Two-Stage Majoritarian Choice

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Abstract

We propose a class of decisive collective choice rules that rely on an exogenous linear ordering to partition the majority relation into two acyclic relations. The first relation is used to obtain a shortlist of the feasible alternatives while the second is used to make a final choice.

In combination with *faithfulness* to the underlying majority relation, rules in this class are characterized by two desirable rationality properties: Sen's *expansion consistency* and a version of Manzini and Mariotti's *weak WARP*. The rules also satisfy natural adaptations of Arrow's *independence of irrelevant alternatives* and May's *positive responsiveness*.

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1 Introduction

In many collective choice settings, rules that recommend more than one alternative are inappropriate. When it comes to selecting a political leader or a public policy, for instance, it is essential to be decisive. May (1952) shows that majority voting is the only reasonable way to decide between two alternatives.[1] With more alternatives, no rule that is faithful to the majority opinion can choose rationally. The root of the problem is the well-known Condorcet (1785) paradox: the majority relation may involve cycles. Arrow (1951) shows that the problem persists even with rules that are not majoritarian: barring dictatorship, there is no way to make rational and Pareto-efficient choices that satisfy the *independence of irrelevant alternatives* (IIA). We take Arrow's result as good reason not to give up on majority voting. The goal, as we see it, is to design collective choice rules that are decisive, faithful to the majority view, and as rational as possible.

We propose a class of rules that not only meet these objectives but also exhibit a number of other desirable features—including adaptations of Arrow's *IIA* and May's *positive responsiveness*. Not least among the virtues of these rules is their simplicity. Each uses a linear ordering to partition the majority relation into two acyclic relations. Then, as in Manzini and Mariotti's (2007) *rational shortlist methods*, the first relation is used to pare down the set of feasible alternatives before the second is used to make a final choice. While the linear orderings used by our rules are exogenous in principle, many choice settings suggest a natural way to order the alternatives.

2 The problem

Given a finite universe of social alternatives X, let $\mathcal{X} = 2^X \setminus \{\emptyset\}$ denote the set of *agendas* and $\mathcal{T}$ the set of *tournaments* on X.[2] We interpret each tournament $T \in \mathcal{T}$ to be the majority relation induced by an underlying profile of agent preferences over X (McGarvey, 1953). Given a tournament T and an agenda A, the problem is to recommend one alternative in A.

Our object of interest is a *choice rule*, that is, a mapping $f : \mathcal{T} \times \mathcal{X} \to X$ such that $f(T; A) \in A$ for each $T \in \mathcal{T}$ and $A \in \mathcal{X}$. For each tournament $T \in \mathcal{T}$, $f(T; \cdot) : \mathcal{X} \to X$ defines a *choice function*.

We require that our choice rules be faithful to each tournament $T \in \mathcal{T}$:

Faithfulness. *For all* $T \in \mathcal{T}$ *and* $a, b \in X, aTb$ *implies* $f(T; \{a, b\}) = a$.

To put it differently, we require binary choices to be consistent with majority rule.

Given a binary relation R on X, let $\max(R; A) = \{a \in A \mid \nexists b \in A : bRa\}$ denote the set of maximal elements of R in A. Let $\mathcal{P}$ denote the set of linear orderings on X.[3] A choice function $f(T; \cdot)$ is *rational* if there is some linear ordering $P \in \mathcal{P}$ such that $\{f(T; A)\} = \max(P; A)$ for all

[1] In the sequel, we assume that the majority relation is decisive. This assumption is fairly innocuous for large electorates; and it is automatically satisfied when voter preferences are strict and the number of voters is odd.

[2] A tournament T is an *asymmetric* ($\nexists a, b : aTb$ and bTa) and *total* ($\forall a, b : aTb, bTa$, or $a = b$) binary relation.

[3] A linear ordering P is an asymmetric, total and *transitive* ($\forall a, b, c : aPbPc \Rightarrow aPc$) binary relation.

agendas $A \in \mathcal{X}$. If f is faithful, then $f(T;\cdot)$ cannot be rational unless T is a linear ordering. The question is whether there are faithful choice rules for which the choice function $f(T;\cdot)$ is rational when the tournament T is a linear ordering and not *too* irrational otherwise.

Some of the simplest faithful choice rules rely on an exogenous linear ordering $P \in \mathcal{P}$. The idea is to give an edge to alternatives that are ranked higher by P and thus guarantee a single-valued choice when the alternatives are difficult to distinguish (as they are in a Condorcet cycle).

