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Estimating the Tobit model with serial correlation : an operational approach*

by

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I. INTRODUCTION

Several authors have discussed recently the limited dependent variable model [Tobin (1958)], when there is serial correlation between the residual errors. Robinson (1982) has shown that the Pseudo-Maximum Likelihood estimators obtained by maximizing the log-likelihood function which Ignores the Serial correlation (PMLIS) are consistent. Robinson shows also that these estimators are asymptotically normally distributed and he derives their asymptotic covariance matrix. Dagenais (1982) has derived an expression for the likelihood function of the Tobit model with first order serial correlation, which illustrates that the computation of this function involves the evaluation of multivariate normal integrals. The dimensions of these integrals correspond to the length of the runs of limit observations. Tests for serial dependence in such models have also been discussed by Gourieroux *et al.* (1982) and Robinson *et al.* (1985). Robinson *et al.* (1985) note also that the above mentioned PMLIS estimators obtained for Tobit models with serial correlation have relatively large mean-squared errors, in finite samples. Since, on the other hand, the computation of true maximum likelihood estimators is not an operational procedure in that case, unless the sample contains only short runs of limit observations, it would be desirable to have on hand an alternative estimating procedure which would be computationally no more difficult than the PMLIS approach mentioned above, but could reduce substantially the variability of the resulting estimators.

ABSTRACT

Several authors have discussed recently the limited dependent variable regression model with serial correlation between the residuals. The pseudo-maximum likelihood estimators obtained by ignoring serial correlation altogether, have been shown to be consistent. We present alternative pseudo-maximum likelihood estimators which are obtained by ignoring serial correlation only selectively. Monte Carlo experiments on a model with first order serial correlation suggest that our alternative estimators have substantially lower mean-squared errors in medium size and small samples, especially when the serial correlation coefficient is high. The same experiments also suggest that the true level of the confidence intervals established with our estimators by assuming asymptotic normality, is somewhat lower than the intended level. A final experiment suggests that our estimators also behave well in very large samples. Although the paper focuses on models with only first order serial correlation, the generalization of the proposed approach to serial correlation of higher order is discussed briefly, in the conclusion.

The purpose of this paper is to suggest such a procedure for Tobit models with first order serial correlation. Starting from the true likelihood function of the model and viewing the likelihood which ignores serial correlation as an approximation to the true function, a different approximation is then derived and Alternative Pseudo-Maximum Likelihood (APML) estimators are proposed, in Section II, which maximize this latter function. Since the asymptotic properties of the APML estimators appear to be difficult to establish and that, moreover, small sample properties are of utmost interest, Monte Carlo experiments are performed to compare these alternative estimators to the PMLIS estimators, in Section III. Using the estimators of the classical multiple regression model with serial correlation [Beach and MacKinnon (1978)] as a yardstick, it is found that these new estimators behave rather well in small samples and have substantially smaller mean-squared errors than the PMLIS estimators. Then, in Section IV, an estimator of the asymptotic covariance matrix of the APML estimators is proposed. The Monte Carlo experiments suggest that the true confidence level of the intervals established with the above covariance matrix, assuming asymptotic normality of the APML estimators, are somewhat lower than the selected level, in medium size samples. In Section V, a final experiment is described which suggests that the APML estimators perform at least as well as the PMLIS estimators, in very large samples. In conclusion, the extension of the proposed procedure to models involving second or higher order serial correlation is very briefly discussed.

II. THE SUGGESTED PROCEDURE

Let us assume the following model :

(1) $$Y_t = X_t\beta + u_t , \qquad \text{if } X_t\beta + u_t = I_t > L_t ,$$

(2) $$Y_t = L_t , \qquad \text{if } X_t\beta + u_t = I_t \leq L_t ,$$

(3) $$u_t = \rho\, u_{t-1} + \varepsilon_t \qquad (-1 < \rho < 1;\ t = 1,\ldots,N) ,$$

where Y_t is the limited dependent variable, X_t is a 1×K vector of exogenous explanatory variables, I_t is a latent variable, L_t is an observed exogenous variable and β is a K×1 vector of parameters. Furthermore, the ε's are $N(0, \sigma_\varepsilon^2)$ random variables and the u's are normally distributed with zero mean and covariance matrix :

(4) $$E(uu') = \Sigma = \sigma^2 \begin{pmatrix} 1 & \rho & \cdots & \rho^{N-1} \\ \rho & 1 & & \rho^{N-2} \\ \vdots & \vdots & & \vdots \\ \rho^{N-1} & \rho^{N-2} & \cdots & 1 \end{pmatrix} = \sigma^2\, \Omega ,$$

where $u' = (u_1,\ldots,u_N)$ and $\sigma^2 = \sigma_\varepsilon^2/(1-\rho^2)$.

The unknown parameters to be estimated are the elements of the column vector θ, where $\theta' = (\beta', \sigma^2, \rho)$.

As mentioned above, our suggested approach consists essentially in obtaining for the true likelihood function of the model a "closer" approximation than the likelihood under serial independence. The procedure can be explained by first expressing the true likelihood function of the model as follows :

(5) $$\tilde{\mathcal{L}} = \ell(Y_1,\ldots,Y_N) = f(\tilde{Y}_1)\ F(\tilde{Y}_2|\tilde{Y}_1) \quad ,$$

where $\ell(Y_1,\ldots,Y_N)$ is the joint mixed density and probability function associated with the observed sample. $\tilde{Y}_1$ is the vector of elements contained in the subset (S_1) of nonlimit observations (such that $Y_t > L_t$; $t\varepsilon S_1$), $\tilde{Y}_2$ is the vector of elements contained in the subset (S_2) of limit observations (such that $Y_t = L_t$; $t\varepsilon S_2$), $f(\tilde{Y}_1)$ is the joint density of the elements of $\tilde{Y}_1$ and $F(\tilde{Y}_2|\tilde{Y}_1)$ is the joint probability of $\tilde{Y}_2$, conditional on $\tilde{Y}_1$.

