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Solutions de rang k et invariants de Riemann pour les systèmes de type hydrodynamique multidimensionnels

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SOMMAIRE

Dans ce travail, nous adaptons la méthode des symétries conditionnelles afin de construire des solutions exprimées en termes des invariants de Riemann. Dans ce contexte, nous considérons des systèmes non elliptiques quasilinéaires homogènes (de type hydrodynamique) du premier ordre d'équations aux dérivées partielles multidimensionnelles. Nous décrivons en détail les conditions nécessaires et suffisantes pour garantir l'existence locale de ce type de solution. Nous étudions les relations entre la structure des éléments intégraux et la possibilité de construire certaines classes de solutions de rang k. Ces classes de solutions incluent les superpositions non linéaires d'ondes de Riemann ainsi que les solutions multisolitoniques. Nous généralisons cette méthode aux systèmes non homogènes quasilinéaires et non elliptiques du premier ordre. Ces méthodes sont appliquées aux équations de la dynamique des fluides en $(3+1)$ dimensions modélisant le flot d'un fluide isentropique. De nouvelles classes de solutions de rang 2 et 3 sont construites et elles incluent des solutions double- et triple-solitoniques. De nouveaux phénomènes non linéaires et linéaires sont établis pour la superposition des ondes de Riemann. Finalement, nous discutons de certains aspects concernant la construction de solutions de rang 2 pour l'équation de Kadomtsev-Petviashvili sans dispersion.

SUMMARY

In this work, the conditional symmetry method is adapted in order to construct solutions expressed in terms of Riemann invariants. Nonelliptic quasilinear homogeneous systems of multidimensional partial differential equations of hydrodynamic type are considered. A detailed description of the necessary and sufficient conditions for the local existence of these types of solutions is given. The relationship between the structure of integral elements and the possibility of constructing certain classes of rank-k solutions is discussed. These classes of solutions include nonlinear superpositions of Riemann waves and multisolitonic solutions. This approach is generalized to first-order inhomogeneous hyperbolic quasilinear systems. These methods are applied to the equations describing an isentropic fluid flow in $(3+1)$ dimensions. Several new classes of rank-2 and rank-3 solutions are obtained which contain double and triple solitonic solutions. New nonlinear and linear superpositions of Riemann waves are described. Finally, certain aspects of the construction of rank-2 solutions through an application to the dispersionless Kadomtsev-Petviashvili equation are discussed.

PRÉFACE

Cette thèse contient cinq articles dont quatre ont déjà été publiés (les chapitres 1,2,3 et 4) et un vient d'être soumis (chapitre 5). Puisque ces articles sont présentés sous leur forme originale, certains éléments de terminologie et notions de base sont sujets à répétition pour la compréhension des publications. Les sujets des cinq articles sont reliés et développent différents aspects d'une problématique commune. Cependant, chaque article peut être lu de façon indépendante. Dans les chapitres 2 et 3, nous présentons l'aspect méthodologique de la méthode des symétries conditionnelles ainsi que la méthode des invariants de Riemann qui servent à construire des solutions de rang k pour les systèmes de type hydrodynamique en plusieurs dimensions. Dans les chapitres 2, 3, 4 et 5, nous faisons une analyse détaillée de ces méthodes en les appliquant à divers systèmes d'équations de la dynamique des fluides en plusieurs dimensions. Le chapitre 6 contient les remarques finales ainsi que les perspectives futures de développement. L'article présenté dans le chapitre 2 a été nommé parmi les meilleurs articles de l'année 2007 par le *Journal of Physics A : Mathematical and Theoretical* en physique mathématique. La préparation de cette thèse a également donné lieu à de multiples communications rapportées dans les comptes rendus de conférence [**19, 49, 60**]. De plus, une adaptation du logiciel *symmgrp.max*, introduit en 1991 pour l'étude du groupe de symétrie des équations différentielles, pour le logiciel libre *Maxima* a été effectuée [**59**] et utilisée dans le chapitre 4.

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DÉDICACE

À mes parents.

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INTRODUCTION

Ce travail représente une description et analyse formelle des solutions de rang k obtenues pour les systèmes non dégénérés hyperboliques quasilinéaires du premier ordre en plusieurs dimensions. Nous considérons dans cette thèse les systèmes de l équations aux dérivées partielles (EDPs) de type hydrodynamique homogènes et non homogènes de la forme

$$\sum_{i=0}^{p}\sum_{\alpha=1}^{q} a^{i\mu}_{\alpha}(u^1,\ldots,u^q)\frac{\partial u^\alpha}{\partial x^i}(x^0,x^1,\ldots,x^n)=b^\mu(u^1,\ldots,u^q),\quad \mu=1,\ldots,l, \tag{0.0.1}$$

sujets aux conditions initiales

$$t=0 \quad : \quad u(0,x^1,\ldots,x^n)=u_0(x^1,\ldots,x^n), \tag{0.0.2}$$

où $x=(x^0=t,x^1,\ldots,x^n)\in\mathbb{R}^p$, $(p=n+1)$ et $u=u(x)=(u^1(x),\ldots,u^q(x))\in\mathbb{R}^q$ représentent respectivement les variables indépendantes et les valeurs des fonctions dépendantes. Nous supposons que le système est bien défini ($l=q$) dans la plupart des cas. L'espace des variables indépendantes sera noté par X et représente l'espace-temps. Nous notons l'espace des valeurs des variables dépendantes, appelé dans la littérature l'espace hodographe, par U. Nous faisons l'hypothèse que les coefficients de l'équation (0.0.1), $a^{i\mu}_{\alpha}(u)$ et $b^\mu(u)$, sont différentiables dans un ouvert de l'espace $X\times U$, ou analytiques où cela est nécessaire. Nous faisons l'hypothèse que les données initiales sont suffisamment différentiables ou analytiques, selon les contextes, pour que nous puissions utiliser les théorèmes classiques d'existence.

Les équations de type (0.0.1) possèdent un caractère général puisqu'elles apparaissent dans plusieurs domaines de la physique tels que la relativité générale, la théorie des champs non linéaires, la mécanique des fluides, la théorie de la plasticité, etc. Il est en général essentiel de faire une analyse systématique du point de vue de la théorie des groupes pour construire des classes de solutions particulières. Dans cette thèse, nous cherchons certaines classes de solutions de rang k du système (0.0.1) en termes des invariants de Riemann. Ces solutions ont plusieurs applications dans les domaines cités ci-dessus et peuvent représenter des superpositions d'ondes multiples et, dans certains cas, des solutions multisolitoniques.

L'approche méthodologique (algébrique et géométrique) est basée sur une adaptation de la méthode des symétries conditionnelles et de la méthode des caractéristiques généralisées. L'origine de cette dernière méthode est basée sur le travail de Bernhard Riemann [**105**] datant de 1859 où le problème de la propagation et de la superposition des ondes simples (appelées par la suite les ondes de Riemann) a été établi pour la première fois. B. Riemann a considéré le système homogène hyperbolique décrivant le flot d'un fluide idéal en deux variables dépendantes et deux variables indépendantes suivant

$$\rho_t + v\rho_x + \rho v_x = 0, \quad v_t + vv_x + \frac{\kappa}{\rho}\rho_x = 0, \quad \kappa = \frac{2}{\gamma - 1}, \tag{0.0.3}$$

où $\rho > 0$ représente la densité, v la vélocité du flot unidimensionnel et γ est l'exposant adiabatique du milieu de propagation. Sous forme matricielle, ces équations s'écrivent

$$\begin{pmatrix} \rho \\ v \end{pmatrix}_t + \begin{pmatrix} v & \rho \\ \kappa/\rho & v \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{0.0.4}$$

Classiquement, pour les systèmes bidimensionnels, les invariants de Riemann r^1, r^2 sont des fonctions de t et x permettant d'écrire le système (0.0.4) sous une forme diagonale. B. Riemann a ainsi obtenu la solution d'onde double sous la forme [**105**]

$$u = \kappa^{1/2}(r^1 - r^2 + u_0), \quad \rho = \rho_0 e^{r^1 + r^2}, \quad u_0, \rho_0 \in \mathbb{R}, \tag{0.0.5}$$

où les fonctions r^1, r^2 satisfont les équations quasi linéaires en termes des invariants de Riemann,

$$\frac{\partial r^1}{\partial t} + \left[\kappa^{1/2}(r^1 - r^2 + 1 + u_0)\right] \frac{\partial r^1}{\partial x} = 0, \quad \frac{\partial r^2}{\partial t} + \left[\kappa^{1/2}(r^1 - r^2 - 1 + u_0)\right] \frac{\partial r^2}{\partial x} = 0. \tag{0.0.6}$$

Spécifiquement, B.Riemann a étudié la superposition de deux ondes simples se propageant dans des directions opposées et a observé que les dérivées de cette solution tendent vers l'infini après un certain temps $t = T$, ce qui peut permettre le développement d'ondes de choc. Depuis cette étude, plusieurs auteurs [**20, 21, 62, 63, 73, 76, 107, 110, 112, 114**] ont généralisé cette idée et l'ont appliquée à plusieurs domaines différents de la physique.

Les systèmes du type (0.0.1) qui ont été les plus étudiés jusqu'à présent sont les systèmes en deux variables indépendantes

$$u_t + a(u)u_x = b(u), \quad a \in \mathbb{R}^{l \times q}, \quad b \in \mathbb{R}^l. \tag{0.0.7}$$

Toutefois, aucune théorie complète et satisfaisante n'existe à ce jour pour ces systèmes. Par exemple, aucun théorème d'existence globale et d'unicitié pour les problèmes aux données initiales n'a pu être démontré, sauf pour certains cas particuliers. Ces cas particuliers incluent, entre autres, les conditions nécessaires et suffisantes pour l'existence globale dans le temps d'une solution lisse pour les systèmes hyperboliques de lois de conservation en $(1 + 1)$ dimensions obtenus précédemment par Glimm [**41**], Lax [**42**] et Smoller [**109**], par exemple. Même pour des données initiales suffisamment petites, les solutions des systèmes hyperboliques quasi linéaires ne peuvent être définies sur une période de temps arbitraire. Ces solutions « explosent » après un intervalle de temps fini : en général, les dérivées premières d'une solution deviennent non bornées après un certain temps $T > 0$, et une solution pour $t > T$ n'existe pas. Dans le contexte de problèmes physiques, ce phénomène est appelé la catastrophe du gradient et survient même lorsque l'intuition physique nous mènerait à croire en l'existence d'une solution au-delà du temps T. En d'autres mots, la difficulté majeure dans ce problème concerne la construction de solutions globales de problèmes de Cauchy pour les systèmes hyperboliques quasi linéaires. Il a été démontré (voir [**20, 66**]) que pour des données initiales analytiques suffisamment petites, il existe un intervalle fini $[t_0, T]$ dans lequel la catastrophe du gradient ne survient pas. Ceci nous permet de considérer des problèmes de propagation et de superposition non linéaire d'ondes dans cet intervalle en utilisant la méthode des caractéristiques. À l'aide de cette méthode, l'existence, l'unicité et la dépendance continue de la solution sur les données initiales ont été démontrées par plusieurs auteurs (voir par exemple [**77, 104, 109**]). Les résultats obtenus sont significatifs dans le sens que la solution a été construite partout dans ce domaine où son existence était prévue.

L'expression (0.0.6) pour les invariants de Riemann ne s'étend pas facilement aux systèmes multidimensionnels. Par exemple, pour un système en $(2+1)$ dimensions,

$$u_t + a^1(u)u_x + a^2(u)u_y = 0, \quad a^i \in \mathbb{R}^{l\times q}, \tag{0.0.8}$$

la généralisation la plus naturelle des solutions en invariants de Riemann consiste à chercher une solution sous la forme $u = f(r^1(t,x,y), r^2(t,x,y))$, où les invariants r^1, r^2 satisfont des systèmes diagonaux

$$\frac{\partial r^A}{\partial t} + v_1^A(r^1,r^2)\frac{\partial r^A}{\partial x} = 0, \quad \frac{\partial r^A}{\partial y} + v_2^A(r^1,r^2)\frac{\partial r^A}{\partial x} = 0, \quad A = 1,2. \tag{0.0.9}$$

Cette méthode, appelée méthode des réductions hydrodynamiques, implique les conditions de compatibilité suivantes sur les fonctions v^A et v^B [**36, 111**],

$$\frac{\partial_B v_1^A}{v_1^B - v_1^A} = \frac{\partial_A v_2^B}{v_2^B - v_2^A}, \quad A \neq B = 1,2, \tag{0.0.10}$$

où ∂_A et ∂_B représentent les dérivées par rapport à r^A et r^B respectivement.

Étant donné ces résultats, l'approche proposée dans cette thèse consiste à étudier les solutions du système (0.0.1) sous la forme implicite

$$\begin{aligned} &u = f\left(r^1(x,u), \ldots, r^k(x,u)\right), \\ &r^A(x,u) = \lambda_i^A(u)x^i, \quad \lambda^A : \mathbb{R}^q \to \mathbb{R}^p, \quad A = 1,\ldots,k<p, \end{aligned} \tag{0.0.11}$$

où $f : \mathbb{R}^k \to \mathbb{R}^q$ est une fonction à déterminer et les fonctions $r^A(x,u)$ sont appelées les invariants de Riemann associés aux vecteurs d'onde linéairement indépendants λ^A. La matrice jacobienne de la relation (0.0.11) peut s'écrire sous la forme

$$\partial u = \left(\frac{\partial u^\alpha}{\partial x^i}\right) = \left(\mathcal{I}_q - \frac{\partial f}{\partial r}\frac{\partial r}{\partial u}\right)^{-1}\frac{\partial f}{\partial r}\lambda, \tag{0.0.12}$$

ou, de façon équivalente

$$\partial u = \left(\frac{\partial u^\alpha}{\partial x^i}\right) = \frac{\partial f}{\partial r}\left(\mathcal{I}_k - \frac{\partial r}{\partial u}\frac{\partial f}{\partial r}\right)^{-1}\lambda. \tag{0.0.13}$$

Ici, les matrices apparaissant dans (0.0.12) et (0.0.13) sont définies par

$$\begin{aligned} &\frac{\partial f}{\partial r} = \left(\frac{\partial f^\alpha}{\partial r^A}\right) \in \mathbb{R}^{q\times k}, \quad \lambda = (\lambda_i^A) \in \mathbb{R}^{k\times p}, \\ &\frac{\partial r}{\partial u} = \eta_i x^i, \quad \eta_i = \left(\frac{\partial \lambda_i^A}{\partial u^\alpha}\right) \in \mathbb{R}^{k\times q}, \quad a = 0, \ldots, n, \end{aligned} \tag{0.0.14}$$

et les matrices $\mathcal{I}_k$ et $\mathcal{I}_q$ représentent les matrices identité de dimension $k\times k$ et $q\times q$ respectivement. L'équivalence des expressions (0.0.12) et (0.0.13) est établie en considérant l'identité

$$\frac{\partial f}{\partial r} - \frac{\partial f}{\partial r}\frac{\partial r}{\partial u}\frac{\partial f}{\partial r} = \frac{\partial f}{\partial r}\left(\mathcal{I}_k - \frac{\partial r}{\partial u}\frac{\partial f}{\partial r}\right) = \left(\mathcal{I}_q - \frac{\partial f}{\partial r}\frac{\partial r}{\partial u}\right)\frac{\partial f}{\partial r}.$$

Nous dirons qu'une solution de la forme (0.0.11) est une solution de rang k s'il existe un voisinage ouvert de l'origine à l'intérieur duquel u peut être exprimé explicitement en termes des variables indépendantes et le rang de sa matrice jacobienne est égal à k,

$$\operatorname{rang}\left(\frac{\partial u^\alpha}{\partial x^i}\right) = k. \tag{0.0.15}$$

Il est important de noter que les conditions d'inversibilité des matrices

$$M_1 = \mathcal{I}_q - \frac{\partial f}{\partial r}\frac{\partial r}{\partial u} \in \mathbb{R}^{q\times q}, \quad M_2 = \mathcal{I}_k - \frac{\partial r}{\partial u}\frac{\partial f}{\partial r} \in \mathbb{R}^{k\times k},$$

apparaissant dans les expressions (0.0.12) et (0.0.13) restreignent le domaine de définition de la matrice jacobienne ∂u. Toutefois, l'inverse des matrices M_1 et M_2 est bien définie dans un ouvert de l'origine $x = 0$, puisqu'en tenant compte des expressions (0.0.11) et (0.0.14), nous avons

$$M_1 = \mathcal{I}_q - \sum_{i=0}^{p}\frac{\partial f}{\partial r}\eta_i x^i,$$

et similairement pour M_2. Notons également que les régions d'inversibilité des matrices M_1 et M_2 coincident puisque par la relation de Weinstein-Aronzjain

[113]

$$\det(\mathcal{I}_k - PQ) = \det(\mathcal{I}_q - QP), \quad P \in \mathbb{R}^{k\times q}, \quad Q \in \mathbb{R}^{q\times k}, \tag{0.0.16}$$

nous avons que $\det(M_1) = \det(M_2)$. La relation (0.0.16) peut être obtenue en considérant l'égalité

$$\det\left[\begin{pmatrix} \mathcal{I}_k & P \\ Q & \mathcal{I}_q \end{pmatrix}\begin{pmatrix} \mathcal{I}_k & 0 \\ -Q & \mathcal{I}_q \end{pmatrix}\right] = \det\left[\begin{pmatrix} \mathcal{I}_k & 0 \\ -Q & \mathcal{I}_q \end{pmatrix}\begin{pmatrix} \mathcal{I}_k & P \\ Q & \mathcal{I}_q \end{pmatrix}\right]. \tag{0.0.17}$$

Ainsi, nous pouvons construire des solutions locales de la forme (0.0.11) dans un ouvert de l'origine dans lequel $\det(M_i) \neq 0$, $i = 1, 2$. Il est commode pour étudier les solutions sous la forme (0.0.11) de réécrire le système initial (0.0.1) sous la forme de traces

$$\mathrm{Tr}\left[\mathcal{A}^\mu(u)\partial u\right] = b^\mu, \quad \mu = 1, \ldots, l, \tag{0.0.18}$$

où les matrices $\mathcal{A}^\mu$ sont des matrices de dimension $p\times q$. Par exemple, les équations d'Euler unidimensionnelles (0.0.4) s'écrivent dans cette formulation

$$\mathrm{Tr}\left[\begin{pmatrix} 1 & 0 \\ v & \rho \end{pmatrix}\begin{pmatrix} \rho_t & \rho_x \\ v_t & v_x \end{pmatrix}\right] = 0, \ \mathrm{Tr}\left[\begin{pmatrix} 0 & 1 \\ \kappa/\rho & v \end{pmatrix}\begin{pmatrix} \rho_t & \rho_x \\ v_t & v_x \end{pmatrix}\right] = 0. \tag{0.0.19}$$

En insérant l'expression de la matrice jacobienne (0.0.12) dans le système initial (0.0.18), nous réduisons le problème des solutions de rang k à

$$\mathrm{Tr}\left[\mathcal{A}^\mu\left(\mathcal{I}_q - \frac{\partial f}{\partial r}\frac{\partial r}{\partial u}\right)^{-1}\frac{\partial f}{\partial r}\lambda\right] = b^\mu, \quad \mu = 1, \ldots, l, \tag{0.0.20}$$

ou, en utilisant l'expression équivalente (0.0.13),

$$\mathrm{Tr}\left[\mathcal{A}^\mu\frac{\partial f}{\partial r}\left(\mathcal{I}_k - \frac{\partial r}{\partial u}\frac{\partial f}{\partial r}\right)^{-1}\lambda\right] = b^\mu, \quad \mu = 1, \ldots, l. \tag{0.0.21}$$

Nous démontrons dans les chapitres 2 - 5, que les solutions de rang k de la forme (0.0.11) sont invariantes sous une algèbre abélienne de symétrie conditionnelle de dimension $p - k$. À partir de leurs propriétés d'invariance, nous montrons que le théorème de Cayley-Hamilton permet d'obtenir des conditions d'existence de ce type de solutions pour les systèmes homogènes et non homogènes respectivement.

Nous présentons ainsi dans l'annexe C l'algorithme de Leverrier-Faddeev permettant de donner explicitement les coefficients du polynôme caractéristique. Nous fournissons également une preuve simple de ce résultat.

Les propriétés de la théorie des groupes appliquées aux systèmes non linéaires d'équations aux dérivées partielles en plusieurs dimensions ainsi que les techniques de construction de classes de solutions particulières constituent un des sujets essentiels de la recherche en mathématiques modernes. La méthode des groupes de Lie appliquée aux équations différentielles [**5, 88, 89**] s'est avérée extrêmement puissante et a mené à des progrès dans ce domaine ainsi qu'à de nouvelles techniques efficaces. Mentionnons à ce titre la méthode des solutions partiellement invariantes [**90, 93, 94**], la méthode non classique [**5, 70**] et la méthode des symétries conditionnelles [**87, 116**]. Cette dernière consiste en l'ajout de contraintes différentielles appropriées permettant d'obtenir, dans certains cas, de plus larges classes de symétries que pour le système original.

Jusqu'à présent, la méthode des symétries conditionnelles (MSC) et la méthode des caractéristiques généralisées (MCG) [**102**] ont été considérées séparément et ont mené dans la plupart des cas à des classes de solutions différentes. Une analyse systématique par la méthode de réduction par symétrie nous permet de construire différentes classes de solutions invariantes. Cette méthode ne se limite pas aux systèmes non elliptiques quasi linéaires. En revanche, la MCG limitée à ces systèmes, s'est avérée plus utile pour produire des solutions décrivant des ondes ou des ondes multiples. Les relations entre les deux méthodes constituent un sujet intéressant et notre intention est de combiner les avantages de ces deux méthodes. Cela constitue le sujet principal de cette thèse.

La méthode présentée dans cette thèse pour construire des solutions de rang k est basée sur le théorème de Cayley-Hamilton ainsi que sur une variation de la MSC présentée dans [**26, 51, 86**]. L'idée principale est de sélectionner les contraintes différentielles à utiliser pour garantir que les solutions invariantes s'expriment en termes d'invariants de Riemann. Il est intéressant de noter que ces contraintes sont moins restrictives que les conditions requises par la MCG. Cela est dû au fait que la MCG demande que l'élément intégral (voir Annexe A) soit une combinaison linéaire des éléments intégraux simples obéissant les relation d'onde et de dispersion, ce qui n'est pas exigé par la MSC. Comme résultat, nous obtenons dans certains cas de plus larges classes de solutions que les k-ondes de Riemann obtenues par la MCG. Ces résultats peuvent être difficiles à obtenir à l'aide d'autres méthodes.

Plusieurs généralisations différentes de la méthode des invariants de Riemann et ses applications peuvent être trouvées dans la littérature [**29, 62, 63, 107**]. Parmi celles-ci, les superpositions non linéaires des k-ondes décrites dans [**84, 96**] peuvent être considérées comme étant des analogues non dispersives des solutions « n-gap » du système (0.0.1). Ces solutions sont celles d'un système diagonalisable pour les invariants de Riemann r^A généralisant l'expression (0.0.9) :

$$r^A_{x^i} + v^A_{i(j)}(r^1, \ldots, r^k) r^A_{x^j} = 0, \quad v^A_{i(j)} = -\lambda^A_i / \lambda^A_j, \tag{0.0.22}$$

où $A = 1, \ldots, k, \quad i \neq j = 1, \ldots, p$. Dans ce contexte, l'étude des structures de Poisson de type géométrique différentiel spécial a été effectuée pour la première fois par B. Dubrovin et S.P. Novikov [**27, 28**]. Ces auteurs ont développé un formalisme général hamiltonien pour les systèmes de type hydrodynamique reliés à l'étude des équations de Whitham décrivant l'évolution de solutions multiphases lentement modulées de systèmes d'équations non linéaires. Ils ont également démontré que les structures de Poisson locales de type hydrodynamique sont générées par des métriques plates pseudo-Riemanniennes et peuvent être réduites à des formes constantes par des changements de coordonnés locales sur les variétés correspondantes. Suivant cette approche hamiltonienne, S.P. Tsarev [**111**] a développé la théorie d'intégration pour les systèmes diagonalisables hamiltoniens de type hydrodynamique basée sur une approche géométrique différentielle. En particulier, cette technique spécifique implique des contraintes différentielles sur les fonctions v^A_{ij} analogues aux contraintes obtenues (0.0.10) dans le cas des 2-ondes. Elles adoptent la forme

$$\frac{\partial_B v^A_{i(j)}}{v^A_{i(j)} - v^A_{j(i)}} - \frac{\partial_A v^B_{i(j)}}{v^B_{i(j)} - v^B_{j(i)}} = 0, \quad i \neq j = 1, \ldots, p, \quad A \neq B = 1, \ldots, k, \tag{0.0.23}$$

où nous n'utilisons pas de sommation sur les indices. Comme pour le cas des k-ondes de Riemann, si le système (0.0.23) est satisfait pour les fonctions v^A_{ij}, alors l'intégrale générale du système (0.0.22) peut être obtenue en résolvant le système [**53, 102**]

$$\lambda_i^A(r^1, \ldots, r^k)x^i = \psi^A(r^1, \ldots, r^k) \qquad (0.0.24)$$

par rapport aux variables $r^1, \ldots, r^k$, où les fonctions $\psi^A(r^1, \ldots, r^k)$ satisfont les relations

$$\frac{\partial \psi^A}{\partial r^B} = \alpha_{AB}(r)\psi^A + \beta_{AB}(r)\psi^B, \quad A \neq B = 1, \ldots, k. \qquad (0.0.25)$$

Cette idée a été poursuivie par Ferapontov [**33**] et Pavlov [**96**] où une généralisation des structures de Poisson non locales a été effectuée. Cette généralisation est générée par des métriques pseudo-Riemanniennes arbitraires avec courbure de Riemann constante. Cela joue un rôle crucial dans la théorie des systèmes hydrodynamiques, surtout pour les équations de Whitham. Plus récemment, d'autres généralisations géométriques différentielles non locales des structures de Poisson ont été introduites par plusieurs auteurs (voir par exemple, [**27, 28, 29, 36, 34, 37, 83, 95, 111**] et références incluses).

En revanche, l'approche proposée dans cette thèse ne demande pas l'utilisation des conditions différentielles (0.0.23). Donc, elle n'impose pas de contraintes de ce type sur les fonctions v_{ij}^A lorsque les 1-formes sont linéairement indépendantes et $k < p$. Cependant, si nous enlevons ces hypothèses et si les λ^A peuvent être linéairement dépendants et $k \geq p$, alors l'approche présentée dans les travaux [**28, 111**] est utile et constitue une méthode efficace pour classifier les systèmes intégrables de type hydrodynamique.

Dans le contexte de la généralisation de la méthode des invariants de Riemann, il est important d'utiliser la méthode des caractéristiques généralisées développée par [**8, 54, 102**] afin de construire des classes de solutions pour les systèmes non elliptiques (0.0.1). Nous présentons dans le chapitre 2, section 2.2, un bref résumé des principaux concepts de cette méthode. Une caractéristique de cette approche est la formulation de la notion d'onde et de superpositions élastiques et non élastiques. Cette formulation a été rendue possible en considérant l'aspect algébrique des équations aux dérivées partielles où l'élément général intégral a été représenté comme une combinaison linéaire des éléments intégraux simples (voir Annexe A, sections A.1 et A.2). Ces éléments simples sont liés à des champs de vecteurs générant des courbes caractéristiques séparément dans l'espace des variables indépendantes X et dépendantes U. L'introduction des éléments simples s'est avérée très utile pour la construction des solutions d'ondes de Riemann ; c'est-à-dire que chaque élément simple correspond à une solution d'onde de Riemann. Cela a permis de généraliser la méthode des invariants de Riemann et de trouver des classes de solution pour les systèmes multidimensionnels. Il a été démontré **[46, 99]** qu'à partir des éléments simples intégraux, nous pouvons construire des solutions plus générales représentant des k-ondes. Nous illustrons cet effet par un exemple de superposition de deux ondes de Riemann pour un système en une variable spatiale.

Il a été démontré **[64, 65]** que si les données initiales sont suffisament petites, il existe un intervalle $[t_0, T]$ pour lequel la catastrophe du gradient de la solution $r^s(t, x)$, $s = 1, 2$, du système écrit en invariants de Riemann

$$r^s_t + v^s(r^1, r^2) r^s_x = 0, \quad s = 1, 2, \tag{0.0.26}$$

n'a pas lieu. Puisque chaque fonction $r^s(t, x)$ est constante le long d'une courbe caractéristique

$$\mathcal{C}^{(s)} \quad : \quad \frac{dx}{dt} = v^s(r^1(t, x), r^2(t, x)) \tag{0.0.27}$$

appropriée du système (0.0.26), les données initiales peuvent être choisies dans l'espace X pour la fonction r^s de telle façon que les supports des dérivées r^s_x soient compacts et disjoints

$$\begin{aligned} \exists a_s, b_s \in \mathbb{R} \quad t = t_0 \quad : \quad & \operatorname{supp} r^s_x(t_0, x) \subset [a_s, b_s], \quad s = 1, 2, \\ & \operatorname{supp} r^1_x(t_0, x) \cap \operatorname{supp} r^2_x(t_0, x) = \emptyset. \end{aligned} \tag{0.0.28}$$

Alors, pour tout temps $t \in [t_0, T]$, le support $\operatorname{supp} r^s_x(t, x)$ est contenu dans une bande entre les caractéristiques des familles $\mathcal{C}^{(s)}$ passant par les extrémités de l'intervalle $[a_s, b_s]$. De plus, si les données initiales sont suffisamment petites, alors dans l'intervalle $[t_0, T]$, les conditions suivantes sont satisfaites **[20]**

$$\exists C > 0 \quad \forall (t, x), (t, x') \in [t_0, T] \times \mathbb{R} \quad v^1(t, x) - v^2(t, x') \geq C. \tag{0.0.29}$$

Cela signifie que chaque caractéristique de la famille $\mathcal{C}^{(1)}$ possède une tangente avec inclinaison (mesurée par rapport à l'axe des x positifs) inférieure à toute caractéristique de la famille $\mathcal{C}^{(2)}$. Il est évident que, dans ce cas, les bandes contenant $\operatorname{supp} r_x^s(t,x)$ divisent le reste de l'espace X en quatre régions disjointes.

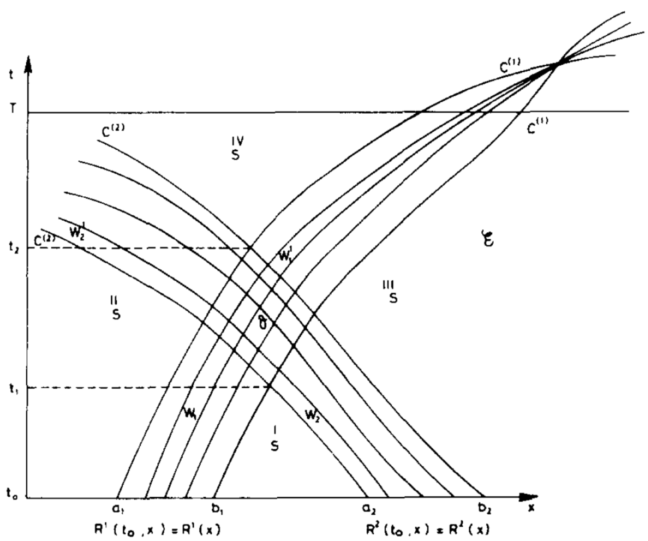

FIG. 0.1. La propagation et superposition de deux ondes ω_1 et ω_2. Si les caractéristiques d'une famille se croisent, nous choisissons une valeur particulière de temps T afin d'exclure la possibilité d'une catastrophe du gradient.

Dans la région

$$G := X \setminus \left\{\operatorname{supp} r_x^1(t,\cdot) \cup \operatorname{supp} r_x^2(t,\cdot)\right\}, \tag{0.0.30}$$

la solution du système (0.0.26) est constante.

Supposons que t_1 et t_2 sont les moments pour lesquels $\operatorname{supp} r_x^1(t,x)$ et $\operatorname{supp} r_x^2(t,x)$ possèdent seulement un point en commun. Pour le temps $t \in (t_0, t_1)$, nous avons

$$\operatorname{supp} r_x^1(t,x) \cap \operatorname{supp} r_x^2(t,x) = \emptyset, \quad t \in (t_0, t_1). \tag{0.0.31}$$

Donc, la solution r^s peut être interprétée comme la propagation de deux ondes séparées simples qui n'interagissent pas. Pour les temps $t \in [t_1, t_2]$, les caractéristiques des familles $\mathcal{C}^{(1)}$ et $\mathcal{C}^{(2)}$ contenant $\operatorname{supp} r_x^s(t,x)$ se croisent entre elles, c'est-à-dire que

$$\operatorname{supp} r_x^1(t,x) \cap \operatorname{supp} r_x^2(t,x) \neq \emptyset, \quad t \in [t_1, t_2]. \tag{0.0.32}$$

Nous interprétons cela comme une superposition de deux ondes (onde double). Pour les temps $t > t_2$, en vertu des conditions (0.0.26) et (0.0.27), les bandes contenant les supports des ondes simples se séparent à nouveau, c'est-à-dire que nous avons la relation

$$\operatorname{supp} r^1_x(t,x) \cap \operatorname{supp} r^2_x(t,x) = \emptyset, \quad t > t_2. \tag{0.0.33}$$

Cela signifie que la solution $r^s(t,x)$ décroît exactement en deux ondes simples se propageant séparément.

Sous les hypothèses ci-dessus, il a été démontré [**55, 56, 101**] pour les superpositions de deux ondes pouvant être décrites en termes des invariants de Riemann, que la solution décroît de façon exacte en deux ondes simples de même type que celles posées dans les données initiales. Dans ce cas, nous avons une loi de conservation pour le nombre et le type d'ondes simples, c'est-à-dire que nous pouvons constater des interactions élastiques des ondes simples. L'interprétation de ces superpositions dans le cas où nous avons plus de deux ondes simples ($k > 2$) est effectuée de façon analogue. Elle est toutefois plus compliquée car la région $\mathcal{D} \subset X$ est divisée par les supports des fonctions $r^1, \ldots, r^k$ en 2^k sous-régions. Ce cas plus général a été discuté dans [**55, 56, 102**]. Dans ces articles, il a été démontré que le nombre et le type des ondes sont également conservés dans le cas d'interactions de plusieurs ondes simples. De plus, ces solutions décroissent de façon exacte en des ondes simples du même type que celles posées dans les données initiales. Des exemples de telles solutions pour les équations de l'hydrodynamique peuvent être trouvées dans la littérature du sujet [**54, 102**].

Étant donné ces résultats, notre objectif est de construire une extension de cette théorie et de vérifier son efficacité pour le cas des symétries conditionnelles appliquées aux systèmes de type hydrodynamique multidimensionnels. L'approche proposée dans cette thèse est basée sur une version de la méthode des symétries conditionnelles qui nous permet de construire les solutions de rang k exprimables en termes d'invariants de Riemann. Notre approche se situe plus fondamentalement dans les aspects de l'invariance sous l'action de groupes de Lie pour les équations différentielles. Cela nous permet d'obtenir de nouveaux résultats sur la solvabilité des systèmes quasi linéaires hyperboliques en plusieurs dimensions.

Nous présentons maintenant un aperçu des résultats principaux obtenus dans cette thèse.

Dans les articles présentés dans les chapitres 1 et 2, nous établissons pour la première fois les propriétés d'invariance de groupe des solutions exprimées en termes d'invariants de Riemann pour les systèmes quasilinéaires d'EDPs du premier ordre. L'analyse de ce type de solution, que nous appelons solutions de rang k, du point de vue de la méthode des symétries conditionnelles nous a permis de généraliser la méthode des invariants de Riemann aux systèmes multidimensionnels et d'établir une procédure générale permettant leur construction. À l'aide des méthodes de la théorie des groupes et du théorème de Cayley-Hamilton, nous avons pu obtenir une proposition fournissant les conditions nécessaires et suffisantes pour garantir leur existence. L'application de cette procédure nous a permis d'obtenir de nouvelles classes de solutions pour plusieurs systèmes de type hydrodynamique. Notamment, nous avons construit des solutions de rang k pour les équations décrivant le flot d'un fluide idéal isentropique en $(3+1)$ dimensions données par

$$\mathrm{D}a + \kappa^{-1} a \operatorname{div} \vec{u} = 0, \quad \mathrm{D}\vec{u} + \kappa a \nabla a = 0, \tag{0.0.34}$$

où D est la dérivée convective

$$\mathrm{D} = \frac{\partial}{\partial t} + (\vec{u} \cdot \nabla),$$

le champ de vecteur représentant la vélocité du fluide est donné par $\vec{u} = (u^1, u^2, u^3)$ alors que a est la vitesse du son dans le milieu considéré. La constante κ est reliée à l'exposant adiabatique du fluide γ par la relation $\kappa = 2(\gamma - 1)^{-1}$ et caractérise par conséquent le milieu de propagation. Sous forme matricielle, ce système s'écrit

$$u_t + a^1(u) u_{x^1} + a^2(u) u_{x^2} + a^3(u) u_{x^3} = 0, \tag{0.0.35}$$

où les matrices $a^i(u)$ sont données par

$$a^i(u) = \begin{pmatrix} u^i & \delta_{i1}\kappa^{-1}a & \delta_{i2}\kappa^{-1}a & \delta_{i3}\kappa^{-1}a \\ \delta_{i1}\kappa a & u^i & 0 & 0 \\ \delta_{i2}\kappa a & 0 & u^i & 0 \\ \delta_{i3}\kappa a & 0 & 0 & u^i \end{pmatrix}, \quad i = 1, 2, 3, \tag{0.0.36}$$

et $\delta_{ij} = 1$ si $i = j$ et 0 sinon. Les vecteurs d'onde du système (0.0.34) s'écrivent sous la forme $(\lambda_0, \vec{\lambda})$, où λ_0 est la vélocité de phase de l'onde et $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ la direction de l'onde considérée. Deux types de vecteurs d'onde sont obtenus en résolvant la relation de dispersion du système isentropique (0.0.34)

$$\det\left(\lambda_0 \mathcal{I}_4 + \lambda_i a^i(u)\right) = \left(\lambda_0 + \vec{u}\cdot\vec{\lambda}\right)\left[\left(\lambda_0 + \vec{u}\cdot\vec{\lambda}\right)^2 - a^2|\vec{\lambda}|^2\right] = 0. \tag{0.0.37}$$

Les vecteurs d'onde sont donc de type potentiel (E) ou rotationnel (S) définis par

$$\begin{aligned} \lambda^E &= \left(\lambda_0^E, \vec{\lambda}^E\right) = \left(\vec{e}\cdot\vec{u} + \varepsilon a, -\vec{e}\right), \quad \varepsilon = \pm 1, \\ \lambda^S &= \left(\lambda_0^S, \vec{\lambda}^S\right) = \left(\det\left(\vec{u}, \vec{e}, \vec{m}\right), -\vec{e}\times\vec{m}\right). \end{aligned} \tag{0.0.38}$$

où $\vec{e}$ est un vecteur unitaire et $\vec{m}$ est un vecteur arbitraire. En étudiant les interactions entre 2 ou 3 ondes des types E et S, nous avons retrouvé les classes de solutions obtenues précédemment par la méthode des caractéristiques généralisées [**100**] en plus d'en obtenir de nouvelles dans les cas ES, SS, EES, ESS et SSS (voir tables 3.1, 3.2). Ces nouvelles solutions sont plus générales que celles qui étaient déjà connues puisque plusieurs d'entre elles dépendent de fonctions arbitraires de plusieurs arguments, ce qui n'était pas le cas pour les solutions construites à l'aide de la MCG qui sont limitées à k fonctions arbitraires d'une variable [**99**]. Par exemple, nous avons été en mesure de trouver la solution de rang 2 suivante sous la forme implicite

$$\begin{aligned} &a = \kappa^{-1}(A_1 r^1 + B_1) + a_0, \quad u^1 = \sin g(r^2, r^3), \\ &u^2 = -\cos g(r^2, r^3), \quad u^3 = A_1 r^1 + B_1 + u_0^3, \quad A_1, B_1 \in \mathbb{R}, \end{aligned} \tag{0.0.39}$$

où les fonctions r^1, r^2, r^3 sont les invariants de Riemann satisfaisant les relations

$$\begin{aligned} r^1 &= \frac{((1+\kappa^{-1})B_1 + a_0 + u_0^3)t - x^3}{1 - (1+\kappa^{-1})A_1 t}, \\ r^2 &= t - x^1 \sin g(r^2, r^3) + x^2 \cos g(r^2, r^3), \\ r^3 &= \Psi\left[\frac{1}{A_1}(A_1(\kappa a_0 - u_0^3)t + x^3 - \kappa a_0 - B_1)((1+\kappa)A_1 t - \kappa)^{-\kappa/\kappa+1}\right]. \end{aligned} \tag{0.0.40}$$

Cette solution dépend de deux fonctions arbitraires $\Psi(\cdot)$ et $g(r^2, r^3)$ dont l'une dépend de deux variables. Ceci démontre que les conditions imposées par la MSC sont moins restrictives que celles requises par la MCG. Cette dépendance sur une ou plusieurs fonctions arbitraires représente une particularité très importante des solutions de rang k. Par un choix judicieux de ces fonctions, nous pouvons modifier les propriétés géométriques du flot considéré. C'est donc dire qu'il est possible de satisfaire une plus grande variété de données initiales physiquement admissibles et surtout, nous pouvons choisir ces fonctions de telle sorte que le flot ne possède aucune singularité. Ce résultat est d'une grande importance puisque, jusqu'à présent, les solutions exprimées en termes d'invariants de Riemann admettent souvent une catastrophe du gradient après un certain temps fini T, même pour des données initiales arbitrairement petites. Dans le contexte de la MSC, nous avons donc montré qu'il est possible de construire des solutions exprimées en termes d'invariants de Riemann qui demeurent bornées pour tout temps $t > 0$. Nous donnons ainsi dans le chapitre 2 une liste de solutions implicites et explicites possédant un comportement solitonique de type « bump », « kink » ou d'onde multiple. Par exemple, nous obtenons une solution du système hydrodynamique (0.0.34) modélisant des ondes concentriques non stationnaires dans un milieu aquatique. Ce résultat a été obtenu sous une autre forme que la solution implicite (0.0.11), c'est-à-dire que les invariants de Riemann r^A sont représentés sous deux formes. Les invariants r^1 et r^2 sont donnés sous la forme $r^s(x,u) = \lambda_i^s(x,u)x^i$, $s = 1, 2$, tandis que l'invariant supplémentaire r^3 obéit une EDP de la forme

$$\frac{\partial r^3}{\partial t} + v(r^1, r^2, r^3)\frac{\partial r^3}{\partial x^3} = 0.$$

L'existence de cette solution, définie par les équations (2.7.15), (2.7.16) et (2.7.17), a été déterminée précédemment par un traitement numérique [**13**] et sa forme analytique est obtenue pour la première fois.

Nous présentons dans le chapitre 3 plusieurs nouvelles classes de solutions bornées pour les équations (0.0.34) qui sont exprimées en termes de la fonction elliptique $\wp$ de Weierstrass. Ces solutions sont obtenues en considérant certaines réductions par symétrie spécifiques de l'équation pour les champs non linéaires de Klein-Gordon, qui possède de larges classes de solutions de type solitonique. Cette approche nous a donc permis de trouver plusieurs propriétés qualitatives et quantitatives des solutions en invariants de Riemann et ainsi de construire des solutions physiquement plausibles. C'est le cas, entre autres, de la solution de rang 3 du système (0.0.34) résultant de l'interaction de 3 ondes potentielles. Nous avons démontré que ce type d'interaction est nécessairement linéaire et n'est possible que lorsque l'angle φ_{ij} entre les vecteurs de direction $\vec{\lambda}^i$ et $\vec{\lambda}^j$ des ondes considérées satisfait la relation [**48, 100**]

$$\vec{\lambda}^i \cdot \vec{\lambda}^j = \cos\varphi_{ij} = \frac{1-\gamma}{2}, \quad i \neq j = 1,2,3. \tag{0.0.41}$$

Puisque l'exposant adiabatique γ est caractéristique à chaque milieu, l'équation (0.0.41) permet d'identifier un milieu inconnu en y étudiant les interactions linéaires d'ondes de propagation. Sous cette hypothèse, nous avons pu construire la solution de rang 3 suivante (équation 3.2.16)

$$\begin{aligned} a &= \sum_{i=1}^{3} \frac{C_i}{\left(\wp(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^2) + \frac{1}{3}\right)^{1/2}}, \quad \vec{u} = \kappa \sum_{i=1}^{3} \frac{C_i \vec{\lambda}^i}{\left(\wp(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^2) + \frac{1}{3}\right)^{1/2}}, \\ r^i &= -(1+\kappa)\frac{C_i}{\left(\wp(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^2) + \frac{1}{3}\right)^{1/2}} t + \vec{\lambda}^i \cdot \vec{x}, \quad i = 1,2,3, \end{aligned} \tag{0.0.42}$$

exprimée en termes de la fonction elliptique $\wp(r^i, g_2, g_3)$ de Weierstrass satisfaisant l'équation différentielle ordinaire du premier ordre

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

où les constantes réelles g_2, g_3 sont appelés les invariants de la fonction elliptique $\wp$. En utilisant un résultat récent de W. Duke et Ö. Imamoglu [**31**] concernant les zéros de la fonction $\wp$ de Weierstrass, nous sommes parvenus à montrer que cette solution demeure bornée même lorsque les premières dérivées partielles des invariants de Riemann deviennent infinies après un certain temps fini. Nous présentons également dans la table 3.3 une liste de solutions exprimées en termes de la fonction de Weierstrass pouvant être soumises à une analyse semblable. Ces solutions sont de type solitonique et représentent des ondes périodiques, des « bumps » et des « kinks ». De plus, nous construisons la solution générale de rang 2 du système isentropique (0.0.34) dans le cas où la vitesse du son a ne dépend que du temps t. La forme implicite de cette solution (équation (2.8.13)) est donnée par

$$\begin{aligned} u^1(t,x,y) &= C_1(x-u^1t) + \frac{\partial h}{\partial r^1}(x-u^1t, y-u^2t), \\ u^2(t,x,y) &= C_1(y-u^2t) - \frac{\partial h}{\partial r^2}(x-u^1t, y-u^2t), \\ a(t) &= A_1((1+C_1t)^2 + B_1t^2)^{-1/\kappa}, \quad A_1 \in \mathbb{R}^+, \end{aligned} \tag{0.0.43}$$

où la fonction h est une fonction des invariants $r^1 = x - u^1t$ et $r^2 = y - u^2t$ et satisfait l'équation de Monge-Ampère non homogène

$$h_{r^1r^1}h_{r^2r^2} - h_{r^1r^2} = B_1, \quad B_1 = -1, 0, 1. \tag{0.0.44}$$

En utilisant une demie transformée de Legendre, nous avons construit plusieurs classes de solutions de l'équation (0.0.44) afin d'obtenir des solutions implicites et explicites du fluide isentropique (0.0.34) (voir [**19**], Table 1).

Les chapitres 4 et 5 proposent deux extensions permettant l'applicabilité de cette approche aux systèmes de type hydrodynamique soumis à des forces extérieures. D'une part, nous construisons dans le chapitre 4 des solutions exprimées en termes d'invariants de Riemann pour le système non homogène d'équations décrivant un flot rotatif en eau peu profonde. Ces équations sont utilisées en géophysique pour modéliser plusieurs phénomènes atmosphériques et océaniques [**98**] et la solution de problèmes de Cauchy est habituellement traitée numériquement [**108**]. Elles sont données par

$$\begin{aligned} &u_t + uu_x + vu_y + gh_x = 2\Omega v, \\ &v_t + uv_x + vv_y + gh_y = -2\Omega u, \\ &h_t + uh_x + vh_y + h(u_x + v_y) = 0, \end{aligned} \tag{0.0.45}$$

où u, v modélisent la vélocité du flot bidimensionnel, h est la hauteur du fluide et Ω est la vélocité angulaire autour de l'axe des z. À travers une analyse de l'algèbre de symétrie de ces équations, nous sommes parvenus à déterminer une transformation ponctuelle localement inversible amenant ce système vers une forme homogène équivalente. Chaque solution de rang k du système homogène nous permet d'obtenir localement une solution du système (0.0.45) exprimée en termes d'invariants de Riemann. À titre d'exemple, la figure 4.1 illustre le comportement de type solitonique (« bump ») de la solution

$$\begin{aligned}
u &= \left(u_0 + \sqrt{g}(2\operatorname{sech}^2(r1) - \operatorname{sech}^2(r^2))\right)\tan(\Omega t) \\
&\quad - (v_0 + \sqrt{3g}\operatorname{sech}^2(r^2)) + \Omega(y - x\tan(\Omega t)), \\
v &= (v_0 + \sqrt{3g}\operatorname{sech}^2(r^2))\tan(\Omega t) + \sqrt{g}(2\operatorname{sech}^2(r^1) - \operatorname{sech}^2(r^2)) \\
&\quad - \Omega(x + y\tan(\Omega t)), \\
h &= (\operatorname{sech}^2(r^1) + \operatorname{sech}^2(r^2))^2\sec^2(\Omega t),
\end{aligned}$$

où les fonctions r^1, r^2 satisfont les équations implicites

$$\begin{aligned}
r^1 &= \frac{1}{2\Omega}\left(u_0 + 3\sqrt{g}\operatorname{sech}^2(r^1)\right)\tan(\Omega t) + \frac{1}{2}\left(y - x\tan(\Omega t)\right), \\
r^2 &= \frac{1}{2\Omega}\left(\frac{u_0}{2} + \frac{\sqrt{3}}{2}v_0 + 3\sqrt{g}\operatorname{sech}^2(r^2)\right)\tan(\Omega t) \\
&\quad - \frac{1}{4}\left(y - x\tan(\Omega t)\right) - \frac{\sqrt{3}}{4}\left(x + y\tan(\Omega t)\right).
\end{aligned}$$

D'autre part, nous donnons dans le chapitre 5 les conditions explicites garantissant l'existence de solutions de rang k exprimées en termes d'invariants de Riemann pour les systèmes non homogènes à l'aide de la MSC, du théorème de Cayley-Hamilton et de l'algorithme de Leverrier-Faddeev (voir Annexe C) . Ces conditions nous ont également permis d'obtenir plusieurs nouvelles classes de solutions pour les équations d'Euler en (3 + 1)-dimensions sous l'influence de la force gravitationnelle et d'une force de Coriolis.

Finalement, après une discussion sur les méthodes de construction des solutions solitoniques, nous illustrons dans le chapitre 6 la procédure de construction des solutions de rang 2 pour l'équation de Kadomtsev-Petviashvili sans dispersion et plusieurs nouvelles classes de solutions sont obtenues. Les travaux en cours et perspectives futures y sont également discutés.

Chapitre 1

RIEMANN INVARIANTS AND RANK-K SOLUTIONS OF HYPERBOLIC SYSTEMS

Référence complète : A.M. Grundland et B. Huard, Riemann Invariants and Rank-k Solutions of Hyperbolic Systems, *Journal of Nonlinear Mathematical Physics*, 13(3), 393-419, 2006.

Résumé

Dans cet article, nous employons une "méthode directe" pour obtenir des solutions de rang k pour tout système hyperbolique d'équations différentielles quasilinéaires du premier ordre en plusieurs dimensions. Nous discutons en détail les conditions nécessaires et suffisantes pour l'existence de ces types de solutions écrites en termes d'invariants de Riemann. La caractéristique la plus importante de cette approche est l'introduction de contraintes différentielles de premier ordre supplémentaires et compatibles avec le système d'EDPs original, ce qui conduit à une généralisation de la méthode des invariants de Riemann pour la résolution de systèmes d'EDPs en plusieurs dimensions. Nous démontrons l'utilité de cette approche par plusieurs exemples de systèmes de type hydrodynamique et de nouvelles classes de solutions sont obtenues.

Abstract

In this paper we employ a "direct method" in order to obtain rank-k solutions of any hyperbolic system of first order quasilinear differential equations in many

dimensions. We discuss in detail the necessary and sufficient conditions for existence of these type of solutions written in terms of Riemann invariants. The most important characteristic of this approach is the introduction of specific first order side conditions consistent with the original system of PDEs, leading to a generalization of the Riemann invariant method of solving multi-dimensional systems of PDEs. We have demonstrated the usefulness of our approach through several examples of hydrodynamic type systems ; new classes of solutions have been obtained in a closed form.

1.1. Introduction

This work has been motivated by a search for new ways of constructing multiple Riemann waves for nonlinear hyperbolic systems. Riemann waves and their superpositions were first studied two centuries ago in connection with differential equations describing a compressible isothermal gas flow, by D. Poisson [**103**] and later by B. Riemann [**105**]. Since then many different approaches to this topic have been developed by various authors with the purpose of constructing solutions to more general hydrodynamic-type systems of PDEs. For a classical presentation we refer reader to a treatise by R. Courant and D. Hilbert [**21**] and for a modern approach to the subject, see e.g. [**65, 99, 107**] and references therein. A review of most recent developments in this area can be found in [**23, 30, 76**].

The task of constructing multiple Riemann waves has been approached so far through the method of characteristics. It relies on treating Riemann invariants as new independent variables (which remain constant along appropriate characteristic curves of the basic system). This leads to the reduction of the dimensionality of the initial system which has to be subjected however to the additional differential constraints, limiting the scope of resulting solutions.

We propose here a new (though a very natural) way of looking at solutions expressible in terms of Riemann invariants, namely from the point of view of their group invariance properties. We show that this approach (initiated in [**26, 51**]) leads to the larger classes of solutions, extending beyond Riemann multiple waves.

We are looking for the rank-k solutions of first order quasilinear hyperbolic system of PDEs in p independent variables x^i and q unknown functions u^α of the form

$$\Delta^{\mu i}_{\alpha}(u)\, u^\alpha_i = 0, \quad \mu = 1, \ldots, l. \tag{1.1.1}$$

We denote by U and X the spaces of dependent variables $u = (u^1, \ldots, u^q) \in \mathbb{R}^q$ and independent variables $x = (x^1, \ldots, x^p) \in \mathbb{R}^p$, respectively. The functions $\Delta^{\mu i}_{\alpha}$ are assumed to be real valued functions on U and are components of the tensor products $\Delta^{\mu i}_{\alpha}\partial_i \otimes du^\alpha$ on $X \times U$. Here, we denote the partial derivatives by $u^\alpha_i = \partial_i u^\alpha \equiv \partial u^\alpha / \partial x^i$ and we adopt the convention that repeated indices are summed unless one of them is in a bracket. For simplicity we assume that all considered functions and manifolds are at least twice continuously differentiable in order to justify our manipulations. All our considerations have a local character. For our purposes it suffices to search for solutions defined on a neighborhood of the origin $x = 0$. In order to solve (1.1.1), we look for a map $f : X \to J^1(X \times U)$ annihilating the contact 1-forms, i.e.

$$f^*(du^\alpha - u^\alpha_i\, dx^i) = 0. \tag{1.1.2}$$

The image of f is in a submanifold of the first jet space J^1 over X given by (1.1.1) for which J^1 is equipped with coordinates $x^i, u^\alpha, u^\alpha_i$.

This paper is organized as follows. Section 1.2 contains a detailed account of the construction of rank-1 solutions of PDEs (1.1.1). In section 1.3 we discuss the construction of rank-k solutions, using geometric and group invariant properties of the system (1.1.1). Section 1.4 deals with a number of examples of hydrodynamic type systems which illustrate the theoretical considerations. Several new classes of solutions in implicit and explicit form are obtained. Section 1.5 contains a comparison of our results with the generalized method of characteristics for multi-dimensional systems of PDEs.

1.2. The rank-1 solutions

It is well known **[21]** that any hyperbolic system (1.1.1) admits rank-1 solutions

$$u = f(r), \quad r(x,u) = \lambda_i(u)\, x^i, \tag{1.2.1}$$

where $f = (f^\alpha)$ are some functions of r and a wave vector is a nonzero function

$$\lambda(u) = (\lambda_1(u), \ldots, \lambda_p(u)) \tag{1.2.2}$$

such that

$$\ker(\Delta^i \lambda_i) \neq 0. \tag{1.2.3}$$

Solution (1.2.1) is called a Riemann wave and the scalar function $r(x)$ is the Riemann invariant associated with the wave vector λ.

The function f is a solution of (1.1.1) if and only if the condition

$$\left(\Delta^{\mu i}_{\alpha}(f)\lambda_i(f)\right) f'^\alpha = 0, \quad f'^\alpha = \frac{df^\alpha}{dr} \tag{1.2.4}$$

holds, i.e. if and only if f' is an element of $\ker(\Delta^i \lambda_i)$. Note that equation (1.2.4) is a system of first order ordinary differential equations (ODEs) which defines f up to reparametrization. The image of a solution (1.2.1) is a curve in U space defined by the map $f : \mathbb{R} \to \mathbb{R}^q$ satisfying the set of ODEs (1.2.4). The extent to which expression (1.2.4) constrains the function f depends on the dimension of $\ker(\Delta^i \lambda_i)$. For example, if $\Delta^i \lambda_i = 0$ then there is no constraint on the function f at all and no integration is involved. The rank-1 solutions have the following common properties :

1. The Jacobian matrix is decomposable (in matrix notation)

$$\partial u = \left(1 - \frac{\partial f}{\partial r}\frac{\partial r}{\partial u}\right)^{-1} \frac{\partial f}{\partial r}\,\lambda, \tag{1.2.5}$$

or equivalently

$$\partial u = \frac{\partial f}{\partial r}\left(1 - \frac{\partial r}{\partial u}\frac{\partial f}{\partial r}\right)^{-1} \lambda, \tag{1.2.6}$$

where we have

$$\begin{gathered}\partial u = (u^\alpha_i) \in \mathbb{R}^{q\times p}, \quad \frac{\partial f}{\partial r} = \left(\frac{\partial f^\alpha}{\partial r}\right) \in \mathbb{R}^q, \\ \frac{\partial r}{\partial u} = \left(\frac{\partial r}{\partial u^\alpha}\right) = \frac{\partial \lambda_i}{\partial u^\alpha} x^i \in \mathbb{R}^q, \quad \lambda = (\lambda_i) \in \mathbb{R}^p.\end{gathered} \tag{1.2.7}$$

This property follows directly from differentiation of (1.2.1). The inverses $\left(1-\frac{\partial f}{\partial r}\frac{\partial r}{\partial u}\right)^{-1}$ or $\left(1-\frac{\partial r}{\partial u}\frac{\partial f}{\partial r}\right)^{-1}$ are well-defined functions, since $\partial r/\partial u = 0$ at $x = 0$. From equations (1.2.5) or (1.2.6), it can be noted that ∂u has rank at most equal to 1.

2. The graph of the rank-1 solution $\Gamma = \{x, u(x)\}$ is (locally) invariant under the linearly independent vector fields

$$X_a = \xi_a^i(u)\partial_i, \quad a = 1, \dots, p-1 \tag{1.2.8}$$

acting on $X \times U$ space. Here the vectors

$$\xi_a(u) = \left(\xi_a^1(u), \dots, \xi_a^p(u)\right)^T \tag{1.2.9}$$

satisfy the orthogonality conditions

$$\lambda_i\, \xi_a^i = 0, \quad a = 1, \dots, p-1 \tag{1.2.10}$$

for a fixed wave vector λ for which (1.2.3) holds. The vector fields (1.2.8) span a Lie vector module g over functions on U which constitutes an infinite-dimensional Abelian Lie algebra. The algebra g uniquely defines a module Λ (over the functions on U) of 1-forms $\lambda_i(u)\, dx^i$ annihilating all elements of g. A basis of Λ is given by

$$\lambda = \lambda_i(u)\, dx^i, \quad \xi_a^i\, \lambda_i = 0 \tag{1.2.11}$$

for all indices $a = 1, \dots, p-1$. The set $\{r = \lambda_i(u)x^i, u^1, \dots, u^q\}$ is the complete set of invariants of the vector fields (1.2.8).

3. It should be noted that rescaling the wave vector λ produces the same solution due to the homogeneity of the original system (1.1.1).

4. Due to the orthogonality conditions (1.2.10), together with property (1.2.5) or (1.2.6), any rank-1 solution is a solution of the overdetermined system of equations composed of system (1.1.1) and the differential constraints

$$\xi_a^i(u)\, u_i^\alpha = 0, \quad a = 1, \dots, p-1. \tag{1.2.12}$$

The side equations (1.2.12) mean that the characteristics of the vector fields (1.2.8) are equal to zero.

5. One can always find nontrivial solutions of (1.2.4) if (1.1.1) is an underdetermined system ($l < q$) or if it is properly determined ($l = q$) and hyperbolic. Here, a weaker assumption can be imposed on the system (1.1.1). Namely, it is sufficient to require that eigenvalues of the matrix $(\Delta^i \lambda_i)$ are real functions.

The method of construction of rank-1 solutions to (1.1.1) can be summarized as follows. First, we seek a wave vector $\lambda = (\lambda_1, \ldots, \lambda_p)$ such that

$$\operatorname{rank}\left(\Delta^{\mu i}_{\alpha} \lambda_i\right) < l. \tag{1.2.13}$$

For each such choice of λ_i we look for the solutions γ^α of the wave relations

$$\left(\Delta^{\mu i}_{\alpha} \lambda_i\right) \gamma^\alpha = 0, \quad \mu = 1, \ldots, l. \tag{1.2.14}$$

Functions $f^\alpha(r)$ are required to satisfy the ODEs

$$f'^\alpha(r) = \gamma^\alpha(f(r)). \tag{1.2.15}$$

Alternatively, the system of equations (1.2.4) is linear in the variables λ_i. Nonzero solutions λ_i exist if and only if

$$\operatorname{rank}\left(\Delta^{\mu i}_{a} \left(f(r)\right) f'^\alpha(r)\right) < p. \tag{1.2.16}$$

If (1.2.16) is satisfied for some function $f(r)$ then one can easily find $\lambda_i(r)$ satisfying equations (1.2.4). Using $u = f(r)$ one can define $\lambda_i(u)$ (not uniquely in general). If $l < p$ then (1.2.16) is identically satisfied for any function $f(r)$ and this approach does not require any integration.

1.3. The rank-k solutions

This section is devoted to the construction of rank-k solutions of a multidimensional system of PDEs (1.1.1). These solutions may be considered as nonlinear superpositions of rank-1 solutions.

Suppose that we fix k linearly independent wave vectors $\lambda^1, \ldots, \lambda^k$, $1 \leq k < p$ with Riemann invariant functions

$$r^A(x, u) = \lambda^A_i(u) x^i, \quad A = 1, \ldots, k. \tag{1.3.1}$$

The equation

$$u = f\left(r(x, u)\right), \quad r(x, u) = \left(r^1(x, u), \ldots, r^k(x, u)\right) \tag{1.3.2}$$

then defines a unique function $u(x)$ on a neighborhood of $x = 0$. The Jacobian matrix of (1.3.2) is given by

$$\partial u = \left(\mathbb{I} - \frac{\partial f}{\partial r}\frac{\partial r}{\partial u}\right)^{-1}\frac{\partial f}{\partial r}\lambda, \tag{1.3.3}$$

or equivalently

$$\partial u = \frac{\partial f}{\partial r}\left(\mathbb{I} - \frac{\partial r}{\partial u}\frac{\partial f}{\partial r}\right)^{-1}\lambda, \tag{1.3.4}$$

where $f = (f^\alpha)$, f^α are arbitrary functions of $r = (r^A)$ and

$$\begin{aligned} \partial u &= (u_i^\alpha) \in \mathbb{R}^{q\times p}, \quad \frac{\partial f}{\partial r} = \left(\frac{\partial f^\alpha}{\partial r^A}\right) \in \mathbb{R}^{q\times k}, \\ \lambda &= \left(\lambda_i^A\right) \in \mathbb{R}^{k\times p}, \quad \frac{\partial r}{\partial u} = \left(\frac{\partial r^A}{\partial u^\alpha}\right) = \frac{\partial \lambda_i^A}{\partial u^\alpha}x^i \in \mathbb{R}^{k\times q}. \end{aligned} \tag{1.3.5}$$

We assume here that the inverse matrices appearing in expressions (1.3.3) or (1.3.4), denoted by

$$\Phi^1 = \left(\mathbb{I} - \frac{\partial f}{\partial r}\frac{\partial r}{\partial u}\right) \in \mathbb{R}^{q\times q}, \quad \Phi^2 = \left(\mathbb{I} - \frac{\partial r}{\partial u}\frac{\partial f}{\partial r}\right) \in \mathbb{R}^{k\times k} \tag{1.3.6}$$

respectively, are invertible in some neighborhood of the origin $x = 0$. This assumption excludes the gradient catastrophe phenomenon for the function u.

Note that the rank of the Jacobian matrix (1.3.3) or (1.3.4) is at most equal to k. Hence the image of the rank-k solution is a k-dimensional submanifold $\mathcal{S}$ which lies in a submanifold of J^1.

If the set of vectors

$$\xi_a(u) = \left(\xi_a^1(u), \ldots, \xi_a^p(u)\right)^T, \quad a = 1, \ldots, p-k, \tag{1.3.7}$$

satisfies the orthogonality conditions

$$\lambda_i^A\,\xi_a^i = 0 \tag{1.3.8}$$

for $A = 1, \ldots, k$, $a = 1, \ldots, p-k$ then by virtue of (1.3.3) or (1.3.4) we have

$$Q_a^\alpha(x, u^{(1)}) \equiv \xi_a^i\,(u)\,u_i^\alpha = 0, \quad a = 1, \ldots, p-k, \quad \alpha = 1, \ldots, q. \tag{1.3.9}$$

Therefore rank-k solutions, given by (1.3.2), are obtained from the overdetermined system (1.1.1) subjected to differential constraints (DCs) (1.3.9)

$$\Delta_\alpha^{\mu i}(u)u_i^\alpha = 0, \quad \xi_a^i(u)u_i^\alpha = 0, \quad a = 1, \ldots, p-k. \tag{1.3.10}$$

Note that the conditions (1.3.9) are more general than the one required for the existence of Riemann k-wave solutions (see expression (1.5.1) and discussion in Section 1.5).

Let us note also that there are different approaches to the overdetermined system (1.3.10) employed in different versions of Riemann invariant method for multi-dimensional PDEs. The essence of our approach lies in treating the problem from the point of view of the conditional symmetry method (for description see e.g. [**92**]). Below we proceed with the adaptation of this method for our purpose.

The graph of the rank-k solution $\Gamma = \{x, u(x)\}$ of (1.3.10) is invariant under the vector fields

$$X_a = \xi_a^i(u)\partial_i, \quad a = 1, \dots, p-k \tag{1.3.11}$$

acting on $X \times U \subset \mathbb{R}^p \times \mathbb{R}^q$. The functions $\{r^1, \dots, r^k, u^1, \dots, u^q\}$ constitute a complete set of invariants of the Abelian Lie algebra $\mathcal{A}$ generated by the vector fields (1.3.11).

In order to solve the overdetermined system (1.3.10) we subject it to several transformations, based on the set of invariants of $\mathcal{A}$, which simplify its structure considerably. To achieve this simplification we choose an appropriate system of coordinates on $X \times U$ space which allows us to rectify the vector fields X_a, given by (1.3.11). Next, we show how to find the invariance conditions in this system of coordinates which guarantee the existence of rank-k solutions in the form (1.3.2).