One natural approach uses P as a tie-breaking device to make a selection from a *Condorcet-consistent choice correspondence*, that is, a mapping $F : \mathcal{T} \times \mathcal{X} \to \mathcal{X}$ such that, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$: (i) $F(T;A) \subseteq A$; and (ii) $F(T;A) = \{a\}$ if aTb for all $b \in A \setminus \{a\}$.[4] Formally, the choice rule F_P generated by the choice correspondence F and the tie-breaking device $P \in \mathcal{P}$ is defined, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, by $\{F_P(T;A)\} = \max(P; F(T;A))$.

Another approach uses P to define a succession of binary elimination votes that, in turn, determine the choice from each agenda. For any agenda $A = \{a_1, ..., a_m\} \in \mathcal{X}$, label the alternatives so that $a_1 P ... P a_m$. Then, define $w_0(T;A) = a_m$ and, for $k = 1, ..., m-1$, recursively define

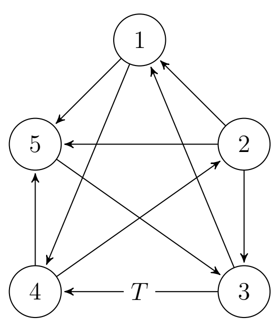
$$w_k(T;A) = \begin{cases} w_{k-1}(T;A) & \text{if } w_{k-1}(T;A) T a_{m-k}, \\ a_{m-k} & \text{otherwise.} \end{cases}$$

The first vote eliminates a_m or a_{m-1}. At any subsequent vote, the winner $w_{k-1}(T;A)$ from the previous vote is put up against the next alternative a_{m-k} from the list. The *successive elimination* rule s_P induced by $P \in \mathcal{P}$ is defined, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, by $s_P(T;A) = w_{m-1}(T;A)$.

Both of these approaches lead to choice rules that are lacking in basic features of rationality:

Example 1 (Selection from the uncovered set). *The* uncovered set *choice correspondence $UC : \mathcal{T} \times \mathcal{X} \to \mathcal{X}$ (Landau, 1951; Fishburn, 1977; Miller, 1977) is defined, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, by $UC(T;A) = \{a \in A \mid \forall b \in A \setminus \{a\} : \text{(i) } aTb \text{ or (ii) } aTcTb \text{ for some } c \in A\}$.*

Clearly, UC is Condorcet-consistent. For $X = \{1,2,3,4,5\}$, consider the tournament T below:



For the linear ordering $P = 1, ..., 5$ (with the alternatives listed in decreasing order of P, i.e.,

[4] In Fishburn (1977), Condorcet-consistent choice correspondences are called C1 social choice functions.

$1P2P3P4P5$), it follows that:

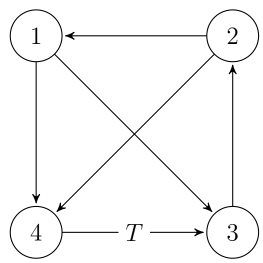
$$UC_P(T;\{1,2,3,4\}) = 2 = UC_P(T;\{2,5\}) \text{ but } UC_P(T;\{1,2,3,4,5\}) = 1.$$

Thus, alternative 2 is chosen from $\{1,2,3,4\}$ and $\{2,5\}$ but not their union.[5,6] *Moreover,*

$$UC_P(T;\{1,2\}) = 2 = UC_P(T;\{1,2,3,4\}) \text{ but } UC_P(T;\{1,2,4\}) = 1.$$

So, 2 is chosen over 1 from $\{1,2\}$ and $\{1,2,3,4\}$ but not the intermediate agenda $\{1,2,4\}$.[7]

Example 2 (Successive elimination). *For $X = \{1,2,3,4\}$, consider the tournament T below:*



For the successive elimination procedure induced by the linear ordering $P = 1,...,4$:

$$s_P(T;\{1,4\}) = s_P(T;\{1,2,3\}) = 1 \text{ but } s_P(T;\{1,2,3,4\}) = 2.$$

So, 1 is chosen from the agendas $\{1,4\}$ and $\{1,2,3\}$ but not their union. Moreover,

$$s_P(T;\{1,2\}) = s_P(T;\{1,2,3,4\}) = 2 \text{ but } s_P(T;\{1,2,3\}) = 1.$$

Thus, 2 is chosen over 1 from $\{1,2\}$ and $\{1,2,3,4\}$ but not the intermediate agenda $\{1,2,3\}$.[8]

