Université de Montréal

CLASSIFICATION DE SYSTÈMES INTÉGRABLES EN COORDONNÉES CYLINDRIQUES EN PRÉSENCE DE CHAMPS MAGNÉTIQUES

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Sommaire

En absence de champ magnétique, le mouvement quadratiquement intégrable en trois dimensions d'une particule classique ou quantique soumise à un potentiel scalaire peut être caractérisé à l'aide du Hamiltonien et de deux intégrales de mouvement quadratiques en composantes de la quantité de mouvement. Il existe alors onze types de tels systèmes intégrables, chacun correspondant à un système de coordonnées dans lequel la séparation des variables dans l'équation de Hamilton-Jacobi ou de Schrödinger est possible. On trouve alors, dans chaque cas, une forme plus explicite pour chacune des intégrales de mouvement ainsi que pour le potentiel scalaire.

Il semble logique de s'intéresser à l'influence d'un champ magnétique pour chacune des situations. Il s'agit alors de postuler les mêmes types d'intégrales de mouvement trouvées pour chacun des types de mouvement, ajouter un potentiel vecteur au Hamiltonien et refaire le travail de classification. Ce mémoire s'intéresse particulièrement au cas des coordonnées cylindriques. Il est à noter qu'avec la présence d'un champ magnétique, les intégrales ne garantissent pas la séparation complète des variables, mais une séparation partielle est possible dans plusieurs cas.

Mots-clés : Hamiltonien, Intégrabilité, Classification, Quantité conservée, Intégrale de mouvement, Potentiel scalaire, Champ magnétique, Classique, Quantique, 3D, Espace euclidien, Séparation de variables, Crochet de Poisson.

Summary

In absence of a magnetic field, the quadratically integrable three-dimensional motion of a classical or quantum particle subject to a scalar potential can be characterized by the Hamiltonian and two integrals of motion which are quadratic in the components of momentum. There exist eleven types of such integrable systems, each corresponding to a system of coordinates in which separation of variables in the Hamilton-Jacobi or Schrödinger equation is possible. In each case, it is then possible to find a more explicit form for each of the integrals of motion and for the scalar potential.

It seems logical to look at the influence of a magnetic field for each of these situations. The process consists of postulating the same type of integrals of motion previously found for each type of motion, add a vector potential to the Hamiltonian and redo the classification work. This master's thesis will mostly treat the case of cylindrical coordinates. It is important to note that in the presence of a magnetic field, these integrals do not guarantee a complete separation of variables, but a partial separation is possible in many cases.

Keywords: Hamiltonian, Integrability, Classification, Conserved quantity, Integrals of motion, Scalar potential, Magnetic field, Classical, Quantum, 3D, Euclidean space, Separation of variables, Poisson bracket.

Table des matières

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Introduction

Ce mémoire introduit d'abord la notion d'intégrabilité quadratique dans le cas d'un système à trois dimensions où une particule est soumise à un potentiel scalaire. Il expose en premier lieu une méthode systématique de classification des symétries applicable à un tel système. Cette méthode consiste à postuler deux intégrales de mouvement (aussi appelées quantités conservées) quadratiques en quantité de mouvement, et imposer que celles-ci Poisson-commutent entre elles et avec le Hamiltonien qui représente le système (c'est-à-dire que les crochets de Poisson à considérer sont nuls). Les résultats d'un telle classification, obtenus en 1967, sont abordés dans le premier chapitre. On note la présence d'onze classes différentes de paires d'intégrales de mouvement, avec leur potentiel scalaire associé. Chacune de ces classes se rapporte à un système de coordonnées distinct dans lequel la séparation des trois variables dans l'équation de Hamilton-Jacobi est possible. Dans le second chapitre, qui est un article co-rédigé par le présent auteur, Libor Šnobl et Pavel Winternitz, on utilise ces résultats et on se concentre sur la forme des intégrales de mouvement de type cylindrique (celles dans la classe où la séparation de variables en coordonnées cylindriques est possible). On tente alors de trouver les conditions pour lesquelles de telles intégrales peuvent exister si la particule est maintenant soumise à un champ magnétique (représenté dans le Hamiltonien par un potentiel vecteur) en plus d'un potentiel scalaire. Il est alors nécessaire d'exprimer toutes les quantités intéressantes en coordonnées cylindriques en effectuant les transformations appropriées. Il faut ensuite calculer les crochets de Poisson avec les quantités transformées et résoudre des systèmes d'équations pour éventuellement trouver des conditions sur le potentiel scalaire, le champ magnétique et les intégrales de mouvement. Une brève conclusion fait part de quelques propriétés intéressantes de certains résultats trouvés et du travail qu'il reste à effectuer dans cette direction. Une partie des calculs un peu plus laborieux est présentée en annexe.

Contribution de l'auteur

Le projet a d'abord commencé par une tentative de vérification des résultats obtenus par Makarov *et al.* en 1967, dans le cadre d'un stage d'été avec Pavel Winternitz en 2016. Plus tard, l'objectif a convergé vers l'idée de classification avec l'incorporation d'un champ magnétique, alors que le travail était déjà terminé en coordonnées cartésiennes et grandement avancé en coordonnées sphériques, grâce à la contribution d'Antonella Marchesiello et de Libor Šnobl. Au moment du dépôt initial de ce mémoire, un progrès avait déjà été publié par Sébastien Bertrand et Libor Šnobl en ce qui a trait aux coordonnées paraboliques circulaires, sphéroïdales allongées et sphéroïdales aplaties. Les étapes de résolution employées par l'auteur du présent mémoire s'inspirent partiellement de ces divers travaux. Les solutions obtenues ont été presque entièrement trouvées par l'auteur et vérifiées par les co-auteurs de l'article. Les logiciels Mathematica et Maple ont été utilisés pour les calculs laborieux et pour une vérification supplémentaire. L'article n'a toujours pas été soumis au moment du dépôt initial du mémoire.

Chapitre 1

INTÉGRABILITÉ DES SYSTÈMES PHYSIQUES CLASSIQUES

1.1. Motivation

Il existe une grande quantité de systèmes physiques qu'il est possible de solutionner complètement par intégration directe. De tels systèmes auront généralement un nombre élevé de symétries spatiales ou temporelles et, par conséquent, un grand nombre de quantités conservées dans le temps. En mécanique classique, on définit l'énergie totale d'un système par son Hamiltonien H, qui est la somme de l'énergie cinétique et de l'énergie potentielle. Une quantité conservée X, communément appelée intégrale de mouvement, se caractérisera par le fait qu'elle Poisson-commute avec le Hamiltonien, c'est-à-dire:

$$\{H,X\}_{C.P.} = 0 \tag{1.1.1}$$

où la définition du crochet de Poisson est la suivante [**1**] :

$$\{A,B\}_{C.P.} = \sum_{i=1}^{3} \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right), \tag{1.1.2}$$

où les q_i sont des coordonnées spatiales et les p_i les quantités de mouvement correspondantes.

1.2. Intégrabilité et superintégrabilité

Un système est dit **intégrable** s'il contient au moins autant de quantités conservées distinctes (linéairement indépendantes entre elles) que de degrés de liberté. Pour un système physique à n dimensions, il faut alors trouver $n-1$ quantités fonctionnellement indépendantes qui Poisson-commutent entre elles et avec le Hamiltonien pour montrer qu'il s'agit d'un système intégrable. Il est à noter que le Hamiltonien est aussi une quantité conservée, pour un grand total de n. Ce dernier doit aussi être fonctionnellement indépendant de chacune des $n-1$ autres quantités. Toutes les intégrales doivent être " globales ", c'est-à-dire des fonctions bien définies dans l'espace de phase.

Il est possible qu'un système ait plus de quantités conservées que de degrés de liberté. On par alors d'un système **superintégrable**. Pour n degrés de liberté, il peut exister jusqu'à un total de $2n-1$ quantités conservées distinctes. Les nouvelles quantités au-delà de n ne vont pas nécessairement Poisson-commuter entre elles et avec les autres. Un système à $n+1$ quantités conservées est dit minimalement superintégrable, et un système à $2n-1$ quantités conservées est dit maximalement superintégrable. Des exemples typiques de systèmes maximalement superintégrables sont l'oscillateur harmonique en n dimensions et le système de Kepler-Coulomb avec potentiel $V(r)=\frac{\alpha}{r}$ [**2**].