Let us assume that the k by k matrix

$$\Pi = \left(\lambda_i^A\right), \quad 1 \leq A, i \leq k < p \tag{1.3.12}$$

built from the components of the wave vectors λ^A is invertible. Then the linearly independent vector fields

$$\begin{aligned} X_{k+1} &= \partial_{k+1} - \sum_{A,j=1}^{k} \left(\Pi^{-1}\right)_A^j \lambda_{k+1}^A \partial_j, \\ &\vdots \\ X_p &= \partial_p - \sum_{A,j=1}^{k} \left(\Pi^{-1}\right)_A^j \lambda_p^A \partial_j, \end{aligned} \tag{1.3.13}$$

have the required form (1.3.11) for which the orthogonality conditions (1.3.8) are satisfied. The change of independent and dependent variables

$$\bar{x}^1 = r^1(x,u), \ldots, \bar{x}^k = r^k(x,u), \quad \bar{x}^{k+1} = x^{k+1}, \ldots, \bar{x}^p = x^p, \bar{u}^1 = u^1, \ldots, \bar{u}^q = u^q \tag{1.3.14}$$

permits us to rectify the vector fields X_a and get

$$X_{k+1} = \partial_{\bar{x}^{k+1}}, \ldots, X_p = \partial_{\bar{x}^p}. \tag{1.3.15}$$

Note that a p-dimensional submanifold is transverse to the projection $(x,u) \to x$ at $x = 0$ if and only if it is transverse to the projection $(\bar{x},\bar{u}) \to \bar{x}$ at $\bar{x} = 0$. The transverse p-dimensional submanifolds invariant under $X_{k+1}, \ldots, X_p$ are defined by the implicit equation of the form

$$\bar{u} = f(\bar{x}^1, \ldots, \bar{x}^k). \tag{1.3.16}$$

Hence, expression (1.3.16) is the general integral of the invariance conditions

$$\bar{u}_{\bar{x}^{k+1}} = 0, \ldots, \bar{u}_{\bar{x}^p} = 0. \tag{1.3.17}$$

The system (1.1.1) is subjected to the invariance conditions (1.3.17) and, when written in terms of new coordinates $(\bar{x},\bar{u}) \in X \times U$, takes the form

$$\Delta^\mu \left(\Phi^1\right)^{-1} \frac{\partial \bar{u}}{\partial \bar{x}} \lambda = 0, \quad , \bar{u}_{\bar{x}^{k+1}} = 0, \ldots, \bar{u}_{\bar{x}^p} = 0, \tag{1.3.18}$$

or

$$\Delta^\mu \frac{\partial \bar{u}}{\partial \bar{x}} \left(\Phi^2\right)^{-1} \lambda = 0, \quad , \bar{u}_{\bar{x}^{k+1}} = 0, \ldots, \bar{u}_{\bar{x}^p} = 0, \tag{1.3.19}$$

where the matrices Φ^1 and Φ^2 are given by

$$\left(\Phi^1\right)^\alpha_\beta = \delta^\alpha_\beta - \bar{u}^\alpha_A \frac{\partial r^A}{\partial \bar{u}^\beta}, \quad \left(\Phi^2\right)^A_B = \delta^A_B - \frac{\partial r^A}{\partial \bar{u}^\alpha} \bar{u}^\alpha_B. \tag{1.3.20}$$

The above considerations characterize geometrically the solutions of the overdetermined system (1.3.10) in the form (1.3.2). Let us illustrate these considerations with some examples.

Example 1. Let us assume that there exist k independent relations of dependence for the matrices $\Delta^1, \ldots, \Delta^p$ such that the conditions

$$\Delta^{\mu i}_\alpha \lambda^A_i = 0, \quad A = 1, \ldots, k \tag{1.3.21}$$

hold. Suppose also that the original system (1.1.1) has the evolutionary form and each of the q by q matrices $A^1, \ldots, A^n$ is scalar, i.e.

$$\Delta^0 = \mathbb{I}, \quad \Delta^{i\alpha}_{\beta} = a^i(u)\delta^{\alpha}_{\beta}, \quad i = 1, \ldots, n \tag{1.3.22}$$

for some functions $a^1, \ldots, a^n$ defined on U, where $p = n+1$ and for convenience we denote the independent variables by $x = (t = x^0, x^1, \ldots, x^n) \in X$. Then the system (1.1.1) is particularly simple and becomes

$$u_t + a^1(u)u_1 + \ldots + a^n(u)u_n = 0. \tag{1.3.23}$$

The corresponding wave vectors

$$\begin{aligned} &\lambda^1 = (-a^1(u), 1, 0, \ldots, 0), \\ &\vdots \\ &\lambda^n = (-a^n(u), 0, \ldots, 0, 1) \end{aligned} \tag{1.3.24}$$

are linearly independent and satisfy conditions (1.3.21).

A vector function $u(x,t)$ is a solution of (1.3.23) if and only if the vector field

$$X = \partial_t + a^i(u)\partial_i$$

defined on $\mathbb{R}^{n+q+1}$ is tangent to the $(n+1)$-dimensional submanifold $\mathcal{S} = \{u = u(x,t)\} \subset \mathbb{R}^{n+q+1}$. The solution is thus identified with the $(n+1)$-dimensional submanifold $\mathcal{S} \subset \mathbb{R}^{n+q+1}$ which is transverse to $\mathbb{R}^{n+q+1} \to \mathbb{R}^{n+1} : (x,t,u) \to (x,t)$ and is invariant under the vector field X. The functions $\{r(x,t,u) = (r^1 = x^1 - a^1(u)t, \ldots, r^n = x^n - a^n(u)t), u^1, \ldots, u^q\}$ are invariants of X, such that $dr^1 \wedge \ldots \wedge dr^n \wedge du^1 \wedge \ldots \wedge du^q \neq 0$. If we define $\bar{t} = t, \bar{u} = u$, then $(r, \bar{t}, \bar{u})$ are coordinates on $\mathbb{R}^{n+q+1}$ and the vector field X can be rectified

$$X = \partial_{\bar{t}}.$$

The general solution is

$$\mathfrak{S} = \{F(r, \bar{u}) = 0\}$$

where $F : \mathbb{R}^{n+q} \to \mathbb{R}^q$ satisfies the condition

$$\det\left(\frac{\partial F}{\partial r}\frac{\partial r}{\partial \bar{u}} + \frac{\partial F}{\partial \bar{u}}\right) \neq 0$$

but is otherwise arbitrary. Note that it may be assumed that

$$\frac{\partial r}{\partial u}(x_0, t_0, u_0) = 0,$$

in which case the transversality condition is

$$\det\left(\frac{\partial F}{\partial \bar{u}}(x_0, t_0, u_0)\right) \neq 0.$$

Hence the general solution of (1.3.23) near (x_0, t_0, u_0) is

$$\mathfrak{S} = \{\bar{u} = f(r)\},$$

where $f : \mathbb{R}^n \to \mathbb{R}^q$ is arbitrary. Thus the equation

$$u = f(x^1 - a^1(u)t, \dots, x^n - a^n(u)t), \tag{1.3.25}$$

defines a unique function $u(x, t)$ on a neighborhood of the point (x_0, t_0, u_0) for any f. Note that

$$t = 0, \quad u(x, 0) = f(x^1, \dots, x^n),$$

so the function f is simply the Cauchy data on $\{t = 0\}$.

Example 2. Another interesting case to consider is when the matrix Φ^1 (or Φ^2) is a scalar matrix. Then system (1.3.18) is equivalent to the quasilinear system in k independent variables $\bar{x}^1, \dots, \bar{x}^k$ and q dependent variables $\bar{u}^1, \dots, \bar{u}^q$. So, we have

$$B^A(\bar{u})\bar{u}^\alpha_A = 0, \tag{1.3.26}$$

where

$$B^A = \Delta^i \lambda^A_i. \tag{1.3.27}$$

If $k \geq 2$ then Φ^1 is a scalar if and only if

$$\frac{\partial r^1}{\partial u} = 0, \dots, \frac{\partial r^k}{\partial u} = 0 \tag{1.3.28}$$

and consequently, if and only if the vector fields $\lambda^1, \dots, \lambda^k$ are constant wave vectors.

Finally, a more general situation occurs when the matrix Φ^1 (or Φ^2) satisfies the conditions

$$\frac{\partial \Phi^1}{\partial \bar{x}^{k+1}} = 0, \dots, \frac{\partial \Phi^1}{\partial \bar{x}^p} = 0. \tag{1.3.29}$$

Then the system (1.3.18) is independent of variables $\bar{x}^{k+1}, \ldots, \bar{x}^p$. The conditions (1.3.29) hold if and only if

$$\frac{\partial^2 r}{\partial u \partial \bar{x}^{k+1}} = 0, \ldots, \frac{\partial^2 r}{\partial u \partial \bar{x}^p} = 0. \tag{1.3.30}$$

Using (1.3.1) and (1.3.12) we get

$$\frac{\partial \lambda_i^A}{\partial u} = \sum_{l,B=1}^{k} \frac{\partial \Pi_l^A}{\partial u} \left(\Pi^{-1}\right)_B^l \lambda_i^B. \tag{1.3.31}$$

Equation (1.3.31) can be rewritten in the simpler form

$$\frac{\partial}{\partial u}\left(\sum_{B=1}^{k} \left(\Pi^{-1}\right)_B^l \lambda_i^B\right) = 0, \quad 1 \leq l \leq k < i \leq p. \tag{1.3.32}$$

Thus system (1.3.18) is independent of variables $\bar{x}^{k+1}, \ldots, \bar{x}^p$ if the k by $p-k$ matrix $\left(\lambda_i^B\right)$, $1 \leq B \leq k < i \leq p$ is equal to the matrix ΠC, where C is a constant k by $(p-k)$ matrix. In this case (1.3.18) is a system not necessarily quasilinear, in k independent variables $\bar{x}^1, \ldots, \bar{x}^k$ and q dependent variables $\bar{u}^1, \ldots, \bar{u}^q$.

Let us now derive the necessary and sufficient conditions for existence of solutions in the form (1.3.2) of the overdetermined system (1.3.10). Substituting (1.3.3) or (1.3.4) into (1.1.1) yields

$$\operatorname{Tr}\left[\Delta^\mu \left(\mathbb{I} - \frac{\partial f}{\partial r}\frac{\partial r}{\partial u}\right)^{-1} \frac{\partial f}{\partial r}\lambda\right] = 0, \tag{1.3.33}$$

or equivalently

$$\operatorname{Tr}\left[\Delta^\mu \frac{\partial f}{\partial r}\left(\mathbb{I} - \frac{\partial r}{\partial u}\frac{\partial f}{\partial r}\right)^{-1} \lambda\right] = 0, \tag{1.3.34}$$

respectively, where

$$\Delta^\mu = \left(\Delta_\alpha^{\mu i}\right) \in \mathbb{R}^{p \times q}, \quad \mu = 1, \ldots, l. \tag{1.3.35}$$

Given the system of PDEs (1.1.1) (i.e. functions $\Delta_\alpha^{\mu i}(u)$) it follows that equations (1.3.33) (or (1.3.34)) are conditions on the functions $f^\alpha(r)$ and $\lambda_i^A(u)$ (or $\xi_a^i(u)$). Since $\partial r/\partial u$ depends explicitly on x it may happen that these conditions have only trivial solutions (i.e. f=const) for some values of k. We discuss a set of conditions following from (1.3.33) or (1.3.34) which allow the system (1.3.10) to possess the nontrivial rank-k solutions.

Let g be a $(p-k)$-dimensional Lie vector module over $C^\infty(X\times U)$ with generators X_a given by (1.3.11). Let Λ be a k-dimensional module generated by $k<p$ linearly independent 1-forms

$$\lambda^A = \lambda^A_i(u)dx^i, \quad A=1,\ldots,k$$

which are annihilated by $X_a\in g$. It is assumed here that the vector fields X_a and λ^A are related by the orthogonality conditions (1.3.8) and form a basis of g and Λ, respectively. For $k>1$, it is always possible to choose a basis λ^A of the module Λ of the form

$$\lambda^A = dx^{i_A} + \lambda^A_{i_a}\, dx^{i_a}, \quad A=1,\ldots,k \tag{1.3.36}$$

where (i_A, i_a) is a permutation of $(1,\ldots,p)$. Here we split the coordinates x^i into x^{i_A} and x^{i_a}. Then from (1.3.1) we obtain the relation

$$\frac{\partial r^A}{\partial u^\alpha} = \frac{\partial \lambda^A_{i_a}}{\partial u^\alpha} x^{i_a}. \tag{1.3.37}$$

Substituting (1.3.37) into equations (1.3.33) or (1.3.34) yields, respectively

$$\mathrm{Tr}\left(\Delta^\mu(\mathbb{I}-Q_a x^{i_a})^{-1}\frac{\partial f}{\partial r}\lambda\right)=0, \tag{1.3.38}$$

or

$$\mathrm{Tr}\left(\Delta^\mu\frac{\partial f}{\partial r}(\mathbb{I}-K_a x^{i_a})^{-1}\lambda\right)=0, \tag{1.3.39}$$

where we use the following notation

$$Q_a = \frac{\partial f}{\partial r}\eta_a \in \mathbb{R}^{q\times q}, \quad K_a = \eta_a\frac{\partial f}{\partial r} \in \mathbb{R}^{k\times k}, \tag{1.3.40}$$

$$\eta_a = \left(\frac{\partial \lambda^A_{i_a}}{\partial u^\alpha}\right) \in \mathbb{R}^{k\times q}, \quad i_a = 1,\ldots,p-1. \tag{1.3.41}$$

The functions r^A and x^{i_a} are all independent in the neighborhood of the origin $x=0$. The functions $\Delta^\mu, \frac{\partial f}{\partial r}, \lambda, Q_a$ and K_a depend on r only. For these specific functions, equations (1.3.38) (or (1.3.39)) must be satisfied for all values of coordinates x^{i_a}. In order to find appropriate conditions for $f(r)$ and $\lambda(u)$ let us notice that, according to the Cayley-Hamilton theorem, for any n by n invertible matrix M, $(M^{-1}\det M)$ is a polynomial in M of order $(n-1)$. Hence, one can replace equation (1.3.38) by

$$\mathrm{Tr}\left(\Delta^\mu\, Q\,\frac{\partial f}{\partial r}\,\lambda\right)=0, \tag{1.3.42}$$

where we introduce the following notation

$$Q = (\mathbb{I} - Q_a x^{i_a})^{-1} \det(\mathbb{I} - Q_a x^{i_a}).$$

Taking equation (1.3.42) and all its x^{i_a} derivatives (with r=const) at $x^{i_a} = 0$, yields

$$\mathrm{Tr}\left(\Delta^\mu \frac{\partial f}{\partial r}\lambda\right) = 0, \tag{1.3.43}$$

$$\mathrm{Tr}\left(\Delta^\mu Q_{(a_1} \dots Q_{a_s)} \frac{\partial f}{\partial r}\lambda\right) = 0, \tag{1.3.44}$$

where $s = 1, \dots, q-1$ and $(a_1, \dots, a_s)$ denotes symmetrization over all indices in the bracket. A similar procedure for equation (1.3.39) yields (1.3.43) and the trace condition

$$\mathrm{Tr}\left(\Delta^\mu \frac{\partial f}{\partial r} K_{(a_1}, \dots, K_{a_s)}\lambda\right) = 0, \tag{1.3.45}$$

where now $s = 1, \dots, k-1$.

Equation (1.3.43) represents an initial value condition on a surface in X space given by $x^{i_a} = 0$. Equations (1.3.44) (or (1.3.45)) correspond to the preservation of (1.3.43) by flows represented by the vector fields (1.3.11). Note that X_a can be put into the form

$$X_a = \partial_{i_a} - \lambda^A_{i_a}\partial_A, \quad \xi^i_a \cdot \lambda^A_i = 0, \quad A = 1, \dots, k. \tag{1.3.46}$$

By virtue of (1.3.40), (1.3.41), equations (1.3.44) or (1.3.45) take the unified form

$$\mathrm{Tr}\left(\Delta^\mu \frac{\partial f}{\partial r}\eta_{(a_1}\frac{\partial f}{\partial r} \dots \eta_{a_s)}\frac{\partial f}{\partial r}\lambda\right) = 0, \tag{1.3.47}$$

where either $\max s = q-1$ or $\max s = k-1$.

The vector fields X_a and the Lie module g spanned by the vector fields $X_1, \dots, X_{p-k}$ are called the conditional symmetries and the conditional symmetry module of (1.1.1), respectively if X_a are Lie point symmetries of the original system (1.1.1) supplemented by the DCs (1.3.9) **[92]**.

Let us now associate the system (1.1.1) and the conditions (1.3.9) with the subvarieties of the solution spaces

$$\mathcal{B}_\Delta = \{(x, u^{(1)}) : \Delta^{\mu i}_\alpha(u)\, u^\alpha_i = 0, \quad \mu = 1, \dots, l\},$$

and

$$\mathcal{B}_Q = \{(x, u^{(1)}) : \xi_a^i(u) u_i^\alpha = 0, \quad a = 1, \ldots, p-k, \quad \alpha = 1, \ldots, q\},$$

respectively. We have the following.

Proposition 1. *A nondegenerate first order hyperbolic system of PDEs (1.1.1) admits a (p-k)-dimensional Lie vector module g of conditional symmetries if and only if (p-k) linearly independent vector fields $X_1, \ldots, X_{p-k}$ satisfy the conditions (1.3.43) and (1.3.47) on some neighborhood of (x_0, u_0) of $\mathcal{B} = \mathcal{B}_\Delta \cap \mathcal{B}_Q$.*

Proof . The vector fields X_a constitute the conditional symmetry module g for the system (1.1.1) if they are Lie point symmetries of the overdetermined system (1.3.10). This means that the first prolongation of X_a has to be tangent to the system (1.3.10). Hence g is a conditional symmetry module of (1.1.1) if and only if the equations

$$\mathrm{pr}^{(1)} X_a (\Delta^{\mu i}_{\alpha}(u)\, u_i^\alpha = 0, \quad \mathrm{pr}^{(1)} X_a \left(\xi_b^i(u) u_i^\alpha\right) = 0, \quad a = 1, \ldots, p-k \tag{1.3.48}$$

are satisfied on J^1 whenever the equations (1.3.10) hold. Now we show that if the conditions (1.3.43) and (1.3.47) are satisfied then the symmetry criterion (1.3.48) is identically equal to zero.

In fact, applying the first prolongation of the vector fields X_a

$$\mathrm{pr}^{(1)} X_a = X_a + \xi^i_{a,u^\beta} u_j^\beta u_i^\alpha \frac{\partial}{\partial u_j^\alpha}$$

to the original system (1.1.1) yields

$$\mathrm{pr}^{(1)} X_a \left(\Delta^{\mu i}_{\alpha} u_i^\alpha\right) = \Delta^{\mu i}_{\alpha} \xi^j_{a,u^\beta} u_i^\beta u_j^\alpha = 0, \tag{1.3.49}$$

whenever equations (1.3.10) hold. On the other hand, carrying out the differentiations of (1.3.8) gives

$$\xi^j_{a,u^\beta} \lambda_j^B = -\xi_a^j \lambda^B_{j,u^\beta}. \tag{1.3.50}$$

Comparing (1.3.49) and (1.3.50) leads to

$$\Omega^{\mu A}_{\ \ B} \xi_a^j Z_A(\lambda_j^B) = 0, \tag{1.3.51}$$

where we introduce the following notation

$$\Omega^{\mu A}_{\ \ B} = \Delta^{\mu i}_{\alpha} Z_B^\alpha \lambda_i^A. \tag{1.3.52}$$

Here the new vector fields Z_B are defined on U

$$Z_A = Z_A^\alpha \frac{\partial}{\partial u^\alpha} \in T_u U. \tag{1.3.53}$$

It is convenient to write equation (1.3.51) in the equivalent form

$$\mathrm{Tr}\,(\Delta^\mu Z\theta_a Z\lambda) = 0, \mu = 1, \ldots, l \tag{1.3.54}$$

where the following notation has been used

$$\theta_a = \lambda^A_{i,u^\beta}\xi^i_a. \tag{1.3.55}$$

The assumption that system (1.1.1) is hyperbolic implies that there exist the real-valued vector fields λ^A and γ_A defined on U for which the wave relation

$$\left(\Delta^{\mu i}_\alpha \lambda^A_i\right)\gamma^\alpha_{(A)} = 0, \quad A = 1, \ldots, k \tag{1.3.56}$$

is satisfied and that the U space is spanned by the linearly independent vector fields

$$\gamma_A = \gamma^\alpha_A\, \partial_{u^\alpha} \in T_u U. \tag{1.3.57}$$

Hence, one can represent the vector fields Z_A through the basis generated by the vector fields $\{\gamma_1, \ldots, \gamma_k\}$, i.e.

$$Z_A = h^B_A \gamma_B. \tag{1.3.58}$$

Using equations (1.3.3) and (1.3.6) we find the coefficients

$$h^B_A = ((\phi^1)^{-1})^B_A.$$

This means that the submanifold $\mathcal{S}$, given by (1.3.2), can be represented parametrically by

$$\frac{\partial f^\alpha}{\partial r^A} = h^B_A \gamma^\alpha_B. \tag{1.3.59}$$

On the other hand, comparing (1.3.3) and (1.3.58) gives

$$u^\alpha_i = Z^\alpha_A \lambda^A_i. \tag{1.3.60}$$

Applying the invariance criterion (1.3.48) to the side conditions (1.3.9) we obtain

$$\mathrm{pr}^{(1)} X_a(Q^\alpha_b) = \xi^i_{[b}\xi^j_{a],u^\beta} u^\beta_i u^\alpha_j. \tag{1.3.61}$$

The bracket $[a, b]$ denotes antisymmetrization with respect to the indices a and b. By virtue of equations (1.3.50) and (1.3.60), the right side of (1.3.61) is identically equal to zero. Substituting (1.3.58) into equation (1.3.54) and taking into account equation (1.3.36) and (1.3.59) we obtain that for any value of $x \in X$ the resulting formulae coincide with equations (1.3.43) and (1.3.47). Hence, the infinitesimal symmetry criterion (1.3.48) for the overdetermined system (1.3.10) is identically satisfied whenever conditions (1.3.43) and (1.3.47) hold.

The converse also holds. The assumption that the system (1.1.1) is nondegenerate means that it is locally solvable and takes a maximal rank at every point $(x_0, u_0^{(1)}) \in \mathcal{B}_\Delta$. Therefore [**89**] the infinitesimal symmetry criterion is a necessary and sufficient condition for the existence of symmetry group G of the overdetermined system (1.3.10). Since the vector fields X_a form an Abelian distribution, it follows that the conditions (1.3.43) and (1.3.47) hold. That ends the proof since the solutions of the original system (1.1.1) are invariant under the Lie algebra generated by $(p-k)$ vectors fields $X_1, \dots, X_{p-k}$.

□

Note that the set of solutions of the determining equations obtained by applying the symmetry criterion to the overdetermined system (1.3.10) is different than the set of solutions of the determining equations for the initial system (1.1.1). Thus the system (1.3.10) admits other symmetries than the original system (1.1.1). So, new reductions for the system (1.1.1) can be constructed, since each solution of system (1.3.10) is a solution of system (1.1.1).

In our approach the construction of solutions of the original system (1.1.1) requires us to solve first the system (1.3.47) for λ_i^A as functions of u^α and then find $u = f(r)$ by solving (1.3.43). Note that the functions $f^*(\lambda_i^A)$ are the functions $\lambda_i^A(f)$ pulled back to the surface $\mathcal{S}$. The $\lambda_i^A(f)$ then become functions of the parameters $r^1, \dots, r^k$ on $\mathcal{S}$. For simplicity of notation we denote $f^*(\lambda_i^A)$ by $\lambda_i^A(r^1, \dots, r^k)$.

The system composed of (1.3.43) and (1.3.47) is, in general, nonlinear. So, we cannot expect to solve it in a closed form, except in some particular cases. But nevertheless, as we show in section 1.4, there are physically interesting examples for which solutions of (1.3.43) and (1.3.47) lead to the new solutions of (1.1.1) which depend on some arbitrary functions. These particular solutions of (1.3.43) and (1.3.47) are obtained by expanding each function λ_i^A into a polynomial in the dependent variables u^α and requiring that the coefficients of the successive powers of u^α vanish. We then obtain a system of first order PDEs for the coefficients of the polynomials. Solving this system allows us to find some particular classes of solutions of the initial system (1.1.1) which can be constructed by applying the symmetry reduction technique.

1.4. Examples of applications

We start with considering the case of rank-2 solutions of the system (1.1.1) with two dependent variables ($q = 2$). Then (1.3.47) adopts the simplified form.

$$\mathrm{Tr}\left(\Delta^\mu \frac{\partial f}{\partial r}\eta_a \frac{\partial f}{\partial r}\lambda\right) = 0. \tag{1.4.1}$$

By virtue of (1.3.43), equation (1.4.1) can be transformed to

$$\mathrm{Tr}\left[\Delta^\mu \frac{\partial f}{\partial r}\left(\eta_a \frac{\partial f}{\partial r} - \mathbb{I}\,\mathrm{Tr}\left(\eta_a \frac{\partial f}{\partial r}\right)\right)\lambda\right] = 0. \tag{1.4.2}$$

Using the Cayley Hamilton identity, we get the relation

$$AB - \mathbb{I}\ \mathrm{Tr}\,AB = (B - \mathbb{I}\ \mathrm{Tr}\,B)(A - \mathbb{I}\ \mathrm{Tr}\,A) \tag{1.4.3}$$

for any 2 by 2 matrices $A, B \in \mathbb{R}^{2\times 2}$. Now we can rewrite (1.4.2) in the equivalent form

$$-\,\mathrm{Tr}\left[\Delta^\mu \frac{\partial f}{\partial r}\left(\frac{\partial f}{\partial r} - \mathbb{I}\,\mathrm{Tr}\,\frac{\partial f}{\partial r}\right)(\eta_a - \mathbb{I}\,\mathrm{Tr}\,\eta_a)\lambda\right] = 0. \tag{1.4.4}$$

So we have

$$\det\left(\frac{\partial f}{\partial r}\right)\mathrm{Tr}\,[\Delta^\mu(\eta_a - \mathbb{I}\,\mathrm{Tr}\,\eta_a)\lambda] = 0. \tag{1.4.5}$$

The rank-2 solutions require that the condition $\det \partial f/\partial r \neq 0$ be satisfied (otherwise $q = 2$ can be reduced to $q = 1$). As a consequence of this, we obtain the following condition

$$\mathrm{Tr}\,[\Delta^\mu(\eta_a - \mathbb{I}\,\mathrm{Tr}\,\eta_a)\lambda] = 0, \quad \mu = 1, \ldots, l, \tag{1.4.6}$$

which coincides with the result obtained earlier for this specific case **[51]**. One can look first for solutions $\lambda(u)$ of (1.4.6) and then find $f(r)$ by solving (1.3.43). Note that equations (1.4.6) form a system of $l(p-2)$ equations for $2(p-2)$ functions $\lambda_{ia}^A(u)$. This indicates that they should have solutions (say, for generic systems) if (1.1.1) is not overdetermined.

Example 3. We are looking for rank-2 solutions of the (2+1) hydrodynamic type equations

$$u_t^i + u^j u_j^i + A_k^{ij} u_j^k = 0, \quad i, j, k = 1, 2 \tag{1.4.7}$$

where A^i are some matrix functions of u^1 and u^2. Using the condition representing the tracelessness of the matrices $\Delta_\alpha^{1i} u_i^\alpha$ and $\Delta_\alpha^{2i} u_i^\alpha$, it is convenient to rewrite the system (1.4.7) in the following form

$$\begin{aligned} &\mathrm{Tr}\left[\begin{pmatrix} 1 & u^1 + A_1^{11} & u^2 + A_1^{12} \\ 0 & A_2^{11} & A_2^{12} \end{pmatrix} \begin{pmatrix} u_t^1 & u_t^2 \\ u_x^1 & u_x^2 \\ u_y^1 & u_y^2 \end{pmatrix}\right] = 0, \\ &\mathrm{Tr}\left[\begin{pmatrix} 0 & u^1 & u^2 \\ 1 & u^1 + A_2^{21} & u^2 + A_2^{22} \end{pmatrix} \begin{pmatrix} u_t^1 & u_t^2 \\ u_x^1 & u_x^2 \\ u_y^1 & u_y^2 \end{pmatrix}\right] = 0. \end{aligned} \tag{1.4.8}$$

Let $\mathcal{F}$ be a smooth orientable surface immersed in 3-dimensional Euclidean $(x, y, t) \in X$ space. Suppose that $\mathcal{F}$ can be written in the following parametric form

$$u = f(r^1, r^2) = (u^1(r^1, r^2), u^2(r^1, r^2)), \tag{1.4.9}$$

such that the Jacobian matrix is different from zero

$$J = \det\left(\frac{\partial f^\alpha}{\partial r^A}\right) = \det\begin{pmatrix} \partial u^1/\partial r^1 & \partial u^1/\partial r^2 \\ \partial u^2/\partial r^1 & \partial u^2/\partial r^2 \end{pmatrix} \neq 0. \tag{1.4.10}$$

Without loss of generality, it is possible to choose a basis λ^A of module Λ of the form

$$\lambda_i^A = \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}^T = \begin{pmatrix} \varepsilon & a^1 & b^1 \\ \varepsilon & a^2 & b^2 \end{pmatrix}^T, \quad \varepsilon = \pm 1, \tag{1.4.11}$$

where a^A and b^A are functions of u^1 and u^2 to be determined.

The rank-2 solution can be constructed from the most general solution of equations (1.4.6) for $\lambda^A = (-1, a^A, b^A)$, $A = 1, 2$. These equations lead to a system of four PDEs with four dependent variables $a^A, b^A, A = 1, 2$ and two independent variables u^1 and u^2,

$$\begin{aligned}
&-(A_2^{11}a^2 + A_2^{12}b^2)\frac{\partial a^1}{\partial u^1} + ((u^1 + A_1^{11})a^2 + (u^2 + A_1^{12})b^2 - 1)\frac{\partial a^1}{\partial u^2} \\
&\quad + (A_2^{11}a^1 + A_2^{12}b^1)\frac{\partial a^2}{\partial u^1} - ((u^1 + A_1^{11})a^1 + (u^2 + A_1^{12})b^1 - 1)\frac{\partial a^2}{\partial u^2} = 0, \\
&-(A_2^{11}a^2 + A_2^{12}b^2)\frac{\partial b^1}{\partial u^1} + ((u^1 + A_1^{11})a^2 + (u^2 + A_1^{12})b^2 - 1)\frac{\partial b^1}{\partial u^2} \\
&\quad + (A_2^{11}a^1 + A_2^{12}b^1)\frac{\partial b^2}{\partial u^1} - ((u^1 + A_1^{11})a^1 + (u^2 + A_1^{12})b^1 - 1)\frac{\partial b^2}{\partial u^2} = 0, \\
&(1 - (u^1 + A_2^{21})a^2 - (u^2 + A_2^{22})b^2)\frac{\partial a^1}{\partial u^1} + (A^{21_1}a^2 + A_1^{22}b^2)\frac{\partial a^1}{\partial u^2} \\
&\quad - (1 - (u^1 + A_2^{21})a^1 - (u^2 + A_2^{22})b^1)\frac{\partial a^2}{\partial u^1} - (A_1^{21}a^1 + A_1^{22}b^1)\frac{\partial a^2}{\partial u^2} = 0, \\
&(1 - (u^1 + A_2^{21})a^2 - (u^2 + A_2^{22})b^2)\frac{\partial b^1}{\partial u^1} + (A^{21_1}a^2 + A_1^{22}b^2)\frac{\partial b^1}{\partial u^2} \\
&\quad - (1 - (u^1 + A_2^{21})a^1 - (u^2 + A_2^{22})b^1)\frac{\partial b^2}{\partial u^1} - (A_1^{21}a^1 + A_1^{22}b^1)\frac{\partial b^2}{\partial u^2} = 0.
\end{aligned} \tag{1.4.12}$$

Finally, a rank-2 solution of (1.4.8) is obtained from the explicit parametrization of the surface $\mathcal{F}$ in terms of the parameters r^1 and r^2, by solving equations (1.3.43) in which λ^A adopt the form (1.4.6)

$$\begin{aligned}
&((u^1 + A_1^{11})a^1 + (u^2 + A_1^{12})b^1 - 1)\frac{\partial u^1}{\partial r^1} + ((u^1 + A_1^{11})a^2 + (u^2 + A_1^{12})b^2 - 1)\frac{\partial u^1}{\partial r^2} \\
&\quad + (A_2^{11}a^1 + A_2^{12}b^1)\frac{\partial u^2}{\partial r^1} + (A_2^{11}a^2 + A_2^{12}b^2)\frac{\partial u^2}{\partial r^2} = 0, \\
&(A_1^{21}a^1 + A_1^{22}b^1)\frac{\partial u^1}{\partial r^1} + (A_1^{21}a^2 + A_1^{22}b^2)\frac{\partial u^1}{\partial r^2} + ((u^1 + A_2^{21})a^1 + (u^2 + A_2^{22})b^1 - 1)\frac{\partial u^2}{\partial r^1} \\
&\quad + ((u^1 + A_2^{21})a^2 + (u^2 + A_2^{22})b^2 - 1)\frac{\partial u^2}{\partial r^2} = 0,
\end{aligned} \tag{1.4.13}$$

while the quantities r^1 and r^2 are implicitly defined as functions of y, x, t by equation (1.3.1).

In the case when equation (1.4.2) does admit two linearly independent vector fields λ^A with $\varepsilon = -1$, there exists a class of rank-2 solutions of equations (1.4.12) and (1.4.13) invariant under the vector fields

$$X_1 = \partial_t + u^1\partial_x, \quad X_2 = \partial_t + u^2\partial_y. \tag{1.4.14}$$

Following the procedure outlined in Section 3 we assume that the functions f^1 and f^2 appearing in equation (1.3.2) are linear in u^2. Then the invariance conditions take the form

$$x - u^1 t = g(u^1) + u^2 h(u^1), \quad y - u^2 t = a(u^1) + u^2 b(u^1), \tag{1.4.15}$$

where a, b, g and h are some functions of u^1.

One can show that if $h = 0$, then the solution of the system (1.4.12), (1.4.13) is defined implicitly by the relations

$$x - u^1 t = g(u^1), \quad y - u^2 t = a(u^1) + u^2 g_{,u^1}, \tag{1.4.16}$$

where a and g are arbitrary functions of u^1. Note that in this case the functions u^1 and u^2 satisfy the following system of equations

$$\begin{aligned} &u^1_t + u^1 u^1_x + u^2 u^1_y + A^{11}_1 (u^1_x - u^2_y) + A^{12}_1 u^1_y = 0, \\ &u^2_t + u^1 u^2_x + u^2 u^2_y + A^{21}_1 (u^1_x - u^2_y) + A^{22}_1 u^1_y = 0, \end{aligned} \tag{1.4.17}$$

for any functions A^{ij}_k of two variables u^1 and u^2.
If the function h of u^1 does not vanish anywhere ($h \neq 0$) then the rank-2 solution is defined implicitly by equations (1.4.15) and satisfies the following system of PDEs

$$\begin{aligned} &u^1_t + u^1 u^1_x + u^2 u^1_y + A^{12}_2 [u^2_y - u^1_x + l(u^1) u^2_x + m(u^1) u^1_y] = 0, \\ &u^2_t + u^1 u^2_x + u^2 u^2_y + A^{22}_2 [u^2_y - u^1_x + l(u^1) u^2_x + m(u^1) u^1_y] = 0, \end{aligned} \tag{1.4.18}$$

where A^{12}_2 and A^{22}_2 are any functions of two variables u^1 and u^2. Given the functions l and m of u^1, we can prescribe the functions a and b in expression (1.4.15) to find

$$h = \int l\, b_{,u^1} du^1, \quad g = \int [b - hm - a_{,u^1}] du^1. \tag{1.4.19}$$

For instance, consider a rank-2 solution of equations (1.4.12) and (1.4.13) invariant under the vector fields

$$X_1 = \partial_t + u^1 \partial_x, \quad X_2 = \partial_t - u^2 \partial_y \tag{1.4.20}$$

with the wave vectors λ^A which are the nonzero multiples of $\lambda^1 = (u^1, -1, 0)$ and $\lambda^2 = (u^2, 0, -1)$. Then the solution is defined by the implicit relations

$$\begin{aligned} x - u^1 t &= g(u^1), \\ y + u^2 t &= h(u^1) + u^2 g_{,u^1}. \end{aligned} \tag{1.4.21}$$

and satisfies the following system of equations

$$\begin{aligned} u^1_t + u^1 u^1_x + u^2 u^1_y + b(u^1, u^2) u^1_y &= 0, \\ u^2_t + u^1 u^2_x + u^2 u^2_y + c(u^1, u^2) u^1_y &= 0, \end{aligned} \tag{1.4.22}$$

where b and c are arbitrary functions of u^1 and u^2.

Thus, putting it all together, we see that the constructed solutions correspond to superpositions of two rank-1 solutions (i.e. simple waves) with local velocities u^1 and u^2, respectively. According to [**53**], if we choose the initial data ($t = 0$) for the functions u^1 and u^2 sufficiently small and such that their first derivatives with respect to x and y will have compact and disjoint supports, then asymptotically there exists a finite time $t = T > 0$ for which rank-2 solution decays in the exact way in two rank-1 solutions, being of the same type as in the initial moment.

Example 4. Consider the overdetermined hyperbolic system in $(2 + 1)$ dimensions ($p = 3$)

$$\begin{aligned} &\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + ka \operatorname{grad} a = 0 \\ &\frac{\partial a}{\partial t} + (\vec{u} \cdot \nabla) a + k^{-1} a \operatorname{div} \vec{u} = 0, \\ &\frac{\partial a}{\partial x} = 0, \quad \frac{\partial a}{\partial y} = 0, \end{aligned} \tag{1.4.23}$$

describing the nonstationary isentropic flow of a compressible ideal fluid. Here we use the following notations : $\vec{u} = (u^1, u^2)$ is the flow velocity, $a(t) = \left(\frac{\gamma p}{\rho}\right)^{1/2} \neq 0$ is the sound velocity which depends on t only, where ρ, p denote the density and pressure respectively, $k = 2(\gamma - 1)^{-1}$ and $\gamma \in \mathbb{R}$ is the polytropic exponent.

The system (1.4.23) can be written in an equivalent form as

$$\mathrm{Tr}\left[\begin{pmatrix} 1 & u^1 & u^2 \\ 0 & 0 & 0 \\ 0 & ka & 0 \end{pmatrix}\begin{pmatrix} u_t^1 & u_t^2 & a_t \\ u_x^1 & u_x^2 & 0 \\ u_y^1 & u_y^2 & 0 \end{pmatrix}\right] = 0,$$

$$\mathrm{Tr}\left[\begin{pmatrix} 0 & 0 & 0 \\ 1 & u^1 & u^2 \\ 0 & 0 & ka \end{pmatrix}\begin{pmatrix} u_t^1 & u_t^2 & a_t \\ u_x^1 & u_x^2 & 0 \\ u_y^1 & u_y^2 & 0 \end{pmatrix}\right] = 0, \tag{1.4.24}$$

$$\mathrm{Tr}\left[\begin{pmatrix} 0 & k^{-1}a & 0 \\ 0 & 0 & k^{-1}a \\ 1 & u^1 & u^2 \end{pmatrix}\begin{pmatrix} u_t^1 & u_t^2 & a_t \\ u_x^1 & u_x^2 & 0 \\ u_y^1 & u_y^2 & 0 \end{pmatrix}\right] = 0.$$

We are interested here in the rank-2 solutions of (1.4.24). So, we require that conditions (1.3.43) and (1.3.47) be satisfied. This demand constitutes the necessary and sufficient condition for the existence of a surface $\mathcal{F}$ written in a parametric form (1.4.9) for which equation (1.4.10) holds. In our case, $p = q = 3$ and $k = 2$, conditions (1.3.43) and (1.3.47) become

$$\mathrm{Tr}\left(\Delta^\mu \frac{\partial f}{\partial r}\lambda\right) = 0, \quad \mu = 1, 2, 3, \tag{1.4.25}$$

and

$$\mathrm{Tr}\left(\Delta^\mu \frac{\partial f}{\partial r}\left(\eta_1 \frac{\partial f}{\partial r}\eta_2 + \eta_2 \frac{\partial f}{\partial r}\eta_1\right)\frac{\partial f}{\partial r}\lambda\right) = 0, \tag{1.4.26}$$

respectively. Here, we assume the following basis for the wave vectors

$$\lambda_i^A = \begin{pmatrix} \lambda_0^1 & \lambda_1^1 & \lambda_2^1 \\ \lambda_0^2 & \lambda_1^2 & \lambda_2^2 \end{pmatrix}^T = \begin{pmatrix} -1 & v^1 & w^1 \\ -1 & v^2 & w^2 \end{pmatrix}^T, \tag{1.4.27}$$

where v^A and w^A are some functions of u^1 and u^2 to be determined. The 2 by 3 matrices η_a and the 3 by 2 matrix $\partial f/\partial r$ take the form

$$\eta_a = \frac{\partial \lambda_{i_a}^A}{\partial u^\alpha} = \begin{pmatrix} \partial\lambda_{i_a}^1/\partial u^1 & \partial\lambda_{i_a}^1/\partial u^2 & \partial\lambda_{i_a}^1/\partial a \\ \partial\lambda_{i_a}^2/\partial u^1 & \partial\lambda_{i_a}^2/\partial u^2 & \partial\lambda_{i_a}^2/\partial a \end{pmatrix}, \quad a = 1, 2$$

$$\frac{\partial f}{\partial r} = \begin{pmatrix} \partial u^1/\partial r^1 & \partial u^1/\partial r^2 \\ \partial u^2/\partial r^1 & \partial u^2/\partial r^2 \\ \partial a/\partial r^1 & \partial a/\partial r^2 \end{pmatrix}. \tag{1.4.28}$$

Equations (1.4.25) lead to the following differential conditions

$$\begin{aligned}
&\frac{\partial u^1}{\partial r^1}+\frac{\partial u^1}{\partial r^2}+(u^1-kav^2)\frac{\partial u^2}{\partial r^1}+(u^1-kaw^2)\frac{\partial u^2}{\partial r^2}+u^2(\frac{\partial a}{\partial r^1}+\frac{\partial a}{\partial r^2})=0,\\
&v^1\frac{\partial u^1}{\partial r^1}+w^1\frac{\partial u^1}{\partial r^2}+u^1(v^1\frac{\partial u^2}{\partial r^1}+w^1\frac{\partial u^2}{\partial r^2})+(u^2v^1+kav^2)\frac{\partial a}{\partial r^1}\\
&\qquad+(u^2w^1+kaw^2)\frac{\partial a}{\partial r^2}=0,\\
&k(v^2\frac{\partial u^1}{\partial r^1}+w^2\frac{\partial u^1}{\partial r^2})-(a-kv^2u^1)\frac{\partial u^2}{\partial r^1}\\
&\qquad-(a-w^2ku^1)\frac{\partial u^2}{\partial r^2}+(av^1+kv^2u^2)\frac{\partial a}{\partial r^1}+(aw^1+kw^2u^2)\frac{\partial a}{\partial r^2}=0.
\end{aligned}\tag{1.4.29}$$

Assuming that we have found v^A and w^A as functions of u^1 and u^2, we have to solve (1.4.26) for the unknown functions u^1 and u^2 in terms of r^1 and r^2. The resulting expressions in the equations (1.4.26) are rather complicated, hence we omit them here. Various rank-2 solutions are determined by a specification of functions v^A and w^A in terms of u^1 and u^2. By way of illustration we show how to obtain a solution which depends on one arbitrary function of two variables.

Let us suppose that we are interested in the rank-2 solutions invariant under the vector fields

$$X_1=\partial_t+u^1\partial_x,\quad X_2=\partial_t+u^2\partial_y.\tag{1.4.30}$$

So, the functions $r^1=x-u^1t$ and $r^2=y-u^2t$ are the Riemann invariants of these vector fields. Under this assumption, equations (1.4.25) and (1.4.26) can be easily solved to obtain the Jacobian matrix

$$J=\frac{\partial(u^1,u^2)}{\partial(r^1,r^2)}\neq 0\tag{1.4.31}$$

which has the characteristic polynomial with constant coefficients. This means that the trace and determinant of J are constant,

$$\begin{aligned}
&(i)\quad u^1_{r^1}+u^2_{r^2}=2C_1,\\
&(ii)\quad u^1_{r^1}u^2_{r^2}-u^1_{r^2}u^2_{r^1}=C_2.
\end{aligned}\qquad C_1,C_2\in\mathbb{R}\tag{1.4.32}$$

The trace condition (1.4.32(i)) implies that there exists a function h of r^1 and r^2 such that the conditions

$$u^1=C_1r^1+h_{r^2},\quad u^2=C_1r^2-h_{r^1},\tag{1.4.33}$$

hold. The determinant condition (1.4.32(ii)) requires that the function $h(r^1, r^2)$ satisfies the Monge-Ampère equation

$$h_{r^1r^1}h_{r^2r^2} - h^2_{r^1r^2} = C, \quad C \in \mathbb{R}. \tag{1.4.34}$$

Hence, the general integral of the system (1.4.23) has the implicit form defined by the relations between the variables t, x, y, u^1 and u^2

$$\begin{aligned} u^1 &= C_1(x - u^1t) + \frac{\partial h}{\partial r^2}(x - u^1t, y - u^2t), \\ u^2 &= C_1(y - u^2t) + \frac{\partial h}{\partial r^1}(x - u^1t, y - u^2t), \\ a &= a_0\left((1 + C_1t)^2 + Ct^2\right)^{-1/k}, \quad a_0 \in \mathbb{R} \end{aligned} \tag{1.4.35}$$

where the function h obeys (1.4.34).

Note that the Gaussian curvature K expressed in curvilinear coordinates $(t, r^1, r^2) \in \mathbb{R}^3$ of the surface $\mathcal{S} = \{t = h(r^1, r^2)\}$ is not constant and is given by

$$K(r^1, r^2) = \frac{C}{1 + h^2_{r^1} + h^2_{r^2}}. \tag{1.4.36}$$

For example, a particular nontrivial class of solution of (1.4.23) can be obtained if we assume that $C = 0$. In this case the general solution of (1.4.23) depends on three parameters, $a_0, C_1, m \in \mathbb{R}$ and takes the form

$$\begin{aligned} u^1 &= C_1(x - u^1t) + (1 - m)\left(\frac{x - u^1t}{y - u^2t}\right)^m, \\ u^2 &= C_1(y - u^2t) - m\left(\frac{y - u^2t}{x - u^1t}\right)^{1-m}, \\ a(t) &= \frac{a_0}{(1 + C_1t)^{2/k}}. \end{aligned} \tag{1.4.37}$$

Note that if $C = 0$ and $C_1 = 0$ then the Jacobian matrix J is nilpotent and the divergence of the vector $\vec{u}$ is equal to zero. Then the expression

$$\begin{aligned} u^1 &= (1 - m)\left(\frac{x - u^1t}{y - u^2t}\right)^m, \quad a = a_0, \\ u^2 &= -m\left(\frac{y - u^2t}{x - u^1t}\right)^{1-m}, \end{aligned} \tag{1.4.38}$$

defines a solution $\vec{u} = (u^1, u^2)$ to incompressible Euler equations

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = 0, \quad \operatorname{div} \vec{u} = 0, \quad a = a_0. \tag{1.4.39}$$

Note that for $m = 2$, the explicit form of (1.4.38) is

$$u^1 = -\frac{y^2 + 2tx \pm \sqrt{y^2 + 4tx}}{2t^2}, \quad u^2 = \frac{y \pm \sqrt{y^2 + 4tx}}{t}. \tag{1.4.40}$$

Example 5. Now let us consider a more general case of to the autonomous system (1.4.23) in $p = n+1$ independent $(t, x^i) \in X$ and $q = n+1$ dependent $(a, u^i) \in U$ variables. We look for the rank-k solutions, when $k = n$. The change of variables in the system (1.4.23) under the point transformation

$$\bar{t} = t, \quad \bar{x}^1 = x^1 - u^1 t, \ldots, \bar{x}^n = x^n - u^n t, \quad \bar{a} = a, \quad \bar{u} = u \tag{1.4.41}$$

leads to the following system

$$\begin{aligned} &\frac{D\bar{u}}{D\bar{t}} = 0, \\ &\frac{D\bar{a}}{D\bar{x}} = 0, \quad \frac{D\bar{a}}{D\bar{t}} + k^{-1}\bar{a}\,\mathrm{Tr}\left(B^{-1}\frac{D\bar{u}}{D\bar{x}}\right) = 0, \quad a \neq 0 \end{aligned} \tag{1.4.42}$$

where the total derivatives are denoted by

$$\frac{D}{D\bar{t}} = \frac{\partial}{\partial t} + \bar{u}^i_{\bar{t}}\frac{\partial}{\partial \bar{u}^i}, \quad \frac{D}{D\bar{x}^j} = \frac{\partial}{\partial \bar{x}^j} + \bar{u}^i_{\bar{x}^j}\frac{\partial}{\partial \bar{u}^i}, \quad j = 1, \ldots, n \tag{1.4.43}$$

and the n by n nonsigular matrix B has the form

$$B = \mathbb{I} + t\frac{\partial \bar{u}}{\partial \bar{x}}. \tag{1.4.44}$$

The general solution of the first equation in (1.4.42) is

$$\bar{u} = f(\bar{x}), \quad \bar{x} = (\bar{x}^1, \ldots, \bar{x}^n) \tag{1.4.45}$$

for some function $f : \mathbb{R}^n \to \mathbb{R}^n$. The second equation in (1.4.42) can be written in an equivalent form

$$\frac{\partial}{\partial \bar{t}}\left(\ln|\bar{a}(t)|^k\right) + \mathrm{Tr}\left[(\mathbb{I} + \bar{t}Df(\bar{x}))^{-1}Df(\bar{x})\right] = 0, \tag{1.4.46}$$

where the Jacobian matrix is denoted by

$$Df(\bar{x}) = \frac{\partial f}{\partial \bar{x}}(\bar{x}). \tag{1.4.47}$$

Differentiation of equation (1.4.46) with respect to $\bar{x}$ yields

$$\frac{\partial^2}{\partial \bar{x}\partial \bar{t}}\left(\ln\det\left(\mathbb{I} + \bar{t}Df(\bar{x})\right)\right) = 0 \tag{1.4.48}$$

with general solution

$$\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)=\alpha(\bar{x})\beta(\bar{t}) \tag{1.4.49}$$

for some functions $\alpha:\mathbb{R}^n\to\mathbb{R}$ and $\beta:\mathbb{R}\to\mathbb{R}$. Evaluating (1.4.49) at the initial data $t=0$ implies $\alpha(\bar{x})=\beta(0)^{-1}$. Therefore

$$\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)=\frac{\beta(\bar{t})}{\beta(0)}. \tag{1.4.50}$$

So, we have

$$\frac{\partial}{\partial x}\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)=0. \tag{1.4.51}$$

Now, let us write the determinant in the form of the characteristic polynomial

$$\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)=\bar{t}^nP_n(\varepsilon,\bar{x}),\quad \varepsilon=\frac{1}{\bar{t}} \tag{1.4.52}$$

where

$$\det\left(\varepsilon\mathbb{I}+Df(\bar{x})\right)=\varepsilon^n+p_{n-1}(\bar{x})\varepsilon^{n-1}+\ldots+p_1(\bar{x})\varepsilon+p_0(\bar{x}). \tag{1.4.53}$$

Equation (1.4.51) holds if and only if the coefficients of the characteristic polynomial $p_0,\ldots,p_{n-1}$ are constants. So, equation (1.4.46) implies that

$$\frac{\partial}{\partial\bar{t}}\ln|\bar{a}(\bar{t})|^k+\frac{\partial}{\partial\bar{t}}\ln|\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)|=0.$$

Then we have,

$$\frac{\partial}{\partial\bar{t}}\left(|\bar{a}(\bar{t})|^k\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)\right)=0. \tag{1.4.54}$$

Solving equation (1.4.54) we obtain

$$\bar{a}(\bar{t})=\gamma\left(\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)\right)^{-1/k},\quad 0\neq\gamma\in\mathbb{R}. \tag{1.4.55}$$

Thus, the general solution of system (1.4.23) is

$$u=f(\bar{x}),\quad a(\bar{t})=\gamma\left[1+p_{n-1}\bar{t}+\ldots+p_0\bar{t}^n\right]^{-1/k}, \tag{1.4.56}$$

for any differentiable function $f:\mathbb{R}^n\to\mathbb{R}^n$ and takes the form of a constant characteristic polynomial on the Cauchy data $t=0$

$$P_n(\varepsilon)=\varepsilon^n+p_{n-1}\varepsilon^{n-1}+\ldots+p_1\varepsilon+p_0. \tag{1.4.57}$$

Note that the function a is constant if and only if

$$P_n(\varepsilon) = \varepsilon^n. \tag{1.4.58}$$

This fact holds if and only if the Jacobian matrix $Df(\bar{x})$ is nilpotent.

Note that in the particular case when $p = 2$, the general explicit solution of (1.4.23) is given by

$$u(x,t) = (\beta + \alpha x)(1 + \alpha t)^{-1}, \quad a(t) = \gamma(1 + \alpha t)^{-1/k}, \quad \alpha, \beta, \gamma \in \mathbb{R}. \tag{1.4.59}$$

An extension of this solution to the (n+1)-dimensional space X is as follows

$$u(x,t) = (\mathbb{I} + t\alpha)^{-1}(\beta + \alpha x), \quad a(t) = \gamma(\det(\mathbb{I} + \alpha t))^{-1/k}, \tag{1.4.60}$$

where β is any constant n-component vector and α is any n by n constant matrix. In this case the Jacobian matrix is constant

$$Df(\bar{x}) = \alpha \tag{1.4.61}$$

for any $\bar{x} \in \mathbb{R}^n$.

Finally, a similar computation can be performed for the case in which the vector function $\vec{u} = (u^1, u^2, u^3)$ satisfies the overdetermined system (1.4.39). In the above notation, an invariant solution under the vector fields

$$X_a = \partial_t + u^a \partial_{(a)}, \quad a = 1, 2, 3 \tag{1.4.62}$$

is given by $\bar{u} = f(\bar{x})$ and is a divergence free solution

$$\operatorname{div} \vec{u} = 0 \tag{1.4.63}$$

if and only if the trace condition

$$\operatorname{Tr}\left(B^{-1}\frac{\partial \bar{u}}{\partial \bar{x}}\right) = 0, \quad B = \mathbb{I} + t\frac{D\bar{u}}{D\bar{x}}(\bar{x}) \tag{1.4.64}$$

holds. But

$$\frac{D\vec{u}}{D\bar{x}} = \frac{\partial B}{\partial t}. \tag{1.4.65}$$

Therefore $\operatorname{div} \vec{u} = 0$ if and only if

$$\operatorname{Tr}\left(B^{-1}\frac{\partial B}{\partial t}\right) = 0, \tag{1.4.66}$$

or equivalently, if and only if

$$\frac{\partial}{\partial t}(\det B) = 0. \tag{1.4.67}$$

This means that the Jacobian matrix $Df(\bar{x})$ has to be a nilpotent one and takes the form

$$Df(\bar{x}) = \begin{pmatrix} 0 & f^1_{\bar{x}^2} & f^1_{\bar{x}^3} \\ 0 & f^2_s & -f^2_s \\ 0 & f^2_s & -f^2_s \end{pmatrix}, \tag{1.4.68}$$

where f^1 is an arbitrary function of two variables $\bar{x}^2$ and $\bar{x}^3$ and f^2 is an arbitrary function of one variable $s = \bar{x}^2 - \bar{x}^3$. Note that if $f^1_{\bar{x}^2} \neq f^1_{\bar{x}^3}$ then the Jacobian matrix $Df(\bar{x})$ has rank 2 (otherwise f^1 is any function of s and $Df(\bar{x})$ has rank 1). As a consequence, the matrix B has the form

$$B = \begin{pmatrix} 1 & tf^1_{\bar{x}^2} & tf^1_{\bar{x}^3} \\ 0 & 1 + tf^2_s & -tf^2_s \\ 0 & tf^2_s & 1 - tf^2_s \end{pmatrix}, \quad \det B = 1. \tag{1.4.69}$$

So, the condition (1.4.67) is identically satisfied. Thus, the general solution of system (1.4.39) is implicitly defined by the equations

$$u^1 = f^1\big(x^2 - tf^2(x^2 - x^3), x^3 - tf^2(x^2 - x^3)\big), \quad u^2 = u^3 = f^2(x^2 - x^3), \quad a = a_0, \tag{1.4.70}$$

where the functions f^1 and f^2 are arbitrary functions of their arguments. Equations (1.4.70) define a rank-2 solution but, according to the formula for the corresponding principle [**53**], it is not a double Riemann wave.

Obviously, other choices of the wave vectors λ^A (and the related vector fields X_a) lead to different classes of solutions. The problem of the classification of these solutions remains still open but some results are known (see e.g. the functorial properties of systems of equations determining Riemann invariants [**55, 56**]).

1.5. Conclusions

In this paper we have developed a new method which serves as a tool for constructing rank-k solutions of multi-dimensional hyperbolic systems including Riemann waves and their superpositions. The most significant characteristic of this approach is that it allows us to construct regular algorithms for finding solutions written in terms of Riemann invariants. Moreover, this approach does not refer to any additional considerations, but proceeds directly from the given system of PDEs.

Riemann waves and their superposition described by multi-dimensional hyperbolic systems have been studied so far only in the context of the generalized method of characteristics (GMC) [**9, 53, 99**]. The essence of this method can be summarized as follows. It requires the supplementation of the original system of PDEs (1.1.1) with additional differential constraints for which all first derivatives are decomposable in the following form

$$\frac{\partial u^\alpha}{\partial x^i}(x) = \sum_{A=1}^{k} \xi^A(x)\gamma_A^\alpha(u)\lambda_i^A(u), \tag{1.5.1}$$

where

$$\begin{aligned} &\left(\Delta_\alpha^{\mu i}(u)\lambda_i^A\right)\gamma_{(A)}^\alpha = 0, \quad A = 1, \ldots, k \\ &\operatorname{rank}\left(\Delta_\alpha^{\mu i}(u)\lambda_i^A\right) < \min(l, q). \end{aligned} \tag{1.5.2}$$

Here, $\xi^A \neq 0$ are treated as arbitrary scalar functions of x and we assume that the vector fields $\{\gamma_1, \ldots, \gamma_k\}$ are locally linearly independent. The necessary and sufficient conditions for the existence of k-wave solutoins (when $k > 1$) of the system (1.5.1) in terms of Riemann invariants impose some additional differential conditions on the wave vectors λ^A and the corresponding vector fields γ_A, namely [**99**]

$$\begin{aligned} &[\gamma_A, \gamma_B] \in \operatorname{span}\{\gamma_A, \gamma_B\}, \\ &\mathcal{L}_{\gamma_B}\lambda^A \in \operatorname{span}\{\lambda^A, \lambda^B\}, \quad \forall A \neq B = 1, \ldots, k, \end{aligned} \tag{1.5.3}$$

where $[\gamma_A, \gamma_B]$ denotes the commutator of the vector fields γ_A and γ_B, $\mathcal{L}_{\gamma_B}$ denotes the Lie derivatives along the vector fields γ_B.

Due to the homogeneity of the wave relation (1.5.2) we can choose, without loss of generality, a holonomic system for the fields $\{\gamma_1, \ldots, \gamma_k\}$ by requiring a proper length for each vector γ_A such that

$$[\gamma_A, \gamma_B] = 0, \quad \forall A \neq B = 1, \ldots, k. \tag{1.5.4}$$

It determines a k-dimensional submanifold $\mathcal{S} \subset U$ obtained by solving the system of PDEs

$$\frac{\partial f^\alpha}{\partial r^A} = \gamma_A^\alpha(f^1, \ldots, f^k) \tag{1.5.5}$$

with solution $\pi : F \to U$ defined by

$$\pi : (r^1, \ldots, r^k) \to \left(f^1(r^1, \ldots, r^k), \ldots, f^q(r^1, \ldots, r^k)\right). \tag{1.5.6}$$

The wave vectors λ^A are pulled back to the submanifold $\mathcal{S}$ and then λ^A become functions of the parameters $r^1, \ldots, r^k$. Consequently, the requirements (1.5.1) and (1.5.3) take the form

$$\frac{\partial r^A}{\partial x^i}(x) = \xi^A(x)\lambda_i^A(r^1, \ldots, r^k), \tag{1.5.7}$$

$$\frac{\partial \lambda^A}{\partial r^B} \in \text{span}\{\lambda^A, \lambda^B\}, \quad \forall A \neq B = 1, \ldots, k \tag{1.5.8}$$

respectively. It has been shown [**99**] that the conditions (1.5.5) and (1.5.8) ensure that the set of solutions of system (1.1.1) subjected to (1.5.1), depends on k arbitrary functions of one variable. It has also been proved [**102**] that all solutions, i.e. the general integral of the system (1.5.7) under conditions (1.5.8) can be obtained by solving, with respect to the variables $r^1, \ldots, r^k$, the system in implicit form

$$\lambda_i^A(r^1, \ldots, r^k)x^i = \psi^A(r^1, \ldots, r^k), \tag{1.5.9}$$

where ψ^A are arbitrary functionnally independent differentiable functions of k variables $r^1, \ldots, r^k$. Note that solutions of (1.5.7) are constant on $(p-k)$-dimensional hyperplanes perpendicular to wave vectors λ^A satisfying conditions (1.5.8).

As one can observe, both methods discussed here exploit the invariance properties of the initial system of equations (1.1.1). In the GMC, they have the purely geometric character for which a form of solution is postulated by subjecting the original system (1.1.1) to the side conditions (1.5.1). In contrast, in the case of the approach proposed here we augment the system (1.1.1) by differential constraints (1.3.9).

There are, however, at least two basic differences between the GMC and our proposed approach. Riemann multiple waves defined from (1.5.1), (1.5.5) and

(1.5.8) constitute a more limited class of solutions than the rank-k solutions postulated by our approach. This difference results from the fact that the scalar functions ξ^A appearing in expression (1.5.1) (which describe the profile of simple waves entering into a superposition) are substituted in our case (see expressions (1.3.3) or (1.3.4)) with a q by q or k by k matrix Φ^1 or Φ^2, respectively. This situation consequently allows for a much broader range of initial data. The second difference consists in fact that the restrictions (1.5.5) and (1.5.8) on the vector fields γ_A and λ^A, ensuring the solvability of the problem by GMC, are not necessary in our approach. This makes it possible for us to consider more general configurations of Riemann waves entering into an interaction.

A number of different attempts to generalize the Riemann invariants method and its various applications can be found in the recent literature on the subject (see e.g. [**26, 35, 36, 38, 111**]). For instance, the nonlinear k-waves superposition $u = f(r^1, \ldots, r^k)$ described in [**97**] can be regarded as dispersionless analogues to "n-gap solution" of (1.1.1) which require the resolution of a set of commuting diagonal systems for the Riemann invariants r^A, i.e.

$$r^A_{x^i} = \mu^A_{i(j)}(r) r^A_{x^j}, \quad A = 1, \ldots, k, \quad i \neq j = 1, \ldots, p. \tag{1.5.10}$$

That specific technique involves differential constraints on the functions μ^A_{ij} of the form [**111**]

$$\frac{\partial_j \mu^A_{i(j)}}{\mu^A_{i(j)} - \mu^A_{j(i)}} = \frac{\partial_j u^B_{i(j)}}{\mu^B_{i(j)} - \mu^B_{j(i)}}, \quad i \neq j, \quad A \neq B = 1, \ldots, k, \tag{1.5.11}$$

no summation. As in the case of Riemann k-waves if the system (1.5.11) is satisfied for the functions μ^A_{ij} then the general integral of the system (1.5.10) can be obtained by solving system (1.5.9) with respect to the variables $r^1, \ldots, r^k$.

In contrast, our proposed approach does not require the use of differential equations (1.5.10) and therefore does not impose constraints on the functions μ^A_{ij} when the 1-forms λ^A are linearly independent and $k < p$.

However, if one removes these assumptions and λ^A can be linearly dependent and $k \geq p$ then the approach presented in [**36**] is a valuable one and provides an effective tool for classification criterion of integrable systems.

In order to verify the efficiency of our approach we have used it for constructing rank-2 solutions of several examples of hydrodynamic type systems. The proposed approach proved to be a useful tool in the case of multi-dimensional hydrodynamic type systems (1.1.1), since it leads to new interesting solutions.

The examples illustrating our method clearly demonstrate its usefulness as it has produced several new and interesting results. Let us note that the outlined approach to rank-k solutions lends itself to numerous potential applications which arise in physics, chemistry and biology. It has to be stressed that for many physical systems, (e.g. nonlinear field equations, Einstein's equations of general relativity and the equations of continuous media, etc) there have been very few, if any, known examples of rank-k solutions. The approach proposed here offers a new and promising way to investigate and construct such type of solutions.

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Chapitre 2

CONDITIONAL SYMMETRIES AND RIEMANN INVARIANTS FOR HYPERBOLIC SYSTEMS OF PDES

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Résumé

Cet article contient une analyse des solutions de rang k exprimées en termes d'invariants de Riemann, obtenues à partir d'une combinaison de la méthode des symétries conditionnelles (CSM) et de la méthode des caractéristiques généralisée (GMC) pour les systèmes quasilinéaires hyperboliques multidimensionnels. Une variation de la méthode des symétries conditionnelles est proposée. Nous construisons un module de Lie de champs de vecteurs représentant des symétries d'un système surdéterminé composé du système original et certaines contraintes différentielles du premier ordre. Nous démontrons que ce système déterminé admet des solutions de rang k exprimées en termes d'invariants de Riemann. Il est démontré que les contraintes différentielles imposées par la CSM sont plus faibles que dans le cas de la MCG, ce qui nous permet de satisfaire un plus grand éventail de données initiales. Finalement, plusieurs exemples d'application de cette approche aux équations d'un fluide isentropique multidimensionnel sont donnés. Parmi celles-ci, plusieurs solutions de type solitoniques (« bumps », « kinks », et ondes multiples) sont obtenues et nous montrons qu'elles demeurent bornées même lorsque les invariants de Riemann admettent la catastrophe du gradient.

Abstract

This paper contains an analysis of rank-k solutions in terms of Riemann invariants, obtained from interrelations between two concepts, that of the symmetry reduction method and of the generalized method of characteristics (GMC) for first order quasilinear hyperbolic systems of PDEs in many dimensions. A variant of the conditional symmetry method (CSM) for obtaining this type of solutions is proposed. A Lie module of vector fields, which are symmetries of an overdetermined system defined by the initial system of equations and certain first order differential constraints, is constructed. It is shown that this overdetermined system admits rank-k solutions expressible in terms of Riemann invariants. It was demonstrated that the differential constraints required by the CSM are weaker than the ones required by the GMC, which allows to ease the restrictions on the initial data. Consequently, we can consider more diverse configurations of wave superpositions than those admitted by the GMC. Finally, examples of applications of the proposed approach to the fluid dynamics equations in (k+1) dimensions are discussed in detail. Several new soliton-like solutions (among them kinks, bumps and multiple wave solutions) have been obtained. These solutions remain bounded even when the Riemann invariants admit a gradient catastrophe. Some physical interpretations of these results are discussed.

