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Generalized Mahler measure of a family of polynomials

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Abstract

In this thesis we consider a variation of the Mahler measure where the defining integral is performed over a more general torus. Our work is based on a tempered family of polynomials originally studied by Boyd, $P_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + k$ with $k \in \mathbb{R}_{\geq 4}$. For the $k = 4$ case we use special values of the Bloch–Wigner dilogarithm to obtain the Mahler measure of P_4 over an arbitrary torus $\mathbb{T}^2_{a,b} = \{(x,y) \in \mathbb{C}^* \times \mathbb{C}^* : |x| = a, |y| = b\}$ with $a, b \in \mathbb{R}_{>0}$. Next we establish a relation between the Mahler measure of P_8 over a torus $\mathbb{T}^2_{a,\sqrt{a}}$ and its standard Mahler measure. The combination of this relation with results due to Lalín, Rogers, and Zudilin leads to a formula involving the generalized Mahler measure of this polynomial given in terms of $L'(E,0)$. In the end, we propose a strategy to prove some similar results for the general case $k > 4$ over $\mathbb{T}^2_{a,b}$ with some restrictions on a, b.

Keywords : Mahler measure, Bloch–Wigner dilogarithm, L-functions of elliptic curves, arbitrary torus, regulator.

Résumé

Le présent mémoire traite une variation de la mesure de Mahler où l'intégrale de définition est réalisée sur un tore plus général. Notre travail est basé sur une famille de polynômes tempérée originellement étudiée par Boyd, $P_k(x, y) = x + \frac{1}{x} + y + \frac{1}{y} + k$ avec $k \in \mathbb{R}_{\geq 4}$. Pour le $k = 4$ cas, nous utilisons des valeurs spéciales du dilogarithme de Bloch-Wigner pour obtenir la mesure de Mahler de P_4 sur un tore arbitraire $\mathbb{T}^2_{a,b} = \{(x, y) \in \mathbb{C}^* \times \mathbb{C}^* : |x| = a, |y| = b\}$ avec $a, b \in \mathbb{R}_{>0}$. Ensuite, nous établissons une relation entre la mesure de Mahler de P_8 sur un tore $\mathbb{T}^2_{a,\sqrt{a}}$ et sa mesure de Mahler standard. La combinaison de cette relation avec des résultats de Lalín, Rogers et Zudilin conduit à une formule impliquant les mesures de Mahler généralisées de ce polynôme données en termes de $L'(E, 0)$. Au final, nous proposons une stratégie pour prouver des résultats similaires dans le cas général $k > 4$ sur $\mathbb{T}^2_{a,b}$ avec certaines restrictions sur a, b.

Mots clés : La mesure de Mahler, la dilogarithme de Bloch–Wigner, les fonctions L des courbes elliptiques, un tore d'intégration variable, régulateur.

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Introduction

0.1. Mahler measure

The (logarithmic) Mahler measure of a non-zero rational function $P \in \mathbb{C}(x_1, \ldots, x_n)$ is defined by

$$\mathrm{m}(P) = \mathrm{m}(P(x_1, \ldots, x_n)) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where $\mathbb{T}^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^* \times \mathbb{C}^* \times \cdots \times \mathbb{C}^* : |x_1| = \cdots = |x_n| = 1\}$. We define the Mahler measure as $M(P) = e^{\mathrm{m}(P)}$.

In the early 80's Smyth [**42**] discovered the following remarkable identity, which is one of the initial formulas for multi-variable cases:

$$\mathrm{m}(x + y + 1) = L'(\chi_{-3}, -1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2), \tag{0.1.1}$$

where

$$L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \quad \text{with} \quad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \pmod 3, \\ -1 & n \equiv -1 \pmod 3, \\ 0 & n \equiv 0 \pmod 3, \end{cases} \tag{0.1.2}$$

is a Dirichlet L-function. Smyth also extended the above example to three variables (the calculation can be found in the Appendix of [**16**]):

$$\mathrm{m}(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3).$$

Interest in Mahler measure of several variable polynomials arose in connection to the identities proved by Smyth [**42**].

When $n = 1$, Jensen's formula (see 1.1.1) gives the identity

$$M(Q) = |c_0| \prod_{|\beta_i| \geq 1} |\beta_i|, \tag{0.1.3}$$

where $Q(x) = c_0 \prod_{i=1}^{n}(x - \beta_i)$ is a polynomial in $\mathbb{C}[x]$. This quantity was introduced first by D.H. Lehmer [**33**] in early 1930's while investigating methods to find new large primes. He was interested, after Pierce [**36**], in the factors of the integers $\Delta_m = \prod_{i=1}^{n}(\beta_i^m - 1)$ associated to a monic polynomial $Q \in \mathbb{Z}[x]$ defined as $Q(x) = \sum_{i=0}^{n} c_{n-i}x^i = c_0 \prod_{i=1}^{n}(x - \beta_i)$. Lehmer checked that Δ_m grows with m roughly like $M(Q)^m$. It is easy to see that for $Q \in \mathbb{Z}[x]$ we have $M(Q) \geq 1$. A polynomial in $\mathbb{Z}[x]$ is called *primitive* if the coefficients have no non-trivial common factor. If Q is a primitive polynomial such that $M(Q) = 1$, then a classical theorem of Kronecker [**26**] establishes a relation between the roots of Q and the fact that $M(Q) = 1$.

Theorem 0.1.1 (Kronecker [**26**]). *If P is a primitive polynomial and $P(0) \neq 0$ then $M(P) = 1$ occurs only if all the roots of P are roots of unity.*

Proof. Let

$$P(x) = \sum_{r=0}^{d} a_r x^r$$

is a primitive polynomial with $a_d a_0 \neq 0$, and the roots of P are $\alpha_1, \alpha_2, \ldots, \alpha_d$. Then we consider the family of polynomials

$$P_n(x) = \prod_{j=1}^{d}(x - \alpha_j^n) = x^d + \sum_{r=0}^{d-1} b_{r,n} x^r.$$

Condition $M(P) = 1$ implies that $|a_d| = 1$ and $|\alpha_j| \leq 1$ for $j = 1, 2, \ldots, d$. Indeed, from (0.1.3) we get $1 = M(P) \geq |a_d|$, and as $a_d \in \mathbb{Z} \setminus \{0\}$ we have $a_d = \pm 1$. Similarly, if any of the roots of P has modulus greater than 1, then by (0.1.3) we get $M(P) > 1$, which contradicts $M(P) = 1$. Now, since $|a_d| = 1$, a_0 is a non-zero integer and $|a_0| = |a_d| \cdot \left|\prod_{j=1}^{d} \alpha_j\right| \leq 1$, we also get that $|a_0| = 1$ and $|\alpha_j| = 1$ for $j = 1, 2, \ldots, d$. This implies that the coefficients of P are bounded above by 2^d. Let $K/\mathbb{Q}$ be the splitting field of P. As Galois conjugates of the roots of P_n are also roots of P_n, we obtain that $\tau(P_n(x)) = P_n(x)$ for all $\tau \in \mathrm{Gal}(K/\mathbb{Q})$, and therefore, $P_n(x) \in \mathbb{Q}[x]$ for all $n \in \mathbb{N}$. Now, notice that each α_j is an algebraic integer, because either P or $-P$ is a monic polynomial in $\mathbb{Z}[x]$ such that $P(\alpha_j) = 0$. Then, each α_j^n is also an algebraic integer for $n \geq 1$, and so are the coefficients of P_n, because they are elementary symmetric polynomials in $\alpha_1^n, \alpha_2^n, \ldots, \alpha_d^n$. In other words, each $b_{r,n}$ is an element of $\overline{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}$, where $\overline{\mathbb{Z}}$ is the ring of all algebraic integers. Therefore, $P_n(x) \in \mathbb{Z}[x]$ for all $n \geq 1$. Also, the fact that $\left|\alpha_j^n\right| = 1$ for all $n \in \mathbb{N}$ and $j \in \{1, 2, \ldots, d\}$ implies the coefficients of P_n are bounded by 2^d for all n. Since the coefficients of P_n are integral, there are finitely many choices for them. This shows that the family $\{P_n\}_{n\geq 1}$ has finitely many elements, and therefore, by pigeonhole principle we must have $P_l = P_t$ for some $1 \leq l < t$. Let $R_n = \{\alpha_j^n : 1 \leq j \leq d\}$ be the set of roots of P_n for all $n \geq 1$. Then, the two sets R_l and R_t must be equal up to a permutation, i.e

$$\alpha_j^l = \alpha_{\sigma(j)}^t,$$

for $j = 1, 2, \ldots, d$ and $\sigma \in S_d$, where S_d is the symmetric group of degree d (group of all permutations on d symbols). Let m be the order of σ in S_d. Then we have

$$\alpha_j^{l^m} = \left(\alpha_j^l\right)^{l^{m-1}} = \left(\alpha_{\sigma(j)}^t\right)^{l^{m-1}} = \left(\alpha_{\sigma(j)}^l\right)^{tl^{m-2}} = \left(\alpha_{\sigma(\sigma(j))}^t\right)^{tl^{m-2}} = \cdots = \alpha_{\sigma^m(j)}^{t^m} = \alpha_j^{t^m},$$

for $1 \leq j \leq d$. This implies that

$$\alpha_j^{t^m - l^m} = 1,$$

i.e. each α_j is a root of unity, and thus deducing Theorem 0.1.3. □

In other words, if Q is a primitive polynomial then we get $M(Q) = 1$ only when $Q(x)$ is a power of x times a product of cyclotomic polynomials in x. In light of this Lehmer proposed the following question regarding the Mahler measure of a single variable polynomial:
Is there a constant $C > 1$ such that for every polynomial $P \in \mathbb{Z}[x]$ with $M(P) > 1$, we have $M(P) \geq C$?

The smallest value he was able to find was

$$M(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = 1.17628081\ldots,$$

which is still the smallest known positive value of $M(P)$. Lehmer's question remains open nowadays.

We can define the Mahler measure of an algebraic number ϑ in the following way. Let $P_\vartheta(x) \in \mathbb{Z}[x]$ be the minimal polynomial of ϑ. Then Mahler measure of ϑ is defined as $M(\vartheta) := M(P_\vartheta)$. In fact,

$$h(\vartheta) = \frac{\log(M(\vartheta))}{\text{degree of } P_\vartheta},$$

where $h(\vartheta)$ is the *Weil* (or *absolute*) height of ϑ. In the Appendix we will provide a derivation of the above fact.

Cassaigne and Maillot [**34**] generalized the formula found by Smyth to $\mathrm{m}(a+bx+cy)$ for arbitrary complex constants $a, b,$ and c. If $|a|, |b|,$ and $|c|$ are the lengths of the sides of a planar triangle while $\alpha, \beta,$ and γ are the respective opposite angles then

$$\pi \mathrm{m}(a+bx+cy) = \alpha \log|a| + \beta \log|b| + \gamma \log|c| + D\left(\frac{|a|}{|b|}e^{ic}\right),$$

where D is the Bloch–Wigner dilogarithm (for definition see (0.3.1)). Alternatively, when $|a|, |b|,$ and $|c|$ are not sides of a triangle we have

$$\mathrm{m}(a+bx+cy) = \log\max\{|a|, |b|, |c|\}.$$

Boyd [**17**] systematically examined families of polynomials associated to elliptic curves, and found numerical connections between their Mahler measure and special values of their L-functions. For example, Boyd considered the following family of two-variable polynomials,

$$P_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + k,$$

where k is a parameter in $\mathbb{C}$.

We know that for $k \neq 0, 4$, $C_k : P_k = 0$ is a genus one curve which is birationally equivalent to an elliptic curve $E_{N(k)}$, where $N(k)$ denotes the conductor. For k integral Boyd numerically found many formulas of the form

$$\mathrm{m}(P_k(x,y)) \stackrel{?}{=} r_k L'(E_{N(k)}, 0), \tag{0.1.4}$$

where r_k is a rational number of low height, and the question mark stands for a numerical formula that is true for at least 20 decimal places. In this thesis we consider the cases where $k \in \mathbb{R}_{>4}$. The computation of Mahler measure for the case $k = 4$ is simpler.

Deninger [**23**] bridged the gap by showing how to interpret $\mathrm{m}(P)$ as a Deligne period of mixed motives when $P(x,y)$ does not vanish on $\mathbb{T}^n$. In fact, identities such as Smyth's and Boyd's (conjectured) can be demystified by combining Deninger's method and the Beĭlinson conjectures. Rodriguez-Villegas [**37**] further investigated this connection, and proved some of these formulas involving P_k when the associated elliptic curves have complex multiplication and $k^2 \in \mathbb{Z}$. The case $k = 4$, for which C_k is a genus 0 curve, is very similar to that of Smyth's. Indeed by direct manipulations of the integral defining $\mathrm{m}(P_4)$ Boyd [**17**] showed that

$$\mathrm{m}(P_4) = 2L'(\chi_{-4}, -1),$$

where χ_{-4} is the quadratic Dirichlet character of conductor 4. A detailed computation of $\mathrm{m}(P_4)$ can be found in [**22**], where it is shown that

$$\pi \mathrm{m}(P_4) = 4D(i),$$

where D is the Bloch–Wigner dilogarithm (defined in (0.3.1)). We note down in Table 0.1 some identities of this kind which were proved by Bosman[**14**], Bertin [**3, 4, 8, 9**], Touafek [**44, 45, 46**] et al. The family of polynomials considered in that table are:

$$Q_t(x,y) = y^2 + (x^4 + tx^3 + 2tx^2 + tx + 1)y + x^4,$$
$$T_p(x,y) = (x+1)y^2 + (x^2 + px + 1)y + (x^2 + x),$$
$$M_j(x,y) = y^2(x+1)^2 + y(x^2 + jx + 1) + (x+1)^2,$$

with $t, j \in \mathbb{C}$ and $p \in \mathbb{Z}$. Recall that χ_{-4} is the quadratic Dirichlet character of conductor 4 and χ_{-3} is defined in (0.1.2).

Identities	Author(s)	Year
$\mathrm{m}(Q_8(x,y)) = 4L'(\chi_{-4}, -1)$	J. Bosman	2004
$\mathrm{m}(Q_{-1}(x,y)) = 2L'(\chi_{-3}, -1)$	J. Bosman	2004
$\mathrm{m}(M_6(x,y)) = \frac{8}{3}L'(\chi_{-4}, -1)$	N. Touafek	2008
$\mathrm{m}(T_3(x,y)) = 2L'(\chi_{-3}, -1)$	M. J. Bertin and W. Zudilin	2015

Table 0.1. Identities of the form $\mathrm{m}(P) = rL'(\chi, s)$ with $r \in \mathbb{Q}$

Further identities like (0.1.4) were proved by Bertin and Zudilin [**8, 9**], Brunault [**18, 19, 20**], Lalín [**7, 28, 29, 30**], Rodriguez-Villegas [**37, 38**], Mellit [**35**], Rogers and Zudilin [**39, 40**] et al. Some of those results are gathered in Table 0.2. The family of polynomials considered in that table are:

$$P_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + k,$$
$$R_m(x,y) = (1+x)(1+y)(x+y) - mxy,$$
$$T_p(x,y) = (x+1)y^2 + (x^2 + px + 1)y + (x^2 + x),$$

with $k, m \in \mathbb{C}$ and $p \in \mathbb{Z}$. Here E_N represents an elliptic curve of conductor N.

Identities	Author(s)	Year
$\mathrm{m}(P_{4\sqrt{2}}(x,y)) = L'(E_{64},0)$	F. Rodriguez-Villegas	1997
$\mathrm{m}(P_{4/\sqrt{2}}(x,y)) = L'(E_{32},0)$	F. Rodriguez-Villegas	1997
$\mathrm{m}(P_1(x,y)) = L'(E_{15},0)$	M. Rogers and W. Zudilin	2010
$\mathrm{m}(P_5(x,y)) = 6L'(E_{15},0)$	M. Lalín	2010
$\mathrm{m}(P_{2i}(x,y)) = L'(E_{40},0)$	A. Mellit	2011
$\mathrm{m}(P_2(x,y)) = L'(E_{24},0)$	M. Rogers and W. Zudilin	2012
$\mathrm{m}(R_4(x,y)) = 2L'(E_{20},0)$	M. Rogers and W. Zudilin	2012
$\mathrm{m}(P_i(x,y)) = 2L'(E_{17},0)$	W. Zudilin	2014
$\mathrm{m}(P_3(x,y)) = 2L'(E_{21},0)$	F. Brunault, M. Lalín, D. Samart and W. Zudilin	2015
$\mathrm{m}(P_{12}(x,y)) = 2L'(E_{48},0)$	F. Brunault	2015
$\mathrm{m}(T_1(x,y)) = L'(E_{14},0)$	M. J. Bertin and W. Zudilin	2015

Table 0.2. Some proven identities of Mahler measure of Boyd's families of polynomials

We will now look into a different perspective for studying Mahler measures of several variable polynomials.

0.1.1. Periods and Mahler measure

Kontsevich and Zagier [**25**] defined "Periods" in the following way.

Definition 0.1.2. *A* ***period*** *is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients over domains in $\mathbb{R}^n$ given by inequalities with rational coefficients.*

We can replace "rational" with "algebraic" in the above definition to obtain a period, because the algebraic functions occurring in the integrand can be replaced by rational functions by introducing more variables. It is easy to see that the set of all periods $\mathcal{P}$ is countable. For example, π and $\zeta(s)$ (where $s \geq 2$ is an integer) are periods. Indeed,

$$\pi = \int_0^1 \frac{4}{x^2+1}dx,$$

and $\zeta(3)$ has an integral representation as

$$\zeta(3) = \iiint_{0<x<y<z<1} \frac{dxdydz}{(1-x)yz}.$$

It is conjectured that $\frac{1}{\pi}, e$ and γ (Euler–Mascheroni constant) are *not* periods.

Periods are intended to bridge the gap between algebraic numbers and transcendental numbers. The class of algebraic numbers is too narrow to include many common mathematical constants, while the set of transcendental numbers is not countable, and its members are not generally computable.

For many purposes it is convenient to widen our previous definition and consider also elements of the **extended period ring** $\hat{\mathcal{P}} = \mathcal{P}[\frac{1}{2i\pi}]$, which is an algebra. In particular, it follows from the definition that the (logarithmic) Mahler measures of polynomials are elements of $\hat{\mathcal{P}}$. Zagier noted that L-functions of elliptic curves at some special integral values produce examples of periods, and it is also conjectured that any two integral representations of a period can be obtained from each other just by using additivity of integrals, changes of variables, or Stokes' theorem. In other words, proven identities in Table 0.2 regarding the relationships between Mahler measure of polynomials and special values of the L-functions of certain elliptic curves are some of such examples.

In a series of papers [**5, 6**], Bertin derived identities between the Mahler measure of a polynomial defining an elliptic $K3$ surface and $L(f,3)$ for some associated newform f of weight 3. In fact, Kontsevich and Zagier argued in [**25**] that, for a modular form f of weight $k \geq 2$, $L(f,m) \in \hat{\mathcal{P}}$ for all $m \geq k$ (as well as for the critical values $0 < m < k$). If f is also a Hecke eigenform and k is even with

$$L^*(f,s) = \int_0^\infty f(it)t^{s-1}dt = -L^*(f,k-s) = -\int_0^\infty f(it)t^{k-s-1}dt,$$

then $L'(f,k/2) \in \hat{\mathcal{P}}$.

Periods can also be seen as the values of integrals of algebraically defined differential forms over certain chains in algebraic varieties. If these forms and chains depend on parameters then the integrals, considered as functions of the parameters, typically satisfy linear differential equations with algebraic coefficients, such as Picard–Fuchs differential equations for elliptic curves whose solutions describe the periods of the elliptic curves. Special values of the solutions of these differential equations at algebraic arguments produce examples of periods. For example, consider the family of elliptic curves over $\mathbb{C}$ given by the Legendre equation

$E_\lambda : y^2 = x(x-1)(x-\lambda)$ with $\lambda \in \mathbb{C}$. The period integrals of this family are

$$\Omega_1(\lambda) = \int_\lambda^1 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}, \quad \Omega_2(\lambda) = \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

They are examples of periods when $\lambda \in \mathbb{Q} \setminus \{0,1\}$. For example, if $\lambda \in \mathbb{Q}_{<1}$ is a positive rational, then we have

$$\begin{aligned} 2\Omega_1(\lambda) = 2\int_\lambda^1 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} &= 2i \int_\lambda^1 \frac{dx}{\sqrt{x(1-x)(x-\lambda)}} \\ &= i \iint_{\{\lambda \le x \le 1\} \cap \{v^2 x(1-x)(x-\lambda) \le 1\}} dx dv, \end{aligned}$$

and therefore, this is a period according to Definition 0.1.2. The period integrals also satisfy the Picard–Fuchs differential equation for E_λ

$$\lambda(\lambda-1)\Omega''(\lambda) + (2\lambda-1)\Omega'(\lambda) + \frac{1}{4}\Omega(\lambda) = 0.$$

Also,

$$\Omega_2(\lambda) = \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = \pi F\left(\frac{1}{2}, \frac{1}{2}, 1; \lambda\right),$$

where $F(a,b,c;x)$ is the Euler–Gauss hypergeometric function. It is defined as

$$F(a,b,c;x) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n n!} x^n \quad (\text{where } (t)_n := t(t+1)\cdots(t+n-1), (0)_n = 1), \tag{0.1.5}$$

for a complex variable x, and $a,b,c \in \mathbb{C}$ with $c \notin \mathbb{Z}_{\le 0}$. The series converges absolutely for all $|x| < 1$. It has a continuation as a single-valued function of x in the complex plane from which a line joining 1 to ∞ is deleted. If $a,b,c \in \mathbb{Q}$, and $c \notin \mathbb{Z}_{\le 0}$, then the differential equation satisfied by the Euler–Gauss hypergeometric function is also of Picard–Fuchs type. The next theorem shows that, given certain conditions on a,b,c and x, $F(a,b,c;x)$ admits an integral representation **[2]**.

Theorem 0.1.3 ([**1**], Theorem 2.2.1)**.** *If $|x| < 1$, $a,b,c \in \mathbb{C}^*$ with $c \notin \mathbb{Z}_{\le 0}$ and $\min\{\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c-a)\} > 0$, then we can express $F(a,b,c;x)$ as*

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 y^{a-1}(1-y)^{c-a-1}(1-xy)^{-b} dy, \tag{0.1.6}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Here it is understood that $\arg y = \arg(1-y) = 0$, and $(1-xy)^{-b}$ has its principal value.

A proof of Theorem 0.1.3 is provided in the Appendix for the sake of commpleteness. Since the integral in (0.1.6) is analytic in $\mathbb{C}\setminus[1,\infty)$, the above integral representation may be viewed as the analytic continuation of F, as a function of x, outside the unit disc, but only when $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$ [**1**]. Given the conditions on a, b, c in the statement of Theorem 0.1.3, if we also consider $\mathrm{Re}(c-a-b) > 0$, then the series in (0.1.5) also converges absolutely when $|x| = 1$. In fact, as x approaches 1, the equation (0.1.6) yields ([**1**], Theorem 2.2.2)

$$F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{0.1.7}$$

Indeed, for $a, b, c \in \mathbb{C}^*$ with $c \notin \mathbb{Z}_{\leq 0}$, and $\min\{\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c-a), \mathrm{Re}(c-a-b)\} > 0$, we have

$$\lim_{x\to 1^-} F(a, b, c; x) = \lim_{x\to 1^-} \left[\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 y^{a-1}(1-y)^{c-a-1}(1-xy)^{-b} dy\right],$$

where $|x| < 1$ and x tends to 1 in such a way that $\frac{|1-x|}{1-|x|}$ remains bounded. Then, an application of *Abel's Limit Theorem* (see Theorem (A.2.1) in the Appendix) yields that the above limiting value of the power series $F(a, b, c; x)$ is in fact $F(a, b, c; 1)$.

For x algebraic with $|x| < 1$, Zagier [**25**] showed that

$$F(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 y^{a-1}(1-y)^{c-a-1}(1-xy)^{-b} dy \in \frac{1}{\pi}\mathcal{P} \subset \hat{\mathcal{P}},$$

when $a, b, c \in \mathbb{Q}^*$ with $c \notin \mathbb{Z}_{\leq 0}$. For example, if we evaluate $F(a, b, c; x)$ at $a = b = \frac{1}{2}, c = 2$, and $x = 1$ using (0.1.7), we obtain $F\left(\frac{1}{2}, \frac{1}{2}, 2; 1\right) = \frac{4}{\pi} \in \hat{\mathcal{P}}$. Rodriguez-Villegas [**37**] showed that this type of hypergeometric functions is directly related to the Mahler measure of multi-variable polynomials.

It would be interesting to obtain connections among Mahler measure of polynomials, periods and special values of L-functions of elliptic curves or modular forms or characters. Including these particular relationships Kontsevich and Zagier gave a more detailed and thorough discussions on periods in [**25**].

0.2. Mahler measure over arbitrary tori

Let $\mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{R}_{>0})^n$ and $\mathbb{T}^n_{\mathbf{a}} = \mathbb{T}^n_{a_1,\ldots,a_n} := \{(x_1, \ldots, x_n) \in \mathbb{C}^* \times \mathbb{C}^* \times \cdots \times \mathbb{C}^* : |x_1| = a_1, \ldots, |x_n| = a_n\} = \mathbb{T}_{a_1} \times \cdots \times \mathbb{T}_{a_n}$, where $\mathbb{T}_{a_i} := \{x \in \mathbb{C} : |x| = a_i\}$. Then the

Mahler measure of a non-zero rational function $P \in \mathbb{C}(x_1, \ldots, x_n)$ over $\mathbb{T}^n_{\mathbf{a}}$ is defined by

$$\mathrm{m}_{a_1,\ldots,a_n}(P) = \mathrm{m}_{a_1,\ldots,a_n}(P(x_1,\ldots,x_n)) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n_{\mathbf{a}}} \log|P(x_1,\ldots,x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

The formula found by Cassaigne and Maillot in [**34**] can be interpreted as $\mathrm{m}_{a,b,c}(x+y+z)$. Some cases of the formulas due to Vandervelde [**47**] for the Mahler measure of $a_0xy + a_1x + a_2y + a_3$ may also be viewed in this context.

Lalín and Mittal [**31**] explored this definition to obtain relations between polynomials mentioned in Boyd's paper [**17**], namely

$$\begin{aligned} R_{-2}(x,y) \quad &:= (1+x)(1+y)(x+y) + 2xy, \\ S_{2,-1}(x,y) \quad &:= y^2 + 2xy - x^3 + x, \end{aligned}$$

over $\mathbb{T}^2_{q^2,q}$ and $\mathbb{T}^2_{q,q}$ respectively for some values of $q \in \mathbb{R}_{>0}$.

Our goal is to investigate this definition for other polynomials, specifically for the particular family of polynomials due to Boyd $P_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + k$, where $k \in \mathbb{R}_{\geq 4}$. To prove the theorems in this thesis we will follow a direction similar to the one used in [**31**]. The results we got here depend on the genus of the family of curves $C_k : P_k(x,y) = 0$. In particular they involve L-functions of elliptic curves when the genus is 1, and the Bloch–Wigner dilogarithm when it is 0. Rogers and Zudilin [**40**], Lalín [**30**], Lalín and Rogers [**32**] showed formulas of the form $\mathrm{m}(P_k(x,y)) = r_k L'(E_{N(k)}, 0)$ for $k = 1, 5, 8, 16$, where $r_k \in \mathbb{Q}^*$. We explore the $k = 8$ case over the torus $\mathbb{T}^2_{a,\sqrt{a}}$. The calculations for $k = 5, 16$ in a similar setting will follow from the previous case. In fact the calculations that we have provided for the case $k = 8$ also hold for all $k > 4$.

Let $a_0 = \left(5 - 2\sqrt{2}\right) + \sqrt{\left(5 - 2\sqrt{2}\right)^2 - 1} = 4.0991954\ldots$. In this thesis, we explicitly showed that for $a \in \left[\frac{1}{a_0}, a_0\right]$

$$\mathrm{m}_{a,\sqrt{a}}(yP_8(x,y)) = \mathrm{m}(yP_8(x,y)) = 4L'(E_{N(8)}, 0), \tag{0.2.1}$$

where $E_{N(8)}(X,Y) := Y^2 - X\left(X^2 + \left(\frac{8^2}{4} - 2\right)X + 1\right)$ is an elliptic curve of conductor $N(8) = 24$. Notice that $P_8(x,y) = P_8(y,x)$. Therefore, we have $\mathrm{m}(P_8(x,y)) = \mathrm{m}(P_8(y,x))$ from the definition of Mahler measure. However, it is easy to see that $\mathrm{m}_{a,\sqrt{a}}(P_8(x,y)) \neq$

$\mathrm{m}_{a,\sqrt{a}}(P_8(y,x))$ when $a \neq 1$. Indeed, we have

$$\mathrm{m}_{a,\sqrt{a}}(P_8(x,y)) = \mathrm{m}(P_8(x,y)) - \frac{1}{2}\log a = \mathrm{m}(P_8(y,x)) - \frac{1}{2}\log a,$$
$$\mathrm{m}_{a,\sqrt{a}}(P_8(y,x)) = \mathrm{m}(P_8(y,x)) - \log a = \mathrm{m}(P_8(x,y)) - \log a,$$

for certain values of $a \in \mathbb{R}_{>0}$. On the other hand, we have

$$\mathrm{m}_{a,\sqrt{a}}(yP_8(x,y)) = \mathrm{m}(yP_8(x,y)) = \mathrm{m}(yP_8(y,x)) = \mathrm{m}_{a,\sqrt{a}}(yP_8(y,x)).$$

Therefore, we obtain an equality between a symmetric case, where the Mahler measure is considered over $\mathbb{T}^2_{1,1}$, and a non-symmetric case, where the domain of the integral is $\mathbb{T}^2_{a,\sqrt{a}}$ with $a \neq 1$.

The results we got are restricted to the conditions on a for the case $k > 4$ due to technical difficulties involving the study of the integration path, such as if it is closed or not, etc. These results are similar to some earlier formulas from **[28]** that involve a single varying parameter and relate the Mahler measure of some polynomials to polylogarithms.

By changing variables, namely $x \mapsto ax$ and $y \mapsto \sqrt{a}y$, we get the same results in terms of non-tempered polynomials (see Definition 1.1.5). The fact that we cannot apply K-theory framework in this case makes it very interesting on its own.

0.3. Main Theorems

$P_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + k$ is a family of Laurent polynomials in x and y with $k \in \mathbb{C}$. We restrict ourselves to those cases when $k \in \mathbb{R}_{\geq 4}$. In this thesis we prove two theorems involving two particular cases when $k = 4$, corresponding to a genus 0 curve, and $k = 8$, where we have a genus 1 curve.

We generalize some of the results numerically found by Boyd [**17**] concerning the connection between Mahler measure of $P_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + k$ and special values of L-functions of elliptic curves associated to P_k, where x and y take values in an arbitrary torus. Our family of polynomials is a tempered family (for more details see Definition 1.1.5 Section 1.1.3). Therefore, Deninger's [**23**] observation regarding the general conjectures of Bloch–Beĭlinson about predicting the connection between the standard Mahler measures of some two-variable polynomials $P(x,y)$, when their zero loci define genus one curves, and the special values of L-functions of the corresponding elliptic curves holds here.

We now state the theorems which we will prove in the next chapters, and then in later sections we will predict and, in some cases, prove similar results for a more general case, namely when $k > 4$.

Let a and b be positive real numbers. We consider the torus $\mathbb{T}^2_{a,b} = \{(x,y) \in \mathbb{C}^* \times \mathbb{C}^* : |x| = a, |y| = b\}$.

