mathbb{R}}^N$ denote the one-point compactification of $\mathbb{R}^N$.

**Theorem 1.** *Let $\Omega$ be an open set in $\mathbb{R}^N$ with $\overline{\mathbb{R}}^N \setminus \Omega$ connected. Then, the family $\mathcal{U}(\Omega)$ of universal functions with respect to $\Omega$ is residual in $\mathcal{H}(\Omega)$.*

The following harmonic analog of Runge's theorem on polynomial approximation is due to Walsh [12].

**Lemma 1.** *[2, Corollary 2.6.5] Let $K$ be a compact subset of $\mathbb{R}^N$ such that $\mathbb{R}^N \setminus K$ is connected. If $h$ is harmonic on $K$ and $\epsilon > 0$, then there exists a harmonic polynomial $p$ such that $|p - h| < \epsilon$ on $K$.*

We need to know that the rate of convergence of a homogeneous expansion of a harmonic function can be estimated in terms of the size of the function. The next lemma serves this purpose.

**Lemma 2.** *Given $0 < \mu < 1$, $M > 0, R > 0$ and $\epsilon > 0$, there exists an $n_o$ depending only on these constants and the dimension $N$, such that if $v$ is harmonic and bounded by $M$ in a ball $B(y, R)$ of $\mathbb{R}^N$, then*

$$|S_n(v, y)(x) - v(x)| < \epsilon,$$

*for all $n \geq n_o$ and all $x \in B(y, \mu R)$.*

*Démonstration.* It is sufficient to consider the unit ball $B$ centered at the origin. Suppose, then, that $v$ is harmonic and bounded by $M$ in $B$. Let $v = \sum v_j$ be the homogeneous expansion of $v$. The lemma follows immediately from the estimate

$$|v_j(x)| \leq C(j + 1)^{N-2}|x|^j M,$$

on page 43 of [2]. □

An important ingredient in the proof of Theorem 1 is the following lemma, which asserts the equiconvergence of homogeneous expansions of a harmonic function over a compact set $K$ as the center $y$ of the expansion varies over a compact set $L$.

**Lemma 3.** *Let $K$ and $L$ be any two compact subsets of $\mathbb{R}^N$. Let $v \in \mathcal{H}(\mathbb{R}^N)$ and $\epsilon > 0$. Then, there is an $n_o$, such that for all $n > n_o$,*

$$\max_{x \in K\, y \in L} |S_n(v, y)(x) - v(x)| < \epsilon.$$

*Démonstration.* Set

$$R = 2 \max\{|x - y| : x \in K, y \in L\}$$

and

$$M = \max\{|v(x)| : x \in \overline{B}(y, R), y \in L\}.$$

The lemma follows immediately from Lemma 2, with $\mu = 1/2$.

□

We now prove Theorem 1.

*Démonstration.* As in [9], there exists a countable collection $\{K_k\}$ of compact subsets of $\mathbb{R}^N \setminus \Omega$ such that each $\mathbb{R}^N \setminus K_k$ is connected and, if $K$ is any compact subset of $\mathbb{R}^N \setminus \Omega$ for which $\mathbb{R}^N \setminus K$ is connected, then $K \subset K_k$, for some $k$. Let $\{L_m\}$ be an exhaustion of $\Omega$ by compact sets such that $\mathbb{R}^N \setminus L_m$ is connected, for each $m$.

Let $\{p_i\}$ be an enumeration of the family of harmonic polynomials with rational coefficients. Then,

$$\mathcal{U}(\Omega) = \bigcap_{i,j,k,m} \bigcup_{n} U_{i,j,k,m,n},$$

where $U_{i,j,k,m,n}$ consists of those functions $v \in \mathcal{H}(\Omega)$ such that

$$\max_{x \in K_k \, y \in L_m} |S_n(v,y)(x) - p_i(x)| < 1/j.$$

We claim that each $U_{i,j,k,m,n}$ is open. Fix $v \in U_{i,j,k,m,n}$ and let $\{v_\alpha\}$ be a sequence in $\mathcal{H}(\Omega)$ which converges to $v$ uniformly on compact subsets of $\Omega$. Then, the partial derivatives also converge uniformly on compact subsets of $\Omega$. It follows that the polynomials $S_n(v_\alpha, y)(x)$ converge to the polynomial $S_n(v,y)(x)$ uniformly on $L_m \times K_k$ as $\alpha \to \infty$. Thus, the sequence $\{v_\alpha\}$ is eventually in $U_{i,j,k,m,n}$. We have shown that each sequence converging to a member of $U_{i,j,k,m,n}$ is eventually in $U_{i,j,k,m,n}$ and so $U_{i,j,k,m,n}$ is open.

To conclude the proof, it is sufficient to show that, for each $i, j, k, m$,

$$\bigcup_{n} U_{i,j,k,m,n}$$

is dense in $\mathcal{H}(\Omega)$. Suppose, then, that $u \in \mathcal{H}(\Omega)$, $L$ is a compact subset of $\Omega$ and $\epsilon > 0$. We seek an $n$ and a function $v \in U_{i,j,k,m,n}$ such that $\max_{x \in L} |v(x) - u(x)| < \epsilon$. We may assume that $L \subset L_m$ and $1/j < \epsilon$. Set

$$w(x) = \begin{cases} u(x) & \text{if } x \in L_m \\ p_i(x) & \text{if } x \in K_k. \end{cases}$$

Set $\Pi = L_m \cup K_k$, and choose $\delta$ with $0 < 2\delta < 1/j$. By the harmonic polynomial Runge Lemma 1, there exists a $v$ harmonic on $\mathbb{R}^N$, such that $\max_{x \in \Pi} |v(x) - w(x)| < \delta$. Now, by Lemma 3 we may choose $n$ so large that

$$\max_{x \in K_k \, y \in L_m} |S_n(v,y)(x) - v(x)| < \delta.$$

Thus, $v \in U_{i,j,k,m,n}$ and $\max_{x \in L} |v(x) - u(x)| < \epsilon$. □

In Theorem 1 we assume that the complement of $\Omega$ in $\overline{\mathbb{R}}^N$ is a continuum (compact connected set). In the next theorem, we assume that the complement of $\Omega$ in $\mathbb{R}^N$ is a continuum. In particular, in the previous theorem, the complement of $\Omega$ in $\mathbb{R}^N$ is unbounded, whereas in the following theorem it is bounded.

Fix a point $y \in \Omega$. A harmonic function $v \in \mathcal{H}(\Omega)$ is said to be *universal at* $y$ with respect to $\mathcal{H}(\Omega)$ if, for each compact set $K \subset \mathbb{R}^N$ disjoint from $\Omega$ and with $\mathbb{R}^N \setminus K$ connected and for each $u \in \mathcal{H}(K)$, there is an increasing sequence of natural numbers $\{n_k\}$, such that

$$\max_{x\in K} |S_{n_k}(v,y)(x) - u(x)| \to 0 \qquad (k \to \infty).$$

We denote by $\mathcal{U}(\Omega, y)$ the family of universal functions at $y$ with respect to $\mathcal{H}(\Omega)$.

**Theorem 2.** *Let $\Omega$ be an open set in $\mathbb{R}^N$ with $\mathbb{R}^N \setminus \Omega$ a continuum and let $y \in \Omega$. Then, the family $\mathcal{U}(\Omega, y)$ of universal functions at $y$ with respect to $\Omega$ is residual in $\mathcal{H}(\Omega)$.*

The following harmonic analog of Runge's theorem on rational approximation is due to Walsh [12, p. 518] (see also [5, Théorème 2.1.4] and [2, Theorem 2.6.4]).

**Lemma 4.** *Let $K$ be a compact subset of an open set $\Omega$ in $\mathbb{R}^N$ such that every bounded component of $\mathbb{R}^N \setminus K$ contains a point of $\mathbb{R}^N \setminus \Omega$. If $h$ is harmonic on $K$ and if $\epsilon > 0$, then there exists $v \in \mathcal{H}(\Omega)$ such that $|v - h| < \epsilon$ on $K$.*

We now prove Theorem 2.

*Démonstration.* By the harmonic polynomial Runge Lemma 1, it is sufficient to consider the case $K = \mathbb{R}^N \setminus \Omega$. Let $\{p_i\}$ be an enumeration of the family of harmonic polynomials with rational coefficients. Then,

$$\mathcal{U}(\Omega, y) = \bigcap_{i,j} \bigcup_{n} U_{i,j,n},$$

where $U_{i,j,n}$ consists of those functions $v \in \mathcal{H}(\Omega)$ such that

$$\max_{x\in K} |S_n(v,y)(x) - p_i(x)| < 1/j.$$

As in the proof of Theorem 1, each $U_{i,j,n}$ is open and to conclude the proof, it is sufficient to show that, for each $i$, $j$,

$$\bigcup_{n} U_{i,j,n}$$

is dense in $\mathcal{H}(\Omega)$. Suppose, then, that $u \in \mathcal{H}(\Omega)$, $L$ is a compact subset of $\Omega$ and $\epsilon > 0$. We seek an $n$ and a function $v \in U_{i,j,n}$ such that $\max_{x\in L} |v(x) - u(x)| < \epsilon$. We may assume that $y$ is in the interior of $L$ and $\mathbb{R}^N \setminus L$ has precisely two components, one of which is unbounded, while the other is bounded and contains $K$. We may also assume that $1/j < \epsilon/2$. Set

$$w(x) = \begin{cases} u(x) & \text{if } x \in L \\ p_i(x) & \text{if } x \in K. \end{cases}$$

