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Public Decisions:

Solidarity and the Status Quo*

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Abstract

A public decision model specifies a fixed set of alternatives A, a variable population, and a fixed set of admissible preferences over A, common to all agents. We study the implications, for any social choice function, of the principle of *solidarity,* in the class of *all* such models. The principle says that when the environment changes, all agents not responsible for the change should all be affected in the same direction: either all weakly win, or all weakly lose. We consider two formulations of this principle: *population-monotonicity* (Thomson, 1983); and *replacement-domination* (Moulin, 1987). Under weak additional requirements, but regardless of the domain of preferences considered, each of the two conditions implies (i) coalition-strategy-proofness; (ii) that the choice only depends on the set of preferences that are present in the society and not on the labels of agents, nor on the number of agents having a particular preference; (iii) that there exists a *status quo point,* i.e. an alternative always weakly Pareto-dominated by the alternative selected by the rule. We also prove that *replacement-domination* is generally at least as strong as *population-monotonicity.*

Key words. Population-monotonicity, replacement-domination, solidarity, strategy-proofness, coalition-strategy-proofness, public decision, status quo.

1 Introduction

Several authors have studied the implications of solidarity conditions in particular models of pure public choice.[1] This paper unifies results obtained in each of these particular models into a general theory, applicable to a large class of pure public decision models.

Solidarity is a general principle of justice. It says that when circumstances change, all agents not responsible for the change should all be affected in the same direction: either they all weakly win, or they all weakly loose. We investigate here two particular formulations of this principle. Population-monotonicity (Thomson, 1983a, 1983b) applies to the arrival and departure of agents. Replacement-domination (Moulin, 1987) applies to changes in preferences. We restrict attention to models of pure public decision.[2]

A model of public decision specifies a fixed set of alternatives A, a variable population, and a fixed common set of admissible preferences over A. This definition implies the three following important assumptions. (i) The set of alternatives A is fixed and does not depend on the population. (ii) Each admissible preference is defined over the fixed set A. (iii) The set of admissible preferences is common to all agents and fixed.

In particular, alternatives are "anonymous", in the sense that they do not contain agent-specific provisions, such as transfers, or the allocation of commodity bundles to particular agents, and that the set of admissible preferences is the same for each agent. This excludes any resource allocation problem with any type of private

[1]See [2] , [3] , [4] , [5] , [9] , [10] , [11] , [12] , [13] , [14] , [21] , [27] and [25] .

[2]Besides models of public decision, *population-monotonicity* and *replacement-domination* have been studied in a very large number of contexts, including baragaining theory, coalitional games, quasi-linear cost allocation problems, fair allocation in economies with private goods, with production, with individible objects, and with single-peaked-preferences. For a survey on *population-monotonicity*, see Thomson (1995) and Thomson (1999b). For a survey on *replacement-domination*, see Thomson (1999a).

consumption, or phenomena such as crowding.

For example, the citizens of a nation choose the location of their capital, the colors and design of the national flag, the philosophical and moral principles underlying the Constitution and the laws governing the nation. A company's executives and board choose a name, an image, etc.

Pure public decision models also serve as benchmarks in the study of models of non-pure public decision, for situations where assumptions *(i)*, *(ii)* and *(iii)* can only be thought of as approximations.

All existing studies on solidarity conditions in pure public choice rely on a particular underlying geometric structure. For example, Thomson (1993) and Ching and Thomson (1997) model the set of alternatives as a segment over which agents have single-peaked preferences (see section 2). Other models depart from this benchmark by analyzing different underlying geometric structures. Our starting point is the observation that, although, in each of these models, solidarity conditions characterize a certain model-specific class of social choice functions, many properties common to all these functions do not depend on the geometry, nor on any other specificity of the model. As we show, solidarity conditions, together with weak additional requirements, have the same implications in any model of pure public decision, regardless of its specific underlying geometry, and regardless of any other type of specificity in the model.

In this paper, we first establish that solidarity with respect to changes in preferences is generally a stronger requirement than solidarity with respect to the arrival and departure of agents, in the following sense. Any social choice function f for a pure public choice problem (with a variable population), that satisfies replacement-domination, together with weak additional conditions, must be population-monotonic (section 4). In addition, any social choice function f for a pure public choice problem, that satisfies at least one of our conditions of solidarity, together with weak additional conditions, must have the following properties. (a) The decision only depends on which preferences are represented in the population by at least one agent, not on how many agents represent each of these preferences, nor on their labels. In other words, neither labels nor numbers matter (section 4). We call this property "represented-types-only". (b) The social function admits at least one status quo point, i.e. an alternative always Pareto-dominated by the choice of the function (section 5). In particular, whenever the status quo point is Pareto-efficient, either it is chosen, or an alternative Pareto-indifferent to it is selected. (c) The social choice function satisfies coalition-strategy-proofness, which means that no coalition of agents can manipulate the choice so as to benefit all members of the coalition (section 6). In particular, it satisfies the weaker condition of strategy-proofness.

We prove that all these implications are general. They hold regardless of the specifics of the particular model under consideration, such as the cardinality of the set of alternatives, the cardinality of the set of admissible preferences, the richness of this set, and whether or not it has any kind of geometric structure. In particular, single-peakedness of the preferences is not required for any of the above implications to hold. Even completeness of the preferences is not required for most of them to hold. We then turn to the particular geometric models studied in the literature on pure public decisions (section 7). We show that in almost all of these models, (existing or new) characterizations of solidarity can be obtained as corollaries of our previously listed general implications of solidarity. Last, we verify that no further unexpected general logical relations hold among the conditions we study (section 8).

2 A class of models

Let A be a set of alternatives, finite or infinite, with generic element denoted by a. Let $\mathcal{N}$ be an infinite set of *potential agents* with generic agent denoted by i. This set can be either countable or uncountable. A *population* is a non-empty finite subset $N \subset \mathcal{N}$ of the set of potential agents. This set can be interpreted as the set of agents actually present in the economy. Agents have preferences over the alternatives. A *preference* R is a binary relation on A that is reflexive and transitive. We do not require preferences to be complete. We say that two alternatives a and b are comparable for the preference R if either $a\ R\ b$ or $b\ R\ a$. Let P and I be the associated strict preference and indifference relations. Preferences may be restricted to belong to a certain set, which represents the constraints imposed on the model by the situation to which the model applies. One important assumption is that these constraints affect all agents in a symmetric way, so that this set is common to all agents. Throughout the paper, let $\mathcal{R}$ be a set of admissible preferences, common to all agents. A pure public choice model with a variable population is a triple $(\mathcal{N}, A, \mathcal{R})$.

For all population N, a preference profile for N is a list $R_N = (R_i)_{i \in N} \in \mathcal{R}^N$ of $|N|$ preferences indexed by N. Let $\mathcal{U}(\mathcal{R})$ be the union of all $\mathcal{R}^N$ for all (non-empty and finite) populations N. This is the set of all admissible preference profiles, for all populations. A choice function f prescribes an outcome for any population and any profile of admissible preferences. It is a therefore a mapping

$$f : \mathcal{U}(\mathcal{R}) \rightarrow A.$$

Our critical assumptions in the above setup are the following. (i) The set of alternatives A is fixed and does not depend on the population. (ii) Each preference

$R \in \mathcal{R}$ is defined over the fixed set A. (iii) The set of admissible preferences $\mathcal{R}$ is common to all agents and fixed. This excludes in particular any resource allocation problem with any type of private consumption, or phenomena such as crowding. We do not impose any additional assumptions on A and $\mathcal{R}$. In particular, these sets may be finite or infinite. Preferences may or may be geometric structured, and may or may not satisfy regularity assumptions.