We denote the Bloch–Wigner dilogarithm by $D(z)$, which is defined for $z \in \mathbb{C}$ as

$$D(z) = \operatorname{Im}\left(\operatorname{Li}_2(z) + i \arg(1-z) \log|z|\right), \text{ where } \operatorname{Li}_2(z) = -\int_0^z \frac{\log(1-v)}{v} dv. \tag{0.3.1}$$

Note that if $z \in \mathbb{R}_{>1}$ then $D(z)$ is still well-defined. Indeed, the function $\operatorname{Li}_2(z)$, extended to $\mathbb{C} \setminus [1, \infty)$, jumps by $2i\pi \log|z|$ as z crosses the cut. Thus the modified function $\operatorname{Li}_2(z) + i\arg(1-z)\log|z|$ is continuous, where arg denotes the branch of argument lying between $-\pi$ and π. The Bloch–Wigner dilogarithm is continuous in $\mathbb{C} \cup \{\infty\}$, with $D(\infty) = D(0) = D(1) = 0$. In fact, it is real-analytic in $\mathbb{C} \setminus \{0, 1\}$.

Now we can state our first theorem.

Theorem 0.3.1. *Let a and b be positive real numbers. If $-1 \leq \frac{(1+ab)(a-b)}{2\sqrt{ab}(a+b)} \leq 1$ and $ab \neq 1$, we define $\sin\alpha := \frac{(1+ab)(a-b)}{2\sqrt{ab}(a+b)}$ with $\alpha \in \left[-\frac{\pi}{2}, 0\right)$, when $b > a$, and $\sin\beta := \frac{(1+ab)(a-b)}{2\sqrt{ab}(a+b)}$ with $\beta \in \left(0, \frac{\pi}{2}\right)$, when $b < a$. Then, for $k = 4$, the values of*

$$R_{a,b} := \frac{1}{2}\left[\mathrm{m}_{a,b}\left(P_4(x,y)\right) + \log b\right]$$

are given by the following table:

Condition 1	*Condition 2*	*Extra conditions*	*Values*
$ab \neq 1$	$b = a$		$\frac{1}{\pi}\left(2D\left(i\sqrt{ab}\right) - \left(\log\sqrt{ab}\right)\tan^{-1}\left(\frac{2\sqrt{ab}}{ab-1}\right)\right)$
	$b > a$	$-1 \leq \sin\alpha < 0$	$\frac{1}{\pi}\left(D\left(i\sqrt{ab}e^{-i\alpha}\right) + D\left(i\sqrt{ab}e^{i\alpha}\right) - \left(\log\sqrt{ab}\right)\tan^{-1}\left(\frac{2\sqrt{ab}\cos\alpha}{ab-1}\right)\right)$
		$\frac{(1+ab)(a-b)}{2\sqrt{ab}(a+b)} < -1$	0
	$b < a$	$0 \leq \sin\beta < 1$	$\frac{1}{\pi}\left(D\left(i\sqrt{ab}e^{-i\beta}\right) + D\left(i\sqrt{ab}e^{i\beta}\right) - \left(\log\sqrt{ab}\right)\tan^{-1}\left(\frac{2\sqrt{ab}\cos\beta}{ab-1}\right)\right)$
		$\frac{(1+ab)(a-b)}{2\sqrt{ab}(a+b)} \geq 1$	0
$ab = 1$	$b = a$		$\frac{2}{\pi}D(i)$
	$b > a$		$\frac{1}{\pi}\left(D\left(-e^{-2i\cot^{-1}\left(\sqrt{\frac{b}{a}}\right)}\right) + D\left(e^{2i\cot^{-1}\left(\sqrt{\frac{b}{a}}\right)}\right)\right)$
	$b < a$		$\frac{1}{\pi}\left(D\left(-e^{-2i\cot^{-1}\left(\sqrt{\frac{b}{a}}\right)}\right) + D\left(e^{2i\cot^{-1}\left(\sqrt{\frac{b}{a}}\right)}\right)\right)$

Table 0.3. Values of $R_{a,b}$

Recall that $a_0 = \left(5 - 2\sqrt{2}\right) + \sqrt{\left(5 - 2\sqrt{2}\right)^2 - 1} = 4.0991954\ldots$. We have the following theorem:

Theorem 0.3.2. *If $k = 8$ and $\frac{1}{a_0} \leq a \leq a_0$ then*

$$\mathrm{m}_{a,\sqrt{a}}\left(yP_8(x,y)\right) = \mathrm{m}\left(yP_8(x,y)\right). \tag{0.3.2}$$

Moreover,

$$\mathrm{m}_{a,\sqrt{a}}\left(P_8\left(x,y\right)\right)=\mathrm{m}\left(P_8\left(x,y\right)\right)-\frac{1}{2}\log a=4L'(E_{N(8)},0)-\frac{1}{2}\log a, \tag{0.3.3}$$

where

$$E_{N(8)}(X,Y):=Y^2-X\left(X^2+\left(\frac{8^2}{4}-2\right)X+1\right)$$

is the Weierstrass form of an elliptic curve of conductor $N(8)=24$.

In fact, it is enough to prove (0.3.2) in order to deduce Theorem 0.3.2.

Proof of (0.3.2) implies (0.3.3). In order to prove this implication we consider the equality

$$\mathrm{m}\left(P_8\left(x,y\right)\right)=4\mathrm{m}\left(P_2\left(x,y\right)\right), \tag{0.3.4}$$

which was proven by Lalín and Rogers in **[32]** building on work of Kurokawa and Ochiai in **[27]**. Later Rogers and Zudilin showed in **[39]** that

$$\mathrm{m}\left(P_2\left(x,y\right)\right)=L'(E_{N(8)},0). \tag{0.3.5}$$

Combining (0.3.4) and (0.3.5) we obtain

$$\mathrm{m}(P_8(x,y))=4\mathrm{m}(P_2(x,y))=4L'(E_{N(8)},0). \tag{0.3.6}$$

Now, from the definition of Mahler measure we have

$$\begin{aligned}
\mathrm{m}(yP_8(x,y)) &= \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\log\left|yP_8\left(x,y\right)\right|\frac{dx}{x}\frac{dy}{y}\\
&= \frac{1}{(2\pi i)^2}\iint_{|x|=1,|y|=1}\log\left|yP_8\left(x,y\right)\right|\frac{dx}{x}\frac{dy}{y}\\
&= \frac{1}{(2\pi i)^2}\iint_{|x|=1,|y|=1}\log\left|P_8\left(x,y\right)\right|\frac{dx}{x}\frac{dy}{y}\\
&= \mathrm{m}(P_8(x,y)),
\end{aligned} \tag{0.3.7}$$

where the penultimate step follows from the fact that $\log|y| = \log 1 = 0$ when $(x,y) \in \mathbb{T}^2$. From the definition of Mahler measure over the torus $\mathbb{T}^2_{a,\sqrt{a}}$ we also have

$$\begin{aligned}
\mathrm{m}_{a,\sqrt{a}}(yP_8(x,y)) &= \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2_{a,\sqrt{a}}} \log|yP_8(x,y)|\frac{dx}{x}\frac{dy}{y} \\
&= \frac{1}{(2\pi i)^2}\iint_{|x|=a,|y|=\sqrt{a}} \log|yP_8(x,y)|\frac{dx}{x}\frac{dy}{y} \\
&= \frac{1}{(2\pi i)^2}\iint_{|x|=a,|y|=\sqrt{a}} \log|P_8(x,y)|\frac{dx}{x}\frac{dy}{y} + \log\left(\sqrt{a}\right) \\
&= \mathrm{m}_{a,\sqrt{a}}(P_8(x,y)) + \frac{1}{2}\log a, \qquad (0.3.8)
\end{aligned}$$

where again the penultimate equality follows from the fact that $\log|y| = \log(\sqrt{a}) = \frac{1}{2}\log a$ when $(x,y) \in \mathbb{T}^2_{a,\sqrt{a}}$, and $\frac{1}{(2\pi i)^2}\iint_{|x|=a,|y|=\sqrt{a}}\frac{dx}{x}\frac{dy}{y} = 1$. Therefore, if we can show that, for $\frac{1}{a_0} \le a \le a_0$,

$$\mathrm{m}_{a,\sqrt{a}}\left(yP_8(x,y)\right) = \mathrm{m}\left(yP_8(x,y)\right),$$

then a combination of the equalities in (0.3.6), (0.3.7) and (0.3.8) yields

$$\mathrm{m}_{a,\sqrt{a}}\left(P_8(x,y)\right) = \mathrm{m}\left(P_8(x,y)\right) - \frac{1}{2}\log a = 4L'(E_{N(8)},0) - \frac{1}{2}\log a,$$

and thus proving (0.3.2) implies (0.3.3). □

Let

$$E_{N(k)} : Y^2 = X^3 + \left(\frac{k^2}{4} - 2\right)X^2 + X,$$

where $k \neq 0, \pm 4$. Rogers and Zudilin [**40**] showed that

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) = L'(E_{N(1)},0).$$

Note that $E_{N(16)}, E_{N(5)}$, and

$$E_{N(1)} : Y^2 = X^3 - \frac{7}{4}X^2 + X$$

are isogenous elliptic curves of conductor 15. Following the discussions in [**32**] we can explicitly describe these isogenies.

Let $E : y^2 = x(x^2 + ax + b)$ and $E' : y'^2 = x'(x'^2 - 2ax' + (a^2 - 4b))$ be two elliptic curves (over $\mathbb{C}$) with $a^2 - 4b \neq 0$. There is a well-known isogeny of degree 2

$$\psi : E \longrightarrow E'$$

defined by

$$(x, y) \longmapsto \begin{cases} \left(\frac{y^2}{x^2}, \frac{y(b-x^2)}{x^2}\right) & \text{when } (x, y) \neq (0, 0), \\ O_{E'} & \text{when } (x, y) = (0, 0), \end{cases}$$

and $\psi(O_E) = O_{E'}$, where O_E and $O_{E'}$ are the identity elements of the additive groups $(E(\mathbb{C}), +)$ and $(E'(\mathbb{C}), +)$, respectively. By using this we get the following isogenies [**32**]:

(1) $\psi_{1,n} : E_{N\left(2\left(n+\frac{1}{n}\right)\right)} \longrightarrow E_{N(4n^2)}$, which is defined by

$$(X, Y) \longmapsto \left(\frac{X(n^2X+1)}{n^2+X}, -\frac{n^3Y(X^2+2n^2X+1)}{(n^2+X)^2}\right),$$

(2) $\psi_{2,n} : E_{N\left(2\left(n+\frac{1}{n}\right)\right)} \longmapsto E_{N\left(\frac{4}{n^2}\right)}$ given by

$$(X, Y) \longmapsto \left(\frac{X(n^2+X)}{n^2X+1}, -\frac{Y(n^2X^2+2X+n^2)}{n(n^2X+1)^2}\right).$$

Now if we take $n = \frac{1}{2}$, then (1) gives an isogeny $\psi_{1,\frac{1}{2}} : E_{N(5)} \longrightarrow E_{N(1)}$, which is defined by

$$(X, Y) \longmapsto \left(\frac{X(X+4)}{4X+1}, -\frac{Y(2X^2+X+2)}{(4X+1)^2}\right).$$

From (2) we obtain an isogeny $\psi_{2,\frac{1}{2}}$ from $E_{N(5)}$ onto $E_{N(16)}$ given by

$$(X, Y) \longmapsto \left(\frac{X(1+4X)}{X+4}, -\frac{8Y(X^2+8X+1)}{(X+4)^2}\right).$$

We can construct an isogeny from $E_{N(16)}$ onto $E_{N(1)}$ by composing $\psi_{1,\frac{1}{2}}$ with the unique dual isogeny of $\psi_{2,\frac{1}{2}}$.

In her paper [**30**] Lalín showed that

$$\frac{\mathrm{m}(P_5(x, y))}{6} = \mathrm{m}(P_1(x, y)) = \frac{\mathrm{m}(P_{16}(x, y))}{11}.$$

Therefore, we can follow a similar manipulation as in the proof of our theorem, for the case $k = 8$, to obtain that, for $b_0 = 5 + 2\sqrt{6}$ and $c_0 = \frac{7-2\sqrt{5}+\sqrt{(7-2\sqrt{5})^2-4}}{2}$, we have

$$\mathrm{m}_{a,\sqrt{a}}\left(P_{16}\left(x, y\right)\right) = 11L'(E_{N(16)}, 0) - \frac{1}{2}\log a, \qquad \text{for } a \in \left(\frac{1}{b_0}, b_0\right),$$

and

$$\mathrm{m}_{a,\sqrt{a}}\left(P_5\left(x, y\right)\right) = 6L'(E_{N(5)}, 0) - \frac{1}{2}\log a, \qquad \text{for } a \in \left(\frac{1}{c_0}, c_0\right),$$

where $E_{N(16)}$ and $E_{N(5)}$ are isogenous elliptic curves of conductor $N(16) = N(5) = 15$.

0.4. Structure of this Thesis

A remarkable breakthrough involving the Mahler measure of several variable polynomials was obtained by Deninger [**23**], where he interpreted the Mahler measure of certain Laurent polynomials $P(x, y)$ as a regulator map r evaluated in a certain K-theory group at some homology class. In other words, he showed that

$$\mathrm{m}(P) = \frac{1}{2\pi} r(\xi)([\gamma]),$$

where ξ is a certain element in an appropriate group in K-theory, and $[\gamma]$ is an equivalence class of paths (a homology class) in the curve $P(x, y) = 0$.

In Chapter 1 we describe this regulator map explicitly. In fact, we establish that the Mahler measure is the integral of a certain differential form η, which corresponds to the regulator map. The differential form η is closed in its domain of definition. For the case $k = 4$, the curve $P_4 = 0$ is of genus 0 and η is exact. The integral of η over a path is then given by a version of the dilogarithm which gives rise to special values of Dirichlet L-functions. In essence of this, we recall some properties of the Bloch–Wigner dilogarithm and its relation with the closed differential form η. In the case $k > 4$, the genus is 1. But we see that η is not exact here. In this case a theorem of Bloch [**11**] relates the regulator to the elliptic dilogarithm, which is conjectured to yield special values of the associated L-function. We then describe the framework connecting the Mahler measures of polynomials and the values of L-functions of their associated elliptic curves using the elliptic regulator. We start with the discussion of Newton polygons and tempered polynomials. These definitions are necessary for some technical conditions coming from K-theory. We continue to describe some properties of the elliptic dilogarithm by introducing the diamond operator, which is necessary to evaluate the regulator. Later we state the theorem of Bloch [**11**] which relates the elliptic dilogarithm with the elliptic regulator, and therefore, with the Mahler measure. We close the chapter with a discussion on Mahler measure over an arbitrary torus and its relationship to the regulator.

In Chapter 2 we prove Theorem 0.3.1, where we consider the polynomial $P_4(x, y)$ over $\mathbb{T}^2_{a,b}$, for $a, b \in \mathbb{R}_{>0}$. Following the discussion in Chapter 1, we concentrate on expressing η as an exact differential form to calculate the regulator in this case, and relate it to the Bloch–Wigner dilogarithm. In order to do so, we use a change of variables to factorize $P_4(x, y)$ as a product of linear polynomials in two variables and a monomial. We also show that the linear polynomials have the same Mahler measure by using another change of variables. This change of variables leads to a change on the integration torus from $\mathbb{T}^2_{a,b}$ to $\mathbb{T}^2_{c,d}$, where c, d can be expressed as $c = \sqrt{ab}$ and $d = \sqrt{\frac{b}{a}}$. We do this because it is easier to express η in an exact form when it is evaluated in linear polynomials. We state some results regarding the relation between η and the Bloch–Wigner dilogarithm D that we need to deduce the theorem. Note that the integral of an exact form over a path is determined completely by the endpoints of that path (by Stokes' theorem). Therefore, we calculate the endpoints of the integration paths for different values of c and d. After obtaining the Mahler measures of those linear polynomials using these endpoints, we end the chapter by concluding the proof of Theorem 0.3.1 by establishing a relation between these Mahler measures and $\mathrm{m}_{a,b}(P_4(x, y))$.

In Chapter 3 we consider the case $k > 4$. The first part of Chapter 3 deals with the proof of Theorem 0.3.2. As noted above, the differential form η is not exact in this case. In favorable cases (when the integration path is closed), the integral of η can be computed by means of the elliptic dilogarithm. This allows us to relate the Mahler measure of P_k over an arbitrary torus to the standard Mahler measure of P_k. We first concentrate on calculating the Mahler measure of P_8 over $\mathbb{T}^2_{a,\sqrt{a}}$ with $a \in \mathbb{R}_{>0}$. We choose P_8 because $\mathrm{m}(P_8)$ is proven to be related to $L'(E_{N(8)}, 0)$ in this case [**32, 39**]. To achieve this we consider a birational transformation ϕ between P_8 and $E_{N(8)}$. In this context, we factorize $yP_8(x, y)$ in $\overline{\mathbb{C}(x)}[y]$, and argue that if $a + \frac{1}{a} < 6$ then we can restrict the integral to only one of the well-defined roots of $yP_8(x, y)$ in $\overline{\mathbb{C}(x)}$. Next we determine the values of a such that the integration path is closed. As the integral of η only depends on the homology class, we describe the homology class of the integration path in $H_1(E_{N(8)}, \mathbb{Z})$ using the invariant holomorphic differential ω of $E_{N(8)}$ and the birational transformation ϕ. This allows us to apply the theorem by Bloch to evaluate the Mahler measure in terms of elliptic dilogarithm, and relate it with $L'(E_{N(8)}, 0)$. We then conclude our proof by evaluating the integral. In the next part of this chapter, we describe some interesting results obtained by a similar calculation regarding the Mahler measure of P_k over $\mathbb{T}^2_{a,b}$, when $k > 4$ and $a, b \in \mathbb{R}_{>0}$. We close the chapter with an explicit calculation of $\mathrm{m}(P_8)$ using the elliptic dilogarithm and the diamond operator, for the sake of completeness.

In Chapter 4 we conclude this thesis with some comments on further questions related to our results which can be pursued in the future.

The last chapter is an appendix where we include additional information on the homology cycle mentioned in Chapter 3, which we believe provides a valuable perspective to the discussion in that chapter in spite of not being strictly necessary in the proof. We also provide a statement of the *Abel's Limit Theorem*, a proof of Theorem 0.1.3, and a derivation of the relation between the (logarithmic) Mahler measure of an algebraic number and its *Weil* height.

Chapter 1

Prerequisites

We briefly review some necessary background in this chapter before proving the theorems in the next ones.

1.1. The Bloch–Wigner dilogarithm and The Elliptic Regulator

In this section we first discuss the relationship between Mahler measure and dilogarithm when $k = 4$, and later for the case $k > 4$, we describe how Mahler measure and the regulator are related.

1.1.1. Jensen's Formula and applications

We recall a special case of Jensen's formula. Let $z_0 \in \mathbb{C}$. Then

$$\frac{1}{2\pi i}\int_{\mathbb{T}^1} \log|z - z_0|\frac{dz}{z} = \begin{cases} \log|z_0| & |z_0| \geq 1, \\ & \\ 0 & |z_0| \leq 1. \end{cases}$$

Let C be a curve over $\mathbb{C}$ which defines a compact Riemann surface, and let $\mathbb{C}(C)$ be its field of fractions. For $f, g \in \mathbb{C}(C)^*$ we define

$$\eta\left(f, g\right) := \log|f| d \arg g - \log|g| d \arg f, \tag{1.1.1}$$

where $d \arg x$ is defined by $\mathrm{Im}(\frac{dx}{x})$. Note that, η is a real C^∞ differential 1-form on $C \setminus S$, where S contains all the zeroes and poles of f and g. The following lemma consists of some useful properties of η which we will be using in later sections.

Lemma 1.1.1. *Let $f, g, h, v \in \mathbb{C}(C)^*$ and $a, b \in \mathbb{C}^*$. Then we have*

(1) $\eta(f, g) = -\eta(g, f)$, i.e. η is anti-symmetric,

(2) $\eta(fg, hv) = \eta(f,h) + \eta(g,h) + \eta(f,v) + \eta(g,v)$,

(3) $\eta(a,b) = 0$,

(4) η *is a closed differential form.*

PROOF. The equality in (1) holds because

$$\eta(f,g) = \log|f|d\arg g - \log|g|d\arg f = -(\log|g|d\arg f - \log|f|d\arg g) = -\eta(g,f).$$

By definition of η we have

$$\begin{aligned}\eta(fg,hv) &= \log|fg|d\arg(hv) - \log|hv|d\arg(fg)\\ &= (\log|f| + \log|g|)(d\arg h + d\arg v) - (\log|h| + \log|v|)(d\arg f + d\arg g).\end{aligned}$$

Expanding the last line gives the equality in (2).

As a, b are complex constants we have $d\arg a = d\arg b = 0$, and thus proving (3).

From the definition of η in (1.1.1) we have

$$d\eta(f,g) = \operatorname{Im}(d\log f \wedge d\log g) = \operatorname{Im}\left(\frac{df}{f} \wedge \frac{dg}{g}\right).$$

But as C is a Riemann surface, it has complex dimension 1. On the other hand, $(d\log f \wedge d\log g)$ is a complex 2-form, and any complex 2-form of a Riemann surface is 0, i.e.

$$d\eta(f,g) = 0.$$

This implies that η is a closed form, and thus proving (4). $\square$

Let $P(x,y)$ be a Laurent polynomial in two variables. We can always multiply $P(x,y)$ by suitable power of y to get a polynomial in $\mathbb{C}(x)[y]$. Therefore, we can assume that $P(x,y) \in \mathbb{C}(x)[y]$ is a polynomial of degree d in y, where $d > 0$. Then we can factorize $P(x,y)$ over $\overline{\mathbb{C}(x)}$ to get

$$P(x,y) = P^*(x)(y - y_1(x))(y - y_2(x)) \cdots (y - y_d(x)),$$

where $P^*(x) \in \mathbb{C}[x]$ and $y_j := y_j(x)$ is an algebraic function of x for $j \in \{1, 2, \ldots, d\}$.

We apply Jensen's formula with respect to the variable y in the standard Mahler measure formula for $P(x,y)$, and we obtain

$$\begin{aligned} \mathrm{m}\left(P(x,y)\right)-\mathrm{m}\left(P^*(x)\right) =& \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\log\left|P\left(x,y\right)\right|\frac{dx}{x}\frac{dy}{y}-\mathrm{m}\left(P^*(x)\right) \qquad (1.1.2)\\ =& \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\left(\sum_{j=1}^{d}\log\left|y-y_j\left(x\right)\right|\right)\frac{dx}{x}\frac{dy}{y}\\ =& \frac{1}{2\pi i}\left(\sum_{j=1}^{d}\int_{|x|=1,|y_j(x)|\geq 1}\log\left|y_j\left(x\right)\right|\frac{dx}{x}\right)\\ =& -\frac{1}{2\pi}\sum_{j=1}^{d}\int_{|x|=1,|y_j(x)|\geq 1}\eta\left(x,y_j\right), \qquad (1.1.3)\end{aligned}$$

where η is defined by (1.1.1), and

$$\eta(x,y_j)=\log|x|d\arg y_j-\log|y_j|d\arg x=i\log|y_j(x)|\frac{dx}{x}.$$

Here we used the fact that $\log|x|=\log 1=0$ and $\frac{dx}{x}=d(\log|x|+i\arg x)$, where we consider $\arg(x)\in[-\pi,\pi)$.

Following the discussion in **[37]** we define $H_1(E_{N(k)},\mathbb{Z})^-$ to be the subgroup of elements in $H_1(E_{N(k)},\mathbb{Z})$ which change signs under complex conjugation. Also let $H_1(E_{N(k)},\mathbb{Z})^+$ be the subgroup of elements in $H_1(E_{N(k)},\mathbb{Z})$ which are invariant under complex conjugation. Therefore, we have

$$H_1\left(E_{N(k)},\mathbb{Z}\right)=H_1\left(E_{N(k)},\mathbb{Z}\right)^+\oplus H_1\left(E_{N(k)},\mathbb{Z}\right)^-,$$

a direct sum of two free $\mathbb{Z}$-modules, as it follows from their definitions that these two components are mutually exclusive.

Remark 1.1.2. *If we integrate $\eta(x,y)$ over a path in $H_1(E_{N(k)},\mathbb{Z})^+$, we will get 0. Indeed, the path we are considering stays invariant under complex conjugation and $\overline{\eta(x,y)}=-\eta(x,y)$. Therefore, we are interested in showing the path $\{|x|=1,|y_j(x)|\geq 1\}$ is closed, and a cycle in $H_1(E_{N(k)},\mathbb{Z})^-$ rather than just in $H_1(E_{N(k)},\mathbb{Z})$.*

1.1.2. Bloch–Wigner dilogarithm and Mahler measure

Following the discussion in the introduction we know that $C_4 : P_4 = 0$ is a genus 0 curve. There is a certain relation between η and D (defined in (0.3.1)) which we can use to calculate the integral in (1.1.2) to obtain the Mahler measure in this case.

Lemma 1.1.3. *Let C be a curve over $\mathbb{C}$ which defines a compact Riemann surface. Then, for $x, 1-x \in \mathbb{C}(C)^*$, we have*

$$\eta\left(x, 1-x\right) = dD\left(x\right). \tag{1.1.4}$$

Proof. Recall that

$$\mathrm{Li}_2(v) = -\int_0^v \log(1-u)\frac{du}{u}, \tag{1.1.5}$$

where we choose the branch of $\log(1-u)$ defined in $\mathbb{C} \setminus [1, \infty)$ such that $\log(1-0) = 0$. We will first prove (1.1.4) when $C = \mathbf{P}^1(\mathbb{C})$, i.e. we will show that if $C = \mathbf{P}^1(\mathbb{C})$ with parameter t, i.e. $\mathbb{C}(\mathbf{P}^1(\mathbb{C})) = \mathbb{C}(t)$, then

$$\eta\left(t, 1-t\right) = dD\left(t\right),$$

where $t \not\equiv 0, 1, \infty$.

We now identify $\mathbf{P}^1(\mathbb{C})$ with $\mathbb{C} \cup \{\infty\}$. For $t, \epsilon \in \mathbf{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, we know that $\int_\epsilon^t \eta(u, 1-u)$ is path-independent. Indeed, it follows from Stokes' theorem using the fact that $\eta(t, 1-t)$ is a closed C^∞ differential 1-form on $\mathbf{P}^1(\mathbb{C}) \setminus T$, where $T = \{\text{poles and zeros of } t \text{ and } 1-t \text{ in } \mathbf{P}^1(\mathbb{C})\} = \{0, 1, \infty\}$. Indeed, if γ_1, γ_2 are two paths joining ϵ and t avoiding all points of singularity of η in $\mathbf{P}^1(\mathbb{C})$ and the branch cut $[1, \infty)$, then

$$\int_{\gamma_1} \eta(u, 1-u) - \int_{\gamma_2} \eta(u, 1-u) = \int_{\gamma} \eta(u, 1-u) = \iint_{\Omega} d\eta(u, 1-u) = 0,$$

where $\gamma = \gamma_1 \cup (-\gamma_2)$ is a closed path passing through ϵ and t, and Ω be the region closed by γ containing no singularities of η. In other words, we obtain

$$\int_{\gamma_1} \eta(u, 1-u) = \int_{\gamma_2} \eta(u, 1-u).$$

We restrict ϵ to $(0, 1)$ and define

$$h(t) = \lim_{\epsilon \to 0} \int_\epsilon^t \eta(u, 1-u),$$

where the integral is taken over a path connecting ϵ and t avoiding the point 0. Then, by the definition of η we have

$$h(t) = \lim_{\epsilon \to 0} \int_\epsilon^t \log|u| d\arg(1-u) - \lim_{\epsilon \to 0} \int_\epsilon^t \log|1-u| d\arg(u),$$

where we are assuming that $\arg z \in [-\pi, \pi)$. We also have

$$\log(1-u) = \log|1-u| + i\arg(1-u),$$
$$\frac{du}{u} = d\log|u| + id\arg u. \tag{1.1.6}$$

We substitute $v = t$ in (1.1.5), and combine it with (1.1.6) to get that

$$\mathrm{Im}(\mathrm{Li}_2(t)) = -\int_0^t \arg(1-u)d\log|u| - \int_0^t \log|1-u|d\arg u, \tag{1.1.7}$$

where $\log(1-u)$ is defined in $\mathbb{C}\setminus[1,\infty)$. As the value of the integral in $h(t)$ is the same for homologous paths, we choose a path which does not intersect the branch cut $[1,\infty)$ to evaluate $h(t)$ in terms of $\mathrm{Im}(\mathrm{Li}_2(t))$, and therefore, in terms of $D(t)$.

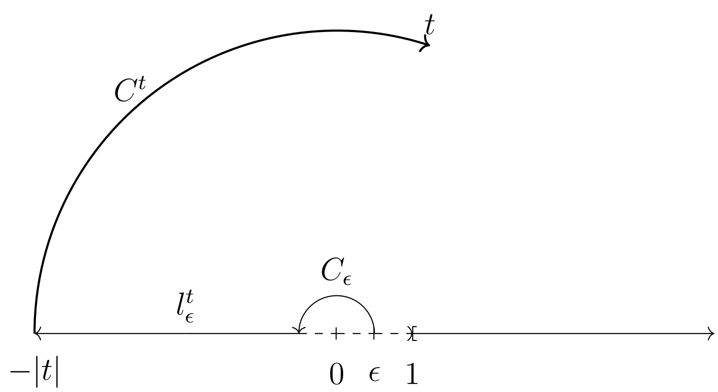

Figure 1.1. Integration path γ^t_ϵ

Let γ^t_ϵ be a curve which starts at ϵ and ends at t, defined as follows. γ^t_ϵ consists of three components (see Figure 1.1):

(1) a semicircle $\overset{\curvearrowleft}{C_\epsilon}$ of radius ϵ centered at 0, starting at ϵ and ending at $-\epsilon$ (we consider this arc to avoid the pole of $\eta(u, 1-u)$ at 0),

(2) a line $\overleftarrow{l^t_\epsilon}$, starting from $-\epsilon$ and ending at $-|t|$,

(3) an arc $\overset{\curvearrowright}{C^t}$ of radius $|t|$ centered at 0, which connects $-|t|$ and t.

Note that $\arg u$ is constant on $\overleftarrow{l^t_\epsilon}$, and therefore, on that line $d\arg u = 0$ with $\arg(1-u) = 0$. Also, $|u|$ is constant on $\overset{\curvearrowright}{C^t}$ and $\overset{\curvearrowleft}{C_\epsilon}$, which implies $d\log|u| = 0$. In particular,

$$\lim_{\epsilon\to 0}\int_{\gamma^t_\epsilon} \arg(1-u)d\log|u| = 0.$$

On $\overset{\frown}{C}_\epsilon$, we have

$$\left| \int_{\overset{\frown}{C}_\epsilon} \log|1-u| d\arg u \right| \leq \pi \log(1+\epsilon),$$

which goes to 0 as $\epsilon \to 0$. Therefore, we get

$$\operatorname{Im}(\mathrm{Li}_2(t)) = -\lim_{\epsilon\to 0} \int_{\gamma_\epsilon^t} \log|1-u| d\arg u = h(t) - \lim_{\epsilon\to 0} \int_{\gamma_\epsilon^t} \log|u| d\arg(1-u). \tag{1.1.8}$$

But again, for ϵ close to 0, we have $\arg(1-u) = 0$ on $\overleftarrow{l_\epsilon^t}$ and

$$\left| \int_{\overset{\frown}{C}_\epsilon} \log|u| d\arg(1-u) \right| \leq \frac{K\epsilon\log\epsilon}{(1-\epsilon)^2},$$

where $d\arg(1-u) = \operatorname{Im}\left(\frac{d(1-u)}{1-u}\right)$ and $K \in \mathbb{R}_{>2\pi}$ is a constant which does not depend on ϵ. This implies that

$$\lim_{\epsilon\to 0} \int_{\gamma_\epsilon^t} \log|u| d\arg(1-u) = \int_{\overset{\frown}{C^t}} \log|u| d\arg(1-u) = \arg(1-t)\log|t|,$$

which combined with (1.1.8) and (0.3.1) gives

$$h(t) = \lim_{\epsilon\to 0} \int_\epsilon^t \eta(u, 1-u) = \operatorname{Im}(\mathrm{Li}_2(t)) + \arg(1-t)\log|t| = D(t).$$

In other words, when $C = \mathbf{P}^1(\mathbb{C})$ we have $\eta(t, 1-t) = dD(t)$, where $t, 1-t$ belong to the domain of definition of η.