The rationality properties violated by the choice rules from Examples 1 and 2 are the following:

Expansion Consistency. *For all $T \in \mathcal{T}$ and $A, B \in \mathcal{X}$:*

$$f(T;A) = f(T;B) \text{ implies } f(T;A \cup B) = f(T;A) = f(T;B).$$

[5] This is true even though the uncovered set correspondence UC satisfies $UC(T;A) \cap UC(T;B) \subseteq UC(T;A \cup B)$ for all $A, B \in \mathcal{X}$. In fact, Moulin (1986) offers a characterization of UC based on that property.

[6] The same choice pattern can also arise if we start with the *top cycle* correspondence TC (as defined in Section 4 below). If we modify T so that $4T'1$, $TC_P(T';\{1,2,3,4\}) = 2 = TC_P(T';\{2,5\})$ but $TC_P(T';\{1,2,3,4,5\}) = 1$.

[7] To see that this choice pattern cannot arise if we start with TC, suppose $TC_P(T;A) = a = TC_P(T;\{a,b\})$ and $TC_P(T;B) = b$ for $\{a,b\} \subseteq B \subseteq A$. Since $TC_P(T;\{a,b\}) = a$ and $TC_P(T;B) = b$, bPa. Since $a \in TC(T;A)$ and $b = c_1T...Tc_n = a$ for some $c_1,...,c_n \in B$, $b \in TC(T;A)$. Since bPa, this contradicts $TC_P(T;A) = a$.

[8] The same choice patterns arise under the *amendment procedure* a_P (Miller, p. 779; Moulin, 1986, p. 287). Following our convention (that higher-ranked alternatives in P are more privileged), the linear ordering $P = 1,2,3,4$ corresponds to the tree $\Gamma_4(4,3,2,1)$ in Moulin. For the tournament T given in Example 2, the corresponding choice function gives $a_P(T;A) = s_P(T;A)$ for all $A \in \mathcal{X}$.

Weak WARP. *For all $T \in \mathcal{T}$, distinct $a, b \in X$, and $A, B \in \mathcal{X}$ such that $\{a, b\} \subseteq B \subseteq A$:*

$$f(T; \{a, b\}) = a = f(T; A) \text{ implies } f(T; B) \neq b.$$

Expansion Consistency dates back to Sen (1971). Weak WARP was introduced by Manzini and Mariotti (2007). Both properties weaken Samuelson's (1938) *weak axiom of revealed preference* (WARP), which requires $f(T; B) = a$ if $f(T; A) = a$ and $a \in B \subseteq A$. Since WARP characterizes rational choice in our setting, it is incompatible with the requirement that f is faithful to T.

3 Two-stage majoritarian rules

We propose a class of choice rules that satisfy Faithfulness, Expansion Consistency and Weak WARP. Like the rules from Examples 1 and 2, each relies on an exogenous linear ordering $P \in \mathcal{P}$. The function of P is to partition the given tournament $T \in \mathcal{T}$ into two acyclic binary relations $T \cap P$ and $T \setminus P$. The first of these relations is used to obtain a preliminary shortlist of the feasible alternatives in $A \in \mathcal{X}$; and the second is used to make a final choice. Formally, the *two-stage majoritarian choice rule f_P based on $P \in \mathcal{P}$* is defined, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, by

$$\{f_P(T; A)\} = \max(T \setminus P; \max(T \cap P; A)). \tag{1}$$

For each tournament $T \in \mathcal{T}$, the choice function $f_P(T; \cdot)$ defines a *rational shortlist method* in the sense of Manzini and Mariotti (2007). Formally, a choice function $c : \mathcal{X} \to X$ is a rational shortlist method if there is a pair of asymmetric binary relations (P_1, P_2) (called *rationales*) on X such that $\{c(A)\} = \max(P_2; \max(P_1; A))$ for all $A \in \mathcal{X}$. To ensure that choice is single-valued, this model imposes non-trivial restrictions on the rationales (Lemma 2 of Dutta and Horan, 2015). When the rationales are built by splitting the tournament T into acyclic relations using a linear ordering P, these restrictions are satisfied *regardless* of T or P.