1.3. Cas d'une particule soumise à un potentiel scalaire

Pour la suite, on s'intéressera au mouvement d'une particule classique de masse $m=1$ dans un espace euclidien tridimensionnel, pour l'instant soumise seulement à un potentiel scalaire $W(\vec{x})$ qui dépend uniquement de sa position. Un tel système est décrit par le Hamiltonien suivant:

$$H=\frac{1}{2}\vec{p}^{\,2}+W(\vec{x}). \tag{1.3.1}$$

On postulera l'existence de quantités conservées au plus quadratiques en quantité de mouvement, c'est-à-dire ayant la forme suivante [**3**]:

$$X = \sum_{j=1}^{3} h^j(\vec{x}) p_j p_j + \sum_{j,k,l=1}^{3} \frac{1}{2} |\epsilon_{jkl}| n^j(\vec{x}) p_k p_l + \sum_{j=1}^{3} s^j(\vec{x}) p_j + m(\vec{x}) \tag{1.3.2}$$

où les fonctions $h^j(\vec{x})$, $n^j(\vec{x})$, $s^j(\vec{x})$ ($j = 1,2,3$) et $m(\vec{x})$ prennent des valeurs réelles. Le calcul du crochet de Poisson (ou du commutateur quantique, dans le cas quantique) avec le Hamiltonien, qui doit être nul, donnera des équations pour divers ordres en quantité de mouvement. On en trouvera dix à l'ordre 3, six à l'ordre 2, trois à l'ordre 1 et une à l'ordre 0, pour un total de vingt [**4**]. Des conditions supplémentaires pourront être obtenues à partir du crochet de Poisson entre deux quantités conservées distinctes X_1 et X_2, qui doit aussi être nul.

1.3.1. Classification et séparation des variables

Le travail de Makarov *et al.* a démontré que dans la situation présente, il existe onze classes distinctes de paires d'intégrales quadratiques en quantité de mouvement, et à chacune d'elles, on associe un système de coordonnées, et la séparation complète des variables est possible dans tous les cas. [**4**] Il a été confirmé par la suite que l'absence de champ magnétique dans cette situation impliquait toujours la séparation des variables.

Le cas qui nous intéressera ici sera le cas cylindrique, duquel on tire les intégrales suivantes:

$$\begin{aligned} X_1 &= (p_\phi)^2 + m_1(\phi); \\ X_2 &= (p_Z)^2 + m_2(Z), \end{aligned} \tag{1.3.3}$$

où r, ϕ et Z sont les coordonnées cylindriques usuelles. On considèrera l'ajout d'un potentiel vecteur au Hamiltonien et on postulera alors des quantités conservées ayant les mêmes termes dominants quadratiques en quantité de mouvement, sans écarter la possibilité de la présence de termes linéaires et libres. Les détails seront explicités au chapitre suivant.

1.3.2. Potentiel vecteur et champ magnétique

La pertinence de l'ajout d'un potentiel vecteur vient de la nécessité d'exprimer le mouvement d'une particule soumise à des forces qui proviennent de potentiels ayant des dépendances explicites en sa vitesse. De tels potentiels existent dans plusieurs problèmes de physique cosmique, mais l'exemple le plus notable et récurrent est la présence d'un champ magnétique. On le note alors :

$$\vec{B}(\vec{x}) = \nabla \times \vec{A}(\vec{x}), \tag{1.3.4}$$

où $\vec{A}(\vec{x})$ est le potentiel vecteur, qui apparaît explicitement dans le Hamiltonien :

$$H = \frac{1}{2}\big(\vec{p} + \vec{A}(\vec{x})\big)^2 + W(\vec{x}), \tag{1.3.5}$$

et il est alors entendu qu'on a normalisé la masse et la charge de la particule à 1 et -1 respectivement, pour simplifier la notation. Un travail considérable a déjà été effectué en deux dimensions [**5**,**6**], et quelques cas tridimentionnels spécifiques ont aussi été traités [**7**–**9**].

Chapitre 2

CYLINDRICAL TYPE INTEGRABLE CLASSICAL SYSTEMS IN A MAGNETIC FIELD

Ce chapitre est une version préliminaire d'un article co-rédigé par l'auteur du présent mémoire ainsi que par Libor Šnobl et Pavel Winternitz. En date du dépôt, l'article n'a pas encore été soumis à une revue scientifique.

2.1. Abstract

We present all second order classical integrable systems of the cylindrical type in a three dimensional Euclidean space $\mathbb{E}_3$ with a nontrivial magnetic field. The Hamiltonian and integrals of motion have the form

$$
\begin{aligned}
H &= \frac{1}{2}\left(\vec{p}+\vec{A}(\vec{x})\right)^2 + W(\vec{x}),\\
X_1 &= (p_\phi^A)^2 + s_1^r(r,\phi,Z)p_r^A + s_1^\phi(r,\phi,Z)p_\phi^A + s_1^Z(r,\phi,Z)p_Z^A + m_1(r,\phi,Z),\\
X_2 &= (p_Z^A)^2 + s_2^r(r,\phi,Z)p_r^A + s_2^\phi(r,\phi,Z)p_\phi^A + s_2^Z(r,\phi,Z)p_Z^A + m_2(r,\phi,Z).
\end{aligned}
$$

Infinite families of such systems are found, in general depending on arbitrary functions or parameters. This leaves open the possibility of finding superintegrable systems among the integrable ones (i.e. systems with 2 or 3 aditional independent integrals.)

2.2. Introduction

This article is part of a research program the aim of which is to identify, classify and solve all superintegrable classical and quantum finite-dimensional Hamiltonian systems. We recall that a superintegrable system is one that allows more integrals of motion than degrees of freedom. For a review of the topic we refer to [**2**]. The best known superintegrable systems are given by the Kepler-Coulomb [**1**,**10**,**11**] and the harmonic oscillator potentials [**1**,**12**,**13**]. A finite-dimensional classical Hamiltonian system in a $2n$-dimensional phase space is called integrable (or Liouville integrable) if it allows n integrals of motion $\{X_0 = H, X_1, ..., X_{n-1}\}$ (including the Hamiltonian). These n integrals must be well defined functions on the phase space. They must be in involution (Poisson commute pairwise, i.e. $\{X_i, X_j\}_{P.B.} = 0$) and be functionally independent. The system is superintegrable if there exist further integrals $\{Y_1, ..., Y_k\}$, $1 \leq k \leq n-1$, that are also well defined functions on the phase space. The entire set $\{X_0 = H, X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_k\}$ must be functionally independent and satisfy

$$\{H, X_j\}_{P.B.} = 0, \quad \{X_i, X_j\}_{P.B.} = 0, \quad \{H, Y_a\}_{P.B.} = 0, \tag{2.2.1}$$

$$i,j = 0,...,n-1, \quad a = 1,...,k, \quad 1 \leq k \leq n-1.$$

Notice that $\{Y_a, X_i\}_{P.B.} = 0$, $0 \leq i \leq n-1$, and $\{Y_a, Y_b\}_{P.B.} = 0$ is not required. Moreover, the Poisson brackets $Z_{ai} = \{Y_a, X_i\}_{P.B.}$ and $Z_{ab} = \{Y_a, Y_b\}_{P.B.}$ generate a non–Abelian polynomial algebra.

A systematic search for "natural" Hamiltonians of the form

$$H = \frac{1}{2}\vec{p}^{\,2} + W(\vec{x}) \tag{2.2.2}$$

that are superintegrable in the n-dimensional Euclidean space E_n started a long time ago [**4**, **14**–**16**] for $n = 2$ and $n = 3$. The integrals of motion X_i and Y_a were restricted to being second order polynomials in the components p_i of the momenta. Second order integrals of motion were shown to be related to the separation of variables in the Hamilton-Jacobi equation (and also the Schrödinger equation). All second order superintegrable systems in E_2 and E_3 were found [**4**, **14**, **17**]. Later developments for the Hamiltonian (2.2.2) and second order superintegrability include extensions to E_n for n arbitrary, to general Riemannian, pseudo-Riemannian, and complex-Riemannian spaces [**18**–**31**].

More general Hamiltonians and their integrability and superintegrability properties are also being studied, in particular Hamiltonians with scalar and vector potentials both in E_2 [**5**, **6**, **32**–**38**] and E_3 [**3**, **7**–**9**, **39**–**41**].

In this article we concentrate on the case of a particle moving in an electromagnetic field in E_3. It is described by a Hamiltonian with a scalar and vector potential, as in [**3**, **7**–**9**, **40**].

As opposed to previous articles, here we consider the "cylindrical case" when we have two second order integrals of motion of the "cylindrical type". In the absence of the vector potential the Hamiltonian would allow the separation of variables in cylindrical coordinates so that the potential in (2.2.2) would have the form

$$W(\vec{r}) = W_1(r) + \frac{1}{r^2}W_2(\phi) + W_3(Z), \tag{2.2.3}$$

with the transformations $x = r\cos(\phi)$, $y = r\sin(\phi)$ and $z = Z$.

2.3. Formulation of the problem

Let us consider a moving particle in an electromagnetic field, in a three-dimensional space. In cartesian coordinates, this simple system is described by the following Hamiltonian:

$$H = \frac{1}{2}\big(\vec{p} + \vec{A}(\vec{x})\big)^2 + W(\vec{x}) \tag{2.3.1}$$

where $\vec{p} = (p_1,p_2,p_3) \equiv (p_x,p_y,p_z)$ are the components of linear momentum, and $\vec{x} = (x_1,x_2,x_3) \equiv (x,y,z)$ are the cartesian spatial coordinates. The vector potential $\vec{A}(\vec{x}) = \big(A_1(\vec{x}),A_2(\vec{x}),A_3(\vec{x})\big) \equiv \big(A_x(\vec{x}),A_y(\vec{x}),A_z(\vec{x})\big)$ and scalar potential $W(\vec{x})$ depend only on the position $\vec{x}$. For practical reasons, the mass and electric charge of the particle have been set to 1 and -1, respectively.