Choose a point $z_o$ in the intersection of $\Omega$ with the bounded component of $\mathbb{R}^N \setminus L$ such that

$$|z_o - y| > \max_{x \in K} |x - y|.$$

Then, for $\Pi = L \cup K$, and $0 < 2\delta < 1/j$, there exists, by Lemma 4, a $v_o$ harmonic on $\mathbb{R}^N \setminus \{z_o\}$, such that $\max_{x\in\Pi} |v_o(x) - w(x)| < \delta$. Now, since $v_o$ is harmonic in the ball $B(y, |z_o - y|)$ and $K$ is a compact subset of this ball, we may choose $n$ so large that

$$\max_{x \in K} |S_n(v_o, y)(x) - v_o(x)| < \delta.$$

Thus, $\max_{x\in L} |v_o(x) - u(x)| < \epsilon/2$. However, $v_o$ may not be in $\mathcal{H}(\Omega)$, because of a possible singularity at $z_o$. A second application of Lemma 4 will get rid of this singularity in $\Omega$. Indeed, by Lemma 4, given $\eta > 0$, there exists a function $v \in \mathcal{H}(\Omega)$ such that $|v - v_o| < \eta$ on $L$. If $\eta < \epsilon/2$, then $\max_{x\in L} |v(x) - u(x)| < \epsilon$. Moreover, since $y$ lies in the interior of $L$, by choosing $\eta$ sufficiently small, we may assure that all of the partial derivatives of $v$ of order less than or equal to $n$ at $y$ are so close to those of $v_o$, that $v \in U_{i,j,n}$. □

In Theorem 2, we considered functions which are universal with respect to a specific point $y \in \Omega$. We shall now consider functions which are universal with respect to *all* points in $\Omega$. The price we shall pay is to restrict the class of compacta $K$ on which we have overconvergence to those compacta in the complement of $\Omega$ which do not meet the boundary of $\Omega$. Let us say that a function $v \in \mathcal{H}(\Omega)$ is *weakly* universal pointwise with respect to $\Omega$ if for each point $y \in \Omega$ and each compact set $K \subset \mathbb{R}^N$ disjoint from $\overline{\Omega}$ and with $\mathbb{R}^N \setminus K$ connected and for each $u \in \mathcal{H}(K)$, there is an increasing sequence of natural numbers $\{n_k\}$, such that

$$\max_{x \in K} |S_{n_k}(v, y)(x) - u(x)| \to 0 \quad (k \to \infty).$$

The word *weakly* has been introduced to signify that $K$ is not allowed to meet the boundary of $\Omega$. Let $\mathcal{U}_p(\Omega)$ denote the family of functions $v \in \mathcal{H}(\Omega)$ which are weakly universal pointwise with respect to $\Omega$. We shall show that the family $\mathcal{U}_p(\Omega)$ is residual in $\mathcal{H}(\Omega)$. The idea of the proof is that for a fixed point $y$, we can allow $y$ to move in a small ball centered at $y$. The set $\Omega$ can be covered by countably many such balls.

**Theorem 3.** *Let $\Omega$ be an open set in $\mathbb{R}^N$ with $\mathbb{R}^N \setminus \Omega$ a continuum. Then, the family $\mathcal{U}_p(\Omega)$ of functions which are weakly universal pointwise with respect to $\Omega$ is residual in $\mathcal{H}(\Omega)$.*

*Démonstration.* Let $K$ be a compact subset of $\mathbb{R}^N \setminus \overline{\Omega}$. For each $y \in \Omega$, there is a $z \in \mathbb{R}^N \setminus \overline{\Omega}$, $0 < \mu < 1$, and $\delta > 0$, such that for each $y' \in \overline{B}(y, \delta)$, we have that $K \subset B(y', \mu R)$ and $z \notin \overline{B}(y', R)$. We may assume that $\overline{B}(y, \delta) \subset \Omega$. Since $\Omega$ is Lindelöf, there exist $y_m, z_m, \mu_m, R_m, \delta_m$, $m = 1, 2 \cdots$, such that

$$\Omega = \bigcup_{m=1}^{\infty} B(y_m, \delta_m),$$

and for each $m$ and each $y \in \overline{B}(y_m, \delta_m)$, we have $K \subset B(y, \mu_m R_m)$ and $z_m \notin \overline{B}(y, R_m)$. Let $\{K_k\}$ be an exhaustion of $\mathbb{R}^N \setminus \overline{\Omega}$ by compact sets, for each of which $\mathbb{R}^N \setminus K_k$ is connected. For each $K_k$, we have an associated countable set of data $y_{km}, z_{km}, \mu_{km}, R_{km}, \delta_{km}$, $m = 1, 2 \cdots$. For simplicity, we shall denote $B(y_{km}, \delta_{km})$ by $B_{km}$.

Let $\{p_i\}$ be an enumeration of the family of harmonic polynomials with rational coefficients. Then, since for each $k$, the family $\{B_{km}\}$ is a cover of $\Omega$,

$$\mathcal{U}_p(\Omega) \supset \bigcap_{i,j,k,m} \bigcup_{n} U_{i,j,k,m,n},$$

where $U_{i,j,k,m,n}$ consists of those functions $v \in \mathcal{H}(\Omega)$ such that

$$\max_{x \in K_k\, y \in B_{km}} |S_n(v, y)(x) - p_i(x)| < 1/j.$$

As in the proof of Theorem 1, each $U_{i,j,k,m,n}$ is open and to conclude the proof, it is sufficient to show that, for each $i, j, k, m$,

$$\bigcup_{n} U_{i,j,k,m,n}$$

is dense in $\mathcal{H}(\Omega)$. Suppose, then, that $u \in \mathcal{H}(\Omega)$, $L$ is a compact subset of $\Omega$ and $\epsilon > 0$. We seek an $n$ and a function $v \in U_{i,j,k,m,n}$ such that $\max_{x \in L} |v(x) - u(x)| < \epsilon$. We may assume that $\mathbb{R}^N \setminus L$ has precisely two components, one of which is unbounded, while the other is bounded and contains $\mathbb{R}^N \setminus \Omega$. We may assume that $1/j < \epsilon$. Set

$$w(x) = \begin{cases} u(x) & \text{if } x \in L \\ p_i(x) & \text{if } x \in K_k. \end{cases}$$

For $\Pi = L \cup K_k$, there exists, by Lemma 4, a $v$ harmonic on $\mathbb{R}^N \setminus \{z_{km}\}$, such that $\max_{x\in\Pi} |v(x) - w(x)| < 1/(2j)$. Now, for each $y \in \overline{B}_{km}$, the function $v$ is harmonic in the ball $\overline{B}(y, R_{km})$ and uniformly bounded by some $M_{km}$ over all of these balls. Moreover $K_k$ is a compact subset of each $B(y, \mu_{km}R_{km})$. Thus, by Lemma 2, we may choose $n$ so large that

$$\max_{x\in K_k\, y\in B_{km}} |S_n(v, y)(x) - v(x)| < 1/(2j).$$

Thus, $v \in U_{i,j,k,m,n}$ and $\max_{x\in L} |v(x) - u(x)| < \epsilon$. □

**Acknowlegment.** We thank V. Nestoridis for pointing out that the results [7] and [3] on universal approximation did not allow the compact set $K$ to meet the boundary. The case where $K$ is allowed to meet the boundary was obtained after 25 years in [10].

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# Chapitre 2

# Universal Series from Fundamental Solutions of the Laplace Operator

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## 2.1 abstract

We investigate the approximation of solutions of the Laplace equation on compact subsets of $\mathbb{R}^N$, by universal series whose terms are translates of the fundamental solution of the same equation. The singularities of the fundamental solutions lie on a prescribed set outside the domain of approximation. We also show that, in the case of translates, the sequence $a = (a_n)_{n\in\mathbb{N}}$ of coefficients of the universal series may be chosen in $\cap_{p>1}\ell^p(\mathbb{N})$.

## 2.2 Introduction

Let $R$ be a fundamental solution of the Cauchy–Riemann operator and $R^{(k)}$ its derivative of order $k$. Stefanopoulos [12] recently proved the existence of series of the form

$$\sum_{k=1}^{\infty} c_k R(z - s_k), \; \sum_{k=0}^{\infty} c_k R^{(k)}(z - s)$$

having universal approximation properties for the space of functions that are holomorphic in a neighborhood of a fixed compact set $K$ with connected complement, or on the complement $\Omega$ of a closed half line. where $s_k, k = 1, 2, \cdots$, and $s$ are fixed points lying outside $K$ or $\Omega$ respectively. Our aim is to prove analogous results in the case of harmonic functions. We denote by $\Phi$ the standard fundamental solution of the Laplace equation. The method that we use is based on the involutive property of the Kelvin transform and the overconvergence results obtained by Armitage, Gauthier and Tamptsé in [1] and [6]. For $E$ a subset of $\mathbb{R}^N$, we denote by $\mathcal{H}(E)$ the space of harmonic functions in the neighborhood of $E$. For $K$ a compact subset of $\mathbb{R}^N$, we endow the space $\mathcal{H}(K)$ with the topology of uniform convergence on $K$, and if $\Omega$ is an open subset of $\mathbb{R}^N$, we endow $\mathcal{H}(\Omega)$ with the topology of uniform convergence on compact subsets of $\Omega$.

Let $x = (x_1, x_2, \cdots, x_N)$ be a point in $\mathbb{R}^N$ and $|x| = (x_1^2 + x_2^2 + \cdots + x_N^2)^{1/2}$ the euclidian norm of $x$. In the sequel, we shall denote by $\Phi$ the harmonic function on $\mathbb{R}^N \setminus \{0\}$ defined by :

$$\Phi(x) = \begin{cases} [(2-N)NV_N]^{-1}|x|^{2-N} & \text{if } N \geq 3 \\ (2\pi)^{-1} \ln |x| & \text{if } N = 2. \end{cases}$$

where $V_N$ is the volume of the unit ball in $\mathbb{R}^N$. Then, in the distributional sense, we have

$$\triangle\Phi = \delta$$

where $\triangle$ is the Laplace operator and $\delta$ is the Dirac distribution at the origin. $\Phi$ is a fundamental solution of the Laplace operator in $\mathbb{R}^N$ with pole at the origin of the space. We recall that the same is true for all functions of the form $\Phi + h$, where $h$ is harmonic in the whole space $\mathbb{R}^N$. In this sense, most results we shall establish will be valid also for these functions.

