Université de Montréal

Classification analytique de systèmes différentiels linéaires déployant une singularité irrégulière de rang de Poincaré 1

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Thèse présentée à la Faculté des études supérieures et postdoctorales
en vue de l'obtention du grade de Philosophiæ Doctor (Ph.D.)
en mathématiques,
option mathématiques pures

Avril 2010

Université de Montréal

Faculté des études supérieures et postdoctorales

Cette thèse intitulée

Classification analytique de systèmes différentiels linéaires déployant une singularité irrégulière de rang de Poincaré 1

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RÉSUMÉ

Cette thèse traite de la classification analytique du déploiement de systèmes différentiels linéaires ayant une singularité irrégulière. Elle est composée de deux articles sur le sujet : le premier présente des résultats obtenus lors de l'étude de la confluence de l'équation hypergéométrique et peut être considéré comme un cas particulier du second ; le deuxième contient les théorèmes et résultats principaux.

Dans les deux articles, nous considérons la confluence de deux points singuliers réguliers en un point singulier irrégulier et nous étudions les conséquences de la divergence des solutions au point singulier irrégulier sur le comportement des solutions du système déployé. Pour ce faire, nous recouvrons un voisinage de l'origine (de manière ramifiée) dans l'espace du paramètre de déploiement ϵ. La monodromie d'une base de solutions bien choisie est directement reliée aux matrices de Stokes déployées. Ces dernières donnent une interprétation géométrique aux matrices de Stokes, incluant le lien (existant au moins pour les cas génériques) entre la divergence des solutions à $\epsilon = 0$ et la présence de solutions logarithmiques autour des points singuliers réguliers lors de la résonance. La monodromie d'intégrales premières de systèmes de Riccati correspondants est aussi interprétée en fonction des éléments des matrices de Stokes déployées.

De plus, dans le second article, nous donnons le système complet d'invariants analytiques pour le déploiement de systèmes différentiels linéaires $x^2y' = A(x)y$ ayant une singularité irrégulière de rang de Poincaré 1 à l'origine au-dessus d'un voisinage fixé $\mathbb{D}_r$ dans la variable x. Ce système est constitué d'une partie formelle, donnée par des polynômes, et d'une partie analytique, donnée par une classe d'équivalence de matrices de Stokes déployées. Pour chaque valeur du paramètre ϵ dans un secteur pointé à l'origine d'ouverture plus grande que 2π, nous recouvrons l'espace de la variable, $\mathbb{D}_r$, avec deux secteurs et, au-dessus de chacun, nous choisissons une base de solutions du système déployé. Cette base sert à définir les matrices de Stokes déployées. Finalement, nous prouvons un théorème de réalisation des invariants qui satisfont une condition nécessaire et suffisante, identifiant ainsi l'ensemble des modules.

Mots-clés : phénomène de Stokes, systèmes differentiels linéaires, singularité irrégulière, déploiement, monodromie, classification analytique, réalisation, espace des modules, équation hypergéometrique, équation différentielle matricielle de Riccati.

ABSTRACT

This thesis deals with the analytic classification of unfoldings of linear differential systems with an irregular singularity. It contains two papers related to this subject : the first paper presents results concerning the confluence of the hypergeometric equation and may be viewed as a particular case of the second one ; the second paper contains the main theorems and results.

In both papers, we study the confluence of two regular singular points into an irregular one and we give consequences of the divergence of solutions at the irregular singular point for the unfolded system. For this study, a full neighborhood of the origin is covered (in a ramified way) in the space of the unfolding parameter ϵ. Monodromy of a well chosen basis of solutions around the regular singular points is directly linked to the unfolded Stokes matrices. These matrices give a complete geometric interpretation to the well-known Stokes matrices : this includes the link (existing at least for the generic cases) between the divergence of the solutions at $\epsilon = 0$ and the presence of logarithmic terms in the solutions for resonant values of ϵ. Monodromy of first integrals of related Riccati systems are also interpreted in terms of the elements of the unfolded Stokes matrices.

The second paper goes further into the subject, giving the complete system of analytic invariants for the unfoldings of nonresonant linear differential systems $x^2y' = A(x)y$ with an irregular singularity of Poincaré rank 1 at the origin over a fixed neighborhood $\mathbb{D}_r$ in the space of the variable x. It consists of a formal part, given by polynomials, and an analytic part, given by an equivalence class of unfolded Stokes matrices. For each parameter value ϵ taken in a sector pointed at the origin of opening larger than 2π, we cover the space of the variable, $\mathbb{D}_r$, with two sectors and, over each of them, we construct a well chosen basis of solutions of the unfolded differential system. This basis is used to define the unfolded Stokes matrices. Finally, we give a realization theorem for the invariants satisfying a necessary and sufficient condition, thus identifying the set of modules.

Key words : Stokes phenomenon, linear differential systems, irregular singularity, unfolding, monodromy, analytic classification, realization, moduli space, hypergeometric equation, Riccati matrix differential equation.

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c.-à-d.	: c'est-à-dire
cf.	: confer
FIG.	: figure (en français et en anglais)
i.e.	: id est (en anglais)
TAB.	: tableau (ou *table*, en anglais)

REMERCIEMENTS

Tout d'abord, j'aimerais remercier sincèrement ma directrice de recherche, Christiane Rousseau. Le sujet original qu'elle m'a proposé a capté mon intérêt. Les discussions que nous avons eues lors de nos rencontres hebdomadaires, sa lecture attentive de mes écrits et ses recommandations m'ont permis d'en arriver à ce point. Je la remercie aussi de m'avoir offert du financement en cette dernière année de doctorat et de m'avoir donné l'opportunité de participer à des congrès où j'ai présenté mes résultats.

Je remercie le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG) et le Fonds québécois de la recherche sur la nature et les technologies (FQRNT) pour le financement qu'ils m'ont accordé au niveau de la maîtrise et du doctorat. Je remercie la Faculté des études supérieures et postdoctorales et le Département de mathématiques et de statistique pour le soutien financier dont j'ai bénéficié (bourse d'admission, bourse Banque Nationale, bourse de passage accéléré au doctorat, bourse de fin d'études doctorales). Je suis également reconnaissante au Département de mathématiques et de statistique de m'avoir permis de donner des démonstrations et une charge de cours, ce qui fut une très belle expérience d'enseignement.

Finalement, je remercie mon mari, ma famille et mes amis pour leur soutien et leurs encouragements. Je remercie les chercheurs que j'ai rencontrés lors de congrès et avec qui j'ai eu l'occasion de discuter. Je remercie également les professeurs, le personnel et les collègues du Département de mathématiques et de statistique de l'Université de Montréal qui rendent ce milieu agréable et propice à la réussite.

INTRODUCTION

0.1. Mise en contexte

Les équations différentielles les plus simples ont des solutions formelles divergentes au voisinage des singularités. Certaines séries divergentes ont été utilisées par les anciens pour faire des calculs reliés à l'astronomie et à la physique. Les approximations étaient très rapprochées des valeurs attendues ou expérimentales. Maintenant, ces calculs peuvent être justifiés en associant des sommes aux séries divergentes. Typiquement, ces sommes sont analytiques sur des secteurs dans le plan complexe. Cependant, elles ne coïncident pas nécessairement sur l'intersection de deux d'entre eux. Ce défaut de recollement est appelé le phénomène de Stokes.

Cette thèse par articles s'inscrit dans un programme de recherche qui consiste à comprendre comment le phénomène de Stokes encode la géométrie complexe des solutions de systèmes différentiels, en se concentrant sur les germes de systèmes différentiels linéaires non résonants ayant une singularité irrégulière de rang de Poincaré k. L'approche utilisée consiste à étudier les perturbations génériques qui séparent le point singulier irrégulier en points singuliers réguliers. Ce processus appelé déploiement est le processus inverse de la confluence. Les perturbations s'effectuant à l'aide d'un paramètre ϵ prenant ses valeurs au voisinage de 0, nous parlons de système confluent à la limite $\epsilon = 0$ et de système déployé lorsque $\epsilon \neq 0$.

Notre étude se concentre sur le déploiement de singularités de rang de Poincaré $k = 1$. Comme point de départ, nous avons considéré la confluence de l'équation hypergéométrique. Nous présentons une méthode permettant de considérer les valeurs du paramètre pour lesquelles il y a résonance et de les inclure de manière continue dans l'étude. Le phénomène de Stokes (observé à la confluence) s'interprète au niveau du comportement des solutions de l'équation déployée. En particulier, le phénomène de Stokes gouverne la présence de solutions logarithmiques au voisinage des singularités régulières lors de la résonance.

La mesure du phénomène de Stokes s'effectue à l'aide de matrices de Stokes qui sont, à équivalence près, des invariants de classification analytique des systèmes considérés à $\epsilon = 0$. Dans cette thèse, nous nous intéressons au système complet d'invariants analytiques (c.-à-d. l'ensemble des invariants qui caractérisent complètement la relation d'équivalence analytique) des systèmes déployés. Précisons que deux germes de systèmes $y_1' = B_1(\epsilon, x)y_1$ et $y_2' = B_2(\epsilon, x)y_2$ sont localement analytiquement équivalents s'il existe une matrice inversible de germes de fonctions analytiques en (ϵ, x) à l'origine, $T(\epsilon, x)$, telle que la substitution $y_1 = T(\epsilon, x)y_2$ transforme l'un en l'autre ; ils sont formellement équivalents si $T(\epsilon, x)$ est une matrice inversible de séries formelles en (ϵ, x).

0.2. But de la thèse

On considère une famille de germes de systèmes différentiels linéaires s'écrivant

$$(x^2 - \epsilon)y' = B(\epsilon, x)y, \tag{0.2.1}$$

avec

- $B(\epsilon, x)$: matrice de germes de fonctions analytiques en $(0, 0)$,
- $B(0, 0) = diag\{\lambda_1, ..., \lambda_n\}$, avec $\Re(\lambda_1) > \Re(\lambda_2) > ... > \Re(\lambda_n)$,
- $x \in (\mathbb{C}, 0)$,
- $y \in \mathbb{C}^n$.

La famille (0.2.1) correspond à un déploiement générique des germes de systèmes

$$x^2 y' = B(0, x)y. \tag{0.2.2}$$

Le but de la présente thèse est d'identifier le système complet d'invariants analytiques des systèmes (0.2.1), de l'interpréter et d'en étudier la réalisation (c.-à-d. la question d'existence, pour tout système complet d'invariants donné, d'un système

(0.2.1) caractérisé par ces invariants). Pour ce faire, nous prendrons le paramètre de déploiement ϵ sur un secteur recouvrant tout un voisinage de l'origine dans le plan complexe.

0.3. Caractère distinctif de la thèse

La présente thèse s'inscrit dans un plus grand projet d'étude des invariants analytiques des déploiements des germes de systèmes ayant une singularité irrégulière de rang de Poincaré k à l'origine, s'écrivant

$$x^{k+1}y' = B(0,x)y, \tag{0.3.1}$$

où $k \in \mathbb{N}^*$, $y \in \mathbb{C}^n$ et $B(0,x)$ est une matrice de germes de fonctions analytiques en $x = 0$. Le cas non résonant considéré dans cette thèse correspond au fait que les valeurs propres de $B(0,0)$ sont distinctes. Plusieurs mathématiciens ont étudié les conditions d'existence d'une matrice de germes de fonctions analytiques à l'origine $Q(x)$ qui conjugue deux systèmes de la forme (0.3.1), via $y_1 = Q(x)y_2$. Le système complet d'invariants analytiques des systèmes non résonants (0.3.1) se retrouve dans l'article [**1**] de W. Balser, W.B. Jurkat et D.A. Lutz. Les invariants formels sont ceux de la forme normale formelle, et les invariants analytiques sont donnés par une classe d'équivalence de matrices de Stokes. Le système complet d'invariants analytiques est réalisable. Une preuve par Y. Sibuya de la réalisation, qui utilise une généralisation du Lemme de Cartan, est donnée dans [**22**] (p. 150).

Le phénomène de Stokes se produit dans le cas générique où les matrices de Stokes sont différentes de l'identité. Dans un déploiement, l'interprétation géométrique des invariants analytiques (dépendant du paramètre de déploiement) mène aux conséquences du phénomène de Stokes dans les systèmes déployés. De nombreux mathématiciens, dont Arnold, Ramis et Bolibruch, ont conjecturé (avec des énoncés qui diffèrent légèrement) que le phénomène de Stokes provient d'une incompatibilité, entre des bases de solutions, qui persiste jusqu'à la confluence des singularités. Des travaux ont permis de donner un sens aux matrices de Stokes via la confluence de singularités. Dans le cas des systèmes (0.3.1) non résonants, il s'agit de ceux de A. Glutsyuk [**6**] ; dans le cas de l'équation hypergéométrique, ce sont ceux de J.-P. Ramis, [**17**], et de C. Zhang, [**24**] et [**25**]. Par le processus de confluence, ils ont relié les matrices de Stokes aux opérateurs de transition entre des bases de solutions particulières des systèmes perturbés. Des questions similaires ont été étudiées par A. Duval ([**4**] et [**5**]) et R. Schäfke [**21**]. Dans toutes ces études, lorsque le paramètre complexe est pris sur des secteurs, ceux-ci ne recouvrent pas tout un voisinage de l'origine, empêchant que les solutions autour des points singuliers réguliers (qui confluent) contiennent des termes logarithmiques. Il est de tradition d'utiliser, au voisinage d'un point singulier régulier, des bases de solutions qui sont des vecteurs propres de la monodromie (la monodromie autour d'un point singulier est un opérateur qui agit sur une solution en lui associant son prolongement analytique le long d'un lacet faisant le tour de ce point). Les solutions formant cette base peuvent ne plus exister lorsqu'il y a résonance, c.-à-d. lorsque la matrice représentant l'opérateur de monodromie a deux valeurs propres égales. Lorsqu'une telle solution n'existe plus, elle peut être remplacée, dans la base de solutions autour du point singulier régulier, par une solution contenant des termes logarithmiques.

La présente thèse se distingue des études précédentes (pour un rang de Poincaré $k = 1$) par le recouvrement, de manière ramifiée, de tout un voisinage de $\epsilon = 0$, permettant l'inclusion, dans un processus continu, des valeurs du paramètre pour lesquelles il y a résonance. Pour $k = 1$ et en dimension $n = 2$ (seulement), la base de solutions que nous choisissons à cet effet équivaut à une base mixte telle que dans [**24**] et [**25**], une base composée de deux solutions qui sont des vecteurs propres de la monodromie à des points singuliers différents.

0.4. Méthodologie

La méthodologie utilisée afin de résoudre le problème considéré repose sur une construction de bases de solutions sur des domaines recouvrant un voisinage de l'origine dans les espaces de la variable x et du paramètre ϵ.

Pour définir les domaines en (ϵ, x), nous considérerons d'abord les solutions des systèmes linéaires dans l'espace projectif complexe. Nous prenons la variable x sur un disque $\mathbb{D}_r$ dont le rayon r est choisi lorsque $\epsilon = 0$ afin de s'assurer, dans les cartes de l'espace projectif, du confinement de certaines solutions. Ensuite, dans l'espace du paramètre ϵ, nous choisissons un secteur S, pointé à l'origine, dont l'ouverture (plus grande que 2π) et le rayon sont déterminés par les invariants formels. Le rayon pourra par la suite être restreint à quelques reprises, entre autres pour construire deux domaines sectoriels en x recouvrant $\mathbb{D}_r$ et variant selon $\epsilon \in S$. Cette construction, détaillée dans **[20]**, est la clé de la définition des bases de solutions que nous utilisons. En effet, afin d'inclure, dans un processus continu, les valeurs du paramètre de déploiement pour lesquelles il y a résonance, il est nécessaire de choisir une base de solutions autrement qu'en prenant des vecteurs propres de la monodromie autour des points singuliers réguliers. Sur l'intersection des domaines sectoriels construits dans la variable x, la base de solutions de la forme normale formelle en (ϵ, x), que nous appelons le modèle, a un comportement bien spécifique (asymptotique) près des points singuliers (ce comportement est le même lorsque $\epsilon = 0$ et c'est ce qui motive la construction des domaines sectoriels en x). Nous choisissons la base de solutions d'un système (0.2.1) comme étant l'unique base de solutions (à normalisation près) qui a ce même comportement spécifique près des points singuliers. Nous prouvons l'existence de cette base de solutions en nous plaçant dans toutes les cartes de l'espace projectif. Dans chaque carte, le système linéaire devient un système de Riccati. Nous considérons, dans chaque système de Riccati ainsi obtenu, les variétés invariantes passant par les points singuliers et leur prolongement analytique (nous utilisons ici le confinement de ces variétés). Par exemple, en dimension $n = 2$, le portrait de phase, dans chacun des deux systèmes de Riccati, est composé d'un col et d'un noeud (qui se confondent en $\epsilon = 0$). En ramenant, dans le système linéaire, la variété invariante du col (dans un système de Riccati), nous obtenons une des deux solutions de la base recherchée (ce procédé est répété avec l'autre système de Riccati afin de compléter la base de solutions).

L'approche unifiée que nous adoptons, en recouvrant tout un voisinage de $\epsilon = 0$, mène à de plus amples informations sur les conséquences du phénomène de Stokes au niveau du comportement des solutions dans un déploiement. Une fois la base de solutions bien choisie, le calcul des invariants analytiques et son interprétation en termes de monodromie en découlent. Afin d'interpréter le phénomène de Stokes mesuré par les invariants analytiques en $\epsilon = 0$, nous utilisons le fait que ces derniers sont la limite, lorsque $\epsilon \to 0$ sur le secteur S, des invariants analytiques en $\epsilon \neq 0$.

Pour une valeur donnée du paramètre ϵ, la réalisation du système complet d'invariants analytiques ne requiert aucune condition. Puisque les invariants sont présentés sur un ouvert ramifié dans l'espace du paramètre, la construction produit une famille ramifiée. La stratégie utilisée afin de trouver une condition nécessaire à la réalisation, soit la correction à une famille uniforme, consiste à comparer les deux présentations de la même dynamique sur l'auto-intersection du secteur S dans l'espace du paramètre. Nous obtenons ainsi une relation, appelée la relation d'auto-intersection, qui doit être satisfaite par le système complet d'invariants analytiques. Pour prouver que cette condition est aussi suffisante à la réalisation, nous généralisons le théorème de réalisation des invariants à $\epsilon = 0$ à un théorème de réalisation pour $\epsilon \neq 0$. Ce dernier est obtenu par la construction d'une base de solutions sur les domaines sectoriels en x et pour $\epsilon \in S$. La relation d'auto-intersection est ensuite utilisée afin de corriger la construction et d'obtenir l'analyticité en ϵ.

0.5. Organisation de la thèse et contribution aux articles

Les articles de la présente thèse sont précédés d'un chapitre de présentation des principaux résultats et sont suivis d'une conclusion. Les deux articles, faisant chacun l'objet d'un chapitre, sont :

- **Article 1** : C. Lambert, C. Rousseau, *The Stokes phenomenon in the confluence of the hypergeometric equation using Riccati equation*, Journal of Differential Equations 244 (2008), n° 10, 2641–2664 ;
- **Article 2** : C. Lambert, C. Rousseau, *Complete system of analytic invariants for unfolded differential linear systems with an irregular singularity of Poincaré rank 1*, 55 pages.

J'ai écrit ces deux articles présentant les résultats de mon projet de recherche. La contribution de Christiane Rousseau aux articles en est une de directrice de recherche, ce qui comprend l'idée du sujet, l'aide à la résolution de problèmes mathématiques ainsi que la lecture commentée de mes écrits. Son accord pour que les articles soient inclus dans la thèse ainsi que l'autorisation des éditeurs se retrouvent à l'annexe A.

Chapitre 1

PRÉSENTATION DES PRINCIPAUX RÉSULTATS

Dans ce chapitre, nous présentons les principaux résultats des deux articles constituant la thèse. Puisque les résultats du premier article peuvent être considérés comme des cas particuliers de certains résultats du second, ils sont présentés en adoptant le point de vue du second article. Nous introduirons et interpréterons d'abord les invariants formels, puis analytiques. Nous terminerons ce chapitre avec les théorèmes de classification analytique et de réalisation.

1.1. Point de départ à la résolution du problème

Justifions dans un premier temps le choix des équations hypergéométriques confluentes comme point de départ à l'étude du déploiement de systèmes de la forme (0.2.2). S'écrivant

$$x^2w''(x) + \{1 + (1 + a + b)x\}w'(x) + abw(x) = 0, \tag{1.1.1}$$

avec des paramètres complexes a et b, les équations hypergéométriques confluentes correspondent à des cas particuliers de systèmes de la forme (0.2.2) avec $n = 2$. En effet, une équation différentielle d'ordre 2 de la forme

$$p(x)w''(x) + a_1(x)w'(x) + a_0(x)w(x) = 0, \tag{1.1.2}$$

avec $p(x)$, $a_1(x)$ et $a_0(x)$ des fonctions analytiques dans un voisinage de 0, se met sous la forme d'un système

$$y' = \frac{1}{p(x)} \begin{pmatrix} 0 & 1 \\ -a_0(x)p(x) & p'(x) - a_1(x) \end{pmatrix} y \tag{1.1.3}$$

via le changement de variables

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} w(x) \\ p(x)w'(x) \end{pmatrix}. \tag{1.1.4}$$

Les solutions d'une équation hypergéométrique confluente et de la famille d'équations hypergéométriques qui la déploie étant connues explicitement, ce cas particulier a servi d'exemple de base aux développement des idées plus générales et abstraites permettant de résoudre le problème considéré.

1.2. Forme prénormale analytique et invariants formels

Avant toute identification des invariants analytiques de la famille de systèmes (0.2.1) qui déploie les systèmes (0.2.2) de manière générique (cf. section 3.3.1), nous introduisons une forme prénormale à partir de laquelle les invariants formels peuvent être calculés. Par le théorème 3.3.4, la famille de systèmes (0.2.1) est analytiquement équivalente à la famille sous forme *prénormale* :

$$(x^2 - \epsilon)y' = \big(\Lambda_0(\epsilon) + \Lambda_1(\epsilon)x + (x^2 - \epsilon)R(\epsilon, x)\big)\, y, \tag{1.2.1}$$

avec

- $\Lambda_0(\epsilon)$, $\Lambda_1(\epsilon)$ des matrices diagonales de germes de fonctions analytiques en $\epsilon = 0$,
- $R(\epsilon, x)$ une matrice de germes de fonctions analytiques en $(\epsilon, x) = (0, 0)$,
- $y \in \mathbb{C}^n$.

Le problème de classification analytique des systèmes (0.2.1) revient donc à celui des systèmes ayant la forme prénormale (1.2.1).

Nous appelons le système

$$(x^2 - \epsilon)y' = (\Lambda_0(\epsilon) + \Lambda_1(\epsilon)x)\, y \tag{1.2.2}$$

le *modèle* associé au système (1.2.1). Le système complet d'invariants formels des systèmes (1.2.1) est entièrement déterminé par $\Lambda_0(\epsilon)$ et $\Lambda_1(\epsilon)$ (cf. théorème 3.4.4). Le modèle permet d'interpréter les invariants de la forme normale formelle des systèmes (0.2.2) qui s'écrit

$$x^2 z' = (\Lambda_0(0) + \Lambda_1(0)x)\, z. \tag{1.2.3}$$

En effet, la matrice $(\Lambda_0(\epsilon) + \Lambda_1(\epsilon)x)$ du modèle est complètement déterminée par les valeurs propres de la matrice de la forme prénormale (1.2.1) aux deux points singuliers $x = \sqrt{\epsilon}$ et $x = -\sqrt{\epsilon}$ (pour $\epsilon \neq 0$). Le modèle permet d'interpréter :

- $\Lambda_0(0)$, en tant que limite de la moyenne des valeurs propres en $x = \pm\sqrt{\epsilon}$;
- $\Lambda_1(0)$, en tant que limite d'un décalage des valeurs propres en $x = \pm\sqrt{\epsilon}$.

Notons que le déploiement générique ainsi que la forme prénormale ont été obtenus (cf. section 3.3) pour le déploiement de systèmes ayant une singularité irrégulière de rang de Poincaré k, avec $k \in \mathbb{N}^*$. La suite ne concerne que le cas $k = 1$.

1.3. Matrices de Stokes déployées

Les invariants analytiques proviennent de la comparaison de transformations vers le modèle sur l'intersection de leur domaine de définition. En effet, en choisissant de manière adéquate (cf. sections 3.4.2, 3.4.3 et 3.4.4) le domaine S (figure 1.1) du paramètre de déploiement et les domaines sectoriels $\Omega_b^{\hat{\epsilon}}$ et $\Omega_h^{\hat{\epsilon}}$ (figure 1.2) dans l'espace de la variable x, nous obtenons (cf. théorème 3.4.21) des transformations $H_b(\hat{\epsilon}, x)$ et $H_h(\hat{\epsilon}, x)$, définies respectivement au-dessus des domaines $\Omega_b^{\hat{\epsilon}}$ et $\Omega_h^{\hat{\epsilon}}$, qui

- ont une limite non singulière quand x s'approche des points singuliers réguliers,
- conjuguent le système (1.2.1) à son modèle au-dessus de leur domaine de définition,
- sont telles que, pour $\bar{\epsilon}$ et $\tilde{\epsilon} = \bar{\epsilon}e^{2\pi i}$ appartenant à l'auto-intersection de S (figure 1.3), $|H_s(\bar{\epsilon}, 0) - H_s(\tilde{\epsilon}, 0)| \leq c|\bar{\epsilon}|$ pour un certain $c \in \mathbb{R}_+$, $s = b, h$,

- sont bornées et ont un inverse borné sur l'auto-intersection de S lorsqu'elles sont évaluées en $x = 0$.

Quand $\hat{\epsilon} \to 0$ et $\hat{\epsilon} \in S$, la transformation $H_h(\hat{\epsilon}, x)$ (respectivement $H_b(\hat{\epsilon}, x)$) vers le modèle converge uniformément sur les compacts de Ω_h^0 (respectivement Ω_b^0) vers une transformation normalisante du système à $\epsilon = 0$ (cf. corollaire 3.4.22 et remarque 3.4.2).

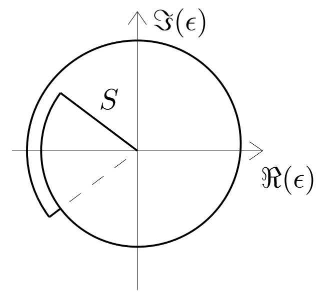

FIG. 1.1. Secteur S dans l'espace du paramètre ϵ.

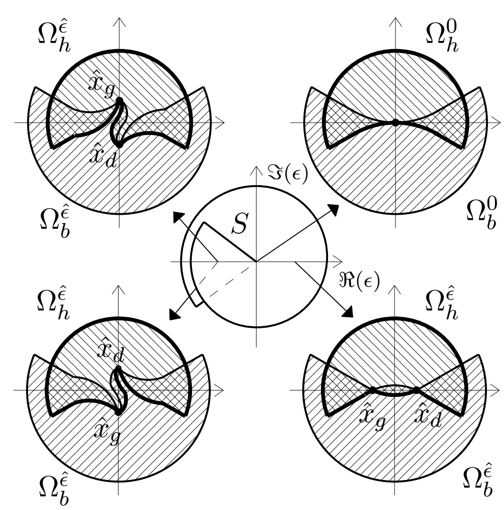

FIG. 1.2. Domaines sectoriels en x des transformations vers le modèle pour quelques valeurs de $\hat{\epsilon} \in S \cup \{0\}$.

L'intersection des domaines sectoriels en x a trois composantes connexes (situées à droite, à gauche et au centre) notées $\Omega_d^{\hat{\epsilon}}$, $\Omega_g^{\hat{\epsilon}}$ et $\Omega_c^{\hat{\epsilon}}$ (figure 1.4). Au-dessus de chacune, $H_b(\hat{\epsilon}, x)^{-1} H_h(\hat{\epsilon}, x)$ est un automorphisme du modèle agissant sur une matrice fondamentale de solutions $F_b(\hat{\epsilon}, x)$ du modèle de la manière suivante (cf.

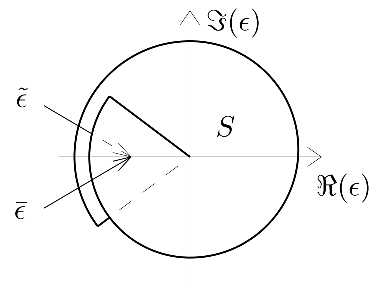

FIG. 1.3. Exemple de valeurs $\bar{\epsilon}$ et $\tilde{\epsilon} = \bar{\epsilon}e^{2\pi i}$ dans l'auto-intersection de S.

théorème 3.4.24) :

$$H_b(\hat{\epsilon}, x)^{-1} H_h(\hat{\epsilon}, x) F_b(\hat{\epsilon}, x) = \begin{cases} F_b(\hat{\epsilon}, x) C_d(\hat{\epsilon}) & \text{sur } \Omega_d^{\hat{\epsilon}n}, \\ F_b(\hat{\epsilon}, x) C_g(\hat{\epsilon}) & \text{sur } \Omega_g^{\hat{\epsilon}}, \\ F_b(\hat{\epsilon}, x) & \text{sur } \Omega_c^{\hat{\epsilon}}, \end{cases} \tag{1.3.1}$$

où $C_d(\hat{\epsilon})$ et $C_g(\hat{\epsilon})$ sont unipotentes, respectivement triangulaire supérieure et triangulaire inférieure, dépendent analytiquement de $\hat{\epsilon} \in S$ et tendent lorsque $\hat{\epsilon} \to 0$ vers les matrices de Stokes. Nous appelons $C_d(\hat{\epsilon})$ et $C_g(\hat{\epsilon})$ les matrices de Stokes déployées.

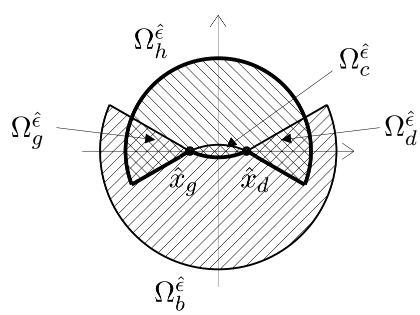

FIG. 1.4. Les composantes connexes de l'intersection des domaines sectoriels $\Omega_b^{\hat{\epsilon}}$ et $\Omega_h^{\hat{\epsilon}}$, cas $\sqrt{\hat{\epsilon}} \in \mathbb{R}_-^*$.

Les transformations $H_b(\epsilon, x)$ et $H_h(\epsilon, x)$ vers le modèle sont uniques à multiplication près par la droite par une matrice diagonale et non singulière $K(\hat{\epsilon})$ qui

- dépend analytiquement de $\hat{\epsilon} \in S$,
- a une limite non singulière à $\epsilon = 0$,

- est telle que, pour $\bar{\epsilon}$ et $\tilde{\epsilon} = \bar{\epsilon}e^{2\pi i}$ appartenant à l'auto-intersection de S (figure 1.3), $|K(\bar{\epsilon}) - K(\tilde{\epsilon})| \leq c|\bar{\epsilon}|$ pour un certain $c \in \mathbb{R}_+$ (cf. proposition 3.4.26).

Ceci induit une équivalence sur les matrices de Stokes déployées : $\{C_d(\hat{\epsilon}), C_g(\hat{\epsilon})\}$ et $\{C_d'(\hat{\epsilon}), C_g'(\hat{\epsilon})\}$ sont équivalentes si

$$C_l'(\hat{\epsilon}) = K(\hat{\epsilon})C_l(\hat{\epsilon})K(\hat{\epsilon})^{-1}, \quad l = g, d. \tag{1.3.2}$$

On a prouvé qu'il existe un représentant $\{C_d(\hat{\epsilon}), C_g(\hat{\epsilon})\}$ de la classe d'équivalence de matrices de Stokes déployées qui est $\frac{1}{2}$-sommable en ϵ (cf. théorème 3.4.53).

1.4. Matrices de Stokes déployées et monodromie de la base de solution choisie

Étant donné que les transformations $H_b(\epsilon, x)$ et $H_h(\epsilon, x)$ vers le modèle sont égales sur $\Omega_c^{\hat{\epsilon}}$ (voir (1.3.1)), nous pouvons définir

$$H(\hat{\epsilon}, x) = \begin{cases} H_b(\hat{\epsilon}, x), & \text{sur } \Omega_b^{\hat{\epsilon}}, \\ H_h(\hat{\epsilon}, x), & \text{sur } \Omega_h^{\hat{\epsilon}}, \end{cases} \tag{1.4.1}$$

une transformation d'un système (1.2.1) vers son modèle pour $\hat{\epsilon} \in S$ et pour x sur le domaine ramifié

$$V^{\hat{\epsilon}} = \Omega_b^{\hat{\epsilon}} \cup \Omega_h^{\hat{\epsilon}} \tag{1.4.2}$$

illustré à la figure 1.5 (et pouvant avoir une forme spiralée autour des points singuliers).

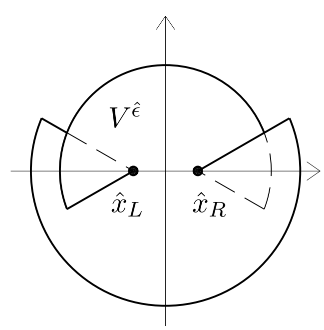

Fig. 1.5. Domaine de $H(\hat{\epsilon}, x)$, dénoté $V^{\hat{\epsilon}}$, cas $\sqrt{\hat{\epsilon}} \in \mathbb{R}_-^*$.

À partir de la transformation $H(\hat{\epsilon}, x)$ vers le modèle et d'une matrice fondamentale de solutions $F_V(\hat{\epsilon}, x)$ du modèle sur $V^{\hat{\epsilon}}$, nous obtenons une matrice fondamentale de solutions du système (1.2.1) donnée par

$$W_V(\hat{\epsilon}, x) = H(\hat{\epsilon}, x)F_V(\hat{\epsilon}, x). \tag{1.4.3}$$

La monodromie de la matrice fondamentale de solutions $W_V(\hat{\epsilon}, x)$ autour des points singuliers est directement reliée aux matrices de Stokes déployées. Plus précisément, pour $l = g, d$, prenons l'opérateur de monodromie $M_{\hat{x}_l}$ associé à un lacet autour du point singulier $x = \hat{x}_l$, tel qu'illustré à la figure 1.6. La représentation de $M_{\hat{x}_l}$ agissant sur la matrice fondamentale de solutions $W_V(\hat{\epsilon}, x)$ est donnée par $C_l(\hat{\epsilon})\hat{D}_l$, où $\hat{D}_l$ est la matrice diagonale représentant l'action de $M_{\hat{x}_l}$ sur la matrice fondamentale de solutions $F_V(\hat{\epsilon}, x)$ du modèle (cf. proposition 3.4.30). Ceci donne une interprétation géométrique aux matrices de Stokes pour $\epsilon = 0$ en

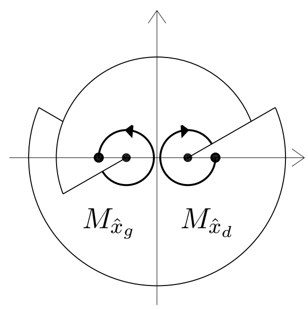

FIG. 1.6. Illustration des lacets définissant les opérateurs de monodromie $M_{\hat{x}_g}$ et $M_{\hat{x}_d}$, cas $\hat{x}_g = \sqrt{\hat{\epsilon}} \in \mathbb{R}_-^*$.

termes de monodromie autour des points singuliers de la matrice fondamentale de solutions $W_V(\hat{\epsilon}, x)$ pour $\hat{\epsilon} \neq 0$ (découlant du fait que si $(C_l(0))_{ij} \neq 0$, alors $(C_l(\hat{\epsilon}))_{ij} \neq 0$ pour $|\hat{\epsilon}|$ assez petit).

1.5. Matrices de Stokes déployées et bases de solutions qui sont des vecteurs propres de la monodromie

Par le théorème 3.4.33 et son corollaire 3.4.35, la matrice de Stokes déployée $C_g(\hat{\epsilon})$ (respectivement $C_d(\hat{\epsilon})$) contribue à établir le lien entre la divergence de solutions à $\epsilon = 0$ et la présence de solutions logarithmiques autour de $x = x_g$ (respectivement $x = x_d$) lors de la résonance (ce lien existe au moins pour les cas génériques). En effet, le nombre de solutions qui sont des vecteurs propres de l'opérateur de monodromie $M_{\hat{x}_l}$ (pour $l = g, d$) est égal au nombre de vecteurs propres de la matrice de Jordan associée à $C_l(\hat{\epsilon})\hat{D}_l$ (qui est la représentation de $M_{\hat{x}_l}$ agissant sur $W_V(\hat{\epsilon}, x)$). Les valeurs du paramètre de déploiement $\hat{\epsilon}$ pour lesquelles il y a résonance sont celles pour lesquelles une matrice $C_l(\hat{\epsilon})\hat{D}_l$ pourrait ne pas être diagonalisable (celles pour lesquelles la matrice $C_l(\hat{\epsilon})\hat{D}_l$ a des valeurs propres multiples). Voici les résultats selon qu'il y a résonance ou non.

- En considérant la matrice triangulaire unipotente $\hat{T}_l$ diagonalisant $C_l(\hat{\epsilon})\hat{D}_l$ dans le cas de non résonance, nous démontrons que $W_V(\hat{\epsilon}, x)\hat{T}_l$ est une matrice fondamentale de solutions qui sont des vecteurs propres de l'opérateur de monodromie $M_{\hat{x}_l}$ (cette matrice fondamentale de solutions est unique à normalisation près) ;
- Lors de la résonance, la matrice $C_l(\hat{\epsilon})\hat{D}_l$ n'est plus diagonalisable avec la $j^{\text{ième}}$ colonne de $\hat{T}_l$ n'existant plus si et seulement si la solution (vecteur propre de la monodromie) correspondant à la $j^{\text{ième}}$ colonne de $W_V(\hat{\epsilon}, x)\hat{T}_l$ n'existe plus. Dans ce cas de non-existence, cette solution doit être remplacée, dans la base de solutions autour de $x = \hat{x}_l$, par une solution contenant des termes logarithmiques. Les conditions pour lesquelles une colonne de $\hat{T}_l$ n'existe plus à la résonance se traduisent par la non-annulation de polynômes en termes des éléments de $\hat{D}_l$ et de $C_l(\hat{\epsilon})$. Dans les cas génériques (où les lignes de Stokes sont distinctes), la non-annulation de ce polynôme pour $|\hat{\epsilon}|$ petit est assuré par la non-annulation du polynôme limite à $\epsilon = 0$. Ce dernier est un polynôme, à coefficients entiers, en les coefficients des matrices de Stokes.

1.6. Systèmes de Riccati

En prenant des cartes dans l'espace projectif complexe, les systèmes déployés s'écrivent comme des équations différentielles matricielles de Riccati, pour $\hat{\epsilon} \in S \cup \{0\}$. En introduisant une variable temporelle, ces équations correspondent à n systèmes différentiels non linéaires que nous avons appelés les n systèmes de Riccati (cf. section 3.4.1).

Dans chaque système de Riccati, nous avons prouvé l'existence, pour $x \in \Omega_s^{\hat{\epsilon}}$ et $s = h, b$, d'une variété invariante de dimension un qui est bornée près des deux points singuliers. C'est en ramenant dans le système linéaire ces n variétés invariantes (une pour chaque système de Riccati) que nous avons obtenu le théorème d'existence des transformations vers le modèle ainsi que leur convergence uniforme sur les compacts de Ω_s^0 (cf. sections 3.4.5 et 3.4.6).

Les matrices de Stokes déployées s'interprètent en termes de monodromie d'intégrales premières dans les systèmes de Riccati. Dans le $j^{\text{ième}}$ système de Riccati, nous donnons l'expression de $n-1$ intégrales premières $\mathcal{H}_q^j$ que nous indexons par $q \in \{1, 2, ..., n\} \backslash \{j\}$. La monodromie d'une intégrale première $\mathcal{H}_q^j$ autour des points singuliers peut être écrite (cf. théorème 3.4.38) comme la composition

- d'une partie sauvage (c.-à-d. de la forme $e^{\frac{2\pi i}{\alpha}}$ avec $\alpha \in \mathbb{C}$, $\alpha \to 0$) et linéaire, ne dépendant que du modèle (c.-à-d. que des invariants formels),
- d'une application dépendant des éléments des lignes q et j de l'inverse des matrices de Stokes déployées et ayant une limite pour $\epsilon = 0$.

On en déduit qu'une ligne j non triviale de l'inverse d'une matrice de Stokes $C_l(0)$ ($l \in \{g, d\}$) est une obstruction à ce que les intégrales premières $\mathcal{H}_q^j$ du $j^{\text{ième}}$ système de Riccati soient des vecteurs propres de la monodromie autour du point singulier $x = \hat{x}_l$, pour $q = 1, 2, ..., j-1, j+1, ..., n$ (cf. corollaire 3.4.39).

1.7. Relation d'auto-intersection

Les invariants formels, $\Lambda_0(\epsilon)$ et $\Lambda_1(\epsilon)$, et les matrices de Stokes déployées, $C_d(\hat{\epsilon})$ et $C_g(\hat{\epsilon})$, satisfont une relation que nous nommons la *relation d'auto-intersection*. Celle-ci provient (cf. section 3.4.10) de l'invariance (à normalisation près) sur l'auto-intersection de S, des matrices de transition entre des bases de solutions qui

- sont des vecteurs propres de la monodromie autour des points singuliers réguliers,
- passent à la limite lorsque $|\epsilon| \to 0$.

En pratique, la relation d'auto-intersection s'écrit

$$Q_b(\bar\epsilon)\bar D_d \bar T_d^{-1}\bar T_g \bar D_d^{-1} = \tilde T_g^{-1}\tilde T_d Q_h(\bar\epsilon), \tag{1.7.1}$$

où

- $\bar\epsilon$ et $\tilde\epsilon = \bar\epsilon e^{2\pi i}$ appartiennent à l'auto-intersection de S (figure 1.3),
- la matrice diagonale $\hat D_d$ représente l'action de $M_{\hat x_d}$ sur la matrice fondamentale de solutions du modèle,
- $\hat T_l$ est la matrice triangulaire unipotente diagonalisant $C_l(\hat\epsilon)\hat D_l$, $l = g, d$,
- $Q_h(\bar\epsilon)$ et $Q_b(\bar\epsilon)$ sont des matrices diagonales non singulières, dépendent analytiquement de $\bar\epsilon \in S_\cap$, ont une limite non singulière lorsque ϵ tend vers 0 et sont telles que

$$|Q_i(\bar\epsilon) - I| < c_i|\bar\epsilon|, \quad c_i \in \mathbb{R},\, \bar\epsilon \in S_\cap,\, i = b, h. \tag{1.7.2}$$

1.8. Théorèmes de classification et de réalisation

Le système complet d'invariants analytiques des systèmes déployés (1.2.1) et les conditions de sa réalisation s'obtiennent finalement des deux théorèmes suivants (qui correspondent aux théorèmes 3.4.61 et 3.5.2).

Théorème de classification analytique

Deux systèmes (1.2.1) sont analytiquement équivalents si et seulement si ils ont les mêmes invariants formels $\Lambda_0(\epsilon)$, $\Lambda_1(\epsilon)$ et des matrices de Stokes déployées équivalentes.

Théorème de réalisation

Soit un système complet d'invariants composé

- **d'un modèle (entièrement déterminé par les invariants formels $\Lambda_0(\epsilon)$, $\Lambda_1(\epsilon)$),**
- **d'une classe d'équivalence de matrices de Stokes déployées (le secteur d'analyticité S de rayon ρ_0 et d'ouverture plus grande que 2π est choisi tel que dans la section 3.4.3, et son rayon ρ_0 peut**

être choisi plus petit afin d'assurer l'analyticité des éléments des matrices de Stokes déployées sur S),

qui satisfont la relation d'auto-intersection. Alors il existe $r > 0$, un rayon $\rho < \min\{\rho_0, \frac{r^2}{2}\}$ de S et un système $(x^2 - \epsilon)y' = B(\epsilon, x)$ ($y \in \mathbb{C}^n$) caractérisé par ces invariants, où $B(\epsilon, x)$ est analytique sur $\mathbb{D}_\rho \times \mathbb{D}_r$.

Chapitre 2

THE STOKES PHENOMENON IN THE CONFLUENCE OF THE HYPERGEOMETRIC EQUATION USING RICCATI EQUATION

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Research supported by NSERC and FQRNT in Canada

Key words : hypergeometric equation, confluence, Stokes phenomenon, divergent series, analytic continuation, summability, monodromy, confluent hypergeometric equation, Riccati equation.

Submitted 8 June 2007 (revised 1 November 2007) and published in the Journal of Differential Equations 244 (2008), n$^{\text{o}}$ 10, 2641–2664

For the purpose of the thesis, this version includes slight modifications from the published version.

Abstract

In this paper we study the confluence of two regular singular points of the hypergeometric equation into an irregular one. We study the consequence of the divergence of solutions at the irregular singular point for the unfolded system. Our study covers a full neighborhood of the origin in the confluence parameter space. In particular, we show how the divergence of solutions at the irregular singular point explains the presence of logarithmic terms in the solutions at a regular singular point of the unfolded system. For this study, we consider values of the confluence parameter taken in two sectors covering the complex plane. In

each sector, we study the monodromy of a first integral of a Riccati system related to the hypergeometric equation. Then, on each sector, we include the presence of logarithmic terms into a continuous phenomenon and view a Stokes multiplier related to a 1-summable solution as the limit of an obstruction that prevents a pair of eigenvectors of the monodromy operators, one at each singular point, to coincide.

2.1. Introduction

The hypergeometric differential equation arises in many problems of mathematics and physics and is related to special functions. It is written

$$X(1-X)\,v''(X) + \{c-(a+b+1)X\}\,v'(X) - ab\,v(X) = 0. \qquad (2.1.1)$$

More precisely, any linear equation of order two ($y''(z)+p(z)y'(z)+q(z)y(z) = 0$) with three regular singular points can be transformed into the hypergeometric equation by a change of variables of the form $y = f(z)v$ and a new independant variable X obtained from z by a Möbius transformation (see for example [**16**] p. 164).

The confluent hypergeometric equation with a regular singular point at $z = 0$ and an irregular one at $z = \infty$ is often written in the form

$$zu''(z) + (c'-z)u'(z) - a'u(z) = 0. \qquad (2.1.2)$$

Solutions of this equation at the irregular point $z = \infty$ are in general divergent and always 1-summable. C. Zhang ([**24**] and [**25**]) and J.-P. Ramis [**17**] showed that the Stokes multipliers related to the confluent equation can be obtained from the limits of the monodromy of the solutions of the nonconfluent equation (2.1.1). They assumed that the bases of solutions of (2.1.1) around the merging singular points ($z = b$ and $z = \infty$) never contain logarithmic terms and they described the phenomenon using two types of limits : first with $\Im(b) \to \infty$, then with $\Re(b) \to \infty$ on the subset $b = b_0 + \mathbb{N}$ for $b_0 \in \mathbb{C}$. They also proved the uniform convergence of the solutions on all compact sets in the case $\Im b \to \infty$. Related questions have been considered by R. Schäfke [**21**].

In this paper, we propose a different approach : we describe the phenomenon in a whole neighborhood of values of the confluence parameter, but we are forced to cover the neighborhood with two sectors on which the presentations are different. We are then able to explain the presence of the logarithmic terms : they occur precisely for discrete values of the confluence parameter when we unfold a confluent equation with at least one divergent solution. On each sector, each divergent solution explains the presence of logarithmic terms at one of the unfolded singular points. The occurrence of logarithmic terms, a discrete phenomenon, is embedded into a continuous phenomenon valid on the whole sector.