Given that the residual errors u_t obey a *first order* Markov process, if $\tilde{Y}_2$ is partitioned into R runs of consecutive limit observations, such that $\tilde{Y}_2 = (\tilde{Y}_{2,1},\ldots,\tilde{Y}_{2,R})$, where $\tilde{Y}_{2,r}$ $(r = 1,\ldots,R)$ designates the vector of elements of $\tilde{Y}_2$ contained in the r'th run, one may also express $\tilde{\mathcal{L}}$ as :

(6) $$\tilde{\mathcal{L}} = \prod_{t\varepsilon S_1} f(Y_t|Y_{pt}) \prod_{r=1}^{R} F(\tilde{Y}_{2,r}|Y_{pr}, Y_{fr}) \quad ,$$

where Y_{pt} designates the value of Y corresponding to the first nonlimit observation preceding Y_t and $f(Y_t|Y_{pt})$ is the conditional density of Y_t, given Y_{pt}. *For the case where t corresponds to the first nonlimit observation* encountered in the sample, Y_{pt} does not exist and in this specific case, $f(Y_t|Y_{pt})$ is defined as : $f(Y_t|Y_{pt}) = f(Y_t)$, where $f(Y_t)$ is the marginal density of Y_t. In addition, $F(\tilde{Y}_{2,r}|Y_{pr}, Y_{fr})$ designates the joint probability associated with the elements of $\tilde{Y}_{2,r}$, conditional on the value (Y_{pr}) of the first nonlimit observation *preceding* the run of limit observations contained in $\tilde{Y}_{2,r}$, and on the value (Y_{fr}) of the first nonlimit observation following the elements in $\tilde{Y}_{2,r}$. Note that in order to compute the joint probability $F(\tilde{Y}_{2,r}|Y_{pr}, Y_{fr})$, one must evaluate a multiple normal integral of as many dimensions as there are elements in $\tilde{Y}_{2,r}$.

One possible approximation of $\tilde{\mathcal{L}}$ is obtained by writing :

(7) $$\tilde{\mathcal{L}}^* = \prod_{t\epsilon S_1} f(Y_t) \prod_{t\epsilon S_2} F(Y_t) \quad ,$$

where, for $t\epsilon S_1$, $f(Y_t)$ designates the marginal density of Y_t and for $t\epsilon S_2$, $F(Y_t)$ designates the marginal probability of Y_t.

The PMLIS estimators of (β, σ^2) are obtained by maximizing $\mathcal{L}^* = \ln \tilde{\mathcal{L}}^*$.

A closer approximation to $\tilde{\mathcal{L}}$ can be obtained by replacing in equation (6) only the term $F(\tilde{Y}_{2,r}|Y_{pr}, Y_{fr})$ by :

(8) $$F^*(\tilde{Y}_{2,r}|Y_{pr}, Y_{fr}) = \prod_{t\epsilon S_r} F(Y_t|Y_{pr}, Y_{fr}) \quad ,$$

where S_r designates the subset of elements of Y contained in the r'th run of nonlimit observations and $F(Y_t|Y_{pr}, Y_{fr})$, for $t\epsilon S_r$, corresponds to the marginal probability of Y_t, given Y_{pr} and Y_{fr}. In that case, $\tilde{\mathcal{L}}^*$ is replaced by $\tilde{\mathcal{L}}^{**}$, where :

(9) $$\tilde{\mathcal{L}}^{**} = \prod_{t\varepsilon S_1} f(Y_t|Y_{pt}) \prod_{r=1}^{R} \prod_{t\varepsilon S_r} F(Y_t|Y_{pr}, Y_{fr}) \quad .$$

Our APML estimators are obtained by maximizing $\mathcal{L}^{**}$: $\ln \tilde{\mathcal{L}}^{**}$, with respect to the elements of θ.

Note that the computation of $\mathcal{L}^{**}$ involves only one-dimensional normal integrals, just as $\mathcal{L}^{*}$. The evaluation of $\mathcal{L}^{**}$ is therefore hardly more costly than that of $\mathcal{L}^{*}$.

III. MONTE CARLO EXPERIMENTS

Given that, on the one hand, it would be difficult to analyze the asymptotic properties of our APML estimators and that, on the other hand, finite sample properties are of utmost interest, Monte Carlo experiments were performed to examine the biases and the mean-squared errors of our estimators. Comparisons are made with the corresponding properties of the PMLIS estimators, as well as with those of the estimators of the classical multiple regression model with serial correlation, which corresponds to the *unrealistic* situation where the latent variable I_t would be known for all observations. Clearly, since these last estimators [designated henceforth as Maximum Likelihood estimators with Latent Variables (MLLV)] are based on a greater amount of information, they are expected to perform better than our APML estimators.

It is found in all cases that, using the MSE as a criterion, the APML estimators perform better than the PMLIS estimators and that, in fact, their performance is surprisingly good in comparison to that of the MLLV estimators.

A. The Data

In order to simulate conditions which resemble those encountered in economic analyses, the model and the data used were derived from an empirical study performed by Dagenais (1964).

The model is a flexible accelerator model [Lovell (1961)] :

(10) $$I_t = .129 + .92\, C_t + .26\, C_{t-1} - .77\, M_{t-1} - .75\, B_{t-1} + u_t ,$$

$$S_t = I_t , \qquad \text{if } I_t < K_t ,$$

$$S_t = K_t , \qquad \text{if } I_t \geq K_t ,$$

$$u_t = \rho\, u_{t-1} + \varepsilon_t ,$$

where :

S_t = ~~shipments~~ production of North American newsprint mills, during half-year t, in million short tons;

C_t = newsprint-consumption of the customers of the North American newsprint mills, in half year t, in million tons;

M_{t-1} = end of period stocks, at the mills, in million tons;

B_{t-1} = end of period stocks, at the customers', in million tons;

K_t = capacity of North American newsprint mills, in period t.

Note that in this application the limit considered is an upper limit. The data can be found in the Appendix[1]. The sample covers the period 1922, first semester to 1960, second semester and contains 76 observations. The values of the β parameters shown in equation (12) are those obtained by Bussière (1983), from the same data, with a slightly different model. The values obtained by Bussière for ρ and σ^2 were .7 and 5700, respectively. In the actual sample, there are 20 limit observations corresponding to the following semiannual periods : 1943-I to 1945-II[2], 1947-I to 1949-I, 1950-II to 1952-I and 1954-II to 1956-II.

B. Design of the Experiments

The main set of experiments was made with the 76 observations shown in the Appendix. For each experiment, σ^2 was set at a given value and so was ρ. Then the series of serially correlated u's were drawn at random and the values of the latent variables were computed. Then, the values of S_t were set at min (I_t, K_t).

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1. The data is similar to the actual data found in Dagenais (1964) except for two observations, namely those referring to 1937, 2nd semester and to 1938, 1st semester. For reasons explained in Dagenais (1964), these observations were very abnormal and could be considered as outliers. They were replaced by more normal values, for performing the present experiments.

2. During World War II, the upper limit imposed on production did not actually correspond to the physical capacity of the mills but to quotas imposed by the Canadian and U.S. governments.