To see this, fix an agenda $A \in \mathcal{X}$. Since the binary relation $T \cap P$ is acyclic, the shortlist $M_A = \max(T \cap P; A)$ must be nonempty. The single-valuedness of $\max(T \setminus P; M_A)$ then follows from the acyclicity and totality of the binary relation $T \setminus P$ on M_A.

This argument holds as long as $T \cap P$ and $T \setminus P$ are acyclic. Since P is also total, more can be said. Letting $P^{-1} = \{(a, b) \in X^2 \mid (b, a) \in P\}$ denote the reverse ordering of P, $\{f_P(T; A)\} = \max(T \setminus P; M_A) = \max(T \cap P^{-1}; M_A) = \max(P^{-1}; \max(T \cap P; A))$. If we interpret aPb as a being "higher-ranked" than b, then the last formula states that $f_P(T; \cdot)$ chooses the lowest-ranked alternative that defeats all higher-ranked alternatives by majority. To illustrate:

Example 3 (Two-stage majoritarian rules). *For $P = 1, ..., 4$, the tournament T from Example 2 gives rationales $P_1 = T \cap P = \{(1, 3), (1, 4), (2, 4)\}$ and $P_2 = T \setminus P = \{(2, 1), (3, 2), (4, 3)\}$.*

To illustrate the resulting two-stage majoritarian rule f_P, first consider the Condorcet cycle

$A = \{1,2,3\}$. *Since* $1P_1 3$, *alternative* 3 *is eliminated in the first stage, leaving the shortlist* $\{1,2\}$. *Since* $2P_2 1$, *alternative* 1 *is eliminated in the second stage, giving the final choice* $f_P(T;A) = 2$.

Letting $f_P^{-1}(T;x) = \{A \in \mathcal{X} \mid f(T;A) = x\}$, *the same kind of reasoning establishes that:*

$$\begin{aligned} f_P^{-1}(T;1) &= \{\{1,3\},\{1,4\}\}, \\ f_P^{-1}(T;2) &= \{\{2,1\},\{2,4\},\{1,2,3\},\{1,2,4\},\{1,2,3,4\}\}, \\ f_P^{-1}(T;3) &= \{\{2,3\},\{2,3,4\}\}, \text{ and} \\ f_P^{-1}(T;4) &= \{\{3,4\}\}. \end{aligned}$$

By definition, any two-stage majoritarian rule f_P satisfies Faithfulness. Since the choice function $f_P(T;\cdot)$ is a rational shortlist method for each $T \in \mathcal{T}$, Manzini and Mariotti's characterization implies that f_P also satisfies Expansion Consistency and Weak WARP.

These same properties are satisfied by other choice rules (like the rule in Example 4 below). What distinguishes two-stage majoritarian rules is a strong version of Weak WARP that applies *across* tournaments. This property relies on Dutta and Horan's taxonomy of WARP violations that are consistent with Weak WARP. Formally, a choice function $c : \mathcal{X} \to X$ exhibits an $\langle a,b\rangle$ *reversal* for $a,b \in X$ if $c(B) = b$ and $c(A) = a$ for some $A, B \in \mathcal{X}$ such that $\{a,b\} \subseteq B \subseteq A$. In turn, c exhibits an $\langle a,b\rangle$ *switch* if $c(B) = b$ and $c(B \cup \{a\}) \notin \{a,b\}$ for some $B \in \mathcal{X}$.

According to WARP, the addition of alternatives can never cause the decision maker to "reverse" her choice to a previously unchosen alternative a. Likewise, the addition of a new alternative a cannot lead her to "switch" from b to a third alternative. While the combination of Expansion Consistency and Weak WARP permits these patterns of choice, it does so only in one direction. For a given tournament T, $\langle a,b\rangle$ reversals preclude $\langle b,a\rangle$ reversals; and $\langle a,b\rangle$ switches preclude $\langle b,a\rangle$ switches.[9] The next property extends these requirements across tournaments:

Inter-Tournament Weak WARP. *For all* $T, T' \in \mathcal{T}$ *and* $a,b \in X$*:*

(i) If $f(T;\cdot)$ *exhibits an* $\langle a,b\rangle$ *reversal, then* $f(T';\cdot)$ *exhibits no* $\langle b,a\rangle$ *reversals; and,*

(ii) If $f(T;\cdot)$ *exhibits an* $\langle a,b\rangle$ *switch, then* $f(T';\cdot)$ *exhibits no* $\langle b,a\rangle$ *switches.*