To help understanding the phenomenon, we give a translation of the hypergeometric equation in terms of a Riccati system in which two saddle-nodes are unfolded with a parameter ϵ. The parameter space is again covered with two sectors $S^{\pm}$. For this Riccati system, we consider on each sector $S^{\pm}$ of the parameter space a first integral which has a limit when $\epsilon \rightarrow 0$, written in the form $I^{\epsilon^{\pm}}(x,y) = H^{\epsilon^{\pm}}(x)\frac{y-\rho_1(x,\epsilon)}{y-\rho_2(x,\epsilon)}$ where $y = \rho_1(x,\epsilon)$ and $y = \rho_2(x,\epsilon)$ are analytic invariant manifolds of singular points and, for $\epsilon = 0$, center manifolds of the saddle-nodes. Then, when we calculate the monodromy of one of these first integrals, we can separate it into two parts : a continuous one which has a limit when $\epsilon \rightarrow 0$ inside the sector $S^{\pm}$ and a wild one which has no limit but which is linear. The wild part is independent of the divergence of the solutions and present in all cases. The divergence of $\rho_1(x,0)$ corresponds to the analytic invariant manifold of one singular point being ramified at the other in the unfolding of one saddle-node. For particular values of ϵ for which one singular point is a resonant node, this forces the node to be nonlinearisable (i.e. to have a nonzero resonant monomial), in which case logarithmic terms appear in $I^{\epsilon^{\pm}}$. This is called the parametric resurgence phenomenon in [**19**]. The divergence of $\rho_2(x,0)$ corresponds to a similar phenomenon with the pair of singular points coming from the unfolding of the other saddle-node. Finally, we translate our results in the case of a universal deformation.

2.2. Solutions of the hypergeometric equation

In this paper, we study the confluence of the singular points 0 and 1; the confluent hypergeometric equation has an irregular singular point at the origin. We make the change of variables $X = \frac{x}{\epsilon}$ in (2.1.1) to bring the singular point at $X = 1$ to a singular point at $x = \epsilon \neq 0$. We consider small values of ϵ and we limit the values of c to

$$c = 1 - \frac{1}{\epsilon}. \tag{2.2.1}$$

Let $v(\frac{x}{\epsilon})$ be denoted by $w(x)$. Then (2.1.1) becomes

$$x(x-\epsilon)\, w''(x) + \{1 - \epsilon + (a+b+1)x\}\, w'(x) + ab\, w(x) = 0. \tag{2.2.2}$$

We will then let $\epsilon \to 0$. We want to study what happens in a neighborhood of $\epsilon = 0$. The confluence parameter ϵ will be taken in two sectors, the union of which is a small pointed neighborhood of the origin in the complex plane.

Remark 2.2.1. *Although not explicitly written, our study is still valid if we let $a(\epsilon)$ and $b(\epsilon)$ be analytic functions of ϵ.*

Definition 2.2.2. *Given $\gamma \in (0, \frac{\pi}{2})$ fixed, we define*

- $S^+ = \{\epsilon \in \mathbb{C} : 0 < |\epsilon| < r(\gamma),\ \arg(\epsilon) \in (-\pi + \gamma, \pi - \gamma)\}$,
- $S^- = \{\epsilon \in \mathbb{C} : 0 < |\epsilon| < r(\gamma),\ \arg(\epsilon) \in (-2\pi + \gamma, -\gamma)\}$.

Remark 2.2.3. *γ can be chosen arbitrary small, but $r(\gamma)$ will depend on γ and $r(\gamma) \to 0$ as $\gamma \to 0$. In particular, we will ask $a + b + \frac{1}{\epsilon} \notin -\mathbb{N}$, $a + \frac{1}{\epsilon} \notin -\mathbb{N}$ and $b + \frac{1}{\epsilon} \notin -\mathbb{N}$ on S^+ and $2 - a - b - \frac{1}{\epsilon} \notin -\mathbb{N}$, $a - \frac{1}{\epsilon} \notin -\mathbb{N}$ and $b - \frac{1}{\epsilon} \notin -\mathbb{N}$ on S^- (in this paper $\mathbb{N} = \{0, 1, ...\}$).*

2.2.1. Bases for the solutions of the hypergeometric equation (2.2.2) at the regular singular points $x = 0$ and $x = \epsilon$

The fundamental group of $\mathbb{C}\backslash\{0, \epsilon\}$ based at an ordinary point acts on a solution (valid at this base point) by giving its analytic continuation at the end of a loop. In this way we have monodromy operators around each singular point. We can extend it to act on any function of solutions.

Notation 2.2.4. *The monodromy operator M_0 (resp. M_ϵ) is the one associated to the loop which makes one turn around the singular point $x = 0$ (resp. $x = \epsilon$) in* *the positive direction (and which does not surround any other singular point). In this paper, since we use bases of solutions whose Taylor series are convergent in a disk of radius ϵ centered at a singular point, it will be useful to define M_0 (resp. M_ϵ) with the fundamental group based at a point belonging to the line joining $-\epsilon$ and 0 (resp. ϵ and 2ϵ).*

As the hypergeometric equation is linear of second order, the space of solutions is of dimension 2. Given a basis for the space of solutions, the monodromy operator M_0 (resp. M_ϵ) acting on this basis is linear and is represented by a two-dimensional matrix.

As elements of a basis $\mathcal{B}_0$ (resp. $\mathcal{B}_\epsilon$) around the singular point $x = 0$ (resp. $x = \epsilon$), it is classical to use solutions which are eigenvectors of the monodromy operator M_0 (resp. M_ϵ) whenever these solutions exist. However, none of these bases is defined on the whole of a sector S^+ or S^-. This is why we later switch to mixed bases. C. Zhang ([**24**] and [**25**]) also used mixed bases but he has not pushed the study as far as we do.

Definition 2.2.5. *The hypergeometric series ${}_kF_j(a_1, a_2, ...a_k, c_1, c_2, ..., c_j; x)$ is defined by*

$$ {}_kF_j(a_1, a_2, ...a_k, c_1, c_2, ..., c_j; x) = 1 + \sum_{n=1}^{\infty} \frac{(a_1)_n (a_2)_n ... (a_k)_n}{(c_1)_n (c_2)_n ... (c_j)_n n!} x^n \tag{2.2.3} $$

with

$$ \begin{cases} (a)_0 = 1, \\ (a)_n = a(a+1)(a+2)...(a+n-1) \end{cases} \tag{2.2.4} $$

and for $c_1, ..., c_j \notin -\mathbb{N}$.

A basis $\mathcal{B}_0 = \{w_1(x), w_2(x)\}$ of solutions of (2.2.2) around the singular point $x = 0$ is well known (see [**14**] pp. 67–71 for details) :

$$ \begin{cases} w_1(x) & = {}_2F_1\left(a, b, 1 - \frac{1}{\epsilon}; \frac{x}{\epsilon}\right) \\ & = \left(1 - \frac{x}{\epsilon}\right)^{1-\frac{1}{\epsilon}-a-b} {}_2F_1\left(1 - \frac{1}{\epsilon} - a, 1 - \frac{1}{\epsilon} - b, 1 - \frac{1}{\epsilon}; \frac{x}{\epsilon}\right), \\ w_2(x) & = \left(\frac{x}{\epsilon}\right)^{\frac{1}{\epsilon}} {}_2F_1\left(a + \frac{1}{\epsilon}, b + \frac{1}{\epsilon}, 1 + \frac{1}{\epsilon}; \frac{x}{\epsilon}\right) \\ & = \left(\frac{x}{\epsilon}\right)^{\frac{1}{\epsilon}} \left(1 - \frac{x}{\epsilon}\right)^{1-\frac{1}{\epsilon}-a-b} {}_2F_1\left(1 - a, 1 - b, 1 + \frac{1}{\epsilon}; \frac{x}{\epsilon}\right). \end{cases} \tag{2.2.5} $$

The solution $w_1(x)$ exists if $1 - \frac{1}{\epsilon} \notin -\mathbb{N}$ whereas $w_2(x)$ exists if $1 + \frac{1}{\epsilon} \notin -\mathbb{N}$.

Similarly, a basis $\mathcal{B}_\epsilon = \{w_3(x), w_4(x)\}$ of solutions of (2.2.2) around the singular point $x = \epsilon$ is given by :

$$\begin{cases} w_3(x) & = {}_2F_1\left(a, b, a+b+\frac{1}{\epsilon}; 1-\frac{x}{\epsilon}\right), \\ w_4(x) & = \left(\frac{x}{\epsilon}\right)^{\frac{1}{\epsilon}}\left(1-\frac{x}{\epsilon}\right)^{1-\frac{1}{\epsilon}-a-b} {}_2F_1\left(1-a, 1-b, 2-\frac{1}{\epsilon}-a-b; 1-\frac{x}{\epsilon}\right). \end{cases} \tag{2.2.6}$$

The solution $w_3(x)$ exists if $a+b+\frac{1}{\epsilon} \notin -\mathbb{N}$ whereas $w_4(x)$ exists if $2-\frac{1}{\epsilon}-a-b \notin -\mathbb{N}$.

In particular, $w_2(x)$ and $w_3(x)$ exist for all $\epsilon \in S^+$ and $w_1(x)$ and $w_4(x)$ exist for all $\epsilon \in S^-$, provided $r(\gamma)$ is sufficiently small.

Traditionally, in order to get a basis when $1-\frac{1}{\epsilon} \in -\mathbb{N}$, $a \notin -\mathbb{N}$ and $b \notin -\mathbb{N}$ (resp. $2-\frac{1}{\epsilon}-a-b \in -\mathbb{N}$, $1-a \notin -\mathbb{N}$ and $1-b \notin -\mathbb{N}$), the solution $w_1(x)$ in $\mathcal{B}_0$ (resp. $w_4(x)$ in $\mathcal{B}_\epsilon$) is replaced by some other solution $\tilde{w}_1(x)$ (resp. $\tilde{w}_4(x)$) which contains logarithmic terms. Similarly, we have $\tilde{w}_2(x)$ and $\tilde{w}_3(x)$ for specific value of ϵ in S^- (see for example [**7**]).

The problem with this approach is that the basis $\mathcal{B}_0 = \{w_1(x), w_2(x)\}$ (resp. $\mathcal{B}_\epsilon = \{w_3(x), w_4(x)\}$) does not have a limit when the parameter tends to a value for which there are logarithmic terms at the origin (resp. at $x = \epsilon$). For $\epsilon \in S^+$, there are values of ϵ for which $w_1(x)$ or $w_4(x)$ may not be defined, whereas $w_2(x)$ or $w_3(x)$ may not be defined for some values of ϵ in S^-. This means that $\mathcal{B}_0$ and $\mathcal{B}_\epsilon$ are not optimal bases to describe the dynamics for all values of ϵ in the sectors $S^\pm$. We will rather consider the bases $\mathcal{B}^+ = \{w_2(x), w_3(x)\}$ for $\epsilon \in S^+$ and $\mathcal{B}^- = \{w_4(x), w_1(x)\}$ for $\epsilon \in S^-$. With these bases we will explain the occurence of logarithmic terms (a phenomenon occuring for discrete values of the confluence parameter) in a continuous way. The following lemma will allow us to consider only one of the bases, namely $\mathcal{B}^+$ with $\epsilon \in S^+$.

Lemma 2.2.6. *The equation (2.2.2) is invariant under*

$$\begin{cases} c' = 1 - c + a + b, \\ \epsilon' = \frac{1}{1-c'}, \\ x' = \epsilon'\left(1 - \frac{x}{\epsilon}\right), \\ a' = a, \\ b' = b, \end{cases} \tag{2.2.7}$$

which transforms S^+ into S^- and $\mathcal{B}^+$ into $\mathcal{B}^-$.

2.2.2. The confluent hypergeometric equation and its summable solutions

Taking the limit $\epsilon \to 0$ in (2.2.2), we obtain a confluent hypergeometric equation :

$$x^2\, w''(x) + \{1 + (1 + a + b)x\}\, w'(x) + ab\, w(x) = 0. \tag{2.2.8}$$

A basis of solutions around the origin is

$$\begin{cases} \hat{g}(x) = {}_2F_0(a, b; -x), \\ \hat{k}(x) = e^{\frac{1}{x}} x^{1-a-b}\, {}_2F_0(1 - a, 1 - b; x) = e^{\frac{1}{x}} x^{1-a-b} \hat{h}(x). \end{cases} \tag{2.2.9}$$

Remark 2.2.7. *The confluent equation in the literature is often studied with the irregular singular point at infinity :*

$$zu''(z) + (c' - z)u'(z) - au(z) = 0. \tag{2.2.10}$$

The following transformation applied to (2.2.10) yields the confluent equation (2.2.8) :

$$\begin{cases} z = \frac{1}{x}, \\ u\left(\frac{1}{x}\right) = x^a w(x), \\ c' = a + 1 - b. \end{cases} \tag{2.2.11}$$

The following theorem is well-known, one can refer for instance to **[15]**.

Theorem 2.2.8. *The series $\hat{g}(x)$ is divergent if and only if $a \notin -\mathbb{N}$ and $b \notin -\mathbb{N}$. It is 1-summable in all directions except $\mathbb{R}_-$. The series $\hat{h}(x)$ is divergent if and only if $1 - a \notin -\mathbb{N}$ and $1 - b \notin -\mathbb{N}$. It is 1-summable in all directions except $\mathbb{R}_+$.*

The Borel sums of these series, denoted $g(x)$ and $h(x)$, are thus defined in the sectors illustrated in Figure 2.1.

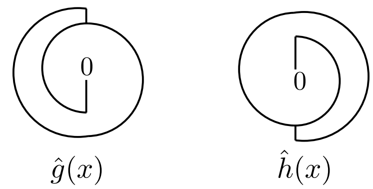

FIG. 2.1. Domains of the Borel sums of the confluent series $\hat{g}(x)$ and $\hat{h}(x)$.

As illustrated in Figure 2.1, we have one Borel sum $g(x)$ in the region $\Re(x) > 0$. When extending $g(x)$ to the region $\Re(x) < 0$ by turning around the origin in the positive (resp. negative) direction, we get a sum $g^+(x)$ (resp. $g^-(x)$). The functions $g^+(x)$ and $g^-(x)$ are different in general and never coincide if the series is divergent. Similarly, we consider $h(x)$ defined in the region $\Re(x) < 0$. When we extend it by turning around the origin in the positive (resp. negative) direction, we obtain the sum $h^+(x)$ (resp. $h^-(x)$). We define

$$\begin{cases} k^+(x) = e^{\frac{1}{x}} x^{1-a-b} h^+(x), \\ k^-(x) = e^{\frac{1}{x}} x^{1-a-b} h^-(x) \end{cases} \tag{2.2.12}$$

for $\Re(x) > 0$, and

$$k(x) = e^{\frac{1}{x}} x^{1-a-b} h(x) \tag{2.2.13}$$

for $\Re(x) < 0$.

Since $g^+(x)$ and $g^-(x)$ have the same asymptotic expansion $\hat{g}(x)$, their difference is a solution of (2.2.8) which is asymptotic to 0 in the region $\Re(x) < 0$, and thus there exists $\lambda \in \mathbb{C}$ such that

$$g^+(xe^{2\pi i}) - g^-(x) = \lambda k(x) \quad \text{if } \arg(x) \in \left(\frac{-3\pi}{2}, \frac{-\pi}{2} \right). \tag{2.2.14}$$

Similarly, there exists $\mu \in \mathbb{C}$ such that

$$k^+(x) - e^{2\pi i(1-a-b)} k^-(xe^{-2\pi i}) = \mu g(x) \quad \text{if } \arg(x) \in \left(\frac{-\pi}{2}, \frac{\pi}{2} \right). \tag{2.2.15}$$

Remark 2.2.9. *For all $n \in \mathbb{Z}$, it is possible to construct a function $g_n(x)$, corresponding to the Borel sum of the divergent series $\hat{g}(x)$ in the regions* $\arg(x) \in (\frac{-\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n)$. *Then, $g_n^+(x)$ (resp. $g_n^-(x)$) denotes its analytic continuation in the positive (resp. negative) direction around the origin, defined in the region* $\arg(x) \in (\frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n)$ *(resp.* $\arg(x) \in (\frac{-3\pi}{2} + 2\pi n, \frac{-\pi}{2} + 2\pi n)$*). Since $g_{n+1}^+(xe^{2\pi i}) = g_n^+(x)$, $g_{n+1}^-(xe^{2\pi i}) = g_n^-(x)$ and $g_{n+1}(xe^{2\pi i}) = g_n(x)$, the subscript n is not necessary and the functions $g(x)$, $g^+(x)$ and $g^-(x)$ are univalued. But what is important is that, when considering $g^+(x)$, the $+$ does not refer to the values of* $\arg(x)$, *but to the fact that $g^+(x)$ has been obtained by analytic continuation of $g(x)$ when turning in the positive direction. Similar relations for $h^+(x)$, $h^-(x)$ and $h(x)$ imply that these functions are also univalued. On the other hand, x^{1-a-b} is a multivalued function, which becomes univalued as soon as* $\arg(x)$ *is determined.*

Definition 2.2.10. *In the relations (2.2.14) and (2.2.15), we call λ and μ the Stokes multipliers associated respectively to the solutions $g(x)$ and $k(x)$.*

Their values are calculated in [**15**]. Using the change of variable (2.2.11), we have

$$\lambda = -\frac{2\pi i e^{i\pi(1-a-b)}}{\Gamma(a)\Gamma(b)} \tag{2.2.16}$$

and

$$\mu = -\frac{2i\pi}{\Gamma(1-a)\Gamma(1-b)}. \tag{2.2.17}$$

Notation 2.2.11. *Let us write*

$$H^0(x) = \begin{cases} \frac{k(x)}{g^-(x)} & \text{if } \Re(x) < 0, \\ \frac{k^+(x)}{g(x)} & \text{if } \Re(x) > 0 \end{cases} \tag{2.2.18}$$

and

$$H^{0'}(x) = \begin{cases} \frac{k^-(x)}{g(x)} & \text{if } \Re(x) > 0, \\ \frac{k(x)}{g^+(x)} & \text{if } \Re(x) < 0 \end{cases} \tag{2.2.19}$$

with $H^0(x)$ (resp. $H^{0'}(x)$) analytic in the complex plane minus a cut with values in $\mathbb{CP}^1$, as illustrated in Figure 2.2. On purpose we leave the ambiguity in the argument. In this form, $H^0(x)$ and $H^{0'}(x)$ are multivalued. They will become univalued when $\arg(x)$ *is specified.*

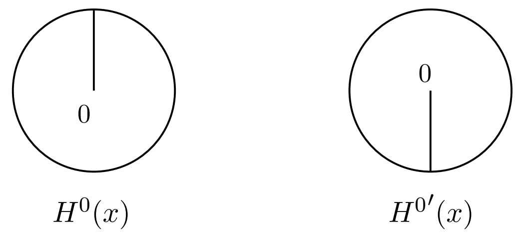

FIG. 2.2. Domains of $H^0(x)$ and $H^{0'}(x)$, with arbitrary radius.

Proposition 2.2.12. *The Stokes multiplier of $g(x)$ is*

$$\lambda = \tfrac{1}{H^{0'}(x)} - \tfrac{1}{H^0(x)} \quad \text{if } \arg(x) \in \left(\tfrac{-3\pi}{2}, \tfrac{-\pi}{2}\right), \tag{2.2.20}$$

while the Stokes multiplier of $k(x)$ is

$$\mu = H^0(x) - e^{2\pi i(1-a-b)} H^{0'}(xe^{-2\pi i}) \quad \text{if } \arg(x) \in \left(\tfrac{-\pi}{2}, \tfrac{\pi}{2}\right). \tag{2.2.21}$$

PROOF. We have

$$\begin{aligned} \lambda &= \tfrac{g^+(xe^{2\pi i})}{k(x)} - \tfrac{g^-(x)}{k(x)} \\ &= \tfrac{g^+(x)}{k(x)} - \tfrac{g^-(x)}{k(x)} \\ &= \tfrac{1}{H^{0'}(x)} - \tfrac{1}{H^0(x)} \quad \text{if } \arg(x) \in \left(\tfrac{-3\pi}{2}, \tfrac{-\pi}{2}\right) \end{aligned} \tag{2.2.22}$$

and

$$\begin{aligned} \mu &= \tfrac{k^+(x)}{g(x)} - e^{2\pi i(1-a-b)} \tfrac{k^-(xe^{-2\pi i})}{g(x)} \\ &= \tfrac{k^+(x)}{g(x)} - e^{2\pi i(1-a-b)} \tfrac{k^-(xe^{-2\pi i})}{g(xe^{-2\pi i})} \\ &= H^0(x) - e^{2\pi i(1-a-b)} H^{0'}(xe^{-2\pi i}) \quad \text{if } \arg(x) \in \left(\tfrac{-\pi}{2}, \tfrac{\pi}{2}\right). \end{aligned} \tag{2.2.23}$$

□

In view of this proposition, it will seem natural in the next section to study the monodromy of some quotient of solutions of the hypergeometric equation (2.2.2). But before, let us explore the link between divergent series in particular solutions of the confluent differential equation and analytic continuation of series appearing in solutions of the nonconfluent equation.

2.3. Divergence and Monodromy

2.3.1. Divergence and ramification : first observations

Let us illustrate by an example the link between the divergence of a confluent series and the ramification of its unfolded series.

Example 2.3.1. *The series* $\hat{g}(x) = {}_2F_0(a,b;-x)$ *is non-summable in the direction* $\mathbb{R}_-$*, i.e. on the left side. We unfold* $g(x)$ *with a small* $\epsilon \in \mathbb{R}$*. Let us define*

$$g^{\epsilon}(x) = \begin{cases} w_3(x) = {}_2F_1\left(a,b,a+b+\frac{1}{\epsilon};1-\frac{x}{\epsilon}\right) & \text{if } \epsilon \in S^+, \\ w_1(x) = {}_2F_1\left(a,b,1-\frac{1}{\epsilon};\frac{x}{\epsilon}\right) & \text{if } \epsilon \in S^-. \end{cases} \tag{2.3.1}$$

By continuity, the analytic continuation of $g^{\epsilon}(x)$ *will be ramified at the left singular point and regular at the right singular point (Figure 2.3). For the special values of* ϵ *for which logarithmic terms may exist in the general solution at the left singular point, this will force their existence. Indeed, for these special values of* ϵ*, the general solution at the left singular point either has logarithmic terms or is not ramified (for more details, refer to the proof of Theorem 2.3.5* (1)*).*

This example illustrates that a direction of non-summability for a confluent series determines which merging singular point is "pathologic" (with ϵ in $S^{\pm}$) for an unfolded solution, as illustrated in Figure 2.3. Although subtleties are needed to adapt Example 2.3.1 to the other solution $k(x) = e^{\frac{1}{x}}x^{1-a-b}h(x)$ because of the ramification of x^{1-a-b}, we have a similar phenomenon if we define adequately the pathology. For example, if $\epsilon \in S^+$, the singular point $x = 0$ will be defined pathologic for the solution $w_3(x)$ if the analytic continuation of this solution is not an eigenvector of the monodromy operator M_0. This will be studied more precisely in Section 2.3.3 using the results we will obtain in the next two sections.

2.3.2. Limit of quotients of solutions on $S^{\pm}$

We will later see that a divergent series in the basis of solutions at the confluence necessarily implies the presence of an obstruction that prevents an eigenvector of M_0 to be an eigenvector of M_{ϵ}. As a tool for our study, we will consider the behavior of the analytic continuation of some functions of the particular solutions $w_i(x) \in \mathcal{B}^{\pm}$ when turning around singular points. A first motivation

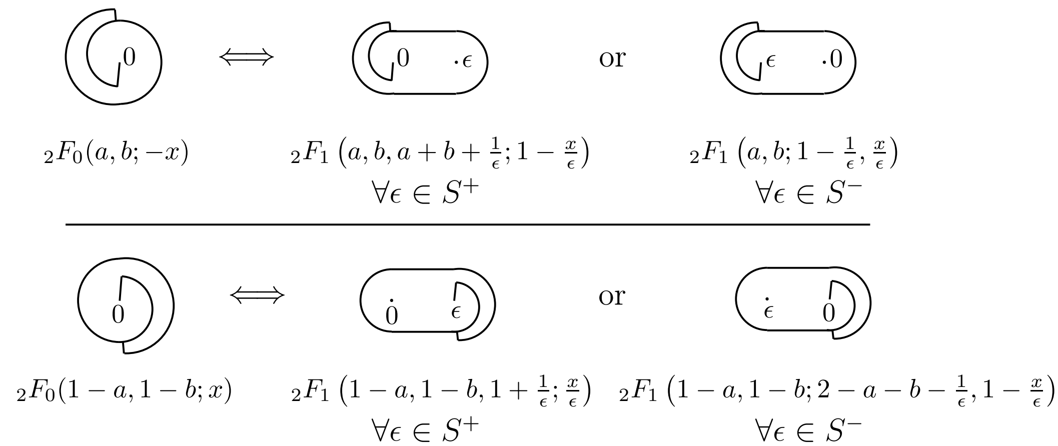

FIG. 2.3. Link between ramification of the analytic continuation of the hypergeometric series in the unfolded case and divergence (ramification) of the associated confluent series.

for studying these functions comes from Proposition 2.2.12. We will also see in Section 2.4 that these quantities have the same ramification as first integrals of a Riccati system related to the hypergeometric equation, these first integrals having a limit when $\epsilon \to 0$ on $S^{\pm}$. The first integrals are defined by

$$H^{\epsilon^+}(x) = \frac{\kappa^+(\epsilon) w_2(x)}{w_3(x)} \quad \text{if } \epsilon \in S^+ \tag{2.3.2}$$

and

$$H^{\epsilon^-}(x) = \frac{\kappa^-(\epsilon) w_4(x)}{w_1(x)} \quad \text{if } \epsilon \in S^- \tag{2.3.3}$$

with

$$\kappa^+(\epsilon) = \epsilon^{1-a-b} e^{\pi i (a+b-1+\frac{1}{\epsilon})}, \quad \kappa^-(\epsilon) = \epsilon^{1-a-b} e^{-\pi i (a+b-1+\frac{1}{\epsilon})}. \tag{2.3.4}$$

$H^{\epsilon^{\pm}}(x)$ are first defined in $B(0,\epsilon) \cap B(\epsilon,\epsilon)$ and then analytically extended as in Figures 2.4 and 2.5. The coefficients $\kappa^{\pm}$ in the functions $H^{\epsilon^{\pm}}(x)$ are chosen so that $H^{\epsilon^{\pm}}(x)$ have the limit $H^0(x)$ when $\epsilon \to 0$ inside $S^{\pm}$. More precisely, for $\epsilon \in S^+$, we replace $f(x) = (\frac{x}{\epsilon})^{\frac{1}{\epsilon}}(1-\frac{x}{\epsilon})^{1-\frac{1}{\epsilon}-a-b}$ by $\kappa^+(\epsilon) f(x)$, so that the limit when $\epsilon \to 0$ and $\epsilon \in S^+$ exists and corresponds to $e^{\frac{1}{x}} x^{1-a-b}$. The limit is uniform on any simply connected compact set which does not contain 0. The constant

$\kappa^+(\epsilon)$ (resp. $\kappa^-(\epsilon)$) is the natural one to consider for $\epsilon \in S^+$ (resp. $\epsilon \in S^-$) when the analytic continuation of $\kappa^+(\epsilon)f(x)$ (resp. $\kappa^-(\epsilon)f(x)$) is done like in Figure 2.4 (resp. Figure 2.5).

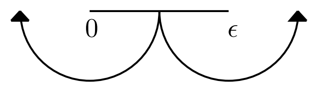

FIG. 2.4. Analytic continuation of $\kappa^+(\epsilon)\left(\frac{x}{\epsilon}\right)^{\frac{1}{\epsilon}}\left(1-\frac{x}{\epsilon}\right)^{1-\frac{1}{\epsilon}-a-b}$ for $\epsilon \in S^+$.

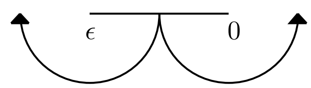

FIG. 2.5. Analytic continuation of $\kappa^-(\epsilon)\left(\frac{x}{\epsilon}\right)^{\frac{1}{\epsilon}}\left(1-\frac{x}{\epsilon}\right)^{1-\frac{1}{\epsilon}-a-b}$ for $\epsilon \in S^-$.

Proposition 2.3.2. *When $\epsilon \to 0$ and $\epsilon \in S^+$ (resp. $\epsilon \in S^-$), $H^{\epsilon^+}(x)$ (resp. $H^{\epsilon^-}(x)$) converges uniformly to $H^0(x)$ on any simply connected compact subset of the domain of $H^0(x)$ illustrated in Figure 2.2. More precisely, we have the uniform limits on compact subsets :*

$$\begin{cases} \lim_{\substack{\epsilon\to 0\\ \epsilon\in S^+}} \kappa^+(\epsilon)w_2(x) = k^+(x), \\ \lim_{\substack{\epsilon\to 0\\ \epsilon\in S^+}} w_3(x) = g(x), \end{cases} \qquad \begin{cases} \lim_{\substack{\epsilon\to 0\\ \epsilon\in S^-}} \kappa^-(\epsilon)w_4(x) = k^+(x), \\ \lim_{\substack{\epsilon\to 0\\ \epsilon\in S^-}} w_1(x) = g(x). \end{cases} \tag{2.3.5}$$

PROOF. The hypergeometric functions appearing in $w_k(x)$ ($k = 1, 2, 3, 4$) and having the limit $h(x)$ or $g(x)$ are ramified as illustrated in Figure 2.3, which suggests to take sectors like in Figure 2.2 when considering the quotient of these functions.

We first prove the uniform convergence of $w_3(x)$ to $g(x)$ on simply connected compact subsets of the domain $\{x, |\arg(x)| < \frac{3\pi}{2}\}$ as $\epsilon \to 0$ in S^+. This proof has been inspired by **[24]**. Let us suppose that $a - b \notin \mathbb{Z}$. The analytic continuation of $w_3(x)$ is (see **[14]** pp. 67–71)

$$w_3(x) = \frac{\Gamma(a+b+\frac{1}{\epsilon})\Gamma(b-a)}{\Gamma(b)\Gamma(b+\frac{1}{\epsilon})}w_5(x) + \frac{\Gamma(a+b+\frac{1}{\epsilon})\Gamma(a-b)}{\Gamma(a)\Gamma(a+\frac{1}{\epsilon})}w_6(x) \tag{2.3.6}$$

with

$$\begin{cases} w_5(x) = \left(\frac{\epsilon}{x}\right)^a \ {}_2F_1\left(a, a+\frac{1}{\epsilon}, a+1-b; \frac{\epsilon}{x}\right), \\ w_6(x) = \left(\frac{\epsilon}{x}\right)^b \ {}_2F_1\left(b, b+\frac{1}{\epsilon}, b+1-a; \frac{\epsilon}{x}\right). \end{cases} \tag{2.3.7}$$

The function ${}_2F_1(a, a+\frac{1}{\epsilon}, a+1-b; \frac{\epsilon}{x})$ converges uniformly on simply connected compact subsets to ${}_1F_1(a, a+1-b; \frac{1}{x})$ and we have

$$\lim_{\substack{\epsilon\to 0\\ \epsilon\in S^+}} \frac{\epsilon^a\Gamma(a+b+\frac{1}{\epsilon})}{\Gamma(b+\frac{1}{\epsilon})} = 1. \tag{2.3.8}$$

The same relations apply with a and b interchanged so $w_3(x)$ converges uniformly on simply connected compact subsets to

$$g(x) = \frac{\Gamma(b-a)}{\Gamma(b)}x^{-a}\ {}_1F_1\left(a, a+1-b; \frac{1}{x}\right) + \frac{\Gamma(a-b)}{\Gamma(a)}x^{-b}\ {}_1F_1\left(b, b+1-a; \frac{1}{x}\right). \tag{2.3.9}$$

Let us suppose now that $a-b=-m$ with $m \in \mathbb{N}$. We take t small, we let $a = b-m+t$. We first show that $\lim_{t\to 0} w_3(x)$ exists with x on a simply connected compact subset of the domain $\{x, |\arg(x)| < \frac{3\pi}{2}\}$. We write $w_3(x)$ as

$$\begin{aligned} w_3(x) = {} & (a-b)\Gamma(b-a)\Gamma(a-b)\Gamma(a+b+\tfrac{1}{\epsilon}) \\ & \times\left[\frac{w_5(x)}{\Gamma(b)\Gamma(b+\frac{1}{\epsilon})\Gamma(a-b+1)} - \frac{w_6(x)}{\Gamma(a)\Gamma(a+\frac{1}{\epsilon})\Gamma(b-a+1)}\right] \end{aligned} \tag{2.3.10}$$

and take the limit $t \to 0$ with $a = b-m+t$. The part inside brackets has a zero at $t=0$ since

$$\begin{aligned} \lim_{t\to 0} \frac{w_5(x)}{\Gamma(a-b+1)} & = \left(\frac{\epsilon}{x}\right)^b \frac{(b-m)_m(b-m+\frac{1}{\epsilon})_m}{m!}\ {}_2F_1\left(b, b+\frac{1}{\epsilon}, m+1; \frac{\epsilon}{x}\right) \\ & = \frac{\Gamma(b)\Gamma(b+\frac{1}{\epsilon})w_6(x)}{\Gamma(a)\Gamma(a+\frac{1}{\epsilon})\Gamma(b-a+1)}. \end{aligned} \tag{2.3.11}$$

The left part of (2.3.10) has a simple pole at $t=0$ so $\lim_{t\to 0} w_3(x)$ exists. Since $w_3(x)$ is an analytic function of t on a punctured neighborhood of $t=0$, we have that $w_3(x)$ converges uniformly on simply connected compact subsets to $\lim_{t\to 0} w_3(x)$ when $t \to 0$. Similarly, $g(x)$ converges uniformly on simply connected compact subsets to $\lim_{t\to 0} g(x)$ since

$$\lim_{t\to 0} \frac{{}_1F_1(a, a+1-b; \frac{1}{x})}{x^a\Gamma(a-b+1)} = \frac{\Gamma(b)\,{}_1F_1(b, b+1-a; \frac{1}{x})}{x^b\Gamma(a)\Gamma(b-a+1)}. \tag{2.3.12}$$

Hence, when $\epsilon \to 0$ in S^+, $\lim_{t\to 0} w_3(x)$ converges uniformly on simply connected compact subsets (of $\{x, |\arg(x)| < \frac{3\pi}{2}\}$) to $\lim_{t\to 0} g(x)$. Interchanging a and b leads to the case $b-a \in -\mathbb{N}$.

Now, $w_2(x)$ (as in (2.2.5)) converges uniformly to $k(x)$ on simply connected compact subsets of the domain $\{x, |\arg(-x)| < \frac{3\pi}{2}\}$ to $k(x)$. Indeed, we can decompose $\kappa^+(\epsilon)w_2(x)$ as

$$\left(e^{\frac{\pi i}{\epsilon}}\left(\frac{x}{\epsilon}\right)^{\frac{1}{\epsilon}}\left(1-\frac{x}{\epsilon}\right)^{-\frac{1}{\epsilon}}\right)\left((x-\epsilon)^{1-a-b}\,{}_2F_1\left(1-a,1-b,1+\frac{1}{\epsilon};\frac{x}{\epsilon}\right)\right). \quad (2.3.13)$$

The first part converges to $e^{\frac{1}{x}}$. The second part converges to $x^{1-a-b}\,{}_2F_0(1-a, 1-b; x)$. The fact that ${}_2F_1(1-a, 1-b, 1+\frac{1}{\epsilon}; \frac{x}{\epsilon})$ converges uniformly on simply connected compact subsets to ${}_2F_0(1-a, 1-b; x)$ can be obtained from the convergence of $w_3(x)$ to $g(x)$ by a change of coordinates. The case $\epsilon \in S^-$ is similar. ☐

2.3.3. Divergence and nondiagonal form of the monodromy operator in the basis $\mathcal{B}^+$

It is clear that $w_2(x)$ is an eigenvector of the monodromy operator M_0 with eigenvalue $e^{\frac{i\pi}{\epsilon}}$, and that $w_3(x)$ is an eigenvector of M_ϵ with eigenvalue 1. In general, eigenvectors of the monodromy operators M_0 and M_ϵ should not coincide. In the generic case, the analytic continuation of an eigenvector of the monodromy operator M_0 is not an eigenvector of M_ϵ. If we are in the generic case and this persists to the limit $\epsilon = 0$, then at the limit we have a nonzero Stokes multiplier. The results stated in the next theorem tell us whether or not the analytic continuation of $w_3(x)$ (resp. $w_2(x)$) is an eigenvector of M_0 (resp. M_ϵ). This is done in the two covering sectors $S^\pm$ of a small neighborhood of ϵ, and it includes the presence of logarithmic terms : we will detail this last part in Theorem 2.3.5 below.

Notation 2.3.3. *Let $w_{(\delta,\theta)}(x)$ be the analytic continuation of $w(x)$ when starting on $(0, \epsilon)$ and turning of an angle θ around $x = \delta$, with $\delta \in \{0, \epsilon\}$ (see Figure 2.6). In short, $w_{(\delta,\pi)}(x)$ can be obtained from the action of the monodromy operator around $x = \delta$ applied on $w_{(\delta,-\pi)}(x)$.*

Theorem 2.3.4.

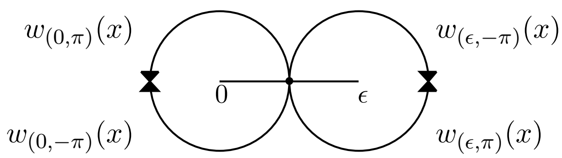

FIG. 2.6. Analytic continuation of $w(x)$.

- *If* $\epsilon \in S^+$, *then*

$$\begin{pmatrix} \kappa^+(\epsilon) w_{2,(0,\pi)} \\ w_{3,(0,\pi)} \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi i}{\epsilon}} & 0 \\ \lambda^+(\epsilon) & 1 \end{pmatrix} \begin{pmatrix} \kappa^+(\epsilon) w_{2,(0,-\pi)} \\ w_{3,(0,-\pi)} \end{pmatrix} \tag{2.3.14}$$

and

$$\begin{pmatrix} \kappa^+(\epsilon) w_{2,(\epsilon,\pi)} \\ w_{3,(\epsilon,\pi)} \end{pmatrix} = \begin{pmatrix} e^{2\pi i(1-a-b-\frac{1}{\epsilon})} & \mu^+(\epsilon) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \kappa^+(\epsilon) w_{2,(\epsilon,-\pi)} \\ w_{3,(\epsilon,-\pi)} \end{pmatrix}, \tag{2.3.15}$$

with

$$\mu^+(\epsilon) = \frac{-2\pi i}{\Gamma(1-a)\Gamma(1-b)} \frac{\epsilon^{1-a-b}\Gamma(1+\frac{1}{\epsilon})}{\Gamma(a+b+\frac{1}{\epsilon})} \tag{2.3.16}$$

and

$$\lambda^+(\epsilon) = \frac{-2\pi i e^{\pi i(1-a-b)}}{\Gamma(a)\Gamma(b)} \frac{\epsilon^{a+b-1}\Gamma(a+b+\frac{1}{\epsilon})}{\Gamma(1+\frac{1}{\epsilon})}. \tag{2.3.17}$$

Hence, when it is nonzero, the coefficient $\lambda^+(\epsilon)$ *(resp.* $\mu^+(\epsilon)$*) represents the obstruction that prevents* $w_3(x)$ *(resp.* $w_2(x)$*) of being an eigenvector of the monodromy operator around* $x = 0$ *(resp.* $x = \epsilon$*).*

- *If* $\epsilon \in S^-$, *then*

$$\begin{pmatrix} \kappa^-(\epsilon) w_{4,(\epsilon,\pi)} \\ w_{1,(\epsilon,\pi)} \end{pmatrix} = \begin{pmatrix} e^{2\pi i(1-\frac{1}{\epsilon}-a-b)} & 0 \\ \lambda^-(\epsilon) & 1 \end{pmatrix} \begin{pmatrix} \kappa^-(\epsilon) w_{4,(\epsilon,-\pi)} \\ w_{1,(\epsilon,-\pi)} \end{pmatrix} \tag{2.3.18}$$

and

$$\begin{pmatrix} \kappa^-(\epsilon) w_{4,(0,\pi)} \\ w_{1,(0,\pi)} \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi i}{\epsilon}} & \mu^-(\epsilon) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \kappa^-(\epsilon) w_{4,(0,-\pi)} \\ w_{1,(0,-\pi)} \end{pmatrix} \tag{2.3.19}$$

with

$$\mu^-(\epsilon) = \frac{-2\pi i}{\Gamma(1-a)\Gamma(1-b)} \frac{(\epsilon e^{\pi i})^{1-a-b}\Gamma(2-\frac{1}{\epsilon}-a-b)}{\Gamma(1-\frac{1}{\epsilon})} \tag{2.3.20}$$

and

$$\lambda^-(\epsilon) = \frac{-2\pi i}{\Gamma(a)\Gamma(b)} \frac{(\epsilon)^{a+b-1}\Gamma(1-\frac{1}{\epsilon})}{\Gamma(2-\frac{1}{\epsilon}-a-b)}. \tag{2.3.21}$$

Hence, when it is nonzero, the coefficient $\lambda^-(\epsilon)$ (resp. $\mu^-(\epsilon)$) represents the obstruction that prevents $w_1(x)$ (resp. $w_4(x)$) of being an eigenvector of the monodromy operator around $x=\epsilon$ (resp. $x=0$).

Then, with the limit taken for any path in S^+ or in S^-, we have

$$\lim_{\epsilon\to 0}\mu^{\pm}(\epsilon)=\mu \tag{2.3.22}$$

and

$$\lim_{\epsilon\to 0}\lambda^{\pm}(\epsilon)=\lambda, \tag{2.3.23}$$

which are precisely the Stokes multipliers associated to the solutions $k(x)$ and $g(x)$ and given by (2.2.16) and (2.2.17).

PROOF. Let $\epsilon\in S^+$. To make analytic continuation of the solutions $w_2(x)$ and $w_3(x)$, we need to make further restrictions on the values of ϵ, but we will shortly show the validity of the result without these hypotheses. We have (see for example **[14]** pp. 67–71)

- if $2-\frac{1}{\epsilon}-a-b\notin -\mathbb{N}$,

$$\begin{aligned} w_2(x) &= \tfrac{\Gamma(1-\frac{1}{\epsilon}-a-b)\Gamma(1+\frac{1}{\epsilon})}{\Gamma(1-a)\Gamma(1-b)}w_3(x)+\tfrac{\Gamma(a+b-1+\frac{1}{\epsilon})\Gamma(1+\frac{1}{\epsilon})}{\Gamma(a+\frac{1}{\epsilon})\Gamma(b+\frac{1}{\epsilon})}w_4(x)\\ &= D(\epsilon)w_3(x)+E(\epsilon)w_4(x); \end{aligned} \tag{2.3.24}$$

- if $1-\frac{1}{\epsilon}\notin -\mathbb{N}$,

$$\begin{aligned} w_3(x) &= \tfrac{\Gamma(\frac{1}{\epsilon})\Gamma(a+b+\frac{1}{\epsilon})}{\Gamma(b+\frac{1}{\epsilon})\Gamma(a+\frac{1}{\epsilon})}w_1(x)+\tfrac{\Gamma(a+b+\frac{1}{\epsilon})\Gamma(-\frac{1}{\epsilon})}{\Gamma(a)\Gamma(b)}w_2(x)\\ &= A(\epsilon)w_1(x)+B(\epsilon)w_2(x). \end{aligned} \tag{2.3.25}$$

These relations allow the calculation of the monodromy of $w_2(x)$ (resp. $w_3(x)$) around $x=\epsilon$ (resp. $x=0$). The explosion of the coefficients (coefficients becoming infinite) for specific values of ϵ corresponds to the presence of logarithmic terms in the general solution around the singular point $x=\epsilon$ (resp. $x=0$). We have, in the region $B(0,\epsilon)\cap B(\epsilon,\epsilon)$ (with the hypothesis that $2-\frac{1}{\epsilon}-a-b\notin -\mathbb{N}$),

$$\begin{aligned}&\kappa^+(\epsilon)w_2(x)\\ &\quad=\kappa^+(\epsilon)(D(\epsilon)w_3(x)+E(\epsilon)w_4(x))\\ &\quad=\kappa^+(\epsilon)\left(D(\epsilon)\,{}_2F_1(a,b,a+b+\tfrac{1}{\epsilon};1-\tfrac{x}{\epsilon})\right.\\ &\qquad\left.+E(\epsilon)(\tfrac{x}{\epsilon})^{\frac{1}{\epsilon}}(1-\tfrac{x}{\epsilon})^{1-\frac{1}{\epsilon}-a-b}\,{}_2F_1(1-a,1-b,-\tfrac{1}{\epsilon}+2-a-b;1-\tfrac{x}{\epsilon})\right).\end{aligned}\tag{2.3.26}$$

Since $w_{3,(\epsilon,-\pi)}=w_{3,(\epsilon,\pi)}$, we obtain

$$\kappa^+(\epsilon)w_{2,(\epsilon,\pi)}=e^{2\pi i(1-a-b-\frac{1}{\epsilon})}\kappa^+(\epsilon)w_{2,(\epsilon,-\pi)}+\mu^+(\epsilon)w_{3,(\epsilon,-\pi)}\tag{2.3.27}$$

with

$$\begin{aligned}\mu^+(\epsilon)&=D(\epsilon)\epsilon^{1-a-b}e^{\pi i(a+b-1+\frac{1}{\epsilon})}\left(1-e^{2\pi i(1-a-b-\frac{1}{\epsilon})}\right)\\ &=-D(\epsilon)\epsilon^{1-a-b}\left(e^{\pi i(1-a-b-\frac{1}{\epsilon})}-e^{-\pi i(1-a-b-\frac{1}{\epsilon})}\right).\end{aligned}\tag{2.3.28}$$

Since $\sin(z)=\frac{e^{iz}-e^{-iz}}{2i}$ and $\Gamma(z)\sin(\pi z)=\frac{\pi}{\Gamma(1-z)}$, we can simplify the latter expression :

$$\begin{aligned}\mu^+(\epsilon)&=-2iD(\epsilon)\epsilon^{1-a-b}\sin\left(\pi\left(1-a-b-\tfrac{1}{\epsilon}\right)\right)\\ &=-2i\tfrac{\Gamma(1-\frac{1}{\epsilon}-a-b)\Gamma(1+\frac{1}{\epsilon})}{\Gamma(1-a)\Gamma(1-b)}\epsilon^{1-a-b}\sin\left(\pi\left(1-a-b-\tfrac{1}{\epsilon}\right)\right)\\ &=-2\pi i\tfrac{\Gamma(1+\frac{1}{\epsilon})}{\Gamma(1-a)\Gamma(1-b)}\epsilon^{1-a-b}\tfrac{1}{\Gamma(a+b+\frac{1}{\epsilon})}.\end{aligned}\tag{2.3.29}$$

Remark that this expression is defined even if $2-\frac{1}{\epsilon}-a-b\in-\mathbb{N}$, so we have removed the indeterminacy!

In the particular case $a+b\in\mathbb{Z}$,

$$\mu^+(\epsilon)=\frac{-2i\pi}{\Gamma(1-a)\Gamma(1-b)}\epsilon^{1-a-b}r(a+b)\tag{2.3.30}$$

with

$$r(\gamma)\;=\frac{\Gamma(1+\frac{1}{\epsilon})}{\Gamma(\gamma+\frac{1}{\epsilon})}\;=\begin{cases}\prod_{j=1}^{\gamma-1}\frac{1}{\frac{1}{\epsilon}+j}, & \gamma>1,\\ \prod_{j=\gamma}^{0}(\frac{1}{\epsilon}+j), & \gamma<1,\\ 1, & \gamma=1.\end{cases}\tag{2.3.31}$$

Finally,

$$\lim_{\substack{\epsilon\to 0\\ \epsilon\in S^+}}\epsilon^{1-a-b}\frac{\Gamma(\frac{1}{\epsilon}+1)}{\Gamma(\frac{1}{\epsilon}+a+b)}=1.\tag{2.3.32}$$

Hence

$$\lim_{\substack{\epsilon\to 0\\ \epsilon\in S^+}} \mu^+(\epsilon) = -\frac{2i\pi}{\Gamma(1-a)\Gamma(1-b)} = \mu. \tag{2.3.33}$$

Let ϵ_n such that $2 - \frac{1}{\epsilon_n} - a - b = -n$, $n \in \mathbb{N}$. Recall that we have supposed $\epsilon \neq \epsilon_n$ to obtain $\mu^+(\epsilon)$. Since $\mu^+(\epsilon)$ is analytic in a punctured disk $B(\epsilon_n, \rho)\backslash\{\epsilon_n\}$ (for some well chosen $\rho \in \mathbb{R}_+$), and $\lim_{\epsilon\to\epsilon_n} \mu^+(\epsilon)$ exists, then $\mu^+(\epsilon)$ is analytic in $B(\epsilon_n, \rho)$. Hence, the result obtained is valid without the restriction $2-\frac{1}{\epsilon}-a-b \notin -\mathbb{N}$.