3 Conditions

To define our conditions, it is useful to define the Pareto-domination relation. For all $R_N \in \mathcal{U}(\mathcal{R})$, let *a weakly Pareto-dominate b for* R_N, if for all $i \in N$, we have $a\ R_i\ b$. This is denoted $a\ R_N\ b$. Let a and b be *Pareto-indifferent for* R_N if for all $i \in N$, we have $a\ I_i\ b$. This is denoted $a\ I_N\ b$. Let *a Pareto-dominate b* if $a\ R_N\ b$ and not $a\ I_N\ b$. This is denoted by $a\ P_N\ b$. Let $a \in A$ be *Pareto-efficient for* R_N if there exists no $b \in A$ such that $b\ P_N\ a$. The *Pareto solution* is the correspondence $\Pi : \mathcal{U}(\mathcal{R}) \twoheadrightarrow A$ that assigns to each profile R_N the set of Pareto-efficient alternatives for R_N. A highly desirable property, for a social choice function is that, for any profile, the function select a Pareto-efficient outcome for this profile.

A social choice function f satisfies **Pareto-efficiency** if for all $R_N \in \mathcal{U}(\mathcal{R})$, we have $f(R_N) \in \Pi(R_N)$.

Let us now present two formulations of the principle of solidarity. The principle says that when changes in the economy occur, all agents who are not directly responsible for these changes should be affected in the same direction: either they all weakly win or they all weakly loose. The first formulation applies to changes in population. It says that when a new agent joins the economy, the agents who were present before the change and whose preferences were kept fixed should all weakly loose, or they should all weakly win.

A social choice function f satisfies **population-monotonicity** if, for all profile $R_N \in \mathcal{U}(\mathcal{R})$, for all agent $i \in \mathcal{N} \setminus N$, and for all preference $R'_i \in \mathcal{R}$, either we have $f(R'_i, R_N)\ R_N\ f(R_N)$, or we have $f(R_N)\ R_N\ f(R'_i, R_N)$.

If f is Pareto-efficient, the choice $f(R_N)$ is Pareto-efficient for the initial profile R_N, and an increase in population by exactly one agent cannot lead to a Pareto-improvement for R_N. From this observation, it follows that population-monotonicity and Pareto-efficiency generally imply the following stronger condition.

A social choice function f satisfies **population-monotonicity$_+$** if, for all two profiles $R_N, R'_M \in \mathcal{U}(\mathcal{R})$ satisfying $M \cap N = \emptyset$, we have $f(R_N)\ R_N\ f(R'_M, R_N)$.

Lemma 1 *Let f be a social choice function that satisfies population-monotonicity and Pareto-efficiency. Then it satisfies population-monotonicity$_+$.*

Proof. Let $R_N, R'_M \in \mathcal{U}(\mathcal{R})$ satisfying $M \cap N = \emptyset$. Let $i \in M$. By population-monotonicity, either $f(R'_i, R_N)\ R_N\ f(R_N)$ or $f(R_N)\ R_N\ f(R'_i, R_N)$. Since we have $f(R_N) \in \Pi(R_N)$, the previous statement implies $f(R_N)\ R_N\ f(R'_i, R_N)$. Introducing one-by-one each of the agents in M, we obtain the desired conclusion.□

The second formulation of the principle of solidarity applies to a change in the preference of exactly one agent within a same population. It says that when the preference of one agent changes, all the other agents whose preferences are kept fixed should either all weakly loose or they should all weakly win.

A social choice function f satisfies **replacement-domination** if, for all profile $R_N \in \mathcal{U}(\mathcal{R})$, for all agent $i \in \mathcal{N} \setminus N$, and for all two preferences $R'_i, R''_i \in \mathcal{R}$, either $f(R'_i, R_N)\ R_N\ f(R''_i, R_N)$ or $f(R''_i, R_N)\ R_N\ f(R'_i, R_N)$.

We will study separately the implications of each of these two formulations of the principle of solidarity. Population-monotonicity restricts the behavior of a social choice function f across populations, but replacement-domination does not require any sort of consistency across populations. Since in our model, the population is variable, it is natural, when studying replacement-domination to impose a condition that restricts the social choice across populations with similar compositions. For all $R_N, R'_M \in \mathcal{U}(\mathcal{R})$, we say that R'_M is a replica of R_N if there exists an integer k such that for all preference $R \in \mathcal{R}$, the number of agents in R'_M having preference R is exactly k times the number of agents in R_N having the same preference R, i.e. $|\{j \in M : R'_h = R\}| = k\,|\{i \in N : R_i = R\}|$. Our next condition, which we use when studying replacement-domination requires a social choice function to select, up to Pareto-indifference, the same alternative for any economy an all of its replicas.

A social choice function f satisfies **replication-indifference** if, for all profiles $R_N, R'_M \in \mathcal{U}(\mathcal{R})$ such that R'_M is a replica of R_N, we have $f(R_N)\ I_N\ f(R'_M)$.

In particular, this condition restricts a social choice function to select, up to Pareto-indifference, the same alternative for any two economy such that one is obtained from the other by relabeling agents. This last weaker requirement is the condition of anonymity.

A social choice function f satisfies **anonymity** if, for all $R_N, R'_M \in \mathcal{U}(\mathcal{R})$ such that R'_M is a replica of R_N and such that $|M| = |N|$, we have $f(R_N)\ I_N\ f(R'_M)$.

In this paper, we investigate the implications of population-monotonicity and Pareto-efficiency on the one hand, and replacement-domination, Pareto-efficiency and replication-indifference, on the other hand. The next three sections are devoted to an analysis of the implications of each of these two combinations of axioms.

4 Represented-types-only

In this section, we obtain two types of results. First, we observe that solidarity and additional conditions imply *represented-types-only*, a significantly stronger indifference condition than *anonymity* (Lemma 2). *Represented-types-only* requires that the choice for any profile only depend on the preferences that are present in the profile, not on the labels or number of the agents who have each of these preferences, up to Pareto-indifference for these preferences. Second, we establish a general relation between the two *solidarity* conditions. We prove that *replacement-domination* together with *Pareto-efficiency* and *replication-indifference* implies *population-monotonicity*, regardless of the set of admissible preferences (Theorem 1). This establishes as a general result the observed pattern, in the literature on solidarity in public decision models, that *replacement-domination* is at least as strong as *population-monotonicity.*

A social choice function f satisfies **represented-types-only** if, for all $R_N, R'_M \in \mathcal{U}(\mathcal{R})$ such that $\{R_i : i \in N\} = \{R'_h : h \in M\}$, we have $f(R_N)\ I_N\ f(R'_M)$.

Lemma 2 *Let f satisfy Pareto-efficiency. (i) If f satisfies population-monotonicity, then it satisfies represented-types-only. (ii) If f satisfies replacement-domination and replication-indifference, then it satisfies represented-types-only.*

Proof. Implication (i) Let $R_N, R'_M \in \mathcal{U}(\mathcal{R})$ such that $\{R_i : i \in N\} = \{R'_i : i \in M\}$. Let $R''_L \in \mathcal{U}(\mathcal{R})$ satisfying $L \cap N = \emptyset$ and $L \cap M = \emptyset$ and $\{R''_h : h \in N''\} = \{R_i : i \in N\}$. By *population-monotonicity*$_+$, we have $f(R_N)\ R_N\ f(R''_L, R_N)$. Since $f(R''_L, R_N) \in \Pi(R''_L, R_N) = \Pi(R_N)$, we have in fact $f(R_N)\ I_N\ f(R''_L, R_N)$. Reproducing three times the same argument, we obtain $f(R''_L, R_N)\ I''_L\ f(R''_L)$, $f(R''_L)\ I''_L\ f(R''_L, R'_M)$ and $f(R''_L, R'_M)\ I'_M\ f(R'_M)$. Since I_N, I'_M and I''_L define the same transitive relation, we have $f(R_N)\ I_N\ f(R'_M)$, the desired conclusion.

Implication (ii). Let $R_N, R'_M \in \mathcal{U}(\mathcal{R})$ such that $\{R_i : i \in N\} = \{R'_i : i \in M\}$.

