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On Limits to the Use of Linear Markov Strategies in Common Property Natural Resource Games[1]

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Abstract

We derive conditions that must be satisfied by the primitives of the problem in order for an equilibrium in linear Markov strategies to exist in some common property natural resource differential games. These conditions impose restrictions on the admissible form of the natural growth function, given a benefit function, or on the admissible form of the benefit function, given a natural growth function.

Key Words: common property, natural resources, differential games, linear Markov strategies

JEL classification: C73, D90, Q20

1 Introduction

For some differential games, it can be shown that there exist equilibrium decision rules that are linear in the current value of the state variables. These types of strategies, called linear Markov strategies, are attractive because of their simplicity and ease of interpretation. They also greatly facilitate the computation of the equilibrium and of its properties. For these reasons, the analysis is often restricted to this class of equilibria, when they exist. Notable examples in common property resource games are Clemhout and Wan [1], Lehvari and Mirman [5], Plourde and Yeung [7], Fischer and Mirman [3, 4], Long and Shimomura [6] and Dockner *et al.* [2].

The use of linear Markov strategies is however limited by the restrictions that must be imposed on the primitives of the model in order for such an equilibrium to exist. In this paper, we derive necessary conditions to the use of linear Markov strategies in natural resource differential games.

The model is presented in section 2. In section 3, we derive restrictions that must be imposed on the natural growth function, given the frequently assumed constant elasticity utility function. In section 4, we derive restrictions that must be imposed on the utility function, given a specific natural growth function. In section (5), we briefly discuss an extension to the case where benefit is derived from the remaining resource stock as well as the flow of consumption. We end with some concluding remarks in section 6.

2 The model

Consider a natural resource that is commonly owned and exploited by n economic agents. Denote by $x(t)$ the stock of the resource at time t and by $c_i(t)$ the rate of harvest of agent i, $i = 1, \ldots, n$. If $g(x(t))$ is the natural growth function of the resource stock, then the state variable $x(t)$ evolves according to the differential equation

$$\dot{x}(t) = g(x(t)) - \sum_{i=1}^{n} c_i(t). \tag{1}$$

It is assumed that agent i derives an instantaneous net benefit $u(c_i(t))$ from his harvest, with $u'(c_i(t)) > 0$ and $u''(c_i(t)) < 0$.

By assumption, we restrict attention to equilibria in stationary linear Markov strategies. Stationary Markov strategies in this context are decision rules that specify an agent's harvest rate as a function of the current resource stock: $c_i(t) = \phi_i(x(t))$. A linear strategy for agent i is a strategy of the form $\phi_i(x(t)) = \delta_i x(t)$, with $\delta_i > 0$ a constant.

An equilibrium in linear Markov strategies, if it exists, will necessarily have the property that a best response of agent i to linear strategies being played by each of his $n-1$ rivals is also a linear strategy. The question then is: What are the minimal restrictions that need to be put on the primitives of the problem (the natural growth function $g(x(t))$ and the utility function $u(c_i)$) in order for this property to be satisfied?

At equilibrium it will be the case that, taking as given the vector of decision rules $\phi_j(x) = \delta_j x$, $j \neq i$ of his $(n-1)$ rivals, agent i's own decision rule, $c_i = \phi_i(x)$, maximizes

$$\int_0^\infty e^{-r_i t} u(c_i) dt \tag{2}$$

subject to

$$\dot{x} = g(x) - c_i - x \sum_{j \neq i} \delta_j \tag{3}$$

$$x(0) = x_0 \quad \text{given} \tag{4}$$

$$c_i \geq 0, \quad \lim_{t \to \infty} x(t) \geq 0, \tag{5}$$

where r_i is agent i's discount rate.

The current value Hamiltonian associated to this problem is

$$H(x, c_i, \lambda_i) = u(c_i) + \lambda_i [g(x) - c_i - x \sum_{j \neq i} \delta_j], \tag{6}$$

where λ_i is the shadow value of the resource stock for agent i.