A similar calculation gives, with $w_{2,(0,\pi)} = e^{\frac{2\pi i}{\epsilon}} w_{2,(0,-\pi)}$,

$$w_{3,(0,\pi)} = w_{3,(0,-\pi)} + \lambda^+(\epsilon)\kappa^+(\epsilon)w_{2,(0,-\pi)} \tag{2.3.34}$$

with $\lambda^+(\epsilon) = B(\epsilon)e^{-\pi i(a+b-1+\frac{1}{\epsilon})}\epsilon^{a+b-1}\left(e^{\frac{2\pi i}{\epsilon}} - 1\right)$.

And then

$$\lambda^+(\epsilon) = -2\pi i e^{\pi i(1-a-b)}\frac{1}{\Gamma(a)\Gamma(b)}\epsilon^{a+b-1}\frac{\Gamma(a+b+\frac{1}{\epsilon})}{\Gamma(1+\frac{1}{\epsilon})}, \tag{2.3.35}$$

which, for $a + b \in \mathbb{Z}$, yields

$$\lambda^+(\epsilon) = \frac{-2\pi i e^{\pi i(1-a-b)}\epsilon^{a+b-1}}{\Gamma(a)\Gamma(b)}\frac{1}{r(a+b)}. \tag{2.3.36}$$

Hence,

$$\lim_{\substack{\epsilon\to 0\\ \epsilon\in S^+}} \lambda^+(\epsilon) = \frac{-2\pi i e^{i\pi(1-a-b)}}{\Gamma(a)\Gamma(b)} = \lambda. \tag{2.3.37}$$

Finally, Lemma 2.2.6 and equation (2.2.1) relates the case $\epsilon' \in S^-$ to the case $\epsilon \in S^+$, and we have, denoting $w_i(x)$ by $w_i(x, \epsilon)$,

$$\begin{aligned} \kappa^+(\epsilon) &= \left(e^{\pi i \frac{\epsilon'}{\epsilon}}\right)^{a+b-1} \kappa^-(\epsilon'), \\ w_2(x, \epsilon) &= w_4(x', \epsilon'), \\ w_3(x, \epsilon) &= w_1(x', \epsilon'). \end{aligned} \tag{2.3.38}$$

□

Theorem 2.3.5.

(1) If the series $\hat{g}(x)$ is divergent, then, for all $\epsilon \in S^+$ (resp. for all $\epsilon \in S^-$), $w_3(x)$ (resp. $w_1(x)$) is not an eigenvector of the monodromy operator M_0 (resp. M_ϵ). In particular, this forces the existence of logarithmic terms at

$x=0$ (resp. $x=\epsilon$) for all special values of ϵ for which logarithmic terms may exist.

(2) *Conversely, for fixed a and b, if $w_3(x)$ (resp. $w_1(x)$) is not an eigenvector of the monodromy operator M_0 (resp. M_ϵ) for some $\epsilon \in S^+$ (resp. for some $\epsilon \in S^-$), then the series $\hat{g}(x)$ is divergent.*

(3) *If the series $\hat{h}(x)$ is divergent, then, for all $\epsilon \in S^+$ (resp. for all $\epsilon \in S^-$), $w_2(x)$ (resp. $w_4(x)$) is not an eigenvector of the monodromy operator M_ϵ (resp. M_0). In particular, this forces the existence of logarithmic terms at $x=\epsilon$ (resp. $x=0$) for all special values of ϵ for which logarithmic terms may exist.*

(4) *Conversely, for fixed a and b, if $w_2(x)$ (resp. $w_4(x)$) is not an eigenvector of the monodromy operator M_ϵ (resp. M_0) for some $\epsilon \in S^+$ (resp. for some $\epsilon \in S^-$), then the series $\hat{h}(x)$ is divergent.*

Proof. Let $\epsilon \in S^+$ (the proof for $\epsilon \in S^-$ is similar). With Theorem 2.2.8, we have that $\hat{g}(x)$ is divergent if and only if $\lambda \neq 0$. Since $\lim_{\epsilon\to 0} \lambda^+(\epsilon) = \lambda$, we have $\lambda^+(\epsilon) \neq 0$ for $\epsilon \in S^+$ provided the radius of S^+ is sufficiently small. If $w_3(x)$ were an eigenvector of the monodromy operator M_0, then we would have $\lambda^+(\epsilon) = 0$ which is a contradiction. If $\lambda^+(\epsilon) \neq 0$, then the analytic continuation of $w_3(x)$ is ramified around $x=0$. When $1-\frac{1}{\epsilon} \in -\mathbb{N}$, $w_2(x)$ is not ramified around $x=0$ and either $w_1(x)$ is a polynomial or it has logarithmic terms. Since the analytic continuation of $w_3(x)$ is ramified at $x=0$ and since it is a linear combination of $w_1(x)$ and $w_2(x)$, we are forced to have $w_1(x)$ with logarithmic terms. The argument is similar for $w_2(x)$.

To prove the converse, we use the expressions (2.3.16) and (2.3.17) : for $\epsilon \in S^+$ and a and b fixed, we have $\lambda^+(\epsilon) \neq 0$ if and only if $\lambda \neq 0$ as well as $\mu^+(\epsilon) \neq 0$ if and only if $\mu \neq 0$. □

Hence, the singular direction $\mathbb{R}_-$ (resp. $\mathbb{R}_+$) of the 1-summable series $\hat{g}(x)$ (resp. $\hat{h}(x)$) is directly related to the presence of logarithmic terms at the *left* (resp. *right*) singular point for specific values of the confluence parameter.

Remark 2.3.6. *The necessary condition (1) in Theorem 2.3.5 is still valid when a and b are analytic functions $a(\epsilon)$ and $b(\epsilon)$. A counterexample to the converse (2) when $a(\epsilon)$ and $b(\epsilon)$ are not constant, is given by*

$$\begin{cases} a(\epsilon) = n + \epsilon, \quad n \in -\mathbb{N}, \\ b(\epsilon) = m + \epsilon, \quad m \in \mathbb{N}^*. \end{cases} \tag{2.3.39}$$

Looking at Theorem 2.3.4, it is clear that, even in the convergent case, there is some wild behavior ($e^{\frac{2\pi i}{\epsilon}}$) in the monodromy of the solutions which does not go to the limit. Fortunately, this wild behavior is linear. In the next section, we will separate it from the non linear part in order to get a limit for the latter.

2.3.4. The wild and continous part of the monodromy operator

In this section, we see that the monodromy of $H^{\epsilon^{\pm}}(x)$ can be separated in a wild part (i.e. of the form $e^{\frac{2\pi i}{\alpha}}$ with $\alpha \in \mathbb{C}$, $\alpha \to 0$) and continuous part. This is the advantage of studying the monodromy of $H^{\epsilon^{\pm}}(x)$ instead of the monodromy of each solution. The wild part is present even in the case of convergence of the confluent series $\hat{g}(x)$ and $\hat{h}(x)$ and is purely linear. The continous part leads us to the Stokes coefficients. This is still done in the two covering sectors $S^{\pm}$ of a small neighborhood of ϵ.

Theorem 2.3.7. *Let $H^{\epsilon^{\pm}}_{i,(\delta,\theta)}(x)$ be obtained from analytic continuation of $H^{\epsilon^{\pm}}(x)$ as in notation 2.3.3. The relation between $H^{\epsilon^{\pm}}_{(\epsilon,\mp\pi)}$ and $H^{\epsilon^{\pm}}_{(\epsilon,\pm\pi)}$, as well as the relation between $H^{\epsilon^{\pm}}_{(0,\mp\pi)}$ and $H^{\epsilon^{\pm}}_{(0,\pm\pi)}$ may be separated into*

- *a wild linear part with no limit at $\epsilon = 0$,*
- *a continuous non linear part*

on each of the sectors $S^{\pm}$. More precisely,

- *if $\epsilon \in S^{+}$,*

$$H^{\epsilon^{+}}_{(\epsilon,-\pi)} = e^{2\pi i(a+b-1+\frac{1}{\epsilon})}(H^{\epsilon^{+}}_{(\epsilon,\pi)} - \mu^{+}(\epsilon)) \tag{2.3.40}$$

and

$$\frac{1}{H^{\epsilon^{+}}_{(0,\pi)}} = e^{\frac{-2\pi i}{\epsilon}}\left(\frac{1}{H^{\epsilon^{+}}_{(0,-\pi)}} + \lambda^{+}(\epsilon)\right) \tag{2.3.41}$$

with $\mu^{+}(\epsilon)$ and $\lambda^{+}(\epsilon)$ as in (2.3.16) and (2.3.17).

- *if* $\epsilon \in S^-$,

$$H^{\epsilon^-}_{(0,-\pi)} = e^{\frac{-2\pi i}{\epsilon}}(H^{\epsilon^-}_{(0,\pi)} - \mu^-(\epsilon)) \tag{2.3.42}$$

and

$$\frac{1}{H^{\epsilon^-}_{(\epsilon,\pi)}} = e^{2\pi i(a+b-1+\frac{1}{\epsilon})}\left(\frac{1}{H^{\epsilon^-}_{(\epsilon,-\pi)}} + \lambda^-(\epsilon)\right) \tag{2.3.43}$$

with $\mu^-(\epsilon)$ *and* $\lambda^-(\epsilon)$ *as in (2.3.20) and (2.3.21).*

Proof. The proof is a mere calculation using (2.3.14), (2.3.15), (2.3.18) and (2.3.19). □

Proposition 2.3.8. *To know which invariants are realisable, it is sufficient to look at the product* $\lambda^+(\epsilon)\mu^+(\epsilon)$. *If* a *and* b *are analytic functions of* ϵ, *this last product is analytic in a neighborhood of* $\epsilon = 0$.

Proof. If $\mu^+(\epsilon) \neq 0$, we can take $\mu^+(\epsilon)w_3(x)$ instead of $w_3(x)$ in the expression for $H^{\epsilon^+}(x)$. Then, $\mu^+(\epsilon)$ is replaced by 1 in equation (2.3.40) and $\lambda^+(\epsilon)$ is replaced by $\lambda^+(\epsilon)\mu^+(\epsilon)$ in equation (2.3.41). Similarly if $\lambda^+(\epsilon) \neq 0$. So we can regard our invariants as 1 and $\lambda^+(\epsilon)\mu^+(\epsilon)$, instead of $\lambda^+(\epsilon)$ and $\mu^+(\epsilon)$, in the case where one of them is different from 0. We have

$$\begin{aligned}\lambda^+(\epsilon)\mu^+(\epsilon) &= -\tfrac{4\pi^2 e^{\pi i(1-a-b)}}{\Gamma(1-a)\Gamma(1-b)\Gamma(a)\Gamma(b)}\\ &= -4e^{\pi i(1-a-b)}\sin(\pi a)\sin(\pi b)\\ &= -(1-e^{-2\pi i a})(1-e^{-2\pi i b})\\ &= \lambda^-(\epsilon)\mu^-(\epsilon).\end{aligned} \tag{2.3.44}$$

□

Remark 2.3.9. *If* $\mu^+(\epsilon) \neq 0$ *(resp.* $\lambda^+(\epsilon) \neq 0$*), the product* $\lambda^+(\epsilon)\mu^+(\epsilon) = \lambda^-(\epsilon)\mu^-(\epsilon)$ *is zero precisely when* $a \in -\mathbb{N}$ *or* $b \in -\mathbb{N}$ *(resp.* $1-a \in -\mathbb{N}$ *or* $1-b \in -\mathbb{N}$*), i.e. when* $g(x)$ *(resp.* $k(x)$*) is a convergent solution.*

Remark 2.3.10. *When* $a+b=1$, *we have* $\mu^+(\epsilon) = \lambda^+(\epsilon)$ *and* $\mu^-(\epsilon) = \lambda^-(\epsilon)$ *(and* $\mu = \lambda$*). We will see in Remark 2.4.4 of Section 2.4 that this is the particular case when the formal invariants of the two saddle-nodes of the Riccati equation (2.4.1) vanish.*

2.4. A RELATED RICCATI SYSTEM

2.4.1. First integrals of a Riccati system related to the hypergeometric equation (2.2.2)

We studied the monodromy of

$$H^{\epsilon^{\pm}}(x) = \frac{\kappa^{\pm}(\epsilon)w_i(x)}{w_j(x)} \quad \left(\text{with } (i,j) = \begin{cases} (2,3),\ \epsilon \in S^+, \\ (4,1),\ \epsilon \in S^- \end{cases}\right)$$

instead of the monodromy of each solution $w_k(x)$, for $k = i, j$. To justify this choice, we transform the hypergeometric equation into a Riccati equation (see for instance **[9]** p. 104) and find a first integral of the Riccati system.

Proposition 2.4.1. *The Riccati system*

$$\begin{cases} \dot{x} = x(x-\epsilon), \\ \dot{y} = abx(x-\epsilon) + (-1 + (1-a-b)x)y + y^2 \end{cases} \tag{2.4.1}$$

is related to the hypergeometric equation (2.2.2) with singular points at $\{0, \epsilon, \infty\}$ *with the following change of variable :*

$$y = -x(x-\epsilon)\frac{w'(x)}{w(x)}. \tag{2.4.2}$$

The space of all nonzero solutions $(C_i w_i(x) + C_j w_j(x))$ of the hypergeometric equation is the manifold $\mathbb{CP}^1 \times \mathbb{C}^*$. The next proposition gives the expression of a first integral of the Riccati system which takes values in $\mathbb{CP}^1$. Up to a constant (in $\mathbb{C}^*$), this first integral is related to a general solution of the hypergeometric equation.

Proposition 2.4.2. *Let* $w_j(X)$ *et* $w_i(X)$ *be two linearly independent solutions of the hypergeometric equation (2.2.2). In their shared region of validity we have the following first integral of the Riccati system (2.4.1) :*

$$I^{\epsilon}_{(i,j)} = \frac{w_i(x)}{w_j(x)}\left(\frac{y - \rho_i(x,\epsilon)}{y - \rho_j(x,\epsilon)}\right) \tag{2.4.3}$$

where

$$\rho_i(x,\epsilon) = -x(x-\epsilon)\frac{w_i'(x)}{w_i(x)}. \tag{2.4.4}$$

In order that the limit exists when $\epsilon \in S^+$ goes to zero, we consider the first integral

$$I^{\epsilon^\pm} = \begin{cases} \kappa^+(\epsilon) I^\epsilon_{(2,3)} & \text{if } \epsilon \in S^+, \\ \kappa^-(\epsilon) I^\epsilon_{(4,1)} & \text{if } \epsilon \in S^- \end{cases} \tag{2.4.5}$$

where $\kappa^\pm(\epsilon)$ are defined in (2.3.4). Now let us see why we can work with a simpler expression than this one to study its ramification.

Proposition 2.4.3. *The quotient $H^{\epsilon^\pm} = \kappa^\pm(\epsilon)\frac{w_i(x)}{w_j(x)}$ has the same ramification around $x = 0$ and $x = \epsilon$ as*

$$I^{\epsilon^\pm} = \kappa^\pm(\epsilon)\frac{w_i(x)}{w_j(x)}\left(\frac{y - \rho_i(x,\epsilon)}{y - \rho_j(x,\epsilon)}\right), \tag{2.4.6}$$

namely we can replace $H^{\epsilon^\pm}$ by $I^{\epsilon^\pm}$ in the formulas (2.3.40)–(2.3.43).

Proof. Let us prove that $H^{\epsilon^+} = \kappa^+(\epsilon)\frac{w_i(x)}{w_j(x)}$ has the same ramification as I^{ϵ^+} in the case $\epsilon \in S^+$. We start with the ramification around $x = \epsilon$. We have, with relation (2.3.15),

$$\begin{aligned}
\frac{w'_{2,(\epsilon,-\pi)}(x)}{w_{2,(\epsilon,-\pi)}(x)} &= \frac{\kappa^+(\epsilon)w'_{2,(\epsilon,-\pi)}(x)}{\kappa^+(\epsilon)w_{2,(\epsilon,-\pi)}(x)} \\
&= \frac{e^{2\pi i(a+b+\frac{1}{\epsilon}-1)}(\kappa^+(\epsilon)w'_{2,(\epsilon,\pi)}(x) - \mu^+(\epsilon)w'_{3,(\epsilon,\pi)}(x))}{e^{2\pi i(a+b+\frac{1}{\epsilon}-1)}(\kappa^+(\epsilon)w_{2,(\epsilon,\pi)}(x) - \mu^+(\epsilon)w_{3,(\epsilon,\pi)}(x))} \\
&= \frac{1}{\kappa^+(\epsilon)\frac{w_{2,(\epsilon,\pi)}(x)}{w_{3,(\epsilon,\pi)}(x)} - \mu^+(\epsilon)}\left(\kappa^+(\epsilon)\frac{w'_{2,(\epsilon,\pi)}(x)}{w_{3,(\epsilon,\pi)}(x)} - \mu^+(\epsilon)\frac{w'_{3,(\epsilon,\pi)}(x)}{w_{3,(\epsilon,\pi)}(x)}\right) \\
&= \frac{1}{H^{\epsilon^+}_{(\epsilon,\pi)} - \mu^+(\epsilon)}\left(\frac{w'_{2,(\epsilon,\pi)}(x)}{w_{2,(\epsilon,\pi)}(x)}H^{\epsilon^+}_{(\epsilon,\pi)} - \mu^+(\epsilon)\frac{w'_{3,(\epsilon,\pi)}(x)}{w_{3,(\epsilon,\pi)}(x)}\right).
\end{aligned} \tag{2.4.7}$$

Using (2.4.4), (2.3.40) and (2.4.7), we have

$$\begin{aligned}
I^{\epsilon^+}_{(\epsilon,-\pi)} &= H^{\epsilon^+}_{(\epsilon,-\pi)}\left(\frac{y-\rho_{2,(\epsilon,-\pi)}(x,\epsilon)}{y-\rho_{3,(\epsilon,-\pi)}(x,\epsilon)}\right) \\
&= e^{2\pi i(a+b-1+\frac{1}{\epsilon})}(H^{\epsilon^+}_{(\epsilon,\pi)} - \mu^+(\epsilon))\frac{y+x(x-\epsilon)\frac{w'_{2,(\epsilon,-\pi)}(x)}{w_{2,(\epsilon,-\pi)}(x)}}{y+x(x-\epsilon)\frac{w'_{3,(\epsilon,-\pi)}(x)}{w_{3,(\epsilon,-\pi)}(x)}} \\
&= e^{2\pi i(a+b-1+\frac{1}{\epsilon})}\frac{\left(H^{\epsilon^+}_{(\epsilon,\pi)} - \mu^+(\epsilon)\right)y + x(x-\epsilon)\left(\frac{w'_{2,(\epsilon,\pi)}(x)}{w_{2,(\epsilon,\pi)}(x)}H^{\epsilon^+}_{(\epsilon,\pi)} - \mu^+(\epsilon)\frac{w'_{3,(\epsilon,\pi)}(x)}{w_{3,(\epsilon,\pi)}(x)}\right)}{y+x(x-\epsilon)\frac{w'_{3,(\epsilon,\pi)}(x)}{w_{3,(\epsilon,\pi)}(x)}} \\
&= e^{2\pi i(a+b-1+\frac{1}{\epsilon})}\left(H^{\epsilon^+}_{(\epsilon,\pi)}\frac{y-\rho_{2,(\epsilon,\pi)}(x,\epsilon)}{y-\rho_{3,(\epsilon,\pi)}(x,\epsilon)} - \mu^+(\epsilon)\right) \\
&= e^{2\pi i(a+b-1+\frac{1}{\epsilon})}\left(I^{\epsilon^+}_{(\epsilon,\pi)} - \mu^+(\epsilon)\right).
\end{aligned} \tag{2.4.8}$$

Singular point	Quotient of eigenvalues
$(0,0)$	$\frac{1}{\epsilon}$
$(\epsilon,0)$	$1-\frac{1}{\epsilon}-a-b$
$(0,1)$	$\frac{-1}{\epsilon}$
(ϵ,y_1)	$-1+\frac{1}{\epsilon}+a+b$

TAB. 2.I. Quotient of the eigenvalue in y by the eigenvalue in x of the Jacobian for each singular point.

The proofs for $I^{\epsilon+}_{(0,\pm\pi)}$, $I^{\epsilon-}_{(0,\pm\pi)}$ and $I^{\epsilon-}_{(\epsilon,\pm\pi)}$ are similar to this one. □

2.4.2. Divergence and unfolding of the saddle-nodes

Let us consider the Riccati system (2.4.1) with $\epsilon = 0$. It has two saddle-nodes located at $(0,0)$ and $(0,1)$ (see Figure 2.7). In the unfolding (with maybe $a(\epsilon)$

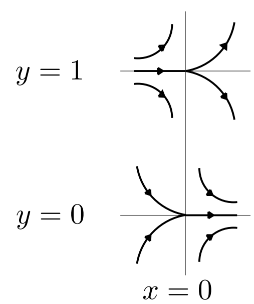

FIG. 2.7. Phase plane, $\epsilon = 0$.

and $b(\epsilon)$), this yields the Riccati system (2.4.1) with the four singular points $(0,0)$, $(\epsilon,0)$, $(0,1)$ and (ϵ,y_1) as illustrated in Figures 2.8 and 2.9, with $y_1 = 1+\epsilon(a+b-1)$.

The quotient of the eigenvalue in y by the eigenvalue in x of the Jacobian, for each singular point, is given in Table 2.I.

Remark 2.4.4. *By summing the quotient of the eigenvalues at the corresponding saddle and node, we get the formal invariant of the saddle-node at* $(0,0)$ *(resp. at* $(0,1)$*), which is* $1-a-b$ *(resp.* $a+b-1$*).*

The curves $y-\rho_k(x,\epsilon) = 0$ for $k = i,j$ appearing in the first integral (2.4.3) are solution curves (trajectories) of the Riccati system, more precisely analytic

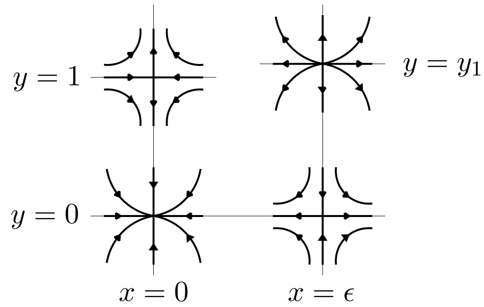

FIG. 2.8. Phase plane if ϵ and $\frac{1}{\epsilon} + a + b \in \mathbb{R}$, $\epsilon > 0$.

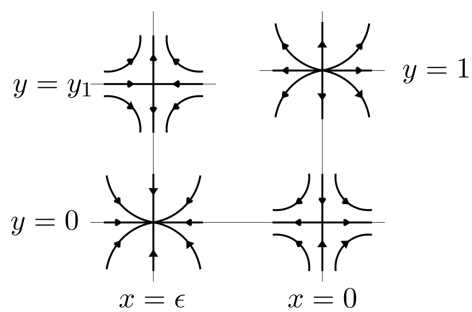

FIG. 2.9. Phase plane if ϵ and $\frac{1}{\epsilon} + a + b \in \mathbb{R}$, $\epsilon < 0$.

invariant manifolds of two of the singular points when $\epsilon \in S^{\pm}$. For example, for $\epsilon \in S^{+}$, $y = \rho_2(x, \epsilon)$ is the invariant manifold of the singular point $(0, 1)$ and $y = \rho_3(x, \epsilon)$ is the invariant manifold of $(\epsilon, 0)$ (see Figure 2.10).

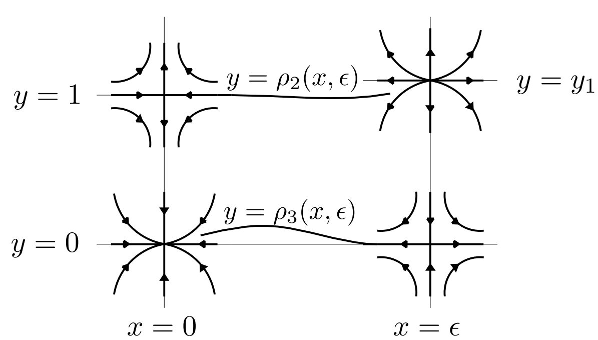

FIG. 2.10. Invariant manifolds $y = \rho_2(x, \epsilon)$ and $y = \rho_3(x, \epsilon)$, case $\epsilon \in \mathbb{R}_{+}^{*}$.

Indeed,

$$\begin{aligned}\rho_2(x,\epsilon) &= -x(x-\epsilon)\frac{w_2'(x)}{w_2(x)}\\ &= 1-\frac{x}{\epsilon}+\{\epsilon(a+b-1)+1\}\frac{x}{\epsilon}+x\left(1-\frac{x}{\epsilon}\right)\frac{(1-a)(1-b)}{1+\frac{1}{\epsilon}}\frac{{}_2F_1(2-a,2-b,2+\frac{1}{\epsilon};\frac{x}{\epsilon})}{{}_2F_1(1-a,1-b,1+\frac{1}{\epsilon};\frac{x}{\epsilon})}\end{aligned} \tag{2.4.9}$$

and $\rho_2(0,\epsilon)=1$. Similarly,

$$\begin{aligned}\rho_3(x,\epsilon) &= -x(x-\epsilon)\frac{w_3'(x)}{w_3(x)}\\ &= -x(x-\epsilon)\frac{ab}{a+b+\frac{1}{\epsilon}}\frac{{}_2F_1(1+a,1+b,1+a+b+\frac{1}{\epsilon};1-\frac{x}{\epsilon})}{{}_2F_1(a,b,a+b+\frac{1}{\epsilon};1-\frac{x}{\epsilon})}\end{aligned} \tag{2.4.10}$$

and $\rho_3(\epsilon,\epsilon)=0$.

The divergence of $\hat{g}(x)$ corresponds to a nonanalytic center manifold at $(0,0)$ for $\epsilon=0$. When we unfold on S^+ (resp. S^-), the invariant manifold of $(\epsilon,0)$ (resp. $(0,0)$) is necessarily ramified at $(0,0)$ (resp. $(\epsilon,0)$) for small ϵ (see Figure 2.11). In the particular case when $1-\frac{1}{\epsilon}\in-\mathbb{N}$ (resp. $a+b+\frac{1}{\epsilon}$) with ϵ small, then $(0,0)$ (resp. $(\epsilon,0)$) is a resonant node. Then necessarily in this case it is non linearisable (the resonant monomial is nonzero) which in practice yields logarithmic terms in the first integral.

Besides, if $\hat{g}(x)$ is convergent, the invariant manifold $y=\rho_3(x)$ (after unfolding in S^+, keeping a and b fixed) is not ramified at $(0,0)$ (recall that if $a\in-\mathbb{N}$ or $b\in-\mathbb{N}$, i.e. if $\hat{g}(x)$ is convergent, then $w_3(x)$ is a polynomial). This correspond to Figure 2.12, an exceptional case.

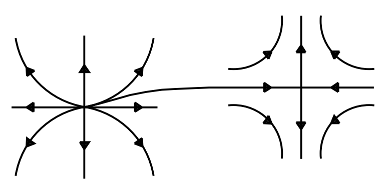

FIG. 2.11. Analytic continuation of an invariant manifold of a saddle when the corresponding analytic center manifold is divergent.

The divergence of $\hat{k}(x)$ has a similar interpretation with the pair of singular points coming from the unfolding of the saddle-node at $(0,1)$. If $\hat{k}(x)$ is divergent then, when we unfold in S^+ (resp. S^-) the invariant manifold of $(0,1)$ (resp. (ϵ,y_1)) is necessarily ramified at (ϵ,y_1) (resp. $(0,1)$). As before, this implies that (ϵ,y_1) (resp. $(0,1)$) is nonlinearisable as soon as it is a resonant node.

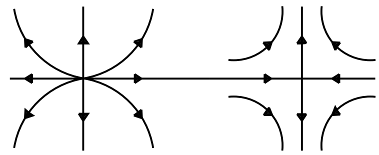

FIG. 2.12. Analytic continuation of an invariant manifold of a saddle when the corresponding analytic center manifold is convergent (this is the case since a and b are fixed).

The general description of this parametric resurgence phenomenon is described in [**19**].

2.4.3. Universal unfolding

As the universal deformation of x^2 is $x^2-\epsilon$, let us translate the previous results in the case of this deformation. When studying the universal unfolding of the Riccati system (2.4.1) evaluated at $\epsilon=0$, the singular points to consider would be at $x=-\sqrt{\epsilon}$ and $x=\sqrt{\epsilon}$ (instead of $x=0$ and $x=\epsilon$).

Proposition 2.4.5. *The unfolded Riccati system (with maybe $a(\epsilon)$ and $b(\epsilon)$)*

$$\begin{cases} \dot{x}=x^2-\epsilon, \\ \dot{y}=a(\epsilon)b(\epsilon)(x^2-\epsilon)+(1+(1-a(\epsilon)-b(\epsilon))x)y+y^2 \end{cases} \tag{2.4.11}$$

is related, with $c=\frac{1}{2\sqrt{\epsilon}}+\frac{a+b+1}{2}$, to the hypergeometric equation with singular points $(-\sqrt{\epsilon},\sqrt{\epsilon},\infty)$

$$(x^2-\epsilon)\,w''(x)+\{-1+(a+b+1)x\}\,w'(x)+ab\,w(x)=0 \tag{2.4.12}$$

with the change of variables

$$y=-(x^2-\epsilon)\frac{w'(x)}{w(x)}. \tag{2.4.13}$$

The product $\lambda^+(\sqrt{\epsilon})\mu^+(\sqrt{\epsilon})$ is an analytic function of ϵ (and not of $\sqrt{\epsilon}$) :

Theorem 2.4.6. *For the family of systems (2.4.11), in which $a(\epsilon)$ and $b(\epsilon)$ are analytic functions of ϵ, the product $L(\epsilon)=\lambda^+(\sqrt{\epsilon})\mu^+(\sqrt{\epsilon})$ is an analytic function of ϵ.*

PROOF. Given $\gamma\in(0,\frac{\pi}{2})$ fixed, we define

- $S^+ = \{\epsilon \in \mathbb{C} : 0 < |\epsilon| < r(\gamma), \arg(\epsilon) \in (\gamma, 4\pi - \gamma)\}$.

The sector S^+ is defined such as $w_2(x)$ and $w_3(x)$ always exist for these values of ϵ. In particular, we ask $-\frac{1}{2\sqrt{\epsilon}} + \frac{3-a+b}{2} \notin -\mathbb{N}$, $-\frac{1}{2\sqrt{\epsilon}} + \frac{a+b+1}{2} \notin -\mathbb{N}$, $-\frac{1}{2\sqrt{\epsilon}} + \frac{a+1-b}{2} \notin -\mathbb{N}$ and $-\frac{1}{2\sqrt{\epsilon}} + \frac{b+1-a}{2} \notin -\mathbb{N}$.

Then, we define

$$H^{\epsilon^+} = \frac{\kappa^+(\sqrt{\epsilon})w_2(x)}{w_3(x)} \tag{2.4.14}$$

with

$$\kappa^+(\sqrt{\epsilon}) = (2\sqrt{\epsilon})^{1-a-b} e^{\pi i(\frac{1}{2\sqrt{\epsilon}} + \frac{a+b+1}{2})}. \tag{2.4.15}$$

The functions $\mu^+(\sqrt{\epsilon})$ and $\lambda^+(\sqrt{\epsilon})$ can be defined as before and the calculations give the same relation

$$L(\epsilon) = \lambda^+(\sqrt{\epsilon})\mu^+(\sqrt{\epsilon}) = -(1 - e^{-2\pi i a(\epsilon)})(1 - e^{-2\pi i b(\epsilon)}). \tag{2.4.16}$$

This product is thus analytic in ϵ if $a(\epsilon)$ and $b(\epsilon)$ are analytic functions of ϵ. □

These results are used in **[3]** to characterize the moduli space of a Riccati equation under orbital equivalence.

Remark 2.4.7. *$L(\epsilon)$ is related to known invariants. Indeed, we have the relation $L(\epsilon) = -4\pi^2 e^{\pi i \alpha(\epsilon)}\gamma(\epsilon)\gamma'(\epsilon)$, where $\alpha(\epsilon) = 1 - a(\epsilon) - b(\epsilon)$ is the formal invariant of the saddle-node family (2.4.11), while $\gamma(\epsilon)$ and $\gamma'(\epsilon)$ are the unfolding of the Jurkat-Lutz-Peyerimhoff invariants γ and γ' (see* **[11]***) obtained with the change of variable (2.2.11) in the system associated to the differential equation (2.2.10).*

2.5. Directions for further research

The hypergeometric equation corresponds to a particular Riccati system. The study of this system allowed us to describe how divergence in the limit organizes the system in the unfolding. Similar phenomena are expected to occur in the more general cases where solutions at the confluence are 1-summable or even k-summable.

2.6. Acknowledgements

We thank the reviewer for useful comments.

Chapitre 3

COMPLETE SYSTEM OF ANALYTIC INVARIANTS FOR UNFOLDED DIFFERENTIAL LINEAR SYSTEMS WITH AN IRREGULAR SINGULARITY OF POINCARÉ RANK 1

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Research supported by NSERC in Canada

Key words : Stokes phenomenon, irregular singularity, unfolding, confluence, divergent series, summability, monodromy, analytic classification, realization, Riccati matrix differential equation.

Abstract

In this, paper, we give a complete system of analytic invariants for the unfoldings of nonresonant linear differential systems with an irregular singularity of Poincaré rank 1 at the origin over a fixed neighborhood $\mathbb{D}_r$. The unfolding parameter ϵ is taken in a sector S pointed at the origin of opening larger than 2π in the complex plane, thus covering a whole neighborhood of the origin. For each parameter value $\epsilon \in S$, we cover $\mathbb{D}_r$ with two sectors and, over each sector, we construct a well chosen basis of solutions of the unfolded linear differential systems. This basis is used to find the analytic invariants linked to the monodromy of the chosen basis around the singular points. The analytic invariants

give a complete geometric interpretation to the well-known Stokes matrices at $\epsilon = 0$: this includes the link (existing at least for the generic cases) between the divergence of the solutions at $\epsilon = 0$ and the presence of logarithmic terms in the solutions for resonance values of the unfolding parameter. Finally, we give a realization theorem for a given complete system of analytic invariants satisfying a necessary and sufficient condition, thus identifying the set of modules.

3.1. Introduction

In this paper, we are interested in the unfolding of linear differential systems written as

$$y' = \frac{A(x)}{x^{k+1}}y, \tag{3.1.1}$$

with $A(x)$ a matrix of germs of analytic functions at the origin such that $A(0)$ has distinct eigenvalues (nonresonant case), $x \in (\mathbb{C}, 0)$, $y \in \mathbb{C}^n$, and k is a strictly positive integer called the Poincaré rank. We investigate the case of Poincaré rank $k = 1$, but a prenormal form, from which formal invariants can be calculated, is obtained in the general case $k \in \mathbb{N}^*$ (Section 3.3).

Most of the time, the solutions of the differential systems (3.1.1) at the irregular singular point $x = 0$ are divergent and the Stokes phenomenon is observed. To understand this phenomenon, the irregular singular point can be split into regular singular points by a deformation depending on a parameter ϵ. A. Glutsyuk [**6**] showed that the Stokes multipliers related to the system (3.1.1) can be obtained from the limits of transition operators of a perturbed system. In the generic deformations of the system (3.1.1) he considered, the parameter ϵ is taken in sectors that do not cover a whole neighborhood of $\epsilon = 0$. In particular, he restricts his study to parameter values for which the bases of solutions of the perturbed system around the regular singular points never contain logarithmic terms. In our previous paper [**12**], we studied the confluence of two regular singular points of the hypergeometric equation into an irregular one. Our approach allowed us to cover a full neighborhood of the origin in the parameter space, the occurrence of logarithmic terms being embedded into a continuous phenomenon. Our description of the geometry however was not uniform in the parameter space.

In this paper, we use the same approach for the unfolding of the systems (3.1.1) : a whole neighborhood of $\epsilon = 0$ is covered, in a ramified way.

One of the main questions of the field is the equivalence problem for systems of the form (3.1.1) : under which conditions does there exist an invertible matrix of germs of analytic functions at the origin, $T(x)$, giving an equivalence between two arbitrary systems of the form (3.1.1) with $y_1 = T(x)y_2$? The complete system of invariants for this equivalence relation contains formal invariants and an equivalence class of Stokes matrices. Many people have worked on it, and a final statement can be found in the paper of W. Balser, W.B. Jurkat and D.A. Lutz [**1**]. In this paper, we give the analog of this complete system of invariants for 1-parameter families of systems that unfold generically the systems (3.1.1), with $k = 1$. Over a fixed neighborhood $\mathbb{D}_r$ in x-space, the complete system of invariants for the unfolded systems consists of formal and analytic invariants. Formal invariants are obtained from the polynomial part of degree k of a prenormal form. The system composed of this polynomial part is a formal normal form which we call the "model system". When ϵ tends to 0, it converges to the usual polynomial formal normal form. $\mathbb{D}_r$ is covered with two sectorial domains converging to sectors when $\epsilon \to 0$. These sectorial domains are chosen so that, on their intersection, solutions of the model have the same behavior when x tends to the singular points as solutions of the formal normal form at $\epsilon = 0$. Analytic invariants are given by an equivalence class of unfolded Stokes matrices (defined in Section 3.4.7), obtained from the monodromy of a well chosen basis of solutions that is the unique basis having the same asymptotic behavior, over the intersection of the sectorial domains and near the singular points, as the "diagonal" basis of the model system. In dimension $n = 2$ and $k = 1$, the well chosen basis corresponds to a "mixed basis" composed of two solutions that are eigenvectors of the monodromy operator at the two different singular points.

Furthermore, we give a geometric interpretation to the Stokes matrices in the unfolded systems : in particular, we link the Stokes matrices to the presence of logarithmic terms in the general solution of the unfolded system for resonance values of the parameter. We also relate these analytic invariants to the monodromy of first integrals of associated Riccati systems. Unfolded Stokes matrices depend analytically on $\hat{\epsilon}$ over a ramified sector around the origin and we show that there exists a representative in their equivalence class which is $\frac{1}{2}$-summable in ϵ.

Finally, we describe the moduli space. We give a necessary and sufficient condition for a given set of invariants to be realizable as the modulus of an equivalence class of differential systems.

3.2. The Stokes phenomenon and invariants, $\epsilon = 0$

We consider the system (3.1.1) and we denote by $\lambda_{1,0}, ..., \lambda_{n,0}$ the distinct eigenvalues of the matrix $A(0)$ that we can assume diagonal after a constant linear change of coordinates in the y variable. There exists a formal transformation $\hat{H}(x)$ such that $\hat{H}(0) = I$ and such that $y = \hat{H}(x)z$ conjugates (3.1.1) with its *formal normal form*

$$z' = \frac{\Lambda_0 + \Lambda_1 x + ... + \Lambda_k x^k}{x^{k+1}} z, \tag{3.2.1}$$

with

$$\Lambda_q = diag\{\lambda_{1,q}, ..., \lambda_{n,q}\}, \quad q = 0, 1, ..., k. \tag{3.2.2}$$

Generally, elements of the matrix $\hat{H}(x)$ are not analytic around $x = 0$. But, there exists a covering of a punctured neighborhood of the origin in x-space by $2k$ sectors Ω_s such that on each of them there exists a unique invertible analytic transformation $H_s(x)$ conjugating (3.1.1) with (3.2.1) and having the asymptotic series $\hat{H}(x)$ in Ω_s. The comparison of these transformations on the intersections of the sectors Ω_s leads to the analytic invariants of the system (3.1.1). In this section, we recall these known results (for instance [**10**] pp. 351–372) in the case $k = 1$, since they will organize our study in the unfolding.

Let us take the system (3.1.1) and its formal normal form (3.2.1) which are written in the case $k = 1$ as

$$y' = \frac{A(x)}{x^2} y \tag{3.2.3}$$

and

$$z' = \frac{\Lambda_0 + \Lambda_1 x}{x^2} z, \tag{3.2.4}$$

with the above assumptions on $A(0)$. We permute the coordinates of $y \in \mathbb{C}^n$ in order to have

$$\Re(\lambda_{1,0}) \geqslant \Re(\lambda_{2,0}) \geqslant ... \geqslant \Re(\lambda_{n,0}) \tag{3.2.5}$$

and, if $\Re(\lambda_{q,0}) = \Re(\lambda_{j,0})$,

$$\Im(\lambda_{q,0}) < \Im(\lambda_{j,0}), \quad q < j. \tag{3.2.6}$$

Then, we have $\arg(\lambda_{q,0} - \lambda_{j,0}) \in]-\frac{\pi}{2}, \frac{\pi}{2}]$ for $q < j$. We rotate slightly the x-plane in the positive direction such that

$$\Re(\lambda_{q,0} - \lambda_{j,0}) > 0, \quad q < j. \tag{3.2.7}$$

From now on, the order of the coordinates of y and the x-coordinate (for $\epsilon = 0$) are fixed. We are now ready to choose the covering sectors in x using the notion of separation rays.

Definition 3.2.1. *When $k = 1$, the* separation rays *in the x-plane corresponding to $\lambda_{q,0}, \lambda_{j,0} \in \mathbb{C}$, $\lambda_{q,0} \neq \lambda_{j,0}$, are the two rays such that*

$$\Re\left(\frac{\lambda_{q,0} - \lambda_{j,0}}{x}\right) = 0. \tag{3.2.8}$$

Definition 3.2.2. *We define two open sectors Ω_D and Ω_U as*

$$\begin{aligned} \Omega_D &= \{x \in \mathbb{C} : |x| < r, -(\pi + \delta) < \arg(x) < \delta\}, \\ \Omega_U &= \{x \in \mathbb{C} : |x| < r, -\delta < \arg(x) < \pi + \delta\}, \end{aligned} \tag{3.2.9}$$

with $\delta > 0$ chosen sufficiently small so that the closure of Ω_D (respectively Ω_U) does not contain any separation rays located in the upper (respectively lower) half plane. Several restrictions on the radius of these sectors will be discussed later. The sectors are illustrated in Figure 3.1 with their intersection $\Omega_L \cap \Omega_R$.

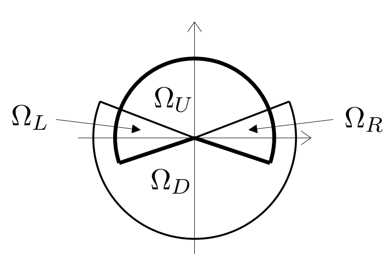

FIG. 3.1. Sectors Ω_D and Ω_U and their intersection $\Omega_L \cup \Omega_R$.

By the sectorial normalization theorem of Y. Sibuya **[22]** (p. 144), if r is chosen sufficiently small, there exists over each sector Ω_s $(s = D, U)$ a unique invertible matrix of analytic functions $H_s(x)$, asymptotic at the origin in Ω_s to a power series $\hat{H}(x)$ independent of s, such that $y = H_s(x)z$ conjugates (3.2.3) with its formal normal form (3.2.4).

The Stokes phenomenon appears when considering the intersection of the sectors Ω_U and Ω_D. Let $F(x)$ be the diagonal fundamental matrix solution of the formal normal form (3.2.4) in the ramified domain $\{x \in \mathbb{C} : -(\pi + \delta) < \arg(x) < \pi + \delta\}$ given by

$$F(x) = x^{\Lambda_1} e^{-\frac{1}{x}\Lambda_0}. \tag{3.2.10}$$

Let $F_s(x)$ be the restriction of $F(x)$ to Ω_s, $s = D, U$. On each connected component of the intersection $\Omega_D \cap \Omega_U$ (Figure 3.1), we have two bases of solutions of (3.2.3) given by $H_D(x)F_D(x)$ and $H_U(x)F_U(x)$, with

$$F_U(x) = \begin{cases} F_D(x), & \text{on } \Omega_R, \\ F_D(x)e^{2\pi i \Lambda_1}, & \text{on } \Omega_L. \end{cases} \tag{3.2.11}$$

Each element of one basis may be expressed as a linear combination of elements of the other basis, giving the existence of matrices C_R and C_L, such that

$$H_D(x)^{-1}H_U(x) = \begin{cases} F_D(x)C_R(F_D(x))^{-1}, & \text{on } \Omega_R, \\ F_D(x)C_L(F_D(x))^{-1}, & \text{on } \Omega_L. \end{cases} \tag{3.2.12}$$

The matrices C_R and C_L are unipotent, respectively upper and lower triangular, and they are called the *Stokes matrices*. The *Stokes phenomenon* occurs when at least one of these Stokes matrices is different from the identity matrix and it reflects the divergence of the formal transformation $\hat{H}(x)$.

As $F(x)K$ is also a fundamental matrix of the normal system (3.2.4) for any nonsingular constant diagonal matrix K, two *Stokes collections* $\{C_R, C_L\}$ and $\{C'_R, C'_L\}$ are said to be *equivalent* if there exists a nonsingular constant diagonal matrix K such that

$$C'_l = KC_lK^{-1}, \quad l = L, R. \tag{3.2.13}$$

The equivalence classes of Stokes collections are analytic invariants for the classification of the systems (3.2.3). The next two theorems are now standard in the literature.

Definition 3.2.3. *Two systems are locally* analytically equivalent *if there exists an invertible matrix of germs of analytic functions at the origin $H(x)$ such that the substitution $y_1 = H(x)y_2$ transforms the system $y_1' = A_1(x)y_1$ into $y_2' = A_2(x)y_2$.*

Theorem 3.2.4. *Two systems (3.2.3) with the same formal normal form (3.2.4) are locally analytically equivalent if and only if their Stokes collections are equivalent in the sense (3.2.13).*

Related to a system (3.2.3), we thus have formal invariants, which are the coefficients of the matrices Λ_0 and Λ_1 in the formal normal form (3.2.4), and analytic invariants, given by the equivalence class of the Stokes collections. The moduli space corresponding to these invariants has been completely described :

Theorem 3.2.5. *Any collection consisting of two unipotent matrices, an upper triangular one and a lower triangular one, can be realized as the Stokes collection of a nonresonant irregular singularity with a preassigned formal normal form.*

Where do these invariants come from ? What do they mean ? The answer appears when unfolding.

3.3. The prenormal form, $k \in \mathbb{N}^*$

In this section, we unfold the systems (3.1.1), with $k \in \mathbb{N}^*$, and introduce a prenormal form in which formal invariants can be calculated from a polynomial part. The transformation from a system (3.1.1) to its prenormal form is analytic.

3.3.1. Generic unfolding

We consider an unfolding of a system (3.1.1) of the form

$$f(\eta, x)y' = A(\eta, x)y, \tag{3.3.1}$$

where $\eta = (\eta_0, ..., \eta_{k-1}) \in \mathbb{C}^k$, $f(\eta, x)$ are germs of analytic functions at the origin such that $f(0, x) = x^{k+1}$ and $A(\eta, x)$ is a matrix of germs of analytic functions at the origin satisfying $A(0, x) = A(x)$. We will restrict ourselves to functions $f(\eta, x)$

such that the unfolding is *generic*. To define this term, we need the following proposition.