Let L be a population such that $|L| = |N| \, |M|$. Let $\underline{R}_L, \overline{R}_L \in \mathcal{R}^L$ such that $\underline{R}_L$ is a replica of R_N and $\underline{R}_L$ is a replica of R'_M. By replication-indifference, we have $f(R_N)$ I_N $f(\underline{R}_L)$ and $f(R'_M)$ I'_M $f(\overline{R}_L)$. There exists a natural integer K and a sequence $\{R^k_L\}_{k=0}^K$ of profiles in $\mathcal{R}^L$ satisfying the following four conditions: (i) $R^0_L = \underline{R}_L$, $R^K_L = \overline{R}_L$. (ii) For all $k = 0, ..., K$, we have $\{R^k_h : h \in L\} = \{R_i : i \in N\}$. (iii) For all $k = 0, ..., K$, for all $R \in \mathcal{R}$, we have $\left|\{h \in L : R^k_h = R\}\right| \geq 2 \, |\{i \in N : R_i = R\}|$. (iv) for all $k = 0, ..., K-1$, the profiles R^k_L and R^{k+1}_L only differ by the preference of exactly one agent $h(k)$. Consider such a sequence. For all $k = 0, ..., K-1$, by *replacement-domination*, either $f\left(R^{k+1}_L\right)$ $R^k_{L \setminus h(k)}$ $f\left(R^k\right)$, or $f\left(R^k\right)$ $R^k_{L \setminus h(k)}$ $f\left(R^{k+1}_L\right)$. By conditions (ii) and (iii), this statement is equivalent to the condition that either $f\left(R^{k+1}_L\right)$ R_N $f\left(R^k_L\right)$, or $f\left(R^k_L\right)$ R_N $f\left(R^{k+1}_L\right)$. Since both $f\left(R^k_L\right) \in \Pi\left(R^k_L\right) = \Pi(R_N)$ and $f\left(R^{k+1}_L\right) \in \Pi\left(R^{k+1}_L\right) = \Pi(R_N)$, we obtain $f\left(R^k_L\right)$ I_N $f\left(R^{k+1}_L\right)$, for all $k = 0, ..., K-1$. Since I_N and I'_M define the same transitive relation, we have $f(R_N)$ I_N $f(R'_M)$, the desired conclusion.$\square$

As an immediate corollary of Lemma 2, under *Pareto-efficiency*, *population-monotonicity* implies *anonymity*. Observe that *replication-indifference* alone implies *anonymity*, but in general, it does not imply *represented-types-only*.

Theorem 1 *Let f satisfy replacement-domination. Suppose further that f satisfies either (i) represented-types-only, or (ii) Pareto-efficiency and replication-indifference. Then f satisfies population-monotonicity.*

Proof. By Lemma 2, it suffices to prove case (i). Let $R_N \in \mathcal{U}(\mathcal{R})$, let $i \in \mathcal{N} \setminus N$, and $R'_i \in \mathcal{R}$. Let $j \in N$ and define $R''_i := R_j$. First, by *represented-types-only*, $f(R_N)$ I_N $f(R''_i, R_N)$. Second, by *replacement-domination* applied to profiles (R''_i, R_N) and (R'_i, R_N), either $f(R''_i, R_N)$ R_N $f(R'_i, R_N)$ or $f(R'_i, R_N)$ R_N $f(R''_i, R_N)$. Since

$f(R_N)\ I_N\ f(R_i'', R_N)$, we obtain that either $f(R_N)\ R_N\ f(R_i', R_N)$, or $f(R_i', R_N)$ $R_N\ f(R_N)$, the desired conclusion.□

Do the conditions of *population-monotonicity* and *Pareto-efficiency* conversely imply *replacement-domination*? The answer is no. Miyagawa (1998, 2001), Ehlers (2002, 2003) and Gordon (2003) each provide models that disprove this claim. In each of these models, the set of social functions satisfying *population-monotonicity* and *Pareto-efficiency* strictly contains the set of functions satisfying *replacement-domination, Pareto-efficiency* and *replication-indifference.*

5 Status quo points

An alternative a^* is a status quo point for a social function f if a^* is always weakly Pareto-dominated (for the relevant preference profile) by any alternative selected by f. In other words, $a^* \in A$ is a *status quo point for* f, if for all $R_N \in \mathcal{U}(\mathcal{R})$, we have $f(R_N)\ R_N\ a^*$.

This definition implies, in particular, that if a^* is a status quo point for f, then either a^* or an alternative that is Pareto-indifferent to a^* is selected whenever a^* is Pareto-efficient. Following the definition, the set of status quo points for f is

$$\bigcap_{R_N \in \mathcal{U}(\mathcal{R})} \{a \in A : f(R_N)\ R_N\ a\}.$$

For a general social function f, this set may contain more than one element, and it may also be empty. Our main results in this section say that, in any model $(A, \mathcal{R})$ satisfying certain minimal requirements, if f satisfies one of the solidarity conditions and additional weak assumptions, then f admits at least one such point. To state the first result of this type, we need the following definition. Given a topology $\mathcal{T}$ on A, a preference R_i over A is *lower-hemi-continuous* for $\mathcal{T}$ if, for all $b \in A$, the set

$\{a \in A : b \ R_i \ a\}$ is closed for $\mathcal{T}$.

Theorem 2 *Suppose that there exists a compact topology $\mathcal{T}$ on A such that all preferences in $\mathcal{R}$ are lower-hemi-continuous for $\mathcal{T}$. Let f satisfy either (i) population-monotonicity$_+$ and anonymity; or (ii) population-monotonicity and Pareto-efficiency; or (iii) replacement-domination, Pareto-efficiency and replication-indifference. Then f admits at least one status quo point.*

Proof. By Lemmas 1 and 2, and Theorem 1, it suffices to prove that (i) is sufficient. First, we show that for any finite subcollection $\mathcal{C} \subseteq \mathcal{U}(\mathcal{R})$, the intersection

$$\bigcap_{R \in \mathcal{C}} \{a \in A : f(R_N) \ \ R_N \ a\}$$

is nonempty. Let $\{R_{N_k}\}_{k=1}^K$ be such a collection. For each $k = 1, ..., K$, let M_k be a population with cardinality $|N_k|$ that does not contain any agents from $M_1, ..., M_{k-1}$. For each $k = 1, ..., K$, let $R'_{M_k} \in \mathcal{R}^{M_k}$ be a profile obtained by relabelling each agent in profile R_{N_k}, while keeping its preference fixed. By *anonymity*, for each $k = 1, ..., K$, we have $f\left(R'_{M_k}\right) \ I_{N_k} \ f(R_{N_k})$. Let $M = M_1 \cup ... \cup M_k$ and $R'_M = \left(R'_{M_1}, ..., R'_{M_K}\right)$. By population-monotonicity$_+$, for each $k = 1, ..., K$, we have $f\left(R'_{M_k}\right) \ R'_{M_k} \ f(R'_M)$, and thus $f\left(R'_{M_k}\right) \ R_{N_k} \ f(R'_M)$ since R'_{M_k} and R_{N_k} define the same relation. Thus for all $k = 1, ..., K$, we have $f(R_{N_k}) \ R_{N_k} \ f(R'_M)$. Thus $f(R'_M)$ is an element of the above finite intersection for the family $\{R_{N_k}\}_{k=1}^K$, which is therefore nonempty. Since each preference in $\mathcal{R}$ is lower-hemi-continuous, then each set $\{a \in A : f(R_N) \ \ R_N \ a\}$ is closed for $\mathcal{T}$. Thus $\{\{a \in A : f(R_N) \ \ R_N \ a\} : R \in \mathcal{U}(\mathcal{R})\}$ is a collection of closed sets that satisfies the finite intersection property. Since $\mathcal{T}$ is compact, then the set of status quo points of f is nonempty.$\square$

The existence of a topology satisfying the conditions of Theorem 3 should be understood as a joint condition on A and $\mathcal{R}$. Indeed, when such a topology exists, then the topology $\mathcal{T}^*$ generated by the collection

$$\{A \backslash \{a \in A : b \; R_i \; a\} : R_i \in \mathcal{R}, \; b \in A\},$$

is compact. All preferences in $\mathcal{R}$ are lower-hemi-continuous for this topology. Therefore the conditions on A and $\mathcal{R}$ in Theorem 3 can be replaced by the compacity of $\mathcal{T}^*$. These conditions are also equivalent to the condition that any subcollection of the above collection admit a finite subcollection that covers A. When A is finite, the conditions on $(A, \mathcal{R})$ in Theorem 2 are obviously satisfied. We thus have the following result.

