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K-theoretic invariants in symplectic topology

par

Lydia Mezrag

Département de mathématiques et de statistique

Faculté des arts et des sciences

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présenté par

Lydia Mezrag

a été évalué par un jury composé des personnes suivantes :

François Lalonde
(président-rapporteur)

Egor Shelukhin
(directeur de recherche)

Octav Cornea
(membre du jury)

Résumé

En employant des méthodes de la théorie de Chern-Weil, Reznikov produit une condition suffisante qui assure la non-trivialité de la projectivisation $\mathbb{P}(E)$ d'un fibré vectoriel complexe en tant que fibré Hamiltonien. Dans le contexte de la quantification géométrique, Savelyev et Shelukhin introduisent un nouvel invariant des fibrés Hamiltoniens avec valeurs dans la K-théorie et étendent le résultat de Reznikov. Cet invariant est donné par l'indice d'Atiyah-Singer d'une famille d'opérateurs Spin^c de Dirac. Dans ce mémoire, on s'intéresse à des fibrés Hamiltoniens résultant d'un produit fibré et d'un produit cartésien d'une collection de fibrés projectifs complexes $\mathbb{P}(E_1), \cdots, \mathbb{P}(E_r)$. En usant des mêmes méthodes que Shelukhin et Savelyev, on définit une famille d'opérateurs Spin^c de Dirac qui agissent sur les sections d'un fibré de Dirac canonique à valeurs dans un fibré pré-quantique. L'indice de famille produit un invariant de fibrés Hamiltoniens avec fibres données par un produit d'espaces projectifs complexes et permet de construire des exemples de fibrés Hamiltoniens non-triviaux.

Mots clés: Fibrés Hamiltoniens, K-théorie, indice de famille d'Atiyah-Singer, quantification géométrique.

Abstract

Using methods of Chern-Weil Theory, Reznikov provides a sufficient condition for the non-triviality of the projectivization $\mathbb{P}(E)$ of a complex vector bundle E as a Hamiltonian fibration. In the setting of geometric quantization, Savelyev and Shelukhin introduce a new invariant of Hamiltonian fibrations and a K-theoretic lift of Reznikov's result. This invariant is given by the Atiyah-Singer index of a family of Spin^c-Dirac operators. In this thesis, we consider Hamiltonian fibrations given by the Cartesian product and the fiber product of a collection of complex projective bundles $\mathbb{P}(E_1), \cdots, \mathbb{P}(E_r)$. Using the same methods as Savelyev and Shelukhin, we define a family of Spin^c-Dirac operators acting on sections of a canonical Dirac bundle with values in a suitable prequantum fibration. The family index gives then an invariant of Hamiltonian fibrations with fibers given by a product of complex projective spaces and allows to construct examples of non-trivial Hamiltonian fibrations.

Keywords: Hamiltonian fibrations, K-theory, Atiyah-Singer family index, geometric quantization.

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Introduction

A symplectic fibration is a locally trivial fiber bundle with fiber a symplectic manifold (M,ω) and structure group $\mathrm{Symp}(M,\omega)$, the group of symplectomorphisms. A Hamiltonian fibration is a special case of a symplectic fibration for which the structure group can be reduced to the subgroup $\mathrm{Ham}(M,\omega)$ of Hamiltonian symplectomorphisms. They are intresting objects of study in symplectic topology. A characterization of Hamiltonian fibrations appears for instance in the works of Guillemin-Lerman-Stenberg [**9**] and Lalonde-McDuff [**19**] building on previous works [**25**] and [**20**] with Polterovich.

A trivial example is given by the trivial bundle $B\times(M,\omega)$ over a topological space B. Another well known example is the complex projective bundle

$$\mathbb{C}P^{n-1}\hookrightarrow\mathbb{P}(E)\to B$$

given by the projectivization of a complex vector bundle $E\to B$ of rank n. Reznikov [**26**] (and later on McDuff [**23**]) showed that this Hamiltonian fibration is not trivial provided for any line bundle μ over B

$$Ch(E\otimes\mu)\neq Ch(\underline{\mathbb{C}}^n)$$

where Ch stands for the Chern character and $\underline{\mathbb{C}}^n$ is the trivial vector bundle of rank n.

Inspired by geometric quantization, Savelyev and Shelukhin [**28**] define a homotopy canonical family of elliptic operators between Hilbert bundles over B and introduce a new invariant of Hamiltonian fibrations given by the Atiyah-Singer family index. This invariant takes values in $K(B)$, the K-theory of the base and leads to an extension of Reznikov's result.

Theorem A. *(Theorem 1.2,* [**28**]*) Let E be a complex vector bundle of rank n over a topological space B. Then $\mathbb{P}(E)$ is not trivial as a Hamiltonian fibration if $E\otimes\mu$ is not stably trivial for any* *line bundle μ over B. That is,*

$$[E \otimes \mu] \neq [\underline{\mathbb{C}}^n] \ \in K(B) \ \forall \mu \in Vect(B).$$

In this thesis, given a collection of complex vector bundles $E_1, \cdots, E_r$ over the topological spaces $B_1, \cdots, B_r$, respectively, we consider Hamiltonian fibrations $\underline{M}$ given by Cartesian products

$$\mathbb{P}(E_1) \times \cdots \times \mathbb{P}(E_r) \to B_1 \times \cdots \times B_r$$

and by fiber products

$$\mathbb{P}(E_1) \times_B \cdots \times_B \mathbb{P}(E_r) \to B$$

when there is a common base space B for the vector bundles $E_1, \cdots, E_r$.

For each symplectic manifold (M, ω), a prequantum space is a 4-tuple $\widehat{M} = (L, M, \nabla, \omega)$ consisting of a Hermitian line bundle $L \to (M, \omega)$ equipped with a unitary connection ∇ with curvature

$$R(\nabla) = -2\pi i \omega.$$

Using the same methods as in [**28**], we can associate to a Hamiltonian fibration $(M, \omega) \hookrightarrow \underline{M} \to B$ a family of Spin^c-Dirac operators $\mathscr{D}_+^1(\widehat{\underline{M}}, \{J_b\}_b)$ and an index

$$[\ker \mathscr{D}_+^1(\widehat{\underline{M}}, \{J_b\}_b)] - [\operatorname{coker} \mathscr{D}_+^1(\widehat{\underline{M}}, \{J_b\}_b)] \ \in K(B).$$

Here $\{J_b\}_b$ is a family of $\omega_{\underline{M}_b}$-compatible almost complex structures parametrised by B and $\widehat{\underline{M}}$ can be viewed as line bundle over $\underline{M}$ or as a family of prequantum spaces parametrized by B.

In this setting, we formulate extensions of Theorem A to Hamiltonian fibrations with fiber a product of complex projective spaces.

Results

We will assume that all topological spaces are paracompact Hausdorff and that the vector bundles have constant rank.

Theorem B. *Let $E_1, \cdots, E_r$ be complex vector bundles over topological spaces $B_1, \cdots, B_r$, respectively. If $(E_1 \boxtimes \cdots \boxtimes E_r) \otimes \mu$ is not stably trivial for any line bundle μ on $B := B_1 \times \cdots \times B_r$ then, the product bundle $\mathbb{P}(E_1) \times \cdots \times \mathbb{P}(E_r)$ over B is not trivial as a Hamiltonian fibration.*

Theorem C. *Let $E_1, \cdots, E_r$ be complex vector bundles over the same topological space B. If $(E_1 \otimes \cdots \otimes E_r \otimes \mu)$ is not stably trivial for any line bundle μ on B, then the fiber product bundle $\mathbb{P}(E_1) \times_B \cdots \times_B \mathbb{P}(E_r)$ over B is not trivial as a Hamiltonian fibration.*

Replacing by $r = 1$ in Theorem C yields Theorem A. Moreover, given complex vector bundles E_1 and E_2 over topological spaces B_1 and B_2, respectively, we can deduce from Theorem B the following.

Corollary D. *Let B_1 and B_2 be connected finite CW-complexes and assume that $K(B_1)$ or $K(B_2)$ is torsion free. If $E_1 \otimes \mu$ is not stably trivial for any line bundle μ on B_1 then, the product bundle $\mathbb{P}(E_1) \times \mathbb{P}(E_2)$ over $B_1 \times B_2$ is not trivial as a Hamiltonian fibration.*

Structure of the thesis

In Chapter 1, we recall basic definitions and elementary properties of vector bundles and principal bundles. We also introduce the notions of connection and curvature on principal bundles.

Chapter 2 aims at defining Spin^c-Dirac operators. We begin by introducing Clifford algebras and natural group structures, namely the Pin, Spin and finally the Spin^c groups. Then, we briefly describe Spin^c-structures and complex spinor bundles in order to define Dirac bundles and Dirac operators. We end the chapter with important properties satisfied by these operators.

Chapter 3 is devoted to K-theory and introduces all the notation and machinery needed for this thesis. We also discuss briefly the Atiyah-Singer family index.

In Chapter 4, we discuss the prequantization of symplectic manifolds and Hamiltonian fibrations. Then, we introduce the index of a family of Spin^c-Dirac operators associated to a Hamiltonian fibration together with its prequantum lift.

Finally, in Chapter 5 we prove our main results (Theorem B and Theorem C) and provide examples as an application. At the end of the chapter, using Theorem B we prove Corollary D .

The appendix recalls important facts about the Dolbeault-Dirac operator.

Chapter 1

Bundle theory

In this chapter we will discuss general constructions and basic results on bundles with a special emphasis on vector bundles which will be important in defining the K-group. The reader can refer to [**13**] for a thorough discussion on fiber bundles. Unless said otherwise, topological spaces are always paracompact and Hausdorff and vector spaces will always be finite dimensional.

1.1. Fiber bundles

In the most general terms, a fiber bundle is a family of spaces (isomorphic to a model space F) which are parametrized by a space B and "glued together" in a consistent way.

Definition 1.1.1. *A fiber bundle ξ with fiber F is given by a triple (E,B,π) such that*

(1) $\pi : E \to B$ is a continuous surjection.

(2) For any $b \in B$, $\pi^{-1}(b)$ is homeomorphic to F.

(3) Every $b \in B$ admits an open neighbourhood U and a homeomorphism φ_U that makes the following diagram commute

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi_U"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "proj_U"] \\ & U \end{tikzcd} .$$

The topological spaces B and E are called, repectively, the *base space* and the *total space*. We call U a *trivialization neighbourhood (domain)* and we refer to the set of pairs $\{(U_\alpha, \varphi_\alpha)\}$ as a *local trivialization*. *Smooth fiber bundles* are defined similarly by requiring the trivialization maps φ_U to be diffeomorphisms, the mapping π to be a smooth surjection and the topological spaces E, B and F to be smooth.

A *section* of the fiber bundle $\pi : E \to B$ is a continuous map $s : B \to E$ that satisfies $\pi \circ s = \mathrm{Id}_B$. The space of (global) sections of E is denoted by $\Gamma(E)$ or $\Gamma(E,B)$ to emphasize the base space. The space of sections of the tangent bundle of a smooth manifold M consists of vector fields and is denoted by $\Gamma(M)$ or $\mathfrak{X}(M)$.

Concrete examples of fiber bundles will be discussed in the following sections. For now, we will describe two important constructions.

Example 1.1.2. *(Cartesian product)*
Given two fiber bundles $\xi_1 = (E_1,B_1,\pi_1)$ and $\xi_1 = (E_2,B_2,\pi_2)$ the Cartesian product $\xi_1 \times \xi_2$ is the fiber bundle $\pi_1 \times \pi_2 : E_1 \times E_2 \to B_1 \times B_2$. For each $(b_1,b_2) \in B_1 \times B_2$, the fiber is given by $\pi_1^{-1}(b_1) \times \pi_2^{-1}(b_2)$. We will usually refer to it as the product bundle.

Example 1.1.3. *Consider again the product bundle $\pi_1 \times \pi_2 : E_1 \times E_2 \to B_1 \times B_2$ in the previous example and assume $B_1 = B_2 = B$. The fiber product of E_1 and E_2 denoted by $E_1 \times_B E_2$ is the fiber bundle defined by the following. The total space $E_1 \times_B E_2$ is the set of pairs $(e_1,e_2) \in E_1 \times E_2$ such that $\pi_1(e_1) = \pi_2(e_2)$. The projection mapping $\pi : E_1 \times_B E_2 \to B$ is given by $\pi(e_1,e_2) := \pi_1(e_1)$, and the fibers are given by products of the fibers over the same point $\Big(E_1\Big)_b \times \Big(E_2\Big)_b$.*

This construction extends to an arbitrary number of fiber bundles $E_1,\cdots,E_r$ over a common base space B and produces a fiber bundle $E_1 \times_B \cdots \times_B E_r \to B$.

For any open cover $U = \{U_\alpha\}$ of the base space B with local trivializations $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times F$ we have the following homeomorphisms

$$\varphi_\alpha \circ \varphi_\beta^{-1} : U_\alpha \cap U_\beta \times F \to U_\alpha \cap U_\beta \times F$$

which determine mappings to the group of homeomorphisms of F

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Homeo}(F)$$

called *transition* or *clutching functions*. Every fiber bundle is completely determined (up to isomorphism) by the open cover U and the transition functions $\{g_{\alpha\beta}\}$ and can be recovered by setting

$$E = \bigcup_\alpha U_\alpha \times F \Big/ \sim$$

the equivalence relation $\sim$ is defined by $(x,f) \sim (x, g_{\alpha\beta}^{-1}(x)f)$ for all $(x,f) \in U_{\alpha\beta} \times F$. With this point of view, it becomes easier to impose further structure on bundles. For instance, smooth transition functions lead to smooth bundles and holomorphic transition functions to holomorphic bundles.

1.2. Vector bundles

Vector bundles are a special case of fiber bundles where the fiber is given by a vector space. Let $\mathbb{K}$ be a field ($\mathbb{R}$ or $\mathbb{C}$) and B be a topological space.

Definition 1.2.1. *A quasi-vector bundle $\xi = (E,B,\pi)$ is given by*

(1) a $\mathbb{K}$-vector space E_b of finite dimension for all b in B.

(2) a topology on the disjoint union $E = \bigsqcup_{b\in B} E_b$ that induces a topology on each E_b and makes the projection $\pi : E \to B$ a continuous map.

We will usually denote the quasi-vector bundle by its total space E and refer to the triple-notation when necessary. Later on when we discuss families of operators, a quasi-vector bundle is what we will mean by a family of vector spaces.

Consider the quasi-vector bundle $\xi = (E,B,\pi)$ and assume A is a subset of B. Then, the triple $(\pi^{-1}(A), \pi|_{\pi^{-1}(A)}, A)$ defines a quasi-vector bundle called the *restriction of ξ to A* and denoted by $\xi|_A$ or $E|_A$. Its total space is a subspace of E and its fibers are simply the fibers of ξ over A. The bundle $E|_A$ can also be defined more formally as an *induced bundle*.

Let $f : A \to B$ be a continuous map. We can define a quasi-vector bundle $(f^*(E),A,\pi_A)$ consisting of pairs (a,e) in $A\times E$ such that the following diagram commutes

$$\begin{tikzcd} f^*(E) \arrow[r, "\mathscr{F}"] \arrow[d, "\pi_A"] & E \arrow[d, "\pi"] \\ A \arrow[r, "f"] & B \end{tikzcd}.$$

The projection mapping π_A is given by the projection onto the first entry and $\mathscr{F}$ is given by the projection onto the second entry. The fibers are then simply $E_{f(a)}$ for a in A. In other words, $f^*(E)$ is the fiber product space $A \times_B E$.

We call $f^*(E)$ the *pullback bundle of E by f* or the *bundle induced* by f. If f is an inclusion map, then $f^*(E) = E|_A$. One can define similarly $f^*(E)$ for a general fiber bundle.

Example 1.2.2. *Let V be a vector space and B be a topological space. The quasi-vector bundle defined by $E = B\times V$ with the natural projection π_B onto B is called trivial quasi-vector bundle or simply trivial vector bundle.*

Example 1.2.3. *Let B be the n-dimensional sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. For each $b \in B$ define E_b to be the vector space orthogonal to b. The family of vector spaces $E = \bigsqcup_b E_b$ parametrized by*

B is a subspace of $S^n \times \mathbb{R}^{n+1}$ and can be endowed with the induced topology which makes E into a quasi-vector bundle.

Definition 1.2.4. *A morphism $(\mathscr{F}, f)$ of quasi-vector bundles $\mu = (E, B, \pi)$ and $\nu = (E^{'}, B^{'}, \pi^{'})$ is a pair of continuous mappings $\mathscr{F} : E \to E^{'}$ and $f : B \to B^{'}$ such that $\mathscr{F}$ restricts to a $\mathbb{K}$-linear morphism on each fiber and the following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{\mathscr{F}} & E^{'} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^{'}} \\ B & \xrightarrow{f} & B^{'} \end{array}.$$

If the base spaces of μ and ν are equal, a morphism $\mathscr{F}$ between μ and ν will denote the pair $(\mathscr{F}, Id_B)$. An isomorphism $\mathscr{F}$ is then a bijective map such that the inverse $\mathscr{F}^{-1}$ is continuous.

Definition 1.2.5. *A quasi-vector bundle $\xi := (E, \pi, B)$ is called vector bundle if for every $b \in B$, there exists a neighbourhood U of b such that $\xi|_U$ is a trivial bundle. In other words, there exists a finite dimensional vector space V and a homeomorphism $\varphi_U : \pi^{-1}(U) \to U \times V$ such that*

(1) $\varphi_y : \pi^{-1}(y) \to V$ is $\mathbb{K}$-linear for all y in U.

(2) the following diagram commutes

$$\begin{array}{ccc} \xi|_U & \xrightarrow{\varphi_U} & U \times V \\ & \searrow^{\pi} & \downarrow{\scriptstyle proj_U} \\ & & U \end{array}.$$

Morphisms of vector bundles are given by Definition 1.2.4. A morphism $\mathscr{F}$ of vector bundles over a common base space B is a morphism $(\mathscr{F}, Id_B)$.

The *rank* of a vector bundle (E, B, π) is a locally constant function from B to $\mathbb{Z}_{\geq 0}$ defined by $b \mapsto \dim(E_b)$. A vector bundle is said to be of *rank* or of *dimension* n when the fibers have constant dimension n as vector spaces (i.e. the rank is a constant function on B with value n). This happens for instance when the base space B is connected.

Remark 1.2.6. *A vector bundle of rank n is a fiber bundle with fiber a $\mathbb{K}$-vector space isomorphic to $\mathbb{K}^n$.*

Example 1.2.7. *Let us return to Example 1.2.3. The quasi-vector bundle E is in fact the vector bundle tangent to the sphere TS^n. To show this, it suffices to verifiy the local triviality condition. Let $b \in B$ and consider a neighbourhood of this point*

$$U_b = \{x \in S^{n+1} : \langle x, b \rangle \neq 0\}$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^{n+1}$. Then, for all $x \in U_b$ we can define the orthogonal projection of x onto E_b by

$$y = x - \langle x,b\rangle b.$$

The map $\varphi_{U_b} : \pi^{-1}(U_b) \subset TS^n \to U_b \times \mathbb{R}^n$ defined by $(b,x) \mapsto (b,y)$ is a homeomorphism.

Example 1.2.8. *Consider the product bundle $\pi : Gr_k^n(\mathbb{K}) \times \mathbb{K}^n \to Gr_k^n(\mathbb{K})$ over the Grassmanian of k-dimensional subspaces of $\mathbb{K}^n$. The tautological k-dimensional vector bundle $\Pi : \gamma_k^n \to Gr_k^n(\mathbb{K})$ is the subbundle of $Gr_k^n(\mathbb{K}) \times \mathbb{K}^n$ which consists of pairs of the form $\{([x],x) : x \in [x]\}$.*

The case $\mathbb{K} = \mathbb{C}$ and $k = 1$ is a line bundle $\gamma_1^{n+1} \to \mathbb{CP}^n$. We will denote this bundle by $O_{\mathbb{P}^n}(-1)$ and its dual bundle by $O_{\mathbb{P}^n}(1)$.

Example 1.2.9. *(External Whitney sum and external product) Let $E \to B$ and $F \to C$ be vector bundles and let $proj_1 : B \times C \to B$ and $proj_2 : B \times C \to C$ be the projection onto the first and the second factor, respectively. The bundle defined by*

$$E \boxplus F := proj_1^*(E) \oplus proj_2^*(F) \to B \times C$$

is a vector bundle called the external Whitney sum of E and F. Analogously, we can define the external tensor product of E and F by

$$E \boxtimes F := proj_1^*(E) \otimes proj_2^*(F) \to B \times C$$

Proposition 1.2.10. *(Proposition 2.7,* **[14]***) Let E and F be vector bundles over B and let $\mathscr{G} : E \to F$ be a morphism of vector bundles. If $\mathscr{G}_b : E_b \to F_b$ is bijective for all b in B, then $\mathscr{G}$ is an isomorphism of vector bundles.*

Let $\mathscr{F} : \xi \to \eta$ be a morphism of vector bundles with base space B. We say that $\mathscr{F}$ is of *constant rank* or a *strict homomorphism* if $\mathscr{F}_b : \pi_\xi^{-1}(b) \to \pi_\eta^{-1}(b)$ is of constant rank for all $b \in B$.