Proposition 3.3.1. *After a translation $X = x+b(\eta)$, with $b(\eta)$ a germ of analytic map such that $b(0) = 0$, any linear differential system (3.3.1) may be written as*

$$q^*(\eta, X)y' = A^*(\eta, X)y, \tag{3.3.2}$$

with $A^*(\eta, X)$ a matrix of germs of analytic functions at the origin satisfying $A^*(0, X) = A(0, x)$ and with $q^*(\eta, X) = X^{k+1} + \epsilon_{k-1}(\eta)X^{k-1} + \epsilon_{k-2}(\eta)X^{k-2}... + \epsilon_0(\eta)$, where $\epsilon_j(\eta)$ are germs of holomorphic functions at the origin such that $\epsilon_j(0) = 0$, $j = 0, 1, ..., k-1$.

Proof. Given a particular $f(\eta, x)$, there exist, from Weierstrass preparation theorem, a unique invertible germ of analytic functions at the origin $u(\eta, x)$ and a unique Weierstrass polynomial $q(\eta, x) = x^{k+1} + \alpha_k(\eta)x^k + \alpha_{k-1}(\eta)x^{k-1}... + \alpha_0(\eta)$ such that $f(\eta, x) = u(\eta, x)q(\eta, x)$, where $\alpha_j(\eta)$ are germs of analytic functions at the origin satisfying $\alpha_j(0) = 0$ for $j = 0, 1, ..., k$. This yields the system

$$q(\eta, x)y' = \frac{A(\eta, x)}{u(\eta, x)}y. \tag{3.3.3}$$

The change of variable $X = x + \frac{\alpha_k(\eta)}{k+1}$ yields the result. □

Definition 3.3.2. *An unfolding is* generic *if the analytic map $\eta = (\eta_0, ..., \eta_{k-1}) \mapsto \epsilon = (\epsilon_0(\eta), ..., \epsilon_{k-1}(\eta))$ defined in Proposition 3.3.1 has an analytic inverse.*

We restrict our study to generic unfoldings of systems (3.1.1). From the equation (3.3.2), the genericity condition allows us to take $\epsilon = (\epsilon_0, ..., \epsilon_{k-1})$ as our new parameter. Let us change the notation of the variable X by x and from now on we do not make any more coordinate change on x. We write the generic unfoldings of the differential linear systems (3.1.1) as

$$p(\epsilon, x)y' = B(\epsilon, x)y, \tag{3.3.4}$$

with

$$p(\epsilon, x) = x^{k+1} + \epsilon_{k-1}x^{k-1} + ... + \epsilon_0, \tag{3.3.5}$$

$\epsilon = (\epsilon_0, ..., \epsilon_{k-1}) \in \mathbb{C}^k$ and $B(\epsilon, x)$ a matrix of germs of analytic functions at the origin satisfying $B(0, x) = A(x)$ as in (3.1.1).

3.3.2. Equivalence classes of generic families of linear systems unfolding (3.1.1)

In this paper, we are interested in equivalence classes of systems (3.3.4). We use the same terminology as the one used for the classification of the systems (3.1.1), since it agrees with it when $\epsilon = 0$:

Definition 3.3.3. *Two systems $y' = A(\epsilon, x)y$ and $z' = B(\epsilon, x)z$ are locally* analytically equivalent *(respectively* formally equivalent*) if there exists an invertible matrix of germs of analytic functions of (ϵ, x) at the origin (respectively an invertible matrix of formal series in (ϵ, x)) denoted $T(\epsilon, x)$ such that the substitution $y = T(\epsilon, x)z$ transforms one system into the other.*

We search for a complete system of analytic invariants for the systems (3.3.4) under analytic equivalence. First, we choose a representative of each equivalence class called the *prenormal form*.

3.3.3. Prenormal form

The families of systems (3.3.4) have singularities at $x = x_l$, for x_l such that $p(\epsilon, x_l) = 0$. When looking at solutions around these singularities, we need to evaluate the eigenvalues of $B(\epsilon, x_l)$. With the next theorem, we express them as the values at x_l of polynomials of degree less than or equal to k.

Theorem 3.3.4. *The family of systems (3.3.4) is analytically equivalent to a family in the prenormal form*

$$p(\epsilon, x)y' = B(\epsilon, x)y, \tag{3.3.6}$$

where

$$B(\epsilon, x) = \Lambda(\epsilon, x) + p(\epsilon, x)R(\epsilon, x), \tag{3.3.7}$$

$$\Lambda(\epsilon, x) = diag\{\lambda_1(\epsilon, x), ..., \lambda_n(\epsilon, x)\}, \tag{3.3.8}$$

$$\lambda_i(\epsilon, x) = \lambda_{i,0}(\epsilon) + \lambda_{i,1}(\epsilon)x + ... + \lambda_{i,k}(\epsilon)x^k, \tag{3.3.9}$$

$\lambda_{j,q}(\epsilon)$ are germs of analytic functions at the origin, $p(\epsilon,x)$ is given by (3.3.5) and $R(\epsilon,x)$ is a matrix of germs of analytic functions at the origin.

Proof. As $A(0)$ in (3.1.1) is a diagonal matrix, $B(0,0) = A(0)$ is also diagonal with distinct eigenvalues. We take x in a neighborhood $\mathbb{D}_r$ of the origin such that the eigenvalues of $A(x)$ are distinct. Let us prove that there exists $P(\epsilon,x)$ a matrix of germs of analytic functions at the origin that diagonalizes $B(\epsilon,x)$ for $x \in \mathbb{D}_r$ and for ϵ sufficiently small. $P(0,0)$ can be any nonsingular diagonal matrix, let us take $P(0,0) = I$. For ϵ small and $x \in \mathbb{D}_r$, the eigenvalues of $B(\epsilon,x)$ are distinct and are analytic functions $\nu_i(\epsilon,x)$ of (ϵ,x) by the implicit function theorem. Also, there exists a unique analytic eigenvector $v_i(\epsilon,x)$ relative to the eigenvalue $\nu_i(\epsilon,x)$ having the i^{th} component equal to one (this is obtained with the implicit function theorem, taking $F_i(w,\epsilon,x) = 0$, where $F_i(w,\epsilon,x) = B_i(\epsilon,x)v_i$, $w = (w_1,...,w_{n-1})$, $v_i = (w_1,...,w_{i-1},1,w_i,...,w_{n-1})$ and where $B_i(\epsilon,x)$ is the matrix obtained by removing the i^{th} line of $(B(\epsilon,x) - \nu_i(\epsilon,x)I)$). We then take the i^{th} column of $P(\epsilon,x)$ equal to $v_i(\epsilon,x)$.

Finally, by taking $z = P(\epsilon,x)^{-1}y$, the new system $p(\epsilon,x)z' = B^*(\epsilon,x)z$ satisfies $B^*(\epsilon,x) = diag\{\nu_1(\epsilon,x),...,\nu_n(\epsilon,x)\} + p(\epsilon,x)P(\epsilon,x)^{-1}\frac{\partial P(\epsilon,x)}{\partial x}$ and is analytically equivalent to the original system. Dividing $\nu_i(\epsilon,x)$ by $p(\epsilon,x)$, we get $\nu_i(\epsilon,x) = c_i(\epsilon,x)p(\epsilon,x) + \lambda_{i,0}(\epsilon) + \lambda_{i,1}(\epsilon)x + ... + \lambda_{i,k}(\epsilon)x^k$, from which the result follows. □

Remark 3.3.5. *The polynomial part $\Lambda(\epsilon,x)$ of the prenormal form is completely characterized by $n(k+1)$ quantities $\lambda_{j,q}(\epsilon)$ (with $q = 0,1,...,k$ and $j = 1,2,...,n$). For ϵ fixed such that the singular points are nonresonant, the collection of the well-known formal invariants at all singular points contains also $n(k+1)$ elements (for instance the collection of the eigenvalues of the residue matrices if the singular points are all distinct).*

For the rest of the paper, we only discuss systems in prenormal form (3.3.6).

3.4. Complete system of invariants in the case $k = 1$

This section leads to the complete description of the analytic equivalence classes of generic families of systems in the prenormal form (3.3.6), limiting ourselves to the case $k = 1$. Let us write these systems as

$$(x^2 - \epsilon)y' = B(\epsilon, x)y, \tag{3.4.1}$$

where

$$B(\epsilon, x) = \Lambda(\epsilon, x) + (x^2 - \epsilon)R(\epsilon, x), \tag{3.4.2}$$

with

$$\begin{aligned} \Lambda(\epsilon, x) &= diag\{\lambda_1(\epsilon, x), ..., \lambda_n(\epsilon, x)\}, \\ &= \Lambda_0(\epsilon) + \Lambda_1(\epsilon)x, \end{aligned} \tag{3.4.3}$$

and

$$\Lambda_q(\epsilon) = diag\{\lambda_{1,q}(\epsilon), ..., \lambda_{n,q}(\epsilon)\}, \quad q = 0, 1. \tag{3.4.4}$$

The quantity $\lambda_{j,0}(0) = \lambda_j(0, 0)$ correspond to $\lambda_{j,0}$ defined in Section 3.2. Hence, relation (3.2.7) may be written as

$$\Re(\lambda_q(0, 0) - \lambda_j(0, 0)) > 0, \quad q < j. \tag{3.4.5}$$

This ordering on the eigenvalues of $\Lambda(\epsilon, x)$ at $(\epsilon, x) = 0$ will be kept for $\epsilon \neq 0$ and $|x| \leq \sqrt{|\epsilon|}$ by taking ϵ sufficiently small (see Remark 3.4.9).

We like to call

$$(x^2 - \epsilon)z' = \Lambda(\epsilon, x)z \tag{3.4.6}$$

the *model system*. When $\epsilon = 0$, it corresponds to the formal normal form.

In the systems (3.4.1) and (3.4.6), the irregular singular point at $\epsilon = 0$ splits into two regular singular points when $\epsilon \neq 0$ (in the present context, these points are Fuchsian).

Notation 3.4.1. *We denote the zeros of $x^2 - \epsilon$ by*

$$x_L = \sqrt{\epsilon} \quad \textit{and} \quad x_R = -\sqrt{\epsilon}. \tag{3.4.7}$$

These points are respectively at the left and at the right of the origin when $\sqrt{\epsilon} \in \mathbb{R}_-$ (this will make sense with Definition 3.4.10).

The model system has a fundamental matrix of solutions given by

$$F(\epsilon,x)=diag\{f_1(\epsilon,x),...,f_n(\epsilon,x)\}=\begin{cases}(x-x_R)^{\mathcal{U}_R}(x-x_L)^{\mathcal{U}_L}, & \epsilon\neq 0,\\ x^{\Lambda_1(0)}\exp\left(-\frac{\Lambda_0(0)}{x}\right), & \epsilon=0,\end{cases} \tag{3.4.8}$$

with

$$\mathcal{U}_l=\frac{1}{2x_l}\Lambda(\epsilon,x_l)=\frac{1}{2x_l}\Lambda_0(\epsilon)+\frac{1}{2}\Lambda_1(\epsilon)=diag\{\mu_{1,l},...,\mu_{n,l}\},\quad l=L,R. \tag{3.4.9}$$

The functions $f_j(\epsilon,x)$ will be at the core of the construction of the sectorial domains in the x-space done in Section 3.4.4.

Remark 3.4.2. *The solutions $f_j(\epsilon,x)$ of the model system given by (3.4.8) are analytic in (ϵ,x) for ϵ in a punctured neighborhood of $\epsilon=0$ and for x in a simply connected domain that does not contain any singular point $x=x_l$, for $l=L,R$. These functions converge uniformly on compact sets to $f_j(0,x)$ when $\epsilon\to 0$.*

Let us immediately state notations related to formal invariants that we will frequently use in this paper.

Notation 3.4.3. *We define*

$$D_R=e^{-2\pi i\mathcal{U}_R},\qquad D_L=e^{2\pi i\mathcal{U}_L}. \tag{3.4.10}$$

and

$$\begin{aligned}\Delta_{sj,l}&=(D_l)_{ss}(D_l^{-1})_{jj},\quad l=L,R,\\ &=\begin{cases}e^{2\pi i(\mu_{s,l}-\mu_{j,l})}, & l=L,\\ e^{2\pi i(\mu_{j,l}-\mu_{s,l})}, & l=R,\end{cases}\end{aligned} \tag{3.4.11}$$

with U_l and $\mu_{j,l}$ given by (3.4.9). We have

$$D_R^{-1}D_L=e^{2\pi i\Lambda_1(\epsilon)}, \tag{3.4.12}$$

with $\Lambda_1(\epsilon)$ given by (3.4.4). We will see that D_L (respectively D_R) is the matrix representing the monodromy around $x=x_L$ in the positive direction (respectively around $x=x_R$ in the negative direction) when acting on the fundamental matrix of solutions (3.4.8) of the model system. $e^{2\pi i\Lambda_1(\epsilon)}$ represents the monodromy around both singular points, in the positive direction.

The model system (3.4.6) corresponding to a system (3.4.1) contains all the information on the formal invariants :

Theorem 3.4.4. *Two systems (3.4.1) are formally equivalent if and only if they have the same model system. Hence, the complete system of formal invariants of the systems (3.4.1) is given by the n (degree 1) polynomials $\lambda_i(\epsilon, x)$ in the polynomial part of the prenormal form.*

Proof. By the Poincaré-Dulac Theorem applied to the nonlinear system

$$\begin{cases} \dot{y} = B(\epsilon, x)y, \\ \dot{x} = x^2 - \epsilon, \\ \dot{\epsilon} = 0, \end{cases} \tag{3.4.13}$$

there exists an invertible formal transformation $Y = T(\epsilon, x)y$ at $(\epsilon, x) = (0, 0)$ eliminating nondiagonal terms in (3.4.1) and yielding a diagonal $R(\epsilon, x)$ in (3.4.2). Then, the transformation $z = e^{-\int_0^x R(\epsilon,x)dx}Y$ leads to the model. Hence, letting $J(\epsilon, x) = e^{-\int_0^x R(\epsilon,x)dx}T(\epsilon, x)$, the invertible transformation $z = J(\epsilon, x)y$ conjugates formally a system (3.4.1) to its model.

Let us take two systems of the form (3.4.1) with the same model system, each of them formally conjugated to the model with $J^i(\epsilon, x)$. The transformation $\mathcal{Q}(\epsilon, x) = (J^1(\epsilon, x))^{-1}J^2(\epsilon, x)$ leads a formal equivalence between the two systems.

On the other hand, let us suppose that two systems $(x^2 - \epsilon)y_1' = B^1(\epsilon, x)y_1$ and $(x^2-\epsilon)y_2' = B^2(\epsilon, x)y_2$, with $B^i(\epsilon, x) = \Lambda^i(\epsilon, x)+(x^2-\epsilon)R^i(\epsilon, x)$, are formally equivalent via $y_1 = \mathcal{Q}(\epsilon, x)y_2$, each of them formally conjugated to its model with $z_i = J^i(\epsilon, x)y_i$. We obtain that $P(\epsilon, x) = J^1(\epsilon, x)\mathcal{Q}(\epsilon, x)(J^2(\epsilon, x))^{-1}$ is an invertible formal transformation from the second model system $(x^2 - \epsilon)z_2' = \Lambda^2(\epsilon, x)z_2$ to the first model system $(x^2 - \epsilon)z_1' = \Lambda^1(\epsilon, x)z_1$. Formally, we thus have

$$(x^2 - \epsilon)\frac{\partial}{\partial x}P(\epsilon, x) + P(\epsilon, x)\Lambda^2(\epsilon, x) = \Lambda^1(\epsilon, x)P(\epsilon, x). \tag{3.4.14}$$

By considering this equality for each power of $\epsilon^p x^q$, we obtain that $\Lambda^1(\epsilon, x) = \Lambda^2(\epsilon, x)$ (and that $P(\epsilon, x)$ is a diagonal matrix depending only on ϵ). Hence, the two systems have the same model system. □

Around each singular point, the system (3.4.1) has a well-known basis of solutions (given by eigenvectors of the monodromy operator) that we present in

Theorem 3.4.33, but the problem with this basis is that it is not defined for an infinite set of resonance values of ϵ which accumulate at $\epsilon = 0$. We want to give a unified treatment which highlights the fact that the Stokes phenomenon at $\epsilon = 0$ organizes, in the unfolding, the form of solutions at the resonance values of the parameter. Thus, we rather use a new basis that is defined for all parameter values in a sector of opening greater than 2π in the universal covering of the ϵ-space punctured at $\epsilon = 0$. To find this particular basis, we choose to consider the solutions of the linear systems in the complex projective space.

3.4.1. The projective space

The system (3.4.1) is invariant under $y \to cy$, with $c \in \mathbb{C}^*$. Taking charts in the complex projective space, it gives n particular Riccati matrix differential equations. We introduce t by $\frac{dx}{dt} = \dot{x} = x^2 - \epsilon$ and replace them by n systems of ordinary differential equations (indexed by j)

$$\begin{cases} \frac{dx}{dt} & = \quad x^2 - \epsilon, \\ \frac{d}{dt}\frac{(y)_q}{(y)_j} & = \quad (\lambda_q(\epsilon, x) - \lambda_j(\epsilon, x))\frac{(y)_q}{(y)_j} \\ & \quad\ +(x^2-\epsilon)\sum_{i=1}^{n}\frac{(y)_i}{(y)_j}\left((R(\epsilon,x))_{qi} - (R(\epsilon,x))_{ji}\frac{(y)_q}{(y)_j}\right), \quad q \neq j, \end{cases} \tag{3.4.15}$$

that we call the *Riccati systems*.

Notation 3.4.5. *Let v be a n-dimensional column vector. We define*

$$[v]_j = \left(-\frac{(v)_1}{(v)_j}, \ldots, -\frac{(v)_{j-1}}{(v)_j}, -\frac{\widehat{(v)_j}}{(v)_j}, -\frac{(v)_{j+1}}{(v)_j}, \ldots, -\frac{(v)_n}{(v)_j}\right)^T, \tag{3.4.16}$$

where $(v)_i$ is the i^{th} component of the column vector v and where the hat denotes omission.

Remark 3.4.6. *Following Notation 3.4.5, the j^{th} Riccati system associated to the linear system (3.4.1) may be written as*

$$\begin{cases} \frac{d}{dt}x & = x^2 - \epsilon, \\ \frac{d}{dt}[y]_j & = -T_j^0(\epsilon, x) + T_j^1(\epsilon, x)[y]_j + \left(T_j^2(\epsilon, x)[y]_j\right)[y]_j, \end{cases} \tag{3.4.17}$$

with, denoting I the $(n-1)\times(n-1)$ identity matrix,

$$\begin{aligned} &T_j^0(\epsilon,x)\ : j^{th} \textit{ column of } B(\epsilon,x) \textit{ except } \left(B(\epsilon,x)\right)_{jj};\\ &T_j^1(\epsilon,x)\ : \big(B(\epsilon,x) \textit{ without } j^{th} \textit{ column and } j^{th} \textit{ line}\big) - \left(B(\epsilon,x)\right)_{jj} I;\\ &T_j^2(\epsilon,x)\ : j^{th} \textit{ line of } B(\epsilon,x) \textit{ except } \left(B(\epsilon,x)\right)_{jj}. \end{aligned} \tag{3.4.18}$$

3.4.2. Radius of the sectors in the x-space when $\epsilon = 0$

In order to obtain a basis of solutions of the linear system (3.4.1), we will find in Section 3.4.5 particular solutions (defined for ϵ in a ramified sector and for x in sectorial domains) of the Riccati systems (3.4.17). To ensure that these solutions will converge uniformly on compact sets to solutions $[y]_j = G_{j,s}(0,x)$ (defined over the sectors Ω_s given by (3.2.9) for $s = D, U$), we choose in this section the radius of Ω_s.

Let us first define the solution $[y]_j = G_{j,s}(0,x)$. When $\epsilon = 0$, if the radius r of Ω_s is chosen sufficiently small, there exists a unique fundamental matrix of solutions of the system (3.4.1) that can be written as

$$W_s(0,x) = H_s(0,x)F_s(0,x), \quad \text{on } \Omega_s,\ s = D, U, \tag{3.4.19}$$

where $F_s(0,x)$ is the restriction of $F(0,x)$ given by (3.4.8) to the sectorial domain Ω_s, and where $H_s(0,x)$ is an invertible matrix of functions which are analytic on Ω_s and continuous on its closure, satisfying $H_s(0,0) = I$. $H_s(0,x)$ links the system to its formal normal form and is obtained by the sectorial normalization theorem of Y. Sibuya **[22]** (p. 144), as mentioned in Section 3.2.

Notation 3.4.7. *The solution corresponding to the j^{th} column of $W_s(0,x)$ in the j^{th} Riccati system passes through $(x,[y]_j) = (0,0)$ and is tangent to the x direction, we denote it as $[y]_j = G_{j,s}(0,x)$.*

Let us now specify how we restrict the radius of Ω_s.

Proposition 3.4.8. *Let us define the region*

$$\mathcal{V}^j = \left\{(x,[y]_j) \in \mathbb{C}\times\mathbb{CP}^{n-1} : \left|\frac{(y)_i}{(y)_j}\right| \le |x|,\ \forall i \in \{1,\dots,n\}\backslash\{j\}\right\}. \tag{3.4.20}$$

The boundary of $\mathcal{V}^j$ *is* $\bigcup_{\substack{i=1\\i\neq j}}^n \mathcal{V}_i^j$, *with*

$$\mathcal{V}_i^j = \left\{(x,[y]_j) \in \mathbb{C}\times\mathbb{CP}^{n-1} : \left|\frac{(y)_i}{(y)_j}\right| = |x|, \ \left|\frac{(y)_k}{(y)_j}\right| \leq |x| \ \textit{if}\ k \neq i,j\right\}, \quad i \neq j. \tag{3.4.21}$$

The radius r *of* Ω_s, $s = D, U$, *can be chosen sufficiently small so that the graph* $[y]_j = G_{j,s}(0,x)$ *is confined inside* $\mathcal{V}^j$, *for all* $j \in \{1,...,n\}$.

PROOF. We consider (3.4.15) for $\epsilon = 0$. We have

$$\left|\frac{d}{dt}|x|\right| = \frac{|\Re(\bar{x}\dot{x})|}{|x|} = \frac{|\Re(\bar{x}x^2)|}{|x|} \leq |x|^2 \tag{3.4.22}$$

and

$$\frac{1}{|x|}\left|\frac{d}{dt}\left|\frac{(y)_i}{(y)_j}\right|\right| \geq |\Re(\lambda_{i,0}(0)-\lambda_{j,0}(0))| - v_{ij}(x), \tag{3.4.23}$$

with

$$\begin{aligned} v_{ij}(x) = \ & |\lambda_{i,1}(0)-\lambda_{j,1}(0)||x| + |x|^2 \textstyle\sum_{\substack{k=1\\k\neq j}}^n |(R(0,x))_{ik}| \\ & + |x|\left(|(R(0,x))_{ij}| + \textstyle\sum_{k=1}^n |(R(0,x))_{jk}|\right). \end{aligned} \tag{3.4.24}$$

Let us choose $0 < \eta < 1$. As $|\Re(\lambda_{i,0}(0)-\lambda_{j,0}(0))| > 0$, we can take the radius r of Ω_D and Ω_U sufficiently small so that

$$v_{ij}(x) + |x| < (1-\eta)|\Re(\lambda_{i,0}(0)-\lambda_{j,0}(0))|, \quad x \in \Omega_D \cup \Omega_U,\ i,j \in \{1,...,n\},\ i \neq j. \tag{3.4.25}$$

This implies

$$\left|\frac{d}{dt}|x|\right| < \left|\frac{d}{dt}\left|\frac{(y)_i}{(y)_j}\right|\right|, \quad \text{for} \begin{cases} (x,[y]_j) \in \mathcal{V}_i^j, & \\ x \in \Omega_s, & s = D, U, \\ i,j \in \{1,...,n\}, & i \neq j. \end{cases} \tag{3.4.26}$$

Since the graph $[y]_j = G_{j,s}(0,x)$ contains the point $(x,[y]_j) = (0,0)$ and is tangent to the x-plane, it is confined inside $\mathcal{V}^j$ (if a solution parametrized by a curve in complex time living on the graph $[y]_j = G_{j,s}(0,x)$ were to intersect a boundary component of $\mathcal{V}^j$, then (3.4.26) would not be satisfied). We introduced the parameter η in order to have in the unfolding a similar property (see Proposition 3.4.15). □

3.4.3. Sector in the parameter space

Let us specify the sector on the universal covering of the ϵ-space punctured at the origin with which we will work.

Remark 3.4.9. *We take ϵ sufficiently small in order to have :*

$$\Re((\lambda_q(\epsilon,x)-\lambda_j(\epsilon,x))>0,\quad |x|\leq\sqrt{|\epsilon|},\, q<j,\, l=L,R. \tag{3.4.27}$$

Hence, we have the same ordering of the eigenvalues of $\Lambda(\epsilon,x_l)$ as the one for $\Lambda(0,0)$ given by (3.4.5).

Definition 3.4.10. *We define the sector S, of opening larger than 2π and covering completely a punctured neighborhood of $\epsilon=0$, as*

$$S=\{\hat{\epsilon}\in\mathbb{C}\,:\,0<|\hat{\epsilon}|<\rho,\,\arg(\hat{\epsilon})\in(\pi-2\gamma,3\pi+2\gamma)\} \tag{3.4.28}$$

(see Figure 3.2). In (3.4.28), any $\gamma>0$ such that $\gamma(1+2\frac{\gamma}{\pi})<\theta_0$ can be chosen,

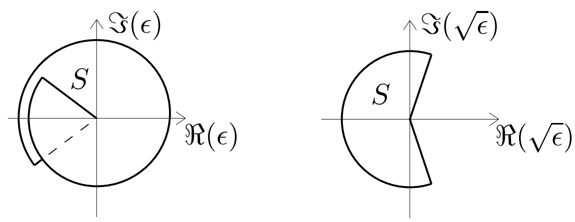

FIG. 3.2. Sector S in terms of the parameters ϵ and $\sqrt{\epsilon}$.

with θ_0 the maximum angle in $(0,\frac{\pi}{2})$ such that

$$\Re(e^{\pm i\theta_0}(\lambda_q(0,0)-\lambda_j(0,0)))\geq 0,\quad q<j, \tag{3.4.29}$$

with $\lambda_j(\epsilon,x)$ as in (3.4.3) (θ_0 exists because of (3.4.5)). Once γ is chosen, the radius ρ is chosen to ensure that there exists $C>0$ for which

$$\Re(e^{\pm i\gamma(1+2\frac{\gamma}{\pi})}(\lambda_q(\epsilon,\hat{x}_l)-\lambda_j(\epsilon,\hat{x}_l)))>C>0,\quad q<j,\quad l=L,R,\quad \hat{\epsilon}\in S. \tag{3.4.30}$$

We will restrict a few other times the value of ρ (in particular, to construct the sectorial domains in the x-variable in Section 3.4.4 and to ensure that Proposition 3.4.15 is true).

Notation 3.4.11. *We denote the auto-intersection of S as $S_\cap$. For values of the parameter in $S_\cap$, we denote*

$$\tilde{\epsilon} = \bar{\epsilon} e^{2\pi i} \in S_\cap \tag{3.4.31}$$

(see Figure 3.3).

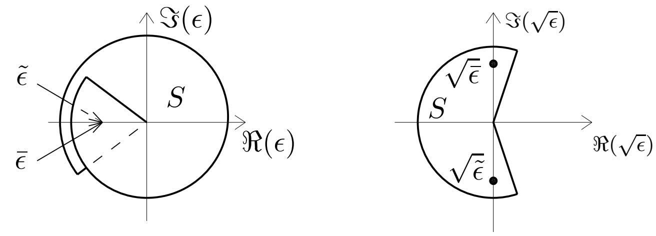

FIG. 3.3. Example of values of $\bar{\epsilon}$ and $\tilde{\epsilon}$ in $S_\cap$ (in terms of ϵ and $\sqrt{\epsilon}$).

Notation 3.4.12. *We frequently write the hat symbol over some quantities to recall the dependence on $\hat{\epsilon} \in S$ (for example $\hat{x}_L$). When we use the hat symbol for values of the parameter in $S_\cap$, we mean that $\hat{\epsilon}$ could either be $\bar{\epsilon}$ or $\tilde{\epsilon}$.*

3.4.4. Sectorial domains in x

For the rest of Section 3.4, x belongs to a disk of radius r determined by Proposition 3.4.8. Let us now explain the construction of the sectorial domains in the complex plane for the x-variable. The boundary of these domains will be defined from solutions of the equation

$$\dot{x} = (x^2 - \epsilon), \tag{3.4.32}$$

allowing complex time. More precisely, passing to the t-variable, we have

$$t(x) = \begin{cases} \frac{1}{2\sqrt{\epsilon}} \ln\left(\frac{x-\sqrt{\epsilon}}{x+\sqrt{\epsilon}}\right), & \epsilon \neq 0, \\ -\frac{1}{x}, & \epsilon = 0. \end{cases} \tag{3.4.33}$$

For $\epsilon = 0$, we cover the disk of radius r with two sectorial domains Ω_D^0 and Ω_U^0 (see Figure 3.4) included respectively inside the sectors Ω_D and Ω_U defined by (3.2.9). The sectorial domains Ω_D^0 and Ω_U^0 correspond respectively, in the t-variable, to the sectorial domains Γ_D^0 and Γ_U^0 illustrated in Figure 3.5.

When $\epsilon \neq 0$, as the function $t(x)$ given by (3.4.33) is multivalued, its inverse function $x(t)$ is periodic of period $T = \frac{\pi i}{\sqrt{\epsilon}}$. Hence, the disk of radius r is sent

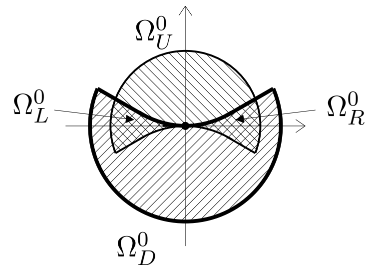

FIG. 3.4. Sectorial domains in the x-variable when $\epsilon = 0$.

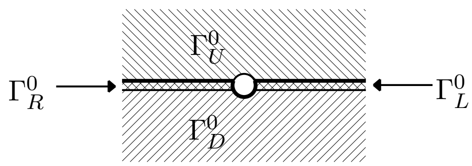

FIG. 3.5. Sectorial domains in the t-variable when $\epsilon = 0$.

to the exterior of a sequence of deformed circles (of initial radius r^{-1} for $\epsilon = 0$) repeated with period T. To cover the disk, we take two strips ($\Gamma_D^{\hat{\epsilon}}$ and $\Gamma_U^{\hat{\epsilon}}$, see Figure 3.6) in the direction of T of width larger than $\frac{T}{2}$, such that their union is a strip (with a hole) of width w, $T < w < 2T$, containing $\frac{\pm\pi i}{2\sqrt{\epsilon}}$. The singular points in the t-variable are located at infinity in the direction perpendicular to the line of holes. The intersection of the two domains $\Gamma_D^{\hat{\epsilon}}$ and $\Gamma_U^{\hat{\epsilon}}$ consists of three connected sets : $\Gamma_L^{\hat{\epsilon}}$ and $\Gamma_R^{\hat{\epsilon}}$ linking a part of the boundary to a singular point, and $\Gamma_C^{\hat{\epsilon}}$ linking the two singular points (coming from the periodicity).

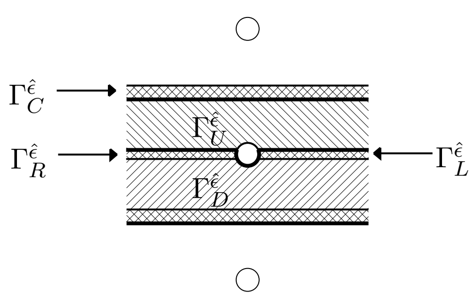

FIG. 3.6. Sectorial domains in the t-variable when $\sqrt{\epsilon} \in \mathbb{R}^*$.

For most values of $\hat{\epsilon} \in S$, the line of holes is slanted and we need to slant the strips. If we take pure slanted strips as in Figure 3.7, we get domains that do not converge when $\hat{\epsilon} \to 0$ to the sectorial domains at $\epsilon = 0$ (Figure 3.5). Hence, we take a part of the boundary horizontal on a length $\frac{c}{\sqrt{|\epsilon|}}$ for some fixed $c > 0$ independent of $\hat{\epsilon}$, as illustrated in Figure 3.8.

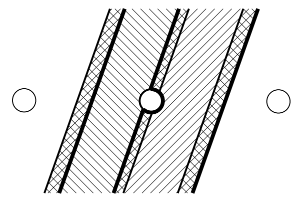

FIG. 3.7. Incorrectly slanted sectorial domains in the t-variable.

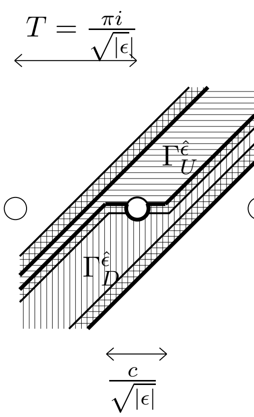

FIG. 3.8. Correctly slanted sectorial domains in the t-variable.

Then, we define the sectorial domain $\Omega_s^{\hat{\epsilon}}$ in the x-variable as the one corresponding, via (3.4.33), to the sectorial domain in the t-variable $\Gamma_s^{\hat{\epsilon}}$, $s \in \{U, D, L, R, C\}$ (Figures 3.10 and 3.11). The points $\hat{x}_R$ and $\hat{x}_L$ are not in the sectorial domains $\Omega_s^{\hat{\epsilon}}$ but in their closure. The region $\Omega_L^{\hat{\epsilon}}$ (respectively $\Omega_R^{\hat{\epsilon}}$) has the singular point $\hat{x}_L$ (respectively $\hat{x}_R$) in its closure and $\Omega_C^{\hat{\epsilon}}$ has both (Figure 3.11). Note that the point $x = 0$ belongs to $\Omega_C^{\hat{\epsilon}}$.

In the x-variable, the difference between $\Omega_s^{\hat{\epsilon}}$ and Ω_s^0 ($s = D, U$) is mainly located inside a disk of radius $c'\sqrt{|\epsilon|}$ (Figure 3.12), due to the non-horizontal part of the boundary of the sectorial domains in the t-variable. Quantitative details and proofs can be found in [**20**]. The construction is possible for all $\hat{\epsilon} \in S$, provided the radius ρ of S is sufficiently small. Indeed, reducing ρ amounts to increase the distance between the holes.

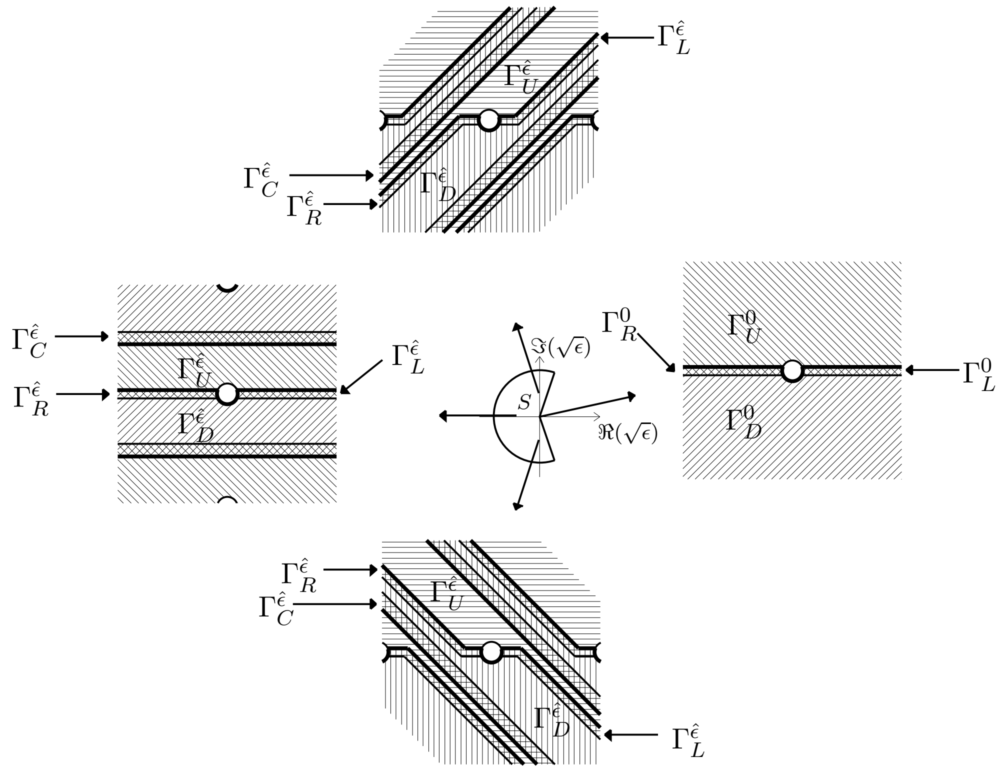

FIG. 3.9. Sectorial domains in the t-variable for some values of $\hat\epsilon \in S \cup \{0\}$, with $\gamma = \frac{\pi}{4}$.

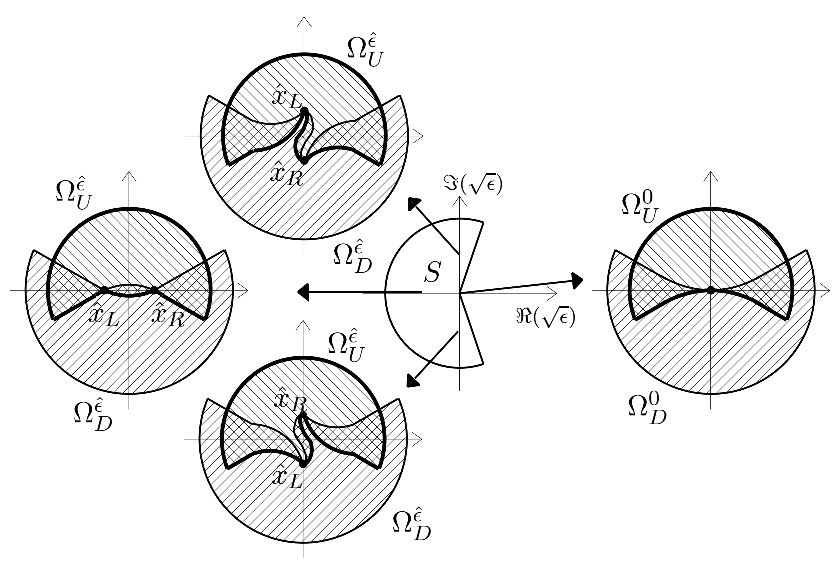

FIG. 3.10. Sectorial domains in the x-variable for some values of $\hat\epsilon \in S \cup \{0\}$.

The angle of the slope is chosen as follows. We take

$$\hat\theta = \frac{2\gamma}{\pi}(\pi - \arg(\sqrt{\hat\epsilon})), \tag{3.4.34}$$

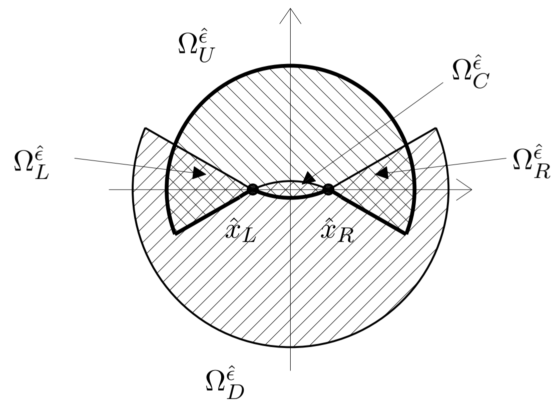

FIG. 3.11. The connected components of the intersection of the sectorial domains $\Omega_D^{\hat\epsilon}$ and $\Omega_U^{\hat\epsilon}$, case $\sqrt{\hat\epsilon} \in \mathbb{R}_-^*$.

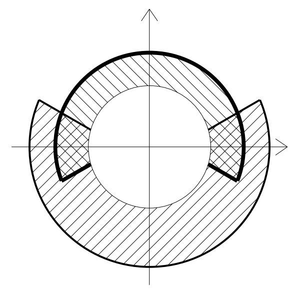

FIG. 3.12. Difference between the sectorial domains $\Omega_s^{\hat\epsilon}$ and Ω_s^0 mainly located inside a small disk of radius $c'\sqrt{|\epsilon|}$.

with γ as chosen in Definition 3.4.10. Then, on the trajectories in the x-plane corresponding to $t = Ce^{i\hat\theta} + C'$ near the singular points, with $C' \in \mathbb{C}$ fixed for each trajectory and $C \in \mathbb{R}$, we have

$$\lim_{\substack{x(t)\to\hat x_l\\ t=Ce^{i\hat\theta}+C'}} (x-\hat x_R)^{\hat\mu_{j,R}-\hat\mu_{q,R}}(x-\hat x_L)^{\hat\mu_{j,L}-\hat\mu_{q,L}} = 0, \quad \text{for } \begin{cases} q>j, & \text{if } l=R,\\ q<j, & \text{if } l=L, \end{cases} \tag{3.4.35}$$

(this is obtained from the fact that $\Re(e^{i\hat\theta}\sqrt{\hat\epsilon})<0$ and that $|\hat\theta| < \gamma(1+2\frac{\gamma}{\pi})$ with γ satisfying (3.4.30)). The limits (3.4.35) yield, with $f_j(\epsilon,x)$ given by (3.4.8),

$$\lim_{\substack{x\to\hat x_l\\ x\in\Omega_s^{\hat\epsilon}}} \frac{f_j(\epsilon,x)}{f_q(\epsilon,x)} = 0, \quad \text{for } \begin{cases} q>j, & \text{if } l=R,\\ q<j, & \text{if } l=L. \end{cases} \tag{3.4.36}$$

Note that we have the same behavior when $\epsilon = 0$:

$$\lim_{\substack{x\to 0\\ x\in\Omega_s^0}} \frac{f_j(0,x)}{f_q(0,x)} = 0, \quad \text{for} \begin{cases} q > j, & \text{if } l = R,\\ q < j, & \text{if } l = L. \end{cases} \tag{3.4.37}$$

3.4.5. Invariant manifolds in the projective space

In this section, we find an invariant manifold $[y]_j = G_{j,s}(\hat{\epsilon}, x)$ of the j^{th} Riccati system (3.4.17) that converges when $\hat{\epsilon} \to 0$ (in S) to the invariant manifold $[y]_j = G_{j,s}(0, x)$ (Notation 3.4.7).

The Jacobian of the j^{th} Riccati system at the singular point $(\hat{x}_l, 0)$, $l = L, R$, has eigenvalues

$$2\hat{x}_l;\ \lambda_1(\epsilon,\hat{x}_l) - \lambda_j(\epsilon,\hat{x}_l);\ \ldots;\ (\widehat{\lambda_j(\epsilon,\hat{x}_l) - \lambda_j}(\epsilon,\hat{x}_l));\ \ldots;\ \lambda_n(\epsilon,\hat{x}_l) - \lambda_j(\epsilon,\hat{x}_l). \tag{3.4.38}$$

For $q \neq j$, the quotient of the eigenvalue in $-\frac{(y)_q}{(y)_j}$ (see Notation 3.4.5) over the one in x gives $\hat{\mu}_{q,l} - \hat{\mu}_{j,l}$, with $\hat{\mu}_{j,l}$ given by (3.4.9).

Definition 3.4.13. *We define the* resonant values *of $\hat{\epsilon}$ as those for which $\hat{\mu}_{q,l} - \hat{\mu}_{j,l} \in \mathbb{N}^*$ for $q \neq j$, $l = L, R$. These are true resonances of the nonlinear Riccati system : they are exactly the values for which there is an obstruction to eliminate the terms $(y)_j(x-\hat{x}_l)^m \frac{\partial}{\partial (y)_q}$ in (3.4.1) when localizing the system at $x = \hat{x}_l$. The parameter $\hat{\epsilon}$ has been taken inside a sector which avoids half of these resonances.*

Remark 3.4.14. *All resonance values of the unfolding parameter ϵ can be integrated in a continuous study : the consideration of half of them on the sector S is sufficient since the change of parameter $\hat{\varepsilon} = \hat{\epsilon}e^{2\pi i}$, under which the unfolded systems are invariant, gives the new parameter $\hat{\varepsilon}$ in a sector including the other half of the resonance values.*

When $\hat{\epsilon} \in S$, the eigenvalues of the Jacobian, listed in (3.4.38), are separated in two distinct groups by a real line passing through the origin. It gives, locally, the existence of invariant manifolds that are tangent to the invariant subspaces of the linearization operator of the vector field at the singular points $(\hat{x}_l, 0)$. We will need the following proposition to extend these local invariant manifolds.

Proposition 3.4.15. *For $\hat{\epsilon} \in S$, let us define the region*

$$\mathcal{V}^j_{\hat{\epsilon}} = \mathcal{V}^j_{\hat{\epsilon},+} \cap \mathcal{V}^j_{\hat{\epsilon},-}, \tag{3.4.39}$$

with

$$\mathcal{V}^j_{\hat{\epsilon},\pm} = \left\{ (x,[y]_j) \in \mathbb{C} \times \mathbb{CP}^{n-1} : \left| \frac{(y)_i}{(y)_j} \right| \leq |x \pm \sqrt{\hat{\epsilon}}|,\ i \in \{1,2,...,n\} \backslash \{j\} \right\}. \tag{3.4.40}$$

The boundary of $\mathcal{V}^j_{\hat{\epsilon},\pm}$ is $\bigcup_{\substack{i=1 \\ i\neq j}}^n \mathcal{V}^j_{\hat{\epsilon},\pm,i}$, with, for $i \neq j$,

$$\mathcal{V}^j_{\hat{\epsilon},\pm,i} = \{ (x,[y]_j) \in \mathbb{C} \times \mathbb{CP}^{n-1} : \left| \frac{(y)_i}{(y)_j} \right| = |x \pm \sqrt{\hat{\epsilon}}|,\ \left| \frac{(y)_k}{(y)_j} \right| \leq |x \pm \sqrt{\hat{\epsilon}}| \ \text{if}\ k \neq i,j \}. \tag{3.4.41}$$

We can take the radius of S sufficiently small so that

$$\left| \frac{d}{dt} |x \pm \sqrt{\hat{\epsilon}}|^2 \right| < \left| \frac{d}{dt} \left| \frac{(y)_i}{(y)_j} \right|^2 \right|, \quad \text{for} \begin{cases} (x,[y]_j) \in \mathcal{V}^j_{\hat{\epsilon},\pm,i}, & \\ x \in \Omega^{\hat{\epsilon}}_s, & s = D, U, \\ \hat{\epsilon} \in S, & \\ i,j \in \{1,...,n\}, & i \neq j. \end{cases} \tag{3.4.42}$$

Proof. Similarly to the proof of Proposition 3.4.8, we consider (3.4.15) and we have, either with the upper or the lower sign,

$$\left| \frac{1}{2} \frac{d}{dt} |x \pm \sqrt{\hat{\epsilon}}|^2 \right| \leq |x \pm \sqrt{\hat{\epsilon}}|^2 |x \mp \sqrt{\hat{\epsilon}}|. \tag{3.4.43}$$

On $\mathcal{V}^j_{\hat{\epsilon},\pm,i}$, we have

$$\frac{1}{2|x \pm \sqrt{\hat{\epsilon}}|^2} \left| \frac{d}{dt} \left| \frac{(y)_i}{(y)_j} \right|^2 \right| \geq |\Re(\lambda_{i,0}(\epsilon) - \lambda_{j,0}(\epsilon))| - v^{\pm}_{ij}(\hat{\epsilon}, x), \tag{3.4.44}$$

with

$$\begin{aligned} v^{\pm}_{ij}(\hat{\epsilon},x) = \ & |\lambda_{i,1}(\epsilon) - \lambda_{j,1}(\epsilon)||x| + |x \pm \sqrt{\hat{\epsilon}}|^2 \textstyle\sum_{\substack{k=1 \\ k\neq j}}^n |(R(\epsilon,x))_{ik}| \\ & + |x \mp \sqrt{\hat{\epsilon}}| \left(|(R(\epsilon,x))_{ij}| + \textstyle\sum_{k=1}^n |(R(\epsilon,x))_{jk}| \right). \end{aligned} \tag{3.4.45}$$

Let us take α such that

$$\alpha \leq \eta |\Re(\lambda_{i,0}(0) - \lambda_{j,0}(0))|, \quad \forall i \neq j, \tag{3.4.46}$$

with η as chosen in Proposition 3.4.8. We restrict the radius of S to $\rho > 0$ such that

$$\big||\Re(\lambda_{i,0}(\epsilon) - \lambda_{j,0}(\epsilon))| - |\Re(\lambda_{i,0}(0) - \lambda_{j,0}(0))|\big| < \frac{\alpha}{2} \tag{3.4.47}$$

and such that

$$\big|v_{ij}^{\pm}(\hat{\epsilon}, x) + |x \mp \sqrt{\hat{\epsilon}}| - |x| - v_{ij}(0, x)\big| < \frac{\alpha}{2}, \quad \forall i \neq j, \tag{3.4.48}$$

implying

$$v_{ij}^{\pm}(\hat{\epsilon}, x) + |x \mp \sqrt{\hat{\epsilon}}| < |\Re(\lambda_{i,0}(\epsilon) - \lambda_{j,0}(\epsilon))|, \quad \forall \hat{\epsilon} \in S, \quad \forall i \neq j. \tag{3.4.49}$$

This yields (3.4.42). □

Using Proposition 3.4.15, we now define the graph $[y]_j = G_{j,s}(\hat{\epsilon}, x)$ as consisting of the union of all solutions, parametrized by curves in complex time of the j^{th} Riccati system, that are confined inside the region $\mathcal{V}_{\hat{\epsilon}}^j$ when restricted to the sectors $\Omega_s^{\hat{\epsilon}}$:

Theorem 3.4.16. *In the j^{th} Riccati system, there exists, over $\Omega_s^{\hat{\epsilon}}$, a one-dimensional invariant manifold given as a graph $[y]_j = G_{j,s}(\hat{\epsilon}, x)$, passing through the two singular points $(x, [y]_j) = (\hat{x}_l, 0)$, $l = L, R$, and located inside the region $\mathcal{V}_{\hat{\epsilon}}^j$ over the sector $\Omega_s^{\hat{\epsilon}}$. $G_{j,s}(\hat{\epsilon}, x)$ depends analytically on $(\hat{\epsilon}, x)$ for $\hat{\epsilon} \in S$ and $x \in \Omega_s^{\hat{\epsilon}}$.*

Proof. We always take x inside the sectorial domain $\Omega_s^{\hat{\epsilon}}$ and we omit the lower index s within the proof : we write simply $G_j(\hat{\epsilon}, x)$.