Corollary 1 *Let A be finite. Let f satisfy either (i) population-monotonicity$_+$ and anonymity, (ii) population-monotonicity and Pareto-efficiency, or (iii) replacement-domination, Pareto-efficiency and replication-indifference. Then f admits a status quo point.*

When A is infinite, but $\mathcal{R}$ is finite (and the conditions on $(A, \mathcal{R})$ of Theorem 2 are violated), then conditions (ii) or (iii) of Theorem 2 are still sufficient for the existence of a status quo point for f. In fact, we will prove a stronger result in Theorem 3.

At this point, a few remarks are in order. First, it is clear that under the general assumptions of Theorem 2, the status quo point need not be unique. Second, there are models $(A, \mathcal{R})$ (even satisfying the assumptions of Theorem 2) such that *all* social choice functions (not necessarily satisfying solidarity) admit a trivial status quo point, and for which Theorem 2 holds in a trivial way. For an example illustrating both remarks, suppose that A contains an element a^* that is at least weakly worse than all other elements in A for all preferences in $\mathcal{R}$, so that a^* is weakly Pareto-dominated for $\mathcal{R}$ by all other alternatives. Then all social choice functions admit a^*

as a trivial status quo point. Moreover, if several such alternatives exist, then all social choice functions admits several trivial status quo points. Third, even under the assumptions of Theorem 2, a social choice function may admit a unique status quo point a^*, that is nevertheless Pareto-dominated by another alternative b^* for the entire set of admissible preferences. Miyagawa (1998, 2001) studies a particular model $(A, \mathcal{R})$ that satisfies the assumptions of Theorem 2, where this phenomenon occurs. The remarks brings up two natural questions. First, under what conditions is the set of status quo points (at least essentially) a singleton? Second, which conditions on $(A, \mathcal{R})$ ensure that all social choice functions satisfying our conditions admit a status quo point not Pareto-dominated by another alternative for the entire domain $\mathcal{R}$? Our next theorem partially answers both of these questions, by providing sufficient conditions. The following definition and lemma are useful. For all profiles R_L, $R_N \in \mathcal{U}(R)$, say that the profile R_L is *at least as rich as* R_N if for all $a, b \in A$, $(a \; R_L \; b) \Rightarrow (a \; R_N \; b)$.

Lemma 3 *Let f satisfy Pareto-efficiency and either (i) population-monotonicity; or (ii) replacement-domination and replication-indifference. Then for all R_L^* and R_N in $\mathcal{U}(\mathcal{R})$ such that R_L^* is at least as rich as R_N, we have $f(R_N) \; R_N \; f(R_L^*)$.*

Proof. By Theorem 1, it suffices to prove that (i) is sufficient, which we do next. Let $R'_M \in \mathcal{U}(\mathcal{R})$ such that $M \cap N = \emptyset$ and $\{R'_i : i \in M\} = \{R^*_i : i \in L\}$. By *population-monotonicity*$_+$, we have $f(R'_M) \; R'_M \; f(R'_M, R_N)$. Since R_L^* is at least as rich as R_N, then R'_M is at least as rich as R_N. Therefore $f(R'_M) \; R_N \; f(R'_M, R_N)$. Since $f(R'_M, R_N) \in \Pi(R'_M, R_N)$ and by the two previous relations, we have in fact $f(R'_M, R_N) \; I_N \; f(R'_M)$. By *population-monotonicity*$_+$, we have $f(R_N) \; R_N \; f(R'_M, R_N)$. By the two previous relations, we have $f(R_N) \; R_N \; f(R'_M)$. By *represented-types-only*, $f(R'_M) \; I'_M \; f(R_L^*)$. Since R'_M is at least as rich as R_N, this implies $f(R'_M) \; I_N \; f(R_L^*)$. Therefore $f(R_N) \; R_N \; f(R_L^*)$.□

An immediate consequence of Lemma 3 is that if f is a social choice function satisfying the assumptions of the lemma, and R_L and R_N are equally rich, then $f(R_L)$ and $f(R_N)$ are Pareto-indifferent for both R_N and R_L. In other words, the social choice essentially only depends on the richness of the preference profile, an indifference property even stronger than represented-types-only.

For all (finite or infinite) set of preferences $\mathcal{R}^*$, we define in the obvious way, by analogy with these notions for a preference profile, the relations of weak-Pareto-domination for $\mathcal{R}^*$, Pareto-domination for $\mathcal{R}^*$, and Pareto-indifference for $\mathcal{R}^*$, and the set $\Pi(\mathcal{R}^*)$ of Pareto-efficient alternatives for $\mathcal{R}^*$. For all two (finite or infinite) sets of preferences $\mathcal{R}'$ and $\mathcal{R}''$, we extend the definition of the relation "at least as rich as" in the obvious way. Observe that when two sets $\mathcal{R}'$ and $\mathcal{R}''$ are as rich as each other, then in particular $\Pi(\mathcal{R}') = \Pi(\mathcal{R}'')$.

The following result applies to a set of admissible preferences $\mathcal{R}$ that contains at least one finite subset as rich as itself. Obviously, any finite domain $\mathcal{R}$ belongs to this category (proving the theorem in this special case is very easy). In particular, any model with a finite set of alternatives A belongs to this category. Other models in this class are domains that contain two strict preferences such that one is obtained reversing the other, like the single-peaked domain on a segment.

Theorem 3 *Let the set $\mathcal{R}$ contain a finite subset as rich as $\mathcal{R}$. Let f satisfy Pareto-efficiency and either (i) population-monotonicity; or (ii) replacement-domination and replication-indifference. Then f admits a status quo point that is Pareto-efficient for $\mathcal{R}$. This point is unique, up to Pareto-indifference for $\mathcal{R}$.*

Proof. By Theorem 1, it suffices to prove that (i) is sufficient, which we do next. Let $R_L^* \in \mathcal{U}(\mathcal{R})$ be a finite profile as rich as $\mathcal{R}$. Let $a^* := f(R_L^*)$. Let $R_N \in \mathcal{U}(\mathcal{R})$. It is clear that R_L^* is at least as rich as R_N. Therefore by Lemma 3, $f(R_N)\ R_N\ a^*$. Since this is true for all $R_N \in \mathcal{U}(\mathcal{R})$, then a^* is a status quo point for f. Since R_L^*

is as rich as $\mathcal{R}$, then $\Pi(R_L^*) = \Pi(\mathcal{R})$. Since $a^* \in \Pi(R_L^*)$, then $a^* \in \Pi(\mathcal{R})$. We now show that a^* is the unique status quo point for f in $\Pi(\mathcal{R})$, up to Pareto-indifference for $\mathcal{R}$. Let b^* such a point. Since b^* is a status quo point for f, we have $a^*\ R_L^*\ b^*$. Since R_L^* is at least as rich as $\mathcal{R}$, then a^* weakly Pareto-dominates b^* for $\mathcal{R}$. Since $b^* \in \Pi(\mathcal{R})$, then a^* and b^* are Pareto-indifferent for $\mathcal{R}$, thus a^* is unique up to Pareto-indifference for $\mathcal{R}$.□