The physical quantity related to the vector potential is the magnetic field

$$\vec{B}(\vec{x}) = \nabla \times \vec{A}(\vec{x}). \tag{2.3.2}$$

Our aim is to find all integrals of motion which are at most quadratic in the momenta. They are of the form [**3**]:

$$X = \sum_{j=1}^{3} h^j(\vec{x})p_j^A p_j^A + \sum_{j,k,l=1}^{3} \frac{1}{2}|\epsilon_{jkl}|n^j(\vec{x})p_k^A p_l^A + \sum_{j=1}^{3} s^j(\vec{x})p_j^A + m(\vec{x}) \tag{2.3.3}$$

where we have defined

$$p_j^A = p_j + A_j(\vec{x}) \tag{2.3.4}$$

and $h^j(\vec{x})$, $n^j(\vec{x})$, $s^j(\vec{x})$ ($j = 1,2,3$) and $m(\vec{x})$ are real valued functions. They must satisfy the determining equations provided by the fact that the Poisson bracket of the integral with the Hamiltonian must vanish, i.e.

$$\{H,X\}_{P.B.} = 0 \tag{2.3.5}$$

using the coefficients in front of each individual combination of powers in momenta. Those equations in cartesian coordinates are listed in previous papers [**3**, **7**–**9**, **40**].

It is possible to express the $h^j(\vec{x})$ and $n^j(\vec{x})$ functions as polynomials depending on 20 real constants α_{ab}, which allows us to say that the highest order terms of the integral X are elements of the universal enveloping algebra of the Euclidean Lie algebra

$$X = \sum_{1\leq a\leq b\leq 6} \alpha_{ab} Y_a^A Y_b^A + \sum_{j=1}^{3} s^j(\vec{x}) p_j^A + m(\vec{x}) \tag{2.3.6}$$

where

$$Y^A = (p_1^A, p_2^A, p_3^A, l_1^A, l_2^A, l_3^A), \qquad l_j^A = \sum_{l\leq k,l\leq 3} \epsilon_{jkl} x_k p_l^A. \tag{2.3.7}$$

We shall consider two integrals of motion X_1 and X_2 of the cylindrical type, in the sense that they imply separation of variables in cylindrical coordinates in the case of a vanishing magnetic field. Their exact form in the adequate system of coordinates will be specified below.

We use the following relations between cartesian and cylindrical coordinates:

$$x = r\cos(\phi), \qquad y = r\sin(\phi), \qquad z = Z. \tag{2.3.8}$$

Given the structure of the canonical 1-form

$$\lambda = p_x \mathrm{d}x + p_y \mathrm{d}y + p_z \mathrm{d}z = p_r \mathrm{d}r + p_\phi \mathrm{d}\phi + p_Z \mathrm{d}Z, \tag{2.3.9}$$

we obtain the following transformations for the linear momentum:

$$p_x = \cos(\phi) p_r - \frac{\sin(\phi)}{r} p_\phi, \quad p_y = \sin(\phi) p_r + \frac{\cos(\phi)}{r} p_\phi, \quad p_z = p_Z \tag{2.3.10}$$

and similarly for the components of the vector potential. On the other hand, the components of the magnetic field are the components of the 2-form $B = \mathrm{d}A$,

$$\begin{aligned} B &= B^x(\vec{x}) \mathrm{d}y \wedge \mathrm{d}z + B^y(\vec{x}) \mathrm{d}z \wedge \mathrm{d}x + B^z(\vec{x}) \mathrm{d}x \wedge \mathrm{d}y \\ &= B^r(r,\phi,Z) \mathrm{d}\phi \wedge \mathrm{d}Z + B^\phi(r,\phi,Z) \mathrm{d}Z \wedge \mathrm{d}r + B^Z(r,\phi,Z) \mathrm{d}r \wedge \mathrm{d}\phi. \end{aligned} \tag{2.3.11}$$

This leads to the following transformations:

$$
\begin{aligned}
B^x(\vec{x}) &= \frac{\cos(\phi)}{r} B^r(r,\phi,Z) - \sin(\phi) B^\phi(r,\phi,Z) \\
B^y(\vec{x}) &= \frac{\sin(\phi)}{r} B^r(r,\phi,Z) + \cos(\phi) B^\phi(r,\phi,Z) \\
B^z(\vec{x}) &= \frac{1}{r} B^Z(r,\phi,Z).
\end{aligned}
\tag{2.3.12}
$$

We can now rewrite both the Hamiltonian and the general form of an integral of motion in cylindrical coordinates.

2.4. Hamiltonian and integrals of motion in the cylindrical case

We will first write down the general form of the Hamiltonian and integrals in cylindrical coordinates, and then restrict to the case of two integrals of motion which correspond to the so-called cylindrical case.

2.4.1. Determining equations in cylindrical coordinates

In cylindrical coordinates, the Hamiltonian (2.3.1) takes the following form:

$$
H = \frac{1}{2}\left((p_r^A)^2 + \frac{(p_\phi^A)^2}{r^2} + (p_Z^A)^2\right) + W(r,\phi,Z), \tag{2.4.1}
$$

where

$$
p_r^A = p_r + A_r(r,\phi,Z), \quad p_\phi^A = p_\phi + A_\phi(r,\phi,Z), \quad p_Z^A = p_Z + A_Z(r,\phi,Z). \tag{2.4.2}
$$

The integral of motion (2.3.3) now reads as follows

$$
\begin{aligned}
X = {} & h^r(r,\phi,Z)(p_r^A)^2 + h^\phi(r,\phi,Z)(p_\phi^A)^2 + h^Z(r,\phi,Z)(p_Z^A)^2 + \\
& + n^r(r,\phi,Z) p_\phi^A p_Z^A + n^\phi(r,\phi,Z) p_r^A p_Z^A + n^Z(r,\phi,Z) p_\phi^A p_r^A + \\
& + s^r(r,\phi,Z) p_r^A + s^\phi(r,\phi,Z) p_\phi^A + s^Z(r,\phi,Z) p_Z^A + m(r,\phi,Z).
\end{aligned}
\tag{2.4.3}
$$

The functions h^r, ..., n^Z can be obtained from the h^j and n^j via their transformations into cylindrical coordinates, and are expressed in terms of the same 20 constants α_{ab}.

Computing the Poisson bracket $\{H,X\}_{P.B.}$ in the cylindrical coordinates we obtain terms of order 3, 2, 1 and 0 in the components of $\vec{p}^A$. The third order terms provide the following determining equations

$$
\begin{aligned}
&\partial_r h^r = 0, && \partial_\phi h^r = -r^2 \partial_r n^Z, && \partial_Z h^r = -\partial_r n^\phi, \\
&\partial_r h^\phi = -\frac{1}{r^2}\partial_\phi n^Z - \frac{2}{r^3} h^r, && \partial_\phi h^\phi = -\frac{1}{r} n^Z, && \partial_Z h^\phi = -\frac{1}{r^2}\partial_\phi n^r - \frac{1}{r^3} n^\phi, \\
&\partial_r h^Z = -\partial_Z n^\phi, && \partial_\phi h^Z = -r^2 \partial_Z n^r, && \partial_Z h^Z = 0, \\
&\partial_\phi n^\phi = -r^2(\partial_Z n^Z + \partial_r n^r).
\end{aligned} \tag{2.4.4}
$$

In the second order terms we use equations (2.4.4) and rewrite derivatives of the vector potential $\vec{A}$ in terms of the magnetic field $\vec{B}$, to obtain

$$
\begin{aligned}
\partial_r s^r &= n^\phi B^\phi - n^Z B^Z, \\
\partial_\phi s^r &= r^2(n^r B^\phi - 2h^\phi B^Z - \partial_r s^\phi) - n^\phi B^r + 2h^r B^Z, \\
\partial_r s^Z &= n^Z B^r - \partial_Z s^r - n^r B^Z + 2h^Z B^\phi - 2h^r B^\phi, \\
\partial_\phi s^\phi &= -n^r B^r + n^Z B^Z - \frac{1}{r} s^r, \\
\partial_\phi s^Z &= r^2(2h^\phi B^r - n^Z B^\phi - \partial_Z s^\phi) - 2h^Z B^r + n^\phi B^Z, \\
\partial_Z s^Z &= n^r B^r - n^\phi B^\phi.
\end{aligned} \tag{2.4.5}
$$

The first and zeroth order terms imply

$$
\begin{aligned}
\partial_r m &= s^Z B^\phi - s^\phi B^Z + n^\phi \partial_Z W + n^Z \partial_\phi W + 2h^r \partial_r W, \\
\partial_\phi m &= s^r B^Z - s^Z B^r + r^2(n^r \partial_Z W + 2h^\phi \partial_\phi W + n^Z \partial_r W), \\
\partial_Z m &= s^\phi B^r - s^r B^\phi + 2h^Z \partial_Z W + n^r \partial_\phi W + n^\phi \partial_r W,
\end{aligned} \tag{2.4.6}
$$

and

$$
s^r \partial_r W + s^\phi \partial_\phi W + s^Z \partial_Z W = 0, \tag{2.4.7}
$$

respectively.