Let us take the first Riccati system and fix $\epsilon_0 \in S$. The choice of S allows to separate, by a real line passing through the origin, the eigenvalue $2\hat{x}_R$ from the other eigenvalues at $(\hat{x}_R, 0)$ given by (3.4.38). From the Hadamard-Perron theorem for holomorphic flows (see **[10]** p. 106), there exist holomorphic invariant manifolds $\mathcal{W}_{\hat{x}_R,1}^{+}$ and $\mathcal{W}_{\hat{x}_R,1}^{-}$ tangent to the invariant subspaces of the linearization operator of the vector field at $(\hat{x}_R, 0)$. We denote by $[y]_1 = G_1(\epsilon_0, x)$ the unique one-dimensional invariant manifold $\mathcal{W}_{\hat{x}_R,1}^{+}$. Near $x = \hat{x}_R$, it is the unique invariant manifold contained inside the region $\mathcal{V}_{\epsilon_0}^j$ (defined by (3.4.39)) and its extension cannot escape from $\mathcal{V}_{\epsilon_0}^j$, by Proposition 3.4.15.

Similarly, in the n^{th} Riccati system, we take $[y]_n = G_n(\epsilon_0, x)$ as the extension of the unique holomorphic one-dimensional invariant manifold $\mathcal{W}^-_{\hat{x}_L,n}$ passing through $(\hat{x}_L, 0)$.

Now, let us take the j^{th} Riccati system, with $1 < j < n$. Around $x = \hat{x}_R$ (respectively $x = \hat{x}_L$), we have two invariant manifolds $\mathcal{W}^+_{\hat{x}_R,j}$ and $\mathcal{W}^-_{\hat{x}_R,j}$ of dimension j and $n-j$ (respectively $\mathcal{W}^+_{\hat{x}_L,j}$ and $\mathcal{W}^-_{\hat{x}_L,j}$ of dimension $j-1$ and $n-j+1$) tangent to the corresponding invariant subspaces of the linearization operator of the vector field. We analytically extend the invariant manifold $\mathcal{W}^+_{\hat{x}_R,j}$ tangent to $(x, \frac{(y)_1}{(y)_j}, ..., \frac{(y)_{j-1}}{(y)_j})$ at $(\hat{x}_R, 0)$ towards the singular point $x = \hat{x}_L$. Proposition 3.4.15 implies that any solution (with complex time) of this extended invariant manifold cannot exit $\mathcal{V}^j_{\epsilon_0}$ by its part of the boundary consisting of the $\mathcal{V}^j_{\epsilon_0,\pm,i}$ for $i \geq j+1$. Near $x = \hat{x}_L$, the extension of $\mathcal{W}^+_{\hat{x}_R,j}$ must then intersect the invariant manifold $\mathcal{W}^-_{\hat{x}_L,j}$, which is tangent to $(x, \frac{(y)_{j+1}}{(y)_j}, ..., \frac{(y)_n}{(y)_j})$, along a unique one-dimensional invariant manifold denoted $[y]_j = G_j(\epsilon_0, x)$.

In each Riccati system, we thus have one-dimensional invariant manifolds $[y]_j = G_j(\epsilon_0, x)$ confined inside $\mathcal{V}^j_{\epsilon_0}$. Near $\epsilon_0 \neq 0$, $\mathcal{W}^\pm_{\hat{x}_l,j}$ depends analytically on ϵ, implying that the unique solution $[y]_j = G_j(\hat{\epsilon}, x)$ is analytic in $\hat{\epsilon}$ for $\hat{\epsilon} \in S$. □

Remark 3.4.17. *The invariant manifolds $[y]_1 = G_{1,s}(\hat{\epsilon}, x)$ and $[y]_n = G_{n,s}(\hat{\epsilon}, x)$ are uniform respectively near $\hat{x}_R$ and near $\hat{x}_L$, whereas $[y]_j = G_{j,s}(\hat{\epsilon}, x)$ is ramified at the two singular points. More precisely, $G_{j,U}(\hat{\epsilon}, x) = G_{j,D}(\hat{\epsilon}, x)$ over $\Omega^{\hat{\epsilon}}_C$ (Figure 3.11) for $j = 1, 2, ..., n$, $G_{1,U}(\hat{\epsilon}, x) = G_{1,D}(\hat{\epsilon}, x)$ over $\Omega^{\hat{\epsilon}}_R$ and $G_{n,U}(\hat{\epsilon}, x) = G_{n,D}(\hat{\epsilon}, x)$ over $\Omega^{\hat{\epsilon}}_L$.*

Solutions in the invariant manifold $[y]_j = G_{j,s}(\hat{\epsilon}, x)$ behave differently from the other solutions of the j^{th} Riccati system, since they are the only ones that are bounded when $x \to \hat{x}_R$ and $x \to \hat{x}_L$ over $\Omega^{\hat{\epsilon}}_s$. The fact that an invariant manifold $[y]_j = G_{j,s}(\hat{\epsilon}, x)$ is bounded over the region $\mathcal{V}^j_{\hat{\epsilon}}$ leads to its uniform convergence on compact sets of Ω^0_s :

Theorem 3.4.18. *The invariant manifold $[y]_j = G_{j,s}(\hat{\epsilon}, x)$ converges uniformly on compact subsets of Ω^0_s, when $\hat{\epsilon} \to 0$, $\hat{\epsilon} \in S$, to the invariant manifold at $\epsilon = 0$ $[y]_j = G_{j,s}(0, x)$ (see Notation 3.4.7), for $s = D, U$.*

Proof. Let us take a simply connected compact subset of Ω_s^0. For $|\epsilon|$ sufficiently small, it does not contain neither $\hat{x}_R$ nor $\hat{x}_L$, nor the spiraling part of $\Omega_s^{\hat{\epsilon}}$. Proposition 3.4.15 implies that the invariant manifold $[y]_j = G_{j,s}(\hat{\epsilon}, x)$ satisfies

$$|(G_{j,s}(\hat{\epsilon}, x))_i| < \min\{|x-\hat{x}_R|, |x-\hat{x}_L|\}, \quad \text{with} \begin{cases} x \in \Omega_s^{\hat{\epsilon}}, \quad s = D, U, \\ \hat{\epsilon} \in S, \\ i = 1, 2, ..., n-1. \end{cases} \tag{3.4.50}$$

This implies the desired convergence to a bounded solution of the system for $\epsilon = 0$ that can only be $[y]_j = G_{j,s}(x, 0)$. □

3.4.6. Basis of the linear system (3.4.1)

In this section, we establish the correspondence between the invariant manifold $[y]_j = G_{j,s}(\hat{\epsilon}, x)$ of the j^{th} Riccati system (3.4.17) and multiples (by a complex constant) of a particular solution of the linear system (3.4.1). We show that these n particular solutions form a basis of solutions of the linear system which is valid for all values of $\hat{\epsilon} \in S$ and $x \in \Omega_s^{\hat{\epsilon}}$.

Notation 3.4.19. *Let $F_D(\hat{\epsilon}, x)$ be the restriction to $\Omega_D^{\hat{\epsilon}}$ of the fundamental matrix of solutions of the model system $F(\epsilon, x)$ (given by (3.4.8)), and let $F_U(\hat{\epsilon}, x)$ be its analytic continuation to $\Omega_U^{\hat{\epsilon}}$, passing through $\Omega_R^{\hat{\epsilon}}$.*

Remark 3.4.20. *The solution $F_s(\hat{\epsilon}, x)$ is uniform over $\Omega_s^{\hat{\epsilon}}$, $s = D, U$, and according to Notation 3.4.19, we have*

$$F_U(\hat{\epsilon}, x) = \begin{cases} F_D(\hat{\epsilon}, x), & \text{on } \Omega_R^{\hat{\epsilon}}, \\ F_D(\hat{\epsilon}, x)e^{2\pi i \Lambda_1(\epsilon)}, & \text{on } \Omega_L^{\hat{\epsilon}}, \\ F_D(\hat{\epsilon}, x)\hat{D}_R^{-1}, & \text{on } \Omega_C^{\hat{\epsilon}}, \end{cases} \tag{3.4.51}$$

with $\hat{D}_R$ given by (3.4.10) and $\Lambda_1(\epsilon)$ by (3.4.4), satisfying (3.4.12).

Theorem 3.4.21. *Let $s = D, U$. There exists a fundamental matrix of solutions of (3.4.1) that can be written as*

$$W_s(\hat{\epsilon}, x) = H_s(\hat{\epsilon}, x)F_s(\hat{\epsilon}, x), \quad (\hat{\epsilon}, x) \in S \times \Omega_s^{\hat{\epsilon}}, \tag{3.4.52}$$

with $H_s(\hat\epsilon, x)$ analytic on $S \times \Omega_s^{\hat\epsilon}$, satisfying

$$|H_s(\bar\epsilon, 0) - H_s(\tilde\epsilon, 0)| \le c|\bar\epsilon|, \quad \text{for some } c \in \mathbb{R}_+, \quad \bar\epsilon,\, \tilde\epsilon = \bar\epsilon e^{2\pi i} \in S_\cap \cup \{0\}, \tag{3.4.53}$$

$$|H_s(\hat\epsilon, 0)| \text{ and } |H_s(\hat\epsilon, 0)^{-1}| \text{ are bounded}, \quad \hat\epsilon \in S_\cap, \tag{3.4.54}$$

and

$$\lim_{\substack{x \to \hat x_l \\ x \in \Omega_s^{\hat\epsilon}}} H_s(x, \hat\epsilon) = \mathcal{K}_l(\hat\epsilon), \quad \hat\epsilon \in S,\ l = L, R, \tag{3.4.55}$$

where $\mathcal{K}_l(\hat\epsilon)$ is an invertible diagonal matrix depending analytically on $\hat\epsilon \in S$ with a nonsingular limit at $\epsilon = 0$ (independent of s).

Proof. The proof is valid for $s = D$ or $s = U$. For our needs, we write $[y]_j = G_{j,s}(\hat\epsilon, x)$ on $\Omega_s^{\hat\epsilon}$ as

$$\begin{cases} \frac{-(y)_k}{(y)_j} & = g_{kj,s}(\hat\epsilon, x), \\ -1 & = g_{jj,s}(\hat\epsilon, x). \end{cases} \tag{3.4.56}$$

With (3.4.1), we can write

$$(y')_j = \frac{\lambda_j(\epsilon, x)}{x^2 - \epsilon}(y)_j + \sum_{k=1}^{n} (R(\epsilon, x))_{jk}(y)_k. \tag{3.4.57}$$

Dividing by $(y)_j$, the known solutions of the j^{th} Riccati system appear in the right hand side :

$$\frac{(y')_j}{(y)_j} = -\frac{\lambda_j(\epsilon, x)}{x^2 - \epsilon} g_{jj,s}(\hat\epsilon, x) - \sum_{k=1}^{n} (R(\epsilon, x))_{jk} g_{kj,s}(\hat\epsilon, x). \tag{3.4.58}$$

The integration of equation (3.4.58) allows to recover $(y)_j$ and relation (3.4.56) leads to the other $(y)_k$, thus yielding a solution $w_{j,s}(\hat\epsilon, x)$ of the linear system (3.4.1) (and all its multiples by a complex constant) that can be written as

$$w_{j,s}(\hat\epsilon, x) = f_{j,s}(\hat\epsilon, x) h_{j,s}(\hat\epsilon, x), \tag{3.4.59}$$

with $f_{j,s}(\hat\epsilon, x)$ the j^{th} diagonal element of $F_s(\hat\epsilon, x)$ (see Notation 3.4.19), and with

$$(h_{j,s}(\hat\epsilon, x))_k = -e^{-\int_0^x \sum_{p=1}^{n} (R(\epsilon, x))_{jp} g_{pj,s}(\hat\epsilon, x) dx} g_{kj,s}(\hat\epsilon, x), \tag{3.4.60}$$

where the integration path is taken inside $\Omega_s^{\hat\epsilon}$. Such a path can be found in the t-variable (see Section 3.4.4) since $t(0) \in \Gamma_C^{\hat\epsilon}$. With the n Riccati systems, we obtain in this way n solutions $w_{j,s}(\hat{\epsilon},x)$ of the linear system (3.4.1) defined for $\hat{\epsilon} \in S$ and $x \in \Omega_s^{\hat{\epsilon}}$. We take

$$W_s(\hat{\epsilon},x) = [w_{1,s}(\hat{\epsilon},x) \ldots w_{n,s}(\hat{\epsilon},x)] \tag{3.4.61}$$

and

$$H_s(\hat{\epsilon},x) = [h_{1,s}(\hat{\epsilon},x) \ldots h_{n,s}(\hat{\epsilon},x)] \tag{3.4.62}$$

to obtain (3.4.52) from (3.4.59). The limit (3.4.55) follows from

$$\lim_{\substack{x\to\hat{x}_l \\ x\in\Omega_s^{\hat{\epsilon}}}} g_{kj,s}(x,\hat{\epsilon}) = 0, \quad k \neq j, \quad l = L, R, \tag{3.4.63}$$

and

$$(\mathcal{K}_l(\hat{\epsilon}))_{jj} = \lim_{x\to\hat{x}_l} (h_{j,s}(\hat{\epsilon},x))_j = e^{-\int_0^{\hat{x}_l} \sum_{p=1}^{n} (R(\epsilon,x))_{jp} g_{pj,s}(\hat{\epsilon},x)dx}, \tag{3.4.64}$$

which is independent of s since the integration path in (3.4.64) may be taken inside $\Omega_C^{\hat{\epsilon}}$ (see Remark 3.4.17).

The solutions $w_{1,s}(\hat{\epsilon},x), \ldots, w_{n,s}(\hat{\epsilon},x)$ form a basis of solutions since the columns of $F_s(\hat{\epsilon},x)$ are linearly independent and since $K_l(\hat{\epsilon})$ in (3.4.55) is invertible.

The property (3.4.54) comes from (3.4.50). Let us now prove (3.4.53). From its definition, $H_s(\hat{\epsilon},0)F_s(\hat{\epsilon},0)$ is a solution of (3.4.1) at $x=0$, so

$$\Lambda(\epsilon,0)H_s(\hat{\epsilon},0) - H_s(\hat{\epsilon},0)\Lambda(\epsilon,0) = \epsilon\left(H_s'(\hat{\epsilon},0) - R(\epsilon,0)H_s(\hat{\epsilon},0)\right). \tag{3.4.65}$$

With $\bar{\epsilon}$ and $\tilde{\epsilon}$ in $S_\cap$ (see Notation 3.4.11), we thus have

$$\begin{aligned} &\Lambda(\epsilon,0)(H_s(\bar{\epsilon},0) - H_s(\tilde{\epsilon},0)) - (H_s(\bar{\epsilon},0) - H_s(\tilde{\epsilon},0))\Lambda(\epsilon,0) \\ &\quad = \epsilon\left(H_s'(\bar{\epsilon},0) - H_s'(\tilde{\epsilon},0) - R(\epsilon,0)(H_s(\bar{\epsilon},0) - H_s(\tilde{\epsilon},0))\right), \end{aligned} \tag{3.4.66}$$

yielding, for some $k \in \mathbb{R}_+$,

$$|\left(H_s(\bar{\epsilon},0) - H_s(\tilde{\epsilon},0)\right)_{jq}| \leq k|\epsilon|, \quad j \neq q,\ \bar{\epsilon} \in S_\cap \cup \{0\},\ i = 1,2, \tag{3.4.67}$$

by the boundedness of $|H_s'(\bar{\epsilon},0) - H_s'(\tilde{\epsilon},0)|$, $|R(\epsilon,0)|$ and $|H_s(\bar{\epsilon},0) - H_s(\tilde{\epsilon},0)|$ over $S_\cap \cup \{0\}$ (recall that $\Lambda(\epsilon,0)$ has distinct eigenvalues for $\epsilon \in S \cup \{0\}$). Relation (3.4.53) comes from (3.4.67) and from the fact that the diagonal elements of $H_s(\bar{\epsilon},0) - H_s(\tilde{\epsilon},0)$ are zeros (since $(H_s(\hat{\epsilon},0))_{jj} = 1$). □

We have seen that the solutions in the invariant manifold $[y]_j = G_{j,s}(\hat{\epsilon}, x)$ converge uniformly on compact sets of Ω_s^0. This property remains for the corresponding solutions of the linear system :

Corollary 3.4.22 (of Theorem 3.4.18)**.** *The fundamental matrix $W_s(\hat{\epsilon}, x)$ converges (uniformly on compact sets of Ω_s^0) to the fundamental matrix $W_s(0, x)$ defined in (3.4.19), $s = D, U$.*

Proof. From (3.4.52) and the convergence of $F(\hat{\epsilon}, x)$ to $F(0, x)$, it suffices to prove the desired convergence of $H_s(\hat{\epsilon}, x)$. This is immediate, since each column has an expression in terms of the solution $[y]_j = G_{j,s}(\hat{\epsilon}, x)$ as in (3.4.60), using the notation (3.4.56). □

Remark 3.4.23. *The transformation $y = H_s(\hat{\epsilon}, x)z$ (with $H_s(\hat{\epsilon}, x)$ given by Theorem 3.4.21) conjugates the system (3.4.1) to its model (3.4.6) over $\Omega_s^{\hat{\epsilon}}$, for $\hat{\epsilon} \in S \cup \{0\}$.*

The bases $W_D(\hat{\epsilon}, x)$ and $W_U(\hat{\epsilon}, x)$ defined respectively on $\Omega_D^{\hat{\epsilon}}$ and $\Omega_U^{\hat{\epsilon}}$ will allow the calculation of the analytic invariants of the linear system.

3.4.7. Definition of the unfolded Stokes matrices

In this section, we define the unfolded Stokes matrices by comparing the fundamental matrices of solutions $W_D(\hat{\epsilon}, x)$ and $W_U(\hat{\epsilon}, x)$ on the connected components of the intersection of $\Omega_D^{\hat{\epsilon}}$ and $\Omega_U^{\hat{\epsilon}}$ (Figure 3.11).

Theorem 3.4.24. *There exist matrices $C_R(\hat{\epsilon})$ and $C_L(\hat{\epsilon})$ such that*

$$H_D(\hat{\epsilon}, x)^{-1} H_U(\hat{\epsilon}, x) = \begin{cases} F_D(\hat{\epsilon}, x) C_R(\hat{\epsilon}) (F_D(\hat{\epsilon}, x))^{-1}, & \text{on } \Omega_R^{\hat{\epsilon}}, \\ F_D(\hat{\epsilon}, x) C_L(\hat{\epsilon}) (F_D(\hat{\epsilon}, x))^{-1}, & \text{on } \Omega_L^{\hat{\epsilon}}, \\ I, & \text{on } \Omega_C^{\hat{\epsilon}}. \end{cases} \tag{3.4.68}$$

$C_R(\hat{\epsilon})$ and $C_L(\hat{\epsilon})$ are respectively an upper triangular and a lower triangular unipotent matrix. They depend analytically on $\hat{\epsilon} \in S$ and converge when $\hat{\epsilon} \to 0$ ($\hat{\epsilon} \in S$) to the Stokes matrices defined by (3.2.12).

Proof. As $W_D(\hat\epsilon,x)$ and $W_U(\hat\epsilon,x)$ are two fundamental matrices of solutions on the intersection of $\Omega_D^{\hat\epsilon}$ and $\Omega_U^{\hat\epsilon}$ (see Theorem 3.4.21), there exist matrices expressing the fact that columns of $W_U(\hat\epsilon,x)$ are linear combinations of columns of $W_D(\hat\epsilon,x)$ on the intersection parts $\Omega_L^{\hat\epsilon}$, $\Omega_R^{\hat\epsilon}$ and $\Omega_C^{\hat\epsilon}$. With (3.4.51) and (3.4.52), these relations become equivalent to

$$H_D(\hat\epsilon,x)^{-1}H_U(\hat\epsilon,x)=\begin{cases} F_D(\hat\epsilon,x)C_R(\hat\epsilon)(F_D(\hat\epsilon,x))^{-1} & \text{on } \Omega_R^{\hat\epsilon}\\ F_D(\hat\epsilon,x)C_L(\hat\epsilon)(F_D(\hat\epsilon,x))^{-1} & \text{on } \Omega_L^{\hat\epsilon}\\ F_D(\hat\epsilon,x)C_0(\hat\epsilon)(F_D(\hat\epsilon,x))^{-1} & \text{on } \Omega_C^{\hat\epsilon}.\end{cases} \tag{3.4.69}$$

Then, taking the limit $x\to\hat x_L$ on $\Omega_L^{\hat\epsilon}$, $x\to\hat x_R$ on $\Omega_R^{\hat\epsilon}$ and both limits on $\Omega_C^{\hat\epsilon}$ leads, with (3.4.36) and (3.4.55), to $C_0(\hat\epsilon)=I$ and to the unipotent triangular form of the matrices $C_R(\hat\epsilon)$ and $C_L(\hat\epsilon)$. Since $W_s(\hat\epsilon,x)$ and $F_s(\hat\epsilon,x)$ converge uniformly on compact sets of Ω_s^0 (see Corollary 3.4.18 and Remark 3.4.2), so does $H_s(\hat\epsilon,x)$. Then, the matrices $C_R(\hat\epsilon)$ and $C_L(\hat\epsilon)$ must converge to the Stokes matrices when $\hat\epsilon\to 0$, $\hat\epsilon\in S$. □

Definition 3.4.25. *We call $C_R(\hat\epsilon)$ and $C_L(\hat\epsilon)$ (defined by (3.4.68)) the* unfolded Stokes matrices *and $\{C_R(\hat\epsilon),C_L(\hat\epsilon)\}$ an* unfolded Stokes collection.

Proposition 3.4.26. *A fundamental matrix of solutions of (3.4.1) that can be written as (3.4.52), with $H_s(\hat\epsilon,x)$ analytic on $S\times\Omega_s^{\hat\epsilon}$, satisfying (3.4.53), (3.4.54) and with a limit when $x\to\hat x_l$, $x\in\Omega_s^{\hat\epsilon}$ that is bounded, invertible and independent of s, is unique up to right multiplication by any nonsingular diagonal matrix $K(\hat\epsilon)$ depending analytically on $\hat\epsilon\in S$ with a nonsingular limit at $\epsilon=0$ and such that*

$$|K(\bar\epsilon)-K(\tilde\epsilon)|\le c|\bar\epsilon|\quad \text{over } S_\cap,\text{ for some } c\in\mathbb{R}_+. \tag{3.4.70}$$

Proof. Let us suppose that we have two fundamental matrices of solutions that can be written as $H_s(\hat\epsilon,x)F_s(\hat\epsilon,x)$ and $H_s^*(\hat\epsilon,x)F_s(\hat\epsilon,x)$ with properties listed in the proposition. Having two bases of solutions over $\Omega_C^{\hat\epsilon}$, there exists a matrix $K(\hat\epsilon)$ such that

$$H_s^*(\hat\epsilon,x)F_s(\hat\epsilon,x)=H_s(\hat\epsilon,x)F_s(\hat\epsilon,x)K(\hat\epsilon),\quad x\in\Omega_C^{\hat\epsilon}. \tag{3.4.71}$$

Since the limits when $x \to \hat{x}_l$, $l = L, R$, of $H_s(\hat{\epsilon}, x)$ and of $H_s^*(\hat{\epsilon}, x)$ are bounded and invertible, $K(\hat{\epsilon})$ must be a diagonal matrix. Then, we have

$$H_s(\hat{\epsilon}, x)^{-1} H_s^*(\hat{\epsilon}, x) = K(\hat{\epsilon}), \quad x \in \Omega_C^{\hat{\epsilon}}, \tag{3.4.72}$$

and in particular

$$H_s(\hat{\epsilon}, 0)^{-1} H_s^*(\hat{\epsilon}, 0) = K(\hat{\epsilon}). \tag{3.4.73}$$

From (3.4.73), (3.4.53) and (3.4.54), we obtain (3.4.70). □

As the uniqueness of $W_s(\hat{\epsilon}, x)$ is ensured by the choice of a nonsingular diagonal matrix $K(\hat{\epsilon})$ having properties listed in Proposition 3.4.26, it is natural to adopt the following definition :

Definition 3.4.27. *Two unfolded Stokes collections written as* $\{C_R(\hat{\epsilon}), C_L(\hat{\epsilon})\}$ *and* $\{C_R'(\hat{\epsilon}), C_L'(\hat{\epsilon})\}$ *(see Definition 3.4.25) are* equivalent *if*

$$C_l'(\hat{\epsilon}) = K(\hat{\epsilon}) C_l(\hat{\epsilon}) K(\hat{\epsilon})^{-1}, \quad l = L, R, \tag{3.4.74}$$

for some nonsingular diagonal matrix $K(\hat{\epsilon})$ *depending analytically on* $\hat{\epsilon} \in S$ *with a nonsingular limit at* $\epsilon = 0$ *and such that (3.4.70) is satisfied.*

Using results obtained from the study of the monodromy of the solutions, we will prove in Section 3.4.13 that these equivalence classes of unfolded Stokes collections constitute the analytic part of the complete system of invariants for the systems (3.4.1).

3.4.8. Unfolded Stokes matrices and monodromy in the linear system

In this section, we show how the unfolded Stokes matrices are linked to the monodromy operator acting on $W_s(\hat{\epsilon}, x)$, how they give information on the existence of the bases of solutions composed of eigenvectors of the monodromy operator, and how they provide a meaning to the Stokes matrices at $\epsilon = 0$.

To study the action of the monodromy operator, we consider the ramified domain

$$V^{\hat{\epsilon}} = \Omega_D^{\hat{\epsilon}} \cup \Omega_U^{\hat{\epsilon}}, \tag{3.4.75}$$

illustrated in Figure 3.13, which could have a (non illustrated) spiraling part around $\hat{x}_R$ and $\hat{x}_L$.

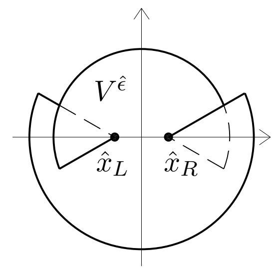

FIG. 3.13. Domain of $H(\hat{\epsilon}, x)$, denoted $V^{\hat{\epsilon}}$, case $\sqrt{\hat{\epsilon}} \in \mathbb{R}_-^*$.

Notation 3.4.28. *Let $F_V(\hat{\epsilon}, x)$ be the analytic continuation of $F_D(\hat{\epsilon}, x)$ from $\Omega_D^{\hat{\epsilon}}$ to $V^{\hat{\epsilon}}$ (through $\Omega_C^{\hat{\epsilon}}$).*

The well chosen basis of solutions we consider on this domain is the analytic continuation of $W_D(\hat{\epsilon}, x)$ from $\Omega_D^{\hat{\epsilon}}$ to $V^{\hat{\epsilon}}$, that we write as

$$W_V(\hat{\epsilon}, x) = [w_1(\hat{\epsilon}, x) \ldots w_n(\hat{\epsilon}, x)] = H(\hat{\epsilon}, x) F_V(\hat{\epsilon}, x), \tag{3.4.76}$$

where

$$H(\hat{\epsilon}, x) = \begin{cases} H_D(\hat{\epsilon}, x), & \text{on } \Omega_D^{\hat{\epsilon}}, \\ H_U(\hat{\epsilon}, x), & \text{on } \Omega_U^{\hat{\epsilon}}, \end{cases} \tag{3.4.77}$$

which is well-defined because of (3.4.68).

The fundamental group of $\mathbb{C} \backslash \{x_R, x_L\}$ based at a nonsingular point acts on a solution (valid at this base point) by giving its analytic continuation at the end of a loop. In this way we have monodromy operators around each singular point $x = x_l$. We can extend this action of the fundamental group to any function of the solutions. When the monodromy operator acts on a fundamental matrix of solutions W, its is represented by a matrix acting by right multiplication on W.

Notation 3.4.29. *We denote $M_{\hat{x}_R}$ (respectively $M_{\hat{x}_L}$) the monodromy operator associated to the loop which makes one turn around the singular point $x = \hat{x}_R$ (respectively $x = \hat{x}_L$) in the negative (respectively positive) direction and which does not surround any other singular point, with the fundamental group based, independently of $\hat{\epsilon} \in S$, at a point belonging to $\Omega_R^{\hat{\epsilon}}$ (respectively $\Omega_L^{\hat{\epsilon}}$) and taken on $\Omega_D^{\hat{\epsilon}}$ (see Figure 3.14).*

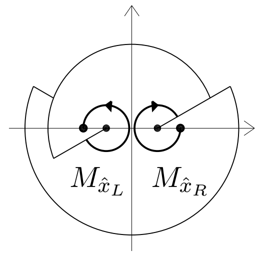

FIG. 3.14. Illustration of the definition of the monodromy operators $M_{\hat{x}_L}$ and $M_{\hat{x}_R}$, case $\hat{x}_L = \sqrt{\hat{\epsilon}} \in \mathbb{R}_-^*$.

Proposition 3.4.30. *For $l = L, R$, the action of the monodromy operator $M_{\hat{x}_l}$ on $W_V(\hat{\epsilon}, x)$ is represented by the matrix $\hat{m}_l$ satisfying*

$$\hat{m}_l = C_l(\hat{\epsilon})\hat{D}_l, \tag{3.4.78}$$

where $C_l(\hat{\epsilon})$ is the unfolded Stokes matrix defined by (3.4.68) and $\hat{D}_l$, given by (3.4.10), is the matrix representing the action of the monodromy operator $M_{\hat{x}_l}$ on the fundamental matrix of solutions $F_V(\hat{\epsilon}, x)$ of the model system.

PROOF. Starting on $\Omega_R^{\hat{\epsilon}}$, the operator $M_{\hat{x}_R}$ acting on $W_V(\hat{\epsilon}, x) = H_D(\hat{\epsilon}, x)F_D(\hat{\epsilon}, x)$ gives $H_U(\hat{\epsilon}, x)F_D(\hat{\epsilon}, x)\hat{D}_R$. Starting on $\Omega_L^{\hat{\epsilon}}$, the operator $M_{\hat{x}_L}$ acting on $W_V(\hat{\epsilon}, x) = H_D(\hat{\epsilon}, x)F_D(\hat{\epsilon}, x)$ gives $H_U(\hat{\epsilon}, x)F_D(\hat{\epsilon}, x)\hat{D}_L$. As we have (3.4.68), equation (3.4.78) is verified for $l = L, R$. □

Notation 3.4.31. *The j^{th} row (respectively column) of a matrix will be called* trivial *if it corresponds to the j^{th} row (respectively column) of the identity matrix.*

Remark 3.4.32. *Relation (3.4.78) gives a geometric meaning to zeros in unfolded Stokes matrices $C_l(\hat{\epsilon})$. For example, if a permutation P is such that $PC_l(\hat{\epsilon})P^{-1}$ is in a block diagonal form, it indicates a decomposition of the solution space into invariant subspaces under the action of the monodromy operator $M_{\hat{x}_l}$. A trivial j^{th} column (see Notation 3.4.31) of $C_l(\hat{\epsilon})$ points out that $w_j(\hat{\epsilon}, x)$ is eigenvector of $M_{\hat{x}_l}$. A trivial unfolded Stokes matrix $C_l(\hat{\epsilon})$ would imply that all the elements of $W_V(\hat{\epsilon}, x)$ are eigenvectors of $M_{\hat{x}_l}$.*

Via the Jordan normal form of the monodromy matrix $C_l(\hat{\epsilon})\hat{D}_l$, we will now express how the elements of the unfolded Stokes matrices are linked to the existence of the solutions that are eigenvectors of the monodromy operator around the singular points. This will give a geometric interpretation to the elements of $C_l(\epsilon)$ and, in particular, of their limits, the elements of $C_l(0)$.

Theorem 3.4.33. *$t \in \mathbb{C}^n$ is an eigenvector of the monodromy matrix $\hat{m}_l$ if and only if $W_V(\hat{\epsilon}, x)t$ is a solution eigenvector of the monodromy operator $M_{\hat{x}_l}$ with the same eigenvalue. Hence, for $l = L, R$, the number of independent solutions which are eigenvectors of $M_{\hat{x}_l}$ is equal to the number of Jordan blocks in the Jordan matrix associated to $\hat{m}_l = C_l(\hat{\epsilon})\hat{D}_l$. The values for which the monodromy matrix $\hat{m}_l$ may not be diagonalizable, are the resonance values of $\hat{\epsilon}$ specified in Definition 3.4.13 (which exactly correspond to multiple eigenvalues of $\hat{m}_l$).*

For nonresonance values of $\hat{\epsilon}$, let $\hat{T}_l$ be the unipotent triangular matrix diagonalizing the monodromy matrix $\hat{m}_l = C_l(\hat{\epsilon})\hat{D}_l$:

$$(\hat{T}_l)^{-1}\hat{m}_l\hat{T}_l = \hat{D}_l. \tag{3.4.79}$$

The fundamental matrix of solutions

$$W_{\hat{x}_l}(x) = W_V(\hat{\epsilon}, x)\hat{T}_l \tag{3.4.80}$$

is composed of eigenvectors of the monodromy operator around $x = \hat{x}_l$. A fundamental matrix having this property is unique up to its normalization : the j^{th} column of $W_{\hat{x}_l}$ is a nonzero multiple of the Floquet solution (for example **[23]** *p. 25) given by*

$$\hat{w}_{j,l}(x) = (x - \hat{x}_l)^{\hat{\mu}_{j,l}}\hat{g}_{j,l}(x), \tag{3.4.81}$$

with $\hat{\mu}_{j,l}$ given by (3.4.9) and $\hat{g}_{j,l}(x) = e_j + O(|x - \hat{x}_l|)$ an analytic function of x in a region containing $x = \hat{x}_l$ but no other singular point.

When $\hat{\epsilon}$ is a resonance value, the matrix $\hat{m}_l = C_l(\hat{\epsilon})\hat{D}_l$ is no more diagonalizable with no j^{th} eigenvector if and only it the j^{th} Floquet solution $\hat{w}_{j,l}(x)$ does not exist and has to be replaced, in the basis of solutions around $x = \hat{x}_l$, by a solution containing logarithmic terms.

Proof. By Proposition 3.4.30, we have $M_{\hat{x}_l}W_V(\hat{\epsilon},x) = W_V(\hat{\epsilon},x)\hat{m}_l$. Let $t \in \mathbb{C}^n$ and $\beta \in \mathbb{C}$. The first assertion of the theorem is obtained from

$$\begin{aligned} \hat{m}_l t = \beta t \quad &\Longleftrightarrow W_V(\hat{\epsilon},x)\hat{m}_l t = \beta W_V(\hat{\epsilon},x)t \\ &\Longleftrightarrow M_{\hat{x}_l}W_V(\hat{\epsilon},x)t = \beta W_V(\hat{\epsilon},x)t. \end{aligned} \tag{3.4.82}$$

To prove the uniqueness (up to normalization) of $W_{\hat{x}_l}(x)$, let us suppose that W^* is such that $M_{\hat{x}_l}W^* = W^*\hat{D}_l$. Since we have two bases of solutions, there exists a nonsingular matrix K such that $W_{\hat{x}_l}(x) = W^*K$. Since $M_{\hat{x}_l}W_{\hat{x}_l} = W_{\hat{x}_l}\hat{D}_l$, we must have $\hat{D}_l K = K\hat{D}_l$. Since $\hat{\epsilon}$ is not a resonance value, the eigenvalues of $\hat{D}_l$ are distinct and K can only be diagonal. □

Remark 3.4.34. *For nonresonance values of $\hat{\epsilon}$, (3.4.79) implies that the unfolded Stokes matrices are equal to the multiplicative commutator of the matrices $\hat{T}_l$ and $\hat{D}_l$:*

$$C_l(\hat{\epsilon}) = \hat{T}_l\hat{D}_l\hat{T}_l^{-1}\hat{D}_l^{-1} = [\hat{T}_l, \hat{D}_l]. \tag{3.4.83}$$

The unfolded Stokes matrix $C_R(\hat{\epsilon})$ (respectively $C_L(\hat{\epsilon})$) is linked to the presence of logarithmic terms in solutions around $x = \hat{x}_R$ (respectively $x = \hat{x}_L$) :

Corollary 3.4.35. *There exist polynomials in terms of the elements of the unfolded Stokes matrices $C_l(\hat{\epsilon})$ and the elements of $\hat{D}_l$ indicating, when they are nonzero at a resonance value, the nonexistence of a Floquet solution $\hat{w}_{j,l}(x)$ at the resonance.*

In generic cases, the obstruction to the existence of Floquet solutions can be forced by the special form of the Stokes matrix $C_l = C_l(0)$. This is the case when

- $(C_R)_{12} \neq 0$ *: $\hat{w}_{2,R}(x)$ does not exist at the resonance $\hat{\mu}_{1,R} - \hat{\mu}_{2,R} \in \mathbb{N}^*$;*
- $(C_L)_{n(n-1)} \neq 0$ *: $\hat{w}_{n-1,L}(x)$ does not exist at the resonance $\hat{\mu}_{n,L} - \hat{\mu}_{n-1,L} \in \mathbb{N}^*$;*
- $\arg(\lambda_{s,0} - \lambda_{j,0})$ *are distinct for all $s \neq j$: a nonvanishing s^{th} polynomial in terms of the elements of the Stokes matrices C_l with integer coefficients yields an obstruction to the existence of $\hat{w}_{j,l}(x)$ at the resonance $\hat{\mu}_{s,l} - \hat{\mu}_{j,l} \in \mathbb{N}^*$, with $s > j$ if $l = L$ and $s < j$ if $l = R$.*

PROOF. The polynomials of the corollary could be obtained by analytic or algebraic arguments, by counting the number of eigenvectors of $C_l(\hat{\epsilon})\hat{D}_l$. We present the proof in the analytic way. Recall that the matrices $\hat{T}_l$ are triangular and unipotent. Since $\hat{T}_l = C_l(\hat{\epsilon})\hat{D}_l\hat{T}_l\hat{D}_l^{-1}$(see (3.4.83)), elements $(\hat{T}_l)_{ij}$, for $i \neq j$, can be calculated from the recurrent equations

$$(\hat{T}_l)_{ij}(1-\hat{\Delta}_{ij,l}) = (C_l(\hat{\epsilon}))_{ij} + \sum_{\substack{i<k<j,\, l=R \\ j<k<i,\, l=L}} (C_l(\hat{\epsilon}))_{ik}(\hat{T}_l)_{kj}\hat{\Delta}_{kj,l}, \tag{3.4.84}$$

with $\hat{\Delta}_{sj,l}$ given by (3.4.11). At the resonance, $\hat{\Delta}_{sj,l} = 1$ for some s, j, l. Conditions to the nonexistence of the j^{th} column of $\hat{T}_l$ at the resonance can be calculated from (3.4.84) : they are given by polynomials in terms of elements of $\hat{D}_l$ and of elements of the unfolded Stokes matrices. For generic cases, these polynomials have a limit at $\epsilon = 0$ and the conditions can be formulated with polynomials in the elements of the Stokes matrices at $\epsilon = 0$: the nonvanishing of the polynomials for small $\hat{\epsilon}$ is ensured by the nonvanishing of the limit polynomial at $\epsilon = 0$ which depends on C_l. This is the case for the conditions to the existence of

- the second column of $\hat{T}_R$;
- the $(n-1)^{th}$ column of $\hat{T}_L$;
- all columns if the Stokes lines are distinct (i.e. $\arg(\lambda_{s,0}-\lambda_{j,0})$ are distinct for all $s \neq j$). In that case, the resonance $\hat{\Delta}_{ij,l} = 1$ is distinct from the resonance $\hat{\Delta}_{kj,l} = 1$ for $k \neq i$. On the sequence $\hat{\epsilon}_n \to 0$ corresponding to the resonance $\hat{\Delta}_{ij,l} = 1$, the limit of $\left(\frac{\hat{\Delta}_{kj,l}}{1-\hat{\Delta}_{kj,l}}\right)$ is 0 or -1, hence the polynomials at the limit have integer coefficients (independent of $\hat{\epsilon}$).

□

Example 3.4.36. *Let us consider the case $n = 3$, with distinct arguments of $\lambda_2 - \lambda_3$, $\lambda_1 - \lambda_2$ and $\lambda_1 - \lambda_3$. Equation (3.4.84) gives*

$$\begin{aligned}
&(\hat{T}_R)_{12}(1-\hat{\Delta}_{12,R}) = (C_R(\hat{\epsilon}))_{12},\\
&(\hat{T}_R)_{13}(1-\hat{\Delta}_{13,R}) = (C_R(\hat{\epsilon}))_{13} + (C_R(\hat{\epsilon}))_{12}(C_R(\hat{\epsilon}))_{23}\left(\tfrac{\hat{\Delta}_{23,R}}{1-\hat{\Delta}_{23,R}}\right),\\
&(\hat{T}_R)_{23}(1-\hat{\Delta}_{23,R}) = (C_R(\hat{\epsilon}))_{23},\\
&(\hat{T}_L)_{21}(1-\hat{\Delta}_{21,L}) = (C_L(\hat{\epsilon}))_{21},\\
&(\hat{T}_L)_{31}(1-\hat{\Delta}_{31,L}) = (C_L(\hat{\epsilon}))_{31} + (C_L(\hat{\epsilon}))_{21}(C_L(\hat{\epsilon}))_{32}\left(\tfrac{\hat{\Delta}_{21,L}}{1-\hat{\Delta}_{21,L}}\right),\\
&(\hat{T}_L)_{32}(1-\hat{\Delta}_{32,L}) = (C_L(\hat{\epsilon}))_{32}.
\end{aligned} \tag{3.4.85}$$

Decreasing values of $\hat{\epsilon}$ such that $\hat{\mu}_{1,R} - \hat{\mu}_{3,R} \in \mathbb{N}^*$ and $\hat{\mu}_{3,L} - \hat{\mu}_{1,L} \in \mathbb{N}^*$ are approaching the ray $\arg(\sqrt{\epsilon}) = \arg(\lambda_{3,0} - \lambda_{1,0})$. The following comes from the inequalities $\arg(\lambda_{1,0} - \lambda_{2,0}) < \arg(\lambda_{1,0} - \lambda_{3,0}) < \arg(\lambda_{2,0} - \lambda_{3,0})$. When $\hat{\epsilon} \to 0$ on resonance values

- *$\hat{\mu}_{1,R} - \hat{\mu}_{3,R} \in \mathbb{N}^*$ making $\hat{\Delta}_{13,R} = 1$, we have $\Im(\hat{\mu}_{2,R} - \hat{\mu}_{3,R}) > 0$ and then $\left(\frac{\hat{\Delta}_{23,R}}{1-\hat{\Delta}_{23,R}}\right) = \left(\frac{1}{\hat{\Delta}_{32,R}-1}\right)$ tends to -1, since $\hat{\Delta}_{32,R} \to 0$;*
- *$\hat{\mu}_{3,L} - \hat{\mu}_{1,L} \in \mathbb{N}^*$ making $\hat{\Delta}_{31,L} = 1$, we have $\Im(\hat{\mu}_{1,L} - \hat{\mu}_{2,L}) > 0$ and then $\left(\frac{\hat{\Delta}_{21,L}}{1-\hat{\Delta}_{21,L}}\right) = \left(\frac{1}{\hat{\Delta}_{12,L}-1}\right)$ tends to -1, since $\hat{\Delta}_{12,L} \to 0$.*

These limits imply that the right hand side of the equations (3.4.85) at the resonance is minus an element of the inverse of the unfolded Stokes matrices. We immediately see that

- *if $(C_R)_{12} \neq 0$, $(\hat{T}_R)_{12}$ (and hence $\hat{w}_{2,R}(x)$) does not have a limit at the resonances $\hat{\mu}_{1,R} - \hat{\mu}_{2,R} \in \mathbb{N}^*$ making $\hat{\Delta}_{12,R} = 1$;*
- *if $(C_R)_{23} \neq 0$, $(\hat{T}_R)_{23}$ (and hence $\hat{w}_{3,R}(x)$) does not have a limit at the resonances $\hat{\mu}_{2,R} - \hat{\mu}_{3,R} \in \mathbb{N}^*$ making $\hat{\Delta}_{23,R} = 1$;*
- *if $(C_R)_{13} - (C_R)_{12}(C_R)_{23} \neq 0$, $(\hat{T}_R)_{13}$ (and hence $\hat{w}_{3,R}(x)$) does not have a limit at the resonances $\hat{\mu}_{1,R} - \hat{\mu}_{3,R} \in \mathbb{N}^*$ making $\hat{\Delta}_{13,R} = 1$;*
- *if $(C_L)_{21} \neq 0$, $(\hat{T}_L)_{21}$ (and hence $\hat{w}_{1,L}(x)$) does not have a limit at the resonances $\hat{\mu}_{2,L} - \hat{\mu}_{1,L} \in \mathbb{N}^*$ making $\hat{\Delta}_{21,L} = 1$;*
- *if $(C_L)_{32} \neq 0$, $(\hat{T}_L)_{32}$ (and hence $\hat{w}_{2,L}(x)$) does not have a limit at the resonances $\hat{\mu}_{3,L} - \hat{\mu}_{2,L} \in \mathbb{N}^*$ making $\hat{\Delta}_{32,L} = 1$;*

- *if* $(C_L)_{31}-(C_L)_{21}(C_L)_{32}\neq 0$, $(\hat{T}_L)_{31}$ *(and hence* $\hat{w}_{1,L}(x)$*) does not have a limit at the resonances* $\hat{\mu}_{3,L}-\hat{\mu}_{1,L}\in\mathbb{N}^*$ *making* $\hat{\Delta}_{31,L}=1$.