An equilibrium must satisfy, for $i = 1 \ldots, n$, the following set of necessary conditions, in addition to (3) and (4):

$$[u'(c_i) - \lambda_i]c_i = 0, \quad u'(c_i) - \lambda_i \leq 0, \quad c_i \geq 0 \tag{7}$$

$$\frac{\dot{\lambda}_i}{\lambda_i} = r_i - g'(x) + \sum_{j \neq i} \delta_j \tag{8}$$

$$\lim_{t\to\infty} e^{-r_i t}\lambda_i x = 0, \quad \lim_{t\to\infty} e^{-r_i t}\lambda_i \geq 0 \quad \lim_{t\to\infty} x(t) \geq 0. \tag{9}$$

Assume $\phi_i(x) = \delta_i x$ to be a solution, with $\delta_i > 0$. Then, for any $x > 0$, it will be the case that $\dot{c}_i = \delta_i \dot{x}$ and hence

$$\frac{\dot{c}_i}{c_i} = \frac{\dot{x}}{x}. \tag{10}$$

It also follows that (3) can be rewritten as

$$\frac{\dot{x}}{x} = \frac{g(x)}{x} - \sum_{j \neq i} \delta_j - \delta_i. \tag{11}$$

Furthermore, from (7) and (8), along an interior solution,

$$\frac{\dot{c}_i}{c_i} = \frac{1}{\eta(c_i)}\left[g'(x) - \sum_{j \neq i} \delta_j - r_i\right], \tag{12}$$

where $\eta(c_i)$ is the elasticity of marginal utility[1], given by

$$\eta(c_i) = \left[-\frac{c_i u''(c_i)}{u'(c_i)}\right]. \tag{13}$$

Therefore, substituting from (11) and (12) into (10), we find that the following condition must be satisfied in order for $c_i = \delta_i x$ to be a best response:

$$\frac{1}{\eta(\delta_i x)}\left[g'(x) - \sum_{j \neq i} \delta_j - r_i\right] - \left[\frac{g(x)}{x} - \sum_{j \neq i} \delta_j - \delta_i\right] = 0, \tag{14}$$

[1]The reciprocal, $1/\eta(c_i)$, can be interpreted as the instantaneous elasticity of intertemporal substitution.

where δ_i and the δ_j's are constants that remain to be determined.

It follows that, for any given utility function $u(c_i)$, the growth function $g(x)$ must satisfy the following first-order linear differential equation in x:

$$xg'(x) - \eta(\delta_i x)g(x) = \left[\sum_{j\neq i} \delta_j + r_i\right] x - \left[\sum_{j\neq i} \delta_j + \delta_i\right] \eta(\delta_i x)x. \tag{15}$$

Alternatively, given a growth function $g(x)$, the marginal utility function $u'(c_i)$ must satisfy the following first-order linear differential equation in c_i:

$$\left[\frac{g(c_i/\delta_i)}{(c_i/\delta_i)} - \sum_{j\neq i} \delta_j - \delta_i\right] c_i u''(c_i) + \left[g'(c_i/\delta_i) - \sum_{j\neq i} \delta_j - r_i\right] u'(c_i) = 0. \tag{16}$$

3 Admissible growth functions, given a utility function

Typically, in this type of problem, attention is restricted to the class of utility functions that exhibit a constant elasticity of marginal utility. Denoting by $\theta > 0$ this elasticity, the utility function may then take the form:

$$u(c_i) = \frac{c_i^{1-\theta}}{1-\theta} \tag{17}$$

or

$$u(c_i) = \ln c_i, \tag{18}$$

which is the limiting case of (17) for $\theta = 1$.[2]

In that case, $\eta(c_i) = \theta$, a constant, and (15) has as a unique general solution:

$$g(x) = \begin{cases} \left[r_i - \theta\delta_i + (1-\theta)\sum_{j\neq i} \delta_j\right] \dfrac{x}{1-\theta} + kx^\theta & \text{if } \theta \neq 1 \\ (r_i - \delta_i)\, x \ln x + kx & \text{if } \theta = 1\ , \end{cases} \tag{19}$$

where k is the constant of integration.

[2]A more general representation of this utility function is $u(c_i) = a(c_i^{1-\theta})/(1-\theta) + b$ or $u(c_i) = a \ln c_i + b$ for $\theta = 1$. In the present context, there is no loss of generality in setting $a = 1$ and $b = 0$.