For any tournament $T \in \mathcal{T}$, requirement (i) implies that $f(T;\cdot)$ cannot exhibit both $\langle a,b\rangle$ and $\langle b,a\rangle$ reversals for any pair of alternatives $a,b \in X$. Cherepanov et al. (2013, Proposition A.5) show that this property (which they call *Irreversibility*) is equivalent to Weak WARP.[10]

In combination with Faithfulness and Expansion Consistency, Inter-Tournament Weak WARP characterizes two-stage majoritarian rules. To state our result formally:

Theorem. *A choice rule* $f : \mathcal{T} \times \mathcal{X} \to X$ *is a two-stage majoritarian choice rule if and only if it satisfies Faithfulness, Expansion Consistency and Inter-Tournament Weak WARP.*

[9] Horan (2016) uses similar "one-way" properties to characterize some special classes of rational shortlist methods.

[10] When it is restricted to a fixed tournament $T \in \mathcal{T}$, requirement (ii) is implied by Expansion Consistency.

Proof. Since the necessity of the axioms is straightforward, we only establish sufficiency. The result is immediate if $|X| \leq 2$. So, suppose $|X| \geq 3$ without loss of generality.

Following Dutta and Horan, first define binary relations P_2^T and P_1^T on X such that, for all $a, b \in X$: $bP_2^T a$ if $f(T; \cdot)$ exhibits an $\langle a, b\rangle$ reversal; and $aP_1^T b$ if $f(T; \cdot)$ exhibits an $\langle a, b\rangle$ switch. Since f satisfies Faithfulness, Expansion Consistency, and Weak WARP (by the observation in the main text), $P_1^T \subseteq T$ and $P_2^T \subseteq T$ (by Proposition 1 of Dutta and Horan).

Step 1. *Given $aTbTcTa$ and $aT'cT'bT'a$, $f(T; \{a, b, c\}) = c$ implies $f(T'; \{a, b, c\}) = a$.*

Since f is faithful, $f(T; \{a, b, c\}) = c$ implies $bP_2^T c$ and $aP_1^T b$. By part (i) of Inter-Tournament Weak WARP, $bP_2^T c$ rules out $f(T'; \{a, b, c\}) = b$. Otherwise, $cP_2^{T'} b$. Similarly, by part (ii) of Inter-Tournament Weak WARP, $aP_1^T b$ rules out $f(T'; \{a, b, c\}) = c$. Otherwise, $bP_1^{T'} a$. □

Next, define the binary relation R on X such that, for all $a, b \in X$: aRb if there is some $T' \in \mathcal{T}$ and $c \in X$ such that $aT'cT'bT'a$ and $f(T'; \{a, b, c\}) = a$. We write aIb if neither aRb nor bRa.

The following observations will be useful: by Step 1, (1) aRb implies that there are $T, T' \in \mathcal{T}$ such that $bP_2^{T'} a$ and $aP_1^T b$; and so, by Inter-Tournament Weak WARP, (2) R is asymmetric.

Step 2. *R is transitive.*

Suppose $xRyRz$. Consider $T \in \mathcal{T}$ such that $xTyTzTx$. If $f(T; \{x, y, z\}) \neq x$, yRx or zRy. Since $xRyRz$, this contradicts the asymmetry of R. So, $f(T; \{x, y, z\}) = x$ and xRz. □

Step 3. *There are exactly two distinct $a, b \in X$ such that aIb; and a, bRc for all $c \in X \setminus \{a, b\}$.*

First, suppose there are two pairs $\{x, y\}$ and $\{z, w\}$ (with $x \neq z, w$ and $z \neq x, y$) such that xIy and zIw. Consider $T, T' \in \mathcal{T}$ such that $xTzTwTx$ and $xT'wT'zT'x$. Since zIw, Step 1 implies z, wRx. Next, consider $T, T' \in \mathcal{T}$ such that $xTyTzTx$ and $xT'zT'yT'x$. Then, x, yRz since xIy. But, xRz and zRx contradicts the asymmetry of R. So, aIb for at most one pair $\{a, b\}$.