5.1 Counterexample

The assumptions on $(A, \mathcal{R})$ in Theorems 2 and 3 are weak. We now show that when they do not hold, the conclusion of the theorem fails. Let $A = [0,1]$. For all $a \in A$, let R^a be the preference relation such that for all $x \in A$ we have $a\ P^a\ b$ and for all $x, y \in A \setminus \{a\}$, we have $x\ I^a\ y$. Let a be the "peak" of preference R^a. Let $\mathcal{R} := \{R^a : a \in A\}$. For the usual topology on $[0,1]$, for which it is a compact set, none of the preferences in $\mathcal{R}$ is lower-hemi-continuous (notice however that they are all upper-hemi-continuous). In fact, any topology of A for which all preferences in this domain are lower-hemi-continuous must have all singletons $\{a\}$ as open sets. It it clear that such a topology cannot be compact. Therefore the conditions of Theorem 2 are not satisfied. Moreover, the only subset of preferences as rich as $\mathcal{R}$ is $\mathcal{R}$ itself, and is not finite. Therefore the conditions of Theorem 3 are not satisfied either. For all $R_N \in \mathcal{U}(\mathcal{R})$, the set of Pareto-efficient alternatives for R_N is the set of peaks in profile R_N. Therefore, the Pareto set for any profile in $\mathcal{U}(\mathcal{R})$ is a finite subset of $[0,1]$. Let $f : \mathcal{U}(\mathcal{R}) \to A$ that selects the highest peak in $[0,1)$ whenever the profile contains at least one peak in this set and otherwise selects 1, when 1 is the common peak of all agents in the profile. For all $R_N \in \mathcal{U}(\mathcal{R})$,

$$\begin{aligned} f(R_N) &\equiv \max\{x \in \Pi(R_N) : x < 1\} \text{ if } \{R_i : i \in N\} \neq \{R^1\} \\ f(R_N) &\equiv 1 \text{ if } \{R_i : i \in N\} = \{R^1\}. \end{aligned}$$

This social choice function is *Pareto-efficient* and *population-monotonic.* It even satisfies *replacement-domination.* But it has no status quo point.

6 Coalition-strategy-proofness

In this section, we analyze the relation between solidarity and the two following important conditions of robustness to preference manipulation. A social choice function satisfies **strategy-proofness** if no agent can benefit from misrepresenting her true preferences, regardless of the preferences reported by all other agents:

For all $R_N \in \mathcal{U}(\mathcal{R})$, all $i \notin N$, all $R_i, R'_i \in \mathcal{R}$, we do not have $f(R'_i, R_N)\ P_i\ f(R_i, R_N)$.

A social choice function satisfies **coalition-strategy-proofness** if no coalition of players can jointly benefit from jointly misrepresenting their preferences, regardless of the preferences reported by all agents outside the coalition:

For all $R_N, R_M, R'_M \in \mathcal{U}(\mathcal{R})$ such that $M \cap N = \emptyset$, we do not have $f(R'_M, R_N)\ P_M\ f(R_M, R_N)$.

The most important and useful result in this section, Corollary 2, says that in a domain of complete preferences, under *Pareto-efficiency*, a social choice function satisfies *population-monotonicity* iff it satisfies *strategy-proofness* and *represented-types-only,* and iff it satisfies *coalition-strategy-proofness* and *represented-types-only.* For a general set of (not necessarily complete) preferences, only an implication holds, which is our first result in this section.

Theorem 4 *Let f satisfy Pareto-efficiency. If f further satisfies either* (i) *population-monotonicity or* (ii) *replacement-domination and replication-indifference, then f is coalition-strategy-proof.*

Proof. By Theorem 1, it suffices to prove case (i). Let $R_N, R_M, R'_M \in \mathcal{U}(\mathcal{R})$ such that $M \cap N = \emptyset$. We now prove that we do not have $f(R'_M, R_N)\ P_M\ f(R_M, R_N)$. Let L be a population such that $L \cap N = L \cap M = \emptyset$ and $|L| = |M|$. Let $R_L \in \mathcal{R}^L$ be a profile obtained by relabelling agents in profile R_M, while keeping preferences fixed and let $R'_L \in \mathcal{R}^L$ be a profile obtained by relabelling agents in profile R'_M, while keeping preferences fixed. First, by *population-monotonicity*$_+$, $f(R_M, R_N)$ $R_M\ f(R'_L, R_M, R_N)$. By *anonymity*, $f(R'_L, R_M, R_N)\ I_M\ f(R_L, R'_M, R_N)$. Therefore $f(R_M, R_N)\ R_M\ f(R_L, R'_M, R_N)$. Second, by *population-monotonicity*$_+$, we have $f(R'_M, R_N)\ R_N\ f(R_L, R'_M, R_N)$ and $f(R'_M, R_N)\ R'_M\ f(R_L, R'_M, R_N)$. Moreover, since $f(R_L, R'_M, R_N) \in \Pi(R_L, R'_M, R_N)$, it is not the case that $f(R'_M, R_N)$ $P_L\ f(R_L, R'_M, R_N)$. Since P_L and P_M define the same relation, this is equivalent to say that we do not have $f(R'_M, R_N)\ P_M\ f(R_L, R'_M, R_N)$. This last statement and $f(R_M, R_N)\ R_M\ f(R_L, R'_M, R_N)$ imply that we do not have $f(R'_M, R_N)\ P_M$ $f(R_M, R_N)$, the desired conclusion.□

When preferences are incomplete, the condition of strategy-proofness can be strengthen as follows. A social choice function f satisfies **strong-strategy-proofness** if each agents weakly prefers to report his true preferences, regardless of the preferences reported by all other agents:

For all $R_N \in \mathcal{U}(\mathcal{R})$, all $i \notin N$, all $R_i, R'_i \in \mathcal{R}$, we have $f(R_i, R_N)\ R_i\ f(R'_i, R_N)$.

Our next result provides a partial converse to Theorem 4.

Lemma 4 *If f satisfies strong-strategy-proofness and represented-types-only, then f satisfies population-monotonicity.*

Proof. Let $R_N \in \mathcal{U}(\mathcal{R})$, let $i \in \mathcal{N} \backslash N$ and let $R_i' \in \mathcal{R}$. We show that $f(R_N)$ R_N $f(R_i', R_N)$. This is shown in two steps. Let $j \in N$ and define $R_i := R_j$. First, by *represented-types-only*, $f(R_N)\ I_j\ f(R_i, R_N)$. Second, by *strong-strategy-proofness*, we have $f(R_i, R_N)\ R_i\ f(R_i', R_N)$. Since $R_i = R_j$, this is in fact equivalent to $f(R_i, R_N)\ R_j\ f(R_i', R_N)$. This last statement and $f(R_N)\ I_j\ f(R_i, R_N)$ imply that $f(R_N)\ R_j\ f(R_i', R_N)$. Since this holds for all $j \in N$, we obtain $f(R_N)\ R_N$ $f(R_i', R_N)$the desired conclusion.□

In the case where $\mathcal{R}$ is a set of complete preferences, this notion and the generally weaker notion previously defined coincide. Thus for complete preferences, *strong-strategy-proofness* can be replaced with *strategy-proofness*, in Lemma 4. This observation and Theorem 4 yields the following important equivalence.

Corollary 2 *Let $\mathcal{R}$ be a set of complete preferences. Let f satisfy Pareto-efficiency. The three following requirements on f are equivalent.*

i). Population-monotonicity.

ii). Strategy-proofness and represented-types-only.

iii). Coalition-strategy-proofness and represented-types-only.