2.4.2. Reduction to the cylindrical case

The integrals of motion corresponding to the cylindrical case, i.e. the case which allows separation of variables in cylindrical coordinates for a vanishing magnetic field, read [**4**]:

$$
\begin{aligned}
X_1 &= (p_\phi^A)^2 + s_1^r(r,\phi,Z)p_r^A + s_1^\phi(r,\phi,Z)p_\phi^A + s_1^Z(r,\phi,Z)p_Z^A + m_1(r,\phi,Z),\\
X_2 &= (p_Z^A)^2 + s_2^r(r,\phi,Z)p_r^A + s_2^\phi(r,\phi,Z)p_\phi^A + s_2^Z(r,\phi,Z)p_Z^A + m_2(r,\phi,Z).
\end{aligned}
\tag{2.4.8}
$$

For such integrals with specific values for the h and n coefficients, all of them being either 0 or 1, it follows that system (2.4.4) is satisfied trivially for both X_1 and X_2. The system (2.4.5) applied to both integrals gives the following equations:

$$
\begin{aligned}
&\partial_r s_1^r = 0, && \partial_\phi s_1^\phi = -\frac{s_1^r}{r},\\
&\partial_\phi s_1^r = -r^2(\partial_r s_1^\phi + 2B^Z), && \partial_\phi s_1^Z = r^2(-\partial_Z s_1^\phi + 2B^r),\\
&\partial_r s_1^Z = -\partial_Z s_1^r, && \partial_Z s_1^Z = 0,
\end{aligned}
\tag{2.4.9}
$$

$$
\begin{aligned}
&\partial_r s_2^r = 0, && \partial_\phi s_2^\phi = -\frac{s_2^r}{r},\\
&\partial_\phi s_2^r = -r^2\partial_r s_2^\phi, && \partial_\phi s_2^Z = -r^2\partial_Z s_2^\phi - 2B^r,\\
&\partial_r s_2^Z = -\partial_Z s_2^r + 2B^\phi, && \partial_Z s_2^Z = 0.
\end{aligned}
\tag{2.4.10}
$$

The systems (2.4.6) and (2.4.7) reduce to

$$
\begin{aligned}
\partial_r m_1 &= s_1^Z B^\phi - s_1^\phi B^Z,\\
\partial_\phi m_1 &= s_1^r B^Z - s_1^Z B^r + 2r^2\partial_\phi W,\\
\partial_Z m_1 &= s_1^\phi B^r - s_1^r B^\phi,
\end{aligned}
\tag{2.4.11}
$$

$$
\begin{aligned}
\partial_r m_2 &= s_2^Z B^\phi - s_2^\phi B^Z,\\
\partial_\phi m_2 &= s_2^r B^Z - s_2^Z B^r,\\
\partial_Z m_2 &= s_2^\phi B^r - s_2^r B^\phi + 2\partial_Z W,
\end{aligned}
\tag{2.4.12}
$$

and

$$
s_i^r\partial_r W + s_i^\phi\partial_\phi W + s_i^Z\partial_Z W = 0 \qquad (i = 1,2),
\tag{2.4.13}
$$

respectively.

Let us now consider the Poisson bracket $\{X_1,X_2\}_{P.B.}$, which must also vanish for an integrable system. This provides further equations for every order in the momenta. First, for the second order, we have

$$\partial_\phi s_2^\phi = 0, \qquad \partial_\phi s_2^r = 0, \qquad \partial_Z s_1^r = 0, \qquad \partial_\phi s_2^Z = \partial_Z s_1^\phi - 2B^r. \tag{2.4.14}$$

From those, we can already conclude, looking again at system (2.4.10), that $s_2^r = 0$. The first order terms in the same Poisson bracket $\{X_1,X_2\}_{P.B.}$ imply

$$\begin{aligned} s_2^Z \partial_Z s_1^r + s_2^\phi \partial_\phi s_1^r &= 0, \\ -s_1^\phi (2B^r + \partial_\phi s_2^Z) + s_2^Z \partial_Z s_1^Z - s_1^Z \partial_Z s_2^Z & \\ + s_2^\phi \partial_\phi 2_1^Z + s_1^r (2B^\phi - \partial_r s_2^Z) + 2\partial_Z m_1 &= 0, \\ -s_2^Z (2B^r - \partial_Z s_1^\phi) + s_2^\phi \partial_\phi s_1^\phi & \\ -s_1^Z \partial_Z s_2^\phi - s_1^r \partial_r s_2^\phi - 2\partial_\phi m_2 &= 0. \end{aligned} \tag{2.4.15}$$

From the zeroth order term we obtain

$$\begin{aligned} -s_1^r \partial_r m_2 + s_2^\phi \partial_\phi m_1 - s_1^\phi \partial_\phi m_2 + s_2^Z \partial_Z m_1 - s_1^Z \partial_Z m_2 & \\ + B^r (s_2^\phi s_1^Z - s_1^\phi s_2^Z) + B^\phi s_1^r s_2^Z - B^Z s_1^r s_2^\phi = 0. & \end{aligned} \tag{2.4.16}$$

2.5. Partial solution of determining equations and reduction to functions of one variable

The second order terms in momenta from the aforementioned vanishing Poisson brackets, i.e. systems (2.4.9), (2.4.10) and (2.4.14) provide a system of equations for the functions $s_j^{r,\phi,Z}$ and the magnetic field $B^{r,\phi,Z}$ which can be easily solved. The solution is expressed in terms of 5 functions of one variable each: $\sigma(r)$, $\rho(r)$, $\tau(\phi)$, $\psi(\phi)$ and $\mu(Z)$. We shall call them the auxiliary functions:

$$\begin{aligned} s_1^r &= \frac{\mathrm{d}}{\mathrm{d}\phi}\psi(\phi), & s_1^\phi &= -\frac{\psi(\phi)}{r} - r^2\mu(Z) + \rho(r), & s_1^Z &= \tau(\phi), \\ s_2^r &= 0, & s_2^\phi &= \mu(Z), & s_2^Z &= -\frac{\tau(\phi)}{r^2} + \sigma(r) \end{aligned} \tag{2.5.1}$$

$$B^r = -\frac{r^2}{2}\frac{\mathrm{d}}{\mathrm{d}Z}\mu(Z) + \frac{1}{2r^2}\frac{\mathrm{d}}{\mathrm{d}\phi}\tau(\phi), \qquad B^\phi = \frac{\tau(\phi)}{r^3} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\sigma(r),$$

$$B^Z = \frac{-\psi(\phi)}{2r^2} + r\mu(Z) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\rho(r) - \frac{1}{2r^2}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\psi(\phi). \tag{2.5.2}$$

Equations (2.5.1) and (2.5.2) are the general solutions of equations (2.4.9), (2.4.10) and (2.4.14). We use them to eliminate the functions $\vec{s}_1$,$\vec{s}_2$ and $\vec{B}$ from the as yet unsolved PDEs (2.4.11-2.4.13) and (2.4.15-2.4.16). Using (2.4.11-2.4.12) and (2.4.15) we end up with one equation for each possible first derivative of both m_1 and m_2, one direct condition on $\mu(Z)$ and $\psi(\phi)$, and two equations which are further conditions on $m_{1,Z}$ and $m_{2,\phi}$

$$\begin{aligned}
&\big(-r^3\mu(Z) + r\rho(r) - \psi(\phi)\big)\big(\psi''(\phi) + r^2\rho'(r)\big) + \big(r^3\mu(Z) + r\rho(r)\big)\psi(\phi)\\
&-\psi(\phi)^2 + r^3\tau(\phi)\sigma'(r) + 2r^6\mu(Z)^2 - 2r^4\rho(r)\mu(Z) + 2\tau(\phi)^2 - 2r^3 m_{1,r} = 0,\\
&\psi'(\phi)\big(2r^3\mu(Z) - r^2\rho'(r) - \psi(\phi) - \psi''(\phi)\big)\\
&+\tau(\phi)\big(r^4\mu'(Z) - \tau'(\phi)\big) + 4r^4 W_\phi - 2r^2 m_{1,\phi} = 0,\\
&\big(\tau'(\phi) - r^4\mu'(Z)\big)\big(-r^3\mu(Z) + r\rho(r) - \psi(\phi)\big)\\
&-\psi'(\phi)\big(r^3\sigma'(r) + 2\tau(\phi)\big) - 2r^3 m_{1,Z} = 0,\\
&r^3\mu(Z)\psi''(\phi) + r^3\sigma'(r)\big(r^2\sigma(r) - \tau(\phi)\big) - 2r^6\mu(Z)^2\\
&+r^5\mu(Z)\rho'(r) + r^3\mu(Z)\psi(\phi) + 2r^2\sigma(r)\tau(\phi) - 2\tau(\phi)^2 - 2r^5 m_{2,r} = 0,\\
&\big(r^4\mu'(Z) - \tau'(\phi)\big)\big(r^2\sigma(r) - \tau(\phi)\big) - 2r^4 m_{2,\phi} = 0,\\
&-r^4\mu(Z)\mu'(Z) + \mu(Z)\tau'(\phi) + 4r^2 W_Z - 2r^2 m_{2,Z} = 0,
\end{aligned}$$

$$\begin{aligned}
&\mu(Z)\psi''(\phi) = 0,\\
&\big(-r^4\mu(Z) + r^2\rho(r) - r\psi(\phi)\big)\mu'(Z) + \mu(Z)\tau'(\phi) + 2m_{1,Z} = 0,\\
&\tau'(\phi)\big(r^2\sigma(r) - \tau(\phi)\big) + r^4\tau(\phi)\mu'(Z) + r^3\mu(Z)\psi'(\phi) + 2r^4 m_{2,\phi} = 0.
\end{aligned} \tag{2.5.3}$$