For the general case, note that there is a one-to-one correspondence between $\mathbb{C}(C) \cup \{\infty\}$ and $\{$rational maps $C \to \mathbf{P}^1(\mathbb{C})$ defined over $\mathbb{C}\}$. More precisely, for $x \in \mathbb{C}(C)^*$, there exists a rational map $\tau : C \longrightarrow \mathbf{P}^1(\mathbb{C})$ defined by

$$P \longmapsto \begin{cases} [x(P), 1] & \text{if } x \text{ is regular at } P, \\ [1, 0] & \text{if } x \text{ has a pole at } P, \end{cases}$$

which induces an injection

$$\tau^* : \mathbb{C}(t) \longrightarrow \mathbb{C}(C), \qquad \tau^*(f) = f \circ \tau,$$

where $\mathbb{C}(t)$ and $\mathbb{C}(C)$ are the function fields of $\mathbf{P}^1(\mathbb{C})$ and C, respectively. Then, for $f \equiv t$, we have $t \circ \tau = x$. Therefore, we have the following diagram:

$$\begin{tikzcd} C \arrow[r, "\tau"] \arrow[dr, "x"'] & \mathbf{P}^1(\mathbb{C}) \arrow[d, "D"] \\ & \mathbb{C} \cup \{\infty\} \end{tikzcd}$$

Let S be a set containing poles and zeros of x and $1-x$. Then restricting the map τ to $C \setminus S \longrightarrow \mathbf{P}^1(\mathbb{C}) \setminus T$ we obtain (1.1.4), i.e.

$$\eta(x,1-x) = \eta(t\circ\tau,(1-t)\circ\tau) = \tau^*\eta(t,1-t) = \tau^* dD(t) = dD(t\circ\tau) = dD(x),$$

where $T = \{0,1,\infty\}$, and $\eta(t,1-t) = dD(t)$ is (1.1.4) when $C = \mathbf{P}^1(\mathbb{C})$. $\square$

Therefore, if we can write

$$\eta\left(x,y_j\right) = \sum_k a_{j_k}\eta\left(z_{j_k},1-z_{j_k}\right) = \sum_k a_{j_k} dD(z_{j_k}), \tag{1.1.9}$$

where $z_{j_k}, (1-z_{j_k}) \in \mathbb{C}(C)^*$ are algebraic functions of x, and the sum is finite. Then, from (1.1.3) we obtain

$$\begin{aligned}
\mathrm{m}\left(P\left(x,y\right)\right) - \mathrm{m}\left(P^*\left(x\right)\right) &= -\frac{1}{2\pi}\sum_{j=1}^{d}\int_{|x|=1,|y_j(x)|\geq 1}\eta\left(x,y_j\right) \\
&= -\frac{1}{2\pi}\sum_{j=1}^{d}\int_{|x|=1,|y_j(x)|\geq 1}\sum_k a_{j_k}\eta\left(z_{j_k},1-z_{j_k}\right) \\
&= -\frac{1}{2\pi}\sum_{j=1}^{d}\int_{|x|=1,|y_j(x)|\geq 1}\sum_k a_{j_k} dD(z_{j_k}) \\
&= -\frac{1}{2\pi}\sum_{j=1}^{d}\sum_k a_{j_k} D\left(z_{j_k}\right)|_{\partial\{|x|=1,|y_j(x)\geq 1\}},
\end{aligned} \tag{1.1.10}$$

where $\partial\{|x|=1,|y_j|\geq 1\}$ is the set of boundary points of $\{|x|=1,|y_j|\geq 1\}$.

Remark 1.1.4. *As mentioned in* **[15]** *and* **[48]***, we may have some extra terms of the form* $\eta\left(c,z\right)$ *in* (1.1.9)*, where c is a constant complex number and z is some algebraic function. But we can still reach a closed formula by integrating $\eta\left(c,z\right)$ directly (i.e. by integrating* $\log|c| d\arg z$*). Also if ν is a constant such that $|\nu|=1$, then we have* $\eta\left(\nu,z\right) = \log|\nu| d\arg z = 0$.

Now we note down some relations and properties of the Bloch–Wigner dilogarithm D (for more details see [**51**]) :

- Note that, for $z \in \mathbb{C}$,

$$D\left(\bar{z}\right) = -D\left(z\right), \tag{1.1.11}$$

  which we will use frequently in the proof of Theorem 0.3.1. This property of D follows from its definition, and shows that $D(r) = 0$ for all $r \in \mathbb{R}$.

- *Five-term Relationship* : For $x, y \in \mathbb{C} \cup \{\infty\}$

$$D(x) + D(y) + D(1 - xy) + D\left(\frac{1-x}{1-xy}\right) + D\left(\frac{1-y}{1-xy}\right) = 0,$$

  where we recall the convention $D(\infty) = 0$. This equality is obtained from a similar relationship involving $\mathrm{Li}_2(t)$.

- We can obtain the following relations from the "*Five-term Relationship*" :

(1) If we take $y = \frac{1}{x}$ in the "*Five-term Relationship*", we get

$$D\left(\frac{1}{x}\right) = -D(x).$$

(2) Evaluating the "*Five-term Relationship*" at $y = 1$ we obtain

$$D(1 - x) = -D(x).$$

(3) If we take $x = y$ in the "*Five-term Relationship*", and use the above relations, we get

$$2D(x) + 2D(-x) = D(x^2).$$

- The Bloch–Wigner dilogarithm also satisfies

$$-2\int_0^{\theta} \log|2\sin u| du = D\left(e^{2i\theta}\right) = \sum_{m=1}^{\infty} \frac{\sin(2m\theta)}{m^2}.$$

1.1.3. The Elliptic Regulator

We will now recall the definition of the regulator map on the second K-group of an elliptic curve E, given by Bloch and Beĭlinson. Then we will explain its relation with the elliptic dilogarithm, and recover its relationship with Mahler measure.

Let F be a field. By a theorem of Matsumoto, the second K-group of F can be described as

$$K_2(F) \cong \Lambda^2 F^{\times}/\{x \otimes (1 - x) : x \in F, x \neq 0, 1\}.$$

Given a Laurent polynomial $P(x, y) = \sum_{(i,j)\in\mathbb{Z}^2} a_{ij} x^i y^j$, let $\Delta(P)$ be its Newton polygon, which is the convex hull of the points in $(i, j) \in \mathbb{Z}^2$ such that the coefficient of $x^i y^j$ is non-zero in $P(x, y)$. The Newton polygons $\Delta(P_k)$ for $P_k(x, y) = x + \frac{1}{x} + y + \frac{1}{y} + k$, are *convex lattice polygons* with only one interior point ([**37**]). Indeed, if we consider the convex hull of the points (i,j) such that $x^i y^j$ in $P_k(x,y)$ has non-zero coefficient, we have the square of vertices

$$(1,0),(0,-1),(-1,0),(0,1),$$

with the sole interior point $(0,0)$ (see Figure 1.2). A one-variable polynomial can be associated for each side of the corresponding Newton Polygon $\Delta(P)$, whose coefficients are identified with the coefficients of $P(x,y)$ associated to the points that lie on that side. Let τ denote a side of $\Delta(P)$. We parametrize a side clockwise around Δ and in such a way that $\tau(0),\tau(1),\ldots$ are the consecutive lattice points in τ. To every side we then associate a one-variable polynomial

$$P_\tau(u)=\sum_{l\geq 0} a_{\tau(l)}u^l \in \mathbb{C}[u],$$

where

$$a_{\tau(l)} = a_{i_{\tau(l)}j_{\tau(l)}}$$

for $\tau(l)=\left(i_{\tau(l)},j_{\tau(l)}\right)\in\tau$. For example, the Newton polygons $\Delta(P_k)$ for $P_k(x,y)=x+\frac{1}{x}+y+\frac{1}{y}+k$ with $k\in\mathbb{C}$ have four vertices, namely

$$(1,0),(0,-1),(-1,0),(0,1).$$

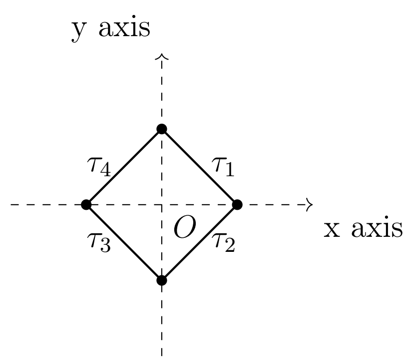

Figure 1.2. Newton Polygon of P_k

Let the sides of the Newton polygon $\Delta(P_k)$ be τ_1,τ_2,τ_3, and τ_4 (see Figure 1.2). The corresponding one-variable polynomials associated to the sides of the Newton polygon $\Delta(P_k)$ are the same for every side, namely $(1+u)$, as each side τ contains only two points (the vertices) such that coefficients of $P_4(x,y)$ associated to them are non-zero (see Table 1.1). Note that the standard Mahler measure of $(1+u)$ is 0, i.e.

$$\mathrm{m}(1+u)=0.$$

Sides (τ_i)	Corresponding Polynomials	Points ($\tau_i(l)$)	Coefficients of P_k	$P_{\tau_i}(u)$
τ_1	$x+y$	$\tau_1(0)=(0,1)$ $\tau_1(1)=(1,0)$	1 1	$1+u$
τ_2	$x+\frac{1}{y}$	$\tau_2(0)=(1,0)$ $\tau_2(1)=(0,-1)$	1 1	$1+u$
τ_3	$\frac{1}{x}+\frac{1}{y}$	$\tau_3(0)=(0,-1)$ $\tau_3(1)=(-1,0)$	1 1	$1+u$
τ_4	$\frac{1}{x}+y$	$\tau_4(0)=(-1,0)$ $\tau_4(1)=(0,1)$	1 1	$1+u$

Table 1.1. Sides of Newton polygons $\Delta(P_k)$ and associated polynomials P_{τ_i}

We can similarly construct the Newton polygon $\Delta(P_{k,a,b})$ of $P_{k,a,b}(x,y) = ax + \frac{1}{ax} + by + \frac{1}{by} + k$, where $k \in \mathbb{C}$, and $a, b \in \mathbb{C}^*$ with $|a| \neq |b|$. $\Delta(P_{k,a.b})$ is still a square with vertices $(1,0), (0,-1), (-1,0)$, and $(0,1)$, but the coefficients are different. Let the sides of $\Delta(P_{k,a,b})$ be τ^1, τ^2, τ^3, and τ^4.

The corresponding one-variable polynomials associated to the sides τ^1, τ^2, τ^3 and τ^4 of the Newton polygon $\Delta(P_{k,a,b})$ are $b + au, a + \frac{u}{b}, \frac{1}{b} + \frac{u}{a}$ and $\frac{1}{a} + bu$, respectively (see Table 1.2). Notice that the standard Mahler measures of P_{τ^i} are generally not all zero. Indeed, as $|a| \neq |b|$, we have that at least one of the following is non-zero:

$$\mathrm{m}(b+au) = \max\{\log|a|, \log|b|\}, \quad \mathrm{m}\left(a+\frac{u}{b}\right) = \max\{-\log|b|, \log|a|\},$$

$$\mathrm{m}\left(\frac{1}{b}+\frac{u}{a}\right) = \max\{-\log|a|, -\log|b|\}, \quad \mathrm{m}\left(\frac{1}{a}+bu\right) = \max\{\log|b|, -\log|a|\}.$$

Now we have the following definition due to Rodriguez-Villegas (for more details see [**37**]):

Sides (τ^i)	Corresponding Polynomials	Points ($\tau^i(l)$)	Coefficients of $P_{k,a,b}$	$P_{\tau^i}(u)$
τ^1	$ax+by$	$\tau^1(0)=(0,1)$ $\tau^1(1)=(1,0)$	b a	$b+au$
τ^2	$ax+\frac{1}{by}$	$\tau^2(0)=(1,0)$ $\tau^2(1)=(0,-1)$	a $\frac{1}{b}$	$a+\frac{u}{b}$
τ^3	$\frac{1}{ax}+\frac{1}{by}$	$\tau^3(0)=(0,-1)$ $\tau^3(1)=(-1,0)$	$\frac{1}{b}$ $\frac{1}{a}$	$\frac{1}{b}+\frac{u}{a}$
τ^4	$\frac{1}{ax}+by$	$\tau^4(0)=(-1,0)$ $\tau^4(1)=(0,1)$	$\frac{1}{a}$ b	$\frac{1}{a}+bu$

Table 1.2. Sides of Newton polygons $\Delta(P_{k,a,b})$ and associated polynomials P_{τ^i}

Definition 1.1.5. *$P(x,y)$ is called* ***tempered*** *if the polynomials constructed from the sides of $\Delta(P)$ have Mahler measure zero.*

For example, $\{P_k(x,y)\}_{k\in\mathbb{C}}$ is a tempered family of polynomials, but $\{P_{k,a,b}(x,y)\}_{k\in\mathbb{C}}$ is a non-tempered family. We will see that this condition plays a role in understanding the K-theory framework of the regulator.

For a field F with discrete valuation ν and maximal ideal $\mathcal{M}$, the tame symbol is given by

$$(x,y)_\nu \equiv (-1)^{\nu(x)\nu(y)}\frac{x^{\nu(y)}}{y^{\nu(x)}} \pmod{\mathcal{M}}$$

(see **[37]**). Note that in particular, $(x,y)_\nu = 1$ if $\nu(x)=\nu(y)=0$.

For a curve C over $\mathbb{C}$ which is a compact Riemann surface, its field of fractions is denoted by $\mathbb{C}(C)$. A point $P\in C(\mathbb{C})$ defines a valuation ν_P on $\mathbb{C}(C)$, which is determined by the order of the rational functions at the point $P\in C(\mathbb{C})$. We follow the notation in **[37]** to denote the tame symbol given by ν_P as $(\cdot,\cdot)_P$. We also have the residue map, which is a linear form determined by P,

$$\mathrm{Res}_P : H^1(C\setminus\{P\},\mathbb{R})\to\mathbb{R}$$

(for more details see [**37**]).

Let $\psi : U \to \mathbb{C}$ be a local chart of C with $P \in U$ such that $\psi(P) = 0$. We define γ_P as the pre-image by ψ of a small circle in $\mathbb{C}$ centered at zero and oriented counterclockwise. Therefore, γ_P is a closed path (oriented counterclockwise) in $C(\mathbb{C})$ with P in its interior region (the region bounded by the path). Then, for $\xi \in H^1(C \setminus \{P\}, \mathbb{R})$, we have

$$\mathrm{Res}_P(\xi) = \frac{1}{2\pi}\xi([\gamma_P]) \in \mathbb{R},$$

where $[\gamma_P]$ is the homology class of γ_P in $H_1(C \setminus \{P\}, \mathbb{Z})$. In fact, since the construction is local, Res_P extends to a linear form mapping $H^1(C \setminus S, \mathbb{R})$ to $\mathbb{R}$ for any finite set S with Res_P is identically zero if $P \notin S$.

We also have the following lemma which relates the differential form η to the tame symbol.

Lemma 1.1.6 (see [**37**])**.** *Let $P \in C(\mathbb{C})$, $x, y \in \mathbb{C}(C)^*$, and $S \subset C(\mathbb{C})$ a finite set containing poles and zeroes of x and y. Then*

$$\mathrm{Res}_P(\eta(x, y)) = \log |(x, y)_P|.$$

The proof of this is an application of Jensen's formula, and follows from properties of log and η (Lemma 1.1.1). Note that, for a closed path γ in $C \setminus S$, the map

$$\gamma \mapsto \int_\gamma \eta(x,y)$$

only depends on the homology class $[\gamma] \in H_1(C \setminus S, \mathbb{Z})$, and therefore determines an element in $H^1(C \setminus S, \mathbb{R})$, say $\bar{r}(x, y)$. From (1.1.4) we also have $\eta(x, 1 - x) = 0$ in $H^1(C \setminus S, \mathbb{R})$, i.e.

$$\int_\gamma \eta(x, 1 - x) = 0 \qquad \forall\ [\gamma] \in H_1(C \setminus S, \mathbb{Z}).$$

Given a finite set $S \subset C$ we can define

$$K_{2,S}(C) = \bigcap_{P \notin S} \ker \lambda_P \subset K_2(\mathbb{C}(C)),$$

where $\lambda_P : K_2(\mathbb{C}(C)) \to \mathbb{C}^*$ is the corresponding map of the tame symbol $(\cdot, \cdot)_P$ (see [**37**]). Now by the Lemma 1.1.6 and the discussion above we have the following commutative diagram for every $P \in S$:

$$\begin{tikzcd} K_{2,S}(C) \arrow[r, "\bar{r}"] \arrow[d, "\lambda_P"] & H^1(C \setminus S, \mathbb{R}) \arrow[d, "\mathrm{Res}_P"] \\ \mathbb{C}^* \arrow[r, "\log|\cdot|"] & \mathbb{R} \end{tikzcd}$$

Finally, for an elliptic curve E over $\mathbb{Q}$ we can define a tame symbol corresponding to a point $T \in E(\bar{\mathbb{Q}})$ as a map $K_2(\mathbb{Q}(E)) \to \mathbb{Q}(T)^*$. We also have an exact sequence

$$0 \to K_2(E) \otimes \mathbb{Q} \to K_2(\mathbb{Q}(E)) \otimes \mathbb{Q} \to \coprod_{T \in E(\overline{\mathbb{Q}})} \mathbb{Q}(T)^* \times \mathbb{Q},$$

where the last arrow corresponds to the coproduct of the tame symbols (for more details see [**32**]).

We will interpret $H^1(E, \mathbb{R})$ as the dual of the first homology group of E with coefficients in $\mathbb{Z}$, namely $H_1(E, \mathbb{Z})$. Let $[\gamma] \in H_1(E, \mathbb{Z})$. Now we can define the regulator map.

Definition 1.1.7. *The regulator map of Bloch* [**12**] *and Beĭlinson* [**10**] *is given by*

$$\begin{aligned} r_E : K_2(E) \otimes \mathbb{Q} &\to H^1(E, \mathbb{R}) \\ \{x, y\} &\to \left\{ [\gamma] \to \int_\gamma \eta(x, y) \right\}. \end{aligned}$$

Remark 1.1.8. *We note down some observations regarding the regulator map.*

- *Let $\mathcal{E}$ be the Néron model of the elliptic curve E. Then the regulator is actually defined over $K_2(\mathcal{E})$. But from* [**11**]*, we also have that $K_2(\mathcal{E}) \otimes \mathbb{Q}$ is a subgroup of $K_2(E) \otimes \mathbb{Q}$ determined by finitely many extra conditions.*
- *Note that the regulator map is trivial for the classes in $H_1(E_{(N(k))}, \mathbb{Z})^+$. Therefore it suffices to consider the regulator as a function on $H_1(E_{N(k)}, \mathbb{Z})^-$. In other words, we consider an element in $H_1(E_{N(k)}, \mathbb{Z})^-$ rather than in $H_1(E_{(N(k))}, \mathbb{Z})^+$ while integrating η, which is an alternative way of stating the reason given in Remark 1.1.2 in terms of the regulator map.*

Rodriguez-Villegas [**37**] proved that the condition of $P(x,y)$ being tempered is equivalent to the triviality of tame symbols in K-theory, thus giving us a way to produce elements in $K_{2,\emptyset}(E)$, where E is an elliptic curve over $\mathbb{Q}$. We can therefore define a map

$$\tilde{r} : K_{2,\emptyset}(E) \to \mathbb{R}$$

by $\varphi \mapsto \frac{1}{2\pi}\bar{r}(\varphi)(c_0)$, where $c_0 \in H_1(E, \mathbb{Z})$ is the cycle determined by the connected component of $E(\mathbb{R})$. Following a similar process we may obtain a formula like (1.1.16) for cases when the polynomial is tempered.

Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the complex upper-half plane. We will now concentrate on the integral of $\eta(x, y)$. Recall that if $E/\mathbb{Q}$ is an elliptic curve, then we have the following sequence of isomorphisms,

$$\begin{array}{ccccc} E(\mathbb{C}) & \xrightarrow{\sim} & \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) & \xrightarrow{\sim} & \mathbb{C}^\times/q^{\mathbb{Z}} \\ T=(\wp(s),\wp'(s)) & \to & s \bmod \Lambda_\tau & \to & z=e^{2\pi i s}, \end{array} \tag{1.1.12}$$

where $\wp$ is the Weierstrass function, Λ_τ is the lattice $\mathbb{Z}+\tau\mathbb{Z}$, $\tau\in\mathbb{H}$, and $q=e^{2\pi i\tau}$. In other words, *Uniformization Theorem* implies that there exists a lattice $\Lambda=\omega_1\mathbb{Z}+\omega_2\mathbb{Z}\subset\mathbb{C}$ with $\frac{\omega_2}{\omega_1}\in\mathbb{C}\setminus\mathbb{R}$ (by swapping ω_1 and ω_2 if necessary we may assume that $\frac{\omega_2}{\omega_1}\in\mathbb{H}$) such that,

(1) $E(\mathbb{C})$ corresponds to the lattice Λ,

(2) $\tau=\frac{\omega_2}{\omega_1}\in\mathbb{H}$,

(3) Λ and Λ_τ are homothetic, i.e. $\Lambda=\omega_1\Lambda_\tau$.

Therefore $\mathbb{C}/\Lambda$ is isomorphic to $\mathbb{C}/\Lambda_\tau$, and this isomorphism induces the first isomorphism in (1.1.12).

Although we will not be using the *elliptic dilogarithm* and *diamond operator* explicitly in our proofs, we will state their definitions and a theorem due to Bloch because we believe they provide a valuable perspective to the discussion about Mahler measure of several variable polynomials in general. We also include some additional calculations from **[32]** to obtain values of the standard Mahler measure of $P_k(x,y)=x+\frac{1}{x}+y+\frac{1}{y}+k$ for $k>4$ in terms of *elliptic dilogarithm* in Section 3.6 for completion.

Now we recall the definition of the *elliptic dilogarithm*, due to Bloch **[12]**.

Definition 1.1.9. *The* ***elliptic dilogarithm*** *is a function on $E(\mathbb{C})$ given as follows. For $T\in E(\mathbb{C})$ corresponding to $z\in\mathbb{C}^\times/q^{\mathbb{Z}}$,*

$$D^E(T):=\sum_{n\in\mathbb{Z}}D(q^nz), \tag{1.1.13}$$

where D is the Bloch–Wigner dilogarithm defined by (0.3.1)*.*

Let $\mathbb{Z}[E(\mathbb{C})]$ be the group of divisors on E, and let

$$\mathbb{Z}[E(\mathbb{C})]^-\cong\mathbb{Z}[E(\mathbb{C})]/\{(T)+(-T):T\in E(\mathbb{C})\}. \tag{1.1.14}$$

Let $f,g\in\mathbb{C}(E)^\times$. We define the diamond operation by

$$\begin{array}{rcl} \diamond:\Lambda^2\mathbb{C}(E)^\times & \to & \mathbb{Z}[E(\mathbb{C})]^- \\ (f)\diamond(g) & = & \displaystyle\sum_{i,j}m_in_j(P_i-Q_j), \end{array}$$

where

$$(f) = \sum_i m_i(P_i) \text{ and } (g) = \sum_j n_j(Q_j).$$

We have the following result.

Theorem 1.1.10. *(Bloch [**12**]) The elliptic dilogarithm D^E extends by linearity to a map from $\mathbb{Z}[E(\mathbb{Q})]^-$ to $\mathbb{C}$. Let $x, y \in \mathbb{Q}(E)$ and $\{x, y\} \in K_2(E)$. Then*

$$r_E(\{x, y\})[\gamma] = D^E((x) \diamond (y)),$$

where $[\gamma]$ is a generator of $H_1(E, \mathbb{Z})^-$.

Remark 1.1.11. *The result above also implies*

$$D^E((f) \diamond (1 - f)) = 0$$

for any $f \in \mathbb{Q}(E)$.

For two elements $u, v \in \mathbb{Z}[E(\mathbb{C})]^-$, we write

$$u \sim v \quad \text{when} \quad D^E(u) = D^E(v).$$

In particular, Remark 1.1.11 implies that $u \sim v$ if

$$u - v = \sum c_i(x_i) \diamond (1 - x_i) \quad \text{for some} \quad x_i \in \mathbb{C}(E)^\times, c_i \in \mathbb{Z}. \tag{1.1.15}$$

Deninger [**23**] was first to write a formula of the form

$$\mathrm{m}(P) = \frac{1}{2\pi} r(\{x, y\})[\gamma], \tag{1.1.16}$$

where r is the elliptic regulator corresponding to the polynomial P.

Rodriguez-Villegas was able to prove identities between two Mahler measures in [**38**] after a thorough study of the properties of $\eta(x, y)$ and elliptic dilogarithm.

1.2. Arbitrary Tori and Mahler measure

We follow an analysis similar to the one in [**31**] if the integration torus is arbitrary. We continue to consider that $P(x, y)$ is a Laurent polynomial and $P^*(x) \in \mathbb{C}[x]$. For simplicity we take $d = 2$, where d is the degree of y in $P(x, y)$.

Let $x = ax'$ and $y = by'$. Then $y_j = y_j(x) = by'_j$ is an algebraic function of x for $j \in \{1,2\}$. We have

$$\begin{aligned}
\mathrm{m}_{a,b}\left(P(x,y)\right) - \mathrm{m}_{a,b}\left(P^*(x)\right) =& \frac{1}{(2\pi i)^2}\iint_{|x|=a,|y|=b} \log\left|P\left(x,y\right)\right| \frac{dx}{x}\frac{dy}{y} - \mathrm{m}_{a,b}\left(P^*(x,y)\right)\\
=& \frac{1}{(2\pi i)^2}\iint_{|x'|=|y'|=1} \log\left|P\left(ax',by'\right)\right| \frac{dx'}{x'}\frac{dy'}{y'} - \mathrm{m}_{a,b}\left(P^*(x)\right)\\
=& \frac{1}{(2\pi i)^2}\iint_{|x'|=|y'|=1} \left(\sum_{j=1}^{2}\log\left|by' - y_j\left(ax'\right)\right|\right) \frac{dx'}{x'}\frac{dy'}{y'}\\
=& \frac{1}{(2\pi i)^2}\iint_{|x'|=|y'|=1} \left(\sum_{j=1}^{2}\log\left|y' - \frac{y_j\left(ax'\right)}{b}\right|\right) \frac{dx'}{x'}\frac{dy'}{y'}\\
&+ \frac{2}{(2\pi i)^2}\iint_{|x'|=|y'|=1} (\log b)\frac{dx'}{x'}\frac{dy'}{y'}\\
=& 2\log b + \frac{1}{2\pi i}\left(\sum_{j=1}^{2}\int_{|x'|=1,|y_j(ax')|\geq b}\log\left|\frac{y_j\left(ax'\right)}{b}\right|\frac{dx'}{x'}\right)\\
=& 2\log b + \frac{1}{2\pi i}\left(\sum_{j=1}^{2}\int_{|x'|=1,|y'_j|\geq 1}\log|y'_j|\frac{dx'}{x'}\right),
\end{aligned} \tag{1.2.1}$$

where the penultimate equality follows from Jensen's formula and the fact that $\int_{|z|=1}\frac{dz}{z} = 2\pi i$. Now using a similar derivation as in (1.1.3) we get

$$\eta(x', y'_j) = i\log|y'_j|\frac{dx'}{x'}$$

for $j = 1,2$. Then (1.2.1) becomes

$$\begin{aligned}
\mathrm{m}_{a,b}\left(P(x,y)\right) - \mathrm{m}_{a,b}\left(P^*(x)\right) =& 2\log b - \frac{1}{2\pi}\int_{|x'|=1,|y'_1|\geq 1}\eta\left(x',y'_1\right) - \frac{1}{2\pi}\int_{|x'|=1,|y'_2|\geq 1}\eta\left(x',y'_2\right)\\
=& 2\log b - \frac{1}{2\pi}\int_{|x|=a,|y_1|\geq b}\eta\left(x/a, y_1/b\right)\\
&- \frac{1}{2\pi}\int_{|x|=a,|y_2|\geq b}\eta\left(x/a, y_2/b\right),
\end{aligned} \tag{1.2.2}$$

where the penultimate step follows from Lemma 1.1.1. Each of the terms in the last line can be further simplified using (2) of Lemma 1.1.1 as

$$\begin{aligned}
-\frac{1}{2\pi}\int_{|x|=a,|y_j|\geq b}\eta\left(x/a,y_j/b\right) =& -\frac{1}{2\pi}\int_{|x|=a,|y_i|\geq b}\left(\eta\left(x,y_j\right) - \eta\left(a,y_i\right) - \eta\left(x,b\right)\right)\\
=& -\log b - \frac{1}{2\pi}\int_{|x|=a,|y_j|\geq b}\left(\eta\left(x,y_j\right) - \log\left(a\right)d\arg y_j\right),
\end{aligned}$$

where

$$\frac{1}{2\pi}\int_{|x|=a,|y_j|\geq b}\eta(x,b)=\frac{1}{2\pi}\int_{|x|=a,|y_j|\geq b} i\log|b|\frac{dx}{x}=-\log b.$$

As mentioned in the beginning, if $P(x,y)$ is tempered then we reduce to the classical case to evaluate

$$-\frac{1}{2\pi}\int_{|x|=a,|y_j|\geq b}\eta\left(x,y_j\right). \tag{1.2.3}$$

If we can show that $\{|x|=a,|y_j|\geq b\}$ is a closed path, which can be characterized as a cycle in $H_1(E,\mathbb{Z})^-$ (where E is the corresponding elliptic curve of $P(x,y)=0$), the integration becomes simpler due to Deninger [**23**].

Now for the term with $d\arg y_j$,

$$\frac{\log a}{2\pi}\int_{|x|=a,|y_j|\geq b} d\arg y_j \tag{1.2.4}$$

leads to a multiple of $\log a$ if $\{|x|=a,|y_i|\geq b\}$ is a closed path.

If we have a genus 0 curve (such as $C_4 : P_4(x,y)=0$) then, instead of proceeding as in the direction above, we may be able to use Lemma 1.1.3 to relate the Bloch–Wigner dilogarithm and η for evaluating the Mahler measure as in (1.1.10). The evaluation is much simpler in this case.