For each experiment, a method combining the control variate and the two-antithetic-variate techniques was adopted, as suggested by Mikhail (1972). One hundred samples were first drawn. Then for each sample, a corresponding sample with the u vector replaced by -u was generated. For every sample, the PMLIS, APML and MLLV estimators were computed. Control variates were obtained for the estimators of β and σ^2 by using MLLV estimators, assuming ρ to be known. A control variate was also obtained for the estimator of ρ by assuming the u's to be known and applying ordinary least squares to equation (3).

Six different experiments (designated as experiments E-1 to E-6) were made for ρ = .5, .7, .95 and R^2 = .5, .95, where R^2 corresponds to the theoretical coefficient of determination of the following regression model :

(11) $$Y^* = X^*\beta + \varepsilon \quad ,$$

where $$Y^{*\prime} = (Y_1 \sqrt{1-\rho^2},\ \ Y_2 - \rho\, Y_1, \ldots, Y_N - \rho\, Y_{N-1})$$

$$X^{*\prime} = (X_1' \sqrt{1-\rho^2},\ \ X_2' - \rho\, X_1', \ldots, X_N' - \rho\, X_{N-1}') \ .$$

R^2 is defined as :

$$R^2 = \beta' x^{*\prime} x^* \beta / (\beta' x^{*\prime} x^* \beta + N\, \sigma_\varepsilon^2) \quad ,$$

where x^* corresponds to X^* with the variables expressed in mean deviation form.

Table 1 gives the biases of the different parameters for the three estimation procedures considered, namely PMLIS, APML and MLLV, when the sample size is 76, for different values of ρ and R^2.

In order to perform our experiments, the model was reparametrized in terms of $w = \ln \sigma^2$ and $z = \frac{1}{2} \ln [(1+\rho)/1-\rho)]$, to avoid the recourse to constrained maximization procedures. Our results concerning σ^2 and ρ are therefore given in terms of w and z, in Table 1.

One notices that the biases of the β parameters are virtually null in the case of the MLLV procedure except for β_o, when $\rho = .95$. As $\rho \to 1$, β_o becomes indeterminate in the multiple regression model with serial correlation. Hence, it is not surprising that the results become less accurate, when ρ is high. Furthermore, the estimators of z and w are biased downwards and in some cases, this bias is not negligible.

The biases associated with the PMLIS procedure are often relatively large. We have underlined once all cases for which the bias is larger, in absolute value, than 10% of the corresponding parameter. Cases for which this percentage was over 25% have been underlined twice and cases for which the percentage was over 50% were boxed. The PMLIS biases are in many cases much larger than the corresponding APML biases, as can be verified readily from the |PMLIS|/|APML| column, which gives the absolute value of the ratios of the two biases. Only for the B_{t-1} variable are the PMLIS biases smaller than the APML biases, in three cases. These cases are identified by a * in Table 1. But the APML biases are themselves relatively negligible, in two of these cases, being smaller than 1% of the value of the parameter.

TABLE I

BIASES (EXPERIMENTS E-1 TO E-6)

...bles	Parameters values	PMLIS	APML	|PMLIS|/|APML|	MLLV	Variables	Parameters values	PMLIS	APML	|PMLIS|/|APML|	MLLV
	$\rho = .5$ $R^2 = .5$					E-2	$\rho = .5$ $R^2 = .9853$				
...tant	β_0=129.	-459.3836	49.6554	9.25	-0.7870	constant	β_0=129.	-19.7562	1.5172	13.02	-0.0000
	β_1=.92	-0.4913	0.0499	9.85	0.0013	C_t	β_1=.92	-0.0281	0.0023	12.22	0.0000
	β_2=.26	0.4493	-0.0296	15.18	-0.0011	C_{t-1}	β_2=.26	-0.0251	-0.0023	10.91	0.0000
	β_3=-.77	.8532	-0.2174	8.52	-0.0001	M_{t-1}	β_3=-.77	0.1036	-0.0114	9.09	0.0000
	β_4=-.75	0.0874	-0.0533	1.64	-0.0004	B_{t-1}	β_4=-.75	0.0017	0.0027	0.63*	-0.0000
	z =.55	N.A.	-0.2447	N.A.	-0.0890	(1)	z =.55	N.A.	-0.1628	N.A.	-0.1143
	w =12.47	-0.6145	-0.2081	2.95	-0.1511		w =8.27	-0.2866	-0.2279	1.26	-0.1773
	$\rho = .7$ $R^2 = .5$					E-4	$\rho = .7$ $R^2 = .9658$				
...tant	β_0=129.	-343.6106	73.5197	4.67	2.8688	constant	β_0=129.	-24.3703	1.1084	21.99	-0.0000
	β_1=.92	-0.3453	0.0938	3.68	0.0007	C_t	β_1=.92	-0.0366	0.0042	8.71	1.0000
	β_2=.26	0.3237	-0.0718	4.51	-0.0010	C_{t-1}	β_2=.26	0.0322	-0.0019	17.47	0.0000
	β_3=-.77	.4469	-0.2988	4.84	-0.0062	M_{t-1}	β_3=-.77	0.1309	-0.0163	8.03	0.0000
	β_4=-.75	0.0332	-0.0511	0.65*	-0.0015	B_{t-1}	β_4=-.75	-0.0004	-0.0016	0.25*	0.0000
	z =.8673	N.A.	-0.2875	N.A.	-0.1488	(1)	z =.8673	N.A.	-0.2082	N.A.	-0.1544
	w =11.99	-0.6609	-0.3630	1.82	-0.2684		w =8.65	-0.4426	-0.3507	1.26	-0.2762
	$\rho = .95$ $R^2 = .5$					E-6	$\rho = .95$ $R^2 = .98$				
...tant	β_0=129.	-272.172	103.527	2.63	2.657	constant	β_0=129.	-8.5195	0.8949	9.52	-0.1913
	β_1=.92	-0.249	0.026	9.58	0.000	C_t	β_1=.92	-0.0146	-0.0003	48.67	-0.0000
	β_2=.26	0.227	-0.025	9.08	0.000	C_{t-1}	β_2=.26	0.0123	0.0003	41.00	-0.0000
	β_3=-.77	1.048	-0.095	11.03	0.002	M_{t-1}	β_3=-.77	0.0518	0.0007	74.00	-0.0001
	β_4=-.75	0.089	-0.063	1.41	0.000	B_{t-1}	β_4=-.75	0.0018	0.0002	9.00	0.0001
	z =1.8318	N.A.	-0.454	N.A.	-0.387	(1)	z =1.8318	N.A.	-0.4315	N.A.	-0.4256
	w =12.15	-1.502	-0.865	1.74	-0.770		w =8.26	-1.2709	-0.8834	1.44	-0.8544

(1) For computational purposes, the model was reparametrized in terms of $w = \ln \sigma^2$ and $z = .5 \ln [(1+\rho)/(1-\rho)]$.

