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Applications des structures algébriques associées aux systèmes intégrables

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Sommaire

Cette thèse en trois parties regroupe des travaux de recherches sous la thématiques des symétries sous-jacentes aux systèmes intégrables et des structures algébriques qui les encodent. Une première partie illustre comment les fonctions spéciales que sont les polynômes orthogonaux apparaissent dans la théorie de la représentation des diverses structures algébriques associées à des symétries. La seconde partie se concentre sur une généralisation algébrique de l'opérateur de Heun classique menant à de nouvelles structures algébriques qui trouvent des applications en traitement de signal et dans l'étude des systèmes intégrables. La dernière partie concerne l'élaboration d'un cadre théorique dans le langage de la théorie de l'information algorithmique permettant de poser une définition mathématique de la notion d'émergence.

Mots clés

- Structures algébriques
- Symétries
- Intégrabilité
- Polynômes orthogonaux
- Opérateur de Heun
- Émergence

Summary

This thesis in three parts groups research work under the theme of the symmetries underlying integrable systems and the algebraic structures that encodes them. A first part illustrates how orthogonal polynomials, a type of special function, appear in the representation theory of various algebraic structures associated to symmetries. The second part focuses on an algebraic generalization of the classical Heun operator that leads to new algebraic structures with applications in signal processing and in the study of integrable systems. The last part concerns the formulation of a framework in the language of algorithmic information theory the enables a mathematical definition for the notion of emergence.

Keywords

- Algebraic structures
- Symmetries
- Integrability
- Orthogonal polynomials
- Heun operators
- Emergence

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Liste des sigles et des abréviations

ECOC	Ensemble complet d'opérateurs qui commutent
AIT	Théorie de l'information algorithmique, de l'anglais *Algorithmic Information Theory*

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Introduction

Les symétries sont d'une importance fondamentale à la physique et sont intrinsèquement liées à la présence de structures dans les systèmes physiques et leurs dynamiques. La notion de symétrie est généralement introduite comme une opération qui laisse un certain objet mathématique invariant. Cette définition se prête naturellement à des descriptions mathématiques en termes de structures algébriques, souvent des groupes, qui agissent sur un objet abstrait. Dans un contexte physique, cet objet mathématique est partie intégrante d'un modèle théorique qui représente le système physique.

L'identification de symétries dans un système s'apparente à la description rigoureuse de structures ou contraintes dans l'ensemble des solutions de la dynamique. C'est cet aspect des symétries qui en rend l'application en physique fructueuse. En effet, l'identification de symétries dans la dynamique d'un système contraint son évolution et donc ses solutions. De cette manière, les systèmes hautement symétriques se prêtent plus facilement à un traitement théorique. Lorsqu'un modèle est suffisamment contraint, il devient parfois possible d'en obtenir par des manipulations symboliques des expressions explicites pour les quantités d'intérêt. On réfère alors au modèle comme étant exactement soluble. Les systèmes intégrables sont des exemples importants de modèles dont la dynamique est exactement soluble.

Tel que mentionné, l'ensemble des symétries associées à un système exactement soluble peut-être encodé sous la forme de structures algébriques. Les modèles apparaissent alors comme des représentations de ces structures. Cette thèse en huit articles aborde l'étude et l'application des structures algébriques encodant les symétries de systèmes exactement solubles. Les résultats sont regroupés en trois parties sous une même thématique. La première partie concerne les fonctions spéciales, et plus particulièrement les polynômes orthogonaux, qui apparaissent dans les représentations de structures algébriques. La seconde partie se concentre sur l'étude de généralisations algébriques de l'opérateur de Heun. Finalement, la dernière partie construit un cadre théorique général dans l'étude des structures exhibées par les systèmes physiques et propose une définition mathématique au concept d'émergence.

Dans l'optique de mettre en contexte les travaux présentés dans cette thèse, un survol de la notion d'intégrabilité est maintenant présenté. Ce survol de la notion d'intégrabilité, autant en mécanique classique que quantique, permet de mettre en évidence les structures algébriques qui y sont associées.

Intégrabilité classique

La notion d'intégrabilité prend racine dans l'étude de la solvabilité des équations différentielles de la mécanique classique. En effet, la mécanique Newtonienne fut initialement formulée comme un système d'équations différentielles couplées donné, pour un système de N particules ponctuelles, par

$$F_i = m_i a_i, \quad i \in 1, 2, \ldots, N.$$

La solution d'un tel système d'équation étant obtenue par intégrations successives on appelle un tel système pour lequel cette intégration est tractable un système intégrable. Cette conception vague et élémentaire de l'intégrabilité et des systèmes intégrables s'est généralisée et formalisée substantiellement pour donner naissance au concept moderne d'intégrabilité, applicable également à la mécanique quantique.

On voudrait formuler une définition de l'intégrabilité en mécanique classique qui soit valide en mécanique quantique. Or, il s'avère qu'une telle définition dans un contexte général n'est pas connue. Concrètement, il est souvent possible de se limiter à une définition opérationnelle dans des contextes restreints, mais il est pertinent de tenter de mettre en valeur un fil directeur derrière les diverses notions d'intégrabilité, tant classiques que quantiques. Différentes notions connexes à l'intégrabilité seront donc présentées en tentant de maintenir une continuité dans les concepts. En commençant par un contexte dynamique général, on cherchera à converger vers une notion d'intégrabilité dans la formulation hamiltonienne de la mécanique classique et de l'importance des symétries dans ce contexte.

Systèmes dynamiques

Avant de se restreindre à la mécanique classique hamiltonienne, une discussion dans le cadre plus général des systèmes dynamiques est pertinente pour introduire les notions centrales. La description mathématique d'un système dynamique comprend premièrement une variété M, l'espace de phase, dont les points indexent les états possibles du système. La dynamique est encodée par l'action sur M d'un groupe G continu d'automorphisme $g_t \in G$ à 1-paramètre encodant l'évolution temporelle:

$$\begin{aligned} g_t : M &\longrightarrow M, \quad \forall\, t \in \mathbb{R} \\ x &\longmapsto x(t). \end{aligned}$$

Cette définition est suffisante pour décrire complètement la dynamique d'un système déterministe. L'étude d'un tel système cherche alors à obtenir une description explicite de la dynamique qui met à jour la structure de celle-ci ou du moins en offre une caractérisation qualitative. Cependant, la présente définition ne garantit pas la tractabilité d'une telle étude. En effet, pour une condition initiale donnée x_0, son orbite $G(x_0)$ sous l'action du groupe d'automorphisme G ne possède pas nécessairement de structure au delà que d'être un sous-ensemble de M. Par exemple, pour certains systèmes toutes les conditions initiales génériques correspondent à des orbites denses dans M et donc à une dynamique chaotique. Il est alors impossible d'identifier des régimes différents de la dynamique ou encore d'en prédire les caractéristiques pour des conditions initiales spécifiées. L'étude théorique d'un tel système demeure ainsi très limitée.

Supposons maintenant qu'il existe une fonction $F : M \longrightarrow \mathbb{R}$ qui soit constante sur les orbites de G dans M. On sait alors que pour une condition initiale x_0, l'orbite $G(x_0)$ sera contenu dans la préimage de $F(x_0)$:

$$G(x_0) \subseteq F^{-1}(F(x_0)).$$

Cette contrainte sur la dynamique peut permettre une classification qualitative des évolutions possibles étant donnée les états initiaux. La tractabilité et la pertinence d'une telle classification dépend de la forme du recouvrement de M par les préimages par F des points de $\mathbb{R}$. Ceci est le reflet de l'existence de régimes chaotiques dans les systèmes dynamiques. En effet, F peut-être interprétée comme une quantité conservée par la dynamique, mais l'existence d'une telle quantité conservée ne permet pas nécessairement une simplification dans l'étude du système. Par exemple, une fonction F dont les préimages seraient des ensembles fractals ne permettrait pas de simplification. Par contre, si les préimages de F spécifient une foliation de M, de sorte que la préimage de chaque point de $\mathbb{R}$ soit une sous-variété N de

M, alors la dynamique est effectivement contrainte à $N \hookrightarrow M$ et donc de dimensionnalité réduite. Cette idée de foliation de l'espace de phase est à la base des différentes notions d'intégrabilité en mécanique classique.

Mécanique Hamiltonienne

On se restreint maintenant à des systèmes dont la dynamique est hamiltonienne. Dans ce contexte, l'espace de phase M est une variété symplectique avec une forme symplectique bilinéaire ω agissant sur le fibré tangent[1]. La dynamique est spécifiée [**8**] par le biais d'une fonction distinguée $H : M \longrightarrow \mathbb{R}$ générant l'évolution temporelle infinitésimale. En effet, puisque H est une fonction sur M, on en obtient aisément une 1-forme par la dérivé extérieure[2]

$$H \longmapsto dH \in T^*M,$$

et puisque M est symplectique, cette 1-forme est associée à un champ de vecteurs par l'isomorphisme suivant, défini à partir de la forme symplectique

$$\Phi : TM \longrightarrow T^*M : X \longmapsto \omega(\,.\,, X).$$

On a alors le champ vectoriel hamiltonien $\Phi^{-1} \circ dH$ générant un flot hamiltonien sur M. La dynamique sur M constitue un groupe de difféomorphismes à 1-paramètre G avec

$$G : \mathbb{R} \longrightarrow Diff(M) : t \longmapsto g_t, \quad g_t : M \longrightarrow M.$$

Ainsi, un flot hamiltonien spécifie une dynamique par l'équation suivante

$$\frac{d}{dt}\, g_t x\Big|_{t=0} = \Phi^{-1} \circ dH\Big|_x \quad \forall\, x \in M.$$

Cette dynamique peut également être exprimée comme une action de G sur les observables directement. En effet, les observables d'un système en mécanique classique correspondent aux fonctions sur l'espace de phase M. Ainsi, pour une observable représentée par une fonction $F : M \longrightarrow \mathbb{R}$, l'évolution du système implique une évolution de cet observable par

$$F(t) = F \circ g_t.$$

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[1]Soit le recollement lisse des espaces tangents à M.

[2]Il s'agit de la différentielle de H dans le formalisme des formes différentielles.

En introduisant le crochet de Poisson comme suit

$$\{f, h\} \equiv \omega(X_f, X_h), \quad X_f = \Phi^{-1}(df), \quad X_h = \Phi^{-1}(dh),$$

on peut exprimer [**8**] l'évolution infinitésimale de F sous l'action de H par

$$\frac{d}{dt}F(t) = \{F, H\}.$$

Il est maintenant possible de formuler une définition de l'intégrabilité classique. Soit un système hamiltonien sur un espace de phase M de dimension $2n$ ayant pour Hamiltonien H. On sait que pour tout observable invariant sous l'évolution temporelle, on aura que la fonction F associée à cet observable sera telle que

$$\{F, H\} = 0,$$

de sorte que sous l'évolution dynamique, la valeur initiale $f = F(x_0)$ sera conservée. Supposons maintenant un ensemble $\{F_i\}$ de n fonctions sur M indépendantes telles que

$$\{F_i, H\} = 0 \quad i = 1, 2, \ldots, n,$$

c'est-à-dire étant toutes associées à un observable conservé par la dynamique. Chacune de ces quantités conservées définit une foliation de M contraignant la dynamique [**1**]. En demandant également que

$$\{F_i, F_j\} = 0, \quad i, j = 1, 2, \ldots, n,$$

on s'assure que l'ensemble des $\{F_i\}$ définisse une foliation globale de M en sous-variétés invariantes sous la dynamique, chacune de dimension n. Le système est alors complètement intégrable au sens de Liouville.

Structures algébriques associées

La présence du crochet de Poisson dans cette définition de l'intégrabilité est le reflet d'une structure algébrique. Plus particulièrement, l'ensemble $\mathcal{F}$ des fonctions lisses sur l'espace de phase M forme une algèbre de Poisson. En effet, une structure naturelle d'espace vectoriel sur $\mathbb{R}$ existe pour $\mathcal{F}$ provenant de celle du codomaine $\mathbb{R}$ et le produit en chaque point font de $\mathcal{F}$ une algèbre associative

$$\alpha f_1 + f_2 \in \mathcal{F}, \quad f_1 f_2 \in \mathcal{F}, \quad \forall\, f_1, f_2 \in \mathcal{F},\, \alpha \in \mathbb{R}.$$

Ensuite, la structure du crochet de Poisson fait de $\mathcal{F}$ une algèbre de Lie. Ces structures définies sur $\mathcal{F}$ forment ensemble une algèbre de Poisson. Aussi, sachant que les fonctions lisses sur M, soit les éléments de $\mathcal{F}$, sont les observables de la mécanique classique, cette structure de Poisson est définie sur les observables. Qui plus est, l'application

$$\Phi^{-1} \circ d : \mathcal{F} \longrightarrow TM : f \longmapsto X_f$$

a pour image les champs vectoriels hamiltoniens [**1**], c'est-à-dire les champs vectoriels qui sont les générateurs infinitésimaux de symplectomorphismes hamiltoniens, aussi appelés transformations canoniques. Ainsi, des quantités invariantes $\{F_i\}$ avec $\{F_i, H\} = 0$ génèreront des transformations de symétrie de l'espace de phase M laissant les surfaces de contour de H invariantes. Le groupe des symplectomorphismes associé s'apparente alors au groupe de symétrie de la dynamique.

L'algèbre de Poisson $\mathcal{F}$ génère, par l'application exponentielle, le groupe des symplectomorphismes hamiltoniens. La sous-algèbre de $\mathcal{F}$ générée par l'Hamiltonien H et les quantités conservées $\{F_i\}$ constitue alors l'algèbre de symétrie du système qui génère le groupe de symétrie par exponentiation. La structure de groupe que forment les symétries d'un système physique peut être encodée [**11**] au niveau algébrique par la construction de l'algèbre enveloppante universelle. Formellement, cette structure se construit en prenant le quotient de l'algèbre tensorielle $T(\mathcal{F})$ par les relations algébriques

$$U(\mathcal{F}) = T(\mathcal{F})/\langle [X, Y] - X \otimes Y + Y \otimes X \rangle_{X,Y \in \mathcal{F}},$$

et peut être vue comme l'algèbre des polynômes sous le produit tensorielle en termes des éléments de l'algèbre de Lie $\mathcal{F}$. Le quotient assure que les relations du crochet de Lie demeurent présentes. L'aspect important de cette construction réside en ce qu'elle constitue une algèbre de Hopf, encodant la structure de groupe de transformation généré par l'algèbre de Lie associée. Aussi, la structure de $U(\mathcal{F})$ est enrichie d'homomorphismes distingués, soit le coproduit Δ, l'antipode S et la counité ϵ, de même que des relations de compatibilité entre ceux-ci. Ces homomorphismes sont le reflet [**11**] de l'existence du produit de groupe, de l'inverse et de l'identité, respectivement, dans la structure de groupe généré par $\mathcal{F}$.

Intégrabilité quantique

On cherche maintenant à cerner une notion d'intégrabilité dans le contexte de la mécanique quantique. Une première définition, inspirée de la notion classique, permettra de mettre en valeur les subtilités de la question. Il sera ensuite question d'exposer l'origine de l'équation de Yang-Baxter constituant les fondements pour la plupart des versions de l'intégrabilité en quantique pour ensuite se tourner sur la formalisation algébrique de cette équation reposant sur les algèbres de Hopf, une formalisation proposant un traitement plus général des symétries qu'il ne l'est possible en utilisant des groupes.

Un système quantique est décrit par un espace de Hilbert $\mathcal{H}$ indexant les états possibles du système et sur lequel agissent des opérateurs linéaires auto-adjoints représentant les observables. Ces opérateurs forment l'algèbre des observables $\mathcal{O}$. Les éléments de cette algèbre d'opérateurs génèrent des groupes à 1-paramètre d'automorphismes unitaires $U(s)$ de $\mathcal{H}$ par l'application exponentielle

$$exp : X \longmapsto U(s) \equiv e^{-isX}, \quad \forall\, X \in \mathcal{O}.$$

La dynamique est encodée par un opérateur distingué, l'Hamiltonien, qui génère l'évolution temporelle du système. Cette évolution temporelle comme action sur les états peut être vue comme un automorphisme de l'algèbre des observables de sorte que pour un observable O, on ait l'évolution suivante

$$\frac{d}{dt}O(t) = i[H, O(t)],$$

correspondant à l'approche de Heisenberg à la mécanique quantique, soit celle utilisée dans cette discussion.

Une première définition

Le formalisme algébrique de la mécanique quantique se rapproche suffisamment de celui de la mécanique classique pour suggérer d'importer directement la notion d'intégrabilité classique au contexte quantique. Dans cette optique [**18**], pour un système avec N degrés de liberté, on propose comme critère d'intégrabilité du système l'existence d'un ensemble $\{Q_i\}$ de N quantités conservées qui commutent mutuellement entre-elles, c'est-à-dire

$$[H, Q_i] = 0, \quad [Q_i, Q_j] = 0, \quad i, j = 1, 2, \ldots, N, \quad i \neq j.$$

Cette définition peut sembler satisfaisante, mais plusieurs difficultés demeurent. En effet, la diagonalisation simultanée des N quantités conservées devrait générer une base de $\mathcal{H}$ dont chaque vecteur est étiqueté uniquement par les valeurs propres des opérateurs, c'est-à-dire que ces quantités conservées forment un ensemble complet d'opérateurs qui commutent (ECOC). Cependant, un théorème de von Neumann [**17**] démontre que pour tout ensemble $\{O_i\}$ d'opérateurs hermitiens bornés qui commutent entre-eux, il existe un opérateur $\mathcal{O}$ tel que les opérateurs de cet ensemble soient tous fonctions de cet opérateur $O_i = f_i(\mathcal{O})$. On aurait alors réduit l'ECOC à un seul opérateur. Inversement, en choisissant judicieusement des projecteurs sur les états propres de l'Hamiltonien, il est en général possible [**18**] d'obtenir des opérateurs qui commutent entre-eux et avec l'Hamiltonien. Ces remarques remettent en question, dans le contexte de l'intégrabilité, le concept même d'ECOC dans le sens où la complétude, ou maximalité, de l'ensemble n'est pas bien définie. Plusieurs raffinements de cette définition existent. Entre autres, la notion d'indépendance fonctionnelle présente dans les définitions classiques de l'intégrabilité n'a pas d'équivalent dans la présente définition. Aussi, il est d'usage de demander l'indépendance algébrique des charges conservées. Par exemple, dans [**13**], en plus de demander une forme d'indépendance algébrique, les opérateurs admissibles pour les quantités conservées sont contraints d'être éléments de l'algèbre enveloppante universelle $U(\mathfrak{h}_N)$ ou de sa complétion par des séries convergentes. Ici, $\mathfrak{h}_n$ est l'algèbre d'Heisenberg, soit l'algèbre de Lie générée par les opérateurs de position et de quantité de mouvement. Cependant, même ces définitions plus précises ne sont pas valides en général [**2**]. Cette première tentative illustre la difficulté du problème et fait ressortir la nécessité d'introduire des contraintes sur les opérateurs admissibles, tout en soulevant l'importance minimale qu'occupe l'espace de Hilbert dans ces considérations.

Équation de Yang-Baxter

Ces observations suggèrent la pertinence, dans le but de cerner une notion quantique de l'intégrabilité, de travailler avec une formulation de la mécanique quantique basée sur des structures algébriques et de voir l'espace de Hilbert que comme une représentation de cette structure. L'ensemble des opérateurs admissibles sur l'espace de Hilbert devrait ainsi [**18**] être restreint par l'introduction de structures additionnelles afin de différentier les quantités conservées ayant des conséquences qualitatives sur la dynamique de celles ne constituant que des reparamétrisations ingénieuses presque toujours possibles. La contrainte utilisée ici consiste essentiellement en ce que les opérateurs admissibles soient les éléments d'une algèbre de Hopf spécifique. Ces algèbres de Hopf apparaissent dans une vaste collection de modèles intégrables. La nécessité d'une telle structure pour beaucoup de modèles intégrables s'obtient en étudiant ce que la première définition de l'intégrabilité proposée implique au niveau algébrique.

La présentation donnée ici suit celle dans [**14**] de même que [**10**]. Aussi, supposons un système intégrable auquel est associé un ECOC $\{Q_i\}$ agissant sur un espace de Hilbert $\mathcal{H}$ de sorte que

$$[Q_i, Q_j] = 0, \quad \forall\, i, j.$$

On assume que la complétude de cet ensemble soit définie dans un sens pertinent pour la situation sans pour autant demander une définition explicite. L'ensemble pourrait être infini et l'étude de ses propriétés algébriques est simplifiée par l'utilisation d'une fonction génératrice, la matrice de transfert, donnée par

$$\ln \mathcal{T}(\lambda) = \sum_{i=0}^{\infty} Q_i (\lambda - \xi)^n,$$

de sorte que l'on ait

$$Q_i = \frac{1}{n!} \frac{d^n}{d\lambda^n} \ln \mathcal{T}(\lambda) \Big|_{\lambda=\xi}. \tag{0.1}$$

La commutativité des charges conservées implique alors la commutativité des matrices de transfert pour différents paramètres

$$[\mathcal{T}(\lambda), \mathcal{T}(\mu)] = 0.$$

En ligne avec la méthode de la diffraction quantique inverse [**16, 15**], plutôt que de chercher directement de tels opérateurs $\mathcal{T}(\lambda)$, on introduit le problème auxiliaire pour les opérateurs de monodromie $T(\lambda)$ agissant sur $V_0 \otimes \mathcal{H}$, où V_0 est l'espace auxiliaire, de sorte qu'une solution $T(\lambda)$ génère une matrice de transfert en prenant la trace sur V_0

$$\mathcal{T}(\lambda) = \mathrm{Tr}_{V_0} T(\lambda).$$

La condition de commutativité des matrices de transfert s'exprime alors en terme des matrices de monodromie comme

$$[\mathrm{Tr}_{V_0} T(\lambda), \mathrm{Tr}_{V_0} T(\mu)] = 0.$$

Ensuite, puisque la trace est multiplicative sur les membres d'un produit tensoriel, on peut réécrire l'expression précédente comme

$$\begin{aligned}
[\operatorname{Tr}_{V_0} T(\lambda), \operatorname{Tr}_{V_0} T(\mu)] &= \operatorname{Tr}_{V_0 \otimes V_0} \left(T(\lambda) \otimes T(\mu) - T(\mu) \otimes T(\lambda)\right) \\
\implies \quad \operatorname{Tr}_{V_0 \otimes V_0} \left(T(\lambda) \otimes T(\mu)\right) &= \operatorname{Tr}_{V_0 \otimes V_0} \left(T(\mu) \otimes T(\lambda)\right).
\end{aligned}$$

Finalement, puisque la trace est invariante sous un isomorphisme de l'espace vectoriel $V_0 \otimes V_0$, on obtient que les matrices de monodromie doivent être entrelacées[3]

$$R(\lambda, \mu)\, T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda)\, R(\lambda, \mu). \tag{0.2}$$

L'opérateur $R(\lambda, \mu)$ agissant sur $V_0 \otimes V_0$ est connu dans la littérature sous le nom de matrice R.[4]

La relation (0.2) n'est pas suffisante seule et nécessite une condition de compatibilité supplémentaire sur la matrice R. Cette condition prend son origine en ce qu'il existe deux façons possibles de ré-ordonner un produit tensoriel de trois matrices de monodromie dans l'ordre inverse en utilisant les matrices R. Explicitement, si l'on considère le triple produit tensoriel $T(\lambda) \otimes T(\mu) \otimes T(\nu)$ agissant sur $V_0 \otimes V_0 \otimes V_0 \otimes \mathcal{H}$ de même que les trois matrices R, notées R_{12}, R_{13} et R_{23}, agissant pour R_{ij} sur le $i$ème et le $j$ème terme de $V_0 \otimes V_0 \otimes V_0$, alors il existe deux isomorphismes différents construits avec les matrices R qui relient $T(\lambda) \otimes T(\mu) \otimes T(\nu)$ à $T(\nu) \otimes T(\mu) \otimes T(\lambda)$, selon l'ordre dans lequel les permutations sont appliquées. Or, puisque que l'on demande l'isomorphisme de ces deux espaces

$$T(\lambda) \otimes T(\mu) \otimes T(\nu) \cong T(\nu) \otimes T(\mu) \otimes T(\lambda),$$

on obtient la condition de compatibilité suivante

$$R_{23}(\mu, \nu) R_{13}(\lambda, \nu) R_{12}(\lambda, \mu) = R_{12}(\lambda, \mu) R_{13}(\lambda, \nu) R_{23}(\mu, \nu). \tag{0.3}$$

Cette équation d'importance [**3, 12**] est connue sous le nom *d'équation de Yang-Baxter* et occupe une place centrale dans l'étude d'un grand nombre de systèmes intégrables quantiques. En effet, une solution à l'équation de Yang-Baxter associée à une représentation de la relation d'entrelacement (0.2) correspond essentiellement à un modèle intégrable quantique.

[3] *intertwined*

[4] littéralement *R-matrix*

C'est cette notion de l'intégrabilité quantique, caractérisée par une dynamique formant une solution à l'équation de Yang-Baxter, que l'on retiendra.

Algèbres de Hopf

Les relations (0.2) et (0.3) sont naturelles dans le contexte des algèbres de Hopf. Ces structures algébriques permettent une définition de l'intégrabilité en quantique, d'autant plus qu'elles possèdent des équivalents classiques bien définis. On donne maintenant une définition sommaire d'une algèbre de Hopf comme suit. Premièrement, une bialgèbre est une algèbre qui est également une coalgèbre[5]:

Definition 1 (Bialgèbre)**.** Une bialgèbre A sur $\mathbb{C}$ est un espace vectoriel A doté d'un produit associatif $\nabla : A \otimes A \to A$ et d'une unité $\eta : \mathbb{C} \to A$ de même qu'un coproduit coassociatif $\Delta : A \to A \otimes A$ et une counité $\epsilon : A \to \mathbb{C}$ avec les conditions de compatibilité suivantes

$$\Delta \circ \nabla = \nabla \otimes \nabla \, \circ \, (1 \otimes \tau \otimes 1) \, \circ \, \Delta \otimes \Delta, \quad \tau(x \otimes y) \equiv y \otimes x,$$

$$\nabla \circ \epsilon = \epsilon \otimes \epsilon, \quad \Delta \circ \eta = \eta \otimes \eta, \quad \epsilon \circ \eta = 1.$$

La coassociativité du coproduit peut être exprimée par la relation suivante

$$(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta.$$

Ainsi la définition d'une algèbre de Hopf est alors

Definition 2 (Algèbre de Hopf)**.** Une algèbre de Hopf $\mathcal{A}$ est une bialgèbre munie d'un homomorphisme supplémentaire, l'antipode S, avec la relation suivante

$$\nabla \circ (S \otimes 1) \circ \Delta = \eta \circ \epsilon = \nabla \circ (1 \otimes S) \circ \Delta.$$

Il est pertinent d'élaborer sur les propriétés du coproduit Δ. Pour une algèbre de Lie $\mathfrak{g}$, ayant pour algèbre enveloppante universelle $U(\mathfrak{g})$, ce coproduit est donné explicitement par

$$\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}) \tag{0.4}$$

$$X \longmapsto X \otimes 1 + 1 \otimes X. \tag{0.5}$$

Le coproduit permet alors d'induire une action de $U(\mathfrak{g})$ sur le produit tensoriel de représentations de cette algèbre, permettant alors de voir de tels produits de représentations comme une représentation de $U(\mathfrak{g})$. Cette première propriété est de grande importance lorsque sont

[5]Ici au sens de la théorie des catégories.

considérés des systèmes composites. Par exemple, il est standard de demander que les espaces de Hilbert associés à des systèmes physiques soient des représentations projectives du groupe des rotations $SO(3)$. Aussi, on voudra que l'espace de Hilbert d'un système composite formé de tels sous-systèmes constitue également une représentation du groupe des rotations. Or, $SO(3)$ étant un groupe de Lie, on a directement cette propriété puisque les sous-systèmes sont associés à des représentations eux-mêmes. L'action d'un générateur X de $SO(3)$ sur le système composite sera alors donnée par le coproduit $\Delta(X)$. Le résultat, bien connu, de cette opération donne les lois d'additions pour les moments angulaires. Il est pertinent de souligner que la forme symétrique du coproduit dans (0.4) ne provient pas de la définition d'une algèbre de Hopf, mais bien du fait que $\mathfrak{g}$ soit une algèbre de Lie. Aussi, des déformations non-triviales du coproduit sont possibles, menant à de nouvelles structures algébriques. Les algèbres quantiques [**4**] sont précisément des algèbres de Hopf n'ayant pas un coproduit symétrique.

Ces algèbres de Hopf sont justement les structures algébriques nécessaires pour poser les relations dérivées plus haut. Essentiellement, les matrices de monodromie $T(\lambda)$ sont définies à partir de l'exponentiation d'une somme formelle d'éléments de l'algèbre associative générée par les opérateurs de l'ensemble initial $\{Q_i\}$. Aussi, ces matrices de monodromie seront des éléments de type groupe[6], en tant qu'éléments d'une algèbre de Hopf $\mathcal{A}$, de sorte que leurs coproduits seront de la forme $\Delta(T(\lambda,\mu)) = T(\lambda) \otimes T(\mu)$, pour un certain $T(\lambda,\mu) \in \mathcal{A}$, telle que la relation (0.2) puisse s'écrire

$$R\,\Delta(T) = \tau \circ \Delta(T)\,R.$$

Ainsi, la matrice R s'interprète comme l'isomorphisme d'entrelacement établissant l'équivalence entre le produit tensoriel de deux représentations V_1 et V_2 de $\mathcal{A}$ et sa permutation

$$V_1 \otimes V_2 \cong V_2 \otimes V_1.$$

De plus, cette interprétation des matrices R comme les opérateurs d'entrelacs d'algèbres de Hopf implique directement [**9, 10**] l'équation de Yang-Baxter (0.3). Ces structures demeurent également pertinentes dans un contexte classique. En effet [**5**], les solutions à l'équation de Yang-Baxter classique correspondent à des bialgèbres de Lie, et donc également des algèbres de Hopf associées aux structures de Poisson énoncées plus haut. Cependant, le passage

[6] group-like

aux algèbres de Hopf, sans que des groupes ou des algèbres de Lie soient spécifiés constitue une généralisation non-triviale de la notion de symétrie. Certaines algèbres de Hopf à la base de plusieurs systèmes intégrables quantiques ne sont reliées à l'algèbre enveloppante d'une algèbre de Lie $\mathfrak{g}$ que par une déformation $U_q(\mathfrak{g})$, appelée quantification en analogie avec la procédure similaire de quantification par déformation. Ces algèbres déformés sont des algèbres ou groupes quantiques [**19**] qui permettent d'exprimer des transformations de symétrie ne s'exprimant pas en terme de groupes de Lie.

L'existence de la structure d'algèbre de Hopf derrière l'intégrabilité d'un système classique permet le passage naturel au contexte quantique tout en préservant la notion d'intégrabilité [**7**]. On identifie alors une des charges conservées de l'ensemble $\{Q_i\}$ comme l'Hamiltonien et toutes ces charges sont générées par l'équation (0.1). La structure algébrique dans laquelle est alors construit le problème implique l'intégrabilité de celui-ci et donc sa solvabilité. Par ailleurs, les travaux fondateurs de Hans Bethe sur les systèmes intégrables quantiques ont menés à la technique de l'ansatz de Bethe qui repose également [**6**] sur cette structure d'algèbre de Hopf associée à l'équation de Yang-Baxter. Cette connection est à la base de la méthode de diffraction quantique inverse[7] qui permet de formuler la version moderne qu'est l'ansatz de Bethe algébrique.

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Partie 1

Approches algébriques aux polynômes orthogonaux

Introduction

Les expressions explicites obtenues dans l'étude de modèles exactement solubles ou de systèmes intégrables s'expriment souvent en termes de fonctions spéciales. La solvabilité de ces modèles s'explique par la présence de symétries qui peuvent être encodées mathématiquement au travers de structures algébriques. Les modèles s'interprètent alors comme les réalisations de ces structures abstraites. Il est alors attendu que les fonctions spéciales se manifestent dans la théorie des représentations de structures algébriques encodant les symétries d'un système.

Les polynômes orthogonaux forment un ensemble important de fonctions spéciales et se retrouvent dans de nombreux domaines de la physique et des mathématiques. Ces polynômes se définissent par un ensemble $\{P_n(x)\}_{n\in I\subseteq\mathbb{N}_0}$ de polynômes réels $P_n(x)$ en $x \in D \subseteq \mathbb{R}$ de degré n doté d'une forme bilinéaire $\mathcal{L}$ non dégénérée de sorte que

$$\mathcal{L}\left\{P_n(x), P_m(x)\right\} \propto \delta_{n,m}.$$

Cette partie de la thèse se concentre sur la connexion entre les polynômes orthogonaux et la théorie de la représentation de certaines structures algébriques. Le chapitre 1 repose sur le fait que les coefficients de Racah de la superalgèbre de Lie $\mathfrak{osp}(1|2)$ s'exprime en termes des polynômes de Bannai-Ito. Au travers d'une réalisation du problème de Racah de $\mathfrak{osp}(1|2)$ avec des opérateurs de Dunkl, une expansion asymptotique est utilisée pour obtenir une fonction génératrice pour les polynômes de Bannai-Ito. En analogie avec l'identification des polynômes de Krawtchouk en tant qu'éléments de matrices du groupe de rotations agissant sur les états de l'oscillateur harmonique multidimensionnel, le chapitre 2 construit les coreprésentations unitaires de groupe quantique $SU_q(3)$ et identifie subséquemment les éléments de matrices avec les polynômes de q-Krawtchouk bivariés. Cette partie de la thèse se conclut au chapitre 3 avec la construction de représentations pour une algèbre quadratique simple.

La diagonalisation d'une combinaison linéaire des générateurs de l'algèbre fait apparaître plusieurs polynômes orthogonaux classiques en tant que vecteurs propres. Dans le cas des représentations de dimension finie, ces vecteurs de base sont donnés par des parapolynômes, qui ne figurent pas dans les classifications standards des polynômes orthogonaux.

Chapitre 1

Generating function for the Bannai-Ito polynomials

G. Bergeron, L. Vinet and S. Tsujimoto (2018). Generating function for the Bannai-Ito polynomials. *Proceedings of the American Mathematical Society* **146** (12), 5077-5090.

Abstract

A generating function for the Bannai-Ito polynomials is derived using the fact that these polynomials are known to be essentially the Racah or $6j$ coefficients of the $\mathfrak{osp}(1|2)$ Lie superalgebra. The derivation is carried in a realization of the recoupling problem in terms of three Dunkl oscillators.

1.1. Introduction

In a previous paper [**2**], generating functions for the dual -1 Hahn polynomials were derived using the Clebsch-Gordan problem of the $\mathfrak{osp}(1|2)$ Lie superalgebra. In the present case, we exploit again the fact that $\mathfrak{osp}(1|2)$ is the dynamical algebra of a parabosonic or Dunkl oscillator. The generating function of the Bannai-Ito polynomials is found by using the wavefunctions of this system and recalling [**5**] that the Racah coefficients for $\mathfrak{osp}(1|2)$ are given in terms of these polynomials.

Related approaches using wavefunction realizations of dynamical algebra to derive identities for orthogonal polynomials have been presented previously [**8, 9, 12, 13**]. In particular, [**3**] uses manipulations of wavefunctions similar to the ones that will be presented here to derive an integral representation of recoupling coefficients.

1.1.1. The Bannai-Ito polynomials

The Bannai-Ito polynomials, introduced in [**1**], denoted here by $B_n(x)$, depend on four parameters $\{r_1, r_2, \rho_1, \rho_2\}$ and can be defined [**11**], see also [**14**], as the functions diagonalizing the difference operator

$$\frac{(x-\rho_1)(x-\rho_2)}{2x}(I-P_x)+\frac{(x-r_1+1/2)(x-r_2+1/2)}{2x+1}(P_xD_x-I),$$

where P_x is the reflection operator acting on functions of x as $P_xf(x)=f(-x)$ and D_x is the forward shift operator acting as $D_xf(x)=f(x-1)$ and with the eigenvalues λ_n given by

$$\lambda_n=\begin{cases}\dfrac{n}{2} & \text{for } n \text{ even},\\ r_1+r_2-\rho_1-\rho_2-\dfrac{n+1}{2} & \text{for } n \text{ odd}.\end{cases}$$

They satisfy a three-term recurrence relation

$$xB_n(x)=B_{n+1}(x)+(\rho_1-a_n-c_n)B_n(x)+a_{n-1}c_nB_{n-1}(x),$$

with coefficients

$$a_n=\begin{cases}\dfrac{(n+2\rho_1-2r_1+1)(n+2\rho_1-2r_2+1)}{4(n+\rho_1+\rho_2-r_1-r_2+1)} & \text{for } n \text{ even},\\[2ex] \dfrac{(n+2\rho_1+2\rho_2-2r_1-2r_2+1)(n+2\rho_1+2\rho_2+1)}{4(n+\rho_1+\rho_2-r_1-r_2+1)} & \text{for } n \text{ odd},\end{cases} \tag{1.1}$$

$$c_n=\begin{cases}\dfrac{-n(n-2r_1-2r_2)}{4(n+\rho_1+\rho_2-r_1-r_2)} & \text{for } n \text{ even},\\[2ex] \dfrac{-(n+2\rho_2-2r_2)(n+2\rho_2-2r_1)}{4(n+\rho_1+\rho_2-r_1-r_2)} & \text{for } n \text{ odd},\end{cases} \tag{1.2}$$

and initial conditions $B_{-1}(x)=0$, $B_0(x)=1$. The possible choices of truncation conditions for the recurrence relation are, for N even,

$$2(r_i-\rho_k)=N+1,\quad i,k=1,2, \tag{1.3}$$

and, for N odd,

$$\rho_1+\rho_2=-(N+1)/2,\quad \text{or}\quad r_1+r_2=(N+1)/2. \tag{1.4}$$

In this work, the truncation conditions used are

$$2(r_2-\rho_1)=N+1,\ \text{ for } N \text{ even},\ \rho_1+\rho_2=-(N+1)/2,\ \text{ for } N \text{ odd}. \tag{1.5}$$

The orthogonality of the Bannai-Ito polynomials

$$\sum_{S=0}^{N} w_S B_n(x_S) B_m(x_S) = h_N \delta_{nm}, \quad S=0,\dots,N, \tag{1.6}$$

is with respect to a discrete measure of weights w_S on the grid x_S, for $S=2s+p\in\{0,...,N\}$ and $p\in\{0,1\}$, with normalization h_N where

$$w_S=\frac{(-1)^p(\rho_1-r_1+1/2)_{s+p}(\rho_1-r_2+1/2)_{s+p}(\rho_1+\rho_2+1)_s}{(\rho_1+r_1+1/2)_{s+p}(\rho_1+r_2+1/2)_{s+p}(1)_s(\rho_1-\rho_2+1)_s}, \tag{1.7}$$

with $(a)_m=a(a+1)...(a+m-1)$ the rising Pochhammer symbol, and

$$x_S=\frac{(-1)^S(S+2\rho_1+1/2)-1/2}{2}, \tag{1.8}$$

$$h_N=\begin{cases}\dfrac{(2\rho_1+1)_{N/2}(r_1-\rho_2+1/2)_{N/2}}{(\rho_1-\rho_2+1)_{N/2}(\rho_1+r_1+1/2)_{N/2}}, & N \text{ even},\\[2ex] \dfrac{(2\rho_1+1)_{(N+1)/2}(r_1+r_2)_{(N+1)/2}}{(\rho_1+r_1+1/2)_{(N+1)/2}(\rho_1+r_2+1/2)_{(N+1)/2}}, & N \text{ odd},\end{cases} \tag{1.9}$$

where $N=|\rho_2+r_2|+r_2-\rho_2-2\rho_1-1$.

1.1.2. The $\mathfrak{osp}(1|2)$ algebra

The $\mathfrak{osp}(1|2)$ algebra is generated by two odd elements $K_\pm$ and one even element K_0, relative to a $\mathbb{Z}_2$-grading. The presentation used in this paper makes this grading explicit by the introduction of a grade involution operator R that commutes/anticommutes with the even/odd elements of the algebra. This presentation, also referred to as the $\mathfrak{sl}_{-1}(2)$ algebra [**10**] in the literature, is given by the four generators K_0, $K_\pm$ and R together with the relations

$$[K_0,K_\pm]=\pm K_\pm,\quad [K_0,R]=0,\quad \{K_+,K_-\}=2K_0,\quad \{K_\pm,R\}=0,\quad R^2=1, \tag{1.10}$$

with $[a,b]=ab-ba$ and $\{a,b\}=ab+ba$. The Casimir operator for the algebra as presented in (1.10) is given by

$$C=(K_+K_--K_0+1/2)R. \tag{1.11}$$

The irreducible positive-discrete series representations of $\mathfrak{osp}(1|2)$ are then labeled by two numbers (μ, ϵ) where $\mu \geq 0$ and $\epsilon = \pm 1$. The actions of the generators on the orthonormal basis vectors $|n, \mu, \epsilon\rangle$ with $n \in \mathbb{N}$ are

$$\begin{aligned}
&K_0 \; |n, \mu, \epsilon\rangle = (n + \mu + 1/2) \; |n, \mu, \epsilon\rangle \,, && R \, |n, \mu, \epsilon\rangle = \epsilon \, (-1)^n \; |n, \mu, \epsilon\rangle \,, \\
&K_+ \; |n, \mu, \epsilon\rangle = \sqrt{[n+1]_\mu} \; |n+1, \mu, \epsilon\rangle \,, && K_- \; |n, \mu, \epsilon\rangle = \sqrt{[n]_\mu} \; |n-1, \mu, \epsilon\rangle \,,
\end{aligned} \tag{1.12}$$

where $[n]_\mu = n + \mu(1 - (-1)^n)$. In these representations, the Casimir (1.11) assumes the value

$$C|n, \mu, \epsilon\rangle = -\epsilon\mu|n, \mu, \epsilon\rangle.$$

1.1.3. Realization as a dynamical algebra

The presentation (1.10) of $\mathfrak{osp}(1|2)$ can be realized [**6, 7**] in terms of operators acting on functions of a real variable x. Let P_x denote the parity operator acting on functions as $P_x f(x) = f(-x)$. The $\mathbb{Z}_2$-Dunkl derivative is defined by

$$\mathfrak{D}_x = \partial_x + \frac{\mu}{x}(1 - P_x).$$

The $\mathfrak{osp}(1|2)$ algebra is realized under the following identification of the generators:

$$K_0 = -\frac{1}{2}\mathfrak{D}_x^2 + \frac{1}{2}x^2, \quad K_\pm = \frac{1}{\sqrt{2}}(x \mp \mathfrak{D}_x), \quad R = P_x. \tag{1.13}$$

This casts $\mathfrak{osp}(1|2)$ as the dynamical algebra of the parabose oscillator [**10**] whose Hamiltonian H is the operator that realizes K_0. It follows that the position operator and its associated eigenvectors are

$$X = \frac{1}{\sqrt{2}}(K_+ + K_-), \quad X|x, \mu, \epsilon\rangle = x|x, \mu, \epsilon\rangle. \tag{1.14}$$

The representation basis (1.12) corresponds to the energy eigenstates with eigenvalues $E = n + \mu + 1/2$ and can be modeled by the wavefunctions $\Psi_n^{\mu,\epsilon}(x)$ defined through

$$\Psi_n^{\mu,\epsilon}(x) = \langle x, \mu, \epsilon | n, \mu, \epsilon\rangle. \tag{1.15}$$

1.1.4. The Racah problem of $\mathfrak{osp}(1|2)$

The $\mathfrak{osp}(1|2)$ algebra also forms a Hopf algebra [**4**] where the coproduct Δ is given in the presentation (1.10) as

$$\Delta(K_0) = K_0 \otimes 1 + 1 \otimes K_0, \quad \Delta(R) = R \otimes R, \quad \Delta(K_\pm) = K_\pm \otimes R + 1 \otimes K_\pm. \tag{1.16}$$

This Hopf algebra structure induces an action of $\mathfrak{osp}(1|2)$ on tensor products of modules. Consider the following threefold tensor product of irreducible representations

$$(\mu_1, \epsilon_1) \otimes (\mu_2, \epsilon_2) \otimes (\mu_3, \epsilon_3). \tag{1.17}$$

One can decompose this product of representations in a direct sum of irreducible representations in two different ways, corresponding to the order in which the coproduct is used to induce an action of $\mathfrak{osp}(1|2)$ on (1.17), either $\Delta \otimes 1 \circ \Delta$ or $1 \otimes \Delta \circ \Delta$. Both cases specify an algebra homomorphism $\mathfrak{osp}(1|2) \to \mathfrak{osp}(1|2) \otimes \mathfrak{osp}(1|2) \otimes \mathfrak{osp}(1|2)$ and an associated decomposition of threefold tensor products of representations into direct sums of irreducible representations

$$\begin{aligned}(\mu_1, \epsilon_1) \otimes (\mu_2, \epsilon_2) \otimes (\mu_3, \epsilon_3) &\overset{\Delta\otimes 1\circ\Delta}{\cong} \bigoplus_u (\mu_{(12)3}(u), \epsilon_{(12)3}(u)), \\ &\overset{1\otimes\Delta\circ\Delta}{\cong} \bigoplus_v (\mu_{1(23)}(v), \epsilon_{1(23)}(v)).\end{aligned} \tag{1.18}$$

In fact, by the coassociativity of the coproduct, we have that two irreducible representations connected in such a way are isomorphic

$$(\mu_{(12)3}, \epsilon_{(12)3}) \cong (\mu_{1(23)}, \epsilon_{1(23)}) \tag{1.19}$$

and thus, we will only keep the notation distinguishing the two in the labels when relevant.

The basis constructed as in (1.12) for these representations does not uniquely determine the map (1.18) on the basis vectors themselves, but a canonical choice of supplementary labels exists that removes the degeneracy. One demands that the basis vectors of $(\mu_{(12)3}, \epsilon_{(12)3})$, (respectively $(\mu_{1(23)}, \epsilon_{1(23)})$), diagonalize the intermediate Casimir operator $C_{12} = \Delta(C) \otimes 1$, (resp. $C_{23} = 1 \otimes \Delta(C)$). Thus, denoting the action of $\Delta \otimes 1 \circ \Delta = 1 \otimes \Delta \circ \Delta$ on generators $A \in \mathfrak{osp}(1|2)$ by

$$\Delta \otimes 1 \circ \Delta : A \mapsto \hat{A} \in \mathfrak{osp}(1|2) \otimes \mathfrak{osp}(1|2) \otimes \mathfrak{osp}(1|2), \tag{1.20}$$

and knowing the two modules $(\mu_{(12)3}, \epsilon_{(12)3})$ and $(\mu_{1(23)}, \epsilon_{1(23)})$ are identical as $\mathfrak{osp}(1|2)$ representations, we have that basis vectors for both satisfy

$$\begin{aligned}
&\hat{K}_0 \, |n_{123}\rangle = (n_{123} + \mu_{123} + 1/2) \, |n_{123}\rangle \,, \qquad && \hat{R} \, |n_{123}\rangle = \epsilon_{123} \, (-1)^{n_{123}} \, |n_{123}\rangle \,, \\
&\hat{K}_+ \, |n_{123}\rangle = \sqrt{[n_{123}+1]_{\mu_{123}}} \, |n_{123}+1\rangle \,, && \hat{K}_- \, |n_{123}\rangle = \sqrt{[n_{123}]_{\mu_{123}}} \, |n_{123}-1\rangle \,, \\
&\hat{C} \, |n_{123}\rangle = -\mu_{123} \, \epsilon_{123} \, |n_{123}\rangle \,,
\end{aligned} \tag{1.21}$$

where $|n_{123}\rangle$ stands for either $\left|n_{(12)3}, \mu_{(12)3}, \epsilon_{(12)3}\right\rangle$ or $\left|n_{1(23)}, \mu_{1(23)}, \epsilon_{1(23)}\right\rangle$. The degeneracy is then lifted through the actions of the intermediate Casimirs

$$C_{12} \left|n_{(12)3}, \mu_{(12)3}, \epsilon_{(12)3}\right\rangle = -\mu_{12} \, \epsilon_{12} \left|n_{(12)3}, \mu_{(12)3}, \epsilon_{(12)3}\right\rangle , \tag{1.22a}$$

$$C_{23} \left|n_{1(23)}, \mu_{1(23)}, \epsilon_{1(23)}\right\rangle = -\mu_{23} \, \epsilon_{23} \left|n_{1(23)}, \mu_{1(23)}, \epsilon_{1(23)}\right\rangle . \tag{1.22b}$$

These bases are not the same since $[C_{12}, C_{23}] \neq 0$. The $\mathfrak{osp}(1|2)$ Racah problem consists in determining the overlaps $\mathcal{R}$ between the two bases (1.21)

$$\mathcal{R} = \langle n_{(12)3}, \mu_{(12)3}, \epsilon_{(12)3} \left| n_{1(23)}, \mu_{1(23)}, \epsilon_{1(23)} \right\rangle . \tag{1.23}$$

1.1.5. Outline

We will first explain the realization of the Racah problem in terms of a system of three parabose harmonic oscillators and will indicate how this realization relates to generating functions in section 2. Section 3 gives the explicit expressions of the angular wavefunctions in each parity case of the parameters and a derivation of their asymptotic form in the relevant limits. Finally, section 4 contains the derivation of the generating functions and is followed by a brief conclusion.

1.2. Realization of the Racah decomposition

The Racah problem of $\mathfrak{osp}(1|2)$ can be expressed within the dynamical algebra realization by considering three uncoupled parabose oscillators in the Cartesian coordinates $\{x, y, z\}$. The total Hamiltonian for this system is simply the sum of the separate Hamiltonians

$$H_{xyz} = H_x + H_y + H_z = K_0 \otimes 1 \otimes 1 + 1 \otimes K_0 \otimes 1 + 1 \otimes 1 \otimes K_0 = \hat{K}_0.$$

The Schrödinger equation $H_{xyz}|\psi\rangle = E_{xyz}|\psi\rangle$ manifestly separates in the Cartesian coordinates. In [**7**], it was shown that it also separates in spherical coordinates. This separation is associated to the symmetries generated by the intermediate Casimir operators C_{12} and C_{23}. In fact, the spherical wavefunctions are constructed [**4**] using the basis (1.21). Not surprisingly then, the Racah problem is directly related to the different possible choices in the construction of the spherical coordinates.

1.2.1. Spherical coordinates realization

The position operator X introduced in (1.14) can naturally be extended to a set of three operators acting on threefold tensor product of irreducible representations as

$$X = X \otimes 1 \otimes 1, \quad Y = 1 \otimes X \otimes 1, \quad Z = 1 \otimes 1 \otimes X,$$

where the X operator in the right-hand side is the one defined in (1.14). From these, one can define the radial operator $X^2+Y^2+Z^2$. It commutes [**6**] with the intermediate Casimirs C_{12} and C_{23}. Thus, the two bases introduced in (1.21) do not differ in their radial parts and the Racah problem is entirely determined by the angular wavefunctions. We may as well take the radius to be fixed and consider the Racah problem on a fixed eigenspace of the radial operator $\hat{X}^2$.

The angular wavefunctions will be defined as the functions satisfying (1.22a) or (1.22b) under the action of the $\mathfrak{osp}(1|2)$ algebra in the coordinate realization and under the constraint $x^2+y^2+z^2 = 1$, where x, y and z are the eigenvalues of the X, Y and Z operators, respectively. As such, these functions are defined on the two-dimensional sphere and can be parametrized by two angles θ and ϕ. We choose these angles to be related to the Cartesian coordinates as usual through

$$x = \sin\theta\cos\phi, \quad y = \sin\theta\sin\phi, \quad z = \cos\theta. \tag{1.24}$$

Using these relations, the realization (1.13) of $\mathfrak{osp}(1|2)$ can be expressed as differential operators in the angular coordinates [**7**]. The angular wavefunctions are then given by

$$\mathcal{Y}_{n_{(12)3}}^{\mu_{(12)3},\epsilon_{(12)3}}(\theta,\phi) = \left\langle \theta,\phi \,\middle|\, n_{(12)3},\mu_{(12)3},\epsilon_{(12)3} \right\rangle, \quad \text{with} \quad x^2+y^2+z^2=1.$$

A similar expression is defined for the other basis with a different set of angular variables $\{\alpha,\beta\}$ by

$$\mathcal{Z}_{n_{1(23)}}^{\mu_{1(23)},\epsilon_{1(23)}}(\alpha,\beta) = \left\langle \alpha,\beta \,\middle|\, n_{1(23)},\mu_{1(23)},\epsilon_{1(23)} \right\rangle, \quad \text{with} \quad x^2+y^2+z^2=1.$$

It is possible to relate the second set of variables to the first by observing that a permutation of the terms in the threefold tensor product of irreducible representations (1.17) maps the basis (1.22a) to (1.22b). Explicitly, this permutation is the cycle $(1\,2\,3)$ acting on the Cartesian coordinates $\{x, y, z\}$. In terms of the angular variables, this corresponds to the relations

$$\sin\alpha\cos\beta = \sin\theta\sin\phi, \quad \sin\alpha\sin\beta = \cos\theta, \quad \cos\alpha = \sin\theta\cos\phi. \tag{1.25}$$

In view of (1.19), the decomposition of these angular wavefunctions onto each other exists and will have the Racah coefficients as overlaps

$$\mathcal{Z}^{\mu_{1(23)},\epsilon_{1(23)}}_{n_{1(23)}}(\alpha(\theta,\phi),\beta(\theta,\phi)) = \sum \mathcal{R}\,\mathcal{Y}^{\mu_{(12)3},\epsilon_{(12)3}}_{n_{(12)3}}(\theta,\phi), \tag{1.26}$$

where $\alpha(\theta,\phi)$ and $\beta(\theta,\phi)$ are obtained from (1.25).

1.2.2. Exact form of the decomposition

Let us now make details explicit. First consider a basis vector of (1.17), which we here denote by
$|n_1,\mu_1,\epsilon_1\rangle \otimes |n_2,\mu_2,\epsilon_2\rangle \otimes |n_3,\mu_3,\epsilon_3\rangle$. In view of (1.18), we may write a decomposition of the form

$$|n_1,\mu_1,\epsilon_1\rangle \otimes |n_2,\mu_2,\epsilon_2\rangle \otimes |n_3,\mu_3,\epsilon_3\rangle = \sum_u \mathcal{C}_u \, |n_{123},\mu_{123},\epsilon_{123}\rangle_u \,. \tag{1.27}$$

For this equality to hold given the action of $\hat{K}_0$ and $\hat{R}$ and knowing that the representation parameters μ_{123} and ϵ_{123} cannot depend on the basis label n_{123} one obtains the following relations

$$n_{123} = n_1 + n_2 + n_3 - N, \quad \mu_{123} = \mu_1 + \mu_2 + \mu_3 + 1 + N, \quad \epsilon_{123} = \epsilon_1\epsilon_2\epsilon_3(-1)^N, \tag{1.28}$$

where $N \in [0, n_1 + n_2 + n_3] \subset \mathbb{N}$. The difference between the two bases (1.22a) and (1.22b) arises when considering the operators C_{12} and C_{23}. Being intermediate Casimirs, these operators satisfy

$$[C_{12}, \Delta(A)\otimes 1] = 0 = [C_{23}, 1\otimes\Delta(A)] \quad \forall \quad A \in \mathfrak{osp}(1|2).$$

Demanding their diagonalisation as in (1.22a) or (1.22b) requires the decomposition (1.27) to solve the Clebsch-Gordan problem [**2**], if one focuses only on the first or second pair of terms in the tensor product (1.17). It is known that the parameters involved in the

Clebsch-Gordan decomposition into the product representation $(\mu_i, \epsilon_i) \otimes (\mu_j, \epsilon_j)$ must verify, for $n_i + n_j \geq q \in \mathbb{N}$,

$$n_{ij} = n_i + n_j - q, \quad \mu_{ij} = \mu_i + \mu_j + q + 1/2, \quad \epsilon_{ij} = \epsilon_i \epsilon_j (-1)^q, \tag{1.29}$$

corresponding to the diagonalization of the intermediate Casimir C_{ij}. Rewriting (1.28) in view of (1.29), one has that the labels of basis vectors in (1.27) with non-vanishing overlap and where $|n_{123}, \mu_{123}, \epsilon_{123}\rangle$ diagonalizes C_{ij} are related by the following equations

$$n_{123} = n_{ij} + n_k - l, \qquad \mu_{123} = \mu_{ij} + \mu_k + 1/2 + l, \qquad \epsilon_{123} = \epsilon_{ij}\epsilon_k(-1)^l, \tag{1.30}$$

$$l = N - q, \qquad N \in [0, n_1 + n_2 + n_3] \subset \mathbb{N}, \qquad \implies q \in [0, N] \subset \mathbb{N}. \tag{1.31}$$

where $i, j \in \{(1,2),(2,3)\}$ and $i, j \neq k \in \{1,2,3\}$ index the terms of the threefold tensor product (1.17).

For given values of μ_1, μ_2, μ_3, ϵ_1, ϵ_2, ϵ_3 and N, the parameters μ_{123} and ϵ_{123} are fixed and, since $l \geq 0$, there are $N+1$ ways of choosing q. Thus, the decomposition of the tensor product of three irreducible $\mathfrak{osp}(1|2)$ representations can be expressed as

$$(\mu_1, \epsilon_1) \otimes (\mu_2, \epsilon_2) \otimes (\mu_3, \epsilon_3) \cong \bigoplus_{N=0}^{\infty} \bigoplus_{q=0}^{N} (\mu_{123}(N), \epsilon_{123}(N))_q,$$

where q indexes as in (1.29) the possible eigenvalues of the intermediate Casimir C_{ij}.

Consider now the Racah coefficients $\mathcal{R}$ as given in (1.23) where both basis vectors come from one of the two different decompositions (1.18) of the same threefold tensor product of irreducible representations (1.17). As the two modules in consideration are identical as $\mathfrak{osp}(1|2)$ modules, the Racah coefficients vanish if the labels of the basis vectors differ. The only free parameter in non-zero coefficients is the value of the intermediate Casimirs. Thus, writing as K and S those free parameters indexing the values of the intermediate Casimirs for the two basis vectors in the overlap, the Racah decomposition will explicitly be written as

$$|n_{123}, \mu_{123}, \epsilon_{123}, \mu_{12}(S)\rangle = \sum_{K=0}^{N} \mathcal{R}^{\mu_1,\mu_2,\mu_3}_{S,K,N} \, |n_{123}, \mu_{123}, \epsilon_{123}, \mu_{23}(K)\rangle \, .$$

This equation can be rewritten in terms of the wavefunctions. As said, the overlap between two such wavefunctions is directly proportional to the Bannai-Ito polynomials [**4**]:

$$\mathcal{Z}_S^N(\alpha(\theta,\phi),\beta(\theta,\phi)) = \sum_{K=0}^{N} \mathcal{R}_{S,K,N}^{\mu_1,\mu_2,\mu_3} \mathcal{Y}_K^N(\theta,\phi), \tag{1.32}$$

$$\mathcal{R}_{S,K,N}^{\mu_1,\mu_2,\mu_3} = \Phi_S^N \sqrt{\frac{w_S}{h_N u_1 u_2 \ldots u_K}} B_K(x_S;\rho_1,\rho_2,r_1,r_2) \tag{1.33}$$

with w_S, x_S and h_N as in (1.7) and (1.9) and where the B_K are the Bannai-Ito polynomials. The u_i are given by $u_i = a_{n-1}b_n$ with a_n and b_n as in (1.1) and (1.2). The choice of phase Φ_S^N is different than in [**4**]. In this work, writing $N = 2n+t \in \mathbb{N}$ and $S = 2s+p$ with $p,t \in \{0,1\}$ the phase is given by

$$\Phi_S^N = (-1)^{n+t(1-p)}. \tag{1.34}$$

The connection between the parameters of the threefold tensor product (1.17) and the parameters of the Bannai-Ito polynomials in (1.33) is as follows

$$\begin{aligned} &\rho_1 = \frac{\mu_2+\mu_3}{2}, \quad \rho_2 = \frac{\mu_1+\mu}{2}, \quad r_1 = \frac{\mu_3-\mu_2}{2}, \quad r_2 = \frac{\mu-\mu_1}{2} \\ &\mu = (-1)^N(N+1+\mu_1+\mu_2+\mu_3). \end{aligned} \tag{1.35}$$

1.2.3. Generating function from the Racah problem

The wavefunction realization of the Racah decomposition (1.32) leads to a functional decomposition with coefficients proportional to the Bannai-Ito polynomials [**5**]. To obtain generating functions, one needs to reduce the right-hand side of (1.32) to a power series of a single variable. As shall be explicit, the angular wavefunctions are polynomials of trigonometric functions which reduces, under some asymptotic expansion, to their leading terms. Monomials are obtained from the expansion by the simultaneous introduction of a suitable relation between the angle variables. However, this procedure must be carried while preventing the trivialization of the left-hand side of (1.32).