In general, $\ker\mathscr{F}$ and $\operatorname{coker}\mathscr{F}$ are not vector bundles as the local triviality condition may fail. The following result provides a sufficient condition ensuring that they are vector bundles.

Theorem 1.2.11. *[Theorem 8.2,* **[13]***] For a morphism of vector bundles $\mathscr{F} : \xi \to \eta$, if $\mathscr{F}$ is of constant rank, then $\ker\mathscr{F}$, $\operatorname{coker}\mathscr{F}$ and $\operatorname{Im}\mathscr{F}$ are vector bundles.*

1.3. Principal bundles, connection and curvature forms

We will discuss briefly some special cases of fiber bundles that we will encounter in this thesis.

1.3.1. Principal bundles

Let G be a topological group. A continuous right action of G on a topological space M is a continuous map

$$\psi : M \times G \to M$$

that satifies the following properties:

(1) $\psi(x,e) = x$; e is the identity element of G.

(2) $\psi(\psi(x,g),h) = \psi(x, \psi(g,h))$.

We denote $\psi(x,g)$ by xg or $\psi_g(x)$ for all $x \in M$ and all $g \in G$. A left action can be defined in a similar way.

Recall that the *stabilizer* of a point $x \in M$ is given by

$$\mathrm{Stab}(x) = \{g \in G : xg = x\}$$

and the *orbit* of $x \in M$ is the set

$$\mathrm{Orb}(x) = \{xg \in M : g \in G\}.$$

The action of G on M is *free* if for any $x \in M$ the stabilizer is trivial (i.e. $\mathrm{Stab}(x) = e$). In other words, the mapping $G \times M \to M \times M$ given by $(g,x) \mapsto (\psi_g(x),x)$ is injective. The action is called *transitive* if there is only one orbit (i.e. for all $x, y \in M$ there is an element $g \in G$ for which $x = yg$).

A topological space with a G-action is called *G-space*. Let M and N be G-spaces. Then a mapping $f : M \to N$ is said to be *G-equivariant* if f intertwines the action of G on M with the action of G on N. Depending on the type of action (left or right) this condition can be phrased as

(1) $f(xg) = f(x)g$ (right-right).

(2) $f(gx) = gf(x)$ (left-left).

(3) $f(xg) = g^{-1}f(x)$ (right-left).

(4) $f(gx) = f(x)g^{-1}$ (left-right).

Definition 1.3.1. *A fiber bundle $\pi : P \to M$ with a continuous right action of G on P is called a principal G-bundle if the G-action preserves the fibers and acts freely and transitively on them (simply transitive action). Moreover, the local trivializations*

$$\varphi_U : \pi^{-1}(U) \xrightarrow{\cong} U \times G$$

are G-equivariant. The group G is called the structure group of P.

For any local trivializing neighbourhood (U, φ_U), we can define a mapping

$$g_U := \mathrm{proj}_G \circ \varphi_U : \pi^{-1}(U) \to G.$$

The G-equivariance property of φ_U means that

$$\varphi_U(pg) = (\pi(p), g_U(p)g)$$

for all $p \in P$ and $g \in G$.

Every principal bundle $\pi : P \to M$ can be fully described by an open cover $U = \{U_\alpha\}_\alpha$ and transition functions

$$g_{\alpha\beta} := g_\alpha \circ g_\beta^{-1} : U_\alpha \cap U_\beta \to G$$

satisfying the cocycle condition

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1. \tag{1.1}$$

Example 1.3.2. *A trivial principal G-bundle is simply a product G-bundle*

$$\pi : M \times G \to M.$$

Example 1.3.3. *(Principal bundle of orthonormal frames)*
Let E be a Riemannian vector bundle of dimension n over an oriented manifold M. Consider the bundle of oriented orthonormal bases in E. That is, the bundle whose fiber at each point $x \in M$ is given by the set of oriented orthonormal bases of the vector space E_x. This is a principal $SO(n)$-bundle denoted by $P_{SO}(E)$.

The action of $SO(n)$ on an (oriented) orthonormal basis $p = (v_1, \cdots, v_n)$ of E_x is defined by

$$pg = \left(\sum_k v_k a_{kj} \right)_{j=1}^{n}$$

where $g \mapsto A = (a_{jk})$ is given by the matrix representation of $SO(n)$.

1.3.2. Associated bundles

Let $\pi : P \to M$ be a principal G-bundle and F be a topological space with $\mathrm{Homeo}(F)$ the group of its homeomorphisms. The group $\mathrm{Homeo}(F)$ endowed with the compact-open topology is a topological group (that is, a group together with a topology that makes the group operations continuous).

Given a continuous homomorphism $\rho : G \to \text{Homeo}(F)$, we can define a new fiber bundle on M with fiber F. Indeed, consider the free left action of G on $P \times F$ given by

$$\psi_g(p,f) := (pg^{-1}, \rho(g)f) \; ; \; g \in G, \; (p,f) \in P \times F.$$

Define $P \times_\rho F := P \times F / \sim_\rho$ where $(p_1,f_1) \sim_\rho (p_2,f_2)$ if and only if (p_1,f_1) and (p_2,f_2) belong to the same orbit.

The projection $P \times F \to P \xrightarrow{\pi} M$ descends to a mapping

$$\pi_\rho : P \times_\rho F \to M.$$

Definition 1.3.4. *The fiber bundle $F \hookrightarrow P \times_\rho F \xrightarrow{\pi_\rho} M$ is called the bundle associated to P by ρ.*

Remark 1.3.5. *The transition functions of $P \times_\rho F$ are given by*

$$\rho \circ g_{\alpha\beta} : U_\alpha \cap U_\beta \to Homeo(F).$$

Note that by imposing more structure on the fiber F and the homomorphism ρ we could define associated bundles with more structure. For instance, we could have a smooth bundle with transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to Diff(F)$$

where $Diff(F)$ denotes the group of diffeomorphisms of F. Associated bundles can also be defined similarly with a more general left-action on F.

Example 1.3.6. *Let E be an oriented Riemannian vector bundle over M and let $P = P_{SO(n)}(E)$ be the $SO(n)$-bundle of positively oriented orthonormal frames. Denote by $\rho_n : SO(n) \to SO(\mathbb{R}^n)$ the standard representation of the group $SO(n)$. We write $\bigotimes^k \mathbb{R}^n$ for the k-fold tensor product and $\Lambda^k \mathbb{R}^n$ for the k-fold exterior product of the vector space $\mathbb{R}^n$. Then*

i. $E = P \times_{\rho_n} \mathbb{R}^n$.

ii. $\Lambda^k E = P \times_{\Lambda^k \rho_n} \Lambda^k \mathbb{R}^n$.

iii. $\bigotimes^k E = P \times_{\bigotimes^k \rho_n} \bigotimes^k \mathbb{R}^n$.

1.3.3. Connections on Principal bundles

We will now see how the structure group of a principal bundle produces by its infinitesimal action vector fields and a canonical distribution. A choice of "connection" on the principal bundle specifies then a complementary distribution in the tangent bundle of this principal bundle. The reader can refer to Chapter 1 of [**27**] or Chapter 6 of [**29**].

Suppose G is a Lie group with Lie algebra $\mathfrak{g}$. Recall that the *exponential* of $A \in \mathfrak{g}$ denoted by e^A is given by $\gamma(1)$ where γ is the unique homomorphism $\mathbb{R} \to G$ (*one-parameter subgroup of* G) that satisfies

$$\frac{d}{dt}\Big|_{t=0} \gamma(t) = A.$$

The mapping $\mathfrak{g} \to G$ defined by $A \mapsto e^A$ is called the *exponential mapping* (see [**12**], Chapter 2 §1).
We can associate to any element $A \in \mathfrak{g}$ a vector field $\underline{A}$ on P defined at any point $p \in P$ by

$$(\underline{A})_p := \frac{d}{dt}\Big|_{t=0} pe^{tA} \ \in T_pP$$

The vector field $\underline{A}$ is called the *fundamental* (or *Killing*) *vector field* generated by A. Define for all $p \in P$ the map

$$j_p : G \to P \ ; \ g \mapsto pg.$$

Then, differentiating along the curve $\gamma(t) = e^{tA}$, we have that

$$(j_p)_* A = \frac{d}{dt}\Big|_{t=0} pe^{tA} = \underline{A}_p.$$

The map $\mathfrak{g} \to \mathfrak{X}(P)$ given by $A \longmapsto \underline{A}$ is a Lie algebra homomorphism.
Denote by r_g the right translation by $g \in G$ and by $Ad : G \to \mathrm{Aut}(G)$, the adjoint representation of G given at each $g \in G$ by $Ad_g : h \mapsto ghg^{-1}$.

Proposition 1.3.7. *Let G be a Lie group acting on a smooth manifold M on the right. Then*

$$(r_g)_* \underline{A} = \underline{Ad_{g^{-1}} A}. \tag{1.2}$$

Remark 1.3.8.

(1) The fundamental vector field $\underline{A}$ vanishes at a point $p \in P$ if and only if A is an element of the Lie algebra $Lie(Stab(p))$.

(2) If the group G acts smoothly from the right on the manifold P, then

$$ker\Big((j_p)_*\Big)_e = Lie(Stab(p)).$$

Consider a principal G-bundle $\pi : P \to M$. The differential map

$$\pi_* : T_pP \to T_{\pi(p)}M$$

is surjective for all $p \in P$.

The *vertical tangent space* at p, denoted by V_p is the subspace $\ker(\pi_*)_p \subset T_pP$.

Proposition 1.3.9. *For all $A \in \mathfrak{g}$ the Killing vector field $\underline{A}$ is vertical for all $p \in P$ and the map $(j_p)_* : \mathfrak{g} \to V_p$ is an isomorphism.*

Definition 1.3.10. *For a smooth principal G-bundle $P \to M$, a subbundle H of TP is called a horizontal distribution on P if $TP = V \bigoplus H$ as vector bundles.*

In general H is not canonically defined unlike the vertical subbundle V. This motivates introducing a new structure to specify the horizontal distribution.

Definition 1.3.11. *An Ehresmann connection on a principal bundle P is a smooth right G-invariant horizontal distribution H, that is*

(1) For all $p \in P$ $T_pP = V_p \bigoplus H_p$.

(2) $(r_g)_* H_p = H_{pg}$.

Suppose H is a horizontal distribution on the total space P of a principal G-bundle $\pi : P \to M$. From Proposition 1.3.9, the vertical tangent space V_p can be canonically identified with the Lie algebra $\mathfrak{g}$.

Let $\sigma : T_pP \to V_p$ be the projection onto the vertical tangent space V_p along a horizontal distribution H_p. We can then define a $\mathfrak{g}$-valued 1-form α on P by

$$\alpha_p := (j_p)_*^{-1} \circ \sigma : T_pP \longrightarrow \mathfrak{g} \tag{1.3}$$

Theorem 1.3.12. *Let H be a right-invariant horizontal distribution on P and let α be the $\mathfrak{g}$-valued 1-form defined in* (1.3)*, then the following holds:*

(1) For all $A \in \mathfrak{g}$ and for all $p \in P$, $\alpha_p(\underline{A}_p) = A$.

(2) (G-equivariance) For all $g \in G$, $(r_g)^* \alpha = (Ad_{g^{-1}})\alpha$.

(3) α is smooth.

Definition 1.3.13. *A connection 1-form on a principal G-bundle $P \to M$ is a $\mathfrak{g}$-valued 1-form α on P that satisfies conditions (1)-(3) in Theorem 1.3.12.*

For a vector bundle endowed with an (affine) connection ∇, there is a natural way of defining a horizontal distribution on the frame bundle. The following theorem describes the correspondence between a metric connection ∇ on a vector bundle and a connection 1-form on its orthonormal frame bundle.

Theorem 1.3.14 (Proposition 4.4, [**22**]). *A connection 1-form α on the orthonormal frame bundle $P_{SO}(E)$ of a smooth Riemannian vector bundle E determines a unique connection ∇ by*

$$\nabla e_i = \sum_{j=1}^{n} \widetilde{\alpha}_{ji} \otimes e_j \tag{1.4}$$

for a local family of sections $e = (e_1, \cdots, e_n)$ in $\Gamma(P_{SO}(E))$ and a matrix of 1-forms $\widetilde{\alpha} = e^*\alpha$. This connection is compatible with the metric (metric connection). Conversly, any metric connection ∇ satisfying (1.4) *determines a unique connection 1-form.*

Definition 1.3.15. *Let P be a principal G-bundle with connection 1-form α. The curvature 2-form of α is a $\mathfrak{g}$-valued 2-form given by*

$$\Omega = d\alpha + \frac{1}{2}[\alpha, \alpha]$$

where $[\alpha, \alpha](v,w) = [\alpha(v), \alpha(w)]$ for all $v,w \in \Gamma(TP)$.

The curvature 2-form Ω measures the failure of integrability of the horizontal distribution.

Chapter 2

Dirac operators

2.1. Clifford algebras

Consider a finite dimensinal vector space V over a commutative field $\mathbb{K}$ endowed with a quadratic form Q. We call (V,Q) a *quadratic space* over $\mathbb{K}$.

Let

$$\mathscr{T}(V) = \bigoplus_{k=0}^{\infty} \bigotimes^{k} V$$

be the tensor algebra over V and denote by $\mathscr{I}_Q(V)$ the two-sided ideal generated by elements $v \otimes v + Q(v)1$ for $v \in V$.

Definition 2.1.1. *The Clifford Algebra $Cl(V,Q)$ associated to the quadratic space (V,Q) is an associative algebra (with unit) over $\mathbb{K}$ defined by the quotient*

$$Cl(V,Q) = \mathscr{T}(V)/\mathscr{I}_Q(V).$$

There is a natural embedding $V \overset{j}{\hookrightarrow} Cl(V,Q)$ given by the composition $V \hookrightarrow \mathscr{T}(V) \hookrightarrow Cl(V,Q)$ which can be used to identify V with a linear space in $Cl(V,Q)$. In fact, the Clifford algebra $Cl(V,Q)$ is generated by $V \subset Cl(V,Q)$ and the identity with the relation

$$v.v = -Q(v)1. \tag{2.1}$$

Moreover, if the characteristic of $\mathbb{K}$ is different from 2, then for all v, $w \in V$

$$v.w + w.v = -2Q(v,w)1. \tag{2.2}$$

Here $2Q(v,w) = Q(v+w) - Q(v) - Q(w)$ is the polarization of the quadratic form. The relation (2.1) gives a useful characterisation of Clifford algebras up to isomorphisms.

Proposition 2.1.2. *(The universal property of Clifford Algebras)*
Let $f : V \to \mathscr{A}$ be a linear map into a unital associative $\mathbb{K}$-Algebra $\mathscr{A}$ such that

$$f(v)^2 = -Q(v)1 \tag{2.3}$$

for all $v \in V$. Then there exists a unique $\mathbb{K}$-algebra homomorphism

$$\widehat{f} : Cl(V,Q) \to \mathscr{A}$$

that extends f and satisfies $f = \widehat{f} \circ j$.

PROOF. Every linear map $f : V \to \mathscr{A}$ extends to a unique algebra homomorphism $F : \mathscr{T}(V) \to \mathscr{A}$. By property (2.3), F vanishes on the ideal $\mathscr{I}_Q(V)$. Hence, F descends to an algebra morphism

$$\widehat{f} : Cl(V,Q) \to \mathscr{A}$$

satisfying $f = \widehat{f} \circ j$. Uniqueness is a consequence of the fact that $\widehat{f}$ is uniquely determined on $V \subset Cl(V,Q)$ which is a generating set for this Clifford algebra.

Assume $\mathscr{B}$ is a unital associative $\mathbb{K}$-algebra such that there is an embedding $V \overset{\iota}{\hookrightarrow} \mathscr{B}$ and any linear mapping $g : V \to \mathscr{A}$ satisfying property (2.3) extends to an algebra homomorphism $\widehat{g} : \mathscr{B} \to \mathscr{A}$ such that $g = \widehat{g} \circ \iota$.

Set $\mathscr{A} := Cl(V,Q)$ and $f := j$. Then, there exists a morphism $\widehat{j} : \mathscr{B} \to Cl(V,Q)$ such that $j = \widehat{j} \circ \iota$. A similar argument shows that $\iota = \widehat{\iota} \circ j$ which implies the following

$$\iota = \widehat{\iota} \circ j = (\hat{\iota} \circ \hat{j}) \circ \iota$$

$$j = \widehat{j} \circ \iota = (\widehat{j} \circ \widehat{i}) \circ j.$$

The mappings $\widehat{\iota} \circ \widehat{j}$ and $\widehat{j} \circ \widehat{\iota}$ correspond to the identity on V, hence, to the identity mappings $Id_{\mathscr{B}}$ and $Id_{Cl(V,Q)}$ on $\mathscr{B}$ and $Cl(V,Q)$, respectively. □

Proposition 2.1.2 shows that the Clifford algebras $Cl(V,Q)$ are universal objects with respect to maps f as defined above. It can be used as an axiomatic definition of $Cl(V,Q)$.

As a consequence of Proposition 2.1.2, there exists a unique automorphism

$$\alpha : Cl(V,Q) \to Cl(V,Q)$$

called *the parity automorphism* which extends the map $v \mapsto -v$ on V. Since $\alpha^2 = Id$, there is a decomposition of the Clifford Algebra

$$Cl(V,Q) = Cl^0(V,Q) \oplus Cl^1(V,Q)$$

where for $i = 0,1$, we denote the eigenspaces of α by

$$Cl^i(V,Q) = \{\phi \in Cl(V,Q) | \alpha(\phi) = (-1)^i\phi\};\ i = 0,\ 1\}.$$

We note that

$$Cl^i(V,Q)\ .\ Cl^j(V,Q) \subseteq Cl^{i+j}(V,Q)$$

where the indices are taken modulo 2. This defines a $\mathbb{Z}_2$-grading on $Cl(V,Q)$. We call $Cl^0(V,Q)$ the *even part* of the Clifford algebra $Cl(V,Q)$ and $Cl^1(V,Q)$ the *odd part*.
Recall that if $\mathscr{A}$ and $\mathscr{B}$ are $\mathbb{Z}_2$-graded algebras

$$\mathscr{A} = \mathscr{A}^0 \oplus \mathscr{A}^1 \text{ and } \mathscr{B} = \mathscr{B}^0 \oplus \mathscr{B}^1,$$

then we can define a "$\mathbb{Z}_2$-graded" multiplication on elements of pure degree (even or odd) by

$$(a \otimes b).(a^{'} \otimes b^{'}) = (-1)^{\deg(b)\deg(a^{'})}(aa^{'} \otimes bb^{'}).$$

The resulting algebra denoted by $\mathscr{A}\widehat{\otimes}\mathscr{B}$ is called the $\mathbb{Z}_2$-*graded tensor product*.

Proposition 2.1.3. *The Clifford algebra of a direct sum $(V_1 \oplus V_2, Q_1 \oplus Q_2)$ of two quadratic spaces is isomorphic to the graded tensor product of their Clifford algebras*

$$Cl(V_1 \oplus V_2, Q_1 \oplus Q_2) = Cl(V_1, Q_1)\widehat{\otimes}Cl(V_2, Q_2)$$

The proof is a simple application of the universal property of Clifford algebras applied to the mapping

$$f : V_1 \oplus V_2 \to Cl(V_1, Q_1)\widehat{\otimes}Cl(V_2, Q_2)$$

$$(v_1,v_2) \mapsto j_1(v_1) \otimes 1 + 1 \otimes j_2(v_2)$$

Proposition 2.1.3 implies that the dimension of $Cl(V,Q)$ is 2^n for a n-dimensional vector space V.

Note that any basis $\{e_i\}_1^n$ of V generates the Clifford algebra $Cl(V,Q)$ multiplicatively and any pair of elements e_i,e_j satisfies the relation

$$e_i.e_j + e_j.e_i = -2Q(e_i,e_j)1.$$

Hence, the 2^n elements $1, e_{i_1}, \cdots, e_{i_k}$ for $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq k \leq n$ span $Cl(V,Q)$.

The following example discusses a special type of Clifford algebras defined on real vector spaces.

Example 2.1.4. *Consider the vector space $V = \mathbb{R}^{r+s}$ endowed with the pseudo-Euclidean quadratic form*

$$Q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2. \tag{2.4}$$

In the sequel, we will denote this quadratic form by $Q_{r,s}$ and the corresponding Clifford algebra $Cl(V,Q_{r,s})$ will be denoted by $Cl_{r,s}$. From the discussion following Proposition 2.1.3, the Clifford algebra $Cl_{r,s}$ is generated by a Q-orthogonal basis of $\mathbb{R}^{r+s} \subset Cl_{r,s}$ subject to the relations

$$e_i.e_j + e_j.e_i = \begin{cases} -2\delta_{ij} & \text{for } i \leq r \\ 2\delta_{ij} & \text{for } i > r \end{cases}$$

It is easy to see from the discussion above that $Cl_{1,0} = Cl_1 \cong \mathbb{C}$ and $Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R}$. One can for instance choose linear mappings

$$f : \mathbb{R}^{1,0} \to \mathbb{C}$$

and

$$g : \mathbb{R}^{0,1} \to \mathbb{R} \oplus \mathbb{R}$$

given by $f(e_1) = i$ and $g(e_1) = (1, -1)$ and apply Proposition 2.1.2 to conclude.