When preferences are complete, strategy-proofness and strong-strategy-proofness coincide, so that together with weak additional conditions, either condition of solidarity implies the strong notion, as a consequence of Theorem 5. How much completeness is needed in the set $\mathcal{R}$ for solidarity (together with weak additional assumptions) to imply[3] *strong-strategy-proofness*? To formulate our next result, we need the following definition. A set of preferences $\mathcal{R}$ is *weakly complete* if for all list of preferences $R^1, ..., R^k, R^l \in \mathcal{R}$, with $l > 1$, and for all $a \in \Pi\left(R^1, ..., R^l\right)$ and all $b \in \Pi\left(R^1, ..., R^k\right)$, such that for all $j = 1, ..., k$, we have $a\ R^j\ b$ then a and b are comparable for R^l.

Theorem 5 *Let $\mathcal{R}$ be weakly complete. Let f be a social choice function that satisfies Pareto-efficiency. (i) The social function f is population-monotonic iff f is strongly-strategy-proof and satisfies represented-types-only. (ii) If f satisfies replacement-domination and replication-indifference, then f is strongly-strategy-proof.*

Proof. Equivalence (i). The converse implication is implied by implication (iii) of Theorem 3 We now prove the direct implication. Let $R_N \in \mathcal{U}(\mathcal{R})$, let $i \in \mathcal{N} \backslash N$ and $R_i, R_i' \in \mathcal{R}$. We now prove that $f(R_i, R_N)\ R_i\ f(R_i', R_N)$. By Theorem $4(i)$, f is *strategy-proof.* Therefore it suffices to prove that the two alternatives are comparable for R_i. Let $l \in \mathcal{N} \backslash (N \cup \{i\})$, let $R_l' := R_i'$ and $R_l := R_i$. First, by *population-monotonicity*$_+$, $f(R_i, R_N)\ R_i\ f(R_l', R_i, R_N)$. By *anonymity*, $f(R_l', R_i, R_N)\ I_i\ f(R_l, R_i', R_N)$. Together, these two relations yield $f(R_i, R_N)\ R_i$ $f(R_l, R_i', R_N)$. Second, by *population-monotonicity*$_+$, $f(R_i', R_N)\ R_N\ f(R_l, R_i', R_N)$ and $f(R_i', R_N)\ R_i'\ f(R_l, R_i', R_N)$. Since $f(R_i', R_N) \in \Pi(R_i', R_N)$, $f(R_l, R_i', R_N) \in \Pi(R_l, R_i', R_N)$, then by weak-completeness of $\mathcal{R}$, the alternatives $f(R_i', R_N)$ and $f(R_l, R_i', R_N)$ are comparable for preference $R_l = R_i$. Since comparability for R_i is transitive, $f(R_i, R_N)$ and $f(R_i', R_N)$ are comparable for R_i, the desired conclusion. Implication (ii) follows from equivalence (i) and Theorem 1.□

[3] Similarly, we can define a stronger notion of coalition-strategy-proofness, that coincides with the notion used here only when preferences are complete. A second question, analogous to the one we ask here for strong strategy-proofness, arises for coalition-strategy-proofness. We do not have an answer to this second question, except that the conditions of Theorem 4 are not sufficient (see also note 5 in section 7).

For an application of Theorem 5 to a specific model with incomplete preferences, see Corollary 8 in the next section.

7 Implications for particular models

In this section, we examine several particular models of pure public choice. In some of them, solidarity conditions have been studied, in other they have not. We show that many of the existing results can be obtained as corollaries of the general implications obtained in this paper and known results on strategy-proofness. We provide a new characterization on one domain where solidarity was never studied, the domain of single-plateaued preferences over a segment.

7.1 From strategy-proofness to solidarity

For each of the particular models we consider next, we show how our results in previous sections yield characterizations of solidarity. In each model, we follow an identical reasoning, which we explain once and for all. For each model, we start from an existing characterization of the rules that satisfy strategy-proofness, Pareto-efficiency and possibly another property X. Let F be the class of such rules. Corollary 2 then tells us that the rules that satisfy population-monotonicity, Pareto-efficiency and X form a subclass $G \subseteq F$, which is exactly the set of rules in F which, in addition, satisfy represented-types-only. When F is known, identifying the subclass G is very easy. In most cases, the class F can be described as a parametrized family with a parameter in a certain set. Further imposing represented-types-only simply further restricts the set of admissible parameters to a subset. Finally, Theorem 1 tells us that the class of rules that satisfy replacement-domination, Pareto-efficiency and replication-indifference, and X form a subset $H \subseteq G$. In many cases, G and H turn out to be equal, but this is not always the case.

A natural starting point is to consider the set $\mathcal{R}_{GS}$ of all complete strict orderings over a set of alternatives A_{GS} containing at least three elements. Solidarity conditions were never analyzed in this model. Theorem 3 enables us to do it. Unsurprinsingly, this leads to a negative result. Gibbard (1973) and Satterthwaite (1975) proved that on such a domain, any social choice rule that satisfies Pareto-efficiency and strategy-proofness must be dictatorial. This well-known result, together with our Corollary 2 and Theorem 1 implies the following.

Corollary 3 *There exists no social choice function $\mathcal{U}(\mathcal{R}_{GS}) \rightarrow A_{GS}$ that satisfies Pareto-efficiency and population-monotonicity. There is no social choice function $\mathcal{U}(\mathcal{R}_{GS}) \rightarrow A_{GS}$ that satisfies Pareto-efficiency, replication-indifference and replacement-domination.*

This negative result motivates the search for existence results for solidarity conditions on restricted set of admissible preferences. Let A be the interval $[0, 1]$. A preference R_i is single-peaked if there is a number $p(R_i) \in [0, 1]$, the "peak", such that for all $a, b \in [0, 1]$, if $a < b \leq p(R_i)$ or $p(R_i) \leq b < a$, then $b \; P_i \; a$. Let $\mathcal{R}_{SP}$ be the set of continuous single-peaked preferences over $[0, 1]$. Moulin (1980), Barberà and Jackson (1991) and Ching (1992) characterized a family of rules, called the *generalized median voters schemes*, as the social choice functions that are Pareto-efficient and strategy-proof on this domain. Thomson (1993) and Ching and Thomson (1997) defined a subfamily, the *target rules.* Each rule in this family is identified with a target in $[0, 1]$ and is defined as follows. For any population of agents and any preference profile for these agents, if the target is located between the most extreme peaks in the profile, the rule selects the target; otherwise, the rule selects among all peaks in the profile, the one closest to the target. The aforementioned characterizations of strategy-proofness and Pareto-efficiency, together with our Corollary 2 and Theorem 1, imply the following results.

Corollary 4 *(Ching and Thomson, 1997; Thomson*[4]*, 1993). The only social choice functions* $\mathcal{U}(\mathcal{R}_{SP}) \rightarrow [0,1]$ *that satisfy Pareto-efficiency and population-monotonicity are the target rules. The only social choice functions* $\mathcal{U}(\mathcal{R}_{SP}) \rightarrow [0,1]$ *that satisfy Pareto-efficiency, replication-indifference and replacement-domination are the target rules.*

The above model of social choice on a segment can easily be extended to a more general setting where the location set A is a tree. Schummer and Vohra (2002) define a tree as a "closed connected subset of an Euclidean space, that is composed of the union of a finite number of closed lines of finite length." The definition of single-peaked preferences over a segment naturally extends to a tree. Let A_T be a tree. Let $\mathcal{R}_T$ be the set of single-peaked preferences over A_T. Target rules in this model are the natural generalization of target rules on a segment. A version of Corollary 4 was established in the case of a tree, respectively by Klaus (1999, 2001) and Vohra (1998). Similar results (Corollary 5) are easily obtained as implications of the characterization of strategy-proof location rules on networks, by Schummer and Vohra (2002), together with our Corollary 2 and Theorem 1.