In view of the form of the wavefunctions given in section 3.1, one is led to consider the expansion $|\theta| \to 0$. To prevent a trivialization we introduce, as follows, the finite variable $z = \cos\alpha$ and use (1.25) under the asymptotic expansion to obtain the following

$$\sin\alpha = \sqrt{1-z^2}, \quad \sin\beta = \frac{1}{\sqrt{1-z^2}}, \quad \cos\beta = i\frac{z}{\sqrt{1-z^2}}, \tag{1.36}$$

The finiteness of z together with (1.25) implies $\mathfrak{Im}(\phi) \to \infty$. With $\mathfrak{Im}(\phi) \geq 0$, one demands that the following limits be defined and give

$$\cosh \mathfrak{Im}(\phi) \sin\theta \to \lambda, \quad \sinh \mathfrak{Im}(\phi) \sin\theta \to \lambda,$$

such that compatibility with (1.25) and (1.36) is maintained and

$$z = \lambda e^{-i\mathfrak{Re}(\phi)}. \tag{1.37}$$

Using (1.25) and (1.36), the following useful relation can be obtained under the asymptotic limit

$$\sin\phi \approx i\cos\phi. \tag{1.38}$$

Under this asymptotic limit, the decomposition (1.32) will take the form of a generating function for the sum of two Bannai-Ito polynomials

$$\mathcal{Z}_S^N(z) = \sum_{K=0}^{N} \mathcal{R}_{S,K,N}^{\mu_1,\mu_2,\mu_3} \mathcal{Y}_K^N(z), \tag{1.39}$$

where $\mathcal{Y}_K^N(z)$ is a sum of two monomials of the z variable. It should be noted, in view of (1.37) and since the parameters λ and $\mathfrak{Re}\,(\phi)$ are not fixed, that z can be any complex number.

1.3. Wavefunctions and their asymptotic forms

As the Racah problem is fully contained in the overlaps of angular wavefunctions, one does not need a set of basis functions that reflects the full degeneracy of the Hamiltonian H_{xyz}. We shall use instead functions of definite parity on which the total Casimir is diagonal. This is justified by remembering that we have $\mathcal{R} \neq 0$ only when the overlap is between two basis vectors from the same eigenspace of the total Hamiltonian. These functions form a basis of the irreducible representations (1.19) and are sufficient for our purpose but do not reflect the full degeneracy of the initial Shrödinger equation. This can be seen from the fact that the operators R_i , $i \in \{1, 2, 3\}$ commute with the total Hamiltonian, but not with the total Casimir, see [**4**].

1.3.1. Angular Wavefunctions

The explicit form of the basis functions used in this work can be obtained by solving the relevant system of Dunkl differential equations. We assume $\epsilon_{123} = 1$ for the rest of this work. In this case, the angular wavefunctions $\mathcal{Y}_K^N(\theta,\phi)$ for $K = 0, ..., N$ satisfy the following equations

$$\begin{aligned}
\hat{C}\hat{R}\,\mathcal{Y}_K^N(\theta,\phi) &= -(N+\mu_1+\mu_2+\mu_3+1)\mathcal{Y}_K^N(\theta,\phi),\\
\hat{R}\,\mathcal{Y}_K^N(\theta,\phi) &= (-1)^N\mathcal{Y}_K^N(\theta,\phi),\\
C_{12}\,\mathcal{Y}_K^N(\theta,\phi) &= -(-1)^K(K+\mu_1+\mu_2)\mathcal{Y}_K^N(\theta,\phi),
\end{aligned}$$

where these operators are defined on (1.17) using (1.20) and the realization (1.13).

The solutions [**4**] correspond to (a subset of) the wavefunctions built on the basis (1.22a) and are given, writing $N = 2n + t$, $n \in \mathbb{N}$ and $K = 2k + p \in \{0, ..., N\}$ with $p, t \in \{0, 1\}$, by

$$\begin{aligned}
\mathcal{Y}_K^N(\theta,\phi) = A_K\Big\{&B_K\cos^t\theta\sin^{2k+2p}\theta P_{n-k-p}^{(2k+2p+\mu_1+\mu_2,\mu_3-1/2+t)}(\cos 2\theta)\mathcal{F}_K^+(\phi)\\
&+(-1)^t B_K^{-1}\cos^{1-t}\theta\sin^{2k+1}\theta P_{n-k-1}^{(2k+1+\mu_1+\mu_2,\mu_3+1/2-t)}(\cos 2\theta)\mathcal{F}_K^-(\phi)\Big\},
\end{aligned}\tag{1.40}$$

where A_K and B_K are

$$\begin{aligned}
A_K &= (-1)^{tK}\sqrt{\frac{(n-k+p(t-1))!\Gamma(n+k+\mu_1+\mu_2+\mu_3+3/2+pt)}{\Gamma(n+k+\mu_1+\mu_2+1+pt)\Gamma(n-k+\mu_3+1/2+p(t-1))}},\\
B_K &= \left(\frac{n-k+\mu_3-1/2+t}{n+k+\mu_1+\mu_2+1}\right)^{(p-t)/2}.
\end{aligned}$$

and where the $\mathcal{F}_K$ functions are as follows

$$\begin{aligned}
\mathcal{F}_K^+(\phi) = \xi_K^+\Big\{&E_K P_{k+p}^{(\mu_2-1/2,\mu_1-1/2)}(\cos 2\phi)\\
&-(-1)^p E_K^{-1}\cos\phi\sin\phi P_{k+p-1}^{(\mu_2+1/2,\mu_1+1/2)}(\cos 2\phi)\Big\},
\end{aligned}\tag{1.41}$$

$$\begin{aligned}
\mathcal{F}_K^-(\phi) = \xi_K^-\Big\{&F_K\sin\phi P_k^{(\mu_2+1/2,\mu_1-1/2)}(\cos 2\phi)\\
&+(-1)^p F_K^{-1}\cos\phi P_k^{(\mu_2-1/2,\mu_1+1/2)}(\cos 2\phi)\Big\},
\end{aligned}\tag{1.42}$$

with

$$\xi_K^+ = \sqrt{\frac{(k+p)!\Gamma(k+\mu_1+\mu_2+1+p)}{2\Gamma(k+\mu_1+1/2+p)\Gamma(k+\mu_2+1/2+p)}}, \qquad E_K = \left(\frac{k+1}{k+\mu_1+\mu_2+1}\right)^{p/2},$$

$$\xi_K^- = \sqrt{\frac{k!\Gamma(k+\mu_1+\mu_2+1)}{2\Gamma(k+\mu_1+1/2)\Gamma(k+\mu_2+1/2)}}, \qquad F_K = \left(\frac{k+\mu_1+1/2}{k+\mu_2+1/2}\right)^{p/2}.$$

A second wavefunction basis is obtained by reparametrizing the sphere in terms of the angular coordinates α, β as per (1.25). These wavefunctions, denoted $\mathcal{Z}_S^N(\alpha,\beta)$ for $S = 0, ..., N$, now satisfy the following equations

$$\begin{aligned}
\hat{C}\hat{R}\,\mathcal{Z}_S^N(\alpha,\beta) &= -(N+\mu_1+\mu_2+\mu_3+1)\mathcal{Z}_S^N(\alpha,\beta),\\
\hat{R}\,\mathcal{Z}_S^N(\alpha,\beta) &= (-1)^N\mathcal{Z}_S^N(\alpha,\beta),\\
C_{23}\,\mathcal{Z}_S^N(\alpha,\beta) &= -(-1)^S(S+\mu_2+\mu_3)\mathcal{Z}_S^N(\alpha,\beta)
\end{aligned}$$

and realize the basis defined by (1.22b). They can be written [**4**] in terms of the first basis of wavefunctions $\mathcal{Y}_K^N$ as

$$\mathcal{Z}_S^N(\alpha,\beta) = \begin{cases} (\,1\,2\,3\,)\,\mathcal{Y}_K^N(\pi-\alpha,\beta), & \text{for } N \text{ even},\\ (\,1\,2\,3\,)\,\mathcal{Y}_K^N(\alpha,\beta), & \text{for } N \text{ odd}. \end{cases} \tag{1.43}$$

where $(\,1\,2\,3\,)$ is the permutation cycle acting on the parameters (μ_1,μ_2,μ_3). This follows from the fact that this permutation induces a mapping from the basis of $(\mu_{(12)3}, \epsilon_{(12)3})$ to the basis of $(\mu_{1(23)}, \epsilon_{1(23)})$ when acting on the terms of the threefold tensor product (1.17).

1.3.2. Asymptotic Expansion

We now derive the asymptotic expansion introduced in section 2.3 of the angular wavefunctions (1.40). One will need the leading term and the value at 1 of Jacobi polynomials given by

$$P_n^{(a,b)}(x) \to 2^{-n}\binom{2n+a+b}{n}x^n, \qquad P_n^{(a,b)}(1) = \binom{n+a}{n}.$$

The polynomials in the $\mathcal{F}_K$ functions (1.41), (1.42) have for variable $\cos 2\phi \to \infty$ and only their leading terms will remain. Using (1.38) in (1.41) or (1.42) while considering only the leading term leads to

$$\mathcal{F}_K^+(\phi) \to \xi_K^+ E_K \binom{2k+2p+\mu_2+\mu_1-1}{k+p} \Psi_+ \cos^{2k+2p}\phi, \tag{1.44}$$

$$\mathcal{F}_K^-(\phi) \to \xi_K^- F_K \binom{2k+\mu_1+\mu_2}{k} \Psi_- \cos^{2k+1}\phi, \tag{1.45}$$

with Ψ_+ and Ψ_- given by

$$\Psi_\pm = \begin{cases} \left[1 - i(-1)^p \frac{k+p}{k+p+\mu_1+\mu_2}\left(\frac{k+\mu_1+\mu_2+1}{k+1}\right)^p\right] & \text{for the case } +, \\ \\ \left[i + (-1)^p \left(\frac{k+\mu_2+1/2}{k+\mu_1+1/2}\right)^p\right] & \text{for the case } -. \end{cases} \tag{1.46}$$

Consider now the full wavefunctions (1.40) under the asymptotic expansion. The remaining Jacobi polynomials are evaluated at 1 as their arguments $\cos 2\theta \to 1$. The remaining cosine terms also simply become $\cos\theta = 1$. The sine terms approach zero, but will be compensated by the $\mathcal{F}_K$ functions which are divergent under the asymptotic expansion. Thus, leaving the sine terms, one is led to the following expressions for the asymptotic wavefunctions for $N = 2n+t$

$$\begin{aligned} \mathcal{Y}_K^N(\theta,\phi) \to A_K \Bigg\{ & B_K \binom{n+k+p+\mu_1+\mu_2}{n-k-p} \mathcal{F}_K^+(\phi) \sin^{2k+2p}\theta \\ & + (-1)^t B_K^{-1} \binom{n+t+k+\mu_1+\mu_2}{n-k-1+t} \mathcal{F}_K^-(\phi) \sin^{2k+1}\theta \Bigg\}. \end{aligned} \tag{1.47}$$

We now remind the reader that from (1.25) and (1.36) we have $\cos\phi\sin\theta = z$. By construction, this z variable remains finite in the asymptotic expansion. Using (1.44) and (1.45) to rewrite (1.47) in terms of the z variable leads to, for $N = 2n+t$

$$\begin{aligned} \mathcal{Y}_K^N(z) = A_K \Bigg\{ & \\ & \xi_K^+ B_K E_K \binom{n+k+p+\mu_1+\mu_2}{n-k-p} \binom{2k+2p+\mu_2+\mu_1-1}{k+p} \Psi_+ z^{2k+2p} \\ & + (-1)^t \xi_K^- B_K^{-1} F_K \binom{n+t+k+\mu_1+\mu_2}{n+t-k-1} \binom{2k+\mu_1+\mu_2}{k} \Psi_- z^{2k+1} \Bigg\}. \end{aligned} \tag{1.48}$$

1.4. Generating functions

In this section, we derive the main result, that is, the generating functions for the Bannai-Ito orthogonal polynomials. The wavefunctions in their asymptotic form being the sum of two monomials have not quite been brought to a monomial form. Thus, two degrees of the Bannai-Ito polynomials will appear in the coefficient of each power of the z variable. This will not yield a proper generating function. However, once this intermediate result is obtained, it proves possible to disentangle the resulting power series with a trick involving analysis of the complex phase of each term. The next two subsections illustrate how the proper generating functions can be found using this two-step approach.

1.4.1. Intermediate result

The asymptotic expansion given in section 2.3 is constructed so that the trigonometric functions of the α and β variables remain finite[1]. Thus, there is no expansion to be made on the left-hand side of (1.32) as defined in (1.43) to obtain (1.39). One only needs to rewrite the functions in terms of the new variable z through the use of (1.36). Using standard trigonometric relations for double angles and (1.36), we have

$$\cos 2\alpha = 2z^2 - 1, \qquad \cos 2\beta = \frac{z^2+1}{z^2-1}.$$

The functions of $\beta(z)$ in $\mathcal{Z}_S^N(z)$, amounting to the $\mathcal{F}_K$ functions in (1.41) and (1.42), now depend on the parameter $S = 2s + p \in \{0, ..., N\}$, $p \in \{0, 1\}$ and have their parameters permuted by $(\,1\,2\,3\,)$ acting on $\{\mu_1, \mu_2, \mu_3\}$. These functions are expressed in terms of the new variable z as

$$\mathcal{F}_S^+(z) = \xi_S^+ \Bigg[E_S P_{s+p}^{(\mu_3-1/2,\mu_2-1/2)}\left(\frac{z^2+1}{z^2-1}\right) \\ - \frac{iz}{1-z^2}(-1)^p E_S^{-1} P_{s+p-1}^{(\mu_3+1/2,\mu_2+1/2)}\left(\frac{z^2+1}{z^2-1}\right)\Bigg], \quad (1.49)$$

$$\mathcal{F}_S^-(z) = \frac{\xi_S^-}{\sqrt{1-z^2}} \Bigg[F_S P_s^{(\mu_3+1/2,\mu_2-1/2)}\left(\frac{z^2+1}{z^2-1}\right) \\ + iz(-1)^p F_S^{-1} P_s^{(\mu_3-1/2,\mu_2+1/2)}\left(\frac{z^2+1}{z^2-1}\right)\Bigg]. \quad (1.50)$$

[1] Omitting the two poles $\{z = 1, z = -1\}$ of the trigonometric functions of β.

Similarly, permuting the parameters, the angular wavefunctions (1.43) are expressed in terms of z when $N = 2n + t$, $n \in \mathbb{N}$ and $t \in \{0, 1\}$ as

$$\mathcal{Z}_S^N(z) = A_S \Big[z^t B_S P_{n-s-p}^{(2s+2p+\mu_2+\mu_3, \mu_1-1/2+t)}(2z^2-1)\mathcal{F}_S^+(z)(1-z^2)^{s+p} \\ - z^{1-t} B_S^{-1} P_{n+t-s-1}^{(2s+1+\mu_2+\mu_3, \mu_1+1/2-t)}(2z^2-1)\mathcal{F}_S^-(z)(1-z^2)^{s+1/2} \Big], \quad (1.51)$$

where one must not forget to introduce the reflection in the α coordinate when N is even.

1.4.2. Proper generating functions

We now turn to the problem of disentangling the quasi generating functions (1.51). Assuming z to be a real variable, by observing (1.48) and (1.46), one can note that the phase information of the asymptotic wavefunctions $\mathcal{Y}_K^N(z)$ is given by Ψ_+ for even powers of z and Ψ_- for odd powers of z.

To disentangle the generating functions, we want to keep only the powers of z coming from values of K of the same parity. Thus, we only want to keep the terms with $p = 0$ for the even powers of z and the terms with $p = 1$ for the odd powers of z. In this case, the matched Ψ terms become

$$\Psi = \begin{cases} \left[1 - i\frac{k}{k+\mu_1+\mu_2}\right] & \text{for even powers of } z, \\ \\ \left[i - \frac{k+\mu_2+1/2}{k+\mu_1+1/2}\right] & \text{for odd powers of } z. \end{cases} \quad (1.52)$$

The remaining mismatched cases of Ψ are simply given by $\Psi = [1 + i]$.

The disentangling procedure rests on the fact that an orthogonal coordinate system of the complex plane can be devised such that one of the components of the vectors in these coordinates is independent of the mismatched terms. More precisely, rotating the complex plane under the multiplication by $e^{i\pi/4}$, one maps the matching terms to some vectors on the unit circle and the remaining ones are purely imaginary. Taking the real part of the result, we obtain an expression that only involves one degree of the Bannai-Ito polynomials per power of z. Let us now calculate the change in the normalization of each asymptotic function that this procedure induces. The rotation in the complex plane leads to

$$e^{i\frac{\pi}{4}} : \Psi_U \mapsto \Psi'_U = i\sqrt{2}, \quad \Psi_A \mapsto \Psi'_A = e^{i\frac{\pi}{4}}\Psi_A.$$

Taking the real part, the desired terms remain whereas the undesired ones vanish, leading to the required disentanglement. The real part of the rotated Ψ_A is

$$\mathfrak{Re}\left(e^{i\frac{\pi}{4}}\Psi_A\right) = \begin{cases} \dfrac{1}{\sqrt{2}}\left(1+\dfrac{k}{k+\mu_1+\mu_2}\right) & \text{for even powers of } z, \\ \\ \dfrac{-1}{\sqrt{2}}\left(1+\dfrac{k+\mu_2+1/2}{k+\mu_1+1/2}\right) & \text{for odd powers of } z. \end{cases}$$

Using the above, the transformed asymptotic wavefunctions, written $\widetilde{\mathfrak{Y}}^N_K(z)$, are then monomials in z

$$\widetilde{\mathfrak{Y}}^N_K(z) = C^{\mu_1,\mu_2,\mu_3}_{K,N}\, z^K, \tag{1.53}$$

where the coefficients are as follows, for $K = 2k+p$ and $N = 2n+t$ with $p,t \in \{0,1\}$,

$$C^{\mu_1,\mu_2,\mu_3}_{K,N} = \frac{(-1)^p}{2\sqrt{k!(n-k+pt-p)!}}\left[\frac{\Gamma(n+k+\mu_1+\mu_2+1+p+t-pt)}{\Gamma(k+\mu_1+1/2+p)\Gamma(k+\mu_2+1/2+p)}\right.$$
$$\left.\times\frac{\Gamma(n+k+\mu_1+\mu_2+\mu_3+3/2+pt)}{\Gamma(n-k+\mu_3+1/2+t(1-p))\Gamma(k+\mu_1+\mu_2+1)}\right]^{1/2}. \tag{1.54}$$

Acting with the same transformation on (1.51) leads to the proper generating function. Writing $S = 2s+p \in \{0,...,N\}$ with $p \in \{0,1\}$, for $N = 2n+t \in \mathbb{N}$ with $t \in \{0,1\}$, one arrives at

$$\begin{aligned}\widetilde{\mathcal{Z}}^N_S(z) = \sum_{u=0,1} z^{t+u}(1-z^2)^s\Big[&(1-z^2)^{p-u} \\ &\times U^u_S\, P^{(2s+2p+\mu_2+\mu_3,\mu_1-1/2+t)}_{n-s-p}(2z^2-1)P^{(\mu_3-1/2+u,\mu_2-1/2+u)}_{s+p-u}\left(\tfrac{z^2+1}{z^2-1}\right) \\ &- z\, L^u_S\, P^{(2s+1+\mu_2+\mu_3,\mu_1+1/2-t)}_{n+t-s-1}(2z^2-1)P^{(\mu_3+1/2-u,\mu_2-1/2+u)}_{s}\left(\tfrac{z^2+1}{z^2-1}\right)\Big].\end{aligned} \tag{1.55}$$

where

$$U^u_S = \frac{(-1)^{pu}}{\sqrt{2}}A_S B_S E^{1-2u}_S \xi^+_S, \qquad L^u_S = \frac{(-1)^{u(p+1)}}{\sqrt{2}}A_S B^{-1}_S F^{1-2u}_S \xi^-_S.$$

The proper generating function decomposition is then expressed as

$$\widetilde{\mathcal{Z}}^N_S(z) = \sum_{K=0}^{N} \mathcal{R}^{\mu_1,\mu_2,\mu_3}_{S,K,N}\, \widetilde{\mathfrak{Y}}^N_K(z) = \sum_{K=0}^{N} \mathcal{R}^{\mu_1,\mu_2,\mu_3}_{S,K,N} C^{\mu_1,\mu_2,\mu_3}_{K,N}\, z^K, \tag{1.56}$$

with $C_{K,N}^{\mu_1,\mu_2,\mu_3}$ as in (1.54) and where the Racah coefficients $\mathcal{R}_{S,K,N}^{\mu_1,\mu_2,\mu_3}$ as given in (1.33) are proportional to the Bannai-Ito polynomials.

1.5. Conclusion

We have derived generating functions for the Bannai-Ito orthogonal polynomials by exploiting the fact that these polynomials present themselves as the Racah coefficients for the $\mathfrak{osp}(1|2)$ Lie superalgebra. This derivation was done using an appropriate asymptotic expansion of Dunkl oscillators wavefunctions.

As the Bannai-Ito polynomials can be obtained as a $q \to -1$ limit of the Askey-Wilson polynomials or q-Racah polynomials, one could ask if their generating functions admit as limits the generating function derived here. However, all of the possible generating functions for these polynomials have limits with different truncation conditions for the Bannai-Ito polynomials than the ones used in this paper and do not correspond to to the result we derived.

This generating function for the Bannai-Ito polynomials might have interesting combinatorial interpretations [**1**]. Various orthogonal polynomials are obtained as limits of the Bannai-Ito polynomials. It would be interesting to investigate how generating functions for these polynomials can be recovered from the one obtained here.

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Chapitre 2

$SU_q(3)$ corepresentations and bivariate q-Krawtchouk polynomials

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Abstract

The matrix elements of unitary $SU_q(3)$ corepresentations, which are analogues of the symmetric powers of the natural repesentation, are shown to be the bivariate q-Krawtchouk orthogonal polynomials, thus providing an algebraic interpretation of these polynomials in terms of quantum groups.

Introduction

A fruitful connection exists between Lie groups and algebras and the theory of orthogonal polynomials. Algebraic interpretations for these orthogonal polynomials enable simple derivations of their properties and often lead to new identities. Similar connections between the theory of quantum groups and (mostly univariate) q-orthogonal polynomials have been established [**17**]. The results of this paper pursue such a connection in multivariate situations by giving an algebraic interpretations of the quantum bivariate q-Krawtchouk polynomials in terms of the quantum group $SU_q(3)$.

In the classical case, the Krawtchouk polynomials of the Tratnik type form a family of multivariate Krawtchouk polynomials constructed from univariate Krawtchouk polynomials using a construction developed in [**22**] that applies to all polynomials of the $(q = 1)$ Askey scheme. These are orthogonal polynomials with respect to the multinomial distribution. Many Lie-theoretic interpretations have been given for the Krawtchouk polynomials. The multivariate Krawtchouk polynomials in d variables were shown [**9**] to be matrix elements of the $SO(d+1)$ Lie group and identified as well [**12, 13**] as overlaps of anti-automorphisms of $\mathfrak{sl}(d+1)$-modules.

In the context of quantum groups and algebras, interpretations analogous to the classical ones have been given for the q-Krawtchouk polynomials, which are orthogonal with respect to the q-binomial distribution. Koornwinder obtained [**16**] the univariate q-Krawtchouk polynomials as the matrix elements of unitary corepresentations of the $SU_q(2)$ quantum group. In a complementary way based on the quantum algebras, the q-Krawtchouk polynomials were seen [**25, 8**] to arise as matrix elements of a class of $U_q(\mathfrak{sl}(2))$ automorphisms. These two approaches are essentially dual one to another [**4**].

A family of multivariate q-Krawtchouk polynomials were first derived by Gasper and Rahman in [**6**] where they constructed q-deformations of Tratnik's polynomials. We will thus refer to these multivariate extensions of the q-Krawtchouk as being of the Tratnik type. An interpretation of the bivariate and multivariate q-Krawtchouk polynomials based on the quantum algebra viewpoint was obtained in [**7**]. With an eye to generalizations and in view of the fact that for $q = 1$, Lie groups rather than algebras provide a most natural framework, it seems appropriate to examine how the multivariate q-Krawtchouk polynomials can be obtained and analyzed in a quantum group framework. In this paper, we build upon the quantum group approach of Koornwinder to obtain the bivariate quantum q-Krawtchouk polynomials of the Tratnik type as the matrix element of unitary $SU_q(3)$ corepresentations. Within this quantum group approach, the structure of the unitary elements of $U_q(\mathfrak{sl}(3))$ constructed in [**7**] is explained from the representation theory of $SU_q(3)$.

This paper is organized as follows. In section 2.1, a presentation of the $SU_q(3)$ algebra is first given and the construction of its unitary representations is reviewed. Symmetric $SU_q(3)$ corepresentations are then constructed at the beginning of section 2.2, followed by the derivation of their matrix elements and a proof of the unitarity of the corepresentations. A generating function for the matrix elements is then obtained. In section 2.3, the matrix elements are evaluated in irreducible $SU_q(3)$ representations and identified as bivariate q-Krawtchouk polynomials, which follows from Soibelman's tensor product theorem. Finally, section 2.4 illustrates how evaluating the matrix elements in reducible $SU_q(3)$ representations leads to identities for orthogonal polynomials. This is followed by a brief conclusion.

2.1. The $SU_q(3)$ Hopf algebra and its representations

We first give in this section a presentation of the $SU_q(3)$ quantum group, a Hopf $*$-algebra, and discuss how its representations are constructed.

2.1.1. The coordinate ring $A(SU_q(3))$

The coordinate ring $A(SL_q(3;\mathbb{C}))$ is a $\mathbb{C}$-algebra $A = \mathbb{C}[x_{ij}; 1 \leq i,j \leq 3]$ with the relations

$$\begin{aligned} &x_{ik}x_{jk} = qx_{jk}x_{ik}, \qquad && x_{ki}x_{kj} = qx_{kj}x_{ki}, && \forall \quad i<j, \\ &x_{il}x_{jk} = x_{jk}x_{il}, && x_{ik}x_{jl} - qx_{il}x_{jk} = x_{jl}x_{ik} - \frac{1}{q}x_{jk}x_{il}, && \forall \quad i<j, \quad k<l, \end{aligned} \tag{2.1}$$

$$\sum_{\sigma\in S_3} (-q)^{l(\sigma)} x_{1\sigma(1)}x_{2\sigma(2)}x_{3\sigma(3)} = 1.$$

A Hopf algebra structure is given with the following coproduct Δ, counit ϵ and antipode S

$$\Delta(x_{ij}) = \sum_{k=1}^{3} x_{ik}\otimes x_{kj}, \quad \epsilon(x_{ij}) = \delta_{ij}, \quad S(x_{ij}) = (-q)^{i-j}\xi_{ji}, \quad 1\leq i,j\leq 3, \tag{2.2}$$

where ξ_{ij} denotes the (i,j) quantum minor, that is the quantum determinant of x with the i^{th} row and the j^{th} column removed:

$$\xi_{ij} = \sum_{\tau\in S_2} (-q)^{l(\tau)} x_{i_1 j_{\tau(1)}} x_{i_2 j_{\tau(2)}}, \quad i_1 < i_2 \in \{1,2,3\}\setminus\{i\}, \quad j_1<j_2\in\{1,2,3\}\setminus\{j\}.$$

Morevover, a unique conjugate linear anti-homomorphism $* : A(SL_q(3;\mathbb{C})) \to A(SL_q(3;\mathbb{C})) : x\mapsto x^*$ exists such that

$$x_{ij}^* = S(x_{ji}) = (-q)^{j-i}\xi_{ij}, \quad \forall \quad i,j. \tag{2.3}$$

This $*$-operation makes $A(SL_q(3))$ into the $*$-Hopf algebra $A(SU_q(3))$ which we will refer to as simply $SU_q(3)$.

2.1.2. $SU_q(3)$ representations

The $SU_q(3)$ $*$-representations used in this paper were constructed in [**1**]. However, to obtain our main results, we will make use of a theorem of Soibelman [**21**] on the construction of modules over quantum groups. Thus, we review how $*$-representations of unitary quantum groups are constructed [**15**] using this result. Following [**19**], one first defines [**23**] the infinite dimensional representations τ_α with $\alpha \in U(1)$, of $SU_q(2)$ [**24**] where its generators $\{t_{ij} \,:\, i,j = 1,2\}$ act on a Hilbert space H with orthonormal basis $\{|n\rangle\,, n \in \mathbb{N}\}$ as follows

$$\tau_\alpha(t_{12})\,|n\rangle = -q^{n+1}\alpha^{-1}\,|n\rangle\,, \qquad\qquad \tau_\alpha(t_{21})\,|n\rangle = q^n\alpha\,|n\rangle\,,$$
$$\tau_\alpha(t_{11})\,|n\rangle = \sqrt{1-q^{2n}}\,|n-1\rangle\,, \quad \tau_\alpha(t_{11})\,|0\rangle = 0, \quad \tau_\alpha(t_{22})\,|n\rangle = \sqrt{1-q^{2(n+1)}}\,|n+1\rangle\,.$$

From these representations, one can build $SU_q(3)$ representations. Indeed, consider the two canonical embeddings $\varphi_i \,:\, U_q(\mathfrak{sl}(2)) \hookrightarrow U_q(\mathfrak{sl}(3))$, $i = 1,2$. These embeddings define by duality the projections $\varphi_i^* \,:\, SU_q(3) \longrightarrow SU_q(2)$ such that irreducible $*$-representations of $SU_q(3)$ are given by the maps $\pi_i \equiv \tau_\alpha \circ \varphi_i^*$ with $\alpha = -1$ acting on $V_{s_i} \cong H$. Explicitly, these elementary representations are specified by

$$\pi_1\begin{pmatrix} x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{pmatrix}\cdot|k\rangle = \begin{pmatrix} \sqrt{1-q^{2k}}|k-1\rangle & q^{k+1}\,|k\rangle & 0\\ -q^k|k\rangle & \sqrt{1-q^{2k+2}}|k+1\rangle & 0\\ 0 & 0 & |k\rangle\end{pmatrix}, \tag{2.4}$$

and

$$\pi_2\begin{pmatrix} x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{pmatrix}\cdot|k\rangle = \begin{pmatrix} |k\rangle & 0 & 0\\ 0 & \sqrt{1-q^{2k}}|k-1\rangle & q^{k+1}\,|k\rangle\\ 0 & -q^k|k\rangle & \sqrt{1-q^{2k+2}}\,|k+1\rangle\end{pmatrix}. \tag{2.5}$$

All $SU_q(3)$ representations can be constructed from the elementary representations π_1 and π_2 using the following tensor product theorem [**19**, Thm 6.2.7]. Denoting by $\mathbb{C}[G]_q$ the quantised algebra of functions [**3**] on G, a connected and simply connected simple compact Lie group with associated Weyl group W, one has

Theorem 1 (Tensor product theorem)**.** For any unitarizable irreducible $\mathbb{C}[G]_q$ $*$-representation V, there exists a unique element $w \in W$ of the Weyl group and a unique element $\tau \in T$ of the distinguished maximal torus such that

$$V \cong V_w \otimes V_\tau, \qquad V_w = V_{s_{i_1}} \otimes V_{s_{i_2}} \otimes \cdots \otimes V_{s_{i_k}},$$

where $w = s_{i_1} s_{i_2} \ldots s_{i_k}$ is a reduced decomposition of w. The tensor product does not depend (up to a unitary equivalence) on the choice of reduced decomposition for w.

The map $\pi_w : SU_q(3) \to End(V_w)$ associated to the representation in the above theorem is specified by

$$\pi_w = (\pi_{i_1} \otimes \pi_{i_2} \otimes \cdots \otimes \pi_{i_k}) \circ \Delta^{(k-1)}, \qquad w = s_{i_1} s_{i_2} \ldots s_{i_k},$$

where the repeated coproduct $\Delta^{(k-1)}$ is defined through

$$\Delta^{(1)} = \Delta, \quad \Delta^{(k)} = \underbrace{1 \otimes \cdots \otimes 1}_{k-1 \text{ times}} \otimes \Delta \circ \Delta^{(k-1)}.$$

The representation on V_τ is a one-dimensional representation of the form $\rho_\tau : SU_q(3) \to \mathbb{C} : x_{ij} \mapsto \alpha_i(\tau)\delta_{ij}$ with $\alpha_i(\tau) \in U(1)$ such that $\alpha_1(\tau)\alpha_2(\tau)\alpha_3(\tau) = 1$. Thus, it only contributes a global phase to our result and will not be further considered. Take now the case where $w = s_2 s_1$. One has

$$\pi_{21} \equiv \pi_{s_2 s_1} = (\pi_2 \otimes \pi_1) \circ \Delta. \tag{2.6}$$

The explicit action on the generators is easily obtained from (2.6) using (2.2), (2.4) and (2.5).

The representation π_{121} is obtained similarly and, upon tensoring with one-dimensional representations, is the most general representation in the sense that the intersection of the kernels of these representation is trivial, c.f. e.g. [**15**, Sect. 5]. It follows from Theorem 1 that π_{121} and π_{212} are equivalent, so it suffices to consider one of them.

2.2. Unitary $SU_q(3)$ corepresentations

We now turn to the construction of unitary $SU_q(3)$ corepresentations in analogy with the $GL(3)$ coaction on functions on $\mathbb{C}^3$. Consider the space $\mathcal{F}^{(1)}(\mathbb{C}^3)$ of linear functions on $\mathbb{C}^3$ with orthonormal basis $\{z_i\}_{i=1,2,3}$. By identifying these basis elements with a fixed column of the $SU_q(3)$ quantum group as follows

$$z_i = x_{ij}, \qquad \text{for } j \text{ fixed}, \tag{2.7}$$

a natural [**20**] coaction is defined using the coproduct (2.2) as

$$\Delta : \mathcal{F}^{(1)}(\mathbb{C}^3) \to SU_q(3) \otimes \mathcal{F}^{(1)}(\mathbb{C}^3) : z_i \mapsto \Delta(x_{ij}) = \sum_{k=1}^{3} x_{ik} \otimes x_{kj} \equiv \sum_{j=1}^{3} x_{ik} \otimes z_k. \tag{2.8}$$

The algebra $\mathcal{F}(\mathbb{C}^3)$ of polynomial functions on $\mathbb{C}^3$ is identified with the tensor algebra of $\mathcal{F}^{(1)}(\mathbb{C}^3)$ as follows

$$\mathcal{F}(\mathbb{C}^3) = T(\mathcal{F}^{(1)}(\mathbb{C}^3)), \quad T(V) \equiv \bigoplus_{n=0}^{\infty} V^{\otimes n}, \quad V^{\otimes n} \equiv \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}, \quad V^{\otimes 0} \cong \mathbb{C}. \tag{2.9}$$

The identification (2.7) together with (2.1) establishes the following relations

$$\mathcal{R} = \{\, z_i z_j \sim q z_j z_i \mid i < j,\ i, j = 1, 2, 3 \,\}. \tag{2.10}$$

Denote the quotient [**3**, Ch. 7] of the tensor algebra (2.9) by the relations (2.10) as $\mathrm{Sym}_q(\mathbb{C}^3)$. A natural grading on $\mathrm{Sym}_q(\mathbb{C}^3)$ is inherited from the one of $\mathcal{F}(\mathbb{C}^3)$ as the relations (2.10) preserve this grading. Explicitly,

$$\mathrm{Sym}_q(\mathbb{C}^3) = \bigoplus_{n=0}^{\infty} \mathcal{F}_q^{(n)}(\mathbb{C}^3), \qquad \mathcal{F}_q^{(n)}(\mathbb{C}^3) \equiv \left(\mathcal{F}^{(1)}(\mathbb{C}^3)\right)^{\otimes n} \Big/ \mathcal{R}\,. \tag{2.11}$$

The coproduct being an homomorphism, the coaction (2.8) is extended to $\mathrm{Sym}_q(\mathbb{C}^3)$ as follows

$$\Delta(z_1^{m_1} z_2^{m_2} z_3^{m_3}) = \Delta(z_1)^{m_1} \Delta(z_2)^{m_2} \Delta(z_3)^{m_3}, \qquad \forall\, z_1^{m_1} z_2^{m_2} z_3^{m_3} \in Sym_q(\mathbb{C}^3). \tag{2.12}$$

Being constructed from the coproduct, (2.12) defines an $SU_q(3)$ corepresentation. From (2.8), it is easily seen that the coaction (2.12) preserves the natural grading (2.11) on $\mathrm{Sym}_q(\mathbb{C}^3)$. Thus, $\mathrm{Sym}_q(\mathbb{C}^3)$ as an $SU_q(3)$ corepresentation decomposes as a direct sum of corepresentations as follows

$$\Delta : \mathrm{Sym}_q(\mathbb{C}^3) \longrightarrow \bigoplus_{n=0}^{\infty} \Delta\left(\mathcal{F}_q^{(n)}(\mathbb{C}^3)\right).$$

In view of (2.10), a basis for $\mathcal{F}_q^{(N)}(\mathbb{C}^3)$ is given by

$$\mathcal{B}_N = \left\{ z^{\vec{m}} \,\Big|\, |\vec{m}| = N \right\}, \quad \text{for} \quad z^{\vec{m}} \equiv z_1^{m_1} z_2^{m_2} z_3^{m_3} \quad \text{and} \quad |\vec{m}| \equiv m_1 + m_2 + m_3. \tag{2.13}$$

2.2.1. Matrix elements

On the basis (2.13), the matrix elements of the corepresentation $\mathcal{F}_q^{(N)}(\mathbb{C}^3)$ are given in the following proposition.

Proposition 1 (Matrix elements)**.** Let $0 < N \in \mathbb{N}$. The matrix elements $h^{(N)}_{\vec{m},\vec{n}}$ of the $SU_q(3)$ corepresentation $\mathcal{F}_q^{(N)}(\mathbb{C}^3)$ with coaction Δ are given in the basis $\mathcal{B}_N$ by

$$\Delta(z^{\vec{m}}) = \sum_{|\vec{n}|=N} h^{(N)}_{\vec{m},\vec{n}} \otimes z^{\vec{n}}, \qquad h^{(N)}_{\vec{m},\vec{n}} = \sum_{\substack{|\overline{a}_j|=m_j\\ |\underline{a}_j|=n_j}} Q(a) \begin{bmatrix} \vec{m} \\ a \end{bmatrix}_{q^{-2}} \prod_{k=1}^{3} \left(\prod_{i=1}^{3} x_{ik}^{a_{ik}} \right), \tag{2.14}$$

with $|\overline{a}_i| \equiv \sum_{j=1}^{3} a_{ij}$, $|\underline{a}_j| \equiv \sum_{i=1}^{3} a_{ji}$ and

$$\begin{bmatrix} \vec{m} \\ a \end{bmatrix}_q \equiv \begin{bmatrix} m_1 \\ \overline{a}_1 \end{bmatrix}_q \begin{bmatrix} m_2 \\ \overline{a}_2 \end{bmatrix}_q \begin{bmatrix} m_3 \\ \overline{a}_3 \end{bmatrix}_q, \qquad \begin{bmatrix} m \\ \overline{a}_i \end{bmatrix}_q \equiv \frac{(q;q)_m}{(q;q)_{a_{i1}} (q;q)_{a_{i2}} (q;q)_{a_{i3}}},$$

and where

$$Q(a) = q^{-f(a)}, \quad f(a) = a_{13}(a_{21} + a_{22} + a_{32}) + a_{31}(a_{12} + a_{22} + a_{23}) + a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32} \tag{2.15}$$

is a manifestly symmetric function of the matrix of indices.

Proof. First, compute the matrix elements of the coaction on powers of a single generator of $\mathcal{F}^{(1)}(\mathbb{C}^3)$. From (2.8) and (2.12), one has

$$\Delta(z_i^m) = \Delta(x_{ij})^m = \left(\sum_{k=1}^{3} x_{ik} \otimes x_{kj} \right)^m,$$

and, knowing that $(x_{ik} \otimes x_{kj})(x_{il} \otimes x_{lj}) = q^2 (x_{il} \otimes x_{lj})(x_{ik} \otimes x_{kj})$, $\forall\, k < l$, one has that [**16**]

$$\Delta(z_i^m) = \sum_{|\overline{a}_i|=m} \begin{bmatrix} m \\ \overline{a}_i \end{bmatrix}_{q^{-2}} x_{i1}^{a_{i1}} x_{i2}^{a_{i2}} x_{i3}^{a_{i3}} \otimes z_1^{a_{i1}} z_2^{a_{i2}} z_3^{a_{i3}},$$

with the understanding that $\overline{a}_i = (a_{i1}, a_{i2}, a_{i3})$. The coaction (2.12) of a generic basis elements in (2.13), is then

$$\Delta(z^{\vec{m}}) = \prod_{i=1}^{3} \Delta(z_i^{m_i}) = \sum_{\substack{|\overline{a}_1|=m_1\\ |\overline{a}_2|=m_2\\ |\overline{a}_3|=m_3}} \begin{bmatrix} \vec{m} \\ a \end{bmatrix}_{q^{-2}} \prod_{i=1}^{3} \left(\prod_{k=1}^{3} (x_{ik} \otimes x_{kj})^{a_{ik}} \right), \tag{2.16}$$

where all the products are to be expanded from left to right and from innermost to outermost. Then, observe that

$$\prod_{i=1}^{3}\left(\prod_{k=1}^{3} x_{ik}^{a_{ik}}\right) = x_{11}^{a_{11}}x_{12}^{a_{12}}x_{13}^{a_{13}}x_{21}^{a_{21}}x_{22}^{a_{22}}x_{23}^{a_{23}}x_{31}^{a_{31}}x_{32}^{a_{32}}x_{33}^{a_{33}}$$
$$= x_{11}^{a_{11}}x_{21}^{a_{21}}x_{31}^{a_{31}}x_{12}^{a_{12}}x_{22}^{a_{22}}x_{32}^{a_{32}}x_{13}^{a_{13}}x_{23}^{a_{23}}x_{33}^{a_{33}} = \prod_{k=1}^{3}\left(\prod_{i=1}^{3} x_{ik}^{a_{ik}}\right). \tag{2.17}$$

Writing $n_k = \sum_{i=1}^{3} a_{ik}$ with $|\vec{n}| = \sum_{k=1}^{3} n_k$, one finds with (2.15) that

$$\prod_{i=1}^{3}\left(\prod_{k=1}^{3} x_{kj}^{a_{ik}}\right) = Q(a)\prod_{k=1}^{3}\left(x_{kj}^{a_{1k}+a_{2k}+a_{3k}}\right) = Q(a) z_1^{n_1} z_2^{n_2} z_3^{n_3} \equiv Q(a) z^{\vec{n}}. \tag{2.18}$$

Using (2.17) and (2.18) in (2.16) yields

$$\Delta(z^{\vec{m}}) = \sum_{|\vec{n}|=N} \sum_{\substack{|\overline{a}_j|=m_j \\ |\underline{a}_j|=n_j}} \left[Q(a) \begin{bmatrix} \vec{m} \\ a \end{bmatrix}_{q^{-2}} \prod_{k=1}^{3}\left(\prod_{i=1}^{3} x_{ik}^{a_{ik}}\right)\right] \otimes \prod_{k=1}^{3} x_{kj}^{n_k},$$

where we introduced $\underline{a}_j = (a_{1j}, a_{2j}, a_{3j})$ with the sums over $|\overline{a}_j| = m_j$ and $|\underline{a}_j| = n_j$ are sums over all the $\{a_{ij}\}_{i,j=1,2,3}$ satisfying $\sum_{i=1}^{3} a_{ij} = n_j$ and $\sum_{j=1}^{3} a_{ij} = m_i$. From this expression, one directly identifies the matrix elements of $\mathcal{F}_q^{(N)}(\mathbb{C}^3)$. □

2.2.2. Unitarity

Unitary corepresentations can be constructed from the above corepresentations through normalization. We have:

Theorem 2 (Unitarity)**.** The following $SU_q(3)$ corepresentation is unitary

$$\Delta : z^{\vec{m}} \begin{bmatrix} N \\ \vec{m} \end{bmatrix}_{q^{-2}}^{\frac{1}{2}} \longmapsto \sum_{|\vec{n}|=N} t_{\vec{m},\vec{n}}^{(N)} \otimes z^{\vec{n}} \begin{bmatrix} N \\ \vec{n} \end{bmatrix}_{q^{-2}}^{\frac{1}{2}},$$

with the matrix elements given by

$$t_{\vec{m},\vec{n}}^{(N)} = \sqrt{\begin{bmatrix} N \\ \vec{m} \end{bmatrix}_{q^{-2}} \begin{bmatrix} N \\ \vec{n} \end{bmatrix}_{q^{-2}}^{-1}} \sum_{\substack{|\overline{a}_j|=m_j \\ |\underline{a}_j|=n_j}} Q(a) \begin{bmatrix} \vec{m} \\ a \end{bmatrix}_{q^{-2}} \prod_{k=1}^{3}\left(\prod_{i=1}^{3} x_{ik}^{a_{ik}}\right), \tag{2.19}$$

and $Q(a)$ as in (2.15).

Proof. A right $SU_q(3)$ comodule is constructed similarly as the left comodule constructed in (2.12). Identifying the right comodule generators as $w_j = x_{ij}$ for i fixed, one has now

$$\Delta(w^{\vec{m}}) = \prod_{j=1}^{3} \Delta(x_{ij}^{m_j}) = \sum_{|\underline{b}_i|=m_i} \begin{bmatrix} \vec{m} \\ b \end{bmatrix}_{q^{-2}} \prod_{j=1}^{3} \left(\prod_{k=1}^{3} (x_{ik} \otimes x_{kj})^{b_{kj}} \right),$$

where the notation for the indices b is the same as in Proposition 1. Writing $n_k = b_{k1}+b_{k2}+b_{k3}$ such that

$$\prod_{j=1}^{3} \left(\prod_{k=1}^{3} x_{ik}^{b_{kj}} \right) = Q(b^\top) \prod_{k=1}^{3} \left(\prod_{j=1}^{3} x_{ik}^{b_{kj}} \right) = Q(b^\top) \prod_{k=1}^{3} x_{ik}^{n_k} = Q(b^\top) w^{\vec{n}},$$

to obtain, given that (2.15) is symmetric,

$$\Delta(w^{\vec{m}}) = \sum_{|\vec{n}|=N} w^{\vec{n}} \otimes \tilde{h}_{\vec{n},\vec{m}}^{(N)}, \qquad \tilde{h}_{\vec{n},\vec{m}}^{(N)} = \sum_{\substack{|\bar{b}_i|=n_i \\ |\underline{b}_i|=m_i}} Q(b) \begin{bmatrix} \vec{m} \\ b \end{bmatrix}_{q^{-2}} \prod_{j=1}^{3} \left(\prod_{k=1}^{3} x_{kj}^{b_{kj}} \right). \tag{2.20}$$

The generators of the left and right comodules are related by the $*$ operation in the following way

$$(w_j)^* = (x_{ij})^* = S(x_{ji}) = S(z_j) \qquad \Longrightarrow \qquad (w^{\vec{m}})^* = S(z^{\vec{m}}).$$

Thus, knowing that $\Delta \circ S(x) = (S \otimes S) \circ \tau \circ \Delta$ for $\tau(x \otimes y) \equiv y \otimes x$, one has on the one hand

$$\Delta\,(w^{\vec{m}})^* = \Delta \circ S(z^{\vec{m}}) \longmapsto \sum_{|\vec{n}|=N} S(z^{\vec{n}}) \otimes S(h_{\vec{m},\vec{n}}) = \sum_{|\vec{n}|=N} (w^{\vec{n}})^* \otimes S(h_{\vec{m},\vec{n}}). \tag{2.21}$$

On the other hand, Δ being a $*$-homomorphism, one has

$$\Delta(w^{\vec{m}})^* \longmapsto \sum_{|\vec{n}|=N} (w^{\vec{n}})^* \otimes (\tilde{h}_{\vec{n},\vec{m}})^*. \tag{2.22}$$

Knowing that the $w^{\vec{n}}$ are linearly independent, it follows from (2.21) and (2.22) that

$$S\left(h_{\vec{m},\vec{n}}^{(N)}\right) = \left(\tilde{h}_{\vec{n},\vec{m}}^{(N)}\right)^*. \tag{2.23}$$

Comparing (2.20) with the matrix elements $h_{\vec{n},\vec{m}}^{(N)}$ of the left coaction (2.14), one can see that they only differ by the q-trinomial coefficient. However, it is easy to show that

$$\begin{bmatrix} N \\ \vec{m} \end{bmatrix}_{q^{-2}} \begin{bmatrix} \vec{m} \\ a \end{bmatrix}_{q^{-2}} = \begin{bmatrix} N \\ \vec{n} \end{bmatrix}_{q^{-2}} \begin{bmatrix} \vec{n} \\ a \end{bmatrix}_{q^{-2}},$$

so that with the proper normalization of the matrix elements, one has

$$t^{(N)}_{\vec{n},\vec{m}} \equiv \sqrt{\begin{bmatrix} N \\ \vec{n} \end{bmatrix}_{q^{-2}} \begin{bmatrix} N \\ \vec{m} \end{bmatrix}_{q^{-2}}}^{-1} h^{(N)}_{\vec{n},\vec{m}} = \sum_{\substack{|\overline{a}_j|=n_j \\ |\underline{a}_j|=m_j}} Q(a) \sqrt{\begin{bmatrix} \vec{n} \\ a \end{bmatrix}_{q^{-2}} \begin{bmatrix} \vec{m} \\ a \end{bmatrix}_{q^{-2}}} \prod_{k=1}^{3} \left(\prod_{i=1}^{3} x_{ik}^{a_{ik}} \right)$$

$$= \sqrt{\begin{bmatrix} N \\ \vec{m} \end{bmatrix}_{q^{-2}} \begin{bmatrix} N \\ \vec{n} \end{bmatrix}_{q^{-2}}}^{-1} \tilde{h}^{(N)}_{\vec{n},\vec{m}} \equiv \tilde{t}^{(N)}_{\vec{n},\vec{m}}.$$

This normalization is equivalent to normalizing the basis elements of the left and right corepresentations as follows

$$\underline{w}^{\vec{m}} \equiv \sqrt{\begin{bmatrix} N \\ \vec{m} \end{bmatrix}_{q^{-2}}} w^{\vec{m}}, \qquad \underline{z}^{\vec{m}} \equiv z^{\vec{m}} \sqrt{\begin{bmatrix} N \\ \vec{m} \end{bmatrix}_{q^{-2}}}. \tag{2.24}$$

With this normalization and (2.23), one has

$$S\left(t^{(N)}_{\vec{m},\vec{n}}\right) = \left(t^{(N)}_{\vec{n},\vec{m}}\right)^*, \tag{2.25}$$

which establishes [**3**, Ch. 4.1] the unitarity of the corepresentations. □

A direct corollary of Theorem 2 is the orthonormality of the matrix elements. Indeed, writing the $SU_q(3)$ product as $\nabla : SU_q(3) \otimes SU_q(3) \to SU_q(3)$, the hexagonal relation from the Hopf algebra structure gives

$$\sum_{|\vec{n}|=N} t_{\vec{m},\vec{n}}\, S\left(t_{\vec{n},\vec{p}}\right) = \nabla \circ (1 \otimes S) \circ \Delta\ \left(t_{\vec{m},\vec{p}}\right) = \eta \circ \epsilon\ \left(t_{\vec{m},\vec{p}}\right) = \delta_{\vec{m},\vec{p}}, \tag{2.26}$$

where the last equality relies on the fact that the counit ϵ vanishes on off-diagonal generators of $SU_q(3)$ and also that the single term in (2.19) containing only diagonal generators has coefficient one. Upon using (2.25) in (2.26), one obtains

$$\sum_{|\vec{n}|=N} t_{\vec{m},\vec{n}}\, t^*_{\vec{p},\vec{n}} = \delta_{\vec{m},\vec{p}}, \qquad \sum_{|\vec{m}|=N} t^*_{\vec{m},\vec{n}}\, t_{\vec{m},\vec{p}} = \delta_{\vec{n},\vec{p}}, \tag{2.27}$$

where a similar argument is used to obtain the second identity.

2.3. Matrix elements in $SU_q(3)$ representations

In this section, the matrix elements (2.19) are shown to be q-Krawtchouk polynomials. Following [**16**], we first identify the matrix elements in the elementary representations π_1 and π_2 as univariate quantum q-Krawtchouk polynomials. Then, the matrix elements in the π_{21} representation are identified as the Tratnik type bivariate quantum q-Krawtchouk polynomials. Finally, it is shown the matrix elements in the π_{121} can be expressed in terms of the same polynomials.

2.3.1. Elementary representations

Consider the matrix elements (2.19) in the π_1 and π_2 representations. That these matrix elements are given in terms of univariate quantum q-Krawtchouk polynomials is a corollary of a previous result of Koornwinder [**16**], as these $SU_q(3)$ repesentations where constructed from an $SU_q(2)$ representation. However, as we now work in an $SU_q(3)$ representation instead of the algebra itself a derivation in the current context is given. The univariate quantum q-Krawtchouk polynomials [**14**] are defined[1] by

$$k_n(x;p,N,q) = (-1)^n \left(q^{-N};q\right)_n q^{n(n-1)/2}{}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-x} \\ q^{-N} \end{matrix}\middle| q\right) pq^{n+1}, \tag{2.28}$$

and are orthogonal with respect to the weight $w_x(p)^2$ for $x \in \{0,\cdots,N\}$, where

$$w_x(p) = \left[(-1)^{N-x} q^{x(x-1)/2} \begin{bmatrix} N \\ x \end{bmatrix}_q \frac{(pq;q)_{N-x}}{(q;q)_N} p^{-N} q^{-N(N+1)/2}\right]^{1/2},$$

with normalization

$$\Theta_n(p) = \frac{q^{-n(n-1)/2}}{(q^{-N};q)_n}\left[(-1)^n q^{n(n+1)/2-Nn} \begin{bmatrix} N \\ n \end{bmatrix}_q \frac{(q;q)_N}{(qp;q)_n}\right]^{1/2}.$$

One has the following proposition.

Proposition 2. The matrix elements (2.19) evaluated in the elementary representations π_i with $i=1,2$ as defined in (2.4) and (2.5), are shift operators given by

$$\pi_i\left(t^{(N)}_{\vec{m},\vec{n}}\right)|k\rangle = (\delta_{i,2}\delta_{m_1,n_1} + \delta_{i,1}\delta_{N-m_1-m_2,N-n_1-n_2}) t_{m_i,n_i,T_i}(k)\,|k+T_i-m_i-n_i\rangle\,, \tag{2.29}$$

$$t_{m,n,T}(k-T+n) \equiv (-1)^{n-m} w_n(q^{-2(k+1)})\Theta_m(q^{-2(k+1)})k_m(q^{-2n};q^{-2(k+1)},T,q^2). \tag{2.30}$$

[1]Part of the normalization of [**14**] is here included in the polynomial $k_n(x;p,N,q)$ themselves.

PROOF. Looking at (2.4), the monomials in the $SU_q(3)$ generators in (2.19) evaluate to

$$\pi_1\left(\prod_{i,k=1}^{3} x_{ik}^{a_{ik}}\right)|k\rangle = \delta_{a_{33},m_3}\prod_{i=1}^{2}\delta_{a_{3i},0}\,\delta_{a_{i3},0}$$
$$\times(-1)^{a_{21}}q^{a_{12}(k+a_{22}+1)+a_{21}(k+a_{22})}\sqrt{(q^{2k+2};q^2)_{a_{22}}\,(q^{2(k+a_{22})};q^{-2})_{a_{11}}}\,|k+a_{22}-a_{11}\rangle\,.$$

Using the above and writing $T = N - m_3 = N - n_3$, the matrix elements simplifies to

$$\pi_1\left(t^{(N)}_{\vec{m},\vec{n}}\right)|k\rangle = \sum_{a_{21}=0}^{m_2}\begin{bmatrix}m_1\\a_{11}\end{bmatrix}_{q^{-2}}\begin{bmatrix}m_2\\a_{21}\end{bmatrix}_{q^{-2}} q^{-a_{21}(a_{12})}(-1)^{a_{21}}q^{a_{12}(k+a_{22}+1)+a_{21}(k+a_{22})}$$
$$\times\sqrt{\begin{bmatrix}N\\\vec{m}\end{bmatrix}_{q^{-2}}\begin{bmatrix}N\\\vec{n}\end{bmatrix}_{q^{-2}}^{-1}}\sqrt{(q^{2k+2};q^2)_{a_{22}}\,(q^{2(k+a_{22})};q^{-2})_{a_{11}}}\,|k+a_{22}-a_{11}\rangle\,.$$

Taking the following parametrization of the summation indices

$$\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = \begin{pmatrix}T-n_2-i & n_2-m_2+i\\ i & m_2-i\end{pmatrix},$$

one has

$$\pi_1\left(t^{(N)}_{\vec{m},\vec{n}}\right)|k\rangle = (-1)^{T-n_2}q^{(n_2-m_2)(k+m_2+1)+n_1(n_1+1)}$$
$$\times\sqrt{(q^{2(k+1)};q^2)_{m_2}\,(q^{2(k+m_2-n_1+1)};q^2)_{n_1}}\,\frac{(q^{-2m_1};q^2)_{n_1}}{(q^2;q^2)_{n_1}}\,|k-n_1+m_2\rangle$$
$$\times\sqrt{\begin{bmatrix}N\\\vec{m}\end{bmatrix}_{q^{-2}}\begin{bmatrix}N\\\vec{n}\end{bmatrix}_{q^{-2}}^{-1}}\sum_{i=0}^{m_2}\frac{(q^{-2n_1};q^2)_i\,(q^{-2m_2};q^2)_i}{(q^{2(n_2-m_2+1)};q^2)_i\,(q^{-2(k+m_2)};q^2)_i\,(q^2;q^2)_i}q^{2i}.$$

Using Jackson's identity [**5**, (III.5)], one can write the ${}_3\phi_2$ as a ${}_2\phi_1$. Then, reversing the order of summation using [**5**, exer. 1.4 (ii)] , after shifting the parameter k by $-n_2$ and using q-Pochhammer symbol identities, leads to

$$\pi_1\left(t^{(N)}_{\vec{m},\vec{n}}\right)|k\rangle = (-1)^{m_2}q^{k(n_2+m_2)+n_2(m_2+1)-2n_1n_2}\,|k-n_1+m_2\rangle$$
$$\times\begin{bmatrix}T\\n_1\end{bmatrix}_{q^2}\sqrt{\begin{bmatrix}N\\\vec{m}\end{bmatrix}_{q^{-2}}\begin{bmatrix}N\\\vec{n}\end{bmatrix}_{q^{-2}}^{-1}\frac{(q^{2(k-n_1+1)};q^2)_{n_1}}{(q^{2(k-n_1+1)};q^2)_{m_2}}}\,{}_2\phi_1\left(\left.\begin{matrix}q^{-2m_2},q^{-2n_2}\\q^{-2(n_1+n_2)}\end{matrix}\right|q^2\right)q^{-2k}.$$

We now use one of Heine's transformation formulas [**5**, (III.3)] to obtain

$$\pi_1\left(t^{(N)}_{\vec{m},\vec{n}}\right)|k-T+n_1\rangle = \delta_{N-m_1-m_2,N-n_1-n_2}(-1)^{n_1}q^{n_1^2-n_1T}$$
$$\times\left[(-1)^{T-n_1+m_1}q^{2T(k+1)-T(T+1)-m_1(m_1-1)+n_1(n_1-1)}\frac{\left(q^{-2k};q^2\right)_{T-n_1}}{(q^{-2k};q^2)_{m_1}}\right]^{1/2}$$
$$\times\begin{bmatrix}T\\ n_1\end{bmatrix}_{q^2}\sqrt{\begin{bmatrix}N\\ \vec{m}\end{bmatrix}_{q^{-2}}\begin{bmatrix}N\\ \vec{n}\end{bmatrix}_{q^{-2}}^{-1}}\;{}_2\phi_1\left(\left.\begin{matrix}q^{-2m_1},q^{-2n_1}\\ q^{-2T}\end{matrix}\right|q^2\right)q^{-2k+2m_1}|k-m_1\rangle\,. \quad (2.31)$$

Proceeding similarly, one obtains for π_2

$$\pi_2\left(t^{(N)}_{\vec{m},\vec{n}}\right)|k-T+n_2\rangle = \delta_{m_1,n_1}(-1)^{n_2}q^{n_2^2-n_2T}\begin{bmatrix}T\\ n_2\end{bmatrix}_{q^2}\sqrt{\begin{bmatrix}N\\ \vec{m}\end{bmatrix}_{q^{-2}}\begin{bmatrix}N\\ \vec{n}\end{bmatrix}_{q^{-2}}^{-1}}$$
$$\times\left[(-1)^{T-n_2+m_2}q^{2T(k+1)-T(T+1)-m_2(m_2-1)+n_2(n_2-1)}\frac{\left(q^{-2k};q^2\right)_{T-n_2}}{(q^{-2k};q^2)_{m_2}}\right]^{1/2}$$
$$\times\,{}_2\phi_1\left(\left.\begin{matrix}q^{-2m_2},q^{-2n_2}\\ q^{-2T}\end{matrix}\right|q^2\right)q^{-2k+2m_2}|k-m_2\rangle\,. \quad (2.32)$$

From (2.31) and (2.32), one directly finds (2.29) and (2.30), which concludes the proof. □

2.3.2. Product representations

We now demonstrate that the matrix elements in the representation π_{21} are espressed in terms of bivariate quantum q-Krawtchouk polynomials of the Tratnik type. These polynomials are defined in [**6**] as

$$K_{n,m}(x,y;u,v,N,q) = k_n(x;v^{-2},x+y,q)k_m(x+y-n;u^{-2},N-n,q), \quad (2.33)$$

with $k_n(x;p,N,q)$ as in (2.28). They are orthogonal with respect to the weight $W^{(N)}_{n_1,n_2}(u,v)^2$ where

$$W^{(N)}_{n_1,n_2}(u,v) = \left[(-1)^{N-n_1}q^{2v(n_1+n_2)}q^{n_1(n_1-1)}\right.$$
$$\left.\times\begin{bmatrix}N\\ \vec{n}\end{bmatrix}_{q^2}\left(q^{-2v};q^2\right)_{n_2}\left(q^{-2u};q^2\right)_{N-n_1-n_2}q^{2N(u+1)}q^{-N(N+1)}\right]^{1/2}, \quad (2.34)$$

with the normalization given by

$$N^{(N)}_{m_1,m_2}(u,v) = q^{-m_1(m_1-1)-m_2(m_2-1)} q^{-m_1(u+1)} \frac{(q^2;q^2)_{N-m_1-m_2}}{(q^2;q^2)_N} \times \left[(-1)^{m_1+m_2} \begin{bmatrix} N \\ \vec{m} \end{bmatrix}_{q^2} \frac{q^{2N(m_1+m_2)+4m_1+2m_2-2m_1m_2-m_1(m_1-1)-m_2(m_2-1)}}{(q^{-2u};q^2)_{m_2}(q^{-2v};q^2)_{m_1}} \right]^{1/2}. \quad (2.35)$$

With these notations, one has the following proposition.