In fact, the Clifford algebras $Cl_{r,s}$ can be explicitly described as matrix algebras over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (quaternions). One can refer to Chapter 1 of [**22**] for more details on the classification of such algebras. More examples can be constructed using the following proposition which is again a direct consequence of the universal property of Clifford algebras.

Proposition 2.1.5. *The complexification of the Clifford algebra $Cl(V,Q)$ of a real quadratic space (V,Q) is isomorphic to the Clifford algebra of the complexification of (V,Q)*

$$Cl(V_{\mathbb{C}}, Q_{\mathbb{C}}) \cong Cl(V,Q) \otimes \mathbb{C}.$$

Proposition 2.1.6. *The Clifford algebra $Cl(V,Q)$ is isomorphic as a vector space to the exterior algebra ΛV.*

PROOF. (Proposition 2.1.6)
Define the mapping

$$\Phi : V \to V^*, \ v \mapsto \iota_v Q(\cdot,\cdot).$$

Here ι denotes the interior product and Q is defined by the relation (2.2)). The mapping

$$f : V \to \mathrm{End}(\Lambda V), \ f(v)\alpha = v \wedge \alpha - \iota(\Phi(v))\alpha \tag{2.5}$$

satisfies the relation

$$f(v)^2\alpha = -Q(v,v)\alpha,\ \forall v \in V\ \forall \alpha \in \Lambda V.$$

Proposition 2.1.2 implies the existence of an algebra homorphism

$$c : Cl(V,Q) \to \mathrm{End}(\Lambda V).$$

The composition of $\widehat{f}$ with the evaluation mapping at the identity yields a morphism of vector spaces

$$\sigma = \mathrm{eval}(1) \circ c : Cl(V,Q) \to \Lambda V.$$

Let $e_1, \cdots, e_n$ be a Q-orthogonal basis of V. For a sequence $1 \leq i_1 < \cdots < i_k \leq n$ the mapping σ coincides with the mapping

$$1 \mapsto 1\ ;\ e_{i_1}.\cdots.e_{i_k} \mapsto e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

This shows that σ is an isomorphism. $\square$

Remark 2.1.7. *For an arbitrary sequence of vectors $v_1, \cdots, v_k \in V$,*

$$\sigma(v_1.\cdots.v_k) = v_1 \wedge \cdots \wedge v_k \ mod \bigoplus_{l\geq 1} \Lambda V^{k-2l}.$$

This means that the Clifford multiplication can be viewed as a "lower order perturbation" of the exterior product.

We will introduce another important mapping. The Clifford Algebra $Cl(V,Q)$ inherits an antiautomorphism called *the transpose* from the tensor algebra $\mathscr{T}(V)$ which is defined as follows. Consider the involution $v_1 \otimes \cdots \otimes v_k \mapsto v_k \otimes \cdots \otimes v_1$ on $\mathscr{T}(V)$ obtained by reversing the order of the elements $v_1, \cdots, v_k$. Since this map leaves the ideal $\mathscr{I}_Q(V)$ invariant, it descends to a morphism

$$(\ )^t : Cl(V,Q) \to Cl(V,Q)$$

which satisfies $(\xi.\phi)^t = \phi^t.\xi^t$. Composing this map with α yields an antiautomorphism

$$\overline{(\ )} : Cl(V,Q) \to Cl(V,Q)\ ;\ \phi \mapsto \overline{\phi} := \alpha(\phi^t).$$

2.1.1. The Pin and Spin groups

We will discuss natural group structures within Clifford algebras. In order to simplify the notation, we will drop the Clifford multiplication sign (.) in this section. All vector spaces are defined over $\mathbb{R}$ or $\mathbb{C}$.

Consider the group of units of the Clifford algebra $Cl(V,Q)$ defined by

$$Cl(V,Q)^{\times} = \{\phi \in Cl(V,Q) : \exists\, \phi^{-1} \text{ with } \phi\phi^{-1} = \phi^{-1}\phi = 1\}.$$

This group is equipped with a natural representation

$$\widetilde{Ad} : Cl^{\times}(V,Q) \to Aut(Cl(V,Q)) \;,\; \widetilde{Ad}(\phi)[x] = \alpha(\phi)x\phi^{-1} \text{ for any } x \in Cl(V,Q)$$

called the *twisted adjoint representation*.

Definition 2.1.8. *The Clifford group $\Gamma(V,Q)$ is a subgroup of $Cl^{\times}(V,Q)$ defined by*

$$\begin{aligned}\Gamma(V,Q) &= \{\phi \in Cl^{\times}(V,Q) : \alpha(\phi)x\phi^{-1} \in V \;\forall x \in V\}\\ &= \{\phi \in Cl^{\times}(V,Q) : \widetilde{Ad}(\phi)(V) = V\}.\end{aligned}$$

Proposition 2.1.9. *The twisted adjoint $\widetilde{Ad}$ satisfies the following properties:*

(1) For any $\phi \in \Gamma(V,Q)$, $\widetilde{Ad}(\alpha(\phi)) = \widetilde{Ad}$.

(2) For any $v \in V$ for which $Q(v) \neq 0$ (anisotropic element)

$$\widetilde{Ad}(v)[w] = w - 2\frac{Q(v,w)}{Q(v)}v.$$

That is, $\widetilde{Ad}(v)$ is the reflection about the hyperplane in V orthogonal to v.

(3) If Q is non-degenerate, then the kernel of $\widetilde{Ad} : \Gamma(V,Q) \to Aut(V)$ is $\mathbb{K}^*.1$, the multiplicative group of non-zero multiples of the identity in $Cl(V,Q)$.

PROOF.

(1) For all $w \in V$ we have that $\alpha(w) = -w$. Applying α to $\widetilde{Ad}(\phi)[w]$ yields

$$\begin{aligned}-\widetilde{Ad}(\phi)[w] &= \alpha(\alpha(\phi)w\phi^{-1})\\ &= -\phi w \alpha(\phi)^{-1}.\end{aligned}$$

In other words,

$$\widetilde{Ad}(\phi)[w] = \widetilde{Ad}(\alpha(\phi))[w].$$

(2) Since $Q(v) \neq 0$ and $v.v = -Q(v)1$, we have that

$$v^{-1} = \frac{-1}{Q(v)}v.$$

Then from this fact and the relation (2.2) we can compute the following

$$\begin{aligned} Q(v)\widetilde{Ad}(v)w &= Q(v)(-vwv^{-1})w \\ &= Q(v)(vw\frac{v}{Q(v)}) \\ &= (-2Q(v,w) - wv)v \\ &= w - 2Q(v,w)v. \end{aligned}$$

(3) If Q is non-degenerate then, we can choose a basis $v_1, \cdots, v_n$ for V such that for all i, $Q(v_i) \neq 0$ and for all $i \neq j$, $Q(v_i,v_j) = 0$. Assume $\phi \in \ker(\widetilde{Ad})$ then

$$\alpha(\phi)v = v\phi,\ \forall v \in V.$$

Writing $\phi = \phi_0 + \phi_1$, the even-odd decomposition of ϕ, yields

$$\phi_0 v = v\phi_0\ ;\ -\phi_1 v = v\phi_1\ \forall v \in V. \tag{2.6}$$

Using (2.2), ϕ_0 can be expressed as

$$\phi_0 = a_0 + v_1 a_1$$

where a_0 and a_1 are polynomial expressions of $v_2, \cdots, v_n$. Applying α shows that a_0 is even and a_1 is odd.

Let $v = v_1$ in (2.6) then,

$$\begin{aligned} v_1 a_0 + v_1^2 a_1 &= a_0 v_1 + v_1 a_1 v_1 \\ &= v_1 a_0 - v_1^2 a_1. \end{aligned}$$

This implies that $Q(v_1)a_1 = 0$ and $a_1 = 0$ (since $Q(v_1) \neq 0$). Hence ϕ_0 doesn't contain v_1.

By induction, one can show that ϕ_0 doesn't involve any of the generators $v_1, \cdots, v_n$. This means that $\phi_0 \in \mathbb{K}1$. Similarly, one can show that ϕ_1 doesn't involve any of the terms $v_1, \cdots, v_n$ and since ϕ_1 is odd, it must vanish. By assumption $\phi \neq 0$ hence, we may conclude that $\phi \in \mathbb{K}^*1$. □

Consider the mapping

$$N : Cl(V,Q) \to Cl(V,Q),\ N(\phi) = \phi.\overline{\phi} \tag{2.7}$$

called the *norm mapping*. Note that $\alpha(\phi^t) = \alpha(\phi)^t$.

From now on, we will assume that Q is a non-degenerate quadratic form.

Remark 2.1.10. *Consider the positive definite quadratic form $Q_{(k,0)}$. For any $v \in V$,*

$$N(v) = -v^2 = Q(v)1.$$

Hence $N(v)$ is the square of the Euclidean norm of v in $Cl_{(k,0)}$.

Proposition 2.1.11. *For any ϕ in the Clifford group $\Gamma(V,Q)$, $N(\phi) \in \mathbb{K}^*$.*

PROOF. For any $\phi \in \Gamma(V,Q)$, by definition we have that $\alpha(\phi)v\phi^{-1} \in V$ for any $v \in V$.

Since the transpose mapping acts as an identity on V, applying the transpose antiautomorphism yields

$$(\phi^{-1})^t v(\alpha(\phi))^t = (\phi^t)^{-1} v\alpha(\phi^t)$$

and

$$(\phi^t)^{-1} v\alpha(\phi^t) = \alpha(\phi)v\phi^{-1}.$$

Hence,

$$\begin{aligned} v &= \phi^t \alpha(\phi)v\phi^{-1}(\alpha(\phi^t))^{-1} \\ &= \alpha(\overline{\phi}\phi)v(\overline{\phi}\phi)^{-1} \\ &= \widetilde{Ad}(\overline{\phi}\phi)[v]. \end{aligned}$$

This means that $\overline{\phi}\phi \in \ker \widetilde{Ad}$ and by Proposition 2.1.9, this corresponds to an element in $\mathbb{K}^*.1$ (because $\overline{\phi}\phi \in \Gamma(V,Q)$). By applying the transpose antiautomorphism which preserves $\Gamma(V,Q)$, we deduce that $\alpha(\overline{\phi}\phi) = N(\phi) \in \mathbb{K}^*.1$. $\square$

Proposition 2.1.12. *The mapping $N : \Gamma(V,Q) \to \mathbb{K}^*$ is a homomorphism and*

$$N(\alpha(\phi)) = N(\phi) \; \forall \phi \in \Gamma(V,Q).$$

PROOF. Consider elements φ and ϕ in $\Gamma(V,Q)$. Then,

$$\begin{aligned} N(\varphi\phi) &= (\varphi\phi)\alpha(\phi^t\varphi^t) \\ &= \varphi(\phi\alpha(\phi^t))\alpha(\varphi^t) \\ &= \varphi N(\phi)\alpha(\varphi^t) \\ &= N(\varphi)N(\phi). \end{aligned}$$

This shows that N is a homomorphism. Moreover, for any $\phi \in \Gamma(V,Q)$

$$\begin{aligned} N(\alpha(\phi)) &= \alpha(\phi\overline{\phi}) \\ &= N(\phi). \end{aligned}$$

□

If we denote by $O(V,Q)$ the group of isometries of V with respect to Q and by $SO(V,Q)$ the corresponding special group, we obtain the following.

Theorem 2.1.13. *The twisted adjoint representation defines a short exact sequence*

$$1 \to \mathbb{K}^*.1 \to \Gamma(V,Q) \xrightarrow{\widetilde{Ad}} O(V,Q) \to 1. \tag{2.8}$$

Lemma 2.1.14. *(Cartan-Dieudonné) Let Q be a non-degenerate quadratic form and V a finite dimensional vector space. Then, any element $g \in O(V,Q)$ can be expressed as a product of reflections,*

$$g = \rho_1 \circ \cdots \circ \rho_k$$

for $k \leq dim(V)$.

PROOF. (Theorem 2.1.13)
We know from point (3) of Proposition 2.1.9 that $\ker \widetilde{Ad}$ corresponds to $\mathbb{K}^*1$. It only remains to show that $\widetilde{Ad}(\Gamma(V,Q)) \subset O(V,Q)$.

For any $v,w \in V$ and any $\phi \in \Gamma(V,Q)$

$$\begin{aligned} -2Q(\widetilde{Ad}(\phi)v,\widetilde{Ad}(\phi)w)1 &= \left(\widetilde{Ad}(\phi)v\right)\left(\widetilde{Ad}(\phi)w\right) + \left(\widetilde{Ad}(\phi)w\right)\left(\widetilde{Ad}(\phi)v\right) \\ &= \left(\widetilde{Ad}(\phi)v\right)\left(\widetilde{Ad}(\alpha(\phi))w\right) + \left(\widetilde{Ad}(\phi)w\right)\left(\widetilde{Ad}(\alpha(\phi))v\right) \\ &= \alpha(\phi)(vw + wv)\alpha(\phi^{-1}) \\ &= -2Q(v,w)1. \end{aligned}$$

The second equality above follows from point (1) of Proposition 2.1.9.

Surjectivity follows from the Cartan-Dieudonné Theorem. Indeed, for Q non-degenerate, any element $A \in O(V,Q)$ can be written as a product of k reflections

$$A = A_1 \cdots A_k,\ k \leq \dim(V).$$

Then, point (2) of Proposition 2.1.9 implies that any reflection in V through an anisotropic vector ($Q(v) \neq 0$) corresponds to an element in $\widetilde{Ad}(\Gamma(V,Q))$.

Moreover, $\widetilde{Ad} : \Gamma(V,Q) \to O(V,Q)$ is a homomorphism thus, any element $A \in O(V,Q)$ can be expressed as $\widetilde{Ad}(a)$ for some element $a = v_1 \cdots v_k \in \Gamma(V,Q)$. In other words, $\widetilde{Ad}$ is surjective and finally

$$1 \to \mathbb{K}^*.1 \to \Gamma(V,Q) \xrightarrow{\widetilde{\mathrm{Ad}}} O(V,Q) \to 1.$$

□

Note that the homomorphism $\widetilde{Ad}$ when restricted to the subgroup of even elements in $\Gamma(V,Q)$ denoted by $\Gamma^0(V,Q)$ induces the short exact sequence

$$1 \to \mathbb{K}^*.1 \to \Gamma^0(V,Q) \xrightarrow{\widetilde{\mathrm{Ad}}} SO(V,Q) \to 1. \tag{2.9}$$

We are now ready to introduce two important groups.

Definition 2.1.15. *The Pin and Spin groups are defined by*

$$Pin(V,Q) = \{\phi \in \Gamma(V,Q) : N(\phi) = 1\}$$
$$Spin(V,Q) = Pin(V,Q) \cap \Gamma^0(V,Q)$$

Remark 2.1.16.

(1) Since $N : \Gamma(V,Q) \to \mathbb{K}^*.1$ is a homomorphism, the groups $Pin(V,Q)$ and $Spin(V,Q)$ are normal subgroups.

(2) Observe that from Remark 2.1.10 the groups Pin and Spin can be equivalently defined as

$$Pin(V,Q) = \{v_1 \cdots v_k \in \Gamma(V,Q) : v_i \in V \, and \; Q(v_i) = \pm 1 \; \forall i\}$$
$$Spin(V,Q) = \{v_1 \cdots v_k \in Pin(V,Q) : k \text{ is even}\}$$

A natural question to ask is whether or not $\mathrm{Pin}(V,Q)$ and $\mathrm{Spin}(V,Q)$ are mapped onto $O(V,Q)$ and $SO(V,Q)$ respectively by the homomorphism $\widetilde{Ad}$. This is not true for an arbitrary field $\mathbb{K}$. Fields where surjectivity holds are called *spin fields*, $\mathbb{R}$ and $\mathbb{C}$ are examples of such fields.
Denote $Pin(\mathbb{R}^{r+s}, Q_{r,s})$ and $Spin(\mathbb{R}^{r+s}, Q_{r,s})$ by $Pin(r,s)$ and $Spin(r,s)$, respectively.

Theorem 2.1.17. *(Theorem 2.10,* [**22**]*)*
Assume $\mathbb{K} = \mathbb{R}$. Then, we have the short exact sequences

$$\begin{gathered} 1 \to \mathbb{Z}_2 \to Pin(r,s) \to O(r,s) \to 1 \\ 1 \to \mathbb{Z}_2 \to Spin(r,s) \to SO(r,s) \to 1. \end{gathered} \tag{2.10}$$

Remark 2.1.18. *For all $n \geq 3$ the mapping $\widetilde{Ad} : Spin(n) \to SO(n)$ is the universal covering homomorphism of $SO(n)$. Here $Spin(n) := Spin(n,0)$.*

2.1.2. The Spin^c-group

The last group we are introducing is of particular interest in this thesis.
Consider the complexification $Cl_n^c = Cl_n \otimes_{\mathbb{R}} \mathbb{C}$ of the Clifford algebra $Cl_n := Cl_{(n,0)}$. Then, one can define an analogue for each group introduced previously and extend the results obtained.
The parity automorphism α and the the transpose antiautomorphism $(.)^t$ are defined on Cl_n^c by

$$\alpha(x \otimes z) = \alpha(x) \otimes z$$

$$(x \otimes z)^t = (x)^t \otimes z$$

for generators $x \in Cl^n$ and $z \in \mathbb{C}$. As before, for any $\phi \in Cl_n^c$ the norm homomorphism is defined by

$$N(\phi) = \phi\alpha(\phi^t)$$

and the Clifford group $\Gamma_n = \Gamma(\mathbb{R}^n, \|.\|^2)$ becomes the subgroup Γ_n^c of invertible elements in Cl_n^c for which $\alpha(x)yx^{-1} \in \mathbb{R}^n$ for all $y \in \mathbb{R}^n$.

Propositions 2.1.11 and 2.1.12 hold with $\mathbb{K} = \mathbb{C}$ and the short exact sequence in Theorem 2.1.13 becomes

$$1 \to \mathbb{C}^* \to \Gamma_n^c \to O(n) \to 1. \tag{2.11}$$

By defining Pin^c as the kernel of $N : \Gamma_n^c \to \mathbb{C}^*$, $n \geq 1$ we obtain the short exact sequence

$$1 \to U(1) \to \mathrm{Pin}^c(n) \to O(n) \to 1. \tag{2.12}$$

The $\mathrm{Spin}^c(n)$ group can then be defined as the subgroup of $\mathrm{Pin}^c(n)$ that maps onto $SO(n)$

$$1 \to U(1) \to \mathrm{Spin}^c(n) \to SO(n) \to 1. \tag{2.13}$$

In fact, $U(1)$ denotes the subgroup $1 \otimes S^1$ of Cl_n^c. A possibly more concrete way to view $\mathrm{Spin}^c(n)$ is as the subgroup of Cl_n^c generated by $\mathrm{Spin}(n)$ and $U(1)$, that is

$$\mathrm{Spin}^c(n) \equiv \mathrm{Spin}(n) \times_{\mathbb{Z}_2} U(1) \tag{2.14}$$

which is simply the set of equivalence classes of pairs (x,z) in $\mathrm{Spin}(n) \times U(1)$ subject to the equivalence relation $(x,z) \sim (-x,-z)$.
Indeed, we have the inclusion

$$\mathrm{Pin}(n) \times_{\mathbb{Z}_2} U(1) \hookrightarrow Cl_n^c \tag{2.15}$$

induced from the inclusion of $\mathrm{Pin}(n)$ in Cl_n and $U(1)$ in $\mathbb{C}$. This factors then through a homomorphism

$$f : \mathrm{Pin}(n) \times_{\mathbb{Z}_2} U(1) \to Pin^c(n). \tag{2.16}$$

Furthermore, from the short exact sequences

$$1 \to U(1) \to \mathrm{Pin}(n) \times_{\mathbb{Z}_2} U(1) \to \mathrm{Pin}(n)/\mathbb{Z}_2 \to 1 \tag{2.17}$$

and (2.12) we obtain the commuting diagram

$$\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r] & \mathrm{Pin}^c(n) \arrow[r] & O(n) \arrow[r] & 1 \\
1 \arrow[r] & U(1) \arrow[r] \arrow[u] & \mathrm{Pin}(n) \times_{\mathbb{Z}_2} U(1) \arrow[r] \arrow[u] & \mathrm{Pin}(n)/\mathbb{Z}_2 \arrow[r] \arrow[u] & 1
\end{tikzcd}.$$

The third vertical homomorphism follows from the double covering of $O(n)$ by $\mathrm{Pin}(n)$. The five-lemma implies then the isomorphism

$$\mathrm{Pin}^c(n) \cong \mathrm{Pin}(n) \times_{\mathbb{Z}_2} U(1)$$

which induces when restricted to $\mathrm{Spin}^c(n)$ the isomorphism

$$\mathrm{Spin}^c(n) \cong \mathrm{Spin}(n) \times_{\mathbb{Z}_2} U(1).$$

Remark 2.1.19. *The covering map $Spin(n) \to SO(n)$ induces a homomorphism*

$$Spin^c(n) \to SO(n) \times U(1),\ (x,z) \mapsto (\widetilde{Ad}(x), z^2). \tag{2.18}$$

Similarly, there is an embedding $U(k) \hookrightarrow SO(2k) \times U(1)$ given by $g \mapsto (g, det(g))$. For $n = 2k$, it would be natural to ask if such a morphism lifts to $Spin^c(n)$. It turns out (Proposition 5.2.14, [**27**]*) that such a lift exists and the following diagram commutes*

$$\begin{tikzcd}
Spin^c(2k) \arrow[r] & SO(2k) \times U(1) \\
U(k) \arrow[u, "G"] \arrow[ur, "g"] &
\end{tikzcd} \tag{2.19}$$

for some homomorphism $G : U(k) \to Spin^c(2k)$.