If there is no such pair, then R is a linear ordering (since it is asymmetric by observation (2) after Step 1 and transitive by Step 2). Suppose aRb and bRc for all $c \in X \setminus \{a, b\}$. By Step 1, there is some $d \in X \setminus \{a, b\}$ such that dRb, which is a contradiction. So, there is exactly one pair $\{a, b\}$ such that aIb. By Step 1, it follows that a, bRc for all $c \in X \setminus \{a, b\}$. □

Step 4. *For the pair $\{a, b\}$ identified in Step 3 and any $T \in \mathcal{T}$, neither $aP_1^T b$ nor $aP_2^T b$.*

If $aP_1^T b$, then $f(T; B) = b$ and $f(T; B \cup \{a\}) = x \notin \{a, b\}$ for some $B \in \mathcal{X}$. So, $xP_2^T b$. By Inter-Tournament Weak WARP and observation (1), this contradicts bRx.

If $aP_2^T b$, then $f(T; A) = a$ and $f(T; B) = b$ for some $A, B \in \mathcal{X}$ such that $\{a, b\} \subseteq A \subseteq B$. By Expansion Consistency, $f(T; A') = a$ where $A' = A \cup \{x \in B \setminus A : aTx\}$. Then, $f(T; A' \cup \{x\}) \notin \{x, a\}$ for some $x \in B \setminus A'$. Otherwise, $f(T; B) \in (B \setminus A') \cup \{a\} \neq b$ by Expansion Consistency. So, $xP_1^T a$. By Inter-Tournament Weak WARP and observation (1), this contradicts aRx. □

Given Steps 2-3, complete R into a linear order P by defining aPb and xPy if xRy for $x, y \in X$.

Step 5. $P_1^T \subseteq T \cap P$ *and* $P_2^T \subseteq T \setminus P$.

First suppose $xP_1^T y$. By Step 4, $\{x, y\} \neq \{a, b\}$. By way of contradiction, suppose $x \; T \setminus P \; y$. Since $\{x, y\} \neq \{a, b\}$, yRx. By observation (1), $yP_1^{T'}x$ for some $T' \in \mathcal{T}$. Since $xP_1^T y$, this contradicts Inter-Tournament Weak WARP. Next suppose $xP_2^T y$. By the same kind of reasoning: if $xT \cap Py$, then $yP_2^{T'}x$ for some $T' \in \mathcal{T}$, which contradicts Inter-Tournament Weak WARP. □

Since f satisfies Expansion Consistency and Weak WARP, the characterization of Manzini and Mariotti implies that $f(T; \cdot)$ is a rational shortlist method for each $T \in \mathcal{T}$. Given Step 5, Proposition 2 of Dutta and Horan then implies that $(T \cap P, T \setminus P)$ is a minimal representation of $f(T; \cdot)$ for each $T \in \mathcal{T}$.[11] Since P is a linear ordering, this completes the proof. ■

4 Further remarks

4.1 Flexibility and Pareto sub-optimality

The *top cycle* correspondence $TC : \mathcal{T} \times \mathcal{X} \to \mathcal{X}$ (Good, 1971; Schwartz, 1972; Smith, 1973) is defined by $TC(T; A) = \{a \in A \mid \forall b \in A \setminus \{a\} : a = c_1 T ... T c_n = b \text{ for some } c_1, ..., c_n \in A\}$.

For all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, the set of alternatives chosen by some two-stage majoritarian rule coincides with the top cycle, that is, $TC(T; A) = \{f_P(T; A) : P \in \mathcal{P}\}$. In one direction, note that $f_P(T; A) = f_P(T; TC(T; A))$. In the other, fix a path $a = a_1 T ... T a_m$ from $a \in TC(T; A)$ that covers $TC(T; A)$. Then, $f_P(T; A) = a$ for any linear ordering $P \in \mathcal{P}$ such that $a_m P ... P a_1$. Since $TC(T; A) = \{s_P(T; A) : P \in \mathcal{P}\}$ as well (Miller, 1977), this means that two-stage majoritarian rules provide the same *flexibility* to the designer as successive elimination rules (Example 2).

It is well known that $TC(T; A)$ may contain alternatives that are Pareto dominated at preference profiles consistent with T (provided that $|A| \geq 4$).[12] Given their flexibility, this means that all two-stage majoritarian rules make Pareto sub-optimal choices for some $T \in \mathcal{T}$ and $A \in \mathcal{X}$.

4.2 Connection to Arrow and May

Two-stage majoritarian rules satisfy natural adaptations of Arrow's (1951) *independence of irrelevant alternatives* (IIA) and May's (1953) *positive responsiveness*.[13] To state the first property, let $T|_A$ denote the restriction of the tournament $T \in \mathcal{T}$ to the agenda $A \in \mathcal{X}$.