Proposition 3. The matrix elements (2.19) evaluated in π_{21} as defined in (2.6) are shift operators specified by

$$\pi_{21}\left(t^{(N)}_{\vec{m},\vec{n}}\right)|u,v\rangle = t^{(21)}_{\vec{m},\vec{n}}(u,v)\,|u-m_2+N-n_1-n_2, v+n_2-m_1\rangle, \quad (2.36)$$

where

$$t^{(21)}_{\vec{m},\vec{n}}(u-N+n_1+n_2, v-n_2) = (-1)^{m_1-n_2} \times W^{(N)}_{n_1,n_2}(u,v) N^{(N)}_{m_1,m_2}(u,v) K_{m_1,m_2}(n_1,n_2;q^{u+1},q^{v+1},N,q^2), \quad (2.37)$$

are the normalized and weighted bivariate quantum q-Krawtchouk polynomials of the Tratnik type.

Proof. From (2.6), one has

$$\pi_{21}\left(t^{(N)}_{\vec{m},\vec{n}}\right)|u,v\rangle = (\pi_2\otimes\pi_1)\circ\Delta\left(t^{(N)}_{\vec{m},\vec{n}}\right)|u,v\rangle = \sum_{|\vec{k}|=N} \pi_2\left(t^{(N)}_{\vec{m},\vec{k}}\right)|u\rangle \otimes \pi_1\left(t^{(N)}_{\vec{k},\vec{n}}\right)|v\rangle. \quad (2.38)$$

Using (2.29) of Proposition 2 in (2.38), one sees the Kronecker deltas δ_{m_1,k_1} and $\delta_{k_1+k_2,n_1+n_2}$ remove the sums and one has

$$k_1 = m_1, \quad k_2 = (n_1+n_2-m_1),$$

which implies $T_1 = n_1+n_2$ and $T_2 = N-m_1$. One can thus identify (2.38) as the shift operator in (2.36). Then, shifting u by n_1+n_2-N and v by n_2 in (2.38) one has that

$$t^{(21)}_{\vec{m},\vec{n}}(u-N+n_1+n_2, v-n_2) = t_{m_2,n_1+n_2-m_1,N-m_1}(u-N+n_1+n_2)\,t_{m_1,n_1,n_1+n_2}(v-n_2). \quad (2.39)$$

Using (2.30) from Proposition 2 in (2.39), one obtains by involved but direct computations

$$t^{(21)}_{\vec{m},\vec{n}}(u-N+n_1+n_2, v-n_2) = \\ (-1)^{m_1-n_2}\, W^{(N)}_{n_1,n_2}(u,v) N^{(N)}_{m_1,m_2}(u,v) K_{m_1,m_2}(n_1,n_2;q^{u+1},q^{v+1},N,q^2),$$

which concludes the proof. □

The expression in (2.37) for the scalar factor of the matrix elements (2.19) evaluated in π_{21} has been obtained in a related but different approach in [**9**], see also [**11**]. There, (2.37) arises as an expression for the matrix elements in symmetric representations of unitary elements of $U_q(\mathfrak{su}(3))$ constructed from q-exponentials. This correspondance is expected in view of the duality [**2**], [**3**, Ch. 7] between quantum algebras and groups and has been discussed [**4**] in the context of q-special functions. The parameters of the polynomials in (2.37) is discrete. However, one recovers the usual q-Krawtchouk polynomials with continuous parameters by extension with analytic continuation.

2.3.3. Representation corresponding to the longest Weyl group element

We now study the matrix elements in π_{121}. In this case, one has

$$\begin{aligned}\pi_{121}\left(t^{(N)}_{\vec{m},\vec{n}}\right)|t,u,v\rangle &= (\pi_1\otimes\pi_{21})\circ\Delta\left(t^{(N)}_{\vec{m},\vec{n}}\right)|t,u,v\rangle \\ &= \sum_{|k|=N}\pi_1\left(t^{(N)}_{\vec{m},\vec{k}}\right)|t\rangle\otimes\pi_{21}\left(t^{(N)}_{\vec{k},\vec{n}}\right)|u,v\rangle\,.\end{aligned}$$

Using (2.29), (2.30), (2.36) and (2.37) in the above, one obtains

$$\pi_{121}\left(t^{(N)}_{\vec{m},\vec{n}}\right)|t,u,v\rangle = \sum_{k_1=0}^{m_1+m_2}(-1)^{k_1-m_1}t_{m_1,k_1,T_1}(t)t^{(21)}_{\vec{k},\vec{n}}(u,v) \\ \times|t+m_2-k_1, u+N-m_1-m_2-n_1-n_2+k_1, v+n_2-k_1\rangle\,. \quad (2.40)$$

As $t_{m_1,k_1,T_1}(t)$ does not depend on the variables, one can identify the scalar coefficients of (2.40) as bivariate q-Krawtchouk polynomials normalized by a factor expressed as a univariate q-Krawtchouk polynomials. Thus, studying the matrix elements in π_{121} leads to the same polynomials. Indeed, the unitarity of the corepresentations is equivalent to the orthogonality relation of the Tratnik bivariate q-Krawtchouk polynomials (2.33):

$$\delta_{\vec{n},\vec{p}}\,\langle a',b',c'|\,a,b,c\rangle = \sum_{|\vec{m}|=N} \langle a',b',c'|\,\pi_{121}\left(t^{(N)}_{\vec{m},\vec{n}}\right)^{*}\pi_{121}\left(t^{(N)}_{\vec{m},\vec{p}}\right)|a,b,c\rangle\,,$$

which, after shifting b by p_1+p_2 and c by $-p_2$, is given by

$$\begin{aligned}
&\delta_{\vec{n},\vec{p}}\,\langle a',b',c'|\,a,b+p_1+p_2,c-p_2\rangle = \\
&\qquad\sum_{|\vec{m}|=N}\sum_{k_1=0}^{m_1+m_2}\sum_{l_1=0}^{m_1+m_2}(-1)^{k_1+l_1}t_{m_1,k_1,T_1}(a)t_{m_1,l_1,T_1}(a-k_1+l_1)\\
&\qquad\times t^{(21)}_{\vec{k},\vec{p}}(b+p_1+p_2,c-p_2)t^{(21)}_{\vec{l},\vec{n}}(b+n_1+n_2+k_1-l_1,c-n_2-k_1+l_1)\\
&\qquad\qquad\times\langle a',b',c'|\,a-k_1+l_1,b+n_1+n_2+k_1-l_1,c-n_2-k_1+l_1\rangle.
\end{aligned}$$

Noticing that the overlap $\langle a'|\,a-k_1+l_1\rangle$ fixes the difference $l_1-k_1\equiv s$, one can rearrange the sums to obtain

$$\begin{aligned}
&\delta_{\vec{n},\vec{p}}\,\langle a-s,b',c'|\,a,b+p_1+p_2,c-p_2\rangle = \\
&\qquad\sum_{|\vec{k}|=N}\sum_{m_1=0}^{k_1+k_2}t^{(21)}_{\vec{k},\vec{p}}(b+p_1+p_2,c-p_2)t^{(21)}_{\vec{k}+s(\vec{e}_1-\vec{e}_2),\vec{n}}(b+n_1+n_2-s,c-n_2+s)\\
&\qquad\qquad\times(-1)^s t_{m_1,k_1,T_1}(a)t_{m_1,k_1-s,T_1}(a-s)\langle b',c'|b+n_1+n_2-s,c-n_2+s\rangle.
\end{aligned}$$

Using the univariate q-Krawtchouk dual orthogonality relation [**14**] in the sum over m_1 forces s to vanish so that one has

$$\begin{aligned}
&\delta_{\vec{n},\vec{p}}\langle b',c'|b+p_1+p_2,c-p_2\rangle = \\
&\qquad\sum_{|\vec{k}|=N}t^{(21)}_{\vec{k},\vec{p}}(b+p_1+p_2,c-p_2)t^{(21)}_{\vec{k},\vec{n}}(b+n_1+n_2,c-n_2)\langle b',c'|b+n_1+n_2,c-n_2\rangle,
\end{aligned}$$

which one can recognize as the dual orthogonality relation of the bivariate q-Krawtchouk polynomials of the Tratnik type.

2.4. Reducible tensor products

The algebraic interpretation of the multivariate q-Krawtchouk polynomials presented in this paper can be used to derive identities for these polynomials. Consider for example the reducible tensor product of the $SU_q(2)$ representations $\tau_\alpha \otimes \tau_\beta$ with $\alpha, \beta \in S^1$. It is known [**18**] to decompose into the direct integral

$$\rho = \int_{S^1}^{\oplus} \tau_\gamma d\gamma,$$

with the intertwiner $\Lambda : \tau_\alpha \otimes \tau_\beta \to \rho$ acting as follows on the basis vectors [**10**]

$$\Lambda \, |v, t\rangle = \sum_{w \in \mathbb{N}} \alpha^{t-w} \beta^{w-v} \, \bar{p}_v(q^{2w}; q^{2|t-v|}; q^2) \, \frac{\gamma^{-v}}{\sqrt{2\pi}} \otimes |w\rangle \, ,$$

where the Clebsch-Gordan coefficients, given by

$$\bar{p}_v(q^{2w}; q^{2|t-v|}; q^2) = (-1)^{v+w} \sqrt{\frac{q^{2(w-v)(|t-v|+1)} \, (q^{2|t-v|+2}; q^2)_\infty \, (q^{2|t-v|+2}; q^2)_v}{(q^2; q^2)_v \, (q^2; q^2)_w}} \\ \times \, {}_2\phi_1 \left(\left. \begin{matrix} q^{-2v}, 0 \\ q^{2|t-v|+2} \end{matrix} \right| q^2 \right) q^{2w+2},$$

are the weighted and normalized Wall polynomials [**14**]. This result can be used in two ways to calculate the matrix elements in the π_{211} representation. One has, on the one hand,

$$\pi_{211} \left(t^{(N)}_{\vec{m},\vec{n}} \right) |u, v, t\rangle = \int_{S^1}^{\oplus} d\kappa \sum_{k_1=0}^{N} \sum_{w' \in \mathbb{N}} (\sqrt{2\pi})^{-1/2} \, |u - m_2 + n_3, w'\rangle \\ \times \alpha^{t+n_2-k_1-w'} \beta^{w'-v-n_1-n_2+k_1+m_1} \kappa^{m_1+k_1-n_1-n_2-v} t^{(21)}_{\vec{m},(k_1,n_1+n_2-k_1)}(u, v) t_{k_1,n_1,T_1}(t) \\ \times \bar{p}_{v+n_1+n_2-k_1-m_1}(q^{2w'}; q^{2|t+m_1-v-n_1|}; q^2),$$

and, on the other hand,

$$\pi_{211} \left(t^{(N)}_{\vec{m},\vec{n}} \right) |u, v, t\rangle = \\ \int_{S^1}^{\oplus} d\gamma \sum_{w \in \mathbb{N}} \frac{\alpha^{t-w} \beta^{w-v} \gamma^{-v}}{\sqrt{2\pi}} \, \bar{p}_v(q^{2w}; q^{2|t-v|}; q^2) \, \pi_{21} \left(t^{(N)}_{\vec{m},\vec{n}} \right) |u, w\rangle = \\ \int_{S^1}^{\oplus} d\gamma \sum_{w \in \mathbb{N}} \frac{\alpha^{t-w} \beta^{w-v} \gamma^{-v}}{\sqrt{2\pi}} \, \bar{p}_v(q^{2w}; q^{2|t-v|}; q^2) \, t^{(21)}_{\vec{m},\vec{n}}(u, w) \, |u - m_2 + n_3, w + n_2 - m_1\rangle \, .$$

Taking the inner product on $S^1 \otimes V_\alpha \otimes V_\gamma$ of both expression with $\frac{\gamma^{-v}}{\sqrt{2\pi}} \otimes |u - m_2 + n_3, w + n_2 - m_1\rangle$, for a fixed but arbitrary w, one gets

$$\bar{p}_v(q^{2w}; q^{2|t-v|}; q^2)\, t^{(21)}_{\vec{m},\vec{n}}(u, w) = \sum_{k_1=0}^{N} \alpha^{m_1-k_1} \beta^{k_1-n_1} t_{k_1,n_1,n_1+n_2}(t)$$
$$\times\, \bar{p}_{v+n_1+n_2-k_1-m_1}(q^{2(w+n_2-m_1)}; q^{2|t+m_1-v-n_1|}; q^2) t^{(21)}_{\vec{m},(k_1,n_1+n_2-k_1)}(u, v).$$

Upon taking $\alpha = \beta = \gamma = -1$ while shifting u by $n_1 + n_2 - N$, v by $-n_1 - n_2$, w by $-n_2$ and t by $-n_2$, one can use (2.30) and (2.37) to express the above in terms of the polynomials, obtaining

$$\bar{p}_{v-n_1-n_2}(q^{2w-2n_2}; q^{2|t+n_1-v|}; q^2)\, K_{m_1,m_2}(n_1, n_2; q^{u+1}, q^{w+1}, N, q^2) =$$
$$\sum_{j=0}^{N} C^{(N)}_{\vec{m},\vec{n},j}(u, v, t)\, \bar{p}_{v-j-m_1}(q^{2(w-n_2-m_1)}; q^{2|t+m_1-v|}; q^2)$$
$$\times\, k_j(q^{-2n_1}; q^{-2(t+1)}, n_1 + n_2, q^2) K_{m_1,m_2}(j, n_1 + n_2 - j; q^{u+1}, q^{v-j+1}, N, q^2), \quad (2.41)$$

where

$$C^{(N)}_{\vec{m},\vec{n},j}(u, v, t) =$$
$$(-1)^{m_1-n_1}\, w_{n_1}(q^{-2(t+1)}) \Theta_j(q^{-2(t+1)}) \frac{W^{(N)}_{j,n_1+n_2-j}(u, v-j)}{W^{(N)}_{n_1,n_2}(u, w)} \sqrt{\frac{(q^{-2w}; q^2)_{m_1}}{(q^{-2v+2j}; q^2)_{m_1}}}.$$

It follows that (2.41) provides an identity for the product of a bivariate quantum q-Krawtchouk polynomial and a Wall polynomial.

Conclusion

This paper identified the matrix elements of the $SU_q(3)$ symmetric corepresentations as the bivariate quantum q-Krawtchouk polynomials of the Tratnik type. This was done by first constructing the symmetric unitary corepresentations and obtaining abstract expressions for the matrix elements and then evaluating these expressions in $SU_q(3)$ representations. These results thus provide an algebraic interpretation for the bivariate q-Krawtchouk polynomials within the quantum group setting. Moreover, this identification of the matrix elements evaluated in $SU_q(3)$ representations is complete in the sense that it held for the most general irreducible $*$-representation. Finally, this paper illustrated how the quantum group interpretation could be exploited to obtain properties of the q-Krawtchouk polynomials.

The results presented here should admit a generalization to the generic case of $SU_q(N)$, that would yield a similar algebraic interpretation of the multivariate quantum q-Krawtchouk polynomials of the Tratnik type with valuable outcomes. In view of the relation between reducibility of $SU_q(3)$ representations and an identity for orthogonal polynomials, this algebraic interpretation might prove useful to obtain identities for the polynomials.

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Chapitre 3

Orthogonal polynomials and the deformed Jordan plane

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Abstract

We consider the unital associative algebra $\mathcal{A}$ with two generators $\mathcal{X}$, $\mathcal{Z}$ obeying the defining relation $[\mathcal{Z}, \mathcal{X}] = \mathcal{Z}^2 + \Delta$. We construct irreducible tridiagonal representations of $\mathcal{A}$. Depending on the value of the parameter Δ, these representations are associated to the Jacobi matrices of the para-Krawtchouk, continuous Hahn, Hahn or Jacobi polynomials.

3.1. Introduction

This paper is devoted to the study of irreducible tridiagonal representations of the two-generated algebra $\mathcal{A}$ which is a deformation of the Jordan plane. It is shown how the para-Krawtchouk polynomials appear quite naturally in this context, along with the other families of classical orthogonal polynomials (OPs) of the Jacobi, continuous Hahn and Hahn type.

The algebra $\mathcal{A}$ over $\mathbb{R}$, with generators $\mathcal{X}$, $\mathcal{Z}$ and satisfying

$$[\mathcal{Z}, \mathcal{X}] = \mathcal{Z}^2 + \Delta \tag{3.1}$$

with Δ a parameter, is a special case of the most general two-generated quadratic algebra $\mathcal{Q}$ with defining relation

$$\alpha_1\mathcal{X}^2 + \alpha_2\mathcal{X}\mathcal{Z} + \alpha_3\mathcal{Z}\mathcal{X} + \alpha_4\mathcal{Z}^2 + \alpha_5\mathcal{X} + \alpha_6\mathcal{Z} + \alpha_7 = 0. \tag{3.2}$$

This algebra has been of interest to various communities. Ring theorists have provided classifications [**17, 8**] of the special cases it entails and studied their properties. The algebra $\mathcal{Q}$ has also been related to non-commutative probability theory [**3**] and is related to martingale polynomials associated to quadratic harnesses [**4**]. On the physics side, $\mathcal{Q}$ describes various 1D asymmetric exclusion models [**5, 6, 19**].

Recently, the last two authors have begun connecting $\mathcal{Q}$ and its various isomorphism classes to families of special functions. In [**18**], by studying tridiagonal representations of the q-oscillator algebra $\mathcal{X}\mathcal{Z} - q\mathcal{Z}\mathcal{X} = 1$, they have identified how they encompass the recurrence relations of the big q-Jacobi, the q-Hahn and the q-para-Krawtchouk polynomials. The case of the q-Weyl algebra $\mathcal{X}\mathcal{Z} - q\mathcal{Z}\mathcal{X} = 0$ has also been studied in [**22**]. The present paper will add to this program by considering an interesting special case of (3.2) and identifying the orthogonal polynomials that can be interpreted from this algebra.

Since their introduction in [**20**], para-polynomials have been the object of growing interest. Four families have been defined and studied offering para-versions of the polynomials of Krawtchouk, q-Krawtchouk, Racah and q-Racah type. While they do not fall in the category of classical orthogonal polynomials [1], they are understood as non-standard truncations of infinite-dimensional families of classical OPs [**11, 15, 16**]. In addition to their natural occurence in the study of perfect state transfer and fractional revival in quantum spin chains [**20, 12, 14**], recent advances have identified these para-polynomials as the basis for finite-dimensional representations of degenerations of the Sklyanin algebra [**7, 1, 2**]. They have also appeared in the study of the Dunkl oscillator in the plane [**9**]. The main goal of this paper is to show that these para-Krawtchouk polynomials as well as the Jacobi, continuous Hahn and Hahn polynomials arise in representations of the two-generated algebra $\mathcal{A}$.

When $\Delta \neq 0$, $\mathcal{A}$ as defined in (3.1) is a deformation of the Jordan plane (with $\mathcal{X}$ and $\mathcal{Z}$ viewed as noncommutative coordinates). Three cases will be distinguished depending on

[1] They obey a three term recurrence relation but a higher order difference equation.

whether $\Delta = 0$, $\Delta > 0$ or $\Delta < 0$. These three cases will be studied separately and provide a complete picture of the connection between the algebra (3.1) and orthogonal polynomials.

The presentation is organized as follows. Section 3.2 will introduce the tridiagonal representations of the algebra $\mathcal{A}$ and the non-degeneracy condition. Standardized versions of $\mathcal{A}$ corresponding to $\Delta = 0$, $\Delta < 0$, $\Delta > 0$ will then be examined in the following sections. The case $\Delta = 0$ will be studied in section 3.3 and the Jacobi OPs will appear, while the case $\Delta > 0$ and the continuous Hahn polynomials will be the object of section 3.4. Section 3.5 will focus on the case $\Delta < 0$ and will feature both the Hahn and the para-Krawtchouk polynomials. Some concluding remarks and perspectives will close the paper.

3.2. Tridiagonal representations of the algebra $\mathcal{A}$

Consider a tridiagonal representation of $\mathcal{A}$ where $\mathcal{X} \mapsto X$ and $\mathcal{Z} \mapsto Z$. The actions of X, Z on a semi-infinite orthonormal basis $|n\rangle, n = 0, 1, 2, \dots$ are taken to be of the form

$$X |n\rangle = c_n |n-1\rangle + b_n |n\rangle + a_n |n+1\rangle, \tag{3.3a}$$

$$Z |n\rangle = u_n |n-1\rangle + v_n |n\rangle + w_n |n+1\rangle, \tag{3.3b}$$

with $c_0 = u_0 = 0$. To ensure that such a representation is irreducible we shall assume that the off-diagonal coefficients are non-zero for $n > 0$. Acting with (3.1) on the basis $|n\rangle$ and using the above definitions, one obtains

$$\begin{aligned}(ZX - XZ - Z^2 - \Delta)|n\rangle &= (c_n u_{n-1} - c_{n-1}u_n - u_{n-1}u_n)|n-2\rangle \\ &+ (b_n u_n - b_{n-1}u_n + c_n v_{n-1} - u_n v_{n-1} - c_n v_n - u_n v_n)|n-1\rangle \\ &+ (-\Delta - a_{n-1}u_n + a_n u_{n+1} - v_n^2 + c_n w_{n-1} - u_n w_{n-1} - c_{n+1}w_n - u_{n+1}w_n)|n\rangle \\ &+ (a_n v_{n+1} - a_n v_n + b_n w_n - b_{n+1}w_n - v_n w_n - v_{n+1}w_n)|n+1\rangle \\ &+ (a_n w_{n+1} - a_{n+1}w_n - w_n w_{n+1})|n+2\rangle.\end{aligned} \tag{3.4}$$

For the actions in (3.3) to define a representation of $\mathcal{A}$, each side of the above equation must vanish. As the basis vectors are orthonormal, one obtains the following conditions on the coefficients of (3.3) that define the representations:

$$0 = c_n u_{n-1} - c_{n-1} u_n - u_{n-1} u_n, \tag{3.5}$$

$$0 = b_n u_n - b_{n-1} u_n + c_n v_{n-1} - u_n v_{n-1} - c_n v_n - u_n v_n, \tag{3.6}$$

$$0 = -\Delta - a_{n-1} u_n + a_n u_{n+1} - v_n^2 + c_n w_{n-1} - u_n w_{n-1} - c_{n+1} w_n - u_{n+1} w_n, \tag{3.7}$$

$$0 = a_n v_{n+1} - a_n v_n + b_n w_n - b_{n+1} w_n - v_n w_n - v_{n+1} w_n, \tag{3.8}$$

$$0 = a_n w_{n+1} - a_{n+1} w_n - w_n w_{n+1}. \tag{3.9}$$

3.2.1. General solutions to the recurrence relations

We now determine the general solutions to the above system of recurrence equations. Dividing (3.5) by $u_n u_{n-1}$, one obtains

$$\frac{c_n}{u_n} - \frac{c_{n-1}}{u_{n-1}} = 1.$$

This implies

$$\phi_n = \phi_0 + n, \qquad \phi_n \equiv \frac{c_n}{u_n}. \tag{3.10}$$

Equation (3.9) can be solved similarly. Dividing by $w_n w_{n+1}$, one has

$$\delta_n = \delta_0 - n, \qquad \delta_n \equiv \frac{a_n}{w_n}. \tag{3.11}$$

Rewriting (3.6) and (3.8) in terms of ϕ_n and δ_n and dividing by u_n or w_n, respectively, one obtains

$$b_{n-1} - b_n = (\phi_n - 1) v_{n-1} - (\phi_n + 1) v_n, \tag{3.12}$$

$$b_{n+1} - b_n = (\delta_n - 1) v_{n+1} - (\delta_n + 1) v_n. \tag{3.13}$$

To solve for v_n, shift the index of (3.12) and add (3.13) to find

$$0 = (\delta_n - \phi_{n+1} - 2) v_{n+1} - (\delta_n - \phi_{n+1} + 2) v_n. \tag{3.14}$$

Substituting the solutions (3.10) and (3.11) in (3.14) leads to

$$0 = (\delta_0 - \phi_0 - 2(n+2) + 1) v_{n+1} - (\delta_0 - \phi_0 - 2n + 1) v_n \tag{3.15}$$

$$= \mu_{n+2} v_{n+1} - \mu_n v_n, \tag{3.16}$$

with $\mu_n \equiv (\delta_0 - \phi_0 - 2n + 1)$. Multiplying the above by μ_{n+1} as an integrating factor, one can solve the recurrence to obtain

$$v_n = \frac{(\delta_0 - \phi_0 - 1)(\delta_0 - \phi_0 + 1)v_0}{(\delta_0 - \phi_0 - 2n + 1)(\delta_0 - \phi_0 - 2n - 1)}. \tag{3.17}$$

To find b_n, substract instead (3.12) with shifted index from (3.13) and get

$$b_{n+1} - b_n = \frac{1}{2}(\delta_n + \phi_n + 1)(v_{n+1} - v_n), \tag{3.18}$$

which, upon using (3.10) and (3.11), can be solved immediately and yields

$$b_n = \frac{1}{2}(\delta_0 + \phi_0 + 1)(v_n - v_0) + b_0. \tag{3.19}$$

Finally, (3.7) is written as follows in terms of ϕ_n and δ_n using (3.10) and (3.11), as

$$\Delta + v_n^2 = (\delta_0 - \phi_0 - 2(n+1))\kappa_{n+1} - (\delta_0 - \phi_0 - 2(n-1))\kappa_n, \tag{3.20}$$

with

$$\kappa_n \equiv u_n w_{n-1}. \tag{3.21}$$

Multipliying both sides by $(\delta_0 - \phi_0 - 2n)$ as an integrating factor, one can reduce the above to

$$(\delta_0 - \phi_0 - 2n)(\delta_0 - \phi_0 - 2n + 2)\kappa_n = (\delta_0 - \phi_0)(\delta_0 - \phi_0 + 2)\kappa_0 + \sum_{k=0}^{n-1}(\Delta + v_k^2)(\delta_0 - \phi_0 - 2k). \tag{3.22}$$

The sum over k in (3.22) can be reexpressed[2] as

$$\sum_{k=0}^{n-1}(\Delta + v_k^2)(\delta_0 - \phi_0 - 2k) = \frac{n(\delta_0 - \phi_0 - n + 1)(\Delta(\delta_0 - \phi_0 - 2n + 1)^2 + v_0^2(\delta_0 - \phi_0 - 1)^2)}{(\delta_0 - \phi_0 - 2n + 1)^2}. \tag{3.23}$$

From (3.22) and (3.23), recalling that u_0 was required to vanish so that $\kappa_0 = u_0 w_{-1} = 0$, one has

$$\kappa_n = \frac{n(\delta_0 - \phi_0 - n + 1)(\Delta(\delta_0 - \phi_0 - 2n + 1)^2 + v_0^2(\delta_0 - \phi_0 - 1)^2)}{(\delta_0 - \phi_0 - 2n + 1)^2(\delta_0 - \phi_0 - 2n)(\delta_0 - \phi_0 - 2n + 2)}. \tag{3.24}$$

[2]This is done by noticing the sum to be telescopic or via the polygamma function of the first order.

3.2.2. The linear pencil $\mathcal{X} + \mu\mathcal{Z}$

The algebra $\mathcal{A}$ is invariant under the affine transformation

$$\mathcal{X} \longmapsto \mathcal{X} + \mu\mathcal{Z}, \qquad \mu \in \mathbb{R}.$$

As a result, one expects the transformed solutions for the coefficients in (3.3) to be given by (3.10), (3.11) and (3.19) with modified parameters. Indeed one finds the parameters to be replaced by

$$\phi_0 \longmapsto \phi_0 - \mu, \qquad \delta_0 \longmapsto \delta_0 - \mu, \qquad b_0 \longmapsto b_0 - \mu v_0.$$

Thus, the diagonalization of the linear pencil $\mathcal{X} + \mu\mathcal{Z}$ amounts to the diagonalization of $\mathcal{X}$ up to a shift in the parameters.

3.2.3. Representations on polynomials

Denoting by $\langle x|$ the dual eigenvectors:

$$\langle x|\, X = x\,\langle x|\,,$$

one can look for the polynomials $q_n(x) \equiv \langle x\,|n\rangle$ that diagonalize X

$$Xq_n(x) \equiv x\,q_n(x) = c_n q_{n-1}(x) + b_n q_n(x) + a_n q_{n+1}(x). \tag{3.25}$$

By appropriate renormalization, one obtains a monic recurrence relation

$$Xp_n(x) \equiv xp_n(x) = a_{n-1}c_n p_{n-1}(x) + b_n p_n(x) + p_{n+1}(x), \qquad p_n(x) = \left(\prod_{i=0}^{n-1} a_i\right) q_n(x). \tag{3.26}$$

The families of polynomials $p_n(x)$ that diagonalize X can be determined by identifying the coefficients $a_{n-1}c_n$ and b_n.

From (3.10), (3.11), (3.21) and (3.24), one has that

$$a_{n-1}c_n = (n+\phi_0)(n-\delta_0-1)\frac{n(n+\phi_0-\delta_0-1)(\Delta(2n+\phi_0-\delta_0-1)^2 + v_0^2(\phi_0-\delta_0+1)^2)}{(2n+\phi_0-\delta_0-1)^2(2n+\phi_0-\delta_0)(2n+\phi_0-\delta_0-2)} \tag{3.27}$$

and from (3.19) and (3.17), that

$$b_n = \frac{1}{2}\frac{(\delta_0+\phi_0+1)(\phi_0-\delta_0+1)(\phi_0-\delta_0-1)v_0}{(2n+\phi_0-\delta_0-1)(2n+\phi_0-\delta_0+1)} + \tilde{b}_0, \qquad \tilde{b}_0 \equiv b_0 - \frac{1}{2}(\delta_0+\phi_0+1)v_0. \tag{3.28}$$

Finite-dimensional representations of dimension $N+1$ are obtained if $w_N = 0$ since it follows that $a_N = 0$ from (3.9). This implies that $\kappa_{N+1} = 0$. From (3.24), we see that this is achieved for any value of Δ by

$$N = (\delta_0 - \phi_0). \tag{3.29}$$

If $\Delta \neq 0$, one finds an additionnal pair of solutions given by

$$N+1 = -\frac{1}{2}\left[\phi_0 - \delta_0 - 1 \pm (\phi_0 - \delta_0 + 1)v_0\sqrt{-\Delta^{-1}}\right]. \tag{3.30}$$

3.3. The case $\Delta = 0$: Jacobi polynomials

With Δ vanishing, the coefficient $a_{n-1}c_n$ (3.27) simplifies to

$$a_{n-1}c_n = \frac{n(n+\phi_0)(n-\delta_0-1)(n+\phi_0-\delta_0-1)(\phi_0-\delta_0+1)^2 v_0^2}{(2n+\phi_0-\delta_0-1)^2(2n+\phi_0-\delta_0)(2n+\phi_0-\delta_0-2)}. \tag{3.31}$$

Setting $v_0 = 2(\phi_0 - \delta_0 + 1)^{-1}$, one identifies the basis vector to be proportionnal to the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with parameters

$$\alpha = -\delta_0 - 1, \qquad \beta = \phi_0. \tag{3.32}$$

With $\tilde{b}_0 = 0$, the coefficient b_n of (3.28) is given by

$$b_n = \frac{(\beta^2 + \alpha^2)}{(2n+\beta+\alpha)(2n+\beta+\alpha+2)}. \tag{3.33}$$

Comparing the expressions (3.31) and (3.33) for the coefficients using for instance [**13**], we conclude:

Proposition 4. In the case $\Delta = 0$, the eigenfunctions $p_n(x)$ of X (3.26) are the monic Jacobi polynomials

$$p_n^{(\alpha,\beta)}(x) = \frac{2^n n!}{(n+\alpha+\beta+1)_n} P_n^{(\alpha,\beta)}(x).$$

with parameters α, β given in (3.32).

The only truncation condition possible is (3.29). However, it yields singular expressions in (3.31) and (3.33) for $n \leq N$.

3.4. The case $\Delta > 0$: Continuous Hahn polynomials

If $\Delta \neq 0$, upon scaling the generators of the algebra according to

$$\tilde{\mathcal{X}} = \Omega\mathcal{X}, \qquad \tilde{\mathcal{Z}} = \Omega\mathcal{Z},$$

we obtain

$$[\tilde{\mathcal{Z}}, \tilde{\mathcal{X}}] = \tilde{\mathcal{Z}}^2 + \Omega^2\Delta. \tag{3.34}$$

In view of (3.34), one can choose Ω so that $\Delta = \pm\frac{1}{4}$. In this section, we shall consider the case $\Delta = +\frac{1}{4}$. The coefficient $a_{n-1}c_n$ (3.27) is then given by

$$a_{n-1}c_n = (n+\phi_0)(n-\delta_0-1)\frac{n(n+\phi_0-\delta_0-1)((2n+\phi_0-\delta_0-1)^2/4+v_0^2(\phi_0-\delta_0+1)^2)}{(2n+\phi_0-\delta_0-1)^2(2n+\phi_0-\delta_0)(2n+\phi_0-\delta_0-2)}. \tag{3.35}$$

Writing

$$\phi_0+1 = a+c, \qquad -\delta_0 = b+d, \qquad v_0 = -i\frac{(a-b-c+d)}{2(a+b+c+d)}, \tag{3.36}$$

one can factorize the term with v_0:

$$\tfrac{1}{4}(2n+\phi_0-\delta_0-1)^2 + v_0^2(\phi_0-\delta_0+1)^2 = (n+a+d-1)(n+b+c-1).$$

With (3.36) and the above, (3.35) becomes

$$\begin{aligned} a_{n-1}c_n &= (n+a+c-1)(n+b+d-1) \\ &\times \frac{n(n+a+b+c+d-2)(n+a+d-1)(n+b+c-1)}{(2n+a+b+c+d-1)(2n+a+b+c+d-2)^2(2n+a+b+c+d-3)}. \end{aligned} \tag{3.37}$$

Using (3.36) and taking $\tilde{b}_0 = \frac{i}{4}(a+b-c-d)$, the coefficient b_n (3.28) is found to be

$$\begin{aligned} b_n = i\Big[&-\frac{(n+a+b+c+d-1)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)} \\ &+\frac{n(n+b+c-1)(n+b+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)} + a\Big]. \end{aligned} \tag{3.38}$$

The coefficients (3.37) and (3.38) can be identified in [**13**] and one arrives at:

Proposition 5. In the case $\Delta > 0$, the eigenfunctions $p_n(x)$ of X (3.26) are the monic continuous Hahn polynomials $P_n^{(a,b,c,d)}(x)$ with parameters given in (3.36).

3.4.1. Finite-dimensional representations and orthogonal polynomials

Using (3.36), condition (3.29) becomes

$$N-1=-a-c-b-d, \tag{3.39}$$

which leads to expressions for (3.35) and (3.38) that are ill-defined for $n<N$. However, this can be resolved using limits and one thus obtains the para-Krawtchouk polynomials [**20**].

Condition (3.30) reads

$$N+1=-\frac{1}{2}\left[(a+b+c+d-2)\pm(a-b-c+d)\right]=\begin{cases}-a-d+1\\ -b-c+1\end{cases} \tag{3.40}$$

and corresponds to the truncation of the continuous Hahn polynomials to Hahn polynomials.

However, for each of these truncations (3.39) and (3.40) to define real polynomials, the operator X has to be scaled by an imaginary number; this is equivalent to setting $\Delta\to-\Delta$ which corresponds to the situation $\Delta<0$ that is the object of the next section.

3.5. The case $\Delta<0$: Hahn and para-Krawtchouk polynomials

When $\Delta<0$, polynomials of a real variable are obtained only if (3.29) or (3.30) are satisfied. We begin by treating the latter case.

3.5.1. Hahn polynomials

In view of (3.34), we may take $\Delta=-\frac{1}{4}$ without loss of generality. Expressing the parameters as follows

$$\phi_0=\beta,\qquad -\delta_0=\alpha+1,\qquad v_0=-\frac{(\alpha+\beta+2N+2)}{2(\alpha+\beta+2)},\qquad \tilde{b}_0=\frac{1}{4}(2N-\alpha+\beta), \tag{3.41}$$

so that (3.30) is satisfied, one obtains

$$a_{n-1}c_n=\frac{n(n+\alpha)(n+\beta)(n+\alpha+\beta)(n+\alpha+\beta+N+1)(N-n+1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, \tag{3.42}$$

as well as

$$b_n=\frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}+\frac{n(n+\alpha+\beta+N+1)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}. \tag{3.43}$$

The coefficients given by (3.42) and (3.43) are found in [**13**].

Proposition 6. In the case $\Delta < 0$, the eigenfunctions $p_n(x)$ of X (3.26) related to the finite-dimensional representation condition (3.30) are given in terms of the monic Hahn polynomials $Q_n^{(\alpha,\beta)}(x)$ for the choice of parameters given in (3.41).

As previously mentionned, these polynomials can also be obtained as a truncation of the recurrence defined by (3.35) and (3.38). Indeed, setting

$$\alpha = a + c - 1, \qquad \beta = b + d - 1, \tag{3.44}$$

with one of (3.40), the coefficient (3.35) and (3.38) become proportional to (3.42) and (3.43), respectively. Hence, the action of iX when $\Delta = +\frac{1}{4}$ also leads to the recurrence relation of the monic Hahn polynomials.

3.5.2. Para-Krawtchouk polynomials

We shall finally indicate how a family of finite-dimensional representations of $\mathcal{A}$ relates to para-Krawtchouk polynomials. Consider the condition (3.29). Although leading to singular expressions for certain values of n, well-defined polynomials are obtained by carefully taking limits. Mindful of (3.34), it is convenient in this case to take $\Delta = -1$. Let $N = 2j + p$ with j an integer and $p = 0, 1$ depending on the parity of N, and set

$$\phi_0 + 1 = -j + e_1 t, \quad -\delta_0 = -j + e_2 t + 1 - p, \quad v_0 = \tfrac{(\gamma+p-1)}{(-2j+e_1t+e_2t-p+1)}, \quad e_1 = e_2 = 1. \tag{3.45}$$

The parameters e_1 and e_2 are chosen equal in order to simplify the expressions. The more general solutions can be recovered using isospectral deformations [**15, 10**]. With the above parametrization, it can be seen that (3.29) is verified in the limit where $t \to 0$. With the parameters as in (3.45), the coefficient $a_{n-1}c_n$ (3.27) becomes

$$\begin{aligned} a_{n-1}c_n = {}& (n-j+t-1)(n-j+t-p) \\ & \times \frac{n(n-2j+2t-p-1)(N-2n+p+\gamma)(N-2n-p+2-\gamma)}{(2n-2j+2t-p-1)^2(2n-2j+2t-p)(2n-2j+2t-p-2)}. \end{aligned} \tag{3.46}$$

Taking the limit $t \to 0$ and treating the cases for $p = 0, 1$ separately, one finds that the results can be combined as follows

$$\lim_{t\to 0} a_{n-1}c_n = \frac{n(N+1-n)(N-2n+p+\gamma)(N-2n-p+2-\gamma)}{4(2n-N+p-1)(2n-N-p-1)}. \tag{3.47}$$

For the coefficient b_n, setting $\tilde{b}_0 = \frac{1}{2}(N + \gamma - 1)$ and inserting (3.45) in (3.28), one finds

$$b_n = \frac{1}{2}\frac{(p-1)(-2j-p+2t-1)(\gamma+p-1)}{(2n-2j-p+2t-1)(2n-2j-p+2t+1)} + \frac{1}{2}(N+\gamma-1). \tag{3.48}$$

Treating the cases $p = 0$ or $p = 1$ separately and taking the limit $t \to 0$, one sees that the results can be written jointly as

$$\lim_{t\to 0} b_n = -\frac{(N-n)(N-2n-2+p+\gamma)}{2(2n-N-p+1)} - \frac{n(N-2n+2-p-\gamma)}{2(2n-N+p-1)}. \tag{3.49}$$

The coefficients given by (3.47) and (3.49) are recognized in [**15**] as the coefficients for the recurrence relation of the monic para-Krawtchouk polynomials.

Proposition 7. In the case $\Delta < 0$, the eigenfunctions $p_n(x)$ of X (3.26) in the finite-dimensional representation (3.29) of $\mathcal{A}$ are the monic para-Krawtchouk polynomials.

3.6. Conclusion

We have studied tridiagonal representations of the algebra $\mathcal{A}$ with defining relation $[\mathcal{Z}, \mathcal{X}] = \mathcal{Z}^2 + \Delta$. Depending on the value of Δ, in these representations, the linear pencil $X + \mu Z$ entailed the recurrence relations of the Jacobi ($\Delta = 0$), continuous Hahn ($\Delta > 0$), Hahn and para-Krawtchouk ($\Delta < 0$) polynomials.

In the wake of this work, two research avenues present themselves. One is the exploration of the tridiagonal representations of the algebra $[\mathcal{Z}, \mathcal{X}] = \mathcal{Z}^2 + \alpha\mathcal{X}$, another class of the general quadratic algebra (3.2). It is expected that the tridiagonal representations will lead to the Wilson, Racah and para-Racah polynomials in a similar fashion.

Another related direction is the study of the so-called meta algebras, poised to describe both polynomial and rational functions of a given type., as shown in [**21**] for functions of the Hahn type. The meta-Hahn algebra is in fact obtained by adjoining to $\mathcal{A}$ an additional generator. As it turns out, the extra generator offers a rationale for considering tridiagonal representations. This suggests in particular that the work on the q-oscillator algebra [**18**] should be revisited in order to bring to the fore the associated rational functions.

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Partie 2

Opérateurs de Heun algébriques

Introduction

L'opérateur de Heun ordinaire est un opérateur différentiel du deuxième ordre définit par

$$M = x(x-1)(x-d)\frac{d^2}{dx^2} + (\rho_2 x^2 + \rho_1 x + \rho_0)\frac{d}{dx} + r_1 x + r_0. \tag{3.50}$$

Une propriété caractéristique de l'opérateur de Heun est d'agir sur les polynômes en x en augmentant leur degré par un. En se basant sur cette observation, il est possible de généraliser l'opérateur de Heun. Ces opérateurs de Heun généralisés sont alors définis comme les opérateurs différentiels ou aux différences du second ordre les plus généraux sur un domaine donné qui satisfont la propriété d'agir sur les polynômes sur ce même domaine en augmentant leur degré par un.

Chaque polynôme orthogonal du schéma d'Askey-Wilson est associé à un problème bispectral. En effet, chacun de ces polynômes diagonalise un opérateur de récurrence X à trois termes agissant sur le degré du polynôme, de même qu'un opérateur différentiel ou aux différences Y agissant sur la variable du polynôme. Ainsi, dans l'algèbre générée par ces deux opérateurs, la combinaison quadratique la plus générale

$$W = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_4 Y + \tau_0,$$

forme alors un opérateur du second ordre. Cet opérateur W est tridiagonal dans la base des polynômes associés et agit sur ceux-ci en augmentant leur degré par un. L'opérateur W généralise alors la propriété caractéristique de l'opérateur de Heun ordinaire. C'est ce qui motive la notion d'opérateur de Heun algébrique, défini comme la combinaison quadratique la plus générale de la paire d'opérateurs associée à tout problème bispectral.

Cette seconde partie de la thèse aborde le thème des opérateurs de Heun algébriques. Le chapitre 4 introduit la notion d'opérateur de Heun algébrique et établit un rapprochement avec la théorie de polynômes orthogonaux. Cette notion est ensuite mise à profit à l'étude d’un problème classique en traitement de signal. Dans le chapitre 5, on se concentre sur l’étude des structures algébriques associées aux opérateurs de Heun algébriques rattachés aux polynômes de Racah et de Bannai-Ito. Finalement, les chapitres 6 et 7 examinent les structures algébriques générées par une spécialisation des opérateurs de Heun algébriques sans terme diagonal. Ces structures sont reconnues en tant qu’algèbres de type Sklyanin et des représentations sont construites sur différentes familles de polynômes orthogonaux.

Chapitre 4

Signal processing, orthogonal polynomials, and Heun equations

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Abstract

A survey of recents advances in the theory of Heun operators is offered. Some of the topics covered include: quadratic algebras and orthogonal polynomials, differential and difference Heun operators associated to Jacobi and Hahn polynomials, connections with time and band limiting problems in signal processing.

4.1. Introduction

This lecture aims to present an introduction to the algebraic approach to Heun equation. To offer some motivation, we shall start with an overview of a central problem in signal treatment, namely that of time and band limiting. Our stepping stone will be the fact that Heun type operators play a central role in this analysis thanks to the work of Landau, Pollack and Slepian [**19**], see also the nice overview in [**6**]. After reminding ourselves of the standard Heun equation, we shall launch into our forays. We shall recall that all polynomials of the Askey scheme are solutions to bispectral problems and we shall indicate that all their properties can be encoded into quadratic algebras that bear the name of these families. We shall use the Jacobi polynomials as example. We shall then discuss the tridiagonalization procedure designed to move from lower to higher families of polynomials in the Askey hierarchy.

This will be illustrated by obtaining the Wilson/Racah polynomials from the Jacobi ones or equivalently by embedding the Racah algebra in the Jacobi algebra. We shall then show that the standard Heun operator can be obtained from the most general tridiagonalization of the hypergeometric (the Jacobi) operator. This will lead us to recognize that an algebraic Heun operator can be associated to each of entries of the Askey tableau. We shall then proceed to identify the Heun operator associated to the Hahn polynomials. It will be seen to provide a difference version of the standard Heun operator. We shall have a look at the algebra this operator forms with the Hahn operator and of its relation to the Racah algebra. We shall then loop the loop by discussing the finite version of the time and band limiting problem and by indicating how the Heun-Hahn operator naturally provides a tridiagonal operator commuting with the non-local limiting operators. We shall conclude with a summary of the lessons we will have learned.

4.2. Motivation and background

4.2.1. Time and band limiting

A central problem in signal processing is that of the optimal reconstruction of a signal from limited observational data. Several physical constraints arise when sampling a signal. We will here focus on those corresponding to a limited time window and to to a cap on the detection of frequencies. Consider a signal represented as a function of time by

$$f : \mathbb{R} \longrightarrow \mathbb{R},$$

and suppose f can only be observed for a finite time interval

$$W = [-T, T] \subset \mathbb{R}.$$

This time limiting can be expressed as multiplication by a step function W defined by

$$\chi_W(t) = \begin{cases} 1, & \text{if } -T \leq t \leq T, \\ 0, & \text{otherwise.} \end{cases}$$

Now, suppose the measurements are limited in their bandwidth. This corresponds to an upper bound on accessible frequencies. Let us express this band limiting as multiplication by a step function χ_N of the Fourier transform of the signal f, where

$$\chi_N(n) = \begin{cases} 1, & \text{if } 0 \le n \le N, \\ 0, & \text{otherwise.} \end{cases}$$

This defines the time limiting operator χ_W

$$\chi_W : \mathcal{C}(\mathbb{R}) \longrightarrow \mathcal{C}(\mathbb{R}),$$

acting by multiplication on functions of time and the band limiting operator χ_N

$$\chi_N : \mathcal{C}(\mathbb{R}) \longrightarrow \mathcal{C}(\mathbb{R}),$$

acting by multiplication on functions of frequencies. Thus, the available data on f is limited to $\chi_N \, F \, \chi_W \, f$, where F denotes the Fourier transform. The time and band limiting problem consists in the optimal reconstruction of f from the limited available data $\chi_N \, F \, \chi_W \, f$.

In this context, the best approximation of f requires finding the singular vectors of the operator

$$E = \chi_N \, F \, \chi_W,$$

which amounts to the eigenvalue problems for the following operators

$$E^*E = \chi_W F^{-1} \chi_N F \chi_W, \quad \text{and} \quad EE^* = \chi_N F \chi_W F^{-1} \chi_N.$$

For F the standard Fourier transform, one has

$$\begin{aligned} \left[EE^* \tilde{f}\right](l) &= \chi_N \int_{-T}^{T} e^{ilt} \left(\int_{0}^{N} \tilde{f}(k) e^{-ikt} dk \right) dt \\ &= \chi_N \int_{0}^{N} \tilde{f}(k) \left(\int_{-T}^{T} e^{i(l-k)t} dt \right) dk, \\ &= \int K_T(l,k) \tilde{f}(k) dk, \end{aligned} \tag{4.1}$$

where

$$K_T(l,k) = \int_{-T}^{T} e^{i(l-k)T} dt = \frac{\sin(l-k)T}{(l-k)},$$

which is the integral operator with the well-known *sinc kernel*. It is known, that non local operators such as E^*E have spectra that are not well-suited to numerical analysis. This makes difficult obtaining solutions to the time and band limiting problem. However, a remarkable observation of Landau, Pollak and Slepian [**20, 12, 13, 17, 18**] is that there is a differential operator D with a well-behaved spectrum that commutes with the integral operator E^*E. This reduces the time and band limiting problem to the numerically tractable eigenvalue problem of D. In the above example, this operator D is a special case of the Heun operator. The algebraic approach presented here will give indications (in the discrete-discrete case in particular) as to why this "miracle" happens.

4.2.2. The Heun operator

Let us first remind ourselves of basic facts regarding the usual Heun operator [**9**]. The Heun equation is the Fuchsian differential equation with four regular singularities. The standard form is obtained through homographic transformations by placing the singularities at $x = 0, 1, d$ and ∞ and is given by

$$\frac{d^2}{dx^2}\psi(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-d}\right)\frac{d}{dx}\psi(x) + \frac{\alpha\beta x - q}{x(x-1)(x-d)}\psi(x) = 0,$$

where

$$\alpha + \beta - \gamma - \delta + 1 = 0,$$

to ensure regularity of the singular point at $x = \infty$. This Heun equation can be written in the form

$$M\psi(x) = \lambda\psi(x)$$

with M the Heun operator given by

$$M = x(x-1)(x-d)\frac{d^2}{dx^2} + (\rho_2 x^2 + \rho_1 x + \rho_0)\frac{d}{dx} + r_1 x + r_0, \tag{4.2}$$

with

$$\rho_2 = -(\gamma + \delta + \epsilon), \qquad \rho_1 = (\gamma + \delta)d + \gamma + \epsilon,$$

$$\rho_0 = -\gamma d,$$

$$r_1 = -\alpha\beta, \qquad r_0 = q + \lambda.$$

One can observe that M sends any polynomial of degree n to a polynomial of degree $n+1$. Indeed, the Heun operator can be defined as the most general second order differential operator with this property.

4.3. The Askey scheme and bispectral problems

A pair of linear operators X and Y is said to be bispectral if there is a two-parameter family of common eigenvectors $\psi(x,n)$ such that one has

$$X\psi(x,n) = \omega(x)\psi(x,n)$$
$$Y\psi(x,n) = \lambda(n)\psi(x,n),$$

where, X acts on the variable n and Y, on the variable x. One should note that the above does not imply that $[X,Y]=0$ as the x and n variables constitute different representations. Thus, one must be careful to use the same representation for all operators when computing products of operators. For the band-time limiting problem associated to sinc kernel, one has the two-parameter family of eigenfunctions given by $\psi(t,n)=e^{itn}$ with the bispectral pair identified as

$$X = -\frac{d^2}{dn^2}, \qquad \omega(t) = t^2,$$
$$Y = -\frac{d^2}{dt^2}, \qquad \lambda(n) = n^2.$$

In this case, both operators are differential operators. However, bispectral pairs are realized in terms of various combinations of continuous and discrete operators. These bispectral problems admit two representations, corresponding to the two spectral parameters x and n. As X and Y do not commute, products of these operators must be taken within the same representation for all terms in the product.

A key observation is that each family of hypergeometric polynomials of the Askey scheme defines a bispectral problem. Indeed, these polynomials are the solution to both a recurrence relation and a differential or difference equation. By associating X with the recurrence relation and Y with the differential or difference equation, one forms a bispectral problem as follows. In the x-representation, X acts a multiplication by the variable and Y as the differential or difference operator while in the n-representation, X acts as a three-term difference operator over n and Y as multiplication by the eigenvalue. The family of common eigenvectors are the orthogonal polynomials.

As a relevant example, consider the (monic) Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$ defined as follows [**10**] These polynomials are the eigenvectors of the hypergeometric operator D_x given by

$$D_x \equiv x(x-1)\frac{d^2}{dx^2} + (\alpha + 1 - (\alpha+\beta+2)x)\frac{d}{dx}, \tag{4.3}$$

such that

$$D_x\, \hat{P}_n^{(\alpha,\beta)}(x) = \lambda_n\, \hat{P}_n^{(\alpha,\beta)}(x),$$

with eigenvalues given by $\lambda_n = -n(n+\alpha+\beta+1)$. They form an orthogonal set:

$$\int\limits_0^1 \hat{P}_n^{(\alpha,\beta)}(x)\hat{P}_m^{(\alpha,\beta)}(x)x^\alpha(1-x)^\beta dx = h_n\delta_{n,m}, \tag{4.4}$$

where

$$h_n = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta_2)}u_1u_2\cdots u_n.$$

The Jacobi polynomials also satisfy the three-term recurrence relation given by

$$x\hat{P}_n^{(\alpha,\beta)}(x) = \hat{P}_{n+1}^{(\alpha,\beta)}(x) + b_n\hat{P}_n^{(\alpha,\beta)}(x) + u_n\hat{P}_{n-1}^{(\alpha,\beta)}(x), \tag{4.5}$$

where

$$\begin{aligned} u_n &= \frac{n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, \\ b_n &= \frac{1}{2} + \frac{\alpha^2-\beta^2}{4}\left(\frac{1}{2n+\alpha+\beta} - \frac{1}{2n+\alpha+\beta+2}\right). \end{aligned}$$

Taking

$$X = x, \qquad Y = D_x,$$

for the x-representation and

$$X = T_n^+ + b_n\cdot 1 + u_nT_n^-, \qquad Y = \lambda_n, \qquad \text{where} \qquad T_n^\pm f_n = f_{n\pm1},$$

for the n-representation, the Jacobi polynomials provide a two-parameter set of common eigenvectors of X and Y and hence of the bispectral problem they define. This construction arises similarly for all the orthogonal polynomials in the Askey scheme.

4.3.1. An algebraic description

The properties of the orthogonal polynomials of the Askey scheme can be encoded in an algebra as follows. For any such polynomials, take the X operator to be the multiplication by the variable and the Y operator as the differential or difference equation they satisfy. Consider then the associative algebra generated by K_1, K_2 and K_3 where

$$K_1 \equiv X, \qquad K_2 \equiv Y, \qquad K_3 \equiv [K_1, K_2]. \tag{4.6}$$

Upon using these definitions for the generators, one can derive explicitly the commutation relations to obtain that $[K_2, K_3]$ and $[K_3, K_1]$ are quadratic expressions in K_1 and K_2. Once these relations have been identified, the algebra can be posited abstractly and the properties of the corresponding polynomials follow from representation theory.

Sitting at the top of the Askey scheme, the Wilson and Racah polynomials [**10**] are the most general ones and the algebra encoding their properties encompasses the other. As the algebraic description is insensitive to truncation, both the Wilson and Racah polynomials are associated to the same algebra. This algebra is known as the Racah-Wilson or *Racah algebra* and is defined [**5**] as the associative algebra over $\mathbb{C}$ generated by $\{K_1, K_2, K_3\}$ with relations

$$[K_1, K_2] = K_3 \tag{4.7}$$

$$[K_2, K_3] = a_1\{K_1, K_2\} + a_2 K_2^2 + bK_2 + c_1 K_1 + d_1 I \tag{4.8}$$

$$[K_3, K_1] = a_1 K_1^2 + a_2\{K_1, K_2\} + bK_1 + c_2 K_2 + d_2 I, \tag{4.9}$$

where $a_1, a_2, b, c_1, c_2, d_1$ and d_2 are structure parameters and where $\{A, B\} = AB + BA$ denotes the anti-commutator. One can show that the Jacobi identity is satisfied. The Racah algebra naturally arises in the study of classical orthogonal polynomials but has proved useful in the construction of integrable models and in representation theory [**4, 5**].

Other polynomials of the Askey scheme can be obtained from the Racah or Wilson polynomials by limits and specializations. The associated algebras can be obtained from the Racah algebra in the same way. In particular, the *Jacobi algebra* [**3**] constitutes one such specialization where $a_1, c_1, d_1, d_2 \to 0$. Indeed, taking

$$\begin{aligned} A_1 &= Y = D_x \equiv x(x-1)\frac{d^2}{dx^2} + (\alpha+1-(\alpha+\beta+2)x)\frac{d}{dx}, \\ A_2 &= X = x, \quad A_3 \equiv [A_1, A_2] = 2x(x-1)\frac{d}{dx} - (\alpha+\beta+2)x + \alpha + 1, \end{aligned} \tag{4.10}$$

one finds the following relations for the Jacobi algebra

$$[A_1, A_2] = A_3 \tag{4.11}$$

$$[A_2, A_3] = a_2 A_2^2 + dA_2 \tag{4.12}$$

$$[A_3, A_1] = a_2\{A_1, A_2\} + dA_1 + c_2 A_2 + e_2, \tag{4.13}$$

where $a_2 = 2$, $d = -2$, $c_2 = -(\alpha+\beta)(\alpha+\beta+2)$ and $e_2 = (\alpha+1)(\alpha+\beta)$.

4.3.2. Duality

The bispectrality of the polynomials in the Askey scheme is related to a notion of duality where the variable and the degree are exchanged. In the algebraic description, this corresponds to exchanging the X and Y operator. Let us make details explicit in the finite-dimensional case where the polynomials satisfy both a second order difference equation and a three-term recurrence relation [**8**].