## 2.3 Review of some abstract results in universal series

We recall here some of the results we shall use in our work. For more details on universal series, we refer to [7], [9],[4].

In the sequel we shall write $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

**Definition 1.** *Let $X$ be a metrisable topological vector space (we assume that the topology of $X$ is induced by a translation invariant metric). Let $(x_j)_{j\in\mathbb{N}_0}$ be a sequence of points of $X$. The series $\sum_{j=0}^{\infty} x_j$ is said to be universal in $X$, if the sequence of its partial sums $\sum_{j=0}^{n} x_j, n = 0, 1, 2, \cdots$ is dense in $X$.*

Let $(x_j)_{j\in\mathbb{N}_0}$ be a fixed sequence in $X$, a sequence $a = (a_0, a_1, \cdots) \in \mathbb{R}^{\mathbb{N}_0}$ belongs to the class $\mathcal{U}$ if the series $\sum_{j=0}^{\infty} a_j x_j$ is universal in $X$. Thus each sequence $(a_j)_{j\in\mathbb{N}_0} \in \mathcal{U}$ generates an unrestricted universal series, provided that $\mathcal{U} \neq \emptyset$. $\mathcal{U}$ is the set of unrestricted universal series (with respect to the sequence $(x_j)_{j\in\mathbb{N}_0}$). We mention here an important result regarding general universal series.

**Proposition 1.** *[7] Let $X$ and $\mathcal{U}$ be as above.*

- *If $\mathcal{U}$ is not empty, then it is automatically $G_\delta$ and dense in $\mathbb{R}^{\mathbb{N}_0}$, endowed with the product topology.*
- *$\mathcal{U}$ is not empty if and only if, for every $p \in \mathbb{N}_0$, the linear span of $x_p, x_{p+1}, \cdots$ is dense in $X$.*

Of interest is whether universal series exists, whose sequence of coefficients lie in specific subspaces of $\mathbb{R}^{\mathbb{N}_0}$. Let $A$ be a vector subspace of $\mathbb{R}^{\mathbb{N}_0}$, endowed with a compatible translation invariant metric $\rho_A$. Assume that the following hold :

**(A1)** $(A, \rho_A)$ is complete.

**(A2)** The projections $\pi_k : A \longrightarrow \mathbb{R}$, where $\pi_k((a_j)_{j\in\mathbb{N}_0}) = a_k$ are continuous, for all $k \in \mathbb{N}_0$.

**(A3)** Let $G = \left\{(a_j)_{j\in\mathbb{N}_0} \in \mathbb{R}^{\mathbb{N}_0} : \{j : a_j \neq 0\} \text{ is finite}\right\}$. Then $G \subset A$.

**(A4)** Let $(e_j)_{j\in\mathbb{N}_0}$be the canonical basis of $\mathbb{R}^{\mathbb{N}_0}$. then for every $a = (a_j)_{j\in\mathbb{N}_0} \in A$

$$\rho_A\left(a, \sum_{j=0}^{k} a_j e_j\right) \longrightarrow 0, \quad k \longrightarrow \infty.$$

**Theorem 4.** *[9] [4] Under the previous assumptions the following are equivalent :*

**(1)** $\mathcal{U} \cap A \neq \emptyset$.

**(2)** *For every* $p \in \mathbb{N}_0, x \in X$ *and* $\epsilon > 0$, *there exist* $n > p$ *and* $a_p, a_{p+1}, \cdots, a_n \in \mathbb{R}$ *such that*

$$d\left(\sum_{j=p}^{n} a_j x_j, x\right) < \epsilon \quad and \quad \rho_A\left(\sum_{j=p}^{n} a_j e_j, 0\right) < \epsilon.$$

**(3)** *For every* $x \in X$ *and* $\epsilon > 0$, *there exist* $n \in \mathbb{N}$ *and* $a_0, a_1, \cdots, a_n$ *such that*

$$d\left(\sum_{j=0}^{n} a_j x_j, x\right) < \epsilon \quad and \quad \rho_A\left(\sum_{j=0}^{n} a_j e_j, 0\right) < \epsilon.$$

**(4)** $\mathcal{U} \cap A$ *is* $G_\delta$ *and dense in* $A$.

**Remark 1.** **(a)** *If the series* $\sum_{j=0}^{\infty} a_j x_j$ *is universal in* $X$, *then, for every* $p \in \mathbb{N}_0$, *the series* $\sum_{j=p}^{\infty} a_j x_j$ *is universal in* $X$. *Thus, if* $a = (a_j)_{j \in \mathbb{N}_0} \in \mathcal{U}$, *then* $a^p = (a_j)_{j \geq p} \in \mathcal{U}$, *for each* $p \in \mathbb{N}_0$.

**(b)** *For every* $a = (a_j)_{j \in \mathbb{N}_0} \in \mathcal{U} \cap A, z \in X$, *there exists an increasing sequence* $(n_k)_{k \in \mathbb{N}_0}$ *in* $\mathbb{N}_0$ *such that*

$$\sum_{j=0}^{n_k} a_j x_j \longrightarrow z \quad k \longrightarrow \infty.$$

Since we shall also study in $\mathbb{R}^N$ universal series of derivatives which are in the form $\sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha D^\alpha \Phi$, we must formulate these results for generalized sequences. Let $N \in \mathbb{N}$, $N \geq 2$ be a fixed integer, we denote by $\Pi_\alpha \mathbb{R}_\alpha$ ( where each $\mathbb{R}_\alpha$, is a copy of $\mathbb{R}$) the vector space

$$\Pi_\alpha \mathbb{R}_\alpha = \{a = (a_\alpha)_{\alpha \in \mathbb{N}_0^N}\}.$$

The space $\Pi_\alpha \mathbb{R}_\alpha$ is metrisable, explicitly, if we set

$$\rho(a, b) = \sum_{j=0}^{\infty} \frac{1}{N_j 2^j} \sum_{|\alpha|=j} \frac{|a_\alpha - b_\alpha|}{1 + |a_\alpha - b_\alpha|},$$

where $N_j = card\{a_\alpha : |\alpha| = j\}$, then $\rho$ is a complete metric on $\Pi_\alpha\mathbb{R}_\alpha$, hence $\Pi_\alpha\mathbb{R}_\alpha$ is a Baire space. We note that the subspace

$$G = \left\{a = (a_\alpha)_{\alpha\in\mathbb{N}_0^N} \in \Pi_\alpha\mathbb{R}_\alpha : \{\alpha : a_\alpha \neq 0\} \text{ is finite}\right\}$$

is dense in $\Pi_\alpha\mathbb{R}_\alpha$.

Let $(X, d)$ be a topological vector space, endowed with a translation invariant metric $d$. Let $u = \{u_\alpha, \alpha \in \mathbb{N}_0^N\}$ be a fixed generalized sequence in $X$, we denote by $\widetilde{\mathcal{U}}$ the class of generalized sequences $a = \{a_\alpha, \alpha \in \mathbb{N}_0^N\}$ in $\Pi_\alpha\mathbb{R}_\alpha$ such that the partial sums $\sum_{j=0}^n \sum_{|\alpha|=j} a_\alpha u_\alpha, n = 0, 1, 2, \cdots$ form a dense subset of $X$. We deduce from Theorem 4 the following theorem

**Theorem 5.** *Under the previous assumptions the following are equivalent.*

(1) $\widetilde{\mathcal{U}} \neq \emptyset$.

(2) *There exists real numbers $b_\alpha, \alpha \in \mathbb{N}_0^N$ such that for each $p \in \mathbb{N}_0$, the linear span of $\sum_{|\alpha|=p} b_\alpha u_\alpha, \sum_{|\alpha|=p+1} b_\alpha u_\alpha, \cdots$ is dense in $X$.*

(3) *For every $p \in \mathbb{N}_0, x \in X$ and $\epsilon > 0$, there exist $n > p$ and real numbers $c_\alpha, p \leq |\alpha| \leq n$ such that*

$$d\left(\sum_{j=p}^{n}\sum_{|\alpha|=j} c_\alpha u_\alpha, x\right) < \epsilon \quad \text{and} \quad \rho(c_p^n, 0) < \epsilon,$$

*where $(c_p^n)_\alpha = c_\alpha$ if $p \leq |\alpha| \leq n$ and $0$ otherwise.*

(4) *$\widetilde{\mathcal{U}}$ is $G_\delta$ and dense in $\Pi_\alpha\mathbb{R}_\alpha$.*

*Démonstration.* (1)⇒(2) For $a \in \widetilde{\mathcal{U}}, p \in \mathbb{N}_0$ and $x \in X$, there exist $n > p$ such that the sum $\sum_{j=0}^n \sum_{|\alpha|=j} a_\alpha u_\alpha$ approximates $x + \sum_{j=0}^p \sum_{|\alpha|=j} a_\alpha u_\alpha$. Thus, we can take $b_\alpha = a_\alpha$. Let us note that we have also just shown that if the series $\sum_{j=0}^\infty \sum_{|\alpha|=j} a_\alpha u_\alpha$ is universal in $X$, then for all $p \in \mathbb{N}$ the series $\sum_{j=p}^\infty \sum_{|\alpha|=j} a_\alpha u_\alpha$ are also universal in $X$.

(2)⇒ (1) Assume that (2) hold, set $x_j = \sum_{|\alpha|=j} b_\alpha u_\alpha; j = 0, 1, \cdots$. Then, the result follows from Proposition 1.