From (2.4.13) and (2.4.16) we obtain 3 further equations

$$
\begin{aligned}
\big(r^2\sigma(r)-\tau(\phi)\big)W_Z+r^2\mu(Z)W_\phi&=0,\\
\big(-r^3\mu(Z)+r\rho(r)-\psi(\phi)\big)W_\phi+r(\psi'(\phi)W_r+\tau(\phi)W_Z)&=0,\\
2r^4\big(r^3\mu(Z)-r\rho(r)+\psi(\phi)\big)m_{2,\phi}+2r^3\big(r^2\sigma(r)-\tau(\phi)\big)m_{1,Z}&\\
+r^3\mu(Z)\psi'(\phi)\psi''(\phi)+2r^5\mu(Z)m_{1,\phi}-2r^5\tau(\phi)m_{2,Z}&\\
+\Big(r^3\big(r^2\sigma(r)-\tau(\phi)\big)\sigma'(r)-2r^6\mu(Z)^2+r^5\mu(Z)\rho'(r)&\\
+r^3\mu(Z)\psi(\phi)+2r^2\sigma(r)\tau(\phi)-2\tau(\phi)^2-2r^5m_{2,r}\Big)\psi'(\phi)&\\
-\Big(r^2\big(r^3\mu(Z)-r\rho(r)+\psi(\phi)\big)\sigma(r)&\\
+\tau(\phi)\big(r\rho(r)-\psi(\phi)\big)\Big)\big(r^4\mu'(Z)-\tau'(\phi)\big)&=0.
\end{aligned}
\tag{2.5.4}
$$

Before summing up the results of this section in the form of a reduced system of determining equations let us analyze the PDEs (2.5.3) and (2.5.4). First of all, $m_{1,Z}$ and $m_{2,\phi}$ appear in (2.5.3) twice each. Since the two values must coincide we obtain two constraints on the auxiliary functions. A further constraint $\mu(Z)\psi''(\phi)=0$ is explicit in (2.5.3). The remaining 6 equations in (2.5.3) are used to express all first order derivatives $m_{1,a}$, $m_{2,a}$ ($a=r,\phi,Z$) in terms of W_ϕ, W_Z and the auxiliary functions. Assuming that the functions m_1 and m_2 are sufficiently smooth we impose the Clairaut compatibility conditions $\partial_a\partial_b m_i=\partial_b\partial_a m_i$ on their second derivatives. This gives us a further set of equations

$$
\begin{aligned}
m_{1,r\phi}: \quad & \psi'(\phi)\big(-3\psi''(\phi) + r^3\rho''(r) - r^3\mu(Z) - r^2\rho'(r) + r\rho(r) - 4\psi(\phi)\big) \\
& + \tau'\big(\phi)(r^3\sigma'(r) + 2\tau(\phi)\big) - 2r^4\tau(\phi)\mu'(Z) - 4r^5 W_{r\phi} - 8r^4 W_\phi \\
& + \big(r\rho(r) - \psi(\phi)\big)\psi'''(\phi) = 0, \\
m_{1,rZ}: \quad & -r^4\mu'(Z)\psi''(\phi) + r^4\psi'(\phi)\sigma''(r) - 6\tau(\phi)\psi'(\phi) \\
& + \tau'(\phi)\big(-r^2\rho'(r) + 2r\rho(r) - 3\psi(\phi)\big) = 0, \\
m_{1,\phi Z}: \quad & \tau''(\phi)\big(r^3\mu(Z) - r\rho(r) + \psi(\phi)\big) + \psi''(\phi)\big(r^3\sigma'(r) + 2\tau(\phi)\big) \\
& + r^5\tau(\phi)\mu''(Z) + \psi'(\phi)\big(r^4\mu'(Z) + 3\tau'(\phi)\big) + 4r^5 W_{\phi Z} = 0, \qquad (2.5.5) \\
m_{2,r\phi}: \quad & -r^3\mu'(Z)\big(r\sigma'(r) + 2\sigma(r)\big) + \mu(Z)\psi'(\phi) = 0, \\
m_{2,rZ}: \quad & r\mu'(Z)\big(-2r^3\mu(Z) + r^2\rho'(r) + \psi(\phi)\big) \\
& - 4r^3 W_{rZ} + 2\mu(Z)\tau'(\phi) = 0, \\
m_{2,\phi Z}: \quad & r^2\big(\tau(\phi) - r^2\sigma(r)\big)\mu''(Z) + \tau''(\phi)\mu(Z) + 4r^2 W_{\phi Z} = 0.
\end{aligned}
$$

Equations (2.5.5) can be solved for the second mixed derivatives of the potential $W_{r\phi}$, W_{rZ} and $W_{\phi Z}$ in terms of W_ϕ and the auxiliary functions. The identities for the mixed third order derivatives of W are satisfied identically as a consequence of the compatibility of the second order ones.

Finally we substitute the first order derivatives $m_{1,a}$, $m_{2,a}$ from (2.5.3) into (2.5.4) and obtain a system of linear inhomogeneous algebraic equations for the first order derivatives W_r, W_ϕ, W_Z. Implementing the procedure described above we obtain the reduced system of determining equations presented in the following Section 2.6.

2.6. Reduced determining system

The determining system now reduces to two conditions on the auxiliary functions, three equations from (2.5.5) that involve mixed second derivatives of W, and a linear algebraic system involving all first derivatives of W. We list them all here:

$$\psi'(\phi)\big(r^3\sigma'(r) + 2\tau(\phi)\big) - \tau'(\phi)\big(r\rho(r) - \psi(\phi)\big) = 0, \qquad (2.6.1a)$$

$$\mu(Z)\psi'(\phi) + r^3\sigma(r)\mu'(Z) = 0 \qquad (2.6.1b)$$

$$
\begin{aligned}
W_{r\phi} &= -\frac{2}{r}W_\phi + \frac{1}{4r^5}\Big(\psi'(\phi)\big(-3\phi''(\phi) + r^3\rho''(r) - r^3\mu(Z) - r^2\rho'(r) + r\rho(r) - 4\psi(\phi)\big) \\
&\quad + \tau'(\phi)\big(r^3\sigma'(r) + 2\tau(\phi)\big) - 2r^4\tau(\phi)\mu'(Z) - \psi'''(\phi)\big(\psi(\phi) - r\rho(r)\big)\Big), \\
W_{\phi Z} &= -\frac{1}{4r^2}\Big(r^2\mu''(Z)\big(\tau(\phi) - r^2\sigma(r)\big) + \tau''(\phi)\mu(Z)\Big), \qquad (2.6.2) \\
W_{rZ} &= \frac{1}{4r^3}\Big(r\mu'(Z)\big(-2r^3\mu(Z) + r^2\rho'(r) + \psi(\phi)\big) + 2\mu(Z)\tau'(\phi)\Big),
\end{aligned}
$$

$$
\underbrace{\begin{pmatrix} 0 & r^2\mu(Z) & r^2\sigma(r) - \tau(\phi) \\ \psi'(\phi) & \rho(r) - r^2\mu(Z) - \frac{\psi(\phi)}{r} & \tau(\phi) \\ 0 & 4r^7\mu(Z) & -4r^5\tau(\phi) \end{pmatrix}}_{M} \cdot \underbrace{\begin{pmatrix} W_r \\ W_\phi \\ W_Z \end{pmatrix}}_{\nabla W} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ \alpha(r,\phi,Z) \end{pmatrix}}_{\vec{\alpha}}, \qquad (2.6.3)
$$

where

$$
\begin{aligned}
\alpha(r,\phi,Z) = &-\psi'(\phi)\Big(\big(-r^5\sigma(r) + r^3\tau(\phi)\big)\sigma'(r) - r^5\mu(Z)\rho'(r) + 2\tau(\phi)^2 \\
&- 2r^2\sigma(r)\tau(\phi) + r^3\mu(Z)\big(r^3\mu(Z) + r\rho(r) - 2\psi(\phi)\big)\Big) \\
&- \tau'(\phi)\Big(\big(-r\rho(r) + \psi(\phi)\big)\tau(\phi) - r^2\sigma(r)\big(r^3\mu(Z) - r\rho(r) + \psi(\phi)\big)\Big) \\
&- r^4\mu'(Z)\tau(\phi)\big(r\rho(r) - \psi(\phi)\big). \qquad (2.6.4)
\end{aligned}
$$

The rank of the matrix M can be either 3, 2 or 1. We rule out the rank 0 case since it leads to vanishing magnetic field, as seen directly from (2.5.2).