In finite dimension, both the X and Y operator will admit a finite eigenbasis. Let us denote the eigenbasis of X by $\{e_n\}$ and the one of Y by $\{d_n\}$ for $n = 0, 1, 2, \ldots, N$. One first notices that Y will be tridiagonal in the X eigenbasis and likewise for X in the Y eigenbasis. Explicitly, one has

$$\begin{aligned} Xe_n &= \lambda_n e_n, & Yd_n &= \mu_n d_n, \\ Xd_n &= a_{n+1}d_{n+1} + b_n d_n + a_n d_{n-1}, & Ye_n &= \xi_{n+1}e_{n+1} + \eta_n e_n + \xi_n e_{n-1}, \\ & & n &= 0, 1, \ldots, N \end{aligned} \tag{4.14}$$

where $\{a_n\}$, $\{b_n\}$, $\{\xi_n\}$ and $\{\eta_n\}$ for $n = 0, 1, \ldots, N$ are scalar coefficients. As both the X and Y eigenbases span the same space, one can expand one basis onto the other as follows

$$e_s = \sum_{n=0}^{N} \sqrt{w_s}\,\phi_n(\lambda_s) d_n, \tag{4.15}$$

where $\phi_n(x)$ are the polynomials associated to the algebra defined by the following recurrence relation

$$a_{n+1}\phi_{n+1}(x) + b_n\phi_n(x) + a_n\phi_{n-1}(x) = x\phi_n(x), \quad \phi_{-1} = 0, \quad \phi_0 = 1,$$

which verify the orthogonality relation

$$\sum_{s=0}^{N} w_s \phi_n(\lambda_s)\phi_m(\lambda_s) = \delta_{n,m},$$

so that the reverse expansion is easily seen to be

$$d_n = \sum_{s=0}^{N} \sqrt{w_s}\phi_n(\lambda_s)e_s.$$

Consider now the *dual* set of polynomials $\chi_n(x)$ defined by the following recurrence relation

$$\xi_{n+1}\chi_{n+1}(x) + \eta_n\chi_n(x) + \xi_n\chi_{n-1}(x) = x\chi_n(x), \quad \chi_{-1} = 0,\ \chi_0 = 1,$$

which are orthogonal with respect to the dual weights $\tilde{w}_s$:

$$\sum_{s=0}^{N} \tilde{w}_s \chi_n(\mu_s)\chi_m(\mu_s) = \delta_{n,m}. \tag{4.16}$$

These dual polynomials provide an alternative expansion of one basis onto the other. One has

$$d_s = \sum_{n=0}^{N} \sqrt{\tilde{w}_s}\chi_n(\mu_s)e_n. \tag{4.17}$$

One readily verifies this expansion by applying Y to obtain

$$\begin{aligned} Yd_s &= \sum_{n=0}^{N} \sqrt{\tilde{w}_s}\chi_n(\mu_s)Ye_n = \sum_{n=0}^{N} \sqrt{\tilde{w}_s}\chi_n(\mu_s)[\xi_{n+1}e_{n+1} + \eta_n e_n + \xi_n e_{n-1}] \\ &= \sum_{n=0}^{N} \sqrt{\tilde{w}_s}[\xi_{n+1}\chi_{n+1}(\mu_s) + \eta_n\chi_n(\mu_s) + \xi_n\chi_{n-1}(\mu_s)]e_n = \mu_s d_s. \end{aligned}$$

Using the orthogonality of the polynomials $\{\chi_n(\mu_s)\}$ given by (4.16), the expansion (4.17) is inverted as

$$e_n = \sum_{s=0}^{N} \sqrt{\tilde{w}_s}\chi_N(\mu_s)d_s.$$

Comparing the above with the first expansion in (4.15), knowing the $\{d_n\}$ to be orthogonal, one obtains

$$\sqrt{w_s}\phi_n(\lambda_s) = \sqrt{\tilde{w}_n}\chi_s(\mu_n), \tag{4.18}$$

a property known as Leonard duality [**14**], see also [**21**] for an introduction to Leonard pairs.

4.4. Tridiagonalization of the hypergeometric operator

Tridiagonalization enables one to construct orthogonal polynomials with more parameters from simpler ones and thus to build a bottom-up characterization of the families of the Askey scheme from this bootstrapping. In particular, properties of the Wilson and Racah polynomials can be found from the tridiagonalization of the hypergeometric operator [**3**]. Moreover, by considering the most general tridiagonalization, one recovers the complete Heun operator [**7**].

4.4.1. The Wilson and Racah polynomials from the Jacobi polynomials

In the canonical realization of the Jacobi algebra in terms of differential operators presented in (4.10), one of the generators is the hypergeometric operator (4.3) and the other is the difference operator in the degree corresponding to the recurrence relation (4.5). We consider the construction of an operator in the algebra which is tridiagonal in the eigenbases of both operators.

Let $Y = D_x$ be the hypergeometric operator and $X = x$ be multiplication by the variable. Define M in the Jacobi algebra as follows

$$M = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_0, \tag{4.19}$$

where $\tau_i,\ i = 0,1,2,3$ are scalar parameters. Knowing that X leads to the three-term recurrence relation of the Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$:

$$X\,\hat{P}_n^{(\alpha,\beta)}(x) = x\hat{P}_n^{(\alpha,\beta)}(x) = \hat{P}_{n+1}^{(\alpha,\beta)}(x) + b_n\hat{P}_n^{(\alpha,\beta)}(x) + u_n\hat{P}_{n-1}^{(\alpha,\beta)}(x),$$

and is obviously tridiagonal, it is clear from (4.19) that M will also be tridiagonal in the eigenbasis of Y that the Jacobi polynomials form. One has

$$M\hat{P}_n^{(\alpha,\beta)}(x) = \xi_{n+1}\hat{P}_{n+1}^{(\alpha,\beta)}(x) + \eta_n\hat{P}_n^{(\alpha,\beta)}(x) + b_n u_n\hat{P}_{n-1}^{(\alpha,\beta)}(x), \tag{4.20}$$

where

$$\begin{aligned}
\xi_n &= \tau_1\lambda_{n-1} + \tau_2\lambda_n + \tau_3,\\
\eta_n &= (\tau_1+\tau_2)\lambda_n b_n + \tau_3 b_n,\\
b_n &= \tau_1\lambda_n + \tau_2\lambda_{n-1} + \tau_3.
\end{aligned}$$

If $\tau_1+\tau_2 = 0$, then M simplifies to $M = \tau_1[X,Y] + \tau_3 X$, which is a first order differential operator. In order for M to remain a second order operator, one demands that $\tau_1+\tau_2 \neq 0$. In this case, normalizing M so that $\tau_1+\tau_2 = 1$, one obtains explicitly

$$\begin{aligned}
M = x^2(x-1)\frac{d^2}{dx^2} &+ x[\alpha+1-2\tau_2-(\alpha+\beta-2\tau_2)x]\frac{d}{dx}\\
&- [\tau_2(\alpha+\beta+2)-\tau_3]x + (\alpha+1)\tau_2 + \tau_0,
\end{aligned} \tag{4.21}$$

We now construct a basis in which M is diagonal. In the realization (4.10), where the algebra acts on functions of x, X is multiplication by x and its inverse is defined by

$$X^{-1} : f(x) \longmapsto \frac{1}{x}f(x).$$

With this definition, one can invert the expression for M given by (4.19) to obtain

$$Y = \tau_1 X^{-1}M + \tau_2 MX^{-1} + (2\tau_1\tau_2 - \tau_0)X^{-1} - (2\tau_1\tau_2 + \tau_3). \tag{4.22}$$

Observing that (4.22) has the same structure as (4.19) under the transformation $X \mapsto X^{-1}$, the eigenfunctions of M can be constructed as follows. Introduce the variable $y = 1/x$ and conjugate M and Y by a monomial in y to obtain

$$\tilde{Y} = y^{\nu-1}Yy^{1-\nu}, \qquad \tilde{M} = y^{\nu-1}My^{1-\nu}.$$

Then, by demanding that

$$\tau_3 = (4+\alpha+\beta-\nu)(\tau_2+\nu-1) - \nu\tau_2,$$

the conjugated operators take the following form,

$$-\tilde{Y} = y^2(y-1)\frac{d^2}{dy^2} + y(a_1 y + b_1)\frac{d}{dy} + c_1 y + d_1,$$
$$-\tilde{M} = y(y-1)\frac{d^2}{dy^2} + (a_2 y + b_2)\frac{d}{dy} + d_2,$$

with all the new parameters being simple expressions in terms of α, β, τ_0, τ_2 and ν. Up to a global sign, one recognizes $\tilde{M}$ as the hypergeometric operator in terms of the variable y, while $\tilde{Y}$ is similar to M. As the Jacobi polynomials diagonalizes the hypergeometric operator, the eigenvectors satisfying

$$M\psi_n(x) = \tilde{\lambda_n}\psi_n(x) \tag{4.23}$$

are easily found to be

$$\psi_n(x) = x^{\nu-1}\hat{P}_n^{(\tilde{\alpha},\tilde{\beta})}(1/x), \qquad \tilde{\lambda_n} = n(n+\tilde{\alpha}+\tilde{\beta}+1),$$
$$\tilde{\beta} = \beta, \qquad \tilde{\alpha} = 2(\tau_2+\nu) - \alpha - \beta - 7.$$

It follows from the recurrence relation of the Jacobi polynomials (4.5) that X^{-1} is tridiagonal in the basis $\psi_n(x)$ as it corresponds to multiplication by the variable. Thus, a glance at (4.22) confirms that Y is tridiagonal in the $\psi_n(x)$ basis.

In order to relate this result with the Wilson and Racah orthogonal polynomials, consider the expansion of $\psi_n(x)$ in terms of $\hat{P}_k^{(\alpha,\beta)}(x)$. One has

$$\psi_n(x) = \sum_{k=0}^{\infty} G_k(n)\hat{P}_k^{(\alpha,\beta)}(x). \tag{4.24}$$

By factoring the expansion coefficients as $G_k(n) = G_0(n)\Xi_k Q_k(n)$, one finds using (4.20) and (4.23) that, for a unique choice of Ξ_k, Q_k satisfies the following three-term recurrence relation

$$\tilde{\lambda_n} Q_k(n) = B_k Q_{k+1}(n) + U_k Q_k(n) + F_k Q_{k-1}(n),$$

where

$$\begin{aligned}
B_k &= u_{k+1}(\tau_1\lambda_{k+1} + \tau_2\lambda_k + \tau_3),\\
U_k &= \lambda_k b_k + \tau_3 b_k,\\
F_k &= \tau_1\lambda_{k-1} + \tau_2\lambda_k + \tau_3.
\end{aligned} \tag{4.25}$$

The recurrence relation allows to identify the factor $Q_k(n)$ of the expansion coefficient in (4.24) as four parameters Wilson polynomials $W_n(x; k_1, k_2, k_3, k_4)$. In this construction, two of these parameters are inherited from the Jacobi polynomials while, after scaling, the tridiagonalisation introduced two free parameters.

The Racah polynomials occur in this setting when a supplementary restriction is introduced. Indeed, a glance at (4.21) shows that the generic M operator maps polynomials of degree n into polynomials of degree $n+1$. However, one can see from (4.20) that if

$$\xi_{N+1} = \tau_1\lambda_N + \tau_2\lambda_{N+1} + \tau_3 = 0,$$

both Y and M preserve the space of polynomials of degree less than or equal to N. This truncation condition is satisfied when $\nu = N+1 = 2-2\tau_2$. In this case, the eigenvectors of M are

$$\psi_n(x) = x^N \hat{P}_n^{(N-\alpha-\beta-4,\beta)}(1/x),$$

which are manifestly polynomials of degree $N-n$. One then considers again the expansion of the basis element $\psi_n(x)$ into $\hat{P}_k^{(\alpha,\beta)}(x)$ to obtain

$$\psi_n(x) = \sum_{k=0}^{N} R_{n,k}\hat{P}_k^{(\alpha,\beta)}(x),$$

where the expansion coefficients $R_{n,k}$ can be shown to be given in terms of the Racah polynomials. Using the orthogonality of the Jacobi polynomials given in (4.4), one obtains

$$R_{n,k}h_k = \int_0^1 \psi_n(x)\hat{P}_k^{(\alpha,\beta)}(x)\, x^\alpha(1-x)^\beta dx,$$

an analog of the Jacobi-Fourier transform of Koornwinder [**11**], giving an integral representation of the Racah polynomials.

It was stated earlier that the properties of the orthogonal polynomials in the Askey scheme are encoded in their associated algebras. This can be seen from the construction of the Wilson and Racah polynomials from the Jacobi polynomials by the tridiagonalization procedure which corresponds algebraically to an embedding of the Racah algebra in the Jacobi algebra. This is explicitly given by

$$K_1 = A_1, \qquad K_2 = \tau_1 A_2 A_1 + \tau_2 A_1 A_2 + \tau_3 A_2, \tag{4.26}$$

where A_1, A_2 are the Jacobi algebra generators as in (4.10). One shows that K_1 and K_2 as defined in (4.26) verify the relations (4.7) of the Racah algebra assuming that A_1 and A_2 verify those of the Jacobi relations as given in (4.11). Thus, the embedding (4.26) encodes the tridiagonalization result abstractly.

The tridiagonalisation (4.19) used to derive higher polynomials from the Jacobi polynomials is not the most general tridiagonal operator that can be constructed from the Jacobi algebra generators. Indeed, consider the addition in (4.19) of a linear term in Y, given by (4.3):

$$M = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_4 Y + \tau_0. \tag{4.27}$$

It is straightforward to see that M as given by (4.27) is equal to the Heun operator (4.2). Expressed as in (4.27), the Heun operator is manifestly tridiagonal on the Jacobi polynomials, which offers a simple derivation of a classical result. For the finite dimensional situation see [**15**].

4.5. The Algebraic Heun operator

The emergence of the standard Heun operator from the tridiagonalization of the hypergeometric operator suggests that Heun-type operators can be associated to bispectral problems. In particular, knowing all polynomials in the Askey scheme to define bispectral problems, there should be Heun-like operators associated to each of these families of polynomials. Guided by this observation, consider a set of polynomials in the Askey scheme and let X and Y be the generators of the associated algebra as in (4.6). As before, X is the recurrence operator and Y, the difference or differential operator. The corresponding Heun-type operator W is defined as

$$W = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_4 Y + \tau_0, \tag{4.28}$$

and will be referred to as an *algebraic Heun operator* [**8**]. The operator W associated to a polynomial family will have features similar to those of the standard Heun operator which arises in the context of the Jacobi polynomials. To illustrate this, a construction that parallels the one made for the Jacobi polynomials is presented

4.5.1. A discrete analog of the Heun operator

The standard Heun operator can be defined as the most general degree increasing second order differential operator. In analogy with this, one defines the *difference Heun operator* as:

Definition 3 (Difference Heun operator)**.** The difference Heun operator is the most general second order difference operator on a uniform grid which sends polynomials of degree n to polynomials of degree $n+1$.

We now obtain an explicit expression for the difference Heun operator on the finite grid $G = \{0, 1, \dots, N\}$. Let $T^{\pm}$ be shift operators defined by

$$T^{\pm} f(x) = f(x \pm 1), \tag{4.29}$$

and take W to be a generic second order difference operator with

$$W = A_1(x)T^{+} + A_2(x)T^{-} + A_0(x)I. \tag{4.30}$$

By demanding that W acting on 1, x and x^2 yields polynomials of one degree higher, one obtains that

$$A_0(x) = \tilde{\pi}_1(x) - \tilde{\pi}_3(x), \qquad A_1(x) = \frac{\tilde{\pi}_3(x) - \tilde{\pi}_2(x)}{2}, \qquad A_2(x) = \frac{\tilde{\pi}_3(x) + \tilde{\pi}_2(x)}{2}, \tag{4.31}$$

where the $\tilde{\pi}_i(x)$ are arbitrary polynomials of degree i for $i = 1, 2, 3$. Thus, in general, $A_i(x)$ for $i = 0, 1, 2$ are third degree polynomials with $A_1(x)$ and $A_2(x)$ having the same leading coefficient. Moreover, the restriction of the action of W to the finite grid G implies that A_1 has $(x - N)$ as a factor and A_2 has x as a factor. Hence, one has

$$\begin{aligned}
A_1(x) &= (x - N)(\kappa x^2 + \mu_1 x + \mu_0), \\
A_2(x) &= x(\kappa x^2 + \nu_1 x + \nu_0), \\
A_0(x) &= -A_1(x) - A_2(x) + r_1 x + r_0,
\end{aligned}$$

for $\mu_0, \mu_1, \nu_0, \nu_1, r_0, r_1$ and κ arbitrary parameters. Then, it is easy to see that

$$W[x^n] = \sigma_n \, x^{n+1} + O(x^n),$$

for a certain σ_n depending on the parameters. We shall see next that this difference Heun operator coincides with the algebraic Heun operator associated to the Hahn algebra.

4.5.2. The algebraic Heun operator of the Hahn type

The Hahn polynomials P_n are orthogonal polynomials belonging to the Askey scheme. As such, an algebra encoding their properties is obtained as a specialization of the Racah algebra (4.7) by taking $a_2 \to 0$. One obtains the Hahn algebra, generated by $\{K_1, K_2, K_3\}$ with the following relations

$$\begin{aligned}
&[K_1, K_2] = K_3, \\
&[K_2, K_3] = a\{K_1, k_2\} + bK_2 + c_1 K_1 + d_1 I, \\
&[K_3, K_1] = aK_1^2 + bK_1 + c_2 K_2 + d_2 I.
\end{aligned} \tag{4.32}$$

A natural realization of the Hahn algebra is given in terms of the bispectral operators associated to the Hahn polynomials P_n, namely,

$$\begin{aligned}
&X = K_1 = x, \qquad (4.33)\\
&Y = K_2 = B(x)T^+ + D(x)T^- - (B(x) - D(x))I,
\end{aligned}$$

with

$$B(x) = (x-N)(x+\alpha+1), \qquad D(x) = x(x-\beta-N-1),$$

and where $T^{\pm}$ is as in (4.29). The action of Y is diagonal in the basis given by the Hahn polynomial P_n and is

$$YP_n(x) = \lambda P_n(x), \qquad \lambda_n = n(n+\alpha+\beta+1).$$

One checks that X and Y satisfy the Hahn algebra relations (4.32) with the structure constants expressed in terms of α, β and N.

Upon identifying the algebra associated to the Hahn polynomials, one can introduce the algebraic Heun operator W of the Hahn type [**22**] using the generic definition (4.28). In this realization, one finds that W can be written as

$$W = A_1(x)T^+ + A_2(x)T^- + A_0(x)I,$$

where

$$
\begin{aligned}
A_1(x) &= (x-N)(x+\alpha+1)((\tau_1+\tau_2)x+\tau_2+\tau_4),\\
A_2(x) &= x(x-\beta-N-1)((\tau_1+\tau_2)x+\tau_4-\tau_2),\\
A_0(x) &= -A_1(x)-A_2(x)+((\alpha+\beta+2)\tau_2+\tau_3)x+\tau_0-N(\alpha+1)\tau_2.
\end{aligned}
$$

As announced, the operator defined above coincides, upon identification of parameters, with the difference Heun operator W given in (4.30) and (4.31) and defined through its degree raising action on polynomials. That the difference Heun operator is tridiagonal on the Hahn polynomials then follows as a direct result. This parallels the construction in the Jacobi algebra that led to a simple proof of the standard Heun operator being tridiagonal on the Jacobi polynomials. Moreover, in the limit $N \to \infty$, the difference Heun operator W goes to the standard Heun operator, which further supports the appropriateness of the abstract definition (4.28) for the algebraic Heun operator.

To conclude this algebraic analysis, let us consider the algebra generated by Y and W in the context of the Hahn algebra. By introducing a third generator given by $[W,Y]$ and using the relation of the Hahn algebra in (4.32), one finds that the algebra thus generated closes as a cubic algebra with relations given by

$$
\begin{aligned}
[Y,[W,Y]] &= g_1Y^2+g_2\{Y,W\}+g_3Y+g_4W+g_5I,\\
[[W,Y],W] &= e_1Y^2+e_2Y^3+g_2W^2+g_1\{Y,W\}+g_3W+g_6Y+g_7I,
\end{aligned}
$$

where the structure constants depend on the parameters of the Hahn polynomials and the parameters of the tridiagonalization (4.28). One can recognize the above as a generalization of the Racah algebra (4.7) with the following two additional terms:

$$
e_1Y^2+e_2Y^3.
$$

The conditions for these terms to vanish are given by

$$
\tau_1+\tau_2=0, \qquad \tau_2\pm\tau_4=0.
$$

When these equalities are satisfied, the operator W simplifies to W_+ or W_- with

$$
W_\pm = \pm\frac{1}{2}[X,Y] \pm \gamma X - \frac{Y}{2} \pm \epsilon I.
$$

Moreover, any pair from the set $\{Y, W_+, W_-\}$ satisfies the Racah algebra relations given by (4.7). Thus, the choice of a pair of operators specifies an embedding of the Racah algebra in the Hahn algebra, which is analogous to the embedding given in (4.26). These embeddings encode abstractly the construction of the Racah polynomials starting from the Hahn polynomials and provide another example where higher polynomials are constructed from simpler ones.

4.6. Application to time and band limiting

We now return to the problem of time and band limiting. Consider a finite dimensional bispectral problem as the one associated to the Hahn polynomials. Denote by $\{e_n\}$ and $\{d_n\}$ for $n = 1, 2, \ldots, N$ the two eigenbases of this bispectral problem such that

$$X : \{e_n\} \to \{e_n\}, \qquad\qquad X e_n = \lambda_n e_n,$$
$$Y : \{d_n\} \to \{d_n\}, \qquad\qquad Y d_n = \mu_n d_n.$$

In this context, X can be thought of being associated to discrete time and Y to frequencies. Suppose now that the spectrum of both X and Y are restricted. These restrictions can be modelled as limiting operators in the form of two projections π_1 and π_2 given by

$$\pi_1 e_n = \begin{cases} e_n & \text{if } n \leq J_1, \\ 0 & \text{if } n > J_1, \end{cases} \qquad\qquad \pi_2 d_n = \begin{cases} d_n & \text{if } n \leq J_2, \\ 0 & \text{if } n > J_2, \end{cases} \tag{4.34}$$
$$\pi_1^2 = \pi_1, \qquad\qquad \pi_2^2 = \pi_2,$$

Simultaneous restrictions on the eigensubspaces of X and Y accessible to sampling lead to the two limiting operators

$$V_1 = \pi_1 \pi_2 \pi_1 = E_1 E_2, \quad V_2 = \pi_2 \pi_1 \pi_2 = E_2 E_1,$$

with

$$E_1 = \pi_1 \pi_2, \qquad E_2 = \pi_2 \pi_1.$$

Here, the limiting operator V_1 and V_2 are symmetric and are diagonalizable. A few limit cases are simple. When there are no restriction, $J_1 = J_2 = N$, in which case $V_1 = V_2 = I$. If the restriction is on only one of the spectra, for instance if $J_2 = N$, then $V_1 = V_2 = \pi_1$

having $J_1 + 1$ unit eigenvalues and the other $N - J_1$ equal to zero. However, the case where J_1 and J_2 are arbitrary is much more complicated.

In the generic case, the eigenbasis expansions (4.15) and (4.17) can be used to evaluate the action of π_2 on an eigenvector of X. One has,

$$\pi_2 e_n = \sum_{s=0}^{J_2} \sqrt{w_n}\phi_s(\lambda_n) d_s = \sum_{s=0}^{J_2}\sum_{t=0}^{N} \sqrt{w_n \tilde{w}_s}\phi_s(\lambda_n)\chi_t(\mu_s) e_t.$$

Similarly, one can evaluate the action of π_1 on eigenvectors of Y and obtain

$$V_1 e_n = \pi_1\pi_2\pi_1 e_n = \sum_{t=0}^{J_1}\sum_{s=0}^{J_2} \sqrt{w_n \tilde{w}_s}\phi_s(\lambda_n)\chi_t(\mu_s) e_t = \sum_{t=0}^{J_1} K_{t,n} e_t, \tag{4.35}$$

with

$$\begin{aligned} K_{t,n} &= \sum_{s=0}^{J_2} \sqrt{w_n \tilde{w}_s}\phi_s(\lambda_n)\chi_t(\mu_s) \\ &= \sum_{s=0}^{J_2} \sqrt{w_n w_t}\phi_s(\lambda_n)\phi_s(\lambda_t) \\ &= \sum_{s=0}^{J_2} \sqrt{\tilde{w}_s}\chi_n(\mu_s)\chi_t(\mu_s), \end{aligned} \tag{4.36}$$

where the Leonard duality relation (4.18) has been used to obtain the last two equalities. The operator V_1 in (4.35) is the discrete analog of the integral operator (4.1) that restricts both in time and frequency, with (4.36) being the discrete kernel. As in the initial continuous case, V_1 and V_2 are non-local operator and the problem of finding their eigenvectors is numerically difficult. However, if there exists a tridiagonal matrix M that commutes with both V_1 and V_2, then M would admit eigenvectors that are shared with V_1 and V_2. This renders the discrete time and band limiting problem well controlled. In this context, the tridiagonal matrix M is the discrete analog of a second order differential operator and plays the role of the differential operator found by Landau, Pollak and Slepian for the continuous time and band limiting problem.

Tridiagonal matrices that commute with the limiting operators π_1 and π_2 in (4.34) will also commute with V_1 and V_2. One then wants to find for an M such that

$$[M, \pi_1] = [M, \pi_2] = 0. \tag{4.37}$$

Taking M to be an algebraic Heun operator with

$$M = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_4 Y,$$

and using (4.37), one finds the following conditions

$$\tau_2 = \tau_1, \qquad \tau_1(\lambda_{J_1} + \lambda_{J_1+1}) + \tau_4 = 0, \qquad \tau_1(\mu_{J_2} + \mu_{J_2+1}) + \tau_3 = 0.$$

Except for the Bannai-Ito spectrum, it is always possible to find τ_3 and τ_4 satisfying the above [**8**], see also [**16**]. Hence, the algebraic Heun operator provides the commuting operator that enables efficient solutions to the time and band limiting.

Conclusion

This lecture has offered an introduction to the concept of algebraic Heun operators and its applications. This construct stems from the observation that the standard Heun operator can be obtained from the tridiagonalization of the hypergeometric operator.The key idea is to focus on operators that are bilinear in the generators of the quadratic algebras associated to orthogonal polynomials. The Heun type operators obtained in this algebraic fashion, coincide with those arising from the definition that has Heun operators raising by one the degree of arbitrary polynomials. This has been illustrated for the discrete Heun operator in its connection to the Hahn polynomials. This notion of algebraic Heun operator tied to bispectral problems has moreover been seen to shed light on the occurence of commuting operators in band and time limiting analyses. The exploration of these algebraic Heun operators and the associated algebras has just begun [**22, 2, 1**] but the results found so far let us believe that it could lead to significant new advances.

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Chapitre 5

The Heun-Racah and Heun-Bannai-Ito algebras

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Abstract

The Heun-Racah and Heun-Bannai-Ito algebra are introduced. Specializations of these algebras are seen to be realized by the operators obtained by applying the algebraic Heun construct to the bispectral operators of the Racah and Bannai-Ito polynomials. The study supplements the results on the Heun-Askey-Wilson algebra and completes the description of the Heun algebras associated to the polynomial families at the top of the Askey scheme, its q-analog and the Bannai-Ito one.

5.1. Introduction

Systematic generalizations of the standard Heun operator were recently introduced in [**18**]. In this approach, an algebraic Heun operator is associated to any bispectral pair [**16**] of operator. To do so, one considers the algebra generated by such a pair and defines the algebraic Heun operator as the generic bilinear combination of the operators in the bispectral pair. Such a construction has proved useful [**9**] in the theory of time and band limiting [**23, 21**] as well as in the study of entanglement in fermionic chains [**7, 8**] and the related algebraic structures are now being studied. Of particular interest are the algebraic Heun operators constructed from the bispectral problems arising in the theory of orthogonal polynomials.

This paper aims to introduce the Heun-Racah and the Heun-Bannai-Ito algebras and the associated algebraic Heun operators. Let us first review the recent results in this setting.

Orthogonal polynomials of the Askey scheme [**20**] are naturally associated to bispectral pairs. Indeed, all these polynomials are the eigenfunctions of both a three-term recurrence operator X, acting on the degree, and a differential or difference operator Y, acting on the variable. Thus, X and Y form a bispectral pair. One then considers the algebra generated by the pair X and Y in either the variable or the degree representation. The resulting structures are quadratic algebras canonically associated to each polynomials and after which they are named. These provide an algebraic approach to the theory of orthogonal polynomials as one recovers the properties of the corresponding polynomial through the representation theory of the associated algebra. Furthermore, the structure of the Askey scheme with other families of polynomials appearing as limits and specializations of higher polynomials is reflected algebraically through specializations and contractions. It is in this algebraic setting that one defines the associated algebraic Heun operator as

$$W = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_4 Y + \tau_0, \tag{5.1}$$

where $\tau_i \in \mathbb{C}$ for $i = 0, 1, 2, 3, 4$ are arbitrary parameters. Following the naming convention of the algebras associated to polynomials in the Askey scheme, operators constructed as in (5.1) will be referred to in terms of the name of the associated polynomials. For instance, if one considers the Racah polynomials and the associated Racah algebra, the algebraic Heun operator W in this setting is referred to as the Heun-Racah operator.

The algebraic Heun operator can be seen to generalize the Heun operator in many ways. It corresponds [**17**] to the standard Heun operator when constructed from the Jacobi algebra. In this case, Y is the hypergeometric operator while X is multiplication by the variable. Moreover, the standard Heun operator can be characterized as the most general second order differential operator that sends polynomials to polynomials one degree higher. The algebraic Heun operators constructed from the other entries in the Askey scheme are observed to satisfy analogs of this property. The Heun-Hahn operator introduced in [**27**] corresponds to the most general second-order difference operator on the uniform grid that sends polynomials on the grid to polynomials one degree higher. Likewise, the most general degree raising second order q-difference operator has been identified as an algebraic Heun operator in [**4**] and seen to correspond to a q-deformed Heun operator as considered in [**24**]. Finally, a q-Heun operator on the Askey grid was introduced in [**3**]. Such operators has been diagonalized recently in [**2**] by a generalization of the algebraic Bethe ansatz, called the modified algebraic Bethe ansatz, introduced to solve spin chains with generic boundaries in [**5, 6**].

A relevant property of the algebraic Heun operator W is characterized as follows. Taking the realization where X acts as the recurrence operator and Y, as multiplication by the eigenvalue, it is easily seen that W, as given by (5.1), will be tridiagonal on the corresponding orthogonal polynomials. Correspondingly, in the finite-dimensional setting where the bispectral pair is taken to be a Leonard pair, it was proven in [**22**] that all tridiagonal operators take the form of (5.1). There is a manifest relation between the construction (5.1) of W and the tridiagonalization approach [**19, 11**] to the study of orthogonal polynomials. From an algebra standpoint, tridiagonalization amounts to the construction of morphisms between the algebras associated to the polynomials of the Askey scheme. By considering the subalgebra generated by certain W and either X or Y, an embedding of the algebra of higher polynomials into the algebra of lower ones can be found. Naturally, this identification cannot be made when W is constructed from the algebra of a polynomial sitting at the top of the Askey scheme as the resulting structures lie beyond the algebraic framework of the Askey scheme. In general, one is lead to the study of algebras, referred to in terms of the underlying polynomials, for instance, if X and Y are the generators of the Racah algebra, the algebra generated by the pair X, W or Y, W shall be called the Heun-Racah algebra. Characterizations of these Heun algebras have been done in [**3**] for the Heun-Askey-Wilson algebra, in [**27**] for the Heun-Hahn algebra and in [**9**] for the Heun algebras of the Lie type which encompasses the cases of the Krawtchouk, Meixner, Meixner-Pollaczek, Laguerre and Charlier polynomials. In this paper, a similar characterization is made of the Heun-Racah and the Heun-Bannai-Ito algebras.

The presentation is as follows. In section 5.2, the Racah algebra is introduced and some key results are reviewed. The Heun-Racah operator is then constructed in section 5.3 as the most general second degree difference operator on the Racah grid that sends polynomials to polynomials one degree higher. As the algebraic Heun operator (5.1) of the Racah type is a bilinear combination of the Racah operator, it can be realized as a difference operator on the Racah grid using the canonical realization of the Racah algebra. This realization is shown to coincide with the Heun-Racah operator. Section 5.4 defines the Heun-Racah algebra abstractly and gives the conditions on the parameters that define a specialization that has an embedding in the Racah algebra. This embedding allows a realization of this specialized Heun-Racah algebra to be induced from the canonical realization of the Racah algebra. The realization thus obtained is seen to be in terms of the Heun-Racah operator. Similarly, after reviewing the Bannai-Ito algebra in section 5.5, the Heun-Bannai-Ito algebra, together with the Heun-Bannai-Ito operator, are introduced in section 5.6. It is shown that there is a specialization, obtained by imposing conditions on the parameters, that can be embedded in the Bannai-Ito algebra and realized in terms of the Heun-Bannai-Ito operator. A brief conclusion follows.

5.2. The Racah algebra

The Racah algebra $\mathcal{R}$ is the quadratic algebra defined as a unital associative algebra over $\mathbb{C}$ that is generated by $\{K_1, K_2, K_3\}$ with the following relations

$$\begin{aligned} &[K_1, K_2] = K_3, \\ &[K_2, K_3] = a_1\{K_1, K_2\} + a_2 K_2^2 + bK_2 + c_1 K_1 + d_1, \\ &[K_3, K_1] = a_1 K_1^2 + a_2\{K_1, K_2\} + bK_1 + c_2 K_2 + d_2, \end{aligned} \tag{5.2}$$

with a_i, c_i and for $i = 1, 2$ being arbitrary parameters in $\mathbb{R}$ and where b and d_i for $i = 1, 2$ are central elements. Throughout this paper, $[A, B] \equiv AB - BA$, $\{A, B\} \equiv AB + BA$ and I denote, respectively, the commutator, the anti-commutator and the identity element. To simplify notations and allow the construction of a Poincaré-Birkhoff-Witt type basis, a third generator K_3 is introduced in this presentation, although it is not algebraically independent from the others. A relevant observation is that the relations (5.2) are fixed [**14**] by considering the most general quadratic associative algebra generated by an independent pair of generators that admits ladder representations [**13**] and demanding compatibility with the following Jacobi identity

$$[K_1, [K_2, K_3]] + [K_3, [K_1, K_2]] + [K_2, [K_3, K_1]] = 0. \tag{5.3}$$

The Racah algebra is known to admit a cubic Casimir element C. In the above presentation, this central element is given by

$$
\begin{aligned}
C = a_1\{K_1^2, K_2\} + a_2\{K_1, K_2^2\} + (a_1a_2 + b)\{K_1, K_2\} \\
+ (a_1^2 + c_1)K_1^2 + (a_2^2 + c_2)K_2^2 + K_3^2 + (a_1b + 2d_1)K_1 + (a_2b + 2d_2)K_2. \quad (5.4)
\end{aligned}
$$

5.2.1. The equitable presentation of the Racah algebra

The Racah algebra is known to admit another presentation that displays explicitly the permutation symmetry of the generators. The relation between this second presentation and the one given in (5.2) is built [**13**] upon the reduced form of the Racah algebra that admits only three free parameters. The reduced Racah algebra $\tilde{\mathcal{R}}$ can be defined as the unital associative algebra generated by R_1, R_2 and R_3 with the following relations

$$
\begin{aligned}
[R_1, R_2] &= R_3, \\
[R_2, R_3] &= R_2^2 + \{R_1, R_2\} + dR_2 + e_1, \qquad (5.5) \\
[R_3, R_1] &= R_1^2 + \{R_1, R_2\} + dR_1 + e_2,
\end{aligned}
$$

where d and e_i for $i = 1, 2$ are central elements. The associated Casimir element (5.4) is given by

$$
C = \{R_1^2, R_2\} + \{R_1, R_2^2\} + R_1^2 + R_2^2 + R_3^2 + (d + 1)\{R_1, R_2\} + (2e_1 + d)R_1 + (2e_2 + d)R_2.
$$

Provided the parameters a_1 and a_2 in the Racah algebra (5.2) are non-vanishing, this reduced form is arrived at under the following affine transformation of the generators

$$
K_1 \mapsto a_2R_1 - \frac{c_2}{2a_2}I, \quad K_2 \mapsto a_1R_2 - \frac{c_1}{2a_1}I, \quad K_3 \mapsto a_1a_2R_3, \qquad (5.6)
$$

where

$$
d = \frac{a_2a_1b - a_1^2c_2 - a_2^2c_1}{a_1^2a_2^2}, \qquad e_1 = \frac{-2a_1c_1b + a_2c_1^2 + 4a_1^2d_1}{4a_1^4a_2}, \qquad e_2 = \frac{-2a_2bc_2 + a_1c_2^2 + 4a_2^2d_2}{4a_1a_2^4}.
$$

From the presentation (5.5) of the (reduced) Racah algebra, one obtains the equitable presentation by introducing four generators as follows

$$
V_1 = -2R_1, \qquad V_2 = -2R_2, \qquad V_3 = 2(R_1 + R_2 + d), \qquad P = 2R_3,
$$

such that one has that

$$V_1 + V_2 + V_3 = 2d, \quad \text{and} \quad [V_1, V_2] = [V_2, V_3] = [V_3, V_1] = 2P. \tag{5.7}$$

The relations (5.5) can be written in terms of the new generators as

$$\begin{aligned} &[V_1, P] = V_2V_1 - V_1V_3 + 4e_2, \qquad [V_2, P] = V_3V_2 - V_2V_1 - 4e_1, \\ &[V_3, P] = V_1V_3 - V_3V_2 + 4(e_1 - e_2). \end{aligned} \tag{5.8}$$

The presentation of the (reduced) Racah algebra given by (5.7) and (5.8) is referred to [**15**] as the equitable presentation, as it makes manifest the $\mathbb{Z}_3$ symmetry of the Racah algebra given by the cyclic permutations of the generators. One concludes that for non-vanishing a_1 and a_2 there is an isomorphism $\chi : \mathcal{R} \longrightarrow \tilde{\mathcal{R}}$ that identifies the equitable presentation given by (5.7) and (5.8) with the Racah algebra as presented in (5.2). Explicitly, this map is

$$\chi : \quad K_1 \mapsto -\frac{a_2}{2}V_1 - \frac{c_2}{2a_2}I, \quad K_2 \mapsto -\frac{a_1}{2}V_2 - \frac{c_1}{2a_1}I, \quad K_3 \mapsto \frac{a_1a_2}{2}P. \tag{5.9}$$

5.2.2. Difference operator realization

The Racah grid λ is a two-parameter quadratic grid indexed by x given by

$$\lambda(x) = x(x + \gamma + \delta + 1), \qquad x = 0, 1, \ldots, N, \qquad N \in \mathbb{N} \tag{5.10}$$

where γ and δ are real parameters. Defining the shift operators acting on functions of x as

$$T^+ f(x) \mapsto f(x + 1), \quad T^- f(x) \mapsto f(x - 1), \tag{5.11}$$

one introduces the forward and backward difference operators as

$$\Delta = T^+ - I, \quad \nabla = I - T^-. \tag{5.12}$$

With these notations, the Racah operator Y takes the following form

$$Y = B(x)\Delta - D(x)\nabla, \tag{5.13}$$

where

$$B(x) = \frac{(x + \alpha + 1)(x + \beta + \delta + 1)(x + \gamma + 1)(x + \gamma + \delta + 1)}{(2x + \gamma + \delta + 1)(2x + \gamma + \delta + 2)}, \tag{5.14}$$

$$D(x) = \frac{x(x - \alpha + \gamma + \delta)(x - \beta + \gamma)(x + \delta)}{(2x + \gamma + \delta)(2x + \gamma + \delta + 1)}. \tag{5.15}$$

This operator is diagonalized [**20**] by the four-parameters Racah polynomials $R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ defined for $n = 0, 1, 2, \ldots, N$ with $N \in \mathbb{N}$ and where either

$$\alpha + 1 = -N, \qquad \beta + \delta + 1 = -N \quad \text{or} \quad \gamma + 1 = -N.$$

The eigenvalue equation is then

$$(B(x)\Delta - D(x)\nabla)\ R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = n(n+\alpha+\beta+1)R_n(\lambda(x);\alpha,\beta,\gamma,\delta).$$

These Racah polynomials being orthogonal polynomials in the Askey scheme, they also satisfy [**20**] a three-term recurrence relation of the following form

$$\lambda(x)R_n(\lambda(x)) = A_nR_{n+1}(\lambda(x)) - (A_n + C_n)R_n(\lambda(x)) + C_nR_{n-1}(\lambda(x)),$$

where the coefficients A_n and C_n only depend [**20**] on the parameters $\alpha,\beta,\gamma,\delta$ and the degree n. The left-hand side of the above can be understood as an operator that acts by multiplication on functions on λ. Denoting this recurrence operator as X

$$X = \lambda(x),$$

a realization of the Racah algebra (5.2) in terms of second-order difference operators is given by

$$K_1 \longmapsto Y, \qquad K_2 \longmapsto X. \tag{5.16}$$

In this realization, the central elements in the relations (5.2) are proportional to the identity element. The scaling of these central elements and the parameters are determined by the parameters of the associated Racah operator as follows

$$\begin{aligned}
a_1 &= -2, \qquad c_1 = -(\gamma+\delta)(\gamma+\delta+2), \qquad d_1 = -(\alpha+1)(\gamma+1)(\beta+\delta+1)(\gamma+\delta)I,\\
a_2 &= -2, \qquad c_2 = -(\alpha+\beta)(\alpha+\beta+2), \qquad d_2 = -(\alpha+1)(\gamma+1)(\alpha+\beta)(\beta+\delta+1)I,
\end{aligned}$$

$$\begin{aligned}
b &= 2\left[\beta(\delta-\alpha) - (\alpha+\beta)(\gamma+\delta+2) - 2(\gamma+1)(\delta+1)\right]I,\\
C &= (\alpha+1)(\gamma+1)(\beta+\delta+1)\Big[2\beta\delta - 2\alpha + \beta(\alpha+1)(\gamma-1) + (\alpha-1)(\gamma+1)(\delta+1)\Big]I.
\end{aligned}$$

With these observations, the bispectral problem associated to the Racah algebra is completely specified. Moreover, the Racah polynomials are seen to span finite-dimensional representations of the Racah algebra under the realization (5.16).

5.3. The Heun-Racah operator

This section is concerned with the construction of a generalization of the Heun differential operator on the Racah grid (5.10). The key property of the standard differential Heun operator is that it is the most general second-order differential operator that sends polynomials of degree n to polynomials of degree $n+1$. By requiring an equivalent property for operators on the Racah grid, one obtains the desired generalization.

Consider the vector space $\mathcal{P}$ of polynomials on the Racah grid $\lambda(x)$ as given by (5.10). One then considers difference operators expressed in terms of the shift operators (5.11). In this setting, a generic second-order difference operator on $\mathcal{P}$ can be written as

$$W = A_1(x)T^{+} + A_2(x)T^{-} + A_0(x)I. \tag{5.17}$$

With the forward Δ and backward ∇ difference operators defined in (5.12), one can write (5.17) as the second-order difference operator

$$W = A_1(x)\Delta - A_2(x)\nabla + [A_1(x) + A_2(x) + A_0(x)]\, I. \tag{5.18}$$

The Heun-Racah operator is now defined as the most general degree increasing second-order difference operator W such that

$$W : \mathcal{P} \to \mathcal{P} : p_n(\lambda) \mapsto q_{n+1}(\lambda), \tag{5.19}$$

for p_n and q_{n+1} arbitrary polynomials in $\mathcal{P}$ of degree n and $n+1$, respectively.

5.3.1. Parametrization of the Heun-Racah operator

The condition (5.19) determines the form of the functions $A_i(x)$ for $i = 0, 1, 2$ in (5.18). This can be seen by acting with W on monomials in $\mathcal{P}$ and demanding that (5.19) holds. Observing that

$$\Delta \cdot \lambda(x) = 2x + \gamma + \delta + 2, \qquad \nabla \cdot \lambda(x) = 2x + \gamma + \delta,$$

one has

$$\begin{cases} W\cdot 1 = A_0(x)+A_1(x)+A_2(x) = p_1(\lambda(x)), \\ W\cdot\lambda(x) = (\lambda(x)+2x+\gamma+\delta+2)A_1(x)+(\lambda(x)-(2x+\gamma+\delta))A_2(x)+\lambda(x)A_0(x), \\ W\cdot\lambda(x)^2 = (\lambda(x)+2x+\gamma+\delta+2)^2A_1(x)+(\lambda(x)-(2x+\gamma+\delta))^2A_2(x)+\lambda(x)^2A_0(x), \end{cases} \tag{5.20}$$

with $W\cdot\lambda(x) = p_2(\lambda(x))$ and $W\cdot\lambda(x)^2 = p_3(\lambda(x))$ where p_1, p_2 and p_3 being arbitrary polynomials of first, second and third degree, respectively. From (5.20), one finds that

$$\begin{aligned} A_0(x) &= p_1(\lambda(x)) - A_2(x) - A_1(x), \\ \theta(x)(A_1(x)-A_2(x)) &= p_2(\lambda(x)) - \lambda(x)p_1(\lambda(x)) - 2A_1(x), \\ \theta^2(x)(A_1(x)+A_2(x)) &= p_3(\lambda(x)) - 2\lambda(x)p_2(\lambda(x)) + \lambda^2(x)p_1(\lambda(x)) - 4(\theta(x)+1)A_1(x), \end{aligned}$$

where $\theta(x) = \nabla\lambda(x) = 2x+\gamma+\delta$. Introducing the polynomials π_1, π_2 and π_3

$$\begin{aligned} &\pi_1(\lambda(x)) = p_1(\lambda(x)), \qquad \pi_2(\lambda(x)) = \frac{p_2(\lambda(x)) - \lambda(x)p_1(\lambda(x))}{2}, \\ &\pi_3(\lambda(x)) = \frac{p_3(\lambda(x)) - 2\lambda(x)p_2(\lambda(x)) + \lambda(x)^2p_1(\lambda(x))}{2}, \end{aligned} \tag{5.21}$$

parametrized as follows

$$\pi_1(z) = \sum_{i=0}^{1} t_i z^i, \qquad \pi_2(z) = \sum_{i=0}^{2} u_i z^i, \qquad \pi_3(z) = \sum_{i=0}^{3} v_i z^i, \tag{5.22}$$

one solves for the $A_i(x)$ to get

$$\begin{aligned} &A_1(x) = \frac{\pi_3(\lambda(x)) + \theta(x)\pi_2(\lambda(x))}{(\theta(x)+1)(\theta(x)+2)}, \qquad A_2(x) = \frac{\pi_3(\lambda(x)) - (\theta(x)+2)\pi_2(\lambda(x))}{\theta(x)(\theta(x)+1)}, \\ &A_0(x) = \pi_1(\lambda(x)) - A_1(x) - A_2(x). \end{aligned} \tag{5.23}$$

It follows from (5.23) that specifying the coefficients of the polynomials π_1, π_2 and π_3 fully determines the functions $A_i(x)$, $i = 0, 1, 2$ in (5.18) upon fixing the the grid $\lambda(x)$. Together, these polynomials admit nine free parameters. However, in order for the Heun-Racah operator to act on the finite grid $\lambda(x)$, one must have that $(x - N)$ is a factor of $A_1(x)$ and x is a factor of $A_2(x)$. The second condition is uniquely satisfied by demanding that

$$v_0 = u_0(\gamma+\delta+2).$$

To satisfy the first one, one finds that the following must hold:

$$v_1 = -\frac{2u_0}{N} - (N+\gamma+\delta+1)^2 N^2 v_3 - (N+\gamma+\delta+1)Nv_2 \\ - \Big[2N(N+1) + (3N+\gamma+\delta+1)(\gamma+\delta)\Big]Nu_2 - (2N+\gamma+\delta)u_1.$$

Thus, with these constraints and a fixed grid $\lambda(x)$, the Heun-Racah operator admits seven free parameters. In the parametrization (5.22) of (5.23), the remaining parameters are $t_0, t_1, u_0, u_1, u_2, v_2$ and v_3.

5.3.2. Sufficiency of the construction

It remains to show that the operator W specified by (5.18) and (5.23) satisfies the property (5.19) in general. To do so, one first computes the action of W on a generic monomial $\lambda(x)^n \in \mathcal{P}$ to obtain

$$W \cdot \lambda(x)^n = (\lambda(x)+\theta(x)+2)^n A_1(x) + (\lambda(x)-\theta(x))^n A_2(x) + \lambda(x)^n A_0(x). \tag{5.24}$$

Expanding the binomials, the above can be written as

$$W \cdot \lambda(x)^n = \sum_{k=1}^{n} \binom{n}{k} \lambda(x)^{n-k}\chi_k + \lambda(x)^n\, \pi_1(\lambda(x)), \tag{5.25}$$

$$\chi_k = \Big[(\theta(x)+2)^k A_1(x) + (-\theta(x))^k A_2(x)\Big], \tag{5.26}$$

where the last term in (5.25) is manifestly a polynomial in $\lambda(x)$ of degree $n+1$ as π_1 is a linear function by construction. Using binomial expansions, one has that

$$(-\theta(x))^{k-1} = \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j(\theta(x)+1)^j, \qquad (\theta(x)+2)^{k-1} = \sum_{j=0}^{k-1} \binom{k-1}{j} (\theta(x)+1)^j.$$

The above identities with (5.23) in (5.26), leads to

$$\chi_k = \sum_{j=0}^{k-1} \binom{k-1}{j} (\theta(x)+1)^{j-1}\Bigg[\Big[\pi_3(\lambda(x)) + \theta(x)\pi_2(\lambda(x))\Big] \\ + (-1)^{j-1}\Big[\pi_3(\lambda(x)) - (\theta(x)+2)\pi_2(\lambda(x))\Big]\Bigg],$$

$$= \sum_{\substack{j \text{ even} \\ 0\leq j\leq k-1}} \binom{k-1}{j} 2\pi_2(\lambda(x))(\theta(x)+1)^j$$

$$+ \sum_{\substack{j \text{ odd} \\ 0\leq j\leq k-1}} \binom{k-1}{j} 2\Big[\pi_3(\lambda(x)) - \pi_2(\lambda(x))\Big](\theta(x)+1)^{j-1}.$$

It is readily verified that $(\theta(x)+1)^2 = 4\lambda(x) + (c+d+1)^2$ is a linear function of $\lambda(x)$ such that even powers of $(\theta(x)+1)$ can be identified as polynomials in $\lambda(x)$. Thus, it can be seen that χ_k is a polynomial in $\lambda(x)$ of degree

$$\deg(\chi_k) = \begin{cases} \frac{k}{2}+2 & \text{for } k \text{ even,} \\ \frac{k+3}{2} & \text{for } k \text{ odd.} \end{cases}$$

In particular, the degree of χ_k is less than $k+1$ for $k>2$. Thus, one can conclude from the above and (5.25) that the operator W acts on monomials as follows

$$W \cdot \lambda(x)^n = (t_1 + 2n\, u_2 + n(n-1)\, v_3)\lambda(x)^{n+1} + O(\lambda(x)^n), \tag{5.27}$$

where t_1, u_2 and v_3 are parameters of the Heun-Racah operator as labelled in (5.22). This result implies that W as defined by (5.18) and (5.23) is the most generic second order difference operator on the grid $\lambda(x)$ that satisfies property (5.19) provided that

$$t_1 \neq 0, \qquad u_2 \neq 0, \quad \text{or} \quad v_3 \neq 0.$$

5.3.3. Specialization as the Racah operator

The Heun-Racah operator was constructed as the most general second-order operator on the Racah grid that satisfies the degree raising property (5.19). We now consider specializations that preserve the space of polynomials of degree n in $\mathcal{P}$. From (5.27), one easily identifies the necessary constraints to be

$$t_1 = u_2 = v_3 = 0. \tag{5.28}$$

Furthermore, if the above constraints are satisfied, normalizing the Heun-Racah operator so that the numerators of the functions $A_1(x)$ and $A_2(x)$ are monic polynomials corresponds to setting

$$v_2 = 1. \tag{5.29}$$

Finally, from (5.25), one can identify the term of W proportional to the identity as $t_0 I$. This term vanishes if

$$t_0 = 0. \tag{5.30}$$

Upon demanding that (5.28), (5.29) and (5.30) are satisfied, the Heun-Racah operator (5.18) takes the form of the Racah operator. The remaining free parameters in (5.22) can be expressed in terms of those of the Racah operator on the same grid $\lambda(x)$ of size N as follows

$$u_0 = \frac{(\alpha+1)(\gamma+1)(\beta+\delta+1)}{2}, \quad u_1 = \frac{\alpha+\beta+2}{2}.$$

5.3.4. Relation with the algebraic Heun operator

The Heun-Racah operator W given by (5.18) together with (5.23) can be written as a bilinear combination of the Racah algebra generators in the canonical realization (5.16) in terms of difference operators. This bilinear expression is shown to coincides with the canonical construction of the algebraic Heun operator as follows. Recall, that the Racah generators in this realization are given by

$$K_2 = X = \lambda(x), \qquad K_1 = Y = B(x)\Delta - D(x)\nabla,$$

where $\lambda(x)$ is the Racah grid (5.10) of size N and the coefficients $B(x)$ and $D(x)$ are given by (5.14). Consider the operator

$$W = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_4 Y + \tau_0 I, \tag{5.31}$$

as in the introduction. A direct computation leads to

$$\begin{aligned} W = \Big[\lambda(x)(\tau_1+\tau_2) + \tau_4 + \tau_2(\theta(x)+2)\Big] B(x)\Delta \\ - \Big[\lambda(x)(\tau_1+\tau_2) + \tau_4 - \tau_2\theta(x)\Big] D(x)\nabla \\ + \tau_2\Big[(\theta(x)+2)B(x) - \theta(x)D(x)\Big] + \tau_3\lambda(x) + \tau_0, \end{aligned} \tag{5.32}$$

where, again, $\theta(x) = 2x + \gamma + \delta$. Equation (5.32) defines an operator of the form (5.18) with

$$\begin{aligned}
A_1(x) &= \Big[\lambda(x)(\tau_1+\tau_2) + \tau_4 + \tau_2(\theta(x)+2)\Big] B(x),\\
A_2(x) &= \Big[\lambda(x)(\tau_1+\tau_2) + \tau_4 - \tau_2\theta(x)\Big] D(x),\\
A_0(x) &= \tau_3\lambda(x) + \tau_0 - \Big[\lambda(x)(\tau_1+\tau_2)+\tau_4\Big]\Big(B(x)+D(x)\Big).
\end{aligned} \tag{5.33}$$

Using (5.23), one can express the polynomials π_1, π_2 and π_3 in terms of $A_i(x)$, $i = 0, 1, 2$ and $\theta(x)$ as follows

$$\begin{aligned}
&\pi_1(\lambda(x)) = A_0(x) + A_1(x) + A_2(x), \qquad \pi_3(\lambda(x)) = \frac{(\theta(x)+2)^2 A_1(x) + \theta^2(x) A_2(x)}{2},\\
&\pi_2(\lambda(x)) = \frac{(\theta(x)+2)A_1(x) - \theta(x)A_2(x)}{2}.
\end{aligned} \tag{5.34}$$

The above allows one to relate the operator (5.32) with the Heun-Racah operator. Indeed, comparing the terms, one finds that (5.32) can be identified with the Heun-Racah operator defined by (5.18) and (5.23) provided the parameters (5.22) are given by

$$\begin{aligned}
&u_0 = (\tau_2(\gamma+\delta+2)+\tau_4)\,\phi_{\alpha,\beta,\gamma,\delta} && t_0 = 2\tau_2\phi_{\alpha,\beta,\gamma,\delta} + \tau_0,\\
&u_1 = (\tau_1+\tau_2)\phi_{\alpha,\beta,\gamma,\delta} + \tau_2\psi_{\alpha,\beta,\gamma,\delta} + \tau_4(\alpha+\beta+2)/2 && t_1 = \tau_2(2+\alpha+\beta)+\tau_3,\\
&u_2 = (\tau_1+\tau_2)(\alpha+\beta+2)/2 + \tau_2, && v_3 = \tau_1+\tau_2,\\
&v_2 = (\tau_1+\tau_2)\psi_{\alpha,\beta,\gamma,\delta} + 2\tau_2(\alpha+\beta+3)+\tau_4,
\end{aligned} \tag{5.35}$$

where

$$\begin{aligned}
\phi_{\alpha,\beta,\gamma,\delta} &= (\alpha+1)(\gamma+1)(\beta+\delta+1)/2,\\
\psi_{\alpha,\beta,\gamma,\delta} &= \alpha\left(\beta + \frac{\gamma+\delta}{2} + 2\right) + \beta\left(\frac{\gamma-\delta}{2}+2\right) + (\gamma\delta+\gamma+\delta+3).
\end{aligned}$$

We remind the reader that the realization (5.16) of the Racah algebra admits two free parameters, once the grid $\lambda(x)$ of size N is specified. Moreover, the construction (5.31) for the algebraic Heun operator introduces five additional parameters. Thus, the seven free parameters in (5.22) of the Heun-Racah operator (5.18) are in correspondence with those of the algebraic Heun operator of the Racah type in the canonical realization (5.16)

5.4. The Heun-Racah algebra

The Heun-Racah algebra $\mathcal{HR}$ is introduced as the unital associative algebra over $\mathbb{C}$ generated by X, W, Z with the following relations

$$\begin{aligned}
&[W,X] = Z,\\
&[X,Z] = x_0 + x_1 X + x_2 X^2 + x_3 X^3 + x_4 W + x_5\{X,W\},\\
&[Z,W] = y_0 + y_1 X + y_2 X^2 + y_3 X^3 + (x_1 - x_3 x_4)W + x_5 W^2 + (x_2 - x_3 x_5)\{X,W\}\\
&\qquad\qquad + 3x_3 XWX,
\end{aligned} \tag{5.36}$$

where $x_i \in \mathbb{R}$ for $i = 3,4,5$ and $y_3 \in \mathbb{R}$ are free parameters and where x_i, y_i for $i = 0,1,2$ are central elements. The constraints on the last three coefficients in (5.36) ensure compatibility with the following Jacobi identity

$$[[X,Z],W] + [[Z,W],X] + [[W,X],Z] = 0.$$

One readily notices that the relations (5.36) reduce to the relations (5.2) of the Racah algebra if

$$x_3 = y_2 = y_3 = 0, \qquad x_2 \propto I \qquad \text{and} \qquad y_1 \propto I. \tag{5.37}$$

It is verified that the element $\Omega \in \mathcal{HR}$ given by

$$\begin{aligned}
\Omega = e_1 X + e_2 W + e_3\{X,W\} + e_4 XWX + e_5 WXW + e_6 X^2 + e_7 W^2 - Z^2\\
+ [XW,WX] + e_8 X^3 + e_9 X^4,
\end{aligned} \tag{5.38}$$

is central when the coefficients are as follows

$$\begin{aligned}
&e_1 = x_5 y_1 + x_4 y_2/3 + x_4 x_5 y_3/6 - y_0, && e_4 = 4x_3 x_5 - x_2, && e_7 = -2x_4,\\
&e_2 = x_2 x_4 - 3x_0 - x_3 x_4 x_5, && e_5 = -3x_5, && e_8 = (5x_5 y_3 + y_2)/3,\\
&e_3 = x_3 x_4 + x_2 x_5 - x_3 x_5^2 - x_1, && e_9 = y_3/2 && ,\\
&e_6 = x_5^2 y_3/6 + 4x_5 y_2/3 + x_4 y_3/.
\end{aligned}$$

5.4.1. Embedding in the Racah algebra

An embedding of a specialization of the Heun-Racah algebra $\mathcal{HR}$ in the Racah algebra $\mathcal{R}$ is possible. This specialization is obtained by imposing conditions on the parameters of the Heun-Racah relations (5.36). Consider the mapping defined by

$$\begin{aligned}\Phi : \mathcal{HR} &\longrightarrow \mathcal{R},\\ X &\longmapsto K_2,\\ W &\longmapsto \tau_1 K_2 K_1 + \tau_2 K_1 K_2 + \tau_3 K_2 + \tau_4 K_1 + \tau_0 I.\end{aligned} \tag{5.39}$$

The mapping $\Phi : \mathcal{HR} \to \mathcal{R}$ is an algebra homomorphism provided, first, that the parameters of the Heun-Racah algebra be the following functions of the parameters of the Racah

$$\begin{aligned}&x_3 = a_2(\tau_1+\tau_2), \qquad x_4 = c_1, \qquad x_5 = a_1,\\ &y_3 = 2a_2^2\,\tau_1\tau_2 - 4a_2\tau_3\,(\tau_1+\tau_2) + 2c_2\,(\tau_1+\tau_2)^2\,,\end{aligned} \tag{5.40}$$

and, second, that the central elements be mapped to those of the Racah algebra as follows

$$x_0 \mapsto \tau_4 d_1 - c_1\tau_0, \qquad x_1 \mapsto (\tau_1+\tau_2)d_1 + \tau_4 b - 2a_1\tau_0 - c_1\tau_3, \qquad x_2 \mapsto b(\tau_1+\tau_2) + \tau_4 a_2 - 2a_1\tau_3,$$

$$\begin{aligned}y_0 \mapsto \Big[a_1 C + b d_1 + (a_1^2 - c_1)d_2\Big]\,\tau_1\tau_2 + \Big[\,(a_2c_1 - d_1)\,\tau_0 - (C + a_2d_1 + a_1d_2)\,\tau_4\Big](\tau_1+\tau_2)&\\ + a_1\tau_0^2 + (d_2\tau_4 - b\tau_0)\tau_4 + (c_1\tau_0 - d_1\tau_4)\tau_3,&\end{aligned}$$

$$\begin{aligned}y_1 \mapsto \Big[b^2 + a_1^2c_2 + 2a_2d_1 - c_1c_2 - a_1(a_2b + 4d_2)\Big]\tau_1\tau_2 - \Big[C + a_2d_1 + a_1d_2\Big](\tau_1+\tau_2)^2&\\ + (c_2\tau_4 - 2a_2\tau_0)\tau_4 + \Big[\,(2a_1a_2 - 2b)\,\tau_0 + (4d_2 - a_1c_2)\,\tau_4 + (a_2c_1 - 2d_1)\tau_3\Big](\tau_1+\tau_2)&\\ + (4a_1\tau_0 - 2b\tau_4 + c_1\tau_3)\,\tau_3,&\end{aligned}$$

$$\begin{aligned}y_2 \mapsto \Big[3d_2 - a_1c_2\Big](\tau_1+\tau_2)^2 + \Big[3a_2b - 3a_1c_2 - a_1a_2^2\Big]\tau_1\tau_2 + \Big[\,(2a_1a_2 - 3b)\,\tau_3 - 3a_2\tau_0 + 3c_2\tau_4\Big](\tau_1+\tau_2)&\\ + 3(a_1\tau_3 - a_2\tau_4)\tau_3,&\end{aligned} \tag{5.41}$$

The above specialization of the Heun-Racah algebra admits a realization in terms of the Heun-Racah operator (5.18). Indeed, comparing (5.39) with (5.31), one can see that the generator W of this specialized Heun-Racah algebra is embedded in the Racah algebra as the algebraic Heun operator of the Racah type. A natural realization of the specialization

(5.40) and (5.41) of the Heun-Racah algebra is obtained from the concatenation of the embedding (5.39) and the canonical realization (5.16) of the Racah algebra. In this realization, one finds that W takes the form of (5.32), which was identified as the Heun-Racah operator with parameters given by (5.35). Moreover, it follows from the above that in the specialization (5.37) of the Heun-Racah algebra to the Racah algebra, the map (5.39) is an affine transformation of the Racah algebra parametrized by τ_0 and τ_4.