This will be useful later on in defining connections on complex spinor bundles.

2.2. Spin^c-structures

Let V be a vector space over a field $\mathbb{K}$ and Q a quadratic form on V. Consider a field $\mathscr{K} \supset \mathbb{K}$, then a $\mathscr{K}$-representation of the Clifford algebra $Cl(V,Q)$ is a $\mathbb{K}$-algebra homomorphism

$$\rho : Cl(V,Q) \to \mathrm{Hom}_{\mathscr{K}}(W,W)$$

where W is a finite dimensional vector space over $\mathscr{K}$ called a $Cl(V,Q)$-*module* over $\mathscr{K}$. To avoid a heavy notation, we will write $\rho(\phi)(w)$ as $\phi.w$ for $\phi \in Cl(V,Q)$ and $w \in W$ and call this operation *Clifford multiplication*.

Example 2.2.1. *The algebra homomorphism*

$$c : Cl(V,Q) \to End(\Lambda V)$$

constructed in the proof of Proposition 2.1.6 is a $\mathbb{K}$-representation of the Clifford algebra $Cl(V,Q)$ and $End(\Lambda V)$ is a Clifford module.

Definition 2.2.2. *A Spin^c-structure on a principal $SO(n)$-bundle $P_{SO(n)} \to M$ with $n > 2$ consists of a principal $U(1)$-bundle $P_{U(1)} \to M$ and a pair $(P_{Spin^c(n)}, \xi)$ where $P_{Spin^c(n)}$ is a principal Spin$^c(n)$-bundle and ξ is a Spin$^c(n)$-equivariant bundle morphism*

$$\xi : P_{Spin^c(n)} \to P_{SO(n)} \times_M P_{U(1)}.$$

This is actually a two-fold covering.

An oriented Riemannian manifold M with a Spin^c-structure on its tangent frame bundle $P_{SO}(M)$ is called a *Spin^c-manifold*.

We would like to mention that there are obstructions to the existence of Spin^c-structures. An orientable manifold M carries a Spin^c-structure if and only if the second Stiefel-Whitney class $w_2(M)$ is the mod 2 reduction of an integral class $u \in H^2(M,\mathbb{Z})$ (Theorem D.2 [**22**]). In this thesis, we are mainly interested in Spin^c-structures on Kähler manifolds which are always Spin^c-manifolds.

Remark 2.2.3. *Any complex (or almost-complex) manifold is a Spin^c-manifold because there is a canonical Spin^c-structure on any complex vector bundle $E \to M$.*

Indeed, recall from (2.19) *the following commutative diagram*

$$\begin{tikzcd} Spin^c(2k) \arrow[r] & SO(2k)\times U(1) \\ U(k) \arrow[u, "G"] \arrow[ur, "g"'] & \end{tikzcd}.$$

Then, a principal $Spin^c$-bundle can be obtained as the bundle associated to the unitary frame bundle $P_{U(k)}(E)$ by the map G

$$P_{Spin^c}(E) = P_{U(k)}(E) \times_G Spin^c(2k)$$

In this case, $P_{U(1)}(E) = P_{U(k)}(E) \times_{det} U(1)$ which is the principal bundle of the complex line bundle $\Lambda^n E$ (unit circle bundle of $\Lambda^n E$).

Suppose the vector bundle $E \to M$ carries a Spin^c-structure

$$\xi : P_{\mathrm{Spin}^c}(E) \to P_{SO}(E) \times_M P_{U(1)}(E).$$

Then, any connection forms on $P_{SO}(E)$ and $P_{U(1)}(E)$ induce a connection form on the fiber product $P_{SO}(E) \times_M P_{U(1)}(E)$ which can be lifted via the covering map ξ to a (unique) connection form on $P_{\mathrm{Spin}^c}(E)$.

2.3. The Dirac Bundle

Consider the action of $SO(n)$ on $(\mathbb{R}^n, Q)$. One of the consequences of the universal property of Clifford algebras is that any $g \in SO(n)$ induces an automorphism in $\mathrm{Aut}(Cl_n)$ which preserves Clifford multiplication. Hence, we obtain a representation of $SO(n)$ on Cl_n by algebra homomorphisms

$$\mu_n : SO(n) \to \mathrm{Aut}(Cl_n). \tag{2.20}$$

Definition 2.3.1. *Let E be an oriented Riemannian n-dimensional vector bundle on M. The associated bundle*

$$Cl(E) = P_{SO}(E) \times_{\mu_n} Cl_n$$

is called the Clifford bundle of E.

The Clifford bundle $Cl(E)$ is in fact a bundle of Clifford algebras parametrized by M. An important example is given by the bundle $Cl(TM)$ whose fibers are given by the Clifford algebras $Cl(T_xM)$ for $x \in M$. It will be denoted by $Cl(M)$ for simplicity.

One can similarly define

$$Cl^c(E) = P_{SO}(E) \times_{\mu_n} Cl^c_n$$

for a Hermitian vector bundle E by extending the representation μ_n to Cl_n^c. As one might suspect, the intrinsic properties of Clifford algebras carry over to Clifford bundles. For instance $Cl(E)$ has an even-odd decomposition

$$Cl(E) = Cl^0(E) \oplus Cl^1(E)$$

and $Cl(E) \cong \Lambda^* E$ as vector bundles. This is in fact an isometry (Proposition 3.5, [**22**]).

We will now consider bundles of modules over a Clifford bundle.

Definition 2.3.2. *Let M be a $Spin^c$-manifold. A complex spinor bundle for M is an associated bundle*

$$\mathscr{S}(M) = P_{Spin^c}(M) \times_\mu W$$

for a complex Cl_n-module W and a representation

$$\mu : Spin^c(n) \to GL(W)$$

given by restriction of the Cl_n-representation to $Spin^c(n) \subset Cl_n \otimes \mathbb{C}$.

Definition 2.3.3. *Let $\mathscr{S}$ be a Clifford module bundle over a Riemannian (or Hermitian) manifold (M,g) equipped with a (fiberwise) Riemannian (or Hermitian) structure h. Assume that Clifford multiplication by unit vectors is an isometry. That is, for all $e \in T_xM$ satisfying $g(e,e) = 1$,*

$$h(c(e)\phi_1, c(e)\phi_2) = h(\phi_1, \phi_2)$$

for all $\phi_1, \phi_2 \in \Gamma(\mathscr{S}_x)$. Then, an h-compatible connection ∇ is called Clifford connection if it is a module connection

$$\nabla(A.\phi) = \nabla^g(A).\phi + A.\nabla(\phi) \tag{2.21}$$

For all $A \in \Gamma(Cl(M))$ and $\phi \in \Gamma(\mathscr{S})$. We call $(\mathscr{S},h,\nabla)$ over (M,g) a Dirac bundle.

Definition 2.3.4. *Let $(\mathscr{S},h,\nabla)$ be a Dirac bundle over (M,g). Then, we can define a first order differential operator $D : \Gamma(\mathscr{S}) \to \Gamma(\mathscr{S})$ as the composition of the following mappings*

$$\Gamma(\mathscr{S}) \xrightarrow{\nabla} \Gamma(T^*M \otimes \mathscr{S}) \xrightarrow{g^{-1}} \Gamma(TM \otimes \mathscr{S}) \xrightarrow{c} \Gamma(\mathscr{S})$$

called the Dirac operator.

In a local orthonormal frame $e_1, \cdots e_n$ on T_xM

$$D\varphi = \sum_{j=1}^{n} e_j.\nabla_{e_j}\varphi \tag{2.22}$$

Since every complex vector space can be viewed as a real vector space together with an almost complex structure J (endomorphism fulfilling $J^2 = -\mathrm{Id}$), in the complex case, the Dirac operator (2.22) can be described locally by:

$$D\varphi = \sum_{j=1}^{n} \{e_j.\nabla_{e_j}\varphi + Je_j.\nabla_{Je_j}\varphi\}$$

for a local orthonormal frame $e_1, Je_1, \cdots, e_n, Je_n$ of $TM \otimes \mathbb{C}$.

Consider a symplectic manifold (M,ω) (a manifold M together with a closed non-degenerate 2-form ω) endowed with an almost complex structure J which is ω-compatible (i.e. $\omega(u,Ju)$ is positive definite for all $u \in TM$ and $\omega(Jv,Jw) = \omega(v,w)$ for all $v,w \in T_xM$). In this case, the vector bundle (TM,J) has a natural Hermitian structure determined by (ω,J) and given by

$$h_J(\cdot,\cdot) = \omega(\cdot,J\cdot)$$

The cotangent bundle inherits an almost complex structure which induces a decomposition of the complexification of T^*M given by

$$T^*M \otimes \mathbb{C} = T^*M^{(1,0)} \oplus T^*M^{(0,1)}$$

where $T^*M^{(1,0)}$ denotes the bundle of holomorphic forms and $T^*M^{(0,1)}$ denotes the bundle of antiholomorphic forms. Define the vector bundle

$$E = \Lambda(T^*M^{(0,1)}) = \bigoplus_{0 \leq q \leq n} E^q \ , \ E^q = \Lambda^{(0,q)}T^*M.$$

We will write

$$E^+ := \bigoplus_{q \text{ even}} E^q$$

and

$$E^- := \bigoplus_{q \text{ odd}} E^q.$$

For any $\xi \in T^{(1,0)}M$ and $\overline{\xi} \in T^{(0,1)}M$, we can define the Clifford multiplication by

$$\begin{aligned} c(\xi) &= \sqrt{2}\,\overline{\xi}^* \wedge . \\ c(\overline{\xi}) &= -\sqrt{2}\iota_{\overline{\xi}} \ . \end{aligned} \tag{2.23}$$

where $\overline{\xi}^* \in T^*M^{(0,1)}$ is the metric dual of ξ. The mapping

$$c : TM \otimes \mathbb{C} \to \mathrm{End}(E)$$

extends to an algebra morphism

$$c : Cl^c(M) \to \mathrm{End}(E)$$

and makes E into a $Cl^c(M)$-module bundle. In fact, (Lemma 5.4, [**7**]), any choice of $U(1)$-invariant connection in K^* the unit circle bundle of the dual canonical line bundle defines with the Levi-Civita connection in $P_{SO}(M)$ a Clifford connection ∇ wich makes E a Dirac bundle. In our case, we choose the canonical connection $\nabla^{\mathrm{ch,n}}$ on $K = T^*M^{(n,0)}$ induced from the Chern connection ∇^{ch} on $T^*M^{(1,0)}$ which is uniquely characterized (Theorem 2.1, [**17**]) by

$$\nabla^{\mathrm{ch}}\omega = 0, \nabla^{\mathrm{ch}}J = 0, T^{(1,1)}_{\nabla^{\mathrm{ch}}} = 0.$$

This connection together with the Levi-Civita connection of h_J defines a Clifford connection on E and a Dirac operator

$$\mathscr{D} : \Gamma(E) \to \Gamma(E).$$

We are actually interested in a twisted version of the module bundle E given by taking the tensor product of E by a Hermitian line bundle (L,∇^L) equipped with a Hermitian connection ∇^L (compatible with the Hermitian structure h^L).

This defines a Clifford connection ∇^{Cl} by setting

$$\nabla^{Cl}(\sigma \otimes \lambda) = \nabla\sigma \otimes \lambda + \sigma \otimes \nabla^L\lambda$$

for any local section $\sigma \in \Gamma(E)$ and $\lambda \in \Gamma(L)$, and a Spin^c Dirac operator

$$\mathscr{D}^1(L,J) : \Gamma(E \otimes L) \to \Gamma(E \otimes L)$$

given locally by

$$\mathscr{D}^1(L,J)(\varphi) = \sum_{j=1}^{2n} e_j.\nabla^{Cl}_{e_j}(\varphi)_x \tag{2.24}$$

for $x \in M$ and an orthonormal local frame $e_1, \cdots, e_{2n}$ in T_xM.

The splitting $E = E^+ \oplus E^-$ induces a splitting of the Dirac operator $\mathscr{D}^1(L,J)$ which takes the form

$$\mathscr{D}^1(L,J) = \begin{pmatrix} 0 & \mathscr{D}^1_+(L,J) \\ \mathscr{D}^1_-(L,J) & 0 \end{pmatrix}$$

for $\mathscr{D}^1_\pm(L,J) = \mathscr{D}^1(L,J)|_{\Gamma(E^\pm \otimes L)} : \Gamma(E^\pm \otimes L) \to \Gamma(E^\mp \otimes L)$.

Remark 2.3.5. *The Hermitian line bundle L will be chosen later on to be the prequantization space of* (M,ω).

2.4. Properties of the Spin^c- Dirac operator

A smooth differential operator of order m on a manifold M is a local linear map $\mathscr{P} : \Gamma(E) \to \Gamma(F)$ for smooth complex vector bundles E and F over M. Locality means that for any neighbourhood $U \subset M$ with local coordinates $(x_1, \cdots, x_n)$ and local trivializations

$$E|_U \xrightarrow{\cong} U \times \mathbb{C}^p \ ; \ F|_U \xrightarrow{\cong} U \times \mathbb{C}^q$$

the operator $\mathscr{P}$ can be expressed as

$$\mathscr{P} = \sum_{|\alpha| \le m} a^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha} \tag{2.25}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of non-negative integers, $|\alpha| = \sum_{i=1}^n \alpha_i$ and $a^\alpha(x)$ is a $q \times p$ matrix of smooth complex-valued functions. Furthermore, there is an index α satisfying $|\alpha| = m$ for which the function a^α is not identically zero. the set of differential operators of order m acting between sections of $E \to M$ and $F \to M$ will be denoted by $\mathscr{P}^m(E,F;M)$ or $\mathscr{P}^m(E,F)$.

Assume M is closed (i.e. compact manifold without boundary) and has a volume form $d\rho$. Let h^E and h^F be Hermitian forms on E and F, respectively. Then, the formal adjoint of a differential operator $\mathscr{P} \in \mathscr{P}^m(E,F)$ is a differential operator $\mathscr{P}^* \in \mathscr{P}^m(F^*,E^*)$ defined by

$$(\mathscr{P}\mu, \nu) = (\mu, \mathscr{P}^*\nu), \ \forall \mu \in \Gamma(E) \ \forall \nu \in \Gamma(F).$$

Here $(.,.)$ denotes the inner product on sections induced from the pointwise inner product. That is

$$\int_M h_x^E(\mathscr{P}\mu(x), \nu(x)) d\rho(x) = \int_M h_x^F(\mu(x), \mathscr{P}^*\nu(x)) d\rho(x).$$

Note that we could drop the compactness condition on the manifold M if we only consider compactly supported smooth sections.

The coefficients $a^\alpha(x)$ with $|\alpha| = m$ transform as a tensor field

$$M \to \bigodot^m TM \otimes \mathrm{Hom}(E,F)$$

where $\bigodot^m TM$ denotes the symmetric tensor product.

Definition 2.4.1. *The section $\sigma \in \Gamma\Big(\bigodot^m TM \otimes Hom(E,F)\Big)$ is called the principal symbol of $\mathscr{P}$.*

By identifying $\bigodot^m TM$ with the space of homogenous polynomials of order m on $T_x^*M \xrightarrow{\pi} M$, we have that for any $\xi \in T^*M$, the principal symbol assigns an element

$$\sigma_\xi : E_{\pi(\xi)} \to F_{\pi(\xi)}.$$

This can be expressed in local coordinates for $\xi = \xi_k dx_k$ as

$$\sigma_\xi(\mathscr{P}) = i^m \sum_{|\alpha|=m} a^\alpha(x)\xi^\alpha.$$

In other words, $\sigma \in \Gamma(T^*M, \mathrm{Hom}(\pi^*E, \pi^*F))$.

Definition 2.4.2. *A differential operator $\mathscr{P}$ of order m on a manifold M is called elliptic if for any non-vanishing $\xi \in T_x^*M$*

$$\sigma_\xi(\mathscr{P}) : E_x \to F_x$$

is a vector space isomorphism.

The Riemannian metric induces an isomorphism of vector bundles TM and T^*M. In the sequel, we will identify the tangent and cotangent bundle.

Proposition 2.4.3. *The Spin^c-Dirac operator defined in* (2.24) *is a selfadjoint operator (i.e. $(\mathscr{D}^1\mu, \nu) = (\mu, \mathscr{D}^1\nu)$) with principal symbol given by*

$$\sigma_\xi(\mathscr{D}^1) = ic(\xi) \tag{2.26}$$

and corresponds to the principal symbol of the Dolbeault operator $\sqrt{2}(\overline{\partial} + \overline{\partial}^*)$.

PROOF. Let μ and ν be sections of $E \otimes L$ and assume (without loss of generality) that μ has compact support contained in an open framed neighbourhood with local frame $\{e_j\}_j$. Then

$$\begin{aligned}
(\mathscr{D}^1\mu, \nu) &= \sum_j \int_M h(e_j.\nabla_{e_j}\mu, \nu)dx \\
&= -\sum_j \int_M h(\nabla_{e_j}\mu, e_j.\nu)dx \\
&= -\sum_j \int_M e_j h(\mu, e_j.\nu) - h(\mu, \nabla_{e_j}(e_j.\nu))dx \\
&= -\sum_j \int_M e_j h(\mu, e_j.\nu) + h(\mu, \nabla_{e_j}e_j) + h(\mu, e_j.\nabla_{e_j}\nu)dx
\end{aligned}$$

The divergence of any vector field χ can be defined in terms of the Lie derivative by

$$div(\chi)dx = \mathscr{L}_\chi dx$$

for a volume form dx. Using this, for any j we have that

$$\int_M e_j h(\mu, e_j.\nu)dx = -\int_M h(\mu, e_j.\nu)\mathrm{div}(f_j)dx$$

and the formal adjoint operator of $\mathscr{D}$ is given by

$$(\mathscr{D}^1)^* = \mathscr{D}^1 + \sum_j \mathrm{div}(e_j)c(e_j) + c(\nabla_{e_j} e_j) \tag{2.27}$$

where c is again the Clifford multiplication. For any $x \in M$ one can choose the frame $\{e_j\}_j$ to be a horizontal orthonormal frame field. This makes the second and the third terms of $(\mathscr{D}^1)^*$ vanish. Let $\{e_j\}_j$ be a local orthonormal frame of TM and set

$$\varepsilon_j = \frac{e_j - iJe_j}{2}\ ,\ \overline{\varepsilon}_j = \frac{e_j + iJe_j}{2}.$$

This gives a local frame for $TM \otimes \mathbb{C}$ with respect to which $\mathscr{D}^1$ can be expressed as

$$\mathscr{D}^1 \varphi = \sum_j \{c(\varepsilon_j)\nabla_{\varepsilon_j} + c(\overline{\varepsilon}_j)\nabla_{\overline{\varepsilon}_j}\}.$$

Then, by definition of the principal symbol we obtain

$$\begin{aligned}
\sigma_\xi(\mathscr{D}^1) &= ic(\xi) \\
&= i\sqrt{2}\{e(\xi^{0,1}) - \iota(g^{-1}(\xi)^{0,1})\} \\
&= \sqrt{2}\{\sigma_\xi(\overline{\partial}) + \sigma_\xi(\overline{\partial}^*)\} \quad \text{(see Chapter 2 of [7] or Appendix A)} \\
&= \sigma_\xi(\sqrt{2}(\overline{\partial} + \overline{\partial}^*))
\end{aligned}$$

$\square$

A consequence of Proposition 2.4.3 is that $\mathscr{D}^1$ is an elliptic operator. For a compact manifold M this ensures that $\ker(\mathscr{D}^1_+)$ and $\ker(\mathscr{D}^1_-)$ are finite dimensional.