(1) ⇒ (3) Let $a \in \widetilde{\mathcal{U}}$ then for all $p \in \mathbb{N}_0, a^p = (a_\alpha)_{|\alpha|\geq p} \in \widetilde{\mathcal{U}}$. If $p$ is sufficiently large, we have $\rho(a^p, 0) < \epsilon/2$. since the series $\sum_{j=p}^\infty \sum_{|\alpha|=j} a_\alpha u_\alpha$ is universal, we can find $n \geq p$ such that $d(\sum_{j=p}^n \sum_{|\alpha|=j} a_\alpha u_\alpha, x) < \epsilon/2$ and $\rho((a_\alpha)_{p\leq|\alpha|\leq n}, a^p) < \epsilon/2$. By the triangle inequality, we have $\rho((a_\alpha)_{p\leq|\alpha|\leq n}, 0) < \epsilon$. We set $c_\alpha = 0, 0 \leq |\alpha| < p$ and $c_\alpha = a_\alpha, p \leq |\alpha| \leq n$ then, we obtain (3).

(3) $\Rightarrow$(4) Since (3) is satisfied, $X$ is separable. Let $(v_m)_{m\in\mathbb{N}}$ be a dense denumerable subset of $X$. For each $m, s \in \mathbb{N}, n \in \mathbb{N}_0$, let

$$E(m,n,s) = \left\{ c \in \Pi_\alpha \mathbb{R}_\alpha : d\left( \sum_{j=0}^{n} \sum_{|\alpha|=j} c_\alpha u_\alpha, v_m \right) < 1/s \right\}.$$

Then, we have $\widetilde{\mathcal{U}} = \cap_{m,s} \cup_{n=0}^{\infty} E(m,n,s)$. Since $\Pi_\alpha \mathbb{R}_\alpha$ is complete, we can use Baire's theorem. For each $m \in \mathbb{N}, n \in \mathbb{N}_0, c \in \Pi_\alpha \mathbb{R}_\alpha$, set

$$\varphi_{m,n}(c) =: d\left( \sum_{j=0}^{n} \sum_{|\alpha|=j} c_\alpha u_\alpha, v_m \right).$$

Then $\varphi_{m,s}$ is a continuous function. Thus $E(m,n,s) = \{c : \varphi_{m,n}(c) < 1/s\}$ is open, therefore, $\widetilde{\mathcal{U}}$ is $G_\delta$ in $\Pi_\alpha \mathbb{R}_\alpha$. It remains to prove that $\cup_n E(m,n,s)$ is dense in $\Pi_\alpha \mathbb{R}_\alpha$.

Let $a \in G, \epsilon > 0$. Since $G$ is dense in $\Pi_\alpha \mathbb{R}_\alpha$, it suffices to find $n \in \mathbb{N}_0, b \in E(m,n,s)$ such that $\rho(a,b) < \epsilon$.

The sum

$$\sum_{j=0}^{\infty} \sum_{|\alpha|=j} a_\alpha u_\alpha,$$

is finite since $a \in G$. Let $x = v_m - \sum_{j=0}^{\infty} \sum_{|\alpha|=j} a_\alpha u_\alpha$. By (3) with $p = 0$, there exist $c \in F, M \in \mathbb{N}_0$ with $c_\alpha = 0, |\alpha| > M$ such that $\rho(c,0) < \epsilon, d(\sum_{j=0}^{M} \sum_{|\alpha|=j} c_\alpha u_\alpha, v_m - \sum_{j=1}^{\infty} \sum_{|\alpha|=j} a_\alpha u_\alpha) < 1/s$. We set $b = a + c$ and we choose $n$ such that $c_\alpha = a_\alpha = 0$ for $|\alpha| > n$ . Then we have $\rho(a,b) = \rho(c,0) < \epsilon$ and $d(\sum_{j=0}^{n} \sum_{|\alpha|=j} b_\alpha u_\alpha, v_m) < 1/s$.

(4)$\Rightarrow$ (1) This is obvious.

□

In the next theorem, we have a characterization of two classes of universal series in the same space $X$.

**Theorem 6.** *Let $X$ be a metrisable topological vector space, $x = (x_j)_{j\in\mathbb{N}_0}$ a fixed sequence of points of $X$, and let $a = (a_j)_{j\in\mathbb{N}_0}$ a sequence of real numbers such that the series $\sum_{j=0}^{\infty} a_j x_j$ is universal in $X$. Let $y = (y_j)_{j\in\mathbb{N}_0}$ be another sequence of points of $X$. The following are equivalent.*

*(i) For each $p \in \mathbb{N}_0$, $x_p$ can be approximated by finite linear combinations of elements of $\{y_j : j \geq p\}$.*

*(ii) There exists a sequence of real numbers $b = (b_j)_{j\in\mathbb{N}_0}$ such that the series $\sum_{j=0}^{\infty} b_j y_j$ is universal in $X$.*

*Démonstration.* (i)⇒ (ii) From Proposition 1, we have to prove that for each $p \in \mathbb{N}_0$, the span $\{y_n, n \geq p\}$ is dense in $X$. But from Proposition 1, the span $\{x_q, q \geq p\}$ is dense in $X$. Thus it suffices to prove that each $x_q, q \geq p$ can be approximated by a finite linear combination of $y_n, n \geq p$ which is the case by using (i).

(ii)⇒ (i) follows from the definition of universal series. □

## 2.4 Approximation by translates and derivatives of fundamental solutions

We now investigate approximation of functions harmonic on neighborhoods of compact subsets of $\mathbb{R}^N$ by translates and derivatives of fundamental solutions of the Laplace equation. In the sequel, the notations $(D^\alpha\Phi)(x-a)$, $D^\alpha\Phi(x-a)$ and $D^\alpha[\Phi(x-a)]$ are equivalent. The following result is known for solutions of differential operators which have a fundamental solution (see [5] p.78) : each homogeneous solution can be written as a superposition of translates of the fundamental solution.

**Proposition 2.** *[2] Let h be a function harmonic in a neighborhood of a compact set K. Then, given $\epsilon > 0$, there exist points $a_1, a_2, \cdots, a_n \in \mathbb{R} \setminus K$ and real numbers $c_1, c_2, \cdots, c_n$ such that*

$$\max_{x \in K} \left| h(x) - \sum_{k=1}^{n} c_k \Phi(x - a_k) \right| < \epsilon.$$

One of the principal results in approximation by translates and derivatives of a fundamental solution of an elliptic operator is theorem 5.3.2 in [13]. The case of the Laplace operator is the following.

**Theorem 7.** *[13] Let K be a compact subset of $\mathbb{R}^N$ and $\sigma$ a subset of $\mathbb{R}^N \setminus K$, such that any harmonic function on $\mathbb{R}^N \setminus K$ vanishing up to order A on $\sigma$ is identically zero. Then for each harmonic function h on K and for each $\epsilon > 0$, there exist points $a_1, \cdots, a_n \in \sigma$, an integer $m \leq A$ and scalars $(c_{\alpha,j})$, $|\alpha| \leq m$, $1 \leq j \leq n$ such that*

$$\left| h(x) - \sum_{|\alpha| \leq m} \sum_{j=1}^{n} c_{\alpha,j} D^\alpha \Phi(x - a_j) \right| < \epsilon, \quad x \in K.$$

As a consequence of this theorem, we have the following corollary which corresponds to the cases $A = 0$ and $A = \infty$ respectively.

**Corollary 1.** *Let $K$ be a compact subset of $\mathbb{R}^N$.*

*(i) Let $\sigma$ be an open subset of $\mathbb{R}^N \setminus K$ such that $\sigma$ meet each complementary component of $K$ and let $E$ be a dense subset of $\sigma$. Then, given $h \in \mathcal{H}(K)$ and $\epsilon > 0$ there exist an integer $n$, points $a_1, \cdots, a_n \in E$ and real numbers $c_1, \cdots, c_n$ such that*

$$\left| h(x) - \sum_{n=1}^{n} c_k \Phi(x - a_k) \right| < \epsilon, \quad x \in K.$$

*(ii) Let the complement of $K$ be connected and let $a \in \mathbb{R}^N \setminus K$. For each $h \in \mathcal{H}(K)$ and $\epsilon > 0$, there exist an integer $n$ and real numbers $c_\alpha$, $|\alpha| \leq n$ such that*

$$\left| h(x) - \sum_{|\alpha| \leq n} c_\alpha D^\alpha \Phi(x - a) \right| < \epsilon, \quad x \in K.$$

## 2.5 Universal series of fundamental solutions

In this section we shall study approximation of functions harmonic on compact subsets of $\mathbb{R}^N$ by universal series of translates and derivatives of a fundamental solution of the Laplace equation.