...IS\|/\|APML\| corresponds to the absolute value of the ratio of the PMLIS bias divided by the APML bias.

* indicates the cases where the APML bias is larger than the PMLIS bias, in absolute value.

— : bias larger than 10% of the parameter value.

= : bias larger than 25% of the parameter value.

☐ : bias larger than 50% of the parameter value.

The APML approach underestimates w and z in all cases and the biases are large in some cases, for z. Note, however, that in four out of six cases, these biases are not very much larger than those found for MLLV, which does yield consistent estimators for w and z.

Table II gives, for the same experiments, the ratios of the mean-squared errors of the three estimators considered, for each parameter. The APML estimators outperform the PMLIS estimators in all cases. The relative advantage of the APML procedure over the PMLIS procedure increases as the serial correlation gets higher. The PMLIS approach produces especially relatively large mean-squared errors when $\rho = .95$. In contrast, the APML estimators do not yield worse results, in comparison to the MLLV estimators, as ρ increases from .5 to .95. Given that the MLLV estimators use information that is actually *unavailable* in situations where the Tobit model is applied, —since the MLLV estimators require the knowledge of the latent variables which, in reality, are *unobservable* for the limit observations,— the fact that the ratio of the MSE of the MLLV estimators over the MSE of the MLLV estimators is, on average, of the order of 1.75, even when $\rho = .95$, and $R^2 = .5$, and of the order of 1.25 when $\rho = .95$ and $R^2 = .98$, appears as a very good performance.

RATIOS OF MEAN-SQUARED ERRORS FOR EXPERIMENTS E-1 - E-6

	E-1 $\rho = .5$ $R^2 = .5$			E-2 $\rho = .5$ $R^2 = .9853$		
	MSE(PMLIS)/MSE(MLLV)	MSE(APML)/MSE(MLLV)	MSE(PMLIS)/MSE(APML)	MSE(PMLIS)/MSE(MLLV)	MSE(APML)/MSE(MLLV)	MSE(PMLIS)/MSE(APML)
constant: β_o	2.48	1.21	2.05	1.44	1.14	1.27
C_t : β_1	2.90	1.29	2.25	2.00	1.11	1.80
C_{t-1} : β_2	2.61	1.28	2.05	2.33	1.19	1.96
M_{t-1} : β_3	4.55	1.46	3.12	2.31	1.23	1.87
B_{t-1} : β_4	1.34	1.32	1.01	1.79	1.32	1.36
z	N.A.	3.10	N.A.	N.A.	1.67	N.A.
w	6.37	1.43	4.44	1.77	1.41	1.25
	E-3 $\rho = .7$ $R^2 = .5$			E-4 $\rho = .7$ $R^2 = .9658$		
constant: β_o	1.94	1.42	1.37	1.46	1.08	1.34
C_t : β_1	3.97	1.77	2.24	3.12	1.15	2.71
C_{t-1} : β_2	4.08	1.83	2.23	4.18	1.17	3.58
M_{t-1} : β_3	6.81	1.64	4.15	4.06	1.28	3.18
B_{t-1} : β_4	1.96	1.48	1.33	3.10	1.27	2.45
z	N.A.	2.67	N.A.	N.A.	1.57	N.A.
w	3.60	1.50	2.40	1.83	1.38	1.33
	E-5 $\rho = .95$ $R^2 = .5$			E-6 $\rho = .95$ $R^2 = .98$		
constant: β_o	1.23	1.33	0.92	1.26	1.03	1.23
C_t : β_1	11.24	1.78	6.30	9.41	1.29	7.30
C_{t-1} : β_2	10.68	1.99	5.36	10.62	1.35	7.87
M_{t-1} : β_3	38.10	2.17	17.55	39.45	1.17	33.73
B_{t-1} : β_4	12.18	2.04	5.96	15.26	1.38	11.05
z	N.A.	1.49	N.A.	N.A.	1.06	N.A.
w	2.22	1.35	1.64	11.90	1.05	11.34

Table III gives, for each of the six experiments concerned, the percentage of limit and nonlimit observations contained in the two hundred samples generated. Table III also gives, for the limit observations, the cumulative distribution of observations contained in runs of different length.

A simple inspection of Table III indicates that the average length of the runs of limit observations tends to increase with ρ and decrease with R^2. In experiment E-5 which is characterized by $\rho = .95$ and $R^2 = .5$, the samples that were generated contained runs of over 30 limit observations. The fact that our procedure still performed well in such an experiment is reassuring. The PMLIS estimator performed rather badly in this case, as can be seen from Table II.

The data used for the explanatory variables in the experiments reported in Tables I and II were highly collinear, since the determinant of their correlation matrix is equal to 0.004. We therefore repeated one of the above experiments, namely E-4, after having orthogonalized the data matrix and modified the β parameters accordingly. The results of this new experiment (E-7) are reported in Table IV. In addition, Table IV gives the results of two further experiments. Experiment E-8 also repeats E-4, with the sample reduced to 25 observations and experiment E-9 corresponds again to E-4, but with ρ set to $-.7$ instead of $+.7$. In E-7, the percentage of limit observations was 24%, and 17% of the observations were limit observations contained in runs of length greater than 4. In experiments E-8 and E-9, the corresponding percentages were 34% and 15% for E-8, 24% and 8% for E-9.