Therefore, given a utility function of the form (17) or (18), a decision rule of the form $\phi_i(x) = \delta_i x$ will be a best response to decision rules of the form $\phi_j(x) = \delta_j x$, $j \neq i$, on the part of i's $n-1$ rivals, only if the growth function is of the form:

$$g(x) = \begin{cases} \alpha x + \beta x^{\theta} & \text{if } \theta \neq 1 \\ \\ \alpha x + \beta x \ln x & \text{if } \theta = 1 \ . \end{cases} \tag{20}$$

Substituting from (20) and (17) or (18) into (15), we get the following system of n equations :

$$\begin{array}{ll} \theta\delta_i + (\theta - 1)\sum_{j\neq i}\delta_j - r_i - (\theta-1)\alpha = 0 & \text{if } \theta \neq 1 \\ \\ \delta_i = r_i - \beta & \text{if } \theta = 1 \end{array} \tag{21}$$

which determines the constant equilibrium values of δ_i, $i = 1, \ldots, n$. In particular, with identical agents (i.e., $r_i = r$, $i = 1, \ldots, n$), the symmetric equilibrium is given by

$$\delta = \begin{cases} \dfrac{r + \alpha(\theta - 1)}{n\theta - (n-1)} & \text{if } \theta \neq 1 \\ \\ r - \beta & \text{if } \theta = 1 \ . \end{cases} \tag{22}$$

The class of functions in (20) exhibits desirable properties for a natural growth function when the parameter values are restricted to $\alpha \geq 0$, $\beta < 0$ and $\theta \geq 1$ (or $\alpha < 0$, $\beta > 0$ and $0 < \theta < 1$). It is then strictly concave, with $g(0) = 0$ and $g(\bar{x}) = 0$, where $\bar{x} = (-\alpha/\beta)^{\frac{1}{\theta-1}}$ in the case of $\theta > 1$ (or $0 < \theta < 1$) and $\bar{x} = e^{-\alpha/\beta}$ in the case of $\theta = 1$.[3] The stock level $\bar{x}$ constitutes a stable steady-state in the absence of harvesting of the resource and captures the idea of the natural carrying capacity of the environment.

A major drawback however is that unless we restrict the growth function to $\beta = 0$, it must depend explicitly on a parameter of the utility function, namely θ, if the decision rule $\phi_i(x) = \delta_i x$ is to be a best response to $\phi_j(x) = \delta_j x$, $j \neq i$. This also means that unless $\beta = 0$

[3] Imposing $\alpha \geq 0$, $\beta \leq 0$ and $\theta \geq 1$ (or $\alpha < 0$, $\beta > 0$ and $0 < \theta < 1$) in fact guarantees the sufficiency of conditions (7), (8) and (9). Note that when $\alpha = \beta = 0$, we have the case of a non renewable resource.

is imposed, heterogeneity over the θ's is not admissible, since the growth function $g(x)$ must be common to all agents, by the very nature of the problem.

4 Admissible utility functions, given a growth function

Conversely, consider the case where the growth function is known to be of one of the forms in (20), with α, β and θ being known exogenous parameters. Then, from (16), we have that the elasticity of marginal utility must be given by:

$$\eta(c_i) = \frac{-c_i u''(c_i)}{u'(c_i)} = \begin{cases} \dfrac{\alpha + \theta\beta\left(\dfrac{c_i}{\delta_i}\right)^{\theta-1} - \sum_{j\neq i}\delta_j - r_i}{\alpha + \beta\left(\dfrac{c_i}{\delta_i}\right)^{\theta-1} - \sum_{j\neq i}\delta_j - \delta_i} & \text{if } g(x) = \alpha x + \beta x^{\theta} \\ \\ \dfrac{\alpha + \beta + \beta\ln\left(\dfrac{c_i}{\delta_i}\right) - \sum_{j\neq i}\delta_j - r_i}{\alpha + \beta\ln\left(\dfrac{c_i}{\delta_i}\right) - \sum_{j\neq i}\delta_j - \delta_i} & \text{if } g(x) = \alpha x + \beta x \ln x\ . \end{cases} \tag{23}$$

It follows that a decision rule of the form $\phi_i(x) = \delta_i x$ can be a best response to decision rules of the form $\phi_j(x) = \delta_j x$, $j \neq i$, on the part of i's $n-1$ rivals only if $\eta(c_i)$ is of the following form:

$$\eta(c_i) = \begin{cases} \dfrac{A + \theta B c_i^{\theta-1}}{C + B c_i^{\theta-1}} & \text{if } g(x) = \alpha x + \beta x^{\theta} \\ \dfrac{D + E\ln c_i}{F + E\ln c_i} & \text{if } g(x) = \alpha x + \beta x\ln x. \end{cases} \tag{24}$$

Hence the utility function will be of the form:

$$u(c_i) = a\int^{c_i} e^{-\int^{z}\frac{\eta(s)}{s}ds}dz + b \tag{25}$$

and the marginal utility function of the form:

$$u'(c_i) = ae^{-\int^{c_i}\frac{\eta(s)}{s}ds}, \tag{26}$$

where $a > 0$ and $\eta(c_i)$ must be given by (24). Strict concavity is assured by imposing $\eta(c_i) > 0$ in (24).