Keywords : Riemann invariants, conditional symmetry method, generalized method of characteristics, rank-k solutions, multiple Riemann waves.

2.1. Introduction

The general properties of nonlinear systems of PDEs in many dimensions and techniques for obtaining their exact solutions remain essential subjects of investigation in modern mathematics. In the case of hyperbolic systems, the oldest, and still useful, approach to this subject has been the method of characteristics which originated from the work of G. Monge [**85**]. In its modern form it is described e.g. in [**23, 62, 102, 107, 112, 118**]. More recently, the development of group theoretical methods, based on the work of S. Lie [**72**], has led to progress in this area, delivering new efficient techniques. However, these two theoretical approaches have remained disconnected and have provided, in most cases, different sets of solutions. The symmetry reduction methods (SRM) certainly have a broader range of application, while the generalized method of characteristics (GMC), though limited to nonelliptic systems, has been more successful in producing wave and multiple wave solutions. Thus, the mutual relation between these two methods is a matter of interest and we have undertaken this subject with the view of combining the strengths of both of them.

The approach to constructing rank-k solutions presented in this paper evolved from our earlier work [**47, 50, 51**], aimed at obtaining Riemann k-waves by means of the conditional symmetry method (CSM). The main idea here has been to select the supplementary differential constraints (DCs), employed by this method, in such a way that they ensure the existence of solutions expressible in terms of Riemann invariants. Interestingly, as we show later, these constraints prove to be less restrictive than the conditions required by the GMC. As a result, we obtain larger classes of solutions than the class of Riemann k-waves obtainable through the GMC.

The organization of this paper is as follows. Section 2.2 gives a brief account of the generalized method of characteristics for first order quasilinear hyperbolic systems of PDEs in many independent and dependent variables. A geometric formulation of the Riemann k-wave problem is presented there. In Section 2.3 we reformulate this problem or rather, more generally, a problem of rank-k solutions expressible in terms of Riemann invariants, in the language of group theoretical approach. The necessary and sufficient conditions for obtaining this type of solutions are determined after an analysis of their group properties. A new version of the conditional symmetry method for construction of these solutions is proposed. Sections 2.4 to 2.7 present an application of the developed approach to the equations describing an ideal nonstationary isentropic compressible fluid. We find rank-1 as well as rank-2 and rank-3 solutions admitted by the system, among them several new types of soliton-like solutions including kinks, bumps and snoidal waves. In Section 2.8, we construct rank-k solutions for the isentropic flow with sound velocity dependent on time only. We show that the general integral of a Cauchy problem for this system depends on k arbitrary functions of k variables. Section 2.9 summarizes the obtained results and contains some suggestions regarding further developments.

2.2. The generalized method of characteristics

The generalized method of characteristics has been designed for the purpose of solving quasilinear hyperbolic systems of first order PDEs in many dimensions.This approach enables us to construct and investigate Riemann waves and their superpositions (i.e. Riemann k-waves), which are admitted by these systems. The main feature of the method is the introduction of new independent variables (called Riemann invariants) which remain constant on certain hyperplanes perpendicular to wave vectors associated with the initial system. This results in a reduction of the dimensionality of the problem. A number of attempts to generalize the Riemann invariants method and its various applications can be found in the recent literature of the subject (see e.g. **[23]** - **[36]**, **[83, 97]** and references therein).

At this point, we summarize the version of the GMC for constructing k-wave solutions developed progressively in **[8, 53, 55, 56, 100, 102]**. Let us consider a quasilinear hyperbolic system of l first order PDEs

$$\mathcal{A}^{\mu i}{}_{\alpha}(u)u^{\alpha}_{i} = 0, \quad \mu = 1,\ldots,l, \quad \alpha = 1,\ldots,q, \quad i = 1,\ldots,p, \tag{2.2.1}$$

in p independent variables $x = (x^1,\ldots,x^p) \in X \subset \mathbb{R}^p$ and q dependent variables $u = (u^1,\ldots,u^q) \in U \subset \mathbb{R}^q$. The term u^{α}_{i} denotes the first order partial derivative of u^{α} with respect to x^i, i.e. $u^{\alpha}_{i} \equiv \partial u^{\alpha}/\partial x^i$. Here we adopt the summation convention over the repeated lower and upper indices, except in the cases in which one index is taken in brackets. The system is properly determined if $l = q$. All considerations have local character, that is, it suffices to look for solutions defined in a neighborhood of $x = 0$. The main steps in constructing k-wave solutions can be presented as follows.

1. Find the real-valued functions $\lambda^A = (\lambda_1^A, \dots, \lambda_p^A) \in X$ and $\gamma_A = (\gamma_A^1, \dots, \gamma_A^q) \in U$ by solving the wave relation associated with the initial system (2.2.1),

$$(\mathcal{A}^{\mu i}_{\alpha}(u)\lambda_i^A)\gamma^{\alpha}_{(A)} = 0, \quad A = 1, \dots, k < p. \tag{2.2.2}$$

Thus we require that the condition

$$\operatorname{rank}(\mathcal{A}^{\mu i}_{\alpha}(u)\lambda_i^A) < \min(l, q) \tag{2.2.3}$$

holds. We assume here the generic case in which the rank does not vary on some open subset $\Omega \subset U$. This step is completely algebraic.

2. Let us assume that we have found k linearly independent functions λ^A and γ_A which are C^1 in Ω. We postulate a form of solution $u(x)$ of the initial system (2.2.1) such that all first order derivatives of u with respect to x^i are decomposable in the following way

$$\frac{\partial u^\alpha}{\partial x^i}(x) = \sum_{A=1}^{k} h^A(x)\gamma_A^\alpha(u)\lambda_i^A(u) \tag{2.2.4}$$

on some open domain $\mathcal{B} \subset X \times U$. Here, $h^A(x)$ are arbitrary functions of x. This step means that the original system (2.2.1) is subjected to the first order differential constraints (2.2.4). Thus we have to solve an overdetermined system composed of (2.2.1) and (2.2.4).

The condition (2.2.4), crucial to the GMC, determines the class of solutions, called Riemann k-waves, resulting from superposition of k simple waves.

3. Before proceeding further, we should verify whether the conditions on the vector functions λ^A and γ_A, which are necessary and sufficient for the existence of solutions of the system composed of (2.2.1) and (2.2.4), are satisfied. These conditions, in accordance with the Cartan theory of systems in involution [**11**], take the form

$$\text{i) } [\gamma_A, \gamma_B] \in \operatorname{span}\{\gamma_A, \gamma_B\}, \quad \text{ii) } \mathcal{L}_{\gamma_B}\lambda^A \in \operatorname{span}\{\lambda^A, \lambda^B\}, \quad A \neq B = 1, \dots, k, \tag{2.2.5}$$

where $\mathcal{L}_{\gamma_B}$ denotes the Lie derivative along the vector field γ_B and the bracket $[\gamma_A, \gamma_B]$ denotes the commutator of the vector fields γ_A, γ_B.

4. Given that the conditions (2.2.5) are satisfied, we can choose, due to the homogeneity of the wave relation (2.2.2), a holonomic system for the vector fields $\{\gamma_1, \ldots, \gamma_k\}$, by requiring a proper length for each vector γ_A such that

$$[\gamma_A, \gamma_B] = 0. \tag{2.2.6}$$

Conditions (2.2.6) determine a k-dimensional submanifold $\mathcal{S} \subset U$ which can be obtained by solving the system of PDEs

$$\frac{\partial u^\alpha}{\partial r^A} = \gamma_A^\alpha(u^1, \ldots, u^q), \quad A = 1, \ldots, k. \tag{2.2.7}$$

The solution of (2.2.7)

$$u = f(r^1, \ldots, r^k) \tag{2.2.8}$$

gives the explicit parametrization of the submanifold $\mathcal{S}$ immersed in the space U.
5. Next we consider the functions $f^*(\lambda^A)$, that is, the functions $\lambda^A(u)$ pulled back to the submanifold $\mathcal{S} \subset U$. The $\lambda^A(u)$ become functions of the parameters $(r^1, \ldots, r^k)$ on $\mathcal{S}$. For simplicity of notation, we denote $f^*(\lambda^A)$ by $\lambda^A(r^1, \ldots, r^k)$.
6. Restricting the equations (2.2.4) and (2.2.5) to the submanifold $\mathcal{S}$ and using the linear independence of the vectors $\{\gamma_1, \ldots, \gamma_k\}$, we obtain

$$\frac{\partial r^A}{\partial x^i} = h^A(x)\lambda_i^A(r^1, \ldots, r^k), \tag{2.2.9}$$

$$\frac{\partial \lambda^A}{\partial r^B} = \alpha_B^A(r^1, \ldots, r^k)\lambda^A + \beta_B^A(r^1, \ldots, r^k)\lambda^B, \quad A \neq B = 1, \ldots, k, \tag{2.2.10}$$

for some real-valued functions α_B^A and $\beta_B^A : \mathcal{S} \to \mathbb{R}$. Here we do not use the summation convention. According to the Cartan theory of systems in involution, the conditions (2.2.7), (2.2.9) and (2.2.10) ensure that the set of solutions of the initial system of PDEs (2.2.1) subjected to the differential constraints (2.2.4) depends on k arbitrary functions of one variable.
7. Next, we look for the most general class of solutions of the linear system of equations (2.2.10) for λ^A as functions of $r^1, \ldots, r^k$. We can perform this analysis by using, for example, the Monge-Darboux method [**53**].
8. From the general solution of (2.2.10) for the functions λ^A, the solution of the system (2.2.9) can be derived in the implicit form

$$\lambda_i^A(r^1, \ldots, r^k)x^i = \psi^A(r^1, \ldots, r^k), \quad A = 1, \ldots, k, \tag{2.2.11}$$

where $\psi^A : \mathbb{R}^k \to \mathbb{R}$ are some functionally independent differentiable functions of k variables $r^1, \ldots, r^k$ such that

$$\frac{\partial \psi^A}{\partial r^B} = \alpha^A_B(r^1, \ldots, r^k)\psi^{(A)} + \beta^A_B(r^1, \ldots, r^k)\psi^{(B)}, \quad A \neq B. \tag{2.2.12}$$

Note that the solutions $r^A(x)$ of (2.2.11) are constant on (p-k)-dimensional hyperplanes perpendicular to the wave vectors λ^A.

9. Finally, the k-wave solution of (2.2.1) is obtained from the explicit parametrization (2.2.8) of the submanifold $\mathcal{S} \subset U$ in terms of the parameters $r^1, \ldots, r^k$, which are now implicitly defined as functions of $x^1, \ldots, x^p$ by the solutions of the system (2.2.11) in the space X .

If the set of implicitly defined relations between the variables u^α, x^i and $(r^1, \ldots, r^k)$,

$$u^\alpha = f^\alpha(r^1, \ldots, r^k), \quad \lambda^A_i(r^1, \ldots, r^k)x^i = \psi^A(r^1, \ldots, r^k), \tag{2.2.13}$$

can be solved in such a way that r and u^α can be given as graphs over an open subset $\mathcal{D} \subset X$, then the functions $u^\alpha = f^\alpha(r^1(x), \ldots, r^k(x))$ constitute a k-wave solution of the quasilinear hyperbolic system (2.2.1). The scalar functions $r^A(x)$ are called the Riemann invariants. For $p = 2$ they coincide with the classical Riemann invariants as they have been usually introduced in the literature of the subject (see e.g. [**21, 62, 107, 114**]).

Finally, let us comment on the Cauchy problem for Riemann k-waves (for a detailed discussion see e.g. [**53, 102, 112**]).

Let us consider q functions $u^1(x), \ldots, u^q(x)$ which take some prescribed values $u_0(\bar{x}) = (u^1_0(\bar{x}), \ldots, u^q_0(\bar{x}))$ on the hyperplane $\mathbb{P} \subset \mathbb{R}^n$ defined by $t = 0$. Here, we use the notation $x = (t, \bar{x}) \in X \subset \mathbb{R}^{n+1}$. It was shown [**53**] that for $0 < t < T$ the initial value problem for the system (2.2.1) has locally exactly one solution in the form of a Riemann k-wave defined implicitly by relations (2.2.8), (2.2.11) and (2.2.12) if the C^2 function $u_0(\bar{x})$ satisfies the following conditions.

i) $u_0(\bar{x})$ is sufficiently small that there exists a time interval $[0, T]$ in which the gradient catastrophe for a solution $u(x)$ of (2.2.1) does not occur.

ii) $u_0(\bar{x})$ is decomposable according to conditions (2.2.4), that is

$$\frac{\partial u^\alpha}{\partial x^j}(\bar{x})\Big|_{\mathbb{P}} = \sum_{A=1}^{k} \xi^A(0,\bar{x})\gamma_A^\alpha(u_0(\bar{x}))\lambda_j^A(u_0(\bar{x}))\Big|_{\mathbb{P}} \tag{2.2.14}$$

on some open domain $\mathcal{E} \subset \mathbb{P} \times U$.

2.3. Conditional Symmetries and Riemann Invariants

Until now, the only way to approach the problem of superposition of many Riemann waves in multi-dimensional space was through the GMC. This method, like all other techniques of solving PDEs, has its limitations. They have motivated the authors to search for the means of constructing larger classes of multiple wave solutions expressible in terms of Riemann invariants. The natural way to do it is to look at these solutions from the point of view of group invariance properties. The feasibility and advantages of such an approach were demonstrated for certain fluid dynamic equations in [**47, 51**]. We have been particularly interested in the construction of nonlinear superpositions of elementary solutions (i.e. rank-1 solutions) of (2.2.1), and the preliminary analysis indicated that the method of conditional symmetry is an especially useful tool for this purpose.

We use the term "conditional symmetry" here as introduced by P.J. Olver and P. Rosenau [**90**]. It evolved from the notion of "nonclassical symmetry" which had originated from the work of G. Bluman and J. Cole [**4**] and was developed by several authors (D. Levi and P. Winternitz [**70**] and Fushchych [**39**] among others). For a review of this subject see e.g. [**5, 16, 92**] and references therein.

The method of conditional symmetry consists in supplementing the original system of PDEs with first order differential constraints for which a symmetry criterion of the given system of PDEs is identically satisfied. Under certain circumstances this augmented system of PDEs admits a larger class of Lie symmetries than the original system of PDEs. For our purpose we adapt here the version of CSM developed in [**45, 50**].

We now reformulate the task of constructing rank-k solutions expressible in terms of Riemann invariants in the language of the group theoretical approach.

Let us consider the nondegenerate system (2.2.1) in its matrix form

$$\mathcal{A}^1(u)u_1 + \ldots + \mathcal{A}^p(u)u_p = 0, \tag{2.3.1}$$

where $\mathcal{A}^1, \ldots, \mathcal{A}^p$ are l by q real-valued matrix functions of u. If we set $l = q$, $p = n+1$ (we denote the independent variables by $t = x^0, x^1, \ldots, x^n$) and $\mathcal{A}^0$ is the identity matrix, then the system has the evolutionary form

$$u_t + \sum_{j=1}^{n} \mathcal{A}^j(u)u_j = 0. \tag{2.3.2}$$

For a fixed set of k linearly independent real-valued wave vectors

$$\lambda^A(u) = (\lambda_1^A(u), \ldots, \lambda_p^A(u)), \quad A = 1, \ldots, k < p,$$

with

$$\ker(\lambda_i^A \mathcal{A}^i) \neq 0, \tag{2.3.3}$$

we define the real-valued functions $r^A : X \times U \to \mathbb{R}$ such that

$$r^A(x,u) = \lambda_i^A(u)x^i, \quad A = 1, \ldots, k. \tag{2.3.4}$$

These functions are Riemann invariants associated with the wave vectors λ^A, as introduced in the previous section.

We postulate the form of solution of (2.3.1) defined implicitly by the following set of relations between the variables u^α, x^i and r^A

$$u = f(r^1(x,u), \ldots, r^k(x,u)), \quad r^A(x,u) = \lambda_i^A(u)x^i, \quad A = 1, \ldots, k. \tag{2.3.5}$$

Equations (2.3.5) determine a unique function $u(x)$ on a neighborhood of $x = 0$ for any $f : \mathbb{R}^k \to \mathbb{R}^q$. The Jacobi matrix of equations (2.3.5) can be presented as

$$\partial u = (u_i^\alpha) = \left(\mathcal{I}_q - \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial u}\right)^{-1} \frac{\partial f}{\partial r}\lambda \in \mathbb{R}^{q\times p}, \tag{2.3.6}$$

or equivalently as

$$\partial u = \frac{\partial f}{\partial r}\left(\mathcal{I}_k - \frac{\partial r}{\partial u} \cdot \frac{\partial f}{\partial r}\right)^{-1} \lambda \in \mathbb{R}^{q\times p}, \tag{2.3.7}$$

where

$$\frac{\partial f}{\partial r}=\left(\frac{\partial f^\alpha}{\partial r^A}\right)\in\mathbb{R}^{q\times k},\quad \lambda=(\lambda_i^A)\in\mathbb{R}^{k\times p}, \tag{2.3.8}$$

$$\frac{\partial r}{\partial u}=\left(\frac{\partial r^A}{\partial u^\alpha}\right)=\left(\frac{\partial \lambda_i^A}{\partial u^\alpha}x^i\right)\in\mathbb{R}^{k\times q},\quad r=(r^1,\dots,r^k)\in\mathbb{R}^k, \tag{2.3.9}$$

and $\mathcal{I}_q$ and $\mathcal{I}_k$ are the q by q and k by k identity matrices respectively, Applying the implicit function theorem, we obtain the following conditions ensuring that r^A and u^α are expressible as graphs over some open subset $\mathcal{D}$ of $\mathbb{R}^p$,

$$\det\left(\mathcal{I}_q-\frac{\partial f}{\partial r}\cdot\frac{\partial\lambda}{\partial u}x\right)\neq 0, \tag{2.3.10}$$

or

$$\det\left(\mathcal{I}_k-\frac{\partial\lambda}{\partial u}x\cdot\frac{\partial f}{\partial r}\right)\neq 0. \tag{2.3.11}$$

The inverse matrix in (2.3.6) (or in (2.3.7)) is well defined, since

$$\frac{\partial r}{\partial u}=0\quad\text{at}\quad x=0. \tag{2.3.12}$$

In our further considerations we assume that the conditions (2.3.10) or (2.3.11) are fulfilled, whenever applicable.

The postulated solution (2.3.5) is a rank-k solution, since the Jacobi matrix of $u(x)$ has a rank equal to k. Its image is a k-dimensional submanifold $\mathcal{S}_k$ in the first jet space $J^1=J^1(X\times U)$.

For a fixed set of k linearly independent wave vectors $\{\lambda^1,\dots,\lambda^k\}$ we define another set of $(p-k)$ linearly independent vectors

$$\xi_a(u)=(\xi_a^1(u),\dots,\xi_a^p(u))^T,\quad a=1,\dots,p-k, \tag{2.3.13}$$

satisfying the orthogonality conditions

$$\lambda_i^A\xi_a^i=0,\quad A=1,\dots,k. \tag{2.3.14}$$

Then, due to (2.3.6) (or (2.3.7)), the graph of the solution $\Gamma=\{(x,u(x))\}$ is invariant under the family of the first order differential operators

$$X_a=\xi_a^i(u)\frac{\partial}{\partial x^i},\quad a=1,\dots,p-k, \tag{2.3.15}$$

defined on $X \times U$. Note that the vector fields X_a do not include vectors tangent to the direction u. So, the vectors fields X_a form an Abelian distribution on $X \times U$, i.e.

$$[X_a, X_b] = 0, \quad a \neq b = 1, \ldots, p-k. \tag{2.3.16}$$

Conversely, if $u(x)$ is a q-component function defined on a neighborhood of $x = 0$ such that the graph $\Gamma = \{(x, u(x))\}$ is invariant under a set of $(p-k)$ vector fields X_a with properties (2.3.14), then $u(x)$ is a solution of equations (2.3.5), for some f. This is so, because the set $\{r^1, \ldots, r^k, u^1, \ldots, u^q\}$ constitutes a complete set of invariants of the Abelian algebra of the vector fields (2.3.15). This geometrically characterizes the solutions $u(x)$ of the equations (2.3.5).

The group-invariant solutions of the system (2.3.1) consist of those functions $u = f(x)$ which satisfy both the initial system (2.3.1) and a set of first order differential constraints

$$\xi_a^i u_i^\alpha = 0, \quad a = 1, \ldots, p-k, \quad \alpha = 1, \ldots, q, \tag{2.3.17}$$

ensuring that the characteristics of the vectors fields X_a are equal to zero.

Note that, in general, the conditions (2.3.17) are weaker than the DCs (2.2.4) required by the GMC, since the latter are submitted to the algebraic condition (2.2.2). Indeed, (2.3.17) implies

$$u_i^\alpha = \Phi_A^\alpha \lambda_i^A, \tag{2.3.18}$$

where Φ_A^α are real-valued matrix functions on the first jet space $J^1 = J^1(X \times U)$,

$$\Phi_A^\alpha = \left[\left(\mathcal{I}_q - \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial u} \right)^{-1} \right]_\beta^\alpha \frac{\partial f^\beta}{\partial r^A} \in \mathbb{R}^{q \times k}, \tag{2.3.19}$$

or

$$\Phi_A^\alpha = \frac{\partial f^\alpha}{\partial r^B} \left[\left(\mathcal{I}_k - \frac{\partial r}{\partial u} \cdot \frac{\partial f}{\partial r} \right)^{-1} \right]_A^B \in \mathbb{R}^{q \times k}, \tag{2.3.20}$$

which do not necessarily satisfy the wave relation (2.2.2). This fact results in easing up the restrictions on initial data at $t = 0$, thus we are able to consider more diverse configurations of waves involved in a superposition than in the GMC case.

We now proceed to solve the overdetermined system composed of the initial system (2.3.1) and the DCs (2.3.17)

$$\mathcal{A}^{i\mu}_{\alpha}(u)u^{\alpha}_{i} = 0, \quad \xi^{i}_{a}(u)u^{\alpha}_{i} = 0, \quad \mu = 1,\dots,l, \quad a = 1,\dots,p-k. \tag{2.3.21}$$

Substituting (2.3.6) (or (2.3.7)) into (2.3.1) yields

$$\operatorname{Tr}\left(\mathcal{A}^{\mu}\left(\mathcal{I}_q - \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial u}\right)^{-1}\frac{\partial f}{\partial r}\lambda\right) = 0, \quad \mu = 1,\dots,l, \tag{2.3.22}$$

or

$$\operatorname{Tr}\left(\mathcal{A}^{\mu}\frac{\partial f}{\partial r}\left(\mathcal{I}_k - \frac{\partial r}{\partial u}\cdot\frac{\partial f}{\partial r}\right)^{-1}\lambda\right) = 0, \quad \mu = 1,\dots,l, \tag{2.3.23}$$

where $\mathcal{A}^1,\dots,\mathcal{A}^l$ are p by q matrix functions of u (i.e. $\mathcal{A}^{\mu} = \left(\mathcal{A}^{\mu i}_{\ \ \alpha}(u)\right) \in \mathbb{R}^{p\times q}$, $\mu = 1,\dots,l$). For the given system of equations (2.2.1), the matrices $\mathcal{A}^{\mu}$ are known functions of u and equations (2.3.22) (or (2.3.23)) constitute conditions on functions $f^{\alpha}(r)$ and $\lambda^{A}(u)$ (or, by virtue of (2.3.14), on $\xi_a(u)$). It is convenient from a computational point of view to split x^i into x^{i_A} and x^{i_a} and to choose a basis for the wave vectors λ^A such that

$$\lambda^A = dx^{i_A} + \lambda^A_{i_a}dx^{i_a}, \quad A = 1,\dots,k, \tag{2.3.24}$$

where (i_A, i_a) is a permutation of $(1,\dots,p)$. Hence, expression (2.3.9) becomes

$$\frac{\partial r^A}{\partial u^{\alpha}} = \frac{\partial \lambda^A_{i_a}}{\partial u^{\alpha}}x^{i_a}. \tag{2.3.25}$$

Substituting (2.3.25) into (2.3.22) (or (2.3.23)) yields

$$\operatorname{Tr}\left(\mathcal{A}^{\mu}\left(\mathcal{I}_q - Q_a x^{i_a}\right)^{-1}\frac{\partial f}{\partial r}\lambda\right) = 0, \quad \mu = 1,\dots,l, \tag{2.3.26}$$

or

$$\operatorname{Tr}\left(\mathcal{A}^{\mu}\frac{\partial f}{\partial r}\left(\mathcal{I}_k - K_a x^{i_a}\right)^{-1}\lambda\right) = 0, \quad \mu = 1,\dots,l, \tag{2.3.27}$$

where

$$Q_a = \frac{\partial f}{\partial r}\eta_a \in \mathbb{R}^{q\times q}, \quad K_a = \eta_a\frac{\partial f}{\partial r} \in \mathbb{R}^{k\times k}, \quad \eta_a = \left(\frac{\partial \lambda^A_{i_a}}{\partial u^{\alpha}}\right) \in \mathbb{R}^{k\times q}, \tag{2.3.28}$$

for i_A fixed and $i_a = 1,\dots,p-1$. Note that the functions r^A and x^{i_a} are functionally independent in a neighborhood of $x = 0$. The matrix functions $\mathcal{A}^{\mu}$, $\partial f/\partial r$, Q_a

and K_a depend on r only. Hence, equations (2.3.26) (or (2.3.27)) have to be satisfied for any value of coordinates x^{i_a}. This requirement leads to some constraints on these matrix functions.

According to the Cayley-Hamilton theorem, for any n by n invertible matrix M, the expression $(M^{-1}\det M)$ is a polynomial in M of order $(n-1)$. Thus, using the tracelessness of the expression $\mathcal{A}^\mu\left(\mathcal{I}_q - Q_a x^{i_a}\right)^{-1}(\partial f/\partial r)\lambda$, we can replace equations (2.3.26) by the following

$$\operatorname{Tr}\left(\mathcal{A}^\mu \mathcal{Q}\frac{\partial f}{\partial r}\lambda\right) = 0, \quad \text{where} \quad \mathcal{Q} = \operatorname{adj}(\mathcal{I}_q - Q_a x^{i_a}) \in \mathbb{R}^{q\times q}. \tag{2.3.29}$$

Here $\operatorname{adj}M$ denotes the classical adjoint of the matrix M. Note that $\mathcal{Q}$ is a polynomial of order $(q-1)$ in x^{i_a}. Taking (2.3.29) and all its partial derivatives with respect to x^{i_a} (with r fixed at $x = 0$), we obtain the following conditions for the matrix functions $f(r)$ and $\lambda(f)$

$$\operatorname{Tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r}\lambda\right) = 0, \quad \mu = 1, \ldots, l, \tag{2.3.30}$$

$$\operatorname{Tr}\left(\mathcal{A}^\mu Q_{(a_1}\ldots Q_{a_s)}\frac{\partial f}{\partial r}\lambda\right) = 0, \tag{2.3.31}$$

where $s = 1, \ldots, q-1$ and $(a_1, \ldots, a_s)$ denotes the symmetrization over all indices in the bracket. A similar procedure can be applied to system (2.3.27) to yield (2.3.30) and

$$\operatorname{Tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r}K_{(a_1}\ldots K_{a_s)}\lambda\right) = 0, \quad \mathcal{K} = \operatorname{adj}(\mathcal{I}_k - K_a x^{i_a}) \in \mathbb{R}^{k\times k}, \tag{2.3.32}$$

where now $s = 1, \ldots, k-1$. Equations (2.3.30) represent the initial value conditions on a surface in the space of independent variables X, given at $x^{i_a} = 0$. Note that equations (2.3.31) (or (2.3.32)) form the conditions required for preservation of the property (2.3.30) by flows represented by the vector fields (2.3.15). Note also that, by virtue of (2.3.24), X_a can be expressed in the form

$$X_a = \partial_{i_a} - \lambda^A_{i_a}\partial_{i_A}. \tag{2.3.33}$$

Substituting expressions (2.3.28) into (2.3.31) or (2.3.32) and simplifying gives the unified form

$$\mathrm{Tr}\left(\mathcal{A}^{\mu}\frac{\partial f}{\partial r}\eta_{(a_1}\frac{\partial f}{\partial r}\dots\eta_{a_s)}\frac{\partial f}{\partial r}\lambda\right)=0,\quad \eta_{a_t}=\left(\frac{\partial\lambda^A_{a_t}}{\partial u^{\alpha}}\right)\in\mathbb{R}^{k\times q},\quad t=1,\dots,s, \tag{2.3.34}$$

where we can choose either $\max(s)=q-1$ or $\max(s)=k-1$, whichever is more convenient.

Let us note that for $k=1$ the results of the two methods, CSM and GMC, overlap. This is due to the fact that conditions (2.2.2) and (2.2.7) coincide with (2.3.30) and conditions (2.2.5) and (2.3.34) are identically equal to zero. In this case, all rank-1 solutions correspond to single Riemann waves. However, for $k\geq 2$ the differences between the two approaches become essential and, as we demonstrate in the following examples, the CSM can provide rank-k solutions which are not Riemann k-waves as defined by the GMC.

We now introduce a change of variables on $\mathbb{R}^p\times\mathbb{R}^q$ which allows us to rectify the vector fields X_a and simplify considerably the structure of the overdetermined system (2.3.21). For this system, in the new coordinates, we derive the necessary and sufficient conditions for existence of rank-k solutions in the form (2.3.5).

Let us assume that there exists an invertible k by k subblock

$$\Lambda=(\lambda^A_B),\quad 1\leq A,B\leq k, \tag{2.3.35}$$

of the matrix $\lambda\in\mathbb{R}^{k\times p}$. Then the independent vector fields

$$X_{k+1}=\frac{\partial}{\partial x^{k+1}}-(\Lambda^{-1})^B_A\lambda^A_{k+1}\frac{\partial}{\partial x^B},\quad \dots\quad ,X_p=\frac{\partial}{\partial x^p}-(\Lambda^{-1})^B_A\lambda^A_p\frac{\partial}{\partial x^B}, \tag{2.3.36}$$

have the required form (2.3.15) for which the orthogonality conditions (2.3.14) are satisfied. We introduce the functions

$$\begin{aligned}&\bar{x}^1=r^1(x,u),\ \dots,\ \bar{x}^k=r^k(x,u),\\&\bar{x}^{k+1}=x^{k+1},\ \dots,\ \bar{x}^p=x^p,\bar{u}^1=u^1,\ \dots,\ \bar{u}^q=u^q,\end{aligned} \tag{2.3.37}$$

as new coordinates on $\mathbb{R}^p\times\mathbb{R}^q$ space which allow us to rectify the vector fields (2.3.36). So, we get

$$X_{k+1}=\frac{\partial}{\partial\bar{x}^{k+1}},\dots,X_p=\frac{\partial}{\partial\bar{x}^p}. \tag{2.3.38}$$

The p-dimensional submanifold invariant under $X_{k+1}, \dots, X_p$ is defined by equations of the form

$$\bar{u} = f(\bar{x}^1, \dots, \bar{x}^k), \tag{2.3.39}$$

for an arbitrary function $f : \mathbb{R}^k \to \mathbb{R}^q$. The expression (2.3.39) is the general solution of the invariance conditions

$$\bar{u}_{\bar{x}^{k+1}}, \dots, \bar{u}_{\bar{x}^p} = 0. \tag{2.3.40}$$

The initial system (2.3.1) described in the new coordinates $(\bar{x}, \bar{u}) \in \mathbb{R}^p \times \mathbb{R}^q$ is, in general, a nonlinear system of first order PDEs,

$$\bar{\mathcal{A}}^i(\bar{x}, \bar{u}, \bar{u}_{\bar{x}})\bar{u}_i = 0, \quad \text{where} \quad \bar{\mathcal{A}}^i = \frac{\partial \bar{x}^i}{\partial x^j}\mathcal{A}^j, \quad i, j = 1, \dots, p. \tag{2.3.41}$$

That is, we have

$$\bar{\mathcal{A}}^1 = \frac{\partial r^1}{\partial x^i}\mathcal{A}^i, \dots, \bar{\mathcal{A}}^k = \frac{\partial r^k}{\partial x^i}\mathcal{A}^i, \bar{\mathcal{A}}^{k+1} = \mathcal{A}^{k+1}, \dots, \bar{\mathcal{A}}^p = \mathcal{A}^p. \tag{2.3.42}$$

The Jacobi matrix in the coordinates $(\bar{x}, \bar{u})$ takes the form

$$\frac{\partial r^i}{\partial x^j} = (\phi^{-1})^i_s \lambda^s_j \in \mathbb{R}^{k \times p}, \quad (\phi^i_j) = \left(\delta^i_j - \frac{\partial r^i}{\partial u^l}\frac{\partial \bar{u}^l}{\partial \bar{x}^j}\right) \in \mathbb{R}^{k \times k}, \tag{2.3.43}$$

whenever the invariance conditions (2.3.40) are satisfied. Augmenting the system (2.3.41) with the invariance conditions (2.3.40) leads to a quasilinear reduced system of PDEs

$$\Delta : \begin{cases} \operatorname{Tr}\left(\bar{\mathcal{A}}^\mu \left(\mathcal{I}_q - \frac{\partial \bar{u}}{\partial \bar{x}} \cdot \frac{\partial r}{\partial u}\right)^{-1} \frac{\partial \bar{u}}{\partial \bar{x}}\lambda\right) = 0, \quad \mu = 1, \dots, l, \\ \bar{u}_{\bar{x}^{k+1}}, \dots, \bar{u}_{\bar{x}^p} = 0, \end{cases} \tag{2.3.44}$$

or

$$\Delta : \begin{cases} \operatorname{Tr}\left(\bar{\mathcal{A}}^\mu \frac{\partial \bar{u}}{\partial \bar{x}} \left(\mathcal{I}_k - \frac{\partial r}{\partial u} \cdot \frac{\partial \bar{u}}{\partial \bar{x}}\right)^{-1} \lambda\right) = 0, \quad \mu = 1, \dots, l, \\ \bar{u}_{\bar{x}^{k+1}}, \dots, \bar{u}_{\bar{x}^p} = 0. \end{cases} \tag{2.3.45}$$

Now we present some examples which illustrate the preceeding construction. If

$$\phi = \mathcal{I}_q - \frac{\partial \bar{u}}{\partial \bar{x}} \cdot \frac{\partial r}{\partial u} \in \mathbb{R}^{q \times q} \tag{2.3.46}$$

is a scalar matrix, then system (2.3.44) is equivalent to the following quasilinear system

$$B^i(\bar{u})\bar{u}_i = 0 \tag{2.3.47}$$

in k independent variables $\bar{x}^1, \ldots, \bar{x}^k$ and q dependent variables $\bar{u}^1, \ldots, \bar{u}^q$, where $B^i = \lambda^i_j \mathcal{A}^j$. In the simplest case, when $k = 1$, the equations (2.3.47) coincide with the system (2.3.44), i.e.

$$\lambda_i(\bar{u})\mathcal{A}^i(\bar{u})\bar{u}_1 = 0, \quad \bar{u}^2, \ldots, \bar{u}^p = 0, \tag{2.3.48}$$

with the general solution

$$\bar{u}(\bar{x}) = f(\bar{x}^1), \tag{2.3.49}$$

where $f : \mathbb{R} \to \mathbb{R}^q$ satisfies the first order ordinary differential equation

$$\lambda_i(f)\mathcal{A}^i(f)f' = 0, \tag{2.3.50}$$

and we have used the following notation $f' = df/d\bar{x}^1$.

If $k \geq 2$ then ϕ is a scalar matrix if and only if the Riemann invariants do not depend on the function u,

$$\frac{\partial r^1}{\partial u}, \ldots, \frac{\partial r^k}{\partial u} = 0. \tag{2.3.51}$$

Consequently, equations (2.3.25) and (2.3.51) imply that the wave vectors $\lambda^1, \ldots, \lambda^k$ are constant. Hence, this solution represents a travelling k-wave.

Consider now a more general situation when the matrix ϕ does not depend on variables $\bar{x}^{k+1}, \ldots, \bar{x}^p$, that is

$$\frac{\partial \phi}{\partial \bar{x}^{k+1}}, \ldots, \frac{\partial \phi}{\partial \bar{x}^p} = 0. \tag{2.3.52}$$

The system (2.3.44) is independent of $\bar{x}^{k+1}, \ldots, \bar{x}^p$ if and only if

$$\frac{\partial^2 r}{\partial u \partial \bar{x}^{k+1}}, \ldots, \frac{\partial^2 r}{\partial u \partial \bar{x}^p} = 0, \tag{2.3.53}$$

or equivalently, due to (2.3.35), if and only if

$$\frac{\partial \lambda^A_i}{\partial u} = \frac{\partial \Lambda^A_m}{\partial u}(\Lambda^{-1})^m_n \lambda^n_i, \quad 1 \leq A \leq k < i < p, \quad m, n = 1, \ldots, k. \tag{2.3.54}$$

So, it follows that

$$\frac{\partial}{\partial u}\left((\Lambda^{-1})^A_m \lambda^m_i\right) = 0, \quad 1 \leq A \leq k < i < p. \tag{2.3.55}$$

Thus, equations (2.3.44) are independent of $\bar{x}^{k+1}, \ldots, \bar{x}^p$ if there exists a k by $(p-k)$ constant matrix C such that

$$(\lambda_i^A) = \Lambda \cdot C, \quad 1 \leq A \leq k < i < p. \tag{2.3.56}$$

In this case, (2.3.44) is a system (not necessarily a quasilinear one) in k independent variables $\bar{x}^1, \ldots, \bar{x}^k$ and q dependent variables $\bar{u}^1, \ldots, \bar{u}^q$.

Let us now proceed to define some basic notions of the conditional symmetry method in the context of Riemann invariants. We associate the original system (2.3.1) and the invariance conditions (2.3.17) with the subvarieties of the solution spaces

$$\mathbb{S}_\Delta = \{(x, u^{(1)}) : \mathcal{A}^{i\mu}_{\alpha} u_i^\alpha = 0, \quad \mu = 1, \ldots, l\}$$

and

$$\mathbb{S}_Q = \{(x, u^{(1)}) : \xi_a^i(u) u_i^\alpha = 0, \quad \alpha = 1, \ldots, q, \quad a = 1, \ldots, p-k\},$$

respectively.

A vector field X_a is called a conditional symmetry of the original system (2.3.1) if X_a is tangent to $\mathbb{S} = \mathbb{S}_\Delta \cap \mathbb{S}_Q$, i.e.

$$\mathrm{pr}^{(1)} X_a \Big|_{\mathbb{S}} \in T_{(x,u^{(1)})}\mathbb{S}, \tag{2.3.57}$$

where $\mathrm{pr}^{(1)} X_a$ is the first prolongation of X_a defined on $J^1(X \times U)$ and is given by

$$\mathrm{pr}^{(1)} X_a = X_a - \xi^i_{a,u^\beta} u_j^\beta u_i^\alpha \frac{\partial}{\partial u_j^\alpha}, \quad a = 1, \ldots, p-k, \tag{2.3.58}$$

and $T_{(x,u^{(1)})}\mathbb{S}$ is the tangent space to $\mathbb{S}$ at some point $(x, u^{(1)}) \in J^1(X \times U)$.

An Abelian Lie algebra L spanned by the vector fields $X_1, \ldots, X_{p-k}$ is called a conditional symmetry algebra of the original system (2.3.1) if the following condition

$$\mathrm{pr}^{(1)} X_a(\mathcal{A}^i u_i)\Big|_{\mathbb{S}} = 0, \quad a = 1, \ldots, p-k, \tag{2.3.59}$$

is satisfied.

Note that every solution of the overdetermined system (2.3.21) can be represented by its graph $\{(x, u(x))\}$, which is a section of J^0. The conditional symmetry algebra L of (2.3.1) defines locally the action of the corresponding Lie group G

on J^0. The symmetry group G transforms certain solutions of (2.3.21) into other solutions of (2.3.21). If the graph of a solution is preserved by G then this solution is called G-invariant.

Assume that L, spanned by the vector fields $X_1, \ldots, X_{p-k}$, is a conditional symmetry algebra of the system (2.3.1). A solution $u = f(x)$ is said to be a conditionally invariant solution of the system (2.3.1) if the graph $\{(x, f(x))\}$ is invariant under the vector fields $X_1, \ldots, X_{p-k}$.

Proposition. *A nondegenerate quasilinear hyperbolic system of first order PDEs (2.3.1) in p independent variables and q dependent variables admits a $(p-k)$-dimensional conditional symmetry algebra L if and only if $(p-k)$ linearly independent vector fields $X_1, \ldots, X_{p-k}$ satisfy the conditions (2.3.30) and (2.3.34) on some neighborhood of (x_0, u_0) of $\mathbb{S}$. The solution of (2.3.1) which are invariant under the Lie algebra L are precisely rank-k solutions of the form (2.3.5).*

Proof : Let us describe the vector fields X_a in the new coordinates $(\bar{x}, \bar{u})$ on $\mathbb{R}^p \times \mathbb{R}^q$. From (2.3.38) and (2.3.59) it follows that

$$\mathrm{pr}^{(1)} X_a = X_a, \quad a = k+1, \ldots, p. \tag{2.3.60}$$

Hence, the symmetry criterion for G to be the symmetry group of the overdetermined system (2.3.44)(or (2.3.45)) requires that the vector fields X_a of G satisfy

$$X_a(\Delta) = 0, \tag{2.3.61}$$

whenever equations (2.3.44)(or (2.3.45)) hold. Thus the symmetry criterion applied to the invariance conditions (2.3.40) is identically equal to zero. After applying this criterion to the system (2.3.41) in new coordinates, carrying out the differentiation and next taking into account the conditions (2.3.30) and (2.3.34) we obtain the equations which are identically satisfied.

The converse is also true. The assumption that the system (2.3.1) be nondegenerate means that it is locally solvable and is of maximal rank at every point $(x_0, u_0^{(1)}) \in \mathbb{S}$. Therefore [**89**], the infinitesimal symmetry criterion is a necessary and sufficient condition for the existence of the symmetry group G of the overdetermined system (2.3.21). Since the vector fields X_a form an Abelian distribution on $X \times U$, it follows that, as we have already shown in this section, conditions

(2.3.30) and (2.3.34) hold. That ends the proof, since the solutions of the overdetermined system (2.3.21) are invariant under the algebra L generated by $(p-k)$ vector fields $X_1, \dots, X_{p-k}$. The invariants of the group G of such vector fields are provided by the functions $\{r^1, \dots, r^k, u^1, \dots, u^q\}$. So the general rank-$k$ solution of (2.3.1) takes the form (2.3.5).

□

The expressions in equations (2.3.30) and (2.3.34) lend themselves to further simplification. Let us recall here that any n by n matrix b is a root of the Cayley-Hamilton polynomial

$$\det(\lambda \mathcal{I}_n - b) = \lambda^n - \sum_{i=1}^{n} p_i(b)\lambda^{n-i}. \tag{2.3.62}$$

Faddeev's approach ([**40**], p.87) provides a recursive method to compute the coefficients $p_i(b)$, based on Newton's formulae

$$kp_k = s_k - p_1 s_{k-1} - \ldots - p_{k-1}s_1, \quad s_k = \operatorname{Tr}(b^k) = \sum_{i=1}^{n} \lambda_i^k, \quad k = 1, \dots, n. \tag{2.3.63}$$

For example, one readily computes

$$\begin{aligned} &p_1 = \operatorname{Tr}(b), \quad p_2 = \frac{1}{2}[\operatorname{Tr}(b^2) - (\operatorname{Tr}(b))^2], \quad p_3 = \frac{1}{6}[(\operatorname{Tr}(b))^3 - 3\operatorname{Tr}(b)\operatorname{Tr}(b^2) + 2\operatorname{Tr}(b^3)],\\ &p_4 = \frac{1}{24}[6\operatorname{Tr}(b^4) - (\operatorname{Tr}(b))^4 - 8\operatorname{Tr}(b)\operatorname{Tr}(b^3) - 3(\operatorname{Tr}(b^2))^2 + 6(\operatorname{Tr}(b))^2\operatorname{Tr}(b^2)], \quad (2.3.64)\\ &\cdots\ p_n = (-1)^{n+1}\det(b). \end{aligned}$$

According to the Cayley-Hamilton theorem one has

$$b^n - \sum_{i=1}^{n} p_i(b) b^{n-i} = 0. \tag{2.3.65}$$

Using the identity (2.3.65), we can simplify the expressions (2.3.30) and (2.3.34) to some degree, depending on the dimension of the matrix b.

As an illustration we present the simplest case of a 2 by 2 matrix, which corresponds to rank-2 solutions for $q = 2$ unknown functions. In this case, the expressions (2.3.30) and (2.3.34) become

$$\operatorname{Tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r}\lambda\right) = 0, \quad \mu = 1, \dots, l, \tag{2.3.66}$$

$$\operatorname{Tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_a \frac{\partial f}{\partial r}\lambda\right) = 0, \quad a = 1, \dots, p-1. \tag{2.3.67}$$

Combining (2.3.66) and (2.3.67) leads to the factorized form

$$\operatorname{Tr}\left(\mathcal{A}^{\mu}\frac{\partial f}{\partial r}\eta_a\frac{\partial f}{\partial r}\lambda\right)=\operatorname{Tr}\left(\mathcal{A}^{\mu}\frac{\partial f}{\partial r}\left(\eta_a\frac{\partial f}{\partial r}-\mathcal{I}_2\operatorname{Tr}\left(\eta_a\frac{\partial f}{\partial r}\right)\right)\lambda\right). \qquad (2.3.68)$$

Note that for any invertible 2 by 2 matrices M and N, the Cayley-Hamilton trace identity has the form

$$MN-\mathcal{I}_2\operatorname{Tr}(MN)=-(N-\mathcal{I}_2\operatorname{Tr}(N))(M-\mathcal{I}_2\operatorname{Tr}(M)). \qquad (2.3.69)$$

Using the above equation, we rewrite (2.3.68) in the equivalent form

$$\begin{aligned}&\operatorname{Tr}\left(\mathcal{A}^{\mu}\frac{\partial f}{\partial r}\left(\eta_a\frac{\partial f}{\partial r}-\mathcal{I}_2\operatorname{Tr}\left(\eta_a\frac{\partial f}{\partial r}\right)\right)\lambda\right)\\&=-\operatorname{Tr}\left(\mathcal{A}^{\mu}\frac{\partial f}{\partial r}\left(\frac{\partial f}{\partial r}-\mathcal{I}_2\operatorname{Tr}\left(\frac{\partial f}{\partial r}\right)\right)(\eta_a-\mathcal{I}_2\operatorname{Tr}(\eta_a))\lambda\right),\end{aligned} \qquad (2.3.70)$$

where the matrices M and N are identified with η_a and $\partial f/\partial r$, respectively. Since we have

$$N^2-\operatorname{Tr}(N)N=-\mathcal{I}_2\det(N),$$

then, if $\det(\partial f/\partial r)\neq 0$, it follows that

$$\operatorname{Tr}\left(\mathcal{A}^{\mu}\left(\eta_a-\mathcal{I}_2\operatorname{Tr}(\eta_a)\right)\lambda\right)=0, \quad \eta_a=\left(\frac{\partial\lambda^A_{i_a}}{\partial u^{\alpha}}\right)\in\mathbb{R}^{2\times 2}, \quad A=1,2. \qquad (2.3.71)$$

For a given system (2.3.1) (i.e. given functions $\mathcal{A}^{\mu}$), the equations (2.3.71) form a bilinear system of $l(p-1)$ PDEs for $2(p-1)$ functions $\lambda^A_{i_a}(u)$. Thus we have eliminated the matrix term $\partial f/\partial r$ in equations (2.3.67). This fact greatly facilitates our task. The proposed procedure for constructing rank-2 conditionally invariant solutions of the system (2.3.1) consists of the following steps.

1. We first look for two linearly independent real-valued wave vectors λ^1 and λ^2 by solving the dispersion relation (2.3.3) associated with the initial system (2.3.1).

2. If such wave vectors do exist, we substitute them into PDEs (2.3.71) and solve this system for λ^A in terms of u.

3. Next, we substitute the most general solutions for λ^1 and λ^2 into equations (2.3.66) and look for a solution $u=f(r^1,r^2)$ of this system. Thus we obtain the explicit parametrization of the 2-dimensional submanifold $\mathcal{S}\subset U$ in terms of r^1 and r^2.

4. We suppose that $u=f(r^1,r^2)$ is the unique solution of PDEs (2.3.66). Then we restrict the wave vectors λ^A to the submanifold $\mathcal{S} \subset U$. That is, the functions $\lambda^A(u)$ are pulled back to $\mathcal{S}$ and become functions of the parameters r^1 and r^2 on $\mathcal{S}$. We denote the function $f^*(\lambda^A)$ by $\lambda^A(r^1, r^2)$.

5. In this parametrization we can implicitly determine the value of the Riemann invariants for each solution λ^A of (2.3.71)

$$r^A(x, f(r^1, r^2)) = \lambda_i^A(r^1, r^2)x^i, \quad A = 1, 2. \tag{2.3.72}$$

6. Finally, we suppose that the set of implicitly defined relations (2.3.5) and (2.3.72) between u^α, r^A and x^i can be solved so that the functions r^A and u^α can be given as graphs over an open subset $\mathcal{D} \subset X$. Then the function

$$u = f(r^1(x), r^2(x)) \tag{2.3.73}$$

is an explicit rank-2 solution of the quasilinear hyperbolic system (2.3.1). The graph of this solution is invariant under $(p-2)$ linearly independent vector fields X_a.

As another illustration, let us consider the rank-3 case when $q = k = 3$. Then the condition (2.3.34) takes the following form

$$\operatorname{Tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r}\left[\eta_{a_1}\frac{\partial f}{\partial r}\eta_{a_2} + \eta_{a_2}\frac{\partial f}{\partial r}\eta_{a_1}\right]\frac{\partial f}{\partial r}\lambda\right) = 0, \quad \mu = 1, \ldots, l, \tag{2.3.74}$$

where

$$\eta_{a_j} = \left(\frac{\partial \lambda^A_{i_{a_j}}}{\partial u^\alpha}\right) \in \mathbb{R}^{3\times 3}, \quad j = 1, 2, \quad A = 1, 2, 3, \tag{2.3.75}$$

and the expressions (2.3.8) become

$$\frac{\partial f}{\partial r} = \left(\frac{\partial f^\alpha}{\partial r^A}\right) \in \mathbb{R}^{3\times 3}, \quad \lambda = (\lambda_i^A(u)) \in \mathbb{R}^{3\times p}, \quad \mathcal{A}^\mu = (\mathcal{A}^{\mu i}{}_\alpha) \in \mathbb{R}^{p\times 3}.$$

We introduce the notation

$$P := \eta_{a_1}\frac{\partial f}{\partial r}, \quad Q := \eta_{a_2}\frac{\partial f}{\partial r}. \tag{2.3.76}$$

Then, combining equations (2.3.30) and (2.3.74), we obtain

$$\operatorname{Tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r}\left[PQ - \mathcal{I}_3 \operatorname{Tr}(PQ) + QP - \mathcal{I}_3 \operatorname{Tr}(QP)\right]\lambda\right) = 0. \tag{2.3.77}$$

If $\det(\partial f/\partial r) \neq 0$ (otherwise the case $q = 3$ can be reduced to $q \leq 2$) then, similarly to the case $q = 2$, we can eliminate the term $\partial f/\partial r$ from (2.3.30) and (2.3.77). The resulting expressions are still quite complicated. Nevertheless, as we show in the examples to follow, our procedure makes the construction of rank-3 solutions feasible.

2.4. The fluid dynamics equations

At this point, we would like to illustrate the proposed approach to constructing rank-k solutions with the example of the fluid dynamics equations. The fluid under consideration is assumed to be ideal, nonstationary, isentropic and compressible. We restrict our analysis to the case in which the dissipative effects, like viscosity and thermal conductivity, are negligible, and no external forces are considered. Under the above assumptions, the classical fluid dynamics model is governed by the system of equations in (3+1) dimensions of the form

$$\begin{aligned} &D\rho + \rho \operatorname{div}\vec{u} = 0, \\ &D\vec{u} + \rho^{-1}\nabla p = 0, \\ &DS = 0. \end{aligned} \tag{2.4.1}$$

Here we have used the following notation : ρ, p and S are the density, pressure and entropy of the fluid, respectively, $\vec{u} = (u^1, u^2, u^3)$ is the vector field of the fluid velocity and D is the convective derivative

$$D = \frac{\partial}{\partial t} + (\vec{u} \cdot \nabla). \tag{2.4.2}$$

Equations (2.4.1) form a quasilinear hyperbolic homogeneous system of five equations in five unknown functions $(\rho, p, \vec{u}) \in \mathbb{R}^5$. The independent variables are denoted by $(x^\mu) = (t, x, y, z) \in X \subset \mathbb{R}^4$, $\mu = 0, 1, 2, 3$. According to [**94, 112**] this system can be reduced to a hyperbolic system of four equations in four unknowns $u = (u^\mu) = (a, \vec{u}) \in U \subset \mathbb{R}^4$ describing an isentropic ideal flow, when the sound velocity a is assumed to be a function of the density ρ only. In this case the state equation of the media $p = f(\rho, S)$ is subjected to the differential constraints

$$\nabla p = a^2(\rho)\nabla\rho, \quad d\ln(a\rho^{-1/\kappa}) = 0, \tag{2.4.3}$$

where $a^2(\rho) = \partial f/\partial\rho$, $\kappa = 2(\gamma-1)^{-1}$ and γ is the adiabatic exponent of the fluid. Under the assumptions (2.4.3), the fluid dynamics model (2.4.1) becomes

$$\begin{aligned} &Da + \kappa^{-1} a \mathrm{div}\vec{u} = 0, \\ &D\vec{u} + \kappa a \nabla a = 0. \end{aligned} \tag{2.4.4}$$

The system of equations (2.4.4) can be written in the equivalent matrix evolutionary form (2.3.2). Here $n = 3$ and the 4 by 4 matrix functions $\mathcal{A}^1$,$\mathcal{A}^2$ and $\mathcal{A}^3$ take the form

$$\mathcal{A}^i = \begin{pmatrix} u^i & \delta_{i1}\kappa^{-1}a & \delta_{i2}\kappa^{-1}a & \delta_{i3}\kappa^{-1}a \\ \delta_{i1}\kappa a & u^i & 0 & 0 \\ \delta_{i2}\kappa a & 0 & u^i & 0 \\ \delta_{i3}\kappa a & 0 & 0 & u^i \end{pmatrix}, \quad i = 1, 2, 3, \tag{2.4.5}$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. The largest Lie point symmetry algebra of these equations has been already investigated in [**43**]. It constitutes a Galilean similitude algebra generated by 12 differential operators

$$\begin{aligned} &P_\mu = \partial_{x^\mu}, \quad J_k = \epsilon_{kij}(x^i\partial_{x^j} + u^i\partial_{u^j}), \quad K_i = t\partial_{x^i} + \partial_{u^i}, \quad i = 1, 2, 3, \\ &F = t\partial_t + x^i\partial_{x^i}, \quad G = -t\partial_t + a\partial_a + u^i\partial_{u^i}. \end{aligned} \tag{2.4.6}$$

In the particular case when the adiabatic exponent is $\gamma = 5/3$, this algebra is generated by 13 infinitesimal differential operators, namely the 12 operators (2.4.6) and a projective transformation

$$C = t(t\partial_t + x^i\partial_{x^i} - a\partial_a) + (x^i - x^0u^i)\partial_{u^i}. \tag{2.4.7}$$

Note that the algebras generated by (2.4.6) and by (2.4.6) with (2.4.7) are fibre preserving. The classification of the subalgebras of these algebras into conjugacy classes is presented in [**43**]. Large classes of solutions of the system (2.4.4), invariant and partially-invariant (with the defect structure $\delta = 1$), have been obtained in [**44**].

The wave vector λ can be written in the form $(\lambda_0, \vec{\lambda})$, where $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ denotes a direction of wave propagation and the eigenvalue λ_0 is a phase velocity of a considered wave. The dispersion relation for the isentropic equations (2.4.4) takes the form

$$\det\left(\lambda_0(u)\mathcal{I}_4+\lambda_i(u)\mathcal{A}^i(u)\right)=(\lambda_0+\vec{u}\cdot\vec{\lambda})^2[(\lambda_0+\vec{u}\cdot\vec{\lambda})^2-a^2\vec{\lambda}^2]=0. \qquad (2.4.8)$$

Solving the dispersion equation (2.4.8), we obtain two types of wave vectors, namely the potential and rotational wave vectors

$$\begin{aligned} &\text{i) } \lambda^E=(\epsilon a+\vec{u}\cdot\vec{e},-\vec{e}), \quad \epsilon=\pm 1, \\ &\text{ii) } \lambda^S=([\vec{u},\vec{e},\vec{m}],-\vec{e}\times\vec{m}), \quad |\vec{e}|^2=1, \end{aligned} \qquad (2.4.9)$$

as introduced in [**100**], where $\vec{e}$ and $\vec{m}$ are unit and arbitrary vectors, respectively. Here, the equation (2.4.9ii) has a multiplicity of 2. The quantity $[\vec{u},\vec{e},\vec{m}]$ denotes the determinant of the matrix based on these vectors, i.e. $[\vec{u},\vec{e},\vec{m}]=\det(\vec{u},\vec{e},\vec{m})$. Several classes of k-wave solutions of the isentropic system (2.4.4), obtained via the GMC, are known [**7, 100**]. Applying the CSM to this system allows us to compare the effectiveness of the two approaches.

2.5. Rank-1 solutions of the fluid dynamics equations

Analyzing the rank-1 solutions associated with the wave vectors λ^E and λ^S we consider separately two cases.

In the first case, the potential wave vectors are the nonzero multiples of

$$\lambda^E=(\epsilon a+\vec{e}\cdot\vec{u},-\vec{e}), \quad |\vec{e}|^2=1, \quad \epsilon=\pm 1.$$

The corresponding vector fields X_i and Riemann invariant $r(x,u)$ are given by

$$X_i=-(a+\vec{e}\cdot\vec{u})^{-1}e_i\frac{\partial}{\partial t}+\frac{\partial}{\partial x^i}, \quad i=1,2,3, \quad r(x,u)=(a+\vec{e}\cdot\vec{u})t-\vec{e}\cdot\vec{x}.$$

We can now consider rank-1 potential solutions, invariant under the vector fields $\{X_1,X_2,X_3\}$. The change of coordinates

$$\bar{t}=t,\ \bar{x}^1=r(x,u),\ \bar{x}^2=x^2,\ \bar{x}^3=x^3,\ \bar{a}=a,\ \bar{u}^1=u^1,\ \bar{u}^2=u^2,\ \bar{u}^3=u^3, \qquad (2.5.1)$$

on $\mathbb{R}^4\times\mathbb{R}^4$ transforms the fluid dynamics equations (2.4.4) into the system

$$\frac{\partial\bar{a}}{\partial\bar{x}^1}=\kappa^{-1}e_i\frac{\partial\bar{u}^i}{\partial\bar{x}^1}, \quad \frac{\partial\bar{u}^i}{\partial\bar{x}^1}=\kappa e_i\frac{\partial\bar{a}}{\partial\bar{x}^1}, \quad i=1,2,3, \qquad (2.5.2)$$

with the invariance conditions

$$\bar{a}_{\bar{t}} = \bar{a}_{\bar{x}^j} = 0, \quad \bar{u}^\alpha_{\bar{t}} = \bar{u}^\alpha_{\bar{x}^j} = 0, \quad j = 2, 3, \quad \alpha = 1, 2, 3.$$

If the unit vector $\vec{e}$ has the form $\vec{e} = (\cos\bar{u}^1 \sin\bar{u}^2, \cos\bar{u}^1\cos\bar{u}^2, \sin\bar{u}^1)$, then the general rank-1 solution is given by

$$\bar{a}(\bar{t}, \bar{x}) = \kappa^{-1}\bar{x}^1 + a^0, \quad \bar{u}^1(\bar{t}, \bar{x}) = -\ln|C\cos\bar{u}^2|, \quad C \in \mathbb{R}$$
$$\bar{u}^2(\bar{t}, \bar{x}) = \bar{u}^2(\bar{x}^1), \quad \bar{u}^3(\bar{t}, \bar{x}) = -\int_0^{\bar{u}^2} \tan(\ln|C\cos s|)\cos s\, ds.$$

In particular, if $\vec{e}$ is a constant unit vector, then we can integrate (2.5.2) and the solution is defined implicitly by the equations

$$\bar{a}(\bar{t}, \bar{x}) = \bar{a}(\bar{x}^1), \quad \bar{u}^i(\bar{t}, \bar{x}) = \kappa e_i \bar{a}(\bar{x}^1) + C_i, \quad C_i \in \mathbb{R}, \quad i = 1, 2, 3.$$

If we choose $\bar{a} = A_1\bar{x}^1$, $A_1 \in \mathbb{R}$ and $C_i = 0$, then the explicit invariant solution has the form

$$a(t, x) = [A_1(1+\kappa)t-1]^{-1}A_1\vec{e}\cdot\vec{x}, \quad \vec{u}(t, x) = [A_1(1+\kappa)t-1]^{-1}\kappa A_1(\vec{e}\cdot\vec{x})\vec{e}. \quad (2.5.3)$$

Note that if the characteristics of one family associated with the eigenvalue $\lambda_0 = a + \vec{e}\cdot\vec{u}$ intersect, then we can choose a particular value of time interval $[t_0, T]$, where $T = (A_1(1+\kappa))^{-1}$, in order to exclude the possibility of a gradient catastrophe. Hence, if the initial data are sufficiently small at $t = t_0$, then the solution (2.5.3) remains a rank-1 solution for the time $t \in [t_0, T)$, and no discontinuities (e.g. shock waves) can appear.

In the second case, we fix a rotational wave vector

$$\lambda^S = ([\vec{u}, \vec{e}, \vec{m}], -\vec{e}\times\vec{m}), \quad |\vec{e}|^2 = 1,$$

and the corresponding vector fields (2.3.36) are given by

$$X_i = [\vec{u}, \vec{e}, \vec{m}]^{-1}(\vec{e}\times\vec{m})_i \frac{\partial}{\partial t} + \frac{\partial}{\partial x^i}, \quad i = 1, 2, 3.$$

Hence, the Riemann invariant associated with λ^S has the form

$$r(x, u) = [\vec{u}, \vec{e}, \vec{m}]t - [\vec{x}, \vec{e}, \vec{m}]. \quad (2.5.4)$$

After substituting (2.5.4) into (2.5.1), the change of coordinates transforms the initial system (2.4.4) into the overdetermined system composed of the following equations

$$\left[\frac{\partial \vec{u}}{\partial \bar{x}^1}, \vec{e}, \vec{m}\right] = 0, \quad (e_i m_j - e_j m_i)\frac{\partial \bar{a}}{\partial \bar{x}^1} = 0, \quad i \neq j = 1,2,3, \tag{2.5.5}$$

and the invariance conditions

$$\bar{a}_{\bar{t}} = \bar{a}_{\bar{x}^j} = 0, \quad \bar{u}^\alpha_{\bar{t}} = \bar{u}^\alpha_{\bar{x}^j} = 0, \quad j = 2,3, \quad \alpha = 1,2,3. \tag{2.5.6}$$

Hence, the sound velocity is constant ($a = a_0$). If $\vec{e}$ and $\vec{m}$ are constant vectors such that $(e_1 m_2 - e_2 m_1) \neq 0$, then we can integrate the system composed of (2.5.5) and (2.5.6). The explicit solution is given by

$$\begin{aligned} &\bar{a}(\bar{t},\bar{x}) = a_0, \quad \bar{u}^1(\bar{t},\bar{x}) = \bar{u}^1(\bar{x}^1), \quad \bar{u}^2(\bar{t},\bar{x}) = \bar{u}^2(\bar{x}^1), \\ &\bar{u}^3(\bar{t},\bar{x}) = (e_1 m_2 - e_2 m_1)^{-1}\left[C - (e_2 m_3 - e_3 m_2)\bar{u}^1 - (e_3 m_1 - e_1 m_3)\bar{u}^2\right], \end{aligned} \tag{2.5.7}$$

where $\bar{u}^1$ and $\bar{u}^2$ are arbitrary functions of the Riemann invariant, which takes the form

$$r(x,u) = Ct - [\vec{x}, \vec{e}, \vec{m}]. \tag{2.5.8}$$

As expected, this result coincides with the solution obtained through the GMC [**100**]. The presence of arbitrary functions in the obtained solution allows us to find bounded solutions valid for all time $t > 0$. For example, the bounded bump-type solution $\bar{u}^i = \operatorname{sech}(A_{(i)} r^i)$, $i = 1,2$, contains no discontinuities.

2.6. Rank-2 solutions

The construction approach outlined in Section 2.4 has been applied to the isentropic flow equations (2.4.4) in order to obtain rank-2 and rank-3 solutions (the latter are presented in the next section). In the case of rank-2 solutions, in order to facilitate computations, we assume that the directions of wave propagation $\vec{\lambda}^A$ are constant, but not their phase velocities λ_0^A.

After considering all possible combinations of the potential and rotational wave vectors (λ^{E_i} and λ^{S_i}, respectively, $i = 1,2,3$) we found eight cases compatible with the conditions (2.3.30) and (2.3.34), leading to eight different classes of solutions. These solutions, in their general form, possess some degree of freedom, that is, depend on one or two arbitrary functions of one or two variables (Riemann invariants), depending on the case. This arbitrariness allows us to change the geometrical properties of the governed fluid flow in such a way as to exclude the presence of singularities. This fact is of a special significance here since, as is well known [**6, 20, 107**], in most cases, even for arbitrary smooth and sufficiently small initial data at $t = t_0$, the magnitude of the first derivatives of Riemann invariants becomes unbounded in some finite time T; thus, solutions expressible in terms of Riemann invariants usually admit a gradient catastrophe. Nevertheless, we have been able to demonstrate that it is still possible in these cases to construct bounded solutions and, in particular, soliton-like solutions, through the proper selection of the arbitrary function(s) appearing in the general solution. To this purpose we submit this arbitrary function(s), say v, to the differential constraint in the form of the nonlinear Klein-Gordon equation in two independent variables r^1 and r^2

$$\Box_{(r^1,r^2)} v = v_{r^1 r^1} - v_{r^2 r^2} - v_{r^3 r^3} = cv^5, \quad c \in \mathbb{R}, \tag{2.6.1}$$

which is known to possess rich families of soliton-type solutions (see e.g. [**1, 115**]). Equation (2.6.1) can be reduced to a second order ODE for v as a function of a Riemann invariant and can very often be explicitly integrated. The analysis of the singularity structure of these ODEs allows us to select soliton-like solutions for v which, in turn, in many cases, lead to the same type of rank-2 and rank-3 solutions of the system (2.4.4). Among them we have various types of algebraic soliton-like solutions (admitting no singularity other than poles), kinks, bumps and doubly periodic solutions which are expressed in terms of Jacobi elliptic functions.

Below we list the obtained rank-2 solutions. Some of the general solutions found (both rank-2 and rank-3 in the next section) coincide with the ones obtained earlier by means of the GMC. Nevertheless, we list them all since we derive from them the particular bounded solutions, which, to our knowledge, are all new.

For convenience, we denote by $(E_iE_j, E_iS_j, S_iS_j, E_iE_jE_k, \ldots, i, j, k = 1, 2, 3)$ the solutions which result from nonlinear superpositions of rank-1 solutions associated with given wave vectors λ^{E_i} or λ^{S_i}. The sign ($+$ or $-$) coincides with the value of $\epsilon = \pm 1$ in equation (2.4.9).