Choice IIA. *For all* $T, T' \in \mathcal{T}$ *and* $A \in \mathcal{X}$: $T|_A = T'|_A$ *implies* $f(T; A) = f(T'; A)$.

In other words, the majority view of infeasible alternatives cannot affect choice. Clearly, two-stage majoritarian rules satisfy this property. Indeed, so do the rules from Examples 1 and 2.

[11] Dutta and Horan call a representation *minimal* if the two rationales neither overlap nor contradict one another.
[12] For an example, see Moulin (1986, p. 274).
[13] Our adaptation of these properties to the setting of choice rules follows Moulin (1986, pp. 278 and 285).

For the second property, say that a binary relation R' on X *improves* an alternative $a \in X$ relative to a binary relation R on X if, for all $x, y \in X \setminus \{a\}$: (i) $aRx \Rightarrow aR'x$; and (ii) $xRy \Leftrightarrow xR'y$.

T-Monotonicity. *For all $T, T' \in \mathcal{T}$ and $A \in \mathcal{X}$ where T' improves $a \in X$ relative to T:*

$$f(T; A) = a \text{ implies } f(T'; A) = a.$$

In words: improving the majority view of a chosen alternative can only reinforce its choice.

To see that two-stage majoritarian rules satisfy this property, recall that $f_P(T; A)$ is the lowest-ranked alternative in A that defeats all higher-ranked alternatives by majority. Improving $f_P(T; A)$ relative to T cannot change this: $f_P(T; A)$ still defeats all higher-ranked alternatives; and every alternative ranked below $f_P(T; A)$ is still defeated by a higher-ranked alternative.

It is known that the rules from Example 2 and footnote 8 also satisfy T-Monotonicity.[14] In contrast, the rules from Example 1 do not. To illustrate, consider $X = \{1, 2, 3, 4\}$ and the ordering $P = 4, 3, 2, 1$. Then, $UC_P(T; X) = 3$ for the tournament T from Example 2 while $UC_P(T'; X) = 4$ for the tournament T' that improves 3 relative to 1.

4.3 Regarding the linear ordering P

Every two-stage majoritarian rule f_P is monotonic with respect to the linear ordering P that defines it. In other words, f_P satisfies the following property:

P-Monotonicity. *For all $T \in \mathcal{T}$, $A \in \mathcal{X}$, and $P, P' \in \mathcal{P}$ where P' improves $a \in X$ relative to P:*

$$f_P(T; A) = a \text{ implies } f_{P'}(T; A) = a.$$

This property justifies the interpretation that alternatives ranked higher by P are privileged. A variation on the argument used to establish T-Monotonicity shows that two-stage majoritarian rules satisfy P-Monotonicity. We note that the rules from Examples 1 and 2 (as well as the related rules discussed in footnotes 6 and 8) satisfy a corresponding property.[15]

The linear ordering P plays a less intrusive role in two-stage majoritarian rules than it does in these other rules. To elaborate, consider the successive elimination rules from Example 2. The key insight is that the choice $s_P(T; A)$ must defeat all higher-ranked alternatives in the agenda A. When $s_P(T; A)$ and $f_P(T; A)$ differ, this means that $f_P(T; A)$ must be ranked lower in terms of P and, consequently, must be preferred over $s_P(T; A)$ by a majority (according to T).

The same reasoning shows that, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, the two-stage majoritarian rule f_P selects an alternative that is weakly preferred by majority over the alternatives selected by the

[14]See Moulin (1988) Exercise 9.4(c) (p. 250) for s_P and the Corollary to Theorem 9.5 (p. 247) for a_P. Horan (2020) shows that a much broader range of binary trees (which he calls "simple agendas") have the same feature.

[15]For UC_P and TC_P, the claim is straightforward. For s_P and a_P, see Moulin (1988) Exercise 9.5 (p. 250). A much broader class of binary trees introduced by Horan (2020) (called "priority agendas") have the same feature.

selection rule TC_P (footnote 6) and the amendment rule a_P (footnote 8). The same is true for the selection rule UC_P from Example 1 when differences in flexibility are taken into account: $f_P(T;A)$ is weakly preferred by a majority over $UC_P(T;A)$ provided that $f_P(T;A) \in UC(T;A)$.