With the parameters of the Heun-Racah algebra as in (5.40), one finds that the image of the central element Ω given in (5.38) under the mapping Φ given by (5.39) and (5.41) is the Casimir element (5.4) of the Racah algebra, up to a central element and scaling. Explicitly, one has

$$\Phi : \Omega \longmapsto u\, C + v,$$

where the coefficients u and v are given by

$$u = \left[(c_1 - a_1^2)\tau_1\tau_2 + a_1\,(\tau_1 + \tau_2)\,\tau_4 - \tau_4^2\right]^{-1},$$

and

$$v = u\Big[\tau_1\tau_2(a_1bd_1 - a_1c_1d_2 - a_2c_1d_1 + a_2a_1^2d_1 - d_1^2) + \Big[(a_1a_2c_1 - bc_1)\tau_0 + (c_1d_2 - 2a_1a_2d_1)\tau_4\Big](\tau_1 + \tau_2) \\ + (2a_1c_1\tau_0 - 2a_1d_1\tau_4)\tau_3 + \Big(2a_2d_1\tau_4 + (2d_1 - a_2c_1)\tau_0\Big)\tau_4 - c_1\tau_0^2\Big] + a_2d_1 - a_1d_2.$$

5.5. The Bannai-Ito algebra

The Bannai-Ito algebra $\mathcal{B}$ is defined [**10**] as the unital associative algebra over $\mathbb{C}$ generated by B_1, B_2 and B_3 with the following relations

$$\{B_1, B_2\} = B_3 + \omega_1, \quad \{B_2, B_3\} = B_1 + \omega_2, \quad \{B_1, B_3\} = B_2 + \omega_3, \tag{5.42}$$

where ω_i for $i = 1, 2, 3$ are central elements. A natural $\mathbb{Z}_2$ grading is given by taking B_1, B_2 to be odd, which implies that B_3 is even. It is observed that this algebra satisfies the following graded Jacobi identity

$$[B_1, \{B_2, B_3\}] + [B_2, \{B_1, B_3\}] + [B_3, \{B_1, B_2\}] = 0.$$

In this presentation, the central Casimir operator is given by

$$Q = B_1^2 + B_2^2 + B_3^2. \tag{5.43}$$

5.5.1. Canonical realization

A realization of the Bannai-Ito algebra in terms of reflection operators acting on univariate polynomials can be constructed [**10**] as follows. One first defines two reflection operators R_1 and R_2 acting on univariate functions as

$$R_1\, f(x) = f(-x), \qquad R_2\, f(x) = f(-x-1), \qquad \Longrightarrow \qquad R_1^2 = R_2^2 = I.$$

The most general symmetrizable first order shift operator that contains reflections and preserves the space of polynomials of a given degree is the Bannai-Ito operator [**25**] given by

$$\tilde{B}_2 = \frac{(x-\rho_1)\,(x-\rho_2)}{2x}(1-R_1) + \frac{\left(-r_1+x+\frac{1}{2}\right)\left(-r_2+x+\frac{1}{2}\right)}{2x+1}(R_2-1). \tag{5.44}$$

This operator is diagonalized by the four parameters Bannai-Ito polynomials [**1, 26**], denoted $B_n(x|\rho_1,\rho_2,r_1,r_2)$, with

$$\tilde{B}_2\, B_n(x|\rho_1,\rho_2,r_1,r_2) = (-1)^n(n+\rho_1+\rho_2-r_1-r_2+1/2)B_n(x|\rho_1,\rho_2,r_1,r_2). \tag{5.45}$$

These polynomials are orthogonal on the finite Bannai-Ito grid x_s defined [**12**], depending on the truncation conditions used, as

$$x_s = \begin{cases} (-1)^s\left(\frac{s}{2}+\rho_j+\frac{1}{4}\right)-\frac{1}{4}, & \text{for } N \text{ even, } 2(r_i+\rho_j)=N+1,\ i,j=1,2, \\ (-1)^s\left(\frac{s}{2}+\rho_2+\frac{1}{4}\right)-\frac{1}{4}, & \text{for } N \text{ odd, } 2(\rho_1+\rho_2)=-N-1, \\ (-1)^s\left(r_1-\frac{s}{2}-\frac{1}{4}\right)-\frac{1}{4}, & \text{for } N \text{ odd, } 2(r_1+r_2)=N+1, \end{cases} \tag{5.46}$$

with $s = 0,1,\ldots,N \in \mathbb{N}$. The Bannai-ito polynomials also satisfy a three-terms recurrence relation of the following form

$$xB_n(x) = B_{n+1}(x) + A_nB_n(x) + C_nB_{n-1}(x), \tag{5.47}$$

where the coefficients depend only [**25**] on n and the parameters ρ_1,ρ_2,r_1,r_2 of the polynomial. The left-hand side of this recurrence relation can be understood as an operator $\tilde{B}_1$ that acts on functions by multiplication as follows

$$\tilde{B}_1 f(x) = xf(x). \tag{5.48}$$

As defined in (5.44) and (5.48), the pair of operators $\tilde{B}_1$, $\tilde{B}_2$ acts on univariate functions of x. Another realization can be given where both operators act on the degrees n. In this case, the action of $\tilde{B}_2$ is given in (5.45) while the action of $\tilde{B}_1$ is defined through the right-hand side of (5.47). Thus, the operators $\tilde{B}_1$, $\tilde{B}_2$ form a bispectral pair.

Introducing the structure operator as $\tilde{B}_3 \equiv \{\tilde{B}_1, \tilde{B}_2\}$, it can be seen that the algebra generated by $\tilde{B}_1, \tilde{B}_2$ and $\tilde{B}_3$ is the algebra (5.42), up to an affine transformation. Explicitly, the following map

$$\begin{aligned} &B_1 \longmapsto 2\tilde{B}_1 + 1/2, \qquad B_2 \longmapsto 2\tilde{B}_2 + (\rho_1 + \rho_2 - r_1 - r_2 + 1/2), \\ &B_3 \longmapsto 4\{\tilde{B}_1, \tilde{B}_2\} + 2\tilde{B}_2 + 4\left(\rho_1 + \rho_2 - r_1 - r_2 + 1/2\right)\tilde{B}_1 \\ &\qquad\qquad\qquad\qquad + \left(\rho_1 + \rho_2 - 4\rho_1\rho_2 - r_1 - r_2 + 4r_1 r_2 + 1/2\right), \end{aligned} \tag{5.49}$$

is an homomorphism. In this realization, the Casimir operator (5.43), together with the central elements in (5.42), are proportional to the identity element. One has

$$Q \longmapsto 2\left(\rho_1^2 + \rho_2^2 + r_1^2 + r_2^2 - 1/8\right) I,$$

$$\omega_1 \mapsto 4\left(\rho_1\rho_2 - r_1 r_2\right) I, \qquad \omega_2 \mapsto 2\left(\rho_1^2 + \rho_2^2 - r_1^2 - r_2^2\right) I, \qquad \omega_3 \mapsto 4\left(\rho_1\rho_2 + r_1 r_2\right) I.$$

5.5.2. Embedding of the Racah algebra in the Bannai-Ito algebra

An embedding of the (reduced) Racah algebra (5.5) into the Bannai-Ito algebra has been shown to exists [**15**]. This embedding is constructed from quadratic combinations of the Bannai-Ito generators as follows. One defines the generators A, B and C as

$$A = \frac{1}{4}\left(B_1^2 - B_1 - \frac{3}{4}\right), \qquad B = \frac{1}{4}\left(B_2^2 - B_2 - \frac{3}{4}\right), \qquad C = \frac{1}{4}\left(B_3^2 - B_3 - \frac{3}{4}\right). \tag{5.50}$$

A direct computation shows that in the sub-algebra generated by A, B and C as defined above, the element

$$\Gamma \equiv B_1 + B_2 + B_3 - 3/2, \tag{5.51}$$

is central, as is the sum of the generators since it can be expressed as

$$A + B + C = \frac{1}{4}(Q - \Gamma - 15/4),$$

where Q is the Casimir operator (5.43) of the Bannai-Ito algebra. The commutators between distinct generators are seen to be equal and define a fourth generator P as follows

$$2P \equiv [A, B] = [B, C] = [C, A]. \tag{5.52}$$

From the relations (5.42) of the Bannai-Ito algebra, one obtains the closure of the algebra generated by A, B, C and P as a quadratic algebra with the following relations:

$$\begin{aligned}
[A,P] &= BA - AC + \frac{1}{16}\frac{\omega_3-\omega_1}{2}\left(\frac{\omega_3+\omega_1}{2}-\Gamma\right),\\
[B,P] &= CB - BA + \frac{1}{16}\frac{\omega_1-\omega_2}{2}\left(\frac{\omega_1+\omega_2}{2}-\Gamma\right),\\
[C,P] &= AC - CB + \frac{1}{16}\frac{\omega_2-\omega_3}{2}\left(\frac{\omega_2+\omega_3}{2}-\Gamma\right).
\end{aligned} \tag{5.53}$$

The relations (5.51), (5.52) and (5.53) can be seen to be identical to the equitable presentation of the Racah algebra given in (5.7) and (5.8). Thus, one can define the embedding $\theta : \tilde{\mathcal{R}} \longrightarrow \mathcal{B}$ of the reduced Racah algebra into the Bannai-Ito algebra as follows

$$\begin{aligned}
\theta : \quad & V_1 \mapsto A, & d &\mapsto \frac{1}{8}(Q-\Gamma-15/4),\\
& V_2 \mapsto B, & e_1 &\mapsto \frac{1}{64}\frac{\omega_3-\omega_1}{2}\left(\frac{\omega_3+\omega_1}{2}-\Gamma\right),\\
& V_3 \mapsto C, & e_2 &\mapsto \frac{1}{64}\frac{\omega_1-\omega_2}{2}\left(\frac{\omega_1+\omega_2}{2}-\Gamma\right).
\end{aligned} \tag{5.54}$$

5.6. The Heun-Bannai-Ito algebra

We now introduce the Heun Bannai-Ito algebra $\mathcal{HB}$ abstractly as the unital associative cubic algebra generated by X, W and Z with the following relations

$$\begin{aligned}
&\{X,W\} \equiv Z,\\
&\{Z,X\} = x_0 I + x_1 X + x_2 X^2 + x_3 X^3 + x_4 W,\\
&\{W,Z\} = y_0 I + y_1 X + y_2 X^2 + y_3 X^3 + (x_1 + x_3 x_4)W + x_2\{X,W\} - x_3 XWX,
\end{aligned} \tag{5.55}$$

where x_i, y_i for $i = 0,1,2$ are central elements and x_i for $i = 3,4$, together with y_3, are parameters in $\mathbb{R}$. The constraints in the last three coefficients of (5.55) ensure compatibility with the following graded Jacobi identity

$$[X,\{Z,W\}] + [W,\{X,Z\}] + [Z,\{W,X\}] = 0.$$

A distinguished central element is identified in this presentation as

$$\Lambda = (x_4y_2 - y_0)X + (x_0 - x_2x_4)W - (x_1 + x_3x_4)Z + \frac{1}{2}x_4y_3X^2 + 2x_4W^2 + Z^2 \\ + [XW, WX] - x_2XWX - y_2X^3 - \frac{y_3}{2}X^4. \quad (5.56)$$

5.6.1. Embedding in the Bannai-Ito algebra

A specialization of the Heun-Bannai-Ito algebra, obtained by imposing conditions on the parameters, admits an embedding into the Bannai-Ito algebra. This embedding can be constructed from the algebraic Heun operator (5.1) of the Bannai-Ito type. To do so, one defines the map ψ on the generators as follows

$$\begin{aligned} \psi : \mathcal{HB} &\longrightarrow \mathcal{B}, \\ X &\longmapsto B_1, \\ W &\longmapsto \tau_1B_1B_2 + \tau_2B_2B_1 + \tau_3B_1 + \tau_4B_2 + \tau_0I, \end{aligned} \quad (5.57)$$

which is an homomorphism, provided that the parameters of the Heun-Bannai-Ito algebra in (5.55) be as follows

$$x_3 = 4\tau_3, \qquad x_4 = 1, \qquad y_3 = 8\tau_3^2 - 2(\tau_1 - \tau_2)^2, \quad (5.58)$$

and that the central elements of the Heun-Bannai-Ito algebra be mapped to those of the Bannai-Ito algebra as follows

$$\begin{aligned} &x_0 \mapsto \tau_4\omega_3 - \tau_0, \qquad x_1 \mapsto 2\tau_4\omega_1 + (\tau_1 + \tau_2)\omega_3 - \tau_3, \qquad x_2 \mapsto 2(\tau_1 + \tau_2)\omega_1 + 4\tau_0, \\ &y_0 \mapsto Q(\tau_1 + \tau_2)\tau_4 + \tau_4\left(-\tau_2\omega_1^2 + \tau_4\omega_2 + 3\tau_3\omega_3\right) - \tau_0(2\tau_4\omega_1 + (\tau_1 + \tau_2)\omega_3 + 3\tau_3) \\ &\qquad + \tau_1\left(\tau_2(\omega_2 - 2\omega_1\omega_3) - \tau_4\omega_1^2\right), \\ &y_1 \mapsto Q(\tau_1 - \tau_2)^2 - 4(\tau_1 + \tau_2)\tau_0\omega_1 - (\tau_1 + \tau_2)^2\omega_1^2 + 4\tau_3\tau_4\omega_1 + 2(\tau_1 + \tau_2)\tau_3\omega_3 \\ &\qquad - 4\tau_0^2 - 3\tau_3^2 + \tau_4^2 + \tau_1\tau_2, \\ &y_2 \mapsto \tau_1^2(-\omega_2) - \tau_1(\tau_4 - 2\tau_2\omega_2) + 2(\tau_1 + \tau_2)\tau_3\omega_1 - \tau_2(\tau_2\omega_2 + \tau_4) + 4\tau_0\tau_3, \end{aligned} \quad (5.59)$$

where Q is the Casimir (5.43) of the Bannai-Ito algebra. Under the map ψ defined in (5.57) and (5.59) with the parameters as in (5.58), the central element Λ given in (5.56) is mapped to the Casimir of the Bannai-Ito algebra, up to a central element and a scaling, such that

$$\psi : \Lambda \longmapsto u\,Q + v,$$

where

$$u = \tau_4^2 + \tau_1\tau_2, \qquad v = -2\tau_0\left(\tau_1\omega_1 + \tau_2\omega_1 - \tau_4\omega_3\right) + \tau_1\tau_4\omega_2 + \tau_4\left(\tau_2\omega_2 - \tau_4\omega_1^2\right) \\ - \tau_1\tau_2\left(\omega_1^2 + \omega_3^2\right) - 3\tau_0^2.$$

5.6.2. The Heun-Bannai-Ito operator

The generalized Heun operator W of the Bannai-Ito type can be introduced as the generic first order reflection operator in the infinite dihedral group D_∞ that has the degree raising property:

$$W p_n(x) \longmapsto q_{n+1}(x), \tag{5.60}$$

for p_n and q_n arbitrary polynomials of degree n and $n+1$, respectively. Consider the general reflection operator W specified by

$$W = A_1(x)R_1 + A_2(x)R_2 + A_0(x)I. \tag{5.61}$$

Acting on the first three monomials in x and demanding that (5.60) holds fully determines the form of the coefficients $A_i(x)$ for $i = 0, 1, 2$. One has

$$\begin{aligned} p_1(x) &\equiv W\cdot 1 = A_1(x) + A_2(x) + A_0(x), \\ p_2(x) &\equiv W\cdot x = x(A_0(x) - A_1(x)) - (x+1)A_2(x), \\ p_3(x) &\equiv W\cdot x^2 = x^2(A_0(x) + A_1(x)) + (x+1)^2 A_2(x), \end{aligned} \tag{5.62}$$

where $p_1(x)$, $p_2(x)$ and $p_3(x)$ are arbitrary polynomials of first, second and third degree, respectively. One solves easily (5.62) for the coefficients $A_i(x)$ to obtain

$$\begin{aligned}
A_0(x) &= \frac{p_3(x) + (2x+1)p_2(x) + x(x+1)p_1(x)}{2x(2x+1)},\\
A_1(x) &= \frac{x(x+1)p_1(x) - p_2(x) - p_3(x)}{2x},\\
A_2(x) &= \frac{p_3(x) - x^2 p_1(x)}{2x+1}.
\end{aligned} \tag{5.63}$$

From the above, one sees that the coefficients of the polynomials $p_1(x)$, $p_2(x)$ and $p_3(x)$ constitute a parametrization of the Heun-Bannai-Ito operator. We will denote these nine parameters as follows

$$p_1(x) = \sum_{i=0}^{1} p_1^{(i)} x^i, \qquad p_2(x) = \sum_{i=0}^{2} p_2^{(i)} x^i, \qquad p_3(x) = \sum_{i=0}^{3} p_3^{(i)} x^i. \tag{5.64}$$

For the Heun-Bannai-Ito operator to act on the finite Bannai-Ito grid x_s given in (5.46), additional constraints exist on the parameters. Depending on the truncation conditions, one has

$$\begin{aligned}
&A_1(x) \propto (x-\rho_j),\ A_2(x) \propto (x - r_i + 1/2), && \text{for } N \text{ even}, \quad 2(r_i+\rho_j) = N+1, \qquad i,j = 1,2,\\
&A_1(x) \propto (x-\rho_1)(x-\rho_2), && \text{for } N \text{ odd}, \quad 2(\rho_1+\rho_2) = -N-1,\\
&A_2(x) \propto (x - r_1 + 1/2)(x - r_2 + 1/2), && \text{for } N \text{ odd}, \quad 2(r_1+r_2) = N+1.
\end{aligned} \tag{5.65}$$

These conditions can be expressed on the parameters (5.64) of the Heun-Bannai-Ito operator as the following constraints. For any constants, a and b, one has that

$$\begin{aligned}
A_1 \propto (x-a) &\implies p_3^{(0)} = a^3(p_1^{(1)} - p_3^{(3)}) + a^2(p_1^{(0)} - p_3^{(2)}) - a\, p_3^{(1)},\\
A_2 \propto (x-b) &\implies p_2^{(0)} = b^3(p_1^{(1)} - p_3^{(3)}) + b^2(p_1^{(0)} + p_1^{(1)} - p_2^{(2)} - p_3^{(2)})\\
&\qquad\qquad + b(p_1^{(0)} - p_2^{(1)} - p_3^{(1)}) - p_3^{(0)},
\end{aligned} \tag{5.66}$$

and, conjunctly with the above, one also has

$$\begin{aligned}
A_1 \propto (x-b) &\implies p_3^{(1)} = (a^2+b^2)(p_1^{(1)} - p_3^{(3)}) + ab(p_1^{(1)} - p_3^{(3)}) + (a+b)(p_1^{(0)} - p_3^{(2)}),\\
A_2 \propto (x-a) &\implies p_2^{(1)} = (a^2+b^2)(p_1^{(1)} - p_3^{(3)}) + ab(p_1^{(1)} - p_3^{(3)})\\
&\qquad\qquad + (a+b)(p_1^{(0)} + p_1^{(1)} - p_2^{(2)} - p_3^{(2)}) + p_1^{(0)} - p_3^{(1)}.
\end{aligned} \tag{5.67}$$

Using two of the above four constraints on the parameters, one can satisfy any case of the truncation conditions displayed in (5.65). Thus, when constrained to the Bannai-Ito grid (5.46), the Heun-Bannai-Ito operator has seven free parameters amongst those of (5.64).

5.6.3. Tridiagonalization in the Bannai-Ito algebra

The Heun-Bannai-Ito operator can be obtained from the tridiagonalization procedure applied to the Bannai-Ito bispectral operators. Consider the following generic $W \in \mathcal{B}$

$$W = \tau_1 B_1 B_2 + \tau_2 B_2 B_1 + \tau_3 B_1 + \tau_4 B_2 + \tau_0 I. \tag{5.68}$$

In the realization (5.49), it can be seen by direct calculations that the above operator takes the form of the Heun-Bannai-Ito operator (5.61). Thus, the Heun-Bannai-Ito operator is one of the generators in the realization of the Heun-Bannai-Ito algebra constructed from the concatenation of the embedding map ψ defined in (5.57) with the realization given in (5.49).

Once the Bannai-Ito grid x_s is specified as in (5.46), the realization (5.49) admits two free parameters. Thus, as the definition (5.68) for W introduced five additional parameters, this realization of W has seven free parameters, as was the case for the Heun-Bannai-Ito operator in (5.64) with (5.65), (5.66) and (5.67). In this case, the parameters of the Heun-Bannai-Ito operator as given by (5.64) can be given in terms of those of the realization (5.49) of the Bannai-Ito algebra together with those of the tridiagonalization (5.68). One obtains

$$\begin{aligned}
{p_3}^{(3)} &= \tau_1 \left(2\rho_1 + 2\rho_2 - 2r_1 - 2r_2 + 5\right) + \tau_2 \left(-2\rho_1 - 2\rho_2 + 2r_1 + 2r_2 - 7\right) + 2\tau_3,\\
{p_3}^{(2)} &= \frac{1}{4}\Big(2\rho_1\tau_2 + 16\rho_1\rho_2\tau_2 + 2\rho_2\tau_2 + 4\rho_1\tau_4 + 4\rho_2\tau_4 + \tau_1 \left(2\rho_1 + 2\rho_2 - 18r_1 - 18r_2 + 21\right)\\
&\qquad + 22r_1\tau_2 - 16r_1r_2\tau_2 + 22r_2\tau_2 - 4r_1\tau_4 - 4r_2\tau_4 + 4\tau_0 - 31\tau_2 + 2\tau_3 + 10\tau_4\Big),
\end{aligned}$$

$$\begin{aligned}
{p_3}^{(1)} &= \left(-3r_2 + r_1\left(4r_2 - 3\right) + 2\right)\tau_1 + \left(r_1\left(5 - 4r_2\right) + 5r_2 - 4\right)\tau_2 - 2\left(r_1 + r_2 - 1\right)\tau_4,\\
{p_3}^{(0)} &= \frac{1}{4}\left(2r_1 - 1\right)\left(2r_2 - 1\right)\left(\tau_1 - 3\tau_2 + 2\tau_4\right),\\
{p_2}^{(2)} &= \tau_1 \left(-2\rho_1 - 2\rho_2 + 2r_1 + 2r_2 - 3\right) + \tau_2 \left(2\rho_1 + 2\rho_2 - 2r_1 - 2r_2 + 5\right) + 2\tau_3,\\
{p_2}^{(1)} &= \frac{1}{4}\Big(- 2\rho_1\tau_2 - 2\rho_2\tau_2 + \tau_1 \left(-2\rho_1 + 16\rho_1\rho_2 - 2\rho_2 + 10r_2 - 2r_1\left(8r_2 - 5\right) - 7\right)\\
&\qquad - 4\rho_1\tau_4 - 4\rho_2\tau_4 - 14r_1\tau_2 - 14r_2\tau_2 + 4r_1\tau_4 + 4r_2\tau_4 + 4\tau_0 + 13\tau_2 + 2\tau_3 - 6\tau_4\Big),
\end{aligned}$$

$$p_2{}^{(0)} = \rho_1\rho_2\left(\tau_1+\tau_2+2\tau_4\right) - \frac{1}{4}\left(2r_1-1\right)\left(2r_2-1\right)\left(\tau_1-3\tau_2+2\tau_4\right),$$

$$p_1{}^{(1)} = \tau_1\left(2\rho_1+2\rho_2-2r_1-2r_2+1\right)+\tau_2\left(-2\rho_1-2\rho_2+2r_1+2r_2-3\right)+2\tau_3,$$

$$\begin{aligned} p_1{}^{(0)} = \frac{1}{4}\Big(&2\rho_1\tau_2+16\rho_1\rho_2\tau_2+2\rho_2\tau_2+4\rho_1\tau_4+4\rho_2\tau_4+\tau_1\left(2\rho_1+2\rho_2-2r_1-2r_2+1\right)+6r_1\tau_2 \\ &-16r_1r_2\tau_2+6r_2\tau_2-4r_1\tau_4-4r_2\tau_4+4\tau_0-3\tau_2+2\tau_3+2\tau_4\Big). \end{aligned}$$

Conclusion

The recently introduced [**18**] notion of the algebraic Heun operator enables the construction of generalized Heun operators from a bispectral pair of operators. In particular, as each polynomial families in the Askey scheme is associated to a bispectral problem, a corresponding generalized Heun operator can be constructed. Furthermore, paralleling the algebraic approach to the Askey scheme, one is lead to the study of algebraic structures that encode the properties of these generalized Heun operators. This paper examined the Racah and Bannai-Ito cases.

The Heun-Racah operator was first constructed as the most general operator on the Racah grid satisfying the Heun property of sending polynomials to polynomials one degree higher. This operator could be identified with the algebraic Heun operator of the Racah type in the canonical realization of the Racah algebra. The Heun-Racah algebra associated to the Racah polynomials was subsequently introduced. This algebra was defined in a generic presentation and a central element was identified. The association with the Racah algebra was made explicit by the identification of a map that embeds a specialization of the Heun-Racah algebra, obtained from conditions on the parameters, as a subalgebra of the Racah algebra. This embedding effectively maps the central element to the Casimir operator of the Racah algebra. Moreover, using this embedding, a realization of the specialized Heun-Racah algebra is induced by the canonical realization of the Racah algebra.

As the Racah polynomials are at the top of the Askey scheme, the algebraic structure that result from the construction of the algebraic Heun operator does not correspond to an algebra associated to polynomials of the Askey scheme. This motivates further examination of the Heun-Racah algebra as a new algebraic structure. Moreover, in view of the limit $q \to 1$ that relates the Askey-Wilson algebra to the Racah algebra, one would expect a similar limit that relates the results in [**3**] with the results presented here.

An analogous examination was made for the Bannai-Ito case. The Heun-Bannai-Ito algebra was first introduced abstractly and a specialization with conditions on the parameters was shown to embed in the Bannai-Ito algebra. In the canonical realization of the Bannai-Ito algebra, this specialization was demonstrated to be realized in terms of the associated Heun-Bannai-Ito operator.

In view of the embedding of the Racah algebra in the Bannai-Ito algebra presented in section 5.5, one could ask if the relation between these two algebraic structures is reflected in the associated Heun algebras. Indeed, this paper has illustrated the following maps between these algebraic structures

$$\begin{tikzcd}
& \mathcal{R} \arrow[dr, "\chi"] & \\
\mathcal{HR} \arrow[ur, "\phi"] \arrow[rr, dashed, "\chi\circ\phi"] \arrow[ddrr, dashed, "\Upsilon"] & & \tilde{\mathcal{R}} \arrow[dd, "\theta"] \\
& & \\
\mathcal{HB} \arrow[rr, "\psi"] & & \mathcal{B},
\end{tikzcd}$$

where χ, ϕ, θ and ψ are defined in (5.9), (5.39), (5.54) and (5.57), respectively and $\Upsilon = \theta \circ \chi \circ \phi$. Furthermore, it can be shown that

$$\begin{aligned}\Upsilon(W_{\mathcal{HR}}) = b_1\psi(W_{\mathcal{HB}})^2 + b_2\{\Gamma, \psi(W_{\mathcal{HB}})\} + [\psi(X_{\mathcal{HB}}), \psi(W_{\mathcal{HB}})] + b_4[\Gamma, \psi(W_{\mathcal{HB}})] + b_5 Q \\ b_6\{\psi(X_{\mathcal{HB}}), \psi(W_{\mathcal{HB}})\} + b_7\{\psi(X_{\mathcal{HB}}), \Gamma\} + b_8\psi(W_{\mathcal{HB}}),\end{aligned}$$

where the subscripts $\mathcal{HR}$ and $\mathcal{HB}$ denote, respectively, generators in the Heun-Racah and Heun-Bannai-Ito algebra and where b_i for $1 \leq i \leq 8$ are coefficients. Finding a specialization of the Heun-Racah algebra that embeds in the Heun-Bannai-Ito algebra would enhance the association between these new algebraic structures and the algebraic structures of the Askey scheme.

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Chapitre 6

Sklyanin-like algebras for (q-)linear grids and (q-)para-Krawtchouk polynomials

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Abstract

S-Heun operators on linear and q-linear grids are introduced. These operators are special cases of Heun operators and are related to Sklyanin-like algebras. The Continuous Hahn and Big q-Jacobi polynomials are functions on which these S-Heun operators have natural actions. We show that the S-Heun operators encompass both the bispectral operators and Kalnins and Miller's structure operators. These four structure operators realize special limit cases of the trigonometric degeneration of the original Sklyanin algebra. Finite-dimensional representations of these algebras are obtained from a truncation condition. The corresponding representation bases are finite families of polynomials: the para-Krawtchouk and q-para-Krawtchouk ones. A natural algebraic interpretation of these polynomials that had been missing is thus obtained. We also recover the Heun operators attached to the corresponding bispectral problems as quadratic combinations of the S-Heun operators.

6.1. Introduction

In the study of orthogonal polynomials (OPs), many of their properties are expressed as structure relations between family members with different parameters, arguments or degrees, examples are the three term recurrence relation, the differential/difference equation, the backward/forward relation, etc. As it turns out, the operators involved in these formulas realize algebras that synthesize much of the characterization of these polynomial ensembles. The present paper relates to this framework.

One such instance that has proven very fruitful is the (algebraic) study of the two bispectral operators associated to hypergeometric OPs. These operators are the recurrence and the differential/difference operators. Let us focus on the developments related to the Askey–Wilson polynomials; since these polynomials sit at the top of the Askey scheme, the gist of their description descends onto all the lower families in the scheme. The two bispectral operators for the Askey–Wilson polynomials do not commute: they form an algebra whose relations have been found by Zhedanov in [**60**] and it is usually referred to as the Askey–Wilson algebra.

This algebra has appeared in a great variety of contexts, such as knot theory [**9**], double affine Hecke algebras and representation theory [**33, 35, 43**], Howe duality [**14, 16**], integrable models [**2, 1, 3, 57**], algebraic combinatorics [**53, 50, 51, 52**], the Racah problem for $U_q(\mathfrak{sl}_2)$ [**21, 26**], etc. The abovementioned connections have some specializations for all entries of the Askey tableau.

The work of Kalnins and Miller [**31, 30, 44**] based on the use of four structure or contiguity operators is another approach that illustrates the use of symmetry techniques in the study of OPs. These operators that shall be referred to as structure operators in the following correspond to the backward and forward operators, as well as to another pair of operators that "factorize" [**27**] the differential/difference operator. It was recently observed [**34**] that for the Askey–Wilson polynomials, these operators realize the relations of the trigonometric degeneration [**20**] of the Sklyanin algebra [**46**]. To our knowledge, the Sklyanin-like algebras similarly connected to other families of OPs have not been described so far and will be the center of attention here.

The differential/difference operator of which the OPs are eigenfunctions belongs to the intersection of the sets of operators involved in the two pictures. A natural question is the following: *what is the most elementary set of operators that encompasses all operators in both of the approaches above?* In the case of the Askey–Wilson polynomials, this answer was given in [**15**]: it is the set of so-called S-Heun operators on the Askey–Wilson grid (these are special types of Heun operators that will be defined in the next section). Operators of the Heun type are related to the tridiagonalization procedure [**28, 24**] and have been given an algebraic formulation [**23, 25**]. They have been identified as Hamiltonians of quantum Euler–Arnold tops [**56**], they have been connected to band-time limiting [**47, 37**] and to the study of entanglement in spin chains [**11, 12**] and they have been studied quite a lot recently [**48, 49, 5, 4, 59, 13, 55, 6, 8, 7**]. As will be shown below, the S-Heun operators allow a factorization of these Heun operators. Let us note that in addition to the unification of the two approaches described above, the S-Heun framework has also led to a novel algebraic interpretation of the q-para-Racah polynomials. The goal of the present paper is to look at the grids of linear type from the S-Heun operators point of view. As a byproduct, an algebraic interpretation of the para-Krawtchouk and q-para-Krawtchouk polynomials will be obtained. These polynomials were first identified in the context of perfect state transfer and fractional revival on quantum spin chains [**58, 41, 19, 61**] and their algebraic interpretation was still lacking.

We will introduce the S-Heun operators on linear grids in Section 6.2. The simplest example of operators of this type will be worked out in Section 6.3 (this will involve differential operators, the Jacobi polynomials and the ordinary Heun operator). Section 6.4 will focus on the S-Heun operators on the discrete linear grid. A new degeneration of the Sklyanin algebra will be presented. Of relevance in this case, the Continuous Hahn polynomials will be seen to truncate to the para-Krawtchouk polynomials under a special condition and an algebraic interpretation of such a truncation will be given. The Heun operator on the uniform grid will also be recovered. The q-linear grid will be examined in Section 6.5 and the previous analysis will be repeated. The degeneration of the Sklyanin algebra that arises will be identified as $U_q(\mathfrak{sl}_2)$. The Big q-Jacobi polynomials will be involved, and they will be observed to reduce to the q-para-Krawtchouk polynomials under a certain condition. The Big q-Jacobi Heun operator will also be recovered as well. Connections between the three grids and the associated S-Heun operators and Sklyanin-type algebras will be presented in

Section 6.6, followed by concluding remarks. The quadratic relations between the S-Heun operators for the three different types of grids are listed in Appendix 6.7.

6.2. S-Heun operators on linear-type grids

S-Heun operators are defined as the most general second order differential/difference operators without diagonal term that obey a degree raising condition. Like Heun operators, they can be defined on different grids. We now introduce the three linear grids that we will use and obtain the S-Heun operators associated to each.

6.2.1. The discrete linear grid

Consider the operator S

$$S = A_1 T_+ + A_2 T_- \tag{6.1}$$

where

$$T_+ f(x) = f(x+1), \qquad T_- f(x) = f(x-1) \tag{6.2}$$

are shift operators, and $A_{1,2}$ are functions in the real variable x. Impose that S maps polynomials of degree n onto polynomials of degree no higher than $n+1$, namely,

$$SP_n(x) = \tilde{P}_{n+1}(x) \tag{6.3}$$

for all $n = 0, 1, 2, \ldots$. This defines the S-Heun operators on the discrete linear grid.

It is sufficient to enforce this raising condition on monomials x^n; for $n = 0$ and $n = 1$, it reads

$$A_1 + A_2 = a_{00} + a_{01}x, \tag{6.4a}$$

$$A_1(x+1) + A_2(x-1) = a_{10} + a_{11}x + a_{12}x^2, \tag{6.4b}$$

for some arbitrary parameters a_{ij}. This can be rewritten as

$$A_1 + A_2 = a_{00} + a_{01}x, \tag{6.5a}$$

$$A_1 - A_2 = a_{10} + (a_{11} - a_{00})x + (a_{12} - a_{01})x^2. \tag{6.5b}$$

Straightforward induction shows that in general one has

$$Sx^n = A_1(x+1)^n + A_2(x-1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k [A_1 + (-1)^{n-k} A_2], \tag{6.6}$$

which is a polynomial of degree $n+1$. Thus, the functions A_1, A_2

$$A_1 = \tfrac{1}{2}\left[(-a_{01}+a_{12})x^2 + (-a_{00}+a_{01}+a_{11})x + (a_{00}+a_{10})\right], \tag{6.7a}$$

$$A_2 = \tfrac{1}{2}\left[(+a_{01}-a_{12})x^2 + (+a_{00}+a_{01}-a_{11})x + (a_{00}-a_{10})\right] \tag{6.7b}$$

satisfy (6.5) and the operator (6.1) meets the degree raising condition.

Proposition 6.2.1. With the functions A_1, A_2 given by (6.7), the operator S in (6.1) is the most general S-Heun operator on the linear grid. S depends on 5 free parameters and spans a 5-dimensional linear space. The elements

$$L = \tfrac{1}{2}\left[T_+ - T_-\right], \tag{6.8a}$$

$$M_1 = \tfrac{1}{2}\left[T_+ + T_-\right], \tag{6.8b}$$

$$M_2 = \tfrac{1}{2}x\left[T_+ - T_-\right], \tag{6.8c}$$

$$R_1 = \tfrac{1}{2}x\left[(1-2x)T_+ + (1+2x)T_-\right], \tag{6.8d}$$

$$R_2 = \tfrac{1}{2}x\left[T_+ + T_-\right]. \tag{6.8e}$$

form a basis for this space.

Using (6.6), one sees that the operator L is a lowering operator (it lowers by one the degree of polynomials in x), the operators M_1, M_2 are stabilizing operators (they do not change the degree) and the operators R_1, R_2 are raising operators (they raise it by one).

6.2.2. The q-linear grid

Condider now the q-linear grid $z = q^x$ (or exponential grid). The S-Heun operators $\hat{S}$ on that grid are of the form

$$\hat{S} = \hat{A}_1(z,q)\hat{T}_+ + \hat{A}_2(z,q)\hat{T}_-, \tag{6.9}$$

with

$$\hat{T}_\pm f(z) = f(q^{\pm 1}z), \tag{6.10}$$

and are taken to map polynomials in z onto polynomials of at most one degree higher: $\hat{S}P_n(z) = \tilde{P}_{n+1}(z)$. Imposing this degree raising condition on the first monomials 1 and z yields

$$\hat{A}_1(z,q) + \hat{A}_2(z,q) = a_{00} + a_{01}z, \tag{6.11a}$$

$$\hat{A}_1(z,q)q + \hat{A}_2(z,q)q^{-1} = a_{10}z^{-1} + a_{11} + a_{12}z. \tag{6.11b}$$

Straightforward induction shows that in general one has

$$\hat{S}z^n = (\hat{A}_1q^n + \hat{A}_2q^{-n})z^n = z^n[\hat{A}_1q + \hat{A}_2q^{-1}]\frac{q^n - q^{-n}}{q - q^{-1}} - z^n[\hat{A}_1 + \hat{A}_2]\frac{q^{n-1} - q^{1-n}}{q - q^{-1}}, \tag{6.12}$$

which is a polynomial of degree $n+1$ in z. Thus, an operator $\hat{S}$ with $\hat{A}_1(z,q)$ and $\hat{A}_2(z,q)$ that satisfies (6.11) will obey the degree raising condition on any monomial. We hence obtain:

$$\hat{A}_1(z,q) = \hat{A}_2(z,q^{-1}) = \frac{1}{(q-q^{-1})z}\left[a_{10} + (a_{11} - a_{00}q^{-1})z + (a_{12} - a_{01}q^{-1})z^2\right]. \tag{6.13}$$

Proposition 6.2.2. With the functions $\hat{A}_1(z,q)$, $\hat{A}_2(z,q)$ given by (6.13), the operator $\hat{S}$ in (6.9) is the most general S-Heun operator on the q-linear grid. $\hat{S}$ depends on 5 free parameters and spans a 5-dimensional linear space. The elements

$$\hat{L} = \frac{1}{(q-q^{-1})}z^{-1}(\hat{T}_+ - \hat{T}_-), \tag{6.14a}$$

$$\hat{M}_1 = \frac{1}{(q-q^{-1})}(-q^{-1}\hat{T}_+ + q\hat{T}_-), \tag{6.14b}$$

$$\hat{M}_2 = \frac{1}{(q-q^{-1})}(\hat{T}_+ - \hat{T}_-), \tag{6.14c}$$

$$\hat{R}_1 = \frac{1}{(q-q^{-1})}z(-q^{-1}\hat{T}_+ + q\hat{T}_-), \tag{6.14d}$$

$$\hat{R}_2 = \frac{1}{(q-q^{-1})}z(\hat{T}_+ - \hat{T}_-). \tag{6.14e}$$

can be chosen as a basis for this space.

Looking at (6.12) and (6.13), one sees that the operator $\hat{L}$ lowers the degrees, and that the $\hat{M}_i$'s and the $\hat{R}_i$'s are respectively stabilizing and raising operators.

6.2.3. The simplest case: differential S-Heun operators

The definition of the S-Heun operators on the real line goes as follows. Consider the first-order differential operator

$$\bar{S} = \bar{A}_1(x)\frac{d}{dx} + \bar{A}_2(x) \tag{6.15}$$

and impose the raising condition $\bar{S}p_n(x) = \tilde{p}_{n+1}(x)$ which demands that $\bar{S}$ sends polynomials into polynomials of one degree higher. The general solution is given by

$$\bar{A}_1(x) = a_{10} + a_{11}x + a_{12}x^2, \qquad \bar{A}_2(x) = a_{20} + a_{21}x. \tag{6.16}$$

This leads to the following set of five linearly independent S-Heun operators [**56**]

$$\bar{L} = \frac{d}{dx}, \qquad \bar{M}_1 = 1, \qquad \bar{M}_2 = x\frac{d}{dx}, \qquad \bar{R}_1 = x, \qquad \bar{R}_2 = x^2\frac{d}{dx}, \tag{6.17}$$

which are once again labelled according to their property of lowering ($\bar{L}$), stabilizing ($\bar{M}$) or raising ($\bar{R}$) the degree of polynomials in the variable x.

These S-Heun operators can also be obtained as a $q \to 1$ limit of the ones defined on the q-linear grid. More precisely, writing $q = e^{\hbar}$ and letting $\hbar \to 0$, one obtains

$$\lim_{q\to 1}\hat{L} = \bar{L}, \qquad \lim_{q\to 1}\hat{M}_1 = \bar{M}_1 - \bar{M}_2, \qquad \lim_{q\to 1}\hat{M}_2 = \bar{M}_2, \qquad \lim_{q\to 1}\hat{R}_1 = \bar{R}_1 - \bar{R}_2, \qquad \lim_{q\to 1}\hat{R}_2 = \bar{R}_2. \tag{6.18}$$

This connects with the definition of the continuous S-Heun operators. These S-Heun operators will also be related to the ordinary Heun operator introduced in the next section.

6.3. The continuous case

The goal of this section is to revisit (mostly known) results with a point of view that will be adopted in the following sections. Here, we are interested in studying the OPs and algebras related to the set of the five S-Heun operators defined in Section 6.2.3.

6.3.1. The stabilizing subalgebra

We first study the subset $\{\bar{L}, \bar{M}_1, \bar{M}_2\}$ of S-Heun operators that stabilize the set of polynomials of a given degree. Let us denote by $\bar{Q}$ the most general quadratic combination of these operators. Using the relations of Appendix 6.7, it is always possible to reduce $\bar{Q}$ to an expression of the form

$$\bar{Q} = \alpha_1 \bar{L}^2 + \alpha_2 \bar{L}\bar{M}_1 + \alpha_3 \bar{L}\bar{M}_2 + \alpha_4 \bar{M}_1^2 + \alpha_5 \bar{M}_1\bar{M}_2 + \alpha_6 \bar{M}_2^2. \tag{6.19}$$

Using the realizations (6.17), the eigenvalue equation for the second-order differential operator $\bar{Q}$ can be brought in the form

$$\begin{aligned} \bar{\mathcal{D}} P_n^{(\alpha,\beta)}(x) &= n(n+\alpha+\beta+1) P_n^{(\alpha,\beta)}(x), \\ \bar{\mathcal{D}} &= (x^2-1)\frac{d^2}{dx^2} + [(\alpha-\beta)+(\alpha+\beta+2)x]\frac{d}{dx}, \end{aligned} \tag{6.20}$$

which is recognized as the differential equation satisfied by the Jacobi polynomials [**32**].

We have thus identified the family of OPs related to these (ordinary) S-Heun operators, and as will be seen in the next subsection, certain combinations of these S-Heun operators provide the structure relations of these polynomials.

6.3.2. Jacobi polynomials and their structure relations

Consider the forward and backward operators for the Jacobi polynomials

$$\bar{\tau} = \bar{L}, \qquad \bar{\tau}^{(\alpha,\beta)^*} = -\bar{L} + (\alpha-\beta)\bar{M}_1 + (\alpha+\beta)\bar{R}_1 + \bar{R}_2. \tag{6.21a}$$

and the contiguity operators

$$\bar{\mu}^{(\alpha)} = -\bar{L} + \alpha\bar{M}_1 + \bar{M}_2, \qquad \bar{\mu}^{(\beta)^*} = \bar{L} + \beta\bar{M}_1 + \bar{M}_2. \tag{6.21b}$$

These four operators act very simply on the Jacobi polynomials:

$$\bar{\tau} P_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{6.22a}$$

$$\bar{\tau}^{(\alpha,\beta)^*} P_n^{(\alpha,\beta)}(x) = 2(n+1) P_{n+1}^{(\alpha-1,\beta-1)}(x), \tag{6.22b}$$

$$\bar{\mu}^{(\alpha)} P_n^{(\alpha,\beta)}(x) = (n+\alpha) P_n^{(\alpha-1,\beta+1)}(x), \tag{6.22c}$$

$$\bar{\mu}^{(\beta)^*} P_n^{(\alpha,\beta)}(x) = (n+\beta) P_n^{(\alpha+1,\beta-1)}(x). \tag{6.22d}$$

The operators $\bar{\mu}^{(\alpha)}$, $\bar{\mu}^{(\beta)^*}$, $\bar{\tau}$, $\bar{\tau}^{(\alpha,\beta)^*}$ built from linear combinations of S-Heun operators are of the type studied by Kalnins and Miller [**30**].

We have mentioned in the introduction that S-Heun operators encompass both the structure operators of Kalnins and Miller and the bispectral operators. Let us indicate how the latter operators appear in this context. First, as mentioned above, the Jacobi differential operator appears as a quadratic combination of the stabilizing generators. We can actually provide a factorization of this operator either as a product of two contiguous operators or as the product of the forward and backward operator:

$$\begin{aligned}\bar{\mathcal{D}} &= \bar{\mu}^{(\alpha+1)}\bar{\mu}^{(\beta)^*} - (\alpha+1)\beta \\ &= \bar{\mu}^{(\beta+1)^*}\bar{\mu}^{(\alpha)} - \alpha(\beta+1) \\ &= \bar{\tau}^{(\alpha+1,\beta+1)^*}\bar{\tau} \\ &= \bar{\tau}\bar{\tau}^{(\alpha,\beta)^*} - (\alpha+\beta).\end{aligned} \tag{6.23}$$

The other bispectral operator $\bar{X}$ is the multiplication by the variable x. It can be directly expressed as $\bar{R}_1$, but since it will appear as a quadratic combination of the S-Heun operators for other grids, we shall write it here as

$$\bar{X} = \bar{R}_1\bar{M}_1. \tag{6.24}$$

We have thus recovered the two bispectral operators as quadratic combinations in the S-Heun operators. This completes the observation that the S-Heun operators are the elementary blocks behind the two factorizations.

6.3.3. The Sklyanin-like algebra realized by the structure operators

We now focus on the algebras that are realized by these sets of operators. On the one hand the pair of bispectral Jacobi operators is known [**17**] to generate the Jacobi algebra that has been well studied [**22**]. On the other hand, the algebra formed by the 4 linear operators $\bar{\mu}^{(\alpha)}$, $\bar{\mu}^{(\beta)^*}$, $\bar{\tau}$, $\bar{\tau}^{(\alpha,\beta)^*}$ can be seen to be a degeneration of the Sklyanin algebra [**46**].

We now give a presentation of this algebra. Denote $\nu = -\frac{1}{2}(\alpha+\beta)$ and set

$$\bar{A} = \bar{M}_2 - \nu\bar{M}_1, \qquad \bar{B} = \bar{R}_2 - 2\nu\bar{R}_1, \qquad \bar{C} = \bar{L}, \qquad \bar{D} = \bar{M}_1. \tag{6.25}$$

These linear combinations of $\bar{\mu}^{(\alpha)}$, $\bar{\mu}^{(\beta)^*}$, $\bar{\tau}$, $\bar{\tau}^{(\alpha,\beta)^*}$ have been chosen in order to simplify the relations.

Proposition 6.3.1. The operators $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ obey the homogeneous quadratic relations

$$\begin{aligned}&[\bar{C},\bar{D}] = 0, \qquad [\bar{A},\bar{C}] = -\bar{C}\bar{D}, \qquad [\bar{A},\bar{D}] = 0, \\ &[\bar{B},\bar{C}] = -2\bar{A}\bar{D}, \qquad [\bar{A},\bar{B}] = \bar{B}\bar{D}, \qquad [\bar{B},\bar{D}] = 0.\end{aligned} \tag{6.26}$$

Remark 6.3.1. One will notice that these relations are actually the relations of the $\mathfrak{sl}_2$ Lie algebra supplemented with a central element D (one recovers $U(\mathfrak{sl}_2)$ by quotienting the above algebra (6.26) by the additional relation $D = 1$). The reason why we wrote these in a quadratic fashion is to make easier the comparison with the other Sklyanin algebras that will be obtained later.

One observes that if ν is an integer or half-integer, the realization (6.25) is associated to a finite dimensional representation of dimension $2\nu + 1$.

6.3.4. Recovering the Heun operator

We now show how to recover the ordinary (differential) Heun operator from the knowledge of the S-Heun operators.

The generic Heun operator $\bar{W}$ can be expressed as the most general tridiagonalization of the hypergeometric operator [**24**]. It has been known to be

$$\bar{W} = Q_3(x)\frac{d^2}{dx^2} + Q_2(x)\frac{d}{dx} + Q_1(x), \tag{6.27}$$

where $Q_3(x)$, $Q_2(x)$ and $Q_1(x)$ are general polynomials of degree 3, 2 and 1 respectively.

Let us consider the most general quadratic combination of S-Heun operators that does not raise the degree of polynomials by more than one. Using the quadratic homogeneous relations of Appendix 6.7, it is always possible to simplify such an expression to

$$\bar{W} = \alpha_1\bar{L}^2 + \alpha_2\bar{L}\bar{M}_1 + \alpha_3\bar{L}\bar{M}_2 + \alpha_4\bar{M}_1^2 + \alpha_5\bar{M}_1\bar{M}_2 + \alpha_6\bar{M}_2^2 + \beta_1\bar{M}_1\bar{R}_2 + \beta_2\bar{M}_2\bar{R}_1 + \beta_3\bar{M}_2\bar{R}_2. \tag{6.28}$$

From the differential expressions of the generators we obtain

$$\begin{aligned}
\bar{W} &= Q_3(x)\frac{d^2}{dx^2} + Q_2(x)\frac{d}{dx} + Q_1(x)\mathcal{I},\\
Q_3(x) &= \alpha_1 + \alpha_3 x + \alpha_6 x^2 + \beta_3 x^3,\\
Q_2(x) &= (\alpha_2 + \alpha_3) + (\alpha_5 + \alpha_6)x + (\beta_1 + \beta_2 + 2\beta_3)x^2,\\
Q_1(x) &= \alpha_4 + \beta_2 x,
\end{aligned} \tag{6.29}$$

where $\mathcal{I}$ is the identity operator: $\mathcal{I}f(x) = f(x)$.

Proposition 6.3.2. The generic Heun operator (6.27) can be obtained as the most general quadratic combination in the S-Heun generators (6.17) that does not raise the degree of polynomials by more than one.

Calling upon the reordering relations of Appendix 6.7, it is seen that the Heun operator generically factorizes as the product of a general S-Heun operator with a stabilizing S-Heun operator:

$$\bar{W} = (\xi_1 \bar{L} + \xi_2 \bar{M}_1 + \xi_3 \bar{M}_2)(\eta_1 \bar{L} + \eta_2 \bar{M}_1 + \eta_3 \bar{M}_2 + \eta_4 \bar{R}_1 + \eta_5 \bar{R}_2) + \kappa. \tag{6.30}$$

6.4. S-Heun operators on the linear grid

We now come to one of the main topics of the paper, namely the S-Heun operators defined on the linear grid.

6.4.1. The stabilizing subset

The subset of S-Heun operators that stabilizes the polynomials of a given degree is $\{L, M_1, M_2\}$. The most general quadratic combination of these operators can always be reduced to an expression of the form

$$Q = \alpha_1 L^2 + \alpha_2 L M_1 + \alpha_3 L M_2 + \alpha_4 {M_1}^2 + \alpha_5 M_1 M_2 + \alpha_6 {M_2}^2 \tag{6.31}$$

using the relations of Appendix 6.7. Substituting the expressions (6.8), one sees that Q is a second-order difference operator. By straightforward manipulations, the eigenvalue equation for Q can be transformed into the difference equation of the Continuous Hahn polynomials [**32**]

$$\begin{gathered}\mathcal{D} P_n(\tilde{x}; a, b, c, d) = n(n + a + b + c + d - 1) P_n(\tilde{x}; a, b, c, d), \\ \mathcal{D} = B(\tilde{x}) T_+^2 - [B(\tilde{x}) + D(\tilde{x})]\mathcal{I} + D(\tilde{x}) T_-^2, \\ B(x) = (c - ix)(d - ix), \qquad D(x) = (a + ix)(b + ix),\end{gathered} \tag{6.32}$$

with $\tilde{x} = i\frac{x}{2}$ and where a, b, c, d are given in terms of the α_i. From this, we recognize that the key family of OPs related to these S-Heun operators is the Continuous Hahn family.

6.4.2. Continuous Hahn polynomials and their structure relations

The following combinations of S-Heun operators

$$\tau = 2L, \tag{6.33a}$$

$$\tau^{(a,b,c,d)*} = \mu_1 L + \mu_2 M_1 + \mu_3 M_2 + \mu_4 R_1 + \mu_5 R_2, \tag{6.33b}$$

with

$$\begin{aligned}
\mu_1 &= \tfrac{1}{2}(1-(a+b+c+d))+(ab+cd),\\
\mu_2 &= \tfrac{1}{2}(a+b-c-d)-(ab-cd),\\
\mu_3 &= \tfrac{1}{2}(c+d-a-b),\\
\mu_4 &= -\tfrac{1}{4},\\
\mu_5 &= \tfrac{1}{2}(a+b+c+d)-\tfrac{3}{4}
\end{aligned} \tag{6.33c}$$

turn out to be the forward and backward operators, while

$$\mu^{(a,b,c,d)} = (d-a)L+(a+d-1)M_1+M_2, \tag{6.33d}$$

$$\mu^{(a,b,c,d)^*} = (c-b)L+(b+c-1)M_1+M_2, \tag{6.33e}$$

will act on polynomials as the contiguity relations. Indeed, these operators have the following actions on the Continuous Hahn polynomials:

$$\tau\ P_n\left(i\tfrac{x}{2},a,b,c,d\right) = i(n+a+b+c+d-1)P_{n-1}\left(i\tfrac{x}{2},a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right), \tag{6.34a}$$

$$\tau^{(a,b,c,d)^*}\ P_n\left(i\tfrac{x}{2},a,b,c,d\right) = -i(n+1)P_{n+1}\left(i\tfrac{x}{2},a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right), \tag{6.34b}$$

$$\mu^{(a,b,c,d)}\ P_n\left(i\tfrac{x}{2},a,b,c,d\right) = (n+a+d-1)P_n\left(i\tfrac{x}{2},a-\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d-\tfrac{1}{2}\right), \tag{6.34c}$$

$$\mu^{(a,b,c,d)^*}\ P_n\left(i\tfrac{x}{2},a,b,c,d\right) = (n+b+c-1)P_n\left(i\tfrac{x}{2},a+\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d+\tfrac{1}{2}\right). \tag{6.34d}$$

The 4 operators $\mu^{(a,b,c,d)}$, $\mu^{(a,b,c,d)^*}$, τ, $\tau^{(a,b,c,d)^*}$ have been studied by Kalnins and Miller in [**30**].

We now indicate how the two bispectral operators are formed from the S-Heun operators. As mentioned above, the Continuous Hahn difference operator can be formed by a quadratic combination of the stabilizing generators. Moreover, we can provide factorizations of this operator, either as a product of two contiguous operators or as the product of the backward and forward operators:

$$\begin{aligned}
\mathcal{D} &= \mu^{(a+\frac{1}{2},b-\frac{1}{2},c-\frac{1}{2},d+\frac{1}{2})}\mu^{(a,b,c,d)^*}-(a+d)(b+c-1)\\
&= \mu^{(a-\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d-\frac{1}{2})^*}\mu^{(a,b,c,d)}-(a+d-1)(b+c)\\
&= \tau^{(a+\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d+\frac{1}{2})^*}\tau\\
&= \tau\ \tau^{(a,b,c,d)^*}+2-(a+b+c+d).
\end{aligned} \tag{6.35}$$

The remaining bispectral operator X is the multiplication by the variable x in this basis: $Xf(x) = xf(x)$. It appears as a quadratic combination in the S-Heun operators

$$X = [M_2, R_2]. \tag{6.36}$$

The framework of S-Heun operators presented here is thus seen to unite the symmetry techniques of Kalnins and Miller and the approach based on the bispectral operators (see [**18**] for more general context).

6.4.3. The Sklyanin-like algebra realized by the structure operators

Let us now look at the algebraic relations obeyed by these operators. On the one hand, the pair of bispectral Continuous Hahn operators realizes the Hahn algebra [**59**]. On the other hand, the algebra formed by the 4 linear operators $\mu^{(a,b,c,d)}$, $\mu^{(a,b,c,d)*}$, τ, $\tau^{(a,b,c,d)*}$ can be seen as a degeneration of the Sklyanin algebra.

This algebra can be presented as follows. Write $\nu = -\frac{1}{2}(a+b+c+d)$ and take

$$\begin{aligned} A &= 2(\nu+1)M_1 - 2M_2, \\ B &= \tfrac{1}{2}(2\nu+1)(2\nu+3)L - R_1 - (4\nu+3)R_2, \\ C &= L, \\ D &= M_1. \end{aligned} \tag{6.37}$$

These are linear combinations of $\mu^{(a,b,c,d)}$, $\mu^{(a,b,c,d)*}$, τ, $\tau^{(a,b,c,d)*}$ that have been chosen in order to simplify the relations.

Proposition 6.4.1. The elements A, B, C, D obey the quadratic relations

$$[C, D] = 0, \qquad [A, C] = \{C, D\}, \qquad [A, D] = \{C, C\}, \tag{6.38a}$$

$$[B, C] = \{D, A\}, \qquad [B, D] = \{C, A\}, \qquad [B, A] = \{B, D\}. \tag{6.38b}$$

We shall refer to these relations as those of the Skl_4 algebra.

The two quadratic Casimir elements are

$$\Omega_1 = D^2 - C^2, \qquad \Omega_2 = A^2 + D^2 - \{B, C\} \tag{6.39}$$

and they take the following values in the realization:

$$\Omega_1 = 1, \qquad \Omega_2 = (2\nu+3)^2. \tag{6.40}$$

Remark 6.4.1. The stabilizing subalgebra of Skl_4 (6.38a), which we shall denote by Skl_3, has been identified in [**29**] as the algebra $T_7|_{(a,b)=(0,0)}$ whose relations are isomorphic to

$$[x,y]=z^2, \qquad [y,z]=0, \qquad [x,z]=zy. \tag{6.41}$$

It enjoys nice properties such as being Koszul, PBW, and being derived from a twisted potential. That the above algebra is Skl_3 is seen by setting $x=\frac{1}{2}A$, $y=D$, $z=C$.