TABLE III

PERCENTAGE OF OBSERVATIONS CONTAINED IN RUNS OF DIFFERENT LENGTHS,

FOR EXPERIMENTS REPORTED IN TABLES I AND II

	E-1 $\rho = .5$ $R^2 = .5$	E-2 $\rho = .5$ $R^2 = .9853$	E-3 $\rho = .7$ $R^2 = .5$	E-4 $\rho = .7$ $R^2 = .9658$	E-5 $\rho = .95$ $R^2 = .5$	E-6 $\rho = .95$ $R^2 = .98$
percentage of nonlimit observations	70.20	76.20	71.73	75.77	71.01	76.35
percentage of limit observations contained in runs of length greater than or equal to : 1	29.80	23.80	28.27	24.03	28.99	23.65
2	24.56	21.79	24.92	22.24	27.68	21.96
3	19.19	19.22	21.64	19.81	26.28	19.95
4	14.57	16.67	19.13	16.91	25.00	16.85
5	11.02	11.88	16.05	13.33	23.63	13.88
6	8.26	6.75	14.01	9.38	22.91	10.06
7	6.68	3.39	11.29	6.42	21.41	6.15
8	5.02	2.56	9.13	5.06	20.17	5.41
9	3.60	2.14	7.50	4.55	19.12	5.25
10	2.71	1.61	7.37	3.96	17.82	5.19
11	1.79	1.16	4.79	3.10	16.57	4.99
12	1.50	0.57	3.63	2.01	15.12	3.61
13	1.11	0.10	3.16	1.06	13.62	2.43
14	0.93	0.10	2.82	0.97	12.94	2.43
15	0.75	0.10	2.64	0.69	12.48	2.43
16	0.45	0.00	2.34	0.39	11.59	2.03
17	0.13	0.00	1.81	0.39	10.22	2.03
18	0.13	0.00	1.70	0.39	9.44	1.92
19	0.13	0.00	1.46	0.27	8.73	1.33
20	0.00	0.00	1.08	0.14	6.98	1.20
21	0.00	0.00	0.95	0.14	6.45	1.20
22	0.00	0.00	0.67	0.00	5.48	0.51
23	0.00	0.00	0.67	0.00	5.33	0.36
24	0.00	0.00	0.52	0.00	5.33	0.36
25	0.00	0.00	0.36	0.00	5.01	0.36
26	0.00	0.00	0.36	0.00	5.01	0.36
27	0.00	0.00	0.19	0.00	4.67	0.36
28	0.00	0.00	0.19	0.00	4.31	0.18
29	0.00	0.00	0.19	0.00	3.39	0.00
30	0.00	0.00	0.00	0.00	1.67	0.00
31	0.00	0.00	0.00	0.00	1.27	0.00
32	0.00	0.00	0.00	0.00	0.66	0.00
33	0.00	0.00	0.00	0.00	0.66	0.00
34	0.00	0.00	0.00	0.00	0.23	0.00
35	0.00	0.00	0.00	0.00	0.23	0.00
36	0.00	0.00	0.00	0.00	0.00	0.00

TABLE IV

RATIOS OF MEAN-SQUARED ERRORS FOR EXPERIMENTS E-7 TO E-9

	E-7 : orthogonal matrix			E-8 : N = 25			E-9 : $\rho = -.7$		
	MSE(PMLIS) / MSE(MLLV)	MSE(APML) / MSE(MLLV)	MSE(PMLIS) / MSE(APML	MSE(PMLIS) / MSE(MLLV)	MSE(APML) / MSE(MLLV)	MSE(PMLIS) / MSE(APML)	MSE(PMLIS) / MSE(MLLV)	MSE(APML) / MSE(MLLV)	MSE(PMLIS) / MSE(APML)
constant : β_o	1.56	1.21	1.29	1.51	1.39	1.09	7.07	1.08	6.54
C_t : β_1	5.07	1.38	3.67	2.07	1.40	1.48	4.55	1.30	3.50
C_{t-1} : β_2	4.88	1.21	4.01	1.71	1.31	1.30	4.64	1.41	3.29
M_{t-1} : β_3	2.71	1.41	1.92	1.75	1.30	1.34	11.43	1.28	8.90
B_{t-1} : β_4	1.30	1.16	1.12	1.47	1.47	1.00	3.68	1.62	2.27
z	N.A.	1.60	N.A.	N.A.	1.58	1.29	N.A.	1.02	N.A.
w	1.78	1.32	1.35	2.01	1.38	1.45	1.31	1.03	1.27

The results of these three experiments are very similar to those of the first 6 experiments.

The APML estimators perform rather well in comparison with the MLLV estimators and they outperform the PMLIS estimators systematically. In particular, the MSE of the PMLIS estimators of the β's are relatively large for experiment E-9 with the negative ρ.

IV. SETTING CONFIDENCE ITERVALS

An estimate of the asymptotic covariance matrix of the APML estimators can also be obtained. Following the procedure suggested by Goldberger, Naga and Odeh (1961), this asymptotic covariance estimator may be expressed as :

$$\hat{V}(\hat{\theta}) = \left(\frac{\partial^2 \mathcal{L}^{**}}{\partial\theta\partial\theta'}\right)^{-1}_{\theta=\hat{\theta}} \hat{V}\left(\frac{\partial \mathcal{L}^{**}}{\partial\theta}\right) \left(\frac{\partial^2 \mathcal{L}^{**}}{\partial\theta\partial\theta'}\right)^{-1}_{\theta=\hat{\theta}} \quad ; \tag{12}$$

where $\hat{\theta}$ is the APML estimator of θ, $\hat{V}(\hat{\theta})$ is its asymptotic covariance matrix estimator and $\hat{V}\left(\frac{\partial \mathcal{L}^{**}}{\partial\theta}\right)$ is an estimator of the asymptotic covariance matrix of the first derivatives of $\mathcal{L}^{**}$ with respect to the elements of θ. In turn, $\hat{V}\left(\frac{\partial \mathcal{L}^{**}}{\partial\theta}\right)$ can be evaluated as follows :

$$\hat{V}\left(\frac{\partial \mathcal{L}^{**}}{\partial\theta}\right) = \left(\sum_{j=1}^{G} A_j\right)_{\theta=\hat{\theta}} \quad ,$$

where $A_j = \left(\sum_{i\varepsilon g_j} \frac{\partial \mathcal{L}_i^{**}}{\partial\theta}\right)\left(\sum_{i\varepsilon g_j} \frac{\partial \mathcal{L}_i^{**}}{\partial\theta}\right)' \quad ,$

where g_j corresponds to the j'th group of observations. These groups are defined with reference to the $\tilde{\mathcal{L}}$ function shown in equation (8). All observations for which the conditional probabilities are not separable, in $\tilde{\mathcal{L}}$, are elements of the same group. The total number of groups is equal to G.

Using $\hat{V}(\hat{\theta})$, 95% confidence intervals were obtained for each parameter, for each of the samples generated in the first 6 experiments described above. Then for each experiment, the proportions of confidence intervals that contained the true parameters were calculated and are reported in Table V. As can be verified from this Table, the confidence intervals obtained for both the APML and the MLLV estimators are somewhat biased downwards and somewhat erratic, with the APML estimators performing again almost as well as the MLLV estimators. The cases where the downward biases of the confidence intervals are the most important are those relating to the constant term in experiments E-5 and E-6.