Chapter 3

K-theory

3.1. The K-group

For a topological space X, the set $(Vect(X), \oplus)$ of isomorphism classes of vector bunles on X endowed with the direct sum operation has the structure of an abelian semigroup. If $\mathscr{M}$ is any abelian semigroup, we can associate to it an abelian group $K(\mathscr{M})$ and a homomorphism $\alpha : \mathscr{M} \hookrightarrow K(\mathscr{M})$ with the following universal property: for any group G and any semigroup homomorphism $f : \mathscr{M} \to G$ there exists a unique group homomorphism $\widetilde{f} : K(\mathscr{M}) \to G$ such the following diagram commutes

$$\begin{tikzcd} \mathscr{M} \arrow[r, "\alpha"] \arrow[d, "f"'] & K(\mathscr{M}) \arrow[dl, "\widetilde{f}"] \\ G & \end{tikzcd} .$$

There are several ways of defining the group $K(\mathscr{M})$:

(1) Consider the free abelian group $F(\mathscr{M})$ generated by the elements of $\mathscr{M}$ and the subgroup $E(\mathscr{M})$ generated by elements of the form $[a]+[b]-([a]\oplus[b])$, where the first addition is in $F(\mathscr{M})$ and the second in $\mathscr{M}$.

The *K-group* can then be defined as the quotient

$$K(\mathscr{M}) := F(\mathscr{M}) \Big/ E(\mathscr{M}). \tag{3.1}$$

(2) Let $\mathscr{M}$ be an abelian monoid and $\Delta : \mathscr{M} \to \mathscr{M} \times \mathscr{M}$ be the diagonal map. The *K-group* $K(\mathscr{M})$ can now be defined as the set of left cosets of $\Delta(\mathscr{M})$ in $\mathscr{M} \times \mathscr{M}$.

Since the interchange of factors in $\mathscr{M} \times \mathscr{M}$ induces an inverse, this quotient is in fact a group. The homomorphism $\alpha : \mathscr{M} \hookrightarrow K(\mathscr{M})$ is defined by the composition

$$\mathscr{M} \hookrightarrow \mathscr{M} \times \{0\} \to \mathscr{M} \times \mathscr{M}/\Delta(\mathscr{M}).$$

For any topological space X, the set $(Vect(X), \oplus)$ of isomorphism classes of vector bundles on X endowed with the direct sum operation is in fact an abelian monoid.

Proposition 3.1.1. *Let $\mathscr{M}$ be an abelian monoid. Then, the K-groups in definitions (1) and (2) are isomorphic.*

PROOF. Denote the group $\mathscr{M} \times \mathscr{M}/\Delta(\mathscr{M})$ defined in (2) by $\mathscr{K}(\mathscr{M})$.

Any semi-group homomorphism $f : \mathscr{M} \to \mathscr{N}$ induces a map

$$\mathscr{K}(f) : \mathscr{M} \to \mathscr{K}(\mathscr{N})$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathscr{M} \arrow[r, "f"] \arrow[d, "\alpha_{\mathscr{M}}"] & \mathscr{N} \arrow[d, "\alpha_{\mathscr{N}}"] \\
\mathscr{K}(\mathscr{M}) \arrow[r, "\mathscr{K}(f)"] & \mathscr{K}(\mathscr{N})
\end{tikzcd}.$$

If $\mathscr{N}$ is an abelian group then, the homomorphism $\alpha_{\mathscr{N}}$ is an isomorphism. Moreover, there exists a group homomorphism $\mathscr{K}(X) \to \mathscr{N}$ that makes the following diagram commute

$$\begin{tikzcd}
\mathscr{M} \arrow[r, "\alpha_X"] \arrow[d, "f"] & \mathscr{K}(X) \arrow[dl, "\alpha_{\mathscr{N}}^{-1} \circ \mathscr{K}(f)"] \\
\mathscr{N} &
\end{tikzcd}.$$

This means in particular that $\mathscr{K}(X)$ is universal with respect to semigroup homomorphisms from $\mathscr{M}$ to abelian groups. ☐

Example 3.1.2. *Consider the monoid $\mathscr{M} := \mathbb{N}$, then $K(\mathscr{M}) = \mathbb{Z}$. Similarily, for the multiplicative monoid $\mathscr{N} := \mathbb{Z} - \{0\}$ we have that $K(\mathscr{N}) = \mathbb{Q} - \{0\}$.*

If $\mathscr{M}$ is a semi-ring then $K(\mathscr{M})$ is a ring. The abelian monoid $Vect(X)$ equipped with the tensor product has the structure of a semi-ring. From now on, we will only be interested in the K-group corresponding to $Vect(X)$ for a paracompact Hausdorff topological space X and will be denoted by $K(X)$. Elements of this group are called *virtual bundles*. Moreover, We will not make a distinction between a vector bundle $E \xrightarrow{\pi} X$ and its isomorphism class in $Vect(X)$.

Example 3.1.3. *Any vector bundle over a point is trivial. Hence, $Vect(pt) \cong \mathbb{N}$ and $K(pt) \cong \mathbb{Z}$.*

For any continuous map $f : X \to Y$, the pullback map of vector bundles $f^* : Vect(Y) \to Vect(X)$ induces a ring morphism

$$f^* : K(Y) \to K(X).$$

An interior tensor product in $K(X)$ is given for E and F in $Vect(X)$ by

$$[E][F] := [E \otimes F] \ \in K(X). \tag{3.2}$$

The projection mappings $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ define an exterior product

$$\begin{aligned} K(X) \otimes K(Y) &\to K(X \times Y) \\ \mu \otimes \nu &\mapsto (p_X^* \mu)(p_Y^* \nu). \end{aligned} \tag{3.3}$$

We will denote both the exterior and interior product of E and F by $[E][F]$. This should be understood from the context and we will always mention the target space.

The following is a consequence of Proposition 3.1.1 and Definition (2) of the K-group.

Corollary 3.1.4. *Every element of $K(X)$ can be expressed as a difference $[E] - [F]$ for some vector bundles E and F on X.*

Lemma 3.1.5. *(Theorem I.6.5,* **[14]***) For any vector bundle E over a compact space X there exists a vector bundle F such that $E \oplus F$ is trivial.*

Proposition 3.1.6. *Suppose X is a compact space. Every element of $K(X)$ is of the form $[H] - [\underline{n}]$ for some vector bundle H and some trivial vector bundle $\underline{n}$ of rank n.*

PROOF. From corollary 3.1.4 every element $x \in \mathscr{K}(X)$ can be written as $[E] - [F]$ for two vector bundles E, F. Denote by F^c the vector bundle in Lemma 3.1.5 that completes F into a trivial vector bundle $\underline{n}$. If we set $H := E \oplus F^c$, we get that

$$\begin{aligned} [E] - [F] &= [E] + [F^c] - ([F] + [F^c]) \\ &= [H] - [\underline{n}] \end{aligned}$$

□

Consider the vector bundles E_i, F_i for $i \in \{1,2\}$ over X. Corollary 3.1.4 implies that

$$\Big([E_1] - [F_1] = [E_2] - [F_2]\Big) \Leftrightarrow \Big(E_1 \oplus F_2 \oplus G \cong E_2 \oplus F_1 \oplus G\Big)$$

for some vector bundle G over X. Write n for the class of the trivial vector bundle $[\underline{n}]$. Then $[E] - n = [F] - m$ is equivalent to

$$E \oplus \underline{m} \oplus G \cong F \oplus \underline{n} \oplus G$$

for some vector bundle G. By completing G to a trivial bundle $\underline{p}$, we obtain that

$$\Big([E]-n=[F]-m\Big) \Leftrightarrow \Big(E\oplus(\underline{m}\oplus\underline{p})\cong F\oplus(\underline{n}\oplus\underline{p})\Big)$$

Hence, for two vector bundles E and F, $[E]=[F]$ if and only if for some trivial vector bundle $\underline{n}$

$$E\oplus\underline{n}\cong F\oplus\underline{n}.$$

This motivates the following definition.

Definition 3.1.7. *Two vector bundles E and F are called stably equivalent if for some $n\in\mathbb{Z}_{\geq 0}$*

$$E\oplus\underline{n}\cong F\oplus\underline{n}$$

This means that $[E]=[F]$ in the K-theory of a topological space X if and only if E and F are stably equivalent.

Remark 3.1.8. *Consider two elements of $K(X)$ given by $[E_1]-[F_1]$ and $[E_2]-[F_2]$. Then, their product in $K(X)$ can be defined by*

$$([E_1]-[F_1])([E_2]-[F_2])=[E_1][E_2]-[E_1][F_2]-[F_1][E_2]+[F_1][F_2].$$

The product of pairs E_i, F_j is given by (3.2). *One can show that this makes $K(X)$ into a commutative ring with identity $\underline{1}$, the trivial line bundle of rank 1.*

Proposition 3.1.9. *If X is the disjoint union of open subspaces $\bigsqcup_{i=1}^{m}X_i$ then,*

$$K(X)\cong K(X_1)\oplus\cdots\oplus K(X_m).$$

PROOF. Every vector bundle $\pi: E\to X$ can be completely described by its restrictions $E|_{X_i}$ for $i\in\{1,\cdots,m\}$. In particular, this means that Vect(X) $\cong Vect(X_1)\times\cdots\times$ Vect(X_m) and that

$$K(X)\cong K(X_1)\oplus\cdots\oplus K(X_m).$$

$\square$

Definition 3.1.10. *The inclusion map of a basepoint $x_0\hookrightarrow X$ induces a homomorphism $K(X)\to\mathbb{Z}$. The kernel of this morphism denoted by $\widetilde{K}(X)$ is called the reduced K-theory of X.*

Similarly, the projection of a given point x_0 in X induces a homomorphism

$$K(x_0)\cong\mathbb{Z}\xrightarrow{p^*}K(X)$$

and a spliting of the short exact sequence

$$0 \to \widetilde{K}(X) \to K(X) \to \mathbb{Z} \to 0.$$

This gives us in turn that

$$K(X) \cong \mathbb{Z} \oplus \widetilde{K}(X).$$

Another possibly more intuitive way of thinking about the reduced K-theory is given by:

Proposition 3.1.11. *(Theorem 3.8, [**13**]) Let E and F be vector bundles on a compact space X. Then $[E] = [F]$ in $\widetilde{K}(X)$ if and only if*

$$E \oplus \underline{n} \cong F \oplus \underline{m} \tag{3.4}$$

for some $n,m \in \mathbb{Z}_{\geq 0}$.

Remark 3.1.12. *Recall that for any vector bundle E over a connected topological space X, the rank of E is a constant function $X \to \mathbb{N}$. This defines a monoid morphism $Vect(X) \to \mathbb{Z}$. The universal property of the K-group ensures the existence of a unique (up to isomorphisms) morphism of abelian groups $rk : K(X) \to \mathbb{Z}$ that makes the following diagram commute*

$$\begin{tikzcd} Vect(X) \arrow[r] \arrow[d, "rk"] & K(X) \arrow[dl, "rk"] \\ \mathbb{Z} & \end{tikzcd}.$$

We call rk the rank mapping. For complex vector bundles, this is in fact a ring morphism and it induces the following split short exact sequence

$$0 \to ker(rk) \to K(X) \xrightarrow{rk} \mathbb{Z} \to 0$$

A right inverse of rk is obtained for instance by the following composition

$$\mathbb{Z} \to Vect(X) \to K(X)$$

$$k \mapsto \ \underline{\mathbb{C}}^k \ \mapsto \ [\underline{\mathbb{C}}^k]$$

Note that $\widetilde{K}(X) \cong ker\Big(rk : K(X) \to \mathbb{Z}\Big)$.

3.2. K as a cohomology theory

We want to use the K-groups to define a cohomology theory and discuss some cohomological properties of the functor K.

A pair (X,Y) where Y is a closed subset of a compact topological space X is called a *compact pair*. We define the *relative K-group* $K(X,Y)$ for a compact pair (X,Y) by

$$K(X,Y) = \widetilde{K}(X/Y).$$

This defines a contravariant functor from the category of compact pairs to the category of compact basepointed spaces. The space Y is considered to be the basepoint of X/Y.

If Y is the empty set then, $X/\emptyset$ is defined as the disjoint union of X with a distinguished point and denoted by X^+,

$$X^+ = X \sqcup \{x_0\}.$$

Observe that for any (non-basepointed) compact space X

$$K(X) \cong \widetilde{K}(X^+) = K(X,\emptyset).$$

For two basepointed compact spaces (X,x_0) and (Y,y_0), we define the *wedge* product $X \vee Y$ by

$$X \vee Y = X \times y_0 \cup x_0 \times Y$$

and the *smash* product $X \wedge Y$ by

$$X \wedge Y = X \times Y \Big/ X \vee Y.$$

We denote by SX the (reduced) suspension of X defined by

$$SX = S^1 \wedge X$$

There is a homeomorphism $S^n \cong S^1 \wedge \cdots \wedge S^1$ so that the n^{th}-iterated suspension S^nX is homeomorphic to $S^n \wedge X$.

Lemma 3.2.1 (Lemma 2.4.2, [**2**])**.** *If (X,Y) is a compact pair, the inclusion mappings $i : Y \hookrightarrow X$ and $j : (X,\emptyset) \hookrightarrow (X,Y)$ induce an exact sequence*

$$K(X,Y) \xrightarrow{j^*} K(X) \xrightarrow{i^*} K(Y) \tag{3.5}$$

Recall that $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$ and $K(Y) \cong \widetilde{K}(Y) \oplus \mathbb{Z}$. Then if Y is a basepointed space, (3.5) yields an exact sequence

$$K(X,Y) \to \widetilde{K}(X) \to \widetilde{K}(Y). \tag{3.6}$$

A natural question to ask now is whether the sequence (3.5) extends to a long exact sequence. It actually does.

Proposition 3.2.2 (Proposition 2.4.4, [**2**])**.** *If (X,Y) is a compact pair of basepointed spaces then* (3.5) *extends to the left to an infinite long exact sequence*

$$\cdots \xrightarrow{i^*} \widetilde{K}(S^2Y) \xrightarrow{\delta} \widetilde{K}(S^1(X/Y)) \xrightarrow{j^*} \widetilde{K}(S^1X) \xrightarrow{i^*} \widetilde{K}(S^1Y) \xrightarrow{\delta} K(X,Y) \xrightarrow{j^*} K(X) \xrightarrow{i^*} K(Y).$$

Definition 3.2.3. *For a compact topological space X with basepoint or a compact pair (X,Y), we define for any $n \geq 0$*

$$\widetilde{K}^{-n}(X) = \widetilde{K}(S^nX)$$

$$K^{-n}(X,Y) = \widetilde{K}^{-n}(X/Y) = \widetilde{K}(S^n(X/Y)).$$

For a compact topological space X, we define for any $n \geq 0$

$$K^{-n}(X) = K^{-n}(X,\emptyset) = \widetilde{K}(S^nX^+).$$

Using the notation above, the sequence 3.2.2 can be expressed as

$$\cdots \xrightarrow{i^*} \widetilde{K}^{-2}(Y) \xrightarrow{\delta} \widetilde{K}^{-1}(X,Y) \xrightarrow{j^*} \widetilde{K}^{-1}(X) \xrightarrow{i^*} \widetilde{K}^{-1}(Y) \xrightarrow{\delta} K^0(X,Y) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(Y). \quad (3.7)$$

Remark 3.2.4.

(1) If Y is a retract of X then, for any $n \geq 0$ we have a split short exact sequence

$$0 \to K^{-n}(X,Y) \to K^{-n}(X) \to K^{-n}(Y) \to 0,$$

hence an isomorphism

$$K^{-n}(X) \cong K^{-n}(X,Y) \oplus K^{-n}(Y).$$

(2) Consider the following projection mappings onto compact basepointed spaces

$$p_X : X \times Y \to X \ ; \ p_Y : X \times Y \to Y.$$

Since X is a retract of (X,Y) and Y is a retract of $X \times Y/X$, the argument in (1) applied twice yields the isomorphism

$$\widetilde{K}^{-n}(X \times Y) \cong \widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(y) \oplus \widetilde{K}^{-n}(X \wedge Y). \quad (3.8)$$

The exact sequence (3.7) extends to an infinite exact sequence in both the left and the right direction by using a fundamental periodicity property of K-theory. Before discussing this, we need to define a pairing for arbitrary $K^n(X)$ groups which is analogous to the exterior product (3.3)

$$m : K(X) \otimes K(Y) \to K(X \times Y).$$

Consider the isomorphism

$$K(X \times Y) \cong K(X) \oplus K(X \times Y, X) \qquad \text{(Remark 3.2.4)}.$$

If $x \in \ker(K(X) \to K(x_0))$ then, $p_X^*(x)$ restricts to zero in $K(X)$. Similarly, if $y \in \ker(K(Y) \to K(y_0))$ then, $p_Y^*(y)$ restricts to zero in $K(Y)$. Hence, $m(x \otimes y)$ restricts to zero in $K(X \vee Y)$ and the pairing

$$m : K^0(X) \otimes K^0(Y) \to K^0(X \times Y)$$

induces a pairing

$$\widetilde{K}^0(X) \otimes \widetilde{K}^0(Y) \to \widetilde{K}^0(X \wedge Y).$$

By replacing X and Y by $S^n X$ and $S^n Y$, and by noting that $S^n(X) \wedge S^m Y = S^{n+m} X \wedge Y$, we obtain a pairing

$$\widetilde{K}^{-n}(X) \otimes \widetilde{K}^{-m}(Y) \to \widetilde{K}^{-n-m}(X \wedge Y).$$

If we replace X and Y by X^+ and Y^+, we get

$$K^{-n}(X) \otimes K^{-m}(Y) \to K^{-m}(X \times Y).$$

More generally, if $(X,Y), (Y,B)$ are pairs then, we have a pairing

$$\widetilde{K}(X/Y) \otimes \widetilde{K}(Y/B) \to \widetilde{K}((X/A) \wedge (Y/B)).$$

In other words,

$$K^0(X,A) \otimes K^0(Y,B) \to K^0((X \times B) \cup (A \times Y)). \tag{3.9}$$

If we define $(X,A) \times (Y,B) := (X \times B) \cup (A \times Y)$ then, we have products

$$K^{-n}(X,A) \otimes K^{-m}(Y,B) \to K^{-n-m}((X,A) \times (Y,B)) \ ; \ n,m \geq 0.$$

This defines a ring structure on K^*. In fact, this product is $\mathbb{Z}_2$-graded.

The simplest form of the Bott Periodicity Theorem states that for any topological space X there is an isomorphism of rings

$$\beta : K(X) \otimes K(S^2) \to K(X \times S^2). \tag{3.10}$$

In fact, $K(S^2) = K^{-2}(pt)$ is isomorphic as a ring to $\mathbb{Z}[H]/(H-1)^2$ where H is the class of the dual tautological line bundle over $S^2 \cong \mathbb{CP}^1$ and the exterior product is an isomorphism

$$m : K(X) \otimes K(S^2) \to K(X \times S^2).$$

We will state a slightly more general version of the Bott Periodicity Theorem. The reader is refered to §2.2 of [**2**].

Theorem 3.2.5. *(Bott Periodicity) For any compact topological space there is an isomorphism*

$$\beta : K^{-n}(X) \to K^{-n-2}(X) \quad \forall n \geq 0$$

given by module multiplication by $H-1 \in K^{-2}(pt)$.

The periodicity isomorphism β commutes with the maps in the long exact sequence (3.7). This yields the following result.

Theorem 3.2.6. *(Theorem 2.4.9* [**2**]*) For any compact topological space X, there is an isomorphism*

$$K^{-n}(X) \to K^{-n-2}(X) \quad \forall n \leq 0$$

given by module multiplication by $H-1 \ \in K^{-2}(pt)$.

The periodicity isomorphism β turns the exact sequence (3.7) into an exact sequence

$$\begin{array}{ccccc} K^0(X,Y) & \longrightarrow & K^0(X) & \longrightarrow & K^0(Y) \\ \uparrow & & & & \downarrow \\ K^1(Y) & \longleftarrow & K^1(X) & \longleftarrow & K^1(X,Y) \end{array} .$$

We can also extend the sequence (3.7) to the right by identifying $K^{-n}(X)$ and $K^{-n-2}(X)$ for all $n \leq 0$. Finally, we define

$$K^*(X) = K^0(X) \oplus K^1(X).$$

3.3. The Atiyah-Singer family index

Let E and F be smooth vector bundles on a compact smooth manifold X. A family of elliptic operators $\mathscr{P}_t : \Gamma(E) \to \Gamma(F)$, $0 \leq t \leq 1$ is said to be *continuous* if the coefficients of the local representations $\mathscr{P}_t = \sum_{|\alpha| \leq m} a_\alpha(x,t) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ are continuous functions of both variables x and t. This can be extended to a continuous family of elliptic operators on a family of vector bundles parametrized by a topological space.

Let A be a Hausdorff topological space and let $\mathrm{Diff}(E;X)$ be the group of diffeomorphisms of the vector bundle E (endowed with the uniform convergence topology). There exists a homomorphism

$$h : \mathrm{Diff}(E;X) \to \mathrm{Diff}(X)$$

which is a mapping onto the diffeomorphism group of the manifold X. Note that its kernel is given by $Aut(E)$ the automorphism group of E.