We now explain that Skl_4 is a degeneration of the Sklyanin algebra. We rewrite the $\tau^{(a,b,c,d)^*}$ in terms of A, B, C, D, using $e_1=a+b+c+d$:

$$\begin{aligned}\tau^{(a,b,c,d)^*}=\tfrac{1}{4}(a+b-c-d)A+\tfrac{1}{4}B+\left[\tfrac{1}{8}(1-e_1)(1+e_1)+ab+cd\right]C&\\ +\left[\tfrac{1}{4}e_1(a+b-c-d)-ab+cd\right]D.&\end{aligned} \tag{6.42}$$

Two analogs of an identity due to Rains [**45**] can be obtained for $\tau^{(a,b,c,d)^*}$. These are the quasi-commutation relations:

$$\tau^{(a+e,b,c,d-e)^*}\tau^{(a-\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d-\frac{1}{2})^*}=\tau^{(a,b,c,d)^*}\tau^{(a-\frac{1}{2}+e,b+\frac{1}{2},c+\frac{1}{2},d-\frac{1}{2}-e)^*}, \tag{6.43}$$

$$\tau^{(a,b+e,c-e,d)^*}\tau^{(a+\frac{1}{2},b-\frac{1}{2},c-\frac{1}{2},d+\frac{1}{2})^*}=\tau^{(a,b,c,d)^*}\tau^{(a+\frac{1}{2},b-\frac{1}{2}+e,c-\frac{1}{2}-e,d+\frac{1}{2})^*}. \tag{6.44}$$

Proposition 6.4.2. Either of the quasi-commutation relation (6.43), (6.44) repackages the relations (6.38) of the Skl_4 algebra.

Proof: Substituting the relation (6.42) into (6.43) and bringing all terms to the rhs, one obtains ($u=b-c$, $v=a-b-c+d$):

$$\begin{aligned}0=\tfrac{e}{4}\Big\{&\tfrac{1}{2}(AB-BA)+u(CB-BC)+\tfrac{1}{2}\left[(2-v)BD+vDB\right]\\ &+u\left[(2-v)AD+vDA-2(1-v)C^2\right]-\tfrac{1}{4}\left[(v^2+4u^2-4v+3)AC-(v^2+4u^2-1)CA\right],\\ &+\tfrac{1}{4}\left[v^3-4u^2v+8u^2-2v^2-v+2\right]CD-\tfrac{1}{4}\left[v^3-4u^2v-4v^2+3v\right]DC\Big\}.\end{aligned} \tag{6.45}$$

The dependence on the free parameter e factors out. Taking $v\to\infty$, one obtains immediately that

$$CD-DC=0. \tag{6.46}$$

Also, taking $u\to 0$ and $v\to 0$, one gets

$$AB-BA=-2BD+\tfrac{3}{2}AC+\tfrac{1}{2}CA-CD. \tag{6.47}$$

Substituting these relations back in (6.45) leads to

$$0 = \tfrac{e}{4}\Big\{u(CB-BC) + \tfrac{v}{2}\left[DB-BD\right] + u\left[(2-v)AD + vDA - 2(1-v)C^2\right] \\ - \tfrac{1}{4}\left[(v^2+4u^2-4v)AC - (v^2+4u^2)CA\right] + \tfrac{1}{4}\left[8u^2+2v^2-4v\right]CD\Big\}. \tag{6.48}$$

Repeating a similar process, the remaining relations of (6.38) are found. A similar derivation starting from (6.44) instead yields the same relations. □

6.4.4. Finite-dimensional representations

It is known that finite-dimensional representations of the Hahn algebra relate to the Hahn polynomials [**22**]. We now wish to obtain finite-dimensional representations of the Skl_4 algebra; looking at (6.37), it is seen that one needs ν to be either an integer or half-integer. It will be shown that this corresponds in fact to a truncation of the Jacobi matrix of the Continuous Hahn polynomials.

Let us write the condition (ν is either an integer or half-integer) as

$$1-(a+b+c+d) = N, \tag{6.49}$$

where N is a positive integer that corresponds to the maximal degree of the truncated family of polynomials.

This truncation condition is known [**41**] to be the one that takes the Wilson polynomials to the para-Racah polynomials. In the present case, we start from the Continuous Hahn OPs so the result of the truncation leads to a different family of para-polynomials.

Proposition 6.4.3. The polynomials that arise from the truncation condition (6.49) form a basis that supports $(N+1)$-dimensional representations of the degenerate Sklyanin algebra Skl_4 and are identified as the para-Krawtchouk polynomials [**58**].

We indicate below how the recurrence relation of the para-Krawtchouk polynomials is obtained from that of the Continuous Hahn polynomials by imposing (6.49).

$N = 2j + 1$ **odd**

In the case where $N = 2j + 1$ is odd (j is a non-negative integer), we parametrize the truncation condition as follows

$$c = -a - j + e_1 t, \qquad b = -d - j + e_2 t \tag{6.50}$$

and then take the limit $t \to 0$. We shall choose $e_1 = e_2$: this will lead to simpler expressions. The more general solutions corresponding to $e_1 \neq e_2$ can be recovered from the simpler solutions by the procedure of isospectral deformations, see for instance [**40**]. Using the chosen parametrization, the recurrence coefficients A_n, C_n appearing in the recurrence relation of the Continuous Hahn polynomials

$$\begin{aligned} (a + ix)P_n(x; a, b, c, d) = A_n P_{n+1}(x; a, b, c, d) + C_n P_{n-1}(x; a, b, c, d) \\ - (A_n + C_n)P_n(x; a, b, c, d), \end{aligned} \tag{6.51}$$

$$P_n(x; a, b, c, d) = \frac{n!}{i^n (a + c)_n (a + d)_n} p_n(x; a, b, c, d)$$

become in the limit $t \to 0$:

$$A_n = -\frac{(n - N)(n + a + d)}{2(2n - N)}, \tag{6.52a}$$

$$C_n = +\frac{n(n - N - a - d)}{2(2n - N)}. \tag{6.52b}$$

Now take γ to be

$$\gamma = (b + c) - (a + d), \tag{6.53}$$

it follows that (6.52) can be rewritten in view of (6.49) as

$$A_n = -\frac{1}{2}\frac{(N - n)(N - 1 - 2n + \gamma)}{2(2n - N)}, \tag{6.54a}$$

$$C_n = -\frac{1}{2}\frac{n(N + 1 - 2n - \gamma)}{2(2n - N)}. \tag{6.54b}$$

These are recognized as the recurrence coefficients of the para-Krawtchouk polynomials in the variable $-\frac{x}{2}$ introduced in [**58**]. These polynomials are defined on the union of two linear lattices and the parameter γ describes the displacement of one lattice with respect to the other.

$N = 2j$ even

In the case where $N = 2j$ is even, we use the parametrization

$$c = -a - j + e_1 t, \qquad b = -d - j + e_1 t + 1 \tag{6.55}$$

and then take the limit $t \to 0$. The recurrence coefficients in the recurrence relation of the Continuous Hahn polynomials become

$$A_n = -\frac{(n-N)(n+a+d)}{2(2n-N+1)}, \tag{6.56a}$$

$$C_n = +\frac{n(n-N-a-d)}{2(2n-N-1)}, \tag{6.56b}$$

and upon writing

$$\gamma = 1 + (b+c) - (a+d), \tag{6.57}$$

we obtain

$$A_n = -\frac{1}{2}\frac{(N-n)(N-2-2n+\gamma)}{2(2n-N+1)}, \tag{6.58a}$$

$$C_n = -\frac{1}{2}\frac{n(N+2-2n-\gamma)}{2(2n-N-1)}. \tag{6.58b}$$

These are the recurrence coefficients of the para-Krawtchouk polynomials in the variable $-\frac{x}{2}$. The expressions for the monic polynomials are given in [**41**].

A remark on the truncation condition

It can be checked that in the realization (6.37), applying the truncation condition (6.49) seems to suggest that the raising operator B annihilates the monomial x^{N+1} and not x^N. A priori, this means that the truncation condition amounts to looking at $(N+2)$-dimensional representations of the algebra Skl_4, which would seem to contradict the fact that the para-Krawtchouk polynomials were truncated to have degrees at most N (and thus to span a space of dimension $N+1$).

Looking at the situation more closely, one observes that B indeed maps para-Krawtchouk polynomial of maximal degree N to a certain polynomial of degree $N+1$. But this polynomial of degree $N+1$ corresponds to the characteristic polynomial of the (upper block of the) truncated Jacobi matrix, hence it is null on the orthogonality grid points. Keeping in mind that the para-Krawtchouk polynomials are the basis vectors for the finite-dimensional representation of Skl_4, this characteristic polynomial thus corresponds to a null vector. Therefore the dimension of the space on which the representation of the Skl_4 algebra acts is indeed $N+1$.

6.4.5. Recovering the associated Heun operator

The Heun operator associated to the Continuous Hahn polynomials was implicitly defined in [**59**]. This operator W_{CH} is the most general second order operator that acts on the discrete linear grid and maps polynomials of degree n into polynomials of degree $n+1$. It can be expressed as

$$W_{CH} = \mathcal{A}_1 T_+ + \mathcal{A}_0 \mathcal{I} + \mathcal{A}_2 T_-, \tag{6.59}$$

where $\mathcal{A}_{1,2}$ are general polynomials of degree 3 with the same leading order coefficient, and $\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 = \pi_1(x)$, with $\pi_1(x)$ a general polynomial of degree 1.

We now consider the most general quadratic combination of S-Heun operators that does not raise the degree of polynomials by more than one. Upon using the quadratic homogeneous relations of Appendix 6.7, this general combination can be brought into the form

$$W = \alpha_1 L^2 + \alpha_2 LM_1 + \alpha_3 LM_2 + \alpha_4 {M_1}^2 + \alpha_5 M_1 M_2 + \alpha_6 {M_2}^2 + \beta_1 M_1 R_2 + \beta_2 M_2 R_1 + \beta_3 M_2 R_2. \tag{6.60}$$

Substituting the expressions of the S-Heun basis operators (6.8), we obtain

$$\begin{aligned}
W &= \mathcal{A}_1 T_+^2 + \mathcal{A}_0 \mathcal{I} + \mathcal{A}_2 T_-^2, \\
\mathcal{A}_1 &= \tfrac{1}{4}[-2\beta_2 x^3 + (\alpha_6 - 3\beta_2 + \beta_3)x^2 + (\alpha_3 + \alpha_5 + \alpha_6 + \beta_1 - \beta_2 + \beta_3)x \\
&\qquad\qquad\qquad\qquad + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \beta_1)], \\
\mathcal{A}_2 &= \tfrac{1}{4}[-2\beta_2 x^3 + (\alpha_6 + 3\beta_2 - \beta_3)x^2 + (\alpha_3 - \alpha_5 - \alpha_6 + \beta_1 - \beta_2 + \beta_3)x \\
&\qquad\qquad\qquad\qquad + (\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 + \alpha_5 - \beta_1)], \\
\mathcal{A}_0 &= (\beta_1 + \beta_2 + \beta_3)x + \alpha_4 - (\mathcal{A}_1 + \mathcal{A}_2).
\end{aligned} \tag{6.61}$$

Proposition 6.4.4. The generic Heun-Continuous Hahn operator (6.59) can be obtained as the most general quadratic combination in the S-Heun generators (6.8) that does not raise the degree of polynomials by more than one.

Using the relations of Appendix 6.7, one can see that the Heun operator generically factorizes as the product of a general S-Heun operator with a stabilizing S-Heun operator:

$$W = (\xi_1 L + \xi_2 M_1 + \xi_3 M_2)(\eta_1 L + \eta_2 M_1 + \eta_3 M_2 + \eta_4 R_1 + \eta_5 R_2) + \kappa. \tag{6.62}$$

6.5. The case of the q-linear grid

We consider now the S-Heun operators associated to the q-linear (or exponential) grid.

6.5.1. The stabilizing subspace

The stabilizing subset of S-Heun operators is $\{\hat{L}, \hat{M}_1, \hat{M}_2\}$. Using the relations of Appendix 6.7, it is always possible to reduce the most general quadratic combination of these operators to

$$\hat{Q} = \alpha_1 \hat{L}^2 + \alpha_2 \hat{L}\hat{M}_1 + \alpha_3 \hat{L}\hat{M}_2 + \alpha_4 \hat{M}_1^2 + \alpha_5 \hat{M}_1 \hat{M}_2 + \alpha_6 \hat{M}_2^2. \tag{6.63}$$

Substituting the expressions (6.14), one recognizes $\hat{Q}$ as a second-order q-difference operator whose eigenvalue problem can be cast as the difference equation

$$\begin{gathered}
\hat{\mathcal{D}} P_n(z;\alpha,\beta,\gamma;\tilde{q}) = (\tilde{q}^{-n} - 1)(1 - \alpha\beta\tilde{q}^{n+1}) P_n(z;\alpha,\beta,\gamma;\tilde{q}), \\
\hat{\mathcal{D}} = B(z)\hat{T}_+^2 - [B(z) + D(z)]\mathcal{I} + D(z)\hat{T}_-^2, \\
B(z) = \frac{\alpha\tilde{q}(z-1)(\beta z - \gamma)}{z^2}, \qquad D(z) = \frac{(z - \alpha\tilde{q})(z - \gamma\tilde{q})}{z^2}
\end{gathered} \tag{6.64}$$

of the Big q-Jacobi polynomials [**32**] in base $\tilde{q} = q^2$, making those the OPs associated to S-Heun operators on the exponential lattice. We note that there is a duality between the Continuous Dual q-Hahn and the Big q-Jacobi polynomials [**36**] that can be pictured as follows: exchanging the degree with the variable in some way takes one family of polynomials into the other (with transformed parameters). Thus, if we were to write the S-Heun operators (6.14) by replacing the variable with the degree in the appropriate way, the Continuous Dual q-Hahn polynomials would arise instead.

6.5.2. Big q-Jacobi polynomials and their structure relations

Focusing on the structure and contiguity relations of the Big q-Jacobi polynomials, we shall show how the set of S-Heun operators spans a space that contains the relevant operators.

Let

$$\hat{\tau} = (q - q^{-1})\hat{L}, \tag{6.65a}$$

$$\hat{\tau}^{(a,b,c,d)*} = \mu_1 \hat{L} + \mu_2 \hat{M}_1 + \mu_3 \hat{M}_2 + \mu_4 \hat{R}_1 + \mu_5 \hat{R}_2, \tag{6.65b}$$

with

$$\begin{aligned}
\mu_1 &= -(q - q^{-1}),\\
\mu_2 &= (a+b)q^{-1} - q(c^{-1} + d^{-1}),\\
\mu_3 &= (a+b) - (c^{-1} + d^{-1}),\\
\mu_4 &= -abq^{-2} + q^2 c^{-1} d^{-1},\\
\mu_5 &= -abq^{-1} + qc^{-1}d^{-1},
\end{aligned} \tag{6.65c}$$

and

$$\hat{\mu}^{(a,b,c,d)} = (q - q^{-1})L - (aq^{-1} - qd^{-1})M_1 - (a - d^{-1})M_2, \tag{6.65d}$$

$$\hat{\mu}^{(a,b,c,d)*} = (q - q^{-1})L - (bq^{-1} - qc^{-1})M_1 - (b - c^{-1})M_2. \tag{6.65e}$$

The actions of these operators on the Big q-Jacobi polynomials $P_n(z;\alpha,\beta,\gamma;q^2)$ is best presented as follows. Let

$$\Phi_n^{(a,b,c,d)}(z;\tilde{q}) = P_n(az; ac\tilde{q}^{-1}, bd\tilde{q}^{-1}, ad\tilde{q}^{-1}; \tilde{q}). \tag{6.66}$$

It is clear that the parameter a is redundant. One has $\Phi_n^{(1,\beta/\gamma,\alpha\tilde{q},\gamma\tilde{q})}(z;\tilde{q}) = P_n(z;\alpha,\beta,\gamma;\tilde{q})$. It is seen that

$$\hat{\tau}\ \Phi_n^{(a,b,c,d)}(z;\tilde{q}) = \frac{aq(1-q^{-2n})(1-abcdq^{2n-2})}{(1-ad)(1-ac)} \Phi_{n-1}^{(aq,bq,cq,dq)}(z;\tilde{q}), \tag{6.67a}$$

$$\hat{\tau}^{(a,b,c,d)*}\ \Phi_n^{(a,b,c,d)}(z;\tilde{q}) = \frac{(ac-q^2)(ad-q^2)}{acdq} \Phi_{n+1}^{(aq^{-1},bq^{-1},cq^{-1},dq^{-1})}(z;\tilde{q}), \tag{6.67b}$$

$$\hat{\mu}^{(a,b,c,d)}\ \Phi_n^{(a,b,c,d)}(z;\tilde{q}) = \frac{q}{d}(1-adq^{-2}) \Phi_n^{(aq^{-1},bq,cq,dq^{-1})}(z;\tilde{q}), \tag{6.67c}$$

$$\hat{\mu}^{(a,b,c,d)*}\ \Phi_n^{(a,b,c,d)}(z;\tilde{q}) = -\frac{q(ad-q^{-2n})(1-bcq^{2n-2})}{c(1-ad)} \Phi_n^{(aq,bq^{-1},cq^{-1},dq)}(z;\tilde{q}). \tag{6.67d}$$

The 4 operators $\hat{\mu}^{(a,b,c,d)}$, $\hat{\mu}^{(a,b,c,d)*}$, $\hat{\tau}$, $\hat{\tau}^{(a,b,c,d)*}$ built from linear combinations of S-Heun operators have been studied by Kalnins and Miller in [**30**].

Let us further indicate how the bispectral operators show up in this context. As mentioned above, the Big q-Jacobi difference operator appears as a quadratic combination of the stabilizing generators. Moreover, one can actually provide factorizations of this operator in terms of contiguity operators as well as backward and forward operators:

$$
\begin{aligned}
\hat{\mathcal{D}} &= \alpha\gamma q^3 \mu^{(q,\frac{\beta}{\gamma q},\alpha q,\gamma q^3)}\mu^{(1,\frac{\beta}{\gamma},\alpha q^2,\gamma q^2)^*} - (1-\gamma q^2)(1-\tfrac{\alpha\beta}{\gamma}) \\
&= \alpha\gamma q^3 \mu^{(q^{-1},\frac{\beta q}{\gamma},\alpha q^3,\gamma q)^*}\mu^{(1,\frac{\beta}{\gamma},\alpha q^2,\gamma q^2)} - (1-\gamma)(1-\tfrac{\alpha\beta q^2}{\gamma}) \\
&= -\alpha\gamma q^3 \hat{\tau}^{(q,\frac{\beta q}{\gamma},\alpha q^3,\gamma q^3)^*}\tau \\
&= -\alpha\gamma q^3 \hat{\tau}\ \hat{\tau}^{(1,\frac{\beta}{\gamma},\alpha q^2,\beta q^2)^*} - (1-q^2)(1-\alpha\beta).
\end{aligned}
\tag{6.68}
$$

The second bispectral operator $\hat{X}$ is the multiplication by the variable z: $\hat{X}f(z) = zf(z)$. It also appears as the quadratic combination of S-Heun operators:

$$
\hat{X} = \hat{M}_2\hat{R}_1 - \hat{M}_1\hat{R}_2. \tag{6.69}
$$

The S-Heun operators thus underscore much of the characterization of the Big q-Jacobi operators.

6.5.3. The Sklyanin-type algebra realized by the structure operators

The pair of bispectral Big q-Jacobi operators is known to realize the Big q-Jacobi algebra [**54, 6**]. The algebra generated by the 4 linear operators $\hat{\mu}^{(a,b,c,d)}$, $\hat{\mu}^{(a,b,c,d)^*}$, $\hat{\tau}$, $\hat{\tau}^{(a,b,c,d)^*}$ is a familiar degeneration of the Sklyanin algebra [**46**].

Denote $q^{-\nu} = (abcd)^{\frac{1}{4}}$ and form

$$
\begin{aligned}
\hat{A} &= q^{-\nu}(\hat{M}_1 + q\hat{M}_2), \\
\hat{B} &= \frac{1}{2(q-q^{-1})}[q^{2\nu}(\hat{R}_1 + q^{-1}\hat{R}_2) - q^{-2\nu}(\hat{R}_1 + q\hat{R}_2)], \\
\hat{C} &= 2\hat{L}, \\
\hat{D} &= q^{\nu}(\hat{M}_1 + q^{-1}\hat{M}_2).
\end{aligned}
\tag{6.70}
$$

Proposition 6.5.1. The operators $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ obey the quadratic relations

$$
\begin{gathered}
\hat{A}\hat{B} = q\hat{B}\hat{A}, \qquad \hat{B}\hat{D} = q\hat{D}\hat{B}, \qquad \hat{C}\hat{A} = q\hat{A}\hat{C}, \qquad \hat{D}\hat{C} = q\hat{C}\hat{D}, \\
[\hat{B},\hat{C}] = \frac{\hat{A}^2 - \hat{D}^2}{q-q^{-1}}, \qquad [\hat{A},\hat{D}] = 0
\end{gathered}
\tag{6.71a}
$$

along with the additional relation

$$\hat{A}\hat{D} = \hat{D}\hat{A} = 1 \tag{6.71b}$$

which define $U_q(\mathfrak{sl}_2)$.

When ν is an integer or a half-integer, one obtains finite-dimensional representations of $U_q(\mathfrak{sl}_2)$ of dimension $2\nu+1$. In that case, the maximal degree of the polynomials obtained from the action of the raising operator $\hat{B}$ is N.

Remark 6.5.1. The $q \to 1$ limit of this realization yields the $\mathfrak{sl}_2$ commutation relations. In fact (6.70) tends to the differential Bargmann realization of $\mathfrak{sl}_2$. Under the limit, the q-linear grid becomes the continuum, and the above combinations of shift operators turn into differential operators.

Remark 6.5.2. The algebra (6.26) has been obtained in [**38**] as a so-called "homogenized $\mathfrak{sl}_2$ algebra" $H(\mathfrak{sl}_2)$. Many algebras of a similar type with 4 generators A, B, C, D, and D central, have been studied in [**39**]. A quantization of $H(\mathfrak{sl}_2)$ which is isomorphic to the algebra with relations (6.71a) and which can be seen as a homogenization of $U_q(\mathfrak{sl}_2)$ has been studied in [**10**].

6.5.4. Finite-dimensional representations

We now wish to obtain finite-dimensional representations of $U_q(\mathfrak{sl}_2)$ corresponding to a particular truncation of the Jacobi matrix of the Big q-Jacobi polynomials. As mentioned previously, this can be accomplished by taking ν to be either an integer or a half-integer. In order to do so, we are led to take [**42**]

$$\sqrt{abcd} = q^{1-N}, \tag{6.72}$$

where N is a positive integer that corresponds to the maximal degree of the truncated family of polynomials.

Proposition 6.5.2. The polynomials that arise from the truncation condition (6.72) form a basis that supports $(N+1)$-dimensional representations of $U_q(\mathfrak{sl}_2)$ in the realization (6.70). The q-para-Krawtchouk polynomials [**54**] are the ones that arise from this truncation condition.

We show below how their recurrence relation is obtained from the one of the Big q-Jacobi polynomials.

$N = 2j + 1$ **odd**

In the case where $N = 2j + 1$ is odd, we write

$$d = a^{-1} q^{-2j+e_1 t}, \qquad b = c^{-1} q^{-2j+e_1 t} \tag{6.73}$$

and then take the limit $t \to 0$. Using this parametrization, the recurrence relation of the Big q-Jacobi polynomials

$$zP_n(z; a, b, c; \tilde{q}) = A_n P_{n+1}(z; a, b, c; \tilde{q}) + C_n P_{n-1}(z; a, b, c; \tilde{q}) + [1 - (A_n + C_n)] P_n(z; a, b, c; \tilde{q}) \tag{6.74}$$

has for coefficients

$$A_n = + \frac{(1 - acq^{2n})(1 - q^{2n-2N})}{(1 + q^{2n-N+1})(1 - q^{4n-2N})}, \tag{6.75a}$$

$$C_n = - \frac{q^{2n-N-1}(1 - q^{2n})(ac - q^{2n-2N})}{(1 + q^{2n-N-1})(1 - q^{4n-2N})} \tag{6.75b}$$

after the use of (6.73) and the limit $t \to 0$. Now letting

$$ac = c_3 q^2 \tag{6.76}$$

it follows that (6.75) can be rewritten as

$$A_n = + \frac{(1 - c_3 q^{2n+2})(1 - q^{2n-2N})}{(1 + q^{2n-N+1})(1 - q^{4n-2N})}, \tag{6.77a}$$

$$C_n = - \frac{q^{2n-N+1}(1 - q^{2n})(c_3 - q^{2n-2N-2})}{(1 + q^{2n-N-1})(1 - q^{4n-2N})}, \tag{6.77b}$$

and one recognizes the recurrence coefficients of the q-para-Krawtchouk polynomials in the base $\tilde{q} = q^2$ introduced in [**54**] when N is odd. These polynomials are defined on the union of two q-linear lattices and the parameter c_3 describes the shift of one lattice with respect to the other.

$N = 2j$ **even**

In the case where $N = 2j$ is even, we take

$$d = a^{-1} q^{-2j+e_1 t}, \qquad b = c^{-1} q^{-2j+e_2 t+2} \tag{6.78}$$

which ensures (6.72) in the limit $t \to 0$. Using this parametrization and after letting $t \to 0$, the recurrence coefficients of the Big q-Jacobi polynomials become

$$A_n = +\frac{(1 - acq^{2n})(1 - q^{2n-2N})}{(1 + q^{2n-N})(1 - q^{4n-2N+2})}, \tag{6.79a}$$

$$C_n = -\frac{q^{2n-N-2}(1 - q^{2n})(ac - q^{2n-2N})}{(1 + q^{2n-N})(1 - q^{4n-2N-2})}, \tag{6.79b}$$

and upon letting

$$ac = c_3 q^2 \tag{6.80}$$

A_n and C_n can be rewritten as

$$A_n = +\frac{(1 - c_3 q^{2n+2})(1 - q^{2n-2N})}{(1 + q^{2n-N})(1 - q^{4n-2N+2})}, \tag{6.81a}$$

$$C_n = -\frac{q^{2n-N}(1 - q^{2n})(c_3 - q^{2n-2N-2})}{(1 + q^{2n-N})(1 - q^{4n-2N-2})}. \tag{6.81b}$$

These are the recurrence coefficients of the q-para-Krawtchouk polynomials in the base $\tilde{q} = q^2$ for N even. For more detail, see [**54**].

A remark on the truncation condition

There is once again an apparent mismatch in the dimensions of the representations of the algebra and those of the representation basis. The same remark as the one made in the preceding section applies here. It can be checked that in the realization (6.70), applying the truncation condition (6.72) seems to suggest that the raising operator $\hat{B}$ annihilates the monomial z^{N+1} and not z^N, which means that the truncation condition leads to representations of the algebra $U_q(\mathfrak{sl}_2)$ of dimension $N+2$. This would contradict the fact that the q-para-Krawtchouk polynomials were truncated to a maximal degree N (and thus span a space of dimension $N+1$).

It can be observed that $\hat{B}$ maps the q-para-Krawtchouk polynomial of degree N to a polynomial of degree $N+1$. The resulting polynomial is the characteristic polynomial of the (upper block of the) truncated Jacobi matrix, hence it is again null on the orthogonality grid points. In the representation basis with which we are working (i.e. where the q-para-Krawtchouk polynomials are the basis elements), this characteristic polynomial corresponds to a null vector. Hence, the dimension of the space on which the realization of the $U_q(\mathfrak{sl}_2)$ algebra acts is indeed $N+1$.

6.5.5. Recovering the related Heun operator

The Heun operator associated to the Big q-Jacobi polynomials is given in [**6**] and had also been introduced previously in [**49**]. This operator W_{BJ} is the most general second order q-difference operator that acts on the q-linear grid and maps polynomials of degree n into polynomials of degree $n+1$. Its expression is

$$W_{BJ} = \mathcal{A}_1\hat{T}_+ + \mathcal{A}_0\mathcal{I} + \mathcal{A}_2\hat{T}_-, \tag{6.82}$$

where

$$\mathcal{A}_1 = \frac{\pi_3(z)}{z^2}, \qquad \mathcal{A}_2 = \frac{\tilde{q}\pi_3(z) + z\pi_2(z)}{z^2} \tag{6.83}$$

and $\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 = \pi_1(z)$, with $\pi_k(z)$ a generic polynomial of degree k and $\tilde{q}$ the base.

Let us consider the most general quadratic combination of S-Heun operators that does not raise the degree of polynomials by more than one. Using the quadratic homogeneous relations of Appendix 6.7, we arrive at

$$W = \alpha_1\hat{L}^2 + \alpha_2\hat{L}\hat{M}_1 + \alpha_3\hat{L}\hat{M}_2 + \alpha_4\hat{M}_1^2 + \alpha_5\hat{M}_1\hat{M}_2 + \alpha_6\hat{M}_2^2 + \beta_1\hat{M}_1\hat{R}_2 + \beta_2\hat{M}_2\hat{R}_1 + \beta_3\hat{M}_2\hat{R}_2. \tag{6.84}$$

Substituting the expressions (6.14) for the generators we obtain

$$\begin{aligned}
&W = \mathcal{A}_1\hat{T}_+^2 + \mathcal{A}_0\mathcal{I} + \mathcal{A}_2\hat{T}_-^2,\\
&\mathcal{A}_1 = \frac{1}{z^2(1-q^2)^2}[(q\alpha_1) + (q^2\alpha_3 - q\alpha_2)z + (q^2\alpha_6 - q\alpha_5 + \alpha_4)z^2 + (q^3\beta_3 - q^2\beta_1 - q^2\beta_2)z^3],\\
&\mathcal{A}_2 = \frac{1}{z^2(1-q^2)^2}[(q^3\alpha_1) + (q^2\alpha_3 - q^3\alpha_2)z + (q^2\alpha_6 - q^3\alpha_5 + q^4\alpha_4)z^2 + (q\beta_3 - q^2\beta_1 - q^2\beta_2)z^3],\\
&\mathcal{A}_0 = \beta_2 z + \alpha_4 - (\mathcal{A}_1 + \mathcal{A}_2).
\end{aligned} \tag{6.85}$$

Proposition 6.5.3. The generic Heun-Big q-Jacobi operator (6.82) (with base q^2) can be obtained as the most general quadratic combination in the S-Heun generators (6.14) that does not raise the degree of polynomials by more than one.

Moreover, using the relations of Appendix 6.7, we see that the Heun operator typically factorizes as the product of a raising S-Heun operator with a stabilizing S-Heun operator:

$$\hat{W} = (\xi_1\hat{L} + \xi_2\hat{M}_1 + \xi_3\hat{M}_2)(\eta_1\hat{L} + \eta_2\hat{M}_1 + \eta_3\hat{M}_2 + \eta_4\hat{R}_1 + \eta_5\hat{R}_2) + \kappa. \tag{6.86}$$

6.6. Connections between the different cases

It is well known that the three grids on which we have defined S-Heun operators can be obtained as limiting cases or contractions of the Askey–Wilson grid. We now observe that this translates into limits/contractions of the associated Sklyanin algebras.

Let us denote the points of the Askey–Wilson grid by

$$\lambda_s = z_s + z_s^{-1}, \qquad z_s = q^s. \tag{6.87}$$

The associated Sklyanin algebra was introduced in [**20**] as the trigonometric degeneration of the Sklyanin algebra [**46**] and was studied from the perspective of S-Heun operators in [**15**]. The defining relations read

$$\begin{gathered}\mathbf{DC} = q\mathbf{CD}, \qquad \mathbf{CA} = q\mathbf{AC}, \qquad [\mathbf{A},\mathbf{D}] = \frac{(q-q^{-1})^3}{4}\mathbf{C}^2,\\ [\mathbf{B},\mathbf{C}] = \frac{\mathbf{A}^2-\mathbf{D}^2}{q-q^{-1}},\\ \mathbf{AB} - q\mathbf{BA} = q\mathbf{DB} - \mathbf{BD} = -\frac{q^2-q^{-2}}{4}(\mathbf{DC}-\mathbf{CA}).\end{gathered} \tag{6.88}$$

The q-linear (or exponential) grid

$$\lambda_s = z_s, \qquad z_s = q^s \tag{6.89}$$

is obtained from the Askey–Wilson one in the asymptotic expansion $z_s \to \infty$ and the same limit takes the Askey–Wilson polynomials into the Big q-Jacobi OPs. At the level of the algebras, this corresponds to the following contraction. Writing

$$\mathbf{A} = \epsilon\hat{A}, \qquad \mathbf{B} = \hat{B}, \qquad \mathbf{C} = \epsilon^2\hat{C}, \qquad \mathbf{D} = \epsilon\hat{D} \tag{6.90}$$

and taking $\epsilon \to 0$, one recovers $U_q(\mathfrak{sl}_2)$:

$$\begin{gathered}\hat{A}\hat{B} = q\hat{B}\hat{A}, \qquad \hat{B}\hat{D} = q\hat{D}\hat{B}, \qquad \hat{C}\hat{A} = q\hat{A}\hat{C}, \qquad \hat{D}\hat{C} = q\hat{C}\hat{D},\\ [\hat{B},\hat{C}] = \frac{\hat{A}^2-\hat{D}^2}{q-q^{-1}}, \qquad [\hat{A},\hat{D}] = 0.\end{gathered} \tag{6.91}$$

We now compare the discrete linear grid to the continuum. A rescaling similar to the one discussed above takes this grid to the real line. This also takes the Continuous Hahn polynomials into the Jacobi ones. From the perspective of the algebras, (6.90) will relate one algebra to the other. The Sklyanin algebra (6.38) associated to the discrete grid is

$$\begin{gathered}[C,D]=0, \qquad [A,C]=\{C,D\}, \qquad [A,D]=\{C,C\}, \\ [B,C]=\{D,A\}, \qquad [B,D]=\{C,A\}, \qquad [B,A]=\{B,D\}\end{gathered} \tag{6.92}$$

and upon writing

$$A=\epsilon\bar{A}, \qquad B=\bar{B}, \qquad C=\epsilon^2\bar{C}, \qquad D=\epsilon\bar{D} \tag{6.93}$$

and taking $\epsilon\to 0$, we recover

$$\begin{gathered}[\bar{C},\bar{D}]=0, \qquad [\bar{A},\bar{C}]=-\bar{C}\bar{D}, \qquad [\bar{A},\bar{D}]=0, \\ [\bar{B},\bar{C}]=-2\bar{A}\bar{D}, \qquad [\bar{A},\bar{B}]=\bar{B}\bar{D}, \qquad [\bar{B},\bar{D}]=0.\end{gathered} \tag{6.94}$$

We recall that the latter algebra is essentially the $\mathfrak{sl}_2$ Lie algebra with a central element D.

We have so far discussed the following contractions, denoted by full arrows:

$$\begin{tikzcd} \boxed{\text{AW grid}} \arrow[r, dotted, "?"] \arrow[d] & \boxed{\text{discrete linear grid}} \arrow[d] \\ \boxed{q\text{-linear grid}} \arrow[r, dotted] & \boxed{\text{continuum}} \end{tikzcd}$$

One could wonder if it is possible to complete the diagram with the dotted arrows. The bottom arrow is easy to add: this amounts to taking the limit $q\to 1$. This limit takes the q-linear grid to the continuum, the Big q-Jacobi polynomials to the Jacobi polynomials, and at the level of the algebra, it takes $U_q(\mathfrak{sl}_2)$ to $\mathfrak{sl}_2$.

The details corresponding to the upper arrow remain to be worked out. It is likely that an intermediary step related to the quadratic grid $\lambda_s=s^2$ should be required. Indeed, it is known that the $q\to 1$ limit of the Askey–Wilson grid leads to the quadratic grid. It should thus be possible to apply the S-Heun construction to the quadratic grid; the related polynomials should be those of Wilson, and the related Sklyanin algebra would stand in between the one of Askey–Wilson type (6.88) and the one of the discrete linear type (6.38).

6.7. Conclusion

The results of this paper are summarized as follows. We have introduced S-Heun operators on linear and q-linear grids. These operators are special cases of second order Heun operators with no diagonal term. On the real line and the discrete and q-linear grids, the sets of five S-Heun operators were constructed and shown to be related to the Jacobi, Continuous Hahn and Big q-Jacobi polynomials respectively. These S-Heun operators were also shown to encompass the bispectral and structure operators for each family of orthogonal polynomials. A presentation of the relations for the four structure operators of Kalnins and Miller was given in each case and identified as realizing degenerations, contractions or limits of the Sklyanin algebra. For the discrete and q-linear grids, the finite-dimensional representations of the Sklyanin-type algebras were obtained from a truncation condition on the Jacobi matrix of the associated polynomials; this yielded the para-Krawtchouk and q-para-Krawtchouk polynomials as bases of the finite representations and provided algebraic interpretations of these sets of OPs that had so far been missing.

The Sklyanin-like algebra related to the discrete linear grid (6.38) has a simple presentaton and a detailed study of its representation theory would be interesting. It would also be instructive to examine the types of Sklyanin algebra that the S-Heun operators on the quadratic grid would lead to. We plan on undertaking this in the near future. Note that we have restricted ourselves to Heun operators defined by actions on polynomials. The exploration of the generalizations that result from the extension to spaces of rational functions have been initiated in [**55**] and should be actively pursued in the S-Heun framework in particular.

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The homogeneous quadratic algebraic relations

The 14 quadratic homogeneous relations associated to all three sets of 5 S-Heun operators are collected here. One notes that all three sets of relations display a similar structure. These relations can be thought of as reordering relations and are especially useful when considering the most general quadratic combinations in the generators.

The continuum

The relations between the S-Heun operators $\bar{L}$, $\bar{M}_1$, $\bar{M}_2$, $\bar{R}_1$, $\bar{R}_2$ defined in (6.17) can be presented as the fourteen following relations:

$$
\begin{aligned}
\bar{M}_1\bar{L} &= \bar{L}\bar{M}_1, & \bar{L}\bar{R}_1 &= 1 + \bar{M}_1\bar{M}_2, & \bar{R}_1\bar{M}_1 &= \bar{M}_2\bar{R}_1 - \bar{M}_1\bar{R}_2,\\
\bar{M}_2\bar{L} &= \bar{L}\bar{M}_2 - \bar{M}_1\bar{L}, & \bar{L}\bar{R}_2 &= \bar{M}_2^2 + \bar{M}_1\bar{M}_2, & \bar{R}_2\bar{M}_1 &= \bar{M}_1\bar{R}_2,\\
\bar{M}_2\bar{M}_1 &= \bar{M}_1\bar{M}_2, & \bar{R}_1\bar{L} &= \bar{M}_1\bar{M}_2, & \bar{R}_1\bar{M}_2 &= \bar{M}_1\bar{R}_2,\\
\bar{M}_1^2 &= 1, & \bar{R}_2\bar{L} &= \bar{M}_2^2 - \bar{M}_1\bar{M}_2, & \bar{R}_2\bar{M}_2 &= \bar{M}_2\bar{R}_2 - \bar{M}_1\bar{R}_2,\\
 & & \bar{R}_2\bar{R}_1 &= \bar{R}_1\bar{R}_2 + \bar{R}_1^2, & \bar{M}_1\bar{R}_1 &= \bar{M}_2\bar{R}_1 - \bar{M}_1\bar{R}_2.
\end{aligned}
\tag{6.95}
$$

The discrete linear grid

Here are the relations between the S-Heun operators L, M_1, M_2, R_1, R_2 that have been defined in (6.8):

$$
\begin{aligned}
M_1L &= LM_1, & & \\
M_2L &= LM_2 - LM_1, & R_2R_1 &= 2{R_2}^2 + R_1R_2 - 4{M_2}^2,\\
M_2M_1 &= M_1M_2 - L^2, & R_1M_1 &= 3M_1R_2 - 2M_2R_2 - 3LM_1,\\
{M_1}^2 &= 1 + L^2, & R_1M_2 &= 2M_2R_2 - 3M_1R_2 + 3LM_2 + M_2R_1,\\
LR_1 &= 1 - 2{M_2}^2 - M_1M_2, & R_2M_1 &= M_1R_2 - LM_1,\\
LR_2 &= 1 + M_1M_2, & R_2M_2 &= M_2R_2 - M_1R_2 + LM_2,\\
R_1L &= 3M_1M_2 - 3L^2 - 2{M_2}^2, & M_1R_1 &= 3M_1R_2 - 2M_2R_2 - 4LM_2.\\
R_2L &= M_1M_2 - L^2, & &
\end{aligned}
\tag{6.96}
$$

The q-linear grid

We remind the reader that the q-number 2 is written as $[2]_q = q + q^{-1}$. The S-Heun operators $\hat{L}$, $\hat{M}_1$, $\hat{M}_2$, $\hat{R}_1$, $\hat{R}_2$ defined in (6.14) obey the fourteen quadratic relations:

$$\begin{aligned}
\hat{M}_1\hat{L} &= [2]_q\hat{L}\hat{M}_1 + \hat{L}\hat{M}_2, & [2]_q\hat{M}_1\hat{M}_2 &= 1 - \hat{M}_1^2 - \hat{M}_2^2,\\
\hat{M}_2\hat{L} &= -\hat{L}\hat{M}_1, & [2]_q\hat{R}_1\hat{R}_2 &= -\hat{R}_1^2 - \hat{R}_2^2,\\
\hat{M}_2\hat{M}_1 &= \hat{M}_1\hat{M}_2, & \hat{R}_1\hat{M}_1 &= -[2]_q^2\hat{M}_1\hat{R}_2 - [2]_q\hat{M}_2\hat{R}_2 + \hat{M}_2\hat{R}_1,\\
\hat{L}\hat{R}_1 &= 1 - \hat{M}_2^2, & \hat{R}_1\hat{M}_2 &= [2]_q\hat{M}_1\hat{R}_2 + \hat{M}_2\hat{R}_2,\\
\hat{L}\hat{R}_2 &= [2]_q\hat{M}_2^2 + \hat{M}_1\hat{M}_2, & \hat{R}_2\hat{M}_1 &= [2]_q\hat{M}_1\hat{R}_2 + \hat{M}_2\hat{R}_2,\\
\hat{R}_1 L &= 1 - \hat{M}_1^2, & \hat{R}_2\hat{M}_2 &= -\hat{M}_1\hat{R}_2,\\
\hat{R}_2\hat{L} &= -\hat{M}_1\hat{M}_2, & \hat{M}_1\hat{R}_1 &= -[2]_q\hat{M}_1\hat{R}_2 - \hat{M}_2\hat{R}_2.
\end{aligned} \tag{6.97}$$

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Chapitre 7

The rational Sklyanin algebra and the Wilson and para-Racah polynomials

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Abstract

The relation between Wilson and para-Racah polynomials and representations of the degenerate rational Sklyanin algebra is established. Second order Heun operators on quadratic grids with no diagonal terms are determined. These special or S–Heun operators lead to the rational degeneration of the Sklyanin algebra; they also entail the contiguity and structure operators of the Wilson polynomials. The finite-dimensional restriction yields a representation that acts on the para-Racah polynomials.

7.1. Introduction

This paper pursues the exploration of the links between Heun operators, Sklyanin algebras and orthogonal polynomials. Originally introduced in the context of quantum integrable systems [**25**], Sklyanin algebras are typically presented in terms of generators verifying homogeneous quadratic relations. These algebras have been the object of much attention from the perspective of algebraic geometry [**28, 34, 18**]. Classes of Heun operators can be defined [**17**] from the property that they increase by no more than one the degree of polynomials defined on certain continuous or discrete domains; they have been the focus of a continued research effort [**29, 2, 33, 10, 30, 3, 5**] with many applications [**26, 21, 7, 8, 9, 4, 1**]. A key observation for our purposes is that a special category of these operators, referred to as S–Heun operators, offers a path towards the identification of interesting Sklyanin-like algebras through the relations they realize. This connects with orthogonal polynomials as these concrete S–Heun operators are recognized as ladder and structure operators for families of bispectral polynomials belonging to the Askey scheme. It is thus observed that these sets of orthogonal polynomials form representation bases for Sklyanin algebras. Furthermore, the finite-dimensional representations of these Sklyanin algebras are found to provide the algebraic setting that had so far been lacking for the orthogonal polynomials of the so-called "para" type.

A first illustration of these connections was achieved in [**11**]. Building on results of Gorsky and Zabrodin [**14**] on the one hand and of Kalnins and Miller [**19**] on the other, this paper focused on S–Heun operators attached to the Askey–Wilson grid. The salient observations were: *i.* that a subset of the S–Heun operators realize the trigonometric degeneration of the original elliptic Sklyanin algebra and *ii.* that this Sklyanin algebra is a basic structure underneath the theory of Askey–Wilson polynomials. Indeed, as was stressed, the Askey–Wilson operator admits a factorization in terms of the S–Heun operators realizing this degenerate Sklyanin algebra and as was also pointed out, the ladder and structure operators for the Askey–Wilson polynomials obtained by Kalnins and Miller actually realize this degenerate algebra. In view of the fact that the Askey–Wilson algebra [**35**] accounts for the bispectrality of the eponym polynomials, a parallel was thus drawn with the dynamical extension of symmetry algebras by the inclusion of ladder operators in the set of generators. Finally, the q-para Racah polynomials were seen to form a basis for the finite-dimensional representation of the degenerate Sklyanin algebra. This set the course for the systematic examination of the Sklyanin-like operators formed by S–Heun operators on lattices admitting orthogonal polynomials.

The study of S–Heun operators on linear and exponential grid and of the Sklyanin algebras they realize was carried out in [**6**]. It allowed to tie the representations of these algebras to the continuous Hahn and big q-Jacobi polynomials and in finite dimensions to the para-Krawtchouk and q-para Krawtchouk polynomials. This analysis confirmed the important role that Sklyanin algebras play in the interpretation of hypergeometric orthogonal polynomials.

We here address the connection that the Wilson polynomials have with Sklyanin algebras. (We recall that these polynomials are at the top of the $q=1$ part of the Askey scheme.) This will call for the determination of the S–Heun operators on quadratic grids. The rational degeneration of the Sklyanin algebra first found by Smirnov [**27**] will be seen to emerge and to be realized by the structure and ladder operators [**23**] of the Wilson polynomials. This will hence attach these polynomials to representations of the rational Sklyanin algebra. In keeping with preceding observations, the finite-dimensional restrictions of these representations will be seen to offer an algebraic interpretation of the para-Racah polynomials [**22**].

7.1.1. The Wilson polynomials and its truncations

As the Wilson polynomials will prove central in deriving subsequent results, some of their known properties are summarized here. The four-parameter Wilson polynomials [**20**] of degree n, denoted $W_n(x^2|a,b,c,d)$, are given by

$$W_n(x^2|a,b,c,d) = (a+b)_n(a+c)_n(a+d)_n\,{}_4F_3\left(\left.\begin{matrix}-n, n+a+b+c+d-1, a+ix, a-ix\\ a+b, a+c, a+d\end{matrix}\right|1\right),$$

where $(a)_n = a(a+1)...(a+n-1)$ are the Pochhammer symbols and $0 < a,b,c,d \in \mathbb{R}$. These polynomials obey the orthogonality relation

$$\int_0^\infty W_n(x^2|a,b,c,d)W_m(x^2|a,b,c,d)\mathrm{d}\omega(x|a,b,c,d) = N_n(a,b,c,d)\delta_{n,m}. \qquad (7.1)$$

The weight $\omega(x|a,b,c,d)$ and normalization $N_n(a,b,c,d)$ are given explicitly in [**20**]. For any admissible set of parameters, the Wilson polynomials form a basis of the space of polynomials on the support of $\omega(x|a,b,c,d)$. Belonging to the Askey–Wilson scheme, they are bispectral, that is, they diagonalize a three-term recurrence operator acting on the degree and a difference operator acting on the variable.

The Wilson polynomials form an infinite set of orthogonal polynomials that can be truncated [**20**] to a finite one by setting the parameters as follows

$$a = \frac{1}{2}(\gamma+\delta+1), \quad b = \frac{1}{2}(2\alpha-\gamma-\delta+1), \quad c = \frac{1}{2}(2\beta-\gamma+\delta+1), \quad d = \frac{1}{2}(\gamma-\delta+1),$$

and imposing any of the conditions

$$\alpha+1=-N, \qquad \beta+\delta+1=-N, \quad \text{or} \quad \gamma+1=-N.$$

One thus obtains the Racah polynomials after taking

$$ix \longmapsto x+\frac{1}{2}(\gamma+\delta+1).$$

An additional truncation can be obtained [**22**] by imposing

$$a+b+c+d=-N+1. \tag{7.2}$$

Indeed, while one is at first sight led to singular expressions, well-defined orthogonal polynomials can nonetheless be obtained through the use of limits and the resulting polynomials, first introduced in [**22**], are the para-Racah polynomials. These polynomials form a three-parameter set of orthogonal polynomials $P_n(x^2|a,c,w)$ of maximal degree N. Explicit expressions can be found by setting $N=2j+p$, where $j\in\mathbb{N}$ and $p=0,1$, depending on the parity of N. The para-Racah polynomial $P_n(x^2|a,c,w)$ obtained from the truncation (7.2) of the Wilson polynomial $W_n(x^2|a,b,c,d)$ is given by

$$P_n(x^2|a,c,w)=\eta_n\sum_{k=0}^{n}A_{n,k}\varphi_k(x^2), \qquad \varphi_k(x^2)\equiv(a-ix)_k(a+ix)_k, \tag{7.3}$$

where

$$A_{n,k}=\begin{cases}\frac{(-n)_k(n-N)_k}{(1)_k(-j)_k(a+c)_k(a-c-j+1-p)_k} & k\leq j\\ \frac{w^{-1}(-n)_k(n-N)_{N-n}(1)_{n+k-1-N}}{(1)_k(-j)_j(1)_{k-j-1}(a+c)_k(a-c-j+1-p)_k} & k>j\\ 0 & k>n,\end{cases} \tag{7.4}$$

with the normalization given by

$$\eta_n=\begin{cases}\frac{(1)_n(-j)_n(a+c)_n(a-c-j+1-p)_n}{(-n)_n(n-N)_n} & n\leq j,\\ \frac{w(1)_n(-j)_j(1)_{n-j-1}(a+c)_n(a-c-j+1-p)_n}{(-n)_n(n-N)_{N-n}(1)_{2n-1-N}} & n>j.\end{cases} \tag{7.5}$$

These polynomials are orthogonal on a discrete measure that has support on the zeros of the characteristic polynomial $P_{N+1}(x^2|a,c,w)$. The corresponding lattice is a quadratic bi-lattice given by

$$x_{2s+t}=\begin{cases}-(s+a)^2 & t=0,\ s=0,1,\ldots,j,\\ -(s+c)^2 & t=1,\ s=0,1,\ldots,j-1+p,\end{cases} \tag{7.6}$$

so that

$$\sum_{s=0}^{N} P_n(x^2|a,c,w)P_m(x^2|a,c,w)\bar{\omega}_s \propto \delta_{n,m}, \tag{7.7}$$

where the weight $\bar{\omega}_s$ is given explicitly in [**22**]. They also satisfy a three-term recurrence relation and a difference equation. However, they do not appear in classifications of classical orthogonal polynomials as their spectrum is doubly-degenerate.

7.1.2. Outline

The remainder of the paper is organized as follows. In section 7.2, the S–Heun operators are introduced and some of their properties are derived. The connection is made with the algebraic Heun operator of the Wilson/Racah type. Section 7.3 focuses on a subset of the S–Heun operators that preserves the degree of polynomials. A stabilizing algebra is defined from the quadratic relations they obey and its representations are constructed. This algebra is extended to a star algebra in section 7.4 for which a universal presentation is obtained; it is subsequently recognized as a Sklyanin-type algebra. Finally, section 7.5 provides a presentation of the rational degenerate Sklyanin algebra introduced in [**27**] and gives an isomorphism with the universal algebra of section 7.4. Using this isomorphism, representations of the rational degenerate Sklyanin algebra on the Wilson and para-Racah polynomials are constructed. A brief conclusion follows.

7.2. Sklyanin–Heun operators on a quadratic grid

The generic algebraic Heun operators on a domain λ have the property that, when acting on polynomials over λ, they raise the degree by at most one. The S–Heun operators are a specialization of these Heun operators without a diagonal term. In this section, we first identify the S–Heun operators on the quadratic grid and then proceed with a brief characterization.

7.2.1. Sklyanin–Heun operators

Let $\lambda = \lambda_x$ be a discrete grid indexed by x and define the shift operators $T^{\pm}$ acting on functions on λ as follows

$$T^{\pm} f(\lambda_x) \equiv f(\lambda_{x\pm 1}).$$

Consider a second order operator S with no diagonal term

$$S = A_1(\lambda_x)T^+ + A_2(\lambda_x)T^-, \tag{7.8}$$

where A_1 and A_2 are functions on λ_x. Demand that S satisfies the degree-raising property

$$S \cdot p_n(\lambda_x) = q_{n+1}(\lambda_x), \tag{7.9}$$

for p_n and q_{n+1} arbitrary polynomials of degree n and $n+1$, respectively. One can determine the coefficients A_1 and A_2 by acting on the first two monomials in λ_x as follows

$$S \cdot 1 = u_0 + u_1\lambda_x, \quad S \cdot \lambda_x = u_2 + u_3\lambda_x + u_4\lambda_x^2. \tag{7.10}$$

One finds

$$\begin{aligned} A_1(\lambda_x) &= \frac{u_2 + u_3\lambda_x + u_4\lambda_x^2 - u_0\lambda_{x-1} - u_1\lambda_{x-1}\lambda_x}{\lambda_{x+1} - \lambda_{x-1}}, \\ A_2(\lambda_x) &= -\frac{u_2 + u_3\lambda_x + u_4\lambda_x^2 - u_0\lambda_{x+1} - u_1\lambda_{x+1}\lambda_x}{\lambda_{x+1} - \lambda_{x-1}}. \end{aligned} \tag{7.11}$$

The S–Heun operators are defined as the set of operators of the form (7.8) with the coefficients given in (7.11). As these coefficients admit five independent parameters, the S–Heun operators form a five-dimensional vector space $\mathcal{SH}$ of operators on λ. A basis for this space can be chosen as follows

$$\begin{aligned} L &= \mathcal{N}(\lambda_x)(T^+ - T^-), \\ M_1 &= \mathcal{N}(\lambda_x)\left[(\lambda_x - \lambda_{x-1})T^+ + (\lambda_{x+1} - \lambda_x)T^-\right], \\ M_2 &= \mathcal{N}(\lambda_x)\left[(\lambda_x + \lambda_{x-1})T^+ - (\lambda_{x+1} + \lambda_x)T^-\right], \\ R_1 &= \mathcal{N}(\lambda_x)\lambda_x\left[(\lambda_x - \lambda_{x-1})T^+ + (\lambda_{x+1} - \lambda_x)T^-\right], \\ R_2 &= \mathcal{N}(\lambda_x)\lambda_x\left[(\lambda_x + \lambda_{x-1})T^+ - (\lambda_{x+1} + \lambda_x)T^-\right], \end{aligned} \tag{7.12}$$

where

$$\mathcal{N}(\lambda_x) \equiv \left[\lambda_{x+1} - \lambda_{x-1}\right]^{-1}.$$

The naming conventions used in (7.12) will be explained in the next subsection.

Remark 1. Acting on the left with T^+ for each of the operators in (7.12), it can be seen that the set of operators $\mathcal{SH}$ can also be understood as the set of first order shift operators of step two over λ that satisfies the property (7.9).

7.2.2. Sufficiency of the construction

As established above, for an operator S of the form (7.8) to satisfy the property (7.9), the expressions (7.11) are necessary conditions. The sufficiency of these conditions follows from the ensuing proposition.

Proposition 8. A generic element $S \in \mathcal{SH}$ satisfies the property (7.9) if the grid λ_x is of one of the following forms

$$\lambda_x = \alpha q^x + \beta q^{-x} + \kappa, \qquad \lambda_x = \alpha x^2 + \beta x + \kappa, \qquad \text{or} \qquad \lambda_x = (-1)^x(\alpha x + \beta) + \kappa, \quad (7.13)$$

for some constants α, β, κ.

PROOF. An element $S \in \mathcal{SH}$ of the form (7.8) is specified by a set of parameters $\{u_i\}_{i=0,1,\cdots,4}$ (7.11). The action of S on a monomial in λ_x can be reduced by linearity to the five cases given by $u_i = \delta_{i,j}$ for $j = 0, 1, \ldots, 4$. Upon inspecting (7.11), one understands that only the operators defined by $u_i = \delta_{i,0}$ or $u_i = \delta_{i,2}$ need to be analyzed; those remaining amount to one of these two operators multiplied by some power of λ_x.

The first case we treat is $u_i = \delta_{i,2}$ and it corresponds to the operator we have denoted L. It can be seen from (7.11) that when $u_i = \delta_{i,3}$ or $u_i = \delta_{i,4}$, the corresponding operator is $\lambda_x L$ or ${\lambda_x}^2 L$, respectively. It follows that for S to satisfy property (7.9), one must have that L decreases the degree of polynomials in λ_x by one. Similarly, it follows from (7.11) that the case $u_i = \delta_{i,1}$ will satisfy (7.9) if the case of $u_i = \delta_{i,0}$, corresponding to the S–Heun operator $\frac{1}{2}(M_1 - M_2)$, is an operator that stabilizes the set of polynomials of a given degree.

Thus, a generic element of the form (7.8) will satisfy (7.9) if the subset of operators generated by the cases $u_i = \delta_{i,j}$ for $j = 0, 2, 3$ preserves the degree of polynomials. As the generators of this subset are all tridiagonal operators, the proposition follows from the results in [**31**] which identifies (7.13) as the possible grids allowing second-order difference equations diagonalized by polynomials. □

On the quadratic grid, it can be shown that a generic element of the vector space spanned by (7.12) satisfies the property (7.9). Indeed, as derived above, the expressions (7.11) for the coefficients are necessary conditions. The derivations so far were grid-independent, but to proceed further, one needs to fix the grid. Let us consider the quadratic grid

$$\lambda_x = x^2. \qquad (7.14)$$

For this choice of grid, one has that proposition 8 holds and the sufficiency of the construction is established.

The leading terms of the actions on monomials in λ_x are now computed for future reference. In the case of L, one obtains

$$L \cdot {\lambda_x}^n = \sum_{k=1}^{n} \binom{n}{k} \lambda_x^{n-k} \sum_{\substack{j \text{ odd} \\ 0 \le j \le k}} \binom{k}{j} (4\,\lambda_x + p^2)^{\frac{j-1}{2}} = n\,{\lambda_x}^{n-1} + O(\lambda_x^{n-2}), \tag{7.15}$$

which is verified to be a degree lowering operator. Moreover, one finds that

$$\begin{aligned}\frac{1}{2}(M_1 - M_2) \cdot {\lambda_x}^n = \\ \sum_{\substack{k=0 \\ 0 \le j \le k}}^{n} \binom{n}{k} \binom{k}{j} \lambda_x^{n-k} \left[\frac{1+(-1)^j}{2} (4\lambda_x + p^2)^{\frac{j}{2}} - \frac{1-(-1)^j}{2} (\lambda_x + 1)(4\lambda_x + p^2)^{\frac{j-1}{2}} \right] \\ = (1-n){\lambda_x}^n + O({\lambda_x}^{n-1}),\end{aligned} \tag{7.16}$$

which preserves the degree of polynomials. The actions of the other generators follow from (7.15) and (7.16) by noting that

$$\frac{1}{2}(M_1 + M_2) = \lambda_x\,L, \qquad R_1 = \lambda_x M_1, \qquad R_2 = \lambda_x M_2. \tag{7.17}$$

With the above observations, it follows that a generic linear combination of the basis elements (7.12) displays the degree raising property (7.9). These calculations enable one to see that the choice of basis (7.12) decomposes the generic special Heun operator into operators that have a prescribed action on polynomials in λ_x. Indeed, L can be identified as a lowering operator, M_1 and M_2 as stabilizing operators while R_1 and R_2 are raising operators.