Chapter 2

Proof of Theorem 0.3.1

In this chapter our goal is to prove Theorem 0.3.1. The methods used to prove the theorem are mostly inspired by those used in [**22**]. We start by considering the factorization of $P_4(x,y)$ due to Boyd (see Section 2A in [**17**]) to obtain an equality as in (1.2.2). We adapt the notation used in the proof of Theorem 12 in [**22**]. We substitute the change of variables $x \mapsto \frac{w}{z}$ and $y \mapsto wz$ in $P_4(x,y) = \left(x + \frac{1}{x} + y + \frac{1}{y} + 4\right)$ to get

$$Q(w,z) = P_4\left(\frac{w}{z}, wz\right) = 4 + wz + \frac{1}{wz} + \frac{w}{z} + \frac{z}{w} = \frac{1}{wz}(1 + iw + iz + wz)(1 - iw - iz + wz). \tag{2.0.1}$$

Therefore we can concentrate on finding the Mahler measure of the new simpler polynomial $Q(w,z)$ for the tori $\mathbb{T}^2_{c,d} = \{(w,z) \in \mathbb{C}^* \times \mathbb{C}^* : |w| = c, |z| = d\}$ (where $c, d \in \mathbb{R}_{>0}$) and later use the above change of variables to discover the values of $P_4(x,y)$ for the tori $\mathbb{T}^2_{a,b} = \{(x,y) \in \mathbb{C}^* \times \mathbb{C}^* : |x| = a, |y| = b\}$. From the change of variable we get that

$$c = \sqrt{ab},\ d = \sqrt{\frac{b}{a}}.$$

Now we have

$$\mathrm{m}_{c,d}(Q(w,z)) = \mathrm{m}_{c,d}\left(\frac{1}{wz}\right) + 2\mathrm{m}_{c,d}(1 + iw + iz + wz), \tag{2.0.2}$$

because we can apply a further change of variables, namely $z \mapsto -z$ and $w \mapsto -w$, to obtain

$$\mathrm{m}_{c,d}(1 - iw - iz + wz) = \mathrm{m}_{c,d}(1 + iw + iz + wz),$$

since it does not alter the Mahler measure. Let $Q_1(w,z) = 1 + iw + iz + wz$. It remains to evaluate $\mathrm{m}_{c,d}\left(\frac{1}{wz}\right)$ and $\mathrm{m}_{c,d}(Q_1(w,z))$ to deduce Theorem 0.3.1.

2.1. Outline of the proof of Theorem 0.3.1

We recall Theorem 0.3.1:

Theorem 0.3.1. *Let a and b be positive real numbers. If $-1 \leq \frac{(1+ab)(a-b)}{2\sqrt{ab}(a+b)} \leq 1$ and $ab \neq 1$, we define $\sin\alpha := \frac{(1+ab)(a-b)}{2\sqrt{ab}(a+b)}$ with $\alpha \in \left[-\frac{\pi}{2}, 0\right)$, when $b > a$, and $\sin\beta := \frac{(1+ab)(a-b)}{2\sqrt{ab}(a+b)}$ with $\beta \in \left(0, \frac{\pi}{2}\right)$, when $b < a$. Then, for $k = 4$, the values of $R_{a,b} := \frac{1}{2}\left[\mathrm{m}_{a,b}\left(P_4(x,y)\right) + \log b\right]$ are given by the following table:*

Condition 1	*Condition 2*	*Extra conditions*	*Values*
$ab \neq 1$	$b = a$		$\frac{1}{\pi}\Big(2D\left(i\sqrt{ab}\right) - \left(\log\sqrt{ab}\right)\tan^{-1}\left(\frac{2\sqrt{ab}}{ab-1}\right)\Big)$
	$b > a$	$-1 \leq \sin\alpha < 0$	$\frac{1}{\pi}\Big(D\left(i\sqrt{ab}e^{-i\alpha}\right) + D\left(i\sqrt{ab}e^{i\alpha}\right) - \left(\log\sqrt{ab}\right)\tan^{-1}\left(\frac{2\sqrt{ab}\cos\alpha}{ab-1}\right)\Big)$
		$\frac{(1+ab)(a-b)}{2\sqrt{ab}(a+b)} < -1$	0
	$b < a$	$0 \leq \sin\beta < 1$	$\frac{1}{\pi}\Big(D\left(i\sqrt{ab}e^{-i\beta}\right) + D\left(i\sqrt{ab}e^{i\beta}\right) - \left(\log\sqrt{ab}\right)\tan^{-1}\left(\frac{2\sqrt{ab}\cos\beta}{ab-1}\right)\Big)$
		$\frac{(1+ab)(a-b)}{2\sqrt{ab}(a+b)} \geq 1$	0
$ab = 1$	$b = a$		$\frac{2}{\pi}D\left(i\right)$
	$b > a$		$\frac{1}{\pi}\left(D\left(-e^{-2i\cot^{-1}\left(\sqrt{\frac{b}{a}}\right)}\right) + D\left(e^{2i\cot^{-1}\left(\sqrt{\frac{b}{a}}\right)}\right)\right)$
	$b < a$		$\frac{1}{\pi}\left(D\left(-e^{-2i\cot^{-1}\left(\sqrt{\frac{b}{a}}\right)}\right) + D\left(e^{2i\cot^{-1}\left(\sqrt{\frac{b}{a}}\right)}\right)\right)$

Table 2.1. Values of $R_{a,b}$

Instead of a direct approach to deduce the theorem we start by finding the values of $\mathrm{m}_{c,d}(Q(w,z))$ for different values of positive real numbers c and d. We can compute $\mathrm{m}_{c,d}\left(\frac{1}{wz}\right)$ and $\mathrm{m}_{c,d}(Q_1(w,z))$ separately, and add them via (2.0.2) to obtain the values of $\mathrm{m}_{c,d}(Q(w,z))$. Though computation of the first term follows directly from the integral, we use the exactness of η to evaluate the second term. In fact, following the discussion in Section 1.2 we can write $\mathrm{m}_{c,d}(Q_1(w,z))$ as

$$\mathrm{m}_{c,d}(Q_1(w,z)) = -\frac{1}{2\pi}\int_{|w|=c,|z_1|\geq d} \left(\eta\left(w,z_1\right) - \log\left(c\right) d\arg z_1\right) = -\frac{1}{2\pi}\int_{|w|=c,|z_1|\geq d} \eta\left(\frac{w}{c}, z_1\right), \tag{2.1.1}$$

where

$$z_1 := z_1\left(w\right) = -\frac{1+iw}{w+i}$$

is a rational function of w, and the last equality in (2.1.1) is obtained by using (2) of Lemma 1.1.1.

Let $w = ce^{i\theta}$ with $\theta \in [-\pi, \pi)$. Our first goal in this case is to determine the values of θ, c and d such that

$$|w| = c \Rightarrow |z_1| \geq d. \tag{2.1.2}$$

Next we want to express $\eta(w, z_1)$ in terms of an exact differential form by writing it as a sum of $\eta(u_j, 1-u_j)$, where u_j are algebraic functions of w, and apply Lemma 1.1.3 to get the required values. For the integral

$$\frac{1}{2\pi}\int_{|w|=c,|z_1|\geq d} \log\left(c\right) d\arg z_1,$$

we propose a simplification of the term $d\arg z_1$ in terms of c and θ, and use the obtained values of θ satisfying (2.1.2) to evaluate the integral. It only remains to deduce a relation between $\mathrm{m}_{a,b}(P_4(x,y))$ and $\mathrm{m}_{c,d}(Q(w,z))$ via the change of variables $x \mapsto \frac{w}{z}$ and $y \mapsto wz$, and thus concluding the proof of Theorem 0.3.1.

2.2. Proof of Theorem 0.3.1

We prove Theorem 0.3.1 as an application of the following results.

Let $w = cw'$ where $w' = e^{i\theta}$ for $\theta \in [-\pi, \pi)$.

Proposition 2.2.1. *For $c, d \in \mathbb{R}_{>0}$, we have*

$$\mathrm{m}_{c,d}\left(\frac{1}{wz}\right) = -\log|cd|.$$

Lemma 2.2.2. *If c, d are positive real numbers, then the condition $|z_1(w)| \geq d$ can be rewritten as:*

(1) $-1 \leq \sin\theta \leq \frac{1+c^2}{2c}\frac{1-d^2}{1+d^2}$, *when* $c \neq 1$,

(2) $\left|\cot\left(\frac{2\theta+\pi}{4}\right)\right| \geq d$, *when* $c = 1$.

Lemma 2.2.3. *For $c \in \mathbb{R}_{>0}$, we can decompose $\eta\left(\frac{w}{c}, z_1\right) = \eta(w', z_1)$ as*

$$\eta(w', z_1) = \eta\left(-icw', 1-(-icw')\right) - \eta\left(icw', 1-icw'\right) - \eta\left(c, \frac{1+icw'}{1-icw'}\right).$$

Therefore, using Lemma 1.1.3 we can rewrite $\eta(w', z_1)$ as

$$\eta(w', z_1) = dD\left(-icw'\right) - dD\left(icw'\right) - \eta\left(c, \frac{1+icw'}{1-icw'}\right),$$

where D is the Bloch–Wigner dilogarithm given in (0.3.1) *as*

$$D\left(z\right) = \operatorname{Im}\left(\operatorname{Li}_2\left(z\right) + i\arg\left(1-z\right)\log|z|\right),$$

with $z \in \mathbb{C}$.

Lemma 2.2.4. *Let $\psi = \theta + \frac{\pi}{2}$. Then, for $c \in \mathbb{R}_{>0}$ and $\theta \in [-\pi, \pi)$ we have*

$$d\arg\left(\frac{1+icw'}{1-icw'}\right) = \frac{2\left(c^{-1}-c\right)\cos\psi}{\left(c^{-1}-c\right)^2 + 4\sin^2\psi}d\psi.$$

Proposition 2.2.5. *If $\left|\frac{1+c^2}{2c}\frac{1-d^2}{1+d^2}\right| \leq 1$, we define $\tau \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin\tau = \frac{1+c^2}{2c}\frac{1-d^2}{1+d^2}$. Then, for $c, d \in \mathbb{R}_{>0}$ and $c \neq 1$, we have*

(1) *if* $\left|\frac{1+c^2}{2c}\frac{1-d^2}{1+d^2}\right| > 1$, *then* $\mathrm{m}_{c,d}(Q_1(w,z)) = 0$,

(2) *if* $\left|\frac{1+c^2}{2c}\frac{1-d^2}{1+d^2}\right| \leq 1$, *then*

$$\mathrm{m}_{c,d}(Q_1(w,z)) = \frac{1}{\pi}\left(D(ice^{-i\tau}) + D(ice^{i\tau}) - (\log c)\tan^{-1}\left(\frac{2\cos\tau}{c-c^{-1}}\right)\right). \tag{2.2.1}$$

Remark 2.2.6. *Note that, if $\frac{1+c^2}{2c}\frac{1-d^2}{1+d^2} = \pm 1$, then $\tau = \pm\frac{\pi}{2}$, respectively. Now, replacing this value in* (2.2.1) *with the fact that $D(\mathbb{R}) = \{0\}$ (see* (1.1.11)*) we get that $\mathrm{m}_{c,d}(Q_1(w,z)) = 0$. Moreover, α and β, defined in the statement of Theorem 0.3.1, are nothing but $\tau = \alpha$ when $\tau \in \left[-\frac{\pi}{2}, 0\right)$, and $\tau = \beta$ when $\tau \in \left(0, \frac{\pi}{2}\right)$.*

Proposition 2.2.7. *If $c = 1$ and $d \in \mathbb{R}_{>0}$, then*

$$\mathrm{m}_{c,d}(Q_1(w,z)) = \frac{1}{\pi}\left(D\left(-e^{-2i\cot^{-1}d}\right) + D\left(e^{2i\cot^{-1}d}\right)\right).$$

Lemma 2.2.8. *The change of variables*

$$x \mapsto \frac{w}{z} \text{ and } y \mapsto wz$$

implies that

$$\mathrm{m}_{c,d}(Q(w,z)) = \mathrm{m}_{a,b}(P_4(x,y)).$$

Proof of Theorem 0.3.1: The proofs of Propositions 2.2.5 and 2.2.7 follow from Lemmas 2.2.2, 2.2.3 and 2.2.4. We conclude the proof of Theorem 0.3.1 by combining the results in Propositions 2.2.1, 2.2.5 and 2.2.7 with the relation in (2.0.2) and the fact that $\mathrm{m}_{c,d}(Q(w,z)) = \mathrm{m}_{a,b}(P_4(x,y))$, which can be obtained from Lemma 2.2.8. Note that the change of variables establishes the following relations among a, b, c and d:

$$c = \sqrt{ab}, \ d = \sqrt{\frac{b}{a}}, \ a = \frac{c}{d} \text{ and } b = cd. \tag{2.2.2}$$

We replace the above values of c, d in the formulas of $\mathrm{m}_{c,d}(Q(w,z))$, obtained from Propositions 2.2.5 and 2.2.7, to deduce the list of values in Table 2.1. □

The rest of this chapter deals with the proofs of the propositions and lemmas mentioned above.

2.3. Evaluation of $\mathrm{m}_{c,d}\left(\frac{1}{wz}\right)$

We start by evaluating $\mathrm{m}_{c,d}\left(\frac{1}{wz}\right)$.

Proof of Proposition 2.2.1: For $c, d \in \mathbb{R}_{>0}$, we have

$$\mathrm{m}_{c,d}\left(\frac{1}{wz}\right) = \frac{1}{(2\pi i)^2}\int_{|w|=c}\int_{|z|=d} \log\left|\frac{1}{wz}\right| \frac{dw}{w}\frac{dz}{z} = -\log|cd|. \tag{2.3.1}$$

□

Next we consider the main term $\mathrm{m}_{c,d}\left(Q_1(w,z)\right)$.

2.4. Evaluation of $\mathrm{m}_{c,d}\left(Q_1(w,z)\right)$

For simplicity we have considered $w = cw'$ with $|w'| = 1$. Now $Q_1(w,z)$ is a non-zero polynomial of degree 1 in z. Recall that $z_1 = z_1(w) = -\frac{1+iw}{w+i}$, a rational function of w, is the solution of $Q_1(w,z)$. Let $w = ce^{i\theta}$ with $-\pi \le \theta < \pi$. To evaluate $\mathrm{m}_{c,d}(Q_1(w,z))$ we use (1.2.2) to get

$$\mathrm{m}_{c,d}(Q_1(w,z)) = -\frac{1}{2\pi}\int_{|w|=c,|z_1|\ge d} \eta\left(\frac{w}{c}, z_1\right). \tag{2.4.1}$$

Our goal is to rewrite $\eta\left(\frac{w}{c}, z_1\right)$ in a similar manner given in (1.1.9), and obtain an expression of the form (1.1.10) to evaluate $\mathrm{m}_{c,d}(Q_1(w,z))$. To begin with, we need to find all the elements of $\partial\{|w| = c, |z_1| \ge d\}$, the set of boundary points of $\{|w| = c, |z_1| \ge d\}$. Note that the set of values of w at the endpoints of the intervals of θ such that $|z_1(w)| \ge d$ is in fact the set $\partial\{|w| = c, |z_1| \ge d\}$.

Proof of Lemma 2.2.2: We have

$$|z_1(w)| \ge d \Leftrightarrow \left|\frac{1+iw}{w+i}\right| \ge d \Leftrightarrow \left|\frac{1+iw}{1-iw}\right| \ge d. \tag{2.4.2}$$

The denominator $|1 - iw|$ is a non-negative real number. We want to consider distinctly the cases when it is always positive and when it can be zero. In order to do so we substitute $w = ce^{i\theta}$ in $|1 - iw|$ and square it to get

$$|1 - iw|^2 = (1 + c\sin\theta)^2 + (c\cos\theta)^2 = 1 + 2c\sin\theta + c^2.$$

Now the term $1 + 2c\sin\theta + c^2$ is 0 only when $c = 1$ and $\sin\theta = -1$. Therefore, we consider the cases $c \ne 1$ and $c = 1$ separately below.

(1) $c \ne 1$: We rewrite (2.4.2) by substituting $w = ce^{i\theta}$ and taking squares on both sides to obtain

$$\begin{aligned}\left|\frac{1+ic(\cos\theta + i\sin\theta)}{1-ic(\cos\theta + i\sin\theta)}\right|^2 \ge d^2 &\Leftrightarrow \frac{(1-c\sin\theta)^2 + (c\cos\theta)^2}{(1+c\sin\theta)^2 + (c\cos\theta)^2} \ge d^2\\ &\Leftrightarrow \frac{1-2c\sin\theta + c^2}{1+2c\sin\theta + c^2} \ge d^2\\ &\Leftrightarrow 1-2c\sin\theta + c^2 \ge d^2\left(1+2c\sin\theta + c^2\right)\\ &\Leftrightarrow 2c\sin\theta\left(1+d^2\right) \le \left(1-d^2\right)\left(1+c^2\right) \qquad (2.4.3)\end{aligned}$$

where we have assumed that $c \ne 1$. Therefore, inequality (2.4.3) gives us a restriction on θ for the condition $|z_1(w)| \ge d$, namely

$$-1 \le \sin\theta \le \frac{1+c^2}{2c}\frac{1-d^2}{1+d^2}. \tag{2.4.4}$$

(2) $c = 1$: We have $w = w' = e^{i\theta}$ with $\theta \in [-\pi, \pi)$. Then the inequality $|z_1(w)| \geq d$ becomes

$$|z_1\left(w\right)| = \left|\frac{1+iw}{w+i}\right| \geq d \Leftrightarrow \left|\frac{1+iw}{w+i}\right| = \left|\frac{1+iw}{1-iw}\right| = \left|\cot\left(\frac{2\theta+\pi}{4}\right)\right| \geq d. \tag{2.4.5}$$

□

In order to use the exactness of η to compute the integral in 2.4.1, we now prove Lemma 2.2.3.

Proof of Lemma 2.2.3: Using properties of η in Lemma 1.1.1 we can decompose $\eta(w', z_1)$ as

$$\begin{aligned}
\eta(w', z_1) &= \eta\left(w', \frac{1+icw'}{1-icw'}\right)\\
&= \eta\left(w', 1+icw'\right) - \eta\left(w', 1-icw'\right)\\
&= \eta\left(-icw', 1-\left(-icw'\right)\right) - \eta\left(-ic, 1+icw'\right) - \eta\left(icw', 1-icw'\right) + \eta\left(ic, 1-icw'\right)\\
&= \eta\left(-icw', 1-\left(-icw'\right)\right) - \eta\left(icw', 1-icw'\right) - \eta\left(c, \frac{1+icw'}{1-icw'}\right) - \eta\left(i, \frac{1+icw'}{1-icw'}\right)\\
&\quad - \eta\left(-1, 1+icw'\right)\\
&= \eta\left(-icw', 1-\left(-icw'\right)\right) - \eta\left(icw', 1-icw'\right) - \eta\left(c, \frac{1+icw'}{1-icw'}\right)\\
&= dD\left(-icw'\right) - dD\left(icw'\right) - \eta\left(c, \frac{1+icw'}{1-icw'}\right),
\end{aligned} \tag{2.4.6}$$

where the penultimate equality follows from Remark 1.1.4, because i and -1 are 4th and 2nd roots of unity respectively. □

We replace the decomposition of $\eta\left(w', z_1\right)$ given by (2.4.6) in

$$\mathrm{m}_{c,d}\left(Q_1\left(w,z\right)\right) = -\frac{1}{2\pi}\int_{|cw'|=c, |z_1|\geq d} \eta\left(w', z_1\right)$$

to get

$$\begin{aligned}
\mathrm{m}_{c,d}\left(Q_1\left(w,z\right)\right) &= -\frac{1}{2\pi}\int_{|w'|=1, |z_1|\geq d}\left(dD\left(-icw'\right) - dD\left(icw'\right) - \eta\left(c, \frac{1+icw'}{1-icw'}\right)\right)\\
&= -\frac{1}{2\pi}\left(D\left(-icw'\right) - D\left(icw'\right)\right)|_{\partial\{|w'|=1, |z_1|\geq d\}} + \frac{\log c}{2\pi}\int_{|w'|=1, |z_1|\geq d} d\arg\left(\frac{1+icw'}{1-icw'}\right),
\end{aligned} \tag{2.4.7}$$

where $\partial\{|w'| = 1, |z_1| \geq d\}$ is the set of boundary points of $\{|w'| = 1, |z_1| \geq d\}$.

2.4.1. Simplification of $d\arg\left(\frac{1+icw'}{1-icw'}\right)$

In this section, we will deal with the simplification of the last term in (2.4.7).

Proof of Lemma 2.2.4: We know that $d\arg z = \operatorname{Im}\left(\frac{dz}{z}\right)$. Then

$$d\arg\left(\frac{1+icw'}{1-icw'}\right) = d\arg\left(1+icw'\right) - d\arg\left(1-icw'\right) = \operatorname{Im}\left(\frac{icdw'}{1+icw'} + \frac{icdw'}{1-icw'}\right)$$
$$= \operatorname{Im}\left(\frac{cd\tilde{w}}{1+c\tilde{w}} + \frac{cd\tilde{w}}{1-c\tilde{w}}\right),$$

where $\tilde{w} = iw'$. Let $\tilde{w} = e^{i\psi}$, i.e. $\psi = \theta + \frac{\pi}{2}$. Then

$$\begin{aligned}
\operatorname{Im}\left(\frac{cd\tilde{w}}{1+c\tilde{w}} + \frac{cd\tilde{w}}{1-c\tilde{w}}\right) &= \operatorname{Im}\left(\frac{ice^{i\psi}d\psi}{1+ce^{i\psi}} + \frac{ice^{i\psi}d\psi}{1-ce^{i\psi}}\right) \\
&= \operatorname{Im}\left(i\left(\frac{ce^{i\psi}d\psi}{1+ce^{i\psi}} + \frac{ce^{i\psi}d\psi}{1-ce^{i\psi}}\right)\right) \\
&= \operatorname{Re}\left(\frac{ce^{i\psi}d\psi}{1+ce^{i\psi}} + \frac{ce^{i\psi}d\psi}{1-ce^{i\psi}}\right) \\
&= \operatorname{Re}\left(\frac{2ce^{i\psi}}{1-c^2e^{2i\psi}}d\psi\right) \\
&= 2\operatorname{Re}\left(\frac{1}{c^{-1}e^{-i\psi} - ce^{i\psi}}d\psi\right).
\end{aligned} \tag{2.4.8}$$

We can simplify the denominator of the last expression as

$$c^{-1}e^{-i\psi} - ce^{i\psi} = \left(c^{-1} - c\right)\cos\psi - i\left(c^{-1} + c\right)\sin\psi.$$

Therefore (2.4.8) becomes

$$\begin{aligned}
2\operatorname{Re}\left(\frac{1}{c^{-1}e^{-i\psi} - ce^{i\psi}}d\psi\right) &= 2\operatorname{Re}\left(\frac{1}{(c^{-1}-c)\cos\psi - i\,(c^{-1}+c)\sin\psi}d\psi\right) \\
&= 2.\frac{1}{2}\left(\frac{1}{(c^{-1}-c)\cos\psi - i(c^{-1}+c)\sin\psi}\right. \\
&\quad \left. + \frac{1}{(c^{-1}-c)\cos\psi + i\,(c^{-1}+c)\sin\psi}\right)d\psi \\
&= \frac{2\,(c^{-1}-c)\cos\psi}{\left(c^{-1}-c\right)^2\cos^2\psi + \left(c^{-1}+c\right)^2\sin^2\psi}d\psi \\
&= \frac{2\,(c^{-1}-c)\cos\psi}{\left(c^{-1}-c\right)^2 + 4\sin^2\psi}d\psi.
\end{aligned} \tag{2.4.9}$$

□

2.4.2. Values of $\mathrm{m}_{c,d}(Q_1(w,z))$ when $c \neq 1$

Proof of Proposition 2.2.5: To evaluate $\mathrm{m}_{c,d}(Q_1(w,z))$, when $c \neq 1$, using (2.4.7) we now consider several cases depending on whether $d = 1$, $d > 1$, or $d < 1$.

Note that, to prove (1) we need to consider only $d > 1$ and $d < 1$, because the case $d = 1$ implies $\frac{1+c^2}{2c}\frac{1-d^2}{1+d^2} = 0$. Therefore, we have the following cases:

d > 1: If $d > 1$, then we have $1 - d^2 < 0$. Therefore, the given condition can be written as

$$\left|\frac{1+c^2}{2c}\frac{1-d^2}{1+d^2}\right| > 1 \Rightarrow \frac{1+c^2}{2c}\frac{1-d^2}{1+d^2} < -1.$$

From (1) of Lemma 2.2.2 we therefore obtain

$$\left\{\theta : -1 \leq \sin\theta \leq \frac{1+c^2}{2c}\frac{1-d^2}{1+d^2}\right\} = \varnothing.$$

In other words, the integration path is empty, i.e. $\{|w| = c, |z_1(w)|\ geqd\} = \varnothing$, with $w = ce^{i\theta}$. This information applied to (2.4.7) concludes

$$\mathrm{m}_{c,d}\left(Q_1\left(w,z\right)\right) = 0.$$

d < 1: In this case we have $1 - d^2 < 0$. Therefore, the interval of values of θ satisfying

$$\left|\frac{1+c^2}{2c}\frac{1-d^2}{1+d^2}\right| > 1, \text{ i.e. } \frac{1+c^2}{2c}\frac{1-d^2}{1+d^2} > 1$$

is

$$\left\{\theta : -1 \leq \sin\theta \leq 1 \leq \frac{1+c^2}{2c}\frac{1-d^2}{1+d^2}\right\} = [-\pi,\pi),$$

where $\frac{w}{c} = w' = e^{i\theta}$. In other words, we have

$$|w'| = 1 \Rightarrow |z_1(w)| \geq d.$$

Now, we conclude our proof of (1) by proving the following proposition.

Proposition 2.4.1. *If the inequality considered above is satisfied, then,*

$$\begin{aligned}\mathrm{m}_{c,d}\left(Q_1(w,z)\right) &= -\frac{1}{2\pi}\int_{|w'|=1,|z_1|\geq d}\eta(w',z_1)\\ &= -\frac{1}{2\pi}\int_{|w'|=1}\eta(w',z_1)\\ &= 0.\end{aligned}$$

PROOF. We start by expanding (the integrand in) $\mathrm{m}_{c,d}(Q_1(w,z))$ (see (2.4.1)) in terms of "log".

Let $z_1(cw') = \frac{1+icw'}{1-icw'}$, a rational function in $w = cw'$ where $w' = e^{i\theta}$ with $-\pi \leq \theta < \pi$. Then we have

$$\begin{aligned}
\mathrm{m}_{c,d}(Q_1(w,z)) =& \frac{1}{(2\pi i)^2} \iint_{|w'|=1,|z|=d} \log|z - z_1(cw')| \frac{dw'}{w'}\frac{dz'}{z'} \\
=& \frac{1}{2\pi i} \int_{|w'|=1,|z_1(cw')|\geq d} \log\left|\frac{1+icw'}{1-icw'}\right| \frac{dw'}{w'} \\
=& \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left|\frac{1+ice^{i\theta}}{1-ice^{i\theta}}\right| d\theta \\
=& \frac{1}{2\pi}\left[\int_{-\pi}^{0} \log\left|\frac{1+ice^{i\theta}}{1-ice^{i\theta}}\right| d\theta\right] + \frac{1}{2\pi}\left[\int_{0}^{\pi} \log\left|\frac{1+ice^{i\theta}}{1-ice^{i\theta}}\right| d\theta\right], \qquad (2.4.10)
\end{aligned}$$

where we used the fact that $|w'| = 1 \Rightarrow |z_1| \geq d$ in the third step. We use two changes of variables, namely $\theta + \frac{\pi}{2} \mapsto \delta$ and $\theta - \frac{\pi}{2} \mapsto \tau$ for the first and second integral in (2.4.10) respectively, and we obtain

$$\begin{aligned}
\mathrm{m}_{c,d}(Q_1(w,z)) =& \frac{1}{2\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\frac{1+ce^{i\delta}}{1-ce^{i\delta}}\right| d\delta\right] + \frac{1}{2\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\frac{1-ce^{i\tau}}{1+ce^{i\tau}}\right| d\tau\right] \\
=& \quad 0.
\end{aligned}$$

$\square$

Now we concentrate on proving (2). As mentioned in the beginning of this proof, we will study the cases $d < 1$, $d = 1$, and $d > 1$ separately.

Case 1: If $d = 1$, then $1 - d^2 = 0$. By (2.4.4), we have $-1 \leq \sin\theta \leq 0$. Therefore,

$$\text{when } -\pi \leq \theta \leq 0, \text{ we have } |z_1(w)| \geq d.$$

Thus we should integrate (2.4.6) between $w'|_{\theta=-\pi} = e^{-i\pi} = -1$ and $w'|_{\theta=0} = e^{i0} = 1$. In other words, we have $\partial\{|w| = c, |z_1| \geq 1\} = \partial\{|w'| = 1, |z_1| \geq 1\} = \{-1,1\}$. As the values of the integrals with integrands of the form $dD(x)$ only depend on the endpoints, from (2.4.7) we have

$$\begin{aligned}
\mathrm{m}_{c,d}\left(Q_1(w,z)\right) =& -\frac{1}{2\pi}\left(D(-icw') - D(icw')\right)|_{-1}^{1} \\
&+ \frac{\log c}{2\pi}\int_{|w'|=1,|z_1|\geq d} d\arg\left(\frac{1+icw'}{1-icw'}\right) \\
=& -\frac{1}{2\pi}(D(-ic) - D(ic) - D(ic) + D(-ic)) \\
&+ \frac{\log c}{2\pi}\int_{|w'|=1,|z_1|\geq d} d\arg\left(\frac{1+icw'}{1-icw'}\right) \\
=& -\frac{1}{2\pi}\left(-4D(ic)\right) + \frac{\log c}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2(c^{-1}-c)\cos\psi}{(c^{-1}-c)^2 + 4\sin^2\psi}d\psi \qquad (2.4.11)
\end{aligned}$$

where the last step follows from (1.1.11) and Lemma 16 of **[22]** which states the following result.

Lemma 2.4.2. *If $R \in \mathbb{R}_{>0}$ and $\upsilon = e^{i\theta}$ with $\theta \in [-\pi, \pi)$ then we have*

$$\int_{-i}^{i} d\arg\left(\frac{1+R\upsilon}{1-R\upsilon}\right) = -2\tan^{-1}\left(\frac{2}{R-R^{-1}}\right).$$

Now we use the change of variables $\sin\psi \mapsto t$ in (2.4.11), and in the following step we consider another change of variable $\frac{2t}{c-c^{-1}} \mapsto u$ to obtain

$$\begin{aligned}
\mathrm{m}_{c,d}\left(Q_1(w,z)\right) =& -\frac{1}{2\pi}\left(-4D(ic)\right) + \frac{\log c}{2\pi}\int_{-1}^{1}\frac{2(c^{-1}-c)}{(c^{-1}-c)^2+4t^2}dt \\
=& -\frac{1}{2\pi}\left(-4D(ic)\right) - \frac{\log c}{2\pi}\int_{-\frac{2}{c-c^{-1}}}^{\frac{2}{c-c^{-1}}}\frac{du}{1+u^2} \\
=& -\frac{1}{2\pi}\left(-4D(ic)\right) - \frac{\log c}{\pi}\tan^{-1}\left(\frac{2}{c-c^{-1}}\right) \\
=& \frac{1}{\pi}\left(2D(ic) - (\log c)\tan^{-1}\left(\frac{2}{c-c^{-1}}\right)\right), \qquad (2.4.12)
\end{aligned}$$

Case 2: We now consider $d > 1$. From (2.4.4) we have

$$\frac{1+c^2}{2c} \geq \frac{1+d^2}{d^2-1} \Leftrightarrow 0 \geq \frac{1+c^2}{2c}\frac{1-d^2}{1+d^2} \geq -1. \qquad (2.4.13)$$

Recall the definition of α given in the statement of Theorem 0.3.1 as $\sin\alpha := \frac{1+c^2}{2c}\frac{1-d^2}{1-d^2}$ with $-\frac{\pi}{2} \leq \alpha < 0$.