Case ($\mathbf{E_1E_2}$). We first discuss the superposition of two potential rank-1 solutions E_i for which the wave vectors have the form

$$\lambda^{E_i^{\pm}} = (\epsilon a + \vec{e}^{\,i} \cdot \vec{u}, -\vec{e}^{\,i}), \quad |\vec{e}^{\,i}|^2 = 1, \quad i = 1, 2, \quad \epsilon = \pm 1. \tag{2.6.2}$$

We assume that the wave vectors $\lambda^{E_1^{\pm}}$ and $\lambda^{E_2^{\pm}}$ are linearly independent. The corresponding vector fields (2.3.36) are given by

$$X_1 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1}\frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1}\frac{\partial}{\partial x^1}, \tag{2.6.3}$$

with

$$\begin{aligned} &\beta_i = e_i^2(a + \vec{e}^{\,1} \cdot \vec{u}) - e_i^1(a + \vec{e}^{\,2} \cdot \vec{u}), \quad i = 1, 2, 3, \\ &\sigma_j = e_1^1 e_j^2 - e_j^1 e_1^2, \quad j = 2, 3, \quad \epsilon = 1. \end{aligned} \tag{2.6.4}$$

The nonscattering rank-2 potential solution $(E_1^+ E_2^+)$ has the form

$$a = a_1(r^1) + a_2(r^2), \quad \vec{u} = \kappa(a_1(r^1)\vec{e}^{\,1} + a_2(r^2)\vec{e}^{\,2}), \tag{2.6.5}$$

where a_1 and a_2 are arbitrary functions of the Riemann invariants

$$\begin{aligned} r^1(x, u) &= (1 + \kappa)a_1(r^1)t - \vec{e}^{\,1} \cdot \vec{x}, \quad |\vec{e}^{\,1}|^2 = 1, \\ r^2(x, u) &= (1 + \kappa)a_2(r^2)t - \vec{e}^{\,2} \cdot \vec{x}, \quad |\vec{e}^{\,2}|^2 = 1, \end{aligned} \tag{2.6.6}$$

respectively, and the wave vectors $\vec{e}^{\,1}$ and $\vec{e}^{\,2}$ have to satisfy the condition

$$\vec{e}^{\,1} \cdot \vec{e}^{\,2} + \kappa^{-1} = 0. \tag{2.6.7}$$

Equation (2.6.7) holds if and only if the angle φ between these vectors is

$$\cos\varphi = \frac{1}{2}(1 - \gamma). \tag{2.6.8}$$

This solution represents a Riemann double wave. Here, the rank-1 solutions E_i^+, $i = 1, 2$, do not influence each other (they superpose linearly). This result coincides with the one obtained earlier by means of the GMC [**100**].

i) In the particular case when $a_i(r^i) = -A_i r^i$, $A_i \in \mathbb{R}$, $i = 1, 2$, the solution (2.6.5) takes the explicit form

$$\begin{aligned} a &= \frac{A_1\vec{e}^{\,1} \cdot \vec{x}}{1 + (1 + \kappa)A_1 t} + \frac{A_2\vec{e}^{\,2} \cdot \vec{x}}{1 + (1 + \kappa)A_2 t} \\ \vec{u} &= \frac{\kappa A_1\vec{e}^{\,1} \cdot \vec{x}}{1 + (1 + \kappa)A_1 t}\vec{e}^{\,1} + \frac{\kappa A_2\vec{e}^{\,2} \cdot \vec{x}}{1 + (1 + \kappa)A_2 t}\vec{e}^{\,2} \end{aligned} \tag{2.6.9}$$

which admits the gradient catastrophe at the time $t = \min\left(A_i^{-1}(1+\kappa)^{-1}\right)$, $i = 1, 2$. Hence, some discontinuities can occur e.g. shock waves which correspond to the formation of a condensation jump from the compression waves related to $\lambda^{E_i^+}$.

ii) The following bounded solution can be obtained using the DC (2.6.1)

$$\begin{aligned} a &= \sum_{i=1}^{2} A_i r^i (1 + B_i (r^i)^2)^{-1/2}, \quad A_i, B_i \in \mathbb{R}, \quad B_i > 0, \\ \vec{u} &= \kappa \left[\sum_{i=1}^{2} A_i r^i (1 + B_i (r^i)^2)^{-1/2} \vec{e}^{\,i} \right], \end{aligned} \tag{2.6.10}$$

where the Riemann invariants are given by

$$r^i = \left[(1+\kappa) A_i r^i (1 + B_i (r^i)^2)^{-1/2}\right] t - \vec{e}^{\,i} \cdot \vec{x}, \quad i = 1, 2. \tag{2.6.11}$$

The result (2.6.10) represents an algebraic kink-type solution which is bounded for $t > 0$ while each r^i possesses a discontinuity at time $T = (A_i(1+\kappa))^{-1}$.

Case $(\mathbf{E_1 S_2})$. In the mixed case $(E_1^+ S_2)$, we consider the superposition of the rank-1 potential solution E_1^+ with the rank-1 rotational solution S_2 associated respectively with the wave vectors

$$\begin{aligned} \lambda^{E_1^+} &= (a + \vec{e}^{\,1} \cdot \vec{u}, -\vec{e}^{\,1}), \\ \lambda^{S_2} &= ([\vec{u}, \vec{e}^{\,2}, \vec{m}^2], -\vec{e}^{\,2} \times \vec{m}^2), \quad |\vec{e}^{\,i}|^2 = 1, \quad i = 1, 2. \end{aligned} \tag{2.6.12}$$

The vector fields (2.3.36) corresponding to the wave vectors (2.6.12) are

$$X_1 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1}\frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1}\frac{\partial}{\partial x^1}, \tag{2.6.13}$$

where

$$\begin{aligned} \beta_i &= -(\vec{e}^{\,2} \times \vec{m}^2)_i (a + \vec{e}^{\,1} \cdot \vec{u}) + e_i^1 [\vec{u}, \vec{e}^{\,2}, \vec{m}^2], \quad i = 1, 2, 3, \\ \sigma_j &= -e_1^1 (\vec{e}^{\,2} \times \vec{m}^2)_j + e_j^1 (\vec{e}^{\,2} \times \vec{m}^2)_1, \quad j = 2, 3. \end{aligned} \tag{2.6.14}$$

The invariant nonscattering rank-2 solution $(E_1^+ S_2)$ has the form

$$a = a_1(r^1) + a_0, \quad \vec{u} = \kappa a_1(r^1) \vec{e}^{\,1} + \vec{u}_2(r^2), \tag{2.6.15}$$

where

$$[\vec{u}_2, \vec{e}^{\,2}, \vec{m}^2] = C_2, \quad |\vec{e}^{\,i}|^2 = 1, \quad i = 1, 2, \tag{2.6.16}$$

and a_1 and u_2^1 are any differentiable functions of r^1 and r^2, respectively, and the relation $u_2^3(r^2) = C_1 u_2^1(r^2)$ holds. Here, $a_0, C_1, C_2 \in \mathbb{R}$ and $\vec{m}^2$ is an arbitrary constant vector. The wave vector λ^{S_2} takes the form

$$\lambda^{S_2} = (C_2, -(e_1^1 e_3^1 + C_1(1-(e_1^1)^2)), -e_2^1(e_3^1 - C_1 e_1^1), (1-(e_3^1)^2 + C_1 e_1^1 e_3^1)). \quad (2.6.17)$$

From (2.6.15), (2.6.16) and (2.6.17), we get

$$[\vec{e}^{\,1}, \vec{e}^{\,2}, \vec{m}^2] = 0, \quad (2.6.18)$$

so the vector $\vec{\lambda}^{E_1^+} = -\vec{e}^{\,1}$ is orthogonal to $\vec{\lambda}^{S_2} = -\vec{e}^{\,2} \times \vec{m}^{\,2}$. Hence, the Riemann invariants are given by

$$\begin{aligned} r^1 &= ((1+\kappa)a_1(r^1) + C_2(C_1 e_1^1 - e_3^1)^{-1})t - \vec{e}^{\,1} \cdot \vec{x}, \quad (2.6.19)\\ r^2 &= C_2 t - (e_1^1 e_3^1 + C_1(1-(e_1^1)^2))x^1 - e_2^1(e_3^1 - C_1 e_1^1)x^2 + (1-(e_3^1)^2 + C_1 e_1^1 e_3^1)x^3. \end{aligned}$$

This solution represents a Riemann double wave.

i) An explicit form of the solution (2.6.15) can be found when $\vec{e}^{\,1} = \vec{e}^{\,2} = (\cos\varphi, \sin\varphi, 0)$ and $\vec{m}^{\,2} = (\sin\varphi, -\cos\varphi, C_1\sin\varphi)$, $C_1 \in \mathbb{R}$, and we choose $a_1(r^1) = A_1 r^1$, $A_1 \in \mathbb{R}$. The Riemann invariants are now given by

$$\begin{aligned} r^1 &= \frac{C_2 t + x^1 C_1 \cos^2\varphi + x^2 C_1 \cos\varphi\sin\varphi}{(C_1\cos\varphi)(A_1(1+\kappa)t - 1)}, \\ r^2 &= C_2 t - C_1 x^1 \sin^2\varphi + x^2 C_1 \sin\varphi\cos\varphi + x^3, \end{aligned} \quad (2.6.20)$$

and the solution becomes

$$\begin{aligned} a &= A_1 \frac{C_2 t + C_1 x^1\cos^2\varphi + C_1 x^2 \cos\varphi\sin\varphi}{(C_1\cos\varphi)(A_1(1+\kappa)t-1)}, \qquad u^3 = C_1 u_2^1(r^2), \\ u^1 &= \frac{\kappa A_1(C_2 t + C_1 x^1\cos^2\varphi + C_1 x^2 \sin\varphi\cos\varphi)}{C_1(A_1(1+\kappa)t-1)} + u_2^1(r^2), \\ u^2 &= \frac{\kappa A_1}{C_1}(C_2\tan\varphi t + C_1 x^1 \sin\varphi\cos\varphi + C_1 x^2\sin^2\varphi) \\ &\quad - \frac{C_2}{C_1\sin\varphi\cos\varphi} - u_2^1(r^2)\cot\varphi, \end{aligned} \quad (2.6.21)$$

where $u_2^1(r^2)$ is an arbitrary function of r^2. Note that a and u^1 admit the gradient catastrophe at the time $T = (A_1(1+\kappa))^{-1}$.

ii) Another interesting case of a conditionally invariant solution occurs when we impose condition (2.6.1) on the functions a_1 and u_2^1. Then the solution is bounded and represents a solitary double wave of the type $(E_1^+ S_2)$

$$\begin{aligned} a &= A_1(1+B_1(r^1-1)^2)^{-1/2} + a_0, \quad A_1, B_1, C_1 \in \mathbb{R}, \quad B_1 > 0, \\ \vec{u} &= \kappa A_1(1+B_1(r^1-1)^2)^{-1/2}\vec{e}^{\,1} + (u_2^1(r^2), E_2 u_2^1(r^2) + F_2, C_1 u_2^1(r^2))^T, \end{aligned} \tag{2.6.22}$$

where

$$u_2^1(r^2) = A_2(1+B_2\cosh D_2(r^2-1))^{-1/2}, \quad A_2, B_2, D_2 \in \mathbb{R}, \quad B_2 > 0, \tag{2.6.23}$$

$$E_2 = -(e_2^1(C_1e_1^1 - e_3^1))^{-1}(C_1e_3^1 + e_1^1)(C_1e_1^1 - e_3^1), \quad F_2 = C_2(e_2^1(C_1e_1^1 - e_3^1))^{-1}.$$

The Riemann invariants take the form

$$r^1 = (1+\kappa)(A_1(1+B_1(r^1-1)^2)^{-1/2} + C_2(C_1e_1^1 - e_3^1)^{-1})t - \vec{e}^{\,1}\cdot\vec{x}, \tag{2.6.24}$$

$$r^2 = C_2 t - (e_1^1e_3^1 + C_1(1-(e_1^1)^2))x^1 - e_2^1(e_3^1 - C_1e_1^1)x^2 + (1-(e_3^1)^2 + C_1e_1^1e_3^1)x^3.$$

The solution remains bounded even though the function r^1 admits the gradient catastrophe at the time $T = (1+B_1)^{3/2}\,[(1+\kappa)A_1B_1]^{-1}$.

Case $(\mathbf{S_1S_2})$: i) Let us assume that

$$\vec{e}^{\,1} = (0,0,1), \quad \vec{m}^1 = (0,1,0), \quad \vec{e}^{\,2} = (1,0,0), \quad \vec{m}^2 = (0,0,1).$$

Then the wave vectors (2.4.9ii) are given by $\lambda^{S_1} = (-u^1, 1, 0, 0)$ and $\lambda^{S_2} = (-u^2, 0, 1, 0)$ and are linearly independent. So we are looking for rank-2 solution (S_1S_2) invariant under the vector fields

$$X_1 = \frac{\partial}{\partial t} + u^1\frac{\partial}{\partial x^1} + u^2\frac{\partial}{\partial x^2}, \quad X_2 = \frac{\partial}{\partial x^3}. \tag{2.6.25}$$

The corresponding Riemann invariants are

$$r^1(x,u) = x^1 - u^1 t, \quad r^2(x,u) = x^2 - u^2 t. \tag{2.6.26}$$

The change of coordinates

$$\begin{aligned} &\bar{t} = t, \quad \bar{x}^1 = x^1 - u^1 t, \quad \bar{x}^2 = x^2 - u^2 t, \quad \bar{x}^3 = x^3, \\ &\bar{a} = a, \quad \bar{u}^1 = u^1, \quad \bar{u}^2 = u^2, \quad \bar{u}^3 = u^3, \end{aligned} \tag{2.6.27}$$

transforms the system (2.4.4) in this case into the equations

$$\begin{aligned} &\frac{\partial \bar{u}^1}{\partial \bar{x}^1} + \frac{\partial \bar{u}^2}{\partial \bar{x}^2} = 0, \quad \frac{\partial \bar{u}^1}{\partial \bar{x}^1}\frac{\partial \bar{u}^2}{\partial \bar{x}^2} - \frac{\partial \bar{u}^1}{\partial \bar{x}^2}\frac{\partial \bar{u}^2}{\partial \bar{x}^1} = 0, \\ &\frac{\partial \bar{a}}{\partial \bar{x}^1} = \frac{\partial \bar{a}}{\partial \bar{x}^2} = \frac{\partial \bar{a}}{\partial \bar{x}^3} = 0, \quad \frac{\partial \bar{u}^i}{\partial \bar{x}^3} = 0, \quad i = 1,2,3. \end{aligned} \tag{2.6.28}$$

The solution of system (2.6.28) has the form

$$\bar{a} = a_0, \quad \bar{u}^1(\bar{t},\bar{x}) = -\frac{\partial\psi}{\partial\bar{x}^2}, \quad \bar{u}^2(\bar{t},\bar{x}) = \frac{\partial\psi}{\partial\bar{x}^1}, \quad \bar{u}^3(\bar{t},\bar{x}) = \bar{u}^3(\bar{x}^1,\bar{x}^2), \tag{2.6.29}$$

where the function ψ satisfies the homogeneous Monge-Ampère equation

$$\psi_{\bar{x}^1\bar{x}^1}\psi_{\bar{x}^2\bar{x}^2} - \psi^2_{\bar{x}^1\bar{x}^2} = 0, \tag{2.6.30}$$

and $\bar{u}^3$ is an arbitrary function of two variables. Note that this solution has rank 2 but it is not a Riemann double wave.

i) The proper selection of the function ψ transforms the solution (2.6.29) into

$$\begin{aligned} &a(t,x) = a_0, \quad u^1 = (1-n)\left(\frac{x^1 - u^1 t}{x^2 - u^2 t}\right)^n, \quad n \in \mathbb{Z}\setminus\{1\}, \\ &u^2 = -n\left(\frac{x^2 - u^2 t}{x^1 - u^1 t}\right)^{1-n}, \quad u^3(t,x) = u^3(x^1 - u^1 t, x^2 - u^2 t). \end{aligned} \tag{2.6.31}$$

For $n = 2$, we obtain an explicit solution of the form

$$\begin{aligned} &a = a_0, \quad u^1 = -2^{-1}t^{-2}[x^1 t + (x^2)^2 \pm x^2((x^2)^2 + 4tx^1)^{1/2}], \\ &u^2 = t^{-1}[x^2 \pm ((x^2)^2 + 4tx^1)^{1/2}], \quad u^3 = u^3(x^1 - u^1 t, x^2 - u^2 t). \end{aligned} \tag{2.6.32}$$

with a singularity at $t = 0$.

ii) Another example worth considering is the case when fluid velocity can be decomposed as follows $\vec{u} = \vec{u}_1(r^1) + \vec{u}_2(r^2)$. Then we get the scattering nonsingular rank-2 solution

$$\begin{aligned} &u^1 = \frac{(C_1 - \lambda_2^1 u_1^2(r^1) - \lambda_3^1 u_1^3(r^1))}{\lambda_1^1} - \left(\frac{\lambda_3^2}{\lambda_1^2}\eta + \frac{\lambda_2^2}{\lambda_1^2}\right)u_2^2(r^2) + \frac{C_2}{\lambda_1^2}, \quad C_1, C_2 \in \mathbb{R}, \\ &u^2 = u_1^2(r^1) + u_2^2(r^2), \quad u^3 = u_1^3(r^1) + \eta u_2^2(r^2), \quad a = a_0, \quad \eta = \frac{\lambda_1^2\lambda_2^1 - \lambda_1^1\lambda_2^2}{\lambda_1^1\lambda_3^2 - \lambda_3^1\lambda_1^2}, \end{aligned} \tag{2.6.33}$$

where we introduced the notation $\lambda^{S_i} = (\lambda_0^i, \vec{\lambda}^i)$, $i = 1, 2$. The above solution is invariant under the vector fields

$$X_1 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1}\frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1}\frac{\partial}{\partial x^1}, \tag{2.6.34}$$

with

$$\sigma_i = \lambda_1^1\lambda_i^2 - \lambda_i^1\lambda_1^2, \quad \beta_j = \lambda_j^2[\vec{u},\vec{e}^1,\vec{m}^1] - \lambda_j^1[\vec{u},\vec{e}^2,\vec{m}^2], \ i = 2,3, \ j = 1,2,3. \tag{2.6.35}$$

Here u_1^2 and u_1^3 are arbitrary functions of r^1, u_2^2 is an arbitrary function of r^2 and $C_i = [\vec{u}_i(r^i), \vec{e}^{\,i}, \vec{m}^{\,i}]$, $i = 1, 2$. The Riemann invariants take the form

$$r^1 = (C_1 + C_2\lambda_1^1/\lambda_1^2)t - \vec{\lambda}^1 \cdot \vec{x},$$
$$r^2 = \left(C_2 + \frac{\lambda_1^2}{\lambda_1^1}C_1 + \left(\lambda_2^2 - \frac{\lambda_1^2\lambda_2^1}{\lambda_1^1}\right) u_1^2(r^1) + \left(\lambda_3^2 - \frac{\lambda_1^2\lambda_3^1}{\lambda_1^1}\right) u_1^3(r^1)\right) t - \vec{\lambda}^2 \cdot \vec{x}. \tag{2.6.36}$$

Note that the Riemann invariant r^2 depends functionally on r^1. This means that the interacting waves influence each other and superpose nonlinearly. The result is a Riemann double wave.

iii) By submitting the arbitrary functions u_1^2, u_1^3 and u_2^2 to the DC (2.6.1) we can construct the rank-2 algebraic kink-type solution of the form

$$\begin{aligned} u^1 &= (\lambda_1^1)^{-1}[C_1 - \lambda_2^1 A_2 r^1(1 + B_2(r^1)^2)^{-1/2} - \lambda_3^1 A_3 r^1(1 + B_3(r^1)^2)^{-1/2}] \\ &\quad - (\lambda_1^2)^{-1}[-C_2 + (\lambda_3^2\eta + \lambda_2^2)A_1 r^2(1 + B_1(r^2)^2)^{-1/2}], \quad A_i, B_i \in \mathbb{R}, \\ u^2 &= A_1 r^2(1 + B_1(r^2)^2)^{-1/2} + A_2 r^1(1 + B_2(r^1)^2)^{-1/2}, \quad B_i > 0, \quad i = 1, 2, 3, \\ u^3 &= A_3 r^1(1 + B_3(r^1)^2)^{-1/2} + \eta A_1 r^2(1 + B_1(r^2)^2)^{-1/2}, \quad a = a_0, \end{aligned} \tag{2.6.37}$$

where the Riemann invariants are given by

$$\begin{aligned} r^1 &= (C_1 + C_2\frac{\lambda_1^1}{\lambda_1^2})t - \vec{\lambda}^1 \cdot \vec{x}, \\ r^2 &= \left[C_2 + C_1\frac{\lambda_1^2}{\lambda_1^1} + \left(\lambda_2^2 - \frac{\lambda_1^2\lambda_2^1}{\lambda_1^1}\right) A_2 r^1(1 + B_2(r^1)^2)^{-1/2}\right. \\ &\quad \left. + \left(\lambda_3^2 - \frac{\lambda_1^2\lambda_3^1}{\lambda_1^1}\right) A_3 r^1(1 + B_3(r^1)^2)^{-1/2}\right] t - \vec{\lambda}^2 \cdot \vec{x}. \end{aligned} \tag{2.6.38}$$

Case $(\mathbf{E_1E_2S_3})$ **.** The nonscattering rank-2 solution $(E_1^+E_2^+S_3)$ invariant under the vector field

$$X = \frac{\partial}{\partial x^3} - \frac{\epsilon_{ijk}\, e_i^1 e_j^2 (\vec{e}^{\,3} \times \vec{m}^3)_k}{\beta_{12}}\frac{\partial}{\partial t} + \frac{\beta_{23}}{\beta_{12}}\frac{\partial}{\partial x^1} + \frac{\beta_{31}}{\beta_{12}}\frac{\partial}{\partial x^2}, \tag{2.6.39}$$

with

$$\begin{aligned} \beta_{ij} &= (e_j^1 e_i^2 - e_i^1 e_j^2)[\vec{u}, \vec{e}^{\,3}, \vec{m}^3] + (e_j^2(\vec{e}^{\,3} \times \vec{m}^3)_i - e_i^2(\vec{e}^{\,3} \times \vec{m}^3)_j)(a + \vec{e}^{\,1} \cdot \vec{u}), \\ &\quad + (e_i^1(\vec{e}^{\,3} \times \vec{m}^3)_j - e_j^1(\vec{e}^{\,3} \times \vec{m}^3)_i)(a + \vec{e}^{\,2} \cdot \vec{u}), \quad i, j = 1, 2, 3, \end{aligned} \tag{2.6.40}$$

has the form

$$
\begin{aligned}
a &= \frac{A_1((e_1^1+e_1^2)x^1+(e_2^1+e_2^2)x^2)}{1-A_1(1+\kappa)t}, \quad u^3 = u_0^3, \\
u^1 &= \frac{-\kappa A_1\left(((e_1^1)^2+(e_1^2)^2)x^1+(e_1^1e_2^1+e_1^2e_2^2)x^2\right)-u_3^1(r^3)}{1-A_1(1+\kappa)t}, \\
u^2 &= \kappa A_1\left(\frac{e_2^1\left(\beta u_3^1(r^3)t-e_1^1x^1-e_2^1x^2\right)}{1-A_1(1+\kappa)t}\right. \\
&\quad \left.+\frac{e_2^2\left(-\beta u_3^1(r^3)t-e_1^2x^1-e_2^2x^2\right)}{1-A_1(1+\kappa)t}\right)+\frac{e_2^2-e_2^1}{e_1^2-e_1^1}u_3^1(r^3),
\end{aligned} \tag{2.6.41}
$$

where $|\vec{e}^{\,1}|^2=|\vec{e}^{\,2}|^2=1$, $\vec{e}^{\,1}\cdot\vec{e}^{\,2}=-\kappa^{-1}$, $e_3^1=e_3^2=0$, $\beta=(1+\kappa^{-1})/(e_1^1-e_1^2)$ and $A_1,u_0^3\in\mathbb{R}$. The Riemann invariants are

$$
r^1=\frac{\beta u_3^1(r^3)t-e_1^1x^1-e_2^1x^2}{1-A_1(1+\kappa)t},\ r^2=\frac{-\beta u_3^1(r^3)t-e_1^2x^1-e_2^2x^2}{1-A_1(1+\kappa)t},\ r^3=x^3-u_0^3t, \tag{2.6.42}
$$

where u_3^1 is an arbitrary function of r^3.

This solution represents a Riemann double wave. It does not admit removable singularities for any choice of $u_3^1(r^3)$, but the functions a,u^1 and u^2 are subject to the gradient catastrophe at the time $T=(A_1(1+\kappa))^{-1}$.

Case ($\mathbf{E_1S_2S_3}$) . The nonscattering rank-2 solution $(E_1^+S_2S_3)$ invariant under the vector field

$$
X=\frac{\partial}{\partial x^3}+\frac{\epsilon_{ijk}e_i^1(\vec{e}^{\,2}\times\vec{m}^{\,2})_j(\vec{e}^{\,3}\times\vec{m}^{\,3})_k}{\beta_{12}}\frac{\partial}{\partial x^1}+\frac{\beta_{23}}{\beta_{12}}\frac{\partial}{\partial x^2}+\frac{\beta_{31}}{\beta_{12}}\frac{\partial}{\partial x^3}, \tag{2.6.43}
$$

with

$$
\begin{aligned}
\beta_{ij} &= [(\vec{e}^{\,2}\times\vec{m}^{\,2})_i(\vec{e}^{\,3}\times\vec{m}^{\,3})_j-(\vec{e}^{\,2}\times\vec{m}^{\,2})_j(\vec{e}^{\,3}\times\vec{m}^{\,3})_i](a+\vec{e}^{\,1}\cdot\vec{u}) \\
&\quad +[e_j^1(\vec{e}^{\,3}\times\vec{m}^{\,3})_i-e_i^1(\vec{e}^{\,3}\times\vec{m}^{\,3})_j][\vec{u},\vec{e}^{\,2},\vec{m}^{\,2}] \\
&\quad +[e_i^1(\vec{e}^{\,2}\times\vec{m}^{\,2})_j-e_j^1(\vec{e}^{\,2}\times\vec{m}^{\,2})_i][\vec{u},\vec{e}^{\,3},\vec{m}^{\,3}],
\end{aligned} \tag{2.6.44}
$$

is given by

$$
\begin{aligned}
&a=A_1\frac{(C_2/\lambda_1^2+C_3/\lambda_1^3)\,t-x^1}{1-A_1(1+\kappa)t}, \quad u^1=\frac{(C_2/\lambda_1^2+C_3/\lambda_1^3)\,(1-A_1t)-\kappa A_1x^1}{1-A_1(1+\kappa)t}, \\
&u^2=C(b\lambda_1^2-\lambda_1^3)(\lambda_2^2x^2+\lambda_3^2x^3), \quad u^3=-\frac{C\lambda_2^2(b\lambda_1^2-\lambda_1^3)}{\lambda_3^2\lambda_1^3}(\lambda_2^2x^2+\lambda_3^2x^3).
\end{aligned} \tag{2.6.45}
$$

The Riemann invariants have the explicit form

$$
\begin{aligned}
r^1 &= \frac{(C_2/\lambda_1^2 + C_3/\lambda_1^3)t - x^1}{1 - A_1(1+\kappa)t}, \quad A_1, C \in \mathbb{R}, \\
r^2 &= \left(\kappa A_1 \frac{(C_2 + \lambda_1^2/\lambda_1^3)t - \lambda_1^2 x^1}{1 - A_1(1+\kappa)t} + C_2 + \frac{\lambda_1^2}{\lambda_1^3} C_3\right) t - \vec{\lambda}^2 \cdot \vec{x}, \\
r^3 &= \left(\kappa A_1 \frac{(\lambda_1^3/\lambda_1^2 + C_3)t - \lambda_1^3 x^1}{1 - A_1(1+\kappa)t} + \frac{\lambda_1^3}{\lambda_1^2} C_2 + C_3\right) t - \lambda_1^3 x^1 - b(\lambda_2^2 x^2 + \lambda_3^2 x^3),
\end{aligned}
\tag{2.6.46}
$$

Here, we introduced the notation $\lambda^{S_i} = (\lambda_0^i, \vec{\lambda}^i)$, $C_i = [\vec{u}_i, \vec{e}^{\,i}, \vec{m}^{\,i}]$, $i = 2, 3$ and $\vec{\lambda}^3 = (\lambda_1^3, b\lambda_2^2, b\lambda_3^2)$, $b \in \mathbb{R}$. Note that a and u^1 both admit the gradient catastrophe at the time $T = (A_1(1+\kappa))^{-1}$ while u^2 and u^3 are stationary. In this case the solution again has a form of Riemann double wave.

Case $(\mathbf{S_1 S_2 S_3})$ **.** The rank-2 solution is invariant under the vector field

$$
X = \frac{\partial}{\partial t} + u^1 \frac{\partial}{\partial x^1} + u^2 \frac{\partial}{\partial x^2} + u^3 \frac{\partial}{\partial x^3}. \tag{2.6.47}
$$

In this case subjecting the initial system (2.4.4) to the DCs (2.3.17) leads to the overdetermined system

$$
a = a_0, \quad \vec{u} + (\vec{u} \cdot \nabla)\vec{u} = 0, \quad \nabla \vec{u} = 0, \quad a_0 \in \mathbb{R}. \tag{2.6.48}
$$

The solution of (2.6.48) is divergence free if and only if

$$
\vec{u} = f(r^1, r^2, r^3), \quad f : \mathbb{R}^3 \to \mathbb{R}^3, \quad r^i = x^i - u^i t, \quad i = 1, 2, 3. \tag{2.6.49}
$$

The Jacobi matrix $Df(r) = (\partial f^\alpha / \partial r^i)$ has to be nilpotent. In fact, the reduced system (2.6.48) mandates that the characteristic polynomial is equal to

$$
\begin{aligned}
&\det(\lambda \mathcal{I}_3 + Df(r)) \\
&\quad = \lambda^3 - \lambda^2 \operatorname{Tr}(f^\alpha_{,r^i}) + \frac{1}{2}[(\operatorname{Tr}(f^\alpha_{,r^i}))^2 - \operatorname{Tr}(f^\alpha_{,r^i})^2]\lambda + \det(f^\alpha_{,r^i}) = \lambda^3.
\end{aligned}
\tag{2.6.50}
$$

In order to satisfy this condition we can select the arbitrary functions f^α in the following way $f^1 = b(r^2, r^3)$ and $f^2 = f^3 = g(r^2 - r^3)$. Then we have

$$
Df(r) = \begin{pmatrix} 0 & b_{r^2} & b_{r^3} \\ 0 & g_s & -g_s \\ 0 & g_s & -g_s \end{pmatrix}, \quad s = r^2 - r^3. \tag{2.6.51}
$$

If $b_{r^2} \neq b_{r^3}$, then $\operatorname{rank} Df(r) = 2$, otherwise f^1 is an arbitrary function of one variable, i.e. $f^1 = h(r^2 - r^3)$, and $\operatorname{rank} Df(r) = 1$. In the rank-2 case the solution has the form

$$\begin{aligned} &u^1(x,t) = b(x^2 - tg(x^2 - x^3), x^3 - tg(x^2 - x^3)), \\ &u^2(x,t) = u^3(x,t) = g(x^2 - x^3), \quad a = a_0, \quad a_0 \in \mathbb{R}, \end{aligned} \tag{2.6.52}$$

where b is an arbitrary function of two variables $(x^2 - u^2 t)$ and $(x^3 - u^3 t)$, and g is an arbitrary function of $(x^2 - x^3)$.

Depending on the choice of the arbitrary functions, the relations (2.6.52) can lead to elementary solutions (constant, algebraic, with one or two poles, trigonometric, hyperbolic) or doubly periodic solutions which can be expressed in terms of the Jacobi's elliptic functions sn, cn and dn. To ensure that the elliptic solutions possess one real and one purely imaginary period and that, for real argument r^i, they are contained in the interval between -1 and $+1$, the moduli k of the elliptic functions have to satisfy the condition $0 < k^2 < 1$. An example of such elliptic solution has been obtained by submitting the arbitrary functions b and g to the DC (2.6.1). It has the explicit form

$$\begin{aligned} &u^1 = A_1[1 + B_1 \mathrm{sn}^2(\beta(x^2 + nx^3) - (n+1)[A_2[1 + B_2 \mathrm{sn}^2(\beta(x^2 - x^3), k)]^{-1/2}], k)]^{-1/2}, \\ &u^2 = u^3 = A_2[1 + B_2 \mathrm{sn}^2(\beta(x^2 - x^3), k)]^{-1/2} \mathrm{sn}(\beta(x^2 - x^3), k), \\ &a = a_0, \quad 0 < k^2 < 1, \quad A_i, B_i, \beta \in \mathbb{R}, \quad B_i > 0, \quad i = 1, 2. \end{aligned} \tag{2.6.53}$$

This is a bounded solution representing a snoidal double wave.

2.7. Rank-3 solutions

Let us now present the rank-3 solutions obtained by way of the procedure analogical to the one described in Section 3 for the rank-2 solutions. Here, the only difference is that the DC of the Klein-Gordon form (2.6.1) include three independent variables r^1, r^2, r^3.

Case ($\mathbf{E_1 E_2 E_3}$) . The rank-3 potential solution $(E_1^+ E_2^+ E_3^+)$ invariant under the vector field

$$X = \frac{\partial}{\partial x^3} - \frac{[\vec{e}^1, \vec{e}^2, \vec{e}^3]}{\beta_3} \frac{\partial}{\partial t} + \frac{\beta_1}{\beta_3} \frac{\partial}{\partial x^1} + \frac{\beta_2}{\beta_3} \frac{\partial}{\partial x^2}, \tag{2.7.1}$$

with $\beta_i = (\vec{e}^2 \times \vec{e}^3)_i(a + \vec{e}^1 \cdot \vec{u}) + (\vec{e}^1 \times \vec{e}^3)_i(a + \vec{e}^2 \cdot \vec{u}) + (\vec{e}^1 \times \vec{e}^2)_i(a + \vec{e}^3 \cdot \vec{u})$, takes the form

$$a = a_1(r^1) + a_2(r^2) + a_3(r^3), \quad \vec{u} = \kappa(\vec{e}^1 a_1(r^1) + \vec{e}^2 a_2(r^2) + \vec{e}^3 a_3(r^3)), \quad (2.7.2)$$

where the Riemann invariants are

$$r^i(x,u) = (1+\kappa)a_i(r^i)t - \vec{e}^i \cdot \vec{x}, \quad \vec{e}^i \cdot \vec{e}^j = -\kappa^{-1}, \quad |\vec{e}^i|^2 = 1, \quad i \neq j = 1,2,3, \quad (2.7.3)$$

and a_i are arbitrary functions of r^i. Note that, just as in the case E_1E_2, the angle φ_{ij} between each pair of wave vectors $\vec{e}^i, \vec{e}^j$, $i \neq j = 1,2,3$, has to satisfy the condition (2.6.8). This nonscattering rank-3 solution coincides with the one obtained previously by the GMC [**100**].

i) After submitting the arbitrary functions a_i to the version of the DC (2.6.1) modified for the case of three dimensions we obtain several bounded solutions. We list here two examples.

An interesting case is the algebraic kink solution

$$a = \sum_{i=1}^{3} A_i r^i (1 + B_i (r^i)^2)^{-1/2}, \quad \vec{u} = \kappa \left[\sum_{i=1}^{3} A_i r^i (1 + B_i (r^i)^2)^{-1/2} \vec{e}^i \right], \quad (2.7.4)$$

where the Riemann invariants are given by

$$r^i = \left[(1+\kappa) A_i r^i (1 + B_i (r^i)^2)^{-1/2} \right] t - \vec{e}^i \cdot \vec{x}, \quad A_i, B_i \in \mathbb{R} \quad i = 1,2,3. \quad (2.7.5)$$

This solution evolves as a triple wave and is bounded even when the Riemann invariants r^i admit the gradient catastrophe at the time $T_i = (1+\kappa)^{-1} A_i^{-1}$.

ii) Another interesting solution describes an algebraic solitary triple wave of a kink type

$$a = \sum_{i=1}^{3} A_i (1 + e^{B_i r^i})^{-1/2}, \quad \vec{u} = \kappa \sum_{i=1}^{3} A_i (1 + e^{B_i r^i})^{-1/2} \vec{e}^i, \quad A_i, B_i \in \mathbb{R} \quad (2.7.6)$$

where the Riemann invariants are given by

$$r^i = \left[(1+\kappa) A_i (1 + e^{B_i r^i})^{-1/2} \right] t - \vec{e}^i \cdot \vec{x}, \quad i = 1,2,3. \quad (2.7.7)$$

The Riemann invariants admit the gradient catastrophe at the time

$$T_i = -2^{5/2}((1+\kappa) A_i B_i)^{-1}, \quad (2.7.8)$$

but the solution remains bounded. In both cases the angle φ_{ij} between the wave vectors $\vec{e}^i$ and $\vec{e}^j$ is given by (2.6.8).

Case ($\mathbf{E_1S_2S_3}$) : In this case we have to distinguish two situations, depending on the choice of wave vectors λ^{E_1}, λ^{S_2} and λ^{S_3}.

First we look for the rank-3 solution $(E_1^+S_2S_3)$ invariant under the vector field

$$X = e_2^1\frac{\partial}{\partial x^1} + e_2^2\frac{\partial}{\partial x^2}, \tag{2.7.9}$$

where we have assumed that the linearly independent wave vectors associated with the waves E_1^+, S_2 and S_3 are given by

$$\lambda^{E_1^+} = (a+u^3, 0, 0, -1), \quad \lambda^{S_2} = (e_2^2u^1 - e_2^1u^2, -e_2^2, e_2^1, 0), \quad \lambda^{S_3} = (-e_3^3u^3, 0, 0, e_3^3). \tag{2.7.10}$$

The corresponding Riemann invariants satisfy the following relations

$$r^1 = ((1+\kappa^{-1})f(r^1) + a_0 + u_0^3)t - x^3, \quad r^2 = t - x^1\sin g(r^2, r^3) + x^2\cos g(r^2, r^3),$$

where r^3 obeys the evolutionary partial differential equation

$$\frac{\partial r^3}{\partial t} + (f(r^1) + u_0^3)\frac{\partial r^3}{\partial x^3} = 0. \tag{2.7.11}$$

The solution then takes the form

$$\begin{aligned} &a = \kappa^{-1}f(r^1) + a_0, \quad u^1 = \sin g(r^2, r^3), \\ &u^2 = -\cos g(r^2, r^3), \quad u^3 = f(r^1) + u_0^3, \quad a_0, u_0^3 \in \mathbb{R}, \end{aligned} \tag{2.7.12}$$

where f is an arbitrary function of r^1 and g is an arbitrary function of r^2 and r^3. This scattering rank-3 solution has been obtained earlier through the GMC [**100**].

i) If $f(r^1) = A_1r^1 + B_1$, then the solution of (2.7.11) can be integrated in a closed form

$$\begin{aligned} &a = \kappa^{-1}(A_1r^1 + B_1) + a_0, \quad u^1 = \sin g(r^2, r^3), \\ &u^2 = -\cos g(r^2, r^3), \quad u^3 = A_1r^1 + B_1 + u_0^3, \quad A_1, B_1 \in \mathbb{R}, \end{aligned} \tag{2.7.13}$$

and the Riemann invariants are given by

$$\begin{aligned} r^1 &= \frac{((1+\kappa^{-1})B_1 + a_0 + u_0^3)t - x^3}{1-(1+\kappa^{-1})A_1 t}, \\ r^2 &= t - x^1 \sin g(r^2, r^3) + x^2 \cos g(r^2, r^3), \\ r^3 &= \Psi\left[\frac{1}{A_1}(A_1(\kappa a_0 - u_0^3)t + x^3 - \kappa a_0 - B_1)((1+\kappa)A_1 t - \kappa)^{-\kappa/\kappa+1}\right], \end{aligned} \tag{2.7.14}$$

where Ψ is an arbitrary function of its argument and g is an arbitrary function of two variables r^2 and r^3. This solution corresponds to a scattering Riemann triple wave.

After subjecting the arbitrary functions f and g, appearing in the solution (2.7.12), to the modified DC (2.6.1) we get several bounded solutions. Below, we present two of them.

ii) A physically interesting subcase of $(E_1^+ S_2 S_3)$ is the solution

$$\begin{aligned} a &= \kappa^{-1} A_1[1 + B_1(1+\cosh(C_1 r^1))]^{-1/2} + a_0, \\ u^1 &= \sin\left[\frac{A_2(R)^{-1/2}\tan y}{(B_2 + \tan^2 y)^{1/2}}\right], \quad A_i, B_i, C_1 \in \mathbb{R}, \quad B_i > 0, \quad i = 1, 2, \\ u^2 &= -\cos\left[\frac{A_2(R)^{-1/2}\tan y}{(B_2 + \tan^2 y)^{1/2}}\right], \\ u^3 &= A_1[1 + B_1(1+\cosh(C_1 r^1))]^{-1/2} + u_0^3, \end{aligned} \tag{2.7.15}$$

Here we introduced the following notation $R = (r^2)^2 + (r^3)^2$ and $y = \frac{1}{2}\ln|D_1 R|$, with $D_1 \in \mathbb{R}$. The Riemann invariants r^1 and r^2 are

$$\begin{aligned} r^1 &= (1+\kappa^{-1})A_1[1 + B_1(1+\cosh(C_1 r^1))]^{-1/2} t - x^3, \\ r^2 &= t - x^1 \sin\left[\frac{A_2(R)^{-1/2}\tan y}{(B_2 + \tan^2 y)^{1/2}}\right] + x^2 \cos\left[\frac{A_2(R)^{-1/2}\tan y}{(B_2 + \tan^2 y)^{1/2}}\right] \end{aligned} \tag{2.7.16}$$

and r^3 satisfies the linear partial differential equation

$$\frac{\partial r^3}{\partial t} + (A_1[1 + B_1(1+\cosh(C_1 r^1))]^{-1/2} + u_0^3)\frac{\partial r^3}{\partial x^3} = 0. \tag{2.7.17}$$

This solution is finite everywhere except at $R = 0$, but has discontinuities for $\ln|D_1 R| = (2n+1)\pi$, $n \in \mathbb{Z}$. It remains bounded even when the Riemann invariants r^1, r^2 and r^3 tend to infinity. Physically, this solution represents nonstationary concentric waves damped by the factor $R^{-1/2}$.

iii) Another solution worth mentioning has the form of an algebraic solitary wave

$$\begin{aligned} &a = \kappa^{-1}A_1[1 + B_1(1 + \cosh C_1 r^1)]^{-1/2} + a_0, \quad B_1, C_1 > 0, \\ &u^1 = \sin(D_1[1 + e^{h(r^2,r^3)}]^{-1/2}), \quad A_1, B_1, C_1, D_1 \in \mathbb{R}, \\ &u^2 = \cos(D_1[1 + e^{h(r^2,r^3)}]^{-1/2}), \\ &u^3 = A_1[1 + B_1(1 + \cosh C_1 r^1)]^{-1/2} + u_0^3, \end{aligned} \tag{2.7.18}$$

where h is an arbitrary function of r^2 and r^3. The Riemann invariants are

$$\begin{aligned} &r^1 = (1 + \kappa^{-1})A_1[1 + B_1(1 + \cosh(C_1 r^1))]^{-1/2} t - x^3, \\ &r^2 = t - x^1 \sin D_1[1 + e^{h(r^2,r^3)}]^{-1/2} + x^2 \cos D_1[1 + e^{h(r^2,r^3)}]^{-1/2}, \end{aligned} \tag{2.7.19}$$

and r^3 satisfies the partial differential equation (2.7.11).

We now consider the case $(E_1^+ S_2 S_3)$ with a different selection of the wave vectors than assumed in (2.7.10), namely we choose

$$\begin{aligned} &\lambda^{E_1^+} = (a + e_1^1 u^1 + e_1^2 u^2, -e_1^1, -e_1^2, 0), \quad |e_1|^2 = 1, \\ &\lambda^{S_2} = (u^2, 0, -1, 0), \quad \lambda^{S_3} = (-u^1, 1, 0, 0). \end{aligned} \tag{2.7.20}$$

This leads to scattering rank-3 solution of the form

$$\begin{aligned} &a = \kappa^{-1} f(r^1) + a_0, \quad u^1 = \sin f(r^1), \quad u^2 = -\cos f(r^1), \\ &u^3 = g(r^2 \cos f(r^1) + r^3 \sin f(r^1)), \quad a_0 \in \mathbb{R}, \end{aligned} \tag{2.7.21}$$

in which g is an arbitrary function of one variable $r^2 \cos f(r^1) + r^3 \sin f(r^1)$. The Riemann invariants are

$$\begin{aligned} &r^1 = (\kappa^{-1} f(r^1) + a_0)t - x^1 \cos f(r^1) - x^2 \sin f(r^1), \\ &r^2 = -t \cos f(r^1) - x^2, \quad r^3 = -t \sin f(r^1) + x^1. \end{aligned} \tag{2.7.22}$$

This triple wave solution coincides with the one obtained through the GMC [**100**].

iv) As previously, we constructed particular solutions from (2.7.21) by requiring that the arbitrary function f satisfies the modified DC (2.6.1). One of the interesting examples is a periodic solution

$$\begin{aligned} a &= \kappa^{-1}A_1(1-B_1\cos C_1r^1)^{-1/2}+a_0,\\ u^1 &= \sin A_1(1-B_1\cos C_1r^1)^{-1/2}, \quad A_1,B_1,C_1\in\mathbb{R},\\ u^2 &= -\cos A_1(1-B_1\cos C_1r^1)^{-1/2}, \quad |B_1|<1,\\ u^3 &= g(r^2\cos A_1(1-B_1\cos C_1r^1)^{-1/2}+r^3\sin A_1(1-B_1\cos C_1r^1)^{-1/2}), \end{aligned} \tag{2.7.23}$$

with the Riemann invariants

$$\begin{aligned} r^1 &= (\kappa^{-1}A_1(1-B_1\cos C_1r^1)^{-1/2}+a_0)t - x^1\cos A_1(1-B_1\cos C_1r^1)^{-1/2}\\ &\quad - x^2\sin A_1(1-B_1\cos C_1r^1)^{-1/2},\\ r^2 &= -t\cos A_1(1-B_1\cos C_1r^1)^{-1/2}-x^2,\\ r^3 &= -t\sin A_1(1-B_1\cos C_1r^1)^{-1/2}+x^1. \end{aligned} \tag{2.7.24}$$

This solution remains bounded even when the Riemann invariants admit a gradient catastrophe.

2.8. Rank-k solutions of fluid dynamics equations

Let us now consider the isentropic flow of an ideal and compressible fluid in the case when the sound velocity depends on time t only. The system (2.4.4) in $(k+1)$ dimensions becomes

$$\begin{aligned} &u_t+(u\cdot\nabla)u=0,\\ &a_t+\kappa^{-1}a\operatorname{div}u=0, \quad a_{x^j}=0, \quad j=1,\dots,k, \quad a>0, \quad \kappa=2(\gamma-1)^{-1}. \end{aligned} \tag{2.8.1}$$

We show that in this case our approach enables us to construct arbitrary rank solutions.

The change of coordinates on $\mathbb{R}^{k+1}\times\mathbb{R}^{k+1}$

$$\bar t=t, \quad \bar x^1=x^1-u^1t,\dots,\bar x^k=x^k-u^kt, \quad \bar a=a, \quad \bar u=u\in\mathbb{R}^k, \tag{2.8.2}$$

transforms (2.8.1) into the system

$$\frac{\partial\bar u}{\partial\bar t}=0, \quad \frac{\partial\bar a}{\partial\bar t}+\kappa^{-1}\bar a\operatorname{Tr}\big((\mathcal{I}_k+\bar tD\bar u(\bar x))^{-1}D\bar u(\bar x)\big)=0, \frac{\partial\bar a}{\partial\bar x}=0, \tag{2.8.3}$$

where $D\bar{u}(\bar{x}) = \partial\bar{u}/\partial\bar{x} \in \mathbb{R}^{k\times k}$ is the Jacobian matrix and $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^k) \in \mathbb{R}^k$. The general solution of the conditions $\partial\bar{u}/\partial\bar{t} = 0$ and $\partial\bar{a}/\partial\bar{x} = 0$ is

$$\bar{u}(\bar{t}, \bar{x}) = f(\bar{x}), \quad \bar{a}(\bar{t}, \bar{x}) = \bar{a}(\bar{t}) > 0 \tag{2.8.4}$$

for any functions $f : \mathbb{R}^k \to \mathbb{R}^k$ and $\bar{a} : \mathbb{R} \to \mathbb{R}$, respectively. Making use of (2.8.4) and of the trace identity

$$\frac{\partial}{\partial\bar{t}}(\ln\det B) = \mathrm{Tr}\left(B^{-1}\frac{\partial B}{\partial\bar{t}}\right), \tag{2.8.5}$$

where $B = (\mathcal{I}_k + \bar{t}Df(\bar{x}))$ and $Df(\bar{x}) = \frac{\partial}{\partial\bar{t}}(\mathcal{I}_k + \bar{t}Df(\bar{x}))$, we obtain from (2.8.3)

$$\frac{\partial}{\partial\bar{t}}\left[\ln\left(|\bar{a}(\bar{t})|^{\kappa}\det\left(\mathcal{I}_k + \bar{t}Df(\bar{x})\right)\right)\right] = 0. \tag{2.8.6}$$

Differentiating (2.8.6) with respect to $\bar{x}$ gives the condition on the flow velocity $f(\bar{x})$

$$\frac{\partial^2}{\partial\bar{x}\partial\bar{t}}\left[\ln\left(\det\left(\mathcal{I}_k + \bar{t}Df(\bar{x})\right)\right)\right] = 0. \tag{2.8.7}$$

Consequently, we have

$$\det\left(\mathcal{I}_k + \bar{t}Df(\bar{x})\right) = \alpha(\bar{x})\beta(\bar{t}), \tag{2.8.8}$$

where α and β are arbitrary functions of their argument. Evaluating (2.8.8) at $\bar{t} = 0$ implies $\alpha(\bar{x}) = \beta(0)^{-1}$. Therefore,

$$\det\left(\mathcal{I}_k + \bar{t}Df(\bar{x})\right) = \frac{\beta(\bar{t})}{\beta(0)},$$

and we obtain

$$\frac{\partial}{\partial\bar{x}}\det\left(\mathcal{I}_k + \bar{t}Df(\bar{x})\right) = 0. \tag{2.8.9}$$

Equation (2.8.9) holds if and only if the coefficients p_n, $n = 0, \ldots, k-1$, of the characteristic polynomial of the matrix $Df(\bar{x})$ are constant. Thus the general solution of (2.8.1) is

$$\bar{u}(\bar{t}, \bar{x}) = f(\bar{x}), \quad \bar{a}(\bar{t}) = A_1\left(1 + p_{k-1}\bar{t} + \ldots + p_0\bar{t}^k\right)^{-1/\kappa}, \quad A_1 \in \mathbb{R}^+, \tag{2.8.10}$$

with the Cauchy data

$$t = 0, \quad u(0, x) = f(x), \quad a(0) = A_1. \tag{2.8.11}$$

In the original coordinates $(x, u) \in \mathbb{R}^p \times \mathbb{R}^q$ this rank-k solution takes the form

$$u = f(x^1 - u^1t, \ldots, x^k - u^kt), \quad a(t) = A_1(1 + p_{k-1}t + \ldots + p_0t^k)^{-1/\kappa}. \tag{2.8.12}$$

Note that the sound velocity a is constant if and only if the Jacobian matrix $Df(\bar{x})$ is nilpotent, i.e.

$$\det\left(-\lambda I_k + Df(\bar{x})\right) = (-\lambda)^k.$$

As an example let us consider the particular solution of (2.8.1) for $k = 2$. It is invariant under the vector fields

$$X_j = \frac{\partial}{\partial t} + u^j \frac{\partial}{\partial x^{(j)}}, \quad j = 1, 2.$$

The requirement that the coefficients p_n of the characteristic polynomial (2.3.64) of the Jacobi matrix $Df(\bar{x})$ are constant means that

$$\det\left(D(f(\bar{x}))\right) = B_1, \quad \mathrm{Tr}\left(Df(\bar{x})\right) = 2C_1, \quad B_1, C_1 \in \mathbb{R},$$

where we denote $B_1 = p_0$ and $2C_1 = p_1$. Solving the above conditions gives us the general rank-2 solution of (2.8.1) which is implicitly defined by

$$\begin{aligned}
u^1(t,x,y) &= C_1(x - u^1 t) + \frac{\partial h}{\partial r^1}(x - u^1 t, y - u^2 t), \\
u^2(t,x,y) &= C_1(y - u^2 t) - \frac{\partial h}{\partial r^2}(x - u^1 t, y - u^2 t), \\
a(t) &= A_1((1 + C_1 t)^2 + B_1 t^2)^{-1/\kappa}, \quad A_1 \in \mathbb{R}^+,
\end{aligned} \tag{2.8.13}$$

where the function h depends on two variables $r^1 = x - u^1 t$ and $r^2 = y - u^2 t$ and satisfies the nonhomogeneous Monge-Ampère equation

$$h_{r^1 r^1} h_{r^2 r^2} - h_{r^1 r^2} = B_1. \tag{2.8.14}$$

Depending on the selection of particular solutions of this equation we obtain Riemann double waves or other types of rank-2 solutions of (2.8.1).

2.9. Summary remarks

The objective of this paper was to develop a new systematic way of constructing rank-k solutions of quasilinear hyperbolic systems of first order PDEs in many dimensions. Specifically, we have been interested in nonlinear superpositions of Riemann waves, which constitute the elementary solutions of these systems and are ubiquitous in the equations of mathematical physics. Interactions of Riemann waves are obviously present in many nonlinear physical phenomena. However there are still only a few examples of multiple rank solutions describing them in multi-dimensional systems. Most of these solutions were obtained through the generalized method of characteristics. The main idea behind our approach has been to look at this type of solutions from a different point of view, namely, to reformulate them in terms of symmetry group theory.

Let us now recapitulate our analysis. We look for rank-k solutions of the system (2.3.1), expressible in terms of Riemann invariants, $u = f(r^1(x,u), \ldots, r^k(x,u))$, where $f : \mathbb{R}^k \to \mathbb{R}^q$. Each Riemann invariant is associated with a specific wave vector involved in the interaction, i.e. $r^A(x,u) = \lambda_i^A(u)x^i$, $A = 1, \ldots, k$, where $\ker(\lambda_i^A A^i(u)) \neq 0$. The basic feature of these solutions is that they remain constant on $(p-k)$-dimensional hyperplanes perpendicular to the set of linearly independent wave vectors $\lambda^1, \ldots, \lambda^k$. In the context of group theory, this means that the graph $\{x, u(x)\}$ of these solutions is invariant under all vector fields $X_a = \xi_a^i(u)\partial_{x^i}$ with $\lambda_i^A \xi_a^i = 0$ for $1 \leq a \leq p-k$. Then $u(x)$ is the solution of (2.3.1) for some function f, because the set $\{r^1, \ldots, r^k, u^1, \ldots, u^q\}$ constitutes a complete set of invariants of the Abelian algebra L of such vector fields. The implicit form of these solutions leads to major difficulties in applying the classical symmetry reduction method to this case. To overcome these difficulties, we rectify the set of vector fields X_a by a change of variables on $X \times U$, choosing Riemann invariants as new independent variables. The initial equations (2.3.1) expressed in the new coordinates, complemented by the invariance conditions for the rectified vector fields X_a, form an overdetermined quasilinear system (2.3.21). Thus the solutions of this system are invariant under the Abelian group corresponding to L. The vector fields X_a constitute the conditional symmetries of the initial system (2.3.1). The consistency conditions for the overdetermined system (2.3.21), that is, the necessary and sufficient conditions for the existence of conditionally invariant solutions of (2.3.1), have been derived here and they take the form of the trace conditions (2.3.30) and (2.3.34). Given these conditions, we were able to devise a specific procedure for constructing solutions in terms of Riemann invariants. We present it for the case of rank-2 solutions, however higher rank solutions can, in principle, be constructed by analogy. The computational difficulties should not be underestimated here and in many cases additional assumptions are needed in order to perform integration or to arrive at compact forms of these solutions. Nevertheless, the implementation of the proposed CSM is still easier than of the GMC. The latter imposes stronger restrictions on the wave vectors λ^A, which contribute to computational complexity as well as narrowing of the range of obtained solutions.

As the application to the isentropic flow equations shows, our approach has proved quite productive. We were able to reconstruct the general rank-2 and rank-3 solutions obtained via the GMC and to deliver several new classes of solutions, namely in the cases E_1S_1, S_1S_2, $E_1E_2S_1$, $E_1S_1S_2$ and $S_1S_2S_3$. For the equations of an isentropic flow with a sound velocity depending on time only (an assumption which simplifies things considerably) we obtained the arbitrary rank solution, together with the Cauchy conditions in a closed form.

Moreover, we present a simple technique which allows us to overcome the main weakness of solutions expressible in terms of Riemann invariants, resulting from the fact that the first derivatives of Riemann invariants, in most cases, tend to infinity after some finite time. We show that a proper selection of the arbitrary functions appearing in the general solution can lead to bounded solutions even in the cases when Riemann invariants admit a gradient catastrophe. We obtained numerous such solutions which, to our knowledge, are all new (we include here only some of them, namely (2.6.10), (2.6.22), (2.6.37), (2.6.53), (2.7.4), (2.7.6), (2.7.15), (2.7.18) and (2.7.23)). Most of these solutions have a soliton-like form and this fact is of note since the integrability properties of soliton theories do not easily generalize to more than two dimensions.

Our technique is applicable to a very wide class of systems, which includes many physically meaningful models. Given the promising results obtained, we expect it may be useful in such areas as nonlinear field equations, general relativity or equations of continuous media. Let us note also that, though the notion of Riemann invariants was originally defined for hyperbolic systems only, it seems that it can be easily adapted to elliptic systems. Since the conditional symmetry method can be applied to these systems, it is worth investigating whether our approach to constructing rank-k solutions can be extended to the elliptic case. Some preliminary analysis suggests that to be feasible.

Chapitre 3

ELLIPTIC SOLUTIONS OF ISENTROPIC IDEAL COMPRESSIBLE FLUID FLOW IN (3+1) DIMENSIONS

Référence complète : R. Conte, A.M. Grundland et B. Huard, Elliptic solutions of isentropic ideal compressible fluid flow in (3+1) dimensions, *Journal of Physics A : Mathematical and Theoretical*, 42(13), 2009.

Résumé

Une adaptation de la méthode des symétries conditionnelles est utilisée pour obtenir de nouvelles classes de solutions elliptiques pour les équations décrivant un fluide idéal isentropique et compressible en $(3 + 1)$ dimensions. Nous portons une attention particulière aux solutions exprimées en termes de la fonction elliptique $\wp$ de Weierstrass et d'invariants de Riemann. Ces solutions sont d'un intérêt physique particulier puisque nous montrons qu'elles demeurent bornées même lorsque les invariants de Riemann admettent la catastrophe du gradient. Nous décrivons en détails une procédure permettant de construire de telles solutions. Finalement, nous présentons plusieurs exemples de solutions de type solitonique représentant des « bumps », des « kinks » ainsi que des ondes multiples.

Abstract

A modified version of the conditional symmetry method, together with the classical method, is used to obtain new classes of elliptic solutions of the isentropic ideal compressible fluid flow in (3+1) dimensions. We focus on those types of

solutions which are expressed in terms of the Weierstrass $\wp$-functions of Riemann invariants. These solutions are of special interest since we show that they remain bounded even when these invariants admit the gradient catastrophe. We describe in detail a procedure for constructing such classes of solutions. Finally, we present several examples of an application of our approach which includes bumps, kinks and multi-wave solutions.

3.1. Introduction

The purpose of this paper is to construct bounded elliptic solutions of a compressible isentropic ideal flow in $(3+1)$ dimensions. Such solutions exist even in the case where the Riemann invariants admit the gradient catastrophe.

Let us first present a brief description of a procedure detailed in [**48**] for constructing rank-k solutions in terms of Riemann invariants for the case of an isentropic compressible ideal fluid in $(3+1)$ dimensions. Such a model is governed by the equations

$$u_t^\alpha + \sum_{\beta=1}^{4}\sum_{j=1}^{3} \mathcal{A}^{j\alpha}_{\ \beta}(u) u_j^\beta = 0, \qquad \alpha = 1,2,3,4, \tag{3.1.1}$$

where $\mathcal{A}^1, \mathcal{A}^2$ and $\mathcal{A}^3$ are 4×4 real-valued matrix functions of the form

$$\mathcal{A}^j = \begin{pmatrix} u^i & \delta_{i1}\kappa^{-1}a & \delta_{i2}\kappa^{-1}a & \delta_{i3}\kappa^{-1}a \\ \delta_{i1}\kappa a & u^i & 0 & 0 \\ \delta_{i2}\kappa a & 0 & u^i & 0 \\ \delta_{i3}\kappa a & 0 & 0 & u^i \end{pmatrix}, \qquad j = 1,2,3,$$

$\kappa = 2(\gamma-1)^{-1}$ and γ is the adiabatic exponent of the medium under consideration. The independent and dependent variables are denoted by $x = (t = x^0, x^1, x^2, x^3) \in X \subset \mathbb{R}^4$ and $u = (a, \vec{u}) \in U \subset \mathbb{R}^4$, respectively, and u_i stands for the first order partial derivatives of u, i.e. $u_i^\alpha \equiv \partial u^\alpha / \partial x^i$, $\alpha = 1, \ldots, 4$, $i = 0,1,2,3$. Here, the quantity a stands for the velocity of sound in the medium and $\vec{u}$ is the velocity vector field of the flow. Throughout this paper, we adopt the summation convention over repeated lower and upper indices.

The purpose of this article is to obtain rank-k solutions of system (3.1.1) expressible in terms of Riemann invariants. To this end, we seek solutions $u(x)$ of

(3.1.1) defined implicitly by the following set of relations between the variables u^α, r^A and x^i,

$$u = f(r^1(x,u),\dots,r^k(x,u)), \quad r^A(x,u) = \lambda_i^A(u)x^i, \quad \ker(\lambda_0^A \mathcal{I}_4 + \mathcal{A}^i(u)\lambda_i^A) \neq 0, \tag{3.1.2}$$

for some function $f : \mathbb{R}^k \to \mathbb{R}^4$ and $A = 1,\dots,k \leq 3$. Such a solution is called a rank-k solution if $\mathrm{rank}\,(u_i^\alpha) = k$. The functions $r^A(x,u)$ are called the Riemann invariants associated with the wave vectors $\lambda^A = (\lambda_0^A, \vec{\lambda}^A) \in \mathbb{R}^4$ of the system (3.1.1). Here, $\vec{\lambda}^A = (\lambda_1^A, \lambda_2^A, \lambda_3^A)$ denotes a direction of wave propagation and the eigenvalue λ_0^A is a phase velocity of the considered wave. Two types of admissible wave vectors for the isentropic equations (3.1.1) are obtained by solving the dispersion relation

$$\det(\lambda_0(u)\mathcal{I}_4 + \lambda_i(u)\mathcal{A}^i(u)) = [(\lambda_0 + \vec{u}\cdot\vec{\lambda})^2 - a^2\vec{\lambda}^2](\lambda_0 + \vec{u}\cdot\vec{\lambda})^2 = 0. \tag{3.1.3}$$

They are called the entropic (E) and acoustic (S) wave vectors and are defined by

$$\text{i) } \lambda^E = (\varepsilon a + \vec{u}\cdot\vec{e}, -\vec{e}), \quad \varepsilon = \pm 1, \quad \text{ii) } \lambda^S = (\det(\vec{u},\vec{e},\vec{m}), -\vec{e}\times\vec{m}), \quad |\vec{e}|^2 = 1, \tag{3.1.4}$$

where $\vec{e}$ and $\vec{m}$ are unit and arbitrary vectors, respectively.

The construction of rank-k solutions through the conditional symmetry method (CSM) is achieved by considering an overdetermined system, consisting of the original system (3.1.1) in 4 independent variables together with a set of compatible first order differential constraints (DCs),

$$\xi_a^i(u)u_i^\alpha = 0, \quad \lambda_i^A\xi_a^i = 0, \quad a = 1,\dots,4-k, \tag{3.1.5}$$

for which a symmetry criterion is automatically satisfied. Such notions as conditional symmetry, conditional symmetry algebra and conditionally invariant solution for the original system (3.1.1) we use in accordance with definitions given in [**48**]. Under the above circumstances, the following result holds :

The isentropic compressible ideal fluid equations (3.1.1) admit a $(4-k)$-dimensional conditional symmetry algebra L if and only if there exists a set of

$(4-k)$ linearly independent vector fields

$$X_a = \xi_a^i(u)\frac{\partial}{\partial x^i}, \quad a = 1,\ldots,4-k, \quad \ker\left(\mathcal{A}^i(u)\lambda_i^A\right) \neq 0, \quad \lambda_i^A\xi_a^i = 0, \quad A = 1,\ldots,k \leq 3, \tag{3.1.6}$$

which satisfy on some neighborhood of $(x_0, u_0) \in X \times U$ the trace conditions

$$\text{i)} \quad \operatorname{tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r}\lambda\right) = 0, \quad \text{ii)} \quad \operatorname{tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_{(a_1}\frac{\partial f}{\partial r}\ldots\eta_{a_s)}\frac{\partial f}{\partial r}\lambda\right) = 0, \quad \mu = 1,\ldots,4, \tag{3.1.7}$$

where

$$\lambda = (\lambda_i^A) \in \mathbb{R}^{k\times 4}, \quad r = (r^1,\ldots,r^k) \in \mathbb{R}^k, \quad \frac{\partial f}{\partial r} = \left(\frac{\partial f^\alpha}{\partial r^A}\right) \in \mathbb{R}^{4\times k},$$

$$\eta_{a_s} = \left(\frac{\partial \lambda_{a_s}^A}{\partial u^\alpha}\right) \in \mathbb{R}^{k\times 4}, \quad s = 1,\ldots,k-1,$$

and $(a_1,\ldots,a_s)$ denotes the symmetrization over all indices in the bracket. Solutions of the system which are invariant under the Lie algebra L are precisely rank-k solutions of the form (3.1.2).

This result is a special case of the proposition in **[48]**. Note that these symmetries are not symmetries of the original system, but they can be used to construct solutions of the overdetermined system composed of (3.1.1) and (3.1.5).