7.2.3. S–Heun operators of the Wilson type and the Heun–Racah operator

As the S–Heun operators are specialized algebraic Heun operators [**17**], they are related to the general algebraic Heun operators associated to the same grid. The Heun–Racah operator W on the quadratic grid introduced in [**5**] admits a quadratic embedding in the set $\mathcal{SH}$ of S–Heun operators on the quadratic grid. In view of Remark 1, it will come as no surprise that this embedding is obtained by first conjugating the Heun–Racah operator W

by a scaling of the grid $\mu : x \to 2x$, such that the shift operators in W act with a step of two. One obtains

$$\mu^{-1} \circ W \circ \mu = R_1(a_1M_1 + a_2M_2 + a_3L) + R_2(a_4M_1 + a_5M_2 + a_6L) + a_7LM_2 + a_8{M_2}^2 + a_9L^2, \tag{7.18}$$

where the coefficients a_i, $i = 1, 2, \ldots, 9$ are given in terms of the parameters $t_0, t_1, u_0, u_1, u_2, v_0, v_1, v_2$ and v_3 of W in [**5**] as

$$a_1 = \frac{t_1 + u_2}{4} - \frac{v_3}{16}, \qquad a_2 = -\frac{t_1}{8} + 8u_0 + u_1 - 2v_1 + \frac{v_3}{16},$$

$$a_4 = \frac{u_2}{4} - a_2, \qquad a_5 = \frac{v_3}{16}, \qquad a_6 = a_3 - 2u_1,$$

$$a_9 = -t_0 - 24u_0 + 16v_0. \qquad a_7 = 8u_0, \qquad a_8 = t_0,$$

$$a_3 = \frac{1}{4}\left(-8t_0 - t_1 - 64u_0 - 3u_2 + 16v_1 + 2v_2\right),$$

The operator X that acts by multiplication by the grid variable λ_x can be written as a quadratic expression in terms of the S–Heun generators:

$$X \equiv x^2 = (R_1 + R_2)(M_1 - L) - \frac{1}{2}R_1M_2 - \frac{1}{2}R_2M_1. \tag{7.19}$$

7.3. The stabilizing subalgebra $\mathfrak{stab}$

By direct computations from the definitions (7.12), it can be seen that the S–Heun generators satisfy homogeneous quadratic relations, with the complete list given in the appendix 7.6.1. From these relations, it is observed that the subset of stabilizing S–Heun operators generated by L, M_1 and M_2 closes as a quadratic algebra to be called $\mathfrak{stab}$ whose relations are

$$[L, M_1] = 2L^2, \qquad [L, M_2] = \{M_1, L\}, \qquad [M_1, M_2] = \{M_2, L\} - 4L^2. \tag{7.20}$$

The Casimir element C is given by

$$C = {M_1}^2 - \{M_2, L\} + 3L^2, \tag{7.21}$$

and is equal to the identity in the realization (7.12) in terms of shift operators. It will prove fruitful to examine the stabilizing algebra (7.20) in this realization. Knowing that it stabilizes polynomials in λ_x of a given degree, one may set up an eigenvalue problem on this space.

7.3.1. Diagonalization of a generic linear element

Consider a generic linear combination of the operators L, M_1, M_2

$$P(s,t) = uL + vM_1 + wM_2, \tag{7.22}$$

parametrized as follows

$$u = \frac{(1+2s)(1+2t)-1}{4}, \qquad v = \frac{1}{2}(1+s+t), \qquad w = \frac{1}{2},$$

with $0 < s, t \in \mathbb{R}$ being arbitrary parameters. It is straightforward to show that, under the invertible transformation

$$\rho : x \mapsto -ix,$$

the operator P is given by

$$\tilde{P} \equiv \rho \circ P \circ \rho^{-1} = -\frac{1}{4ix}\left[(t-ix)(s-ix)\tilde{T}^{+} - (s+ix)(t+ix)\tilde{T}^{-}\right],$$

with $\tilde{T}^{\pm}$ defined by $\tilde{T}^{\pm} f(x) \mapsto f(x \pm i)$. Multiplying each term in the above by $(2ix \pm 1)/(2ix \pm 1)$, one recognizes the off-diagonal terms of the difference operator diagonalized by the continuous dual Hahn polynomials [**20**]. Denoting these polynomials as $S_n(x^2|1/2, s, t)$ one has

$$\tilde{P}\, S_n(x^2|1/2, s, t) = (n - (s+t)/2)\, S_n(x^2|1/2, s, t).$$

Once an element is specified by (7.22), this defines an eigenbasis in terms of the continuous dual Hahn polynomials. However, no meaningful action can be identified for the remaining elements in $\mathfrak{stab}$. We consider instead quadratic combinations in the elements of the algebra.

7.3.2. Action on Wilson polynomials

A natural action of the stabilizing algebra $\mathfrak{stab}$ on the Wilson polynomials arises from the realization (7.12). Indeed, defining the following pair of operators from (7.22)

$$\mu^{(a,b,c,d)} = P(2a-1, 2b-1), \qquad \mu^{*(a,b,c,d)} = P(2c, 2d), \tag{7.23}$$

such that manifestly

$$\mu^{*(a,b,c,d)} = \mu^{(c+1/2, d+1/2, a-1/2, b-1/2)},$$

one has the following proposition:

Proposition 9. The quadratic element $Q \in \mathfrak{stab}$ defined by

$$Q \equiv \mu^{*(a,b,c,d)} \mu^{(a,b,c,d)}, \tag{7.24}$$

where μ and μ^* are given by (7.23) and with $0 < a, b \in \mathbb{R}$ and $1/2 < c, d \in \mathbb{R}$ is realized, up to a constant term, by the Wilson operator conjugated by the grid scaling

$$\phi : x \mapsto -2ix. \tag{7.25}$$

Proof. In the realization (7.12), conjugating Q by the scaling transformation (7.25), it can be seen by direct calculations that the transformed operator $\tilde{Q}$ is given by

$$\tilde{Q} \equiv \phi \circ Q \circ \phi^{-1} = \frac{(a-ix)(b-ix)(c-ix)(d-ix)}{2ix(2ix-1)} \tilde{T}^+ + \frac{(a+ix)(b+ix)(c+ix)(d+ix)}{2ix(2ix+1)} \tilde{T}^-$$
$$- \left[\frac{(a-ix)(b-ix)(c-ix)(d-ix)}{2ix(2ix-1)} + \frac{(a+ix)(b+ix)(c+ix)(d+ix)}{2ix(2ix+1)} \right] + (a+b)(c+d-1).$$

The above operator is identified as the Wilson operator [**20**], up to a constant term. □

Remark 2. The operator X that acts by multiplication by the variable λ_x can be embedded (7.19) in the set $\mathcal{SH}$ of S–Heun operators. In addition, with the operator Q identified as the Wilson operator, the bispectral pair of operators that generates the Racah/Wilson algebra [**15, 13, 12**] admits an embedding in the set $\mathcal{SH}$ of S–Heun operators. Moreover, a quartic embedding of the Heun–Racah operator (7.18) is obtained from the construction of the Heun–Racah operator [**5**] by the tridiagonalization [**16**] of the Racah operator.

The definition of Q in (7.24) naturally provides a factorization of the Wilson operator in terms of $\mu^{*(a,b,c,d)}$ and $\mu^{(a,b,c,d)}$. Moreover, it directly follows from proposition 9 that the operator $\tilde{Q}$ is diagonalized by the Wilson polynomials:

$$\tilde{Q}\, W_n(x^2|a,b,c,d) = [n(n+a+b+c+d-1) + (c+d)(a+b-1)]\; W_n(x^2|a,b,c,d).$$

Introducing a third operator $\tau^{(a,b,c,d)}$ defined by

$$\tau^{(a,b,c,d)} = 4L, \tag{7.26}$$

a presentation of $\mathfrak{stab}$ in terms of the generators $\mu^{(a,b,c,d)}$, $\mu^{*(a,b,c,d)}$ and $\tau^{(a,b,c,d)}$ can be given for generic values of the parameters a, b, c, d. This allows to construct representations of $\mathfrak{stab}$ on the Wilson polynomials.

Proposition 10. A representation of $\mathfrak{stab}$ on the Wilson polynomials $\tilde{W}$ (see (7.28)) is given by the following actions

$$\begin{aligned}
\mu^{(a,b,c,d)}\cdot \tilde{W}_n(x^2|a,b,c,d) &= -(n+a+b-1)\,\tilde{W}_n\left(x^2\middle|a-\tfrac{1}{2},b-\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right),\\
\tau^{(a,b,c,d)}\cdot \tilde{W}_n(x^2|a,b,c,d) &= n(n+a+b+c+d-1)\,\tilde{W}_{n-1}\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right),\\
\mu^{*(a,b,c,d)}\cdot \tilde{W}_n(x^2|a,b,c,d) &= -\sigma(n+a+b-1)\,\tilde{W}_n\left(x^2\middle|a-\tfrac{1}{2},b-\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right)\\
&\quad+\left[\sigma(ab-cd)-\frac{1}{2}(c+d)-\frac{1}{4}\right]n(n+e_1-1)\,\tilde{W}_{n-1}\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right),\\
&\quad-(1-\sigma)(n+c+d-1)\,\tilde{W}_n\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right)
\end{aligned}\tag{7.27}$$

$$\sigma \equiv (a+b-c-d)^{-1}, \qquad e_1 \equiv a+b+c+d,$$

with

$$\tilde{W}_n(x^2|a,b,c,d) \equiv \phi^{-1}\cdot W_n(x^2|a,b,c,d) = W_n\Big(-\frac{x^2}{4}\Big|a,b,c,d\Big), \tag{7.28}$$

and ϕ defined in (7.25).

PROOF. The conjugation of the three operators $\mu^{(a,b,c,d)}$, $\mu^{*(a,b,c,d)}$ and $\tau^{(a,b,c,d)}$ by the scaling map (7.25) yields operators that are identified as the structure and forward shift operators for the Wilson polynomials [**20**]. These structure operators have a known action on the Wilson polynomials [**23**]. Using the identity

$$\mu^{*(a,b,c,d)} = \sigma\,\mu^{(a,b,c,d)} + (1-\sigma)\mu^{(c,d,a,b)} + \left[\sigma(ab-cd)-\frac{1}{2}(c+d)-\frac{1}{4}\right]\tau^{(a,b,c,d)}, \tag{7.29}$$

which is directly verified and applying the scaling (7.25) to the polynomials to get (7.28), one obtains the actions (7.27). As one can use the orthogonality relation (7.1) of the Wilson polynomials to express all polynomials with shifted parameters in (7.27) as sums of Wilson polynomials with the initial parameters, these actions define representations of the stabilizing algebra $\mathfrak{stab}$ on the Wilson polynomials. □

7.4. Extension of $\mathfrak{stab}$ to a star algebra

The construction laid out in the preceding section parallels the structural approach to orthogonal polynomials due to Kalnins and Miller [**23, 19**]. In particular, Miller derives in [**23**] the orthogonality (7.1) of the Wilson polynomials from the structural recurrence relations associated to $\mu^{(a,b,c,d)}$ and $\tau^{(a,b,c,d)}$ by identifying the operator $\mu^{*(a,b,c,d)}$ and deriving an inner product such that this operator is the adjoint of $\mu^{(a,b,c,d)}$. An operator $\tau^{*(a,b,c,d)}$ is then identified as the adjoint of $\tau^{(a,b,c,d)}$. A similar approach in the context of the S–Heun operators can be pursued at the algebraic level.

The representations defined through (7.27) are endowed with a natural inner product inherited from the orthogonality relation (7.1). This enables one to define a star operation, such that $\mu^{*(a,b,c,d)}$ is precisely the adjoint of $\mu^{(a,b,c,d)}$ under the inner product. It follows that $\tilde{Q}$ is a self-adjoint operator. However, the stabilizing algebra is not closed under the star operation. This can be seen by taking the adjoint of $\tau^{(a,b,c,d)}$, a lowering operator, which would involve raising operators that are not contained in the stabilizing algebra $\mathfrak{stab}$. We shall now extend $\mathfrak{stab}$ to its closure under the star operation.

7.4.1. Star operation

With the help of (7.1), one constructs an operator as the adjoint of the forward shift operator. This leads to the backward shift operator for the Wilson polynomials [**20**] with action given by

$$(\phi^{-1} \circ \tau^{*(a,b,c,d)} \circ \phi) \cdot W_n(x^2|a,b,c,d) = W_{n+1}\left(x^2 \middle| a-1/2, b-1/2, c-1/2, d-1/2\right). \tag{7.30}$$

The operator $\tau^{*(a,b,c,d)}$ can then be decomposed in terms of the S–Heun operators as follows

$$\tau^{*(a,b,c,d)} = a_1 L + a_2 M_1 + a_3 M_2 + a_4 R_1 + a_5 R_2, \tag{7.31}$$

with the coefficients given by

$$a_1 = 4e_4 - e_3 + \frac{e_1 - 1}{4}, \quad a_2 = e_3 - \frac{e_2}{2} + \frac{e_1}{8}, \quad a_3 = \frac{e_2}{2} - \frac{5e_1}{8} + \frac{1}{2}, \quad a_4 = \frac{e_1}{4} - \frac{3}{8}, \quad a_5 = -\frac{1}{8},$$

where e_1, e_2, e_3 and e_4 are the elementary symmetric polynomials in the four parameters a, b, c and d:

$$\begin{aligned} e_1 &= a+b+c+d, & e_2 &= ab+ac+ad+bc+bd+cd, \\ e_3 &= abc+abd+acd+bcd, & e_4 &= abcd. \end{aligned} \tag{7.32}$$

Introducing $\tau^{*(a,b,c,d)}$ as a fourth generator together with those of the stabilizing algebra $\mathfrak{stab}$ leads to an algebra closed under the star operation.

Proposition 11. The algebra $\mathfrak{stab}^*$ generated by $\mu^{(a,b,c,d)}$, $\mu^{*(a,b,c,d)}$, $\tau^{(a,b,c,d)}$ and $\tau^{*(a,b,c,d)}$, together with the relations induced from their definitions in terms of S–Heun operators given in (7.23), (7.26) and (7.31) admits the natural star map defined from its canonical action on the generators:

$$* : \tau \longmapsto \tau^*, \tag{7.33}$$

$$\mu \longmapsto \mu^*. \tag{7.34}$$

Proof. The result follows from the results of [**23**] after conjugation of the generators by the scaling map (7.25). □

7.4.2. A universal presentation of $\mathfrak{stab}^*$

The algebra $\mathfrak{stab}^*$ can be presented in terms of quadratic relations by making use of the relations given in the appendix 7.6.1. However, such a presentation obfuscates the structure of the algebra because the parameters a, b, c and d of the Wilson polynomial appear explicitly in the relations. Thus, it does not define uniquely an algebra associated to the quadratic grid.

Recall that the normalized Wilson polynomials are known [**23**] to be fully symmetric under permutations of their four parameters. However, the definitions for the two stabilizing generators given in (7.23) do not make this symmetry manifest, because they contain the specific parameters of the representation. Nervertheless, the permutation symmetry of the polynomials can be made manifest at the level of the algebra to obtain a universal presentation.

Proposition 12. The algebra $\mathfrak{stab}^*$ admits a presentation as a unital associative algebra with four generators U, V, Y and R obeying the following relations

$$\begin{gathered}[V,Y] = -\{U,Y\}, \qquad [U,Y] = -\{Y,Y\}, \qquad [U,V] = \{V,Y\} - 2\{Y,Y\}, \\ [R,Y] = \{U,U\} - \{U,V\} + \{V,Y\}, \qquad [R,V] = 2\{V,Y\} - \{Y,Y\} - \{V,V\} - \{U,R\}, \\ [R,U] = \{U,V\} + 2\{V,Y\} - 2\{U,Y\} - \{V,V\} - \{Y,Y\} - \{R,Y\}.\end{gathered} \tag{7.35}$$

The two Casimir operators are given by

$$Q_1 = U^2 - \{V, Y\} + 3Y^2, \qquad Q_2 = U^2 + V^2 - \{U, V\} - \{U, Y\} - \{R, Y\}. \tag{7.36}$$

Proof. Consider the following generic linear combination of generators

$$u\, \mu^{(a,b,c,d)} + v\, \mu^{*(a,b,c,d)}.$$

Acting with the symmetric group S_4 on the parameters (a, b, c, d), one constructs a fully symmetric element in terms of the S–Heun operators as follows

$$\frac{1}{|S_4|} \sum_{\sigma \in S_4} \left[u\, \mu^{\sigma(a,b,c,d)} + v\, \mu^{*\,\sigma(a,b,c,d)}\right] = \\ \frac{1}{2}[(u-v) - e_1(u+v)]M_1 - \frac{1}{2}(u+v)M_2 + \left[\frac{e_1}{2}(u-v) - \frac{2e_2}{3}(u+v)\right] L.$$

Setting $u = 1$ and either $u = v$ or $u = -v$ in the above yields two independent generators that are manifestly symmetric and can be used instead of μ and μ^* to obtain another presentation of $\mathfrak{stab}^*$. The relations in this new presentation now only involve the elementary symmetric polynomials (7.32). Subsequently, it becomes straightforward to eliminate all remaining parameters in the algebraic relations by further redefining the generators as

$$U = M_1 + e_1\, L, \qquad V = M_2 + e_1\, M_1 + \frac{1}{2}{e_1}^2 L, \qquad Y = L, \tag{7.37}$$

$$R = R_2 + (2e_1 - 3)R_1 + \frac{1}{2}(3{e_1}^2 - 10e_1 + 4)M_2 + \frac{1}{2}(e_1+1)({e_1}^2 - 4e_1 + 2)M_1 \\ + \frac{1}{8}({e_1}^4 - 4{e_1}^3 - 8{e_1}^2 + 24e_1 - 8)L. \tag{7.38}$$

Using the quadratic relations of the S–Heun operators given in the appendix 7.6.1, the relations (7.35), as well as the centrality of the two operators in (7.36), are verified. □

In a realization in terms of S–Heun operators, the Casimir operators (7.36) are proportionnal to the identity and the coefficients are functions of the parameters of the polynomials. One has

$$Q_1 = 1, \qquad Q_2 = (e_1 - 2)(e_1 - 4), \tag{7.39}$$

where e_1 is given in (7.32).

Remark 3. While a universal presentation of $\mathfrak{stab}^*$ has been given in proposition 12, the star structure is not universal and depends explicitly on the representation parameters. This is not surprising because the map (7.33) is constructed using the inner product (7.1) corresponding to a specific realization with fixed parameters. Nevertheless, one can work in a specific realization and write the generators in (7.35) in terms of the structural operators (7.23), (7.26) and (7.31) as follows

$$\begin{aligned}
&U = \frac{1}{|S_4|}\sum_{\sigma\in S_4}\Big[\mu^{\sigma(a,b,c,d)} - \mu^{*\,\sigma(a,b,c,d)}\Big], \qquad V = \frac{1}{|S_4|}\sum_{\sigma\in S_4}\Big[-\mu^{\sigma(a,b,c,d)} - \mu^{*\,\sigma(a,b,c,d)}\Big] + \alpha\, Y,\\
&Y = \frac{1}{4}\tau, \qquad R = 8\tau^{*(e_1,e_2,e_3,e_4)} - (2-3\alpha)V + (1-3\alpha+\beta)U + (1-\beta+\gamma)Y,
\end{aligned} \tag{7.40}$$

where

$$\alpha = \frac{1}{2}{e_1}^2 - \frac{4}{3}e_2, \qquad \beta = -{e_1}^3 + 4e_2e_1 - 8e_3, \quad \text{and} \quad \gamma = \frac{3}{4}\alpha\, {e_1}^2 - {e_1}^2 e_2 + 8e_1e_3 - 32e_4, \tag{7.41}$$

with e_1, e_2, e_3 and e_4 given in (7.32). With the above, one obtains

$$\begin{aligned}
&Y^* = \frac{1}{32}\left[R + (2-3\alpha)V - (1-3\alpha+\beta)U - (1-\beta+\gamma)Y\right],\\
&U^* = -U, \qquad V^* = V + \alpha\,(Y^* - Y),\\
&R^* = \Big[32 + \alpha\,(2-3\alpha)\Big]Y - (2-3\alpha)V - (1-3\alpha+\beta)U + \Big[1-\beta+\gamma-\alpha\,(2-3\alpha)\Big]Y^*.
\end{aligned}$$

7.4.3. The algebra $\mathfrak{stab}^*$ as a Sklyanin algebra

It can be seen from (7.37) and (7.38) that the generators of $\mathfrak{stab}^*$ only depend on the parameters a, b, c, d via the elementary symmetric polynomial $e_1(a, b, c, d)$. Thus, they will be invariant under a commensurate increase and decrease of any pair of parameters. A glance at (7.31) indicates that this will not be the case for $\tau^{*(a,b,c,d)}$. However, a pseudo-commutation relation similar to the one introduced by Rains in [**24**] is obtained.

Proposition 13. In the realization (7.12) the identity

$$\tau^{*(a,b,c+k,d-k)}\,\tau^{*(a+\frac{1}{2},b+\frac{1}{2},c-\frac{1}{2},d-\frac{1}{2})} = \tau^{*(a,b,c,d)}\,\tau^{*(a+\frac{1}{2},b+\frac{1}{2},c-\frac{1}{2}+k,d-\frac{1}{2}-k)}, \tag{7.42}$$

is satisfied. Moreover, at the abstract level (7.42) encodes the algebraic relations of the $\mathfrak{stab}^*$ algebra (7.35).

Proof. Using the definition (7.31), the identity (7.42) is readily verified. The second statement is demonstrated by using (7.40) to express $\tau^{*(a,b,c,d)}$ in terms of the generators (7.37) and (7.38) as

$$8\tau^{*(a,b,c,d)} = R + (2-3\alpha)V - (1-3\alpha+\beta)U - (1-\beta+\gamma)Y,$$

where α, β and γ are given in (7.41). Upon using the above in (7.42), one can pick any one of the parameters a, b, c, d and take the remaining ones to be vanishing. Equating the coefficients of each power of the remaining non-zero parameter in the left- and right-hand side of (7.42) yields a set of relations that is algebraically identical to the relations (7.35). $\square$

That the relations of $\mathfrak{stab}^*$ are encoded in the identity (7.42) identifies the $\mathfrak{stab}^*$ algebra as a Sklyanin-type algebra [**24**].

7.5. The rational degenerate Sklyanin algebra

The rational degenerate Sklyanin algebra $\mathfrak{sK}_r$ is obtained in [**27**] from the Sklyanin algebra [**25**] and is associated to a rational degeneration of an elliptic R-matrix. A presentation can be given as a unital associative algebra generated by four elements S_0, S_3, S_+, S_- obeying the defining relations

$$\begin{aligned} &[S_0, S_-] = -2\{S_-, S_-\}, \qquad [S_0, S_+] = 16\{S_3, S_-\} - 16\{S_-, S_-\} + 2\{S_+, S_-\} - 4\{S_3, S_3\}, \\ &[S_+, S_-] = 2\{S_0, S_3\}, \qquad [S_0, S_3] = 2\{S_3, S_-\} - 8\{S_-, S_-\}, \qquad [S_3, S_\pm] = \pm\{S_0, S_\pm\}. \end{aligned} \tag{7.43}$$

The rational degenerate Sklyanin algebra admits two Casimir operators which are given in the above presentation by

$$C_1 = S_0^2 + S_3^2 + \frac{1}{2}\{S_+, S_-\}, \qquad C_2 = \frac{1}{2}\{S_+, S_-\} + 2\{S_-, S_3\} + S_3^2 - 6\{S_-, S_-\}. \tag{7.44}$$

The presentation (7.43) is recovered from the one in [**27**] upon setting the free parameter $\eta = 1$ and defining $S_\pm = S_1 \pm iS_2$. The following proposition identifies the $\mathfrak{stab}^*$ algebra with the rational degenerate Sklyanin algebra.

Proposition 14. The $\mathfrak{sK}_r$ algebra defined in (7.43) is isomorphic to the $\mathfrak{stab}^*$ algebra defined in (7.35).

PROOF. The following map is readily verified to be an isomorphism of algebras.

$$S_0 = 4Y - 4U, \qquad S_3 = 4U - 2Y - 4V, \qquad S_+ = 16R - 14Y - 8U + 24V, \qquad S_- = -2Y. \tag{7.45}$$

□

7.5.1. A realization in terms of difference operators

A realization of the rational degenerate Sklyanin algebra in terms of difference operators is provided in [**27**]. The Casimir elements are realized as multiples of the identity and are given by

$$C_1 = 16(2s+1)^2 Id, \qquad C_2 = 64s(s+1)Id.$$

The generators thus represented can be written in terms of the S–Heun operators (7.12) as follows

$$S_0 = 4(2s-1)L - 4M_1, \qquad S_3 = -2(2s-1)^2 L + 4(2s-1)M_1 - 4M_2, \qquad S_1 - iS_2 = -2L,$$

$$\begin{aligned} S_1 + iS_2 = -2(4s^2-1)(4s^2-8s-1)L - 8(2s-1)(4s^2-4s-1)M_1 + 8(2s-1)(6s+1)M_2 \\ - 16(4s-1)R_1 + 16R_2. \end{aligned}$$

It is immediate from the above that the realization in terms of S–Heun operators of the $\mathfrak{sK}_r$ algebra involves coefficients that depend on the values of the Casimir operators. A similar observation could be made for the case of the $\mathfrak{stab}^*$ algebra in (7.37) and (7.38). It follows from proposition 14 that the parameters e_1 and s are related by

$$e_1 = 2 - 2s.$$

7.5.2. A family of representations

The identification of the rational degenerate Sklyanin algebra $\mathfrak{sK}_r$ with the $\mathfrak{stab}^*$ algebra directly leads to a family of representations of $\mathfrak{sK}_r$ on the Wilson polynomials.

Proposition 15. A representation of the rational degenerate Sklyanin algebra $\mathfrak{s}\mathfrak{K}_r$ (7.43) on the Wilson polynomials is defined by the following actions

$$\begin{aligned} S_0 \cdot \tilde{W}_n\left(x^2\middle|a,b,c,d\right) &= 4\sigma(n+c+d-1)\,\tilde{W}_n\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right)\\ &\quad+(4\sigma(ab-cd)-e_1)n(n+e_1-1)\,\tilde{W}_{n-1}\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right)\\ &\quad-4\sigma(n+a+b-1)\,\tilde{W}_n\left(x^2\middle|a-\tfrac{1}{2},b-\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right), \end{aligned}$$

$$\begin{aligned} S_3 \cdot \tilde{W}_n\left(x^2\middle|a,b,c,d\right) &= -4(n+a+b-1)\,\tilde{W}_n\left(x^2\middle|a-\tfrac{1}{2},b-\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right)\\ &\quad+\frac{1}{2}(8(ab+cd)-{e_1}^2-1)n(n+e_1-1)\,\tilde{W}_{n-1}\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right)\\ &\quad-4(n+c+d-1)\,\tilde{W}_n\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right), \end{aligned}$$

$$S_- \cdot \tilde{W}_n\left(x^2\middle|a,b,c,d\right) = -\frac{1}{2}n(n+e_1-1)\,\tilde{W}_{n-1}\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right),$$

$$\begin{aligned} S_+ \cdot \tilde{W}_n\left(x^2\middle|a,b,c,d\right) &= 128\tilde{W}_{n+1}\left(x^2\middle|a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right)\\ &\quad+8(6\alpha-1+2\beta\sigma)(n+a+b-1)\,\tilde{W}_n\left(x^2\middle|a-\tfrac{1}{2},b-\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right)\\ &\quad+8(6\alpha-1-2\beta\sigma)(n+c+d-1)\,\tilde{W}_n\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right)\\ &+8\left[(1-6\alpha)(ab+cd)-2\beta\sigma(ab-cd)+\xi\right]n(n+e_1-1)\,\tilde{W}_{n-1}\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right), \end{aligned}$$

where α,β and γ are defined in (7.41) and

$$\xi \equiv \frac{1}{2}(1-2{e_1}^2+{e_1}^4-256e_4),$$

with e_1 and e_4 defined in (7.32).

PROOF. One first derives the action of the symmetrized structure operators on the Wilson polynomials. It can be seen from (7.27) and (7.30) that the expressions in the case of τ and τ^* are fully symmetric under permutations of the parameters such that their actions are invariant under the symmetrization. To obtain similar expressions for μ and μ^*, one uses

(7.29) to write

$$\mu^{(a,b,c,d)} \pm \mu^{*(a,b,c,d)} = (\mu^{(a,b,c,d)} \pm \mu^{(c,d,a,b)}) \mp \frac{1}{2}\Big(c+d+\frac{1}{2}\Big)\tau^{(a,b,c,d)} \\ \pm \sigma\left[\mu^{(a,b,c,d)} - \mu^{(c,d,a,b)} + (ab-cd)\tau^{(a,b,c,d)}\right]. \quad (7.46)$$

The last term in the right-hand side of (7.46) is independent of the parameters as

$$\sigma\left[\mu^{(a,b,c,d)} - \mu^{(c,d,a,b)} + (ab-cd)\tau^{(a,b,c,d)}\right] = Y - U, \tag{7.47}$$

and is thus invariant under the symmetrization. As it is verified that

$$\frac{1}{|S_4|}\sum_{\pi\in S_4}(\mu^{\pi(a,b,c,d)} - \mu^{\pi(c,d,a,b)}) = 0,$$

one can use the invariance of τ under permutations of the parameters to obtain from (7.46) using (7.47) that

$$\frac{1}{|S_4|}\sum_{\pi\in S_4}(\mu^{\pi(a,b,c,d)} - \mu^{*\,\pi(a,b,c,d)}) = \sigma\mu^{(c,d,a,b)} - \sigma\mu^{(a,b,c,d)} + \left[\frac{1}{4}(e_1+1) - \sigma(ab-cd)\right]\tau^{(a,b,c,d)}.$$

Likewise, observing that $\mu^{(a,b,c,d)}+\mu^{(c,d,a,b)}+(ab+cd)\tau^{(a,b,c,d)}$, is symmetric under permutations of the parameters, one can use the invariance of τ and (7.47) in (7.46) to obtain

$$\frac{1}{|S_4|}\sum_{\pi\in S_4}(\mu^{\pi(a,b,c,d)} + \mu^{*\,\pi(a,b,c,d)}) = (1+\sigma)\mu^{(a,b,c,d)} + (1-\sigma)\mu^{(c,d,a,b)} \\ + \left[(ab+cd) + \sigma(ab-cd) - \frac{1}{3}e_2 - \frac{1}{4}(e_1+1)\right]\tau^{(a,b,c,d)}.$$

The actions on the scaled Wilson polynomials (7.28) of τ, τ^* and of the operators in (7.46) are obtained from (7.27) and found to be:

$$\frac{1}{|S_4|}\sum_{\pi\in S_4}(\mu^{\pi(a,b,c,d)} - \mu^{*\,\pi(a,b,c,d)})\cdot\tilde{W}_n(x^2|a,b,c,d) = \\ \left[\frac{1}{4}(e_1+1) - \sigma(ab-cd)\right] n(n+e_1-1)\tilde{W}_{n-1}(x^2|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}) \\ +\sigma(n+a+b-1)\tilde{W}_n(x^2|a-\tfrac{1}{2},b-\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}) \\ -\sigma(n+c+d-1)\tilde{W}_n(x^2|a+\tfrac{1}{2},b+\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}),$$

$$\frac{1}{|S_4|}\sum_{\pi\in S_4}(\mu^{\pi(a,b,c,d)}+\mu^{*\,\pi(a,b,c,d)})\cdot\tilde{W}_n(x^2|a,b,c,d)=$$
$$\Big[(ab+cd)+\sigma(ab-cd)-\frac{1}{3}e_2-\frac{1}{4}(e_1+1)\Big]\,n(n+e_1-1)\tilde{W}_{n-1}(x^2|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2})$$
$$+(\sigma-1)(n+c+d-1)\tilde{W}_n(x^2|a+\tfrac{1}{2},b+\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2})$$
$$-(\sigma+1)(n+a+b-1)\tilde{W}_n(x^2|a-\tfrac{1}{2},b-\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2})$$

$$\frac{1}{|S_4|}\sum_{\sigma\in S_4}\tau^{\sigma(a,b,c,d)}\cdot\tilde{W}_n(x^2|a,b,c,d)=n(n+e_1-1)\,\tilde{W}_{n-1}\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right),$$
$$\frac{1}{|S_4|}\sum_{\sigma\in S_4}\tau^{*\,\sigma(a,b,c,d)}\cdot\tilde{W}_n(x^2|a,b,c,d)=\tilde{W}_{n+1}\left(x^2\middle|a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right).$$

Using (7.40) and the above, one can construct a representation of (7.35) on the Wilson polynomials (7.28). Proposition 15 then follows from proposition 14.

□

Finite-dimensional representations can be obtained by truncating the representations of proposition 15.

Proposition 16. The finite-dimensional representations obtained from truncations of the representations in proposition 15 act on the para-Racah polynomials.

Proof. Looking at the content of proposition 15, it is seen that the only generator that raises the degree is S_+. Using (7.45), (7.40) and (7.38) this degree-raising action can be traced back to the following combination of S–Heun operators

$$R_2+(2e_1-3)R_1.$$

With the help of (7.17),(7.16) and (7.15), one can obtain the leading term of the action of the above operator on a polynomial of degree N in λ_x:

$$R_1\cdot{\lambda_x}^N={\lambda_x}^{N+1}+O({\lambda_x}^N),\qquad R_2\cdot{\lambda_x}^N=(2N-1){\lambda_x}^{N+1}+O({\lambda_x}^N),$$
$$[R_2+(2e_1-3)R_1]\cdot{\lambda_x}^N=2(N-1+e_1){\lambda_x}^{N+1}+O({\lambda_x}^N).\tag{7.48}$$

Demanding that the leading term in the above vanishes is tantamount to truncating the representation of proposition 15 at the degree N. This truncation condition is precisely the one that leads to the para-Racah polynomials (7.2). Thus, the finite-dimensional representations of the rational degenerate Sklyanin algebra obtained under this truncation have for basis the para-Racah polynomials. □

The actions of the generators in these truncated representations are as given in proposition 15, although one has to carry the appropriate limiting process described in [**22**] to deal with the otherwise singular expressions. Proposition 16 provides for the algebraic interpretation of the para-Racah polynomials as the basis elements of the finite-dimensional representations of the rational degenerate Sklyanin algebra.

7.6. Conclusion

This paper has introduced the S–Heun operators associated to the quadratic grid as a special case of the algebraic Heun operator. These operators were shown to form a five-dimensional space. The subset of these operators which stabilizes the space of polynomials of a given degree was identified and the algebra that they realize was examined. The extension of this stabilizing algebra to a star algebra was identified as the rational degenerate Sklyanin algebra. This definition of the rational degenerate Sklyanin algebra through S–Heun operators directly led to the construction of infinite-dimensional representations on the Wilson polynomials as well as finite-dimensional representations on the para-Racah polynomials.

The rational degenerate Sklyanin algebra is known [**27**] to be a one parameter deformation of the Yangian $Y(\mathfrak{sl}_2)$. In the same way that the Yangian $Y(\mathfrak{sl}_2)$ is the quantum algebra that encodes the symmetry of integrable XXX spin-half chains associated with the ordinary rational R-matrix, the rational degenerate Sklyanin algebra can be understood as the symmetry algebra of a generalized XXX chain corresponding to a deformed rational R-matrix, a new integrable model. Thus, it would be of interest to use the representations introduced in section 7.5 to construct explicit realizations of this new integrable model in terms of finite and infinite spin chains. In the finite case, one would expect the para-Racah polynomials to appear as the basis of representations of the symmetry algebra. Interestingly, these para polynomials were first introduced in the context of perfect state transfer on spin chains [**32**] and the advances in this paper suggest they would also find applications as solutions to new integrable spin chain models.

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Appendix

7.6.1. Quadratic relations of the S–Heun operators

The set of homogeneous quadratic algebraic relations satisfied by the S–Heun operators is given below for reference:

$$[L, M_1] = 2L^2, \qquad [L, M_2] = \{M_1, L\}, \qquad [M_1, M_2] = \{M_2, L\} - 4L^2,$$

$$[L, R_1] = M_1^2+L^2+\{M_1, L\}+\frac{1}{2}\{M_2, L\}, \quad [L, R_2] = M_1^2+L^2+\{M_1, L\}+\frac{1}{2}\{M_2, L\}+\{M_1, M_2\},$$

$$[M_1, R_1] = 2M_1^2 - 3L^2 + \{M_1, M_2\} - \frac{1}{2}\{M_1, L\} - \{M_2, L\},$$

$$[M_1, R_2] = M_1^2 + M_2^2 + 7L^2 + 2\{R_2, L\} - \frac{5}{2}\{M_1, L\} - 5\{M_2, L\},$$

$$[R_1, M_2] = 3L^2 - M_1^2 - M_2^2 + 2\{R_1 + R_2, L\} - \{R_1, M_1\} - \{M_1, M_2\} - 5\{M_1, L\} - \frac{9}{2}\{M_2, L\},$$

$$[R_2, M_2] = Y^2 - M_1^2 - M_2^2 + \{R_1, M_1 - M_2\} - \{M_1, M_2\} + \frac{1}{2}\{M_1, L\},$$

$$\begin{aligned}[R_2, R_1] = 2R_1^2 + M_1^2 + 2M_2^2 + 3L^2 + \frac{1}{2}\{R_2 - R_1, L\} - \frac{3}{2}\{R_1 + R_2, M_2\} + \{M_1, M_2\} \\ + \frac{3}{2}\{M_1, L\} - \frac{1}{2}\{M_2, Y\},\end{aligned}$$

$$M_1^2 - \{M_1, M_2\} + 3L^2 = 1, \qquad \{R_1 - R_2, L\} + M_2^2 + \{M_2, L\} - 3L^2 = -3,$$

$$-2\{R_1, L\} - 3L^2 + \{M_1, M_2\} + 2\{M_1 + M_2, L\} = 4,$$

$$0 = M_1^2 + \frac{1}{2}L^2 + \{R_1, M_1 - M_2\} - \frac{5}{2}\{R_1, L\} - 2\{R_2, L\} + \{R_1 + R_2, M_1\} + \frac{1}{4}\{M_1, M_2\} + 6\{M_1, L\} + 4\{M_2, L\}.$$

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Partie 3

Émergence et intégrabilité

Introduction

Cette dernière partie de la thèse comportant un unique chapitre a pour but la construction d'un cadre théorique général permettant d'étudier et de quantifier la connexion entre la présence de structures dans un système physique et la capacité de le modéliser et de l'analyser. Ce travail repose sur les outils de la théorie algorithmique de l'information. Cette théorie peut être appréhendée comme une reformulation de la théorie de l'information de Shannon et des statistiques en termes de la complexité algorithmique. La différence importante réside en ce que la complexité de Kolmogorov joue le rôle de l'entropie de Shannon. Cette différence permet de construire une théorie de l'information dans laquelle toutes les quantités ne sont fonction que des échantillons obtenus, sans référence à une distribution de probabilité sous-jacente. Lorsqu'utilisée en physique, cette approche offre également une solution à l'ambiguïté dans la définition de l'entropie de Shannon provenant du degré de liberté dans le choix de l'alphabet de symboles.

Ce cadre théorique permet par après de formuler une définition mathématique du concept d'émergence. Cette approche comporte l'avantage de permettre une définition objective et rigoureuse de phénomène d'émergence. Le chapitre se termine par plusieurs exemples illustrant comment la définition proposée parvient à capturer la notion intuitive d'émergence.

Chapitre 8

An algorithmic approach to emergence

C. Bédard and G. Bergeron (2021). An algorithmic approach to emergence. *in preparation.*

Abstract

This article proposes a quantitative definition of *emergence*. Our proposal uses algorithmic information theory as a basis for an objective framework for the notion of emergence. Emergence would be marked by sudden drops of the Kolmogorov structure function. Our definition offers some theoretical results, in addition to an extension of the notions of coarse-graining and boundary conditions. Finally, we confront our definition with applications to dynamical systems and thermodynamics.

8.1. Introduction

Emergence is a concept often referred to in the study of complex systems. Coined in 1875 by the philosopher George H. Lewes in his book *Problems of Life and Mind* [**21**], the term has ever since mainly been used in qualitative discussions [**24, 6**]. In most contexts, emergence refers to the phenomenon by which properties of a complex system, composed of a large quantity of simpler subsystems, are not exhibited by those simple systems by themselves, but only through their collective interactions. The following citation from Wikipedia [**1**] reflects this popular idea: "For instance, the phenomenon of life as studied in biology is an emergent property of chemistry, and psychological phenomena emerge from the neurobiological phenomena of living things".

For claims such as the above to have any meaning, an agreed upon definition of emergence must be provided. Current definitions are framed around a *qualitative* evaluation of the "novelty" of properties exhibited by a system with respect to those of its constituent subsystems. This state of matters renders generic use of the term ambiguous and subjective, hence problematic within a scientific discussion. In this paper, we attempt to free the notion of emergence from subjectivity by proposing a mathematical, operational and *quantitative* notion of emergence.

8.1.1. Existing notions of emergence

We review a few of the many appeals to the notion of emergence. One of them goes all the way back to Aristotle's metaphysics [**26**]:

> «*The whole is something over and above its parts, and not just the sum of them all...* »

This common idea is revisited by the theoretical physicist Philip W. Anderson [**2**], who claims that "[...] the whole becomes not only more, but very different from the sum of its parts". In the same essay, he highlights the asymmetry between reducing and constructing:

> «*The ability to reduce everything to simple fundamental laws does not imply the ability to start from those laws and reconstruct the universe. In fact, the more elementary particle physicists tell us about the nature of the fundamental laws, the less relevance they seem to have for the very real problems of the rest of science, much less to those of society.*
>
> *The constructionist hypothesis breaks down when confronted with the twin difficulty of scale and complexity. [...] at each level of complexity, entirely new properties appear, and the understanding of the new behaviours requires research which I think is as fundamental in its nature as any other. [...] At each stage, entirely new laws, concepts, and generalizations are necessary, requiring inspiration and creativity to just as great a degree as the previous one. Psychology is not applied biology, nor biology is applied chemistry.* »

More recently, David Wallace [**34**, Chapter 2] qualifies emergent entities to be "not directly definable in the language of microphysics (try defining a haircut within the Standard Model) but that does not mean that they are somehow independent of that underlying microphysics". The notion of structures, or patterns, often related to the concept of emergence are specified by Dennett's Criterion [**12**] (the criterion was named by Wallace in [**33**]).

> «<u>*Dennett's Criterion:*</u> *An emergent object exhibits patterns. The existence of patterns as real things depends on the usefulness — in particular the*

explanatory power and the predictive reliability — of theories which admit those patterns in their ontology. »

Dennett's criterion, when applied to the notion of temperature, tells us that it should be thought to be an emergent but real concept because it is a *useful pattern*. As Wallace [**34**, Chapter 2] observes, even if temperature is not a fundamental entity of the microphysics, a full scientific description of a gas with no reference to the notion temperature completely misses a fundamental aspect. In this spirit, temperature is as real as it is useful. This notion of useful patterns, or structures, is a key concept that we shall formalize in our approach.

8.1.2. From systems to bit strings

Our proposed approach to emergence relies on algorithmic information theory. To better justify this formalism, we present our epistemological standpoints. We take the realist view that there is a world outside of our perception. This world is made of physical systems, and the goal of science is to understand their properties, their dynamics and their possibilities. This is done through an interplay between the formulation of theories and their experimental challenges. Theories have the purpose of providing simple models to explain the data, a concept which will be explored throughout the paper. Empirical observations, on the other hand, collect data from physical systems. The main epistemological question that we want to address here is how to get from a system, assumed to exist in reality, to a string of symbols that we shall take binary.

Observation starts by an interaction between the physical system we care to learn about and some measurement apparatus. The measurement apparatus then interacts with a computing device (this can be the experimenter) that arranges its memory in a physical representation of a bit string x. However, a scientist who wants to get data about a system will be left with an x, which, clearly, is not only determined by the investigated system. The information in x could reflect properties of other systems with which it has previously interacted, like the environment, the measurement apparatus and the scientist itself! As observed by Gell-Mann and Lloyd [**16**], this introduces several sources of arbitrariness into x, in addition to the level of details of the description and the coding convention that maps the apparatus's configuration into bits. Also, the knowledge and cognitive biases of the scientist impacts *what* is being measured. For Gell-Mann and Lloyd, all this arbitrariness is to be discarded in order to define the (algorithmic) information content of the object through that

$$\text{System} \xrightarrow{\text{Experimentation \& Observation}} \text{Data } x \xrightarrow{\text{Algorithmic Information Theory}} \text{Models \& Boundaries}$$

Fig. 1. From systems to algorithmic models

Systems are comprehended through experimentation and observation, which yield a bit string. Models and their respective boundaries can then be defined for each string through methods from algorithmic information theory.

of x. We don't share this view, as we think that this arbitrariness inhibits a well-posed definition. A subtlety of scientific investigation concerns what to probe of the system in order to push into x the yet-to-be-understood features that it can exhibit. Upon deciding what is to be measured, the ongoing challenge to the scientist lies in managing this arbitrariness so that the string x reflects the relevant properties of the system under observation.

Nonetheless, this subjective connection between the system and the data does not exclude an overall objective modelling of the world. For instance, if we ask a dishonest scientist to give us data about a system, but he elects instead to give us bits at whim, then investigating the data will lead to models of what was happening in that person's brain, which is itself a part of reality. Thus, the string x is always objective data from a real system, although not necessarily the one that was presumed to be under investigation.

Once the data x is fixed, we face the mathematical problem of finding the best explanations for it, which is related to finding its patterns or structures. This is the main investigation of the paper. It can be done in the realm of algorithmic information theory (AIT), a branch of mathematics and logic that offers similar tools as probability theory, but with no need for unexplained randomness. Li and Vitányi, authors of the most cited textbook [**22**] in the field, claim that "Science may be regarded as the art of data compression". And according to the pioneer Gregory Chaitin [**11**], "[A] scientific theory is a computer program that enables you to compute or explain your experimental data". Indeed, even theoretical pen and paper work constitutes symbolic manipulations which are inherently algorithmic. See Figure 1.

A point remains to be addressed. Why work with classical information and computation instead of their quantum counterparts? As quantum computation can be classically emulated [**13**], the quantum gain is only in speed, and not fundamental in terms of what we can or cannot compute. This work is grounded in computability theory, so by leaving aside questions of time complexity, we also leave aside quantum computation.

8.1.3. Outline

This paper is organized as follows. In section 2, we give a review of the basic notions of algorithmic information theory, with a particular focus on nonprobabilistic statistics and connexions in physics. Building on those, we introduce in section 3 an algorithmic definition of emergence and we derive from it some concepts and results. Finally, we illustrate the relevance of the proposed definition by discussing its uses in section 4 through examples. A brief conclusion follows.

8.2. A primer on algorithmic methods

Algorithmic information theory (AIT) originates [**28, 18, 10**] from the amalgamation of Shannon's theory of information [**27**] and Turing's theory of computation [**29**]. Introduced in his seminal paper titled "A Mathematical Theory of Communication", Shannon's theory concerns the ability to communicate a message that comes from a random source. The randomness, formalized in the probabilistic setting, represents ignorance, or unpredictability, of the symbols to come. The entropy is then a functional on the underlying distribution that quantifies an optimal compression of the message. Concretely, this underlying distribution is often estimated through the observed biases in the frequency of the sequences of symbols to transmit. However, noticing such biases is only a single way to compress a message. For instance, if Alice were to communicate the 10^{10} first digits of π to Bob, a pragmatic application of Shannon's information theory would be of no help since the frequencies of the symbols to transmit is uniform (if π is normal, which it is conjectured to be). However, Alice could simply transmit:

$$\text{`}\texttt{The first } 10^{10} \texttt{ digits of } 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\texttt{.}\text{'}$$

Bob then understands the received message as an instruction that he runs on a universal computing device to obtain the desired message. Equipping information theory with universal computation enables message compression by all possible (computable) means. As we will see, the length of the best compression of a message is a natural measure of the information contained in the message.

8.2.1. Algorithmic Complexity

We give the basic definitions and properties of algorithmic complexity. See Ref. [**22**, Chapters 1 – 3] for details, attributions and background on computability theory.

The *algorithmic complexity* $K(x)$ of a piece of data x is the length of the shortest computable description of x. It can be understood as the minimum amount of information required to produce x by any computable process. Per contra to Shannon's notion of information, which supposes an *a priori* random process from which the data has originated, algorithmic complexity is an *intrinsic* measure of information. Because all discrete data can be binary coded, we consider only finite binary strings (referred to as "strings" from now on), *i.e.*,

$$x \in \{0,1\}^* = \{\epsilon, 0, 1, 00, \dots\}\,,$$

where ϵ stands for the empty word. For a meaningful definition, we have to select a universal[1] computing device $\mathcal{U}$ on which we execute the computation to obtain x from the description. The latter is called the program p, and since it is itself a string, the length of p is well defined and noted $|p|$. Therefore,

$$K_{\mathcal{U}}(x) \equiv \min_p \{|p| \colon \mathcal{U}(p) = x\}\,.$$

Note that abstractly, $\mathcal{U}$ is any Turing-complete model of computation, such as Turing machines or recursive functions. Concretely, $\mathcal{U}$ could be thought of as a modern computer or a human with pen and paper. This is the essence of the Church-Turing thesis, according to which, all sufficiently generic approaches to symbolic manipulations are equivalent and encompass physically realizable computations. The invariance theorem for algorithmic complexity guarantees that no other formal mechanism can yield an essentially shorter

[1] In the realm of Turing machines, a universal device expects an input p encoding a pair $p = \langle q, i\rangle$ and simulates the machine of program q on input i.

description. This is because the reference universal computing device $\mathcal{U}$ can simulate any other computing device $\mathcal{V}$ with a constant overhead in program length, *i.e.*, there exists a constant $C_{\mathcal{UV}}$ such that

$$|K_{\mathcal{U}}(x) - K_{\mathcal{V}}(x)| \leq C_{\mathcal{UV}} \tag{8.1}$$

holds uniformly for all x. In such a case, it is customary in this field to use the big-O notation[2] and write $K_{\mathcal{U}}(x) = K_{\mathcal{V}}(x) + O(1)$. Since the ambiguity in the choice of computing devices is lifted (up to an additive constant), we omit the subscript $\mathcal{U}$ in the notation. Algorithmic complexity is in this sense a *universal* measure of the complexity of x.

The *conditional algorithmic complexity* $K(x \,|\, y)$ of x relative to y is defined as the length of the shortest program to compute x, if y is provided as an auxiliary input. Then one defines

$$K(x \,|\, y) \equiv \min_p \{|p| \colon \mathcal{U}(p, y) = x\}.$$

Multiple strings $x_1, \ldots, x_n$ can be encoded into a single one denoted $\langle x_1, \ldots, x_n \rangle$. The algorithmic complexity $K(x_1, \ldots x_n)$ of multiple strings is then defined as

$$K(x_1, \ldots, x_n) \equiv \min_p \{|p| \colon \mathcal{U}(p) = \langle x_1, \ldots, x_n \rangle\}.$$

For technical reasons, we restrict the set of programs resulting in a halting computation to be such that no halting program is a prefix of another halting program, namely, the set of halting programs is a *prefix code*. One way to impose such a constraint on the programs is to have all programs to be *self-delimiting*, meaning that the computational device $\mathcal{U}$ halts its computation after reading the last bit of the program p, but no further. This restriction is not fundamentally needed for our purposes, but it entails an overall richer and cleaner theory of algorithmic information. For instance, the upcoming relation (8.2) holds within an additive constant only if self-delimitation is imposed.

A key property of entropy in Shannon's theory is the chain rule that relates the entropy of a pair to those of the constituents. This is also achieved in the realm of AIT. Let x^* be the[3] shortest program that computes x. Algorithmic complexity satisfies the important

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[2] In general, $O(f(n))$ denotes a quantity that does not exceed $f(n)$ by more than a fixed multiplicative factor.

[3] If there are more than one "shortest program", then x^* is the fastest, and if more than one have the same running time, then x^* is the first in lexicographic order.

chain rule

$$K(x,y) = K(x) + K(y\,|\,x^*) + O(1)\,. \tag{8.2}$$

One obvious procedure to compute the pair of strings x and y is to first compute x out of its shortest program x^*, and then use x^* to compute y, which proves the "$\leq$" part of (8.2).

8.2.2. Nonprobabilistic Statistics

Standard statistics are founded upon probability theory. Curiously, the same person who axiomatized probability theory managed to detach statistics and model selection from its probabilisitic roots. Kolmogorov suggested [**19**] that AIT could serve as a basis for statistics and model selection for individual data. See Ref. [**30**] for a modern review.

In this setting, a *model* of x is defined to be a finite set $S \subseteq \{0,1\}^*$ such that $x \in S$. It is also referred to as an *algorithmic* or *nonprobabilistic statistic*. Any model S can be quantified by its cardinality, noted $|S|$, and by its algorithmic complexity $K(S)$, yielding a quantitative meaning of "simple" and "complex". To define $K(S)$ properly, let again $\mathcal{U}$ be the reference universal computing device. Let p be a program that computes an encoding $\langle x_1, \ldots, x_N\rangle$ of the lexicographical ordering of the elements of S and halts.

$$\mathcal{U}(p) = \langle x_1, \ldots, x_N\rangle\,, \qquad \text{where} \qquad S = \{x_1, \ldots, x_N\}\,.$$

Then, S^* is the shortest such program and $K(S)$ is its length. When S and S' are two models of x of the same complexity α, we say that S is *a better* model than S' if it contains fewer elements. This is because there is less ambiguity in specifying x within a model containing fewer elements. In this sense, more of the distinguishing properties of x are reflected by such a model. Indeed, among all models of complexity $\leq \alpha$, a model of smallest cardinality is *optimal* for this fixed threshold of complexity.

Any string x of length n exhibits two canonical models shown in Table 1. The first is simply $S_{\text{Babel}} \equiv \{0,1\}^n$, which has small complexity as it is easy to describe — a program producing it only requires the information about n. However, it is a large set, containing 2^n elements. It is intuitively a bad model since it does not capture any properties of x, except its length. The other canonical example is $S_x \equiv \{x\}$. This time, S_x has large complexity, namely, it is as hard to describe as x is, but it is a very tiny set with a single element. S_x is a bad model, capturing *everything* about x, even the noise or incidental randomness. This

Model S	Complexity	$K(S)$	Cardinality	$\log\|S\|$
$S_{\text{Babel}} = \{0,1\}^n$	Small	$K(n) + O(1)$	Large	n
$S_x = \{x\}$	Large	$K(x) + O(1)$	Small	0

Tableau 1. Complexity and cardinality of S_{Babel} and S_x

significantly weighs down the description of the model, and is commonly known as *over-fitting.* A good map of Montreal is not Montreal itself!

If S is a model of x, then

$$K(x \mid S) \le \log |S| + O(1)\,,$$

because one way to compute x out of S is to give the $\lceil \log |S| \rceil$ bit-long index of x in the lexicographical ordering of the elements of S, where $\lceil \cdot \rceil$ denotes the ceiling function.[4] This trivial computation of x relative to S is known as the *data-to-model code*[5]. A string x is a *typical* element of its model S if the data-to-model code is essentially the shortest program, *i.e.*, if

$$K(x \mid S) = \log |S| + O(1)\,.$$

In such a case, there is no simple property that singles out x from the other elements of S. Notice also that the data x can always be described by a *two-part description:* The model description and the data-to-model code. Hence,

$$K(x) \le K(S) + \log |S| + O(1)\,. \tag{8.3}$$

In his seminal paper [**14**] on the foundations of theoretical (probabilistic) statistics, Fischer stated: "The statistic chosen should summarize the whole of the relevant information supplied by the sample. This may be called the *Criterion of Sufficiency.*". Kolmogorov suggested an algorithmic counterpart. A model $S \ni x$ is *sufficient* for x if the two-part

[4] Note that the program can be made self-delimiting at no extra cost because the length of the index can be computed from the resource S provided.

[5] Really, it should be called model-to-data.

description with S as a model is an almost-shortest description, namely,

$$K(S) + \log|S| \leq K(x) + O(\log n)^6 .$$

Such models constrains the set of strings to those sharing "the whole of the relevant" properties that characterize x, which is then a typical element of those models.

Finally, a good model should not give more than the relevant information supplied by the data. The simplest sufficient model displays all the relevant properties of the data, and nothing more, thus preventing over-fitting. It is called the *minimal sufficient model* and denoted S_M.

Kolmogorov's structure function

For a given string x, its associated Kolmogorov's structure function explores trade-offs between the complexity and cardinality of possible models. This function maps any complexity threshold to the log-cardinality of the optimal model S within that threshold. It will be applied to investigate more interesting models, between S_{Babel} and S_x.

Definition 8.2.1 (Structure function)**.** The *Structure function* of a string x, $h_x : \mathbb{N} \to \mathbb{N}$, is defined as

$$h_x(\alpha) = \min_{S \ni x} \{\lceil \log|S| \rceil \ : K(S) \leq \alpha\} .$$

We say that optimal models of α bits or less *witness* $h_x(\alpha)$. Extremal points of $h_x(\alpha)$ are essentially determined by S_{Babel} and S_x, as shown in Table 1. It follows that

$$h_x(K(n) + O(1)) \leq \log|S_{\text{Babel}}| = n \qquad \text{and} \qquad h_x(K(x) + O(1)) \leq \log|S_x| = 0 .$$

An upper bound for h_x is prescribed by noticing that a more complex model S', can be built from a previously described one S by including into the description of S' the first bits of index of $x \in S$. In this case, for each bit of index specified, the log-cardinality of the resulting model reduces by one. This implies that the overall[7] slope of the structure function must be ≤ -1. Applying this argument to S_{Babel}, we conclude that the graph of $h_x(\alpha)$ is

[6] Here the $O(\log n)$ refers to $K(K(S), \log|S|)$ since the self-delimited 2-part code implicitly carry the length of each part as its intrinsic information. The optimal one part code x^* in general shall not know about the size of each part.

[7] A knowledgeable reader may frown upon this simple and not-so-precise argument because prefix technicalities demand a more careful analysis as is done in [**31**]. Such an analysis shows that the linear relations as presented here hold up to logarithmic fluctuations.

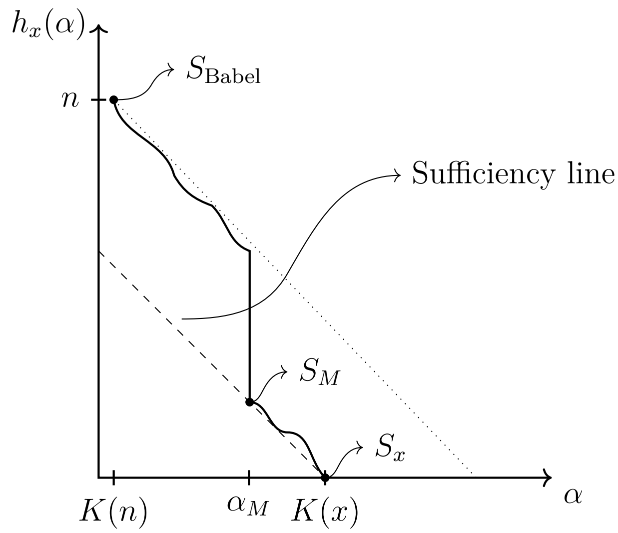

Fig. 2. Kolmogorov's structure function of a string x of length $|x| = n$.

upper bounded by the line $n + K(n) - \alpha$. A lower bound is obtained from applying (8.3) to the model S witnessing $h_x(\alpha)$. In such a case, $K(S) \leq \alpha$ and $\log |S| = h_x(\alpha)$, so

$$K(x) - \alpha \leq h_x(\alpha) .$$

This means that the graph of $h_x(\alpha)$ always sits above the line $K(x) - \alpha$, known as the *sufficiency line*. The above inequality turns into an equality (up to a logarithmic term) if and only if the witness S is a sufficient model, by definition. Thus, this sufficiency line is reached by the structure function when enough bits of model description are available to formulate a sufficient statistics for x. Once the structure function reaches the sufficiency line, it stays near it, within logarithmic precision, because it is then bounded by above and by below by the -1 slope linear regime. The sufficiency line is always reached as S_x is such a sufficient model.

For concreteness, a plot of $h_x(\alpha)$, for some string x of length n, is given in Figure 2. In this example, the string x is such that optimal models of complexity smaller than α_M are not teaching us much about x. Indeed, bits describing those models are used as inefficiently as an enumeration of x. In sharp contrast, S_M is exploiting complex structures in x to efficiently constrain the size of the resulting set. It is fundamentally different from the optimal model of $\alpha_M - 1$ bits as it does not recite trivial properties of x, but rather express some distinguishing property of the data. Indeed, from α_M bits of model, the uncertainty about x is decreased by much more than α_M bits, as x is now known to belong to a much smaller set. In this example, S_M is the minimal sufficient statistics.

The complexity of the minimal sufficient statistics, α_M, is known as the *sophistication* of the string x, which captures the amount of algorithmic information needed to grasp all structures — or regularities — of the string. Technically, here we refer to set-sophistication, as defined in Ref. [**3**], since sophistication has been originally defined [**20**] through total functions as model classes instead of finite sets. Importantly, Vitányi has investigated [**32**] three different classes of model: Finite sets, probability distributions (or statistical ensembles, *c.f.* the following section) and total functions. Although they may appear to be of increasing generality, he shows that they are not. Any model of a particular class defines a model in the other two classes of the same complexity (up to a logarithmic term) and log-cardinality or its analogues.