Therefore, when $-1 \leq \sin\theta \leq \sin\alpha$, we have $|z_1(w)| \geq d$, i.e.

$$\text{when } -\pi-\alpha \leq \theta \leq \alpha < 0, \text{ we have } |z_1(w)| \geq d$$

where $w = ce^{i\theta}$. A similar reasoning as in **Case 1** implies that we have to integrate between $w'|_{\theta=-\pi-\alpha} = e^{i(-\pi-\alpha)} = -e^{-i\alpha}$ and $w'|_{\theta=\alpha} = e^{i\alpha}$, i.e. the set of boundary points $\partial\{|w| = c, |z_1| \geq d\} = \partial\{|w'| = 1, |z_1| \geq d\} = \{-e^{-i\alpha}, e^{i\alpha}\}$. Therefore, from (2.4.7) we have

$$\begin{aligned}
\mathrm{m}_{c,d}\left(Q_1(w,z)\right) =& -\frac{1}{2\pi}\left(D(-icw') - D(icw')\right)|_{-e^{-i\alpha}}^{e^{i\alpha}} \\
&+ \frac{\log c}{2\pi}\int_{|w'|=1,|z_1|\geq d} d\arg\left(\frac{1+icw'}{1-icw'}\right) \\
=& -\frac{1}{2\pi}\left(D(-ice^{i\alpha}) - D(ice^{i\alpha}) - D(ice^{-i\alpha}) + D(-ice^{-i\alpha})\right) \\
&+ \frac{\log c}{2\pi}\int_{|w'|=1,|z_1|\geq d} d\arg\left(\frac{1+icw'}{1-icw'}\right) \\
=& -\frac{1}{2\pi}\left(-2D(ice^{-i\alpha}) - 2D(ice^{i\alpha})\right) \\
&+ \frac{\log c}{2\pi}\int_{-\left(\frac{\pi}{2}+\alpha\right)}^{\frac{\pi}{2}+\alpha} \frac{2\left(c^{-1}-c\right)\cos\psi}{\left(c^{-1}-c\right)^2 + 4\sin^2\psi} d\psi \\
=& \frac{1}{\pi}\left(D(ice^{-i\alpha}) + D(ice^{i\alpha})\right) \\
&- \frac{\log c}{\pi}\tan^{-1}\left(\frac{2\cos\alpha}{c-c^{-1}}\right),
\end{aligned}$$

where the third step is obtained by using the change of variable $\psi = \theta + \frac{\pi}{2}$ and a similar calculation to (2.4.12). So we get, in this particular case,

$$\mathrm{m}_{c,d}\left(Q_1\left(w,z\right)\right) = \frac{1}{\pi}\left(D\left(ice^{-i\alpha}\right) + D\left(ice^{i\alpha}\right) - (\log c)\tan^{-1}\left(\frac{2\cos\alpha}{c-c^{-1}}\right)\right). \tag{2.4.14}$$

Case 3: The last case is when $d < 1$. Let,

$$0 < \frac{1+c^2}{2c} < \frac{1+d^2}{1-d^2} \Leftrightarrow \frac{1+c^2}{2c}\frac{1-d^2}{1+d^2} < 1.$$

Recall the definition of β given in the statement of Theorem 0.3.1 as $\sin\beta := \frac{1+c^2}{2c}\frac{1-d^2}{1-d^2}$ with $0 < \beta < \frac{\pi}{2}$. Therefore, when $-1 \leq \sin\theta \leq \sin\beta$, we have $|z_1\left(w\right)| \geq d$, i.e.

$$\text{when } \theta \in [-\pi, \beta]\bigcup[\pi-\beta, \pi], \text{ we have } |z_1(w)| \geq d,$$

where $\frac{w}{c} = w' = e^{i\theta}$. So, we have to integrate between $\{w'|_{\theta=-\pi} = e^{-i\pi}$ and $w'|_{\theta=\beta} = e^{i\beta}\}$ and $\{w'|_{\theta=\pi-\beta} = e^{i(\pi-\beta)}$ and $w'|_{\theta=\pi} = e^{i\pi}\}$. This implies that $\partial\{|w|=c,|z_1|\geq d\}=\partial\{|w'|=1,|z_1|\geq d\}=\{-e^{-i(\pi-\beta)},e^{i\beta},e^{i\pi}\}$, as $e^{i\pi}=e^{-i\pi}$. Therefore, following the discussion in 2.4, we can rewrite (2.4.7) as

$$\begin{aligned}
\mathrm{m}_{c,d}\left(Q_1(w,z)\right)=&-\frac{1}{2\pi}\left(D(-icw')-D(icw')\right)|_{e^{-i\pi}}^{e^{i\beta}}\\
&-\frac{1}{2\pi}\left(D(-icw')-D(icw')\right)|_{e^{i(\pi-\beta)}}^{e^{i\pi}}\\
&+\frac{\log c}{2\pi}\int_{|w'|=1,|z_1|\geq d}d\arg\left(\frac{1+icw'}{1-icw'}\right)\\
=&-\frac{1}{2\pi}\left(D(-icw')-D(icw')\right)|_{-e^{-i\beta}}^{e^{i\beta}}\\
&+\frac{\log c}{2\pi}\int_{|w'|=1,|z_1|\geq d}d\arg\left(\frac{1+icw'}{1-icw'}\right)\\
=&\frac{1}{\pi}\left(D(ice^{-i\beta})+D(ice^{i\beta})\right)\\
&+\frac{\log c}{2\pi}\int_{|w'|=1,|z_1|\geq d}d\arg\left(\frac{1+icw'}{1-icw'}\right),
\end{aligned}\tag{2.4.15}$$

where the penultimate step holds because $e^{i\pi}=e^{-i\pi}$, and the last step follows from (1.1.11). Now we will calculate the remaining integral in (2.4.15), namely

$$\begin{aligned}
\frac{\log c}{2\pi}\int_{|w'|=1,|z_1|\geq d}d\arg\left(\frac{1+icw'}{1-icw'}\right)=&\frac{\log c}{2\pi}\int_{-\pi}^{\beta}d\arg\left(\frac{1+ice^{i\theta}}{1-ice^{i\theta}}\right)\\
&+\frac{\log c}{2\pi}\int_{\pi-\beta}^{\pi}d\arg\left(\frac{1+ice^{i\theta}}{1-ice^{i\theta}}\right).
\end{aligned}\tag{2.4.16}$$

We use the change of variable $\theta-2\pi\mapsto\tau$ for the second integral in (2.4.16) to get

$$\begin{aligned}
\frac{\log c}{2\pi}\int_{|w'|=1,|z_1|\geq d}d\arg\left(\frac{1+icw'}{1-icw'}\right)=&\frac{\log c}{2\pi}\int_{-\pi}^{\beta}d\arg\left(\frac{1+ice^{i\theta}}{1-ice^{i\theta}}\right)\\
&+\frac{\log c}{2\pi}\int_{-\pi-\beta}^{-\pi}d\arg\left(\frac{1+ice^{i\tau}}{1-ice^{i\tau}}\right)\\
=&\frac{\log c}{2\pi}\int_{-\pi-\beta}^{\beta}d\arg\left(\frac{1+ice^{i\theta}}{1-ice^{i\theta}}\right)\\
=&\frac{\log c}{2\pi}\int_{-\frac{\pi}{2}-\beta}^{\frac{\pi}{2}+\beta}\frac{2\left(c^{-1}-c\right)\cos\gamma}{\left(c^{-1}-c\right)^2+4\sin^2\gamma}d\gamma\\
=&-\frac{\log c}{\pi}\left(\tan^{-1}\left(\frac{2\cos\beta}{c-c^{-1}}\right)\right),
\end{aligned}\tag{2.4.17}$$

where we obtain the penultimate line using a similar calculation to the one in (2.4.11) with a change of variable $\theta+\frac{\pi}{2}\mapsto\gamma$. From (2.4.15) and (2.4.17) we get, in this particular case,

$$\mathrm{m}_{c,d}\left(Q_1(w,z)\right) = \frac{1}{\pi}\left(D(ice^{-i\beta}) + D(ice^{i\beta}) - (\log c)\tan^{-1}\left(\frac{2\cos\beta}{c-c^{-1}}\right)\right). \tag{2.4.18}$$

Combining (2.4.12), (2.4.14), and (2.4.18) with Remark 2.2.6 conclude the proof of Proposition 2.2.5. □

Next we will describe the case when $c = 1$, and deduce Proposition (2.2.7).

2.4.3. Values of $\mathrm{m}_{c,d}\left(Q_1(w,z)\right)$ when $c = 1$

When $c = 1$ we have $w = e^{i\theta}$ and $z_1 = z_1(w) = -\frac{1+iw}{w+i} = -\frac{1+ie^{i\theta}}{i+e^{i\theta}}$. We already obtained in (2.4.5) that if $c = 1$, then

$$|z_1(w)| = \left|\frac{1+iw}{w+i}\right| \geq d \Leftrightarrow \left|\cot\left(\frac{2\theta+\pi}{4}\right)\right| \geq d.$$

Proof of Proposition 2.2.7: We now consider again several cases regarding the values of d as before.

Case 1: $d > 1$

To find the interval(s) of θ such that $|z_1| \geq d$ we first look for solutions of $\left|\cot\left(\frac{2\theta+\pi}{4}\right)\right| = d$ for $\theta \in [-\pi, \pi)$:

$$\begin{aligned}\left|\cot\left(\frac{2\theta+\pi}{4}\right)\right| = d \Leftrightarrow \cot\left(\frac{2\theta+\pi}{4}\right) &= \pm d \\ \Leftrightarrow \frac{2\theta+\pi}{4} &= \cot^{-1}(\pm d) = \pm\cot^{-1} d \\ \Leftrightarrow \theta = \frac{\pm 4\cot^{-1} d - \pi}{2} &= \pm 2\cot^{-1} d - \frac{\pi}{2}.\end{aligned} \tag{2.4.19}$$

Note that $d > 1$ implies that we are considering the case when $0 \leq \cot^{-1}(d) < \frac{\pi}{4}$ with $\theta \in [-\pi, \pi)$. Therefore, if $d < 1$, then

$$\left|\cot\left(\frac{2\theta+\pi}{4}\right)\right| \geq d \Leftrightarrow -\left(2\cot^{-1} d + \frac{\pi}{2}\right) \leq \theta \leq \left(2\cot^{-1} d - \frac{\pi}{2}\right),$$

which implies $\partial\{|w|=1,|z_1|\geq d\}=\left\{-e^{-i\left(2\cot^{-1}d+\frac{\pi}{2}\right)},e^{i\left(2\cot^{-1}d-\frac{\pi}{2}\right)}\right\}$, Now using this restriction on θ and the fact that $\eta\left(1,\frac{1+iw}{1-iw}\right)=0$ in (2.4.7), we have

$$\begin{aligned}
\mathrm{m}_{1,d}\left(Q_1(w,z)\right)=&-\frac{1}{2\pi}\int_{|w|=1,|z_1|\geq d}\left(dD(-iw)-dD(iw)\right)\\
=&-\frac{1}{2\pi}\left(D(-iw)-D(iw)\right)\Big|_{e^{-i\left(2\cot^{-1}d+\frac{\pi}{2}\right)}}^{e^{i\left(2\cot^{-1}d-\frac{\pi}{2}\right)}}\\
=&-\frac{1}{2\pi}\Big(D\left(-e^{2i\cot^{-1}d}\right)-D\left(e^{2i\cot^{-1}d}\right)\\
&-D\left(-e^{-2i\cot^{-1}d}\right)+D\left(e^{-2i\cot^{-1}d}\right)\Big)\\
=&-\frac{1}{2\pi}\left(-2D\left(-e^{-2i\cot^{-1}d}\right)-2D\left(e^{2i\cot^{-1}d}\right)\right)\\
=&\frac{1}{\pi}\left(D\left(-e^{-2i\cot^{-1}d}\right)+D\left(e^{2i\cot^{-1}d}\right)\right),
\end{aligned}\tag{2.4.20}$$

where the penultimate step follows from (1.1.11).

Case 2: $d=1$

Following a similar calculation to (2.4.19) we obtain

$$\left|\cot\left(\frac{2\theta+\pi}{4}\right)\right|\geq 1\Leftrightarrow-\pi\leq\theta\leq 0,\tag{2.4.21}$$

i.e. $\partial\{|w|=1,|z_1|\geq 1\}=\{e^{-i\pi},e^{i0}\}=\{-1,1\}$. In [**17**], Boyd gives the main idea to obtain the Mahler measure for this case. Here we reproduce the steps in the proof of Theorem 12 in [**22**] to obtain the standard Mahler measure of $Q_1(w,z)$. We know that $\eta\left(1,\frac{1+iw}{1-iw}\right)=0$ in (2.4.7). Combining this fact with the restriction on θ in (2.4.21) we have

$$\begin{aligned}
\mathrm{m}(Q_1(w,z))=&-\frac{1}{2\pi}\int_{|w|=1,|z_1|\geq d}\left(dD(-iw)-dD(iw)\right)\\
=&-\frac{1}{2\pi}\left(D(-iw)-D(iw)\right)\Big|_{-1}^{1}\\
=&-\frac{1}{2\pi}\left(D(-i\cdot 1)-D(-i\cdot(-1))-D(i\cdot 1)+D(i\cdot(-1))\right)\\
=&\frac{2}{\pi}D(i),
\end{aligned}\tag{2.4.22}$$

where the penultimate step follows from (1.1.11). In fact, in this case

$$\cot^{-1}d=\cot^{-1}1=\frac{\pi}{4},-\pi+\frac{\pi}{4}.$$

If we replace these values in (2.4.20), then (2.4.22) follows from (1.1.11), i.e.

$$\frac{1}{\pi}\left(D\left(-e^{-2i\cot^{-1}1}\right)+D\left(e^{2i\cot^{-1}1}\right)\right)=\frac{2}{\pi}D(i).$$

Case 3: $d<1$

Note that $d<1$ implies that we are considering the case when $\frac{\pi}{4}\leq\cot^{-1}d<\frac{\pi}{2}$ with $\theta\in[-\pi,\pi)$. Following a calculation similar to the one in (2.4.19) and using the fact that $d<1$ we have

$$\left|\cot\left(\frac{2\theta+\pi}{4}\right)\right|\geq d\Leftrightarrow\theta\in\left[-\pi,2\cot^{-1}d-\frac{\pi}{2}\right]\bigcup\left[\frac{3\pi}{2}-2\cot^{-1}d,\pi\right].$$

Therefore, the set of boundary points

$$\partial\{|w|=1,|z_1|\geq d\}=\left\{e^{-i\pi},e^{i\left(2\cot^{-1}d-\frac{\pi}{2}\right)},e^{-i\left(2\cot^{-1}d-\frac{3\pi}{2}\right)},e^{i\pi}\right\}.$$

We also have $\eta\left(1,\frac{1+iw}{1-iw}\right)=0$ in (2.4.7). Therefore, we obtain

$$\begin{aligned}
\mathrm{m}_{1,d}\left(Q_1(w,z)\right)=&-\frac{1}{2\pi}\int_{|w|=1,|z_1|\geq d}(dD(-iw)-dD(iw))\\
=&-\frac{1}{2\pi}\Bigg[(D(-iw)-D(iw))\Big|_{e^{-i\pi}}^{e^{i\left(2\cot^{-1}d-\frac{\pi}{2}\right)}}\\
&+(D(-iw)-D(iw))\Big|_{e^{-i\left(2\cot^{-1}d-\frac{3\pi}{2}\right)}}^{e^{i\pi}}\Bigg]\\
=&-\frac{1}{2\pi}\Big[D\left(-e^{2i\cot^{-1}d}\right)-D\left(e^{2i\cot^{-1}d}\right)-D(i)+D(-i)\\
&+D(i)-D(-i)-D\left(-e^{-2i\cot^{-1}d}\right)+D\left(e^{-2i\cot^{-1}d}\right)\Big]\\
=&-\frac{1}{2\pi}\left(-2D\left(-e^{-2i\cot^{-1}d}\right)-2D\left(e^{2i\cot^{-1}d}\right)\right)\\
=&\frac{1}{\pi}\left(D\left(-e^{-2i\cot^{-1}d}\right)+D\left(e^{2i\cot^{-1}d}\right)\right),
\end{aligned}\tag{2.4.23}$$

where the penultimate step again follows from (1.1.11).

This conclude the proof of Proposition 2.2.7. □

2.5. Evaluation of $\mathrm{m}_{c,d}(Q(w,z))$

Now that we have obtained the values of $\mathrm{m}_{c,d}\left(Q_1(w,z)\right)$ for different values of c and d, we can use Propositions 2.2.1, 2.2.5, and 2.2.7 to evaluate $\mathrm{m}_{c,d}\left(Q(w,z)\right)$ for each of the cases described above. Rewriting the equalities in (2.0.2) and (2.3.1) we have

$$\frac{1}{2}\left[\mathrm{m}_{c,d}\left(Q(w,z)\right)+\log(cd)\right]=\mathrm{m}_{c,d}\left(Q_1(w,z)\right).$$

We now gather the values of $\mathrm{m}_{c,d}\left(Q_1(w,z)\right)$ which we obtained in Section 2.4, and rewrite them in Table 2.2.

Values of c	Values of d	Extra conditions	Values of $\mathrm{m}_{c,d}\left(Q_1(w,z)\right)$
$c \neq 1$	$d=1$		$\frac{1}{\pi}\left(2D(ic)-(\log c)\tan^{-1}\left(\frac{2}{c-c^{-1}}\right)\right)$
	$d>1$	$\alpha \in \left[-\frac{\pi}{2},0\right)$ $-1 \leq \frac{1+c^2}{2c}\frac{1-d^2}{1+d^2}=\sin\alpha<0$	$\frac{1}{\pi}\Bigg(D\left(ice^{-i\alpha}\right)+D\left(ice^{i\alpha}\right)$ $-(\log c)\tan^{-1}\left(\frac{2\cos\alpha}{c-c^{-1}}\right)\Bigg)$
		$\frac{1+c^2}{2c}\frac{1-d^2}{1+d^2}<-1$	0
	$d<1$	$\beta \in \left(0,\frac{\pi}{2}\right)$ $0 \leq \frac{1+c^2}{2c}\frac{1-d^2}{1+d^2}=\sin\beta<1$	$\frac{1}{\pi}\Bigg(D\left(ice^{-i\beta}\right)+D\left(ice^{i\beta}\right)$ $-(\log c)\tan^{-1}\left(\frac{2\cos\beta}{c-c^{-1}}\right)\Bigg)$
		$\frac{1+c^2}{2c}\frac{1-d^2}{1+d^2}\geq 1$	0
$c=1$	$d=1$		$\frac{2}{\pi}D(i)$
	$d>1$		$\frac{1}{\pi}\Bigg(D\left(-e^{-2i\cot^{-1}d}\right)$ $+D\left(e^{2i\cot^{-1}d}\right)\Bigg)$
	$d<1$		$\frac{1}{\pi}\Bigg(D\left(-e^{-2i\cot^{-1}d}\right)$ $+D\left(e^{2i\cot^{-1}d}\right)\Bigg)$

Table 2.2. Values of $\mathrm{m}_{c,d}\left(Q_1(w,z)\right)$

Now it only remains to establish the relation $\mathrm{m}_{c,d}(Q(w,z))=\mathrm{m}_{a,b}(P_4(x,y))$, where the variables w,x,y, and z are related via the change of variables

$$x\mapsto\frac{w}{z}\text{ and }y\mapsto wz.$$

2.6. Change of variables and proof of Lemma 2.2.8

Lemma 2.2.8 is a corollary to the following result from **[24]** (pg. 52):

Proposition 2.6.1. *Let*

$$P(\boldsymbol{x}) = \sum c_{\boldsymbol{m}} \boldsymbol{x}^{\boldsymbol{m}} \in \mathbb{C}[x_1, x_2, \dots, x_n],$$

where $\boldsymbol{x}^{\boldsymbol{m}} = x_1^{m_1} \cdots x_n^{m_n}$. *Let* A *be an* $n \times n$ *integer matrix with non-zero determinant, and define* $P^{(A)}(\boldsymbol{x}) = \sum c_{\boldsymbol{m}} \boldsymbol{x}^{\boldsymbol{m}A}$. *Then*

$$\mathrm{m}(P(\boldsymbol{x})) = \mathrm{m}(P^{(A)}(\boldsymbol{x})).$$

Note that we can write $\mathrm{m}_{c,d}(Q(w,z))$ as

$$\begin{aligned}
\mathrm{m}_{c,d}(Q(w,z)) &= \frac{1}{(2\pi i)^2} \iint_{|w|=c,|z|=d} \log|Q(w,z)| \frac{dw}{w}\frac{dz}{z} \\
&= \int_0^1 \int_0^1 \log\left|Q(ce^{2i\pi\tau_1}, de^{2i\pi\tau_2})\right| d\tau_1 d\tau_2 \\
&= \int_0^1 \int_0^1 \log\left|P_4(ae^{2i\pi(\tau_1-\tau_2)}, be^{2i\pi(\tau_1+\tau_2)})\right| d\tau_1 d\tau_2,
\end{aligned} \tag{2.6.1}$$

where we use (2.2.2), (2.0.1)) to deduce the last step.

Proof of Lemma 2.2.8: We consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Let $g(\mathbf{u}) = g(u_1, u_2) := P_4(au_1, bu_2) = au_1 + \frac{1}{au_1} + bu_2 + \frac{1}{bu_2} + 4$ be a Laurent polynomial in $\mathbf{u} = (u_1, u_2) \in \mathbb{T}^2$. Then we have

$$\begin{aligned}
g^{(A)}(\mathbf{u}) =& a\frac{u_1}{u_2} + \frac{u_2}{au_1} + bu_1u_2 + \frac{1}{bu_1u_2} + 4 \\
=& \frac{cu_1}{du_2} + \frac{du_2}{cu_1} + cdu_1u_2 + \frac{1}{cdu_1u_2} + 4 \\
=& Q(cu_1, du_2),
\end{aligned} \tag{2.6.2}$$

where we use the relation among a, b, c and d from (2.2.2) in the penultimate line, namely

$$a = \frac{c}{d}, \ b = cd.$$

Let $w = cu_1$ and $z = du_2$, where $(u_1, u_2) \in \mathbb{T}^2$. Then, the relation

$$\mathrm{m}_{c,d}(Q(w,z)) = \mathrm{m}(Q(cu_1, du_2))$$

follows from the definition of Mahler measure. Using (2.2.2), (2.6.2) and Proposition 2.6.1 we get

$$\begin{aligned}
\mathrm{m}_{c,d}(Q(w,z)) = \mathrm{m}(Q(cu_1, du_2)) =& \mathrm{m}(g^{(A)}(u_1,u_2)) \\
=& \mathrm{m}(g(u_1,u_2)) \\
=& \mathrm{m}(P_4(au_1, bu_2)) \\
=& \frac{1}{(2\pi i)^2} \iint_{|u_1|=1,|u_2|=1} \log |P_4(au_1, bu_2)| \frac{du_1}{u_1}\frac{du_2}{u_2} \\
=& \int_0^1 \int_0^1 \log |P_4(ae^{2i\pi\theta_1}, be^{2i\pi\theta_2})| d\theta_1 d\theta_2 \\
=& \mathrm{m}_{a,b}(P_4(x,y)),
\end{aligned}$$

which concludes the proof of Lemma 2.2.8. □

Chapter 3

Proof of Theorem 0.3.2 and generalization

In this section our main goal is to prove Theorem 0.3.2. Later we will consider a more general case, namely $P_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + k$ when $k > 4$. Our methods to prove the theorem are mostly inspired by those used in [**31**].

3.1. The birational transformation

In order to establish a relation between the Mahler measure of P_k and a Weierstrass form of a family of elliptic curves $E_{N(k)}$, we begin with considering a birational transformation connecting these families of curves. Notice that these transformations work for $k \neq 0, \pm 4$, but we will be using it only for the cases $k > 4$. In Section 3.4.2 we use this transformation to integrate the invariant holomorphic differential of $E_{N(k)}$ over certain integration paths which are closed to determine their homology class(es) in $H_1(E_{N(k)}, \mathbb{Z})^-$ (as mentioned in Chapter 2).

Let C_k be the curve defined by $P_k(x,y) = 0$. The change of variables

$$\begin{cases} X &= -\dfrac{1}{xy}, \\[2ex] Y &= \dfrac{(y-x)\left(1+\frac{1}{xy}\right)}{2xy}, \end{cases}$$

and

$$\begin{cases} x &= \dfrac{kX - 2Y}{2X(X-1)}, \\[2ex] y &= \dfrac{kX + 2Y}{2X(X-1)}, \end{cases} \tag{3.1.1}$$

gives a birational transformation

$$\phi : P_k(x,y) \to E_{N(k)}(X,Y), \tag{3.1.2}$$

between

$$P_k(x,y) := x + \frac{1}{x} + y + \frac{1}{y} + k$$

and

$$E_{N(k)}(X,Y) := Y^2 - X\left(X^2 + \left(\frac{k^2}{4} - 2\right)X + 1\right),$$

which is an elliptic curve for $k \neq 0, \pm 4$.

3.2. Outline of the proof of Theorem 0.3.2

Recall that $a_0 = \left[\left(5-2\sqrt{2}\right) + \sqrt{\left(5-2\sqrt{2}\right)^2 - 1}\right]$. Then, the statement of Theorem 0.3.2 is as follows:

Theorem 0.3.2. *If $k = 8$ and $\frac{1}{a_0} \leq a \leq a_0$ then*

$$\mathrm{m}_{a,\sqrt{a}}\left(yP_8\left(x,y\right)\right) = \mathrm{m}\left(yP_8\left(x,y\right)\right).$$

Moreover,

$$\mathrm{m}_{a,\sqrt{a}}\left(P_8\left(x,y\right)\right) = \mathrm{m}\left(P_8\left(x,y\right)\right) - \frac{1}{2}\log a = 4L'(E_{N(8)},0) - \frac{1}{2}\log a,$$

where

$$E_{N(8)}(X,Y) := Y^2 - X\left(X^2 + \left(\frac{8^2}{4} - 2\right)X + 1\right)$$

is the Weierstrass form of an elliptic curve of conductor $N(8) = 24$.

We start by factorizing $yP_8(x,y)$ in $\overline{\mathbb{C}(x)}[y]$ as

$$yP_8\left(x,y\right) = y\left(x + \frac{1}{x} + y + \frac{1}{y} + 8\right) = \left(y - y_1\left(x\right)\right)\left(y - y_2\left(x\right)\right),$$

where

$$\begin{aligned} y_1\left(x\right) =& \frac{-\left(8 + x + \frac{1}{x}\right) - \sqrt{\left(8 + x + \frac{1}{x}\right)^2 - 4}}{2}, \\ y_2\left(x\right) =& \frac{-\left(8 + x + \frac{1}{x}\right) + \sqrt{\left(8 + x + \frac{1}{x}\right)^2 - 4}}{2}. \end{aligned} \tag{3.2.1}$$

Notice that $y_i(x)$ is well-defined for each $i \in \{1,2\}$ because the square root is taken whenever $\left[\left(8 + x + \frac{1}{x}\right)^2 - 4\right] > 0$. In particular, we will show in Lemma 3.3.1 that we can fix a certain branch of the square root for every x where $|x| = a$ with $a \in \mathbb{R}_{>0}$ and $a + \frac{1}{a} < 6$.

Following the discussion in Section 1.2, we can write $\mathrm{m}_{a,b}\left(yP_8\left(x,y\right)\right)$ as

$$\begin{aligned}\mathrm{m}_{a,b}\left(yP_8(x,y)\right) =& 2\log b - \frac{1}{2\pi}\int_{|x|=a,|y_1|\geq b}\eta\left(x/a, y_1/b\right)\\ &- \frac{1}{2\pi}\int_{|x|=a,|y_2|\geq b}\eta\left(x/a, y_2/b\right)\\ =& -\frac{1}{2\pi}\int_{|x|=a,|y_1|\geq b}\left(\eta\left(x,y_1\right) - \log\left(a\right)d\arg y_1\right)\\ &- \frac{1}{2\pi}\int_{|x|=a,|y_2|\geq b}\left(\eta\left(x,y_2\right) - \log\left(a\right)d\arg y_2\right),\end{aligned} \tag{3.2.2}$$

where in our case we have $b=\sqrt{a}$, and

$$\eta(x,y_j) = i\log|y_j|\frac{dx}{x} \tag{3.2.3}$$

for $j=1,2$. To calculate the integral using methods developed by Deninger [**23**], we need to obtain values of a such that $\{|x|=a, |y_j|\geq\sqrt{a}\}$ is a closed path and a cycle in $H_1(E_{N(8)},\mathbb{Z})$. In fact, following the discussion in Remark 1.1.2 we will show that $\{|x|=a, |y_j|\geq\sqrt{a}\}$ is a cycle in $H_1(E_{N(8)},\mathbb{Z})^-$ for those values of a. We can rephrase the closedness of the integration path by identifying $\{|x|=a, |y_j|\geq\sqrt{a}\}$ with $\{|x|=a\}$ for certain values of a.

Note that $y_1\left(x\right)\cdot y_2\left(x\right)=1$. So if $|y_i\left(x\right)|\geq\sqrt{a}$, then $|y_j\left(x\right)|\leq\frac{1}{\sqrt{a}}$, where $i\neq j$ and $\{i,j\}=\{1,2\}$. If we have $a\geq 1$ then, for $i\neq j$,

$$|y_i\left(x\right)|\geq\sqrt{a}\Leftrightarrow|y_j\left(x\right)|\leq\frac{1}{\sqrt{a}}\leq\sqrt{a}.$$

As $P_8\left(x,y\right)$ is invariant under the transformations $x\mapsto\frac{1}{x}$ and $y\mapsto\frac{1}{y}$, it is enough to consider the case when $a>1$ ($a=1$ is done in [**39**]). We will follow a similar approach to this problem as the one taken in [**31**].

In few words, our main aim is to first determine which $y_j(x)$, between $y_1(x)$ and $y_2(x)$, we should consider such that $|y_j(x)|\geq\sqrt{a}$ whenever $|x|=a$; later we will study the behavior of $\min_{x=ae^{i\theta},\theta\in[0,2\pi)}|y_j(x)|$ with different values of a to find the values of a such that $\{|x|=a, |y_j(x)|\geq\sqrt{a}\}$ is a closed path. A comparison of values of $|y_j(x)|$ at $x=a$ with $\sqrt{a}$ yields that it is enough to consider $y_1(x)$ for rest of the proof. Now we can concentrate on obtaining $\min_{\theta\in[0,2\pi)}|y_1\left(ae^{i\theta}\right)|$. Proposition 3.3.3 shows that this minimum is attained at $\theta=\pi$. Therefore, in order to obtain values of a such that the integration path $\{|x|=a,|y_1|\geq\sqrt{a}\}$ is closed, we just need to find intervals of positive reals satisfying

$$|y_1\left(ae^{i\pi}\right)| = |y_1(-a)|\geq\sqrt{a}.$$

But as noted in Remark 1.1.2, we have to show that, for these values of a, the integration path is in fact a representative of certain homology class in $H_1(E_{N(8)}, \mathbb{Z})^-$. Once we prove this fact using Lemma 3.3.5, the only integral in (3.2.2) remains to be calculated is

$$\frac{1}{2\pi}\int_{|x|=a,|y_1|\geq\sqrt{a}} d\arg y_1.$$

Lemma 3.3.6 deals with this integral, and we obtain that the value of this integral is 0 for the above obtained values of a. These results and the relation between elliptic dilogarithm and regulator map (see Theorem 1.1.10) can be combined with the relation between regulator map and Mahler measure (see [**23**]) to deduce the desired result.

3.3. Proof of Theorem 0.3.2

Following the discussions in Sections 3.2 we prove Theorem 0.3.2 as an application of the following results.