TABLE V

ACTUAL LEVELS OF COMPUTED 95% CONFIDENCE INTERVALS

variables	E-1		E-2		E-3		E-4		E-5		E-6	
	APML	MLLV	APML	MLLV	APML	MLLV	APML	MLLV	APML	MLLV	APML	MLLV
constant: β_o	92.5	90.5	87.0	92.0	87.5	88.0	85.0	86.0	67.0	67.5	76.5	78.0
C_t : β_1	92.5	94.0	97.0	95.0	91.0	94.0	97.5	96.0	85.5	91.0	88.5	86.0
C_{t-1} : β_2	95.0	94.0	95.0	94.0	94.0	96.0	95.5	94.0	86.5	88.0	92.5	94.0
M_{t-1} : β_3	91.0	94.0	92.0	91.0	95.5	92.0	90.5	91.0	92.5	96.0	88.0	94.0
B_{t-1} : β_4	90.5	90.5	87.0	89.0	85.0	88.0	91.5	95.0	95.0	94.0	92.0	95.0
average level	92.3	92.6	91.6	92.2	90.6	91.6	92.0	92.4	85.3	87.3	87.5	89.4
averave level excluding the constant	92.3	93.1	92.8	92.3	91.4	92.5	93.8	94.0	89.9	92.3	90.3	92.3
MSE *	9.8	8.7	28.2	12.4	34.7	21.8	27.8	19.8	190.6	164.7	89.8	74.4
MSE excluding the constant	10.6	5.8	19.3	13.3	29.3	15.0	9.8	4.5	42.2	16.8	26.6	20.8

* MSE : average value of $(\text{actual level} - 95.0)^2$, for the five β's.

V. LARGE SAMPLE BEHAVIOR

Since no analytical derivation of the asymptotic properties of our APML estimators is available, it is interesting to examine, through an example, how our estimators behave when the sample gets large, in comparison to the PMLIS estimators which have been shown to be consistent [Robinson (1982)].

For this purpose, we have generated a sample of 5776 observations with autocorrelated error terms and have observed how the values of the APML and PMLIS estimators evolve as more and more of the 5776 observations are taken into account.

The model used for this last experiment was that corresponding to experiment E-4. The values of the exogenous variables appearing in equation (12) – including K_t – corresponded to those of experiment E-4, for the first 76 observations. The values of the explanatory variables for the next 76 observations corresponded again to those of the first 76 observations, but in reverse order : that is, the values of the explanatory variables for observation 77 were the same as those corresponding to observation 76 those of observation 78 corresponded to those of observation 75 ... and those of observation 152 corresponded to those of observation 1. Then, a similar procedure was repeated for the third set of 76 observations, with observation 153 corresponding to 152, 154 to 151, etc. The procedure was continued for 76 sets of 76 observations[1]. Then, a set of 5776 serially correlated normal residual errors was generated and the values of the I_t variable were computed. Finally, the values of Y_t were derived by comparing I_t with K_t.

Afterwards, the values of the APML and PMLIS estimators were computed using the first 76 observations, then the first 152 orservations, etc. until the 10 first sets of 76 observations were used. Then, the estimators were again computed after adding, each time, *2* more sets of 76 observations until the first 20 sets of observations were used. Afterwards, *4* sets of 76 observations were added each time, for every additional computation, until the 5776'th observation was reached. The results are graphed, for each parameter on figures I to VII. On each graph, the abscissa indicates the number of sets of 76 observations used to calculate the estimators and the ordinate shows the difference between the estimates and the true value of the parameter. The interval comprised between the lines designated as SUP (—▲—) and INF (—×—) corresponds to the 99% confidence interval computed by using the estimate of the asymptotic variance of the APML estimator obtained from equation (12).

By examining the graphs, it is readily seen that the difference between the APML estimator and the true value of the parameter always lies within the confidence interval, for all the parameters except for the parameters

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1. The procedure was then stopped for want of space allowed on the computer. Generating a greater number of observations would have required rewriting the program. This was not considered useful since it was felt that using a larger sample would not have been much more informative.

LEGEND

Ordinate Fig.1: difference ÷ 100
Figs.2-7: difference × 10
Abscissa: sets of observations

- ● APML
- □ PMLIS
- ♦ ZERO

△ × } Boundaries of 99% asympt. confidence interval for PMLIS

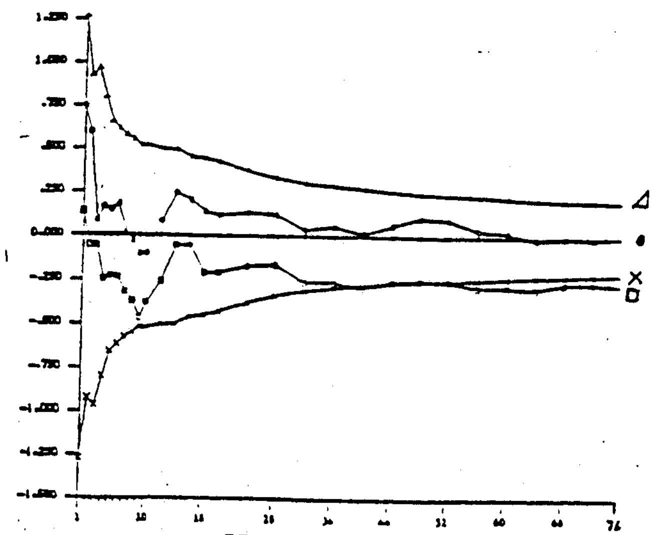


FIGURE 1 : CONSTANT TERM (β_0 = 129)