3.4.9. Stokes matrices and monodromy in the Riccati systems

In this section, we give a meaning to the unfolded Stokes matrices in the corresponding Riccati systems. This allows an interpretation of the Stokes matrices at $\epsilon=0$. This section is not prerequisite to state the complete system of analytic invariants of the systems (3.4.1) .

We will look at the monodromy of first integrals in the Riccati systems. These first integrals are obtained from the basis of the linear system.

Proposition 3.4.37. *For* $x\in V^{\hat{\epsilon}}$, *the* j^{th} *Riccati system has first integrals* $\mathcal{H}_q^j$, *for* $q\in\{1,2,...,n\}\backslash\{j\}$, *that can be written as*

$$\mathcal{H}_q^j=(-1)^{q-j}\frac{\begin{vmatrix}b_1^j(\hat{\epsilon},x,[y]_j) & ... & b_q^j\widehat{(\hat{\epsilon},x,[y]_j)} & ... & b_n^j(\hat{\epsilon},x,[y]_j)\end{vmatrix}}{\begin{vmatrix}b_1^j(\hat{\epsilon},x,[y]_j) & ... & b_j^j\widehat{(\hat{\epsilon},x,[y]_j)} & ... & b_n^j(\hat{\epsilon},x,[y]_j)\end{vmatrix}},\tag{3.4.86}$$

with

$$b_i^j(\hat{\epsilon},x,[y]_j)=(-1)^{i-j}(w_i(\hat{\epsilon},x))_j\left([y]_j-[w_i]_j\right),\tag{3.4.87}$$

and $w_i(\hat{\epsilon},x)$ *the* i^{th} *column of the fundamental matrix of solutions* $W_V(\hat{\epsilon},x)$ *given by (3.4.76) (for* $[w_i]_j$, *see Notation 3.4.5).* $(\mathcal{H}^j)_q$ *has values in* $(\mathbb{CP}^1)$ *for* $q\neq j$.

Proof. Let $w_i(\hat{\epsilon},x)$ be the columns of the fundamental matrix of solutions $W_V(\hat{\epsilon},x)$ given by (3.4.76). The general solution of a linear system (3.4.1) may be expressed as a linear combination $y=\sum_{q=1}^n k_q w_q(\hat{\epsilon},x)$ of the particular solution $w_q(\hat{\epsilon},x)$, with $k_q\in\mathbb{C}$. In particular, the j^{th} component of this general solution y satisfies

$$(y)_j=\sum_{q=1}^n k_q(w_q(\hat{\epsilon},x))_j,\tag{3.4.88}$$

so

$$\sum_{q=1}^n k_q(w_q(\hat{\epsilon},x))_j\frac{y}{(y)_j}=\sum_{q=1}^n k_q w_q(\hat{\epsilon},x),\tag{3.4.89}$$

and

$$\sum_{q=1}^n\frac{k_q}{k_j}\left(w_q(\hat{\epsilon},x)-(w_q(\hat{\epsilon},x))_j\frac{y}{(y)_j}\right)=0.\tag{3.4.90}$$

Solving for $\frac{k_q}{k_j}$, $q \neq j$, and using Notation 3.4.5 and (3.4.87) gives (3.4.86). $\square$

As detailed in the next theorem, elements of the inverse of the unfolded Stokes matrices appear in the expression of the monodromy of the first integrals $\mathcal{H}_q^j$ around $x = \hat{x}_l$.

Theorem 3.4.38. *The monodromy of a first integral $\mathcal{H}_q^j$ around $x = \hat{x}_l$ may be written as the composition of*

- *a wild part (i.e. of the form $e^{\frac{2\pi i}{\alpha}}$ with $\alpha \in \mathbb{C}$, $\alpha \to 0$) depending on the formal invariants,*
- *a map depending on the elements of the inverse of the unfolded Stokes matrices and having a limit for $\epsilon = 0$.*

More precisely, with $\mathcal{H}_j^j = 1$, the monodromy of the first integrals may be expressed as

$$M_{\hat{x}_R}(\mathcal{H}_q^j) = \hat{\Delta}_{jq,R} \frac{\mathcal{H}_q^j + \sum_{p=q+1}^{n} (C_R(\hat{\epsilon})^{-1})_{qp} \mathcal{H}_p^j}{1 + \sum_{p=j+1}^{n} (C_R(\hat{\epsilon})^{-1})_{jp} \mathcal{H}_p^j}, \tag{3.4.91}$$

and

$$M_{\hat{x}_L}(\mathcal{H}_q^j) = \hat{\Delta}_{jq,L} \frac{\mathcal{H}_q^j + \sum_{p=1}^{q-1} (C_L(\hat{\epsilon})^{-1})_{qp} \mathcal{H}_p^j}{1 + \sum_{p=1}^{j-1} (C_L(\hat{\epsilon})^{-1})_{jp} \mathcal{H}_p^j}. \tag{3.4.92}$$

Denoting

$$\mathcal{H}^j = (\mathcal{H}_1^j, ..., \mathcal{H}_n^j)^T, \tag{3.4.93}$$

this is equivalent to

$$M_{\hat{x}_l}(\mathcal{H}^j) = diag\{\hat{\Delta}_{j1,l}, ..., \hat{\Delta}_{jn,l}\} \frac{C_l(\hat{\epsilon})^{-1} \mathcal{H}^j}{[(C_l(\hat{\epsilon})^{-1})_{j1}, ..., (C_l(\hat{\epsilon})^{-1})_{jn}] \mathcal{H}^j}, \tag{3.4.94}$$

with $\hat{\Delta}_{jq,l}$ as defined by (3.4.11).

Proof. In order to calculate the monodromy of the first integrals given by (3.4.86), we need to compute the monodromy of

$$B^j(\hat{\epsilon}, x, [y]_j) = [b_1^j(\hat{\epsilon}, x, [y]_j) \quad ... \quad b_n^j(\hat{\epsilon}, x, [y]_j)], \tag{3.4.95}$$

with $b_i^j(\hat{\epsilon}, x, [y]_j)$ given by (3.4.87). Since the monodromy of $w_q(\hat{\epsilon}, x)$ is given by Proposition 3.4.30, we have

$$M_{\hat{x}_l}(B^j(\hat{\epsilon}, x, [y]_j)) = B^j(\hat{\epsilon}, x, [y]_j) \hat{m}_l, \tag{3.4.96}$$

with $\hat{m}_l$ given by (3.4.78). With $\mathcal{H}^j$ defined in (3.4.93), relation (3.4.90) implies

$$B^j(\hat{\epsilon}, x, [y]_j)\mathcal{H}^j = 0, \tag{3.4.97}$$

and thus, using (3.4.96),

$$B^j(\hat{\epsilon}, x, [y]_j)\hat{m}_l M_{\hat{x}_l}(\mathcal{H}^j) = 0. \tag{3.4.98}$$

Equations (3.4.97) and (3.4.98) imply that

$$M_{\hat{x}_l}(\mathcal{H}_q^j) = \frac{(\hat{m}_l^{-1}\mathcal{H}^j)_q}{(\hat{m}_l^{-1}\mathcal{H}^j)_j}, \tag{3.4.99}$$

leading to the equations of the theorem, using (3.4.78). □

Theorem 3.4.38 yields the following interpretation of the Stokes matrices at $\epsilon = 0$:

Corollary 3.4.39. *The first integral $\mathcal{H}_q^j$ is an eigenvector of the monodromy operator around a singular point $x = \hat{x}_l$ (by this we means $M_{\hat{x}_l}\mathcal{H}_q^j = \hat{\Delta}_{jq,l}\mathcal{H}_q^j$) if and only if the rows j and q in the inverse of the unfolded Stokes matrix $C_l(\hat{\epsilon})$ are trivial (see Notation 3.4.31). Hence, a nontrivial i^{th} row in the inverse of the right (respectively left) Stokes matrix at $\epsilon = 0$ is an obstruction for the first integrals $\mathcal{H}_k^i$ to be eigenvectors of the monodromy operator around the right (respectively left) singular point, for $k \in \{1, ..., n\}\backslash\{i\}$.*

Proof. This is immediate from equations (3.4.91) and (3.4.92). □

The wild part in the monodromy of the first integrals of the Riccati system is due to the definition of the fundamental matrix of solutions of the model system over the considered domain and is not a consequence of the Stokes phenomenon :

Remark 3.4.40. *If we compare first integrals over the intersections of the sectorial domains $\Omega_U^{\hat{\epsilon}}$ and $\Omega_D^{\hat{\epsilon}}$ instead of over the auto-intersection of $V^{\hat{\epsilon}}$ (thus taking Notation 3.4.19 for $F_s(\hat{\epsilon}, x)$ over $\Omega_s^{\hat{\epsilon}}$ instead of Notation 3.4.28 for $F_V(\hat{\epsilon}, x)$ over $V^{\hat{\epsilon}}$), the wild part is only present in the comparison over $\Omega_C^{\hat{\epsilon}}$ (which does not exist at $\epsilon = 0$). When we compare the first integrals over $\Omega_R^{\hat{\epsilon}}$ and $\Omega_L^{\hat{\epsilon}}$, there is no wild part in equations corresponding to (3.4.91), (3.4.92) and (3.4.94).*

3.4.10. Auto-intersection relation and $\frac{1}{2}$-summable representative of the equivalence class of unfolded Stokes matrices

In this section, we compare the two points of view that we have on $S_\cap$, the auto-intersection of S. This will yield a relation that is satisfied for all $\epsilon \in S_\cap$. We call it the auto-intersection relation. It allows to prove the existence of a representative of the equivalence class of unfolded Stokes matrices which is $\frac{1}{2}$-summable in ϵ. Further, it will be a necessary and sufficient condition for the realization of the complete system of analytic invariants.

For $\bar{\epsilon}$ and $\tilde{\epsilon} = \bar{\epsilon}e^{2\pi i}$ in $S_\cap$ (Figure 3.3), we have two different presentations of the dynamics of the same linear differential system. On $S_\cap$, since there is no resonance value, there always exists transition matrices between fundamental matrices of solutions composed of eigenvectors of the monodromy operators around the singular points. We will use these transition matrices to compare the two presentations on $S_\cap$. First, let us take the monodromy operators with the base point taken on the upper (respectively lower) sectorial domain when the corresponding loop surrounds the upper (respectively lower) singular point.

Notation 3.4.41. *In Notation 3.4.29, we defined the monodromy operators $M_{\hat{x}_R}$ and $M_{\hat{x}_L}$, for $\hat{\epsilon} \in S$. Over $S_\cap$, let us denote*

- $M^*_{\tilde{x}_L} = M_{\tilde{x}_L}$,
- $M^*_{\bar{x}_R} = M_{\bar{x}_R}$,
- $M^*_{\bar{x}_L} = M^{-1}_{\bar{x}_L}$,
- $M^*_{\tilde{x}_R} = M^{-1}_{\tilde{x}_R}$.

Hence, the base points of $M^*_{\tilde{x}_L}$ and $M^*_{\bar{x}_R}$ belongs to $\Omega^{\bar{\epsilon}}_D \cap \Omega^{\tilde{\epsilon}}_D$, whereas the base points of $M^*_{\bar{x}_L}$ (respectively $M^*_{\tilde{x}_R}$) are taken on $\Omega^{\bar{\epsilon}}_U \cap \Omega^{\tilde{\epsilon}}_U$ (Figures 3.15 and 3.16).

Definition 3.4.42. *For $l = L, R$, let us take $W_{\hat{x}_l}(x)$ a fundamental matrix of solutions of (3.4.1) composed of eigenvectors of the monodromy operator $M^*_{\hat{x}_l}$, depending analytically on $\hat{\epsilon} \in S_\cap$ and converging uniformly over compact sets of Ω^0_s when $\hat{\epsilon} \to 0$ (and $\hat{\epsilon} \in S_\cap$) to $W_s(0, x)$ defined by (3.4.19), with $s = D$ if $\Im(\hat{x}_l) < 0$ and $s = U$ otherwise. Let $E_{L,\hat{x}_L \to \hat{x}_R}$ be the matrix such that, over a*

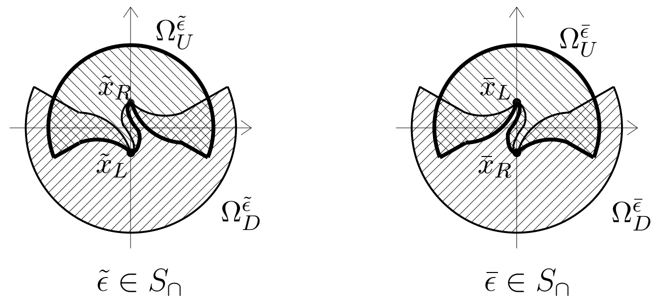

FIG. 3.15. Sectorial domains in the x-variable for $\hat\epsilon \in S_\cap$.

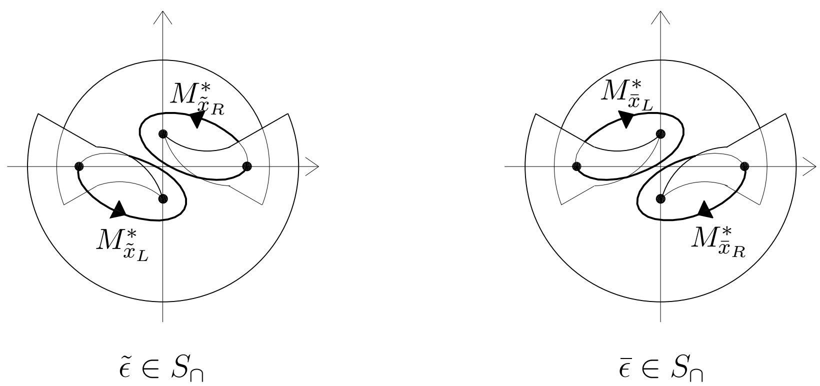

FIG. 3.16. Illustration of the definition of the monodromy operators $M^*_{\tilde x_L}$, $M^*_{\bar x_R}$, $M^*_{\bar x_L}$ and $M^*_{\tilde x_R}$.

fixed compact set of Ω_L^0 *sufficiently far from the singular points,*

$$E_{L,\hat x_L \to \hat x_R} = (W_{\hat x_L}(x))^{-1} W_{\hat x_R}(x). \tag{3.4.100}$$

Let $E_{\hat x_L \to \hat x_R}$ *be the matrix such that, over a fixed compact set of* Ω_R^0 *sufficiently far from the singular points,*

$$E_{R,\hat x_L \to \hat x_R} = (W_{\hat x_L}(x))^{-1} W_{\hat x_R}(x). \tag{3.4.101}$$

We call $E_{L,\hat x_L \to \hat x_R}$ *(respectively* $E_{\hat x_L \to \hat x_R}$*) the* left *(respectively* right*)* transition matrix *from* $\hat x_L$ *to* $\hat x_R$*. These transition matrices are unique up to multiplication on each side by nonsingular diagonal matrices depending analytically on* $\hat\epsilon \in S_\cap$*, with a nonsingular limit at* $\epsilon = 0$ *(coming from the normalization of the chosen fundamental matrices of solutions).*

The following proposition is implicit from the paper [**6**] of A. Glutsyuk. The proof will be useful later.

Proposition 3.4.43. *Let us take two families of systems*

$$(x^2 - \hat{\epsilon})y_i' = B_i(\hat{\epsilon}, x)y_i, \quad i = 1, 2, \tag{3.4.102}$$

having the form (3.4.1) with the same model system and depending on $\hat{\epsilon} \in S_\cap$*. Let*

$$x_U = \bar{x}_L = \tilde{x}_R, \qquad x_D = \bar{x}_R = \tilde{x}_L. \tag{3.4.103}$$

Let us take for each family of systems a right transition matrix from x_D *to* x_U*, i.e.* $E^i_{R,x_D \to x_U}$ *(Definition 3.4.42). The two family of systems (3.4.102) are analytically equivalent, the equivalence depending analytically on* $(\epsilon, x) \in S_\cap \times \mathbb{D}_r$ *and converging uniformly on compact sets of* $\mathbb{D}_r$ *when* $\epsilon \to 0$*, if and only if there exist* $Q_U(\hat{\epsilon})$ *and* $Q_D(\hat{\epsilon})$ *nonsingular diagonal matrices depending analytically on* $\hat{\epsilon} \in S_\cap$*, with a nonsingular limit at* $\epsilon = 0$*, and such that*

$$E^1_{R,x_D \to x_U} Q_U(\hat{\epsilon}) = Q_D(\hat{\epsilon}) E^2_{R,x_D \to x_U}. \tag{3.4.104}$$

PROOF. Let us denote by $W^i_{\hat{x}_l}(x)$, $l = L, R$, the fundamental matrix of solutions taken to calculate the right transition matrices $E^i_{R,x_D \to x_U}$, $i = 1, 2$. Let us take two domains $\mathcal{G}^{\hat{\epsilon}}_U$ and $\mathcal{G}^{\hat{\epsilon}}_D$ covering $\mathbb{D}_r$ (Figure 3.17), such that $\mathcal{G}^{\hat{\epsilon}}_U$ (respectively $\mathcal{G}^{\hat{\epsilon}}_D$) contains x_U but not x_D (respectively x_D but not x_U) and has the limit Ω^0_U (respectively Ω^0_D) when $\hat{\epsilon} \to 0$ in $S_\cap$.

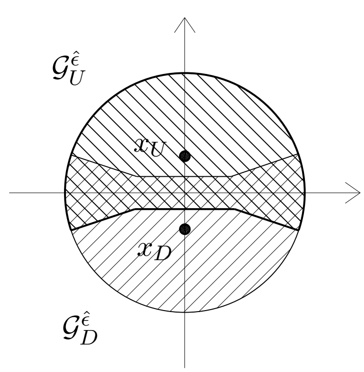

FIG. 3.17. Domains $\mathcal{G}^{\hat{\epsilon}}_U$ and $\mathcal{G}^{\hat{\epsilon}}_D$ and their intersection.

Let us suppose that (3.4.104) is satisfied. The transformation $y_1 = P_{\hat{\epsilon}}(x)y_2$, with

$$P_{\hat{\epsilon}}(x) = \begin{cases} W^1_{x_U}(x)Q_U(\hat{\epsilon})(W^2_{x_U}(x))^{-1}, & \text{on } \mathcal{G}^{\hat{\epsilon}}_U, \\ W^1_{x_D}(x)Q_D(\hat{\epsilon})(W^2_{x_D}(x))^{-1}, & \text{on } \mathcal{G}^{\hat{\epsilon}}_D, \end{cases} \tag{3.4.105}$$

is well-defined on $\mathbb{D}_r$ because of (3.4.104), for any $\hat{\epsilon} \in S_\cap \cup \{0\}$. It conjugates the two systems, depends analytically on $(\hat{\epsilon}, x) \in S_\cap \times \mathbb{D}_r$ and converges uniformly on compact sets of $\mathbb{D}_r$ when $\hat{\epsilon} \to 0$.

On the other hand, let us suppose that the change $y_1 = P_{\hat{\epsilon}}(x)y_2$ yields an analytic equivalence (as in the statement of the proposition) between the two systems. Then, by uniqueness (up to normalization) of $W^i_{\hat{x}_L}(x)$ and $W^i_{\hat{x}_R}(x)$, we must have

$$P_{\hat{\epsilon}}(x)W^2_{x_U}(x) = W^1_{x_U}(x)Q_U(\hat{\epsilon}), \quad \text{over } \mathcal{G}^{\hat{\epsilon}}_U, \tag{3.4.106}$$

and

$$P_{\hat{\epsilon}}(x)W^2_{x_D}(x) = W^1_{x_D}(x)Q_D(\hat{\epsilon}), \quad \text{over } \mathcal{G}^{\hat{\epsilon}}_D, \tag{3.4.107}$$

with $Q_U(\hat{\epsilon})$ and $Q_D(\hat{\epsilon})$ nonsingular diagonal matrices depending analytically on $\hat{\epsilon} \in S_\cap$ and having a nonsingular limit at $\epsilon = 0$. Isolating $P_{\hat{\epsilon}}(x)$ in (3.4.106) and (3.4.107), we get, over a compact in Ω^0_R and for $\hat{\epsilon}$ sufficiently small,

$$W^1_{x_D}(x)Q_D(\hat{\epsilon})(W^2_{x_D}(x))^{-1} = W^1_{x_U}(x)Q_U(\hat{\epsilon})(W^2_{x_U}(x))^{-1}, \tag{3.4.108}$$

which is equivalent to (3.4.104), using (3.4.101). □

Remark 3.4.44. *Taking the left transition matrices instead of the right transition matrices in Proposition 3.4.43 (and taking in the proof a compact set on Ω^0_L instead of Ω^0_R) yields similar result.*

When taking a system of the form (3.4.1), the right transition matrices $E_{R,\tilde{x}_L\to\tilde{x}_R}$ and $E_{R,\bar{x}_R\to\bar{x}_L}$ both correspond to the transition from the lower point to the upper point. By Proposition 3.4.43, we know they satisfy a relation like (3.4.104). We formulate it more precisely in Proposition 3.4.52.

Definition 3.4.45. *For $r(\epsilon)$ analytic in $\epsilon \in S_\cap$, we say that $r(\epsilon)$ is exponentially close to 0 in $\sqrt{\epsilon}$ if it satisfies $|r(\epsilon)| < be^{-\frac{a}{\sqrt{|\epsilon|}}}$ for some $a, b \in \mathbb{R}^*_+$.*

Lemma 3.4.46. *Following its definition given by (3.4.11),*

$$\tilde{\Delta}_{sj,l} = (\tilde{D}_l)_{ss}(\tilde{D}_l^{-1})_{jj}, \quad s < j,\, l = L, R \tag{3.4.109}$$

is exponentially close to 0 *in* $\sqrt{\epsilon}$ *(to prove it, use (3.4.30)). We also have*

$$(\hat{\Delta}_{sj,l})^{-1} = \hat{\Delta}_{js,l} \tag{3.4.110}$$

and

$$\hat{\Delta}_{sj,l}\hat{\Delta}_{ji,l} = \hat{\Delta}_{si,l}. \tag{3.4.111}$$

By (3.4.7), (3.4.10) and (3.4.31), we obtain

$$\tilde{D}_L = \bar{D}_R^{-1}, \quad \tilde{D}_R = \bar{D}_L^{-1} \tag{3.4.112}$$

and, by (3.4.11),

$$\begin{aligned} \bar{\Delta}_{sj,R} &= \tilde{\Delta}_{js,L}, \\ \bar{\Delta}_{sj,L} &= \tilde{\Delta}_{js,R}. \end{aligned} \tag{3.4.113}$$

Hence, $\bar{\Delta}_{sj,l}$ *is exponentially close to* 0 *in* $\sqrt{\epsilon}$ *for* $s > j$ *and* $l = L, R$.

Lemma 3.4.47. *On* $S_\cap$*, elements from the following matrices are exponentially close to* 0 *in* $\sqrt{\epsilon}$ *in the sense of Definition 3.4.45 :*

$$\begin{aligned} &C_L(\bar{\epsilon}) - \bar{T}_L, \qquad I - \tilde{T}_L, \\ &C_R(\tilde{\epsilon}) - \tilde{T}_R, \qquad I - \bar{T}_R. \end{aligned} \tag{3.4.114}$$

$$\begin{aligned} &C_L(\bar{\epsilon})^{-1} - \bar{T}_L^{-1}, \qquad I - \tilde{T}_L^{-1}, \\ &C_R(\tilde{\epsilon})^{-1} - \tilde{T}_R^{-1}, \qquad I - \bar{T}_R^{-1}, \end{aligned} \tag{3.4.115}$$

$$\begin{aligned} &C_L(\tilde{\epsilon}) - \tilde{D}_L\tilde{T}_L^{-1}\tilde{D}_L^{-1}, \\ &C_R(\bar{\epsilon}) - \bar{D}_R\bar{T}_R^{-1}\bar{D}_R^{-1}, \end{aligned} \tag{3.4.116}$$

and

$$\begin{aligned} &I - \bar{D}_R\bar{T}_L\bar{D}_R^{-1}, \qquad I - \bar{D}_L\bar{T}_L\bar{D}_L^{-1}, \qquad I - \bar{D}_R\bar{T}_L^{-1}\bar{D}_R^{-1} \\ &I - \tilde{D}_R\tilde{T}_R\tilde{D}_R^{-1} \qquad I - \tilde{D}_L\tilde{T}_R\tilde{D}_L^{-1} \qquad I - \tilde{D}_L\tilde{T}_R^{-1}\tilde{D}_L^{-1}. \end{aligned} \tag{3.4.117}$$

Proof. The proof follows from Lemma 3.4.46 and (3.4.83). Relation (3.4.84) is used to obtain (3.4.114). Since $\hat{T}_l^{-1} = \hat{D}_l\hat{T}_l^{-1}\hat{D}_l^{-1}C_l(\hat{\epsilon})^{-1}$, we have, for $i \neq j$,

$$(\hat{T}_l^{-1})_{ij}(1-\Delta_{ij,l}) = (C_l(\hat{\epsilon})^{-1})_{ij} + \sum_{\substack{i<k<j,\, l=R\\ j<k<i,\, l=L}} (\hat{T}_l^{-1})_{ik}(C_l(\hat{\epsilon})^{-1})_{kj}\Delta_{ik,l}, \qquad (3.4.118)$$

Relation (3.4.118) leads to (3.4.115). Since $\hat{D}_l\hat{T}_l^{-1}\hat{D}_l^{-1} = \hat{T}_l^{-1}C_l(\hat{\epsilon})$, we have, for $i \neq j$,

$$(\hat{T}_l^{-1})_{ij}(\Delta_{ij,l}-1) = (C_l(\hat{\epsilon}))_{ij} + \sum_{\substack{i<k<j,\, l=R\\ j<k<i,\, l=L}} (\hat{T}_l^{-1})_{ik}(C_l(\hat{\epsilon}))_{kj}, \qquad (3.4.119)$$

Relations (3.4.119) and (3.4.115) yield (3.4.116). Finally, (3.4.117) follows from (3.4.114) and (3.4.115), using (3.4.12) if necessary. □

Definition 3.4.48. *Let the unfolded Stokes matrices and the formal invariants be given, and let $\hat{T}_l$ be obtained by (3.4.79) from them. Let*

$$\begin{aligned} \tilde{N}_L &= \tilde{D}_L\tilde{T}_L^{-1}\tilde{T}_R\tilde{D}_L^{-1}, & \tilde{N}_R &= \tilde{T}_L^{-1}\tilde{T}_R,\\ \bar{N}_L &= \bar{T}_R^{-1}\bar{T}_L, & \bar{N}_R &= \bar{D}_R\bar{T}_R^{-1}\bar{T}_L\bar{D}_R^{-1}. \end{aligned} \qquad (3.4.120)$$

We call the matrix $\hat{N}_L$ (respectively $\hat{N}_R$) the left *(respectively* right*)* transition invariant. *Note that the equivalence classes of unfolded Stokes matrices induce an equivalence class on the transition invariants.*

Corollary 3.4.49 (of Lemma 3.4.47)**.** *On $S_\cap$, the difference between a left (respectively right) transition invariants and a left (respectively right) unfolded Stokes matrix is exponentially close to 0 in $\sqrt{\epsilon}$ in the sense of Definition 3.4.45, i.e.*

$$\tilde{N}_R - C_R(\tilde{\epsilon}), \quad \bar{N}_L - C_L(\bar{\epsilon}), \quad \tilde{N}_L - C_L(\tilde{\epsilon}), \quad \bar{N}_R - C_R(\bar{\epsilon}). \qquad (3.4.121)$$

Remark 3.4.50. *From Corollary 3.4.49, the diagonal entries of the transition invariants $\hat{N}_l$, $l = L, R$, tend to 1 when $\hat{\epsilon} \to 0$ in $S_\cap$. They are thus always different from zero if the radius ρ of the sector S is sufficiently small.*

Definition 3.4.51. *Let the unfolded Stokes matrices and the formal invariants be given. Let $\hat{T}_l$ as obtained by (3.4.79). We say that the* auto-intersection relation *is satisfied if there exist $Q_U(\bar{\epsilon})$ and $Q_D(\bar{\epsilon})$ nonsingular diagonal matrices depending analytically on $\bar{\epsilon} \in S_\cap$, with a nonsingular limit at $\epsilon = 0$, such that*

$$|Q_i(\bar{\epsilon}) - I| < c_i|\bar{\epsilon}|, \quad c_i \in \mathbb{R},\, \bar{\epsilon} \in S_\cap,\, i = U, D, \qquad (3.4.122)$$

and

$$Q_D(\bar{\epsilon})\bar{D}_R\bar{T}_R^{-1}\bar{T}_L\bar{D}_R^{-1} = \tilde{T}_L^{-1}\tilde{T}_R Q_U(\bar{\epsilon}), \tag{3.4.123}$$

which is equivalent to

$$Q_D(\bar{\epsilon})\bar{N}_l = \tilde{N}_l Q_U(\bar{\epsilon}), \quad l = L, R. \tag{3.4.124}$$

because of (3.4.112) and Definition 3.4.48.

Proposition 3.4.52. *The auto-intersection relation (3.4.124) for the family (3.4.1) is satisfied.*

Proof. We proceed similarly as the proof of Proposition 3.4.43, taking

(a) $W_U(\bar{\epsilon},x)\bar{D}_R\bar{T}_L\bar{D}_R^{-1}$ and $W_U(\tilde{\epsilon},x)\tilde{D}_R\tilde{T}_R\tilde{D}_R^{-1}$ as the fundamental matrices of solutions composed of eigenvectors of $M^*_{\tilde{x}_L}$ (to verify, use (3.4.51), (3.4.68) and (3.4.83)),

(b) $W_D(\bar{\epsilon},x)\bar{T}_R$ and $W_D(\tilde{\epsilon},x)\tilde{T}_L$ as the fundamental matrices of solutions composed of eigenvectors of $M^*_{\bar{x}_R}$,

with $W_s(\epsilon,x)$ given by (3.4.52). By Lemma 3.4.47 and Corollary 3.4.22, these solutions converge uniformly to $W_s(0,x)$ (defined by (3.4.19)) on compact sets of Ω_s^0 when $\bar{\epsilon}\to 0$, $\bar{\epsilon}\in S_\cap$, for $s=D$ or $s=U$. The corresponding transition matrices are here given by

$$E_{L,\tilde{x}_L\to\tilde{x}_R} = \tilde{N}_L e^{2\pi i\Lambda_1(\epsilon)}, \qquad E_{R,\tilde{x}_L\to\tilde{x}_R} = \tilde{N}_R, \tag{3.4.125}$$

$$E_{L,\bar{x}_R\to\bar{x}_L} = \bar{N}_L e^{2\pi i\Lambda_1(\epsilon)}, \qquad E_{R,\bar{x}_R\to\bar{x}_L} = \bar{N}_R, \tag{3.4.126}$$

leading to (3.4.124).

Let us now prove (3.4.122) for $i=D$ (the case $i=U$ is similar). We have obtained the existence of nonsingular diagonal matrices $Q_U(\bar{\epsilon})$ and $Q_D(\bar{\epsilon})$ depending analytically on $\bar{\epsilon}\in S_\cap$, with a nonsingular limit at $\epsilon=0$, such that

$$W_U(\bar{\epsilon},x)\bar{D}_R\bar{T}_L\bar{D}_R^{-1} = W_U(\tilde{\epsilon},x)\tilde{D}_R\tilde{T}_R\tilde{D}_R^{-1}Q_U(\bar{\epsilon}) \tag{3.4.127}$$

and

$$W_D(\bar{\epsilon},x)\bar{T}_R = W_D(\tilde{\epsilon},x)\tilde{T}_L Q_D(\bar{\epsilon}). \tag{3.4.128}$$

Extending the solution $W_D(\bar{\epsilon},x)\bar{T}_R$ (respectively $W_D(\tilde{\epsilon},x)\tilde{T}_L$) to $x=0$ along a path in $\Omega_D^{\bar{\epsilon}}$ (respectively $\Omega_D^{\tilde{\epsilon}}$), we obtain

$$W_D(\bar{\epsilon},0)\bar{T}_R = W_D(\tilde{\epsilon},0)\tilde{T}_L Q_D(\bar{\epsilon})\bar{D}_R \tag{3.4.129}$$

or equivalently, because of (3.4.52),

$$H_D(\bar{\epsilon},0)F_D(\bar{\epsilon},0)\bar{T}_R = H_D(\tilde{\epsilon},0)F_D(\tilde{\epsilon},0)\tilde{T}_L Q_D(\bar{\epsilon})\bar{D}_R. \tag{3.4.130}$$

Since $F_D(\bar{\epsilon},0) = F_D(\tilde{\epsilon},0)\bar{D}_R$, we have

$$H_D(\bar{\epsilon},0)F_D(\bar{\epsilon},0)\bar{T}_R F_D(\bar{\epsilon},0)^{-1} = H_D(\tilde{\epsilon},0)F_D(\tilde{\epsilon},0)\tilde{T}_L F_D(\tilde{\epsilon},0)^{-1} Q_D(\bar{\epsilon}). \tag{3.4.131}$$

$F_D(\bar{\epsilon},0)\bar{T}_R F_D(\bar{\epsilon},0)^{-1}$ and $F_D(\tilde{\epsilon},0)\tilde{T}_L F_D(\tilde{\epsilon},0)^{-1}$ are exponentially close in $\sqrt{\epsilon}$ to I. We use (3.4.53) and (3.4.54) in order to obtain (3.4.122) for $i=D$. □

We now show that the auto-intersection relation implies that there exists a representative of the equivalence class of unfolded Stokes matrices which is $\frac{1}{2}$-summable in ϵ.

Theorem 3.4.53. *There exists a representative of the equivalence class of unfolded Stokes matrices which is $\frac{1}{2}$-summable in ϵ.*

Proof. The strategy consists in using the Ramis-Sibuya Theorem (see for instance [**18**]) : if $C(\hat{\epsilon})$ depends analytically on $\hat{\epsilon}$ on a ramified sector around the origin and if the difference on the auto-intersection of the sector is exponentially close to 0 in $\sqrt{\epsilon}$, i.e. $|C(\bar{\epsilon}) - C(\tilde{\epsilon})| < Be^{-\frac{A}{\sqrt{|\epsilon|}}}$ for some positive A and B, then $C(\epsilon)$ is $\frac{1}{2}$-summable in ϵ.

By Proposition 3.4.52, the auto-intersection relation (3.4.124) is satisfied. Hence,

$$Q_D(\bar{\epsilon})\bar{N}_l = \tilde{N}_l Q_D(\bar{\epsilon})Q(\bar{\epsilon}), \tag{3.4.132}$$

with $Q(\bar{\epsilon}) = Q_D(\bar{\epsilon})^{-1}Q_U(\bar{\epsilon})$. We then have

$$(\bar{N}_l)_{ii} = (\tilde{N}_l)_{ii}(Q(\bar{\epsilon}))_{ii}, \tag{3.4.133}$$

Corollary 3.4.49 says that $\bar{N}_l$ (respectively $\tilde{N}_l$) is exponentially close in $\sqrt{\epsilon}$ to $C_l(\bar{\epsilon})$ (respectively $C_l(\tilde{\epsilon})$). Since the unfolded Stokes matrices has 1's on the diagonal, relation (3.4.133) implies that $Q(\bar{\epsilon})$ is exponentially close (in $\sqrt{\epsilon}$) to I. Let $K(\hat{\epsilon})$ be a nonsingular diagonal matrix depending analytically on $\hat{\epsilon} \in S$, with a nonsingular limit at $\epsilon = 0$, such that $K(\tilde{\epsilon})^{-1}K(\bar{\epsilon}) = Q_D(\bar{\epsilon})$ (recall that (3.4.122) is satisfied). Relation (3.4.132) becomes

$$K(\bar{\epsilon})\bar{N}_l K(\bar{\epsilon})^{-1} = K(\tilde{\epsilon})\tilde{N}_l K(\tilde{\epsilon})^{-1} Q(\bar{\epsilon}), \tag{3.4.134}$$

since $Q(\bar{\epsilon})$ is diagonal, and hence commutes with $K(\bar{\epsilon})$. Let us take the representative of the equivalence class of unfolded Stokes matrices $C_l'(\hat{\epsilon}) = K(\hat{\epsilon})C_l(\hat{\epsilon})K^{-1}(\hat{\epsilon})$. Using Corollary 3.4.49 with

$$N_l'(\hat{\epsilon}) = K(\hat{\epsilon})N_l(\hat{\epsilon})K^{-1}(\hat{\epsilon}), \tag{3.4.135}$$

we obtain that $\bar{N}_l'$ (respectively $\tilde{N}_l'$) is exponentially close to $C_l'(\bar{\epsilon})$ (respectively $C_l'(\tilde{\epsilon})$). On the other hand, relation (3.4.134) implies

$$\bar{N}_l' = \tilde{N}_l' Q(\bar{\epsilon}) \tag{3.4.136}$$

with $Q(\bar{\epsilon})$ exponentially close in $\sqrt{\epsilon}$ to I. The difference between the representatives $C_l'(\bar{\epsilon})$ and $C_l'(\tilde{\epsilon})$ is hence exponentially close to 0 in $\sqrt{\epsilon}$, for $l = L, R$. □

Remark 3.4.54. *In dimension $n = 2$, it is always possible to choose an analytic representative of the equivalence classes of unfolded Stokes matrices. All the cases have been enumerated in* [**3**]. *Indeed, in the case of nonvanishing elements $(C_L(\hat{\epsilon}))_{21}$ and $(C_R(\hat{\epsilon}))_{12}$, the auto-intersection relation is equivalent to the analyticity of the product $(C_L(\hat{\epsilon}))_{21}(C_R(\hat{\epsilon}))_{12}$. Preliminary investigation in the case $n = 3$ shows that this could not be the case generically. We study this in more details in* [**13**].

3.4.11. Unfolded Stokes matrices reducible in block diagonal form

We will now state a sufficient condition for the decomposition of a system (3.4.1) in dimension n as the direct product of irreducible systems of lower dimension (this may require a permutation), using the following lemma.

Lemma 3.4.55. *For $\hat{\epsilon} \in S_\cap$, the matrix $P^{-1}C_L(\hat{\epsilon})P$ (respectively $P^{-1}C_R(\hat{\epsilon})P$), with a permutation matrix P, is lower (respectively upper) triangular, unipotent and in a block diagonal form if and only if $P^{-1}\hat{T}_L P$ (respectively $P^{-1}\hat{T}_R P$) has the same form.*

$C_R(\bar{\epsilon})$ and $C_L(\bar{\epsilon})$ have a common block diagonal form with the same permutation matrix P (when staying triangular) if and only if $C_R(\tilde{\epsilon})$ and $C_L(\tilde{\epsilon})$ have the same block diagonal form with the same permutation matrix P (and stay triangular).

Proof. The first assertion comes from the fact that columns of $P^{-1}\hat{T}_l P$ are eigenvectors of $P^{-1}C_l(\hat{\epsilon})\hat{D}_l P$ (note that there are no resonances for $\hat{\epsilon}$ in $S_\cap$). Let us prove the converse. $P^{-1}\hat{T}_l P$ is unipotent, triangular and in a block diagonal form if and only if $P^{-1}\hat{D}_l\hat{T}_l\hat{D}_l^{-1}P$ has the same structure with the same permutation matrix P. Then, the product $(P^{-1}\hat{T}_l P)(P^{-1}\hat{D}_l\hat{T}_l\hat{D}_l^{-1}P)^{-1} = P^{-1}C_l(\hat{\epsilon})P$ (by (3.4.83)) has the desired property.

The second assertion follows directly from (3.4.123) and from the first assertion. □

Theorem 3.4.56. *Let us take any family of systems (3.4.1) with both unfolded Stokes matrices admitting, after conjugation (if necessary) by the same permutation matrix P preserving their triangular form, the same decomposition in diagonal blocks for all $\hat{\epsilon} \in S$ (i.e. $P^{-1}C_l(\hat{\epsilon})P = c^l_{n_1} \oplus c^l_{n_2} \oplus ... \oplus c^l_{n_k}$ for $l = L, R$, with $n_1 + n_2 + ... + n_k = n$). This family of systems is analytically equivalent (with permutation P) to the direct product of families of systems.*

Proof. First, let us take a system (3.4.1) which has unfolded Stokes matrices in block diagonal form with the same positions of the blocks : $C_l(\hat{\epsilon}) = c^l_{n_1}(\hat{\epsilon}) \oplus c^l_{n_2}(\hat{\epsilon}) \oplus ... \oplus c^l_{n_k}(\hat{\epsilon})$ for $l = L, R$, with $n_1 + n_2 + ... + n_k = n$. We will prove that this system is analytically equivalent to a direct product of smaller systems of dimensions n_1, ..., n_k. Looking at (3.4.68), we notice that these relations would still hold if we replace by zero each element $(H_s(\hat{\epsilon}, x))_{ij}$ such that the position (i, j) is outside the diagonal blocks of $C_l(\hat{\epsilon})$. This leads us to define $J_s(\hat{\epsilon}, x)$, for

$x \in \Omega_s^{\hat{\epsilon}}$, by

$$(J_s(\hat{\epsilon}, x))_{ij} = \begin{cases} 0, & \text{if } (C_l(\hat{\epsilon}))_{ij} \text{ is outside the diagonal blocks,} \\ (H_s(\hat{\epsilon}, x))_{ij}, & \text{otherwise.} \end{cases} \tag{3.4.137}$$

$J_s(\hat{\epsilon}, x)$ is in block diagonal form $J_{s,n_1}(\hat{\epsilon}, x) \oplus J_{s,n_2}(\hat{\epsilon}, x) \oplus ... \oplus J_{s,n_k}(\hat{\epsilon}, x)$ and it follows from (3.4.55) that it is invertible. From (3.4.68), we have

$$J_D(\hat{\epsilon}, x)^{-1} J_U(\hat{\epsilon}, x) = \begin{cases} F_D(\hat{\epsilon}, x) C_R(\hat{\epsilon}) (F_D(\hat{\epsilon}, x))^{-1}, & \text{on } \Omega_R^{\hat{\epsilon}}, \\ F_D(\hat{\epsilon}, x) C_L(\hat{\epsilon}) (F_D(\hat{\epsilon}, x))^{-1}, & \text{on } \Omega_L^{\hat{\epsilon}}, \\ I, & \text{on } \Omega_C^{\hat{\epsilon}}. \end{cases} \tag{3.4.138}$$

These relations imply that the transformation

$$\mathcal{Q}(\hat{\epsilon}, x) = \begin{cases} J_D(\hat{\epsilon}, x) H_D(\hat{\epsilon}, x)^{-1}, & x \in \Omega_D^{\hat{\epsilon}}, \\ J_U(\hat{\epsilon}, x) H_U(\hat{\epsilon}, x)^{-1}, & x \in \Omega_U^{\hat{\epsilon}}, \end{cases} \tag{3.4.139}$$

is well-defined on the intersections of the domains and is an analytic function of x in a whole neighborhood of $x = 0$, including the points $\hat{x}_R$ and $\hat{x}_L$. We will now prove that $\mathcal{Q}(\hat{\epsilon}, x)$ is unramified in ϵ. Since it is bounded at $\epsilon = 0$, this will imply the analyticity of $\mathcal{Q}(\epsilon, x)$ at $\epsilon = 0$. To prove that

$$\mathcal{Q}(\tilde{\epsilon}, x) = \mathcal{Q}(\bar{\epsilon}, x), \tag{3.4.140}$$

i.e.

$$J_s(\tilde{\epsilon}, x)^{-1} J_s(\bar{\epsilon}, x) = H_s(\tilde{\epsilon}, x)^{-1} H_s(\bar{\epsilon}, x), \quad s \in \{1, 2\}, \tag{3.4.141}$$

we will consider $x \in \Omega_C^{\tilde{\epsilon}} \cap \Omega_C^{\bar{\epsilon}}$. In this region, we have $J_U(\hat{\epsilon}, x) = J_D(\hat{\epsilon}, x)$ and $H_U(\hat{\epsilon}, x) = H_D(\hat{\epsilon}, x)$. By uniqueness of the Floquet solutions (Theorem 3.4.33), we have

$$H_D(\bar{\epsilon}, x) F_D(\bar{\epsilon}, x) \bar{T}_R K = H_D(\tilde{\epsilon}, x) F_D(\tilde{\epsilon}, x) \tilde{T}_L \tag{3.4.142}$$

with K a nonsingular diagonal matrix. Hence,

$$H_D(\bar{\epsilon}, x) F_D(\bar{\epsilon}, x) = H_D(\tilde{\epsilon}, x) F_D(\tilde{\epsilon}, x) Z, \tag{3.4.143}$$

with $Z = \tilde{T}_L K^{-1} \bar{T}_R^{-1}$. By Lemma 3.4.55, Z is in the block diagonal form $Z_{n_1} \oplus Z_{n_2} \oplus ... \oplus Z_{n_k}$. By definition of $J_D(\hat{\epsilon}, x)$, we have

$$J_D(\bar{\epsilon}, x) F_D(\bar{\epsilon}, x) = J_D(\tilde{\epsilon}, x) F_D(\tilde{\epsilon}, x) Z. \tag{3.4.144}$$

Relations (3.4.143) and (3.4.144) yield (3.4.141). Finally, $\lim_{\epsilon \to 0} \mathcal{Q}(\epsilon, x)$ is bounded, so $\mathcal{Q}(\epsilon, x)$ is an analytic function of (ϵ, x) in a whole neighborhood of $(0, 0)$. The transformation $v = \mathcal{Q}(\epsilon, x)y$ gives a system with the fundamental matrix of solutions $J_s(\hat{\epsilon}, x) F_s(\hat{\epsilon}, x)$ on $\Omega_s^{\hat{\epsilon}}$, and hence with the matrix in block diagonal form $B(\epsilon, x) = B_{n_1}(\epsilon, x) \oplus B_{n_2}(\epsilon, x) \oplus ... \oplus B_{n_k}(\epsilon, x)$.

Finally, let us take a system (3.4.1) in which the unfolded Stokes matrices conjugated by a permutation matrix have the same decomposition in diagonal blocks. We apply the previous result to the system transformed by $y \mapsto Py$. □

3.4.12. Unfolded Stokes matrices with trivial rows or column

We include here the study of the cases when both unfolded Stokes matrices have a trivial row or column (see Notation 3.4.31). When this happens, the system is analytically equivalent to a simpler one. This section is not a prerequisite to obtain the complete system of invariants of the systems (3.4.1).