While the linear ordering P plays a less intrusive role for two-stage majoritarian rules than it does for some other rules, it still has a significant impact on the outcome. Fortunately, there is a natural (or, at least, conventional) way of ordering the alternatives in many choice settings. In the committee setting, for instance, it is customary to use the preference of the chair as a tie-breaker (Robert, 2011, p. 405). In the public policy setting, it is natural to rank competing policies in terms of increasing cost or, in some cases, in terms of decreasing equity. Finally, in legislative settings, it is conventional to order proposals either by the time at which that they were tabled or by their degree of divergence from the *status quo* legislation (Rasch, 2000, p. 15).

4.4 Extensions

By dropping requirement (ii) of Inter-Tournament Weak WARP while maintaining the other properties in our Theorem, one obtains a broader class of choice rules. The main difference is that each tournament $T \in \mathcal{T}$ is assigned its own binary relation P_T, which must be acyclic but is not necessarily a linear ordering. Given a collection of such relations $\mathbf{P} = \{P_T\}_{T \in \mathcal{T}}$, the choice rule $f_{\mathbf{P}}$ is then defined, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, by

$$\{f_{\mathbf{P}}(T;A)\} = \max(T \setminus P_T; \max(T \cap P_T; A)). \tag{2}$$

As with two-stage majoritarian rules, each binary relation P_T partitions the tournament T into acyclic relations.[16] The next example describes an interesting rule in this class.

Example 4. (A variation on two-stage majoritarian rules) *Consider the choice rule g_P that, for any tournament $T \in \mathcal{T}$ and any agenda $A = \{a_1, ..., a_{m+1}\} \in \mathcal{X}$, is defined by*

$$g_P(T;A) = \begin{cases} \max(T; \{a_m, a_{m+1}\}) & \text{if } \max(T;A) \in \{a_m, a_{m+1}\}, \\ f_P(T; A \setminus \{a_m, a_{m+1}\}) & \text{otherwise.} \end{cases}$$

To elaborate, note that the rule f_P selects the lowest-ranked alternative only if it is the Condorcet winner in the agenda. The rule g_P simply extends this to the second lowest-ranked alternative.

Clearly, g_P satisfies Faithfulness, Expansion Consistency, requirement (i) of Inter-Tournament Weak WARP, Choice IIA, T-Monotonicity, and P-Monotonicity. However, the rules g_P and f_P

[16] To establish the sufficiency of the axioms for such a representation, let $P_T = T \setminus P_2^T$ where P_2^T is the binary relation defined in the proof of our theorem. Then, as in the proof of our theorem, Proposition 2 of Dutta and Horan implies that $(T \cap P_T, T \setminus P_T)$ is a minimal representation of $f(T;\cdot)$. Since $f(T;\cdot)$ is a choice function, $T \cap P_T$ is acyclic. By part (i) of Inter-Tournament Weak WARP, $T \setminus P_T = P_2^T$ is also acyclic. (If $a_1 P_2^T a_2 P_2^T ... P_2^T a_n P_2^T a_1$, consider $T \in \mathcal{T}$ such that $a_1 T a_n T ... T a_2 T a_1$. If $f(T;A) = a_i$, then, $a_{i+1} P_2^T a_i$, which contradicts $a_i P_2^T a_{i+1}$.) For an axiomatization of rational shortlist methods where both rationales are acyclic, see Houy (2008).

are distinct if $|X| \geq 3$. In fact, f_P is distinct from $g_{P'}$ for *any* linear ordering $P' \in \mathcal{P}$. To see this, consider the agenda $A = \{1, 2, 3\} \subseteq X$ and tournaments $T, T' \in \mathcal{T}$ such that $1T2T3T1$ and $1T'3T'2T'1$. Then, $f_P(T; A) \neq f_P(T'; A)$ while $g_{P'}(T; A) = g_{P'}(T'; A)$.

We note that the rules from Example 4 provide the same flexibility as two-stage majoritarian rules: for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, $TC(T; A) = \{g_P(T; A) : P \in \mathcal{P}\}$. So, like two-stage majoritarian rules, they sometimes make Pareto sub-optimal choices. This raises the question of whether an efficient choice rule can satisfy the kinds of requirements that we consider. A narrower question is whether there exists a binary relation P_T for each tournament $T \in \mathcal{T}$ such that $f_{\mathbf{P}}(T; \cdot)$ selects within the uncovered set for every agenda $A \in \mathcal{X}$.[17] For this purpose, Manzini and Mariotti (2006, Proposition 5) show that the transitivity of the first rationale $T \cap P_T$ is sufficient.

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