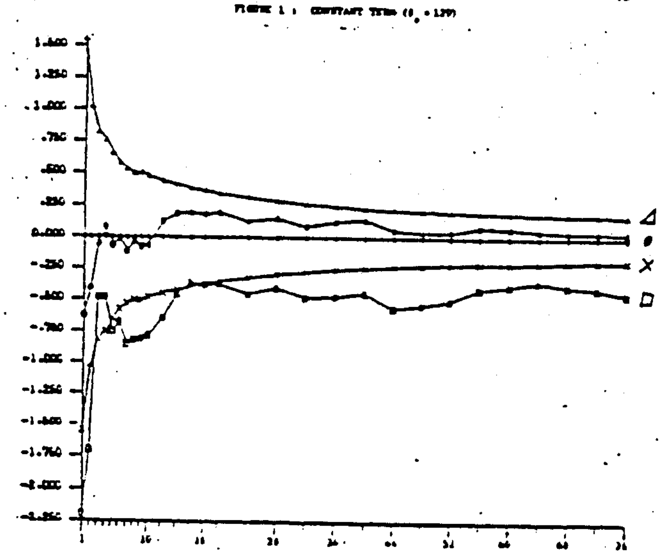
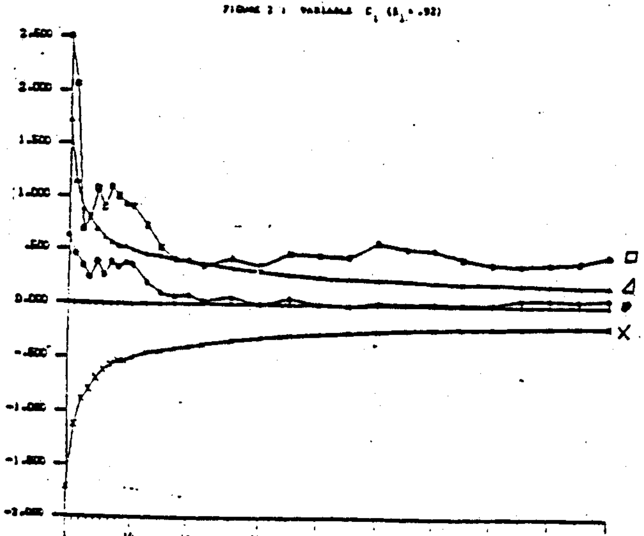


FIGURE 2 : VARIABLE C_t (β_1 = .92)



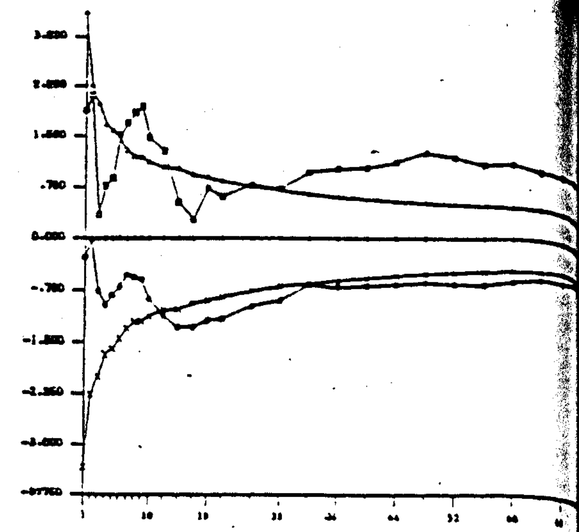


FIGURE 4 : VARIABLE M_{t-1} (β_3 = -.77)

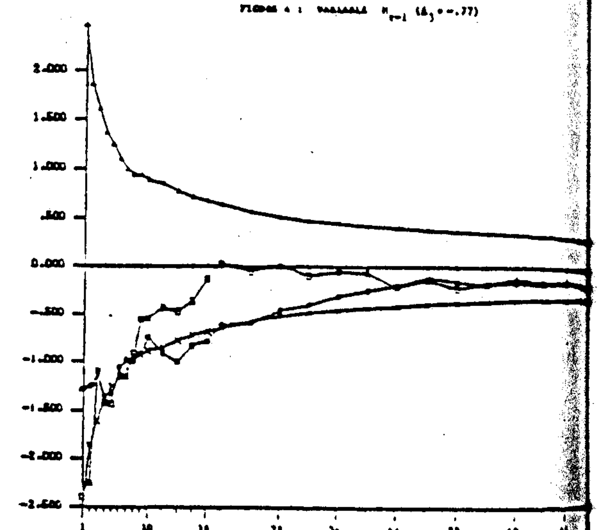


FIGURE 5 : VARIABLE I_{t-1} (β_4 = -.75)

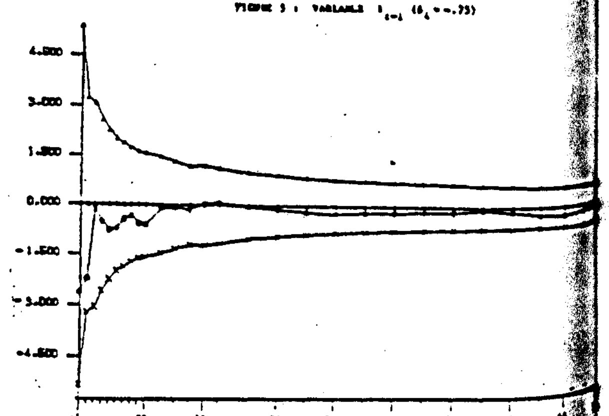
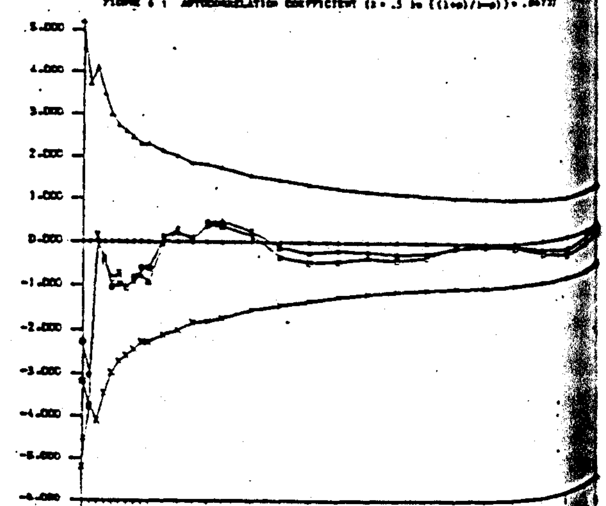


FIGURE 6 : AUTOCORRELATION COEFFICIENT (Z = .5 ln [(1+ρ)/(1-ρ)]) = .0672)



associated with the variables B_{t-1} and M_{t-1}. For the variable B_{t-1}, the APML estimate always lies inside the confidence interval after the 28'th set of 76 observations has been reached. In the case of the M_{t-1} variable, the difference between the parameter estimate and the true value lies systematically slightly below the confidence interval until the whole set of 5776 observations has been used.

A comparison between the consistent PMLIS estimators and the APML estimators shows that, for all parameters except for that associated with the variable B_{t-1} (namely β_4) and for w, the APML estimators are closer to the true parameter values than the PMLIS estimators, when the sample gets large. For β_4 and w, the performances of the APML and PMLIS estimators are very similar when the sample is large, with the APML estimators very slightly closer to the true values when the whole set of 5776 observations is used.

This final experiment therefore suggests that even in very large samples, the APML estimators behave at least as favorably as the consistent PMLIS estimators. This experiment suggests also that our covariance estimators of the APML estimators are fairly reliable, for large samples.