Definition 3.3.1. *Let $X \hookrightarrow Z \xrightarrow{\pi} A$ be a fiber bundle. A smooth vector bundle $\mathscr{E} \xrightarrow{p} Z$ is a vector bundle over Z such that the composition*

$$\mathscr{E} \underset{p}{\to} Z \underset{\pi}{\to} A$$

defines a fiber bundle over A with fiber a smooth vector bundle E over X and structure group $Diff(E;X)$.

Remark 3.3.2. *In* [**22**] *the fiber bundle $\mathscr{E} \underset{\pi\circ p}{\longrightarrow} A$ is called a continuous family of smooth vector bundles parametrized by A. Provided with this data, one can define the fibre bundle Z as the associated bundle $\mathscr{E} \times_h X$ induced by the homomorphism h.*

Let $\mathscr{P}^m(E,F;X)$ be the space of order m differential operators from E to F. It is a Fréchet space given by the natural norms induced from the C^∞ topology on the bundles of smooth sections $\Gamma(E;X)$ and $\Gamma(F;X)$. Let $\overline{\mathscr{P}}^m(E,F;X)$ be the completion of $\mathscr{P}^m(E,F;X)$ with respect to the family of norms $\| \; \|_s$ defined as the norms of the bounded operators

$$\mathscr{P}_s : H_s(E;X) \to H_{s-m}(F;X)$$

induced on Sobolev spaces. We denote by $\mathfrak{D}$ the closed subgroup of Diff(E;X) $\times$ Diff$(F;X)$ given by pairs (Ψ,Φ) such that

$$h(\Psi) = h(\Phi).$$

The group $\mathfrak{D}$ acts on $\mathscr{P}^m(E,F;X)$ by

$$T \mapsto \Psi^{-1} T \Phi$$

and extends to an action on $\overline{\mathscr{P}}^m(E,F;X)$. This action induces a bundle

$$\mathscr{P}^m(E,F;X) \hookrightarrow \mathscr{P}^m(\mathscr{E},\mathscr{F};Z) \to A$$

associated to the principal $\mathfrak{D}$-bundle $\mathscr{E} \oplus \mathscr{F} \to A$.

Definition 3.3.3. *A (continuous) family of differential operators parametrized by A is a continuous section of $\mathscr{P}^m(\mathscr{E},\mathscr{F};Z)$. It is called an elliptic family if at each point a in A the section evaluates to an elliptic operator.*

Remark 3.3.4. *When $Z = X \times A$, $\mathscr{E} = E \times A$ and $\mathscr{F} = F \times A$ a family is simply a continuous map $A \to \mathscr{P}^m(E,F;X)$ and is called a product family.*

Theorem 3.3.5. *Let $\mathscr{P}$ be an elliptic family in $\mathscr{P}^m(\mathscr{E},\mathscr{F};Z)$. Then there exists a finite set of smooth sections $(s_1,\dots,s_N)$ of $\mathscr{F}$ over Z such that the following map is surjective for all a in A*

$$Q_a : \Gamma(\mathscr{E};Z) \oplus \mathbb{C}^N \longrightarrow \Gamma(\mathscr{F};Z)$$

$$(\chi;\lambda_1,\dots,\lambda_N) \longmapsto \mathscr{P}_a(\chi) + \sum_{i=1}^{N} \lambda_i s_i(\xi).$$

Moreover, the vector spaces ker Q_a form a vector bundle over A and define an element in $K(A)$ given by

$$[ker\ Q] - [\underline{\mathbb{C}}^N] \tag{3.11}$$

that depends only on $\mathscr{P}$.

We are now ready to define the index of the family $\mathscr{P}$ as described by Atiyah and Singer [**4**].

Definition 3.3.6. *The element $[ker\ Q] - [\underline{\mathbb{C}}^N]$ in Theorem 3.3.5 is called the (analytic) index of the family of elliptic operators $\mathscr{P}$ and is denoted by $ind(\mathscr{P})$.*

Remark 3.3.7. *If the vector spaces $ker\mathscr{P}_a$ have constant dimension independently of a in A, then $ker\mathscr{P}$ and $coker\mathscr{P}$ are vector bundles and*

$$ind(\mathscr{P}) = [ker\mathscr{P}] - [coker\mathscr{P}] \ \in K(A). \tag{3.12}$$

Note that the index of a family of elliptic operators is homotopy invariant as a consequence of the homotopy invariance of $K(A)$.

We denote by $\mathfrak{F}$ the space of Fredholm operators on an infinite-dimensional separable Hilbert space $\mathscr{H}$.

Theorem 3.3.8. *For any compact Hausdorff space A, the space of Fredholm operators $\mathfrak{F}$ is a classifying space for the K-theory of A. In otherwords, there exists a natural isomorphism*

$$ind : [A,\mathfrak{F}] \to K(A) \tag{3.13}$$

such that for all continuous maps $f : A \to B$ between compact Hausdorff spaces.

$$f^* \circ ind = ind \circ f^*.$$

Chapter 4

Symplectic topology and geometric quantization

4.1. The group Ham and Hamiltonian fibrations

Consider a compact symplectic manifold (M,ω). The non-degeneracy of the symplectic form induces a non-degenerate pairing between the space of vector fields $\mathfrak{X}(M)$ and the space of 1-forms $\Omega^1(M)$ given by

$$\flat : \mathfrak{X}(M) \xrightarrow{\simeq} \Omega^1(M)$$
$$\xi \mapsto \iota_\xi \omega$$

with inverse

$$\sharp : \Omega^1(M) \to \mathfrak{X}(M)$$
$$\mu \to \mu^\sharp.$$

The *group of symplectomorphisms* Symp(M,ω) consists in the subset of diffeomorphisms $f \in$ Diff(M,ω) of (M,ω) that preserve the symplectic form, that is:

$$f^*\omega = \omega$$

A vector field ξ is called a *symplectic vector field* if it generates a flow that preserves the symplectic structure

$$\mathscr{L}_\xi \omega = 0.$$

Here $\mathscr{L}$ denotes the Lie derivative. By the Cartan formula, this is equivalent to $\iota_\xi \omega$ being a closed form. A symplectic vector field is called *Hamiltonian vector field* if $\iota_\xi \omega$ is exact, which means that there exists a function $H \in C^\infty(M)$ called a *Hamiltonian function* such that

$$\iota_\xi \omega = -dH.$$

Definition 4.1.1. *A symplectic isotopy of a symplectic manifold without boundary (M,ω) is a smooth map $[0,1]\times M \to M$ given by $(t,x) \mapsto f_t(x)$ such that for any $t \in [0,1]$, f_t is a symplectomorphism in Symp(M,ω) and $f_0 = Id$.*

Any symplectic isotopy $\{f_t\}_{t\in[0,1]}$ is generated by a smooth family of symplectic vector fields $\{\xi_t\}_{t\in[0,1]}$, that is:

$$\frac{d}{dt} f_t = \xi_t \circ f_t \text{ and } f_0 = Id.$$

A symplectic isotopy $\{f_t\}_{t\in[0,1]}$ is called *Hamiltonian isotopy* if it is generated by Hamiltonian vector fields $\{\xi_t\}_{t\in[0,1]}$ (i.e. the 1-forms $\xi_t^\flat$ are exact).

For every map f in the connected component of the symplectomorphism group denoted by $\mathrm{Symp}_0(M,\omega)$ there exists a smooth family of symplectomorphisms $f_t \in \mathrm{Symp}(M,\omega)$ such that

$$f_0 = Id \text{ and } f_1 = f \text{ (Theorem 10.1.1, [24]).} \tag{4.1}$$

There exists a unique family of vector fields that generates this isotopy. The *Hamiltonian group* $\mathrm{Ham}(M)$ is a subgroup of $\mathrm{Symp}(M)$ consisting of symplectomorphisms f which admit a Hamiltonian isotopy f_t that satisfies (4.1).

Definition 4.1.2. *A Hamiltonian fibration is a fiber bundle $(M,\omega) \hookrightarrow P \xrightarrow{\pi} B$ whose structure group reduces to the Hamiltonian group Ham(M).*

We will now introduce an important group for the prequantization of a Hamiltonian fibration. The reader may find more details in Chapter 2 of [**6**].

Consider a complex line bundle with connection $\pi : (L,\nabla) \to N$ over a connected manifold N. Let K be the curvature of the connection ∇ and $\sigma := \frac{K}{2\pi\sqrt{-1}}$. The group of diffeomorphisms of N which preserve σ denoted by $\mathrm{Diff}(N,\sigma)$ is a Lie group which acts smoothly on (N,σ) (Proposition 2.4.1,[**6**]). Given a smooth σ-preserving action of a Lie group G on N, we would like to lift this action to a connection preserving action on $L^+ = L - \{0\}$ the complement of the zero section of L. Although the action of G

$$\psi : G \times N \to N$$

preserves the curvature K, in the sense that for any g in G we have that $\psi_g^* K = K$, the pullback bundle $\psi_g^*(L,\nabla)$ does not need to be isomorphic to (L,∇). This means that we need to impose an additional condition. If we denote by H the subgroup of G which preserves the isomorphism class of (L,∇) the following proposition shows that up to a $\mathbb{C}^*$-valued function, H can be lifted to a group of bundle diffeomorphisms of L^+.

Proposition 4.1.3 (Proposition 2.4.5, [**6**])**.** *Assume N is connected. The group $\mathscr{Q}(L^+)$ of connection preserving diffeomorphisms of L^+ is a Lie group. Moreover, there is a short exact sequence*

$$1 \to \mathbb{C}^* \to \mathscr{Q}(L^+) \xrightarrow{p} H \to 1 \tag{4.2}$$

where $p(\hat{h}) = h$ for $\hat{h} \in \mathscr{Q}(L^+)$ and $h \in H$ if $\hat{h}$ is a lift of h.

We are intersted in Hermitian line bundles (L,∇) over symplectic manifolds (M,ω) for which the symplectic form ω pulls back to $\frac{-K}{2\pi\sqrt{-1}}$ (see [**10**], §2). In this case, the short exact sequence (4.2) becomes

$$1 \to S^1 \to \mathscr{Q}(\mathscr{P}) \to \mathrm{Ham}(M,\omega) \to 1. \tag{4.3}$$

The group $\mathscr{Q}(\mathscr{P})$ is the automorphism group of a principal S^1-bundle $\mathscr{P}$ that will be defined shortly.

4.2. Prequantization

Proposition 4.2.1 (Proposition 15.1, [**7**])**.** *For a symplectic manifold (M,ω), the integrality condition of the closed 2-form ω is equivalent to the existence of a smooth Hermitian line bundle $L \underset{p}{\to} M$ with a unitary connection whose curvature K satisfies:*

$$K = (-2\pi\sqrt{-1})\; p^*\omega \tag{4.4}$$

Definition 4.2.2. *A prequantization (or prequantum space) of a symplectic manifold (M,ω) is given by:*

(1) A Hermitian line bundle (L,∇) over (M,ω) with a unitary connection.

(2) A curvature form that satisfies the relation

$$R(\nabla) = -2\pi\sqrt{-1}\omega \tag{4.5}$$

We denote a given prequatization of (M,ω) by $\widehat{M}$.

Proposition 4.2.1 implies that the relation 4.5 is equivalent to the vanishing of the class of the symplectic form ω in $H^2(M,\mathbb{R})\Big/H^2(M,\mathbb{Z})$. Symplectic manifolds with this property are called *quantizable*. From now on if not mentionned otherwise all symplectic manifolds will be of this type.

Definition 4.2.3. *The quantomorphism group $\mathscr{Q}(\widehat{M})$ is defined as the identity component of the group Diff(L,∇) of diffeomorphisms of the total space of $L \to M$ that are bundle maps, connection preserving and unitary on the fibers.*

A prequantum space of a symplectic manifold (M,ω) can be described alternatively as a principal S^1-bundle $p : \mathscr{P} \to M$ with connection form α such that

$$d\alpha = -p^*\omega.$$

In this setting, the quantomorphism group $\mathscr{Q}(\mathscr{P},\alpha)$ is defined as the identity component of those diffeomorphisms of $\mathscr{P}$ that preserve the connection form α. In other words, $\mathscr{Q}(\mathscr{P},\alpha) = \mathrm{Cont}_0(\mathscr{P},\alpha)$ the identity component of the group of strict contactomorphisms of $(\mathscr{P},\alpha)$.

This group enters the central S^1-extension of $\mathrm{Ham}(M,\omega)$ (4.3) mentioned in the previous section.

We aim now at defining *prequantization* for a Hamiltonian fibration.

Definition 4.2.4. *A prequantum fibration consists in a Hamiltonian fibration $(M,\omega) \hookrightarrow \underline{M} \xrightarrow{\pi} B$ together with a prequantum lift, which is given by the following:*

(1) A Hermitian line bundle $\underline{L} \to \underline{M}$ with a continuous family of unitary connections ∇_b on the fibers over B.

(2) On each fiber $\underline{L}_b \to \underline{M}_b$ the curvature satisfies the relation

$$R(\nabla_b) = -2\pi\sqrt{-1}\,\omega_b$$

where $\{\omega_b\}_{b\in B}$ denotes the fiberwise symplectic forms.

Moreover, we require the structure group $\mathscr{Q}(\widehat{\underline{M}})$ of $\underline{L} \to \underline{M}$ to be compatible with the Hamiltonian group $Ham(M,\omega)$.

4.3. Quantization

We will discuss in this section quantization of prequantum fibrations as defined in [**28**].

As seen in Chapter 2.3, given a symplectic manifold (M,ω) with prequantum space (L,∇^L), a choice of ω-compatible almost complex structure J on M determines canonically a Dirac bundle

$\mathscr{E} := L \otimes E$, with $E = \Lambda(T^*M)^{(0,1)}$ and a $Spin^c$-Dirac operator $D^1(L,J)$. A quantization of (M,ω) due to Bott [**5**] is given by the index

$$\text{ind } D^1_+(L,J) = [\ker D^1_+(L,J)] - [\text{coker } D^1_+(L,J)] \in \mathbb{Z}. \tag{4.6}$$

Using the Atiyah-Singer family index, we can extend this definition of quantization to a family of symplectic manifolds.

Consider a Hamiltonian fibration $\underline{M} \to B$ with a prequantum lift

$$\underline{\widehat{M}} := (\underline{M}, \underline{L}, \nabla, \{\omega_b\}_{b\in B}).$$

The associated bundle $\mathscr{J}_{\underline{M}}$ with structure group Ham(M) and fibers given for all b in B by the space of ω_b-compatible almost complex structures has contractible fibers. Therefore $\mathscr{J}_{\underline{M}}$ has a global section. Choose such a section $\{j_b\}_{b\in B}$, then by the previous discussion this determines canonically a family of Dirac operators on the fibers. Consider the dual bundle $(T^*)^{vert}\underline{M}$ of the vertical tangent bundle to the fibration $\underline{M} \to B$, then define the following Clifford module bundle fibrations

$$E^+ := \Lambda^{even}(T^*)^{vert}\underline{M}^{(0,1)} \;;\; E^- := \Lambda^{odd}(T^*)^{vert}\underline{M}^{(0,1)}$$

and finally the Atiyah-Singer family $\mathscr{D}^k_+(\underline{\widehat{M}}, \{j_b\}_{b\in B})$ of operators

$$D^k_+(\underline{\widehat{M}}_b, j_b) : \Gamma(\mathscr{E}^+_b) \to \Gamma(\mathscr{E}^-_b)$$

where $\mathscr{E}^{\pm}_b := E^{\pm} \otimes \underline{L}^k$. As seen in Chapter 3.3, we can associate an index in $K(B)$ to this family of operators. Moreover, if $\mathscr{H}$ is a separable (infinite-dimensional) Hilbert space, just as in Theorem 3.3.8 there is an element in $[B, \mathfrak{F}(\mathscr{H})]$ corresponding to this index which we shall define explicitly following Savelyev and Shelukhin [**28**].

Let $\mathscr{H}_0$ and $\mathscr{H}_1$ be the fiberwise completion of the bundles of smooth sections $\Gamma(\mathscr{E}^+)$ and $\Gamma(\mathscr{E}^-)$ with respect to a suitable Sobolev norm. For any fixed integer k the family of Dirac operators $\mathscr{D} := \mathscr{D}^k_+(\underline{\widehat{M}}, \{j_b\}_{b\in B})$ induces a Fredholm map between Hilbert-bundles over B

$$[\mathscr{D}] : \mathscr{H}_0 \to \mathscr{H}_1. \tag{4.7}$$

The structure groups of the bundles $\mathscr{H}_0$ and $\mathscr{H}_1$ are contractible by Kuiper's theorem [**18**], hence they admit a trivialization. Any choice of trivialization being unique up to homotopy (*homotopy canonical*), the trivializations $\phi_0 : \mathscr{H}_0 \to \mathscr{H} \times B$ and $\phi_1 : \mathscr{H}_1 \to \mathscr{H} \times B$ define a map

$$\begin{aligned} f_{\phi_1 \circ [\mathscr{D}] \circ \phi_0^{-1}} : B &\to \mathfrak{F}(\mathscr{H}) \\ b &\mapsto \Big((\phi_1 \circ [\mathscr{D}] \circ \phi_0^{-1})_b : \mathscr{H} \to \mathscr{H} \Big). \end{aligned}$$

Denote the homotopy class of this map by

$$H^k(\underline{\widehat{M}}) := [f_{\phi_1 \circ [\mathscr{D}] \circ \phi_0^{-1}}] \in [B, \mathfrak{F}(\mathscr{H})]. \tag{4.8}$$

The family index defined by $H^k(\underline{\widehat{M}})$ does not depend on the choice of trivializations ϕ_0 and ϕ_1.

Proposition 4.3.1. *(Proposition 3.2,* [**28**]*)*
The family analytic index $H^k(\underline{\widehat{M}})$ of the Atiyah-Singer family $\mathscr{D}_+^k(\underline{\widehat{M}}, \{j_b\}_{b \in B})$ is invariant under isomorphisms of prequantum fibrations and is independent of the choice of the family of almost complex structures $\{j_b\}_{b \in B}$.

Chapter 5

Hamiltonian K-theoretic invariants

5.1. Proof of Theorem B and Theorem C

Recall that the Dirac operator $D^1(L,J)$ coincides with the Dolbeault operator $\sqrt{2}\left(\overline{\partial}_{L\otimes E}+\overline{\partial}^*_{L\otimes E}\right)$ for Kähler manifolds. Then, from the Dolbeault theorem, $\ker D^1(L,J)$ corresponds to $H^*(M,\mathscr{O}(L))$, the cohomology of the sheaf of holomorphic sections of L (Appendix A).

Given Hermitian complex vector spaces $V_1,\cdots,V_r$ endowed with the standard Kähler structure, we can form the exterior product of the corresponding dual tautological line bundles $L=O_{\mathbb{P}(V_1)}(1)\boxtimes\cdots\boxtimes O_{\mathbb{P}(V_r)}(1)$ over $\mathbb{P}(V_1)\times\cdots\times\mathbb{P}(V_r)$ together with the induced connection $\nabla^{ch,r}$.

Lemma 5.1.1. *Let $(L,\nabla^{ch,r})$ be the prequantum lift of $\mathbb{P}(V_1)\times\cdots\times\mathbb{P}(V_r)$. For $J:=j_{std}\times\cdots\times j_{std}$, we have that*

$$ker(D^1_-(L,J))\cong coker(D^1_+(L,J))=0.$$

and

$$ker(D^1_+(L,J))=V_1^*\otimes\cdots\otimes V_r^*.$$

PROOF. For $r=1$, $L=O_{\mathbb{P}(V_1)}(1)$ defined over the complex projective space $\mathbb{P}(V_1)$ and

$$\ker D^1(L,J)=H^*(M,\mathscr{O}(L)).$$

As a consequence of the Kodaira-Nakano Vanishing Theorem (Chapter 1, [**8**]),

$$H^q(\mathbb{P}^n,\mathscr{O}(1))=0\qquad\forall\ 1\le q\le n-1.$$

It is also known from general results in algebraic geometry that the higher order cohomology groups vanish and that the sheaf cohomology of global holomorphic sections of L can be identified with the dual of V_1

$$H^0(\mathbb{P}^n, \mathscr{O}(1)) \cong V_1^*.$$

The general case follows from the Künneth formula (Proposition 9.2.4, [**16**]). Indeed, for $M = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ and for all $k \geq 0$

$$H^k(M, \mathscr{O}(L)) = \bigoplus_{i_1+\cdots+i_r=k} H^{i_1}(\mathbb{P}^{n_1}, \mathscr{O}(1)) \otimes \cdots \otimes H^{i_r}(\mathbb{P}^{n_r}, \mathscr{O}(1)).$$

Then by using the same arguments as before, the only non-vanishing term corresponds to

$$H^0(M, \mathscr{O}(L)) = V_1^* \otimes \cdots \otimes V_r^*.$$

□

Lemma 5.1.2. *Let* $(M,\omega) \hookrightarrow \underline{M} \xrightarrow{\pi} B$ *be a Hamiltonian fibration. Then, for any two prequantum lifts* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *over* $\underline{M}$*, there exists a line bundle* $L \to B$ *such that*

$$\mathscr{L}_1 = \mathscr{L}_1 \otimes \pi^* L. \tag{5.1}$$

PROOF. Any Hamiltonian fibration is completely determined by an open cover $\{U_\alpha\}_\alpha$ and a family of transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Ham}(M,\omega)$$

satisfying the cocycle condition.