For the case of rank-1 entropic solution E, the wave vector λ^E is a non-zero multiple of (3.1.4 i). Therefore, the corresponding vector fields X_i and Riemann invariant r become

$$\begin{aligned} X_i &= -(a + \vec{e}\cdot\vec{u})^{-1}e_i\frac{\partial}{\partial t} + \frac{\partial}{\partial x^i}, \quad i = 1,2,3, \\ r(x,u) &= (a + \vec{u}\cdot\vec{e})t - \vec{e}\cdot\vec{x}, \quad |\vec{e}|^2 = 1, \end{aligned} \tag{3.1.8}$$

where we chose $\varepsilon = 1$ in (3.1.4 i). Rank-1 solutions invariant under the vector fields $\{X_1, X_2, X_3\}$ are obtained through the change of coordinates

$$\bar{t} = t,\ \bar{x}^1 = r(x,u),\ \bar{x}^2 = x^2,\ \bar{x}^3 = x^3,\ \bar{a} = a,\ \bar{u}^1 = u^1,\ \bar{u}^2 = u^2,\ \bar{u}^3 = u^3, \tag{3.1.9}$$

on $\mathbb{R}^4 \times \mathbb{R}^4$. Assuming that the direction of the wave vector $\vec{e}$ is constant, the fluid dynamics equations (3.1.1) transform into the system

$$\frac{\partial \bar{a}}{\partial \bar{x}^1} = \kappa^{-1}e_i\frac{\partial \bar{u}^i}{\partial \bar{x}^1}, \quad \frac{\partial \bar{u}^i}{\partial \bar{x}^1} = \kappa e_i\frac{\partial \bar{a}}{\partial \bar{x}^1}, \quad i = 1,2,3, \tag{3.1.10}$$

with the invariance conditions

$$\bar{a}_{\bar{t}} = \bar{a}_{\bar{x}^j} = 0, \quad \bar{u}^{\alpha}_{\bar{t}} = \bar{u}^{\alpha}_{\bar{x}^j} = 0, \quad j = 2,3, \quad \alpha = 1,2,3. \tag{3.1.11}$$

The general rank-1 entropic E solution takes the form

$$\bar{a}(\bar{t},\bar{x}) = \bar{a}(\bar{x}^1), \quad \bar{u}^i(\bar{t},\bar{x}) = \kappa e_i \bar{a}(\bar{x}^1) + C_i, \quad C_i \in \mathbb{R}, \quad i = 1,2,3, \tag{3.1.12}$$

where the Riemann invariant $\bar{x}^1 = r(x,u)$ is given by

$$r(x,u) = [(1+\kappa)a + \vec{e}\cdot\vec{C}]t - \vec{e}\cdot\vec{x}, \quad \vec{C} = (C_1, C_2, C_3) \in \mathbb{R}^3.$$

A similar procedure can be applied to the rank-1 acoustic solution S. Here, the wave vector λ^S is a non-zero multiple of (3.1.4 ii) and the corresponding vector fields X_i and Riemann invariant are

$$r(x,u) = \det(\vec{u},\vec{e},\vec{m})t - (\vec{e}\times\vec{m})\cdot\vec{x}, \quad X_i = \frac{(\vec{e}\times\vec{m})_i}{\det(\vec{u},\vec{e},\vec{m})}\frac{\partial}{\partial t} + \frac{\partial}{\partial x^i}, \quad i = 1,2,3. \tag{3.1.13}$$

Again, the change of variables (3.1.9) leads to transformed dynamical equations, which we integrate in order to find rank-1 acoustic solution of the form

$$\bar{a}(\bar{t},\bar{x}) = a_0, \quad \bar{u}^1(\bar{t},\bar{x}) = \bar{u}^1(\bar{x}^1), \quad \bar{u}^2(\bar{t},\bar{x}) = \bar{u}^2(\bar{x}^1), \quad C \in \mathbb{R}, \tag{3.1.14}$$
$$\bar{u}^3(\bar{t},\bar{x}) = (e_1m_2 - e_2m_1)^{-1}\left[C - (e_2m_3 - e_3m_2)\bar{u}^1(\bar{x}^1) - (e_3m_1 - e_1m_3)\bar{u}^2(\bar{x}^1)\right].$$

Here $\bar{u}^1$ et $\bar{u}^2$ are arbitrary functions of the Riemann invariant $\bar{x}^1 = r(x,u)$ which has the explicit form

$$r(x,u) = Ct - \det(\vec{x},\vec{e},\vec{m}). \tag{3.1.15}$$

In general, the overdetermined system composed of (3.1.7 i) and (3.1.7 ii) is nonlinear and cannot always be solved in a closed form. Nevertheless, particular rank-k solutions for many physically interesting systems of PDEs are well worth pursuing. These particular solutions of (3.1.7 i) and (3.1.7 ii) can be obtained by assuming that the function f is in the form of a rational function, which may also be interpreted as a truncated Laurent series in the variables r^A. This method can work only for equations having the Painlevé property [**17**]. Consequently, these equations can be very often integrated in terms of known functions.

Applying a version of the conditional symmetry method to the isentropic model (3.1.1), several new classes of solutions have been constructed in a closed form [**47, 48**]. Comparing these results with the ones obtained via the generalized method of caracteristics (GMC) [**100**], it was shown that more diverse classes of solutions are involved in superpositions (i.e. rank-k solutions) than in the case of the GMC [**48**].

This paper is a continuation of the papers [**47, 48**]. The objective is to construct bounded elliptic solutions of the isentropic system (3.1.1) using the version of the CSM proposed in [**48**]. These types of solutions are obtained through a proper selection of differential constraints (DCs) compatible with the initial system of equations (3.1.1). That is, the solution should satisfy both the initial system (3.1.1) and the differential constraints (3.1.5). Among the new results obtained, we have rank-2 and rank-3 periodic bounded solutions expressed in terms of Weierstrass $\wp$-functions. They represent bumps, kinks and multiple waves, all of which depend on Riemann invariants. These solutions remain bounded even when the invariants admit a gradient catastrophe.

The paper is organized as follows. In Section 3.2 we construct rank-2 and rank-3 elliptic solutions of the system, among which multiple waves and doubly periodic solutions are included, and we show that they remain bounded everywhere. Section 3.3 summarizes the results obtained and contains some suggestions for future developments.

3.2. Rank-2 and rank-3 solutions

The construction approach outlined in Section 1 has been applied to the isentropic flow equations (3.1.1) in order to obtain rank-2 and rank-3 solutions. The results of our analysis are summarized in Tables 3.1 and 3.2. Several of them possess a certain amount of freedom. They depend on one or two arbitrary functions of one or two Riemann invariants, depending on the case. The range of the types of solutions obtained depends on different combinations of the vector fields X_a as given in (3.1.6). For convenience, we denote by E_iE_j, E_iS_j, S_iS_j, $E_iE_jE_k$, etc, $i, j, k = 1, 2, 3$, the solutions which are the result of nonlinear superpositions of rank-1 solutions associated with different types of wave vectors (3.1.4 i) and (3.1.4 ii). By r^1, r^2 and r^3 we denote the Riemann invariants which coincide with the group invariants of the differential operators X_a of the solution under consideration.

The arbitrary functions appearing in the solutions listed in Tables 3.1 and 3.2 allow us to change the geometrical properties of the governed fluid flow in such a way as to exclude the presence of singularities. This fact is of special significance since, as is well known [**107, 112**], in most cases, even for arbitrary smooth and sufficiently small initial data at $t = t_0$ the magnitude of the first derivatives of Riemann invariants becomes unbounded in some finite time T. Thus, solutions expressible in terms of Riemann invariants usually admit a gradient catastrophe. Nevertheless, we have been able to show that it is still possible in these cases to construct bounded solutions expressed in terms of elliptic functions, through the proper selection of the arbitrary functions appearing in the general solution. For this purpose it is useful to select DCs corresponding to a certain class of the nonlinear Klein-Gordon equation which is known to possess rich families of bounded solutions [**1**]. We choose elliptic solutions of the Klein-Gordon equation because a group theoretical analysis has already been performed [**115**]. The obtained results can be adapted to the isentropic ideal compressible fluid flow in $(3 + 1)$ dimensions. Thus, we specify the arbitrary function(s) appearing in the general solutions listed in Tables 3.1 and 3.2, say ϕ, to the differential constraint in the form of the Klein-Gordon ϕ^6-field equation in three independent variables r^1, r^2 and r^3 which form the coordinates of the Minkowski space $M(1, 2)$

$$\phi_{r^1r^1} - \phi_{r^2r^2} - \phi_{r^3r^3} = c\phi^5, \quad c \in \mathbb{R}. \tag{3.2.1}$$

Here, we choose r^1 to be timelike and r^2, r^3 to be spacelike coordinates. It is well known (see e.g. [**115**]) that equation (3.2.1) is invariant with respect to the similitude Lie algebra $sim(1, 2)$ involving the following generators

$$\begin{aligned} &D = r^i\partial_{r^i} - \frac{1}{2}\phi\partial_\phi, \quad P_i = \partial_{r^i}, \quad i = 1, 2, 3, \\ &L_{ab} = r^a\partial_{r^b} - r^b\partial_{r^a}, \quad a \neq b = 2, 3, \\ &K_{1a} = -(r^1\partial_{r^a} - r^a\partial_{r^1}), \quad a = 2, 3, \end{aligned} \tag{3.2.2}$$

where D denotes a dilation, P_i represents translations, L_{ab} stands for rotations and K_{1a} for Lorentz boosts. A systematic use of the subgroup structure [**115**] of the invariance group of (3.2.1) allows us to generate all symmetry variables ξ in terms of the Riemann invariants r^1, r^2, r^3. We concentrate here only on the case when symmetry variables are invariants of the assumed subgroups G_i of $Sim(1,2)$ having generic orbits of codimension one. For illustration purposes, we perform a symmetry reduction analysis on four selected members of the list of subalgebras given in ([**115**], Table IV) which involve dilations. For each selected subalgebra in Minkowski space $M(1,2)$, we compute the group invariants ξ of the corresponding Lie subgroup and reduce the equation (3.2.1) to a second order ODE. The application of the symmetry reduction method to equation (3.2.1) leads to solutions of the form

$$\phi(r) = \alpha(r)F(\xi(r)), \quad r = (r^1, r^2, r^3), \tag{3.2.3}$$

where the multiplier $\alpha(r)$ and the symmetry variable $\xi(r)$ are given explicitly by group theoretical considerations and $F(\xi)$ satisfies an ODE obtained by substituting (3.2.3) into equation (3.2.1). The results of our computation are listed below.

$$\begin{aligned}
&1. \quad \{D, P_1\}: \quad \alpha = \{4c[(r^2)^2 + (r^3)^2]\}^{-1/4}, \quad \xi = \frac{1}{2}\arctan\frac{r^3}{r^2}, \quad F'' + F + F^5 = 0,\\
&2. \quad \{D, L_{31}\}: \quad \alpha = \{-c(r^1)^2/4\}^{-1/4}, \quad \xi = \frac{(r^2)^2 + (r^3)^2}{(r^1)^2},\\
&\qquad \xi(1+\xi)F'' + \left(2\xi + \frac{3}{2}\right)F' + \frac{3}{16}F + F^5 = 0,\\
&3. \quad \{D + \frac{1+q}{q}K_{12}, L_{23}\}: \quad \alpha = \{-\frac{(2q+1)}{c}\}^{1/4}(r^1 + r^2)^{q/2},\\
&\qquad \xi = [(r^1)^2 - (r^2)^2 - (r^3)^2](r^1 + r^2)^q,\\
&\qquad F'' + \frac{3q+l}{2q+1}\frac{1}{\xi}F' + F^5 = 0, \quad q = -l/3, l-2, 4-3l, \quad l \in \mathbb{Z}^+,\\
&4. \quad \{D + \frac{1}{2}K_{12}, L_1 - K_{13}\}: \quad \alpha = (9/4C)^{1/4}\{r^2 - (r^1 + r^3)^2/4\}^{-1/2},\\
&\qquad \xi = \frac{6(r^3 - r^1) + 6r^2(r^1 + r^3) - (r^1 + r^3)^3}{8(r^2 - (r^1 + r^3)^2/4)^{3/2}}, \quad (1+\xi^2)F'' + \frac{7}{3}F' + \frac{1}{3}F + F^5 = 0.
\end{aligned} \tag{3.2.4}$$

The parity invariance of (3.2.1) suggests the substitution

$$F(\xi) = [H(\xi)]^{1/2}$$

which transforms the equations listed in (3.2.4) to

$$\{D, P_1\} \qquad H'' = \frac{H'^2}{2H} - 2(H + H^3), \tag{3.2.5}$$

$$\{D, L_{31}\} \qquad H'' = \frac{H'^2}{2H} - \frac{1}{\xi(1+\xi)}\left[\left(2\xi + \frac{3}{2}\right)H' + \frac{3}{8}H + 2H^3\right], \tag{3.2.6}$$

$$\{D + \frac{1+q}{q}K_{12}, L_{23}\} \qquad H'' = \frac{H'^2}{2H} - \left[\frac{m}{\xi}H' + 2H^3\right], m = \frac{3q+l}{2q+1} = (0, 4/3, 2), \tag{3.2.7}$$

$$\{D + \frac{1}{2}K_{12}, L_1 - K_{13}\} \quad H'' = \frac{H'^2}{2H} - \frac{1}{1+\xi^2}\left[\frac{7}{3}\xi H' + \frac{2}{3}H + 2H^3\right], \tag{3.2.8}$$

where the three admissible values for the scalar m come from group theoretical considerations [**115**]. Each of these four equations possesses a first integral

$$K' = \frac{1}{4}Gg^2\frac{(gH)'^2}{gH} - \frac{c_0}{4}(gH)^3 - 3e_0 gH, \tag{3.2.9}$$

in which the four sets of functions G, g and constants e_0, c_0 obey the respective conditions

$$G = -\frac{3c_0}{4}, \ g^2 = \frac{4e_0}{c_0},$$

$$G = -\frac{3c_0}{4}\xi(\xi+1), \ g^2 = -\frac{64e_0}{c_0}\xi,$$

$$G = -\frac{3c_0}{4}, \ (a, e_0, g^2) = (0, 0, k_1), \ (4/3, 0, k_1\xi^{4/3}), \ (2, e_0, -\frac{16e_0}{c_0}\xi^2),$$

$$G = -\frac{3c_0}{4}(\xi^2 + 1), \ g = k_1(1+\xi^2)^{1/3}, \ e_0 = 0,$$

(k_1 denotes an arbitrary nonzero real constant). Under a transformation $(H, \xi) \to (U, \zeta)$ which preserves the Painlevé property,

$$H(\xi) = U(\zeta)/g(\xi), \ \left(\frac{\mathrm{d}\zeta}{\mathrm{d}\xi}\right)^2 = \frac{1}{Gg^2},$$

the equation (3.2.9) becomes autonomous

$$U'^2 - c_0U^4 - 12e_0U^2 - 4K'U = 0, \ c_0 \neq 0.$$

When $K' = 0$, U^{-1} is either a sine, cosine, hyperbolic sine or a hyperbolic cosine function, depending on the signs of the constants, therefore bounded solutions are easily characterized.

When $K' \neq 0$, it is convenient to first integrate this elliptic equation in terms of the Weierstrass function $\wp(\zeta, g_2, g_3)$,

$$U(\zeta) = \frac{K'}{\wp(\zeta) - e_0},\ g_2 = 12e_0^2,\ g_3 = -8e_0^3 - c_0 K'^2,$$
$$\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) = 4\wp^3 - g_2\wp - g_3,$$

which is a real continuous function on some open finite interval for real ζ and real invariants g_2, g_3, (where we abbreviate $\wp(\zeta, g_2, g_3)$ by $\wp(\zeta)$) then to use the classical formulae which connect $\wp$ and various bounded Jacobi functions.

This correspondence is quite easy to write down if one uses the symmetric notation of Halphen [**57**] to represent the Jacobi functions. Halphen introduces three basis functions

$$h_\alpha(u) = \sqrt{\wp(u) - e_\alpha}, \quad \alpha = 1, 2, 3,$$

and the connection between the Weierstrass $\wp$ function and the Jacobi copolar trio $\mathrm{cs}, \mathrm{ds}, \mathrm{ns}$ is given by [**57**, p. 46]

$$\frac{\mathrm{cs}(z|k)}{h_1(u)} = \frac{\mathrm{ds}(z|k)}{h_2(u)} = \frac{\mathrm{ns}(z|k)}{h_3(u)} = \frac{u}{z} = \frac{1}{\sqrt{e_1 - e_3}}, \quad k^2 = \frac{e_2 - e_3}{e_1 - e_3},$$

where k is the modulus of the Jacobi elliptic functions. For full details on Halphen's symmetric notation, see [**75**]. We give here explicit solutions in terms of the Weierstrass function, leaving the conversion to Jacobi's notation to the reader. These solutions are obtained by convenient choices of the normalization constants e_0, c_0, K', k_1.

With the normalization $e_0 = -1/3,\ c_0 = -4/3,\ K' = C$, the general solution of (3.2.5) is

$$F^2(\xi) = \frac{C}{\wp(\xi) + 1/3},\ \zeta = \xi,\ g_2 = \frac{4}{3},\ g_3 = \frac{8}{27} + \frac{4}{3}C^2,\ C \in \mathbb{R}. \qquad (3.2.10)$$

With the normalization $e_0 = k_0^{-2}/48,\ c_0 = -(4/3)k_0^{-2},\ K' = C$, the solution of (3.2.6) has the form

$$F^2(\xi) = \frac{C\xi^{-1/2}}{\wp(\zeta) - \frac{1}{48k_0^2}},\ \zeta = -2k_0 \mathrm{argth}\sqrt{\xi+1},\ g_2 = \frac{1}{192k_0^4},\ g_3 = -\frac{1}{13824k_0^6} + \frac{4C^2}{3k_0^2},$$

with $k_0, C \in \mathbb{R}$.

The three cases for equation (3.2.7) associated with the subalgebra $\{D + \frac{1+q}{q}K_{12}, L_{23}\}$ yield the respective solutions

$$q = -k/3 : F^2(\xi) = \frac{C}{\wp(\xi)},\ \zeta = \xi,\ g_2 = 0,\ g_3 = \frac{4C^2}{3},$$

$$q = 4 - 3k : F^2(\xi) = \frac{C\xi^{-2/3}}{\wp(\zeta)},\ \zeta = 3k_0\xi^{1/3},\ g_2 = 0,\ g_3 = \frac{4C^2}{3k_0^2},$$

$$q = k - 2 : F^2(\xi) = \frac{C\xi^{-1}}{\wp(\zeta) - \frac{1}{12k_0^2}},\ \zeta = k_0 \log\xi,\ g_2 = \frac{1}{12k_0^4},\ g_3 = -\frac{1}{216k_0^6} + \frac{4C^2}{3k_0^2}.$$

Finally, equation (3.2.8) integrates as (equation no 4 in (3.2.4))

$$F^2(\xi) = \frac{C(\xi^2+1)^{-1/3}}{\wp(\zeta)}, \zeta = \xi\ {}_2F_1\left(\frac{1}{2}, \frac{5}{6}; \frac{3}{2}; -\xi^2\right),\ g_2 = 0,\ g_3 = \frac{4C^2}{3k_0^2},$$

where ${}_2F_1$ denotes the hypergeometric function.

Using these results, we construct bounded rank-3 solutions of the equations (3.1.1). For this purpose, for each general solution appearing in Tables 3.1 and 3.2, we introduce the arbitrary functions into the Klein-Gordon equation (3.2.1) and select only the solutions expressed in terms of the Weierstrass $\wp$-function.

For illustration, let us now discuss the case of the rank-3 entropic solution $E_1E_2E_3$ which represents a superposition of three rank-1 entropic solutions E_i given by (3.1.12). We assume that the entropic wave vectors λ^{E_1}, λ^{E_2} and λ^{E_3} are linearly independent and take the form

$$\lambda^{E_i} = (a + \vec{e}^i \cdot \vec{u}, -\vec{e}^i), \quad |\vec{e}^i|^2 = 1, \quad i = 1, 2, 3.$$

Hence the corresponding vector fields X_i and Riemann invariants are given by

$$X = \frac{\partial}{\partial x^3} - \frac{\det(\vec{e}^1, \vec{e}^2, \vec{e}^3)}{\beta_3}\frac{\partial}{\partial t} + \frac{\beta_1}{\beta_3}\frac{\partial}{\partial x^1} + \frac{\beta_2}{\beta_3}\frac{\partial}{\partial x^2}, \quad r^i(x,u) = (a + \vec{e}^i \cdot \vec{u})t - \vec{e}^i \cdot \vec{x}, \tag{3.2.11}$$

where $\beta_i = (\vec{e}^2 \times \vec{e}^3)_i(a + \vec{e}^1 \cdot \vec{u}) + (\vec{e}^1 \times \vec{e}^3)_i(a + \vec{e}^2 \cdot \vec{u}) + (\vec{e}^1 \times \vec{e}^2)_i(a + \vec{e}^3 \cdot \vec{u})$. The rank-3 entropic solutions invariant under the vector field X are obtained through the change of coordinates

$$\bar{t}=t,\bar{x}^1=r^1(x,u),\bar{x}^2=r^2(x,u),\bar{x}^3=r^3(x,u),\bar{a}=a,\bar{u}^1=u^1,\bar{u}^2=u^2,\bar{u}^3=u^3, \tag{3.2.12}$$

on $\mathbb{R}^4\times\mathbb{R}^4$. Specifying the form of the solution as a linear superposition of rank-1 solutions (3.1.12),

$$\bar{a}=\bar{a}_1(r^1)+\bar{a}_2(r^2)+\bar{a}_3(r^3),\quad \vec{u}=\kappa(\vec{e}^1\bar{a}_1(r^1)+\vec{e}^2\bar{a}_2(r^2)+\vec{e}^3\bar{a}_3(r^3)) \tag{3.2.13}$$

the fluid dynamics equations (3.1.1) transform to

$$\sum_{i=1}^{2}\sum_{j=i+1}^{3}[\kappa(\vec{e}^i\cdot\vec{e}^j)^2+(1-\kappa)(\vec{e}^i\cdot\vec{e}^j)-1]\bar{a}_i'(r^i)\bar{a}_j'(r^j)=0, \tag{3.2.14}$$

while the invariance conditions have the form

$$\bar{a}_{\bar{t}}=\bar{u}^1_{\bar{t}}=\bar{u}^2_{\bar{t}}=\bar{u}^3_{\bar{t}}=0. \tag{3.2.15}$$

This solution exists if and only if the three entropic wave vectors $\vec{e}^1,\vec{e}^2,\vec{e}^3$ intersect at a certain specific angle given by

$$\cos\phi_{ij}=-\frac{1}{\kappa},\quad i\neq j=1,2,3,$$

where ϕ_{ij} denotes the angle between the wave vectors $\vec{e}^i$ and $\vec{e}^j$ [**48, 100**]. Imposing the condition that each of the functions $\bar{a}_i(r^i)$, $i=1,2,3$ obeys the ODE

$$F''+F+F^5=0,$$

then according to (3.2.10), the rank-3 entropic solution takes the form

$$\begin{gathered} a=\sum_{i=1}^{3}\frac{C_i}{\left(\wp(r^i,\frac{4}{3},\frac{8}{27}+\frac{4}{3}C_i^2)+\frac{1}{3}\right)^{1/2}},\quad \vec{u}=\kappa\sum_{i=1}^{3}\frac{C_i\vec{\lambda}^i}{\left(\wp(r^i,\frac{4}{3},\frac{8}{27}+\frac{4}{3}C_i^2)+\frac{1}{3}\right)^{1/2}},\\ r^i=-(1+\kappa)\frac{C_i}{\left(\wp(r^i,\frac{4}{3},\frac{8}{27}+\frac{4}{3}C_i^2)+\frac{1}{3}\right)^{1/2}}t+\vec{\lambda}^i\cdot\vec{x},\quad i=1,2,3. \end{gathered} \tag{3.2.16}$$

Making use of an explicit expression for the zeros of the $\wp$-function, we show that the values for which the denominator in solution (3.2.16) vanish are not located on the real axis for a specific choice of the constants of integration C_i. Then we have

Proposition 2. *If the constants of integration C_i are equal to $\sqrt{19}/6$, then the elliptic rank-3 entropic solution (3.2.16) of the isentropic ideal compressible fluid flow equations (3.1.1) is bounded.*

Proof. According to recent results obtained by Duke and Imamoglu in [**31**], the location of the zeros of the $\wp$-function can be given explicitly in terms of generalized hypergeometric functions.

Considering a lattice $\mathcal{L} = \mathbb{Z} + \tau\mathbb{Z}$, $\operatorname{Im}\tau > 0$, the doubly periodic Weierstrass $\wp$-function is defined by

$$\wp(z; 1, \tau) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

where the sum ranges over all $\omega \in \mathcal{L}$. Note that the $\wp$-function assumes every value of the extended complex plane exactly twice in $\mathcal{L}$ and since it is even, its zeros are of the form $\pm z_0$. The value of z_0 can be determined from the following theorem.

Proposition 3. [**31**] *The zeros $\pm z_0$ of the $\wp$-function are given by*

$$z_0 = \frac{1+\tau}{2} + \frac{c_2 s^{1/4} {}_3F_2\left(\frac{1}{3}, \frac{2}{3}, 1; \frac{3}{4}, \frac{5}{4}; s\right)}{{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; 1-s\right)}, \quad c_2 = -\frac{i\sqrt{6}}{3\pi}, \quad |s| < 1, \quad |1-s| < 1, \tag{3.2.17}$$

where ${}_pF_q$ denotes the generalized hypergeometric functions and s is a function of the modular discriminant Δ and the Eisenstein series E_4,

$$s = 1 - \frac{1728\Delta}{E_4^3}, \quad \Delta(\tau) = q \prod_{n \geq 1} (1-q^n)^{24}, \quad E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \quad q = e^{2\pi i \tau}. \tag{3.2.18}$$

It is understood that the principal branch is to be taken in the radical expression $s^{1/4}$.

We now illustrate the application of this theorem by showing that the function $\wp(z, 4/3, 1)$ is always strictly positive for real z. This case is the specific case of solution (3.2.10) for which $C = \sqrt{19}/6$. From given invariants g_2 and g_3, by finding the roots e_1, e_2, e_3 of the cubic polynomial $4t^3 - g_2 t - g_3$, one can determine the values of the periods ω_1 and ω_2. In the case when $g_2 = 4/3$ and $g_3 = 1$, we obtain the periods $\omega_1 = 2.81$ and $\omega_2 = 1.405 + \text{i}2.902$, hence $\tau = \omega_2/\omega_1 = 0.5 + \text{i}1.033$.

For any pair of periods ω_1, ω_2, the lattice $\omega_1\mathbb{Z}+\omega_2\mathbb{Z}$ can be rescaled in such a way that ω_1 is normalized to 1 by using the well-known formula for the $\wp$-function

$$\wp(z;\omega_1,\omega_2) = \wp(z/\omega_1; 1, \omega_2/\omega_1)/\omega_1^2. \tag{3.2.19}$$

Introducing the numeric value of τ into (3.2.18), we can evaluate from (3.2.17) and (3.2.19) the zeros of the Weierstrass function $\wp(z, 4/3, 1)$, $z_0 = \pm(1.405 + \mathrm{i}0.929)$, which are indeed complex. Note that the $\wp$-function always possesses a double pole at $z = 0$ and that it tends to $+\infty$ at this point. Since it is continuous for all real $z \in (0, \omega_1)$, this implies that it must always be strictly positive. Hence, $\wp(z, 4/3, 1) + 1/3 > 0$ on the real interval $(0, \omega_1)$ and the function $(\wp(z, 4/3, 1) + 1/3)^{-1}$ is therefore bounded for all real z. Choosing the constants C_i as values of the initial constant C for which solution (3.2.10) is bounded then guarantees the boundedness of functions a and $\vec{u}$ in solution (3.2.16). Therefore, solution (3.2.16) is bounded everywhere, even when the Riemann invariants r^i admit the gradient catastrophe. A similar analysis can be applied to every solution presented in Table 3.3 to show that they are bounded for some choices of the arbitrary constants. This can be accomplished by the same procedure as presented above through the use of the theorem from [**31**].

□

This solution is physically interesting since it remains bounded for every value of the Riemann invariants r^i. Thus, it represents a bounded solution with periodic flow velocities. Similarly, it is possible to submit the arbitrary functions of the differential equations no $2, 3, 4$ listed in (3.2.4) to obtain other types of bounded solutions. Table 3.3 presents these various types of solutions with their corresponding Riemann invariants. They are all bounded solutions of periodic, bump or kink type. Note that these solutions of (3.1.1) admit gradient catastrophes at some finite time. Hence, some discontinuities can occur like shock waves [**20, 64**]. Note also that the solutions remain bounded even when the first derivatives of r^i tend to infinity after some finite time T. However, after time T, the solution cannot be represented in parametric form by the Riemann invariants and ceases to exist.

3.3. Concluding remarks

The methods presented in this paper can be applied quite broadly and can usually provide at least certain particular solutions of hydrodynamic type equations. The conditional symmetries refer to the symmetries of the overdetermined system obtained by subjecting the original system (3.1.1) to certain differential constraints defined by setting the characteristics of the vector fields X_a to zero. The conditional symmetries are not symmetries of the original system (3.1.1). However, they are used to construct classes of rank-3 solutions of this system which are not obtainable by the classical symmetry approach. Among the new results obtained, we have rank-2 and rank-3 periodic solutions expressed in terms of the Weierstrass $\wp$-function that we have shown to be bounded over the real axis. They represent bumps, kinks and multiple-wave solutions, all of which depend on Riemann invariants. These solutions remain bounded even when the invariants admit the gradient catastrophe.

Among the questions that one may ask is what role do exact analytical solutions play in the physical interpretation. One possible response is that such solutions may display qualitative behaviour which would otherwise be difficult to detect numerically or by approximations. For example, this is true of the doubly periodic properties of certain solutions expressed in terms of the Weierstrass $\wp$-function.

One could also inquire about the stability property of the obtained solutions. Indeed, solutions which possess the property of stability should be observable physically and such analysis could be the starting point for perturbative computations. This task will be undertaken in a future work.

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No	Type	Vector Fields	Riemann Invariants	Solutions
1	E_1S_1	$X_1 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1}\frac{\partial}{\partial x^1}$	$r^1 = ((1+k)\bar{a}_1(r^1) + C_2)t - \vec{e}^1 \cdot \vec{x}$	$\bar{a} = \bar{a}_1(r^1) + a_0, \quad [\vec{u}_2, \vec{e}^2, \vec{m}^2] = C$
		$X_2 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1}\frac{\partial}{\partial x^1}$	$r^2 = Ct - [\vec{x}, \vec{e}^2, \vec{m}^2], \quad [\vec{e}^1, \vec{e}^2, \vec{m}^2] = 0$	$\vec{u} = k\bar{a}_1(r^1) + \vec{u}_2(r^2), \quad \bar{u}_2^3(r^2) = C_1\bar{u}_2^1(r^2)$
		$\beta_i = -(\vec{e}^2 \times \vec{m}^2)_i(a + \vec{e}^1 \cdot \vec{u}) + e_i^1[\vec{u}, \vec{e}^2, \vec{m}^2]$	$C_2 = (C_1 e_1^1 - e_3^1)^{-1}$	$a_0, C, C_1, C_2 \in \mathbb{R}$
		$\sigma_j = -e_1^1(\vec{e}^2 \times \vec{m}^2)_j + e_j^1(\vec{e}^2 \times \vec{m}^2)_1, j = 2, 3$		
$2a$	S_1S_2	$X_1 = \frac{\partial}{\partial t} + u^1\frac{\partial}{\partial x^1} + u^2\frac{\partial}{\partial x^2}$	$r^1 = x^1 - u^1 t$	$\bar{a} = a_0, \quad \bar{u}^1 = -\phi_{r^2}, \quad \bar{u}^2 = \phi_{r^1},$
		$X_2 = \frac{\partial}{\partial x^3}$	$r^2 = x^2 - u^2 t$	$\phi = \varphi(\alpha_1 r^1 + \alpha_2 r^2) + \beta_1 r^1 + \beta_2 r^2 + \gamma,$
				$\bar{u}^3 = \bar{u}^3(r^1, r^2), \quad a_0, \alpha_i, \beta_i, \gamma \in \mathbb{R}, i = 1, 2,$
$2b$	S_1S_2	$X_1 = \frac{\partial}{\partial t} + u^1\frac{\partial}{\partial x^1} + u^2\frac{\partial}{\partial x^2}$	$r^1 = x^1 - u^1 t$	$\bar{a} = a_0, \quad \bar{u}^2 = \bar{u}^3 = g(x^1 - x^2), \quad a_0 \in \mathbb{R},$
		$X_2 = \frac{\partial}{\partial x^3}$	$r^2 = x^2 - u^2 t$	$\bar{u}^1 = b(x^1 - tg(x^1 - x^2), x^2 - tg(x^1 - x^2))$
$2c$	S_1S_2	$X_2 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1}\frac{\partial}{\partial x^1}$	$r^1 = \left(C_1 + \frac{\lambda_1^1}{\lambda_1^2}C_2\right)t - \vec{\lambda}^1 \cdot \vec{x}$	$\bar{a} = a_0, \quad a_0, C_1, C_2 \in \mathbb{R}$
		$X_3 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1}\frac{\partial}{\partial x^1}$	$r^2 = \left(C_2 + \frac{\lambda_1^2}{\lambda_1^1}C_1 + G(r^1)\right)t - \vec{\lambda}^2 \cdot \vec{x}$	$\bar{u}^1 = \frac{1}{\lambda_1^1}(C_1 - \lambda_2^1\bar{u}_1^2(r^1) - \lambda_3^1\bar{u}_1^3(r^1))$
		$\beta_j = \lambda_j^2[\vec{u}, \vec{e}^1, \vec{m}^1] - \lambda_j^1[\vec{u}, \vec{e}^2, \vec{m}^2]$	$\lambda_i^j = -(\vec{e}^j \times \vec{m}^j)_i$	$- \left(\frac{\lambda_3^2}{\lambda_1^2}\eta + \frac{\lambda_2^2}{\lambda_1^2}\right)\bar{u}_2^2(r^2) + \frac{C_2}{\lambda_1^2}$
		$\sigma_i = \lambda_1^1\lambda_i^2 - \lambda_i^1\lambda_1^2$	$G(r^1) = \frac{1}{\lambda_1^1}\left((\lambda_1^1\lambda_2^2 - \lambda_2^1\lambda_1^2)\bar{u}_1^2(r^1)\right.$	$\bar{u}^2 = \bar{u}_1^2(r^1) + \bar{u}_2^2(r^2)$
			$\left.+ (\lambda_1^1\lambda_3^2 - \lambda_1^2\lambda_3^1)\bar{u}_1^3(r^1)\right)$	$\bar{u}^3 = \bar{u}_1^3(r^1) + \eta\bar{u}_2^2(r^2), \quad \eta = \frac{\lambda_1^2\lambda_2^1 - \lambda_1^1\lambda_2^2}{\lambda_1^1\lambda_3^2 - \lambda_3^1\lambda_1^2}$
3	$E_1E_2S_1$	$X = \frac{\partial}{\partial x^3} - \frac{\sigma_1}{\beta_{12}}\frac{\partial}{\partial t} + \frac{\beta_{23}}{\beta_{12}}\frac{\partial}{\partial x^1} + \frac{\beta_{31}}{\beta_{12}}\frac{\partial}{\partial x^2}$	$r^1 = \frac{\beta\bar{u}_3^1(r^3)t - e_1^1x^1 - e_2^1x^2}{1-\alpha(1+\kappa)t}$	$\bar{a} = \frac{\alpha((e_1^1+e_1^2)x^1 + (e_2^1+e_2^2)x^2)}{1-\alpha(1+\kappa)t}, \quad \bar{u}^3 = u_0^3$
		$\sigma_1 = \varepsilon_{ijk}\, e_i^1 e_j^2(\vec{e}^3 \times \vec{m})_k$	$r^2 = \frac{-\beta\bar{u}_3^1(r^3)t - e_1^2x^1 - e_2^2x^2}{1-\alpha(1+\kappa)t}$	$\bar{u}^1 = \frac{-\kappa\alpha\left(((e_1^1)^2+(e_1^2)^2)x^1 + (e_1^1e_2^1 + e_1^2e_2^2)x^2\right) - \bar{u}_3^1(r^3)}{1-\alpha(1+\kappa)t}$
		$\beta_{ij} = (e_j^1e_i^2 - e_i^1e_j^2)[\vec{u}, \vec{e}^3, \vec{m}^3]$	$r^3 = x^3 - u_0^3 t$	$\bar{u}^2 = \kappa\alpha\left(\frac{e_2^1\left(\beta\bar{u}_3^1(r^3)t - e_1^1x^1 - e_2^1x^2\right)}{1-\alpha(1+\kappa)t}\right.$
		$+(e_j^2(\vec{e}^3 \times \vec{m}^3)_i - e_i^2(\vec{e}^3 \times \vec{m}^3)_j)(a + \vec{e}^1 \cdot \vec{u})$	$\beta = (1+\kappa^{-1})/(e_1^1 - e_1^2)$	$\left.+ \frac{e_2^2\left(-\beta\bar{u}_3^1(r^3)t - e_1^2x^1 - e_2^2x^2\right)}{1-\alpha(1+\kappa)t}\right) + \frac{e_2^2 - e_2^1}{e_1^2 - e_1^1}\bar{u}_3^1(r^3)$
		$+(e_i^1(\vec{e}^3 \times \vec{m}^3)_j - e_j^1(\vec{e}^3 \times \vec{m}^3)_i)(a + \vec{e}^2 \cdot \vec{u})$		$\alpha, u_0^3 \in \mathbb{R}$

TAB. 3.1. Rank-2 solutions with the freedom of one, two or three arbitrary functions of one or two variables. Unassigned unknown functions $a(\cdot), u(\cdot), \ldots$ are arbitrary functions of their respective arguments.

No	Type	Vector Fields	Riemann Invariants	Solutions
1	$E_1E_2E_3$	$X_1 = \frac{\partial}{\partial x^3} + \frac{\sigma_1}{\beta_3}\frac{\partial}{\partial t} + \frac{\beta_1}{\beta_3}\frac{\partial}{\partial x^1} + \frac{\beta_2}{\beta_3}\frac{\partial}{\partial x^2}$	$r^i = (1+\kappa)a_i(r^i)t - \vec{e}^i \cdot \vec{x},\ i = 1,2,3$	$\bar{a} = \bar{a}_1(r^1) + \bar{a}_2(r^2) + \bar{a}_3(r^3)$
		$\sigma_1 = -[\vec{e}^1, \vec{e}^2, \vec{e}^3]$	$\vec{e}^i \cdot \vec{e}^j = -1/\kappa,\ i \neq j = 1,2,3$	$\vec{u} = \kappa(\vec{e}^1\bar{a}_1(r^1) + \vec{e}^2\bar{a}_2(r^2) + \vec{e}^3\bar{a}_3(r^3))$
		$\beta_i = (\vec{e}^2 \times \vec{e}^3)_i(a + \vec{e}^1 \cdot \vec{u})$		
		$+(\vec{e}^1 \times \vec{e}^3)_i(a + \vec{e}^2 \cdot \vec{u})$		
		$+(\vec{e}^1 \times \vec{e}^2)_i(a + \vec{e}^3 \cdot \vec{u})$		
$2a$	$E_1S_1S_2$	$X = e_1^2\frac{\partial}{\partial x^1} + e_2^2\frac{\partial}{\partial x^2}$	$r^1 = ((1+k^{-1})f(r^1) + a_0 + u_0^3)t - x^3$	$\bar{a} = k^{-1}f(r^1) + a_0, \quad \bar{u}^1 = \sin g(r^2, r^3)$
			$r^2 = t - x^1 \sin g(r^2, r^3) + x^2 \cos g(r^2, r^3)$	$\bar{u}^2 = -\cos g(r^2, r^3), \quad \bar{u}^3 = f(r^1) + u_0^3$
			$\frac{\partial r^3}{\partial t} + (f(r^1) + u_0^3)\frac{\partial r^3}{\partial x^3} = 0$	$a_0, u_0^3 \in \mathbb{R}$
$2b$	$E_1S_1S_2$	$X = e_1^2\frac{\partial}{\partial x^1} + e_2^2\frac{\partial}{\partial x^2}$	$r^1 = \frac{((1+k^{-1})B + a_0 + u_0^3)t - x^3}{1-(1+k^{-1})At}$	$\bar{a} = k^{-1}(Ar^1 + B) + a_0,$
			$r^2 = t - x^1 \sin g(r^2, r^3) + x^2 \cos g(r^2, r^3)$	$\bar{u}^1 = \sin g(r^2, r^3), \bar{u}^2 = -\cos g(r^2, r^3)$
			$r^3 = \Psi\left[\frac{1}{A}(A(ka_0 - u_0^3)t + x^3 - ka_0 - B)((1+k)At - k)^{-k/k+1}\right]$	$\bar{u}^3 = Ar^1 + B + u_0^3, \quad a_0, u_0^3 \in \mathbb{R}$
$2c$	$E_1S_1S_2$	$X = \frac{\partial}{\partial x^3}$	$r^1 = (k^{-1}f(r^1) + a_0)t - x^1 \cos f(r^1) - x^2 \sin f(r^1)$	$\bar{a} = k^{-1}f(r^1) + a_0, \quad \bar{u}^1 = \sin f(r^1)$
			$r^2 = -t\cos f(r^1) - x^2$	$\bar{u}^2 = -\cos f(r^1), \quad a_0 \in \mathbb{R}$
			$r^3 = -t\sin f(r^1) + x^1$	$\bar{u}^3 = g(r^2 \cos f(r^1) + r^3 \sin f(r^1))$

TAB. 3.2. Rank-3 solutions. Unassigned unknown functions $a(\cdot), u(\cdot), \ldots$ are arbitrary functions of their respective arguments.

No	Riemann invariants	Solution	Type and comments
1	$r^i=-(1+\kappa)\frac{C_i}{\left(\wp\left(r^i,\frac{4}{3},\frac{8}{27}+\frac{4}{3}C_i^4\right)+\frac{1}{3}\right)^{1/2}}t+\vec{\lambda}^i\cdot\vec{x}$	$a=\sum_{i=1}^{3}\frac{C_i}{\left(\wp\left(r^i,\frac{4}{3},\frac{8}{27}+\frac{4}{3}C_i^4\right)+\frac{1}{3}\right)^{1/2}}$	Periodic solution
		$\vec{u}=\kappa\sum_{i=1}^{3}\frac{C_i\vec{\lambda}^i}{\left(\wp\left(r^i,\frac{4}{3},\frac{8}{27}+\frac{4}{3}C_i^4\right)+\frac{1}{3}\right)^{1/2}}$	$C_i\in\mathbb{R}$
$2a$	$r^i=-(1+\kappa)\left(\frac{C_i}{\wp\left(r^i,0,\frac{4C_i^2}{3}\right)}\right)^{1/2}t+\vec{\lambda}^i\cdot\vec{x}$	$a=\sum_{i=1}^{3}\left(\frac{C_i}{\wp\left(r^i,0,\frac{4C_i^2}{3}\right)}\right)^{1/2},\vec{u}=\kappa\sum_{i=1}^{3}\left(\frac{C_i}{\wp\left(r^i,0,\frac{4C_i^2}{3}\right)}\right)^{1/2}\vec{\lambda}^i$	Periodic Solution
			$C_i>0$
$2b$	$r^i=-(1+\kappa)\left(\frac{C_i(r^i)^{-2/3}}{\wp(\zeta_i,0,\frac{4C_i^2}{3k_0^2})}\right)^{1/2}t+\vec{\lambda}^i\cdot\vec{x}$	$a=\sum_{i=1}^{3}\left(\frac{C_i(r^i)^{-2/3}}{\wp(\zeta_i,0,\frac{4C_i^2}{3k_0^2})}\right)^{1/2},\vec{u}=\kappa\sum_{i=1}^{3}\left(\frac{C_i(r^i)^{-2/3}}{\wp(\zeta_i,0,\frac{4C_i^2}{3k_0^2})}\right)^{1/2}\vec{\lambda}^i$	Bump
	$\zeta_i=3k_0(r^i)^{1/3}$		$k_0\in\mathbb{R},C_i>0$
$2c$	$r^i=-(1+\kappa)\left(\frac{C_i(r^i)^{-1}}{(\wp(\zeta_i,12e_0^2,-8e_0^3+16C_i^2e_0)-e_0)}\right)^{1/2}t+\vec{\lambda}^i\cdot\vec{x}$	$a=\sum_{i=1}^{3}\left(\frac{C_i(r^i)^{-1}}{(\wp(\zeta_i,12e_0^2,-8e_0^3+16C_i^2e_0)-e_0)}\right)^{1/2}$	Bump
	$\zeta_i=k_0\ln r^i$	$\vec{u}=\sum_{i=1}^{3}\kappa\left(\frac{C_i(r^i)^{-1}}{(\wp(\zeta_i,12e_0^2,-8e_0^3+16C_i^2e_0)-e_0)}\right)^{1/2}\vec{\lambda}^i$	$e_0,\in\mathbb{R},C_i>0$
3	$r^i=-(1+\kappa)\left(\frac{C_i((r^i)^2+1)^{-1/3}}{\wp\left(\zeta_i,0,\frac{4C_i^2}{3k_0^2}\right)}\right)^{1/2}t+\vec{\lambda}^i\cdot\vec{x}$	$a=\sum_{i=1}^{3}\left(\frac{C_i((r^i)^2+1)^{-1/3}}{\wp\left(\zeta_i,0,\frac{4C_i^2}{3k_0^2}\right)}\right)^{1/2}$	Kink
	$\zeta_i=r^i\ {}_2F_1\left(\frac{1}{2},\frac{5}{6};\frac{3}{2};-(r^i)^2\right)$	$\vec{u}=\kappa\sum_{i=1}^{3}\left(\frac{C_i((r^i)^2+1)^{-1/3}}{\wp\left(\zeta_i,0,\frac{4C_i^2}{3k_0^2}\right)}\right)^{1/2}\vec{\lambda}^i$	$k_0,\in\mathbb{R},C_i>0$

TAB. 3.3. Real solutions for the nonscattering solution $E_1E_2E_3$ which remain bounded for some choices of the arbitrary constants. They are obtained by submitting the arbitrary functions to the various reductions (3.2.5)-(3.2.8) of the Klein-Gordon equation (3.2.1).

Chapitre 4

CONDITIONALLY INVARIANT SOLUTIONS OF THE ROTATING SHALLOW WATER WAVE EQUATIONS

Référence complète : B. Huard, Conditionally invariant solutions of the rotating shallow water wave equations, *Journal of Physics A : Mathematical and Theoretical*, 43(23) : 235205, 2010.

Résumé

Cet article est consacré à l'extension de la méthode des symétries conditionnelles récemment proposée aux systèmes non homogènes pouvant être amenés à une forme homogène sous une transformation ponctuelle locale et inversible. Nous effectuons une analyse systématique des solutions de rang 1 et 2 admises par les équations homogènes d'un flot en eau peu profonde et nous obtenons les solutions associées pour le système rotatif non homogène. Ces solutions dépendent en général de fonctions arbitraires, ce qui nous permet de construire de nouvelles classes de solutions intéressantes.

Abstract

This paper is devoted to the extension of the recently proposed conditional symmetry method to first order nonhomogeneous quasilinear systems which are equivalent to homogeneous systems through a locally invertible point transformation. We perform a systematic analysis of the rank-1 and rank-2 solutions admitted by

the shallow water wave equations in $(2+1)$ dimensions and construct the corresponding solutions of the rotating shallow water wave equations. These solutions involve in general arbitrary functions depending on Riemann invariants, which allow us to construct new interesting classes of solutions.

4.1. Introduction

In this paper, we use the conditional symmetry method in the context of Riemann invariants (**CSM**) as presented in [**48**] to obtain conditionally invariant solutions of the rotating shallow water wave (**RSWW**) equations with a flat bottom topography [**98**]

$$\Delta(\mathbf{x},\mathbf{u}):\begin{cases} u_t+uu_x+vu_y+gh_x=2\Omega v, \quad \Omega\in\mathbb{R},\\ v_t+uv_x+vv_y+gh_y=-2\Omega u,\\ h_t+uh_x+vh_y+h(u_x+v_y)=0,\end{cases} \tag{4.1.1}$$

where we denote by $\mathbf{x}=(t,x,y)$ and $\mathbf{u}=(u,v,h)$ the independent and dependent variables respectively. Here, u and v stand for the velocity vector fields, h represents the height of the fluid layer, g is the gravitational constant and Ω characterizes the constant angular velocity of the fluid around the z-axis induced by a Coriolis force. It can be proved using the chain rule, see [**15**], that if a set of functions $u'(t',x',y'), v'(t',x',y'), h'(t',x',y')$ satisfies the irrotational shallow water wave equations (**SWW**)

$$\Delta'(\mathbf{x},\mathbf{u}):\begin{cases} {u'}_{t'}+u'{u'}_{x'}+v'{u'}_{y'}+g{h'}_{x'}=0,\\ {v'}_{t'}+u'{v'}_{x'}+v'{v'}_{y'}+g{h'}_{y'}=0,\\ {h'}_{t'}+u'{h'}_{x'}+v'{h'}_{y'}+h'({u'}_{x'}+{v'}_{y'})=0,\end{cases} \tag{4.1.2}$$

then the functions $u(t,x,y), v(t,x,y), h(t,x,y)$ defined by

$$\begin{aligned} &t'=-\frac{1}{2\Omega}\cot(\Omega t), \quad x'=\frac{1}{2}\left(y-x\cot(\Omega t)\right), \quad y'=-\frac{1}{2}\left(x+y\cot(\Omega t)\right),\\ &u'=-\frac{1}{2}\big(u\sin(2\Omega t)-v(1-\cos(2\Omega t))-2\Omega x\big),\\ &v'=-\frac{1}{2}\big(u(1-\cos(2\Omega t))+v\sin(2\Omega t)-2\Omega y\big), \quad h'=\frac{h}{2}(1-\cos(2\Omega t)),\end{aligned} \tag{4.1.3}$$

form a solution of the RSWW equations.

The task of constructing invariant solutions of systems (4.1.1) and (4.1.2) using the classical Lie approach was undertaken by several authors. A systematic classification of the subalgebras of the symmetry algebra of the equations describing a rotating shallow water flow in a rigid ellipsoidal bassin was performed in [**69**] and many invariant solutions were obtained. In [**15**], the author introduced the transformation (4.1.3) to generate invariant solutions of (4.1.1) from known invariant solutions of the homogeneous system (4.1.2), previously computed in [**14**] .

The CSM approach to be used in this paper was developed progressively and applied in [**18, 47, 48**] in order to construct rank-2 and rank-3 solutions to the equations governing the flow of an isentropic fluid. The main feature of this approach, which proved to be less restrictive than the generalized method of characteristics [**48**], is that the obtained rank-k solutions can depend on many arbitrary functions of many independent variables, called Riemann invariants. Through a judicious selection of these arbitrary functions, it is possible to construct solutions of the considered homogeneous system which are bounded everywhere, even when the Riemann invariants admit a gradient catastrophe [**18**]. Although the applicability of the CSM approach is technically restricted to first order homogenous hyperbolic quasilinear systems, the objective of the present paper is to apply it to the RSWW equations (4.1.1) through the transformation (4.1.3). Large classes of implicit rank-k solutions are then constructed for the SWW and RSWW equations, including bumps, kinks and periodic solutions.

The paper is organized as follows. We give in Section 4.2 the symmetry algebra of system (4.1.1) and construct the point transformation (4.1.3) relating systems (4.1.1) and (4.1.2). Section 4.3 contains a brief review of the conditional symmetry method in the context of Riemann invariants for homogeneous systems and we present many interesting rank-1 and rank-2 solutions to the SWW-equations (4.1.2) together with corresponding solutions to the RSWW equations (4.1.1). Results and perspectives are summarized in Section 4.4.

The classical Lie symmetry algebra admitted by system (4.1.1) is generated by vector fields of the form

$$X = \xi^1(\mathbf{x},\mathbf{u})\partial_t + \xi^2(\mathbf{x},\mathbf{u})\partial_x + \xi^3(\mathbf{x},\mathbf{u})\partial_y + \eta^1(\mathbf{x},\mathbf{u})\partial_u + \eta^2(\mathbf{x},\mathbf{u})\partial_v + \eta^3(\mathbf{x},\mathbf{u})\partial_h. \tag{4.2.1}$$

The requirement that the generator (4.2.1) leave system (4.1.1) invariant yields an overdetermined system of linear equations for the functions $\xi^i(\mathbf{x},\mathbf{u})$ and $\eta^i(\mathbf{x},\mathbf{u})$, $i = 1,2,3$ [**89**]. Since this step is completely algorithmic and involves tedious computations, many computer programs have been designed to derive these determining equations, see [**58**] for a complete review. The package *symmgrp2009.max* [**12, 59**] for the computer algebra system *Maxima* has been used in this work to obtain the determining equations of the RSWW equations (4.1.1) and solve them partially in a recursive way. Solving them shows that the Lie algebra $\mathcal{L}$ of point symmetries of the RSWW equations (4.1.1) is nine-dimensional and is generated by the following differential generators

$$\begin{aligned}
&P_0 = \partial_t, \quad P_1 = \partial_x, \quad P_2 = \partial_y, \quad L = y\partial_x - x\partial_y + v\partial_u - u\partial_v,\\
&G_1 = -\frac{1}{2\Omega}\cos(2\Omega t)\partial_x + \frac{1}{2\Omega}\sin(2\Omega t)\partial_y + \sin(2\Omega t)\partial_u + \cos(2\Omega t)\partial_v,\\
&G_2 = \frac{1}{2\Omega}\sin(2\Omega t)\partial_x + \frac{1}{2\Omega}\cos(2\Omega t)\partial_y + \cos(2\Omega t)\partial_u - \sin(2\Omega t)\partial_v,\\
&D = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2h\partial_h,\\
&Z_1 = \sin(2\Omega t)\partial_t + \Omega\left[x\cos(2\Omega t) + y\sin(2\Omega t)\right]\partial_x + \Omega\left[y\cos(2\Omega t) - x\sin(2\Omega t)\right]\partial_y\\
&\qquad + \Omega\left[(2\Omega y - u)\cos(2\Omega t) - (2\Omega x - v)\sin(2\Omega t)\right]\partial_u\\
&\qquad - \Omega\left[(2\Omega x + v)\cos(2\Omega t) + (2\Omega y + u)\sin(2\Omega t)\right]\partial_v - 2\Omega h\cos(2\Omega t)\partial_h,\\
&Z_2 = \cos(2\Omega t)\partial_t + \Omega\left[y\cos(2\Omega t) - x\sin(2\Omega t)\right]\partial_x - \Omega\left[x\cos(2\Omega t) + y\sin(2\Omega t)\right]\partial_y\\
&\qquad - \Omega\left[(2\Omega y - u)\sin(2\Omega t) + (2\Omega x - v)\cos(2\Omega t)\right]\partial_u\\
&\qquad + \Omega\left[(2\Omega x + v)\sin(2\Omega t) - (2\Omega y + u)\cos(2\Omega t)\right]\partial_v + 2\Omega h\sin(2\Omega t)\partial_h.
\end{aligned} \tag{4.2.2}$$

The geometrical interpretation of these generators is as follows. The system (4.1.1) is left invariant by translations P_0, P_1, P_2 in the space of independent variables since it is autonomous. The element L generates a rotation of the whole coordinate system while G_1 and G_2 represent helical rotations. The system is also left invariant by the dilation D and the two conformal transformations Z_1 and Z_2. The Levi decomposition $\mathcal{L} = F \ni N$ of the symmetry algebra $\mathcal{L}$ can be exhibited by considering its commutation table (Table 4.1) in the following basis

$$\begin{aligned} &Y_1 = P_2 - 2\Omega G_2, \quad Y_2 = -(P_1 + 2\Omega G_1), \quad Y_3 = P_1 - 2\Omega G_1, \quad Y_4 = P_2 + 2\Omega G_2, \\ &Y_5 = -L, \quad Y_6 = D, \quad Y_7 = P_0 - \Omega L - Z_2, \quad Y_8 = P_0 - \Omega L + Z_2, \quad Y_9 = -\frac{1}{\Omega} Z_1. \end{aligned} \tag{4.2.3}$$

Here $F = \{Y_1, Y_2, Y_3, Y_4, Y_5, Y_6\}$ is a maximal solvable ideal and $N = \{Y_7, Y_8, Y_9\}$ is isomorphic to the simple Lie algebra $su(1, 1)$. Following the procedure presented in **[24, 25]**, we introduce a set of canonical variables associated with the abelian subalgebra $\{Y_1, Y_2, Y_7\}$ and defined by

$$\begin{aligned} &Y_7 t' = 1, && Y_1 t' = 0, && Y_2 t' = 0, \\ &Y_7 x' = 0, && Y_1 x' = 1, && Y_2 x' = 0, \\ &Y_7 y' = 0, && Y_1 y' = 0, && Y_2 y' = 1, \\ &Y_7 u' = Y_7 v' = Y_7 h' = && Y_1 u' = Y_1 v' = Y_1 h' = && Y_2 u' = Y_2 v' = Y_2 h' = 0, \end{aligned} \tag{4.2.4}$$

to bring system (4.1.1) into an equivalent autonomous form. It turns out that the set of variables (4.1.3) satisfies system (4.2.4) so that when expressed in these variables, the vector fields Y_1, Y_2, Y_7 are rectified to the canonical form

$$Y_7 = \partial_{t'}, \quad Y_1 = \partial_{x'}, \quad Y_2 = \partial_{y'}.$$

Moreover, using the chain rule, it is easily found that system (4.1.1) transforms to

$$\begin{aligned} &u'_{t'} + u'u'_{x'} + v'u'_{y'} + gh'_{x'} = 0, \\ &v'_{t'} + u'v'_{x'} + v'v'_{y'} + gh'_{y'} = 0, \\ &h'_{t'} + u'h'_{x'} + v'h'_{y'} + h'(u'_{x'} + v'_{y'}) = 0, \end{aligned}$$

which shows the equivalence between systems (4.1.1) and (4.1.2). The next section demonstrates how the point transformation (4.1.3) can be used to construct implicit solutions of equations (4.1.1) expressed in terms of Riemann invariants.

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9
Y_1	0	0	0	0	$-Y_2$	$-Y_1$	0	$-2\Omega Y_3$	$-Y_1$
Y_2		0	0	0	Y_1	$-Y_2$	0	$-2\Omega Y_4$	$-Y_2$
Y_3			0	0	$-Y_4$	$-Y_3$	$2\Omega Y_1$	0	Y_3
Y_4				0	Y_3	$-Y_4$	$2\Omega Y_2$	0	Y_4
Y_5					0	0	0	0	0
Y_6						0	0	0	0
Y_7							0	$-4\Omega^2 Y_9$	$-2Y_7$
Y_8								0	$2Y_8$
Y_9									0

TAB. 4.1. Commutation relations for the Lie symmetry algebra of the RSWW equations.

4.3. Conditionally invariant solutions of the SWW and RSWW equations

We present in this section a brief description of the CSM approach developed progressively in [**47**] and [**48**] and obtain several rank-1 and rank-2 solutions of the SWW equations in closed form. We illustrate the process of construction of the corresponding solutions for the RSWW equations with several interesting examples. The SWW equations (4.1.2) can be written in matrix evolutionary form as

$$\mathbf{u}_t + a^1(\mathbf{u})\mathbf{u}_x + a^2(\mathbf{u})\mathbf{u}_y = 0, \tag{4.3.1}$$

where a^1, a^2 are 3×3 matrix functions given by

$$a^1 = \begin{pmatrix} u & 0 & g \\ 0 & u & 0 \\ h & 0 & u \end{pmatrix}, \quad a^2 = \begin{pmatrix} v & 0 & 0 \\ 0 & v & g \\ 0 & h & v \end{pmatrix}.$$

The objective is to construct rank-k solutions, $k = 1, 2$, of system (4.3.1) expressible in terms of Riemann invariants. To this end, we look for solutions of (4.3.1) defined implicitly by the relations

$$\begin{aligned} &\mathbf{u} = \mathbf{f}(r^1(\mathbf{x},\mathbf{u}),\dots,r^k(\mathbf{x},\mathbf{u})), \quad r^A(\mathbf{x},\mathbf{u}) = \lambda_i^A(\mathbf{u})x^i, \\ &\det\left(\lambda_0^A\mathcal{I}_3 + a^1(\mathbf{u})\lambda_1^A + a^2(\mathbf{u})\lambda_2^A\right) = 0, \quad A = 1,\dots,k, \end{aligned} \tag{4.3.2}$$

for some function $f : \mathbb{R}^k \to \mathbb{R}^3$, where $\mathcal{I}_3$ is the 3 by 3 identity matrix. A solution of the form (4.3.2) will be called a rank-k solution if $\text{rank}(\partial u) = k$ in some open set $\mathcal{D} \subset \mathbb{R}^3$ around the origin, where ∂u stands for the Jacobian matrix of $\mathbf{u}$ in the original variables. The functions $r^A(\mathbf{x},\mathbf{u})$ are called the Riemann invariants associated with the linearly independent wave vectors $\lambda^A = (\lambda_0^A, \vec{\lambda}^A) = (\lambda_0^A, \lambda_1^A, \lambda_2^A)$, which are obtained by solving the dispersion relation of equation (4.3.1) for the phase velocity λ_0. This relation takes the form

$$\begin{aligned} &\det\left(\lambda_0\mathcal{I}_3 + a^1(\mathbf{u})\lambda_1 + a^2(\mathbf{u})\lambda_2\right) \\ &= (\lambda_0 + \lambda_1 u + \lambda_2 v)(\lambda_0 + \lambda_1 u + \lambda_2 v + \sqrt{gh})(\lambda_0 + \lambda_1 u + \lambda_2 v - \sqrt{gh}) = 0. \end{aligned} \tag{4.3.3}$$

The wave vectors are thus of the entropic (E) and acoustic (S) type defined respectively by

$$\begin{aligned} &\text{i) } \lambda^E = (-\lambda_1 u - \lambda_2 v, \lambda_1, \lambda_2), \\ &\text{ii) } \lambda^{S_\varepsilon} = (-(\lambda_1 u + \lambda_2 v + \varepsilon\sqrt{gh}), \lambda_1, \lambda_2), \quad |\vec{\lambda}|^2 = {\lambda_1}^2 + {\lambda_2}^2 = 1, \quad \varepsilon = \pm 1. \end{aligned} \tag{4.3.4}$$

We associate to each of them the corresponding Riemann invariant

$$\begin{aligned} &\text{i) } r^E = -(\lambda_1 u + \lambda_2 v)t + \lambda_1 x + \lambda_2 y, \\ &\text{ii) } r^{S_\varepsilon} = -(\lambda_1 u + \lambda_2 v + \varepsilon\sqrt{gh})t + \lambda_1 x + \lambda_2 y, \quad |\vec{\lambda}|^2 = 1, \quad \varepsilon = \pm 1. \end{aligned} \tag{4.3.5}$$

The analysis of rank-k solutions for the cases $\varepsilon = \pm 1$ are very similar, hence we restrict ourselves to the positive case.

It is convenient when studying solutions of type (4.3.2) to write system (4.3.1) in the form of a trace equation,

$$\text{Tr}\left[\mathcal{A}^\mu(\mathbf{u})\partial u\right] = 0, \quad \mu = 1,\dots,l, \tag{4.3.6}$$

where $\mathcal{A}^\mu(\mathbf{u})$ are now 3×3 matrix functions of $\mathbf{u}$, defined by

$$\mathcal{A}^1 = \begin{pmatrix} 1 & 0 & 0 \\ u & 0 & g \\ v & 0 & 0 \end{pmatrix}, \quad \mathcal{A}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & u & 0 \\ 0 & v & g \end{pmatrix}, \quad \mathcal{A}^3 = \begin{pmatrix} 0 & 0 & 1 \\ h & 0 & u \\ 0 & h & v \end{pmatrix}.$$

The construction of rank-k solutions through the conditional symmetry method is achieved by considering an overdetermined system, consisting of the original system (4.3.1) together with a set of compatible first order differential constraints (DCs),

$$\xi_a^i(\mathbf{u})u_i^\alpha = 0, \quad \lambda_i^A(\mathbf{u})\xi_a^i(\mathbf{u}) = 0, \quad a = 1, \dots, 3-k, \quad A = 1, \dots, k, \tag{4.3.7}$$

for which a symmetry criterion is automatically satisfied. Here and throughout this work, we use the summation convention over repeated indices. Introducing the functions

$$\begin{aligned} &\bar{x}^1 = r^1(\mathbf{x}, \mathbf{u}), \ \dots, \ \bar{x}^k = r^k(\mathbf{x}, \mathbf{u}), \ \bar{x}^{k+1} = x^{k+1}, \ \dots \\ &\bar{u} = u, \ \bar{v} = v, \ \bar{h} = h, \end{aligned} \tag{4.3.8}$$

as new coordinates on $\mathbb{R}^3 \times \mathbb{R}^3$ space, the Jacobi matrix ∂u now reads

$$\partial u = \frac{\partial f}{\partial r}\left(\mathcal{I}_k - (\eta_0 t + \eta_1 x + \eta_2 y)\frac{\partial f}{\partial r}\right)^{-1}\lambda, \tag{4.3.9}$$

where

$$\begin{aligned} &\lambda = (\lambda_i^A) \in \mathbb{R}^{k\times 3}, \quad r = (r^1, \dots, r^k) \in \mathbb{R}^k, \quad \frac{\partial f}{\partial r} = \left(\frac{\partial f^\alpha}{\partial r^A}\right) \in \mathbb{R}^{3\times k}, \\ &\eta_a = \left(\frac{\partial \lambda_a^A}{\partial u^\alpha}\right) \in \mathbb{R}^{k\times 3}, \quad a = 0, \dots, 2, \end{aligned} \tag{4.3.10}$$

so that system (4.3.6) is now expressed as

$$\operatorname{Tr}\left[\mathcal{A}^\mu(\mathbf{u})\frac{\partial f}{\partial r}\left(\mathcal{I}_k - (\eta_0 t + \eta_1 x + \eta_2 y)\frac{\partial f}{\partial r}\right)^{-1}\lambda\right] = 0, \quad \mu = 1, \dots, l. \tag{4.3.11}$$

Requiring that system (4.3.11) be satisfied for all values of the coordinates (t, x, y), the following result holds (see **[48]** for a general statement and a detailed proof).

Proposition. *The nondegenerate quasilinear hyperbolic system of first order PDEs (4.3.1) admits a* $(3-k)$*-dimensional conditional symmetry algebra* L*,* $k \leq 2$*, if and only if there exists a set of* $(3-k)$ *linearly independent vector fields*

$$X_a = \xi_a^i(u)\frac{\partial}{\partial x^i}, \quad a = 1, \dots, 3-k, \quad \det\left(a^i(u)\lambda_i^A\right) = 0, \quad \lambda_i^A \xi_a^i = 0, \quad A = 1, \dots, k,$$

which satisfy, on some neighborhood of $(x_0, u_0) \in X \times U$*, the trace conditions*

$$k = 1: \quad \text{i)} \quad \operatorname{tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r}\lambda\right) = 0, \quad \mu = 1, \dots, 3, \tag{4.3.12}$$

$$k = 2: \quad \text{i)} \quad \operatorname{tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r}\lambda\right) = 0, \quad \text{ii)} \quad \operatorname{tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_a \frac{\partial f}{\partial r}\lambda\right) = 0, \quad a = 0, 1, 2, \tag{4.3.13}$$

where the relevant matrices are defined in (4.3.10). Solutions of the system which are invariant under the Lie algebra L *are precisely rank-*k *solutions of the form (4.3.2).*

Note that the vector fields X_a, $a = 1, \dots, 3-k$, are not symmetries of the original system. Nevertheless, as we will show, they can be used to build solutions of the overdetermined system composed of (4.3.1) and the differential constraints (4.3.7).