V. CONCLUSION

We have suggested alternative pseudo-maximum likelihood estimators for Tobit models with first order serial correlation. These alternative estimators are not notably more costly to calculate than the pseudo-maximum likelihood estimators ignoring serial correlation, but they appear to be more efficient in terms of MSE. They also appear to behave well in large samples.

We have also suggested a procedure for estimating the asymptotic covariance matrix of the APML estimators and setting confidence intervals. The true confidence levels associated with these intervals appear to be somewhat lower than the intended levels in small samples, but the MSE associated with these levels are not much higher than those associated with the maximum likelihood procedure that would be applicable if the latent variables were always observed.

Our approach could be extended to Tobit models with serial correlatio[n] of second or higher orders. For example, in the case of residual errors with a second order autoregressive scheme, the terms appearing in $\tilde{\mathcal{L}}^{**}$ would be :

a) the joint marginal density of the first Two Consecutive NonLimit (TCNL) observations;

b) the joint conditional density of all the nonlimit observations preceding the TCNL mentioned in a), given these TCNL observations;

c) the joint conditional density of each group of *nonlimit observations preceded by the same TCNL observations, given* these TCNL observations;

d) for every limit observation preceding the first TCNL observations, the integral of the conditional univariate density of this limit observation, given these first TCNL observations and the other nonlimit observations that precede these first TCNL observations;

e) for every other limit observation, the integral of the conditional univariate density of this limit observation, given the preceding TCNL observations and the following TCNL observations (if they exist) and given also all other nonlimit observations comprised between the above two sets of TCNL observations (or between the preceding TCNL and the end of the sample).

APPENDIX

Data used for the Monte Carlo Experiments

The variables are described in Section III, above.

observation number	C_{t-1}	C_t	M_{t-1}	B_{t-1}	K_t
1	1233.0	1364.0	99.0	230.0	1455.0
2	1364.0	1297.0	73.0	248.0	1507.0
3	1297.0	1359.0	98.0	340.0	1545.0
4	1359.0	1473.0	95.0	384.0	1637.0
5	1473.0	1436.0	117.0	353.0	1709.0
6	1436.0	1545.0	97.0	361.0	1731.0
7	1545.0	1646.0	126.0	331.0	1780.0
8	1646.0	1788.0	120.0	294.0	1887.0
9	1788.0	1867.0	152.0	297.0	2036.0
10	1867.0	1872.0	105.0	398.0	2163.0
11	1872.0	1899.0	161.0	342.0	2339.0
12	1899.0	1957.0	123.0	382.0	2436.0
13	1957.0	2064.0	284.0	333.0	2548.0
14	2064.0	2127.0	145.0	359.0	2582.0
15	2127.0	2201.0	204.0	305.0	2697.0
16	2201.0	2083.0	137.0	410.0	2762.0
17	2083.0	1960.0	220.0	350.0	2850.0
18	1960.0	1898.0	192.0	380.0	2924.0
19	1898.0	1785.0	237.0	332.0	2965.0
20	1785.0	1672.0	221.0	343.0	2947.0
21	1672.0	1578.0	269.0	308.0	2957.0
22	1578.0	1522.0	217.0	286.0	2923.0
23	1522.0	1686.0	223.0	232.0	2949.0
24	1686.0	1788.0	174.0	351.0	2932.0
25	1788.0	1962.0	247.0	399.0	2894.0
26	1962.0	1916.0	166.0	481.0	2860.0
27	1916.0	2096.0	281.0	357.0	2866.0
28	2096.0	2121.0	196.0	437.0	2837.0
29	2121.0	2363.0	278.0	347.0	2854.0
30	2363.0	2236.0	145.0	461.0	2814.0
31	2236.0	2259.0	280.0	526.0	2838.0
32	2259.0	2013.0	127.0	441.0	2828.0
33	2013.0	2042.0	287.0	538.0	2843.0
34	2042.0	1942.0	227.0	484.0	2807.0
35	1942.0	2110.0	292.0	428.0	2854.0
36	2110.0	2326.0	271.0	502.0	2889.0
37	2326.0	2403.0	269.0	499.0	2912.0
38	2403.0	2334.0	238.0	589.0	2887.0
39	2334.0	2445.0	272.0	536.0	2951.0
40	2445.0	2143.0	201.0	633.0	2903.0
41	2143.0	2150.0	270.0	718.0	2897.0
42	2150.0	2078.0	168.0	779.0	1949.0
43	2078.0	2177.0	143.0	677.0	2075.0
44	2177.0	1888.0	116.0	601.0	1959.0
45	1888.0	2063.0	131.0	557.0	2025.0
46	2063.0	2120.0	105.0	545.0	2076.0
47	2120.0	2294.0	164.0	441.0	2241.0
48	2294.0	2520.0	143.0	409.0	2715.0
49	2520.0	2692.0	199.0	403.0	2798.0
50	2692.0	2774.0	162.0	456.0	2787.0
51	2774.0	2804.0	166.0	465.0	2858.0
52	2804.0	2836.0	102.0	583.0	2861.0
53	2836.0	2886.0	140.0	569.0	2990.0
54	2886.0	3002.0	118.0	694.0	3064.0
55	3002.0	3080.0	177.0	697.0	3070.0
56	3080.0	3127.0	132.0	674.0	3094.0
57	3127.0	3227.0	168.0	587.0	3217.0
58	3227.0	3215.0	97.0	648.0	3301.0
59	3215.0	3274.0	143.0	688.0	3340.0
60	3274.0	3333.0	109.0	788.0	3413.0
61	3333.0	3346.0	151.0	826.0	3436.0
62	3346.0	3386.0	134.0	918.0	3416.0
63	3386.0	3521.0	183.0	844.0	3534.0
64	3521.0	3505.0	119.0	831.0	3532.0
65	3505.0	3737.0	160.0	799.0	3677.0
66	3737.0	3947.0	131.0	773.0	3804.0
67	3947.0	3951.0	138.0	623.0	3939.0
68	3951.0	3949.0	88.0	660.0	4044.0
69	3949.0	3951.0	139.0	705.0	4142.0
70	3951.0	4097.0	110.0	924.0	4282.0
71	4097.0	4021.0	211.0	1002.0	4467.0
72	4021.0	3853.0	152.0	986.0	4540.0
73	3853.0	3994.0	245.0	972.0	4851.0
74	3994.0	4115.0	201.0	943.0	4926.0
75	4115.0	4262.0	265.0	832.0	5009.0
76	4262.0	4354.0	172.0	954.0	4997.0

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