**Proposition 3.** *Let $K$ be a compact subset of $\mathbb{R}^N$, $\sigma \subset \mathbb{R}^N \setminus K$ an open subset that meets each complementary component of $K$ and let $(a_n)_{n \in \mathbb{N}_0}$ be a countable dense subset of $\sigma$. There exists a sequence of real numbers $(c_n)_{n \in \mathbb{N}_0}$ with the property that for each $h \in \mathcal{H}(K)$, there exists an increasing sequence $(n_k)_{k \in \mathbb{N}_0}$ in $\mathbb{N}_0$ such that*

$$\max_{x \in K} \left| h(x) - \sum_{j=0}^{n_k} c_j \Phi(x - a_j) \right| \longrightarrow 0, \quad (k \longrightarrow \infty).$$

*Moreover, the set $\mathcal{U}$ of such sequences $(c_n)_{n \in \mathbb{N}_0}$ is $G_\delta$ and dense in $\mathbb{R}^{\mathbb{N}_0}$endowed with the cartesian topology.*

*Démonstration.* From Proposition 1, it is enough to prove that, given $p \in \mathbb{N}_0$ the function $h$ can be approximated by a finite linear combination from $\{\Phi(x - a_p), \Phi(x - a_{p+1}), \cdots\}$. This follows from Corollary 1 with $E = (a_n)_{n \geq p}$. □

Let $\overline{\mathbb{R}}^N = \mathbb{R}^N \cup \{\infty\}$ be the Alexandroff compactification of $\mathbb{R}^N$. We consider the inversion on $\overline{\mathbb{R}}^N$ relatively to the unit ball, defined by

$$x^* = \begin{cases} x/|x|^2 & \text{if } x \neq 0, \infty \\ \infty & \text{if } x = 0 \\ 0 & \text{if } x = \infty. \end{cases} \tag{2.1}$$

For a subset $E$ of $\overline{\mathbb{R}}^N$, we denote by $E^* = \{x^*, x \in E\}$ its inverse. Let $h$ be a harmonic function in the neighborhood of a subset $E \subset \mathbb{R}^N \setminus \{0\}$. We consider the Kelvin transform $\mathcal{K}[h]$ of $h$ defined in the neighborhood of $E^*$ by :

$$\mathcal{K}[h](x^*) = |x^*|^{2-N} h(x).$$

We recall that when $N = 2, \mathcal{K}[h](x^*) = h(x)$ and $h$ is harmonic in the neighborhood of $E$ if and only if $\mathcal{K}[h]$ is harmonic in the neighborhood of $E^*$. For $p = \sum_{|\alpha| \leq n} a_\alpha x^\alpha$ a polynomial, we denote by $p(D)$ the differential operator $p(D) = \sum_{|\alpha| \leq n} a_\alpha D^\alpha$. For $m \in \mathbb{N}_0$, we denote by $\mathcal{H}_m(\mathbb{R}^N)$ the vector subspace of $\mathcal{H}(\mathbb{R}^N)$ whose elements are homogeneous polynomials of degree $m$.

**Lemma 5.** *[3, Corollary 5.20] For $p \in \mathcal{H}_m(\mathbb{R}^N)$, we have*

$$p(x^*) = \mathcal{K}[p(D)\Phi](x^*)/\gamma_m$$

*with $\gamma_0 = 1$*

$$\text{and } \gamma_m = \begin{cases} \dfrac{1}{(2-N)NV_N} \Pi_{k=0}^{m-1}(2-N-2k), & N \geq 3 \\ \dfrac{1}{2\pi}(-2)^{m-1}(m-1)!, & N = 2 \end{cases} \quad \text{when } m \geq 1.$$

*Where $V_N$ is the volume of the unit ball in $\mathbb{R}^N$.*

**Theorem 8.** *Let $a \in \mathbb{R}^N, r > 0$. There exists a generalized sequence of real numbers $c = \{c_\alpha \in \mathbb{R}, \alpha \in \mathbb{N}_0^N\}$ such that the series $\sum_{j=0}^{\infty} \sum_{|\alpha|=j} c_\alpha D^\alpha \Phi(x-a)$ is locally uniformly convergent in $\mathbb{R}^N \setminus \overline{B}(a,r)$ and for every compact set $K \subset B(a,r) \setminus \{a\}$ with connected complement and each $h \in \mathcal{H}(K)$ there exists an increasing sequence $(n_k)$ in $\mathbb{N}_0$ such that*

$$\max_{x \in K} \left| h(x) - \sum_{j=0}^{n_k} \sum_{|\alpha|=j} c_\alpha D^\alpha \Phi(x-a) \right| \longrightarrow 0, \quad (k \longrightarrow \infty).$$

*Démonstration.* We may assume that $a = 0$ and $r = 1$. From Theorem 1 of [1] (see also [6]), there exist homogeneous harmonic polynomials $p_m(x^*) = \sum_{|\alpha|=m} a_\alpha (x^*)^\alpha$ such that the series $\sum_{m=0}^{\infty} p_m$ converges locally uniformly to a harmonic function $v$ on $B(0,1)$ and is universal in $\mathcal{H}(K^*)$, for all compact subsets $K^* \subset \mathbb{R}^N \setminus \overline{B}(0,1)$ with connected complement. Thus, for all compact sets $K$ of $B(0,1) \setminus \{0\}$, with connected complement, for all $h \in \mathcal{H}(K)$ and for all $\epsilon > 0$, there exists an integer $n$ such that

$$\max_{x^* \in K^*} \left| \sum_{m=0}^{n} p_m(x^*) - \mathcal{K}(h)(x^*) \right| < \epsilon.$$

and when we use Lemma 5 we have

$$\max_{x\in K}\left|\sum_{m=0}^{n}\sum_{|\alpha|=m}\frac{a_\alpha}{\gamma_m}D^\alpha\Phi(x)-h(x)\right|<\epsilon \text{ if } N=2$$

and

$$\max_{x\in K}\left|\sum_{m=0}^{n}\sum_{|\alpha|=m}\frac{a_\alpha}{\gamma_m}D^\alpha\Phi(x)-h(x)\right|<c\epsilon \text{ if } N\geq 3$$

with $c=(\min_{x^*\in K^*}\Phi(x^*))^{-1}$.

Therefore, if

$$c_\alpha=\begin{cases}\dfrac{a_\alpha}{\gamma_m} & \text{if } N=2\\ \dfrac{a_\alpha}{\gamma_m} & \text{if } N\geq 3\end{cases}$$

the partial sums of the series $\sum_{m=0}^{\infty}c_\alpha D^\alpha\Phi$ are dense in $\mathcal{H}(K)$, for all compact subsets $K\subset B(0,1)\setminus\{0\}$ with connected complement. Since the Kelvin transform preserves uniform convergence on compact sets, these series converge on compact sets of $\mathbb{R}^N\setminus\overline{B}(0,1)$. □

**Corollary 2.** *Let $K$ be a compact subset of $\mathbb{R}^N$ with connected complement and let $a\notin K$. Then, there exists a generalized sequence $c=\{c_\alpha\in\mathbb{R},\alpha\in\mathbb{N}_0^N\}$ such that given $h\in\mathcal{H}(K)$, there exists an increasing sequence $(n_k)$ such that*

$$\max_{x\in K}\left|h(x)-\sum_{j=0}^{n_k}\sum_{|\alpha|=j}c_\alpha D^\alpha\Phi(x-a)\right|\longrightarrow 0,\quad (k\longrightarrow\infty).$$

*Moreover, the family $\widetilde{\mathcal{U}}$ of such generalized sequence is $G_\delta$ and dense in $\Pi_\alpha\mathbb{R}_\alpha$ endowed with the cartesian topology.*

*Démonstration.* Since $K$ is compact, there exists $r>0$ such that $K\subset B(a,r)\setminus\{a\}$. Thus, the result follows from Theorem 8 and Theorem 5. □

A proof similar to the proof of Lemma 2.1 of [8](see also [11]) gives the following lemma.

**Lemma 6.** *Let $a\in\mathbb{R}^N$ and $0<r\leq+\infty$. There exists a sequence of compact sets $K_l\subset B(a,r)\setminus\{a\}$, $l=1,2,\cdots$. with connected complement such that every compact set of $B(a,r)\setminus\{a\}$ with connected complement is contained in some $K_l$.*

**Theorem 9.** *Let $a \in \mathbb{R}^N$. There exists a generalized sequence of real numbers $c = \{c_\alpha \in \mathbb{R}, \alpha \in \mathbb{N}_0^N\}$ such that for each compact set $K \subset \mathbb{R}^N \setminus \{a\}$ with connected complement, and each $h \in \mathcal{H}(K)$, there exists an increasing sequence $(n_k)$ in $\mathbb{N}_0$ such that*

$$\max_{x\in K}\left|h(x) - \sum_{j=0}^{n_k}\sum_{|\alpha|=j} c_\alpha D^\alpha \Phi(x-a)\right| \longrightarrow 0, \quad (k \longrightarrow \infty).$$

*Moreover, the set of $c_\alpha, \alpha \in \mathbb{N}_0^N$ is $G_\delta$ and dense in $\Pi_\alpha \mathbb{R}_\alpha$ endowed with the cartesian topology.*

*Démonstration.* We use Lemma 6 to obtain compact sets $K_l$ with connected complement such that each compact set of $B(a,r) \setminus \{a\}$ with connected complement is contained in some $K_l$. By Corollary 2, for each $l$, we can find a family of generalized sequences $C^l = \{c_\alpha^l, \alpha \in \mathbb{N}_0^N\}$ that is $G_\delta$ and dense in $\Pi_\alpha \mathbb{R}_\alpha$. Thus, $C = \cap_{l\in\mathbb{N}} C^l$ satisfies the theorem.

Indeed, Let $K$ be a compact subset of $\mathbb{R}^N \setminus \{0\}$ with connected complement, $h \in \mathcal{H}(K)$ and $\epsilon > 0$. By the harmonic analog of Runge's theorem (see [14]), there exists a harmonic polynomial $H$ such that

$$|h(x) - H(x)| < \epsilon/2, \quad x \in K, \tag{2.2}$$

and there exists $l \in \mathbb{N}$ such that $K \subset K_l$. Since $H \in \mathcal{H}(K_l)$, for all $c = (c_\alpha) \in C = \cap_{l\in\mathbb{N}} C^l$ there exist $n \in \mathbb{N}$ such that

$$\left|H(x) - \sum_{j=0}^{n}\sum_{|\alpha|=j} c_\alpha D^\alpha \Phi(x)\right| < \epsilon/2, \quad x \in K_l. \tag{2.3}$$

Therefore, via (2.2) and (2.3), we have

$$\left|h(x) - \sum_{j=0}^{n}\sum_{|\alpha|=j} c_\alpha D^\alpha \Phi(x)\right| < \epsilon, \quad x \in K.$$

Thus, the series $\sum_{j=0}^{\infty}\sum_{|\alpha|=j} c_\alpha D^\alpha \Phi$ are universal in $\mathcal{H}(K)$ for all $c \in C$.