Lemma 3.3.1. *If $a+\frac{1}{a}<6$ and $a\in\mathbb{R}_{>0}$, then $\left(8+x+\frac{1}{x}\right)^2-4\notin(-\infty,0)$, where $x=ae^{i\theta}$ and $\theta\in[0,2\pi)$.*

Note that if we consider the case where $a>1$ and $a+\frac{1}{a}>10$, then $|y_1(-a)|<1<\sqrt{a}$ and $|y_2(a)|<\frac{1}{\sqrt{a}}$. Lemma 3.5.1 gives a detailed description of this fact for a general case, namely when $a+\frac{1}{a}>k+2$ for $k>4$. Also in this case ($k=8$) if $a<1$, then we can show that $|y_1(ae^{\frac{5\pi i}{6}})|<\sqrt{a}$. And if $6\leq a+\frac{1}{a}\leq 10$ then $\left[\left(8-a-\frac{1}{a}\right)^2-4\right]\leq 0$. We want to avoid the case where $\left[\left(8-a-\frac{1}{a}\right)^2-4\right]<0$, because then we will not be able to work with a fixed branch of the square root of $\left[\left(8+x+\frac{1}{x}\right)^2-4\right]$. In case of equality we have $y_1(x)=y_2(x)=\pm1$ (as $y_1(x)\cdot y_2(x)=1$), which after replacing in (3.2.3) yields $\eta(x,y_j)=0$ for $j\in\{1,2\}$, and as a result we get $\mathrm{m}_{a,\sqrt{a}}(P_8(x,y))=0$. More details on these results is provided in Section 3.5. Once we fix a branch (principal branch) of square root of $\left(8+x+\frac{1}{x}\right)^2-4$, we have two well-defined the algebraic functions of x, namely $y_1(x)$ and $y_2(x)$, when $|x|=a$.

Lemma 3.3.2. *Let $y_1(x)$ and $y_2(x)$ be defined by (3.2.1), where $|x|=a$ with $a\in\mathbb{R}_{>0}$ and $a+\frac{1}{a}<6$. Then $|y_1(a)|>|y_2(a)|$. We also have $|y_1(a)|>\sqrt{a}$.*

Therefore, we can concentrate on obtaining $\min_{\theta\in[0,2\pi)}|y_1\left(ae^{i\theta}\right)|$ for $|x|=a$.

Proposition 3.3.3. *If $a+\frac{1}{a}<6$ and $a\in\mathbb{R}_{>0}$, then $|y_1(x)|=|y_1\left(ae^{i\theta}\right)|$ attains its minimum at $\theta_{\min}=\pi$, where $\theta\in[0,2\pi)$.*

Note that $a_0 = \left[(5-2\sqrt{2}) + \sqrt{(5-2\sqrt{2})^2 - 1}\right] > 1$.

Lemma 3.3.4. *For $a \in \left[\frac{1}{a_0}, a_0\right]$, the integration path $\{|x| = a, |y_1| \geq \sqrt{a}\}$ is closed.*

Lemma 3.3.5. *Let $a \in \mathbb{R}_{>0}$ be such that $\frac{1}{a_0} \leq a \leq a_0$. Then*

$$\int_{\phi_*(|x|=a)} \omega \in i\mathbb{R},$$

where the integral is performed over the path $\{|x| = a,\ |y_1(x)| \geq \sqrt{a}\}$, where $y_1(x)$ is given by (3.2.1) and satisfies $P_8(x, y_1) = 0$.

If a satisfies the conditions of Lemma 3.3.5, then the fact that the value of the above integral is independent of a implies

$$[\phi_*(|x| = a)] = \pm[\phi_*(|x| = 1)] = \pm[|X| = 1],$$

where $[|X| = 1] \in H_1(E_8, \mathbb{Z})^-$. It can also be shown that the sign is independent of a in this case. This will imply

$$[\phi_*(|x| = a)] = [\phi_*(|x| = 1)] = [|X| = 1] \in H_1(E_8, \mathbb{Z})^-.$$

In particular, the above homology classes are identified as the generator of the component $H_1(E_{N(8)}, \mathbb{Z})^-$ of the group $H_1(E_{N(8)}, \mathbb{Z})$. In fact, for $a \in \left[\frac{1}{a_0}, a_0\right]$ we have

$$\int_{\phi_*(|x|=a)} \omega = -\frac{i}{2} K\left(\frac{1}{2}\right), \tag{3.3.1}$$

where

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$$

is the complete Elliptic Integral of the First Kind. We give a proof of this fact in the Appendix (Lemma A.1.1). The next lemma deals with the value of

$$\frac{1}{2\pi} \int_{|x|=a, |y_1| \geq \sqrt{a}} d \arg y_1.$$

Lemma 3.3.6. *Let y_1 be the root of $yP_8(x, y) = 0$ given by (3.2.1), and let $a \in \mathbb{R}_{>0}$ be such that $a \in \left[\frac{1}{a_0}, a_0\right]$. Then,*

$$\frac{1}{2\pi} \int_{|x|=a, |y_1| \geq \sqrt{a}} d \arg y_1 = \frac{1}{2\pi} \int_{|x|=a} d \arg y_1 = 0.$$

Before we proceed to prove these results, we use them to deduce Theorem 0.3.2.

Proof of Theorem 0.3.2: From (3.2.2) and Lemma 3.3.6 we obtain that

$$\mathrm{m}_{a,\sqrt{a}}\left(yP_8(x,y)\right) = -\frac{1}{2\pi}\int_{|x|=a,|y_1|\geq\sqrt{a}} \eta\left(x,y_1\right).$$

We use Lemmas 3.3.4 and 3.3.5 to get

$$\int_{|x|=a,|y_1|\geq\sqrt{a}} \eta\left(x,y_1\right) = \int_{|x|=a} \eta\left(x,y_1\right) = \frac{1}{2\pi}\int_{\phi_*(|x|=a)} \eta\left(x\circ\phi^{-1}, y\circ\phi^{-1}\right), \tag{3.3.2}$$

where ϕ is the birational transformation in (3.1.2) with $k=8$. But, from Lemma 3.3.5 and (3.3.1) it follows that

$$\frac{1}{2\pi}\int_{\phi_*(|x|=a)} \eta\left(x\circ\phi^{-1}, y\circ\phi^{-1}\right) = \frac{1}{2\pi}\int_{\phi_*(|x|=1)} \eta\left(x\circ\phi^{-1}, y\circ\phi^{-1}\right). \tag{3.3.3}$$

The relation between $P_8(x,y)$ and $E_{N(8)}(X,Y)$ via the birational transformation ϕ implies that

$$\frac{1}{2\pi}\int_{|X|=1} \eta\left(X,Y\right) = \frac{1}{2\pi}\int_{\phi_*(|x|=1)} \eta\left(x\circ\phi^{-1}, y\circ\phi^{-1}\right). \tag{3.3.4}$$

Therefore, using (3.3.2), (3.3.3) and (3.3.4) we can now write, $\forall\, a \in \left[\frac{1}{a_0}, a_0\right]$,

$$\frac{1}{2\pi}\int_{|X|=1} \eta\left(X,Y\right) = \frac{1}{2\pi}\int_{\phi_*(|x|=1)} \eta\left(x\circ\phi^{-1}, y\circ\phi^{-1}\right) = \frac{1}{2\pi}\int_{\phi_*(|x|=a)} \eta\left(x\circ\phi^{-1}, y\circ\phi^{-1}\right).$$

We can now conclude that, $\forall\, a \in \left[\frac{1}{a_0}, a_0\right]$,

$$\begin{aligned}
\mathrm{m}_{a,\sqrt{a}}\left(yP_8(x,y)\right) &= -\frac{1}{2\pi}\int_{\phi_*(|x|=a)} \eta\left(x\circ\phi^{-1}, y\circ\phi^{-1}\right) \\
&= -\frac{1}{2\pi}\int_{\phi_*(|x|=1)} \eta\left(x\circ\phi^{-1}, y\circ\phi^{-1}\right) \\
&= \mathrm{m}\left(yP_8(x,y)\right) = \mathrm{m}\left(P_8(x,y)\right).
\end{aligned}$$

In other words,

$$\mathrm{m}_{a,\sqrt{a}}\left(P_8\right) = \mathrm{m}\left(P_8\right) - \frac{1}{2}\log a. \tag{3.3.5}$$

We conclude the proof by combining (3.3.5) with the results

$$\mathrm{m}\left(P_8\right) = 4\mathrm{m}\left(P_2\right),$$

due to Lalín and Rogers [**32**], and

$$\mathrm{m}\left(P_2\right) = L'(E_{N(8)}, 0),$$

proved by Rogers and Zudilin [**39**]. Combining these results we obtain

$$\mathrm{m}_{a,\sqrt{a}}\left(P_8\right) = \mathrm{m}\left(P_8\right) - \frac{1}{2}\log a = 4L'(E_{N(8)}, 0) - \frac{1}{2}\log a,$$

thus deducing Theorem 0.3.2. □

Next section deals with the proofs of the proposition and lemmas mentioned in this section.

3.4. Proof of the proposition and lemmas

Our first task is to find the values of $a \in \mathbb{R}_{>0}$ such that $\{|x| = a, |y_i\left(x\right)| \geq \sqrt{a}\}$ is a closed path, where $y_i\left(x\right)$ is an algebraic function in x for $i = 1, 2$, and satisfies $P_8\left(x, y_i\right) = 0$. In particular, we will show that $\{|x| = 1, |y_i\left(x\right)| \geq 1\}$ is a closed path. Once we recover those values of a we conclude that the above path is homologous to the closed path $\{|x| = 1, |y_i\left(x\right)| \geq 1\}$, and therefore we will be able to prove Lemmas 3.3.4 and 3.3.5. The importance of considering $E_{N(k)}$ follows from the formula in (1.1.16), found by Deninger [**23**].

3.4.1. The integration path $\{|x| = a, |y_i| \geq \sqrt{a}\}$ for $i = 1, 2$

Based on numerical experiments, Boyd [**17**] hypothesized that

$$\mathrm{m}(P_8(x, y)) = 4\mathrm{m}(P_2(x, y)) = 4L'(E_{N(8)}, 0), \tag{3.4.1}$$

where $E_{N(8)}(X, Y) := Y^2 - X\left(X^2 + \left(\frac{8^2}{4} - 2\right)X + 1\right)$. The relation between the Mahler measures was then proved by Lalín and Rogers [**32**] by establishing functional equations for the function $\mathrm{m}(P_k(x, y))$, and combining them with other functional equations proved by Kurokawa and Ochiai [**27**]. The relationship with the L-function was eventually proved by Rogers and Zudilin [**39**] using a relation between Mahler measure and hypergeometric series. There are also similar results to (3.4.1) when $k = 16$ and $k = 5$ (for more details on the standard Mahler measure in these cases see [**30**], [**32**] and [**40**]).

We can extend our method for the cases when $k = 5, 16$ and obtain a relation involving $\mathrm{m}(P_k)$ and the Mahler measures of the corresponding polynomials considered over $\mathbb{T}^2_{a,\sqrt{a}}$. In particular, if we consider $\mathbb{T}^2_{a,b}$ instead of $\mathbb{T}^2_{a,\sqrt{a}}$ as our integration torus then some restricted results for the cases when $k > 4$ (see Section 3.5) will follow from an argument similar to the one provided below.

As mentioned above, our main aim in this section is to first determine which $y_j(x)$, between $y_1(x)$ and $y_2(x)$, we should consider such that $|y_j(x)| \geq \sqrt{a}$ whenever $|x| = a$; later we will study the behavior of $\min_{x=ae^{i\theta},\theta\in[0,2\pi)} |y_j(x)|$ with different values of a to find the values of a such that $\{|x| = a, |y_j(x)| \geq \sqrt{a}\}$ is a closed path.

Proof of Lemma 3.3.1: After substituting $x = ae^{i\theta}$ the given expression can be expanded to

$$\begin{aligned} h(a,\theta) := \left(8 + ae^{i\theta} + a^{-1}e^{-i\theta}\right)^2 - 4 &= \left(8 + \left(a + a^{-1}\right)\cos\theta + i\left(a - a^{-1}\right)\sin\theta\right)^2 - 4 \\ &= \left(8 + \left(a + a^{-1}\right)\cos\theta\right)^2 - \left(a - a^{-1}\right)^2 \sin^2\theta - 4 \\ &\quad + 2i\left(8 + \left(a + a^{-1}\right)\cos\theta\right)\left(\left(a - a^{-1}\right)\sin\theta\right). \end{aligned}$$

Now $h(a,\theta) \in (-\infty, 0)$ only if both of the following conditions hold.

- $\operatorname{Im}(h(a,\theta)) = 0$, i.e. $2\left(8 + (a + a^{-1})\cos\theta\right)\left((a - a^{-1})\sin\theta\right) = 0$.
  As $a \in \mathbb{R}_{>0}$ and $a + \frac{1}{a} < 6$, we get that the above will happen if and only if
  $$\left(a - a^{-1}\right)\sin\theta = 0,$$
  i.e. if and only if $(a - a^{-1}) = 0$ or $\sin\theta = 0$ or both.

- $\operatorname{Re}(h(a,\theta)) < 0$. Replacing the values of a and θ obtained above, we get $(a - a^{-1})\sin\theta = 0$. But again using the conditions in Proposition 3.3.3, we obtain $\left(8 + (a + a^{-1})\cos\theta\right)^2 - 4 > 0$. In other words, $\operatorname{Re}(h(a,\theta)) > 0$ for all a and θ satisfying the given conditions.

Thus, for $x = ae^{i\theta}$, $\theta \in [0, 2\pi)$ and $a + \frac{1}{a} < 6$, $\left(8 + x + \frac{1}{x}\right)^2 - 4 \notin (-\infty, 0)$. □

Before starting the proof of Lemma 3.3.2 we fix a branch of the square root for the rest of the section in order to make the expressions of $y_j(x)$ in (3.2.1),for $j = 1, 2$, well-defined.

Proof of Lemma 3.3.2: We start by evaluating $|y_j(x)|$ at $\theta = 0$ (viz. $x = a$) for $j \in \{1, 2\}$:

$$|y_1\left(ae^{i0}\right)| = \left|\frac{-\left(8 + a + \frac{1}{a}\right) - \sqrt{\left(8 + a + \frac{1}{a}\right)^2 - 4}}{2}\right| = \frac{\left(8 + a + \frac{1}{a}\right) + \sqrt{\left(8 + a + \frac{1}{a}\right)^2 - 4}}{2},$$

$$|y_2\left(ae^{i0}\right)| = \left|\frac{-\left(8 + a + \frac{1}{a}\right) + \sqrt{\left(8 + a + \frac{1}{a}\right)^2 - 4}}{2}\right| = \frac{\left(8 + a + \frac{1}{a}\right) - \sqrt{\left(8 + a + \frac{1}{a}\right)^2 - 4}}{2},$$

where we use the fact that $a+\frac{1}{a} \geq 2$ because $a \in \mathbb{R}_{>0}$, and therefore, $\left(8+a+\frac{1}{a}\right) > \sqrt{\left(8+a+\frac{1}{a}\right)^2-4}$. We obtain $|y_1(a)| > |y_2(a)|$. It remains to find if $|y_1(a)| \geq \sqrt{a}$.

We denote $g(a) := 4\left(|y_1(a)|^2 - a\right)$. If we can show that $g(a) \geq 0$ for $a \in \mathbb{R}_{>0}$, then we get

- if $a > 1$, then $|y_1(a)| \geq \sqrt{a}$, and simultaneously $|y_2(a)| < \frac{1}{\sqrt{a}} \leq \sqrt{a}$;
- if $a < 1$, then $|y_1(a)| = \left|y_1\left(\frac{1}{a}\right)\right| \geq \frac{1}{\sqrt{a}} > \sqrt{a}$, and simultaneously $|y_2(a)| \leq \sqrt{a}$, where we use the fact that $y_1(x)$ and $y_2(x)$ are invariant under the transformation $x \mapsto \frac{1}{x}$ when $|x| = a$.

We can then concentrate on finding the values of a such that $|y_1(x)| \geq \sqrt{a}$, for all x satisfying $|x| = a$. But observe that

$$g(a) = 4\left(|y_1(a)|^2 - a\right) = 2\left(8+a+\frac{1}{a}\right)^2 - 4 + f(a) - 4a > 2a^2 + 4 - 4a > 0,$$

where $f(s) = 2\left(8+s+\frac{1}{s}\right)\left(\sqrt{\left(8+s+\frac{1}{s}\right)^2-4}\right) > 0$ for any positive real s, and this proves our lemma. □

Our next goal is to prove Proposition 3.3.3. In order to prove it, we start with the following auxiliary lemma.

Lemma 3.4.1. *If* $\operatorname{Re}(z) > 0$ *and* $\arg(z) \in (-\pi, \pi]$, *then* $\operatorname{Re}(\sqrt{z}) \geq \sqrt{\operatorname{Re}(z)}$, *where the square root is taken with the principal branch.*

Proof. Let $z = re^{i\theta}$ where $r = |z|$. We also have $\operatorname{Re}(\sqrt{z}) > 0$ because $\arg(z) \in (-\pi, \pi]$. So we have

$$\operatorname{Re}\left(\sqrt{z}\right) = \sqrt{r}\cos\frac{\theta}{2} = \sqrt{\frac{1}{2}r\left(1+\cos\theta\right)} \geq \sqrt{r\cos\theta} = \sqrt{\operatorname{Re}(z)},$$

where the penultimate step is true because $|\cos\theta| \leq 1$. □

As $\left(8+x+\frac{1}{x}\right)^2 - 4 \notin (-\infty, 0)$, we can assume that, for $t = \left(6 + ae^{i\theta} + a^{-1}e^{-i\theta}\right)$ with $\theta \in [0, 2\pi)$,

$$\arg(t(t+4)) = \arg((t+2)^2 - 4) = \arg\left[\left(8+x+\frac{1}{x}\right)^2 - 4\right] \in (-\pi, \pi].$$

We will now prove Proposition 3.3.3.

Proof of Proposition 3.3.3: We write $|y_1(x)| = |y_1(ae^{i\theta})|$ in terms of t as

$$\begin{aligned} 2|y_1\left(ae^{i\theta}\right)| &= \left|-\left(8+ae^{i\theta}+a^{-1}e^{-i\theta}\right)-\sqrt{\left(8+ae^{i\theta}+a^{-1}e^{-i\theta}\right)^2-4}\right| \\ &= \left|t+2+\sqrt{(t+2)^2-4}\right| \\ &= \left|t+2+\sqrt{t\,(t+4)}\right|. \end{aligned}$$

As mentioned before, we fixed that $\arg(t(t+4)) = \arg((t+2)^2-4) \in (-\pi, \pi]$, which implies $\arg(\sqrt{t(t+4)}) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We also have from Lemma 3.3.1 that $\sqrt{t(t+4)}$ is well-defined and so is $2|y_1(ae^{i\theta})|$. Also note that

$$\begin{aligned} \left|t+2+\sqrt{t\,(t+4)}\right|^2 &= \left[\mathrm{Re}\,(t+2)+\mathrm{Re}\left(\sqrt{t\,(t+4)}\right)\right]^2+\left[\mathrm{Im}\,(t+2)+\mathrm{Im}\left(\sqrt{t\,(t+4)}\right)\right]^2 \\ &\geq \left[\mathrm{Re}\,(t)+2+\mathrm{Re}\left(\sqrt{t\,(t+4)}\right)\right]^2, \end{aligned} \tag{3.4.2}$$

where the equality holds if $\left[\mathrm{Im}\,(t+2)+\mathrm{Im}\left(\sqrt{t\,(t+4)}\right)\right]=0$.

Now to minimize $2|y_1\left(ae^{i\theta}\right)|$ we first need to minimize $\mathrm{Re}\,(t+2)+\mathrm{Re}\left(\sqrt{t\,(t+4)}\right)$. From the proof of Lemma 3.3.1 we have that $\mathrm{Re}\,(t+2), \mathrm{Re}\,(t\,(t+4))$, and $\mathrm{Re}\left(\sqrt{t(t+4)}\right)$ are positive for $a>0, a+\frac{1}{a}<6$ and $\theta\in[0,2\pi)$. Indeed, as we have considered the principal branch of the square root of $(t(t+4))$, we obtain $\arg\left(\sqrt{t(t+4)}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and this implies $\mathrm{Re}\left(\sqrt{t(t+4)}\right) > 0$. Therefore,

$$\begin{aligned} \min_{\theta\in[0,2\pi)}\left|t+2+\sqrt{t\,(t+4)}\right|^2 &\geq \min_{\theta\in[0,2\pi)}\left[\mathrm{Re}\,(t+2)+\mathrm{Re}\left(\sqrt{t\,(t+4)}\right)\right]^2 \\ &\geq \left[\min_{\theta\in[0,2\pi)}(\mathrm{Re}\,(t+2))+\min_{\theta\in[0,2\pi)}\left(\mathrm{Re}\left(\sqrt{t\,(t+4)}\right)\right)\right]^2. \end{aligned} \tag{3.4.3}$$

First we will minimize $\mathrm{Re}\,(t+2)$. If it attains its minimum at $\theta_{\min,t+2}\in[0,2\pi)$, then we get $\theta_{\min,t+2}=\pi$, because

$$\min_{\theta\in[0,2\pi)}\mathrm{Re}\,(t+2) = \min_{\theta\in[0,2\pi)}\left(8+\left(a+a^{-1}\right)\cos\theta\right) = \mathrm{Re}\,(t+2)\,\big|_{\theta_{\min,t+2}=\pi} = 8-a-\frac{1}{a},$$

where $2 \leq a+\frac{1}{a} < 6$. Note that we have $t(t+4) = (t+2)^2-4$, $\mathrm{Re}(t(t+4)) > 0$ and $\arg(t(t+4)) \in [-\pi, \pi)$ in our case. In other words, we can apply Lemma 3.4.1 to $t(t+4)$ to get

$$\mathrm{Re}\left(\sqrt{(t+2)^2-4}\right) \geq \sqrt{\mathrm{Re}\left((t+2)^2\right)-4} = \sqrt{[\mathrm{Re}\,(t+2)]^2-[\mathrm{Im}\,(t+2)]^2-4},$$

and

$$\min_{\theta\in[0,2\pi)} \left[[\operatorname{Re}(t+2)]^2 - [\operatorname{Im}(t+2)]^2 - 4\right] = \left[[\operatorname{Re}(t+2)]^2 - [\operatorname{Im}(t+2)]^2 - 4\right] \Big|_{\cos\theta=-1,\sin\theta=0}.$$

Note that we have $\operatorname{Re}(t+2)\,\big|_{\theta=0} > \operatorname{Re}(t+2)\,\big|_{\theta=\pi}$ and

$$[\operatorname{Im}(t+2)]\,\big|_{\sin\theta=0} = 0 = \left[\operatorname{Im}\left(\sqrt{t\,(t+4)}\right)\right]\Big|_{\sin\theta=0}.$$

Therefore, if $\operatorname{Re}(t(t+4)) = \operatorname{Re}((t+2)^2-4)$ attains its minimum at $\theta_{\min,t(t+4)} \in [0,2\pi)$, then $\theta_{\min,t(t+4)} = \pi$. Thus, if we can minimize $\operatorname{Re}(t+2)$, we will simultaneously minimize $\operatorname{Re}(t\,(t+4))$. In other words, we have $\theta_{\min,t+2} = \theta_{\min,t(t+4)} = \theta_{\min} = \pi$,

$$\left[\operatorname{Im}(t+2) + \operatorname{Im}\left(\sqrt{t\,(t+4)}\right)\right]\Big|_{\theta_{\min}} = 0, \tag{3.4.4}$$

and

$$\operatorname{Re}\left(\sqrt{t(t+4)}\right)\Big|_{\theta_{\min}} = \operatorname{Re}\left(\sqrt{(t+2)^2-4}\right)\Bigg|_{\theta_{\min}} = \sqrt{\operatorname{Re}\left((t+2)^2\right)-4}\,\Bigg|_{\theta_{\min}}.$$

The equations (3.4.2) and (3.4.4) combined with the fact that $\operatorname{Re}(t+2)$ and $\operatorname{Re}\left(\sqrt{t(t+4)}\right)$ are strictly positive when $\theta \in [0,2\pi)$ imply that the equality holds through out (3.4.3), and we get

$$\begin{aligned}
\min_{\theta\in[0,2\pi)} \left|t+2+\sqrt{t\,(t+4)}\right|^2 &= \left[\min_{\theta\in[0,2\pi)} \operatorname{Re}(t+2) + \min_{\theta\in[0,2\pi)} \operatorname{Re}\left(\sqrt{t\,(t+4)}\right)\right]^2 \\
&= \left[\operatorname{Re}(t+2)\,\big|_{\theta_{\min}=\pi} + \operatorname{Re}\left(\sqrt{t\,(t+4)}\right)\big|_{\theta_{\min}=\pi}\right]^2 \\
&= \left[8-a-\frac{1}{a}+\sqrt{\left(8-a-\frac{1}{a}\right)^2-4}\right]^2.
\end{aligned} \tag{3.4.5}$$

Therefore, if $a+\frac{1}{a} < 6$ and $a \in \mathbb{R}_{>0}$, then $|y_1(x)| = \left|y_1\left(ae^{i\theta}\right)\right|$ attains its minimum for $\theta \in [0,2\pi)$ at $\theta_{\min} = \pi$. $\square$

To determine the values of a such that $\{|x| = a, |y_1(x)| \geq \sqrt{a}\}$ is a closed path, we now just have to consider the cases where $|y_1(ae^{i\pi})| = |y_1(-a)| \geq \sqrt{a}$.

Proof of Lemma 3.3.4: Note that it is enough to find the positive real roots of

$$G(u) = 8-u-\frac{1}{u}-2\sqrt{u}+\sqrt{\left(8-u-\frac{1}{u}\right)^2-4}$$

such that $u + \frac{1}{u} < 6$. Indeed, once we find the roots we can determine the interval(s) in the real line where $G(u) \geq 0$, following the conditions on u, and that will give us our required values of a.

By setting $G(u) = 0$ we obtain

$$
\begin{aligned}
& 8 - u - \frac{1}{u} + \sqrt{\left(8 - u - \frac{1}{u}\right)^2 - 4} = 2\sqrt{u} \\
& \Leftrightarrow 8 - u - \frac{1}{u} = 2\sqrt{u} - \sqrt{\left(8 - u - \frac{1}{u}\right)^2 - 4} \\
& \Rightarrow \left(8 - u - \frac{1}{u}\right)^2 = \left(2\sqrt{u} - \sqrt{\left(8 - u - \frac{1}{u}\right)^2 - 4}\right)^2 \\
& \Leftrightarrow \left(8 - u - \frac{1}{u}\right)^2 = 4u + \left(8 - u - \frac{1}{u}\right)^2 - 4 - 2\sqrt{u\left(\left(8 - u - \frac{1}{u}\right)^2 - 4\right)} \\
& \Rightarrow 4\left(u - 1\right)^2 = u\left(\left(8 - u - \frac{1}{u}\right)^2 - 4\right) \\
& \Leftrightarrow \left(8 - u - \frac{1}{u}\right)^2 = 4\left(u + \frac{1}{u} - 1\right) \\
& \Leftrightarrow \left(1 - u - \frac{1}{u}\right)^2 + 18\left(1 - u - \frac{1}{u}\right) + 7^2 = 0 \\
& \Leftrightarrow \left(10 - u - \frac{1}{u}\right)^2 = 9^2 - 7^2 \\
& \Leftrightarrow u + \frac{1}{u} = 10 \pm 4\sqrt{2}.
\end{aligned}
$$

Since we assumed that $u + \frac{1}{u} < 6$, we are left with only one choice, namely

$$
\begin{aligned}
& u + \frac{1}{u} = 10 - 4\sqrt{2} \\
& \Leftrightarrow u = \left(5 - 2\sqrt{2}\right) \pm \sqrt{\left(5 - 2\sqrt{2}\right)^2 - 1} = a_0 \text{ or } u = \frac{1}{a_0},
\end{aligned}
\tag{3.4.6}
$$

where $a_0 = \left[(5 - 2\sqrt{2}) + \sqrt{(5 - 2\sqrt{2})^2 - 1}\right] > 1$.

Notice that $u_0 = \frac{1}{4}\left(17 - \sqrt{41} + \sqrt{314 - 34\sqrt{41}}\right) = 5.10245374\ldots$ is a root of $G(u)$. Indeed, after clearing the fraction in the equation $G(u) = 0$, we square both sides to obtain

$$4u^3 = \left(1 - 8u + u^2 - u\sqrt{\left(8 - u - \frac{1}{u}\right)^2 - 4}\right)^2.$$

Now, we open the square on the right-hand side, and rearrange the terms to isolate the radicals. Next we square both sides, and multiply the equation through a suitable power of u to obtain a polynomial equation in u, namely

$$16u^3(u^4 - 17u^3 + 64u^2 - 17u + 1) = 0.$$

As the domain of the function $G(u)$ under consideration is contained in $\mathbb{R}_{>0}$, it only remains to solve the equation

$$(u^4 - 17u^3 + 64u^2 - 17u + 1) = 0. \tag{3.4.7}$$

As $u > 0$, we consider $U = u + \frac{1}{u}$ to get

$$U^2 - 17U + 62 = 0,$$

which has roots at $\frac{17+\sqrt{41}}{2}$ and $\frac{17-\sqrt{41}}{2}$. Then, the possible solutions of the equation (3.4.7) are

$$u = \frac{1}{4}\left(\sqrt{41} + 17 + \sqrt{314 + 34\sqrt{41}}\right), \text{ or } u = \frac{1}{4}\left(\sqrt{41} + 17 - \sqrt{314 + 34\sqrt{41}}\right),$$
$$\text{or } u = \frac{1}{4}\left(17 - \sqrt{41} + \sqrt{314 - 34\sqrt{41}}\right), \text{ or } u = \frac{1}{4}\left(17 - \sqrt{41} - \sqrt{314 - 34\sqrt{41}}\right).$$

But, only $u_0 = \frac{1}{4}\left(17 - \sqrt{41} + \sqrt{314 - 34\sqrt{41}}\right)$ satisfies the equation $G(u) = 0$.

Now, if $\frac{1}{5} \le u \le 5$ then $u + \frac{1}{u} \le 5 + \frac{1}{5}$, which yields that $G(u) \ge 0$ when $u \in \left[\frac{1}{5}, 5\right]$. In particular, we have $\frac{1}{5} < \frac{1}{a_0} < 1 < a_0 < 5$, and $G(1) = 4 + \sqrt{32} > 0$. Therefore, $G(u) \ge 0 \quad \forall\, u \in [\frac{1}{a_0}, a_0]$.

Therefore, if $a \in [1, a_0]$, then $\{|x| = a, |y_i| \ge \sqrt{a}\}$ is a closed path. But note that $|y_1(x)| = |y_1(x^{-1})|$, and if $a < 1$, then $\sqrt{a} < \frac{1}{\sqrt{a}}$. In this case we have $|y_1(-a)| = |y_1\left(-\frac{1}{a}\right)| \ge \frac{1}{\sqrt{a}} > \sqrt{a}$. So, if $a \in \left[\frac{1}{a_0}, 1\right)$, we still have a closed path, namely $\{|x| = a, |y_i| \ge \sqrt{a}\}$. Thus proving our lemma. $\square$

In conclusion, we can now remove the restriction on a of being greater than 1, and we can proceed to find the values of $\mathrm{m}_{a,\sqrt{a}}\left(P_8(x, y)\right)$ for $a \in \left[\frac{1}{a_0}, a_0\right]$.