Lemma 3.4.57. *For $\hat{\epsilon}$ in $S_\cap$ and $j \in \{1, 2, ..., n\}$, the following properties are equivalent, and they are satisfied for $\bar{\epsilon}$ if and only if they are satisfied for $\tilde{\epsilon}$:*

(1) *the j^{th} solution that is eigenvector of the monodromy operator around $x = \hat{x}_R$ is a multiple of the j^{th} solution that is eigenvector of the monodromy operator around $x = \hat{x}_L$;*

(2) *the j^{th} column of the transition invariants $\hat{N}_L$ and $\hat{N}_R$ (Definition 3.4.48) is trivial;*

(3) *the j^{th} columns of $\hat{T}_R$ and $\hat{T}_L$ are trivial;*

(4) *the j^{th} columns of $C_R(\hat{\epsilon})$ and $C_L(\hat{\epsilon})$ are trivial;*

(5) *the solution $w_j(\hat{\epsilon}, x)$, corresponding to the j^{th} column of $W_V(\hat{\epsilon}, x)$ given by (3.4.76), is eigenvector of the monodromy around both singular points.*

Proof. It follows from (3.4.124). □

Theorem 3.4.58. *A family of systems (3.4.1) with both unfolded Stokes matrices having the j^{th} column trivial for all $\hat{\epsilon} \in S$ is analytically equivalent to a family of system (3.4.1) with an invariant subsystem formed by the equations $i \neq j$ (i.e. the (i, j) entries are null for all $i \neq j$).*

Proof. We follow the same steps as in the proof of Theorem 3.4.56, considering the j^{th} column with nondiagonal elements null (instead of null elements outside diagonal blocks in Theorem 3.4.56), and taking a different definition of $J_s(\hat{\epsilon}, x)$. We take $J_s(\hat{\epsilon}, x) = \mathcal{Q}_s(\hat{\epsilon}, x)H_s(\hat{\epsilon}, x)$, with

$$(\mathcal{Q}_s(\hat{\epsilon}, x))_{ik} = \begin{cases} 1, & \text{if } i = k, \\ \frac{-(H_s(\hat{\epsilon},x))_{ij}}{(H_s(\hat{\epsilon},x))_{jj}}, & \text{if } k = j,\ i \neq k, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4.145}$$

The j^{th} column of $J_s(\hat{\epsilon}, x)$ then has zero nondiagonal elements. The rest follows as in the proof of Theorem 3.4.56, using Lemma 3.4.57 instead of Lemma 3.4.55 (and forgetting about the last part of the proof about the permutation of the y-coordinates). □

Lemma 3.4.59. *For $\hat{\epsilon}$ in $S_\cap$ and $j \in \{1, 2, ..., n\}$, the following properties are equivalent, and they are satisfied for $\bar{\epsilon}$ if and only if they are also for $\tilde{\epsilon}$:*

(1) the j^{th} row of the transition invariants $\hat{N}_L$ and $\hat{N}_R$ is trivial;

(2) the j^{th} rows of $\hat{T}_R$ and $\hat{T}_L$ are trivial;

(3) the j^{th} rows of $C_R(\hat{\epsilon})$ and $C_L(\hat{\epsilon})$ are trivial.

Hence, in the j^{th} Riccati system, the property of a first integral $\mathcal{H}_q^j$ to be an eigenvector of the monodromy around both singular points is conserved in both points of view $\bar{\epsilon}$ and $\tilde{\epsilon}$.

Proof. The first part follows from (3.4.124). The last part comes from Corollary 3.4.39 : a first integral $\mathcal{H}_q^j$ is eigenvector of the monodromy around both singular points if and only if rows q and j of the inverse of two unfolded Stokes matrices are trivial. □

Theorem 3.4.60. *A family of systems (3.4.1) with both unfolded Stokes matrices having the j^{th} row trivial for all $\hat{\epsilon} \in S$ is analytically equivalent to a family of system (3.4.1) where the j^{th} equation is independent of the others, hence integrable (i.e. the (j,i) entries are null for all $i \neq j$).*

Proof. The proof of the analytic equivalence (to a system having (j,i) entries null for all $i \neq j$ with j fixed) is very similar to the proof of Theorem 3.4.56, considering the j^{th} row with nondiagonal elements null (instead of null elements outside diagonal blocks in Theorem 3.4.56), and taking a different definition of $J_s(\hat{\epsilon}, x)$, namely

$$(J_s(\hat{\epsilon}, x))_{ik} = \begin{cases} (H_s(\hat{\epsilon}, x))_{ik}, & i \neq j, \\ 0, & i = j, \neq j \\ 1, & i = j = k. \end{cases} \tag{3.4.146}$$

We then follow the proof of Theorem 3.4.56, using Lemma 3.4.59 instead of Lemma 3.4.55 and forgetting about the last section of the proof that concerns permutation. □

3.4.13. Analytic invariants

We now have the tools to prove that the equivalent unfolded Stokes collections are analytic invariants for the classification of the systems (3.4.1).

Theorem 3.4.61. *Two families of systems of the form (3.4.1) with the same model system (3.4.6) are analytically equivalent if and only if their unfolded Stokes collections are equivalent. In particular, a family (3.4.1) is analytically equivalent to its model if and only if the unfolded Stokes collection is trivial.*

Proof. We consider two systems of the form (3.4.1) :

$$(x^2 - \epsilon)y_i' = B_i(\epsilon, x)y_i, \tag{3.4.147}$$

with

$$B_i(\epsilon,x)=\Lambda(\epsilon,x)+(x^2-\epsilon)R_i(\epsilon,x),\quad i=1,2, \tag{3.4.148}$$

and $\Lambda(\epsilon,x)$ given by (3.4.3). We choose a neighborhood of the origin $\mathbb{D}_r$ common to the two systems for which the modulus is defined. We denote the fundamental matrix of solutions of (3.4.147) given by Theorem 3.4.21 as $H_{i,s}(\hat{\epsilon},x)F_s(\hat{\epsilon},x)$ (for $(\hat{\epsilon},x)\in S\times\Omega_s^{\hat{\epsilon}}$, $s=D,U$).

Let us suppose that these two systems are analytically equivalent via a transformation $y_2=\mathcal{Q}(\epsilon,x)y_1$. By Proposition 3.4.26, we must have

$$H_{2,s}(\hat{\epsilon},x)=\mathcal{Q}(\epsilon,x)H_{1,s}(\hat{\epsilon},x)K(\hat{\epsilon})\quad \text{on } \Omega_s^{\hat{\epsilon}},\quad s=D,U, \tag{3.4.149}$$

with $K(\hat{\epsilon})$ a nonsingular diagonal matrix depending analytically on $\hat{\epsilon}\in S$ with a nonsingular limit at $\epsilon=0$ and such that (3.4.70) is satisfied. Then, on the intersections of $\Omega_D^{\hat{\epsilon}}$ and $\Omega_U^{\hat{\epsilon}}$, we have

$$(H_{2,D}(\hat{\epsilon},x))^{-1}H_{2,U}(\hat{\epsilon},x)=K(\hat{\epsilon})^{-1}(H_{1,D}(\hat{\epsilon},x))^{-1}H_{1,U}(\hat{\epsilon},x)K(\hat{\epsilon}). \tag{3.4.150}$$

This implies that the unfolded Stokes collections given by (3.4.68) are equivalent.

Let us prove the other direction. Let us suppose that the two systems above have equivalent Stokes collections $\{C_R^i(\hat{\epsilon}),C_L^i(\hat{\epsilon})\}$ with a matrix $K(\hat{\epsilon})$ as in Definition 3.4.27, i.e.

$$C_l^2(\hat{\epsilon})=K(\hat{\epsilon})C_l^1(\hat{\epsilon})K(\hat{\epsilon})^{-1},\quad l=L,R. \tag{3.4.151}$$

By taking, for the second system, an adequate normalization of the fundamental matrix of solutions (namely changing from $H_{2,s}(\hat{\epsilon},x)F_s(\hat{\epsilon},x)$ to $H_{2,s}(\hat{\epsilon},x)F_s(\hat{\epsilon},x)K(\hat{\epsilon})$, $s=D,U$), we can, without loss of generality, suppose that

$$C_l^2(\hat{\epsilon})=C_l^1(\hat{\epsilon}),\quad l=L,R. \tag{3.4.152}$$

First, let us suppose that the unfolded Stokes matrices $C_R^i(\hat{\epsilon})$ and $C_L^i(\hat{\epsilon})$ cannot have a block diagonal form (for all $\hat{\epsilon}\in S$) with the same positions of the blocks, neither after conjugation of each of them by the same permutation matrix (that keeps their triangular form). We take

$$\mathcal{Q}(\hat{\epsilon},x)=\begin{cases}H_{2,D}(\hat{\epsilon},x)(H_{1,D}(\hat{\epsilon},x))^{-1}, & \text{on } \Omega_D^{\hat{\epsilon}},\\ H_{2,U}(\hat{\epsilon},x)(H_{1,U}(\hat{\epsilon},x))^{-1}, & \text{on } \Omega_U^{\hat{\epsilon}},\end{cases} \tag{3.4.153}$$

which is well-defined because of (3.4.152). Since $\lim_{x\to\hat{x}_l}\mathcal{Q}(\hat{\epsilon},x)$ is bounded, invertible and independent of s for $l=L,R$ (see (3.4.55)), $\mathcal{Q}(\hat{\epsilon},x)$ is an analytic function of x on the whole neighborhood $\mathbb{D}_r$ of $x=0$ which includes the points $\hat{x}_R$ and $\hat{x}_L$, for $\hat{\epsilon}\in S$. We will now choose carefully $\eta(\hat{\epsilon})$ such that $\eta(\hat{\epsilon})\mathcal{Q}(\hat{\epsilon},x)$ becomes analytic at $\epsilon=0$. We will prove that $\eta(\hat{\epsilon})\mathcal{Q}(\hat{\epsilon},x)$ is uniform in ϵ and bounded near $\epsilon=0$. The transformation $\mathcal{Q}(\bar{\epsilon},x)\mathcal{Q}(\tilde{\epsilon},x)^{-1}$ is an automorphism of the second family of systems (3.4.147). Hence, over each domain $\Omega_s^{\bar{\epsilon}}$, $s=D,U$, we have the following automorphism of the model

$$(H_{2,s}(\bar{\epsilon},x))^{-1}\mathcal{Q}(\bar{\epsilon},x)\mathcal{Q}(\tilde{\epsilon},x)^{-1}H_{2,s}(\bar{\epsilon},x)=D_s(\bar{\epsilon}),\tag{3.4.154}$$

giving $D_s(\bar{\epsilon})$ a diagonal matrix depending on $\bar{\epsilon}$. With relations (3.4.68) applied to the second system, (3.4.154) leads to

$$C_l^2(\bar{\epsilon})D_U(\bar{\epsilon})=D_D(\bar{\epsilon})C_l^2(\bar{\epsilon}),\quad l\in\{R,L\}.\tag{3.4.155}$$

As the diagonal entries of $C_l(\bar{\epsilon})$ are 1's, we have $D_U(\bar{\epsilon})=D_D(\bar{\epsilon})$. The hypothesis that the Stokes matrices have no common reduction to block diagonal form (neither after conjugation by a permutation matrix that keeps their triangular form) implies that this relation can only be satisfied for $D_U(\bar{\epsilon})=\mu(\bar{\epsilon})I$ for some $\mu(\bar{\epsilon})$ analytic function over $S_\cap$. Relation (3.4.154) becomes

$$\mathcal{Q}(\bar{\epsilon},x)\mathcal{Q}(\tilde{\epsilon},x)^{-1}=\mu(\bar{\epsilon})I.\tag{3.4.156}$$

In particular,

$$\mathcal{Q}(\bar{\epsilon},0)\mathcal{Q}(\tilde{\epsilon},0)^{-1}=\mu(\bar{\epsilon})I.\tag{3.4.157}$$

Using properties (3.4.53) and (3.4.54)(which remained valid when we modified $H_{2,s}(\hat{\epsilon},x)$ to $H_{2,s}(\hat{\epsilon},x)K(\hat{\epsilon})$, $s=D,U$), the definition (3.4.153) implies there exists $C\in\mathbb{R}_+$ such that

$$|\mathcal{Q}(\bar{\epsilon},0)\mathcal{Q}(\tilde{\epsilon},0)^{-1}-I|\le C|\epsilon|,\quad \bar{\epsilon}\in S_\cap.\tag{3.4.158}$$

Relation (3.4.157) and (3.4.158) imply there exists $c\in\mathbb{R}_+$ such that

$$|\mu(\bar{\epsilon})-1|\le c|\epsilon|,\quad \bar{\epsilon}\in S_\cap.\tag{3.4.159}$$

Reducing slightly the radius ρ of S and its opening, let $\eta(\hat{\epsilon})$ be an analytic function of $\hat{\epsilon}$ on S satisfying

$$\eta(\bar{\epsilon})^{-1}\eta(\tilde{\epsilon}) = \mu(\bar{\epsilon}). \tag{3.4.160}$$

Of course, such a function can be found with $\lim_{\hat{\epsilon}\to 0}\eta(\hat{\epsilon}) = 1$. Let $\mathcal{Q}^*(\hat{\epsilon},x) = \eta(\hat{\epsilon})\mathcal{Q}(\hat{\epsilon},x)$. From (3.4.156) and (3.4.160), we get

$$\mathcal{Q}^*(\bar{\epsilon},x) = \mathcal{Q}^*(\tilde{\epsilon},x). \tag{3.4.161}$$

Then,

$$\lim_{\epsilon\to 0}\mathcal{Q}^*(\epsilon,x) = H_{2,s}(0,x)(H_{1,s}(0,x))^{-1}, \quad x\in\Omega_s^0,\ s=D,U, \tag{3.4.162}$$

which is finite, so $\mathcal{Q}^*(\epsilon,x)$ is analytic in ϵ at $\epsilon=0$. Hence, $\mathcal{Q}^*(\epsilon,x)$ analytically conjugates the two systems.

Finally, let us suppose that both unfolded Stokes matrices of each system admit, after conjugation if necessary by the same permutation matrix that keeps their triangular form, the same maximal decomposition in diagonal blocks for all $\hat{\epsilon}\in S$. By Theorem 3.4.56, each system is analytically equivalent (with permutation P) to a system decomposed in smaller indecomposable systems. The decomposed systems have equivalent unfolded Stokes collections and the smaller indecomposable systems too. By applying the former argument to each pair of indecomposable systems, we find that they are analytically equivalent. Hence, the two decomposed systems are analytically equivalent, and so are the initial systems. □

3.5. Realization of the analytic invariants

By Section 3.4, the complete system of analytic invariants for the systems (3.4.1) consists of the formal invariants (the model system) and an equivalence class of unfolded Stokes matrices. In this section, we give the realization theorem for these invariants by proceeding in two steps. First, we consider the local realization :

Theorem 3.5.1. *Let a complete system of analytic invariants be given :*

- *a model system (i.e. formal invariants $\lambda_{j,q}(\epsilon)$, $j=1,2,...,n$, $q=0,1$, depending analytically on ϵ at the origin),*

- *an equivalence class (see Definition 3.4.27) of unfolded Stokes matrices $C_R(\hat{\epsilon})$ and $C_L(\hat{\epsilon})$, which are respectively an upper triangular and a lower triangular unipotent matrix depending analytically on $\hat{\epsilon} \in S$ and having a bounded limit when $\hat{\epsilon} \to 0$ on S (the sector S of radius ρ_0 and of opening $2\pi + \gamma_0$ is chosen from the formal invariants as in Section 3.4.3, and ρ_0 can obviously be chosen smaller to ensure the analyticity, over S, of the entries of $C_R(\hat{\epsilon})$ and $C_L(\hat{\epsilon})$).*

Then, there exist $r > 0$, a radius $\rho < \min\{\rho_0, \frac{r^2}{2}\}$ of S and a system $(x^2 - \epsilon)y' = A(\hat{\epsilon}, x)y$ ($y \in \mathbb{C}^n$) characterized by these analytic invariants, with $A(\hat{\epsilon}, x)$ analytic over $S \times \mathbb{D}_r$. The limit of $A(\hat{\epsilon}, x)$ when $\hat{\epsilon} \to 0$ ($\hat{\epsilon} \in S$) is analytic in x over $\mathbb{D}_r$.

We prove Theorem 3.5.1 from Sections 3.5.1 to 3.5.5. Then, we show that the auto-intersection relation (3.4.123) is sufficient for the global realization of the analytic invariants, i.e. :

Theorem 3.5.2. *Let a complete system of analytic invariants as described in Theorem 3.5.1 be given and satisfying the auto-intersection relation (3.4.123). Then, there exist $r > 0$, a radius $\rho < \min\{\rho_0, \frac{r^2}{2}\}$ of S and a system $(x^2 - \epsilon)y' = B(\epsilon, x)y$ ($y \in \mathbb{C}^n$) characterized by these analytic invariants, with $B(\epsilon, x)$ analytic over $\mathbb{D}_\rho \times \mathbb{D}_r$.*

The proof of Theorem 3.5.2 is presented from Sections 3.5.6 to 3.5.9. It uses the ramified system constructed in the proof of Theorem 3.5.1. The auto-intersection relation (3.4.123) will be the key ingredient to prove Theorem 3.5.2, namely to correct the family to a uniform family. It will guarantee the triviality of the abstract vector bundle realizing the family of Stokes matrices.

3.5.1. Introduction to the proof of Theorem 3.5.1

Considering $\hat{\epsilon}$ fixed, we realize the invariants on an abstract vector bundle which we then show to be trivial. For this, using ideas from the proof of the realization theorem at $\epsilon = 0$ in **[22]** (p. 150) and from the proof of Cartan's Lemma in **[8]** (p. 199), we will prove that, for $s = D, U$ and sufficiently small radii ρ of S and r of $\Omega_s^{\hat{\epsilon}}$, there exist matrices $H_s(\hat{\epsilon}, x)$ depending analytically on $(\hat{\epsilon}, x) \in S \times \Omega_s^{\hat{\epsilon}}$, having a limit when $\hat{\epsilon} \to 0$ in S that is analytic in x over Ω_s^0, and such that, for $\hat{\epsilon} \in S \cup \{0\}$,

$$H_D(\hat{\epsilon}, x)^{-1} H_U(\hat{\epsilon}, x) = I + Z(\hat{\epsilon}, x), \quad x \in \Omega_U^{\hat{\epsilon}} \cap \Omega_D^{\hat{\epsilon}}, \tag{3.5.1}$$

where

$$Z(\hat{\epsilon}, x) = \begin{cases} F_D(\hat{\epsilon}, x) C_R(\hat{\epsilon}) F_D(\hat{\epsilon}, x)^{-1} - I & \text{on } \Omega_R^{\hat{\epsilon}}, \\ F_D(\hat{\epsilon}, x) C_L(\hat{\epsilon}) F_D(\hat{\epsilon}, x)^{-1} - I & \text{on } \Omega_L^{\hat{\epsilon}}, \\ 0 & \text{on } \Omega_C^{\hat{\epsilon}}, \end{cases} \tag{3.5.2}$$

with $F_s(\hat{\epsilon}, x)$ a fundamental matrix of solutions of the model system (as in Notation 3.4.19) which is completely determined by the given formal invariants.

Then, we consider

$$W_s(\hat{\epsilon}, x) = H_s(\hat{\epsilon}, x) F_s(\hat{\epsilon}, x), \quad (\hat{\epsilon}, x) \in (S \cup \{0\}) \times \Omega_s^{\hat{\epsilon}}, \ s = D, U. \tag{3.5.3}$$

Relations (3.5.1) implies that

$$W_D'(\hat{\epsilon}, x) W_D(\hat{\epsilon}, x)^{-1} = W_U'(\hat{\epsilon}, x) W_U(\hat{\epsilon}, x)^{-1}, \quad \text{on } \Omega_\cap^{\hat{\epsilon}}, \ \hat{\epsilon} \in (S \cup \{0\}), \tag{3.5.4}$$

so that

$$A(\hat{\epsilon}, x) = \begin{cases} (x^2 - \epsilon) W_D'(\hat{\epsilon}, x) W_D(\hat{\epsilon}, x)^{-1}, & \text{on } \Omega_D^{\hat{\epsilon}}, \\ (x^2 - \epsilon) W_U'(\hat{\epsilon}, x) W_U(\hat{\epsilon}, x)^{-1}, & \text{on } \Omega_U^{\hat{\epsilon}}, \end{cases} \tag{3.5.5}$$

is well-defined and hence analytic over $(\mathbb{D}_r \backslash \{\hat{x}_R, \hat{x}_L\}) \times (S \cup \{0\})$.

We will prove the boundedness of $H_s(\hat{\epsilon}, x)$, $H_s(\hat{\epsilon}, x)^{-1}$ and $H_s'(\hat{\epsilon}, x)$ near $x = \hat{x}_l$, for $\hat{\epsilon} \in (S \cup \{0\})$, $s = D, U$ and $l = L, R$. This implies that $A(\hat{\epsilon}, x)$ is analytic over $S \times \mathbb{D}_r$ and has a limit when $\hat{\epsilon} \to 0$ (with $\hat{\epsilon} \in S$) that is analytic in x over $\mathbb{D}_r$, since

$$\begin{aligned} &(x^2 - \epsilon) W_s'(\hat{\epsilon}, x) W_s(\hat{\epsilon}, x)^{-1} \\ &\quad = (x^2 - \epsilon) H_s'(\hat{\epsilon}, x) H_s(\hat{\epsilon}, x)^{-1} + H_s(\hat{\epsilon}, x) \Lambda(\epsilon, x) H_s(\hat{\epsilon}, x)^{-1}. \end{aligned} \tag{3.5.6}$$

$H_s(\hat{\epsilon}, x)$ will be obtained in Section 3.5.5 from a specific sequence of matrices constructed in Section 3.5.4. This proof needs adequate choice of radii r of $\Omega_s^{\hat{\epsilon}}$ and ρ of S.

3.5.2. Choice of the radius r for the domains in the x-variable

First, we choose r by considering the case $\epsilon = 0$. For $r > 0$, let us take Ω_D and Ω_U as in Definition 3.2.2 (Figure 3.1) and let

$$\begin{aligned}\Omega_{D,\beta} &= \{x \in \mathbb{C} : |x| < r, -(\pi+\delta+\beta) < \arg(x) < \delta+\beta\},\\ \Omega_{U,\beta} &= \{x \in \mathbb{C} : |x| < r, -(\delta+\beta) < \arg(x) < \pi+\delta+\beta\},\end{aligned} \tag{3.5.7}$$

with $\beta > 0$ sufficiently small so that the closure of $\Omega_{s,\beta}$ does not contain more separation rays (Definition 3.2.1) than Ω_s, $s = D, U$ (Figure 3.18). From these domains, we define domains having their part of the boundary other than the part $\{|x| = r\}$ included in some solution curves of the system $\dot{x} = x^2 - \epsilon$ allowing complex time. The procedure explained in Section 3.4.4 yields Ω_s^0 (respectively $\Omega_{s,\beta}^0$) included in Ω_s (respectively $\Omega_{s,\beta}$), for $s = D, U$ (Figure 3.18). In the course of the proof, for domains denoted by the letter Ω, we use the notation

$$\Omega_\cap = \Omega_U \cap \Omega_D = \Omega_L \cup \Omega_C \cup \Omega_R. \tag{3.5.8}$$

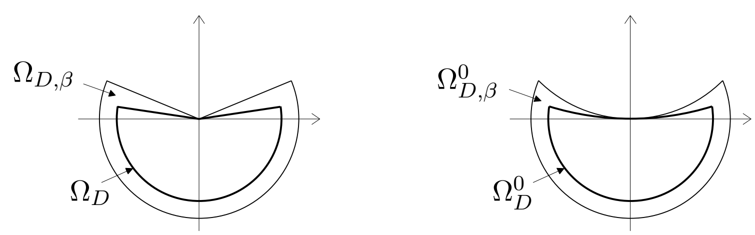

FIG. 3.18. Sectorial domains Ω_D, $\Omega_{D,\beta}$, Ω_D^0 and $\Omega_{D,\beta}^0$.

We now define domains $\Omega_s^0(\nu)$ included in $\Omega_{s,\beta}^0$ and converging when $\nu \to \infty$ to Ω_s^0. In the t-variable (see Section 3.4.4), let us define the neighborhoods $\Gamma_s^0(\nu)$ (Figure 3.19) of Γ_s^0 (which is the domain corresponding to Ω_s^0 in the t-variable) :

$$\Gamma_s^0(\nu) = \{z : \exists t \in \Gamma_s^0 \text{ s.t. } |z-t|\frac{|z|}{|t|} < 2^{-\nu}\theta\}, \quad \nu \geq 1,\ s = D, U. \tag{3.5.9}$$

We choose $\theta > 0$ such that $\Gamma_s^0(1)$ is included in $\Gamma_{s,\beta}^0$ (which is the domain corresponding to $\Omega_{s,\beta}^0$ in the t-variable). In the x-variable, the domains $\Gamma_s^0(\nu)$

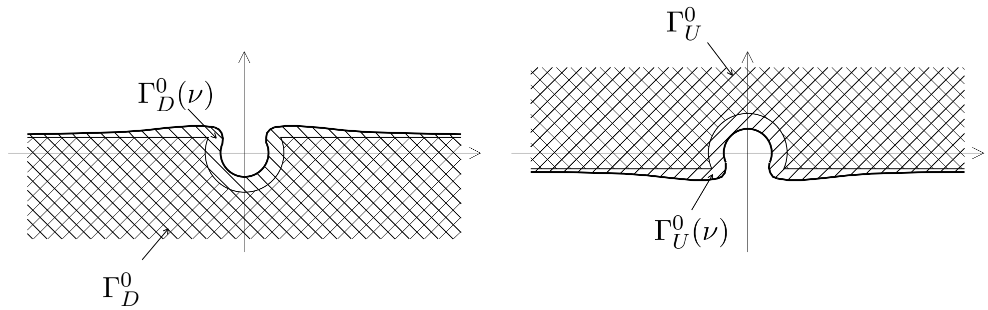

FIG. 3.19. A neighborhood $\Gamma^0_s(\nu)$ of Γ^0_s, $s = D, U$.

correspond to

$$\Omega^0_s(\nu) = \{y : \exists x \in \Omega^0_s \text{ s.t. } |y - x| < 2^{-\nu}\theta|y|^2\}, \quad \nu \geq 0,\ s = D, U. \tag{3.5.10}$$

As illustrated in Figure 3.20, we write the boundary of $\Omega^0_\cap(\nu) = \Omega^0_U(\nu) \cap \Omega^0_D(\nu)$ as

$$\partial\Omega^0_\cap(\nu) = \gamma^0_{\nu,U} \cup \gamma^0_{\nu,D}, \tag{3.5.11}$$

denoting $\gamma^0_{\nu,s} \subset \partial\Omega^0_\cap(\nu)$ the path included in the boundary of $\Omega^0_s(\nu)$, $s = D, U$ starting at $x = -r$ and ending at $x = r$.

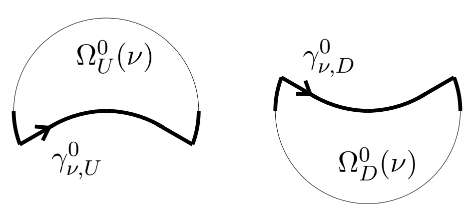

FIG. 3.20. Integration path $\gamma^0_{\nu,s} \subset \partial\Omega^0_s(\nu)$, $s = D, U$.

Asymptotic properties of $Z(0, x)$ imply that $\forall N \in \mathbb{N}^*$ there exists $K^0_N \in \mathbb{R}_+$ such that

$$|Z(0, x)| \leq K^0_N |x|^N, \quad x \in \Omega^0_l(\theta),\ l = L, R. \tag{3.5.12}$$

We take r sufficiently small so that the length of each path $\gamma^0_{\nu,s}$ is bounded by a constant c^0_s such that

$$\int_{\gamma^0_{\nu,s}} |dh| < c^0_s < \min\left\{\frac{\pi\theta}{2^4 K^0_2}, \frac{\pi}{K^0_1}\right\}, \quad \nu \geq 1,\ s = D, U. \tag{3.5.13}$$

3.5.3. Choice of radius ρ of S and sequence of spiraling domains

First, let us take the radius $\rho > 0$ for S such that $\rho < \min\{\rho_0, \frac{r^2}{2}\}$. Restricting ρ if necessary, we construct, as in Section 3.4.4, sectorial domains $\Omega_s^{\hat\epsilon}$ (respectively $\Omega_{s,\beta}^{\hat\epsilon}$) that differ from Ω_s^0 (respectively $\Omega_{s,\beta}^0$) mainly inside a small disk. $\Omega_{s,\beta}^{\hat\epsilon}$ is a neighborhood of $\Omega_s^{\hat\epsilon}$ (see Figure 3.21). As in Figure 3.10, these sectorial domains may spiral around the singular points, depending on the value of $\hat\epsilon$. Nevertheless, $\Omega_s^{\hat\epsilon}$ always stay inside $\Omega_{s,\beta}^{\hat\epsilon}$.

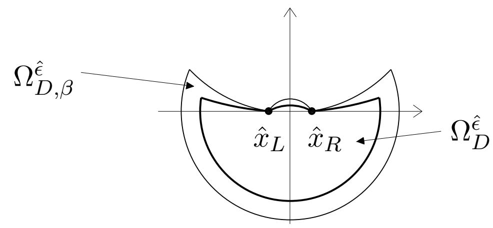

FIG. 3.21. Sectorial domains $\Omega_D^{\hat\epsilon}$ and $\Omega_{D,\beta}^{\hat\epsilon}$, case $\sqrt{\hat\epsilon} \in \mathbb{R}_-^*$.

For $\nu \geq 1$, we define the spiraling domains $\Omega_s^{\hat\epsilon}(\nu)$ which converge when $\nu \to \infty$ to $\Omega_s^{\hat\epsilon}$ and are included in $\Omega_{s,\beta}^{\hat\epsilon}$ for ρ sufficiently small :

$$\Omega_s^{\hat\epsilon}(\nu) = \Omega_s^{\hat\epsilon} \cup_{l=L,R} \{y : \exists x \in \Omega_l^{\hat\epsilon} \text{ s.t. } |y-x| < 2^{-\nu}\theta|y-\hat x_l|^2\}, \quad \hat\epsilon \in S\cup\{0\},\ s = D, U. \tag{3.5.14}$$

The spirals of $\Omega_s^{\hat\epsilon}(\nu)$ near $x = \hat x_l$ are approximately logarithmic.

As illustrated in Figure 3.22, we denote as $\gamma_{\nu,s}^{\hat\epsilon} = \gamma_{\nu,s,L}^{\hat\epsilon} \cup \gamma_{\nu,s,R}^{\hat\epsilon}$ the broken path included in the boundary of $\Omega_s^{\hat\epsilon}(\nu)$, $s = D, U$. The path $\gamma_{\nu,s,L}^{\hat\epsilon}$ starts at $x = -r$ and ends at $x = \hat x_L$, whereas $\gamma_{\nu,s,R}^{\hat\epsilon}$ starts at $x = \hat x_R$ and ends at $x = r$. Remember that they may spiral near the singular points.

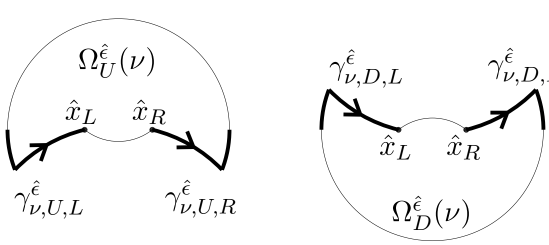

FIG. 3.22. Integration path $\gamma_{\nu,s}^{\hat\epsilon} = \gamma_{\nu,s,L}^{\hat\epsilon} \cup \gamma_{\nu,s,R}^{\hat\epsilon}$, $s = D, U$, case $\sqrt{\hat\epsilon} \in \mathbb{R}_-^*$.

Reducing ρ if necessary, properties of $Z(\hat{\epsilon},x)$ (from (3.5.2)) on $\Omega^{\hat{\epsilon}}_{L,\beta}$ and $\Omega^{\hat{\epsilon}}_{R,\beta}$ imply that, for $N=1,2,3,4$, there exists $K_N\in\mathbb{R}_+$ ($K_N\geq K^0_N$) such that

$$|Z(\hat{\epsilon},x)|\leq K_N|x-\hat{x}_l|^N,\quad (\hat{\epsilon},x)\in(S\cup\{0\})\times\Omega^{\hat{\epsilon}}_l(1),\, l=L,R. \tag{3.5.15}$$

Also,

$$Z(\hat{\epsilon},x)=0\quad (\hat{\epsilon},x)\in S\times\Omega^{\hat{\epsilon}}_C(1). \tag{3.5.16}$$

We reduce ρ in order to have

$$\int_{\gamma^{\hat{\epsilon}}_{\nu,s}}|dh|=c_s\leq\min\left\{\frac{\pi\theta}{2^4K_2},\frac{\pi}{2^2K_1}\right\},\quad \nu\geq1,\,\hat{\epsilon}\in S\cup\{0\},\, s=D,U, \tag{3.5.17}$$

(since the spirals are logarithmic, they have finite length).

3.5.4. Construction of a specific sequence Z^ν, Z^ν_U and Z^ν_D

In this section, starting from $Z^1=Z(\hat{\epsilon},x)$, we construct, for $\nu=2,3,...$, a sequence of matrices Z^ν, Z^ν_U and Z^ν_D such that the following four conditions are satisfied :

(I) $Z^{\nu-1}=Z^\nu_U-Z^\nu_D$, for $(\hat{\epsilon},x)\in(S\cup\{0\})\times\Omega^{\hat{\epsilon}}_\cap(\nu-1)$;

(II) for $s=D,U$,

- $Z^\nu_s(\hat{\epsilon},x)$ is analytic on $S\times\Omega^{\hat{\epsilon}}_s(\nu-1)$,
- $Z^\nu_s(0,x)$ is analytic for $x\in\Omega^0_s(\nu-1)$,
- $|Z^\nu_s|\leq 2^{-(\nu+1)}$ for $(\hat{\epsilon},x)\in(S\cup\{0\})\times\Omega^{\hat{\epsilon}}_s(\nu)$

(III) $I+Z^\nu=(I+Z^\nu_D)(I+Z^{\nu-1})(I+Z^\nu_U)^{-1}$, $(\hat{\epsilon},x)\in(S\cup\{0\})\times\Omega^{\hat{\epsilon}}_\cap(\nu-\delta)$ for some $0<\delta<1$;

(IV)
- $Z^\nu(0,x)$ is analytic over $\Omega^0_\cap(\nu-\delta)$,
- $Z^\nu(\hat{\epsilon},x)=0$ on $S\times\Omega^{\hat{\epsilon}}_C(\nu-\delta)$,

- $Z^\nu(\hat\epsilon, x)$ is analytic on $S \times \Omega^{\hat\epsilon}_{\cap}(\nu-\delta)$ and satisfies, for $N = 1, 2, 3, 4$, $|Z^\nu| \le 2^{-2(\nu-1)} K_N |x - \hat x_l|^N$ for $(\hat\epsilon, x) \in (S \cup \{0\}) \times \Omega^{\hat\epsilon}_l(\nu)$, $l = L, R$.

In order to obtain condition (I), we define the matrices $Z^\nu_D(\hat\epsilon, x)$ and $Z^\nu_U(\hat\epsilon, x)$ for $\nu = 2, 3, \ldots$ by

$$Z^\nu_s(\hat\epsilon, x) = \frac{1}{2\pi i} \int_{\gamma^{\hat\epsilon}_{\nu-1,s}} \frac{Z^{\nu-1}(\hat\epsilon, h)}{h - x} dh, \quad (\hat\epsilon, x) \in (S \cup \{0\}) \times \Omega^{\hat\epsilon}_s(\nu-1),\ s = D, U. \tag{3.5.18}$$

Condition (I) is satisfied since, for $(\hat\epsilon, x) \in (S \cup \{0\}) \times \Omega^{\hat\epsilon}_{\cap}(\nu-1)$,

$$Z^\nu_U(\hat\epsilon, x) - Z^\nu_D(\hat\epsilon, x) = \frac{1}{2\pi i} \int_{\gamma^{\hat\epsilon}_{\nu-1}} \frac{Z^{\nu-1}(\hat\epsilon, h)}{h - x} dh = Z^{\nu-1}(\hat\epsilon, x), \tag{3.5.19}$$

where $\gamma^{\hat\epsilon}_{\nu-1}$ (Figure 3.23) is a union of two paths surrounding $\Omega^{\hat\epsilon}_L(\nu-1)$ and $\Omega^{\hat\epsilon}_R(\nu-1)$:

$$\gamma^{\hat\epsilon}_{\nu-1} = \gamma^{\hat\epsilon}_{\nu-1,U,L}(\gamma^{\hat\epsilon}_{\nu-1,D,L})^{-1} \cup \gamma^{\hat\epsilon}_{\nu-1,U,R}(\gamma^{\hat\epsilon}_{\nu-1,D,R})^{-1}. \tag{3.5.20}$$

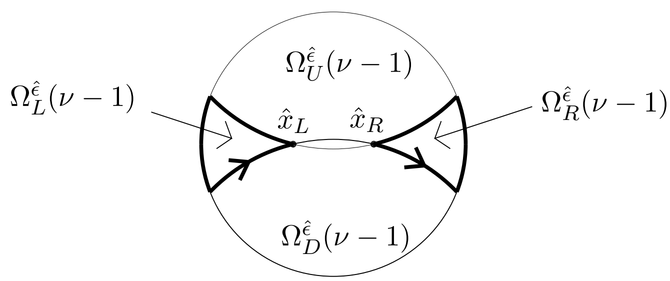

FIG. 3.23. Integration path $\gamma^{\hat\epsilon}_{\nu-1}$, case $\sqrt{\hat\epsilon} \in \mathbb{R}^*_-$.

Let us now prove (II) for $\nu \ge 2$, taking into account that (IV) is satisfied (it is indeed for $\nu = 1$). When integrating in (3.5.18), we have

$$|h - x| \ge 2^{-\nu}\theta |h - \hat x_l|^2, \quad h \in \gamma^{\hat\epsilon}_{\nu-1,s},\ x \in \Omega^{\hat\epsilon}_s(\nu),\ \hat\epsilon \in S \cup \{0\},\ s = D, U,\ l = L, R. \tag{3.5.21}$$

Then, using (IV) as well as relations (3.5.17) and (3.5.21), we have, for $s = D, U$,

$$\begin{aligned} |Z^\nu_s(\hat\epsilon, x)| &\le \tfrac{1}{2\pi} \int_{\gamma^{\hat\epsilon}_{\nu-1,s}} \tfrac{|Z^{\nu-1}(\hat\epsilon,h)|}{|h-x|} |dh|, && (\hat\epsilon, x) \in (S \cup \{0\}) \times \Omega^{\hat\epsilon}_s(\nu), \\ &\le \tfrac{2^{-2(\nu-2)} K_2 c_s}{2\pi 2^{-\nu}\theta} \le 2^{-(\nu+1)}, && (\hat\epsilon, x) \in (S \cup \{0\}) \times \Omega^{\hat\epsilon}_s(\nu). \end{aligned} \tag{3.5.22}$$

Let us now prove condition (IV), taking Z^ν defined by relation (III) (there exists some $0 < \delta < 1$ such that $(I + Z_U^\nu)$ is invertible for $(\hat{\epsilon}, x) \in (S \cup \{0\}) \times \Omega_\cap^{\hat{\epsilon}}(\nu - \delta)$). On each side of (III), multiplying by $(I + Z_U^\nu)$ on the right yields

$$Z_U^\nu + Z^\nu(I + Z_U^\nu) = Z^{\nu-1} + Z_D^\nu + Z_D^\nu Z^{\nu-1}, \quad (\hat{\epsilon}, x) \in (S \cup \{0\}) \times \Omega_\cap^{\hat{\epsilon}}(\nu). \tag{3.5.23}$$

Using condition (I), it yields

$$Z^\nu(I + Z_U^\nu) = Z_D^\nu Z^{\nu-1}, \quad (\hat{\epsilon}, x) \in (S \cup \{0\}) \times \Omega_\cap^{\hat{\epsilon}}(\nu). \tag{3.5.24}$$

Hence,

$$\begin{aligned} |Z^\nu| &\leq |Z_D^\nu||Z^{\nu-1}||(I + Z_U^\nu)^{-1}| && (\hat{\epsilon}, x) \in (S \cup \{0\}) \times \Omega_\cap^{\hat{\epsilon}}(\nu) \\ &\leq |Z_D^\nu||Z^{\nu-1}|\tfrac{1}{1-|Z_U^\nu|} && (\hat{\epsilon}, x) \in (S \cup \{0\}) \times \Omega_\cap^{\hat{\epsilon}}(\nu), \end{aligned} \tag{3.5.25}$$

the last inequality obtained since $|Z_U^\nu| < \frac{1}{2}$. Because of (3.5.16), we have

$$Z^\nu(\hat{\epsilon}, x) = 0 \text{ on } S \times \Omega_C^{\hat{\epsilon}}(\nu). \tag{3.5.26}$$

Finally, we finish the proof of (IV) from condition (II) and the induction hypothesis into (3.5.25) : for $N \leq 4$ and $l = L, R$, we have

$$\begin{aligned} |Z^\nu| &\leq 2^{-(\nu+1)}(2^{-2(\nu-2)}K_N|x - \hat{x}_l|^N)(\tfrac{1}{1-2^{-(\nu+1)}}), && (\hat{\epsilon}, x) \in (S \cup \{0\}) \times \Omega_l^{\hat{\epsilon}}(\nu), \\ &\leq 2^{-2(\nu-1)}K_N|x - \hat{x}_l|^N, && (\hat{\epsilon}, x) \in (S \cup \{0\}) \times \Omega_l^{\hat{\epsilon}}(\nu). \end{aligned} \tag{3.5.27}$$

3.5.5. Construction of $H_D(\hat{\epsilon}, x)$ and $H_U(\hat{\epsilon}, x)$

The sequence of matrices Z^ν, Z_U^ν and Z_D^ν constructed in Section 3.5.4 satisfies condition (III) and hence, for $(\hat{\epsilon}, x) \in (S \cup \{0\}) \times \Omega_\cap^{\hat{\epsilon}}(\nu)$,

$$\begin{aligned} &I + Z(\hat{\epsilon}, x) = I + Z^1 = \\ &\quad \left[(I + Z_D^\nu)...(I + Z_D^3)(I + Z_D^2)\right]^{-1}(I + Z^\nu)\left[(I + Z_U^\nu)...(I + Z_U^3)(I + Z_U^2)\right]. \end{aligned} \tag{3.5.28}$$

Since

$$\lim_{\nu \to \infty} Z^\nu = 0, \quad (\hat{\epsilon}, x) \in (S \cup \{0\}) \times \Omega_\cap^{\hat{\epsilon}}(\nu), \tag{3.5.29}$$

and

$$\prod_{\nu=2}^{\infty} |1 + Z_s^\nu| \leq \prod_{\nu=2}^{\infty}(1 + 2^{-(\nu+1)}), \quad (\hat{\epsilon}, x) \in (S \cup \{0\}) \times \Omega_s^{\hat{\epsilon}}(\nu),\ s = D, U, \tag{3.5.30}$$

the products in brackets are convergent in (3.5.28) when $\nu \to \infty$ and we are led to matrices satisfying (3.5.1) (details in Lemma 4 from the proof of Cartan's Lemma in [**8**] p. 195) :

$$H_s(\hat{\epsilon}, x) = \lim_{\nu\to\infty} (I + Z_s^\nu)...(I + Z_s^3)(I + Z_s^2), \quad (\hat{\epsilon}, x) \in (S \cup \{0\}) \times \Omega_s^{\hat{\epsilon}}(\nu),\ s = D, U. \tag{3.5.31}$$

The boundedness of $H_s(\hat{\epsilon}, x)$ and $H_s(\hat{\epsilon}, x)^{-1}$ when $x \to \hat{x}_l$, $x \in \Omega_s^{\hat{\epsilon}}$, $\hat{\epsilon} \in S \cup \{0\}$, $s = D, U$, $l = L, R$, is obtained from (IV) and from the fact that the limit of the products in brackets in (3.5.28) are invertible and convergent when $\nu \to \infty$.

Let us now prove that $H_s'(\hat{\epsilon}, x) = \frac{\partial H_s(\hat{\epsilon},x)}{\partial x}$ is bounded when $x \to \hat{x}_l$, $x \in \Omega_s^{\hat{\epsilon}}$, $\hat{\epsilon} \in S \cup \{0\}$, $l = L, R$ and $s = D, U$, by proving there exists $K \in \mathbb{R}_+$ such that

$$|H_s(\hat{\epsilon}, \hat{x}_l + t) - H_s(\hat{\epsilon}, \hat{x}_l)| \le K|t|, \quad \hat{x}_l + t \in \Omega_s^{\hat{\epsilon}}. \tag{3.5.32}$$

First, let us prove there exists $k \in \mathbb{R}_+$ such that

$$|Z_s^\nu(\hat{\epsilon}, \hat{x}_l + t) - Z_s^\nu(\hat{\epsilon}, \hat{x}_l)| \le 2^{-\nu} k|t|, \quad \hat{x}_l + t \in \Omega_s^{\hat{\epsilon}}(\nu). \tag{3.5.33}$$

Using (3.5.18), (IV), (3.5.21) and (3.5.17), we have, for t such that $\hat{x}_l + t \in \Omega_s^{\hat{\epsilon}}(\nu)$,

$$\begin{aligned} |Z_s^\nu(\hat{\epsilon}, \hat{x}_l + t) - Z_s^\nu(\hat{\epsilon}, \hat{x}_l)| &= \frac{1}{2\pi} \left| \int_{\gamma_{\nu-1,s}^{\hat{\epsilon}}} Z^{\nu-1}(\hat{\epsilon}, h) \left(\frac{1}{h-(\hat{x}_l+t)} - \frac{1}{h-\hat{x}_l} \right) dh \right| \\ &\le \frac{|t|}{2\pi} \left| \int_{\gamma_{\nu-1,s}^{\hat{\epsilon}}} \frac{|Z^{\nu-1}(\hat{\epsilon},h)|}{|h-(\hat{x}_l+t)||h-\hat{x}_l|} |dh| \right| \\ &\le \frac{|t|}{2\pi} \frac{K_3 2^\nu c_s}{2^{2(\nu-2)}\theta} \\ &\le |t| \frac{K_3}{2^{\nu+1} K_2}, \end{aligned} \tag{3.5.34}$$

thus proving (3.5.33) with $k = \frac{K_3}{2K_2}$. To obtain (3.5.32) from (3.5.33), let us denote shortly

$$Z_s^\nu(\hat{\epsilon}, \hat{x}_l) = \hat{Z}_{s,l}^\nu, \quad Z_s^\nu(\hat{\epsilon}, \hat{x}_l + t) = \hat{Z}_{s,t}^\nu. \tag{3.5.35}$$

From (3.5.31), we have

$$\begin{aligned} &|H_s(\hat{\epsilon}, \hat{x}_l + t) - H_s(\hat{\epsilon}, \hat{x}_l)| \\ &\quad = \lim_{\nu\to\infty} |(I + \hat{Z}_{s,t}^\nu)...(I + \hat{Z}_{s,t}^3)(I + \hat{Z}_{s,t}^2) - (I + \hat{Z}_{s,l}^\nu)...(I + \hat{Z}_{s,l}^3)(I + \hat{Z}_{s,l}^2)|. \end{aligned} \tag{3.5.36}$$

Using (3.5.33) and (II), we can bound (3.5.36) and obtain (3.5.32) from :

$$\begin{aligned}
&|H_s(\hat{\epsilon}, \hat{x}_l + t) - H_s(\hat{\epsilon}, \hat{x}_l)| \\
&\quad \leq \lim_{\nu\to\infty} \textstyle\sum_{i=2}^{\nu} |\hat{Z}_{s,t}^i - \hat{Z}_{s,l}^i| \prod_{q=2}^{i-1} |I + \hat{Z}_{s,t}^q| \prod_{p=i+1}^{\nu} |I + \hat{Z}_{s,l}^p| \\
&\quad \leq \lim_{\nu\to\infty} \textstyle\sum_{i=2}^{\nu} \frac{|I+\hat{Z}_{s,l}^i|}{1-2^{-(i+1)}} |\hat{Z}_{s,t}^i - \hat{Z}_{s,l}^i| \prod_{q=2}^{i-1} |I + \hat{Z}_{s,t}^q| \prod_{p=i+1}^{\nu} |I + \hat{Z}_{s,l}^p| \\
&\quad \leq \lim_{\nu\to\infty} \textstyle\sum_{i=2}^{\nu} \frac{|\hat{Z}_{s,t}^i - \hat{Z}_{s,l}^i|}{1-2^{-(i+1)}} \prod_{p=2}^{\nu} (1 + 2^{-(p+1)}) \\
&\quad \leq \lim_{\nu\to\infty} \textstyle\sum_{i=2}^{\nu} \frac{k|t|}{2^i(1-2^{-(i+1)})} \prod_{p=2}^{\nu} (1 + 2^{-(p+1)}) \\
&\quad \leq \lim_{\nu\to\infty} k|t| \textstyle\sum_{i=2}^{\nu} 2^{-(i-1)} \prod_{p=2}^{\nu} (1 + 2^{-(p+1)}).
\end{aligned} \tag{3.5.37}$$

This section concludes the proof of Theorem 3.5.1. □

3.5.6. Introduction to the proof of Theorem 3.5.2

From now on and until the end of Section 3.5, we present the proof of Theorem 3.5.2, using the proof of Theorem 3.5.1.