8.2.3. Algorithmic information theory in physics

Ideas of using AIT and nonprobabilistic statistics to enhance the understanding of physical concepts are not new. For example, expanding on the famous Landauer principle [], Bennett [**7**] suggested that thermodynamics is more a theory of computation than a theory of probability, so better rooted in AIT than in Shannon information theory. Based on his work, Zurek proposed [**35**] the notion of *physical entropy*, which generalizes thermodynamic entropy to ensure consistency. In the case of a system with microstate x, the physical entropy is defined based on the statistical ensemble P. The latter is very similar to an algorithmic model for x, except that in general a non-uniform probability distribution governs the elements of P, so the amount of information needed to specify an element $x' \in P$, on average, is given by Shannon entropy $H(P) = -\sum_{x'} P(x') \log P(x')$. Important paradoxes, such as the famous Maxwell's demon [**23, 7**] or Gibbs' paradox [**17**], appears when it is realized that the ensemble P, and hence the entropy of the system, depends upon the knowledge d held by the agent, *i.e.*, $P = P_d$. Such knowledge is usually given by macroscopic observations such as temperature, volume and pressure, and defines an ensemble P_d by the principle of maximal ignorance [**25**]. However, a more knowledgeable — or better equipped — agent shall gather more information d' about the microstate, which in turn defines a more precise ensemble $P' \ni x$. This leads to incompatible measures of entropy. Zurek's physical entropy

S_d includes the algorithmic information contained in d as an additional cost to the overall entropy measure of the system,

$$S_d = K(d) + H(P_d)\,.$$

Note that the similarity with Equation (8.3) is not a mere coincidence. Zurek's physical complexity encompasses a two-part description of the microstate. First, describe a model — or an ensemble — for it. Second, give the residual information to get from the ensemble to the microstate, on average. In fact, when the ensemble takes a uniform distribution over all its possible elements, Shannon's entropy $H(P)$ reduces to the log-cardinality of the ensemble, which is, up to a $k_B \ln 2$ factor, Boltzmann's entropy.

With sufficient data d, the physical entropy S_d gets close to the complexity of the microstate $K(x)$. The ensemble P_d is then analogous to a sufficient statistics. Indeed, Baumeler and Wolf suggest [**5**] to take the minimal sufficient statistics as an objective — observer independent — statistical ensemble (they call it *the* macrostate). Gell Man and Lloyd define [**16**] the complexity $K(P_d)$ of such a minimal sufficient ensemble to be the *Effective Complexity* of x. Because of Vitányi's aforementioned equivalence between model classes, effective complexity is essentially the same as sophistication (See also [**4**, lemma 21]). Müller and Szkola [**4**] show that strings of high effective complexity must have very large logical depth.

8.3. Defining emergence

The previous discussion of the Kolmogorov structure function made manifest the fact that a drop, like displayed in Figure 2 at α_m, was associated to distinguished models that accounted for meaningful properties of the data. In this example, all such properties were reflected in the description of S_m. In general, this does not have to be the case. In this spirit, what should one think of a string whose structure function is as displayed in Figure 3? In fact, it is natural to inquire about the properties of a string with many drops in its structure function. With only a few bits of model, not much can be apprehended of x. With slightly more bits, there is a first model, S_1, capturing some useful properties of x, which leads to a more concise two-part description. Allowing even more bits, a second model S_2 is possible. While being more complex, this second model reflects more properties of x in such a way as to yield an even smaller two-part description. This series of models continues as the allowed complexity increases. Eventually, the structure function reaches the minimal sufficient statistics S_M, after which more complex models are of no help in capturing meaningful properties of x.

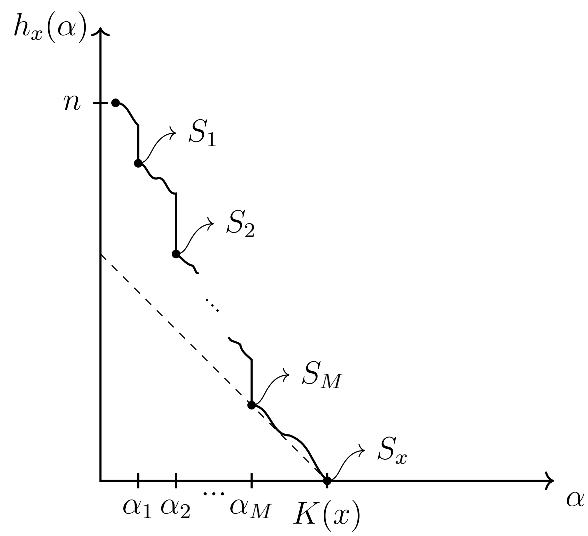

Fig. 3. A structure function with many drops.

Before going further, a points needs to be addressed: Do strings with such a structure function exist? In Ref. [**31**], it is shown that *all shapes are possible*, *i.e.*, for any graph exhibiting the necessary properties mentioned in the previous section, there exists a string whose structure function lies within logarithmic resolution of that graph.

These observations illustrate that models can prove useful when not displaying all relevant properties of the data. Those "partial" models, while not sufficient, enable a more efficient description of the data with respect to all models of lower complexity. Thus, in the same way that a model witnessing the minimal sufficient statistics is understood to capture the meaningful properties of the data, those intermediate models can be thought of as capturing only some of those meaningful properties. It is from this notion that the proposed definition of emergence is constructed.

The main idea of our proposal is to relate emergence to the phenomenon by which the experimental data x exhibits a structure function with many drops. They feature regularities that can be grasped at different levels of complexity and tractability.

8.3.1. Towards a definition

In order to sharply define the models corresponding to drops of the structure function, and to make precise in which sense these are "new" and "understand" more properties, we construct a modified structure function upon which we formalize these notions.

Induced models

As discussed briefly in the previous section, one can construct models canonically from a given model by appending to it bits of the index of the data x in that model.

Definition 8.3.1 (*Induced models*)**.** For a model $S \ni x$ and $i \in \{0, \ldots, \lceil \log |S| \rceil\}$, the induced model $S[i]$ is given by the subset of S whose first i bits of index are the same as those of x.

For concreteness, one way to produce such an $S[i]$ is to first execute the (self-delimiting) program that computes S, and then concatenate the following program:

$$\underbrace{\texttt{The following program has } i+c \texttt{ bits.}}_{K(i)+O(1)\ \texttt{bits}} \tag{8.4}$$

$$\underbrace{\texttt{Among the strings of } S\texttt{, keep those whose index start with}}_{c\ \text{bits}}\ b_1 b_2 b_3 \ldots b_i\,.$$

where the first line of the routine is only for the sake of self-delimitation. Note that this concrete description of $S[i]$ implies

$$K(S[i]) \leq K(S) + i + K(i) + O(1).$$

Furthermore, for every bit of index given, the model $S[i] \ni x$ so defined contains half-fewer elements than S does. Hence,

$$\log|S[i]| = \log|S| - i.$$

As can be seen from the program (8.3.1), specifying i bits of index requires more than i extra bits of model description. Thus, we define $\delta \equiv i + K(i) + c'$, where c' accounts for the constant-size part of program (8.4). A unique inverse of the relation is defined as

$$\bar{\imath}(\delta) = \max_i \{i \colon i + K(i) + c' = \delta\},$$

which represents the number of index bits that can be specified with δ extra bits of model desciption. Note that the difference between δ and $\bar{\imath}(\delta)$ is of logarithmic magnitude. We denote the induced model $S(\delta) \equiv S[\bar{\imath}(\delta)]$. Thus, one has that

$$h_x(K(S)+\delta) \leq \log|S(\delta)| = \log|S| - \bar{\imath}(\delta).$$

Induced structure function

We now introduce our modified structure function. First, we define, for each α, a model of complexity less or equal to α, which has log cardinality very close to $h_x(\alpha)$. Intuitively, we do so by mapping α to the[8] witness of $h_x(\alpha)$ whenever α corresponds to a drop of the structure function. And whenever the structure function is in a -1 slope regime, we map α to an induced model that builds upon the last witness of $h_x(\alpha)$. Formally, let α_0 be the smallest complexity threshold for which $h_x(\alpha_0)$ is defined. Define the sequence of models $\{S^{(\alpha)}\}$ recursively through

$$(S^{(\alpha_0)}, k_{\alpha_0}) = (S, \alpha_0) \text{ with } S \text{ a witness of } h_x(\alpha_0) \tag{8.5}$$

$$(S^{(\alpha)}, k_\alpha) = \begin{cases} (S^{(k_{\alpha-1})}(\alpha - k_{\alpha-1}), k_{\alpha-1}) & \text{if } \log|S^{(k_{\alpha-1})}(\alpha - k_{\alpha-1})| - h_x(\alpha) < Q(\alpha) \\ (S, \alpha) \text{ with } S \text{ a witness of } h_x(\alpha) & \text{otherwise,} \end{cases}$$

where the quantity $Q(\alpha)$ is determined precisely in the proof of Theorem 8.3.3 and can be upper bounded[9] by

$$K(h_x(\alpha)|\alpha) + O(\log\log n)\,.$$

Note that the set of numbers $\{k_\alpha\}$ corresponds to the set of α for which there are significant drops in the structure function. As can be seen from the above definition, a "significant drop" corresponds to a decrease of ε in the structure function, which is beyond what is naturally entailed by inducing the model one bit further.

Definition 8.3.2 (*Induced structure function*)**.** The induced structure function $\tilde{h}_x(\alpha)$ is defined as

$$\tilde{h}_x(\alpha) = \log|S^{(\alpha)}|.$$

It follows from this definition that $\tilde{h}_x$ lies just above h_x, within an additive term smaller than ε. Why define an induced structure function $\tilde{h}_x$, which is very close to the original structure function h_x? An important difference is that the construction of the induced

[8] If the witness of $h_x(\alpha)$ is not unique, we choose the fastest one produced by α-bit programs.

[9] The precise quantity is $\max_{N>B(\alpha-\ell^*(m))}\{K(m|\alpha,N)\} + 2\ell^*(\ell^*(n)) + O(1)\,.$

structure function $\tilde{h}_x(\alpha)$ in (8.5) keeps track of the actual models used at each complexity threshold. This has two advantages. First, a "drop" of the structure function is clearly identified: it corresponds to a point in the construction of the induced structure function where the model used is updated rather than induced. Second, for two neighbouring points α and β in a slope -1 regime, nothing guarantees that the model witnessing $h_x(\alpha)$ and $h_x(\beta)$ are not completely different. They could *a priori* be completely different models, capturing completely different properties about the string x, but it just happens that the difference of their log-cardinality is roughly $\beta - \alpha$. On the other hand, the defining models of $\tilde{h}_x$ are constructed in a way that the -1 slope forces the models to be induced from the same original model. They simply contain more or less of the index of x. Finally, departure from the slope -1 regime in the function $\tilde{h}_x$ indicates that a new model is used, one that intuitively captures other properties of x.

Minimal Partial models as a signature of emergence

We have emphasized in the construction of the induced structure function a difference between the slope -1 regime and the drops of the structure function. Indeed, while the former amounts to induced models, the latter corresponds to relevant yet partial models. These will be central to our proposed definition of emergence.

Definition 8.3.3 (*Minimal partial models*)**.** The minimal partial models are defined as the witnesses of the drops of $\tilde{h}_x$, namely the models $\{S^{(k_\alpha)}\}_{\alpha\in\{\alpha_0,\ldots,K(x)+O(1)\}}$ as defined in (8.5).

In what follows, we denote by S_1, S_2, $\ldots$, S_M the successive minimal partial models with respective complexity $\alpha_1 < \alpha_2 < \ldots < \alpha_M$. Minimal partial models are the interesting models, out of all the optimal models witnessing the structure function.

Definition 8.3.4 (*Emergence*)**.** Emergence is the phenomenon characterized by observation data that display several minimal partial models.

It can be seen that the above definition maintains the generality expected of the notion of emergence, allowing for it to be applied in many different contexts. Moreover, it will be seen to allow for a mathematical treatment of various related notions.

In view of the comments of section 1.2, emergence is a function of the observation string x and not necessarily of the real object that x is purported to represent. For instance, in the case of the dishonest scientist who disregards the object under investigation to give bits at whim, any emergence displayed by x would arise from the system that produced those bits, itself a part of reality.

8.3.2. Quantifying emergence

Under the proposed definition of emergence, we develop quantitative statements. In this section, three theorems are presented. We do not claim complete originality, as these results use ideas that have been previously developed in algorithmic statistics [**15, 31, 30**]. The novelty lies in the formulation in terms of minimal partial models.

The data specifies the minimal partial models

The first theorem confirms a basic intuition. The minimal partial models should be thought of as optimized ways to give the structural information about x. In particular, the following theorem shows that most of their algorithmic information is in fact information *about* x.

Theorem 8.3.1. The minimal partial models S_i can be computed from x and a logarithmic advice,

$$K(S_i|x^*) = K(K(S_i), \lceil \log |S_i| \rceil \mid x^*) + O(1) = O(\log n)\,.$$

Using the chain rule in Eq. (8.2) twice allows to expand $K(x, S_i)$ in two ways:

$$\begin{aligned} K(x, S_i) &= K(x) + K(S_i|x^*) + O(1) \\ &= K(S_i) + K(x|S_i^*) + O(1)\,, \end{aligned}$$

so another way to phrase the theorem is

$$K(x) = K(S_i) + K(x \mid S_i^*) + O(\log n)\,,$$

such that producing S_i in order to get x is not a waste, in fact, it is almost completely a part of the algorithmic information of x.

Proof. We give a program q of length $K(K(S_i), \lceil \log|S_i| \rceil \mid x^*) + O(1)$ that computes S_i out of x^*.

```
q : Compute K(S_i) and ⌈log|S_i|⌉ from x* (if useful)
    Run all p of length K(S_i) in parallel
    If p halts with U(p) = ⟨S⟩:
        If log|S| ≤ ⌈log|S_i|⌉ and U(x*) = x ∈ S:
            Print S and halt.
```

$\square$

Partial understanding

We now justify the use of the term "partial" to qualify the non-sufficient minimal partial models. Intuitively, sharp drops of the structure function should be in correspondance with non-trivial properties of the underlying string. The minimal partial models at those points should encompass an "understanding" of these properties. Naturally, the magnitude of this understanding could be equated to the size of the drop. Theorem 8.3.2 confirms this idea, when "understanding" holds the following meaning.

In the context of AIT, understanding amounts to reducing redundancy, as a good explanation is a simple rule that accounts for a substantial specification of the data. For instance, when one understands a grammar rule of some foreign language, that rule can be referred to in order to explain its many different instantiations. Those instantiations are redundant, and once the grammar rule is specified, this redundancy is reduced.

Definition 8.3.5. The *Redundancy* of a string x of length n is defined to be

$$\mathrm{Red}(x) \equiv n - K(x|n)\,.$$

The redundancy of a string is thus the number of bits of a string that are not irreducible algorithmic information. In other words, it is the compressible part of x. Redundancy could then be thought of as a quantification of how much there is to be understood about x upon learning x^*. Comparing x to x^*, however, is an all or nothing approach and the a purpose of nonprobabilistic statistics is to make sense of partial understanding by studying (two-part) programs for x that interpolate between the "`Print` x" and the x^* explanations. The next definition, in some sense, generalizes redundancy so that it can be relative to an algorithmic model.

Definition 8.3.6. The *Randomness deficiency* of a string x, with respect to the model S is

$$\delta(x|S) \equiv \log|S| - K(x|S)\,.$$

It measures how far x is from being a typical element of the set. Indeed, a typical element would have $K(x|S) = \log|S| + O(1)$ so that $\delta(x|S)$ essentially vanishes. Notice that redundancy can be recovered from

$$\delta(x|S_{\text{Babel}}) = n - K(x|S_{\text{Babel}}) = \text{Red}(x) + O(1)\,.$$

We can then explore how much each minimal partial model reduces the randomness deficiency — or understands — the data x. Define d_i as the height of the drop just before getting to S_i, namely,

$$d_i \equiv \tilde{h}_x(\alpha_i - 1) - \tilde{h}_x(\alpha_i)\,.$$

Theorem 8.3.2. The height of the i-th drop measures how much more S_i reduces the randomness deficiency, compared to S_{i-1}, *i.e.*,

$$\delta(x|S_{i-1}) - \delta(x|S_i) = d_i + O(\log n)\,.$$

Proof. Using the chain rule in Eq. (8.2) twice, which amounts to a bayesian inversion, and Theorem 8.3.1,

$$\begin{aligned}
\delta(x|S_i) &= \log|S_i| - K(x|S_i) \\
&= h_x(\alpha_i) - K(x) - K(S_i|x) + K(S_i) + O(\log n) \\
&= h_x(\alpha_i) - K(x) + \alpha_i + O(\log n)\,. \qquad (8.6)
\end{aligned}$$

With the help of Figure 4, and recalling that if δ extra bits of model description are given, $\bar{\imath}(\delta) = \delta + O(\log n)$ bits of index can be given, observe that

$$\begin{aligned}
h_x(\alpha_{i-1}) - h_x(\alpha_i) &= h_x(\alpha_{i-1}) - h_x(\alpha_i - 1) + h_x(\alpha_i - 1) - h_x(\alpha_i) \\
&= \bar{\imath}(\alpha_i - 1 - \alpha_{i-1}) + d_i \\
&= \alpha_i - \alpha_{i-1} + d_i + O(\log n)\,.
\end{aligned}$$

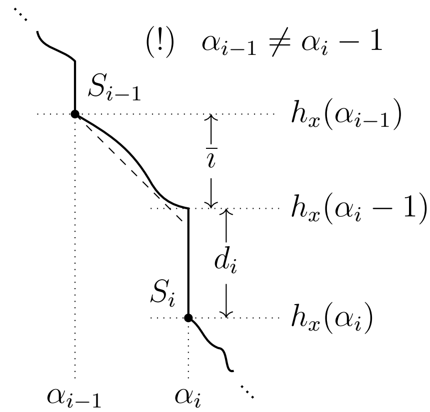

Fig. 4. A visual help for the proof of Thm 8.3.2.

Using Equation (8.6),

$$\begin{aligned}\delta(x|S_{i-1}) - \delta(x|S_i) &= h_x(\alpha_{i-1}) - h_x(\alpha_i) + \alpha_{i-1} - \alpha_i + O(\log n) \\ &= d_i + O(\log n)\,.\end{aligned}$$

□

We can then interpret the algorithmic information in the minimal partial models as being parts of the algorithmic information of x that enables a reduction of the redundancy of x. This reduction of redundancy can be quantified by the sum of all previous drops, and the amount of redundancy left to be reduced is the sum of the drops to come. When the minimal sufficient statistic is described, with only those α_M bits of the algorithmic information in x, the redundancy of x is completely reduced. The remaining information left to specify is then the index of $x \in S_M$, which is itself irreducible algorithmic information in x. However, this information does not contain the relevant structural information about x.

Hierarchy of minimal partial models

The following theorem shows that the algorithmic information in the minimal partial models is organized in a nested structure, namely, the complex minimal partial models can compute the simpler ones with a logarithmic advice.

Theorem 8.3.3. For $j > i$,

$$K(S_i|S_j) = \alpha_i - \alpha_{i-1} + O(\log n)\,.$$

Proof Sketch.

For this proof we use the results of [**3**], which links set-sophistication with Busy Beaver logical depth. This result implies that the shortest programs for the minimal partial models will run for so long that they are mainly consisted of halting information. Once it is shown that S_i and S_j are made of i and j bits (up to logarithmic error), respectively, of irreducible halting information, it becomes necessary that S_i can be computed by S_j (within logarithmic error).

The proof is relegated to Appendix A.

8.3.3. Extending concepts

We revisit the notions of coarse-graining and boundary conditions, broadening their scope.

A notion of coarse-graining

Many approaches to emergence appeal to some notion of coarse-graining. For instance, the relevant quantities of a physical system might correspond to functions over state space. In this case, an important tool consists of averaging those quantities over regions of that space, retaining only the large scale structures, as is done in the method of effective field theories. In the context of algorithmic statistics, coarse-graining will be seen as a special case of what we will call *regraining*. We begin by defining the notion of coarse-graining precisely.

In set theory, coarse-grainings are defined from some mother set Ω. A *partition* $\mathcal{P}$ of Ω is a collection of disjoint and non-empty subsets such that their union gives back Ω. Let $\mathcal{P}_{\text{fine}}$ and $\mathcal{P}_{\text{coarse}}$ be two partitions of Ω. We say that $\mathcal{P}_{\text{fine}}$ is a *refinement* of $\mathcal{P}_{\text{coarse}}$ if every element in $\mathcal{P}_{\text{fine}}$ is a subset of some element of $\mathcal{P}_{\text{coarse}}$. We also say that $\mathcal{P}_{\text{coarse}}$ is a *coarse-graining* of $\mathcal{P}_{\text{fine}}$. In physics, those partitions are usually specified through non-injective functions globally defined on the state space via the pre-images.

The key point of nonprobabilistic statistics is to investigate an individual object x, without needing to refer to other x' in the set of bit strings. Hence, algorithmic models are disconnected from the notion of partition, since a single set is defined specifically for x, with no requirement to define a corresponding set for x'. As such, algorithmic models do not partition bit strings. Still, an algorithmic model $A \ni x$ could be qualified as a *model coarse-graining* of a $B \ni x$ if $B \subseteq A$. This type of "model coarse-graining" in fact occur in the regime of induced models. However, if we compare minimal partial models to each other, even if they are of different cardinality, they are in general not subsets of one another[10]. This motivates the extended notion of *regraining*, which is simply a change from some model $A \ni x$ to $B \ni x$, where neither model needs to be a subset of the other. It is qualified as a *fine* regraining if $|B| < |A|$ and a *coarse* one if $|B| > |A|$. Model coarse-grainings are particular cases of regrainings.

The *optimal regraining* corresponds to jumping along the minimal partial models. The optimal coarse regraining occurs in the direction from S_M to S_1, and corresponds to modelling less and less features of the data x to the benefit of having simpler and simpler models. It is optimal in the sense that this procedure will yield the best possible models over all complexity thresholds. As opposed to the usual approaches, where the coarse-graining occurs by averaging over specific variables (space, momentum, time…), our notion of regraining is parametrized by theory size. Properties of a physical system could be such that the optimal coarse regraining happens by averaging over space configurations, so the algorithmic regraining would boil down to the usual methods.

Boundary conditions

Recall from §8.1.2 that an intrinsic difficulty of scientific investigation is that the recorded data x never perfectly reflects a single system. Even if we leave aside the effect of the measurement apparatus and the scientist on the data, it remains that systems are never completely isolated from an environment. As any interaction mediates an exchange of information, the effect of a large and complex environment will be modelled as random noise[11] in models of small complexity. However, if the string x is sufficiently detailed, some structures of the environmental "noise" shall be grasped by models complex enough. This highlights that some information in x may be explained by models of large complexity but

[10] Although nothing garanties that $S_j \subset S_i$, for $j > i$, S_j cannot be almost entirely composed of elements that are not in S_i. In fact, Theorem 8.3.3 states that S_i can be easily computed from S_j, so with slightly more than α_j bits of model size, an optimal model would be $S_i \cap S_j \ni x$, which, cannot be of too small cardinality unless the structure function exhibits a drop right after α_j.

[11] An example of this situation is given by the dissipation-fluctuation theorem [**8**] that relates dissipative interactions in a system to the statistical fluctuations around its equilibrium point. Indeed, this theorem relates dissipation, an irreversible process that does not preserve information, with noise in the form of statistical fluctuations.

remain unexplained by a simple model. Such information could be thought to reside outside of such a simple model, namely, in its data-to-model code. This suggests to interpret the data-to-model code as the *boundary* of the model.

Definition 8.3.7. The *boundary conditions of the model S* corresponding to the data x is the index of x in S.

In this definition, the scope of the term is broadened from its physical meaning, so that it can be thought as the boundary of a model S, namely, what from the system that generated the observational data $x \in S$, is *not modelled* by S. The remaining structure in x is then viewed as coming from non-typical boundary conditions forced by interactions with an environment. In the case of the minimal sufficient statistics S_M, the typicality of x in S_M captures the fact that the boundary conditions are arbitrary with respect to the model.

The traditional space-time boundary conditions of a system are an example of what is usually relegated to the data-to-model code, as models usually dont aim at explaining them. Another example are the precise values of mechanical friction coefficients. Within classical mechanics, these values come from outside the theory and would thus be a part of the boundary conditions when understood as per definition 8.3.7. However, with more precise observations, one could explain the values of the coefficients from a more precise model that encompasses molecular interactions. More example are provided in the following section.

8.4. Examples

The versatility of the proposed approach to emergence is now illustrated through some examples. This section is not meant to be an exhaustive review of the possible uses of these definitions, but should rather be understood as an illustrative complement to the main exposition.

8.4.1. Simulation of a 2D gas toy model

As a first example, we consider a toy model for a non-interacting 2D gas on a lattice. The gas is taken to be spatially confined on an $L \times L$ grid with a discrete time evolution. Using a pseudorandom number generator, we choose an initial position and momentum for each of the N particles. Each momentum is only a direction in the set $\{l, r, u, d\}$, corresponding to left, right, up and down. The gas then evolves according to simple rules. A single particle, represented by a 1 in the lattice, just keeps its trajectory and momentum, as in Figure 5. When it bounces off a boundary, its momentum gets flipped, as in Figure 6. Intersecting trajectories are represented as displayed in Figure 7.

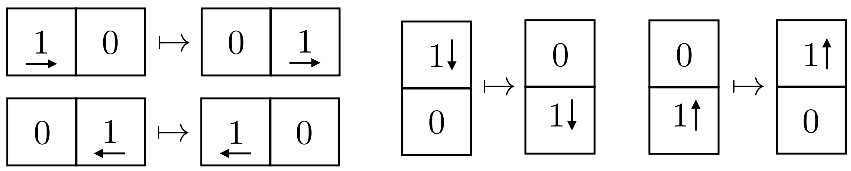

Fig. 5. A gas particle freely moving.

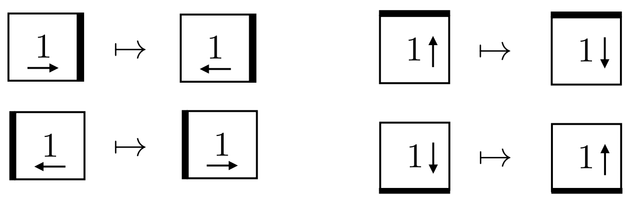

Fig. 6. A particle bouncing off walls.

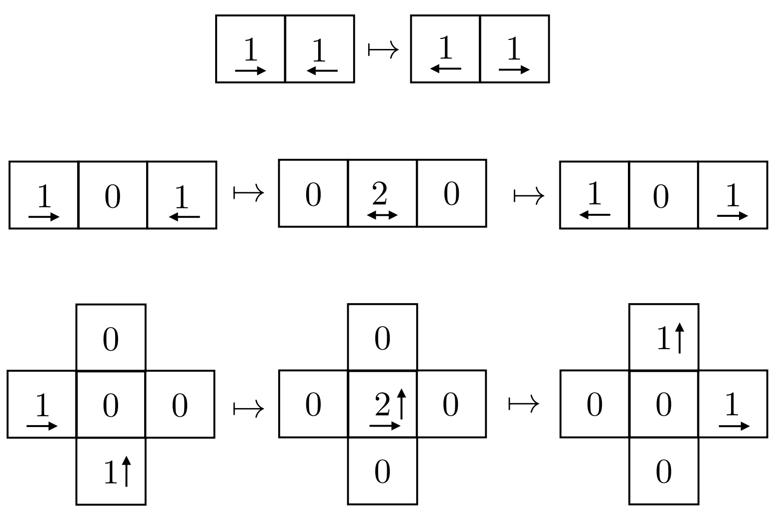

Fig. 7. Particles "collide" as if they go through one another.

At any time (including the initial time), if two or more particles are at the same site, we simply write down the number of particles in the site and keep track of the momenta. As an observation x, we extract for each of the first T time steps the state in configuration space (*i.e.* we ignore momentum). One visual way to encode the state in configuration space is to write in each of the L^2 sites a 0, and write to the left of it, in unary, the number of particles in that site. For instance, a 3×3 grid example of this coding is given in Figure 8.

$$\begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline 0 & 1 & 2 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|}\hline 10 & 0 & 0 \\ \hline 0 & 10 & 110 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \mapsto 1000010110000$$

Fig. 8. Encoding of the configuration state into bits.

At each time step, the bit string corresponding to the configuration state has one 0 for each site of the grid, and one 1 for each particle, for a total length of $L^2 + N$. By concatenation, the observation x so generated is a bit string of length $|x| = (L^2 + N)T$.

Because algorithmic complexity is uncomputable, so is the structure function. However, it can be *upper semi-computed*, which means that there is an algorithm that keeps outputting better upper bounds of the structure function until it eventually reaches the actual structure function. When this happens, the algorithm does not halt, as it keeps looking for better upper bounds, not knowing that this is in vain. In our generic context of finding explanations for observation data, this upper semi-computation is done by scientific research finding simpler and better models. In the specific case of the simulated 2D gas, we got x from the known context of the simulation, which provides important clues to find models other than the obvious S_{Babel} and $\{x\}$.

A first model that comes from the simulation specifies the parameters L and N, external to the gas, together with the final time T. Compared to $\{x\}$ listing everything about the simulation, simplicity is gained by leaving open the initial conditions. This defines the set $S_{\text{Gas}}^{L,N,T}$ of all configuration histories of T iterations, for each possible initial conditions of N particles confined to a $L \times L$ grid. The size of this program is smaller than $K(L, N, T) + O(1)$, since the evolution rules are of constant length.

Even simpler models can be made by pushing into the boundaries the particular values of the external parameters L, N or T. For illustration, we make the argument only for T fully specifying L and N. We now suppose T to be expressed by τ bits in a binary expansion. The model $S_{\text{Gas}}^{L,N,T}$ can then be simplified by producing, for each possible initial condition, *all* histories of length smaller than 2^τ. We denote this set as $S_{\text{Gas}}^{L,N,<2^\tau}$. Its cardinality is 2^τ times bigger, thus adding τ to the log-cardinality axis, but if T were a random number,

$$K(T) = \tau + K(\tau) + O(1)\,, \tag{8.7}$$

exactly τ bits would be saved on the complexity axis, since only τ, as opposed to T would be needed to compute the model. In general however, T is not algorithmically random, but $S_{\text{Gas}}^{L,N,<2^\tau}$ only encodes a basic upper bound for T, leaving it outside of what is modelled. In a similar fashion, other complexity thresholds can be introduced for L and N, pushing again their information into the boundary conditions.

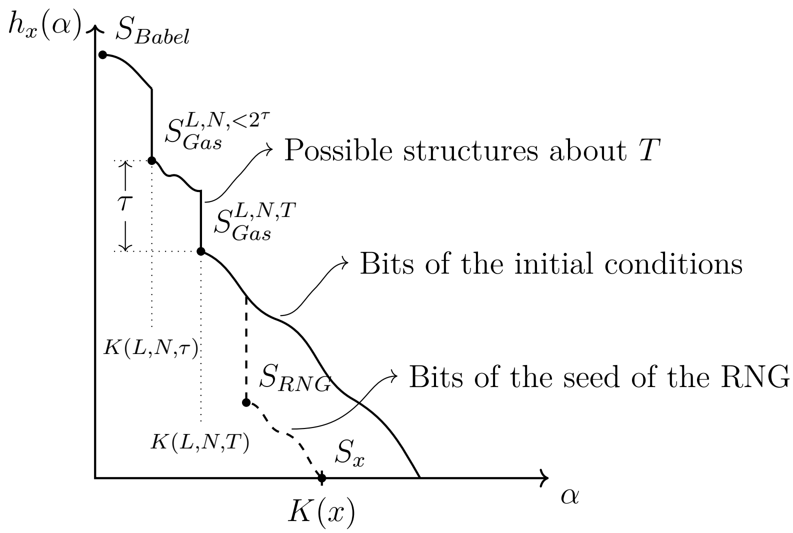

Fig. 9. In solid is the known upper bound of the structure function. In dashed is the hypothesized real structure function.

As presented in Figure 9, the previously discussed upper bound of the structure function is likely to be different from the real structure function. In particular, in the simulation of the gas, the initial conditions were not algorithmically random, as they came from a sufficiently shortly pseudorandom number generator, a program that on an input seed gives a sufficiently long string. This means that this program, together with its seed, is shorter than an enumeration of the initial conditions it generated, which the real structure function will reflect through one more drop at a higher level of complexity. The witness of this drop, S_{RNG}, is a model that explains the initial conditions as coming from the pseudorandom number generator. It is the set of all gas histories compatible with the dynamics previously described, and where the initial conditions have been generated with the pseudorandom program, the seed being relegated to the data-to-model code. If the slope after S_{RNG} remains in a -1 regime, it means that the seed is typical, among allowed seeds. However, this seed comes from another physical system, for instance the programmer itself. Yet again, if the seed is long enough, the structure function could potentially find more drops that again capture more structures. This process will go on until all that can be explained has been explained.

This example makes clear that the notion of boundary conditions really refers to a theory (or an algorithmic model), and are fixed somewhat arbitrarily, when the users of the theory are satisfied with their notion of the system that is being modelled. In this case, if what we wanted to model was the gas, then $S^{L,N,T}_{\mathrm{Gas}}$ was good enough, and it was practical to declare that the initial state was typical. But the reality may be quite different, and what we prescribe as a boundary condition to our theory may in fact be explained by a more complex, deeper theory.

8.4.2. Dynamical systems

In this second example, we review how the notions introduced in this paper appear in the more general setting of dynamical systems. We begin by documenting how the concept of integrability and chaos can be cast in the language of algorithmic information theory. This is followed by an account of how thermodynamics can be seen to emerge, under the proposed definition of emergence, from the application of statistical mechanics to complex dynamical systems.

From integrability to chaos

Consider a generic classical system with Hamiltonian H and where the state space M is indexed[12] by a set of real coordinates $X = \{q_i, p_i\}_{i\in\{1,\ldots,\dim M/2\}} \in M$. Solutions to the dynamics are curves in M describing the evolution of the state in time. Specifying M, H and an initial point $X_0 \in M$ singles out a unique solution curve X_t of the dynamics. As a rudimentary formalization of some observation of the system, consider a bounded observable represented by a function f with $f : M \to [0,1]$. A discrete sequence is constructed from its evaluation $\{f(X_{n\tau})\}_{n\in\{1,\ldots,N\}}$ at a regular time interval $0 < \tau \in \mathbb{Q}$ with negligible $K(\tau)$.

[12]More precisely, M is a symplectic manifold parametrized locally by real coordinates forming an atlas.

As this sequence is to represent a series of measurements, one must restrict its resolution. Indeed, in the laboratory, as well as in numerical simulations, the values measured are always constrained to a finite resolution. For a real number α, we denote by $[\alpha]_k$ the truncation of its binary expansion after the first k bits beyond the decimal point, *i.e.*, $|[\alpha]_k - \alpha| \leq 2^{-k}$. This truncation effectively restricts the resolution to k bits as the measurement function is upper-bounded by 1. Denoting by $f_n^k \equiv [f(X_{n\tau})]_k$ the restricted measurements, the recorded observational data string x is then an encoding of the sequence of measurements:

$$x \equiv \langle \{f_n^k\}_{n\in\{1,\ldots,N\}} \rangle.$$

We now wish to characterize the complexity of the data string x and study its asymptotic behaviour when the length N of the measurement sequence is increasing. First, one must formulate a meaningful upper bound for $K(x)$. To that end, we require f to preserve information, that is

$$K([f(X)]_k \,|\, [X]_k) = O(1).$$

A trivial bound is then given by the bit length of the encoded sequence of measurements, thus

$$K(x) \leq kN + O(\log kN\,).$$

However, the regularity provided by the laws of motion implies that this bound is not strict. Indeed, given the Hamiltonian H and the manifold M, the machinery of symplectic geometry specifies the dynamical evolution as a set of differential equations that we will denote as $\langle H, M \rangle$. These equations can be integrated numerically from the initial conditions X_0 to obtain f_n to a desired precision. These remarks, together with the stated condition on f, imply that

$$K(x) \leq K(\langle M, H \rangle, \tau, k, N, X_0) + O(1)\,.$$

The above can be further simplified in view of studying the asymptotic behaviour in N by observing that the dynamical laws $\langle M, H \rangle$, the time interval τ and the resolution k are fixed and independent of N. Thus, as the length of x is scaled by increasing N, they can be taken to be constant[13]. Hence, one has

$$K(x) \leq K(N) + K(X_0) + O(1)\,. \tag{8.8}$$

Remembering that X_0 encodes the initial conditions, which are a set of real numbers which cannot be constructively specified in general, one is left with a conundrum. Indeed, if X_0 encodes typical real numbers, the upper bound (8.8) is trivial as the right-hand side is infinite. However, only a finite precision in the initial conditions is required in order to integrate the system to a given precision in the final result. Thus, the resolution in X_0 required is only as much as is needed to compute $\{f_n^k\}_{n\in\{0,1,\dots,N\}}$. As such, the asymptotic behaviour of $K(x)$ for $N \to \infty$ is determined by the scaling in the required resolution.

A chaotic dynamical system is often characterized by an exponential divergence in the evolution of nearby initial configurations, namely

$$\frac{|X_t' - X_t|}{|X_0' - X_0|} = e^{\lambda t}\,,$$

Where $|\cdot|$ denotes a metric on M and λ is known as the Lyapunov exponent. In such a chaotic system,

$$|X_0' - X_0| < 2^{-\lambda' n - k} \qquad \Longrightarrow \qquad |X_{n\tau}' - X_{n\tau}| < 2^{-k}\,, \qquad \text{with} \qquad \lambda' = \lambda\tau\ln 2\,,$$

so k bits of precision on $X_{n\tau}$ can be achieved by $k + \lambda' n$ bits of precision on X_0. Therefore, the computation of $X_{N\tau}$ from the initial condition is more efficient, in terms of description length, than straightforward enumeration if $k + \lambda' N \leq kN$, which is

$$\lambda' \leq k - \frac{k}{N}\,. \tag{8.9}$$

This means that for some values of Lyapunov exponent λ and precision k, it could be more efficient to simply recite the observed data $\{f_n\}_{n\in\{1,\dots,N\}}$ as a genuinely random string. However, no matter how large the Lyapunov exponent is, there will always be a regime of precision for which it is more efficient to calculate $\{f_n\}_{n\in\{1,\dots,N\}}$ from enough bits of initial conditions. Concretely, the precision on the initial conditions that can be obtained is bounded by the resolution of measurement devices. A more practical approach accounts for this with a fixed resolution $k' > k$ in the initial conditions and is thus limited to the truncation $[X_0]_{k'}$. This, together with the Lyapunov exponent of the system under consideration, determines

[13]To simplify the analysis, it is tacitly assumed that the dynamical laws are simple in the sense that the coefficients of the differential equations in $\langle H, M\rangle$ are rational numbers.

an interval of predictability within which the observational data x can be compressed. To preserve predictability beyond this interval, one is forced to update[14] his knowledge of the state of the system with a measurement. The phenomenon is well-known within chaos theory and shows up as a fundamental limitation to the predictability of such systems, a common exemple of which is weather.

Dynamical systems can generally be organized by considering the asymptotic of the string of measurements x with $N \to \infty$. At one end of the spectrum lie integrable systems, where k bits of knowledge of X_0 can be used all the way through to compute k bits of f_N. Those are systems where integration can be carried symbolically without an accumulation of errors. On the other side of this spectrum are chaotic systems, where $k + \lambda' N$ bits of X_0 are required to compute k bits of f_N. Similar classification schemes for dynamical systems that account for integrability and the appearance of chaos based on computational complexity have been proposed previously [**9**]. An algorithmic perspective on dynamical systems brings the possibility of considering other types of systems, where $k + g(N)$ bits of X_0 can be used to compute k bits of f_N, with $g(N)$ some *a priori* generic function.

Thermodynamics and statistical mechanics

Statistical mechanics posits the ergodicity of a complex dynamical system in order to obtain a partial, yet useful, description of its behaviour. This partial description is mostly understood to refer to the macroscopic description of a system displaying intractable microscopic descriptions. The generic approach is as follows. Starting again with a Hamiltonian and the associated phase space M, one first investigates the quantities conserved by the time evolution. By fixing those conserved quantities, one establishes constraints that restrict the accessible phase space to a bounded region. Properly defined, those constraints effectively decompose[15] the phase space into a family of submanifolds $F \subseteq M$ that are each preserved by time evolution. The ergodic hypothesis now posits that the curves X_t produced by an initial point $X_0 \in F$ under the time evolution are dense in each submanifold F such that the time average value of an observable $\mathcal{O} : M \to \mathbb{R}$, over such a curve is equal to the average of

[14] It is here assumed that the Lyapunov exponent is constant and unique, which is not always the case.

[15] More precisely, these constraints generate a foliation of phase space that is invariant under the Hamiltonian flow.

the same quantity over a uniform measure on each submanifold F,

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_{t_0}^{t_0+T}\mathcal{O}(X_t)dt=\int\limits_{F_{X_0}\subseteq M}\mathcal{O}(X)d\mu(X),$$

for μ the uniform measure over F. This uniform measure over the submanifolds F is often specified indirectly in terms of the Boltzmann weights of a state $X\in M$ over the submanifolds. With the above in mind, thermodynamics can be seen as the study of the interrelation of a relevant collection of macroscopic observables $\{\mathcal{O}_i\}$, expressing the change in the value of some observables in terms of the change in the value of the others. Such a thermodynamic description of a complex system is partial yet useful and relevant to the scale at which one would like to investigate the system.

Let us now concentrate on how this very generic picture of statistical mechanics and its relation to thermodynamics fits under our proposed definition of emergence. We first define the truncation $[F]_k$ of a submanifold $F\subset M$ to resolution k as the truncation[16] of all coordinates of F to a k-bit resolution. Then, positing the ergodicity of the system under study enables a direct reframing of statistical mechanics in terms of the ideas of this paper. Indeed, the postulated uniform measure on submanifolds of the phase space M amounts to postulating the corresponding microscopic states[17] in a submanifold to be equally likely under time evolution. In other words, for some large enough finite time interval τ, the sequence

$$x_N\equiv\langle\{[X^{(i)}_{n\tau}]_k\}_{i\in\{1,\ldots,\dim F\},\,n\in\{1,\ldots,N\}}\rangle,\quad\text{for}\quad {X^{(i)}}_{i\in\{1,\ldots,\dim F\}}\quad\text{coordinates on F},$$

is a typical sample of the truncated submanifold $[F]_k$. The lower bound on the time interval τ that needs to be satisfied for the above to hold is related to the Lyapunov exponent of the system. Indeed, such a bound corresponds to time intervals satisfying the converse of (8.9). In such a case, x_N is essentially an algorithmically random string. From this observation, it follows that for a sufficiently large time interval, one has that

$$K(x_N)=K([F]_k)+N\log(|[F]_k|),\tag{8.10}$$

[16]Technically this truncation depends on the chart, but we take an encoding of F to include an atlas and one for $[F]_k$ to include a prescription on the choice of chart in which truncate each points.

[17]Possibly with the exception of a measure zero set of states that are not relevant to the averaging of observables.

which indicates that the model $[F]_k$ for the string x_N is an algorithmic sufficient statistics.

The above discussion emphasized how thermodynamics, together with the ergodic hypothesis, amount to postulating that the models in (8.10) associated to the decomposition into invariant submanifolds are sufficient. Indeed, a thermodynamical description of the system at equilibrium is in correspondence with such a decomposition of the phase space, provided that the conserved quantities that define the submanifolds are taken to be the thermodynamical variables.

8.5. Conclusion

We proposed a definition of emergence casted in the language of algorithmic information theory. This field has many times shown its usefulness to mathematically address mathematics itself.

Intuitively, emergence is the appearance of unforeseen dynamics or properties exhibited by a complex system. In most discussions about emergence, the criteria of novelty highly depends upon the field: The aerodynamicist may be stunned by new patterns in fluid dynamics; The biochemist, by new ways in which enzymatic networks interact. In our proposed definition, emergence occurs in *"theory space"*: the thresholds of emergence are marked by the complexity of new models, which enable an over-all shorter expression of the observed data. This is the essence of *understanding new structures*. These models (sets of finite bit strings) are as general as they can be, since they are rooted in universal computation: Any "new pattern in fluid dynamics" or "enzymatic networks interaction" that can be described is amenable to a computational process and thus an algorithmic model.

The development of our proposal was done through the "locally best models" of Kolmogorov's structure function. We called them the minimal partial models. In §8.3, we proved that:

(1) The data specifies almost everything about the minimal partial models;

(2) The magnitude of the drop measures the amount of "new understanding";

(3) Deeper minimal partial models almost specify the shallower ones.

We also extended the notions of coarse-grainings and boundary conditions, freeing them from any specific theory. In §8.4 we considered some applications to a toy model of a gas, dynamical systems and thermodynamics.

The absolute generality of algorithmic information theoretic methods come at the price of uncomputability. For instance, the shapes of Figure 9, in §8.4.1, are only conjectured. No program can return the structure function of a piece of data x. Nevertheless, the definition provides a precise framework to discuss the notion of emergence. A relaxation to the context of limited computational resources may be of interest in order to find concrete utility and applications in real life computations. While some of the results obtained might not hold anymore, the definition itself can still be applied within this limited computational context.

We recognize that the concepts involved in the proof of Theorem 8.3.3 challenge the reconciliation between our mathematical proposal and the youth of our Universe. For a long string, the deep models, namely those that occur at late drops of the structure function, are likely the result of programs that terminate after an unthinkably long computation. They have the largest finite running times among all programs no larger in size, so they solve the halting problem for shorter programs. With a mere 14 billion years old, our Universe seems too young to accommodate such computations unless the information was in the initial conditions. However, assessing the actual computational capabilities of all physical phenomenons in the entire Universe is problematic. It may also be the case that strings that appear in nature are confined to the relatively shallow models. Even in the case of deep emergent models that cannot realistically be modelled, the identification of a string as having such models is not necessarily a deep model itself. This suggests the possibility of structures one can recognize and yet never model.

Facing the realization that models witnessing drops of the structure functions are made of halting information, Vereshchagin and Shen [**30**] wrote "This looks like a failure. [...] [I]f we start with two old recordings, we may get the same information [about their minimal sufficient statistic], which is not what we expect from a restoration procedure. Of course, there is still a chance that some Ω-number [halting information] was recorded and therefore the restoration process indeed should provide the information about it, but this looks like a very special case that hardly should happen for any practical situation." Facing this, they suggest to consider models of more restricted classes or add some additional conditions and look for "strong models".

On the contrary, we think that that the minimal sufficient statistics of two recordings *should* share information, as they inevitably share a very common origin, which the model aims to capture. That this shared information is about the halting problem simply reflects the fact that their plausible common origin is the fruit of a very long computation, and not that the recording has anything to do with an Ω-number, or any representation of the halting problem.

Acknowledgements

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A. Appendix: proof of theorem 8.3.3

Theorem A.1. For $j > i$,

$$K(S_i|S_j) = \alpha_i - \alpha_{i-1} + O(\log n)\,.$$

PROOF.

Definition A.1. The *busy beaver* is a function $\mathrm{B} : \mathbb{N} \to \mathbb{N}$ defined by

$$\mathrm{B}(n) \equiv \max\,\{\mathrm{RT}(p) \colon \mathcal{U}(p) \searrow \text{ and } |p| \leq n\}\,.$$

It is the maximal finite running time of a program of n bits or less.

We say that p is a (τ, ℓ)-*program* for x if $|p| = \ell$ and $\mathrm{B}(\tau - 1) < \mathrm{RT}(p) \leq \mathrm{B}(\tau)$. The latter condition means that there is a τ-bit halting program that runs for at least as long as p runs, but none of length $\tau - 1$ or less. For a string x, define the *time profile* as the boundary of the region

$$\mathcal{L}_x = \{(\tau', \ell') : \tau' \geq \tau,\ \ell' \geq \ell,\ \exists(\tau, \ell)\text{-program for } x\}\,.$$

We say that S is a (α, m)-*two part description* for x if $x \in S$, $\log|S| = \alpha$ and $K(S) + \log|S| = m$. For a string x, define the *description profile* as the boundary of the region

$$\Lambda_x = \{(\alpha', m') : \alpha' \geq \alpha,\ m' \geq m,\ \exists(\alpha, m)\text{-two part description for } x\}\,.$$

The description profile is closely related to Kolmogorov's structure function $h_x(\alpha)$: it is the largest monotonic decreasing lower bound of its affine transformation $h_x(\alpha)+\alpha$.

Proposition A.1. $\exists$ a (α, m)-two part description $\implies$ $\exists$ a (τ, ℓ)-program, with $\tau \leq \alpha + O(1)$ and $\ell \leq m + O(1)$.

PROOF. A two-part description for x is almost a program for x. The first part is given by S^*, the second part is i_x, the index of x in S. To make a well-defined program from the concatenation $S^* i_x$, one needs only a $O(1)$ instruction as a preamble. Note that the second part of the code does not need any additional prefix for self-delimitation, since its length $|i_S^x| = \lceil \log |S| \rceil$ can be computed (by the preamble) from S^*. The resulting program has length $m + O(1)$.

A bound on the running time is obtained on the two parts of the code. The first part runs for time $\leq \mathrm{B}(\alpha)$, the most conservative bound for an α-bit program. The second part is fast: its running time is linear in $|S|$, so for more elaborated models than S_{Babel} (*i.e.*, $\alpha \geq K(n)$) the running time is at most exponential in n, so smaller than $\mathrm{B}(\alpha)$. The conclusion follows from $2\mathrm{B}(\alpha) \leq \mathrm{B}(\alpha + O(1))$. $\square$

Proposition A.2. $\exists$ a (τ, ℓ)-program $\implies$ $\exists$ a (α, m)-two part description, with $\alpha \leq \tau + K(\ell|\bar{q}_\tau)$ and $m \leq \ell + \max_{N > B(\tau-1)}\{K(\tau|\ell, N)\}$, where $\bar{q}_\tau$ is the last halting program of length τ.

PROOF. From a (τ, ℓ)-program p for x, we can define the set of programs

$$M' = \{q \colon |q| = \ell \;\&\; B(\tau - 1) < \mathrm{RT}(q) \leq B(\tau)\} \ni p\,,$$

from which we define the model $M = \{\mathcal{U}(q) \colon q \in M'\} \ni x$ serving as the (α, m)-two-part description.

$$\alpha = K(M) \leq K(M') + O(1) \leq K(\ell, \bar{q}_\tau) + O(1) \leq \tau + K(\ell|\bar{q}_\tau) + O(1)\,.$$

We used that $K(\bar{q}_\tau) \leq \tau + O(1)$, which holds because $\bar{q}_\tau$ is already a self-delimiting program.

Consider any program $q \in M'$. Using the chain rule[18] in two different ways,

$$\begin{aligned} K(q, \bar{q}_\tau, \ell) &= K(\bar{q}_\tau, \ell) + K(q|\bar{q}_\tau, \ell, K(\bar{q}_\tau, \ell)) \\ &= K(q) + K(\bar{q}_\tau, \ell|q, K(q)) \,, \end{aligned}$$

so

$$\begin{aligned} K(q|\bar{q}_\tau, \ell, K(\bar{q}_\tau, \ell)) &= K(q) + K(\bar{q}_\tau, \ell|q, K(q)) - K(\bar{q}_\tau, \ell) \\ &\leq \ell + K(\tau|\ell, \mathrm{RT}(q)) - K(\bar{q}_\tau, \ell) \\ &\leq \ell + \max_{N > B(\tau-1)} \{K(\tau|\ell, N)\} - K(\bar{q}_\tau, \ell), \end{aligned}$$

The second line was obtained by noticing again that q a ℓ-bit self-delimiting program and by observing that $\mathrm{RT}(q)$ can be computed from q, and if τ is given, $\bar{q}_\tau$ can be computed from $\mathrm{RT}(q)$. In the third line, to obtain bounds as tight as possible, while liberating the expression from a dependence in q, we take the largest majorant which still encompasses a possible compression of τ from some halting knowledge, represented by a large number N. A less precise expression would have simply been $K(\tau|\ell)$.

In general, the number of strings s with $K(s|z) \leq b$ is smaller than $2^{b+1} - 1$, because there are only this many programs short enough. Hence,

$$\log |M| \leq \log |M'| \leq \ell - K(\bar{q}_\tau, \ell) + \max_{N > B(\tau-1)} \{K(\tau|\ell, N)\} \,,$$

and so

$$m = K(M) + \log |M| \leq \ell + \max_{N > B(\tau-1)} \{K(\tau|\ell, N)\}.$$

□

Proposition A.2 is about the existence of a model not too far on the up-right of a program. However, we would like to bound the region for the path of the time profile, given the description profile. Given a (α, m)-two-part description that is optimal (*i.e.*, of minimal m for a given α), the time profile cannot admit programs too far on the lower-left of (α, m),

[18] Note that the chain rule as written in Equation (8.2) conditions on x^*. Here we condition on $(x, K(x))$, which is informationally equivalent to x^*. In fact, $(x, K(x))$ is computed from x^* and a $O(1)$ advice which measures x^* before executing it, while x^* can be computed from $(x, K(x))$ by running in parallel all programs of length $K(x)$ until one of them produces x. This program is x^*.

otherwise, the aforementioned proposition would contradict the optimality the (α, m)-two-part description. We know that "lower" is quantified by $\max_{N>B(\tau-1)}\{K(\tau|\ell, N)\}$, while "left", by $K(\ell|\bar{q}_\tau)$. But these quantities are expressed in terms of τ and ℓ, and need to be bounded in terms of α and m:

Proposition A.3.

$$K(\ell|\bar{q}_\tau) \leq \max_{N>B(\alpha-\ell^*(m))}\{K(\alpha|m, N)\} + 2\ell^*(\ell^*(m)) \equiv \delta(\alpha, m),$$

where $\ell^*(m) \equiv \log m + \log\log m + \log\log\log m + ... + \log^* m$, with the sum taken over non-negative terms only with $\log^* m$ defined as the number terms in the sum.

Proposition A.4.

$$\max_{N>B(\tau-1)}\{K(\tau|\ell, N)\} \leq \epsilon(\alpha, m) \leq \max_{N>B(\alpha-\ell^*(m))}\{K(m|\alpha, N)\} + 2\ell^*(\ell^*(n)) \equiv \epsilon(\alpha, m).$$

The course of the Λ_x-profile ensures that the $\mathcal{L}_x$-profile is not too far away. In fact, for each point (α, m) of the boundary of the Λ_x-profile can be drawn the course of $(\alpha+O(1), m+O(1))$, as well as $(\alpha-\delta(\alpha, m), m-\epsilon(\alpha, m))$. A "drop" of the structure function is defined precisely to ensure that the time profile has dropped of one. Therefore, there is a drop when the description profile drops by more than $\epsilon(\alpha, m)+O(1)$, compared to the previous model settled on.

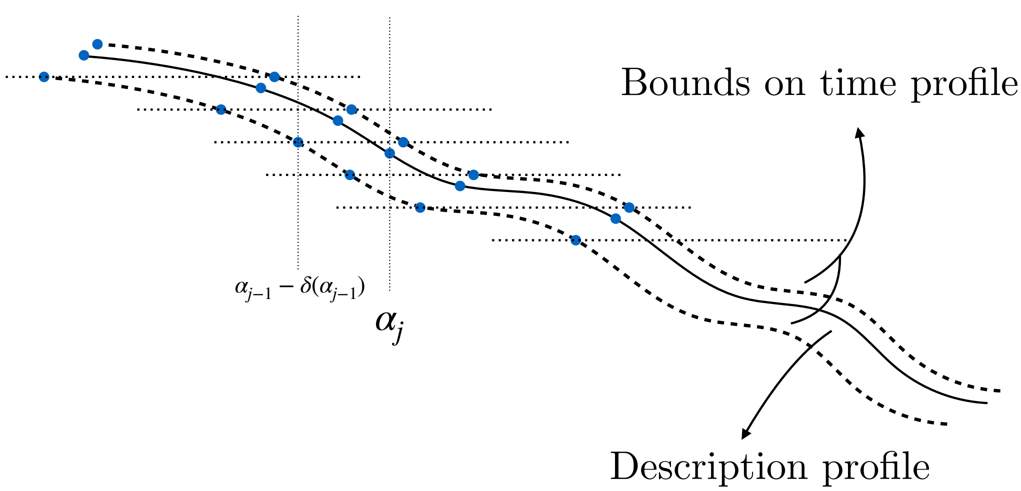

Fig. 10. A visual help for the proof of theorem 8.3.3

A glance at Figure 10 indicates that the model S_j corresponding to the drop has its running time lower bounded by $B(\alpha_{j-1} - \delta(\alpha_{j-1}, m) + O(1))$. Otherwise, the program (determined by the two-part description) of length $m + O(1)$ runs in a time that contradicts the time profile. An upper bound on the running time for S_j^* is easily obtained as $B(\alpha_j)$.

Hence, for $i < j$, S_i contains at least $\xi \equiv \alpha_{i-1} - \delta(\alpha_{i-1})$ bits of irreducible halting information, namely, S_i^* can be used to compute ω_ξ, the number of programs of length ξ or less that halts. Using the fact that $K(\omega_\xi) = \xi$,

$$\alpha_i + K(\xi|S_i^*) = K(S_i) + K(\xi|S_i^*) = K(S_i, \omega_\xi) = K(\omega_\xi) + K(S_i|\omega_\xi, \xi) = \alpha_{i-1} - \delta(\alpha_{i-1}) + K(S_i|\omega_\xi, \xi)\,.$$

The model S_j contains more halting information than S_i, so if ξ is given, S_j^* can be used to compute ω_ξ. Hence,

$$K(S_i|S_j^*) \le K(\xi|S_j^*) + K(S_i|\omega_\xi, \xi) = K(\xi|S_j^*) + K(\xi|S_i^*) + \alpha_i - \alpha_{i-1} + \delta(\alpha_{i-1}) = \alpha_i - \alpha_{i-1} + O(\log n)\,.$$

□

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Conclusion

Les travaux présentés dans cette thèse suggèrent plusieurs avenues de recherches intéressantes qui sont regroupés ici.

- Les travaux réalisés au chapitre 2 suggère la possibilité d'obtenir une expression pour les polynômes de q-Krawtchouk bivariés les plus généraux, en analogie avec les cas classiques. En effet, une expression formelle peut-être obtenue. Cependant, le résultat s'exprime comme un polynôme de variables non commutatives qui ne peut pas être réexprimé en variables abéliennes avec les méthodes utilisées. Il est possible que l'orthogonalité de l'expression, lorsqu'exprimée en termes de variables abéliennes, puisse tout de même être évaluée directement.
- Les résultats du chapitre 3 suggèrent l'identification d'autres polynômes orthogonaux dans les représentations d'algèbres simples. Un premier pas serait l'obtention des polynômes en haut du schéma de Askey, tels que les polynômes de Wilson et de Racah. La simplicité de la structure algébrique utilisée est prometteuse en ce qui a trait à l'application de ces résultats.
- La notion d'opérateur de Heun algébrique s'est avérée d'une grande utilité. Comme illustré au chapitre 4, les opérateurs de Heun généralisés et les nouvelles structures algébriques associées mènent à des solutions numériquement efficaces aux problèmes de limitation en bande et en fréquence. Il est alors d'un grand intérêt pratique d'expliciter les problèmes de traitement de signal dont la solution s'obtient de ces opérateurs de Heun algébriques et des nouvelles structures algébriques associées, telles que celles présentées au chapitre 5.
- Les représentations construites aux chapitres 6 et 7 pour les algèbres de type Sklyanin devraient permettre la diagonalisation de modèles intégrables sur des chaînes de spin.

En particulier, le modèle correspondant au cas du chapitre 7 est un nouveau système intégrable introduit par Smirnov.

- Il serait pertinent de développer des approximations calculables aux méthodes introduites dans le chapitre 8 pour en permettre l'application en pratique. En particulier, une étude explorant quels résultats peuvent être maintenus dans un contexte de ressources de calcul limitées serait nécessaire. Ensuite, un modèle de calcul flexible, tel que des réseaux de neurones, pourrait être utilisé en pratique.

Contributions de l'auteur

Cette annexe détaille les contributions de l'auteur aux travaux présentés dans cette thèse en conformité avec les exigences de la FESP.

- Chapitre 1: Idée originale de LV et ST. Calculs et rédaction par GB.
- Chapitre 2: Idée originale de LV. Supervision par LV et EK. Calculs et rédaction par GB. Révision par LV et EK.
- Chapitre 3: Idée originale par LV et AZ. Calculs initiaux AB et AB supervisé par JG. Complétion calculs GB. Rédaction JG et GB.
- Chapitre 4: Idée originale par LV et AZ. Calculs initiaux LV et AZ. Révision calculs et rédaction GB.
- Chapitre 5: Idée originale par LV, ST et AZ. Supervision et révision NC et LV. Calculs et rédaction GB.
- Chapitre 6: Idée originale par LV et AZ. Calculs et rédaction JG. Collaboration GB.
- Chapitre 7: Idée originale par LV et AZ. Calculs et rédaction par GB. Collaboration JG.
- Chapitre 8: Idée initiale GB. Calculs et rédaction CB et GB.

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