If the rank is 3, then the determinant of M

$$
\det(M) = 4r^9\psi'(\phi)\mu(Z)\sigma(r) \qquad (2.6.5)
$$

is not zero and it implies a unique solution for each first derivative of W. We will explore this case shortly and show that it leads to a contradiction.

If instead the rank is either 2 or 1, then $\det(M) = 0$, and from (2.6.5), there are *a priori* three possible cases:

a) $\psi'(\phi) = 0$,

b) $\psi'(\phi) \neq 0$ and $\mu(Z) = 0$,

c) $\psi'(\phi) \neq 0$, $\mu(Z) \neq 0$ and $\sigma(r) = 0$. However, we observe that this is inconsistent with (2.6.1b), so we can already rule this case out.

We shall first show that we must have $\alpha = 0$ in all these cases, allowing us to simplify further considerations below.

a) $\psi'(\phi) = 0$. This is equivalent to $\psi(\phi) = 0$ since the function ψ has to be constant and thus it can be absorbed into a redefinition of $\rho(r)$ in equations (2.5.1) and (2.5.2). The matrix system (2.6.3) can be written in its reduced row echelon form as the following augmented matrix:

$$\begin{pmatrix} 0 & r^2\mu(Z) & -\tau(\phi) & \frac{\alpha}{4r^5} \\ 0 & \rho(r) & 0 & \frac{\alpha}{4r^5} \\ 0 & 0 & \sigma(r) & -\frac{\alpha}{4r^7} \end{pmatrix} \tag{2.6.6}$$

From (2.6.1) we have

$$\tau'(\phi)\rho(r) = 0, \quad \mu'(Z)\sigma(r) = 0. \tag{2.6.7}$$

Consequently, the expression for α reads

$$\alpha = r^5 \left(\tau'(\phi)\sigma(r)\mu(Z) - \mu'(Z)\rho(r)\tau(\phi)\right). \tag{2.6.8}$$

Equations (2.6.7) give rise to four possible solutions

- $\tau'(\phi) = 0$, $\mu'(Z) = 0$, implying $\alpha = 0$ directly,
- $\rho(r) = 0$, $\sigma(r) = 0$, implying $\alpha = 0$ directly,
- $\tau'(\phi) = 0$, $\sigma(r) = 0$, implying $\alpha = -r^5\mu'(Z)\rho(r)\tau(\phi)$,
- $\rho(r) = 0$, $\mu'(Z) = 0$, implying $\alpha = r^5\tau'(\phi)\sigma(r)\mu(Z)$.

On the other hand, the solvability condition of the linear system (2.6.3), namely that the rank of M and of the corresponding augmented matrix coincide, imply that if either $\rho(r) = 0$ or $\sigma(r) = 0$, the function α must vanish. Thus in the two cases above we find constraints,

- if $\psi'(\phi) = \tau'(\phi) = \sigma(r) = 0$ we must have

$$\mu'(Z)\rho(r)\tau(\phi) = 0, \tag{2.6.9}$$

- if $\psi'(\phi) = \rho(r) = \mu'(Z) = 0$ we must have

$$\tau'(\phi)\sigma(r)\mu(Z) = 0. \tag{2.6.10}$$

b) $\psi'(\phi) \neq 0$ and $\mu(Z) = 0$. In this case equation (2.6.1b) is satisfied trivially. Equation (2.6.1a) we differentiate with respect to r, arriving at

$$\left(r^3\sigma'(r)\right)' = \frac{\tau'(\phi)}{\psi'(\phi)} \left(r\rho(r)\right)', \tag{2.6.11}$$

leading to three distinct possibilities

- $\left(r^3\sigma'(r)\right)' = \left(r\rho(r)\right)' = 0$, i.e.

$$\sigma(r) = \frac{C_\sigma}{r^2} + \tilde{C}_\sigma, \qquad \rho(r) = \frac{C_\rho}{r}. \tag{2.6.12}$$

  Substituting (2.6.12) into equation (2.6.1a) we find

$$2\left(\tau(\phi) - C_\sigma\right)\psi'(\phi) + \left(\psi(\phi) - C_\rho\right)\tau'(\phi) = 0 \tag{2.6.13}$$

  which directly implies that α defined in (2.6.4) vanishes.

- $\left(r^3\sigma'(r)\right)' = \tau'(\phi) = 0$, i.e.

$$\sigma(r) = \frac{C_\sigma}{r^2} + \tilde{C}_\sigma, \qquad \tau(\phi) = C_\tau. \tag{2.6.14}$$

  Substituting (2.6.14) into equation (2.6.1a) we find $C_\sigma = C_\tau$ and that together with equation (2.6.14) implies again that we find $\alpha = 0$ in (2.6.4).

- $\frac{\left(r^3\sigma'(r)\right)'}{\left(r\rho(r)\right)'} = \frac{\tau'(\phi)}{\psi'(\phi)} = \lambda \neq 0$, implying that

$$\rho(r) = \frac{1}{\lambda} r^2\sigma'(r) + \frac{C_\rho}{r}, \qquad \tau(\phi) = \lambda\psi(\phi) + C_\tau. \tag{2.6.15}$$

  However, substituting (2.6.15) into (2.6.1a) and differentiating it with respect to ϕ we arrive at $\lambda\psi'(\phi) = 0$ which contradicts our assumptions $\lambda \neq 0$ and $\psi'(\phi) \neq 0$.

Thus we see that for all solutions of the determining equations we have $\alpha = 0$. In most cases $\alpha = 0$ by virtue of (2.6.1) alone, in two cases the condition that the augmented matrix of the system (2.6.3) and the matrix M have the same rank leads to certain additional constraints, cf. equations (2.6.9) and (2.6.10).

We are now ready to split the classification problem into three main cases according to the rank of the matrix M, which leads to various classes of potentials and magnetic fields.

2.7. Solutions of determining equations for Case 1: $\det(M) \neq 0$ ($\mathbf{rank}(M) = 3$)

Let's begin with the seemingly most complicated case: the case where the determinant of M is not equal to zero, or in other words, the rank of M is 3. We are going to prove that this case leads to an inconsistency and has no solutions.

Recalling (2.6.5), this requires that $\psi'(\phi) \neq 0, \mu(Z) \neq 0$ and $\sigma(r) \neq 0$. From (2.6.1a) we have $\psi''(\phi) = 0$. We can assume that $\psi(\phi) = \psi_1\phi$ where the constant ψ_1 satisfies $\psi_1 \neq 0$, since an additive constant would be absorbed into $\rho(r)$ by a simple redefinition. Looking at equation (2.6.1b), it becomes obvious that $\sigma(r)$ takes the following form:

$$\sigma(r) = \frac{\sigma_0}{r^3}, \quad \sigma_0 \neq 0. \tag{2.7.1}$$

Equation (2.6.1a) then becomes:

$$\psi_1\big(- 3\sigma_0 + 2r\tau(\phi)\big) - \tau'(\phi)\big(r^2\rho(r) - r\psi_1\phi\big) = 0. \tag{2.7.2}$$

Differentiation with respect to r gives:

$$2\psi_1\tau(\phi) - \tau'(\phi)\big(2r\rho(r) + r^2\rho'(r) - \psi_1\phi\big) = 0. \tag{2.7.3}$$

From this point we can separate the variables r and ϕ if $\tau'(\phi) \neq 0$. Notice that this has to be true since $\tau'(\phi) = 0$ would imply that either ψ_1 or $\tau(\phi)$ is zero, from the previous equation. The latter is not possible in view of (2.7.2) since it would imply that $\sigma_0 = 0$, which contradicts our initial hypothesis. This means that we can rewrite (2.7.3) as

$$2r\rho(r) + r^2\rho'(r) = k = \psi_1\phi + \frac{2\psi_1\tau(\phi)}{\tau'(\phi)}, \tag{2.7.4}$$

where k is a constant. Solving for $\rho(r)$, we have

$$\rho(r) = \frac{\rho_0}{r^2} + \frac{k}{r}. \tag{2.7.5}$$

Heading back to (2.6.1a) and separating the expression in its explicit dependencies on r^0 and r^1 using the newly known expression for $\rho(r)$, this ultimately implies that $\sigma_0\psi_1 = 0$, which is a contradiction. This means that the system is inconsistent and admits no solutions. The source of this inconsistency is that W_r, W_ϕ and W_Z can be determined in a unique manner from the algebraic equation (2.6.3). They must however also be first derivatives of a smooth function $W(r,\phi,Z)$ and hence satisfy the Clairaut theorem on mixed derivatives. This contradicts (2.6.1).