Corollary 5 *(Klaus, 1999, 2001; Vohra, 1998). The only social choice functions* $\mathcal{U}(\mathcal{R}_T) \rightarrow A_T$ *that satisfy Pareto-efficiency and population-monotonicity are the target rules. The only social choice functions* $\mathcal{U}(\mathcal{R}_T) \rightarrow A_T$ *that satisfy Pareto-efficiency, replication-indifference and replacement-domination are the target rules.*

What if the location set is a cycle? The definition of single-peaked preferences over a segment naturally extends to a cycle and to a graphs containing cycles. Let A_G be such a graph and let $\mathcal{R}_G$ be the set of single-peaked preferences over A_G. Gordon (2003a) proves directly that there are no rules satisfying either solidarity

[4]Thomson (1993) proves a stronger version of this result, without the requirement of replication-indifference and with a fixed population of $n \geq 3$ agents.

condition and Pareto-efficiency in this model. Again, this result can be obtained as an implication of a result by Schummer and Vohra (2002) showing that a strategy-proof and Pareto-efficient location rule on such a graph must be locally dictatorial, together with our Corollary 2 and Theorem 1.

Corollary 6 *(Gordon, 2003a). There are no social choice functions $\mathcal{U}(\mathcal{R}_G) \to A_G$ that satisfy Pareto-efficiency and population-monotonicity. There are no social choice functions $\mathcal{U}(\mathcal{R}_G) \to A_G$ that satisfy Pareto-efficiency, replication-indifference and replacement-domination.*

We now turn to a model in which results on strategy-proofness exist, but nothing is known on solidarity. This is the domain of single-plateaued preferences, studied by Berga (1998). Let A be the interval $[0,1]$. A preference R_i is single-plateaued if there is are numbers $p^-(R_i), p^+(R_i) \in [0,1]$, $p^-(R_i) \leq p^+(R_i)$ such that $\{a \in A : \forall b \in A,\ a\ R_i\ b\} = [p^-(R_i), p^+(R_i)]$ and, for all $a, b \in [0,1]$, if $a < b \leq p^-(R_i)$ or $p^+(R_i) \leq b < a$, then $b\ P_i\ a$. Let $\mathcal{R}_{PL}$ be the set of continuous single-plateaued preferences over $[0,1]$. Berga (1998) characterizes a certain family of rules as the set of social choice functions that are plateau-only and strategy-proof on this domain. One can define a set of *target rules* as a subfamily of the family described by Berga. Each target rule is identified by a parameter $a \in [0,1]$. For any population N and any profile of preferences R, let $N^-(R)$ be the set of agents in N whose preferences R_i are such that $p^+(R_i) < a$, and let $N^+(R)$ be the set of agents in N whose preferences R_i are such that $a < p^-(R_i)$. If none of these sets is empty, then the target rule selects a. If $N^-(R)$ is not empty, but $N^+(R)$ is, then let $f(R) \equiv \max\{p^-(R_i) : i \in N^-(R)\}$. If $N^+(R)$ is not empty, but $N^-(R)$ is, then let $f(R) \equiv \min\{p^+(R_i) : i \in N^+(R)\}$. If both of the sets are empty, the rule select a. The following result follows immediately from Berga's Theorem 1, together with our Corollary 2 and Theorem 1.

Corollary 7 *The only social choice functions* $\mathcal{U}(\mathcal{R}_{PL}) \to [0,1]$ *that satisfy Pareto-efficiency, population-monotonicity and plateau-only are the social choice functions that are Pareto-indifferent to a target rule. These are also the only social choice functions* $\mathcal{U}(\mathcal{R}_{PL}) \to [0,1]$ *that satisfy Pareto-efficiency, replication-indifference, replacement-domination and plateau-only.*

The previous domains are all complete. The following is an example of a domain that is not complete, but is still weakly complete. This domain was introduced by Ehlers, Peters and Storcken (2001) and was also studied by Ehlers and Klaus (2001). Let A be the set of probability distributions on $[0,1]$, denoted by $\Delta(0,1)$. Each single-peaked preference R_i over (deterministic alternatives of) the segment naturally extend to *incomplete* preferences over $\Delta(0,1)$. Let R_i be a complete preference over deterministic alternatives, and let $\alpha, \beta \in \Delta(0,1)$. Let $\alpha\ R_i\ \beta$ if, for all weak-upper contour set U for preference R_i, the measure of U under α is greater or equal than the measure of U under β. In other words, α is weakly preferred to β under R_i if α weakly first-order stochastically dominates β for preference R_i. Furthermore, let $\alpha\ P_i\ \beta$ if, $\alpha\ R_i\ \beta$ and there exists an upper contour set V for preference R_i such that the measure of V under α is strictly greater than the measure of V under β. In other words, α is strictly preferred to β under P_i if α strictly first-order stochastically dominates β for preference R_i. For any domain of complete preferences over deterministic alternatives, this defines a set of admissible preferences over lotteries, which are incomplete. It is straightforward, however, to prove that all preferences in this set are weakly complete. Ehlers and Klaus (2001) define target rules in this model and show the result stated below, which can also be obtained as a corollary of a result by Ehlers, Peters and Storcken (2002) on strong-strategy-proof social choice functions in this model, together with our Theorem 5.

Corollary 8 *(Ehlers and Klaus, 2001). The only social choice functions* $\mathcal{U}(\mathcal{R}_{SP}) \to$

$\Delta(0,1)$ *that satisfy Pareto-efficiency and population-monotonicity are the target rules. The only social choice functions* $\mathcal{U}(\mathcal{R}_{SP}) \rightarrow \Delta(0,1)$ *that satisfy Pareto-efficiency, replication-indifference and replacement-domination are the target rules.*

Interestingly, the target rules in this model are strongly strategy-proof, and weakly coalition-strategy-proof, but not strongly coalition strategy-proof, which proves that weak completeness is not a sufficient condition for solidarity and Pareto-efficiency (and replication-indifference) to imply strong coalition-strategy-proofness.

7.2 From solidarity to strategy-proofness?

In some models, the implications of solidarity conditions are well understood, even though the implications of strategy-proofness are not. The models studied by Gordon (2003a, the discrete case), Miyagawa (1998, 2001), and Ehlers (2002, 2003) fall in this category[5]. These authors characterize in these models the set of social choice functions that satisfy population-monotonicity and Pareto-efficiency, along with the subset of social choice functions that satisfy replacement-domination, Pareto-efficiency and replication-indifference. Corollary 2 tells us that these characterizations provide a starting point and a hint towards a characterization of strategy-proof and Pareto-efficient social choice. Indeed, strategy-proofness is population-monotonicity without represented-types-only.

[5] Gordon (2003a) studies a model where the set of alternatives is a circle and the set of admissible preferences $\mathcal{R}$ is a finite set of symmetric single-peaked preferences. Gordon provides characterizations of solidarity in this model when $\mathcal{R}$ has a symmetric structure with respect to the circle and its cardinality is sufficiently small. For larger cardinalities, a negative result is obtained. Miyagawa (1998, 2001) considers a model where two locations have to be chosen from an interval, when agents have single-peaked preferences over single locations. Preference comparisons of pairs of locations are solely determined by the preferred location in each pair. Ehlers (2002, 2003) considers a variation of this model, where preferences over pairs of locations are lexicographic. Preference comparisons of pairs of locations are determined first by the preferred location in each pair. Only in case of a tie is the second location taken into account.

8 Axioms independence

In this section, we examine the role of each axiom in Lemma 1, Corollary 2 and 3, by the means of examples.

Example 1 shows a social choice function that satisfies *population-monotonicity, replacement-domination*, *coalition-strategy-proofness, and anonymity,* but not *Pareto-efficiency, represented-types-only,* nor *replication-indifference.* Therefore *population-monotonicity,* without *Pareto-efficiency,* does not imply *represented-types-only* (see Lemma 2(i)). It is easy to construct a *non-anonymous* variant of this example.

Example 1: Let $A = \{a, b\}$. Let $\mathcal{R}$ be the set of strict preferences on A. Let f be such that for all profile R_N, if $|N| = 1$, then f selects the preferred alternative of the unique agent in the population. If $|N| > 1$, then f is constant $f(R) = b$.