Since the given system of invariants satisfy the auto-intersection relation (3.4.123), Theorem 3.4.53 allows us to take, without loss of generality, the unfolded Stokes matrices as $\frac{1}{2}$-summable in ϵ and then, by (3.4.136), the corresponding matrices $\tilde{N}_R$ and $\bar{N}_R$ (Definition 3.4.48) satisfy

$$\bar{N}_R = \tilde{N}_R Q(\bar{\epsilon}), \tag{3.5.38}$$

with $Q(\bar{\epsilon})$ a nonsingular diagonal matrix exponentially close to I in $\sqrt{\epsilon}$. Let

$$(x^2 - \epsilon)v' = A(\hat{\epsilon}, x)v \tag{3.5.39}$$

be the system constructed in the proof of Theorem 3.5.1 by using the $\frac{1}{2}$-summable unfolded Stokes matrices. We will correct the system (3.5.39) by a transformation $y = J(\hat{\epsilon}, x)v$ (defined for $(\hat{\epsilon}, x) \in S \times \mathbb{D}_r$) to obtain a system $(x^2 - \epsilon)y' = B(\epsilon, x)y$ with $B(\epsilon, x)$ analytic in ϵ at $\epsilon = 0$. The condition (3.5.38) will be used in the correction of the family.

3.5.7. The correction to a uniform family

Let $\bar{\epsilon}$ and $\tilde{\epsilon} = \bar{\epsilon}e^{2\pi i}$ in $S_\cap$. Similarly as in Proposition 3.4.52, $\bar{N}_R$ (respectively $\tilde{N}_R$) is the transition matrix $E_{R,\bar{x}_R\to\bar{x}_L}$ (respectively $E_{R,\tilde{x}_L\to\tilde{x}_R}$) between $H_D(\bar{\epsilon}, x)F_D(\bar{\epsilon}, x)\bar{T}_R$ and $H_U(\bar{\epsilon}, x)F_U(\bar{\epsilon}, x)\bar{D}_R\bar{T}_L\bar{D}_R^{-1}$ (respectively $H_D(\tilde{\epsilon}, x)F_D(\tilde{\epsilon}, x)\tilde{T}_L$

and $H_U(\tilde{\epsilon},x)F_U(\tilde{\epsilon},x)\tilde{D}_R\tilde{T}_R\tilde{D}_R^{-1})$. Because the transition matrices satisfy (3.5.38), Proposition 3.4.43 implies that there exists an invertible transformation $P(\bar{\epsilon},x)$ analytic in $(\bar{\epsilon},x)\in S_\cap\times\mathbb{D}_r$ and conjugating the systems $(x^2-\epsilon)v'=A(\bar{\epsilon},x)v$ and $(x^2-\epsilon)v'=A(\tilde{\epsilon},x)v$, i.e.

$$A(\tilde{\epsilon},x)=P(\bar{\epsilon},x)A(\bar{\epsilon},x)P(\bar{\epsilon},x)^{-1}+(x^2-\epsilon)P(\bar{\epsilon},x)'P(\bar{\epsilon},x)^{-1}. \tag{3.5.40}$$

We need to go inside the details of the construction of $P(\bar{\epsilon},x)$ to estimate its growth. $P(\bar{\epsilon},x)$ is as follows :

$$P(\bar{\epsilon},x)=\begin{cases} H_U(\tilde{\epsilon},x)F_U(\tilde{\epsilon},x)\tilde{D}_R\tilde{T}_R\tilde{D}_R^{-1}Q(\bar{\epsilon}) & \\ \qquad\times\left(H_U(\bar{\epsilon},x)F_U(\bar{\epsilon},x)[\bar{D}_R\bar{T}_L\bar{D}_R^{-1}]\right)^{-1}, & x\in\Omega_U^{\bar{\epsilon}}\cap\Omega_U^{\tilde{\epsilon}},\\ H_D(\tilde{\epsilon},x)F_D(\tilde{\epsilon},x)\tilde{T}_L\left(H_D(\bar{\epsilon},x)F_D(\bar{\epsilon},x)\bar{T}_R\right)^{-1}, & x\in\Omega_D^{\bar{\epsilon}}\cap\Omega_D^{\tilde{\epsilon}}.\end{cases} \tag{3.5.41}$$

$P(\bar{\epsilon},x)$ is well-defined (to verify, use (3.5.1), (3.4.51) and (3.4.83)) and can be analytically extended to $\mathbb{D}_r$. It satisfies $P(0,x)=I$ (see Lemma 3.4.47).

In Section 3.5.8, we will show that there exists $\mathcal{K}_1\in\mathbb{R}_+$ such that

$$|P(\bar{\epsilon},x)-I|\le\mathcal{K}_1|\bar{\epsilon}|,\quad(\bar{\epsilon},x)\in(S_\cap\cup\{0\})\times\mathbb{D}_r. \tag{3.5.42}$$

This leads to the proof, sketched in Section 3.5.9, of the existence of $J(\hat{\epsilon},x)$, a nonsingular matrix depending analytically on $(\hat{\epsilon},x)\in S\times B_r$ such that

$$J(\tilde{\epsilon},x)^{-1}J(\bar{\epsilon},x)=P(\bar{\epsilon},x) \tag{3.5.43}$$

on $S_\cap$ and such that $J(\hat{\epsilon},x)$, $J'(\hat{\epsilon},x)$ and $J(\hat{\epsilon},x)^{-1}$ have a bounded limit at $\epsilon=0$ (this proof requires slight reductions of the radius and opening of S).

Let $(x^2-\epsilon)y'=B(\hat{\epsilon},x)y$ be the system obtained by the change $y=J(\hat{\epsilon},x)v$ into (3.5.39). We have

$$B(\hat{\epsilon},x)=J(\hat{\epsilon},x)A(\hat{\epsilon},x)J(\hat{\epsilon},x)^{-1}+(x^2-\epsilon)J(\hat{\epsilon},x)'J(\hat{\epsilon},x)^{-1}. \tag{3.5.44}$$

Replacing (3.5.43) into (3.5.40), we get

$$\begin{aligned}&J(\tilde{\epsilon},x)A(\tilde{\epsilon},x)J(\tilde{\epsilon},x)^{-1}+(x^2-\epsilon)J(\tilde{\epsilon},x)'J(\tilde{\epsilon},x)^{-1}\\ =\ &J(\bar{\epsilon},x)A(\bar{\epsilon},x)J(\bar{\epsilon},x)^{-1}+(x^2-\epsilon)J(\bar{\epsilon},x)'J(\bar{\epsilon},x)^{-1},\end{aligned} \tag{3.5.45}$$

and hence we will have $B(\tilde{\epsilon},x) = B(\bar{\epsilon},x)$ on $S_\cap$ (for x fixed). $B(\epsilon,x)$ will be analytic in ϵ because it will be unramified and because $\lim_{\epsilon\to 0} B(\epsilon,x)$ will exist.

In conclusion, once (3.5.42) and the existence of the desired $J(\hat{\epsilon},x)$ will be proved (in Sections 3.5.8 and 3.5.9), we will have constructed an analytic family of systems with the given complete system of analytic invariants.

3.5.8. Properties of $P(\bar{\epsilon},x)$ near $\bar{\epsilon}=0$

In this section, we show that the conjugating transformation $P(\bar{\epsilon},x)$ satisfies (3.5.42).

3.5.8.1. *Proof of (3.5.42)*

Let us detail how to obtain (3.5.42) for $\bar{\epsilon}\neq 0$ from the construction of $P(\bar{\epsilon},x)$ given by (3.5.41). We will prove that (3.5.42) is satisfied for $x\in(\Omega_U^{\bar{\epsilon}}\cap\Omega_U^{\tilde{\epsilon}})\cup(\Omega_D^{\bar{\epsilon}}\cap\Omega_D^{\tilde{\epsilon}})$. By the Maximum Modulus Theorem, this implies that (3.5.42) is satisfied for $x\in\mathbb{D}_r$.

With the shorter notations

$$\hat{H}_D = H_D(\hat{\epsilon},x) \quad \text{and} \quad \hat{F}_D = F_D(\hat{\epsilon},x), \tag{3.5.46}$$

we have, for $x\in\Omega_D^{\bar{\epsilon}}\cap\Omega_D^{\tilde{\epsilon}}$,

$$\begin{aligned}
|P(\bar{\epsilon},x)-I| &= |\tilde{H}_D\tilde{F}_D\tilde{T}_L\bar{T}_R^{-1}\bar{F}_D^{-1}\bar{H}_D^{-1}-I| \\
&= |\tilde{H}_D\tilde{F}_D(\tilde{T}_L\bar{T}_R^{-1}-I)\bar{F}_D^{-1}\bar{H}_D^{-1}+(\tilde{H}_D\bar{H}_D^{-1}-I)| \\
&\le |\bar{H}_D^{-1}||\tilde{H}_D||\tilde{F}_D||\bar{F}_D^{-1}||\tilde{T}_L\bar{T}_R^{-1}-I| + |\bar{H}_D^{-1}||\tilde{H}_D-\bar{H}_D| \\
&\le |\bar{H}_D^{-1}||\tilde{H}_D||\tilde{F}_D||\bar{F}_D^{-1}|(|\tilde{T}_L-I|+|\bar{T}_R^{-1}-I|+|\tilde{T}_L-I||\bar{T}_R^{-1}-I|) \\
&\quad +|\bar{H}_D^{-1}||\tilde{H}_D-\bar{H}_D|,
\end{aligned} \tag{3.5.47}$$

as well as a similar relation on $\Omega_U^{\bar{\epsilon}}\cap\Omega_U^{\tilde{\epsilon}}$. From Lemma 3.4.47 (and using (3.4.12)), the following matrices appearing in (3.5.47) and in the similar relation on $\Omega_U^{\bar{\epsilon}}\cap\Omega_U^{\tilde{\epsilon}}$ are exponentially close in $\sqrt{\epsilon}$ to I :

$$\tilde{D}_R\tilde{T}_R\tilde{D}_R^{-1}, \qquad \bar{D}_R\bar{T}_L^{-1}\bar{D}_R^{-1}, \qquad \tilde{T}_L, \qquad \bar{T}_R^{-1}. \tag{3.5.48}$$

Hence, in order to obtain the relation (3.5.42) for $x \in (\Omega_U^{\bar{\epsilon}} \cap \Omega_U^{\tilde{\epsilon}}) \cup (\Omega_D^{\bar{\epsilon}} \cap \Omega_D^{\tilde{\epsilon}})$, it suffices to bound $|H_s(\tilde{\epsilon}, x) - H_s(\bar{\epsilon}, x)|$. From (3.5.31), we have

$$\begin{aligned} &|H_s(\tilde{\epsilon}, x) - H_s(\bar{\epsilon}, x)| \\ &\quad = \lim_{\nu\to\infty} |(I + \tilde{Z}_s^\nu)...(I + \tilde{Z}_s^3)(I + \tilde{Z}_s^2) - (I + \bar{Z}_s^\nu)...(I + \bar{Z}_s^3)(I + \bar{Z}_s^2)|. \end{aligned} \tag{3.5.49}$$

We will prove in Section 3.5.8.2 that there exists $k_1 \in \mathbb{R}_+$ such that, for $\nu \geq 2$,

$$|\tilde{Z}_s^\nu - \bar{Z}_s^\nu| \leq 2^{-\nu} k_1 |\bar{\epsilon}|, \quad (\bar{\epsilon}, x) \in S_\cap \times (\Omega_s^{\bar{\epsilon}}(\nu) \cap \Omega_s^{\tilde{\epsilon}})(\nu),\ s = D, U. \tag{3.5.50}$$

Using (3.5.49), (3.5.50) and condition (II) in Section 3.5.4, we then have

$$\begin{aligned} &|H_s(\tilde{\epsilon}, x) - H_s(\bar{\epsilon}, x)| \\ &\quad \leq \lim_{\nu\to\infty} \textstyle\sum_{i=2}^\nu |\tilde{Z}_s^i - \bar{Z}_s^i| \prod_{q=2}^{i-1} |I + \tilde{Z}_s^q| \prod_{p=i+1}^\nu |I + \bar{Z}_s^p| \\ &\quad \leq \lim_{\nu\to\infty} \textstyle\sum_{i=2}^\nu |\tilde{Z}_s^i - \bar{Z}_s^i| \prod_{q=2}^{i-1} |I + \tilde{Z}_s^q| \prod_{p=i+1}^\nu |I + \bar{Z}_s^p| \frac{|I+\tilde{Z}_s^i|}{1-2^{-(i+1)}} \\ &\quad \leq \lim_{\nu\to\infty} \textstyle\prod_{p=2}^\nu (1 + 2^{-(p+1)}) \sum_{i=2}^\nu \frac{|\tilde{Z}_s^i - \bar{Z}_s^i|}{1-2^{-(i+1)}} \\ &\quad \leq \lim_{\nu\to\infty} \textstyle\prod_{p=2}^\nu (1 + 2^{-(p+1)}) \sum_{i=2}^\nu \frac{k_1|\bar{\epsilon}|}{2^i(1-2^{-(i+1)})} \\ &\quad \leq \lim_{\nu\to\infty} k_1|\bar{\epsilon}| \textstyle\prod_{p=2}^\nu (1 + 2^{-(p+1)}) \sum_{i=2}^\nu 2^{-(i-1)}, \end{aligned} \tag{3.5.51}$$

yielding the existence of $\mathcal{K}_1^* \in \mathbb{R}_+$ such that

$$|H_s(\tilde{\epsilon}, x) - H_s(\bar{\epsilon}, x)| \leq \mathcal{K}_1^* |\bar{\epsilon}|, \quad (\bar{\epsilon}, x) \in S_\cap \times (\Omega_s^{\bar{\epsilon}} \cap \Omega_s^{\tilde{\epsilon}}),\ s = D, U. \tag{3.5.52}$$

3.5.8.2. *Property (3.5.50) of* Z_s^ν

Let us now prove (3.5.50), the remaining ingredient of the proof of (3.5.42) for $x \in (\Omega_U^{\bar{\epsilon}} \cap \Omega_U^{\tilde{\epsilon}}) \cup (\Omega_D^{\bar{\epsilon}} \cap \Omega_D^{\tilde{\epsilon}})$. From the definition of $\hat{Z}_s^\nu$ in (3.5.18), we have, for $(\hat{\epsilon}, x) \in S \times \Omega_s^{\bar{\epsilon}}(\nu) \cap \Omega_s^{\tilde{\epsilon}}(\nu)$ and $s = D, U$,

$$\left|\tilde{Z}_s^\nu - \bar{Z}_s^\nu\right| = \left|\frac{1}{2\pi i}\int_{\gamma_{\nu-1,s}^{\tilde{\epsilon}}} \frac{Z^{\nu-1}(\tilde{\epsilon}, h)}{h - x} dh - \frac{1}{2\pi i}\int_{\gamma_{\nu-1,s}^{\bar{\epsilon}}} \frac{Z^{\nu-1}(\bar{\epsilon}, h)}{h - x} dh\right| \tag{3.5.53}$$

The integration paths in (3.5.53) differ near the singular points but have a nonvoid common part. For $s = D, U$, we denote by $i_{\nu,s}^{\bar{\epsilon}}$ the common part of $\gamma_{\nu,s}^{\tilde{\epsilon}}$ and $\gamma_{\nu,s}^{\bar{\epsilon}}$, and by $r_{\nu,s}^{\tilde{\epsilon}}$ and $r_{\nu,s}^{\bar{\epsilon}}$ their respective remaining broken paths (i.e. we have $\gamma_{\nu,s}^{\hat{\epsilon}} = i_{\nu,s}^{\bar{\epsilon}} + r_{\nu,s}^{\hat{\epsilon}}$). Finally, as illustrated in Figure 3.24, we separate the left and right parts of $r_{\nu,s}^{\hat{\epsilon}}$, denoting $r_{\nu,s}^{\hat{\epsilon}} = r_{\nu,s,L}^{\hat{\epsilon}} \cup r_{\nu,s,R}^{\hat{\epsilon}}$. With these notations, we can

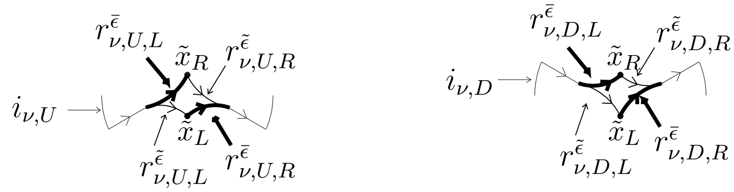

FIG. 3.24. Integration paths $i^{\bar\epsilon}_{\nu,s}$, $r^{\tilde\epsilon}_{\nu,s} = r^{\tilde\epsilon}_{\nu,s,L} \cup r^{\tilde\epsilon}_{\nu,s,R}$ and $r^{\bar\epsilon}_{\nu,s} = r^{\bar\epsilon}_{\nu,s,L} \cup r^{\bar\epsilon}_{\nu,s,R}$, $s = D, U$.

write (3.5.53) as

$$\begin{aligned} \left|\tilde Z^\nu_s - \bar Z^\nu_s\right| &= \left|\frac{1}{2\pi i}\int_{i^{\bar\epsilon}_{\nu-1,s}} \frac{Z^{\nu-1}(\tilde\epsilon,h)-Z^{\nu-1}(\bar\epsilon,h)}{h-x}dh \right. \\ &\left. +\frac{1}{2\pi i}\int_{r^{\tilde\epsilon}_{\nu-1,s}} \frac{Z^{\nu-1}(\tilde\epsilon,h)}{h-x}dh - \frac{1}{2\pi i}\int_{r^{\bar\epsilon}_{\nu-1,s}} \frac{Z^{\nu-1}(\bar\epsilon,h)}{h-x}dh\right|, \end{aligned} \tag{3.5.54}$$

and hence

$$\begin{aligned} \left|\tilde Z^\nu_s - \bar Z^\nu_s\right| &\le \frac{1}{2\pi}\int_{i^{\bar\epsilon}_{\nu-1,s}} \frac{|Z^{\nu-1}(\tilde\epsilon,h)-Z^{\nu-1}(\bar\epsilon,h)|}{|h-x|}|dh| \\ &+\frac{1}{2\pi}\left|\int_{r^{\tilde\epsilon}_{\nu-1,s}} \frac{Z^{\nu-1}(\tilde\epsilon,h)}{h-x}dh\right| + \frac{1}{2\pi}\left|\int_{r^{\bar\epsilon}_{\nu-1,s}} \frac{Z^{\nu-1}(\bar\epsilon,h)}{h-x}dh\right|. \end{aligned} \tag{3.5.55}$$

In order to prove (3.5.50) from (3.5.55), we will bound its last row, and then use induction.

By condition (IV) in Section 3.5.4, we have

$$|Z^{\nu-1}(\hat\epsilon,h)| \le 2^{-2(\nu-2)}K_4|h-\hat x_l|^4, \quad (\hat\epsilon, x) \in (S\cup\{0\}) \times \Omega^{\hat\epsilon}_l(\nu). \tag{3.5.56}$$

Using (3.5.21), we thus have, for $x \in \Omega^{\bar\epsilon}_l(\nu) \cap \Omega^{\tilde\epsilon}_l(\nu)$ $s = D, U$, $l = L, R$ and $\hat\epsilon \in \{\tilde\epsilon, \bar\epsilon\}$,

$$\begin{aligned} \left|\int_{r^{\hat\epsilon}_{\nu-1,s,l}} \frac{Z^{\nu-1}(\hat\epsilon,h)}{h-x}dh\right| &\le \int_{r^{\hat\epsilon}_{\nu-1,s,l}} \frac{|Z^{\nu-1}(\hat\epsilon,h)|}{|h-x|}|dh| \\ &\le \int_{r^{\hat\epsilon}_{\nu-1,s,l}} \frac{2^{-2(\nu-2)}K_4|h-\hat x_l|^2}{2^{-\nu}\theta}|dh|. \end{aligned} \tag{3.5.57}$$

The integration paths $r^{\hat\epsilon}_{\nu,s}$ are located inside a disk of radius $c\sqrt{|\epsilon|}$ for some $c \in \mathbb{R}^*_+$ (Section 3.4.4), yielding

$$\begin{aligned} \left|\int_{r^{\hat\epsilon}_{\nu-1,s,l}} \frac{Z^{\nu-1}(\hat\epsilon,h)}{h-x}dh\right| &\le \int_{r^{\hat\epsilon}_{\nu-1,s,l}} \theta^{-1}2^{4-\nu}K_4(|h|+\sqrt{|\epsilon|})^2|dh| \\ &\le \int_{r^{\hat\epsilon}_{\nu-1,s,l}} \theta^{-1}2^{4-\nu}K_4|\epsilon|(c+1)^2|dh| \\ &= \theta^{-1}2^{4-\nu}K_4|\epsilon|(c+1)^2\int_{r^{\hat\epsilon}_{\nu-1,s,l}}|dh|. \end{aligned} \tag{3.5.58}$$

Thus, a bound for the last row of (3.5.55) is, using (3.5.17) and the fact that the length of the path $r^{\hat{\epsilon}}_{\nu-1,s}$ is smaller than the length of the path $\gamma^{\hat{\epsilon}}_{\nu-1,s}$,

$$\begin{aligned}
&\tfrac{1}{2\pi}\left(\left|\int_{r^{\bar{\epsilon}}_{\nu-1,s}} \tfrac{Z^{\nu-1}(\tilde{\epsilon},h)}{h-x}dh\right| + \left|\int_{r^{\bar{\epsilon}}_{\nu-1,s}} \tfrac{Z^{\nu-1}(\bar{\epsilon},h)}{h-x}dh\right|\right)\\
&\quad\le (2\pi\theta)^{-1}2^{4-\nu}K_4|\epsilon|(c+1)^2\left(\int_{r^{\tilde{\epsilon}}_{\nu-1,s}}|dh| + \int_{r^{\bar{\epsilon}}_{\nu-1,s}}|dh|\right)\\
&\quad\le (2\pi\theta)^{-1}2^{4-\nu}K_4|\epsilon|(c+1)^2\left(\int_{\gamma^{\tilde{\epsilon}}_{\nu-1,s}}|dh| + \int_{\gamma^{\bar{\epsilon}}_{\nu-1,s}}|dh|\right)\\
&\quad\le (2\pi\theta)^{-1}2^{4-\nu}K_4|\epsilon|(c+1)^2 2c_s\\
&\quad= \tfrac{k_1^*}{2^{\nu+5}}|\epsilon|,
\end{aligned} \tag{3.5.59}$$

where

$$k_1^* = \frac{2^5K_4(c+1)^2}{K_2}. \tag{3.5.60}$$

Hence, (3.5.55) becomes

$$\begin{aligned}
\left|\tilde{Z}^\nu_s - \bar{Z}^\nu_s\right| \le \tfrac{1}{2\pi}\int_{i^{\bar{\epsilon}}_{\nu-1,s}} \tfrac{|Z^{\nu-1}(\tilde{\epsilon},h)-Z^{\nu-1}(\bar{\epsilon},h)|}{|h-x|}|dh| + \tfrac{k_1^*}{2^{\nu+5}}|\bar{\epsilon}|,\\
(\bar{\epsilon},x)\in S_\cap\times(\Omega^{\bar{\epsilon}}_l(\nu)\cap\Omega^{\tilde{\epsilon}}_l(\nu)).
\end{aligned} \tag{3.5.61}$$

From (3.5.61), we will prove (3.5.50) for $\nu=2$, $\nu=3$ and $\nu>3$.

Beginning with $\nu=2$, we have, from

$$F_s(\bar{\epsilon},x) = F_s(\tilde{\epsilon},x), \quad x\in\Omega^{\bar{\epsilon}}_s\cap\Omega^{\tilde{\epsilon}}_s,\ s=D,U, \tag{3.5.62}$$

and from (3.5.2),

$$|\tilde{Z}^1-\bar{Z}^1| \le \begin{cases} |F_D(\bar{\epsilon},x)\,(C_R(\tilde{\epsilon})-C_R(\bar{\epsilon}))\,F_D(\bar{\epsilon},x)^{-1}|, & \text{on } \Omega^{\tilde{\epsilon}}_R\cap\Omega^{\bar{\epsilon}}_R,\\ |F_D(\bar{\epsilon},x)\,(C_L(\tilde{\epsilon})-C_L(\bar{\epsilon}))\,F_D(\bar{\epsilon},x)^{-1}|, & \text{on } \Omega^{\tilde{\epsilon}}_L\cap\Omega^{\bar{\epsilon}}_L. \end{cases} \tag{3.5.63}$$

By the $\frac{1}{2}$-summability of the unfolded Stokes matrices, $|C_l(\tilde{\epsilon})-C_l(\bar{\epsilon})|$ is exponentially close to 0 in $\sqrt{\epsilon}$. Then, (3.5.63) implies that there exists $w_1\in\mathbb{R}_+$ such that

$$|\tilde{Z}^1-\bar{Z}^1| \le \frac{w_1}{2^4}K_2|\bar{\epsilon}||x-\bar{x}_l|^2, \quad l=L,R,\ (\bar{\epsilon},x)\in S_\cap\times(\Omega^{\bar{\epsilon}}_l(\nu)\cap\Omega^{\tilde{\epsilon}}_l(\nu)), \tag{3.5.64}$$

with K_2 given by (3.5.15). Using relations (3.5.17) and (3.5.21) and the fact that the length of the path $i^{\bar{\epsilon}}_{\nu,s}$ is smaller than the length of the path $\gamma^{\hat{\epsilon}}_{\nu,s}$, the integral in (3.5.61) for $\nu = 2$ is bounded by

$$\begin{aligned}
\tfrac{1}{2\pi}\int_{i_{1,s}} \tfrac{|Z^1(\tilde{\epsilon},h)-Z^1(\bar{\epsilon},h)|}{|h-x|}|dh| &\le \tfrac{1}{2\pi}\int_{i_{1,s}} \tfrac{w_1K_2|\bar{\epsilon}||h-\bar{x}_l|^2}{2^4|h-x|}|dh| \\
&\le \tfrac{1}{2\pi}\int_{i_{1,s}} \tfrac{w_1K_2|\bar{\epsilon}||h-\bar{x}_l|^2}{2^4 2^{-2}\theta|h-\bar{x}_l|^2}|dh| \\
&\le w_1|\bar{\epsilon}|\tfrac{K_2}{2^3\theta\pi}\int_{i_{1,s}}|dh| \\
&\le w_1|\bar{\epsilon}|\tfrac{K_2c_s}{2^3\theta\pi} \\
&\le \tfrac{1}{2^7}w_1|\bar{\epsilon}|.
\end{aligned} \tag{3.5.65}$$

From (3.5.61) and (3.5.65), we have

$$|\tilde{Z}_s^2 - \bar{Z}_s^2| \le \frac{1}{2^6}k_1|\bar{\epsilon}|, \quad (\bar{\epsilon},x) \in S_\cap \times (\Omega_s^{\bar{\epsilon}}(2)\cap\Omega_s^{\tilde{\epsilon}})(2),\ s = D,U, \tag{3.5.66}$$

with

$$k_1 = \max\{k_1^*, w_1\}. \tag{3.5.67}$$

Relation (3.5.50) is thus satisfied for $\nu = 2$.

Now, let us study $|\tilde{Z}^{\nu-1} - \bar{Z}^{\nu-1}|$ in order to bound (3.5.61) for $\nu \ge 3$. From the equality

$$\tilde{A}\tilde{B}\tilde{C}^{-1} - \bar{A}\bar{B}\bar{C}^{-1} = \Big((\tilde{A}-\bar{A})\tilde{B} + \bar{A}(\tilde{B}-\bar{B}) + \bar{A}\bar{B}\bar{C}^{-1}(\bar{C}-\tilde{C})\Big)\tilde{C}^{-1}, \tag{3.5.68}$$

applied to relation $Z^{\nu-1} = Z_D^{\nu-1}Z^{\nu-2}(I+Z_U^{\nu-1})^{-1}$ coming from (3.5.24), we have (taking $Z^{\nu-1} = ABC^{-1}$, $Z_D^{\nu-1} = A$, $Z^{\nu-2} = B$ and $(I+Z_U^{\nu-1}) = C$)

$$\begin{aligned}
&|\tilde{Z}^{\nu-1} - \bar{Z}^{\nu-1}| \\
&\quad \le \Big(|\tilde{Z}_D^{\nu-1} - \bar{Z}_D^{\nu-1}||\tilde{Z}^{\nu-2}| + |\bar{Z}_D^{\nu-1}||\tilde{Z}^{\nu-2} - \bar{Z}^{\nu-2}| + |\bar{Z}^{\nu-1}||\bar{Z}_U^{\nu-1} - \tilde{Z}_U^{\nu-1}|\Big) \\
&\qquad \times|(I+\tilde{Z}_U^{\nu-1})^{-1}|.
\end{aligned} \tag{3.5.69}$$

Let us remark that, because of (3.5.62), we have

$$|\hat{Z}^\nu| \le 2^{-2(\nu-1)}K_1|x-\bar{x}_l|, \quad (\hat{\epsilon},x) \in S_\cap \times \Omega_l^{\bar{\epsilon}}(\nu)\cap\Omega_l^{\tilde{\epsilon}}(\nu), l = L,R, \tag{3.5.70}$$

coming from condition (IV) of Section 3.5.4.

For $\nu = 3$, equation (3.5.69) yields, with the use of (3.5.64), (3.5.66), (3.5.67), (3.5.70) and $|Z_s^{\nu-1}| \le 2^{-\nu}$ (from (II)),

$$\begin{aligned}
|\tilde{Z}^2 - \bar{Z}^2| &\le k_1|\bar{\epsilon}|K_2|x-\bar{x}_l|^2\left(\tfrac{1}{2^6} + \tfrac{1}{2^3 2^4} + \tfrac{1}{2^2 2^6}\right)\left(\tfrac{1}{1-2^{-3}}\right), \qquad l = L,R, \\
&\le k_1|\bar{\epsilon}|K_2|x-\bar{x}_l|^2\tfrac{1}{2^4}, \qquad l = L,R,
\end{aligned} \tag{3.5.71}$$

for $(\bar{\epsilon}, x) \in S_\cap \times (\Omega_s^{\bar{\epsilon}}(2) \cap \Omega_s^{\tilde{\epsilon}})(2)$. In the same way as when we bounded (3.5.65), we use (3.5.71) to bound the integral in (3.5.61) for $\nu = 3$:

$$\begin{aligned} \tfrac{1}{2\pi}\int_{i_{2,s}} \tfrac{|Z^2(\tilde{\epsilon},h)-Z^2(\bar{\epsilon},h)|}{|h-x|}|dh| &\leq \tfrac{1}{2\pi}\int_{i_{2,s}} \tfrac{k_1|\bar{\epsilon}|K_2|h-\bar{x}_l|^2}{2^4 2^{-3}\theta|h-\bar{x}_l|^2}|dh| \\ &\leq \tfrac{k_1|\bar{\epsilon}|K_2}{2^2\pi\theta}\int_{i_{2,s}}|dh| \\ &\leq \tfrac{k_1|\bar{\epsilon}|K_2}{2^2\pi\theta}\int_{\gamma_{2,s}^{\bar{\epsilon}}}|dh| \\ &\leq \tfrac{1}{2^2}k_1|\bar{\epsilon}|\tfrac{K_2 c_s}{\pi\theta} \\ &\leq \tfrac{1}{2^6}k_1|\bar{\epsilon}|. \end{aligned} \tag{3.5.72}$$

Then, (3.5.72) into (3.5.61) gives

$$|\tilde{Z}_s^3 - \bar{Z}_s^3| \leq \frac{1}{2^5}k_1|\bar{\epsilon}|, \qquad (\bar{\epsilon}, x) \in S_\cap \times (\Omega_s^{\bar{\epsilon}}(3) \cap \Omega_s^{\tilde{\epsilon}}(3)),\ s = D, U. \tag{3.5.73}$$

Relation (3.5.50) is hence satisfied for $\nu = 3$.

We are now ready to prove (3.5.50) for $\nu > 3$ by induction on ν. Let us suppose that we have

$$\begin{aligned} |\tilde{Z}^{\nu-2} - \bar{Z}^{\nu-2}| \leq \tfrac{k_1}{2^{2(\nu-3)}}|\bar{\epsilon}|K_2|x-\bar{x}_l|^2, \\ (\bar{\epsilon}, x) \in S_\cap \times (\Omega_s^{\bar{\epsilon}}(\nu-2) \cap \Omega_s^{\tilde{\epsilon}}(\nu-2)),\ l = L, R, \end{aligned} \tag{3.5.74}$$

and

$$|\tilde{Z}_s^{\nu-1} - \bar{Z}_s^{\nu-1}| \leq \frac{1}{2^{\nu-1}}k_1|\bar{\epsilon}|, \quad (\bar{\epsilon}, x) \in S_\cap \times (\Omega_s^{\bar{\epsilon}}(\nu-1) \cap \Omega_s^{\tilde{\epsilon}}(\nu-1)),\ s = D, U, \tag{3.5.75}$$

(this is indeed satisfied for $\nu = 4$ because of (3.5.71) and (3.5.73)). For $\nu > 3$, relation (3.5.69) yields, using (3.5.74), (3.5.75), (3.5.70) and $|\hat{Z}_s^{\nu-1}| \leq 2^{-\nu}$ (from (II)),

$$\begin{aligned} |\tilde{Z}^{\nu-1} - \bar{Z}^{\nu-1}| \leq\ & k_1|\bar{\epsilon}|K_2|x-\bar{x}_l|^2\left(\tfrac{1}{2^{\nu-1}2^{2(\nu-3)}} + \tfrac{1}{2^{\nu}2^{2(\nu-3)}} + \tfrac{1}{2^{2(\nu-2)}2^{\nu-1}}\right) \\ & \times\left(\tfrac{1}{1-2^{-\nu}}\right), \end{aligned} \tag{3.5.76}$$

and thus, for $(\bar{\epsilon}, x) \in S_\cap \times (\Omega_s^{\bar{\epsilon}}(\nu-1) \cap \Omega_s^{\tilde{\epsilon}}(\nu-1))$ and $l = L, R$,

$$|\tilde{Z}^{\nu-1} - \bar{Z}^{\nu-1}| \leq \tfrac{k_1}{2^{2(\nu-2)}}|\bar{\epsilon}|K_2|x-\bar{x}_l|^2. \tag{3.5.77}$$

In the same way as when we bounded (3.5.65) and (3.5.72), we use (3.5.77) to bound the integral in (3.5.61) for $\nu > 3$:

$$\begin{aligned}
\frac{1}{2\pi}\int_{i^{\bar\epsilon}_{\nu-1,s}} \frac{|Z^{\nu-1}(\tilde\epsilon,h)-Z^{\nu-1}(\bar\epsilon,h)|}{|h-x|}|dh| &\leq \frac{1}{2\pi}\int_{i^{\bar\epsilon}_{\nu-1,s}} \frac{k_1K_2|\bar\epsilon||h-\bar x_l|^2}{2^{2(\nu-2)}|h-x|}|dh| \\
&\leq \frac{1}{2\pi}\int_{i^{\bar\epsilon}_{\nu-1,s}} \frac{k_1K_2|\bar\epsilon||h-\bar x_l|^2}{2^{2(\nu-2)}2^{-\nu}\theta|h-\bar x_l|^2}|dh| \\
&\leq \frac{k_1K_2|\bar\epsilon|}{\pi 2^{\nu-3}\theta}\int_{i^{\bar\epsilon}_{\nu-1,s}} |dh| \\
&\leq \frac{k_1K_2|\bar\epsilon|}{\pi 2^{\nu-3}\theta}\int_{\gamma^{\hat\epsilon}_{\nu-1,s}} |dh| \\
&\leq \frac{1}{2^{\nu-3}}k_1|\bar\epsilon|\frac{K_2c_s}{\pi\theta} \\
&\leq \frac{1}{2^{\nu+1}}k_1|\bar\epsilon|.
\end{aligned} \tag{3.5.78}$$

Then, (3.5.78) and (3.5.61) gives (3.5.50) for $\nu > 3$ (using (3.5.67)).

3.5.9. Construction of $J(\hat\epsilon, x)$

For fixed x, the existence of $J(\hat\epsilon, x)$ follows from the triviality of the vector bundle on the punctured disk in ϵ-space. But, we need to show that $J(\hat\epsilon, x)$ depends analytically on the "parameter" $x \in \mathbb{D}_r$ and also that we can fill the hole at $\epsilon = 0$. So we need to go into the details of the construction of $J(\hat\epsilon, x)$. We do this in a sketchy way since the details are completely similar (and simpler) to those we have done in Sections 3.5.1 to 3.5.5.

S has been taken previously with an opening $2\pi + \gamma_0$. We reduce slightly the opening of S to $2\pi + \gamma$ with $0 < \gamma < \gamma_0$, denoting the sector with the previous opening S^{prev}, such that, for some $\alpha > 0$,

$$S(1) = S \cup \{\epsilon : \exists\hat\epsilon \in S_\cap \text{ s.t. } |\epsilon - \hat\epsilon| < 2^{-1}\alpha|\epsilon|\} \subset S^{prev}. \tag{3.5.79}$$

We write S as the union of two sectors V_U and V_D

$$\begin{aligned}
V_D &= \{\epsilon \in \mathbb{C} \,:\, 0 < |\epsilon| < \rho,\ \arg(\epsilon) \in (\pi - \gamma, 2\pi + \gamma)\}, \\
V_U &= \{\epsilon \in \mathbb{C} \,:\, 0 < |\epsilon| < \rho,\ \arg(\epsilon) \in (2\pi - \gamma, 3\pi + \gamma)\}.
\end{aligned} \tag{3.5.80}$$

We take the following domains converging when $\nu \to \infty$ to V_s and included into S^{prev} :

$$V_s(\nu) = V_s \cup \{\epsilon : \exists\hat\epsilon \in V_U \cap V_D \text{ s.t. } |\epsilon - \hat\epsilon| < 2^{-\nu}\alpha|\epsilon|\}, \quad \nu \geq 1, s = D, U. \tag{3.5.81}$$

We separate the intersection of $V_U(\nu)$ and $V_D(\nu)$ into a left and a right domain :

$$V_\cap(\nu) = V_U(\nu) \cap V_D(\nu) = V_L(\nu) \cup V_R(\nu). \tag{3.5.82}$$

We divide the boundary of $V_\cap(\nu)$ in two parts : as illustrated in Figure 3.25, we denote $t_{\nu,s} = t_{\nu,s,L} \cup t_{\nu,s,R}$ the path included in the boundary of $V_s(\nu)$, $s = D, U$. The path $t_{\nu,s,L}$ begins at $\epsilon = -\rho$ and ends at $\epsilon = 0$, whereas $t_{\nu,s,R}$ begins at $\epsilon = 0$ and ends at $\epsilon = \rho$.

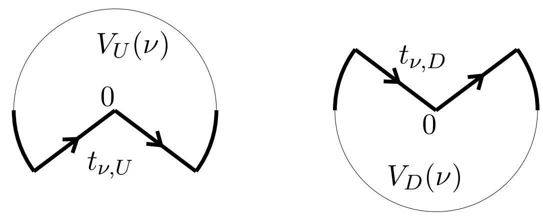

FIG. 3.25. Integration path $t_{\nu,s} = t_{\nu,s,L} \cup t_{\nu,s,R}$, $s = D, U$.

We reduce the radius ρ of S (and hence of V_s and $V_s(\nu)$, $s = D, U$) a last time so that the length of each path $t_{\nu,s}$ is bounded as follows :

$$\int_{t_{\nu,s}} |dh| < l_s < \min\left\{\frac{\pi\alpha}{2^4\mathcal{K}_1}, \frac{\pi}{\mathcal{K}_1}\right\}, \quad s = D, U, \ \nu \geq 1, \tag{3.5.83}$$

with $\mathcal{K}_1$ given by (3.5.42).

Starting from

$$Y^1 = \begin{cases} P(\epsilon, x) - I, & \epsilon \in V_L, \\ 0, & \epsilon \in V_R, \end{cases} \tag{3.5.84}$$

and using (3.5.42), we construct, for $\nu = 2, 3, \ldots$, a sequence of matrices Y^ν, Y_U^ν and Y_D^ν satisfying the conditions :

(i) $Y^{\nu-1} = Y_U^\nu - Y_D^\nu$, $(\epsilon, x) \in V_\cap(\nu - 1) \times \mathbb{D}_r$;

(ii) for $s = D, U$,

- Y_s^ν is analytic for $(\epsilon, x) \in V_s(\nu - 1) \times \mathbb{D}_r$,

- $|Y_s^\nu| \leq 2^{-(\nu+1)}$ for $(\epsilon, x) \in V_s(\nu) \times \mathbb{D}_r$;

(iii) For some $0 < \delta < 1$,

- $I + Y^\nu = (I + Y_D^\nu)(I + Y^{\nu-1})(I + Y_U^\nu)^{-1}$ for $(\epsilon, x) \in V_L(\nu - \delta) \times \mathbb{D}_r$,
- $Y^\nu = 0$ on $V_R(\nu - \delta) \times \mathbb{D}_r$;

(iv)
- Y^ν is analytic for $(\epsilon, x) \in V_L(\nu - \delta) \times \mathbb{D}_r$,

- $Y^\nu(0,x)=0$,

- Y^ν satisfies, with $\mathcal{K}_1$ given by (3.5.42),
  $|Y^\nu|\leq 2^{-2(\nu-1)}\mathcal{K}_1|\epsilon|$ for $(\hat{\epsilon},x)\in V_L(\nu)\times\mathbb{D}_r$.

We can prove that the properties (i) to (iv) are satisfied in a similar (and simpler) way as in Section 3.5.4, by defining the matrices $Y_D^\nu(\epsilon,x)$ and $Y_U^\nu(\epsilon,x)$, for $\nu=2,3,...$, by

$$Y_s^\nu(\epsilon,x)=\frac{1}{2\pi i}\int_{t_{\nu-1,s}}\frac{Y^{\nu-1}(h,x)}{h-\epsilon}dh,\quad (\epsilon,x)\in V_s(\nu-1)\times\mathbb{D}_r,\ s=D,U. \tag{3.5.85}$$

As in Section 3.5.5, the desired $J(\hat{\epsilon},x)$ is given by

$$J(\hat{\epsilon},x)=\begin{cases}J_D(\hat{\epsilon},x), & \hat{\epsilon}\in V_D,\\ J_U(\epsilon,x), & \hat{\epsilon}\in V_U,\end{cases} \tag{3.5.86}$$

with

$$J_s(\epsilon,x)=\lim_{\nu\to\infty}(I+Y_s^\nu)...(I+Y_s^3)(I+Y_s^2),\quad s=D,U. \tag{3.5.87}$$

By (ii), $J(\hat{\epsilon},x)^{-1}$ has a bounded limit at $\epsilon=0$. Since the family$\{J'(\hat{\epsilon},x)\}_{\hat{\epsilon}\in(S\cup\{0\})}$ is bounded, $J'(\hat{\epsilon},x)$ has a bounded limit at $\epsilon=0$. This concludes the proof of Theorem 3.5.2. □

3.6. Discussion and directions for further research

The work presented in this paper brings a new light on the divergence of formal solutions near an irregular singular point of Poincaré rank 1. It gives new perspectives, including a unified point of view in the understanding of the dynamics of the singularities by deformation. We have identified, interpreted and studied the realization of the complete system of analytic invariants of unfolded differential linear systems with an irregular singularity of Poincaré rank 1 (nonresonant case). The meaning of the auto-intersection relation (which is the necessary and sufficient condition for the realization) is still obscure (in dimension $n\geq 3$). We will investigate it in more details in [**13**].

One of the next steps in the large program of understanding singularities by unfolding is the study of analytic invariants of nonresonant linear differential equations with singularities of Poincaré rank k higher than 1. One difference is that there is no more a bijection between the $2k$ Stokes matrices and the $k+1$ singular points in the unfolded systems.

Another direction of research is the existence of universal families. Can we identify canonical representatives of the analytic equivalence classes of unfolded systems ?

CONCLUSION

La présente thèse, concernant la classification analytique des déploiements de systèmes différentiels linéaires ayant une singularité irrégulière (non résonante) de rang de Poincaré $k = 1$, donne un système d'invariants complet, son interprétation ainsi que les conditions nécessaires et suffisantes à sa réalisation.

Le but de la thèse a été atteint par les résultats des articles. Le système complet d'invariants analytiques a été identifié par des invariants formels et analytiques. La partie formelle a été déterminée à partir d'une forme prénormale et se retrouve dans un modèle polynomial qui devient, lorsque $\epsilon \to 0$, la forme normale formelle du système confluent. Le modèle nous a permis alors d'interpréter tous les invariants formels au point singulier irrégulier. Nous avons montré que la partie analytique du système d'invariants compte des classes d'équivalence de matrices de Stokes déployées, dont les représentants peuvent être choisis $\frac{1}{2}$-sommables en ϵ. De plus, nous avons interprété les matrices de Stokes déployées en termes de monodromie d'une base de solutions particulières dans le système linéaire (et d'intégrales premières dans les systèmes de Riccati correspondants). Ceci nous a mené aux conséquences du phénomène de Stokes dans les déploiements, ce qui comprend le lien avec la présence de solutions logarithmiques aux points singuliers réguliers lors de la résonance. Finalement, nous avons résolu la question de la réalisation des invariants. Tout système complet d'invariants analytiques peut être réalisé si et seulement si la relation d'auto-intersection est satisfaite.

Notons que la notion de sommabilité n'est pas disparue par le processus de déploiement. En effet, afin d'expliquer le phénomène de Stokes dans la variable x, nous avons déployé la singularité irrégulière et recherché les invariants analytiques des systèmes déployés. Alors que les invariants formels ont une dépendance analytique en ϵ, les représentants des invariants analytiques peuvent être choisis $\frac{1}{2}$-sommable en ϵ. On a donc introduit de la $\frac{1}{2}$-sommabilité dans le paramètre afin de comprendre la 1-sommabilité dans la variable x.

Sur l'auto-intersection du secteur S dans l'espace du paramètre, les deux points de vue se recollent par une relation que nous avons appelée la relation d'auto-intersection. Elle dépend indirectement des matrices de Stokes déployées et reste alors un peu mystérieuse. Cependant, quelle que soit la dimension n, la relation d'auto-intersection peut être interprétée en termes d'invariance (lors de l'adoption de deux points de vue différents sur l'auto-intersection du secteur S) des matrices de transition entre des bases de solutions particulières qui sont des vecteurs propres de la monodromie autour des points singuliers réguliers. La relation d'auto-intersection peut s'énoncer facilement en fonction des matrices de Stokes déployées en dimension $n = 2$. Ceci vient du fait que les éléments des matrices diagonalisant la monodromie sont dans ce cas des multiples (bien précis) des éléments des matrices de Stokes déployées (ceci est généralement faux en dimension supérieure). L'étude du cas $n = 3$ devrait s'avérer fructueuse afin de dégager des résultats généraux pour toute dimension n.

L'approche utilisée apporte une nouvelle lumière sur le phénomène de Stokes et mérite d'être généralisée à des systèmes déployant une singularité irrégulière (non résonante) de rang de Poincaré $k \in \mathbb{N}^*$, avec $k > 1$. Cette généralisation n'est pas immédiate, nous pouvons déjà y voir une différence notable avec les travaux de cette thèse. En effet, lorsque $k = 1$, nous avons deux matrices de Stokes et chacune est associée à un point singulier régulier des systèmes déployés. Dans le cas général, nous devrons faire le lien entre $2k$ matrices de Stokes et $k + 1$ points singuliers réguliers dans le déploiement. La théorie sera forcément plus élaborée et nous en apprendra davantage sur la dynamique en jeu. Près de 150 ans après sa découverte, le phénomène de Stokes a encore des subtilités à nous faire découvrir.

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