2.8. Solutions of determining equations for Case 2: rank$(M) = 2$

There are two main subcases to consider here: a) $\psi'(\phi) = 0$, and b) $\mu(Z) = 0$ while $\psi'(\phi) \neq 0$, so that we ensure that the determinant (2.6.5) vanishes and thus the rank of M is at most 2.

2.8.1. Case 2a: $\psi'(\phi) = 0$

It is understood again that $\psi(\phi)$ is set to zero. There are several ways for the rank to be equal to 2. We recall the reduced row echelon form of the matrix M

$$\begin{pmatrix} 0 & r^2\mu(Z) & -\tau(\phi) \\ 0 & \rho(r) & 0 \\ 0 & 0 & \sigma(r) \end{pmatrix} \tag{2.8.1}$$

The rank of a matrix is the largest size of its invertible square submatrices. Thus for the rank of the matrix (2.8.1) to be 2, at least one of the three minors involving the second and third column must be non-zero. The possibilities are as follows:

1) $\tau(\phi)\rho(r) \neq 0$, and then $\mu(Z)$ and $\sigma(r)$ are arbitrary;
2) $\mu(Z)\sigma(r) \neq 0$, and then $\tau(\phi)$ and $\rho(r)$ are arbitrary;
3) $\rho(r)\sigma(r) \neq 0$, and then $\mu(Z)$ and $\tau(\phi)$ are arbitrary.

Let us consider these cases one by one.

1) $\tau(\phi)\rho(r) \neq 0$.

From (2.6.1a), we have that $\tau(\phi) = \tau_0$ is a non-zero constant. Now recall that $\rho(r) \neq 0$ implies that $W_\phi = -\frac{1}{4}\mu'(Z)\tau(\phi)$, then notice that $\rho(r)W_\phi = 0$. So $W_\phi = 0$, and $\mu(Z) = \mu_0$ is a constant. It follows that $W_Z = 0$. All of (2.6.2) is then satisfied trivially. The solution for the magnetic field and the potential reads

$$W = W(r),\ B^r = 0,\ B^\phi = \frac{\tau_0}{r^3} + \frac{1}{2}\sigma'(r),\ B^Z = \mu_0 r - \frac{1}{2}\rho'(r). \tag{2.8.2}$$

Recalling (2.3.12), we express this system in cartesian coordinates

$$\begin{aligned} W &= W\left(\sqrt{x^2+y^2}\right), \\ B^x &= -y\left(\frac{\tau_0}{(x^2+y^2)^2} + S\left(\sqrt{x^2+y^2}\right)\right), \\ B^y &= x\left(\frac{\tau_0}{(x^2+y^2)^2} + S\left(\sqrt{x^2+y^2}\right)\right), \\ B^z &= \mu_0 - P\left(\sqrt{x^2+y^2}\right), \end{aligned} \tag{2.8.3}$$

where $S(r) = \frac{\sigma'(r)}{2r}$ and $P(r) = \frac{\rho'(r)}{2r}$.

2) $\mu(Z)\sigma(r) \neq 0$.

The computation is very similar to the previous subcase. This time from (2.6.1b) we see that $\mu'(Z)\sigma(r) = 0$, so $\mu(Z) = \mu_0$ is a non-zero constant. Now recall that $\sigma(r) \neq 0$ implies that $W_Z = \frac{1}{4r^2}\tau'(\phi)\mu(Z)$, and notice that $\sigma(r)W_Z = 0$. So $W_Z = 0$, and $\tau(\phi) = \tau_0$ is a constant. It follows that $W_\phi = 0$, and we have the same solution for $W(r,\phi,Z)$. The magnetic field is also the same, except that now $\rho(r)$ is arbitrary and $\sigma(r)$ is arbitrary and non-zero.

3) $\rho(r)\sigma(r) \neq 0$.

Recall that this directly implies that $\mu(Z) = \mu_0$ and $\tau(\phi) = \tau_0$ are constants. Once again (2.6.2) is the same and the solutions are identical, except that neither $\rho(r)$ nor $\sigma(r)$ can be equal to zero.

Thus the results for the case rank $M = 2$ and $\psi'(\phi) = 0$ take the form (2.8.2) (or, equivalently, (2.8.3)). We notice that for the system (2.8.2) the two quadratic integrals (2.4.8) can be reduced to the first order integrals

$$\tilde{X}_1 = p_\phi^A + \frac{\rho(r)}{2} - \frac{\mu_0 r^2}{2}, \quad \tilde{X}_2 = p_Z^A + \frac{\sigma(r)}{2} - \frac{\tau_0}{2r^2}. \tag{2.8.4}$$

Thus the system (2.8.3) was already encountered in [**3**], cf. equation (76) therein.

2.8.2. Case 2b: $\mu(Z) = 0$, $\psi'(\phi) \neq 0$

Under these assumptions equations (2.6.2) directly imply that the variable Z can be separated from the other two variables r and ϕ in the potential W, i.e.

$$W(r,\phi,Z) = W^{12}(r,\phi) + W^3(Z). \tag{2.8.5}$$

The reduced row echelon form of M becomes

$$\begin{pmatrix} r\psi'(\phi) & r\rho(r)-\psi(\phi) & 0 \\ 0 & 0 & \sigma(r) \\ 0 & 0 & \tau(\phi) \end{pmatrix} \tag{2.8.6}$$

Our assumption $\psi'(\phi) \neq 0$ implies that $r\rho(r) - \psi(\phi) \neq 0$. Thus to have rank $M = 2$ we have two possibilities, namely $\sigma(r) \neq 0$ or $\tau(\phi) \neq 0$. Either of them implies

$$W^3_Z = 0. \tag{2.8.7}$$

Since the separation of the potential (2.8.5) is defined up to an additive constant, we can set $W^3(Z) = 0$.

1) $\sigma(r) \neq 0$.

We first rewrite (2.6.1a) in the following way:

$$r^3\sigma'(r) + 2\tau(\phi) - \frac{\tau'(\phi)}{\psi'(\phi)}(r\rho(r) - \psi(\phi)) = 0. \tag{2.8.8}$$

Differentiation with respect to ϕ leads to the following equation:

$$3\tau'(\phi) + \psi(\phi)\frac{\tau''(\phi)\psi'(\phi) - \tau'(\phi)\psi''(\phi)}{\psi'(\phi)^2} = r\rho(r)\frac{\tau''(\phi)\psi'(\phi) - \tau'(\phi)\psi''(\phi)}{\psi'(\phi)^2}. \tag{2.8.9}$$

If $\tau''(\phi)\psi'(\phi) - \tau'(\phi)\psi''(\phi) \neq 0$ we can separate the variables r and ϕ. If instead this expression vanishes, we directly conclude from (2.8.9) that $\tau'(\phi) = 0$, thus $\tau(\phi) = \tau_0$ is a constant. We study both situations separately.

1.1) $\tau''(\phi)\psi'(\phi) - \tau'(\phi)\psi''(\phi) = 0$, i.e. $\tau(\phi) = \tau_0$.

Equation (2.6.1a) now reads $r^3\sigma'(r) = -2\tau_0$; thus, we have $\sigma(r) = \frac{\tau_0}{r^2} + \sigma_0$. This reduces the system (2.6.1-2.6.3) to the following two equations

$$r\psi'(\phi)W^{12}_r + \big(r\rho(r) - \psi(\phi)\big)W^{12}_\phi = 0, \tag{2.8.10}$$

$$\begin{aligned} &\psi'(\phi)\big(-3\psi''(\phi) + r^3\rho''(r) - r^2\rho'(r) + r\rho(r) - 4\psi(\phi)\big) \\ &+\psi'''(\phi)\big(r\rho(r) - \psi(\phi)\big) - 4r^5W^{12}_{r\phi} - 8r^4W^{12}_\phi = 0. \end{aligned} \tag{2.8.11}$$

The magnetic field takes the form

$$B_\phi = 0, \quad B_r = 0, \quad B_Z = -\rho'(r) - \frac{\psi''(\phi) + \psi(\phi)}{2r^2}. \tag{2.8.12}$$

Thus the motion of the system splits into a motion in the xy-plane under the influence of the potential $W^{12}(r,\phi)$ and the perpendicular magnetic field $B_Z(r,\phi)$

(a problem discussed by McSween and Winternitz in polar coordinates in [**6**]) plus a free motion in the z-direction. The integral X_2 reduces to a first order one

$$\tilde{X}_2 = p_Z^A + \frac{\sigma_0}{2} \tag{2.8.13}$$

and in a suitably chosen gauge becomes simply p_Z.

1.2) $\tau''(\phi)\psi'(\phi) - \tau'(\phi)\psi''(\phi) \neq 0$.

In this case we can separate the variables r and ϕ in (2.8.9), arriving at the equations

$$\frac{3\tau'(\phi)\psi'(\phi)^2}{\tau''(\phi)\psi'(\phi) - \tau'(\phi)\psi''(\phi)} + \psi(\phi) = \rho_0 = r\rho(r), \tag{2.8.14}$$

where ρ_0 is the separation constant. Solving them we find

$$\rho(r) = \frac{\rho_0}{r}, \qquad \tau(\phi) = \tau_0 + \frac{\tau_1}{(\psi(\phi) - \rho_0)^2}. \tag{2.8.15}$$

From equation (2.6.1a) we find $\sigma(r) = \frac{\tau_0}{r^2} + \sigma_0$. The system (2.6.3) implies $W_Z^3 = 0$, i.e. again $W = W^{12}(r,\phi)$.