Let $G^{(i)}_{\alpha\beta}(x)$ be the lift of $g_{\alpha\beta}(x)$ to an automorphism of $\mathscr{L}_i$ for $i = 1,2$.

The short exact sequence

$$1 \to S^1 \to \mathscr{Q}(\widehat{\underline{M}}) \to \mathrm{Ham}(M,\omega) \to 1$$

implies that in appropriate coordinate neighbourhoods $\{U_\alpha\}_\alpha$, the mappings $G^{(1)}_{\alpha\beta} \circ (G^{(2)}_{\alpha\beta})^{-1}(y)$ defined on $\pi^{-1}(U_\alpha \cap U_\beta)$ are smooth S^1-valued invertible functions satisfying the cocycle condition. This defines a Hermitian line bundle on $\underline{M}$ (Theorem 2.1.8, [**6**]) and since these transition functions are constant on the fibers of $\underline{M}$, this bundle is induced from a line bundle L on B. Then

$$\mathscr{L}_1 = \mathscr{L}_2 \otimes \pi^*(L).$$

□

Given vector bundles $E_1 \to B_1, \cdots, E_r \to B_r$ of finite dimensions $n_1, \cdots, n_r$ (respectively), the Cartesian product $\underline{M} := \mathbb{P}(E_1) \times \cdots \times \mathbb{P}(E_r)$ is a Hamiltonian fibration over $B := B_1 \times \cdots \times B_r$. Consider the prequantum lift

$$\widehat{\underline{M}} = O_{\mathbb{P}(E_1)}(1) \boxtimes \cdots \boxtimes O_{\mathbb{P}(E_r)}(1) \to \mathbb{P}(E_1) \times \cdots \times \mathbb{P}(E_r).$$

This is a vector bundle over $\underline{M}$ and a family of dual tautological line bundles parametrized by B when viewed as a fiber bundle. For any $b = (b_1, \cdots, b_r)$ in B

$$\widehat{\underline{M}}_b = O_{\mathbb{P}(E_1)_{b_1}}(1) \boxtimes \cdots \boxtimes O_{\mathbb{P}(E_r)_{b_r}}(1) \to \mathbb{P}(E_1)_{b_1} \times \cdots \times \mathbb{P}(E_r)_{b_r}.$$

is isomorphic to

$$\widehat{\underline{M}}_b = \mathrm{proj}_1^* O_{\mathbb{P}^{n_1}}(1) \otimes \cdots \otimes \mathrm{proj}_r^* O_{\mathbb{P}^{n_r}}(1) \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}.$$

We equip each tautological line bundle with the Chern connection and each fiber $\widehat{\underline{M}}_b$ with the corresponding induced connection. As discussed in the previous section, we can associate to this prequantum fibration a family of Dirac operators $\mathscr{D}^1(\widehat{\underline{M}}, J = \{j_b\}_{b\in B})$.

PROOF. (Theorem B)
To avoid cumbersome notation, we will prove the theorem for $r = 2$. The proof for an arbitrary $r \in \mathbb{N}$ follows exactly the same steps.
The operator $\mathscr{D}^1_b(\widehat{\underline{M}}, J)$ is an elliptic operator in $\mathscr{P}^1\Big(\mathbb{P}(E_{1,b_1}) \times \mathbb{P}(E_{2,b_2}), O_{\mathbb{P}(E_{1,b_1})}(1) \boxtimes O_{\mathbb{P}(E_{2,b_2})}(1) \otimes \Lambda(T^*)^{vert}\Big(\mathbb{P}(E_{1,b_1}) \times \mathbb{P}(E_{2,b_2})\Big)^{(0,1)}\Big)$. For all b in B, Lemma 5.1.1 shows that

$$\ker \mathscr{D}^1_{+,b}(\widehat{\underline{M}}, J) \cong E^*_{1,b_1} \otimes E^*_{2,b_2}.$$

Since for all $b \in B$, $\ker \mathscr{D}^1_{+,b}(\widehat{\underline{M}}, J)$ is a vector space and $\ker \mathscr{D}^1_{-,b}(\widehat{\underline{M}}, J)$ vanishes, ind $\mathscr{D}^1_+(\widehat{\underline{M}}, J)$ is the isomorphism class of a vector bundle. The kernel of $\mathscr{D}^1_+(\widehat{\underline{M}}, J)$ defines a sub-bundle of the space of sections of the vector bundle $\mathscr{E}^+ := O_{\mathbb{P}(E_1)}(1) \boxtimes O_{\mathbb{P}(E_2)}(1) \otimes \Lambda(T^*)^{vert}\Big(\mathbb{P}(E_1) \times \mathbb{P}(E_2)\Big)^{(0,1)}$. There is a morphism of vector bundles $E_1^* \boxtimes E_2^* \to \Gamma(\mathscr{E}^+)$ which reduces to an isomorphism on the fibers of $\ker \mathscr{D}^1_+(\widehat{\underline{M}}, J)$. Hence,

$$E_1^* \boxtimes E_2^* \cong \ker \mathscr{D}^1_+(\widehat{\underline{M}}, J) \quad \text{(Proposition 1.2.10).}$$

We conclude that

$$\mathrm{ind}\, \mathscr{D}^1_+(\widehat{\underline{M}}, J) = [E_1^* \boxtimes E_2^*] = [E_1^*][E_2^*] \in \mathrm{K}(B_1 \times B_2).$$

Here, the term after the second equality stands for the exterior product in K-theory.

Observe that a vector bundle is stably trivial if and only if its dual bundle is stably trivial. The condition $[E_1 \boxtimes E_2] \neq [\underline{\mathbb{C}}^{n_1 n_2}]$ implies then that the fibration $\widehat{\underline{M}}$ is not trivial as a prequantum fibration.

We will now prove that the condition $[E_1 \boxtimes E_2][\mu] \neq [\underline{\mathbb{C}}^{n_1 n_2}]$ for all line bundles μ on $B_1 \times B_2$ implies that $\mathbb{P}(E_1) \times \mathbb{P}(E_2)$ is not trivial as a Hamiltonian fibration.

From Lemma 5.1.2, the central extension of the Hamiltonian group

$$1 \to S^1 \to \mathscr{Q}(O_{\mathbb{P}(E_1)}(1) \boxtimes O_{\mathbb{P}(E_2)}(1)) \to \mathrm{Ham}(\mathbb{P}(E_1) \times \mathbb{P}(E_2)) \to 1$$

implies that the prequantum lift of $\underline{M}$ is unique up to a twist by a line bundle.
Over each point $b \in B$, an arbitrary prequantum fibration $\mathscr{L}$ over $\mathbb{P}(E_1) \times \mathbb{P}(E_2)$ takes the following form

$$\left(O_{\mathbb{P}(E_{1,b_1})}(1) \boxtimes O_{\mathbb{P}(E_{2,b_2})}(1)\right) \otimes \underline{L_b} \to \mathbb{P}(E_{1,b_1}) \times \mathbb{P}(E_{2,b_2}) \to b = (b_1, b_2)$$

Here $L \to B$ is a line bundle. Therefore, $\underline{L_b}$ is isomorphic to the trivial line bundle $\underline{\mathbb{C}}^1$ over $\mathbb{P}(E_{1,b_1}) \times \mathbb{P}(E_{2,b_2})$. The kernel of the operator $\mathscr{D}^1_{+,b}(\mathscr{L}, J_{\mathscr{L}})$ becomes

$$\ker(\mathscr{D}^1_{+,b}(\mathscr{L}, J_{\mathscr{L}})) \cong \mathscr{O}\left(\left(O_{\mathbb{P}(E_{1,b_1})}(1) \boxtimes O_{\mathbb{P}(E_{2,b_2})}(1)\right) \otimes \underline{L_b}\right) \cong E^*_{b_1} \otimes E^*_{b_2} \otimes L_b \tag{5.2}$$

The family of vector spaces $E^*_{b_1} \otimes E^*_{b_2} \otimes L_b$ parametrized by B defines a subbundle of the space of sections of the Dirac bundle $O_{\mathbb{P}(E_1)}(1) \boxtimes O_{\mathbb{P}(E_2)}(1) \otimes \mathscr{E}^+$ isomorphic to $\left(E_1^* \boxtimes E_2^*\right) \otimes L \to B$.

Assume for a contradiction that $[E_1 \boxtimes E_2][\mu] \neq [\underline{\mathbb{C}}^{n_1 n_2}]$ for all line bundles μ on $B = B_1 \times B_2$ (or equivalently that $[E_1^* \boxtimes E_2^*][\mu] \neq [\underline{\mathbb{C}}^{n_1 n_2}]$ for any $\mu \ \in Vect(B)$) and that $\mathbb{P}(E_1) \times \mathbb{P}(E_2)$ is a trivial Hamiltonian fibration. This means that any prequantum lift $\mathscr{L}$ can be realized as a trivial fibration over B twisted by a line bundle $L \to B$ and that ind $\mathscr{D}^1_+(\mathscr{L}, J_L)$ corresponds to the class of a trivial vector bundle in $K(B)$. Then from the isomorphism (5.2) and the discussion above, we have that

$$[E_1^* \boxtimes E_2^*][L] = [\underline{\mathbb{C}}^{n_1 n_2}] \ \in K(B_1 \times B_2).$$

This implies that $\left(E_1 \boxtimes E_2\right) \otimes L^*$ is stably trivial and contradicts the assumptions. □

PROOF. (Theorem C)
Given vector bundles $E_1, \cdots, E_r$ of finite dimensions $n_1, \cdots, n_r$ (respectively) over B, the fiber

product $\underline{M} := \mathbb{P}(E_1) \times_B \cdots \times_B \mathbb{P}(E_r)$ is a Hamiltonian fibration over B. Consider the prequantum lift defined by

$$\widehat{\underline{M}} = O_{\mathbb{P}(E_1)}(1) \otimes \cdots \otimes O_{\mathbb{P}(E_r)}(1) \to \mathbb{P}(E_1) \times_B \cdots \times_B \mathbb{P}(E_r).$$

We can associate to this prequantum fibration a family of Dirac operators $\mathscr{D}^1_+(\widehat{\underline{M}}, J = \{j_b\}_{b \in B})$. The proof follows exactly the same steps as for Theorem 1 but now for $r = 2$, we have the isomorphism of vector bundles

$$\ker \mathscr{D}^1_+(\widehat{\underline{M}}, J) \cong E_1^* \otimes E_2^*$$

and

$$\operatorname{ind} \mathscr{D}^1_+(\widehat{\underline{M}}, J) = [E_1^* \otimes E_2^*] = [E_1^*][E_2^*] \in K(B).$$

Here the exterior product is replaced by the interior product in K-theory.
Then the condition

$$[E_1 \otimes E_2] \neq [\underline{\mathbb{C}}^{n_1 n_2}] \in K(B)$$

implies that the fiber product $\mathbb{P}(E_1) \times_B \mathbb{P}(E_2)$ is not trivial as a prequantum fibration. To prove that $\mathbb{P}(E_1) \times_B \mathbb{P}(E_2)$ is not a trivial Hamiltonian fibration we can argue again by contradiction. Assume $[E_1 \otimes E_2][\mu] \neq [\underline{\mathbb{C}}^{n_1 n_2}]$ for all line bundles μ over B and that $\underline{M}$ is a trivial Hamiltonian fibration. Then Lemma 5.1.2 implies that any prequantum lift of $\underline{M}$ can be expressed as a trivial fibration up to a line bundle L and that

$$[E_1 \otimes E_2][L] = \underline{\mathbb{C}}^{n_1 n_2} \ \in K(B),$$

which contradicts the assumptions. $\square$

5.2. Examples

Example 5.2.1. [**28**] *Let $B = \mathbb{RP}^{n-1}$ for $n = 7$ or 8 and $L = \tau^1_{\mathbb{RP}^{n-1}} \otimes_{\mathbb{R}} \mathbb{C}$, where $\tau^1_{\mathbb{RP}^{n-1}}$ denotes the tautological line bundle on B. Proposition 2.7.7* [**4**] *or Corollary 6.47 in Chapter 6 of* [**14**] *implies that $\widetilde{K}(B) \cong \mathbb{Z}/8\mathbb{Z}$ with generator $H = [L] - 1 \in K(B)$. Consider the vector bundle $F = L \otimes \underline{\mathbb{C}}^m$ for $m \in \mathbb{N}$.*

$$[F] = [L \otimes \underline{\mathbb{C}}^m] = m[L \otimes \underline{\mathbb{C}}^1] = m[L] \in K(B).$$

Case m=4

If $[F] = 4[L]$ is stably trivial then we have the following

$$[F] = 4 \ \in K(B) \Leftrightarrow 4(H+1) = 4 \in K(B).$$

Projecting onto reduced K-theory, we get that

$$[F] = 4 \ \in K(B) \Rightarrow 4H = 0 \in \widetilde{K}(B).$$

This is a contradiction because since H is a generator of $\mathbb{Z}/8\mathbb{Z}$, it has order 8 in $\widetilde{K}(B)$. Hence F is not stably trivial which means by Theorem C that $\widehat{\mathbb{P}(F)}$ is not trivial as a prequantum fibration. The same result holds for $m \notin 8\mathbb{N}$ however, the projectivization of F is a trivial fibration for all m because

$$\mathbb{P}(F) = \mathbb{P}(L \otimes \underline{\mathbb{C}}^m) \cong \mathbb{P}(\underline{\mathbb{C}}^m).$$

Example 5.2.2. *Let $B = B_1 \times B_2$ for $B_1 = B_2 = \mathbb{RP}^{n-1}$, $n = 7$ or 8 and let $L = \tau^1_{\mathbb{RP}^{n-1}} \otimes_{\mathbb{R}} \mathbb{C}$. Consider the vector bundles $F_1 := proj_1^* F$ and $F_2 := proj_2^* F$. Here $F = L \otimes \underline{\mathbb{C}}^m$ for $m \in \mathbb{N}$ and the maps $proj_1$ and $proj_2$ denote the projection mappings onto B_1 and B_2, respectively. Define the $2m$-dimensional vector bundle $E = F_1 \oplus F_2$. Note that the Picard group $Pic(B)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and is generated by the line bundles $L_i = proj_i^* L$ for $i = 1$ or 2. Moreover, from Remark 3.2.4 we have that*

$$\widetilde{K}(B) \cong \widetilde{K}(B_1) \oplus \widetilde{K}(B_2) \oplus \widetilde{K}(B_1 \wedge B_2) \cong \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \widetilde{K}(B_1 \wedge B_2).$$

We denote by H_i the virtual bundles $proj_i^* H$ in $K(B)$ or $\widetilde{K}(B)$ for $i = 1$ or 2 and $H = [L] - 1$.

$$\begin{aligned} [E] &= [F_1] + [F_2] \ \in K(B) \\ &= m\Big([L_1] + [L_2]\Big) \\ &= m\Big(H_1 + H_2 + 2\Big). \end{aligned}$$

By projecting to reduced K-theory, we get that

$$[E] = m\Big(H_1 + H_2\Big) \ \in \widetilde{K}(B)$$

where $[E]$, H_1 and H_2 denote classes in $\widetilde{K}(B)$ (by abuse of notation). Since H_1 and H_2 generate $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$, $[E] \neq 0$ in $\widetilde{K}(B)$ for any non-zero element m in $\mathbb{Z}/8\mathbb{Z}$. This gives examples of non-trivial prequantum fibrations. To find examples of non-trivial Hamiltonian fibrations, we compute the class of $E \otimes L_i$ in K-theory:

$$\begin{aligned} [E \otimes L_1] &= m\Big(\big([L_1] + [L_2]\big)\Big)[L_1] \ \in K(B) \\ &= m\Big(1 + [L_1][L_2]\Big). \end{aligned} \tag{5.3}$$

Assume for a contradiction that $E \otimes L_1$ is stably trivial, then

$$\begin{aligned} [E \otimes L_1] = 2m &\Leftrightarrow m[L_1][L_2] = m \ \in K(B) \\ &\Leftrightarrow m\Big([L_1] - [L_2]\Big) = 0. \end{aligned}$$

The second line is obtained by multiplying both sides of the equality by $[L_2]$. Now, passing to the reduced K-theory we get that

$$m\Big(H_1 - H_2\Big) = 0 \ \in \widetilde{K}(B)$$

which is a contradiction for all $m \neq 0 \in \mathbb{Z}/8\mathbb{Z}$ by the same argument as before. This means that $[E \otimes L_1]$ is not stably trivial for all $m \neq 0 \in \mathbb{Z}/8\mathbb{Z}$. The same result holds if we replace L_1 by L_2. By multiplying both sides of the equality (5.3) *by $[L_2]$ in $K(B)$ and by projecting onto reduced K-theory, we obtain that if $E_1 \otimes L_1 \otimes L_2$ is stably trivial then,*

$$m\Big(H_1 + H_2\Big) = 0 \ \in \widetilde{K}(B)$$

which implies that $m = 0 \in \mathbb{Z}/8\mathbb{Z}$.

Hence, for any line bunle μ over B and $m \notin 8\mathbb{N}$, $[E \otimes \mu]$ is not stably trivial and by Theorem A, the projectivization of E is not trivial as a Hamiltonian fibration.

Example 5.2.3. *Using the same setting as in Example 5.2.2, we want to find examples of non-trivial Hamiltonian fibrations with fibers given by products of complex projective spaces.*

We compute:

$$\begin{aligned} [E^2] &= \Big(m\Big([L_1] + [L_2]\Big)\Big)^2 \ \in K(B) \\ &= m^2\Big([L_1]^2 + [L_2]^2 + 2[L_1][L_2]\Big) \\ &= 2m^2\Big(1 + [L_1][L_2]\Big). \end{aligned}$$

Then for i=1 or 2,

$$[E^2 \otimes L_i] = 2m^2\Big([L_1] + [L_2]\Big) \ \in K(B) \tag{5.4}$$

and

$$\begin{aligned} [E^2 \otimes L_1 \otimes L_2] &= 2m^2\Big(1 + [L_1][L_2]\Big) \ \in K(B) \\ &= [E^2]. \end{aligned}$$

Assume $m = 3$ or 5. Projecting onto the reduced K-theory, (5.4) *yields*

$$[E^2 \otimes L_i] = 2\Big(H_1 + H_2\Big) \ \in \widetilde{K}(B).$$

As shown in Example 5.2.2 since H_1 and H_2 are generators of $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$, $[E^2 \otimes L_i]$ and $[E^2 \otimes L_1 \otimes L_2]$ are non-vanishing in $\widetilde{K}(B)$. This means that for any line bundle ν over B, $E^2 \otimes \nu$ is not stably trivial and by Theorem C, this gives examples of non-trivial Hamiltonian fibrations. For instance, the following fiber bundle is a non-trivial Hamiltonian fibration

$$\mathbb{CP}^9 \times \mathbb{CP}^9 \hookrightarrow \mathbb{P}(G) \times_B \mathbb{P}(G) \to \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1},\ n\text{=7 or 8}$$

for $G := (L \otimes \underline{\mathbb{C}}^5) \boxplus (L \otimes \underline{\mathbb{C}}^5)$ and $L = \tau^1_{\mathbb{RP}^{n-1}} \otimes_{\mathbb{R}} \mathbb{C}$.

Example 5.2.4. *We will now use Theorem B to find other examples of non-trivial Hamiltonian fibrations. Let $B_1 = B_2$ be a projective space $\mathbb{RP}^{n-1}$ for $n = 7, 8$.*
Consider again the vector bundles $F^{(m_i)} = L \otimes \underline{\mathbb{C}}^{m_i}$ of rank m_i and set

$$F_1 = proj_1^* F^{(m_1)},\ F_2 = proj_2^*\Big(F^{(m_2)} \oplus \underline{\mathbb{C}}^1\Big).$$

Then

$$\begin{aligned}[F_1 \otimes F_2] &= m_1[L_1]\Big(m_2[L_2] + 1\Big) \ \in K(B_1 \times B_2)\\ &= m_1 m_2 [L_1][L_2] + m_1[L_1].\end{aligned}$$

Let $F := F_1 \otimes F_2$ then,

$$[F_1 \otimes F_2][L_1] = m_1 m_2 [L_2] + m_1 \ \in K(B_1 \otimes B_2)$$

and

$$[F_1 \otimes F_2][L_2] = m_1 m_2 [L_1] + m_1[L_1][L_2] \ \in K(B_1 \otimes B_2).$$

Assume $m_1 = 1 = m_2$ in $\mathbb{Z}/8\mathbb{Z}$. By projecting onto $\widetilde{K}(B_1 \times B_2)$ we obtain

$$[F_1 \otimes F_2][L_1] = H_2 \ \in \widetilde{K}(B_1 \times B_2) \tag{5.5}$$

and

$$[F_1 \otimes F_2][L_1][L_2] = H_2 \ \in \widetilde{K}(B_1 \times B_2). \tag{5.6}$$

Moreover, note that

$$m_1[L_1](m_2 + [L_2]) = m_1(m_2+1) \Leftrightarrow (m_1[L_2] - m_1(m_2+1)[L_1]) + m_1 m_2 = 0 \ \in K(B_1 \times B_2). \tag{5.7}$$

Finally, if $F_1 \otimes F_2 \otimes L_2$ is stably trivial, then projecting to reduced K-theory yields a contradiction:

$$H_2 - 2H_1 = 0 \in \widetilde{K}(B_1 \times B_2). \tag{5.8}$$

From (5.5), (5.6) *and* (5.8), *we deduce that for any line bundle ν over $B_1 \times B_2$,*

$$[F_1 \otimes F_2][\nu] \neq 0 \in \widetilde{K}(B_1 \times B_2).$$

Hence, by Theorem B the product bundle $\mathbb{P}(F^{(m_1)}) \times \mathbb{P}\left(F^{(m_2)} \oplus \underline{\mathbb{C}}^1\right)$ is a non-trivial Hamiltonian fibration.