Example 2 shows a social choice function that satisfies *Pareto-efficiency, coalition-strategy-proofness, anonymity, and replacement-domination,* but not *population-monotonicity, represented-types-only* nor *replication-indifference.* Therefore *Pareto-efficiency,* without *population-monotonicity,* does not imply *represented-types-only* (see Lemma 2(i)). The example also shows that *Pareto-efficiency* and *replacement-domination,* without *replication-indifference,* do not imply *represented-types-only,* (see Lemma 2(ii)) nor *population-monotonicity* (see Theorem 1(ii)). The example also shows that *coalition-strategy-proofness* without *represented-types-only* does not imply *population-monotonicity (see Corollary 2*(iii)). It is easy to construct a *non-anonymous* variant of this example.

Example 2: Let A and $\mathcal{R}$ as in Example 1. Let f be such that for all population N with even cardinality, for all profile R, if all agents unanimously prefer a to b at R, then let $f(R) = a$ and otherwise, let $f(R) = b$; and for all population N with odd cardinality, for all profile R, if all agents unanimously prefer b to a at R, then let $f(R) = b$ and otherwise, let $f(R) = a$.

Example 3 shows a social choice function satisfying *Pareto-efficiency, replication-indifference* (therefore *anonymity*), *coalition-strategy-proofness,* but not *population-monotonicity, replacement-domination* nor *represented-types-only.* Therefore *Pareto-efficiency* and *replication-indifference,* without *replacement-domination,* do not imply *represented-types-only* (see Lemma 2(ii)), nor *population-monotonicity* (see Theorem 1(ii)).

Example 3: Let A and $\mathcal{R}$ as in Example 1. Let f be such that for all profile R_N, if, at the profile R_N, the cardinality of the set of agents who prefer a is at least as large as the cardinality of the set of agents that prefer b, then $f(R_N) = a$, and otherwise $f(R_N) = b$.

Example 4 shows a social choice function that satisfies *replacement-domination, replication-indifference* (and therefore *anonymity*)*,* but not *population-monotonicity, Pareto-efficiency, represented-types-only* nor *strategy-proofness.* Thus *replacement-domination* and *replication-indifference,* without *Pareto-efficiency,* do not imply *represented-types-only* (see Lemma 2(ii)), *population-monotonicity* (see Theorem 1(ii)), nor *strategy-proofness.*(see Corollary 2(ii)).

Example 4: Let $A = \{a, b, c\}$. Let $\mathcal{R} = \{R^a, R^c\}$ be the set of strict preferences on A such that $a \; P^a \; c \; P^a \; b$ and $c \; P^c \; a \; P^c \; b$. Let f be such that for all profile R_N, if, at the profile R_N, the proportion of agents with preference R^a is exactly $\frac{1}{2}$, then $f(R_N) := a$, if this proportion is exactly $\frac{3}{5}$, then $f(R_N) := c$, otherwise $f(R_N) := b$.

Example 5 shows a function that satisfies *Pareto-efficiency, represented-types-only* (therefore also *anonymity* and *replication-indifference*) but not *population-monotonicity, replacement-domination, nor strategy-proofness.* Therefore *Pareto-efficiency*, without *population-monotonicity* does not imply *strategy-proofness* (see Corollary 2(i)). Similarly, the example shows that *Pareto-efficiency*, and *replication-indifference*, without *replacement-domination*, do not imply *strategy-proofness* (see Corollary 2(ii)). Finally, the example also proves that *represented-types-only*, without *strategy-proofness,* does not imply *population-monotonicity* (see Corollary 2(iii)).

Example 5: Let $A = [0,1]$. Let $\mathcal{R}$ be the set of single-peaked preferences over A (for a definition, see section 6). Let f be such that for all profile R, $f(R)$ is the average of the two most extremes peaks in the profile R.

Example 6 shows a social choice function that satisfies *population-monotonicity, replacement-domination, represented-types-only,* (therefore *replication-indifference* and *anonymity),* but not *strategy-proofness*, nor *Pareto-efficiency.* Thus *population-monotonicity,* without *Pareto-efficiency*, does not imply *strategy-proofness* (see Corollary 2(i)).

Example 6: Let A and $\mathcal{R}$ as in Example 1. Let f be such that for all profile R, if all agents unanimously prefer a to b at R, then let $f(R) = b$ and otherwise, let $f(R) = a$.

Example 7 shows a social choice function that satisfies *replacement-domination* and *Pareto-efficiency,* but not *population-monotonicity*, *strategy-proofness, replication-indifference (* therefore also not *represented-types-only)* nor *Pareto-efficiency.* Therefore *replacement-domination and Pareto-efficiency,* without *replication-indifference*, do not imply *strategy-proofness* (see Corollary 2(ii)).

Example 7: Let $A = \{a, b, c\}$. Let $\mathcal{R} = \left\{R^a, R^b, R^c\right\}$ be the set of strict preferences on A such that $a\ P^a\ c\ P^a\ b$, $b\ P^b\ a\ P^b\ c$ and $c\ P^c\ b\ P^c\ a$. Let f be such that if all agents in the profile have the same preference R^h, then $f(R_N) = h$. If at least preferences R^c and R^a (or all three preferences) are represented at profile R_N, then $f(R_N) = a$. If preferences R^a and R^b are represented at profile R_N, then $f(R_N) = b$ if $|N| = 2$ and $f(R_N) = a$ otherwise. If preferences R^b and R^c are represented at profile R_N, then $f(R_N) = c$ if $|N| = 2$ and $f(R_N) = b$ otherwise.

9 Conclusion

This paper raises mainly two types of open questions for future work.

First, we developed a systematic way to study *solidarity* in public choice models. In any new public choice model that anyone could come up with, our results provide a list of implications that provide a solid starting point towards a characterization on the basis of *solidarity* in any such model. In section 7, we have shown that old and new results on *solidarity* can be obtained as by-products of results on *strategy-proofness.*

An example of a public choice problem that is not well understood, is the model studied by Ehlers (2002, 2003), on the provision of multiple public goods, when agents have lexicographic preferences. In these papers, Ehlers provides a complete answer to the problems we address here, but only for the case of the provision of two goods. How to extend his results to more goods is not obvious, but our results provide steps towards this goal. All of our results apply, and yield several solid starting points. For example, we know that any social choice function that satisfies *population-monotonicity* and *Pareto-efficiency* admits a status quo point. In the model with k goods, this means that there exists a vector of k locations that is always Pareto-dominated by the function, and is selected whenever it is Pareto-efficient. We also know that such a function satisfies *coalition-strategy-proofness* and *represented-types-only.* A similar remark applies to the model studied by Miyagawa (1998, 2001).

Second, for public choice models where *strategy-proofness* is not well understood, we propose the study of solidarity as a preliminary step, that can potentially suggest what kind of functions are *strategy-proof* in these models. For example, in the model of Ehlers (2002, 2003) with two public goods, the set of *Pareto-efficient* and *strategy-proof* social choice functions is not known. But our Corollary 2 says that the intersection of this set with the set of rules that satisfy *represented-types-only* is exactly the class of functions characterized in Ehlers (2003). Therefore, Corollary 2 suggests the conjecture that the set of *Pareto-efficient* and *strategy-proof* functions consist of functions resembling those described by Ehlers, but freed from an obligation to satisfy *represented-types-only.* A similar remark applies to the model studied by Miyagawa (1998, 2001) and to any new model of public choice. In such a model, our work indicated that studying *solidarity* is a natural starting point to study *strategy-proofness.*

Finally, an important question that we leave open is whether anything remains of our strong implications once our two *public choice* assumptions are relaxed. Obvious ways to relax these assumptions are the presence of money transfers, or to let preferences depend on the population, so as to allow phenomena such as ”crowding”.

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