3.4.2. Homology class of the integration path in $H_1(E_{N(8)}, \mathbb{Z})$

We have concluded that the integration path is closed under certain conditions. In this section we study its class in the homology group $H_1\left(E_{N(8)}, \mathbb{Z}\right)$. We substitute $k = 8$ in (3.1.1), and consider the invariant holomorphic differential of the Weierstrass form $E_{N(8)}(X, Y)$, namely ω. Let $S = \{\alpha_1, \alpha_2\}$ be a set of generators of the group $H_1(E_{N(8)}, \mathbb{Z})$, which is a rank 2 free $\mathbb{Z}$-module. We consider the complex analytic isomorphism

$$E_{N(8)}(\mathbb{C}) \longrightarrow \mathbb{C}/\Lambda,$$

defined by $P \mapsto \int_O^P \omega \pmod{\Lambda}$, where $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ with $\int_{\alpha_i} \omega = \omega_i$, for $i = 1, 2$, and O is the identity element of the additive group $\left(E_{N(8)}(\mathbb{C}), +\right)$. This map implies that any path $\int_O^P \omega$ is well-defined up to addition of a complex number of the form $n_1\omega_1 + n_2\omega_2$, for $n_i \in \mathbb{Z}$. Therefore, the integral of ω over a closed path is 0 (mod Λ). Let γ_1 and γ_2 be two closed paths which do not self-intersect. In order to show that the two paths γ_1 and γ_2 define the same homology class, we first need to show that $\int_{\gamma_1} \omega = m \int_{\gamma_2} \omega$, for some positive integer m. In other words, we need to show $[\gamma_1] = m[\gamma_2]$. But as γ_1 and γ_2 are closed and do not self-intersect, then they must be generators, and $m = \pm 1$. In our case, our goal is to show that the homology classes $[\phi_*(|x| = a)], [\phi_*(|x| = 1)]$ and $[|X| = 1]$ are equal as elements of $H_1(E_{N(8)}, \mathbb{Z})^-$, which is a rank 1 free $\mathbb{Z}$-module. In order to do so, we follow a similar argument to the one given in **[31]** to show that the integrals $\int_\gamma \omega$ are in fact elements of $i\mathbb{R}$, where γ is a representative of one of the above mentioned homology classes, and they are also positive multiples of each other.

Recall that $a_0 = \left[(5 - 2\sqrt{2}) + \sqrt{(5 - 2\sqrt{2})^2 - 1}\right]$.

Proof of Lemma 3.3.5: We will prove the above result for $a \in \left(\frac{1}{a_0}, a_0\right)$ and the boundary cases will follow by continuity. Since we have

$$\int_{\phi_*(|x|=a)} \omega = \int_{|x|=a} \phi^*\omega, \tag{3.4.8}$$

we need to find an explicit expression for $\phi^*\omega$. From the change of variables in (3.1.1), we obtain

$$dX = \frac{1}{x^2 y}dx + \frac{1}{y^2 x}dy.$$

By differentiating the expression $P_8(x,y)=0$, we get

$$\left(1-\frac{1}{x^2}\right)dx+\left(1-\frac{1}{y^2}\right)dy=0.$$

Now, we replace dy in the expression of dX to get

$$dX=\left[\frac{1}{x^2y}+\frac{\left(1-\frac{1}{x^2}\right)}{y^2x\left(\frac{1}{y^2}-1\right)}\right]dx.$$

Also we have

$$2Y=\frac{(y-x)\left(1+\frac{1}{xy}\right)}{xy}.$$

Therefore, $\phi^*\omega$ is given by

$$\frac{dX}{2Y}=\frac{\left[\frac{1}{x^2y}+\frac{\left(1-\frac{1}{x^2}\right)}{y^2x\left(\frac{1}{y^2}-1\right)}\right]dx}{\frac{(y-x)\left(1+\frac{1}{xy}\right)}{xy}}=\frac{\left[y+\frac{x\left(1-\frac{1}{x^2}\right)}{\left(\frac{1}{y^2}-1\right)}\right]}{(y-x)(xy+1)}dx.$$

We also have $y_1(x)\cdot y_2(x)=1$. As we are working with $y_1:=y_1(x)$, we replace y with y_1 in the above expression, where $y_1(x)$ is given by (3.2.1). Before doing so we denote

$$\Delta_8:=\left(8+x+\frac{1}{x}\right)^2-4,\ \text{i.e. } y_2(x)-y_1(x)=\sqrt{\Delta_8}.$$

Let $y_2:=y_2(x)$. We now rewrite the expression of $\frac{dX}{2Y}$ with y_1 to get

$$\begin{aligned}\frac{dX}{2Y}&=\frac{\left[y_1+\frac{x\left(1-\frac{1}{x^2}\right)}{\left(\frac{1}{y_1{}^2}-1\right)}\right]}{(y_1-x)(xy_1+1)}dx\\
&=\frac{\left(\frac{1}{y_1}-y_1\right)+\left(x-\frac{1}{x}\right)}{y_1\left(1-\frac{x}{y_1}\right)(1+xy_1)\left(\frac{1}{y_1{}^2}-1\right)}dx\\
&=\frac{\sqrt{\Delta_8}+x-\frac{1}{x}}{(1-xy_2)(1+xy_1)\sqrt{\Delta_8}}dx \qquad (\text{using } y_1\cdot y_2=1)\\
&=\frac{\sqrt{\Delta_8}+x-\frac{1}{x}}{\left(1-x(y_2-y_1)-x^2\right)\sqrt{\Delta_8}}dx\\
&=-\frac{1}{\sqrt{\Delta_8}}\frac{dx}{x}.\end{aligned}\tag{3.4.9}$$

Therefore, we find that

$$\int_{\phi_*(|x|=a)}\omega=\int_{|x|=a}\phi^*\omega=-\int_{|x|=a}\frac{1}{\sqrt{\Delta_8}}\frac{dx}{x}.$$

We now substitute $x = ae^{i\theta}$ for $\theta \in [0, 2\pi)$ in the above expression. We know from Lemma 3.3.1 that we can consider a fixed branch of square root of Δ_8. Therefore, we can write

$$\begin{aligned}
\int_{\phi_*(|x|=a)} \omega = \int_{|x|=a} \phi^*\omega &= -\int_{|x|=a} \frac{1}{\sqrt{\Delta_8}} \frac{dx}{x} \\
&= -i \int_0^{2\pi} \frac{d\theta}{\sqrt{(8 + ae^{i\theta} + a^{-1}e^{-i\theta})^2 - 4}} \\
&= -i \int_{-\pi}^{\pi} \frac{d\tau}{\sqrt{(8 + ae^{i\tau} + a^{-1}e^{-i\tau})^2 - 4}} \\
&= -2i \operatorname{Re}\left[\int_0^{\pi} \frac{d\tau}{\sqrt{(8 + ae^{i\tau} + a^{-1}e^{-i\tau})^2 - 4}} \right],
\end{aligned}$$

where we used the change of variables $\theta - \pi \mapsto \tau$ in the second line. We then have

$$\int_{\phi_*(|x|=a)} \omega = \int_{|x|=a} \phi^*\omega \in i\mathbb{R}.$$

On other hand we know that $\phi : P_8 \mapsto E_{N(8)}$ induces $\phi_* : H_1(C_8, \mathbb{Z}) \mapsto H_1\left(E_{N(8)}, \mathbb{Z}\right)$, where $C_8 : P_8(x, y) = 0$. Using the *Uniformization Theorem* we get that there exists a lattice $\Lambda \subset \mathbb{C}$ such that $E_{N(8)}(\mathbb{C}) \cong \mathbb{C}/\Lambda$ is a complex analytic isomorphism. We therefore get a group isomorphism $H_1\left(E_{N(8)}(\mathbb{C}), \mathbb{Z}\right) \cong H_1(\mathbb{C}/\Lambda, \mathbb{Z})$. But the last term is a rank 2 free $\mathbb{Z}$-module. But we can also write

$$H_1\left(E_{N(8)}(\mathbb{C}), \mathbb{Z}\right) = H_1\left(E_{N(8)}(\mathbb{C}), \mathbb{Z}\right)^+ \oplus H_1\left(E_{N(8)}(\mathbb{C}), \mathbb{Z}\right)^-,$$

which shows that $H_1\left(E_{N(8)}(\mathbb{C}), \mathbb{Z}\right)^+$ and $H_1\left(E_{N(8)}(\mathbb{C}), \mathbb{Z}\right)^-$ are free $\mathbb{Z}$-modules of rank 1.

From Remark 1.1.2 and the above discussion we know that $[\phi_*(|x| = a)] \in H_1\left(E_{N(8)}, \mathbb{Z}\right)^-$. Taking $a = 1$ we also get $[\phi_*(|x| = 1)] \in H_1\left(E_{N(8)}, \mathbb{Z}\right)^-$. But $H_1\left(E_{N(8)}, \mathbb{Z}\right)^-$ is a rank 1 free $\mathbb{Z}$-module, and the closed integration paths $\phi_*(|x| = 1)$ and $\phi_*(|x| = a)$ do not self-intersect. Therefore, we obtain $[\phi_*(|x| = 1)] = c[\phi_*(|x| = a)]$ and $c'[\phi_*(|x| = 1)] = [\phi_*(|x| = a)]$ for some $c, c' \in \mathbb{Z} \setminus \{0\}$. This implies that $c = c'$ and $c \in \{-1, 1\}$. Also, from **[32]** and **[39]**, we have

$$\int_{|X|=1} \omega = \int_{\phi_*(|x|=1)} \omega \in i\mathbb{R}, \qquad \text{for } (x, y) \in \mathbb{T}^2,$$

and $[\phi_*(|x|=1)] = [|X|=1]$. As $a+\frac{1}{a}<6$, the sign of the integral in (3.4.8) is independent of the values of a. In fact, we will argue that $c=c'=1$ in the Appendix. Combining these results we get our desired relations. $\square$

3.4.3. The integral over $d\arg y$

It remains to compute the integral (1.2.4).

Proof of Lemma 3.3.6: We have shown in Lemma 3.3.1 that $\left(8+x+\frac{1}{x}\right)^2-4 \notin (-\infty,0)$, when $|x|=a\in\mathbb{R}_{>0}$ and $a+\frac{1}{a}<6$. This implies that we can consider the principal branch of the square root of $\left(8+x+\frac{1}{x}\right)^2-4$. Therefore, we have a well-defined algebraic function of x when $|x|=a$, namely

$$y_1(x) = -\frac{\left(8+x+\frac{1}{x}+\sqrt{\left(8+x+\frac{1}{x}\right)^2-4}\right)}{2}.$$

The above discussion leads to the fact that $y_1(z)$ is holomorphic on the annulus $\mathbf{A}=\{z\in\mathbb{C}:\frac{1}{a_0}<|z|<a_0\}$. We claim that $y_1(z)$ does not vanish for any $z\in\mathbf{A}$. Suppose it vanishes at least at one point, say x_0. In other words,

$$\begin{aligned} & y_1(x_0)=0 \\ \Leftrightarrow & -\left(8+x_0+\frac{1}{x_0}\right)=\sqrt{\left(8+x_0+\frac{1}{x_0}\right)^2-4} \\ \Rightarrow & \left(8+x_0+\frac{1}{x_0}\right)^2=\left(8+x_0+\frac{1}{x_0}\right)^2-4, \end{aligned}$$

which is impossible. Therefore, $y_1(z)$ is holomorphic and non-vanishing in $\mathbf{A}$.

Let $\gamma_a = \overrightarrow{\tau_1}\bigcup\overrightarrow{l}\bigcup\overleftarrow{l}\bigcup\overleftarrow{\tau_a}$, where $\overleftarrow{\tau_r}$ is the path obtained by traveling the circle $|z|=r$ counter-clockwise with $r\in\left(\frac{1}{a_0},a_0\right)$ and $\overrightarrow{l}$ is the straight line along the imaginary axis starting from i and ending at ia. We also have *Cauchy's Theorem* in our domain of definition $\mathbf{A}$, which says

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}=0,$$

for any f which is non-vanishing and holomorphic in $\mathbf{A}$ and γ is a simple closed path contained completely in $\mathbf{A}$. We take $f=y_1$, i.e. $f(z)=y_1(z)$ and $\gamma=\gamma_a$. Let $I(r)=$

$\frac{1}{2\pi i}\int_{\overleftarrow{\tau_r}}\frac{dy_1}{y_1}$, where $r\in\left(\frac{1}{a_0},a_0\right)$. We have

$$I\left(1\right)=-\frac{1}{2\pi i}\int_{\overrightarrow{\tau_1}}\frac{dy_1}{y_1},\ I\left(a\right)=\frac{1}{2\pi i}\int_{\overleftarrow{\tau_a}}\frac{dy_1}{y_1},\ \text{and}\ \frac{1}{2\pi i}\int_{\gamma_a}\frac{dy_1}{y_1}=I\left(a\right)-I\left(1\right)=0,$$

where the last expression follows from *Cauchy's Theorem* and the fact that integration along the paths $\overrightarrow{l}$ and $\overleftarrow{l}$ cancel each other. Therefore, it is enough to calculate $I\left(1\right)$. But

$$\begin{aligned}
I\left(a\right)=I\left(1\right)=\frac{1}{2\pi i}\int_{\overleftarrow{\tau_1}}\frac{dy_1}{y_1}&=\frac{i}{2\pi}\int_0^{2\pi}\frac{2\sin\theta}{\sqrt{\left(8+2\cos\theta\right)^2-4}}d\theta\\
&=\frac{-i}{2\pi}\int_{-\pi}^{\pi}\frac{2\sin\tau}{\sqrt{\left(8-2\cos\tau\right)^2-4}}d\tau\qquad[\text{Change of variable: }\theta-\pi\mapsto\tau]\\
&=\ 0.
\end{aligned}$$

The last step follows from the fact that the integrand is an odd function in τ, and the previous steps follow from the following calculation:

$$\begin{aligned}
I\left(1\right)&=\frac{1}{2\pi i}\int_{\overleftarrow{\tau_1}}\frac{d\left(8+z+\frac{1}{z}+\sqrt{\left(8+z+\frac{1}{z}\right)^2-4}\right)}{\left(8+z+\frac{1}{z}+\sqrt{\left(8+z+\frac{1}{z}\right)^2-4}\right)}\\
&=\frac{1}{2\pi i}\int_{\overleftarrow{\tau_1}}\frac{z-\frac{1}{z}}{\sqrt{\left(8+z+\frac{1}{z}\right)^2-4}}\frac{dz}{z}\\
&=\frac{i}{2\pi}\int_0^{2\pi}\frac{2\sin\theta}{\sqrt{\left(8+2\cos\theta\right)^2-4}}d\theta\qquad\left[\text{Change of variable: }z\mapsto e^{i\theta}\right]\\
&=\frac{-i}{2\pi}\int_{-\pi}^{\pi}\frac{2\sin\tau}{\sqrt{\left(8-2\cos\tau\right)^2-4}}d\tau,
\end{aligned}$$

where we used the change of variable $\theta-\pi\mapsto\tau$ in the last line.

We know that $d\arg y_1=\operatorname{Im}\left(\frac{dy_1}{y_1}\right)$. Therefore we can rewrite $I\left(1\right)$ as

$$I(1)=\frac{1}{2\pi i}\int_{\overleftarrow{\tau_1}}\frac{dy_1}{y_1}=\frac{1}{2\pi i}\left[\int_{\overleftarrow{\tau_1}}\operatorname{Re}\left(\frac{dy_1}{y_1}\right)+i\int_{\overleftarrow{\tau_1}}\operatorname{Im}\left(\frac{dy_1}{y_1}\right)\right]=0.$$

Therefore, for $a\in\left(\frac{1}{a_0},a_0\right)$ we have

$$\frac{1}{2\pi}\int_{|x|=a}d\arg y_1=\frac{1}{2\pi}\int_{|x|=a}\operatorname{Im}\left(\frac{dy_1}{y_1}\right)=0.$$

By continuity we get $\forall a \in \left[\frac{1}{a_0}, a_0\right]$,

$$\frac{1}{2\pi}\int_{|x|=a} d\arg y_1 = 0.$$

$\square$

3.5. Partial results when $k > 4$

We now describe some partial results that we obtained using a similar method as above, regarding the values of $\mathrm{m}_{a,b}\left(P_k(x,y)\right)$ when $b \in \mathbb{R}_{\geq 1}$, $a \in \mathbb{R}_{>0}$ with b not necessarily a function of a. We also restrict ourselves to $k \in \mathbb{R}_{>4}$.

We can factorize $yP_k(x,y)$ over $\overline{\mathbb{C}(x)}$ as

$$yP_k\left(x,y\right) = \left(y - y_{1,k}\left(x\right)\right)\left(y - y_{2,k}\left(x\right)\right),$$

where

$$\begin{aligned} y_{1,k}\left(x\right) =& \frac{-\left(k + x + \frac{1}{x}\right) - \sqrt{\left(k + x + \frac{1}{x}\right)^2 - 4}}{2}, \\ y_{2,k}\left(x\right) =& \frac{-\left(k + x + \frac{1}{x}\right) + \sqrt{\left(k + x + \frac{1}{x}\right)^2 - 4}}{2}, \end{aligned} \tag{3.5.1}$$

are algebraic functions in x.

Our main goal is to determine when the integration path $\{|x| = a, |y_{i,k}| \geq b\}$ is a closed path for $i \in \{1,2\}$.

Firstly, a discussion similar to the one in the proof of Proposition 3.3.3 implies that the minimum of $|\Delta_k| = \left|\left(k + x + \frac{1}{x}\right)^2 - 4\right|$ is attained at $x = -a$. As described in the proof of Lemma 3.3.5, we consider a restriction on a so that $\Delta_k \notin (-\infty, 0)$. We can even omit the case when $\Delta_k = 0$ for all $x \in \mathbb{T}^1_a$, because in that case we have $y_{1,k}(x) = y_{2,k}(x) = \pm 1$. In other words, we want to consider a fixed branch of the square root of Δ_k for our computation to conclude when $\{|x| = a, |y_{i,k}(x)| \geq b\}$ is a closed path for $i \in \{1,2\}$. In fact, we want to consider the principal branch of the square root of Δ_k.

To do so we need

$$\Delta_k\Big|_{x=-a} = \left(k - a - \frac{1}{a}\right)^2 - 4 = \left(k - a - \frac{1}{a} + 2\right)\left(k - a - \frac{1}{a} - 2\right) \in \mathbb{C}\setminus(-\infty, 0].$$

But as $\Delta_k\Big|_{x=-a} \in \mathbb{R}$, we can restrict our search to the case when $\Delta_k\Big|_{x=-a} > 0$. We claim that if $a + \frac{1}{a} > k+2$, then $\{|x| = a, |y_{i,k}(x)| \geq b\}$ is not a closed path for any $i \in \{1,2\}$. Therefore, according to our claim we will have the liberty to work with $a+\frac{1}{a}<k-2$ as we did in "$k=8$" case. In order to prove our claim, we search for values of x such that $|y_{i,k}(x)|<b$ for $i\in\{1,2\}$. As mentioned above we are restricting our computations to $b\in[1,\infty)$. The following lemma provides us those particular values of x.

Lemma 3.5.1. *Let $y_{1,k}(x)$ and $y_{2,k}(x)$ be defined by (3.5.1), where $|x|=a$ with $a\in\mathbb{R}_{>0}$. We now have the following relations between them:*

- $|y_{1,k}(a)|>|y_{2,k}(a)|$,
- *If $a+\frac{1}{a}>k+2$ then $|y_{1,k}(ae^{i\pi})|=|y_{1,k}(-a)|<1\le b$,*
- $|y_{2,k}(a)|<1\le b$.

Proof. We have proved $|y_{1,k}(a)|>|y_{2,k}(a)|$ in 3.4.1 for the special case $k=8$. The proof for $k>4$ follows a similar direction.

For the second part, we denote $-N=\left(k-a-\frac{1}{a}\right)$. Then, we have $N>2$ as $a+\frac{1}{a}>k+2$. Now we rewrite $|y_{1,k}(-a)|$ in terms of N as

$$|y_{1,k}(-a)|=\left|\frac{\left(k-a-\frac{1}{a}\right)+\sqrt{\left(k-a-\frac{1}{a}\right)^2-4}}{2}\right|=\left|\frac{-N+\sqrt{(-N)^2-4}}{2}\right|=\frac{N-\sqrt{N^2-4}}{2},$$

where the last equality holds because $N>\sqrt{N^2-4}$. Now note that $N-2>0$, and we get

$$(N-2)<\sqrt{(N-2)(N+2)}.$$

In other words,

$$\frac{N-\sqrt{N^2-4}}{2}<1,$$

which proves the result.

The third part follows from a proof similar to the one above by considering $N_1=k+a+\frac{1}{a}>2$, and noting that

$$|y_{2,k}(a)|=\frac{N_1-\sqrt{N_1^2-4}}{2}<1.$$

□

Lemma 3.5.1 implies that if $a+\frac{1}{a}>k+2$, then $\{|x|=a,|y_{1,k}(x)|\ge b\}$ is not a closed path because we have $|y_{1,k}(ae^{i\pi})|=|y_{1,k}(-a)|<b$. Similarly it also shows that $|y_{2,k}(a)|<b$, which combined with the previous statement proves our claim that if $a+\frac{1}{a}>k+2$, then $\{|x|=a,|y_{i,k}(x)|\ge b\}$ is not a closed path for any $i\in\{1,2\}$. In particular, we can restrict ourselves to $a+\frac{1}{a}<k-2$.

Now that we have $a + \frac{1}{a} < k - 2$, we can proceed similarly as we did in $k = 8$ (simply by replacing 8 with k).

For $a + \frac{1}{a} < k - 2$, we write $x = ae^{i\theta}$ to obtain

$$\min_{\theta \in [0,2\pi)} |y_{1,k}(x)| = |y_{1,k}\left(ae^{i\pi}\right)| = |y_{1,k}(-a)|,$$

and

$$\begin{aligned}\min_{\theta \in [0,2\pi)} |y_{2,k}(x)| &= \frac{1}{\max_{\theta \in [0,2\pi)} |y_{1,k}(x)|} \\ &= |y_{1,k}\left(ae^{i0}\right)|^{-1} \\ &= |y_{2,k}\left(ae^{i0}\right)| = |y_{2,k}(a)|.\end{aligned}$$

The first equality can be justified from an argument similar to the proof of Proposition 3.3.3, and the second equality follows from the following lemma.

Lemma 3.5.2. *If $a \in \mathbb{R}_{>0}$ and $a + \frac{1}{a} < k - 2$ then $|y_{1,k}(x)| = |y_{1,k}\left(ae^{i\theta}\right)|$ attains its maximum in $[0, 2\pi)$ at $\theta_{\max} = 0$.*

Proof. We use the expression of $y_{1,k}$ from (3.5.1) and the above mentioned conditions on a to get

$$\begin{aligned}|y_{1,k}(x)| &= \left|\frac{-\left(k + x + \frac{1}{x}\right) - \sqrt{\left(k + x + \frac{1}{x}\right)^2 - 4}}{2}\right| \\ &\le \left|\frac{k + x + \frac{1}{x}}{2}\right| + \left|\frac{\sqrt{\left(k + x + \frac{1}{x}\right)^2 - 4}}{2}\right| \\ &= \frac{\left|k + x + \frac{1}{x}\right|}{2} + \frac{\sqrt{\left|k - 2 + x + \frac{1}{x}\right| \ \left|k + 2 + x + \frac{1}{x}\right|}}{2} \\ &\le \frac{k + a + \frac{1}{a} + \sqrt{\left(k + a + \frac{1}{a}\right)^2 - 4}}{2} = |y_{1,k}(a)|,\end{aligned}$$

where the last inequality follows from the fact that

I. $a + \frac{1}{a} < k - 2$, which implies $\left(k + x + \frac{1}{x}\right)^2 - 4 \notin (-\infty, 0]$ for $x = ae^{i\theta}$ with $\theta \in [0, 2\pi)$ (see Lemma 3.3.1),

II. $|\cos\theta|, |\sin\theta|$ are bounded by 1, and writing $x = ae^{i\theta}$, we have

$$\left|L + ae^{i\theta} + \frac{1}{ae^{i\theta}}\right|^2 = \left(L + \left(a + \frac{1}{a}\right)\cos\theta\right)^2 + \left(a - \frac{1}{a}\right)^2 \sin^2\theta$$
$$\leq \left(L + a + \frac{1}{a}\right)^2,$$

where $L \in \{k-2, k, k+2\}$.

Therefore, $|y_{1,k}(ae^{i\theta})|$ attains its maximum at $\theta_{\max} = 0$, where $\theta \in [0, 2\pi)$. □

- **Main Result** : The conditions $a \in \mathbb{R}_{>0}$ and $a + \frac{1}{a} < k - 2$ implies that if

$$\max\{1, |y_{2,k}(a)|\} < b \leq |y_{1,k}(-a)|,$$

then

$$\mathrm{m}_{a,b}\left(yP_k\left(x,y\right)\right) = \mathrm{m}\left(yP_k\left(x,y\right)\right) \Leftrightarrow \mathrm{m}_{a,b}\left(P_k\left(x,y\right)\right) = \mathrm{m}\left(P_k\left(x,y\right)\right) - \log b,$$

where $k > 4$. The proof of the above results is very similar to the proof of the $k = 8$ case before.

Now let

$$\mathrm{m}_{a,b,1,k} = \frac{1}{2\pi i}\int_{|x|=a, |y_{1,k}(x)|\geq b} \log|y_{1,k}(x)|\frac{dx}{x}$$

and

$$\mathrm{m}_{a,b,2,k} = \frac{1}{2\pi i}\int_{|x|=a, |y_{2,k}(x)|\geq b} \log|y_{2,k}(x)|\frac{dx}{x}.$$

If we let $0 < b \leq |y_{2,k}(a)| < 1$, then, as $y_{1,k}(x) \cdot y_{2,k}(x) = 1$, we can derive that

$$\mathrm{m}_{a,b}(yP_k(x,y)) = \mathrm{m}_{a,b,1,k} + \mathrm{m}_{a,b,2,k} = \mathrm{m}_{a,b,1,k} - \mathrm{m}_{a,b,1,k} = 0.$$

In other words, if $a \in \mathbb{R}_{>0}$, $a + \frac{1}{a} < k - 2$ and $0 < b \leq |y_{2,k}(a)| < 1$ then,

$$\mathrm{m}_{a,b}(P_k(x,y)) = \log\left(\frac{1}{b}\right).$$

3.6. Additional calculation with the diamond operator

We have shown $\mathrm{m}_{a,b}(yP_k(x,y)) = \mathrm{m}(yP_k(x,y))$ for some restricted values of a and b when $k > 4$. Note that, except for computing integrals of ω and $d\arg y$ over some closed paths in section 3.4.2 and 3.4.3 we are not using the birational transformation ϕ explicitly in our proof. In fact, we have shown that, for certain values of a and b, the Mahler measures of this family of polynomials are the same as their standard Mahler measures, and to do so we obtained values of a, b in $\mathbb{R}_{>0}$ such that the integration path $\{|x| = a, |y_{i,k}| \geq b\}$ is closed for $i \in \{1, 2\}$ and homologous to $\{|x| = 1, |y_{i,k}| \geq 1\}$.

Although we have not used the elliptic dilogarithm and the diamond operator in our proof explicitly, we add this section here for the sake of completeness.

We will follow [**32**] and [**30**] to calculate $\mathrm{m}_{a,b}(P_k(x, y))$ in terms of elliptic dilogarithm when $k > 4$ for certain values of a, b (obtained in Section 3.5). As we have the equality $\mathrm{m}_{a,b}(yP_k(x, y)) = \mathrm{m}(yP_k(x, y))$ for certain values of a and b, it is enough to consider the case where $a = b = 1$.

From Definition 1.1.9 we know that for $T \in E_{N(k)}(\mathbb{C})$, corresponding to $z \in \mathbb{C}^\times/q^{\mathbb{Z}}$, the *elliptic dilogarithm* is

$$D^{E_{N(k)}}(T) = D^{E_{N(k)}}(z) := \sum_{n \in \mathbb{Z}} D(q^n z).$$

In our context, it is enough to take into account that

$$r_{E_{N(k)}}(\{x_{N(k)}, y_{N(k)}\})[\gamma] = D^{E_{N(k)}}((x_{N(k)}) \diamond (y_{N(k)})), \tag{3.6.1}$$

where $r_{E_{N(k)}}$ is the regulator map due to Bloch [**12**] and Beĭlinson [**10**], $\{x_{N(k)}, y_{N(k)}\}$ is an element of $K_2(\mathbb{C}(E_{N(k)}))$, $D^{E_{N(k)}}$ is the elliptic dilogarithm in $E_{N(k)}$ constructed by Bloch [**12**], and $[\gamma]$ is a generator of $H_1(E_{N(k)}, \mathbb{Z})^-$, which is a rank 1 free $\mathbb{Z}$-module.

Our goal is to prove equality (0.3.4), namely

$$\mathrm{m}\left(P_8\left(x, y\right)\right) = 4\mathrm{m}\left(P_2\left(x, y\right)\right),$$

following the steps in [**32**]. To begin with we recall two functional equations of $\mathrm{m}(P_k)$:

(1) Kurokawa and Ochiai [**27**] showed that, for $h \in \mathbb{R} \setminus \{0\}$ we have

$$\mathrm{m}(P_{4h^2}) + \mathrm{m}\left(P_{\frac{4}{h^2}}\right) = 2\mathrm{m}\left(P_{2\left(h+\frac{1}{h}\right)}\right). \tag{3.6.2}$$

(2) Lalín and Rogers [**32**] showed that, for $h \neq 0$, and $|h| < 1$ we have

$$\mathrm{m}\left(P_{2\left(h+\frac{1}{h}\right)}\right) + \mathrm{m}\left(P_{2\left(ih+\frac{1}{ih}\right)}\right) = \mathrm{m}\left(P_{\frac{4}{h^2}}\right). \tag{3.6.3}$$

Setting $h = \frac{1}{\sqrt{2}}$ in (3.6.2) and (3.6.3) we obtain

$$\mathrm{m}(P_2) + \mathrm{m}(P_8) = 2\mathrm{m}(P_{3\sqrt{2}}), \tag{3.6.4}$$

and

$$\mathrm{m}(P_{3\sqrt{2}}) + \mathrm{m}(P_{i\sqrt{2}}) = \mathrm{m}(P_8), \tag{3.6.5}$$

respectively. Therefore it remains to obtain a relation between $\mathrm{m}(P_{3\sqrt{2}})$ and $\mathrm{m}(P_{i\sqrt{2}})$ to finish the proof of (0.3.4).

First we consider the torsion group of $E_{N(k)}$ over $\mathbb{Q}$. It is not hard to see that $(E_{N(k)}(\mathbb{Q}))_{\mathrm{tor}}$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ with generator $P = (1, k/2) \in E_{N(k)}(\mathbb{Q})$. Notice that, $2P = (0,0)$, $3P = (1, -k/2)$, and $4P = O$.

From [**32**] and (3.1.1) we also have

$$\begin{aligned} (X) &= 2(2P) - 2O, \\ (x) &= (P) - (2P) - (3P) + O, \\ (y) &= -(P) - (2P) + (3P) + O. \end{aligned} \tag{3.6.6}$$

Applying the diamond operator between (x) and (y) we get

$$(x) \diamond (y) = 8(P). \tag{3.6.7}$$

Note that $Q = \left(-\frac{1}{h^2}, 0\right)$ is a point of order 2 in $E_{N(k)}$ when $k = 2\left(h + \frac{1}{h}\right)$. It is easy to see that $P + Q = \left(-1, h - \frac{1}{h}\right)$, $2P + Q = (-h^2, 0)$ and $3P + Q = (-1, \frac{1}{h} - h)$.

For simplicity we will denote $E_{N(k)}$ as E_k and $r_{E_{N(k)}}$ as r_k in the future.