This class of utility functions includes as a special case that specified in (17) whenever $A = \theta C$ (with $b = 0$ and $a = 1/(1-\theta)$), or that specified in (18) whenever $D = F$ (with $b = 0$ and $a = 1$).

Substituting from (24) into (23), we find that the constant equilibrium solution for δ_i, $i = 1, \ldots, n$, must satisfy, if $g(x) = \alpha x + \beta x^\theta$:

$$\begin{aligned} &\frac{A}{B}\left(1-\theta\frac{C}{A}\right)\beta\delta_i^{1-\theta} - \left(\theta\delta_i + (\theta-1)\textstyle\sum_{j\neq i}\delta_j - r_i - (\theta-1))\alpha\right) = 0 \\ &\frac{C}{A} - \frac{\alpha - \sum_{j\neq i}\delta_j - \delta_i}{\alpha - \sum_{j\neq i}\delta_j - r_i} = 0 \end{aligned} \tag{27}$$

and, if $g(x) = \alpha x + \beta x \ln x$:

$$\delta_i = r_i - \beta + (D - F). \tag{28}$$

In particular, with identical agents, the symmetric equilibrium value of δ will be given, if $g(x) = \alpha x + \beta x^\theta$, by:

$$\begin{aligned} &\frac{A}{B}\left(1-\theta\frac{C}{A}\right)\beta\delta^{1-\theta} - [n\theta - (n-1)]\delta - r - (\theta-1))\alpha = 0 \\ &\frac{C}{A} - \frac{\alpha - n\delta}{\alpha - (n-1)\delta - r} = 0 \end{aligned} \tag{29}$$

and, if $g(x) = \alpha x + \beta x \ln x$, by:

$$\delta = r - \beta + (D - F). \tag{30}$$

Notice that if $A = \theta C$ and $D = F$, in which case, as noted above, the utility function is of the constant elasticity form (17) and (18) respectively, then (27) and (28) reduce to (21) and (29) and (30) reduce to (22). But the admissible class of utility functions is wider than the constant elasticity class. Again, however, an important drawback is that the parameters of the utility function depend explicitly on the exogenous parameters of the growth function, namely α, β and θ.

5 An extension

It is sometimes appropriate to have utility depend not only on the flow of consumption of the resource, but also directly on the stock, because of the flow of amenities it may provide.[4] To capture this, assume that agent i derives an instantaneous benefit $u(c_i(t), x(t))$ from those two sources, with $u_c(c_i(t), x(t)) > 0$, $u_x(c_i(t), x(t)) > 0$, $u_{cc}(c_i(t), x(t)) < 0$ and $u_{xx}(c_i(t), x(t)) < 0$. Then the equivalent of (15) is:

$$xg'(x) - [\eta(\delta_i x, x) - \xi(\delta_i x, x)]g(x) = \left[\sum_{j \neq i} \delta_j + r_i\right] x - \left[\sum_{j \neq i} \delta_j + \delta_i\right] [\eta(\delta_i x, x) - \xi(\delta_i x, x)]x - S(\delta_i x, x)x, \tag{31}$$

where

$$\eta(c_i, x) = -\frac{c_i u_{cc}(c_i, x)}{u_c(c_i, x)} \quad \text{and} \quad \xi(c_i, x) = \frac{x u_{cx}(c_i, x)}{u_c(c_i, x)}$$

are respectively the elasticity of marginal utility of c_i with respect to c_i and x and

$$S(\delta_i x, x) = \frac{u_x(\delta_i x, x)}{u_c(\delta_i x, x)}$$

is the marginal rate of substitution between c_i and x. Given the function $u(c_i(t), x(t))$, the function $g(x)$ must satisfy the first-order linear differential equation (31) if $c_i = \delta_i x$ is to be a best response.

Consider, as an example, the following version of the constant elasticity utility function, with $\sigma > 0$ and $\theta \neq 1$:

$$u(c_i, x) = x^\sigma \frac{c_i^{1-\theta}}{1-\theta} \tag{32}$$

Then $\eta(\delta_i x, x) = \theta$, $\xi(\delta_i x, x) = \sigma$, $S(\delta_i x, x) = \sigma\delta_i/(1-\theta)$ and (31) has as a general

[4] We thank Ngo Van Long for suggesting this extension.