□

**Definition 2.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ and $(K_n)_{n\geq 1}$ a sequence of compact subsets of $\Omega$. We say that the sequence $(K_n)_{n\geq1}$ is an exhaustion of $\Omega$ if :*

$$\Omega = \bigcup_{n=1}^{\infty} K_n \quad \textit{and} \quad K_n \subset \overset{o}{K}_{n+1} \ \forall n \geq 1.$$

It is easy to see that if $\Omega$ is an open set of $\mathbb{R}^N$ and $\overline{\mathbb{R}}^N \setminus \Omega$ is connected, then $\Omega$ has an exhaustion by compact subsets with connected complement. Let $\Omega$ be such an open set and, $(K_n)_{n\geq 1}$ an exhaustion of $\Omega$ by compact subsets with connected complement and let $\sigma \subset \mathbb{R}^N \setminus \Omega$ an open non void set. Let $h \in \mathcal{H}(\Omega)$. Set

$$||h||_n = \max_{x\in K_n} |h(x)|$$

We consider the metric on $\mathcal{H}(\Omega)$ defined by

$$d(h,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{||h-g||_n}{1+||h-g||_n} \qquad h,g \in \mathcal{H}(\Omega).$$

This metric is compatible with +,. and invariant under translation.

We introduce some notions regarding analytic sets.

**Definition 3.** *Let $\sigma$ be an open subset of $\mathbb{R}^N$. A set $A \subset \sigma$ is said to be analytic in $\sigma$ if given $a \in \sigma$, there exists a neighborhood $U$ of $a$ in $\sigma$ and analytic functions $f_1, f_2, \cdots, f_m$ defined in $U$ such that*

$$A \cap U = \{x \in U : f_k(x) = 0 \ \textit{for all} \ 1 \leq k \leq m\}.$$

*By the analytic closure $\tilde{E}$, relative to $\sigma$, of a subset $E$ of $\sigma$, we understand the smallest analytic subset of $\sigma$ which contains $E$. We shall say that $E$ is analytically dense in $\sigma$ if the analytic closure $\tilde{E}$ of $E$, relative to $\sigma$ is $\sigma$.*

In particular, we are interested in analytic sets generated by harmonic functions.

**Proposition 4.** *Let $\sigma$ be an open subset of $\mathbb{R}^N$, and $A$ a subset of $\sigma$.*

**(1)** *$A$ is analytic in $\sigma$ if and only if for each connected component $\sigma_j$ of $\sigma$, $A \cap \sigma_j$ is analytic in $\sigma_j$.*

**(2)** *$A$ is analytically dense in $\sigma$ if and only if for each finite subset $F$ of $A$, $A \setminus F$ is analytically dense in $\sigma$.*

*Démonstration.* Proof of (1) :

$\Longrightarrow$ Suppose $A$ is analytic in $\sigma$. Let $\sigma_j$ be a connected component of $\sigma$ and $a \in \sigma_j$. There exists a neighborhood $U$ of $a$ in $\sigma$ and functions $f_1, f_2, \cdots, f_m$ analytic in $U$ such that

$$A \cap U = \{x \in U : f_k(x) = 0 \ \text{for all} \ 1 \leq k \leq m\}.$$

Therefore

$$(A \cap \sigma_j) \cap (U \cap \sigma_j) = \{x \in U \cap \sigma_j : f_k(x) = 0 \text{ for all } 1 \leq k \leq m\}.$$

Thus $A \cap \sigma_j$ is analytic in $\sigma_j$.

$\Longleftarrow$ Suppose $A \cap \sigma_j$ is analytic in $\sigma_j$ for each component $\sigma_j$ of $\sigma$. Let $a \in \sigma$; there exists a $\sigma_j$ such that $a \in \sigma_j$. Thus, there exists a neighborhood $U$ of $a$ in $\sigma_j$ and functions $f_1, f_2, \cdots, f_m$ analytic in $U$ such that

$$(A \cap \sigma_j) \cap U = \{x \in U : f_k(x) = 0 \text{ for all } 1 \leq k \leq m\}.$$

Hence, $A \cap U = \{x \in U : f_k(x) = 0 \text{ for all } 1 \leq k \leq m\}$, since $U \subset \sigma_j$. Therefore, $A$ is analytic in $\sigma$.

Proof of (2) :

$\Longrightarrow$ Suppose $A$ is analytically dense in $\sigma$. Let $F$ be a finite subset of $A$. We must show that $A \setminus F$ is analytically dense in $\sigma$. It suffices to give the proof with $F = \{a\}$.

a) We begin with the case where $\sigma$ is connected. Let $B$ be an analytic subset of $\sigma$ such that $A \setminus \{a\} \subset B$. Then $B \cup \{a\}$ is an analytic subset of $\sigma$ that contains $A$. Since $A$ is analytically dense in $\sigma$, we have $B \cup \{a\} = \sigma$, and since $\sigma$ is connected, we have $a \in B$ and $B = \sigma$.

b) The case where $\sigma$ is not connected. It suffices to use part (1) of the proposition and the proof is the same as in a) with each $\sigma_j$.

$\Longleftarrow$ This is obvious since $A \setminus F \subset A$. □

**Lemma 7.** *Let $K$ be a compact subset of $\mathbb{R}^N$, with connected complement, $\sigma$ an open non void subset of $\mathbb{R}^N \setminus K$ and $E$ a subset of $\sigma$ which is analytically dense in $\sigma$. Given $h \in \mathcal{H}(K)$ and $\epsilon > 0$, there exists a positive integer $n$, elements $a_0, a_1, \cdots, a_n$ of $E$ and real numbers $c_0, c_1, \cdots, c_n$ such that*

$$\left| h(x) - \sum_{k=0}^{n} c_k \Phi(x - a_k) \right| < \epsilon \quad x \in K.$$

*Démonstration.* This is a consequence of Theorem 7. □

**Lemma 8.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ such that $\overline{\mathbb{R}}^N \setminus \Omega$ is connected, and $\sigma$ an open non void subset of $\mathbb{R}^N \setminus \Omega$. Let $(a_n)_{n \in \mathbb{N}_0}$ be a countable analytically dense subset of $\sigma$. Set*

$$\Phi_n(x) = \Phi(x - a_n) \quad x \in \Omega.$$

*Given $h \in \mathcal{H}(\Omega)$, $\epsilon > 0$ and $p \in \mathbb{N}_0$, there exist real numbers $c_0, c_1, c_2 \cdots, c_m$ such that*

$$d\left(h, \sum_{j=0}^{m} c_j \Phi_{p+j}\right) < \epsilon.$$

*Démonstration.* Let $p \in \mathbb{N}_0$, by Proposition 4, the set $E = (a_n)_{n \geq p}$ is analytically dense in $\sigma$, and the result follows from Lemma 7. □

**Theorem 10.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ such that $\overline{\mathbb{R}}^N \setminus \Omega$ is connected, and $\sigma$ an open non void subset of $\mathbb{R}^N \setminus \Omega$. Let $(a_n)_{n \in \mathbb{N}_0}$ be a countable analytically dense subset of $\sigma$. There exists a sequence of real numbers $(c_n)_{n \in \mathbb{N}_0}$ with the property that given a harmonic function $h \in \mathcal{H}(\Omega)$, there exists an increasing sequence of positive integers $(n_k)$ such that, on each compact subset $K \subset \Omega$, we have*

$$\max_{x \in K} \left| D^\alpha h(x) - D^\alpha \sum_{j=0}^{n_k} c_j \Phi(x - a_j) \right| \longrightarrow 0 \quad (k \longrightarrow \infty).$$

*for each multi-index $\alpha \in \mathbb{N}_0^N$. Moreover, the set of such sequences $(c_n)_{n \in \mathbb{N}_0}$ is $G_\delta$ and dense in $\mathbb{R}^{\mathbb{N}_0}$ endowed with the cartesian topology.*

*Démonstration.* By Lemma 8, for each $p \in \mathbb{N}_0$, the linear span of $\{\Phi(x - a_n) : n \geq p\}$ is dense in $\mathcal{H}(\Omega)$. Hence by Proposition 1, there exists a sequence $(c_n)_{n \geq 0}$ such that the series

$$\sum_{j=0}^{\infty} c_j \Phi_j$$

is universal in $\mathcal{H}(\Omega)$. Thus, for each compact set $K \subset \Omega$, there exists an increasing sequence $(n_k)$ in $\mathbb{N}_0$ such that

$$\max_{x \in K} \left| h(x) - \sum_{j=0}^{n_k} c_j \Phi(x - a_j) \right| \longrightarrow 0 \quad (k \longrightarrow \infty).$$

We deduce that for each compact set $K \subset \Omega$, we have

$$\max_{x \in K} \left| D^\alpha h(x) - D^\alpha \sum_{j=0}^{n_k} c_j \Phi(x - a_j) \right| \longrightarrow 0 \quad (k \longrightarrow \infty),$$

for each multi-index $\alpha \in \mathbb{N}_0^N$. □

**Lemma 9.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ such that $\overline{\mathbb{R}}^N \setminus \Omega$ is connected. Let $a \in \mathbb{R}^N \setminus \Omega$. For each $\alpha \in \mathbb{N}_0^N$, set*

$$\Phi_\alpha(x) = D^\alpha \Phi(x - a) \quad x \in \Omega.$$

*There exists a generalized sequence $c = \{c_\alpha, \alpha \in \mathbb{N}_0^N\}$ such that, for all $h \in \mathcal{H}(\Omega)$, $p \in \mathbb{N}_0$ and $\epsilon > 0$, there exists $m \in \mathbb{N}_0$ such that*

$$d\left(h, \sum_{j=0}^{m} \sum_{|\alpha|=p+j} c_\alpha \Phi_\alpha\right) < \epsilon.$$