Now we use the isomorphism

$$\varphi : E_{2\left(h+\frac{1}{h}\right)} \to E_{2\left(ih+\frac{1}{ih}\right)}, \qquad (X, Y) \mapsto (-X, iY), \tag{3.6.8}$$

to pull some rational functions $u, v \in \mathbb{C}\left(E_{2\left(ih+\frac{1}{ih}\right)}\right)$ back to $\mathbb{C}\left(E_{2\left(h+\frac{1}{h}\right)}\right)$. This implies that

$$r_{2\left(ih+\frac{1}{ih}\right)}(\{u, v\}) = r_{2\left(h+\frac{1}{h}\right)}(\{u \circ \varphi, v \circ \varphi\}). \tag{3.6.9}$$

Recall that $\mathbb{H}$ is the complex upper-half plane. The *Uniformization Theorem* says that there exists a lattice $\Lambda' \subset \mathbb{C}$ such that $E_{3\sqrt{2}}(\mathbb{C}) \cong \mathbb{C}/\Lambda'$ is a complex analytic isomorphism. Let ω_1', ω_2' be complex numbers such that $\Lambda' = \omega_1'\mathbb{Z} + \omega_2'\mathbb{Z}$ with $\frac{\omega_2'}{\omega_1'} \in \mathbb{C} \setminus \mathbb{R}$ (by swapping ω_1' and ω_2' if necessary we may assume that $\frac{\omega_2'}{\omega_1'} \in \mathbb{H}$). We denote $\tau := \frac{\omega_2'}{\omega_1'} \in \mathbb{H}$. Then we combine (1.1.12) and (3.6.8) to get the following commutative diagram :

$$\begin{tikzcd} E_{3\sqrt{2}} \arrow[r, "\varphi"] \arrow[d, "\simeq"] & E_{i\sqrt{2}} \arrow[d, dashed] \\ \mathbb{C}/\Lambda_\tau \arrow[r, "\simeq"] & \mathbb{C}^\times/q^{\mathbb{Z}} \end{tikzcd}$$

where $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$, and $q = e^{2i\pi\tau}$. Note that Λ_τ and Λ' are homothetic, i.e. $\Lambda' = \omega_1'\Lambda_\tau$. The above diagram implies that for $T \in E_{3\sqrt{2}}(\mathbb{C})$, we have

$$D^{E_{i\sqrt{2}}}(\varphi(T)) = D^{E_{3\sqrt{2}}}(T).$$

We write x_1, y_1, X_1, Y_1 for the rational functions in $E_{3\sqrt{2}}$, and x_2, y_2, X_2, Y_2 for the corresponding objects in $E_{i\sqrt{2}}$. The relations among the above set of rational functions in $E_{3\sqrt{2}}$ and the relations among x_2, y_2, X_2, Y_2 in $E_{i\sqrt{2}}$ can be written using (3.1.1) as

$$\begin{cases} x_j &= \dfrac{k_jX_j - 2Y_j}{2X_j(X_j-1)}, \\ \\ y_j &= \dfrac{k_jX_j + 2Y_j}{2X_j(X_j-1)}, \end{cases} \tag{3.6.10}$$

for $j \in \{1, 2\}$ with $k_1 = 3\sqrt{2}$ and $k_2 = i\sqrt{2}$.

We have a commutative triangle in this following diagram:

$$\begin{tikzcd} & P_{3\sqrt{2}} \arrow[d, "\phi"] \\ P_{i\sqrt{2}} \arrow[r, dashed] \arrow[dr, "\phi"] & E_{3\sqrt{2}} \arrow[d, "\varphi"] \\ & E_{i\sqrt{2}} \end{tikzcd}$$

(see (3.1.2) for more details on the birational transformation ϕ). Therefore, it follows from **[37]** that some integer multiples of

$$\xi_1 = \{x_1, y_1\} \text{ and } \xi_2 = \{x_2 \circ \varphi, y_2 \circ \varphi\}$$

are in $K_2(E_{3\sqrt{2}})$. From the above diagram and (3.6.10) we can write $x_2 \circ \varphi$ and $y_2 \circ \varphi$ in terms of X_1, Y_1 as

$$\begin{cases} x_2 \circ \varphi &= \dfrac{-k_2X_1 - 2iY_1}{2X_1(X_1+1)}, \\ \\ y_2 \circ \varphi &= \dfrac{-k_2X_1 + 2iY_1}{2X_1(X_1+1)}. \end{cases} \tag{3.6.11}$$

Combining (3.6.11) and a calculation similar to (3.6.6) we obtain

$$(x_2 \circ \varphi) \diamond (y_2 \circ \varphi) = 8(P + Q). \tag{3.6.12}$$

Also note that we can rewrite (3.6.7) in terms of rational functions in $E_{3\sqrt{2}}$ as

$$(x_1) \diamond (y_1) = 8(P).$$

Now, in order to obtain a relation between $\mathrm{m}(P_{3\sqrt{2}})$ and $\mathrm{m}(P_{i\sqrt{2}})$ we have to find relations between (P) and $(P+Q)$ in $\mathbb{Z}\left[E_{3\sqrt{2}}(\mathbb{C})\right]^-$, where $\mathbb{Z}\left[E_{3\sqrt{2}}(\mathbb{C})\right]$ is the group of divisors on $E_{3\sqrt{2}}$ (see (1.1.14)). In other words, we need to relate the dilogarithm $D^{E_{3\sqrt{2}}}$ evaluated in both elements $(x_1)\diamond(y_1)$ and $(x_2\circ\varphi)\diamond(y_2\circ\varphi)$ of $\mathbb{Z}\left[E_{3\sqrt{2}}(\mathbb{C})\right]^-$. We want to find combinations of tame symbols (also known as Steinberg symbols) $\{f, 1-f\}$ with $f \in \mathbb{C}(E_{3\sqrt{2}})$ such that the corresponding combination $(f)\diamond(1-f)$ yields a linear combination of (P) and $(P+Q)$. From Remark 1.1.11 we know that $\{f, 1-f\}$ is trivial in K-theory. Thus giving us a linear combination involving (P) and $(P+Q)$.

In order to do so we consider the function $f = \frac{\sqrt{2}Y_1 - X_1}{2}$ in $\mathbb{C}(E_{3\sqrt{2}})$. Note that

$$\left(\frac{\sqrt{2}Y_1 - X_1}{2}\right) = (2P) + 2(P+Q) - 3O,$$

$$\left(1 - \frac{\sqrt{2}Y_1 - X_1}{2}\right) = (P) + (Q) + (3P+Q) - 3O.$$

We now apply the diamond operation to get

$$(f)\diamond(1-f) = 6(P) - 10(P+Q).$$

But the discussion on the previous paragraph, Remark 1.1.11 and (1.1.15) yield

$$6(P) \sim 10(P+Q) \Longleftrightarrow 6D^{E_{3\sqrt{2}}}(x_1 \diamond y_1) = 10D^{E_{3\sqrt{2}}}\left((x_2\circ\varphi)\diamond(y_2\circ\varphi)\right)$$

because of the triviality of $(f)\diamond(1-f)$ in K-theory. In other words, using (3.6.1) we get

$$6r_{3\sqrt{2}}(\{x_1, y_1\}) = 6r_{3\sqrt{2}}(\xi_1) = 10r_{3\sqrt{2}}(\xi_2) = 10r_{3\sqrt{2}}(\{x_2\circ\varphi, y_2\circ\varphi\}) = 10r_{i\sqrt{2}}(\{x_2, y_2\}),$$

where last equality follows from (3.6.9). Therefore, we have

$$3\mathrm{m}(P_{3\sqrt{2}}) = 5\mathrm{m}(P_{i\sqrt{2}}). \tag{3.6.13}$$

From (3.6.4), (3.6.5) and (3.6.13) we conclude that

$$\mathrm{m}(P_8) = \frac{8}{5}\mathrm{m}(P_{3\sqrt{2}}) = 4\mathrm{m}(P_2) = 4L'(E_{N(8)}, 0),$$

where the last equality was proved by Rogers and Zudilin [**39**].

Recall that $a_0 = \left[(5-2\sqrt{2}) + \sqrt{(5-2\sqrt{2})^2 - 1}\right]$.

In fact we have proved above that, for $a \in \left[\frac{1}{a_0}, a_0\right]$, we have

$$\mathrm{m}_{a,\sqrt{a}}(P_8) = \mathrm{m}(P_8) - \frac{1}{2}\log a = 4L'(E_{N(8)}, 0),$$

thus proving Theorem 0.3.2.

Chapter 4

Conclusions and further questions

We have used a dependence of b on a or vice-versa while proving the results for $k > 4$ cases. It would be interesting to investigate the cases where we remove this dependence and vary a and b independently.

There are many additional problems which could be addressed. The most immediate ones that we have to investigate are the cases when $k \notin \mathbb{R}$, such as when $k^2 \in \mathbb{Z}_{<0}$. We may be able to obtain a similar string of results for such cases with the help of our method. It will be challenging to consider $k \in \mathbb{C} \setminus \mathbb{R}$ as our method may not work in those cases. Another intriguing problem will be to compute the Mahler measure of non-tempered polynomials over arbitrary tori, as it is not certain that the K-theory framework works in such cases.

It would be also natural to consider the cases where the integration paths are not closed, and they are not easily identifiable as cycles in the homology group.

A different direction would be to consider other families of polynomials due to Boyd.

Finally as mentioned in 0.1.1, it would be interesting to look for periods in terms of the Mahler measure of several variable polynomials and special values of L-functions of elliptic curves or modular forms.

Bibliography

[1] G. E. Andrews, R. Askey et R. Roy : *The Hypergeometric Functions*, pages 61–123. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1999.

[2] W. N. Bailey : *Generalized hypergeometric series.* Cambridge Tracts in Mathematics and Mathematical Physics, No. 32. Stechert-Hafner, Inc., New York, 1964.

[3] Marie José Bertin : Mesure de Mahler d'une famille de polynômes. *J. Reine Angew. Math.*, 569:175–188, 2004.

[4] Marie José Bertin : Mesure de Mahler et régulateur elliptique: preuve de deux relations "exotiques". *In Number theory*, volume 36 de *CRM Proc. Lecture Notes*, pages 1–12. Amer. Math. Soc., Providence, RI, 2004.

[5] Marie José Bertin : Mahler's measure and L-series of $K3$ hypersurfaces. *In Mirror symmetry. V*, volume 38 de *AMS/IP Stud. Adv. Math.*, pages 3–18. Amer. Math. Soc., Providence, RI, 2006.

[6] Marie José Bertin : Mesure de Mahler d'hypersurfaces $K3$. *J. Number Theory*, 128(11):2890–2913, 2008.

[7] Marie-José Bertin et Matilde Lalín : Mahler measure of multivariable polynomials. *In Women in numbers 2: research directions in number theory*, volume 606 de *Contemp. Math.*, pages 125–147. Amer. Math. Soc., Providence, RI, 2013.

[8] Marie José Bertin et Wadim Zudilin : On the Mahler measure of a family of genus 2 curves. *Math. Z.*, 283(3-4):1185–1193, 2016.

[9] Marie José Bertin et Wadim Zudilin : On the Mahler measure of hyperelliptic families. *Ann. Math. Qué.*, 41(1):199–211, 2017.

[10] A. A. Beĭlinson : Higher regulators and values of L-functions. *In Current problems in mathematics, Vol. 24*, Itogi Nauki i Tekhniki, pages 181–238. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.

[11] S. Bloch et D. Grayson : K_2 and L-functions of elliptic curves: computer calculations. *In Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, volume 55 de *Contemp. Math.*, pages 79–88. Amer. Math. Soc., Providence, RI, 1986.

[12] Spencer J. Bloch : *Higher regulators, algebraic K-theory, and zeta functions of elliptic curves*, volume 11 de *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2000.

[13] Enrico Bombieri et Walter Gubler : *Heights in Diophantine geometry*, volume 4 de *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2006.

[14] Johan Bosman : *Boyd's conjecture for a family of genus 2 curves*. Theses, Universiteit Utrecht, 2004.

[15] D. W. Boyd, F. Rodriguez-Villegas et N. M. Dunfield : Mahler's Measure and the Dilogarithm (II). *arXiv Mathematics e-prints*, août 2003.

[16] David W. Boyd : Speculations concerning the range of Mahler's measure. *Canad. Math. Bull.*, 24(4): 453–469, 1981.

[17] David W. Boyd : Mahler's measure and special values of L-functions. *Experiment. Math.*, 7(1):37–82, 1998.

[18] François Brunault : Version explicite du théorème de Beilinson pour la courbe modulaire $X_1(N)$. *C. R. Math. Acad. Sci. Paris*, 343(8):505–510, 2006.

[19] François Brunault : Regulators of Siegel units and applications. *J. Number Theory*, 163:542–569, 2016.

[20] François Brunault : *Value at 2 of the L-function of an elliptic curve*. Theses, Université Paris-Diderot - Paris VII, décembre 2005.

[21] Paul F. Byrd et Morris D. Friedman : *Handbook of elliptic integrals for engineers and scientists*. Die Grundlehren der mathematischen Wissenschaften, Band 67. Springer-Verlag, New York-Heidelberg, 1971. Second edition, revised.

[22] Abhijit Champanerkar, Ilya Kofman et Matilde N. Lalín : Mahler measure and the vol-det conjecture. *J. London Math Soc.*, 99(3):872–900, 2019.

[23] Christopher Deninger : Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbf{Z}^n$-actions. *J. Amer. Math. Soc.*, 10(2):259–281, 1997.

[24] Graham Everest et Thomas Ward : *Heights of polynomials and entropy in algebraic dynamics*. Universitext. Springer-Verlag London, Ltd., London, 1999.

[25] Maxim Kontsevich et Don Zagier : Periods. *In Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.

[26] L. Kronecker : Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten. *J. Reine Angew. Math.*, 53:173–175, 1857.

[27] Nobushige Kurokawa et Hiroyuki Ochiai : Mahler measures via the crystalization. *Comment. Math. Univ. St. Pauli*, 54(2):121–137, 2005.

[28] Matilde N. Lalín : Some examples of Mahler measures as multiple polylogarithms. *J. Number Theory*, 103(1):85–108, 2003.

[29] Matilde N. Lalín : *Some relations of Mahler measure with hyperbolic volumes and special values of L-functions*. ProQuest LLC, Ann Arbor, MI, 2005. Thesis (Ph.D.)–The University of Texas at Austin.

[30] Matilde N. Lalín : On a conjecture by Boyd. *Int. J. Number Theory*, 6(3):705–711, 2010.

[31] Matilde N. Lalín et Tushant Mittal : The Mahler measure for arbitrary tori. *Res. Number Theory*, 4(2):Art. 16, 23, 2018.

[32] Matilde N. Lalín et Mathew D. Rogers : Functional equations for Mahler measures of genus-one curves. *Algebra Number Theory*, 1(1):87–117, 2007.

[33] D. H. Lehmer : Factorization of certain cyclotomic functions. *Ann. of Math. (2)*, 34(3):461–479, 1933.

[34] Vincent Maillot : Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables. *Mém. Soc. Math. Fr. (N.S.)*, (80):vi+129, 2000.

[35] Anton Mellit : Elliptic dilogarithms and parallel lines. *J. Number Theory*, 204:1–24, 2019.

[36] Tracy A. Pierce : The numerical factors of the arithmetic forms $\prod_{i=1}^{n}(1 \pm \alpha_i^m)$. *Ann. of Math. (2)*, 18(2):53–64, 1916.

[37] F. Rodriguez-Villegas : Modular Mahler measures. I. *In Topics in number theory (University Park, PA, 1997)*, volume 467 de *Math. Appl.*, pages 17–48. Kluwer Acad. Publ., Dordrecht, 1999.

[38] F. Rodriguez-Villegas : Identities between Mahler measures. *In Number theory for the millennium, III (Urbana, IL, 2000)*, pages 223–229. A K Peters, Natick, MA, 2002.

[39] Mathew Rogers et Wadim Zudilin : From L-series of elliptic curves to Mahler measures. *Compos. Math.*, 148(2):385–414, 2012.

[40] Mathew Rogers et Wadim Zudilin : On the Mahler measure of $1 + X + 1/X + Y + 1/Y$. *Int. Math. Res. Not. IMRN*, (9):2305–2326, 2014.

[41] C. J. Smyth : Mahler measure and the vol-det conjecture. *Canad. Math. Bull.*, 24(4):447–452, 1981.

[42] C. J. Smyth : On measures of polynomials in several variables. *Bull. Austral. Math. Soc.*, 23(1):49–63, 1981.

[43] Alvin I. Thaler : On the newton polytope. *Proc. Amer. Math. Soc.*, 15(6):944–950, 1964.

[44] Nouressadat Touafek : From the elliptic regulator to exotic relations. *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.*, 16(2):117–125, 2008.

[45] Nouressadat Touafek : Mahler's measure: proof of two conjectured formulae. *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.*, 16(2):127–136, 2008.

[46] Nouressadat Touafek et Mohamed Kerada : Mahler measure and elliptic regulator: some identities. *JP J. Algebra Number Theory Appl.*, 8(2):271–285, 2007.

[47] Sam Vandervelde : A formula for the Mahler measure of $axy + bx + cy + d$. *J. Number Theory*, 100(1):184–202, 2003.

[48] Sam VANDERVELDE : The Mahler measure of parametrizable polynomials. *J. Number Theory*, 128(8): 2231–2250, 2008.

[49] Michel WALDSCHMIDT : *Diophantine approximation on linear algebraic groups*, volume 326 de *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2000. Transcendence properties of the exponential function in several variables.

[50] Edwin WEISS : *Algebraic number theory*. Dover Publications, Inc., Mineola, NY, 1998. Reprint of the 1963 original.

[51] Don ZAGIER : The remarkable dilogarithm. *J. Math. Phys. Sci.*, 22(1):131–145, 1988.

Appendix A

Additional results

A.1. Evaluation of the integral in Lemma 3.3.5

Here we describe an approach to show that the sign of the integral in Lemma 3.3.5 is independent of a if $a \in \left[\frac{1}{a_0}, a_0\right]$, where

$$a_0 = \left[\left(5 - 2\sqrt{2}\right) + \sqrt{\left(5 - 2\sqrt{2}\right)^2 - 1}\right].$$

Moreover, we will show that $c = c' = 1$ as mentioned at the end of the proof of Lemma 3.3.5 in 3.4.2.

Lemma A.1.1. *Let $a \in \mathbb{R}_{>0}$ be such that $\frac{1}{a_0} \leq a \leq a_0$. Then*

$$\int_{\phi_*(|x|=a)} \omega = -\frac{i}{2} K\left(\frac{1}{2}\right),$$

where

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

is the complete Elliptic Integral of the First Kind. Thus if a belongs to the given interval then the sign of the integral is independent of a.

Proof. Recall that we have shown in Lemma 3.3.5 that

$$\int_{\phi_*(|x|=a)} \omega = \int_{|x|=a} \phi^* \omega = -2i \operatorname{Re}\left[\int_0^{\pi} \frac{d\tau}{\sqrt{(8 + ae^{i\tau} + a^{-1}e^{-i\tau})^2 - 4}}\right].$$

We use the change of variable $\frac{(ae^{i\tau} + a^{-1}e^{-i\tau})}{2} \mapsto u$ and the fact that $\operatorname{Re}(z) = \operatorname{Re}(\bar{z})$ to obtain

$$\int_{\phi_*(|x|=a)} \omega = -i \operatorname{Im} \int_{-\frac{a+a^{-1}}{2}}^{\frac{a+a^{-1}}{2}} \frac{du}{\sqrt{(u^2 - 1)(u + 3)(u + 5)}}, \tag{A.1.1}$$

where the integral is over an arc in the complex upper half plane $\mathbb{H}$, joining two real points $-\left(\frac{a+a^{-1}}{2}\right)$ and $\left(\frac{a+a^{-1}}{2}\right)$. We close the curve by connecting these two real points with a straight line along the real line. Let the complete closed path be Γ.

It is easy to see that $1 \leq \frac{a+a^{-1}}{2} < 3$, for a satisfying $a + \frac{1}{a} < 6$ and $a \in \mathbb{R}_{>0}$. Also if S is the set of poles of the denominator of the integrand then $S \subset \{-5, -3, -1, 1\}$. Indeed, notice that the polynomial in u under the square root has four roots u_1, u_2, u_3, and u_4, which satisfy

$$u_1 = 1 > u_2 = -1 > u_3 = -3 > u_4 = -5. \tag{A.1.2}$$

The poles of the integrand are not in the interior of the region covered by Γ but on the real line joining the two points mentioned above. Now note that

$$-3 < -\frac{a+a^{-1}}{2} \leq -1 < 1 \leq \frac{a+a^{-1}}{2} < 3.$$

Therefore, to avoid the poles we modify the integration path by subtracting a semicircle of radius ϵ around each pole. Notice that the integrals over these semicircles approach 0 as $\epsilon \to 0$. In addition, the imaginary parts of the integrals over $\left[-\frac{a+a^{-1}}{2}, -1\right]$ and $\left[1, \frac{a+a^{-1}}{2}\right]$ are zero because the integrand is real in those intervals. Therefore, we get

$$\begin{aligned} \int_{\phi_*(|x|=a)} \omega &= -i \operatorname{Im} \int_{-\frac{a+a^{-1}}{2}}^{\frac{a+a^{-1}}{2}} \frac{du}{\sqrt{(u^2-1)(u+3)(u+5)}} \\ &= -i \operatorname{Im} \int_{-1}^{1} \frac{du}{\sqrt{(u^2-1)(u+3)(u+5)}}. \end{aligned}$$

We will now state a result which we will use to prove our lemma.

Proposition A.1.2 ([**21**], formula 256.00 page 120)**.** *Let $a > b > c > d$ be real numbers and $a \geq r > b$. Then*

$$\begin{aligned} &\int_b^r \frac{dt}{\sqrt{(a-t)(t-b)(t-c)(t-d)}} \\ =&\frac{2}{\sqrt{(a-c)(b-d)}} F\left(\sin^{-1}\left(\sqrt{\frac{(a-c)(r-b)}{(a-b)(r-c)}}\right), \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}\right), \end{aligned}$$

where

$$F(\delta, k) = \int_0^\delta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

Notice that $K(l) = F\left(\frac{\pi}{2}, l\right)$. Now replacing the values of

$$a = r = u_1, b = u_2, c = u_3, d = u_4,$$

(given by (A.1.2)) in Proposition A.1.2, we have

$$\begin{aligned}\int_{\phi_*(|x|=a)} \omega &= -i\frac{2}{\sqrt{(u_1-u_3)(u_2-u_4)}} K\left(\sqrt{\frac{(u_1-u_2)(u_3-u_4)}{(u_1-u_3)(u_2-u_4)}}\right) \\ &= -\frac{i}{2}K\left(\frac{1}{2}\right),\end{aligned}$$

which proves our result. □

Therefore, for $a \in \left[\frac{1}{a_0}, a_0\right]$ we have that

$$\int_{\phi_*(|x|=a)} \omega = -\frac{i}{2}K\left(\frac{1}{2}\right),$$

which implies that $[\phi_*(|x| = a)] = [\phi_*(|x| = 1)] = [|X| = 1]$ is a generator of $H_1(E_{N(8)}, \mathbb{Z})^-$. A similar calculation holds for the more general case $k > 4$ when $a + \frac{1}{a} < k - 2$.

A.2. Abel's Limit Theorem

Theorem A.2.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series such that $\sum_{n=0}^{\infty} a_n$ converges. Then for any $K \geq 1$, $f(z)$ tends to $f(1)$ as z tends to 1 within*

$$D_K = \{z \in \mathbb{C} : |z| < 1 \text{ and } |1 - z| \leq K(1 - |z|)\}.$$

Note that the fact $\sum_{n=0}^{\infty} a_n$ converges implies that the radius of convergence of $f(z)$ is at least 1. In particular, $\sum_{n=0}^{\infty} a_n z^n$ converges when $|z| < 1$.

A.3. Integral representation of $F(a, b, c; x)$

Recall that

$$F(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n \quad (\text{where } (t)_n := t(t+1)\cdots(t+n-1), (0)_n = 1),$$

where x is a complex variable, and $a, b, c \in \mathbb{C}$ with $c \notin \mathbb{Z}_{\leq 0}$. In fact, the series converges absolutely for all $|x| < 1$. We will now prove Theorem 0.1.3 in order to obtain an integral reperesentation of $F(a, b, c; x)$ given by (0.1.6).

Theorem 0.1.3. *If* $|x| < 1$, $a, b, c \in \mathbb{C}^*$ *with* $c \notin \mathbb{Z}_{\leq 0}$ *and* $\min\{\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c-a)\} > 0$, *then we can express* $F(a, b, c; x)$ *as*

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 y^{a-1}(1-y)^{c-a-1}(1-xy)^{-b}dy, \tag{A.3.1}$$

where $\Gamma(\cdot)$ *denotes the Gamma function. Here it is understood that* $\arg y = \arg(1-y) = 0$, *and* $(1-xy)^{-b}$ *has its principal value.*

Proof. In order to obtain (A.3.1) from the definition of $F(a, b, c; x)$, we first notice that

$$\frac{(a)_n}{(c)_n} = \frac{\Gamma(c)\Gamma(a+n)}{\Gamma(a)\Gamma(c+n)} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 y^{a+n-1}(1-y)^{c-a-1}dy,$$

when $a, c \in \mathbb{C}^*$ with $c \notin \mathbb{Z}_{\leq 0}, \mathrm{Re}(a) > 0$ and $\mathrm{Re}(c-a) > 0$. We also have the binomial expansion

$$\sum_{n=0}^{\infty} \frac{(b)_n}{n!} u^n = (1-u)^{-b},$$

where $|u| < 1$ and $b \in \mathbb{C}$ with $\mathrm{Re}(b) > 0$. Therefore, if $|x| < 1$ then combining all of these we obtain

$$\begin{aligned} F(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{n=0}^{\infty} \int_0^1 \frac{(b)_n}{n!} x^n y^n y^{a-1}(1-y)^{c-a-1}dy \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 y^{a-1}(1-y)^{c-a-1} \left\{ \sum_{n=0}^{\infty} \frac{(b)_n}{n!} y^n x^n \right\} dy \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 y^{a-1}(1-y)^{c-a-1}(1-xy)^{-b}dy, \end{aligned}$$

where the interchange of the sum and the integral in the first step follows from *Fubini's Theorem*, i.e. we can take the infinite sum inside the integral because, for $|x|, |y| < 1$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \int_0^1 \left| \frac{(b)_n}{n!} y^{a-1}(1-y)^{c-a-1} y^n x^n \right| dy &\leq \sum_{n=0}^{\infty} \left| \frac{(b)_n}{n!} x^n \right| \int_0^1 y^{\mathrm{Re}(a)-1}(1-y)^{\mathrm{Re}(c-a)-1}dy \\ &= \mathrm{B}\left(\mathrm{Re}(a), \mathrm{Re}(c-a)\right) \sum_{n=0}^{\infty} \left| \frac{(b)_n}{n!} x^n \right| < \infty, \end{aligned}$$

where we used the fact that $\min\{\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c-a)\} > 0$. □

This Euler's integral representation of the hypergeometric function is convergent on the given domain. Here, we assume $|x| < 1$ to avoid the singularity of $(1-xy)^{-b}$. Note that if

$|x| < 1, a, b, c \in \mathbb{C}^*$ with $c \notin \mathbb{Z}_{\leq 0}$ and $\min\{\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c-a), \mathrm{Re}(c-b)\} > 0$, then a derivation similar to (A.3.1) yields

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 y^{b-1}(1-y)^{c-b-1}(1-xy)^{-a} dy,$$

where again it is understood that $\arg y = \arg(1-y) = 0$, and $(1-xy)^{-a}$ has its principal value.

A.4. Mahler measure of an algebraic number

A.4.1. Product formula

Let K be a number field and let $\mathbb{Q}_p$ be the field of p-adic numbers with the usual valuation $|\cdot|_p$. We consider a complete set of inequivalent valuations $|\cdot|_\nu$ of the field K, normalized so that for $\nu|p$, $|\cdot|_\nu = |\cdot|_p$ on $\mathbb{Q}_p$. Then we have the following result.

Proposition A.4.1 (Product Formula, see e.g. [**50**], [**13**]). *Let K_ν be the completion of K with respect to $|\cdot|_\nu$. It is possible to choose a set M_K of representatives of equivalence classes of absolute values on K in such a way that for all $a \in K^\times$*

$$\prod_{\nu \in M_K} |a|_\nu^{d_\nu} = 1,$$

where $d_\nu := [K_\nu : Q_\nu]$.

In the next section we use this proposition to prove that the (logarithmic)Mahler measure of an algebraic number equals the product of the degree of its minimal polynomial and its Weil height from Section 0.1.

A.4.2. Mahler measure and Weil Height

Recall that the Mahler measure of algebraic number is defined as the Mahler measure of its minimal polynomial over $\mathbb{Z}$. The Mahler measure is actually a height function on polynomials with integer coefficients, as there are only a finite number of such polynomials of bounded degree and bounded Mahler measure. In fact, we can relate the Mahler measure of an algebraic number with its *Weil* height. In order to do so we need to consider the Newton polytope of its minimal polynomial with respect to valuations of the smallest algebraic extension of $\mathbb{Q}$ containing the algebraic number.

Let $\mathbb{Q}_\nu$ be the completion of $\mathbb{Q}$ with respect to $|\cdot|_\nu$. Given a polynomial $P \in \mathbb{Q}_\nu[x]$ we can associate a polygon in $\mathbb{R}^2$. To each term of $P(x) = \sum_{i=0}^m b_i x^i$ we assign a point in $\mathbb{R}^2$ in the following manner:

$$\begin{aligned}&\text{if } b_i x^i \neq 0, \text{ take the point } (i, \nu(b_i)),\\ &\text{if } b_i x^i = 0, \text{ take the non-existent point } (i, \infty) = (i, \nu(b_i)).\end{aligned}$$

Now, we consider the (lower) convex hull of the set of points

$$\{(i, \nu(b_i)) : i = 0, 1, \ldots, n\}.$$

The polygon thus determined is called the **Newton polytope of** $P(x)$ **with respect to** ν. Note that the Newton Polygon defined in Section 1.1.3 differs from this definition. In fact, we have considered the exponent polytope (see [**41**]) of a two-variable polynomial in Section 1.1.3 instead of the Newton polytope of the polynomial. In other words, we have considered the convex polygon of a two-variable polynomial constructed by taking the projection of its Newton polytope on its first two coordinates (for more details on Newton polytopes see [**43**]).

Let ϑ be an algebraic number and let $P_\vartheta(x) = \sum_{i=0}^d a_i x^i \in \mathbb{Z}[x]$ be the minimal polynomial of ϑ. We can consider a complete set of inequivalent valuations $|\cdot|_\nu$ of the field $K := \mathbb{Q}(\vartheta)$, normalized so that, for $\nu|p$, $|\cdot|_\nu = |\cdot|_p$ on $\mathbb{Q}_p$ as in A.4.1. Then we get

$$|a_0| = \prod_{p<\infty} |a_0|_p^{-1} = \prod_{p<\infty} \prod_{\nu|p} \max(1, |\vartheta|_\nu^{d_\nu}), \tag{A.4.1}$$

which can be derived from the product formula on $\mathbb{Q}$ (see Proposition A.4.1), and from considering the Newton polytopes of irreducible factors (of degree d_ν) of P_ϑ with respect to ν over $\mathbb{Q}_p$ (as $\nu|p$) (see e.g. [**50**]).

Let M_K be the set of places on K, with representatives chosen in such a way that the product formula holds (see Proposition A.4.1). Then from (0.1.3) and (A.4.1) (see [**13**], [**49**]) we have

$$M(\vartheta) := M(P_\vartheta) = \prod_{\nu \in M_K} \max(1, |\vartheta|_\nu^{d_\nu}),$$

and

$$h(\vartheta) := \frac{\log M(\vartheta)}{d} = \sum_{\nu \in M_K} \log^+ |\vartheta|_\nu^{\frac{d_\nu}{d}},$$

where

$$\log^+ |\rho| = \int_0^1 \log |e^{2i\pi t} - \rho| dt = \max(0, \log |\rho|),$$

for some non-zero complex number ρ. Here $h(\vartheta)$ is the (absolute logarithmic) *Weil* height of ϑ. In sum, we see that the (logarithmic) Mahler measure of an algebraic number is same as the product of its (logarithmic) *Weil* height and the degree of its minimal polynomial, and the Mahler measure is indeed a height function on polynomials.