To construct solutions of the RSWW equations, we assume that a solution of the SWW equations (4.1.2)

$$u = u(\mathbf{r}),\ v = v(\mathbf{r}),\ h = h(\mathbf{r}) \quad \mathbf{r} = (r^1, \dots, r^k),$$

has been obtained from equations (4.3.12) or (4.3.13). Then the Riemann invariants r^A can be expressed as a graph

$$r^A = r^A(\mathbf{x}, \mathbf{u}) = r^A(\mathbf{x}, \Phi(\mathbf{r})) \tag{4.3.14}$$

in the $(\mathbf{r}, \mathbf{x})$ space for some function $\Phi : \mathbb{R}^k \to \mathbb{R}^q$. The change of variables (4.1.3) induces a transformation of the independent variables in this space,

$$t \to -\frac{1}{2\Omega}\cot(\Omega t), \quad x \to \frac{1}{2}(y - x\cot(\Omega t)), \quad y \to -\frac{1}{2}(x + y\cot(\Omega t)), \tag{4.3.15}$$

and we denote by $\tilde{\mathbf{r}} = (\tilde{r}^1, \ldots, \tilde{r}^k)$ the resulting functions in the new variables. Then, according to transformation (4.1.3), the functions

$$\begin{aligned} \tilde{u} &= -u(\tilde{\mathbf{r}}) \cot(\Omega t) - v(\tilde{\mathbf{r}}) + \Omega\left(y + x \cot(\Omega)\right), \\ \tilde{v} &= u(\tilde{\mathbf{r}}) - v(\tilde{\mathbf{r}}) \cot(\Omega t) - \Omega\left(x - y \cot(\Omega t)\right), \\ \tilde{h} &= h(\tilde{\mathbf{r}}) \csc^2(\Omega t), \end{aligned} \tag{4.3.16}$$

form a solution of the RSWW equations (4.1.1). Even though tranformation (4.1.3) is singular at every time $t = \frac{\pi}{2\Omega}(2n+1)$, $n \in \mathbb{N}$, we show that it is possible to obtain implicit solutions defined in a neigborhood of the origin $t = 0$.

4.3.1. Rank-1 solutions

The reduction procedure outlined above has been applied to obtain rank-1 and rank-2 solutions of the SWW equations (4.1.2) and their corresponding solutions of the RSWW system (4.1.1). We present here several rank-1 solutions, also called simple waves, associated with the different types of wave vectors (4.3.4). Note that in the case where $k = 1$, the CSM and the generalized method of characteristics agree [**48**].

i) Similarly, simple entropic-type waves are obtained by considering system (4.1.2) in the new variables

$$\bar{t} = t,\ \bar{x} = r(\mathbf{x}, \mathbf{u}),\ \bar{y} = y, \quad \bar{u} = u, \bar{v} = v, \bar{h} = h,$$

where $r(\mathbf{x}, \mathbf{u}) = -(\lambda_1 u + \lambda_2 v)t + \lambda_1^1 x + \lambda_2^1 y$ and the functions λ_i, $i = 1, 2$, are allowed to depend on u, v, h. Following Proposition 1, we look for solutions invariant under the vector fields

$$X_1 = \lambda_1 \partial_t + (\lambda_1 u + \lambda_2 v)\partial_x, \quad X_2 = \lambda_2 \partial_t + (\lambda_1 u + \lambda_2 v)\partial_y. \tag{4.3.17}$$

The transformed system (4.3.12) reads as

$$g\lambda_1 h_r = 0, \quad g\lambda_2 h_r = 0, \quad (\lambda_1 u_r + \lambda_2 v_r)h = 0. \tag{4.3.18}$$

To obtain a nontrivial solution, we must have $h = h_0 \in \mathbb{R}^+$ together with the relation

$$\lambda_1 u_r + \lambda_2 v_r = 0. \tag{4.3.19}$$

For example, if λ_1 and λ_2 are constant, we can express u in terms of v and obtain the explicit solution

$$u = u_0 - \frac{\lambda_2}{\lambda_1} v(r), \quad v = v(r), \quad h = h_0, \quad r = -u_0\lambda_1 t + \lambda_1 x + \lambda_2 y, \quad \lambda_1 \neq 0, \quad h_0 \in \mathbb{R}^+,$$

where λ_2 is an arbitrary constant and $v(r)$ is an arbitrary function.

When the λ_i are not constant, different choices can lead to solutions for the velocity vector fields $u(r)$ and $v(r)$ which are of distinct nature. For example, consider the choice $\lambda_1 = u$, $\lambda_2 = v$, leading to

$$uu_r + vv_r = \frac{1}{2}(u^2 + v^2)_r = 0 \Rightarrow u^2 + v^2 = C^2, \quad C \in \mathbb{R}.$$

A periodic solution is obtained by choosing

$$u = C\sin r, \quad v = C\cos r, \quad h = h_0, \quad C \in \mathbb{R}, \tag{4.3.20}$$

where the Riemann invariant is given implicitly by

$$r = -C(Ct - x\sin r - y\cos r). \tag{4.3.21}$$

When $\lambda_1 = v$, $\lambda_2 = u$, equation (4.3.19) implies

$$vu_r + uv_r = (uv)_r = 0 \Rightarrow v = \frac{C}{u(r)}, \quad C \in \mathbb{R}.$$

We then get the solution

$$u = u(r), \quad v = \frac{C}{u(r)}, \quad h = h_0 \in \mathbb{R}, \quad r = -2Ct + \frac{C}{u(r)}x + u(r)y, \tag{4.3.22}$$

where $u(r)$ is an arbitrary function of the Riemann invariant r.

ii) Simple acoustic-type waves are obtained by considering system (4.1.2) in the new variables

$$\bar{t} = t,\ \bar{x} = r(\mathbf{x}, \mathbf{u}),\ \bar{y} = y, \quad \bar{u} = u, \bar{v} = v, \bar{h} = h,$$

where $r(\mathbf{x}, \mathbf{u}) = -(\lambda_1 u + \lambda_2 v + \sqrt{gh})t + \lambda_1^1 x + \lambda_2^1 y$, $|\vec{\lambda}|^2 = 1$, and the functions λ_i, $i = 1, 2$, are allowed to depend on u, v, h. Rank-1 solutions of this type are invariant under the vector fields

$$X_1 = \lambda_1\partial_t + (\lambda_1 u + \lambda_2 v + \sqrt{gh})\partial_x, \quad X_2 = \lambda_2\partial_t + (\lambda_1 u + \lambda_2 v + \sqrt{gh})\partial_y. \tag{4.3.23}$$

In this case, the transformed system (4.3.12) is

$$\lambda_1\sqrt{\frac{g}{h}}h_r = u_r, \quad \lambda_2\sqrt{\frac{g}{h}}h_r = v_r, \quad h(\lambda_1 u_r + \lambda_2 v_r) = \sqrt{gh}h_r. \tag{4.3.24}$$

The third equation is automatically satisfied whenever the first two are and $|\vec{\lambda}|^2 = 1$. Note that in order to obtain a solution for $h(r)$, it is necessary that the relation

$$\lambda_1(u, v, h)v_r - \lambda_2(u, v, h)u_r = 0 \tag{4.3.25}$$

be satisfied. Considering different choices for the functions $\lambda_i(u, v, h)$, we obtain several interesting solutions, presented in Table 4.2.

For illustration, we now turn to the construction of the implicit solution of the RSWW equations corresponding to (4.3.20), (4.3.21) using transformation (4.1.3). We first transform the Riemann invariant r to obtain an implicit equation for $\tilde{r}$,

$$\tilde{r} = \frac{C^2}{2\Omega}\cot(\Omega t) + \frac{C}{2}\left[(y - x\cot(\Omega t))\sin\tilde{r} - (x + y\cot(\Omega t))\cos\tilde{r}\right]. \tag{4.3.26}$$

Using equations (4.3.16), we obtain the implicit solution of the RSWW equations

$$\begin{aligned} u &= -C\cos\tilde{r} - C\cot(\Omega t)\sin\tilde{r} + \Omega\left(y + x\cot(\Omega t)\right), \\ v &= C\sin\tilde{r} - C\cot(\Omega t)\cos\tilde{r} - \Omega\left(y + x\cot(\Omega t)\right), \\ h &= h_0\csc^2(\Omega t), \end{aligned} \tag{4.3.27}$$

where $\tilde{r}$ is the solution of the implicit equation (4.3.26). This solution has period π/Ω and goes to infinity at every time $t = k\pi/\Omega$, $k \in \mathbb{N}$. Nevertheless, due to the invariance of equations (4.1.1) with respect to translations in time, it is possible to use a time shift $t \to t + t_0$ so that equations (4.3.26) are well defined in a neighborhood of length π/Ω around $t = 0$. For example, the translation $t \to t + \frac{\pi}{2\Omega}$ gives the solution

$$\begin{aligned} u &= -C\cos\bar{r} + C\tan(\Omega t)\sin\bar{r} + \Omega\left(y - x\tan(\Omega t)\right), \\ v &= C\sin\bar{r} + C\tan(\Omega t)\cos\bar{r} - \Omega\left(y - x\tan(\Omega t)\right), \\ h &= h_0\sec^2(\Omega t), \end{aligned} \tag{4.3.28}$$

where $\bar{r}$ satisfies the equation

$$\bar{r} = -\frac{C^2}{2\Omega}\tan(\Omega t) + \frac{C}{2}\left[(y + x\tan(\Omega t))\sin\bar{r} - (x - y\tan(\Omega t))\cos\bar{r}\right], \tag{4.3.29}$$

which is clearly defined in the interval $\left(-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}\right)$. Note that this process can be applied to every solution presented in Table 4.2 to generate local solutions of the RSWW equations defined around $t = 0$.

4.3.2. Rank-2 solutions

The construction of rank-2 solutions is much more involved than in the case $k = 1$ since it requires us to solve system (4.3.13), which is composed of at most twelve independent nonlinear partial differential equations, compared to only three equations. However, we now show that the task is undertakable and leads to interesting solutions. The results of this analysis are summarized in Table 4.3 and 4.4.

i) We first look for rank-2 solutions resulting from the interaction of two entropic-type solutions. They are invariant under the vector field

$$X = \partial_t + u\partial_x + v\partial_y. \tag{4.3.30}$$

In the variables

$$\begin{aligned} &\bar{t} = t,\ \bar{x}^1 = r^1(\mathbf{x}, \mathbf{u}),\ \bar{x}^2 = r^2(\mathbf{x}, \mathbf{u}),\ \bar{u} = u,\ \bar{v} = v,\ \bar{h} = h, \\ &r^i(\mathbf{x}, \mathbf{u}) = t - \frac{\lambda_1^i}{\lambda_1^i u + \lambda_2^i v} x - \frac{\lambda_2^i}{\lambda_1^i u + \lambda_2^i v} y, \quad i = 1, 2, \end{aligned} \tag{4.3.31}$$

equations (4.3.13 i) read as

$$\lambda_1^1(\lambda_1^2 u + \lambda_2^2 v)h_{r^1} + \lambda_1^2(\lambda_1^1 u + \lambda_2^1 v)h_{r^2} = 0, \tag{4.3.32}$$

$$\lambda_2^1(\lambda_1^2 u + \lambda_2^2 v)h_{r^1} + \lambda_2^2(\lambda_1^1 u + \lambda_2^1 v)h_{r^2} = 0, \tag{4.3.33}$$

$$(\lambda_1^2 u + \lambda_2^2 v)(\lambda_1^1 u_{r^1} + \lambda_2^1 v_{r^1}) + (\lambda_1^1 u + \lambda_2^1 v)(\lambda_1^2 u_{r^2} + \lambda_2^2 v_{r^2}) = 0. \tag{4.3.34}$$

A solution to the first two equations exists if and only if

$$(\lambda_1^1\lambda_2^2 - \lambda_2^1\lambda_1^2)(\lambda_1^1 u + \lambda_2^1 v)(\lambda_1^2 u + \lambda_2^2 v) = 0 \quad \text{or} \quad h = h_0 \in \mathbb{R}^+.$$

The conditions on the functions λ_j^i imply either that the wave vectors are parallel or one of the considered waves has zero velocity. From these conditions, we now show that no rank-2 solution can be built from this type of interaction. When $\vec{\lambda}^2 = k\vec{\lambda}^1$, the Riemann invariants r^1 and r^2 are equal, hence the solution cannot be of rank 2. Therefore we look for solutions with $h = h_0$, a positive constant. Equation (4.3.34) implies that

$$u_{r^1} = -\frac{1}{\lambda_1^1}\frac{\lambda_1^1 u + \lambda_2^1 v}{\lambda_1^2 u + \lambda_2^2 v}\left(\lambda_1^2 u_{r^2} + \lambda_2^2 v_{r^2}\right) - \frac{\lambda_2^1}{\lambda_1^1} v_{r^1}. \tag{4.3.35}$$

We then consider the linear combination

$$\frac{1}{uv}\operatorname{Tr}\left[\mathcal{A}^3 \frac{\partial f}{\partial r}(u\eta_1 + v\eta_2)\frac{\partial f}{\partial r}\lambda\right] = -\frac{2}{uv}\left(\lambda_1^1 u + \lambda_2^1 v\right)\left(\lambda_1^2 u + \lambda_2^2 v\right)\left(\lambda_1^1\lambda_2^2 - \lambda_2^1\lambda_1^2\right) \times$$
$$\left(\left(\lambda_1^1 u + \lambda_2^1 v\right)\left(\lambda_1^2 u_{r^2} + \lambda_2^2 v_{r^2}\right) v_{r^2} + \left(\lambda_1^2 u + \lambda_2^2 v\right)\left(\lambda_1^1 u_{r^2} + \lambda_2^1 v_{r^2}\right) v_{r^1}\right), \tag{4.3.36}$$

implying that a rank-2 solution must satisfy

$$\left(\lambda_1^1 u + \lambda_2^1 v\right)\left(\lambda_1^2 u_{r^2} + \lambda_2^2 v_{r^2}\right) v_{r^2} + \left(\lambda_1^2 u + \lambda_2^2 v\right)\left(\lambda_1^1 u_{r^2} + \lambda_2^1 v_{r^2}\right) v_{r^1} = 0. \tag{4.3.37}$$

When $\lambda_1^2 u_{r^2} + \lambda_2^2 v_{r^2} = 0$, equation (4.3.35) requires that

$$u_{r^1} = -\frac{\lambda_2^1}{\lambda_1^1} v_{r^1}, \quad v_{r^2} = -\frac{\lambda_1^2}{\lambda_2^2} u_{r^2},$$

so that (4.3.37) becomes

$$\frac{1}{\lambda_2^2}(\lambda_1^2 u + \lambda_2^2 v)(\lambda_1^1\lambda_2^2 - \lambda_1^2\lambda_2^1) u_{r^2} v_{r^1} = 0,$$

leading necessarily to a rank-1 solution. Hence we can solve (4.3.37) for v_{r^2}, and the expression (4.3.35) for u_{r^1} implies that

$$\frac{u_{r^2}}{u_{r^1}} = \frac{v_{r^2}}{v_{r^1}} = -\frac{\left(\lambda_1^2 u + \lambda_2^2 v\right)\left(\lambda_1^1 u_{r^2} + \lambda_2^1 v_{r^2}\right)}{\left(\lambda_1^1 u + \lambda_2^1 v\right)\left(\lambda_1^2 u_{r^2} + \lambda_2^2 v_{r^2}\right)}, \tag{4.3.38}$$

hence we must have $v = F(u)$, for an arbitrary function $F : \mathbb{R} \to \mathbb{R}$. But this implies that the Jacobian matrix of the solution is of rank 1, since $h = h_0$. Thus, no rank-2 solution of type E-E exists. For example, consider the simplest case when $\lambda_1^1 = \lambda_2^2 = 1, \lambda_2^1 = \lambda_1^2 = 0$. The Riemann invariants are then given by

$$r^1 = t - \frac{x}{u}, \quad r^2 = t - \frac{y}{v}.$$

Equations (4.3.35) and (4.3.37) become

$$u v_{r^2} + v u_{r^1} = 0, \quad u {v_{r^2}}^2 + v u_{r^2} v_{r^1} = 0. \tag{4.3.39}$$

Solving for v_{r^1} and v_{r^2}, we obtain that the rank of the Jacobian matrix

$$J = \frac{\partial(u,v,h)}{\partial(r^1,r^2)} = \begin{pmatrix} u_{r^1} & u_{r^2} \\ -\frac{v}{u}\frac{{u_{r^1}}^2}{u_{r^2}} & -\frac{v}{u}u_{r^1} \\ 0 & 0 \end{pmatrix} \tag{4.3.40}$$

is equal to one. A particular solution of (4.3.39) is given by

$$\begin{aligned} &u = (-1)^m s^m, \quad m \neq -1, \quad s = \frac{C_1 r^2 + C_2}{C_3 r^1 + C_4}, \quad C_i \in \mathbb{R}, \quad i = 1, \dots, 5, \\ &v = C_5 \exp\left(\frac{C_3}{C_1} m s\right), \quad h = h_0 \in \mathbb{R}^+, \end{aligned} \tag{4.3.41}$$

which is indeed seen to depend on the single variable s.

ii) We now look for superpositions of two solutions of the dispersion relation (4.3.3), one of each type. This type of solution is invariant under the vector field

$$X = \delta\partial_t + \left(\delta u - \lambda_2^1\sqrt{gh}\right)\partial_x + \left(\delta v - \lambda_1^1\sqrt{gh}\right)\partial_y, \quad \delta = \lambda_1^1\lambda_2^2 - \lambda_2^1\lambda_1^2. \tag{4.3.42}$$

Introducing the change of variables

$$\bar{t} = t,\ \bar{x}^1 = r^1(\mathbf{x},\mathbf{u}),\ \bar{x}^2 = r^2(\mathbf{x},\mathbf{u}),\ \bar{u} = u,\ \bar{v} = v,\ \bar{h} = h,$$

with

$$\begin{aligned} r^1(\mathbf{x},\mathbf{u}) &= t - \frac{\lambda_1^1}{\lambda_1^1 u + \lambda_2^1 v}x - \frac{\lambda_1^2}{\lambda_1^1 u + \lambda_2^1 v}y, \\ r^2(\mathbf{x},\mathbf{u}) &= t - \frac{\lambda_1^2}{\lambda_1^2 u + \lambda_2^2 v + \sqrt{gh}}x - \frac{\lambda_2^2}{\lambda_1^2 u + \lambda_2^2 v + \sqrt{gh}}y, \end{aligned} \tag{4.3.43}$$

we show that rank-2 solutions can be built by setting $\lambda_1^2 = 1, \lambda_2^2 = 0$. Supposing that $\lambda_1^1, \lambda_2^1 \neq 0$, equations (4.3.13) require that

$$\begin{aligned} &u_{r^1} = -\frac{{\lambda_1^1}^2 - {\lambda_2^1}^2}{\lambda_2^1({\lambda_1^1}^2 + {\lambda_2^1}^2)}v_{r^2}, \quad v_{r^1} = -\frac{2\lambda_1^1}{{\lambda_1^1}^2 + {\lambda_2^1}^2}v_{r^2}, \\ &h_{r^1} = \frac{\sqrt{gh}}{\lambda_2^1 g}v_{r^2}, \quad h_{r^2} = \frac{\sqrt{gh}}{\lambda_2^1 g}(\lambda_2^1 u_{r^2} - \lambda_1^1 v_{r^2}), \\ &(3{\lambda_2^1}^2 - {\lambda_1^1}^2)(({\lambda_2^1}^2 - {\lambda_1^1}^2)v_{r^2} + 2\lambda_1^1\lambda_2^1 u_{r^2}) = 0, \\ &(({\lambda_2^1}^2 - {\lambda_1^1}^2)v_{r^2} + 2\lambda_1^1\lambda_2^1 u_{r^2})(\sqrt{gh}\lambda_{1,u}^1 + h\lambda_{1,h}^1) = 0, \\ &(({\lambda_2^1}^2 - {\lambda_1^1}^2)v_{r^2} + 2\lambda_1^1\lambda_2^1 u_{r^2})(\sqrt{gh}\lambda_{2,u}^1 + h\lambda_{2,h}^1) = 0. \end{aligned} \tag{4.3.44}$$

It is easily computed from equations (4.3.44) that when $({\lambda_2^1}^2-{\lambda_1^1}^2)v_{r^2}+2\lambda_1^1\lambda_2^1u_{r^2}=0$, the obtained solution will be of rank 1. Thus, we must have $\lambda_1^1=F_1(u-2\sqrt{gh},v)$, $\lambda_2^1=F_2(u-2\sqrt{gh},v)$ where F_1,F_2 are arbitrary functions. Equations (4.3.44) can be solved for specific choices of the arbitrary functions F_1,F_2. Hence we consider the case where $({\lambda_2^1}^2-{\lambda_1^1}^2)v_{r^2}+2\lambda_1^1\lambda_2^1u_{r^2}\neq 0$, together with the relation $\lambda_1^1=\sqrt{3}\varepsilon\lambda_2^1$, $\varepsilon=\pm1$, which leads to

$$u_{r^1}=-\frac{1}{2\lambda_2^1}v_{r^2}, \tag{4.3.45}$$

$$v_{r^1}=-\frac{\sqrt{3}\varepsilon}{2\lambda_2^1}v_{r^2}, \tag{4.3.46}$$

$$h_{r^1}=\frac{1}{\lambda_2^1}\sqrt{\frac{h}{g}}v_{r^2},\quad h_{r^2}=\sqrt{\frac{h}{g}}(u_{r^2}-\sqrt{3}\varepsilon v_{r^2}), \tag{4.3.47}$$

$$\sqrt{gh}\lambda^1_{2,u}+h\lambda^1_{2,h}=0. \tag{4.3.48}$$

When λ_2^1 is a function of v only, system (4.3.45) - (4.3.48) is compatible and can be integrated to yield

$$u=\frac{\sqrt{3}}{3}\varepsilon v(r^1,r^2)+F(r^2),\quad h=\frac{1}{4g}\left(F(r^2)-\frac{2\sqrt{3}}{3}\varepsilon v(r^1,r^2)+h_0\right)^2, \tag{4.3.49}$$

where $v(r^1,r^2)$ is given implicitly by

$$v=G\left(s\right),\quad s=r^2-\frac{\sqrt{3}\varepsilon}{2\lambda_2^1(v)}r^1, \tag{4.3.50}$$

and $G(s),F(r^2)$ are arbitrary functions of their respective argument. The Riemann invariants r^1 and r^2 then satisfy the implicit relations

$$r^1=\lambda_2^1(G(s))\left(\left(2G(s)+\sqrt{3}\varepsilon F(r^2)\right)t-\sqrt{3}\varepsilon x-y\right),\quad s=r^2-\frac{\sqrt{3}\varepsilon}{2\lambda_2^1(v)}r^1,\quad \varepsilon=\pm1,$$
$$r^2=\left(\frac{3}{2}F(r^2)+\frac{h_0}{2}\right)t-x. \tag{4.3.51}$$

Because of the nonlinear coupling of the Riemann invariants (4.3.51), this type of solution is said to be scattering. For different choices of the function λ_2^1 and the profile of v (i.e. $G(s)$), following the construction presented in [**18**], it is possible to construct rank-2 solutions which are bounded everywhere, for example bumps, kinks and periodic solutions, even when the Riemann invariants admit the gradient catastrophe after a finite time. We present in Table 4.5 several solutions of the SWW equations obtained in this way. According to equations (4.1.3), after a time shift $t \to t + \pi/2\Omega$, the RSWW equations admit the following solution

$$
\begin{aligned}
u &= -\left(\frac{\sqrt{3}}{3}\varepsilon G(\tilde{s}) + F(\tilde{r}^2)\right)\tan(\Omega t) - G(\tilde{s}) + \Omega(y + x\tan(\Omega t)), \quad \varepsilon = \pm 1,\\
v &= \frac{\sqrt{3}}{3}\varepsilon G(\tilde{s}) + F(\tilde{r}^2) - G(\tilde{s})\tan(\Omega t) - \Omega(x - y\tan(\Omega t)),\\
h &= \frac{1}{4g}\left(F(\tilde{r}^2) - \frac{2\sqrt{3}}{3}\varepsilon G(\tilde{s}) + h_0\right)^2 \sec^2(\Omega t),
\end{aligned}
\tag{4.3.52}
$$

where the functions $\tilde{r}^1, \tilde{r}^2, \tilde{s}$ now satisfy the implicit relations

$$
\begin{aligned}
\tilde{r}^1 &= \lambda_2^1\left(G(\tilde{s})\right)\Bigg(-\frac{1}{2\Omega}(2G(\tilde{s}) + \sqrt{3}\varepsilon F(\tilde{r}^2))\tan(\Omega t) \\
&\qquad -\frac{\sqrt{3}}{2}\varepsilon(y - x\tan(\Omega t)) + \frac{1}{2}(x + y\tan(\Omega t))\Bigg),\\
\tilde{r}^2 &= -\frac{1}{4\Omega}\left(3F(\tilde{r}^2) + h_0\right)\tan(\Omega t) - \frac{1}{2}(y - x\tan(\Omega t)),\\
\tilde{s} &= \tilde{r}^2 - \frac{\sqrt{3}\varepsilon}{2\lambda_2^1\left(G(\tilde{s})\right)}\tilde{r}^1,
\end{aligned}
\tag{4.3.53}
$$

and $G(\tilde{s})$, $\lambda_2^1(G(\tilde{s}))$ and $F(\tilde{r}^2)$ are arbitrary functions of their respective argument. Equations (4.3.52) and (4.3.53) define a rank-2 solution in the interval $\left(-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}\right)$. From bounded solutions of the SWW equations (see Table 4.5), one can then construct rank-2 solutions of the RSWW equations which are bounded in this interval.

iii) We now turn to the analysis of the interaction of two acoustic type solutions. Therefore, introducing the change of variables

$$
\bar{t} = t,\ \bar{x}^1 = r^1(\mathbf{x}, \mathbf{u}),\ \bar{x}^2 = r^2(\mathbf{x}, \mathbf{u}),\ \bar{u} = u,\ \bar{v} = v,\ \bar{h} = h,
$$

with

$$
r^i(\mathbf{x}, \mathbf{u}) = -(\lambda_1^i u + \lambda_2^i v + \sqrt{gh})t + \lambda_1^i x + \lambda_2^i y, \quad |\vec{\lambda}^i|^2 = 1, \quad i = 1, 2,
$$

the system (4.3.13) is formed of twelve independent equations. Equations (4.3.13 i) are in this case

$$g(\lambda_1^1 h_{r^1} + \lambda_1^2 h_{r^2}) = \sqrt{gh}(u_{r^1} + u_{r^2}), \tag{4.3.54}$$

$$g(\lambda_2^1 h_{r^1} + \lambda_2^2 h_{r^2}) = \sqrt{gh}(v_{r^1} + v_{r^2}), \tag{4.3.55}$$

$$h(\lambda_1^1 u_{r^1} + \lambda_1^2 u_{r^2} + \lambda_2^1 v_{r^1} + \lambda_2^2 v_{r^2}) = \sqrt{gh}(h_{r^1} + h_{r^2}). \tag{4.3.56}$$

A process of elimination of the derivatives of the functions $\lambda_j^i(u, v, h)$ in (4.3.13ii), leads us to a system composed of

$$\begin{aligned} (u_{r^1} + u_{r^2})(\lambda_2^1 u_{r^1} - \lambda_1^1 v_{r^1} + \lambda_2^2 u_{r^2} - \lambda_1^2 v_{r^2}) = 0, \\ (v_{r^1} + v_{r^2})(\lambda_2^1 u_{r^1} - \lambda_1^1 v_{r^1} + \lambda_2^2 u_{r^2} - \lambda_1^2 v_{r^2}) = 0, \end{aligned} \tag{4.3.57}$$

and a third complicated expression which takes a much simpler form depending on the branch of solution chosen in (4.3.57).

a) If $u(r^1, r^2) = F(r^1 - r^2)$ and $v(r^1, r^2) = G(r^1 - r^2)$, then the last equation is automatically satisfied. The solution is obtained from the system

$$(\lambda_1^1 - \lambda_1^2)F' + (\lambda_2^1 - \lambda_2^2)G' = 0, \quad |\vec{\lambda}^i|^2 = 1, \quad h = h_0 \in \mathbb{R}^+. \tag{4.3.58}$$

However, it should be noted that any solution built from this branch reduces to a rank-1 entropic-type solution. Indeed, since $h = h_0$, by equations (4.3.54) and (4.3.55), the Jacobian matrix of the solution in the original variables reads as

$$\begin{vmatrix} F'(r^1 - r^2)(r_t^1 - r_t^2) & -F'(r^1 - r^2)(r_x^1 - r_x^2) \\ G'(r^1 - r^2)(r_t^1 - r_t^2) & -G'(r^1 - r^2)(r_x^1 - r_x^2) \\ 0 & 0 \end{vmatrix},$$

which is manifestly of rank-1. Moreover, it can be easily seen that the resulting rank-1 solution will be a solution of the first type. For example, choosing

$$\lambda_1^1 = 1, \quad \lambda_2^1 = 0, \quad \lambda_1^2 = \frac{1 - u^2}{1 + u^2}, \quad \lambda_2^2 = \frac{2u}{u^2 + 1},$$

we obtain the solution

$$v = v_0 + \frac{1}{2}u^2, \quad h = h_0,$$

where $u = F(s)$ is an arbitrary function of

$$s = r^1 - r^2 = -\frac{F(F^2 - 2v_0)}{1 + F^2}t + \frac{2F^2}{1 + F^2}x - \frac{2F}{1 + F^2}y.$$

b) When $\lambda_2^1 u_{r^1} - \lambda_1^1 v_{r^1} + \lambda_2^2 u_{r^2} - \lambda_1^2 v_{r^2} = 0$, the last equation reduces to

$$\left[2\delta^2 + (\vec{\lambda}^1 \cdot \vec{\lambda}^2) - 1\right] v_{r^1} v_{r^2} = 0, \quad \delta = \begin{vmatrix} \lambda_1^1 & \lambda_2^1 \\ \lambda_1^2 & \lambda_2^2 \end{vmatrix}, \quad |\vec{\lambda}^i|^2 = 1. \tag{4.3.59}$$

The solution is necessarily of rank-1 if $v_{r^1} = 0$ or $v_{r^2} = 0$. We then suppose that v depends essentially on r^1 and r^2, so the wave vectors $\vec{\lambda}^1$ and $\vec{\lambda}^2$ must satisfy the relations

$$2\delta^2 + (\vec{\lambda}^1 \cdot \vec{\lambda}^2) - 1 = 0, \quad |\vec{\lambda}^i|^2 = 1, \quad i = 1, 2. \tag{4.3.60}$$

Writing $\lambda_1^1 = \sin\varphi_1, \lambda_2^1 = \cos\varphi_1, \lambda_1^2 = \sin\varphi_2, \lambda_2^2 = \cos\varphi_2$, equation (4.3.60) implies that the angle $\varphi = |\varphi_1 - \varphi_2| \in [0, 2\pi)$ between the wave vectors $\vec{\lambda}^1$ and $\vec{\lambda}^2$ has to satisfy

$$2\sin^2\varphi + \cos\varphi - 1 = 0,$$

which can be written as

$$-2\left(\cos\varphi + \frac{1}{2}\right)(\cos\varphi - 1) = 0.$$

Therefore, excluding the case where $\varphi = 0$, we obtain

$$\cos\varphi = \vec{\lambda}^1 \cdot \vec{\lambda}^2 = -1/2 \quad \Rightarrow \quad \varphi = |\varphi_1 - \varphi_2| = 2\pi/3, \tag{4.3.61}$$

in accordance with results already obtained for an isentropic fluid flow [**48, 100**]. In this case, since by (4.3.60) and (4.3.61) we must have $\delta = \varepsilon\sqrt{3}/2$, $\varepsilon = \pm 1$, one can show that the system composed of (4.3.54) - (4.3.57) becomes

$$u_{r^1} = \lambda_1^1 \sqrt{\frac{g}{h}} h_{r^1}, \quad u_{r^2} = \lambda_1^2 \sqrt{\frac{g}{h}} h_{r^2}, \quad v_{r^1} = \lambda_2^1 \sqrt{\frac{g}{h}} h_{r^1}, \quad v_{r^2} = \lambda_2^2 \sqrt{\frac{g}{h}} h_{r^2}, \tag{4.3.62}$$

and that the functions λ_j^i must satisfy the equations

$$\lambda_1^1 \lambda_{1,u}^2 + \lambda_2^1 \lambda_{1,v}^2 + \frac{h}{\sqrt{gh}} \lambda_{1,h}^2 = 0, \quad \lambda_1^2 \lambda_{1,u}^1 + \lambda_2^2 \lambda_{1,v}^1 + \frac{h}{\sqrt{gh}} \lambda_{1,h}^1 = 0. \tag{4.3.63}$$

Using (4.3.63) and writing $h = H(r^1, r^2)^2$, the compatibility conditions of equations (4.3.62) yield the relation

$$H_{r^1r^2} = 0 \Rightarrow h(r^1, r^2) = \left(h_1(r^1) + h_2(r^2)\right)^2. \tag{4.3.64}$$

When the velocity vectors $\vec{\lambda}^1$ and $\vec{\lambda}^2$ are constant, integration of (4.3.62) then shows that the velocity vector fields split as a linear sum. Hence, we obtain the nonscattering solution

$$\begin{aligned} &u = u_0 + 2\sqrt{g}\left(\lambda_1^1 h_1(r^1) + \lambda_1^2 h_2(r^2)\right), \quad v = v_0 + 2\sqrt{g}\left(\lambda_2^1 h_1(r^1) + \lambda_2^2 h_2(r^2)\right), \\ &h = \left(h_1(r^1) + h_2(r^2)\right)^2, \end{aligned} \tag{4.3.65}$$

where the functions $h_1(r^1)$ and $h_2(r^2)$ are arbitrary functions of the Riemann invariants

$$\begin{aligned} r^1 &= -\left(\lambda_1^1 u_0 + \lambda_2^1 v_0 + 3\sqrt{g} h_1(r^1)\right) t + \lambda_1^1 x + \lambda_2^1 y, \quad \lambda_j^i \in \mathbb{R}, \quad |\vec{\lambda}^i|^2 = 1, \\ r^2 &= -\left(\lambda_1^2 u_0 + \lambda_2^2 v_0 + 3\sqrt{g} h_2(r^2)\right) t + \lambda_1^2 x + \lambda_2^2 y, \quad \vec{\lambda}^1 \cdot \vec{\lambda}^2 = -1/2, \end{aligned} \tag{4.3.66}$$

so that the angle between the vectors $\vec{\lambda}^1$ and $\vec{\lambda}^2$ is fixed by relation (4.3.61). Once more, these arbitrary functions can be selected as to ensure that the solution remains bounded everywhere, see Table 4.5. By means of transformation (4.1.3), we obtain the solution of the RSWW equations (4.1.1) corresponding to solution (4.3.65). It is given by

$$\begin{aligned} &u = -(u_0 + 2\sqrt{g}(\lambda_1^1 h_1(\tilde{r}^1) + \lambda_1^2 h_2(\tilde{r}^2)) \cot(\Omega t) - (v_0 + 2\sqrt{g}(\lambda_2^1 h_1(\tilde{r}^1) + \lambda_2^2 h_2(\tilde{r}^2))) + \Omega(y + x\cot(\Omega t)), \\ &v = u_0 + 2\sqrt{g}(\lambda_1^1 h_1(\tilde{r}^1) + \lambda_1^2 h_2(\tilde{r}^2)) - (v_0 + 2\sqrt{g}(\lambda_2^1 h_1(\tilde{r}^1) + \lambda_2^2 h_2(\tilde{r}^2))) \cot(\Omega t) - \Omega(x - y\cot(\Omega t)), \\ &h = (h_1(\tilde{r}^1) + h_2(\tilde{r}^2)) \csc^2(\Omega t), \end{aligned} \tag{4.3.67}$$

where the transformed Riemann invariants $\tilde{r}^1, \tilde{r}^2$ satisfy the implicit relations

$$\begin{aligned} \tilde{r}^1 &= \frac{1}{2}\left[\frac{1}{\Omega}(\lambda_1^1 u_0 + \lambda_2^1 v_0 + 3\sqrt{g} h_1(\tilde{r}^1)) \cot(\Omega t) + \lambda_1^1(y - x\cot(\Omega t)) - \lambda_2^1(x + y\cot(\Omega t))\right], \\ \tilde{r}^2 &= \frac{1}{2}\left[\frac{1}{\Omega}(\lambda_1^2 u_0 + \lambda_2^2 v_0 + 3\sqrt{g} h_2(\tilde{r}^2)) \cot(\Omega t) + \lambda_1^2(y - x\cot(\Omega t)) - \lambda_2^2(x + y\cot(\Omega t))\right]. \end{aligned} \tag{4.3.68}$$

Again, it is interesting to note that due to the invariance of equations (4.1.1) with respect to translations in time, it is possible to use a time translation $t \to t + t_0$ so that equations (4.3.68) are well defined for $t = 0$. For example, when functions $h_1(r^1), h_2(r^2)$ are assumed to be hyperbolic functions of their respective argument, i.e. $h_1(r^1) = \operatorname{sech}^2(r^1)$, $h_2(r^2) = \operatorname{sech}^2(r^2)$, and if we choose $\vec{\lambda}^1 = (1, 0)$ and $\vec{\lambda}^2 = (-1/2, \sqrt{3}/2)$, then we obtain after a time shift $t \to t + \pi/2\Omega$ the singular bump-type solution

$$\begin{aligned} u &= \left(u_0 + \sqrt{g}(2\operatorname{sech}^2(r1) - \operatorname{sech}^2(r^2))\right)\tan(\Omega t) - (v_0 + \sqrt{3g}\operatorname{sech}^2(r^2)) + \Omega(y - x\tan(\Omega t)),\\ v &= (v_0 + \sqrt{3g}\operatorname{sech}^2(r^2))\tan(\Omega t) + \sqrt{g}(2\operatorname{sech}^2(r^1) - \operatorname{sech}^2(r^2)) - \Omega(x + y\tan(\Omega t)),\\ h &= (\operatorname{sech}^2(r^1) + \operatorname{sech}^2(r^2))^2\sec^2(\Omega t) \end{aligned} \tag{4.3.69}$$

with

$$\begin{aligned} r^1 &= \frac{1}{2\Omega}\left(u_0 + 3\sqrt{g}\operatorname{sech}^2(r^1)\right)\tan(\Omega t) + \frac{1}{2}\left(y - x\tan(\Omega t)\right), \qquad (4.3.70)\\ r^2 &= \frac{1}{2\Omega}\left(\frac{u_0}{2} + \frac{\sqrt{3}}{2}v_0 + 3\sqrt{g}\operatorname{sech}^2(r^2)\right)\tan(\Omega t) - \frac{1}{4}\left(y - x\tan(\Omega t)\right) - \frac{\sqrt{3}}{4}\left(x + y\tan(\Omega t)\right). \end{aligned}$$

Figure 4.1 illustrates the behavior of the height function $h(t, x, y)$ defined by (4.3.69) and (4.3.70) .

When the λ^i_j are not constant, equations (4.3.63) possess several classes of implicit solutions. Supposing that

$$\lambda^1_1 = \frac{\Psi}{\sqrt{1+\Psi^2}}, \quad \lambda^1_2 = \frac{1}{\sqrt{1+\Psi^2}}, \tag{4.3.71}$$

for some function $\Psi : \mathbb{R}^3 \to \mathbb{R}$, equation (4.3.59) requires that

$$\lambda^2_1 = -\frac{1}{2}\frac{\Psi + \sqrt{3}}{\sqrt{1+\Psi^2}}, \quad \lambda^2_2 = \frac{1}{2}\frac{\sqrt{3}\Psi - 1}{\sqrt{1+\Psi^2}}. \tag{4.3.72}$$

The system (4.3.63) then becomes

$$\begin{aligned} &\Psi\Psi_u + \Psi_v + \frac{h}{\sqrt{gh}}\sqrt{\Psi^2+1}\Psi_h = 0,\\ &-(\sqrt{3}+\Psi)\Psi_u + (\sqrt{3}\Psi - 1)\Psi_v + \frac{2h}{\sqrt{gh}}\sqrt{1+\Psi^2}\Psi_h = 0, \end{aligned} \tag{4.3.73}$$

implying that Ψ must satisfy

$$(1 + \sqrt{3}\Psi)\Psi_u + (\sqrt{3} - \Psi)\Psi_v = 0. \tag{4.3.74}$$

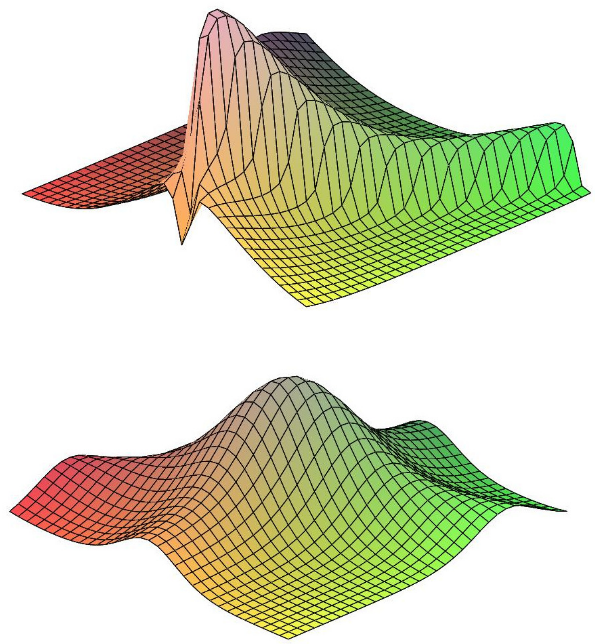

FIG. 4.1. Graph of the height function $h(t,x,y)$ for the rank-2 solution of the S-S type (4.3.69) at times $t=-\pi/5$ and $t=0$.

It is easy to show from (4.3.73) that Φ is either constant or depends essentially on all functions u,v,h. Looking for a solution of the form $\Psi = F(\gamma_1(\Psi)u + \gamma_2(\Psi)v - \phi(h))$, where $\phi(h)$ is some function of h to be determined, we obtain that equations (4.3.73) possess the implicit solution

$$\Psi = F\left((\Psi-\sqrt{3})u + (1+\sqrt{3}\Psi)v - 2\sqrt{1+\Psi^2}\sqrt{gh}\right), \tag{4.3.75}$$

where F is an arbitrary function of its argument. Equations (4.3.73) also possess infinite classes of solutions of the form

$$\Psi = F(s_1, s_2), \quad s_1 = \gamma_1(\Psi)u - 2\sqrt{gh}, \quad s_2 = \gamma_2(\Psi)v - 2\sqrt{gh}. \tag{4.3.76}$$

The compatibility relation (4.3.74) requires that

$$\frac{\partial F}{\partial s_1} = \frac{(\Psi-\sqrt{3})\gamma_2(\Psi)}{(1+\sqrt{3}\Psi)\gamma_1(\Psi)}\frac{\partial F}{\partial s_2} = G(s_1,s_2)\frac{\partial F}{\partial s_2}, \tag{4.3.77}$$

for some function $G(s_1,s_2)$. Equations (4.3.73) then become

$$\gamma_1(\Psi)(\sqrt{3}\Psi^2 - 2\Psi - \sqrt{3}) + \gamma_2(\Psi)(\Psi^2 - 2\sqrt{3}\Psi + 3) + \gamma_1(\Psi)\gamma_2(\Psi)\sqrt{1+\Psi^2}(\sqrt{3}-\Psi) = 0. \tag{4.3.78}$$

For a selected function $G(s_1, s_2)$, solving equations (4.3.78) and

$$G(s_1,s_2)=\frac{(\Psi-\sqrt{3})\gamma_2(\Psi)}{(1+\sqrt{3}\Psi)\gamma_1(\Psi)}$$

gives the explicit expressions for $\gamma_1(\Psi)$ and $\gamma_2(\Psi)$ while integration of (4.3.77) gives the dependence of F on s_1 and s_2. For example, when $G(s_1, s_2) = 1$, then $\Psi = F(\gamma_1(\Psi) + \gamma_2(\Psi) - 4\sqrt{gh})$, with

$$\gamma_1(\Psi)=2\frac{\sqrt{3}\Psi^3-5\Psi^2+\sqrt{3}\Psi+3}{(\sqrt{3}\Psi^2-2\Psi-\sqrt{3})\sqrt{1+\Psi^2}},\quad \gamma_2(\Psi)=2\frac{3\Psi^4-4\sqrt{3}\Psi^3-2\Psi^2+4\sqrt{3}\Psi+3}{(\sqrt{3}\Psi^3-5\Psi^2+\sqrt{3}\Psi+3)\sqrt{1+\Psi^2}},\tag{4.3.79}$$

and F arbitrary.

From any explicit solution of (4.3.63) obtained by specifying the arbitrary function in (4.3.75) or in (4.3.76) and (4.3.77) and using the relations (4.3.71), (4.3.72), the solution for the vector fields $u(r^1, r^2), v(r^1, r^2)$ is obtained by integrating system (4.3.62). However, since the resulting expressions are very involved even in the simplest cases, we will not present a solution of this type in closed form.

iv) Finally, conducting an analysis similar to that of the previous case, we finally look for linear interactions of two acoustic type waves of constant direction for which we choose different signs for ε in (4.3.4 ii). Suppose in this case that the Riemann invariants are given in the form

$$\begin{aligned} r^1 &= -(\lambda_1^1 u + \lambda_2^1 v + \sqrt{gh})t + \lambda_1^1 x + \lambda_2^1 y, \quad \lambda_j^i \in \mathbb{R}, \\ r^2 &= -(\lambda_1^2 u + \lambda_2^2 v - \sqrt{gh})t + \lambda_1^2 x + \lambda_2^2 y, \quad |\vec{\lambda}^i| = 1, i = 1,2. \end{aligned}\tag{4.3.80}$$

Writing $\lambda_1^1 = \sin\varphi_1, \lambda_2^1 = \cos\varphi_1, \lambda_1^2 = \sin\varphi_2, \lambda_2^2 = \cos\varphi_2$, where φ_1, φ_2 are constant, we find that a rank-2 solution invariant under

$$\begin{aligned} X = \sin(\varphi_1-\varphi_2)\partial_t &+ \Big(\sin(\varphi_1-\varphi_2)u + (\cos(\varphi_1)+\cos(\varphi_2))\sqrt{gh}\Big)\,\partial_x \\ &+ \Big(\sin(\varphi_1-\varphi_2)v - (\sin(\varphi_1)+\sin(\varphi_2))\sqrt{gh}\Big)\,\partial_y \end{aligned}\tag{4.3.81}$$

exists if and only if the angle between φ_1 and φ_2 satisfies

$$|\varphi_1-\varphi_2| = \frac{\pi}{3},\tag{4.3.82}$$

in comparison with relation (4.3.61). This nonscattering rank-2 solution of the SWW equations can be presented as

$$u = u_0 + 2\sqrt{g}\left(\lambda_1^1 h_1(r^1) - \lambda_1^2 h_2(r^2)\right), \quad v = v_0 + 2\sqrt{g}\left(\lambda_2^1 h_1(r^1) - \lambda_2^2 h_2(r^2)\right),$$
$$h = \left(h_1(r^1) + h_2(r^2)\right)^2, \quad u_0, v_0 \in \mathbb{R},$$

where the functions $h_1(r^1)$ and $h_2(r^2)$ are arbitrary functions of the Riemann invariants

$$r^1 = -\left(\lambda_1^1 u_0 + \lambda_2^1 v_0 + 3\sqrt{g}h_1(r^1)\right)t + \lambda_1^1 x + \lambda_2^1 y, \quad \lambda_j^i \in \mathbb{R}, \quad \vec{\lambda}^1 \cdot \vec{\lambda}^2 = 1/2,$$
$$r^2 = -\left(\lambda_1^2 u_0 + \lambda_2^2 v_0 - 3\sqrt{g}h_2(r^2)\right)t + \lambda_1^2 x + \lambda_2^2 y, \quad |\vec{\lambda}^i| = 1, \quad i = 1, 2. \tag{4.3.83}$$

so that the angle between $\vec{\lambda}^1$ and $\vec{\lambda}^2$ satisfies (4.3.82). The similarity with solution (4.3.65) is not surprising. It can in fact be obtained by considering the wave vector $\vec{\lambda}^2$ in the opposite direction, that is by setting $\vec{\lambda}^2 \to -\vec{\lambda}^2$ in expressions (4.3.61), (4.3.65) and (4.3.66). The computation of the corresponding solution of the RSWW equations is done analogously to that of the previous case and the result is included in Table 4.4.

4.4. Conclusion

In this work, we have extended the applicability of the conditional symmetry approach in the context of Riemann invariants to a certain class of first order inhomogeneous quasilinear hyperbolic system of the first order, namely those systems that are equivalent to a homogeneous one under an invertible point transformation. Such classes of systems have been characterized recently in the case of systems of two equations in two dependent and independent variables in [**22**] and an algorithm to construct the appropriate point transformation was also given. The key element in this analysis is the presence of an infinite dimensional Lie algebra admitted by every quasilinear homogenous system in two variables. Although this is not true in general for multidimensional systems, we have been able to show that such a transformation exists for the rotating shallow water wave equations and after an analysis of the rank-k solutions of the SWW equations, we used it to construct several of their implicit solutions expressed in terms of Riemann invariants. While several classes of invariant solutions of the RSWW equations are known, these new conditionally invariant solutions possess in general a considerable degree of freedom in the sense that they depend on one or two arbitrary functions of the Riemann invariants. Although it is possible in the case of a homogeneous system to select these arbitrary functions so as to obtain bounded solutions for every value of the Riemann invariants, such solutions could not be constructed here since the point transformation (4.1.3) is singular at times $t = \frac{\pi}{2\Omega}(2n+1), n \in \mathbb{N}$. However, by using invariance under time translation, we have shown that it is possible to construct solutions expressed in terms of Riemann invariants defined in a finite interval around $t = 0$.

One may ask whether rank-k solutions of a given inhomogeneous system in the form (4.3.2) can be constructed without relying on a point transformation bringing it to a homogeneous form. A preliminary analysis shows that this type of solution would possess invariance properties similar to those admitted by homogeneous systems, as expressed in Proposition 1. This study shall be addressed in a future work.

Type	Solution	Riemann invariant	Comments
1.E	$u=u_0-\frac{\lambda_2}{\lambda_1}\varphi(r)$	$r=-u_0\lambda_1 t+\lambda_1 x+\lambda_2 y$	$\varphi:\mathbb{R}\to\mathbb{R}$
	$v=\varphi(r)$		$\lambda_i, u_0\in\mathbb{R}$
	$h=h_0$		$h_0\in\mathbb{R}^+$
2.E	$u=C\sin r$	$r=C(-Ct+x\sin r+y\cos r)$	$C\in\mathbb{R}$
	$v=C\cos r$		$h_0\in\mathbb{R}^+$
	$h=h_0$		
3.E	$u=\varphi(r)$	$r=-2Ct+\frac{C}{\varphi(r)}x+u(r)y$	$\varphi:\mathbb{R}\to\mathbb{R}$
	$v=C/\varphi(r)$		$C\in\mathbb{R}$
	$h=h_0$		$h_0\in\mathbb{R}^+$
4.S	$u=u_0+2\lambda_1\sqrt{g}\varphi(r)$	$r=-(\lambda_1 u_0+\lambda_2 v_0+3\sqrt{gh})t+\lambda_1 x+\lambda_2 y$	$\varphi:\mathbb{R}\to\mathbb{R}$
	$v=v_0+2\lambda_2\sqrt{g}\varphi(r)$		$\lambda_i, u_0, v_0\in\mathbb{R}$
	$h=\varphi(r)^2$		
5.S	$u=u_0-2\sqrt{g}\cos\varphi(r)$	$r=-\left(u_0\sin(\varphi(r))+v_0\cos(\varphi(r))+\sqrt{g}(\varphi(r)+h_0)\right)t$	$\varphi:\mathbb{R}\to\mathbb{R}$
	$v=v_0+2\sqrt{g}\sin\varphi(r)$	$+\sin(\varphi(r))x+\cos(\varphi(r))y$	$u_0, v_0\in\mathbb{R}$
	$h=(\varphi(r)+h_0)^2$		$h_0\in\mathbb{R}^+$
6.S	$u=u_0+\sqrt{2\pi g}S\left(\sqrt{\frac{2\varphi(r)}{\pi}}\right)$	$r=-\Bigg(\sin(\varphi(r))\left(u_0+\sqrt{2\pi g}S\left(\sqrt{\frac{2\varphi(r)}{\pi}}\right)\right)$	$\varphi:\mathbb{R}\to\mathbb{R}^+$
	$v=v_0+\sqrt{2\pi g}C\left(\sqrt{\frac{2\varphi(r)}{\pi}}\right)$	$+\cos(\varphi(r))\left(v_0+\sqrt{2\pi g}C\left(\sqrt{\frac{2\varphi(r)}{\pi}}\right)\right)$	$u_0, v_0\in\mathbb{R}$
	$h=(\sqrt{\varphi(r)}+h_0)^2$	$+\sqrt{g}(\sqrt{\varphi(r)}+h_0)\Bigg)t+\sin(\varphi(r))x+\cos(\varphi(r))y$	$h_0\in\mathbb{R}^+$

TAB. 4.2. Rank-1 solutions of the SWW equation (4.1.2). The functions $S(\cdot)$ and $C(\cdot)$ are the sine and cosine Fresnel integrals.The function $\varphi(r)$ denotes an arbitrary positive function.

Type	Riemann invariants	Solution	Comments
ES	$r^1=\lambda_2^1(v)\left(\left(2G(s)+\sqrt{3}\varepsilon F(r^2)\right)t-\sqrt{3}\varepsilon x-y\right)$	$u=\frac{\sqrt{3}}{3}\varepsilon G(s)+F(r^2)$	$h_0\in\mathbb{R}^+,\varepsilon^2=1$
	$r^2=\left(\frac{3}{2}F(r^2)+\frac{h_0}{2}\right)t-x$	$v=G(s)$	$F,G:\mathbb{R}\to\mathbb{R}$
	$s=r^2-\frac{\sqrt{3}\varepsilon}{2\lambda_2^1(G(s))}r^1$	$h=\frac{1}{4g}\left(F(r^2)-\frac{2\sqrt{3}}{3}\varepsilon G(s)+h_0\right)^2$	$\lambda_2^1:\mathbb{R}\to\mathbb{R}$
SS	$r^1=-\left(\lambda_1^1u_0+\lambda_2^1v_0+3\sqrt{g}h_1(r^1)\right)t+\lambda_1^1x+\lambda_2^1y$	$u=u_0+2\sqrt{g}\left(\lambda_1^1h_1(r^1)+\varepsilon\lambda_1^2h_2(r^2)\right)$	$u_0,v_0,\lambda_j^i\in\mathbb{R},\lvert\vec{\lambda}^i\rvert=1$
	$r^2=-\left(\lambda_1^2u_0+\lambda_2^2v_0+3\varepsilon\sqrt{g}h_2(r^2)\right)t+\lambda_1^2x+\lambda_2^2y$	$v=v_0+2\sqrt{g}\left(\lambda_2^1h_1(r^1)+\varepsilon\lambda_2^2h_2(r^2)\right)$	$h_1,h_2:\mathbb{R}\to\mathbb{R}$
		$h=\left(h_1(r^1)+h_2(r^2)\right)^2$	$\vec{\lambda}^1\cdot\vec{\lambda}^2=-\varepsilon/2,\varepsilon^2=1$

TAB. 4.3. Rank-2 solutions of the SWW equations. Unassigned functions are arbitrary functions of their respective argument.

Type	Riemann invariants	Solution	Comments
ES	$r^1 = \lambda_2^1(G(\tilde{s}))\Big(-\frac{1}{2\Omega}(2G(s)+\sqrt{3}\varepsilon F(r^2))\cot(\Omega t)$	$u = -\Big(\frac{\sqrt{3}}{3}\varepsilon G(s)+F(r^2)\Big)\cot(\Omega t)-G(s)+\Omega(y+x\cot(\Omega t))$	$h_0 \in \mathbb{R}^+, \epsilon^2 = 1$
	$-\frac{\sqrt{3}}{2}\varepsilon(y-x\cot(\Omega t))+\frac{1}{2}(x+y\cot(\Omega t))\Big)$	$v = \frac{\sqrt{3}}{3}\varepsilon G(s)+F(r^2)-G(s)\cot(\Omega t)-\Omega(x-y\cot(\Omega t))$	$F, G : \mathbb{R} \to \mathbb{R}$
	$r^2 = -\frac{1}{4\Omega}(3F(r^2)+h_0)\cot(\Omega t)-\frac{1}{2}(y-x\cot(\Omega t)),$	$h = \frac{1}{4g}\Big(F(r^2)-\frac{2\sqrt{3}}{3}\varepsilon G(s)+h_0\Big)^2\csc^2(\Omega t)$	$\lambda_2^1 : \mathbb{R} \to \mathbb{R}$
	$s = r^2 - \frac{\sqrt{3}\varepsilon}{2\lambda_2^1(G(s))}r^1$		
SS	$r^1 = \frac{1}{2}\Big[\frac{1}{\Omega}(\lambda_1^1 u_0+\lambda_2^1 v_0+3\sqrt{g}h_1(r^1))\cot(\Omega t)$	$u = -(u_0+2\sqrt{g}(\lambda_1^1 h_1(r^1)+\varepsilon\lambda_1^2 h_2(r^2)))\cot(\Omega t),$	$u_0, v_0, \lambda_j^i \in \mathbb{R},$
	$+\lambda_1^1(y-x\cot(\Omega t))-\lambda_2^1(x+y\cot(\Omega t))\Big]$	$-(v_0+2\sqrt{g}(\lambda_2^1 h_1(r^1)+\varepsilon\lambda_2^2 h_2(r^2)))+\Omega(y+x\cot(\Omega t))$	$h_1, h_2 : \mathbb{R} \to \mathbb{R}$
	$r^2 = \frac{1}{2}\Big[\frac{1}{\Omega}(\lambda_1^2 u_0+\lambda_2^2 v_0+3\varepsilon\sqrt{g}h_2(r^2))\cot(\Omega t)$	$v = -(v_0+2\sqrt{g}(\lambda_2^1 h_1(r^1)+\varepsilon\lambda_2^2 h_2(r^2)))\cot(\Omega t)$	$\vec{\lambda}^1 \cdot \vec{\lambda}^2 = -\varepsilon/2$
	$+\lambda_1^2(y-x\cot(\Omega t))-\lambda_2^2(x+y\cot(\Omega t))\Big]$	$+u_0+2\sqrt{g}(\lambda_1^1 h_1(r^1)+\varepsilon\lambda_1^2 h_2(r^2))-\Omega(x-y\cot(\Omega t))$	$\lvert\vec{\lambda}^i\rvert = 1,\ \varepsilon^2 = 1$
		$h = (h_1(\tilde{r}^1)+h_2(\tilde{r}^2))\csc^2(\Omega t)$	

TAB. 4.4. Rank-2 solutions of the RSWW equations. Unassigned functions are arbitrary functions of their respective argument.

No	Riemann invariants	Solution	Comments
1.	$r^1 = (2\tanh^2(s) + \sqrt{3}\tanh^2(r^2))t - \sqrt{3}x - y$ $r^2 = \left(\frac{3}{2}\tanh^2(r^2) + \frac{h_0}{2}\right)t - x$ $s = r^2 - \frac{\sqrt{3}}{2}r^1$	$u = \frac{\sqrt{3}}{3}\tanh^2(s) + \tanh^2(r^2)$ $v = \tanh^2(s)$ $h = \frac{1}{4g}\left(\tanh^2(r^2) - \frac{2\sqrt{3}}{3}\tanh^2(s) + h_0\right)^2$	Anti-bump
2.	$r^1 = (2\,\mathrm{sech}^2(s) + \sqrt{3}\,\mathrm{sech}^2(r^2))t - \sqrt{3}x - y$ $r^2 = \left(\frac{3}{2}\mathrm{sech}^2(r^2) + \frac{h_0}{2}\right)t - x$ $s = r^2 - \frac{\sqrt{3}}{2}r^1$	$u = \frac{\sqrt{3}}{3}\mathrm{sech}^2(s) + \mathrm{sech}^2(r^2)$ $v = \mathrm{sech}^2(s)$ $h = \frac{1}{4g}\left(\mathrm{sech}^2(r^2) - \frac{2\sqrt{3}}{3}\mathrm{sech}^2(s) + h_0\right)^2$	Bump
3.	$r^1 = -\left(u_0 + 3\sqrt{g}\,\mathrm{sech}^2(r^1)\right)t + x$ $r^2 = -\left(-\frac{u_0}{2} + \frac{\sqrt{3}}{2}v_0 + 3\sqrt{g}\,\mathrm{sech}^2(r^2)\right)t - \frac{1}{2}x + \frac{\sqrt{3}}{2}y$	$u = u_0 + 2\sqrt{g}\left(\mathrm{sech}^2(r^1) - \frac{1}{2}\mathrm{sech}^2(r^2)\right)$ $v = v_0 + \sqrt{3g}\,\mathrm{sech}^2(r^2)$ $h = \left(\mathrm{sech}^2(r^1) + \mathrm{sech}^2(r^2)\right)^2$	Bump $u_0, v_0 \in \mathbb{R}$
4.	$r^1 = -\left(u_0 + \frac{3\sqrt{g}A_1 r^1}{\sqrt{1+B_1(r^1)^2}}\right)t + x$ $r^2 = -\left(-\frac{u_0}{2} + \frac{\sqrt{3}}{2}v_0 + \frac{3\sqrt{g}A_2}{\sqrt{1+B_2(r^2)^2}}\right)t - \frac{1}{2}x + \frac{\sqrt{3}}{2}y$	$u = u_0 + 2\sqrt{g}\left(\frac{A_1 r^1}{\sqrt{1+B_1(r^1)^2}} - \frac{A_2 r^2}{2\sqrt{1+B_2(r^2)^2}}\right)$ $v = v_0 + \frac{\sqrt{3g}A_2}{\sqrt{1+B_2(r^2)^2}}$ $h = \left(\frac{A_1 r^1}{\sqrt{1+B_1(r^1)^2}} + \frac{A_2 r^2}{\sqrt{1+B_2(r^2)^2}}\right)^2$	Kink $u_0, v_0, A_1, A_2 \in \mathbb{R}$ $B_1, B_2 \in \mathbb{R}^+$
5.	$r^1 = -\left(u_0 + \frac{3\sqrt{g}A_1}{\wp\left(r^1, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}A_1^4\right)}\right)t + x$ $r^2 = -\left(-\frac{u_0}{2} + \frac{\sqrt{3}}{2}v_0 + \frac{A_2}{\wp\left(r^2, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}A_2^4\right)}\right)t - \frac{1}{2}x + \frac{\sqrt{3}}{2}y$	$u = u_0 + 2\sqrt{g}\left(\frac{A_1}{\wp\left(r^1, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}A_1^4\right)} - \frac{1}{2}\frac{A_2}{\wp\left(r^2, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}A_2^4\right)}\right)$ $v = v_0 + \frac{A_2}{\wp\left(r^2, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}A_2^4\right)}$ $h = \left(\frac{A_1}{\wp\left(r^1, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}A_1^4\right)} + \frac{A_2}{\wp\left(r^2, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}A_2^4\right)}\right)^2$	Periodic $u_0, v_0, A_1, A_2 \in \mathbb{R}$

TAB. 4.5. Examples of rank-2 solutions of the SWW equations which remain constant for some choices of the arbitrary constants. The function $\wp(\cdot, g_2, g_3)$ is the elliptic Weierstrass $\wp$ function with invariants g_2,g_3.

Chapitre 5

CONDITIONAL SYMMETRIES AND RIEMANN INVARIANTS FOR INHOMOGENEOUS QUASILINEAR SYSTEMS

Référence complète : A.M. Grundland et B. Huard, Conditional symmetries and Riemann invariants for inhomogeneous quasilinear systems, Journal of Mathematical Physics, soumis juin 2010.

Résumé

Une nouvelle approche permettant la construction de solutions pour les systèmes quasilinéaires hyperboliques et non homogènes en plusieurs dimensions est présentée. Cette approche est basée sur les méthodes des symétries conditionnelles et des invariants de Riemann. Nous discutons en détails les conditions nécessaires et suffisantes garantissant l'existence de solutions de rang 2 et 3 exprimées en termes d'invariants de Riemann. Cette analyse est effectuée en appliquant le théorème de Cayler-Hamilton à un système algébrique associé au système considéré. Nous obtenons des propositions permettant la construction de telles solutions. Cette approche est appliquée aux équations non homogènes de la dynamique des fluides en présence de la force gravitationnelle et de la force de Coriolis. Plusieurs nouvelles solutions de rang 2 ainsi que leur interprétation physique sont données.

Abstract

A new approach to the solution of quasilinear nonelliptic first-order systems of inhomogeneous PDEs in many dimensions is presented. It is based on a version of

the conditional symmetry and Riemann invariant methods. We discuss in detail the necessary and sufficient conditions for the existence of rank-2 and rank-3 solutions expressible in terms of Riemann invariants. We perform the analysis using the Cayley-Hamilton theorem for a certain algebraic system associated with the initial system. The problem of finding such solutions has been reduced to expanding a set of trace conditions on wave vectors and their profiles which are expressible in terms of Riemann invariants. A couple of theorems useful for the construction of such solutions are given. These theoretical considerations are illustrated by the example of inhomogeneous equations of fluid dynamics which describe motion of an ideal fluid subjected to gravitational and Coriolis forces. Several new rank-2 solutions are obtained. Some physical interpretation of these results is given.

5.1. Introduction

In the last three decades, a number of useful extensions of the classical Lie approach to group-invariant solutions of PDEs have generated a great deal of interest and activity in several fields of research, including mathematics, physics, chemistry and biology. In particular, the method of partially invariant solutions [**90, 93, 94**], the non-classical method [**5, 70**], the conditional symmetry method [**47, 48, 50, 107, 116**], the weak symmetry method [**2, 52**] and the introduction of general "side conditions" or differential constraints [**91**] have been shown to play an essential role in several applications to nonlinear phenomena. In particular, the method of differential constraints incorporates all known methods for determining particular solutions of PDEs. They establish a direct connection between certain classes of solutions and a framework based on the theory of overdetermined nonlinear systems of PDEs. In an attempt to understand certain classes of solutions expressible in terms of Riemann invariants, we have introduced a specific version of the conditional symmetry method [**48**]. The basic idea was to analyze a Lie module of vector fields which are symmetries of an overdetermined system defined by the initial system of equations and certain first-order differential constraints. It was shown [**48**] that this overdetermined system admits rank-k

solutions expressible in terms of Riemann invariants. The method of conditional symmetry has led to a large number of new results.

This paper is a follow-up of our previous investigations which were performed in [**18, 47, 48, 61**]. Here, we are using the Riemann invariants method for inhomogeneous nonelliptic quasilinear first-order systems of l PDEs in $p = n + 1$ independent variables of the form

$$u_t + A^i(u)u_{x^i} = B(u) \tag{5.1.1}$$

where $x = (t, x^1, \ldots, x^n) = (x^0, \bar{x}) \in \mathbb{R}^{n+1}$ and $u = (u^1, \ldots, u^q) \in \mathbb{R}^q$ with the initial condition

$$x_0 = 0 \quad : \quad u(0, \bar{x}) = u_0(\bar{x}) \in \mathbb{R}^q.$$

As usual in mathematical physics, the space of independent variables $(x_0, \bar{x}) \in \mathbb{R}^p$ is called the physical space while the space of values of dependent variables $u = (u^1(x), \ldots, u^q(x))$ is denoted by $U \subset \mathbb{R}^q$ and is called the hodograph space. The terms u_t, u_{x^i} denote the first-order partial derivatives of u with respect to t and x^i, respectively. The matrices $A^i(u)$ are $l \times q$ matrix functions of u and $B(u)$ is an l-component vector. Throughout this paper, we use the summation convention over repeated indices. All our considerations are local. It suffices to search for solutions defined on a neighborhood of $x = 0$. The solutions $u = u(x)$ of the system (5.1.1) are identified with their graphs which are $(n+1)$-dimensional manifolds in the cartesian product $X \times U$.