solution:

$$g(x) = \left[r_i - (\theta - \sigma + \frac{\sigma}{1-\theta})\delta_i + (1 - \theta + \sigma) \sum_{j \neq i} \delta_j \right] \frac{x}{1 - \theta + \sigma} + kx^{\theta - \sigma} \tag{33}$$

where k is the constant of integration. Hence the growth function has to be of the form:

$$g(x) = \alpha x + \beta x^{\theta - \sigma} \tag{34}$$

In particular, if $\sigma = \theta$, then admissible functions must be of the form

$$g(x) = \alpha x + \beta.$$

In that case, the utility function is homogeneous of degree one and hence marginal utility of consumption can be written $u_c(c_i/x, 1)$, which becomes a constant when we set $c_i = \delta_i x$. Therefore, from the first-order condition for the maximization of the Hamiltonian, $\lambda(t)$ must be a constant. It then follows directly from condition (8) that $g'(x)$ must be a constant as well. This will be true of any utility function that is homogeneous of degree one in c and x.

The constant equilibrium values of δ_i, $i = 1, \ldots, n$ are obtained as the solution to the following system of n equations:

$$\left[(\theta - \sigma) + \frac{\sigma}{1 - \theta} \right] \delta_i + (\theta - \sigma - 1) \sum_{j \neq i} \delta_j - r_i - (\theta - \sigma - 1)\alpha = 0. \tag{35}$$

Setting $r_i = r$, the symmetric equilibrium is given by:

$$\delta = \frac{r + (\theta - \sigma - 1)\alpha}{n(\theta - \sigma) - (n - 1) + \sigma/(1 - \theta)}.$$

The limiting case of (32) for $\theta = 1$, $\sigma > 0$, provides an example of the fact that for $g(x)$ to satisfy condition (31), though necessary, is not sufficient for the best response to be linear.

The utility function is then given by:

$$u(c_i, x) = x^{\sigma} \ln c_i, \tag{36}$$

and $\eta(\delta_i x, x) = 1$, $\xi(\delta_i x, x) = \sigma$ and $S(\delta_i x, x) = \sigma\delta_i[\ln x + \ln \delta_i]$. The solution to the differential equation (31) is then:

$$g(x) = \left[\frac{r_i}{\sigma} + (1 - \ln \delta_i)\delta_i + \sum_{j \neq i} \delta_j\right] x - \delta_i x \ln x + kx^{1-\sigma},$$

which means that admissible growth functions must be of the form:

$$g(x) = \alpha x + \beta x \ln x + \gamma x^{1-\sigma}.$$

Substituting for this form of growth function into the differential equation (31), we find that it will be satisfied only if δ_i, $i = 1, \ldots, n$, solves:

$$(1 - \sigma + \sigma \ln \delta_i)\delta_i - \sigma \sum_{j \neq i} \delta_j - r_i + \sigma\alpha + \beta + \sigma(\delta_i + \beta \ln x = 0, \tag{37}$$

which involves x. Clearly, for agent i's best response to be linear in x further requires that $\delta_i = -\beta > 0$, with, in addition, $\alpha = r_i/\sigma - n\beta + \beta \ln(-\beta)]$. This in turn requires $r_i = r$ for all i, since it is inherent to the problem that the growth function is common to all agents.

6 Concluding remarks

The above results place in proper perspective models that rely on linear Markov strategies to study the competition over a common property resource, by showing that the parameters of the utility function and of the growth function cannot be chosen independently of one another, but must satisfy a precise relationship. Assigning specific numerical values to the parameters of functional forms that happen to satisfy this relationship — for instance a unit elasticity of marginal utility or a linear growth function — sometimes tends to obscure this fact.

The papers cited in the introduction all assume specific functional forms for the utility function and the growth function that happen to jointly satisfy this necessary relationship. Clemhout and Wan [1] and Plourde and Yeung [7] both assume a logarithmic utility function as in (18) — hence an elasticity of marginal utility equal to one — combined with a growth function as in (20) with $\theta = 1$. Lehvari and Mirman [5] assume a discrete time version of the same growth function, also combined with a logarithmic utility function. This also applies to Fischer and Mirman [3, 4], although in those cases the growth functions allow for interaction between two types of resources. Long and Shimomura [6] assume the utility function to be homogeneous of degree $h > 0$ (or the log of such a function). Such a utility function exhibits a constant elasticity of $1 - h$ (or of 1 in the case of the logarithmic version). Their growth function is assumed homogeneous of degree one, which in effect means a function such as that in (20) with $\beta = 0$. Dockner *et al.* [2] (chapter 12) study an example with a constant elasticity function of the form (17) and a growth function as in (20) with $\theta \neq 1$.

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