*Démonstration.* Without loss of generality, we suppose $a = 0$. Let $n \in \mathbb{N}$ be such that

$$\sum_{k=n+1}^{\infty} \frac{1}{2^k} < \frac{\epsilon}{2}. \tag{2.4}$$

By Corollary 2, there exists a generalized sequence $c = \{c_\alpha \in \mathbb{R}; \alpha \in \mathbb{N}_0^N\}$ such that the series

$$\sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha \Phi_\alpha$$

is universal in $\mathcal{H}(K_n)$. As we noted in the proof of Theorem 5, for all $p \in \mathbb{N}$, the series $\sum_{k=0}^{\infty} \sum_{|\alpha|=p+k} c_\alpha \Phi_\alpha$ are also universal in $\mathcal{H}(K_n)$. Thus, for all $h \in \mathcal{H}(K_n)$ there exists $m \in \mathbb{N}$ such that

$$\max_{x \in K_n} \left| h(x) - \sum_{k=0}^{m} \sum_{|\alpha|=p+k} c_\alpha \Phi_\alpha(x) \right| < \frac{\epsilon}{2(n+1)}. \tag{2.5}$$

Set $U_p^m(x) = \sum_{k=0}^{m} \sum_{|\alpha|=p+k} c_\alpha \Phi_\alpha(x)$. Since $\|.\|_l \leq \|.\|_n$, for $l \leq n$, we have by (2.4) and (2.5) that

$$\begin{aligned} d\left(h, \sum_{k=0}^{m} \sum_{|\alpha|=p+k} c_\alpha \Phi_\alpha\right) &= \sum_{k=0}^{n} \frac{1}{2^k} \frac{\|h - U_p^m\|_k}{1 + \|h - U_p^m\|_k} + \sum_{k=n+1}^{\infty} \frac{1}{2^k} \frac{\|h - U_p^m\|_k}{1 + \|h - U_p^m\|_k} \\ &< (n+1)\|h - U_p^m\|_n + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

□

**Theorem 11.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ such that $\overline{\mathbb{R}}^N \setminus \Omega$ is connected. Let $a \in \mathbb{R}^N \setminus \Omega$. There exists a generalized sequence $c = \{c_\alpha, \alpha \in \mathbb{N}^N\}$ with the property that, given $h \in \mathcal{H}(\Omega)$, there exists an increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ such that for all $K \subset \Omega$, we have*

$$\max_{x\in K} \left| D^\beta h(x) - D^\beta \sum_{j=0}^{n_k} \sum_{|\alpha|=j} c_\alpha D^\alpha \Phi(x-a) \right| \longrightarrow 0 \quad (k \longrightarrow \infty),$$

*for each multi-index $\beta$. Moreover, the family of such generalized sequences $\{c_\alpha, \alpha \in \mathbb{N}^N\}$ is $G_\delta$ and dense in $\prod_\alpha \mathbb{R}_\alpha$ endowed with the cartesian topology, where each $\mathbb{R}_\alpha$ is a copy of $\mathbb{R}$.*

*Démonstration.* Via Lemma 9, there exists a generalized sequence $b = \{b_\alpha; \alpha \in \mathbb{N}^N\}$ such that for each $p \in \mathbb{N}$ the linear span of

$$\left\{ \sum_{|\alpha|=k} b_\alpha D^\alpha \Phi(x-a) : k \geq p \right\}$$

is dense in $\mathcal{H}(\Omega)$. Thus, with the use of Theorem 5, there exists a generalized sequence $c = \{c_\alpha \in \mathbb{R}; \alpha \in \mathbb{N}^N\}$ such that the series

$$\sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha D^\alpha \Phi(x-a)$$

is universal in $\mathcal{H}(\Omega)$. Therefore, given $h \in \mathcal{H}(\Omega)$, there exists a sequence $(n_k)$ in $\mathbb{N}$ such that

$$\max_{x\in K_n} \left| h(x) - \sum_{j=0}^{n_k} \sum_{|\alpha|=j} c_\alpha D^\alpha \Phi(x-a) \right| \longrightarrow 0 \quad (k \longrightarrow \infty)$$

for all $n \in \mathbb{N}$. Thus, for each compact set $K$ of $\Omega$, we have

$$\max_{x\in K} \left| D^\beta h(x) - D^\beta \sum_{j=0}^{n_k} \sum_{|\alpha|=j} c_\alpha D^\alpha \Phi(x-a) \right| \longrightarrow 0 \quad (k \longrightarrow \infty)$$

for each multi-index $\beta \in \mathbb{N}^N$. □

## 2.6 The restricted case

In this section, we investigate universal series with coefficients in $\ell^p(\mathbb{N})$, $p > 1$. Note that the spaces $\ell^p(\mathbb{N})$, $p > 1$ satisfy the conditions (A1), (A2), (A3) and (A4). Thus to prove the existence of universal series with coefficients in these subspaces of $\mathbb{R}^{\mathbb{N}}$, it suffices to prove assertion (2) of Theorem 4.

**Lemma 10.** *Let $K$ be a compact subset of $\mathbb{R}^N$ and $L$ a closed subset of $\mathbb{R}^N$ disjoint from $K$. Let $Q$ be a dense subset of $L$. Let $M \in \mathbb{N}$ and $a \in L$. Given $\epsilon > 0$, there exists $\eta > 0$ such that, for all points $a_1, a_2, \cdots, a_M \in Q$ satisfying*

$$|a - a_k| < \eta, \qquad k = 1, 2, \cdots, M,$$

*we have*

$$\left| \Phi(x - a) - \frac{1}{M} \sum_{k=1}^{M} \Phi(x - a_k) \right| < \epsilon, \qquad x \in K.$$

*Démonstration.* Let $\overline{B}$ be a closed ball centered at $a$ and disjoint from $K$. The lemma follows immediately from the uniform continuity of $\Phi(x - y)$, for $(x, y) \in K \times \overline{B}$. □

**Lemma 11.** *Let $\Omega$ and $\sigma$ be as in Lemma 8, $(a_n)_{n \in \mathbb{N}}$ be a dense subset of $\sigma$, and $\Phi_n$ as in Lemma 8. Let $p \in \mathbb{R}$, $p > 1$, $K$ be a compact subset of $\Omega$ with connected complement, and $h \in \mathcal{H}(K)$. Given $\epsilon > 0$, there exists $q \in \mathbb{N}$ and real numbers $b_1, b_2, \cdots, b_q$ such that*

$$\max_{x \in K} \left| h(x) - \sum_{k=1}^{q} b_k \Phi_k(x) \right| < \epsilon$$

*and*

$$\sum_{k=1}^{q} |b_k|^p < \epsilon.$$

*Démonstration.* From Corollary 1, there exist real numbers $c_1, c_2, \cdots, c_n$ such that

$$\max_{x \in K} \left| h(x) - \sum_{j=1}^{n} c_j \Phi_j(x) \right| < \epsilon/2.$$

Let $\vartheta \in \mathbb{N}$ be such that

$$\frac{1}{\vartheta^{p-1}} \sum_{j=1}^{n} |c_j|^p < \epsilon.$$

By Lemma 10, we can find distinct points $a_{j,l} : j = 1, 2, \cdots, n; l = 1, 2, \cdots, \vartheta$ such that

$$\left| c_j\Phi(x - a_j) - \frac{1}{\vartheta}\sum_{l=1}^{\vartheta} c_j\Phi(x - a_{j,l})\right| < \frac{\epsilon}{2n} \quad x \in K.$$

Let $q \in \mathbb{N}$ such that the set $\{a_{j,l} : j = 1, 2, \cdots, n; l = 1, 2, \cdots, \vartheta\} \subset \{a_k : k = 1, 2, \cdots, q\}$. We set

$$b_k = \begin{cases} 0 & \text{if } a_k \notin \{a_{j,l} : j = 1, 2, \cdots, n; l = 1, 2, \cdots, \vartheta\}. \\ \dfrac{c_j}{\vartheta} & \text{if } a_k = a_j, l \end{cases}$$

then

$$\sum_{k=1}^{q} |b_k|^p = \vartheta \sum_{j=1}^{n} \left|\frac{c_j}{\vartheta}\right|^p \tag{2.6}$$

$$= \frac{1}{\vartheta^{p-1}} \sum_{j=1}^{n} |c_j|^p < \epsilon \tag{2.7}$$

On other hand, we have

$$\left| h(x) - \sum_{k=1}^{q} b_k\Phi_k(x)\right| \le \left| h(x) - \sum_{j=1}^{n} c_j\Phi_j(x)\right| + \left|\sum_{j=1}^{n} c_j\Phi_j(x) - \sum_{k=1}^{q} b_k\Phi_k(x)\right|$$

$$\le \left| h(x) - \sum_{j=1}^{n} c_j\Phi_j(x)\right| + \sum_{j=1}^{n} \left| c_j\Phi_j(x) - \frac{c_j}{\vartheta}\sum_{l=1}^{\vartheta} \Phi(x - a_{j,l})\right| \tag{2.8}$$

$$\le \frac{\epsilon}{2} + \sum_{k=1}^{n} \frac{\epsilon}{2n} = \epsilon \tag{2.9}$$

for all $x \in K$. □

**Theorem 12.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ such that $\overline{\mathbb{R}}^N \setminus \Omega$ is connected, and $\sigma$ an open non void subset of $\mathbb{R}^N \setminus \Omega$. Let $(a_n)_{n\in\mathbb{N}}$ be a countable dense subset of $\sigma$. Let $p \in \mathbb{R}, p > 1$. There exists a sequence $(c_n)_{n\ge1}$ in $\ell^p(\mathbb{N})$ with the property that given a harmonic function $h \in \mathcal{H}(\Omega)$ there exists an increasing sequence $(n_k)_{k\ge1}$ in $\mathbb{N}$ such that for any compact set $K \subset \Omega$, we have*

$$\max_{x\in K}\left| h(x) - \sum_{n=1}^{n_k} c_n\Phi(x - a_n)\right| \longrightarrow 0 \quad (k \longrightarrow \infty).$$