In our previous analysis of first-order systems of the form (5.1.1) (see e.g. [**48**]), we considered homogeneous systems, i.e. systems with $B(u) = 0$, with coefficients depending only on the unknown functions u. The methodological approach proposed in these works was based on some generalization of the Riemann invariants method. A specific aspect of that approach is the presence of both algebraic and geometric points of view. A conversion of systems of PDEs into algebraic form was made possible by representing the so called integral elements as linear combinations of simple elements [**48**] associated with those vectors fields which generate characteristic curves in the $X \times U$ space. The introduction of those elements proved to be fruitful from the point of view of constructing the rank-k solutions in closed form by means of the Cayley-Hamilton theorem applied directly to the algebraic form of the system. Using a variant of the conditional symmetry method, we have shown that these solutions comprise among others multiple wave solutions as a superposition of two or more single Riemann waves. The results obtained for the homogeneous systems were so promising that it seemed worthwhile to try to extend this approach and check its effectiveness for the case of nonelliptic inhomogeneous partial differential first-order systems. This is, in short, the objective of this paper.

The paper is organized as follows. In Section 5.2, we investigate the group-invariance properties of rank-2 and rank-3 solutions for inhomogeneous systems (5.1.1) and obtain necessary and sufficient conditions for their existence. Section 5.3 illustrates the construction of such solutions for the inhomogeneous Euler equations describing a $(3+1)$-dimensional fluid flow under the influence of gravitational and Coriolis effects. Results and future perspectives are summarized in Section 5.4.

5.2. Rank-k solutions described by inhomogeneous systems of PDEs

A version of the conditional symmetry method [**18, 47, 48**] has been developed recently for homogeneous nonelliptic systems (5.1.1). This method consists of supplementing the original system of PDEs (5.1.1) with first-order differential constraints (DCs) for which a symmetry criterion for the given system of PDEs is identically satisfied. It turns out that under certain circumstances, the so-augmented system of PDEs admits a larger class of Lie symmetries than the original system of PDEs (5.1.1). We now extend this method to inhomogeneous systems (5.1.1) and reformulate the task of constructing rank-k solutions expressible in terms of Riemann invariants in the language of the group analysis of differential equations. For this purpose, we require the solution u of PDEs (5.1.1) to be invariant under the family of commuting first-order differential operators

$$X_a = \xi_a^i(u)\frac{\partial}{\partial x^i}, \quad a = 1, \ldots, p-k, \tag{5.2.1}$$

defined on $X \times U$ and satisfying the conditions

$$\xi_a^i \lambda_i^A = 0, \quad A = 0, \ldots, k-1, \quad a = 1, \ldots, p-k. \tag{5.2.2}$$

Here, the inhomogeneous wave vector λ^0 satisfies the rank condition

$$\text{rank}\left(\lambda_i^0 A^i, B(u)\right) = \text{rank}\left(\lambda_i^0 A^i\right), \tag{5.2.3}$$

while the wave vectors $\lambda^0, \lambda^1, \ldots, \lambda^{k-1}$ are linearly independent vectors satisfying the dispersion relation for system (5.1.1)

$$\text{rank}(\lambda_i^A A^i) < l, \quad A = 1, \ldots, k-1.$$

The group-invariant solutions of the system (5.1.1) then consist of those functions $u = f(x)$ which satisfy both the initial system (5.1.1) and a set of first-order differential constraints

$$\xi_a^i \frac{\partial u^\alpha}{\partial x^i} = 0, \quad a = 1, \ldots, p-k, \tag{5.2.4}$$

ensuring that the characteristics of the vector fields X_a are equal to zero. In this case, if the vector function $u(x)$ is invariant under a set of $(p-k)$ vector fields X_a for which the orthogonality property (5.2.2) is satisfied, then the solution $u(x)$ of (5.1.1) can be defined implicitly by the following set of relations between the variables u^α, x^i and r^A :

$$u = f(r^0, \ldots, r^{k-1}), \quad r^A(x,u) = \lambda_0^A(u)t + \lambda_i^A(u)x^i, \quad A = 0, \ldots, k-1. \tag{5.2.5}$$

Each function $r^A(x,u)$ is called the Riemann invariant associated with the vector λ^A.

A vector field X_a defined on $X \times U$ is said to be a conditional symmetry of system (5.1.1) if it is tangent to $\mathbb{S} = \mathbb{S}_\Delta \cap \mathbb{S}_Q$, where $\mathbb{S}_\Delta$ and $\mathbb{S}_Q$ are submanifolds of the solution spaces defined by

$$\mathbb{S}_\Delta = \left\{\left(x, u^{(1)}\right) : u_t + A^i(u)u_{x^i} = B(u)\right\},$$

$$\mathbb{S}_Q = \left\{\left(x, u^{(1)}\right) : \xi_a^i u_{x^i}^\alpha = 0, \quad \alpha = 1, \ldots, q, \quad a = 1, \ldots, p-k\right\}.$$

An Abelian Lie algebra L spanned by the vector fields $X_1, \ldots, X_{p-k}$ is called a conditional symmetry algebra of the original system (5.1.1) if the following condition

$$\mathrm{pr}^{(1)} X_a \left(u_t + A^i(u) u_{x^i} - B(u) \right) \Big|_{\mathbb{S}} = 0, \quad a = 1, \ldots, p-k,$$

is satisfied, where $\mathrm{pr}^{(1)} X_a$ is the first prolongation of X_a.

When studying solutions of type (5.2.5), it is convenient from the computational point of view to write system (5.1.1) in the form of a trace equation,

$$\mathrm{Tr}\left[\mathcal{A}^\mu(u)\partial u\right] = B^\mu(u), \quad \mu = 1, \ldots, l, \tag{5.2.6}$$

where $\mathcal{A}^\mu(u)$ are $p \times q$ matrix functions of u. The Jacobian matrix of relations (5.2.5) can be expressed in matrix form either as

$$\partial u = (u^\alpha_{x^i}) = \frac{\partial f}{\partial r} \left(\mathcal{I}_k - \left(\eta_0 t + \eta_i x^i\right) \frac{\partial f}{\partial r} \right)^{-1} \lambda, \tag{5.2.7}$$

or as

$$\partial u = \left(\mathcal{I}_q - \frac{\partial f}{\partial r} \left(\eta_0 t + \eta_i x^i\right) \right)^{-1} \frac{\partial f}{\partial r} \lambda, \tag{5.2.8}$$

where

$$\frac{\partial f}{\partial r} = \left(\frac{\partial f^\alpha}{\partial r^A} \right) \in \mathbb{R}^{q \times k}, \quad \lambda = (\lambda^A_i) \in \mathbb{R}^{k \times p}, \quad \eta_i = \left(\frac{\partial \lambda^A_i}{\partial u^\alpha} \right), \quad i = 0, \ldots, n,$$

and $\mathcal{I}_k$ and $\mathcal{I}_q$ are the $k \times k$ and $q \times q$ identity matrices respectively.

A solution of the form (5.2.5) is called a rank-k solution if, in some open set of the origin $x = 0$, we can express u explicitly as a graph over the space of independent and dependent variables and the condition

$$\mathrm{rank}(\partial u) = k$$

holds.

The condition of inversibility of the matrices appearing in equations (5.2.7) and (5.2.8)

$$M_1 = \mathcal{I}_k - \left(\eta_0 t + \eta_i x^i\right) \frac{\partial f}{\partial r}, \quad M_2 = \mathcal{I}_q - \frac{\partial f}{\partial r} \left(\eta_0 t + \eta_i x^i\right),$$

restricts the domain of existence of rank-k solutions. However, since the inverse matrices are well defined at $x = 0$, there exists a neighborhood of the origin in which $\det(M_i) \neq 0$, $i = 1, 2$, allowing us to look for rank-k solutions locally parametrized by equations (5.2.5). It should also be noted that because of the Weinstein-Aronzjain determinant relation

$$\det(\mathcal{I}_k - PQ) = \det(\mathcal{I}_q - QP), \quad P \in \mathbb{R}^{k\times q}, \quad Q \in \mathbb{R}^{q\times k}, \tag{5.2.9}$$

which can be derived from the relation

$$\det\left[\begin{pmatrix} \mathcal{I}_k & P \\ Q & \mathcal{I}_q \end{pmatrix}\begin{pmatrix} \mathcal{I}_k & 0 \\ -Q & \mathcal{I}_q \end{pmatrix}\right] = \det\left[\begin{pmatrix} \mathcal{I}_k & 0 \\ -Q & \mathcal{I}_q \end{pmatrix}\begin{pmatrix} \mathcal{I}_k & P \\ Q & \mathcal{I}_q \end{pmatrix}\right],$$

we have that $\det(M_1) = \det(M_2)$. Hence, the regions of validity for the expressions (5.2.7) and (5.2.8) are the same. Morever, (5.2.9) implies that $\det(M_1)$ is a polynomial of order $s = \min(k, q)$. Inserting (5.2.7) into the original system (5.2.6), we obtain the result

$$\operatorname{Tr}\left[\mathcal{A}^\mu(u)\left(\mathcal{I}_k - \left(\eta_0 t + \eta_i x^i\right)\frac{\partial f}{\partial r}\right)^{-1}\lambda\right] = B^\mu(u),$$

or, upon multiplication by $\det(M_1)$,

$$\operatorname{Tr}\left[\mathcal{A}^\mu(u)\operatorname{adj}\left(\mathcal{I}_k - \left(\eta_0 t + \eta_i x^i\right)\frac{\partial f}{\partial r}\right)\lambda\right] - \det\left[\mathcal{I}_k - \left(\eta_0 t + \eta_i x^i\right)\frac{\partial f}{\partial r}\right]B^\mu(u) = 0, \tag{5.2.10}$$

which is a polynomial expression of degree s in the independent variables (t, x^i). For the system (5.2.10) to be expressed only in terms of Riemann invariants, we must require that every coefficient of t, x^i of this polynomial vanish identically. We then consider equation (5.2.10) and all its partial derivatives up to order s with respect to the independent variables t, x^i taken at the origin and set the resulting expressions to zero. The partial derivatives of the trace equation (5.2.10) are obtained by making use of the Jacobi equations for the derivative of the determinant of any square matrix $M = M(\xi)$ depending on a parameter ξ,

$$\begin{aligned} &\frac{\partial}{\partial \xi}\det[M] = \operatorname{Tr}\left[\operatorname{adj}[M]\frac{\partial M}{\partial \xi}\right], \\ &\frac{\partial}{\partial \xi}\operatorname{adj}[M] = \left(\operatorname{Tr}\left[\operatorname{adj}[M]\frac{\partial M}{\partial \xi}\right]\mathcal{I} - \operatorname{adj}[M]\frac{\partial M}{\partial \xi}\right)M^{-1}. \end{aligned} \tag{5.2.11}$$

A conditionally invariant solution of (5.1.1) under an Abelian algebra L spanned by vector fields $X_1, \ldots, X_{p-k}$ of the form (5.2.1) is obtained by solving the overdetermined system

$$\Delta' : \begin{cases} \mathrm{Tr}\left[\mathcal{A}^\mu(u)\mathrm{adj}\left(\mathcal{I}_k - \left(\eta_0 t + \eta_i x^i\right)\frac{\partial f}{\partial r}\right)\lambda\right] - \det\left[\mathcal{I}_k - \left(\eta_0 t + \eta_i x^i\right)\frac{\partial f}{\partial r}\right] B^\mu(u) = 0, \\ \xi^i_a u^\alpha_{x^i} = 0, \quad a = 1, \ldots, p-k, \quad \alpha = 1, \ldots, q. \end{cases} \tag{5.2.12}$$

Therefore, in the case $k = 2$, we can devise the following proposition.

Proposition. *Consider a fixed set of linearly-independent wave vectors λ^0 and λ^1 associated with a nondegenerate quasilinear nonelliptic first-order system of PDEs (5.1.1) in p independent variables and q dependent variables. This system admits a $(p-2)$-dimensional conditional symmetry algebra L if and only if $(p-2)$ linearly independent vector fields*

$$X_a = \xi^i_a(u)\frac{\partial}{\partial x^i}, \quad \lambda^A_i \xi^i_a = 0, \quad a = 1, \ldots, p-2, \quad A = 0, 1,$$

satisfy the conditions

$$\mathrm{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\lambda\right] = B^\mu, \quad \mu = 1, \ldots, l, \tag{5.2.13a}$$

$$\mathrm{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\lambda\right] = 0, \quad i = 0, \ldots, n, \tag{5.2.13b}$$

$$\mathrm{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\left(\eta_i \frac{\partial f}{\partial r}\eta_j + \eta_j \frac{\partial f}{\partial r}\eta_i\right)\frac{\partial f}{\partial r}\lambda\right] = 0, \quad i \neq j = 0, \ldots, n, \tag{5.2.13c}$$

$$\det\left[\eta_i \frac{\partial f}{\partial r}\right] = 0, \quad \eta_i = \left(\frac{\partial \lambda^A_i}{\partial u^\alpha}\right) \in \mathbb{R}^{2\times q}. \tag{5.2.13d}$$

and such that $\mathrm{rank}(\partial f/\partial r) = 2$ *on some neighborhood of a point $(x_0, u_0) \in \mathbb{S}$. The solution of (5.1.1) which is invariant under the Lie algebra L is precisely a* rank-2 *solution of the form (5.2.5).*

Proof : It was shown in [**48**] that in the new coordinates $\mathbb{R}^p \times \mathbb{R}^q$

$$\bar{x}^1 = r^1(x,u), \bar{x}^2 = r^2(x,u), \bar{x}^3 = x^3, \ldots, \bar{x}^p = x^p, \bar{u}^1 = u^1, \ldots, \bar{u}^q = u^q,$$

the vector fields X_a adopt the rectified form

$$X_a = \frac{\partial}{\partial_{\bar{x}^a}}, \quad a = 3, \ldots, p.$$

Hence, the symmetry criterion for G to be the symmetry group of the overdetermined system (5.2.12) requires that the vector fields X_a of G satisfy

$$X_a(\Delta') = 0,$$

whenever equations (5.2.12) hold. Thus the symmetry criterion applied to the invariance conditions (5.2.4) is identically equal to zero. After applying the symmetry criterion to the system (5.2.10) in new coordinates and taking into account the conditions (5.2.13), we obtain the equations which are identically satisfied.

The converse is also true. The requirement that the system (5.1.1) be nondegenerate means that it is locally solvable and is of maximal rank at every point $(x_0, u_0^{(1)}) \in \mathbb{S}$. Therefore **[89]**, the infinitesimal symmetry criterion is a necessary and sufficient condition for the existence of the symmetry group G of the overdetermined system (5.2.12). Using relations (5.2.11), we show that equations (5.2.13) must hold. Equations (5.2.13a) and (5.2.13b) are easily obtained by considering the system (5.2.10) and all its first-order partial derivatives with respect to t and x^i. The second-order derivatives with respect to x^i and x^j provide us with the condition

$$\mathrm{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\left(\eta_i \frac{\partial f}{\partial r}\eta_j + \eta_j \frac{\partial f}{\partial r}\eta_i\right)\frac{\partial f}{\partial r}\lambda\right] = 0, \quad i,j = 0,\ldots,n,$$

which reduces when $i = j$ to

$$\mathrm{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\lambda\right] = 0, \quad i = 0,\ldots,n. \tag{5.2.14}$$

Considering equations (5.2.13b), we can write (5.2.14) as

$$\begin{aligned}\mathrm{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\lambda\right] &= \mathrm{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\lambda\right] - \mathrm{Tr}\left[\eta_i \frac{\partial f}{\partial r}\right]\mathrm{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\lambda\right] \\ &= \mathrm{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\left(\eta_i \frac{\partial f}{\partial r} - \mathrm{Tr}\left[\eta_i \frac{\partial f}{\partial r}\right]\mathcal{I}_2\right)\lambda\right]. \end{aligned} \tag{5.2.15}$$

According to the Cayley-Hamilton theorem, each matrix $\left(\eta_i \frac{\partial f}{\partial r}\right) \in \mathbb{R}^{2\times 2}$ satifies

$$\left(\eta_i \frac{\partial f}{\partial r}\right)^2 - \mathrm{Tr}\left[\eta_i \frac{\partial f}{\partial r}\right]\left(\eta_i \frac{\partial f}{\partial r}\right) = -\det\left[\eta_i \frac{\partial f}{\partial r}\right]\mathcal{I}_2$$

allowing us to write (5.2.15) as

$$\mathrm{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\lambda\right] = \det\left[\eta_i \frac{\partial f}{\partial r}\right]\mathrm{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\lambda\right] = B^\mu \det\left[\eta_i \frac{\partial f}{\partial r}\right],$$

by (5.2.13a). As a consequence of the assumption $B^\mu \not\equiv 0$, we must have that

$$\det\left[\eta_i \frac{\partial f}{\partial r}\right] = 0.$$

That ends the proof, since the solutions of the overdetermined system (5.2.12) are invariant under the algebra L generated by the $(p-k)$ vector fields $X_1, \ldots, X_{p-2}$. The invariants of the group G of such vector fields are provided by the functions $\{r^0, r^1, u^1, \ldots, u^q\}$. So the general rank-$k$ solution of (5.1.1) takes the form (5.2.5).

Note that in the case $q = k = 2$, equations (5.2.13d) imply that

$$\det\left[\eta_i \frac{\partial f}{\partial r}\right] = \det\left[\eta_i\right] \det\left[\frac{\partial f}{\partial r}\right] = 0.$$

When $\det\left[\frac{\partial f}{\partial r}\right] = 0$, the considered solution reduces to a rank-1 solution. Consequently, when $q = 2$, rank-2 solutions exist only when $\det\left[\eta_i\right] = 0$ for $i = 0, \ldots, n$. Note that in every case, equations (5.2.13d) can be satisfied, for example, by choosing one of the vectors λ^A to be constant.

Analogously, we obtain the following conditions in the case $k = 3$.

Proposition. *Consider a fixed set of linearly-independent wave vectors* λ^0, λ^1 *and* λ^2 *associated with a nondegenerate quasilinear nonelliptic first-order system of PDEs (5.1.1) in* p *independent variables and* q *dependent variables. This system admits a* $(p-3)$*-dimensional conditional symmetry algebra* L *if and only if* $(p-3)$ *linearly independent vector fields*

$$X_a = \xi_a^i(u) \frac{\partial}{\partial x^i}, \quad \lambda_i^A \xi_a^i = 0, \quad a = 1, \ldots, p-3, \quad A = 0, 1, 2,$$

satisfy the conditions

$$\mathrm{Tr}\left[\mathcal{A}^{\mu}\frac{\partial f}{\partial r}\lambda\right]=B^{\mu},\quad \mu=1,\ldots,l, \tag{5.2.16a}$$

$$\mathrm{Tr}\left[\mathcal{A}^{\mu}\frac{\partial f}{\partial r}\eta_{i_1}\frac{\partial f}{\partial r}\lambda\right]=0,\quad i_1=0,\ldots,n, \tag{5.2.16b}$$

$$\mathrm{Tr}\left[\mathcal{A}^{\mu}\frac{\partial f}{\partial r}\left(\eta_{i_1}\frac{\partial f}{\partial r}\eta_{i_2}+\eta_{i_2}\frac{\partial f}{\partial r}\eta_{i_1}\right)\frac{\partial f}{\partial r}\lambda\right]=0,\quad i_1,i_2=0,\ldots,n, \tag{5.2.16c}$$

$$\begin{aligned}\mathrm{Tr}\Bigg[\mathcal{A}^{\mu}\frac{\partial f}{\partial r}\Bigg(&\eta_{i_1}\frac{\partial f}{\partial r}\eta_{i_2}\frac{\partial f}{\partial r}\eta_{i_3}+\eta_{i_1}\frac{\partial f}{\partial r}\eta_{i_3}\frac{\partial f}{\partial r}\eta_{i_2}+\eta_{i_2}\frac{\partial f}{\partial r}\eta_{i_1}\frac{\partial f}{\partial r}\eta_{i_3}\\ &+\eta_{i_2}\frac{\partial f}{\partial r}\eta_{i_3}\frac{\partial f}{\partial r}\eta_{i_1}+\eta_{i_3}\frac{\partial f}{\partial r}\eta_{i_1}\frac{\partial f}{\partial r}\eta_{i_2}+\eta_{i_3}\frac{\partial f}{\partial r}\eta_{i_2}\frac{\partial f}{\partial r}\eta_{i_1}\Bigg)\frac{\partial f}{\partial r}\lambda\Bigg]=0,\end{aligned} \tag{5.2.16d}$$

$$\det\left[\eta_i\frac{\partial f}{\partial r}\right]=0,\quad \eta_i=\left(\frac{\partial\lambda_i^A}{\partial u^{\alpha}}\right)\in\mathbb{R}^{3\times q}, \tag{5.2.16e}$$

and such that $\mathrm{rank}(\partial f/\partial r)=3$ *on some neighborhood of a point* $(x_0,u_0)\in\mathbb{S}$*, where the indices* i_1,i_2,i_3 *take the values* $0,\ldots,n$ *and are not all equal.*

Proof : The proof for sufficiency follows the same steps as in the case $k=2$. We now show the necessity of equations (5.2.16). As previously, equations (5.2.16a-d) represent the first, second and third-order partial derivatives of expression (5.2.10) taken at the origin, respectively. When $i_1=i_2=i_3=i$, equations (5.2.16d) reduce to

$$\mathrm{Tr}\left[\mathcal{A}^{\mu}\frac{\partial f}{\partial r}\eta_i\frac{\partial f}{\partial r}\eta_i\frac{\partial f}{\partial r}\eta_i\frac{\partial f}{\partial r}\lambda\right]=0.$$

For any 3 by 3 matrix M, the Leverrier-Faddeev algorithm and the Cayley-Hamilton theorem allow us to write

$$M^3-\mathrm{Tr}\,[M]M^2+\frac{1}{2}\left(\mathrm{Tr}\,[M]^2-\mathrm{Tr}\left[M^2\right]\right)M-\det[M]\mathcal{I}_3=0. \tag{5.2.17}$$

Hence, taking into account equations (5.2.16) and (5.2.17), we get

$$
\begin{aligned}
&\operatorname{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\lambda\right] - \operatorname{Tr}\left[\eta_i \frac{\partial f}{\partial r}\right] \operatorname{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\lambda\right] \\
&\quad = \operatorname{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\left(\eta_i \frac{\partial f}{\partial r}\right)^2 \left(\eta_i \frac{\partial f}{\partial r} - \operatorname{Tr}\left[\eta_i \frac{\partial f}{\partial r}\right]\mathcal{I}_3\right)\lambda\right] \\
&\quad = \operatorname{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\left(-\frac{1}{2}\left(\operatorname{Tr}\left[\eta_i \frac{\partial f}{\partial r}\right]^2 - \operatorname{Tr}\left[\left(\eta_i \frac{\partial f}{\partial r}\right)^2\right]\right)\eta_i \frac{\partial f}{\partial r} + \det\left[\eta_i \frac{\partial f}{\partial r}\right]\mathcal{I}_3\right)\lambda\right] \\
&\quad = -\frac{1}{2}\left(\operatorname{Tr}\left[\eta_i \frac{\partial f}{\partial r}\right]^2 - \operatorname{Tr}\left[\left(\eta_i \frac{\partial f}{\partial r}\right)^2\right]\right)\operatorname{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_i \frac{\partial f}{\partial r}\lambda\right] + \det\left[\eta_i \frac{\partial f}{\partial r}\right]\operatorname{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\lambda\right] \\
&\quad = B^\mu \det\left[\eta_i \frac{\partial f}{\partial r}\right]
\end{aligned}
$$

which is equal to zero if and only if the relations

$$\det\left[\eta_i \frac{\partial f}{\partial r}\right] = 0, \quad i = 0, \ldots, n$$

hold. Again, the solutions of the overdetermined system (5.2.12) are invariant under the algebra of vector fields $X_1, \ldots, X_{p-3}$ defined by (5.2.1), (5.2.2) and (5.2.16). The invariants of the group G of such vector fields are provided by the functions $\{r^0, r^1, r^2, u^1, \ldots, u^q\}$, therefore the rank-$k$ solution of (5.1.1) is of the form (5.2.5).

5.3. Applications in fluid dynamics

Now we present some examples which illustrate the theoretical considerations presented in Section 5.2. We consider classical equations of hydrodynamics describing a motion in a fluid medium when the gravitational force $\vec{g}$ and Coriolis force $\Omega \times \vec{v}$ occur. We restrict ourselves to the equations of a one-component non-viscous fluid flow. Under these assumptions, our equations are of the type (5.1.1). The matrix form of these equations of hydrodynamics in a noninertial coordinates system is

$$A^0 u_t + A^1 u_x + A^2 u_y + A^3 u_z = B \tag{5.3.1}$$

where

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\kappa p/\rho & 1 \end{pmatrix}, \quad A^i = \begin{pmatrix} v_i & 0 & 0 & 0 & \delta_{i1}/\rho \\ 0 & v_i & 0 & 0 & \delta_{i2}/\rho \\ 0 & 0 & v_i & 0 & \delta_{i3}/\rho \\ 0 & 0 & 0 & v_i & 0 \\ 0 & 0 & 0 & -\kappa p v_i/\rho & v_i \end{pmatrix}, \quad i = 1, 2, 3,$$

$$B = \begin{pmatrix} g_1 - (\Omega_2 v_3 - \Omega_3 v_2) \\ g_2 - (\Omega_3 v_1 - \Omega_1 v_3) \\ g_3 - (\Omega_1 v_2 - \Omega_2 v_1) \\ 0 \\ 0 \end{pmatrix}, \quad u = (\vec{v}, \rho, p), \quad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Here we treat the physical space $X \subset \mathbb{R}^4$ as classical space-time, each of its points having coordinates $(t, \vec{x}) = (t, x, y, z)$ and the space of unknown functions (i.e. the hodograph space) $U \subset \mathbb{R}^5$ having coordinates $(\vec{v}, \rho, p)$. The vector $\vec{\Omega}$ denotes the angular velocity while κ is related to the adiabatic exponent γ of the fluid by the relation $\kappa = 2(\gamma - 1)^{-1}$.

For the homogeneous system (5.1.1), each wave vector λ can be written in the form $\lambda = (\lambda_0, \vec{\lambda})$, where $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ denotes a direction of wave propagation and the eigenvalue λ_0 of the matrix $(\lambda_i A^i)$ is a phase velocity of the considered wave. The dispersion relation for the homogeneous hydrodynamic system of equations (5.3.1) takes the form

$$(\lambda_0 + \vec{v} \cdot \vec{\lambda})^3 \left[(\lambda_0 + \vec{\lambda} \cdot \vec{v})^2 - \frac{\kappa p}{\rho} |\vec{\lambda}|^2 \right] = 0. \qquad (5.3.2)$$

Solving equation (5.3.2), we obtain two types of wave vectors, namely

i) Entropic : $\lambda^E = \left(-\vec{\lambda} \cdot \vec{v}, \vec{\lambda} \right)$,

ii) Acoustic : $\lambda^{A_\varepsilon} = \left(\varepsilon \left(\frac{\kappa p}{\rho} \right)^{1/2} |\vec{\lambda}| - \vec{v} \cdot \vec{\lambda}, \vec{\lambda} \right), \quad \varepsilon = \pm 1,$

as introduced by Lichnerowicz in [**71**]. Algebraic equations defining inhomogeneous wave vectors which satisfy equation (5.2.3) are of the form

$$\textbf{i)}\quad \text{Entropic}: \lambda_{E^0} = \left(-\vec{v}\cdot\vec{g}, \vec{g} - \vec{\Omega}\times\vec{v}\right),$$

$$\textbf{ii)}\quad \text{Acoustic}: \lambda_{A^0_\varepsilon} = \left(\varepsilon\left(\frac{\kappa p}{\rho}\right)^{1/2}|\vec{\lambda}| - \vec{v}\cdot\vec{\lambda}, \vec{\lambda}\right), \quad \varepsilon = \pm 1$$

$$\textbf{iii)}\quad \text{Hydrodynamic}: \lambda_{H^0} = \left(\lambda_0, \vec{\lambda}\right), \quad \lambda_0 + \vec{v}\cdot\vec{\lambda} \neq 0, \left(\frac{\kappa p}{\rho}\right)^{1/2}|\vec{\lambda}|.$$

The sign "+" or "-" means here that the wave goes in the right or the left direction with respect to the medium. Several classes of wave solutions of the hydrodynamic system (5.3.1) have been obtained via the generalized method of characteristics [**46, 54**]. Applying the CSM to this system allows us to compare the effectiveness of these two approaches. Rank-2 solutions obtained through the CSM are summarized in Table 5.1.

	E	A
E^0	$+$	$+$
A^0	$+$	$-$
H^0	$+$	$+$

TAB. 5.1. The existence (+) or absence (-) of nonlinear superposition of waves admitted by the system (5.3.1).

Here we restrict ourselves to the consideration of superpositions of waves which are admissible by the inhomogeneous system (5.3.1). We are interested only in rank-2 solutions which can be written in Riemann invariants. The results of such superpositions are set in Table 5.1, where the signs (+) and (-) indicate when superposition does and does not occur, respectively. We will now show for illustration some physically interesting solutions from among the ones obtained.

Different possibilities of existence of solutions in Riemann invariants for different combinations of wave vectors (homogeneous and inhomogeneous) are denoted by subsystems $E^0_i E_j$, $E^0_i A_\varepsilon$, $A^0_\varepsilon E_i$, $A^0_\varepsilon A_\varepsilon$, $H^0 E_i$, $H^0 A_\varepsilon$, $i, j = 1, 2$. Moreover, we denote by r^0 a Riemann invariant associated with wave vectors $\lambda^0_E, \lambda^0_{A_\varepsilon}, \lambda^0_H$ and by r^1 a Riemann invariant associated with $\lambda^E, \lambda^{A_\varepsilon}$.

The above analysis leads to 36 classes of rank-2 solutions of which we present only the more interesting ones and at the same time those which do not involve too much computation. Let us list some results.

Subsystem $\{E_1^0, E\}$ **:** We look for solutions of the form (5.2.5) defined by

$$\begin{aligned} &\vec{v} = \vec{v}(r^0, r^1), \quad \rho = \rho(r^0, r^1), \quad p = p(r^0, r^1), \\ &r^0 = -\left(\vec{v}\cdot\vec{g}\right)t + (g_1 + \Omega_3 v_2 - \Omega_2 v_3)x + (g_2 + \Omega_1 v_3 - \Omega_3 v_1)y + (g_3 + \Omega_2 v_1 - \Omega_1 v_2)z, \\ &r^1 = -\left(\vec{\lambda}^1 \cdot \vec{v}\right) t + \lambda_1^1 x + \lambda_2^1 y + \lambda_3^1 z, \quad \vec{\lambda}^1 = (\lambda_1^1, \lambda_2^1, \lambda_3^1), \end{aligned} \tag{5.3.3}$$

where the λ_j^1 are allowed to depend on the unknown functions. To satisfy equations (5.2.13d), we require that the coefficients of t, x, y, z in r^0 be constant, say $r^0 = c_0 t + c_1 x + c_2 y + c_3 z$. To allow for a rank-2 solution, the following relations must then hold

$$\vec{g}\cdot\vec{\Omega} = 0, \quad v_1 = \frac{g_2 - c_2 + \Omega_1 v_3}{\Omega_3}, \quad v_2 = \frac{c_1 - g_1 + \Omega_2 v_3}{\Omega_3},$$

implying that this type of solution exists only when the rotation axis of the system is perpendicular to the constant gravitational force. Solving (5.2.13), we obtain the following solution

$$\begin{aligned} &v_1 = \frac{g_2}{\Omega_3} - \frac{c_0}{g_1} - \frac{c_0 \Omega_2 g_2}{g_1 g_3 \Omega_3} + \frac{\Omega_1}{\Omega_3} v_3(r^0, r^1), \quad v_2 = \frac{c_0 \Omega_2}{g_3 \Omega_3} - \frac{g_1}{\Omega_3} + \frac{\Omega_2}{\Omega_3} v_3(r^0, r^1), \\ &\rho = p_{r^0}, \quad p = p(r^0), \end{aligned} \tag{5.3.4}$$

where $p(r^0), v_3(r^0, r^1)$ are arbitrary functions of the Riemann invariants

$$\begin{aligned} &r^0 = \frac{c_0}{g_3}\left(g_3 t + \Omega_2 x - \Omega_1 y\right), \quad \vec{g}\cdot\vec{\Omega} = 0, \\ &r^1 = \left(\frac{\lambda_2^1 g_1 - \lambda_1^1 g_2}{\Omega_3} + \frac{c_0 \lambda_3^1}{g_3}\right) t + \lambda_1^1 x + \lambda_2^1 y + \lambda_3^1 z, \end{aligned} \tag{5.3.5}$$

and $\vec{\lambda}^1$ is a constant vector satisfying $\vec{\lambda}^1 \cdot \vec{\Omega} = 0$. The solution defined by (5.3.4) and (5.3.5) represents a double traveling wave with constant velocities.

Subsystem $\{E_1^0, A_\varepsilon\}$: We look for solutions of the form (5.2.5) defined by

$$\vec{v} = \vec{v}(r^0, r^1), \quad \rho = \rho(r^0, r^1), \quad p = p(r^0, r^1),$$

$$r^0 = -(\vec{v}\cdot\vec{g})\,t + (g_1 + \Omega_3 v_2 - \Omega_2 v_3)x + (g_2 + \Omega_1 v_3 - \Omega_3 v_1)y + (g_3 + \Omega_2 v_1 - \Omega_1 v_2)z$$

$$r^1 = -\left(\vec{\lambda}^1\cdot\vec{v} + \varepsilon\left(\frac{\kappa p}{\rho}\right)^{1/2}|\vec{\lambda}^1|\right)t + \lambda_1^1 x + \lambda_2^1 y + \lambda_3^1 z, \quad \vec{\lambda}^1 = (\lambda_1^1, \lambda_2^1, \lambda_3^1),$$

where the λ_j^1 are allowed to depend on the unknown functions. The solution in this case exists only when $\kappa = 1$. Following the same process as with subsystem $\{E_1^0, E\}$, the rank-2 solution exists when $\vec{g}\cdot\vec{\Omega} = 0$ and is given by

$$\begin{aligned}
v_1 &= C_1 - \frac{\sqrt{A}}{|\vec{\Omega}|}\Omega_1 B(r^1),\\
v_2 &= \frac{1}{\Omega_1\Omega_3}\left[c_1\Omega_1 - g_1\Omega_1 + \Omega_2\left(c_2 - g_2 + \Omega_3\left(C_1 - \frac{\sqrt{A}}{|\vec{\Omega}|}\Omega_1 B(r^1)\right)\right)\right]\\
v_3 &= \frac{1}{\Omega_1}\left[c_2 - g_2 + \Omega_3\left(C_1 - \frac{\sqrt{A}}{|\vec{\Omega}|}\Omega_1 B(r^1)\right)\right]\\
\rho &= e^{r^0/A + B(r^1)}, \quad p = Ae^{r^0/A+B(r^1)}, \quad A, C_1 \in \mathbb{R}.
\end{aligned} \tag{5.3.6}$$

Here $B(r^1)$ is an arbitrary function, the Riemann invariants are given by

$$\begin{aligned}
r^0 &= \frac{c_2 g_1 - c_1 g_2}{\Omega_3}t + c_1 x + c_2 y - \frac{c_1\Omega_1 + c_2\Omega_2}{\Omega_3}z,\\
r^1 &= \left(\frac{\sqrt{A}}{|\vec{\Omega}|}(\vec{\lambda}^1\cdot\vec{\Omega})B(r^1) - \lambda_1^1 C_1 + \sqrt{A}|\vec{\lambda}^1|\right.\\
&\quad \left. + \frac{\lambda_2^1\Omega_1(g_1 - c_1) - (\lambda_2^1\Omega_2 + \lambda_3^1\Omega_3)(c_2 - g_2 + C_1\Omega_3)}{\Omega_1\Omega_3}\right)t + \lambda_1^1 x + \lambda_2^1 y + \lambda_3^1 z
\end{aligned} \tag{5.3.7}$$

and $\vec{\lambda}^1$ is colinear with $\vec{\Omega}$, $\vec{\lambda}^1\times\vec{\Omega} = 0$. Note that this solution admits the gradient catastrophe for a certain time T_0 that we estimate by considering a first-order taylor expansion for $B(r^1) \sim B_0 + B_1 r^1$. We obtain that the first-partial derivative of (5.3.7) with respect to t goes to infinity at time

$$T_0 = \frac{|\vec{\Omega}|}{B_1\sqrt{A}\left(\vec{\lambda}^1\cdot\vec{\Omega}\right)}. \tag{5.3.8}$$

A physically interesting solution can be built by selecting $B(r^1)$ as the Jacobi elliptic function $\mathrm{sn}(r^1, m)$, where $m \in [0, 1]$ is the modulus. For this choice, solution

(5.3.6), becomes

$$
\begin{aligned}
v_1 &= C_1 - \frac{\sqrt{A}}{|\vec{\Omega}|}\Omega_1 \operatorname{sn}(r^1, k),\\
v_2 &= \frac{1}{\Omega_1\Omega_3}\left[c_1\Omega_1 - g_1\Omega_1 + \Omega_2\left(c_2 - g_2 + \Omega_3\left(C_1 - \frac{\sqrt{A}}{|\vec{\Omega}|}\Omega_1 \operatorname{sn}(r^1, m)\right)\right)\right]\\
v_3 &= \frac{1}{\Omega_1}\left[c_2 - g_2 + \Omega_3\left(C_1 - \frac{\sqrt{A}}{|\vec{\Omega}|}\Omega_1 \operatorname{sn}(r^1, m)\right)\right]\\
\rho &= e^{r^0/A + \operatorname{sn}(r^1,m)}, \quad p = Ae^{r^0/A+\operatorname{sn}(r^1,m)}, \quad A \in \mathbb{R}.
\end{aligned}
\tag{5.3.9}
$$

while the invariants adopt the implicit form

$$
\begin{aligned}
r^0 &= \frac{c_2 g_1 - c_1 g_2}{\Omega_3} t + c_1 x + c_2 y - \frac{c_1\Omega_1 + c_2\Omega_2}{\Omega_3} z,\\
r^1 &= \left(\frac{\sqrt{A}}{|\vec{\Omega}|}(\vec{\lambda}^1 \cdot \vec{\Omega})\operatorname{sn}(r^1, m) - \lambda_1^1 C_1 + \sqrt{A}|\vec{\lambda}^1|\right.\\
&\quad \left. + \frac{\lambda_2^1\Omega_1(g_1 - c_1) - (\lambda_2^1\Omega_2 + \lambda_3^1\Omega_3)(c_2 - g_2 + C_1\Omega_3)}{\Omega_1\Omega_3}\right) t + \lambda_1^1 x + \lambda_2^1 y + \lambda_3^1 z.
\end{aligned}
\tag{5.3.10}
$$

Figure 5.1 illustrates the evolution of the density function $\rho(r^0, r^1)$ in (5.3.6) near the gradient catastrophe when the function $B(r^1)$ is chosen to be one the Jacobi elliptic functions $\operatorname{sn}(r^1, 1/2), \operatorname{sn}(r^1, 1/2), \operatorname{sn}(r^1, 1/2)$ for a certain choice of the parameters. It should be noted that for solution (5.3.9), the gradient catastrophe can be estimated to take place, according to formula (5.3.8), around the time

$$
T_0 = \frac{|\vec{\Omega}|}{\sqrt{A}(\vec{\lambda}^1 \cdot \vec{\Omega})}.
$$

Subsystem $\{A_\varepsilon^0, E\}$: We finally look for solutions of the form (5.2.5) defined by

$$
\begin{aligned}
\vec{v} &= \vec{v}(r^0, r^1), \quad \rho = \rho(r^0, r^1), \quad p = p(r^0, r^1),\\
r^0 &= -\left(\vec{\lambda}^0 \cdot \vec{v} + \varepsilon\left(\frac{\kappa p}{\rho}\right)^{1/2}|\vec{\lambda}^0|\right) t + \lambda_1^0 x + \lambda_2^0 y + \lambda_3^0 z\\
r^1 &= -\left(\vec{\lambda}^1 \cdot \vec{v}\right) t + \lambda_1^1 x + \lambda_2^1 y + \lambda_3^1 z, \quad \vec{\lambda}^A = (\lambda_1^A, \lambda_2^A, \lambda_3^A), A = 0, 1,
\end{aligned}
$$

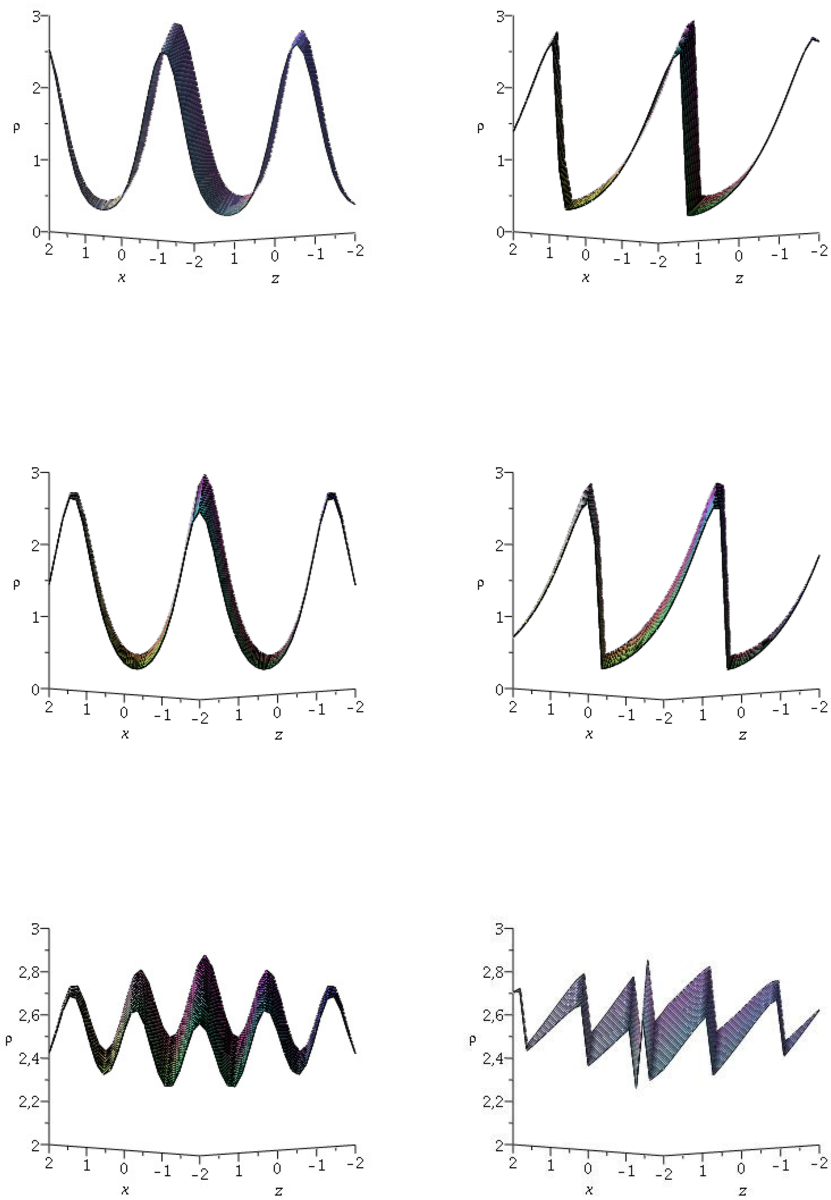

FIG. 5.1. Density distribution for the solution (5.3.6) at time $t = 0$ and near the gradient catastrophe for the elliptic functions $B(r^1) = \mathrm{sn}(r^1, \frac{1}{2}), \mathrm{cn}(r^1, \frac{1}{2}), \mathrm{dn}(r^1, \frac{1}{2})$.

where the λ_j^1 are allowed to depend on the unknown functions. In this case, the solution exists only when $\vec{\lambda}^0 = \varepsilon_1|\vec{\lambda}^0|\vec{\Omega}$, $|\vec{\Omega}| = 1$ and $\vec{g}\cdot\vec{\Omega} = 0$. Under these circumstances, the rank-2 solution takes the form

$$p = p_0, \quad \rho = \rho_0, \quad p_0, \rho_0 \in \mathbb{R},$$
$$v_3 = \frac{1}{\Omega_3 \varepsilon_1}\left[\varepsilon\sqrt{p_0\kappa/\rho} - \varepsilon_1(\Omega_1 v_1 + \Omega_2 v_2) - c_0\right]$$

where the functions v_1, v_2 satisfy the equations

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_{r^0} = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} d_5 \\ d_6 \end{pmatrix} \tag{5.3.11}$$

with the constants being given by

$$d_1 = \frac{\Omega_1\Omega_2}{\varepsilon\Omega_3\sqrt{p_0\kappa/\rho_0}}, \quad d_2 = \frac{{\Omega_2}^2 + {\Omega_3}^2}{\varepsilon\Omega_3\sqrt{p_0\kappa|/\rho_0}}, \quad d_3 = \frac{\Omega_2\left(c_0 - \varepsilon\sqrt{p_0\kappa/\rho_0}\right) + g_1\Omega_3\varepsilon_1}{\varepsilon\Omega_3\sqrt{p_0\kappa/\rho_0}},$$
$$d_4 = -\frac{{\Omega_1}^2 + {\Omega_3}^2}{\varepsilon\Omega_3\sqrt{p_0\kappa/\rho_0}}, \quad d_5 = -\frac{\Omega_1\Omega_2}{\varepsilon\Omega_3\sqrt{p_0\kappa/\rho_0}}, \quad d_6 = \frac{-\Omega_1\left(c_0 - \varepsilon\sqrt{p_0\kappa/\rho_0}\right) + g_2\Omega_3\varepsilon_1}{\varepsilon\Omega_3\sqrt{p_0\kappa/\rho_0}}.$$

The eigenvalues $\mu_\pm$ and eigenvectors $V_\pm$ of matrix $\begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$ are given by

$$\mu_\pm = \pm\varepsilon i\sqrt{\frac{\rho_0}{p_0\kappa}}, \quad V_\pm = \left(\frac{{\Omega_2}^2 + {\Omega_3}^2}{\Omega_1\Omega_2 \mp i\Omega_3}, -1\right)^T.$$

The solution to equations (5.3.11) is then given by

$$\begin{aligned} v_1 &= F_1(r^1)\cos\left(\varepsilon\sqrt{\rho_0/\kappa p_0}r^0\right) + F_2(r^1)\sin\left(\varepsilon\sqrt{\rho_0/\kappa p_0}r^0\right) + c_1 \\ v_2 &= -\frac{\Omega_1\Omega_2}{\Omega_2^2 + \Omega_3^2}\left(F_1(r^1)\cos\left(\varepsilon\sqrt{\rho_0/\kappa p_0}r^0\right) + F_2(r^1)\sin\left(\varepsilon\sqrt{\rho_0/\kappa p_0}r^0\right) + c_1\right) \\ &+ \frac{\varepsilon_1\Omega_3}{\varepsilon(\Omega_2^2 + \Omega_3^2)}\left[F_2(r^1)\cos\left(\varepsilon\sqrt{\rho_0/\kappa p_0}r^0\right) - F_1(r^1)\sin\left(\varepsilon\sqrt{\rho_0/\kappa p_0}r^0\right)\right] \\ &+ \frac{\varepsilon\Omega_2\sqrt{p_0\kappa/\rho_0} - c_0\Omega_2 - \varepsilon_1 g_1\Omega_2\Omega_3}{\varepsilon_1(\Omega_2^2 + \Omega_3^2)} \end{aligned}$$

where $c_0 \in \mathbb{R}$, $F_1(r^1)$, $F_2(r^1)$ are arbitrary functions of the Riemann invariant r^1 and $c_1 \in \mathbb{R}$ is given by

$$c_1 = \frac{\varepsilon_1(g_2\Omega_3^2 + \Omega_2^2(g_2 + g_1\Omega_1)) - c_0\Omega_1\Omega_3 + \varepsilon\Omega_1\Omega_3\sqrt{p_0\kappa/\rho_0}}{\varepsilon_1\Omega_3}.$$

The Riemann invariants then adopt the explicit form

$$r^0 = c_0 t + \varepsilon_1 (\Omega_1 x + \Omega_2 y + \Omega_3 z),$$
$$r^1 = \varepsilon_1 \left(\varepsilon \sqrt{p_0 \kappa / \rho_0} - c_0\right) t - \Omega_1 x - \Omega_2 y - \Omega_3 z.$$

This solution models the superposition of two traveling waves with the same direction but different phase velocities.

5.4. Final remarks

The main result of this paper consists of extending the Riemann invariant method to inhomogeneous quasilinear systems with coefficients depending only on dependent variables u. The conditional symmetry method, together with the Cayley-Hamilton idea, was adapted to partial differential equations in such a way as to allow the applicability of the Riemann invariants method. The conditions (5.2.13) and (5.2.16) are necessary and sufficient for the existence of rank 2 and 3 solutions of the system (5.1.1), respectively. In the proof of propositions 1 and 2, we obtain that the gradient of the matrix functions M_1^{-1} or M_2^{-1} becomes infinite for certain values of x^0 (the solution and its derivatives do not remain bounded). This means that we are dealing with a gradient catastrophe and some discontinuities can arise such as, for example, shock waves. Thus, the existence of Riemann invariants implies that it is not possible to introduce the concept of a weak solution in the broad sense.

These theoretical considerations have been illustrated by the examples of inhomogeneous equations of fluid dynamics in the presence of gravitational and Coriolis external forces. We were able to construct new classes of rank-2 solutions expressible in terms of Riemann invariants in the cases E^0E, E^0A_ε, $A^0_\varepsilon E$, H^0E, H^0A. One of these solutions admits a gradient catastrophe. This results from the fact that the first derivatives of Riemann invariants become infinite after some finite time. These solutions, in their general form, possess some degree of freedom, that is, depend on one or two arbitrary functions of one variable (Riemann invariant) depending on the case. This arbitrariness allows one to change the geometrical properties of the governed fluid flow in such a way as to displace the singularities to a sufficient extent. This fact is of some significance since even for arbitrary smooth and sufficiently small initial data at $t = t_0$, the magnitude of the first derivatives of the Riemann invariants become unbounded in some finite time T. This time can be estimated for each solution.

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Chapitre 6

CONCLUSION

6.1. Remarques finales

Pour conclure, nous aimerions ajouter certaines remarques concernant le problème de la construction de solutions solitoniques, domaine qui se développe très rapidement. En particulier, les solitons incluent les solutions élémentaires des systèmes d'EDPs en deux variables indépendantes et q variables dépendantes

$$u_t + a(u)u_x = b(u), \tag{6.1.1}$$

qui possèdent la forme d'onde de propagation

$$u = \psi(x - ct) \tag{6.1.2}$$

où c est la vitesse de propagation de l'onde. Afin d'assurer le caractère solitonique de la solution (6.1.2), nous demandons que le profil de la fonction $\psi(x-ct)$ tende vers une constante $u^0_\pm$ lorsque $x \to \pm\infty$. De plus, ces solutions doivent avoir une grande stabilité. Ces dernières conditions signifient que les solitons ne perdent pas leur "individualité" lors de leur superposition. Il s'agit alors d'interactions élastiques.

Nous pouvons constater que la matrice des dérivées de la solution solitonique de l'EDP (6.1.1) peut être représentée par des éléments intégraux simples non homogènes

$$\frac{\partial u^\alpha}{\partial x^i} = \gamma^\alpha \lambda_i, \quad (\mathcal{I}_q \lambda_1 + a(u)\lambda_2)\gamma = b(u). \tag{6.1.3}$$

où $\mathcal{I}_q$ est la matrice identité de dimension $q \times q$. Les solitons de l'équation (6.1.1) représentent donc des états simples dans le sens de la définition en accord avec

[**46**]. Ces états simples peuvent décrire dans certains cas des perturbations localisées. Pour chaque temps fixé $t = t_0$, nous supposons que la dérivée de la solution $u_x(t_0, \cdot)$ possède un support compact. La possibilité de décrire les solitons à partir de l'équation (6.1.3) dans le langage des éléments intégraux simples suggère que nous pouvons décrire leurs superpositions dans ce langage. Il est nécessaire de souligner que jusqu'à présent, il n'a pas été possible d'obtenir des résultats positifs par la méthode des caractéristiques généralisées. L'idée la plus simple nous permettant de résoudre ce problème est de procéder par analogie avec le problème de superposition des ondes simples en termes des invariants de Riemann. Ces interactions analogues aux solitons ont aussi un caractère élastique. C'est-à-dire que nous pouvons décrire ces interactions à l'aide de la somme des éléments simples intégraux qui correspondent aux ondes qui entrent dans la superposition. Il semble prometteur de chercher des solutions double-solitoniques sous la forme où la matrice des dérivées des solutions peut être représentée comme la somme de deux éléments simples intégraux non homogènes

$$\begin{aligned} \mathrm{d}u = \frac{\partial u^\alpha}{\partial x^i}\mathrm{d}x^i =&\xi(\theta_1\mathcal{I} - A)^{-1}B \otimes (\theta_1\mathrm{d}t - \mathrm{d}x) \\ &+ (1-\xi)(\theta_2\mathcal{I} - A)^{-1}B \otimes (\theta_2\mathrm{d}t - \mathrm{d}x), \end{aligned} \tag{6.1.4}$$

où $\lambda^s = (\theta_s, -1), \gamma_s = (\theta_s\mathcal{I} - A)^{-1}B,\ s = 1, 2$, et la quantité ξ peut être considérée comme une fonction dépendante de t et x. Il s'avère que les contraintes correspondant à la forme postulée (6.1.4) ne sont pas suffisantes pour extraire les solutions double-solitoniques de l'ensemble des solutions du système initial (6.1.1). La matrice des dérivées des solutions double-solitoniques consiste en la somme de deux éléments simples intégraux non homogènes, mais chaque solution de (6.1.4) ne possède pas la propriété d'être une solution double-solitonique. Illustrons cela avec un exemple.

Considérons l'équation de Sine-Gordon sous la forme d'un système du premier ordre

$$\begin{aligned} v_t &= \sin\varphi, \\ \varphi_x &= v. \end{aligned} \tag{6.1.5}$$

Il est bien connu dans la littérature du sujet ([**1**] et références incluses) que cette équation admet entre autres, des solutions de la type solitonique (par exemple, bumps, kinks) ainsi que des solutions multisolitoniques. La matrice des dérivées pour les solutions double-solitoniques prend la forme

$$\mathrm{d}u = \xi \begin{pmatrix} \frac{\sin\varphi}{\theta_1} \\ -v \end{pmatrix} \otimes (\theta_1 \mathrm{d}t - \mathrm{d}x) + (1-\xi) \begin{pmatrix} \frac{\sin\varphi}{\theta_2} \\ -v \end{pmatrix} \otimes (\theta_2 \mathrm{d}t - \mathrm{d}x) \tag{6.1.6}$$

qui peut être écrite sous la forme équivalente des quatre équations

$$\begin{aligned} &v_t = \sin\varphi, \quad \varphi_t = -\xi v\theta_1 - (1-\xi)v\theta_2, \\ &v_x = -\xi\frac{\sin\varphi}{\theta_1} - (1-\xi)\frac{\sin\varphi}{\theta_2}, \quad \phi_x = v, \end{aligned} \tag{6.1.7}$$

pour cinq fonctions indépendantes $v, \varphi, \xi, \theta_1, \theta_2$. L'équation (6.1.7) est équivalente au système (6.1.5) car les expressions pour v_x et ϕ_t n'imposent aucune contrainte, puisque les variables θ_1, θ_2 et ξ sont arbitraires. Pour extraire les solutions double-solitoniques, il est donc nécessaire d'ajouter des contraintes supplémentaires.

Nous donnons à présent un exemple de solution pour ce type de problème pour certains cas particuliers. Considérons tout d'abord le cas quand les vecteurs d'onde λ^s sont des 1-formes constantes et notons les par $\underset{0}{\lambda}^s$. Considérons le système hyperbolique du premier ordre en p variables indépendantes

$$a^{i\mu}_{\alpha}(u)u^{\alpha}_i = b^{\mu}(u), \quad \mu = 1,\ldots,l, \quad \alpha = 1,\ldots,q, \quad i = 0,\ldots,p. \tag{6.1.8}$$

Nous cherchons les solutions pour lesquelles la matrice des dérivées est sous la forme d'une somme d'éléments simples intégraux non homogènes,

$$\begin{aligned} &\mathrm{d}u = \xi^1 \underset{0}{\gamma_1} \otimes \underset{0}{\lambda}^1 + \xi^2 \underset{0}{\gamma_2} \otimes \underset{0}{\lambda}^2, \quad \underset{0}{\gamma_s} \in T_uU, \quad \underset{0}{\lambda}^s = \underset{0}{\lambda}^s_i(u)\mathrm{d}x^i \in T^*_xX, \quad s = 1,2, \\ &\underset{0}{\gamma_s} = (\underset{0}{\gamma}^1_s, \ldots, \underset{0}{\gamma}^q_s) \in \mathbb{R}^q, \quad \lambda^s = (\lambda^s_1, \ldots, \lambda^s_p) \in \mathbb{R}^p, \end{aligned} \tag{6.1.9}$$

sous la contrainte

$$\xi^1 + \xi^2 = 1, \tag{6.1.10}$$

où les ξ^s sont des variables qui dépendent seulement de x. Afin de ne pas alourdir la notation, nous omettons dans ce qui suit l'indice 0 pour les vecteurs λ^s et γ^s. Nous faisons ici l'hypothèse que les vecteurs γ_1 et γ_2 ainsi que les vecteurs λ_1 et λ_2

sont linéairement indépendants. Les éléments simples non homogènes intégraux vérifient les équations algébriques

$$a^{i\mu}_{\alpha}(u)\gamma^{\alpha}_{(s)}\lambda^s_i = b^\mu(u), \quad s = 1,2. \tag{6.1.11}$$

En considérant la dérivée extérieure du système (6.1.9), nous obtenons l'équation suivante

$$\begin{aligned} &\gamma_1 \otimes \mathrm{d}\xi^1 \wedge \lambda^1 + \gamma_2 \otimes \mathrm{d}\xi^2 \wedge \lambda^2 + \xi^1 \mathrm{d}\gamma_1 \wedge \lambda^1 + \xi^2 \mathrm{d}\gamma_2 \wedge \lambda^2 = 0, \\ &\mathrm{d}\xi^1 + \mathrm{d}\xi^2 = 0, \end{aligned} \tag{6.1.12}$$

qui doit être vérifiée lorsque l'équation (6.1.9) est satisfaite. En vertu de (6.1.9), nous avons les relations suivantes

$$\mathrm{d}\gamma_s = \gamma_{s,u^\alpha}\mathrm{d}u^\alpha = \xi^1\gamma_{s,\gamma_1} \otimes \lambda^1 + \xi^2\gamma_{s,\gamma_2} \otimes \lambda^2, \quad s = 1,2. \tag{6.1.13}$$

En remplaçant les relations (6.1.13) dans le système prolongé (6.1.12), nous obtenons

$$\begin{aligned} &\gamma_1 \otimes \mathrm{d}\xi^1 \wedge \lambda^1 + \gamma_2 \otimes \mathrm{d}\xi^2 \wedge \lambda^2 + \xi^1\xi^2[\gamma_1, \gamma_2] \otimes \lambda^1 \wedge \lambda^2 = 0, \\ &\mathrm{d}\xi^1 + \mathrm{d}\xi^2 = 0. \end{aligned} \tag{6.1.14}$$

Sous l'hypothèse que les covecteurs λ^1 et λ^2 sont linéairement indépendants et que $\xi^1\xi^2 \neq 0$, l'équation (6.1.14) implique que le commutateur des vecteurs γ_1 et γ_2 est engendré par une combinaison linéaire de γ_1 et γ_2

$$[\gamma_1, \gamma_2] = \alpha^1(u)\gamma_1 + \alpha^2(u)\gamma_2, \tag{6.1.15}$$

où les coefficients α^s dépendent seulement de u. La condition (6.1.15) signifie que les hypothèses du théorème de Frobenius sont satisfaites. Ainsi, pour chaque point u_0 dans l'espace des variables dépendantes U, il existe une surface tangente aux vecteurs γ_1 et γ_2 passant par le point $u_0 \in U$. En remplaçant (6.1.15) dans les équations (6.1.14) et en utilisant l'indépendance linéaire des vecteurs γ_1 et γ_2, nous obtenons

$$\begin{aligned} &\mathrm{d}\xi^1 \wedge \lambda^1 + \alpha^1\xi^1\xi^2\lambda^1 \wedge \lambda^2 = 0, \quad \mathrm{d}\xi^1 + \mathrm{d}\xi^2 = 0, \\ &\mathrm{d}\xi^2 \wedge \lambda^2 + \alpha^2\xi^1\xi^2\lambda^1 \wedge \lambda^2 = 0. \end{aligned} \tag{6.1.16}$$

Le lemme de Cartan [**10**] nous permet de démontrer qu'il existe deux fonctions μ^1 et μ^2 telles que

$$\begin{aligned} &\mathrm{d}\xi^1 = \mu^1\lambda^1 + \alpha^1\xi^1\xi^2\lambda^2, \quad \mathrm{d}\xi^1 + \mathrm{d}\xi^2 = 0, \\ &\mathrm{d}\xi^2 = \mu^2\lambda^2 - \alpha^2\xi^1\xi^2\lambda^1. \end{aligned} \tag{6.1.17}$$

En additionnant les équations (6.1.17) et en utilisant le fait que les vecteurs λ^1 et λ^2 sont linéairement indépendants, nous obtenons

$$\mu^1 = \alpha^2\xi^1\xi^2, \quad \mu^2 = -\alpha^1\xi^1\xi^2. \tag{6.1.18}$$

En vertu de (6.1.10), la substitution de (6.1.18) dans l'équation (6.1.17) nous permet d'obtenir la relation suivante

$$\frac{\mathrm{d}\xi^1}{\xi^1(1-\xi^1)} = \alpha^2\lambda^1 + \alpha^1\lambda^2, \quad \xi^2 = 1 - \xi^1. \tag{6.1.19}$$

En prenant la dérivée extérieure de l'équation (6.1.19), nous obtenons la relation

$$\mathrm{d}\alpha^1 \wedge \lambda^2 + \mathrm{d}\alpha^2 \wedge \lambda^1 = 0, \tag{6.1.20}$$

qui doit être satisfaite lorsque l'équation (6.1.9) est vérifiée. En tenant compte de (6.1.9), nous avons

$$\mathrm{d}\alpha^s = \alpha^s_{,u^\alpha}\mathrm{d}u^\alpha = \xi^1\alpha^s_{,\gamma_1}\lambda^1 + (1-\xi^1)\alpha^s_{,\gamma_2}\lambda^2, \quad s = 1, 2. \tag{6.1.21}$$

Remplaçons les relations (6.1.21) dans l'équation (6.1.20) pour obtenir

$$\xi^1\alpha^1_{,\gamma_1} - (1-\xi^1)\alpha^2_{,\gamma_2} = 0. \tag{6.1.22}$$

L'équation (6.1.22) nous mène à considérer les deux cas suivants :

Cas 1 : Lorsque les coefficients par rapport aux puissances de ξ^1 dans (6.1.22) sont identiquement nuls, nous obtenons les relations suivantes

$$\alpha^1_{,\gamma_1} = 0, \quad \alpha^2_{,\gamma_2} = 0. \tag{6.1.23}$$

Cas 2 : Lorsque l'équation (6.1.22) génère des contraintes sur la variable ξ^1, nous obtenons la relation suivante

$$\xi^1 = \frac{\alpha^2_{,\gamma_2}}{\alpha^1_{,\gamma_1} + \alpha^2_{,\gamma_2}}. \tag{6.1.24}$$

En remplaçant (6.1.24) dans l'équation (6.1.19) et en utilisant le fait que les vecteurs λ^1 et λ^2 sont linéairement indépendants, nous obtenons

$$\alpha^1 = \left(\ln \frac{\alpha^2_{,\gamma_2}}{\alpha^1_{,\gamma_1} + \alpha^2_{,\gamma_2}}\right)_{,\gamma_2}, \quad \alpha^2 = -\left(\ln \left(1 - \frac{\alpha^2_{,\gamma_2}}{\alpha^1_{,\gamma_1} + \alpha^2_{,\gamma_2}}\right)\right)_{,\gamma_1}. \tag{6.1.25}$$

Les conditions (6.1.15), (6.1.23) ou (6.1.25) garantissent l'existence de la solution du système (6.1.9). Le degré de liberté du système (6.1.9) dans le premier cas dépend d'une fonction arbitraire d'une variable alors que dans le deuxième cas, la solution possède seulement l'arbitrarité sur les constantes d'intégration.

Sans perte de généralité, en normalisant les vecteurs γ_1 et γ_2 de façon appropriée, nous pouvons obtenir que le commutateur (6.1.15) est égal à zéro. Dans ce cas, les vecteurs γ_1 et γ_2 constituent un système holonomique, c'est-à-dire qu'il existe une surface paramétrique $u = f(r^1, r^2)$, tangente aux champs de vecteurs γ_1 et γ_2, telle que les EDPs

$$\frac{\partial f}{\partial r^s} = \alpha^s(f)\gamma_{(s)}(f), \quad s = 1, 2, \tag{6.1.26}$$

sont satisfaites, où les coefficients α^s dépendent de f. En vertu de l'hypothèse que les vecteurs γ_1 et γ_2 sont linéairement indépendants et en tenant compte de la relation (6.1.26), le système (6.1.9) peut être transformé à la forme

$$\mathrm{d}r^s = \xi^s \alpha^s \lambda^s = 0, \quad \xi^1 + \xi^2 = 1, \quad s = 1, 2, \tag{6.1.27}$$

où les variables α^s et ξ^s deviennent des fonctions des paramètres r^1 et r^2.