5.3. Proof of Corollary D

We will discuss here an application of Theorem B which provides more examples of non-trivial Hamiltonian fibrations.

Let $E_1 \to B_1$ and $E_2 \to B_2$ be complex vector bundles of rank n_1 and n_2, respectively. Suppose that for all line bundles μ over B_1, the vector bundle E_1 satisfies

$$[E_1 \otimes \mu] \neq n_1 \in K(B_1)\ .$$

We want to know if there is a condition that we could impose to ensure that

$$[E_1 \boxtimes E_2][\nu] \neq n_1 n_2 \in K(B_1 \times B_2)$$

for any line bundle ν over $B_1 \times B_2$.

Theorem 5.3.1. *(Theorem 2.7.15, [**2**]) Let A be a space such that $K^*(A)$ is torsion free and let B be a finite cell complex. For a subcomplex $B^{'} \subset B$, we have the following isomorphism*

$$K^*(A) \otimes K^*(B, B^{'}) \to K^*(A \times B, A \times B^{'}). \tag{5.9}$$

Atiyah shows in [**2**] that $K^*(Gr_k^s(V))$ is torsion free for any s-dimensional vector space V and similarily for products of Grassmianns. Using this result and Theorem 5.3.1 we deduce for instance that

$$K^*(\mathbb{P}^n) \otimes K^*(\mathbb{P}^m) \xrightarrow{\cong} K^*(\mathbb{P}^n \times \mathbb{P}^m) \tag{5.10}$$

is an isomorphism. Consider the exterior product bundle $E = E_1 \boxtimes E_2$. Furthermore, assume that $K^*(B_1)$ or $K^*(B_2)$ is torsion free and that B_1 and B_2 are a finite CW-complexes. Then the morphism

$$\begin{aligned} m : \sum_{j+l=k} K^j(B_1) \otimes K^l(B_2) \to K^k(B_1 \times B_2) \\ \sum_{j+l=k} x^{(j)} \otimes y^{(l)} \mapsto \sum_{j+l=k} proj_1^*[x^{(j)}] proj_2^*[y^{(l)}] \end{aligned} \tag{5.11}$$

induced by the product in K-theory is an isomorphism.

PROOF. (Corollary D)

Let $E_1 \to B_1$ and $E_2 \to B_2$ be complex vector bundles of rank n_1 and n_2, respectively.

Assume that for any line bundle $\mu \in \mathrm{Vect}^1(B_1)$

$$[E_1 \otimes \mu] \neq n_1 \ \in K^0(B_1). \tag{5.12}$$

Consider the inclusion mappings $\iota_1 : B_1 \to B_1 \times B_2$ and $\iota_2 : B_2 \to B_1 \times B_2$ defined by $x \mapsto (x, x_2)$ and $x \mapsto (x_1, x)$ (respectively) for $x_1 \in B_1$ and $x_2 \in B_2$ arbitrary points. The maps ι_1 and ι_2 induce ring morphisms

$$\iota_1^* : K^*(B_1 \times B_2) \to K^*(B_1) \ ; \ \iota_2^* : K^*(B_1 \times B_2) \to K^*(B_2).$$

Define the mapping

$$\begin{aligned} \iota_1^* \otimes \iota_2^* : K^0(B_1 \times B_2) \to K^0(B_1) \otimes K^0(B_2) \\ x \mapsto \iota_1^*(x) \otimes \iota_2^*(x). \end{aligned} \tag{5.13}$$

This is a morphism of abelian monoids if we consider only the multiplicative operations $\bullet_{K^*(B_1 \times B_2)}$ and $\bullet_{K^*(B_1) \otimes K^*(B_2)}$ on $K^*(B_1 \times B_2)$ and $K^*(B_1) \otimes K^*(B_2)$, respectively.

Let $x \otimes y \in K^0(B_1) \otimes K^0(B_2)$. Then, we have that

$$\begin{aligned} (\iota_1^* \otimes \iota_2^*) \circ m(x \otimes y) &= (\iota_1^* \otimes \iota_2^*) \Big(proj_1^* x \bullet_{K^0(B_1 \times B_2)} proj_2^* y \Big) \ \in K^0(B_1) \otimes K^0(B_2) \\ &= (\iota_1^* \otimes \iota_2^*)(proj_1^* x) \bullet_{K^0(B_1) \otimes K^0(B_2)} (\iota_1^* \otimes \iota_2^*)(proj_2^* y) \\ &= x \otimes (\iota_2^* \circ proj_1^*) x \bullet_{K^0(B_1) \otimes K^0(B_2)} (\iota_1^* \circ proj_2^*) y \otimes y. \end{aligned} \tag{5.14}$$

We used in the above the fact that $\iota_1^* \otimes \iota_2^*$ is a morphism of commutative monoids $\Big(K(B_1 \times B_2), \bullet_{K(B_1 \times B_2)} \Big)$ and $\Big(K(B_1) \otimes K(B_2), \bullet_{K(B_1) \otimes K(B_2)} \Big)$.

In fact, $\iota_2^* \circ proj_1^*$ and $\iota_1^* \circ proj_2^*$ in (5.14) correspond to the mappings $rk : K^0(B_1) \to \mathbb{Z}$ and $rk : K^0(B_2) \to \mathbb{Z}$, respectively.
For $j \in \{1,2\}$, recall that rk is the rank ring morphism making the following diagram commute:

$$\begin{tikzcd} Vect(B_j) \arrow[r, "rk"] \arrow[d] & \mathbb{Z} \\ K^0(B_j) \arrow[ur, "rk"] & \end{tikzcd} .$$

This implies that

$$(\iota_1^* \otimes \iota_2^*) \circ m(x \otimes y) = rk(y)x \otimes rk(x)y. \tag{5.15}$$

Since $m : K^*(B_1) \otimes K^*(B_2) \to K^*(B_1 \times B_2)$ given by (5.11) is an isomorphism of rings, there exists a mapping

$$\psi : K^*(B_1 \times B_2) \to K^*(B_1) \otimes K^*(B_2)$$

such that

$$\psi \circ m = Id_{K^*(B_1) \otimes K^*(B_2)} \text{ and } m \circ \psi = Id_{K^*(B_1 \times B_2)}. \tag{5.16}$$

From (5.16) we get that

$$(\iota_1^* \otimes \iota_2^*) \circ m(x \otimes y) = \psi \circ m(x \otimes y) \ \in K^0(B_1) \otimes K^0(B_2)$$

for $x \otimes y \in K^0(B_1) \otimes K^0(B_2)$ such that $rk(x) = rk(y) = 1$.

Restricting the isomorphism ψ to $K^0(B_1 \times B_2)$ yields

$$K^0(B_1) \otimes K^0(B_2) + K^1(B_1) \otimes K^0(B_2) \underset{\psi}{\overset{m}{\rightleftarrows}} K^0(B_1 \times B_2) \ .$$

Denote by $\theta_j(k)$ the trivial complex vector bundle of rank k over B_j for $j \in \{1,2\}$ and by $\theta(k)$ the trivial complex vector bundle of rank k over $B_1 \times B_2$. Then, for any line bundle $\nu \in \mathrm{Vect}^1(B_1 \times B_2)$, $[E_1 \boxtimes E_2][\nu] = [\theta(n_1 n_2)]$ is equivalent to

$$\psi([E_1 \boxtimes E_2][\nu]) = [\theta_1(n_1)] \otimes [\theta_2(n_2)] \ \in K^0(B_1) \otimes K^0(B_2). \tag{5.17}$$

Indeed, we know that

$$m([\theta_1(n_1)] \otimes [\theta_2(n_2)]) = [\theta(n_1 n_2)] \ \in K^0(B_1 \times B_2)$$

and by injectivity of m or by applying ψ on the left, we get that

$$\psi([\theta(n_1 n_2)]) = [\theta_1(n_1)] \otimes [\theta_2(n_2)] \ \in K^0(B_1) \otimes K^0(B_2).$$

Using the fact that ψ is a ring morphism, we observe that

$$\begin{aligned}\psi([E_1 \boxtimes E_2][\nu]) &= \psi([E_1 \boxtimes E_2]) \bullet_{K^*(B_1)\otimes K^*(B_2)} \psi([\nu]) \\ &= ([E_1]\otimes[E_2]) \bullet_{K^*(B_1)\otimes K^*(B_2)} (\nu_1^0 \otimes \nu_2^0 + \nu_1^1 \otimes \nu_2^1)\end{aligned}$$

where $\nu_j^k \in K^k(B_j)$ for $j \in \{1,2\}$ and $k \in \{0,1\}$. Hence,

$$\psi([E_1 \boxtimes E_2][\nu]) = \Big([E_1]\nu_1^0 \otimes [E_2]\nu_2^0\Big) + \Big([E_1]\nu_1^1 \otimes [E_2]\nu_2^1\Big). \tag{5.18}$$

Equation (5.17) implies that the second summand after the equality sign must vanish,

$$[E_1]\nu_1^1 \otimes [E_2]\nu_2^1 = 0 \ \in K^1(B_1) \otimes K^1(B_2)$$

and the following must hold

$$[E_1]\nu_1^0 = [\theta_1(n_1)] \in K^0(B_1), \ [E_2]\nu_2^0 = [\theta_2(n_2)] \in K^0(B_2). \tag{5.19}$$

Since rk is a ring morphism, Equation (5.19) implies that

$$rk(\nu_1^0) = rk(\nu_2^0) = 1. \tag{5.20}$$

Moreover,

$$rk(\nu_1^1) = rk(\nu_2^1) = 0. \tag{5.21}$$

In other words, if $[E_1 \boxtimes E_2][\nu] \ \in K^0(B_1 \times B_2)$ is stably trivial then,

$$\psi([E_1 \boxtimes E_2][\nu]) = [E_1]\nu_1^0 \otimes [E_2]\nu_2^0.$$

Using the relations in (5.20) and (5.21), we compute the following

$$\begin{aligned}(\iota_1^* \otimes \iota_2^*) \circ m \circ \psi[\nu] &= (\iota_1^* \otimes \iota_2^*) \circ m(\nu_1^0 \otimes \nu_2^0) + (\iota_1^* \otimes \iota_2^*) \circ m(\nu_1^1 \otimes \nu_2^1) \\ &= \nu_1^0 \otimes \nu_2^0 + 0.\end{aligned}$$

This means that

$$(\iota_1^* \otimes \iota_2^*)[\nu] = \nu_1^0 \otimes \nu_2^0. \tag{5.22}$$

For $\xi := \psi([E_1 \boxtimes E_2][\nu])$, if $[E_1 \boxtimes E_2][\nu] = n_1 n_2 \ \in K^0(B_1 \times B_2)$ then,

$$\begin{aligned}\xi &= [E_1]\nu_1^0 \otimes [E_2]\nu_2^0 \\ &= ([E_1]\otimes[E_2]) \bullet_{K^0(B_1)\otimes K^0(B_2)} (\iota_1^* \otimes \iota_2^*)[\nu] \\ &= [E_1 \otimes \iota_1^*\nu] \otimes [E_2 \otimes \iota_2^*\nu].\end{aligned}$$

As a consequence of this, we have that for any line bundle $\nu \in \mathrm{Vect}^1(B_1 \times B_2)$,

$$\begin{aligned}
[E_1 \boxtimes E_2][\nu] = n_1 n_2 &\Leftrightarrow m \circ \psi\Big([E_1 \boxtimes E_2][\nu]\Big) = n_1 n_2 \\
&\Rightarrow m\Big([E_1 \otimes \nu|_{B_1}] \otimes [E_2 \otimes \nu|_{B_2}]\Big) = n_1 n_2.
\end{aligned}$$

Injectivity of m implies that

$$[E_1 \otimes \nu|_{B_1}] = n_1 \ \in K^0(B_1),$$

which contradicts our assumption (5.12). This means that for any $\nu \in \mathrm{Vect}^1(B_1 \times B_2)$

$$[E_1 \boxtimes E_2][\nu] \neq n_1 n_2 \ \in K^0(B_1 \times B_2).$$

We conclude from Theorem B that $\mathbb{P}(E_1) \times \mathbb{P}(E_2)$ is not trivial as a Hamiltonian fibration.

$\square$

We will now give another example of non-trivial Hamiltonian fibrations as an application of Corollary D.

Example 5.3.2. *Let $E_1 := L \otimes \underline{\mathbb{C}}^3 \boxplus L \otimes \underline{\mathbb{C}}^3 \to \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}$, where L is again the tautological line bundle $\tau^1_{\mathbb{RP}^{n-1}}$ and $n = 7$ or 8. Let E_2 be the tautological k-dimensional vector bundle over the Grassmannian of k-hyperplanes in $\mathbb{C}^n$.*

Since $[E_1 \otimes \mu] \neq 6$ for all line bundles μ over $\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}$ (see Example 5.2.3), the fiber bundle

$$\mathbb{C}P^5 \times \mathbb{C}P^{k-1} \hookrightarrow \mathbb{P}(E_1) \times \mathbb{P}(E_2) \to \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \times Gr_k^n(\mathbb{C})$$

is not a trivial Hamiltonian fibration.

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Appendix A

Here we will recall the Dolbeault complex and define a natural elliptic operator on Kähler manifolds. The reader is refered to [**30**] or [**8**] for more details.

Let E be a holomorphic vector bundle of rank n over a complex manifold M. Denote by $\Omega^{(0,q)}(E)$ the space of smooth sections in $\Gamma(T^*M^{(0,q)} \otimes_{\mathbb{C}} E)$. Every section α in $\Omega^{(0,q)}(E)$ can be expressed in a local holomorphic trivialization (U,φ) as an n-tuple $(\alpha_1,\cdots,\alpha_n)$ of local $(0,q)$-forms. We can define the operator

$$\overline{\partial}_E : \Omega^{(0,q)}(E) \to \Omega^{(0,q+1)}(E) \tag{A.1}$$

as the operator which satisfies on a trivialization neighbourhood U the following

$$\overline{\partial}_E|_U \alpha = (\overline{\partial}\alpha_1,\cdots,\overline{\partial}\alpha_n) \tag{A.2}$$

Definition A.0.1. *The operator $\overline{\partial}_E$ defines a complex*

$$\cdots \to \Omega^{(0,q-1)}(E) \xrightarrow{\overline{\partial}_E} \Omega^{(0,q)}(E) \xrightarrow{\overline{\partial}_E} \Omega^{(0,q+1)} \xrightarrow{\overline{\partial}_E} \cdots \tag{A.3}$$

called the Dolbeault complex of E.

The following theorem which is an analogue of the De Rham Theorem shows a correspondence between the cohomology of this complex and the sheaf cohomology of M (see for instance Corollary 4.38, [**30**]).

Theorem A.0.2. *(Dolbeault) Let E be a holomorphic vector bundle over a complex manifold M. Then, the i^{th} cohomology group of M with values in the sheaf of holomorphic sections of E corresponds to the i^{th} cohomology group of the Dolbeault complex* (A.3) *given by*

$$H^i_{Dolb}(M,E) = \frac{ker\ (\overline{\partial} : \Omega^{(0,i)}(E) \to \Omega^{(0,i+1)}(E))}{im\ (\overline{\partial} : \Omega^{(0,i-1)}(E) \to \Omega^{(0,i)}(E))}.$$

Assume M is endowed with a Hermitian structure h. Denote by J the canonical almost complex structure on TM. For all $x \in M,\ h_x$ is a complex-valued positive definite sesquilinear form on T_xM satisfying the following:

$$h_x(J_x(v),w) = ih(v,w) = -h(v,J_x(w))\ \forall\ v,w \in T_xM.$$

Morover, the real part $g := \text{Re}\ h$ defines a Riemannian structure in M and the imaginary part $\omega :=$ Im h defines a non-degenerate two-form such that

$$g(v,w) = \omega(Jv,w).$$

The two-form ω corresponds to a symplectic form for a Kähler manifold. Assume M is a Kähler manifold. Observe that the mapping

$$T_xM \to T_x^*M^{(0,1)},\ v \mapsto h_x(v,.)$$

is a complex linear isomorphism. We can use this map to define a Hermitian structure $h^{(0,1)}$ on $T_x^*M^{(0,1)}$ which induces then a Hermitian structure h^E on $E = \Lambda(T_x^*M^{(0,1)})$.

Let L be a holomorphic line bundle with Hermitian structure h^L. The tensor product $h^E \otimes h^L$ defines a Hermitian structure in $E \otimes L$.

Consider the mapping that assigns to any $\alpha \in T_x^*M^{(0,1)}$ the following exterior product operation

$$e(\alpha) : E_x^q \otimes L \to E^{q+1} \otimes L_x,\ \nu \mapsto \alpha \wedge \nu.$$

One way of computing the principal symbol of a differential operator $D \in \mathscr{P}^m(E,F;M)$ for complex vector bundles E and F over M is given by the following. Let $\xi \in T_x^*M$ for any point x in M and let f be any smooth function on M that satisfies

$$df(x) = \xi \text{ and } f(x) = 0. \tag{A.4}$$

Then for any section $\alpha \in \Gamma(M,E)$

$$\sigma_D(x,\xi)\alpha(x) = \frac{i^m}{m!}D(f^m\alpha)(x),\ i = \sqrt{-1}. \tag{A.5}$$

Using (A.5) we will compute $\sigma_{\overline{\partial}}$. Let $\alpha \in T^*M^{(0,q)}$ and $\xi \in T_x^*M$. Then for any $f \in C^\infty(M)$ satisfying (A.4),

$$\begin{aligned}\sigma_{\overline{\partial}}(x,\xi)\alpha(x) &= i\overline{\partial}(f\alpha)(x)\\ &= i(\overline{\partial}f)\alpha(x)\\ &= i(\pi^{(0,1)} \circ d)(f)\alpha(x)\\ &= ie(\xi^{(0,1)})\alpha(x).\end{aligned}$$

If $e_1, \cdots, e_n$ is a unitary local frame in TM with respect to h then, $\varepsilon_j = h(e_j,.)$ defines a unitary frame in $T^*M^{(0,1)}$ such that we have the following pairing

$$\langle e_j, \varepsilon_k \rangle = \delta_{jk}.$$

The adjoint of $e(\varepsilon_j)$ is given by the interior product operator $\iota(e_j)$. Write

$$\xi^{(0,1)} = \frac{1}{2}\sum_{j=1}^{n}\Big(\langle e_j, \xi\rangle + i\langle Je_j, \xi\rangle\Big)\varepsilon_j$$

then

$$\begin{aligned}\sigma_{\overline{\partial}^*}(\xi) &= \frac{-i}{2}\sum_{j=1}^{n}\Big(\langle e_j, \xi\rangle - i\langle Je_j, \xi\rangle\Big)\iota(e_j)\\ &= \frac{-i}{2}\iota(g^{-1}\xi).\end{aligned}$$

One can show that the principal symbol of the square of $D := \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$ is given by

$$\sigma_{D^2}(\xi) = \frac{1}{2}\langle g^{-1}\xi, \xi\rangle = \frac{1}{2}\|\xi\|. \tag{A.6}$$

An operator satisfying A.6 is called a *generalized Dirac operator*. We'll refer to D as the *Dolbeault-Dirac operator*. The relation A.6 implies that D^2 and D are elliptic. For a compact manifold M this implies that $\ker D$ is finite dimensional and that $\operatorname{im} D = \ker(D^*)^\perp$. Since $\overline{\partial}^2 = (\overline{\partial}^*)^2 = 0$, we have that

$$\frac{1}{2}D^2 = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$$

or

$$\frac{1}{2}(Du, Du) = (\overline{\partial}u, \overline{\partial}u) + (\overline{\partial}^*u, \overline{\partial}^*u).$$

Hence $Du = 0$ is equivalent to $\overline{\partial} = 0$ and $\overline{\partial}^* = 0$. In other words, u is in the orthogonal complement of the range of $\overline{\partial}$. By the Dolbeault Theorem, we have that

$$\ker D|_{\Gamma(M, E^q \otimes L)} \cong H^q(M, \mathscr{O}(L)).$$