*Moreover, the set of such sequences $(c_n)_{n\in\mathbb{N}}$ is $G_\delta$ and dense in $\ell^p(\mathbb{N})$.*

*Démonstration.* From part (3) of Theorem 4, it is enough to prove that, for all $h \in \mathcal{H}(\Omega)$ and $\epsilon > 0$, there exist $q \in \mathbb{N}$ and real numbers $c_1, c_2, \cdots, c_q$ such that

$$d\left(h, \sum_{j=1}^{q} c_j \Phi_j\right) < \epsilon$$

and

$$\sum_{j=1}^{q} |c_j|^p < \epsilon.$$

Let $n \in \mathbb{N}$ such that $\sum_{j=n+1}^{\infty} \frac{1}{2^j} < \frac{\epsilon}{2}$. From Lemma 11, there exists $q \in \mathbb{N}$ and $c_1, c_2, \cdots, c_q \in \mathbb{R}$ such that

$$\max_{x \in K_n} \left| h(x) - \sum_{j=1}^{q} c_j \Phi_j(x) \right| < \frac{\epsilon}{2n}$$

and

$$\sum_{j=1}^{q} |c_j|^p < \frac{\epsilon}{2n} < \frac{\epsilon}{2}.$$

We set $U_q(x) = \sum_{j=1}^{q} c_j \Phi_j(x)$ then, as in the proof of Lemma 9, we have

$$d\left(h, \sum_{j=1}^{q} c_j \Phi_j\right) < \epsilon.$$

□

We endow the space $\cap_{p>1} \ell^p(\mathbb{N})$ with the distance

$$\rho(a, b) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|a - b\|_{1+\frac{1}{j}}}{1 + \|a - b\|_{1+\frac{1}{j}}}$$

where $\|.\|_p$ is the $\ell^p$ norm. The space $(\cap_{p>1} \ell^p, \rho)$ is a Banach space and satisfied conditions (A1),(A2),(A3) and (A4).

We recall the following Lemma 2.1 of [10] :

**Lemma 12.** *[10] Let $(X, d)$ be a real vector space with translation invariant metric $d$ compatible with $+, .$ and let $x = (x_n)_{n \geq 1}$ be a fixed sequence in $X$. Let $\mathcal{U}$ be the set of scalar sequences defining universal series. Let $p_0 \geq 1$ be fixed. The following are equivalent :*

(1) $\mathcal{U} \cap \ell^p \neq \emptyset$ *for every* $p > p_0$.

(2) $\mathcal{U} \cap (\cap_{p>p_0} \ell^p) \neq \emptyset$.

**Corollary 3.** *Under the hypothesis of Theorem 12, there exists a sequence* $(c_n)_{n\geq 1}$ *in* $\cap_{p>1} \ell^p(\mathbb{N})$ *with the property that given a harmonic function* $h \in \mathcal{H}(\Omega)$*, there exists an increasing sequence* $(n_k)_{k\geq 1}$ *in* $\mathbb{N}$ *such that for any compact set* $K \subset \Omega$*, we have*

$$\max_{x\in K} \left| h(x) - \sum_{n=1}^{n_k} c_n \Phi(x - a_n) \right| \longrightarrow 0 \qquad (k \longrightarrow \infty).$$

*Moreover, the set of such sequences* $(c_n)_{n\in\mathbb{N}}$ *is* $G_\delta$ *and dense in* $\cap_{p>1} \ell^p(\mathbb{N})$.

*Démonstration.* This follows from Theorem 12, Lemma 12 and Theorem 4. □

**Proposition 5.** *Let* $\Omega$ *be an open subset of* $\mathbb{R}^N$*. There does not exist a universal series* $\sum_{n=1}^{\infty} c_n \Phi(x - a_n)$ *in* $\mathcal{H}(\Omega)$*, with* $(c_n)_{n\in\mathbb{N}}$ *in* $\ell^1(\mathbb{N})$ *and* $a_n \in \mathbb{R}^N \setminus \Omega$ *bounded.*

*Démonstration.* Suppose that such a series exists. Let $K$ be a compact subset of $\Omega$. Set $M = \sup_{x\in K, y\in\mathbb{R}^n\setminus\Omega} |\Phi(x - y)|$. We have

$$\left| \sum_{n=1}^{\infty} c_n \Phi(x - a_n) \right| \leq M \sum_{n=1}^{\infty} |c_n,|$$

for all $x \in K$. Thus, since the coefficients belong to $\ell^1(\mathbb{N})$, the series converges uniformly on each compact subset of $\Omega$ to a harmonic function. This contradicts the fact that the series is universal in $\mathcal{H}(\Omega)$.

It is to note that when $N \geq 3$, the Proposition remains true without supposing that the sequence $(a_n)$ is bounded. □

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# Conclusion

Les travaux d'Ostrowski ([4],[5],[6]) dans le cas des fonctions holomorphes montrent une corrélation entre la surconvergence et l'existence des lacunes dans la suite des coefficients de la série, qui sont généralement désignées par lacunes d'Ostrowski et définies de la façon suivante :

**Definition 4.** *Soit* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *une série de puissance de rayon de convergence* 1, $S_m(z) = \sum_{n=0}^{m} a_n z^n, m = 0, 1, 2, \cdots$ *ses sommes partielles. On dit que la série possède une suite* $(p_k, q_k, \theta)$ *de lacunes d'Ostrowski s'il existe des entiers naturels*

$$p_1 < q_1 < p_2 < q_2 < \cdots, \text{ avec } \underline{lim}_{p\to\infty} \frac{q_k}{p_k} > 1$$

*et une constante* $\theta \in [0, 1]$ *telle que* $|a_n| \leq \theta^n$ *pour* $p_k \leq n \leq q_k$.

Nous pouvons résumer dans le théorème suivant les principaux résultats obtenu par Ostrowski ([4],[5],[6]) concernant la surconvergence.

**Theorem 13.** *Soit* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *une série de puissance de rayon de convergence* 1.

*1. Si la série possède des lacunes d'Ostrowski* $p_k, q_k, \theta$, *alors la suite* $\{S_{p_k}(z)\}_{k\in\mathbb{N}}$ *converge compactement dans un domaine contenant tout point de* $\{z \in \mathbb{C}; |z| = 1\}$ *en lequel* $f$ *est holomorphe.*

*2. Si la série possède des lacunes d'Ostrowski* $(p_k, q_k, 0)$ *avec* $\lim_{k\to\infty} \frac{q_k}{p_k} = \infty$, *alors le domaine d'analyticité de* $f$ *est simplement connexe et* $\{S_{p_k}(z)\}_{k\in\mathbb{N}}$ *converge compactement sur ce domaine.*

*3. Toutes séries de puissance surconvergentes possèdent des lacunes d'Ostrowski.*

Une question intéressante serait de savoir s'il existe un analogue des lacunes d'Ostrowski pour le dévellopement homogène des fonctions harmoniques ou plus généralement pour les fonctions analytiques de plusieurs variables.

Dans notre premier article, nous avons désigné par $\mathcal{U}(\Omega)$ la famille des fonctions universelles relativement à $\mathcal{H}(\Omega)$ et par $\mathcal{U}(\Omega, y)$ la famille de fonctions universelles relativement à $y$. Il est évident que $\mathcal{U}(\Omega) \subset \mathcal{U}(\Omega, y)$.

A-t-on $\mathcal{U}(\Omega) = \mathcal{U}(\Omega, y)$ ?

Dans le cas des fonctions holomorphes, la réponse est oui, ce résultat a été obtenu par : Gehlen, Luh, et Müller [1], Melas et Nestoridis [2] et tout récemment Müller, Vlachou et Yavrian [3]. Tous utilisent les lacunes d'Ostrowski.

Dans [7] Seleznev, Motova et Volokhin ont obtenu le résultat suivant :

**Theorem 14.** *Soit $r \in \mathbb{R}, 0 < r < \infty$ et $\mathcal{R}$ un sous ensemble dénombrable dense de $\mathbb{C}$. Toute série*

$$\sum_{n=0}^{\infty} a_n z^n$$

*de rayon de convergence $r$ peut s'écrire comme une somme de deux séries universelles surconvergentes*

$$\sum_{n=0}^{\infty} b_n z^n, \sum_{n=0}^{\infty} c_n z^n$$

*de rayon de convergence $r$ et telles que pour tout $n$, au moins un des coefficients $b_n$ ou $c_n$ appartient à $\mathcal{R}$.*

A t-on un tel résultat dans le cas du développement homogène des fonctions harmoniques ?

La même question se poserait pour les séries universelles dont les termes sont les dérivées de la solution fondamentale.

Dans le cas des séries universelles dont les termes sont les translatées de la solution fondamentale, nous avons montré qu'on pouvait trouver des séries à coefficients dans $\ell^p (p > 1)$. Une conséquence est que la suite des coefficients de la série tend vers 0.

Dans le cas des séries dont les termes sont les dérivées de la solution fondamentale $\left(\sum_{n=0}^{\infty} \sum_{|\alpha|=n} a_\alpha D^\alpha \phi\right)$, est-il possible d'obtenir de telle séries universellement surconvergentes avec les coefficients tendant vers 0 ? c'est à dire :

$$a_\alpha \longrightarrow 0 \quad \text{lorsque} \quad |\alpha| \longrightarrow \infty.$$

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