En supposant que les profils des dérivées des fonctions r^s_x possèdent chacun un support compact et disjoint pour le temps initial $t = t_0$ (représentant par exemple deux solitons localisés), i.e.

$$\exists a^s, b^s \in \mathbb{R} \quad : \quad \operatorname{supp} r^s_x(t_0, x) \subset [a^s, b^s], \quad s = 1, 2, \tag{6.1.28}$$

et

$$\operatorname{supp} r^1_x(t_0, x) \cap \operatorname{supp} r^2_x(t_0, x) = \phi, \tag{6.1.29}$$

où $a^1 < b^1 < a^2 < b^2$, la forme de l'équation (6.1.27) implique que les fonctions $r^s(t, x)$ sont constantes le long des droites données par

$$\underset{0}{\lambda}{}^s_i (x^i - x^i_0) = 0. \tag{6.1.30}$$

En conséquence, le résultat de la superposition de deux solitons consiste en la propagation des deux solitons sans changement dans leur phase[1] et leur profil. Dans ce cas, ces solitons se propagent de façon indépendante, c'est-à-dire qu'ils n'ont aucune influence l'un sur l'autre. Selon le langage introduit dans la littérature du sujet [**107**], ces interactions correspondent à une superposition linéaire. Donc, le cas que nous considérons ne peut pas décrire les solutions double-solitoniques de l'équation de Sine-Gordon (voir par exemple [**106**])

$$\varphi = 4 \arctan \left[v \frac{\sinh \frac{x+t}{\beta}}{\cosh \frac{v(x-t)}{\beta}} \right], \quad \text{où } \beta = (1-v^2)^{1/2}, \quad v \neq 0, 1, \tag{6.1.31}$$

puisque le résultat de la superposition de deux solitons change les phases de façon appropriée. Donc, le problème que nous avons posé, la construction des doubles solitons et des n-solitons à l'aide de la méthode des caractéristiques, reste ouvert jusqu'à présent.

Le parallèle entre la problématique des superpositions des solitons et les superpositions des ondes de Riemann présentées dans ce travail nous motive à poursuivre nos études dans ce domaine. Il est à noter que la problématique de la construction de ces solutions est basée sur la somme des éléments simples intégraux non homogènes, qui ne doivent pas seulement se limiter à des états simples représentant des solitons.

Nous construisons à présent des solutions particulières du système de Kadomtsev-Petviashvili sans dispersion en utilisant les symétries conditionnelles et en exprimant ces solutions en termes des invariants de Riemann. Plus précisément, nous construisons trois classes de solution de rang-2.

6.2. Solutions de l'équation de Kadomtsev-Petviashvili sans dispersion

L'équation de Kadomtsev-Petviashvili

$$(u_t - uu_x + \varepsilon^2 u_{xxx})_x - u_{yy} = 0, \tag{6.2.1}$$

[1]En tant que phase, nous voulons dire $\eta_s = \lambda_1^s t - \lambda_2^s x$, $s = 1, 2$.

apparaît comme une généralisation naturelle bidimensionnelle de l'équation de Korteweg deVries et est utilisée pour modéliser des ondes aquatiques possédant une grande longueur d'onde. L'équation (6.2.1) étant complètement intégrable, ses solutions peuvent être obtenues à l'aide de la méthode de diffusion inverse [**1**]. Lorsque le terme modélisant les effets dispersifs u_{xxx} peut être négligé, nous sommes amenés à considérer les fonctions $u = u(t, x, y)$ satisfaisant l'équation de Kadomtsev-Petviashvili sans dispersion (dKP),

$$(u_t - uu_x)_x - u_{yy} = 0. \tag{6.2.2}$$

Cette équation, également appelée équation de Khokhlov-Zabolotskaya, possède plusieurs applications en accoustique non linéaire, en dynamique des gaz et en géométrie différentielle [**1, 106**]. La méthode des réductions hydrodynamiques [**36**], discutée en introduction, fut l'une des premières approches permettant de construire des solutions de l'équation dKP en termes d'invariants de Riemann [**67, 68, 119**]. Le comportement asymptotique des solutions de l'équation dKP a récemment été décrit par Santini et Manakov dans [**81, 82**], où les auteurs investissent le problème de Riemann-Hilbert non linéaire associé. Finalement, la solution formelle du problème de Cauchy est obtenue dans [**80**] en appliquant la méthode de diffusion inverse et en utilisant la paire de Lax associée à (6.2.2) suivante

$$L_1 = \partial_y + \lambda\partial_x - u_x\partial_\lambda, \quad L_2 = \partial_t - (\lambda^2 + u)\partial_x - (u_y - \lambda u_x)\partial_\lambda. \tag{6.2.3}$$

Afin d'écrire l'équation (6.2.2) sous la forme d'un système quasilinéaire du premier ordre, introduisons $u = \phi_x$ pour obtenir

$$(\phi_{xt} - \phi_x\phi_{xx})_x - \phi_{xyy} = 0 \Rightarrow \phi_{xt} - \phi_x\phi_{xx} - \phi_{yy} = F(y, t).$$

Sans perte de généralité, considérons le cas lorsque $F(y, t) = 0$. En notant $v = \phi_x, w = \phi_y$, l'équation (6.2.2) s'écrit

$$v_y - w_x = 0, \quad w_y - v_t + vv_x = 0, \tag{6.2.4}$$

qui, sous forme matricielle évolutive, devient

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_y + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} 0 & -1 \\ v & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.2.5}$$

Il est également commode dans le contexte des solutions en invariants de Riemann d'écrire le système (6.2.4) sous forme de traces

$$\mathrm{Tr}\left[\mathcal{A}^{\mu}\begin{pmatrix} v_y & v_t & v_x \\ w_y & w_t & w_x \end{pmatrix}\right] = 0, \quad \mathcal{A}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathcal{A}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ v & 0 \end{pmatrix}. \tag{6.2.6}$$

Notons que pour toute solution v, w du système (6.2.4), la composante v satisfait l'équation dKP (6.2.2). Nous effectuons maintenant une analyse des solutions de rang 1 et 2 de telle façon à maximiser le degré de liberté sur la solution v .

6.2.1. Solutions de rang 1

Nous supposons la solution de (6.2.6) sous la forme implicite

$$v = v(r^1), \quad w = w(r^1), \quad r^1 = y + \lambda_1(v, w)t + \lambda_2(v, w)x, \tag{6.2.7}$$

où la fonction r^1 est appelée invariant de Riemann. Cette solution est invariante sous l'algèbre bidimensionnelle de symétrie conditionnelle

$$X_1 = \partial_t - \lambda_1(v, w)\partial_y, \quad X_2 = \partial_x - \lambda_2(v, w)\partial_y. \tag{6.2.8}$$

Le système (6.2.6) pour $v(r^1)$ et $w(r^1)$ s'écrit

$$v_{r^1} - \lambda_2 w_{r^1} = 0, \quad v_{r^1} - \frac{1}{\lambda_1 - \lambda_2 v} w_{r^1} = 0, \quad \lambda_1 \neq \lambda_2 v. \tag{6.2.9}$$

Pour qu'il existe une solution non-constante, en supposant que λ_2 est non nul, on doit avoir

$$\lambda_2 = \frac{1}{\lambda_1 - \lambda_2 v} \Rightarrow \lambda_1 = \frac{1}{\lambda_2} + \lambda_2 v. \tag{6.2.10}$$

L'invariant r^1 est alors de la forme

$$r^1 = y + \left(\frac{1}{\lambda_2} + \lambda_2 v\right) t + \lambda_2 x. \tag{6.2.11}$$

L'équation (6.2.9) peut être intégrée pour certains choix particuliers de $\lambda_2(v,w)$. Par exemple, lorsque $\lambda_2(v,w) = W'(w)/V'(v)$, avec $V'(v) > 0$ sur un certain intervalle $v \in [v_1, v_2]$, alors l'équation (6.2.9) nous donne l'expression pour v

$$V'(v)v_{r^1} = W'(w)w_{r^1} \Rightarrow v = V^{-1}(W(w) + V_0), \quad v \in [v_1, v_2], \tag{6.2.12}$$

où r^1 satisfait l'équation implicite

$$r^1 = y + \left(\frac{V'(v)}{W'(w)} + \frac{W'(w)}{V'(v)} V^{-1}\left(W(w) + V_0\right)\right) t + \frac{W'(w)}{V'(v)} x. \tag{6.2.13}$$

Il est à noter que les fontions $V(v)$, et $W(w)$ sont arbitraires, ce qui nous permet de sélectionner le profil de la fonction v. Par exemple, si $V(v) = \operatorname{arctanh}(\sqrt{v})$, $v > 0$ et $W(w) = w$, nous obtenons la solution solitonique de type « bump »

$$v = \operatorname{sech}^2\left(w(r^1) + V_0\right), \tag{6.2.14}$$

$$r^1 = y - \left(\frac{1}{2}\cosh^2\big(w(r^1) + V_0\big)\coth\big(w(r^1) + V_0\big) + 2\operatorname{sech}^4\big(w(r^1) + V_0\big)\tanh\big(w(r^1) + V_0\big)\right) t$$
$$-2\operatorname{sech}^2\big(w(r^1) + V_0\big)\tanh\big(w(r^1) + V_0\big)\, x. \tag{6.2.15}$$

Pour $w(r^1)$ fixé, en insérant la solution explicite pour r^1 de l'équation (6.2.15) dans l'expression (6.2.14), nous pouvons obtenir une solution explicite de type solitonique de l'équation dKP.

6.2.2. Solutions de rang 2

Cherchons les solutions de (6.2.6) sous la forme implicite

$$v = v(r^1, r^2), \quad w = w(r^1, r^2), \quad r^A = y + \lambda_1^A(v,w)t + \lambda_2^A(v,w)x, \quad A = 1, 2. \tag{6.2.16}$$

Ces solutions sont invariantes sous le générateur de symétrie conditionnelle

$$X = (\lambda_1^1\lambda_2^2 - \lambda_1^2\lambda_2^1)\partial_y + (\lambda_2^1 - \lambda_2^2)\partial_t + (\lambda_1^2 - \lambda_1^1)\partial_x. \tag{6.2.17}$$

La matrice jacobienne ∂u devient

$$\partial u = \left(\mathcal{I}_2 - \frac{\partial f}{\partial r}\frac{\partial r}{\partial u}\right)^{-1}\frac{\partial f}{\partial r}\lambda, \quad \frac{\partial f}{\partial r} = \begin{pmatrix} v_{r^1} & v_{r^2} \\ w_{r^1} & w_{r^2} \end{pmatrix} \in \mathbb{R}^{2\times 2}, \quad \text{rang}(\partial u) = 2,$$

$$\frac{\partial r}{\partial u} = \eta_1 t + \eta_2 x, \quad \eta_a = \begin{pmatrix} \lambda^1_{a,v} & \lambda^1_{a,w} \\ \lambda^2_{a,v} & \lambda^2_{a,w} \end{pmatrix} \in \mathbb{R}^{2\times 2}, \quad \lambda = \begin{pmatrix} 1 & \lambda^1_1 & \lambda^1_2 \\ 1 & \lambda^2_1 & \lambda^2_2 \end{pmatrix} \in \mathbb{R}^{2\times 3}. \tag{6.2.18}$$

Dans les nouvelles variables, l'équation (6.2.6) s'écrit sous la forme

$$\det\left(I_2 - \frac{\partial f}{\partial r}(\eta_1 t + \eta_2 x)\right) \operatorname{Tr}\left[\mathcal{A}^\mu \left(\mathcal{I}_2 - \frac{\partial f}{\partial r}(\eta_1 t + \eta_2 x)\right)^{-1}\frac{\partial f}{\partial r}\lambda\right] = 0, \tag{6.2.19}$$

Par le théorème de Cayley-Hamilton, l'équation (6.2.19) est polynomiale de degré 1 en les variables t et x. En considérant le système (6.2.19) et ses dérivées partielles du premier ordre à l'origine, nous obtenons que les coefficients de ce polynôme sont donnés par

$$\operatorname{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\lambda\right] = 0, \quad \mu = 1,2, \tag{6.2.20}$$

$$\operatorname{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_a\frac{\partial f}{\partial r}\lambda\right] = 0, \quad a = 1,2. \tag{6.2.21}$$

Il est possible de simplifier l'équation (6.2.21) en considérant (6.2.20). En effet,

$$\begin{aligned}\operatorname{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_a\frac{\partial f}{\partial r}\lambda\right] &= \operatorname{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_a\frac{\partial f}{\partial r}\lambda\right] - \operatorname{Tr}\left[\eta_a\frac{\partial f}{\partial r}\right]\operatorname{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\lambda\right] \\ &= \operatorname{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\left(\eta_a\frac{\partial f}{\partial r} - \mathcal{I}\operatorname{Tr}\left[\eta_a\frac{\partial f}{\partial r}\right]\right)\lambda\right] \\ &= \operatorname{Tr}\left[\mathcal{A}^\mu \frac{\partial f}{\partial r}\left(\eta_a\frac{\partial f}{\partial r}\right)^{-1}\lambda\right] = \operatorname{Tr}\left[\mathcal{A}^\mu (\eta_a)^{-1}\lambda\right].\end{aligned} \tag{6.2.22}$$

Cette équation est satisfaite si et seulement si

$$\operatorname{Tr}\left[\mathcal{A}^\mu \operatorname{adj}(\eta_a)\lambda\right] = 0, \tag{6.2.23}$$

où $\operatorname{adj}(M)$ est la matrice adjointe de M. Les équations (6.2.23) forment un système de quatre équations pour les fonctions $\lambda^A_i(v,w)$. Il convient alors de résoudre d'abord les équations (6.2.23) et ensuite substituer chaque solution dans l'équation (6.2.20) pour obtenir v, w en termes de r^1 et r^2.

Pour le système (6.2.6), le système découplé (6.2.23) prend la forme

$$\begin{aligned}&\lambda^1_{a,w} - \lambda^2_{a,w} + \lambda^2_2\lambda^1_{a,v} - \lambda^1_2\lambda^2_{a,v} = 0, \quad a = 1, 2,\\&\lambda^2_{a,v} - \lambda^1_{a,v} + (\lambda^1_1 - \lambda^1_2 v)\lambda^2_{a,w} - (\lambda^2_1 - \lambda^2_2 v)\lambda^1_{a,w} = 0.\end{aligned} \tag{6.2.24}$$

On doit par la suite substituer la solution pour λ^A_i dans le système

$$\begin{aligned}&v_{r^1} + v_{r^2} - \left(\lambda^1_2 w_{r^1} + \lambda^2_2 w_{r^2}\right) = 0,\\&w_{r^1} + w_{r^2} - (\lambda^1_1 - \lambda^1_2 v)v_{r^1} - (\lambda^2_1 - \lambda^2_2 v)v_{r^2} = 0,\end{aligned} \tag{6.2.25}$$

pour obtenir v et w. Notons que lorsque $\lambda^A_1 = \lambda^A_2 v$, $A = 1, 2$, les équations (6.2.24) impliquent que $\lambda^2_1 = \lambda^1_1$ et $\lambda^2_2 = \lambda^1_2$, ce qui ne peut permettre la construction de solution de rang 2.

6.2.2.1. *Superposition linéaire*

Supposons que, en accord avec les équations de l'onde simple (6.2.9) et (6.2.10), les relations suivantes soient satisfaites

$$\text{i) } \lambda^A_1 = \frac{1}{\lambda^A_2} + \lambda^A_2 v, \quad \text{ii) } v_{r^A} - \lambda^A_2 w_{r^A}, \quad A = 1, 2, \quad \lambda^2_2 \neq \lambda^1_2. \tag{6.2.26}$$

Pour $\lambda^A_2(v, w)$ donnés, les équations (6.2.25) sont automatiquement satisfaites. La condition de compatibilité $v_{r^1 r^2} = v_{r^2 r^1}$ de l'équation (6.2.26) implique

$$(\lambda^2_{2,w} + \lambda^1_2\lambda^2_{2,v} - \lambda^1_{2,w} - \lambda^2_2\lambda^1_{2,v})w_{r^1}w_{r^2} + (\lambda^2_2 - \lambda^1_2)w_{r^1 r^2} = 0. \tag{6.2.27}$$

En tenant compte des équations (6.2.24), l'équation (6.2.27) devient

$$(\lambda^2_2 - \lambda^1_2)w_{r^1 r^2} = 0 \quad \Rightarrow w = w_1(r^1) + w_2(r^2). \tag{6.2.28}$$

Les équations (6.2.26ii) impliquent alors

$$v_{r^1} = \lambda^1_2(v, w)w'_1(r^1), \quad v_{r^2} = \lambda^2_2(v, w)w'_2(r^2). \tag{6.2.29}$$

Toutefois, il est facile de montrer que pour les équations (6.2.24), l'hypothèse (6.2.26i) implique que $\lambda^1_2 = \lambda^2_2$. En effet, (6.2.26) implique que le système (6.2.24) s'écrit

$$\begin{pmatrix} \left(\frac{1}{{\lambda_2^1}^2}-v\right)\lambda_2^2 & \frac{1}{{\lambda_2^1}^2}-v & \lambda_2^1\left(v-\frac{1}{{\lambda_2^2}^2}\right) & v-\frac{1}{{\lambda_2^2}^2} \\ -\lambda_2^2 & -1 & \lambda_2^1 & 1 \\ v-\frac{1}{{\lambda_2^1}^2} & \frac{v{\lambda_2^1}^2-1}{{\lambda_2^1}^2\lambda_2^2} & \frac{1}{{\lambda_2^2}^2}-v & \frac{1-v{\lambda_2^2}^2}{\lambda_2^1{\lambda_2^2}^2} \\ 1 & \frac{1}{\lambda_2^2} & -1 & -\frac{1}{\lambda_2^1} \end{pmatrix} \cdot \begin{pmatrix} \lambda^1_{2,v} \\ \lambda^1_{2,w} \\ \lambda^2_{2,v} \\ \lambda^2_{2,w} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \lambda_2^2-\lambda_2^1 \\ 0 \end{pmatrix}. \tag{6.2.30}$$

En considérant une combinaison linéaire appropriée de ces équations, nous obtenons que ce système implique que $\lambda_2^2 = \lambda_2^1$. L'équation dKP n'admet donc pas de superposition linéaire des solutions de rang 1. C'est donc dire que l'interaction de deux ondes de Riemann est nécessairement non linéaire.

6.2.2.2. *Superpositions non linéaires*

Nous déterminons maintenant trois classes de solutions de rang 2 en supposant tout d'abord que le vecteur λ^1 soit donné par $\lambda^1 = (v, 0)$. Les équations (6.2.24) deviennent

$$\lambda^2_{1,w} - \lambda^2_2 = 0, \quad \lambda^2_{2,w} = 0, \quad \lambda^2_{1,v} + v\lambda^2_{1,w} - 1 = 0, \quad \lambda^2_{2,v} + v\lambda^2_{2,w} = 0, \tag{6.2.31}$$

et ont comme solution générale

$$\lambda^2_1 = \frac{C_1}{2}(2w - v^2) + v + C_2, \quad \lambda^2_2 = C_1. \tag{6.2.32}$$

Lorsque $C_1 = 1$ et $C_2 = 0$, le générateur de symétrie conditionnelle (6.2.17) devient

$$X = v\partial_y - \partial_t + \left(w - \frac{v^2}{2}\right)\partial_x, \tag{6.2.33}$$

alors que le système (6.2.25) s'écrit

$$v_{r^1} + v_{r^2} - w_{r^2} = 0, \quad w_{r^1} + w_{r^2} - vv_{r^1} + \left(\frac{1}{2}v^2 - w\right)v_{r^2} = 0. \tag{6.2.34}$$

Afin d'obtenir des solutions particulières du système (6.2.34), nous utilisons l'invariance de ce système par rapport aux groupes engendrés par les générateurs

$$X_1 = r^1\partial_{r^1} + r^2\partial_{r^2}, \quad X_2 = -r^1\partial_{r^2} + 2\partial_v + (v+2)\partial_w.$$

Les invariants du générateur X_1 sont donnés par $\{\xi = r^2/r^1, v, w\}$, ce qui nous permet d'obtenir le système d'EDOs

$$(\xi-1)F'(\xi)+G'(\xi)=0, \quad \left(2\xi F(\xi)+F(\xi)^2-2G(\xi)\right)F'(\xi)-2(\xi-1)G'(\xi)=0, \tag{6.2.35}$$

où nous avons introduit $v = F(\xi), w = G(\xi)$. La solution du système (6.2.35) est alors donnée par

$$F(\xi)=-2\xi-W(Ce^{-\xi}), \quad G(\xi)=\xi F(\xi)+\frac{1}{2}F(\xi)^2+\xi^2-2\xi+1, \quad C\in\mathbb{R}, \tag{6.2.36}$$

où $W(Ce^{\xi})$ est la fonction de Lambert satisfaisant la relation

$$W(Ce^{-\xi})e^{W(Ce^{-\xi})}=Ce^{-\xi}. \tag{6.2.37}$$

La solution du système (6.2.34) s'écrit alors

$$\begin{aligned} v &= -2\frac{r^2}{r^1}-W\left(Ce^{-r^2/r^1}\right) \\ w &= \left(\frac{r^1-r^2}{r^1}\right)^2+\frac{r^2}{r^1}W\left(Ce^{-r^2/r^1}\right)+\frac{1}{2}W\left(Ce^{-r^2/r^1}\right)^2 \end{aligned} \tag{6.2.38}$$

où les invariants r^1, r^2 sont donnés implicitement par

$$\begin{aligned} r^1 &= y-\left(2\frac{r^2}{r^1}+W\left(Ce^{-r^2/r^1}\right)\right)t, \\ r^2 &= y-\left(\left(\frac{r^2}{r^1}\right)^2+4\frac{r^2}{r^1}-1+\left(\frac{r^2}{r^1}+1\right)W\left(Ce^{-r^2/r^1}\right)\right)t+x. \end{aligned} \tag{6.2.39}$$

L'élimination de la fonction $W(\cdot)$ dans les expressions (6.2.39) implique que l'invariant r^2 satisfait la relation polynomiale

$$\left(\frac{r^2}{r^1}\right)^2 t-\frac{r^2}{r^1}(y+2t)+r^1+x+t=0, \tag{6.2.40}$$

ce qui nous permet d'obtenir l'expression suivante pour r^2 lorsque $t \neq 0$,

$$r^2=\frac{r^1}{2t}\left(y+2t+\varepsilon\sqrt{y^2+4t(y-x-r^1)}\right), \quad \varepsilon=\pm 1. \tag{6.2.41}$$

En substituant dans (6.2.38) et (6.2.39), nous obtenons que la solution v de l'équation dKP (6.2.2) pour $t > 0$ est donnée par

$$v=-\frac{y+2t+\varepsilon\sqrt{y^2+4t(y-x-r^1)}}{t}-W\left(C\exp\left[-\frac{y+2t+\varepsilon\sqrt{y^2+4t(y-x-r^1)}}{2t}\right]\right), \tag{6.2.42}$$

où r^1 satisfait la relation transcendante

$$r^1 = -2t - \varepsilon\sqrt{y^2+4t(y-x-r^1)} - tW\left(C\exp\left[-\frac{y+2t+\varepsilon\sqrt{y^2+4t(y-x-r^1)}}{2t}\right]\right). \tag{6.2.43}$$

La fonction $W(s)$ de Lambert possède deux branches réelles autour du point de branchement $s=-1/e$. Dans notre cas, lorsque $C>0$, la solution (6.2.42) est continue pour toute valeur de t,x,y puisque l'argument est toujours positif. Lorsque $C<0$, la solution possède un branchement lorsque

$$-\frac{y+2t+\varepsilon\sqrt{y^2+4t(y-x-r^1)}}{2t} = -\frac{1}{e},$$

c'est-à-dire sur la surface définie par

$$r^1 = -(x+t) + \frac{y+2t}{e} - \frac{t}{e^2}, \tag{6.2.44}$$

et peut par conséquent posséder un caractère ergodique [**74**]. Il est à noter que pour $t=0$, les expressions (6.2.38) et (6.2.39) impliquent que la solution v est définie explicitement par

$$t=0 \quad : \quad r^1 = y, \quad r^2 = y+x, \quad v = -2\frac{y+x}{y} - W\left(Ce^{-(y+x)/y}\right). \tag{6.2.45}$$

La figure 6.1 illustre la solution v définie par (6.2.38) et (6.2.39) pour $x>0$ au temps $t=0$ lorsque $C>0$.

Similairement, une réduction par rapport au groupe généré par X_2 nous permet d'obtenir la solution singulière

$$v = \frac{A-2r^2}{r^1}, \quad w = 1 + \frac{A-2r^2}{r^1} + \frac{(r^2)^2 - Ar^2 + B}{(r^1)^2}, \quad A,B\in\mathbb{R}. \tag{6.2.46}$$

En substituant ces expressions dans la définition de r^1 et r^2, nous obtenons que les invariants doivent satisfaire les relations

$$\begin{aligned} r^1 &= y + \frac{A-2r^2}{r^1}t, \quad t>0,\\ r^2 &= y + \left(1 + 2\frac{A-2r^2}{r^1} + \frac{1}{(r^1)^2}\left(Ar^2 - (r^2)^2 - \frac{1}{2}A^2 + B\right)\right)t + x. \end{aligned} \tag{6.2.47}$$

Notons que, par construction, les fonctions v, w, r^1, r^2 forment l'ensemble complet des invariants du générateur de symétrie conditionnelle X. Ainsi, en substituant la solution (6.2.46) dans l'expression (6.2.33) du générateur X, nous obtenons que

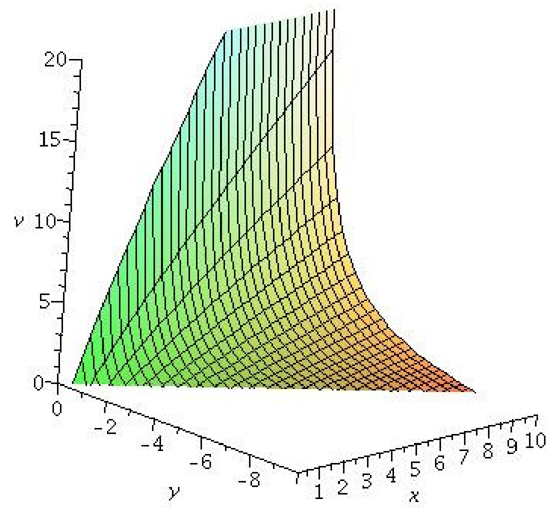

Fig. 6.1. Solution (6.2.38) pour $t = 0$.

les invariants de Riemann r^1 et r^2 sont invariants par rapport au groupe généré par

$$X = \partial_t - \frac{A - 2r^2}{r^1}\partial_y - \left(1 + \frac{A - 2r^2}{r^1} + \frac{1}{{r^1}^2}\left(Ar^2 - {r^2}^2 + B - \frac{A^2}{4}\right)\right)\partial_x. \quad (6.2.48)$$

Pour $t = 0$, les invariants (6.2.47) sont donnés par

$$r^1 = y, \quad r^2 = y + x.$$

Lorsque $t \neq 0$, r^1 est une solution de l'expression rationnelle

$$\frac{{r^1}^2}{4t} + 2r^1 + \frac{t}{{r^1}^2}\left(B - \frac{A^2}{4}\right) + x + t - y - \frac{y^2}{4t} - \frac{A}{2} = 0, \quad (6.2.49)$$

alors que r^2 est donné par

$$r^2 = -\frac{1}{2t}\left({r^1}^2 - yr^1 - At\right). \quad (6.2.50)$$

La solution (6.2.46) est singulière sur l'hyperplan de R^3 défini par la relation $r^1 = 0$. Lorsque $B = A^2/4$, l'expression (6.2.49) est polynomiale d'ordre deux pour r^1 et nous obtenons la forme explicite des invariants en termes de radicaux

$$\begin{aligned} r^1 &= -4t + \varepsilon\sqrt{12t^2 + 4t(y-x) + y^2 + 2At}, \quad \varepsilon = \pm 1, \\ r^2 &= -\frac{y^2}{2t} + 2x - 4y - 14t - \frac{A}{2} + \varepsilon\left(4 + \frac{y}{2t}\right)\sqrt{12t^2 + 4t(y-x) + y^2 + 2At}. \end{aligned} \tag{6.2.51}$$

La branche négative de (6.2.51) ne révèle aucune singularité pour r^1 lorsque $t > 0$. Pour chaque $t > 0$ fixé, l'hyperplan $r^1 = 0$ représentant la courbe singulière de la solution v est décrit dans $\mathbb{R}^2$ par la parabole

$$y^2 + t(2A + 4(y-x)) - 4t^2 = 0. \tag{6.2.52}$$

On obtient alors la solution singulière explicite

$$v = \frac{-28t^2 - y^2 - 2At + 4t(x-2y) + (y+8t)\sqrt{12t^2 + 4t(y-x) + y^2 + 2At)}}{t(4t - \sqrt{12t^2 + 4t(y-x) + y^2 + 2At)}}, A \in \mathbb{R}.$$

Notons que lorsque $t \to 0^+$, l'équation (6.2.52) implique que la courbe $y = 0$ est singulière pour cette solution au temps initial $t = 0$.

Lorsque $B \neq A^2/4$ et $t \neq 0$, r^1 est solution de l'équation polynomiale de degré quatre

$${r^1}^4 + 8{r^1}^3 t + (4t^2 + 4t(x-y) - y^2 - 2At){r^1}^2 - (A^2 - 4B)t^2 = 0. \tag{6.2.53}$$

Une expression pour r^1 peut être obtenue en utilisant la méthode de Ferrari [**117**] comme suit. En posant tout d'abord $r^1 = z - 2t$, l'équation (6.2.53) devient

$$z^4 + pz^2 + qz + s = 0, \tag{6.2.54}$$

où

$$\begin{aligned} p &= 4t(x-y) - y^2 - 20t^2 - 2At, \\ q &= 4t(12t^2 + 2At - 4t(x-y) + y^2), \\ s &= -t^2\left(A^2 + 8At - 4(B - 8t^2 + 4t(x-y) - y^2)\right). \end{aligned} \tag{6.2.55}$$

La méthode de Ferrari propose de remplacer z^4 par $(z^2+\omega)^2 - 2\omega z^2 - \omega^2$ pour écrire (6.2.54) sous la forme

$$(z^2+\omega)^2 + (p - 2\omega)z^2 + qz + s - \omega^2 = 0. \tag{6.2.56}$$

L'objectif est de déterminer ω tel que l'équation puisse être amenée à la forme

$$(z^2+\omega)^2-(\alpha z+\beta)^2=0, \tag{6.2.57}$$

ce qui nous permettrait d'obtenir

$$(z^2+\alpha z+\omega+\beta)(z^2-\alpha z+\omega-\beta)=0. \tag{6.2.58}$$

Pour que de tels α et β existent, le polynôme $(p-2\omega)z^2+qz+s-\omega^2$ doit posséder une racine double, c'est-à-dire que son discriminant

$$q^2+4(p-2\omega)(\omega^2-s)=8\omega^3-4p\omega^2-8s\omega+4ps-q^2 \tag{6.2.59}$$

doit être nul. On obtient une solution réelle ω_1 de ce polynôme à l'aide de la méthode de Cardan. L'équation (6.2.56) s'écrit alors sous la forme (6.2.58), avec

$$\alpha^2=2\omega_1-p \quad \text{et} \quad \begin{cases} \beta=-\frac{q}{2\alpha} & 2\omega_1-p\neq 0 \\ \beta^2=\omega_1^2-s & 2\omega_1-p=0. \end{cases} \tag{6.2.60}$$

Les polynômes

$$p_1=z^2+\alpha z+\omega_1+\beta, \quad p_2=z^2-\alpha z+\omega_1-\beta \tag{6.2.61}$$

possèdent des racines essentiellement complexes lorsque

$$\begin{cases} p_1: & \alpha^2-4(\beta+\omega_1)<0 \\ p_2: & \alpha^2-4(\omega_1-\beta)<0. \end{cases} \tag{6.2.62}$$

Le polynôme (6.2.54) ne possède donc que des racines non réelles lorsque

$$\begin{cases} \alpha^2-4(\beta+\omega_1)<0 \\ \alpha^2-4(\omega_1-\beta)<0 \end{cases} \Rightarrow \begin{cases} \beta>0 \\ \alpha^2-4\omega_1<0 \end{cases} \Rightarrow \begin{cases} \beta>0 \\ 2\omega_1-p>0 \end{cases}. \tag{6.2.63}$$

Dès que l'une de ces conditions est brisée, c'est-à-dire que $\beta\leq 0$ ou $2\omega_1-p\leq 0$, le polynôme (6.2.54) possède au moins deux branches réelles.

Mentionnons qu'il est possible de généraliser les solutions (6.2.38) et (6.2.46) en observant que pour toute fonction $v=V(r^2/r^1)$, les fonctions définies par

$$\begin{aligned} &v=V(r^2/r^1), \quad r^1=y+V(r^2/r^1)t, \\ &r^2=y+\left(\left(\frac{r^2}{r^1}\right)^2+\left(\frac{r^2}{r^1}+1\right)V(r^2/r^1)-2\frac{r^2}{r^1}+1\right)+x, \end{aligned} \tag{6.2.64}$$

définissent une solution de l'équation dKP.

Finalement, il est également possible de construire des solutions du système (6.2.34) sous la forme d'ondes de propagation à l'aide de l'algorithme de la tangente hyperbolique, ou méthode "tanh". Cette méthode introduite par Malfliet et Hereman **[78, 79]** s'avère très efficace pour obtenir des solutions en ondes de propagation pour les systèmes d'EDPs nonlinéaires autonomes

$$G(u, u_{r^1}, u_{r^2}, \ldots) = 0, \tag{6.2.65}$$

où les variables apparaissent seulement de façon polynomiale. Selon cette méthode, nous cherchons des solutions de (6.2.65) sous la forme d'un polynôme en tanh, c'est-à-dire

$$u = \sum_{j=0}^{M} a_j \tanh^j(\xi), \quad \xi = \alpha r^1 + \beta r^2 + \delta, \quad \alpha, \beta, \delta \in \mathbb{R}. \tag{6.2.66}$$

En substituant cette forme de solution dans l'équation (6.2.65), cette dernière devient une expression polynomiale en $\tanh(\xi)$. Cela est dû au fait que les dérivées successives de la fonction tanh s'exprime de façon polynomiale en termes de tanh, par exemple

$$\tanh'(\xi) = \operatorname{sech}^2(\xi) = 1 - \tanh^2(\xi), \quad \tanh''(\xi) = -2\tanh(\xi) + 2\tanh^3(\xi), etc. \tag{6.2.67}$$

puisque $\cosh^2(\xi) - \sinh^2(\xi) = 1$. L'ordre M du polynôme (6.2.66) est déterminé en balançant les termes d'ordre les plus élevés. Finalement, les coefficients a_j sont obtenus en résolvant le système obtenu en demandant que les coefficients de chacune des puissances de tanh soient nuls.

L'application de cette méthode au système (6.2.34), nous permet d'obtenir la solution solitonique de type "kink" suivante

$$\begin{aligned} v &= v_0 + A_1(\tanh\xi - \sqrt{3}\operatorname{sech}\xi), \quad \xi = \alpha r^1 + \beta r^2 + \delta \\ w &= w_0 + A_1\frac{(\alpha+\beta)}{\beta}\left(\tanh\xi - \sqrt{3}\operatorname{sech}\xi\right), \end{aligned} \tag{6.2.68}$$

où les invariants r^1 et r^2 satisfont les relations implicites

$$r^1 = y + \left(v_0 + A_1(\tanh\xi - \sqrt{3}\,\mathrm{sech}\,\xi)\right)t,$$
$$r^2 = y + \Big(w_0 + v_0 - \frac{v_0^2}{2} - \frac{A_1^2}{2}\tanh^2\xi - A_1^2\,\mathrm{sech}^2\,\xi \qquad (6.2.69)$$
$$+A_1^2\sqrt{3}\tanh\xi\,\mathrm{sech}\,\xi + \frac{A_1}{\beta}(2\beta + \alpha - v_0\beta)(\tanh\xi - \sqrt{3}\,\mathrm{sech}\,\xi)\Big)t + x.$$

Compte tenu de la forme de la solution (6.2.68) rappelant la structure d'une onde de propagation, Banerjee et al. [**3**] ont attribué à ce type de solution le nom d'onde solitaire implicite. Ils ont également observé que ces solutions se comportent localement comme des ondes solitaires. Ce comportement solitonique de la solution (6.2.68) est illustré à la figure (6.2). Il est à noter que ces solutions

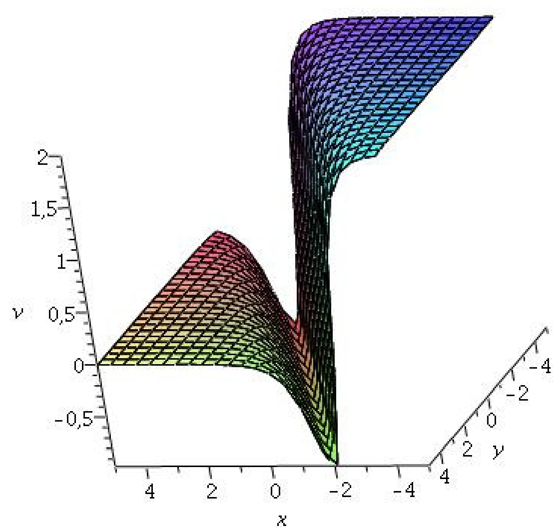

FIG. 6.2. Solution solitonique (6.2.68) de type « kink » de l'équation dKP en $t = 0$.

ont été obtenues en fixant préalablement le vecteur $\lambda^1 = (1, v, 0)$. Il serait possible d'obtenir des classes de solution différentes en considérant d'autres choix.

Cette méthode nous a permis de construire certaines classes de solutions de rang-2 et ainsi de trouver de nouveaux résultats. Ce travail vaut la peine d'être entrepris et est présentement en cours.

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Annexe A

ÉLÉMENTS INTÉGRAUX ET ÉLÉMENTS SIMPLES POUR LES SYSTÈMES QUASILINÉAIRES DU PREMIER ORDRE

Nous utilisons les éléments intégraux de rang k ainsi que les éléments simples de rang 1 pour les systèmes homogènes et non homogènes d'EDPs et nous les classifions selon la définition dans [**46**]. Nous rappelons certains théorèmes et notation concernant les éléments intégraux pour les systèmes quasi linéaires du premier ordre (0.0.1). En chaque point (x_0, u_0) d'un ouvert $\mathcal{D} \subset X \times U$, nous définissons l'hyperplan $\mathcal{L}(x_0, u_0)$ dans $\mathbb{R}^{p\times q}$ consistant en les matrices $\{L_i^\alpha(x_0, u_0)\}$, que nous appelons éléments intégraux, qui satisfont les équations algébriques

$$\sum_{i=0}^{p}\sum_{\alpha=1}^{q} a^{i\mu}_{\ \alpha}(u^1, \ldots, u^q) L_i^\alpha(x, u) = b^\mu(u^1, \ldots, u^q), \quad \mu = 1, \ldots, l, \tag{A.0.1}$$

où $\max\limits_{(x,u)\in\mathcal{D}} \operatorname{rank} L_i^\alpha(x, u) = \min(p, q)$. Si $\underset{0}{L}$ est une solution du système (A.0.1), alors

$$\mathcal{L} = \mathcal{K} + \underset{0}{L}, \tag{A.0.2}$$

où $\mathcal{K} = \{K(x_0, u_0) \in \mathbb{R}^{p\times q} : a^{i\mu}_{\ \alpha}(u_0) K_i^\alpha(x_0, u_0) = 0\}$ est l'espace vectoriel des solutions du système algébrique homogène (A.0.1). La dimension de l'espace $\mathcal{K}(x_0, u_0)$ des éléments intégraux homogènes est donnée par

$$\dim \mathcal{K}(x_0, u_0) = pq - l(x_0, u_0), \tag{A.0.3}$$

où $l(x_0, u_0)$ est le nombre d'équations algébriques indépendantes dans le système (A.0.1) ou le nombre de matrices $a^i = \{a^{i\mu}_{\ \alpha}(u_0)\}$ linéairement indépendantes.

D'après cette définition, une combinaison linéaire

$$\mu^1(x)L_1 + \ldots + \mu^n(x)L_n$$

d'éléments intégraux $L_1, \ldots, L_n$ appartient à $\mathcal{L}$ si et seulement si

$$\sum_{s=1}^{n} \mu^s(x) = 1. \tag{A.0.4}$$

S'il existe au moins une solution du système non homogène (A.0.1), alors

$$\dim \mathcal{K}(x_0, u_0) = \dim \mathcal{L}(x_0, u_0). \tag{A.0.5}$$

A.1. Éléments simples

Parmi les éléments intégraux définis précédemment, les éléments simples permettent la construction des solutions élémentaires, appelées ondes simples ou ondes de Riemann. Un élément intégral $L \in \mathcal{L}(x_0, u_0)$ est dit simple (ou décomposable) s'il existe des vecteurs $\lambda \in \mathbb{R}^p$ et $\gamma \in \mathbb{R}^q$ tels que L peut être écrit sous la forme

$$L_i^\alpha = \gamma^i \lambda_\alpha, \tag{A.1.1}$$

c'est-à-dire que

$$\operatorname{rank} ||L_i^\alpha(u_0, x_0)|| = 1. \tag{A.1.2}$$

Il est utile de considérer λ comme un élément de $T_x X^*$. L'espace X^* est l'espace dual de X, l'espace des formes linéaires : $X^* \ni \lambda : X \to \mathbb{R}$. Dans cette terminologie, L est un élément de l'espace tensoriel produit $T_u U \times T_x X^*$ s'écrivant sous la forme

$$L = \gamma \otimes \lambda \in T_u U \otimes T_x X^*. \tag{A.1.3}$$

Les éléments simples d'un système homogène sont notés par $\gamma \otimes \lambda$ alors qu'on notera $\underset{0}{\gamma} \otimes \underset{0}{\lambda}$ pour un élément simple non homogène tel que

$$a^{i\mu}_{\ \alpha}(u) \underset{0}{\gamma}^\alpha \underset{0}{\lambda}_i = b^\mu(u).$$

Les éléments homogènes sont directement reliés à l'existence de vecteurs caractéristiques. En effet, si $\gamma \otimes \lambda$ est un élément simple d'un système homogène, alors λ est un vecteur caractéristique puisque si $a^{i\mu}_{\ \alpha} \gamma^\alpha \lambda_i = 0$, nous devons avoir que $\operatorname{rank} \left(a^{i\mu}_{\ \alpha} \lambda_i\right) < q$ ou si $l = q$, $\det \left(a^{i\mu}_{\ \alpha} \lambda_i\right) = 0$.

Une solution $u : D \to U$, $D \subset X$ du système (0.0.1) est appelée une onde simple du système homogène (ou état simple du système non homogène) si l'application tangente linéaire du est un élément simple en chaque point $x_0 \in D$.

Théorème 1 ([8]). *L'hodographe $u(D)$ d'une onde simple (ou d'un état simple) pour un système homogène ou non homogène est donné par la courbe dans l'espace hodographe U ayant γ comme vecteur tangent en chaque point.*

Par la définition d'un élément intégral, une application $X \supset D \xrightarrow{u} U$ est une solution si et seulement si $du \in \mathcal{L}$. Nous pouvons ainsi chercher des solutions appartenant à un sous-espace de $\mathcal{L}$. Par exemple, si nous considérons une famile d'éléments intégraux dépendant sur q paramètres $\xi^1, \ldots, \xi^q$, $L(u, x, \xi^1, \ldots, \xi^r) \in \mathcal{L}(u, x)$, alors les solutions de

$$du = L(u, x, \xi^1, \ldots, \xi^r) \tag{A.1.4}$$

existent si et seulement si les conditions de compatibilité

$$0 = d(du) = dL \text{ modulo (A.1.4)} \tag{A.1.5}$$

sont satisfaites. En particulier, nous pouvons choisir

$$L(u, x, \xi^1, \ldots, \xi^r, \mu^1, \ldots, \mu^p) = \xi^1 \gamma_1 \otimes \lambda^1 + \ldots + \xi^r \gamma_r \otimes \lambda^r + \mu^1 \underset{0}{\gamma_1} \otimes \underset{0}{\lambda^1} + \ldots + \mu^n \underset{0}{\gamma_n} \otimes \underset{0}{\lambda^n}, \tag{A.1.6}$$

où $\sum_{s=1}^n \mu^s = 1$ et $\gamma_j \otimes \lambda^j$ sont des éléments simples du système homogène et $\mu^s \underset{0}{\gamma_s} \otimes \underset{0}{\lambda^s}$ sont des éléments simples du système non homogène.

L'interprétation physique de ces deux types d'éléments $\gamma \otimes \lambda$ et $\underset{0}{\gamma} \otimes \underset{0}{\lambda}$ sont différents. D'une part, les éléments simples homogènes sont associés à certains types d'ondes de propagation dans un milieu. D'autre part, les éléments non homogènes décrivent l'état perturbé du milieu et ne peuvent être associés en général à des phénomènes d'onde. Les solutions du type (A.1.6) correspondent alors à des interactions d'ondes de propagation dans un milieu perturbé.

A.2. Classification des systèmes quasi linéaires d'équations différentielles

Une classification des éléments simples est utile pour construire des classes de solutions des systèmes quasi linéaires. L'idée principale de cette classification est de distinguer les sous-espaces de l'espace des éléments intégraux homogènes suivants.

Nous notons par Q_1 l'espace linéaire engendré par tous les éléments simples appartenant à $\mathcal{K}(x_0, u_0)$

$$Q_1 = \{\gamma_k \otimes \lambda^k\}, \tag{A.2.1}$$

où $\{\}$ représente l'espace engendré par les éléments $\gamma_k \otimes \lambda^k$. Évidemment, $Q_1 \subset \mathcal{K}$.

De même, Q_l désigne l'espace vectoriel engendré par tous les éléments intégraux homogènes de rang au plus l,

$$\{q(x_0, u_0) \in \mathcal{K} :< a^i, q >= 0 \text{ and } \operatorname{rank} ||q(x_0, u_0)|| \leq l\}. \tag{A.2.2}$$

Nous avons donc

$$\{0\} \subset Q_1 \subset Q_2 \subset \ldots \subset Q_l = \mathcal{K}. \tag{A.2.3}$$

Ces définitions s'étendent également aux systèmes non homogènes de la façon suivante. Nous notons par $\mathcal{L}_1$ l'hyperplan contenant tous les éléments L_1 de la forme

$$L_1 = \underset{0}{\gamma} \otimes \underset{0}{\lambda}, \tag{A.2.4}$$

où $\underset{0}{\gamma} \in \mathbb{R}^q$ et $\underset{0}{\lambda} \in \mathbb{R}^{p*}$ et

$$< a^i, \underset{0}{\gamma} \otimes \underset{0}{\lambda} >= a^{i\mu}_{\alpha} \underset{0}{\gamma}^{\alpha} \underset{0}{\lambda}_i = b^s. \tag{A.2.5}$$

Tous les éléments de $\mathcal{L}_1$ sont alors de la forme

$$L_1 = \sum_{s=1}^{n} \mu^s \underset{0}{\gamma_s} \otimes \underset{0}{\lambda}^s, \quad \sum_{s=1}^{n} \mu^s = 1, \tag{A.2.6}$$

et $\underset{0}{\gamma_1} \otimes \underset{0}{\lambda}^1, \ldots, \underset{0}{\gamma_n} \otimes \underset{0}{\lambda}^n$ sont des éléments simples non homogènes linéairement indépendants qui engendrent $\mathcal{L}_1$. Nous avons donc

$$\mathcal{L}_1 \subset \mathcal{L}.$$

Les systèmes pour lesquels $\mathcal{L}_1(x_0, u_0) = \mathcal{L}(x_0, u_0)$ sont appelés systèmes $\mathcal{L}_1$.

Notons finalement par $\mathcal{L}_k(x_0, u_0)$ le sous-espace vectoriel engendré par tous les éléments $L \in \mathcal{L}$ tels que

$$\text{rank}\,||L(x_0, u_0)|| \leq k. \tag{A.2.7}$$

Nous avons donc

$$\mathcal{L}_1 \subset \mathcal{L}_2 \subset \ldots \subset \mathcal{L}_k = \mathcal{L}. \tag{A.2.8}$$

A.3. Théorèmes sur les système $\mathcal{L}_1$

Nous donnons maintenant quelques théorèmes exhibant la structure de $\mathcal{L}$. Ceux-ci permettent de déterminer si un système est de type $\mathcal{L}_1$. Considérons un système sous forme évolutive en deux variables indépendantes

$$u_t + a(u)u_x = b(u), \qquad \text{où } a(u) = (a^i_j(u)) \in \mathbb{R}^{q\times q}. \tag{A.3.1}$$

L'équation pour les éléments simples de (A.3.1) s'écrit

$$(\mathcal{I}_q\lambda_0 + a(u))\gamma = b(u). \tag{A.3.2}$$

Nous avons alors le théorème suivant.

Théorème 2. *Considérons le système (A.3.1). Si le vecteur $b(u)$ n'appartient pas à un sous-espace invariant $N \subsetneq U$ de la matrice $a(u)$, alors*

$$\mathcal{L}_1(x_0, u_0) = \mathcal{L}(x_0, u_0).$$

Un sous-espace N est dit invariant si $x \in N$ implique que $a(u)x \in N$.

Notons par N_b le plus petit sous-espace invariant de $a(u)$ contenant le vecteur $b(u)$. L'espace H^{Hyp} est le sous-espace de U engendré par les vecteurs propres réels de $a(u)$. Nous obtenons le résultat suivant.

Théorème 3. *Si les valeurs propres réelles de la matrice $a(u)$ sont distinctes, alors*

$$\dim \mathcal{L}_1 = \dim H^{Hyp} + \dim N_b - \dim (H^{Hyp} \cap N_b). \tag{A.3.3}$$

Corollaire 1. *Supposons que le système (A.3.1) satisfait les hypothèses du théorème 3. Puisque $\dim Q_1 = \dim H^{Hyp}$, l'équation (A.3.3) peut être écrite*

$$\dim \mathcal{L}_1 = \dim Q_1 + \dim N_b - \dim (Q_1 \cap N_b). \tag{A.3.4}$$

Nous avons alors les énoncés suivants.

1. *Si* $N_b \subset H^{Hyp}$*, alors* $\dim \mathcal{L}_1 = \dim Q_1$.

2. *Si* $N_b \not\subset H^{Hyp}$*, alors* $\dim \mathcal{L}_1 > \dim Q_1$*. En fait,* $\dim N_b \geq \dim \left(H^{Hyp} \cap N_b\right)$*. Il suit alors que*

$$\dim \mathcal{L}_1 \geq \dim Q_1. \tag{A.3.5}$$

3. *Si le système est elliptique, c'est-à-dire que* $\dim H^{Hyp} = \dim Q_1 = 0$*, alors*

$$\dim \mathcal{L}_1 = \dim N_b. \tag{A.3.6}$$

Théorème 4. *Supposons que la matrice* $a(u)$ *du le système (A.3.1) possède* r *valeurs propres distinctes* $\mu_1, \ldots, \mu_r$ *avec multiplicités* $k(1), \ldots, k(r)$ *en un point* (x_0, u_0)*. Supposons de plus que le nombre de vecteurs propres linéairement indépendants associés à chaque valeur propre* μ_i *est égal à sa multiplicité* $k(i)$*, pour* $i = 1, \ldots, r$*. Nous avons alors*

$$\dim \mathcal{L}_1 = \dim N_b + \sum_{i=1}^{k} k(i) \left(1 - \chi(H_i)\right), \tag{A.3.7}$$

ou, de façon équivalente

$$\dim \mathcal{L}_1 = \dim N_b + \dim H^{Hyp} - \dim(H^{Hyp} \cap N_b) - \sum_{i=1}^{r} \left(k(i) - 1\right) \chi(H_i). \tag{A.3.8}$$

Ici, les H_i *représentent les sous-espaces engendrés par les vecteurs propres associées à* μ_i *et la fonction* $\chi(H_i)$ *est définie par*

$$\chi(H_i) = \begin{cases} 0 & \textit{lorsque } H_i \cap N_b = \{0\}, \\ 1 & \textit{lorsque } H_i \cap N_b \neq \{0\}. \end{cases} \tag{A.3.9}$$

En somme, les résultats peuvent être résumés comme suit.

Corollaire 2. 1. *Si le système est hyperbolique, alors* $\dim N_b = \dim(N_b \cap H^{Hyp}) = \sum_{i=1}^{r} \chi(H_i)$ *et donc*

$$\dim \mathcal{L}_1 = \sum_{i=1}^{r} k(i) - \sum_{i=1}^{r} k(i)\chi(H_i) + \sum_{i=1}^{r} \chi(H_i) \tag{A.3.10}$$

D'où, puisque pour les systèmes hyperboliques $H^{Hyp} = U$*,*

$$\dim \mathcal{L}_1 = \dim U - \sum_{i=1}^{r} \left(k(i) - 1\right) \chi(H_i). \tag{A.3.11}$$

De plus,

$$\dim \mathcal{L}_1 \leq \dim Q_1 = \dim U.$$

L'inégalité peut apparaître lorsqu'une des valeurs propres de A possède une multiplicité plus grande que un.

2. *Si* $N_b \subset H^{Hyp}$, *alors* $\dim \mathcal{L}_1 \leq \dim Q_1$.
3. *Si* $N_b \cap H^{Hyp} = \{0\}$, *alors* $\dim \mathcal{L}_1 > \dim Q_1$.
4. *Si le système est elliptique,* $\dim H^{Hyp} = \dim Q_1 = 0$, *alors*

$$\dim \mathcal{L}_1 = \dim N_b.$$

Nous terminons cette discussion en considérant les systèmes évolutifs plus généraux en plusieurs variables indépendantes

$$u_t + a^i(u)u_{x^i} = b(u). \tag{A.3.12}$$

Pour une telle étude, il est nécessaire de considérer la matrice $\lambda_i a^i(u)$ plutôt que la matrice $a(u)$. Supposons que $\mu_i = \mu_i(\bar{\lambda})$, $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^p$, sont des fonctions réelles, distinctes et analytiques sur $\mathbb{R}^p$. Supposons, de plus, que le nombre de vecteurs propres linéairement indépendants correspondants à chaque valeur propre μ_i est égal à sa multiplicité $k(i)$, $i = 1, \ldots, r$. Il est possible de généraliser le résultat du théorème 4 de la façon suivante :

$$\dim \mathcal{L}_1 = n \left(\dim N_b + \sum_{i=1}^{r} k(i) \left(1 - \chi(H_i)\right) \right) \tag{A.3.13}$$

ou sous une forme équivalente

$$\dim \mathcal{L}_1 = \dim Q_1 + n \left(\dim N_b - \dim \left(H^{Hyp} \cap N_b \right) - \sum_{i=1}^{r} \left(k(i) - 1\right) \chi(H_i) \right). \tag{A.3.14}$$

Ici, les sous-espaces N_b et H^{Hyp} sont obtenus pour toute direction fixée $\bar{\lambda} \in \mathbb{R}^p$ telle que les racines $\mu_i(\bar{\lambda})$ et leur multiplicité sont distinctes. En particulier, ces hypothèses sont satisfaites si pour tout $\bar{\lambda} \in \mathbb{R}^p$ nous avons $\mu_i(\bar{\lambda}) \neq \mu_j(\bar{\lambda})$, $i \neq j$, les multiplicités $k(i)$ sont indépendantes de $\bar{\lambda}$ et le nombre de vecteurs propres linéairement indépendants est égal à $k(i)$, $i = 1, \ldots, r$.

Annexe B

UN THÉORÈME POUR LES SYSTÈMES HYPERBOLIQUES

Nous souhaitons démontrer le rôle des éléments simples pour les systèmes hyperboliques homogènes du premier ordre

$$a^{i\mu}_{\ \alpha}(u)\frac{\partial u^\alpha}{\partial x^i} = 0, \quad \mu = 1, \ldots, l. \tag{B.0.1}$$

Nous considérons pour ces systèmes le polynôme caractéristique suivant

$$\omega(\xi) = a^{i\mu}_{\ \alpha}(\xi\eta_i + \theta_i), \tag{B.0.2}$$

où $\eta, \theta \in X^*$. Si, pour un certain $\xi^0 \in \mathbb{R}$ nous avons que $\omega(\xi^0) = 0$, alors $\lambda = \xi^0\eta + \theta$ est un covecteur caractéristique. Il existe alors des vecteurs caractéristiques $\underset{0,\alpha}{\gamma}$, où $\alpha = 1, \ldots, r_0$, r_0 étant la multiplicité de ξ^0.

Définition 1 ([**46**]). *Le système (B.0.1) est dit hyperbolique au point* (x_0, u_0) *dans la direction* $\sigma \in X$ *si pour tout* $0 \neq \theta \in X^*$ *tel que*

$$< \theta, \sigma >= 0, \tag{B.0.3}$$

et tout η *tel que*

$$< \eta, \sigma >\neq 0,$$

le polynôme caractéristique (B.0.2) possède :

1. $k \leq q$ *racines réelles dont les multiplicités ne dépendent pas du choix de* θ*.*

2. Les vecteurs caractéristiques $\underset{1,1}{\gamma}, \underset{1,2}{\gamma}, \ldots, \underset{1,r_1}{\gamma}, \underset{2,1}{\gamma}, \ldots, \underset{k,r_k}{\gamma}$ *correspondant aux vecteurs* $\lambda^1, \ldots, \lambda^k$ *engendrent l'espace hodographe* U*.*

Définition 2. *Le système est fortement hyperbolique dans la direction* $\sigma \in X$ *pour tout* θ *satisfaisant (B.0.3) si et seulement si son polynôme caractéristique (B.0.2) possède exactement* q *racines réelles distinctes. Lorsque* $k = q$ *et que les racines* ξ^i *sont distinctes, les vecteurs* $\underset{n}{\gamma}$*,* $n = 1, \ldots, q$*, qui sont associés aux valeurs propres engendrent l'espace hodographe entier* U*.*

Définition 3. *Le système est hyperbolique (respectivement fortement hyperbolique) s'il existe* $\sigma \in X$ *tel que le système est hyperbolique (respectivement fortement hyperbolique) dans la direction* σ*.*

Le théorème suivant permet de relier la notion d'hyperbolicité et les sous-espaces Q_1.

Théorème 5 ([**46**])**.** *Si le système (B.0.1) est hyperbolique, alors tous ses éléments intégraux peuvent être écrits comme une combinaison linéaire d'éléments simples, c'est-à-dire que*

$$\mathcal{K}(x_0, u_0) = Q_1(x_0, u_0). \tag{B.0.4}$$

On conclut donc que l'espace des éléments intégraux est engendré par les éléments simples. Ainsi, chaque élément intégral K est une combinaison linéaire de n éléments simples

$$K = \gamma_1 \otimes \lambda^1 + \ldots + \gamma_q \otimes \lambda^q,$$

où $n \leq pq - l$.

Annexe C

L'ALGORITHME DE LEVERRIER-FADDEEV

Soit A une matrice $n \times n$, son polynôme caractéristique $p(s)$ est défini par

$$p(s) = \det(s\mathcal{I}_n - A) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1}s + a_n. \qquad \text{(C.0.1)}$$

Il est possible de relier celui-ci à la matrice résolvante $(sI - A)^{-1}$ en notant que

$$(s\mathcal{I}_n - A)^{-1} = \frac{N_1 s^{n-1} + N_2 s^{n-2} + \ldots + N_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1}s + a_n} = \frac{N(s)}{p(s)}. \qquad \text{(C.0.2)}$$

La matrice $N(s)$ représente l'adjointe classique de $(s\mathcal{I}_n - A)$ et est polynomiale d'ordre $n-1$ en s. On peut, à partir de l'expression (C.0.2), déterminer récursivement les matrices constantes N_i et les coefficients du polynome caractéristique a_i, $i = 1, \ldots, n$. En effet, en développant

$$(s\mathcal{I}_n - A)N(s) = p(s)I, \qquad \text{(C.0.3)}$$

c'est-à-dire

$$(s\mathcal{I}_n - A)(N_1 s^{n-1} + N_2 s^{n-2} + \ldots + N_n) = (s^n + a_1 s^{n-1} + \ldots + a_n)\mathcal{I}_n \qquad \text{(C.0.4)}$$

on obtient, en comparant les puissances de s,

$$\begin{aligned}
0 &= AN_n + a_n\mathcal{I}_n, \\
N_n &= AN_{n-1} + a_{n-1}\mathcal{I}_n, \\
N_{n-1} &= AN_{n-2} + a_{n-2}\mathcal{I}_n, \\
&\vdots \\
N_2 &= AN_1 + a_1\mathcal{I}_n, \\
N_1 &= \mathcal{I}_n.
\end{aligned} \tag{C.0.5}$$

ou sous forme plus concise

$$N_{i+1} = AN_i + a_i\mathcal{I}_n, \quad i = 0, \ldots, n, \quad N_0 = 0, N_{n+1} = 0, a_0 = 1. \tag{C.0.6}$$

De la première équation de (C.0.5), on tire

$$a_n = -\frac{1}{n}\operatorname{Tr}(AN_n). \tag{C.0.7}$$

Faddeev et Faddeeva [**32**] ont proposé que chacun des coefficients a_i soit donné de telle façon,

$$a_k = -\frac{1}{k}\operatorname{Tr}(AN_k), \quad k = 1, 2, \ldots, n. \tag{C.0.8}$$

On obtient alors le résultat suivant.

Proposition 4 ([**32**]). *Les matrices N_k et les coefficients a_k peuvent être calculés récursivement à l'aide de*

$$\begin{array}{ll}
N_1 = \mathcal{I}_n, & a_1 = -\frac{1}{1}\operatorname{Tr}(AN_1) \\
N_2 = AN_1 + a_1\mathcal{I}_n & a_2 = -\frac{1}{2}\operatorname{Tr}(AN_2) \\
& \vdots \\
N_n = AN_{n-1} + a_{n-1}\mathcal{I}_n & a_n = -\frac{1}{n}\operatorname{Tr}(AN_n) \\
0 = AN_n + a_n\mathcal{I}_n &
\end{array} \tag{C.0.9}$$

À titre d'exemple, calculons les coefficients pour les matrices d'ordre inférieur ou égal à 4. Les expressions pour les coefficients ne dépendent pas de l'ordre de la matrice considérée. Nous n'avons alors qu'à considérer les coefficients $a_1, \ldots, a_4$.

En utilisant (C.0.9), on trouve tout d'abord

$$\begin{aligned} N_1 &= \mathcal{I}_n & a_1 &= -\operatorname{Tr} A\mathcal{I}_n = -\operatorname{Tr}(A), \\ N_2 &= A - \operatorname{Tr}(A)\mathcal{I}_n & a_2 &= -\tfrac{1}{2}\operatorname{Tr}(A[A - \operatorname{Tr}(A)\mathcal{I}_n]) = \tfrac{1}{2}\left(\operatorname{Tr}(A)^2 - \operatorname{Tr}(A^2)\right). \end{aligned} \tag{C.0.10}$$

Pour $k = 3$,

$$\begin{aligned} N_3 &= AN_2 + a_2\mathcal{I}_n = A^2 - \operatorname{Tr}(A)A + \frac{1}{2}\left(\operatorname{Tr}(A)^2 - \operatorname{Tr}\left(A^2\right)\right) \\ a_3 &= -\frac{1}{3}\operatorname{Tr}\left(A\left[A^2 - \operatorname{Tr}(A)A + \frac{1}{2}\left(\operatorname{Tr}(A)^2 - \operatorname{Tr}\left(A^2\right)\right)\right]\right) \\ &= -\frac{1}{6}\left(2\operatorname{Tr}\left(A^3\right) - 3\operatorname{Tr}(A)\operatorname{Tr}\left(A^2\right) + \operatorname{Tr}(A)^3\right). \end{aligned} \tag{C.0.11}$$

Finalement, pour $k = 4$,

$$\begin{aligned} N_4 &= AN_3 + a_3\mathcal{I}_n \\ &= A\left[A^2 - \operatorname{Tr} AA + \frac{1}{2}\left(\operatorname{Tr}(A)^2 - \operatorname{Tr}\left(A^2\right)\right)\right] \\ &\quad - \left[\frac{1}{6}\left(2\operatorname{Tr}\left(A^3\right) - 3\operatorname{Tr}(A)\operatorname{Tr}\left(A^2\right) + \operatorname{Tr}(A)^3\right)\right]\mathcal{I}_n \\ a_4 &= -\frac{1}{4}\operatorname{Tr}(AN_4) \\ &= -\frac{1}{24}\left[6\operatorname{Tr}\left(A^4\right) - 8\operatorname{Tr}(A)\operatorname{Tr}\left(A^3\right) + 6\operatorname{Tr}\left(A^2\right)\operatorname{Tr}(A)^2 - 3\operatorname{Tr}\left(A^2\right)^2 - \operatorname{Tr}(A)^4\right]. \end{aligned} \tag{C.0.12}$$

En poursuivant ce processus récursif, il est possible d'obtenir chacun des coefficients a_n du polynôme caractéristique de $A \in \mathbb{R}^{n\times n}$.

Nous donnons maintenant une preuve simplifiée de la proposition 4.

Preuve : Sans perte de généralité, considérons l'équation (C.0.2) où la matrice A est exprimée sous sa forme de Jordan,

$$A = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_n \end{pmatrix}, \quad J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}, \tag{C.0.13}$$

où $\lambda_1, \ldots, \lambda_n$ sont les valeurs propres de A. La matrice inverse $(sI - A)^{-1}$ s'écrit alors

$$\begin{pmatrix} J_1^* & & \\ & \ddots & \\ 0 & & J_n^* \end{pmatrix}, \quad J_i^* = \begin{pmatrix} \frac{1}{s-\lambda_i} & & * \\ & \ddots & \\ & & \frac{1}{s-\lambda_i} \end{pmatrix}. \tag{C.0.14}$$

Ainsi, l'équation (C.0.2) nous donne

$$\frac{\mathrm{Tr}\,(N(s))}{p(s)} = \mathrm{Tr}\left((s\mathcal{I} - A)^{-1}\right) = \sum_{i=1}^{n} \frac{\mu_i}{s - \lambda_i}, \tag{C.0.15}$$

où $\mu_i = \dim(J_i)$ est égal à la multiplicité de la valeur propre λ_i. Si $p(s) = \prod_{i=1}^{n}(s - \lambda_i)^{\mu_i}$, alors

$$p'(s) = \sum_{i=1}^{n} \mu_j (s - \lambda_j)^{\mu_i - 1} \prod_{k \neq i} (s - \lambda_k)^{\mu_k}, \tag{C.0.16}$$

et donc

$$\frac{p'(s)}{p(s)} = \sum_{i=1}^{n} \mu_i (s - \lambda_i)^{\mu_i - 1} \left[\frac{\prod_{k \neq i}(s - \lambda_k)^{\mu_k}}{\prod_{l=1}^{n}(s - \lambda_l)^{\mu_l}} \right] = \sum_{i=1}^{n} \frac{\mu_i}{s - \lambda_i}. \tag{C.0.17}$$

Nous obtenons alors à partir des équations (C.0.2) et (C.0.15)

$$\mathrm{Tr}\,(N(s)) = p'(s) \Rightarrow \sum_{i=1}^{n} \mathrm{Tr}\,(N_i) s^{n-i} = n s^{n-1} + \sum_{i=1}^{n} (n - i) a_i s^{n-i-1}. \tag{C.0.18}$$

En comparant les puissances de s dans (C.0.18), nous avons

$$\mathrm{Tr}\,(N_i) = (n - i + 1) a_{i-1}, \quad i = 1, \ldots, n. \tag{C.0.19}$$

Prenons la trace de l'équation (C.0.6) pour obtenir

$$\mathrm{Tr}\,(N_{i+1}) = \mathrm{Tr}\,(AN_i) + a_i \,\mathrm{Tr}\,(\mathcal{I}_n), \quad i = 1, \ldots, n, \tag{C.0.20}$$

et donc les relations (C.0.19) impliquent le résultat final,

$$(n - (i + 1) + 1) a_i = \mathrm{Tr}\,(AN_i) + n a_i \Rightarrow a_i = -\frac{1}{i} \mathrm{Tr}\,(AN_i), \quad i = 1, \ldots, n.$$

□