Next, we insert these results into the remaining equations (2.6.2-2.6.3) and find two equations which read

$$\begin{gathered} r\psi'(\phi)W_r + \big(\rho_0 - \psi(\phi)\big)W_\phi = 0, \\ -3\psi'(\phi)\psi''(\phi) - 4\psi'(\phi)\big(\psi(\phi) - \rho_0\big) - \psi'''(\psi(\phi) - \rho_0) \\ -\frac{4\tau_1^2}{\big(\psi(\phi) - \rho_0\big)^5}\psi'(\phi) - 4r^5 W_{r\phi} - 8r^4 W_\phi = 0. \end{gathered} \tag{2.8.16}$$

We can rewrite $\beta(\phi) = \psi(\phi) - \rho_0$ and integrate the second equation once with respect to ϕ, arriving at

$$r\beta'(\phi)W_r^{12} - \beta(\phi)W_\phi^{12} = 0 \tag{2.8.17a}$$

$$\begin{gathered} -\beta(\phi)\beta''(\phi) - \beta'(\phi)^2 - 2\beta(\phi)^2 + \frac{\tau_1^2}{\beta(\phi)^4} \\ -4r^5 W_r^{12} - 8r^4 W^{12}(r,\phi) - f(r) = 0. \end{gathered} \tag{2.8.17b}$$

Substituting for W_r^{12} from (2.8.17a) into (2.8.17b) we find expressions for both W_r^{12} and W_ϕ^{12}. Substituting them into (2.8.16) we find that

$$f(r) = \frac{f_1}{4} + f_2 r^4 \tag{2.8.18}$$

in (2.8.17b). Next, we find solving (2.8.17a) the explicit form of the potential in terms of the yet unknown function $\beta(\phi)$

$$W(r,\phi) = -\frac{f_2}{8} + \frac{\tilde{W}(\phi)}{r^2} + \frac{\beta(\phi)\beta''(\phi) + \beta'(\phi)^2 + \frac{f_1}{4} - \frac{\tau_1^2}{\beta(\phi)^4} + 2\beta(\phi)^2}{8r^4}. \tag{2.8.19}$$

The function $\tilde{W}(\phi)$ is determined by (2.8.17b) and reads

$$\tilde{W}(\phi) = \frac{W_0}{\beta(\phi)^2}, \tag{2.8.20}$$

where W_0 is an arbitrary constant. The potential thus becomes

$$W(r,\phi) = -\frac{f_2}{8} + \frac{W_0}{r^2\beta(\phi)^2} + \frac{\beta(\phi)\beta''(\phi) + \beta'(\phi)^2 + \frac{f_1}{4} - \frac{\tau_1^2}{\beta(\phi)^4} + 2\beta(\phi)^2}{8r^4}. \tag{2.8.21}$$

The sole remaining equation (2.8.17b) becomes an equation for the uknown function $\beta(\phi)$ only

$$\beta'(\phi)\big(7\beta(\phi)\beta''(\phi) + 4\beta'(\phi)^2 + 12\beta(\phi)^2 + f_1\big) + \beta(\phi)^2\beta'''(\phi) = 0. \tag{2.8.22}$$

This equation can be integrated twice, down to a first order ODE. In order to do this we must multiply by $\beta(\phi)$ and integrate, then multiply by $\beta'(\phi)\beta(\phi)$ and integrate again. The result is

$$4\beta(\phi)^4\beta'(\phi)^2 + 4\beta(\phi)^6 - 4\beta_1\beta(\phi)^2 + f_1\beta(\phi)^4 = \beta_2 \tag{2.8.23}$$

where β_1,β_2 are the constants of integration. Substituting $\gamma(\phi) = \beta(\phi)^2$ we can re-express it as

$$\gamma(\phi)\gamma'(\phi)^2 + 4\gamma(\phi)^3 - 4\beta_1\gamma(\phi) + f_1\gamma(\phi)^2 = \beta_2. \tag{2.8.24}$$

In the special case where $\beta_2 = 0$, it is possible to solve this equation and the solution is expressed in terms of elementary analytic functions. If $\beta_2 \neq 0$, the solution is as far as we know not known.

The magnetic field is also expressed in terms of the function $\beta(\phi)$ and reads

$$B_r = -\tau_1\frac{\sqrt{4\beta_1\beta(\phi)^2 + \beta_2 - 4\beta(\phi)^6 - f_1\beta(\phi)^4}}{2r^2\beta(\phi)^5},$$

$$B_\phi = \frac{\tau_1}{r^3\beta(\phi)^2}, \quad B_Z = \frac{2\beta_1\beta(\phi)^2 + \beta_2}{4r^2\beta(\phi)^5}. \tag{2.8.25}$$

(The sign of the square root depends on the choice of the branch of the square root of $\beta'(\phi)$ in (2.8.23).)

Using (2.8.23) potential (2.8.21) simplifies to an explicit function of $\beta(\phi)$,

$$W(r,\phi) = -\frac{f_2}{8} + \frac{W_0}{r^2\beta(\phi)^2} - \frac{4\tau_1^2+\beta_2}{32\beta(\phi)^4 r^4}. \tag{2.8.26}$$

2) $\tau(\phi) \neq 0$.

In this case it is now understood that there is no constraint on $\sigma(r)$ yet. But in the previous case we never actually considered a case where $\tau(\phi)$ would vanish, and there was no division by $\sigma(r)$, so we can follow the same splitting as well as some of the same results. So the first subcase is once again the polar case treated in Ref. [**6**] but with $\tau(\phi) \neq 0$, and the second subcase is again the same as in (2.8.25) and (2.8.26) while taking (2.8.17) into account.

2.9. Solutions of determining equations for Case 3: rank$(M) = 1$

Once again there are only two consistent ways for the determinant of M to vanish, i.e. $\psi'(\phi) = 0$ which without loss of generality becomes $\psi(\phi) = 0$, and $\mu(Z) = 0$ while $\psi'(\phi) \neq 0$.

2.9.1. Case 3a: $\psi'(\phi) = 0$

We have the same reduced row echelon form (2.8.1) for M. This time around we ask the rank to be 1, so every minor of size 2 has to vanish, but there has to remain at least one non-zero entry. There are four possibilities, one for each function to individually be non-zero,

(1) $\mu(Z) \neq 0$, this implies that $\sigma(r) = 0$ and $\rho(r)\tau(\phi) = 0$,

(2) $\mu(Z) = 0$, $\tau(\phi) \neq 0$, this implies that $\rho(r) = 0$,

(3) $\mu(Z) = 0$, $\tau(\phi) = 0$ and $\rho(r) \neq 0$, this implies that $\sigma(r) = 0$,

(4) $\mu(Z) = 0$, $\tau(\phi) = 0$ and $\rho(r) = 0$, this implies that $\sigma(r) \neq 0$.

Let us now consider these cases separately

(1) $\mu(Z) \neq 0$, $\sigma(r) = 0$, $\rho(r)\tau(\phi) = 0$.

We use the fact that $\rho(r)W_\phi = 0$, which further splits the problem into two subcases.

(a) Let's first consider what happens when $\rho(r) = 0$. Plugging everything we know into (2.6.1), (2.6.2) and (2.6.3), we have the remaining four equations:

$$W_{r\phi} = -\frac{2}{r}W_\phi + \frac{1}{2r^5}\tau'(\phi)\tau(\phi) - \frac{1}{2r}\tau(\phi)\mu'(Z), \tag{2.9.1a}$$

$$W_{\phi Z} = -\frac{1}{4}\mu''(Z)\tau(\phi) - \frac{1}{4r^2}\tau''(\phi)\mu(Z), \tag{2.9.1b}$$

$$W_{rZ} = -\frac{r}{2}\mu'(Z)\mu(Z) + \frac{1}{2r^3}\mu(Z)\tau'(\phi), \tag{2.9.1c}$$

$$r^2\mu(Z)W_\phi - \tau(\phi)W_Z = 0. \tag{2.9.1d}$$

We introduce $M'(Z) = \mu(Z)$ and $T'(\phi) = \tau(\phi)$. Integrating (2.9.1b) with respect to Z and ϕ we find an expression for the potential in terms of two functions of two variables each:

$$W(r,\phi,Z) = -\frac{1}{4r^2}\tau'(\phi)M(Z) - \frac{1}{4}T(\phi)\mu'(Z) + F_1(r,\phi) + F_2(r,Z). \tag{2.9.2}$$

This expression for W we substitute into (2.9.1c), finding $F_2(r,Z)$. Inserting it into (2.9.1a) we find $F_1(r,\phi)$. Thus we arrive at the explicit form of the potential

$$\begin{aligned} W(r,\phi,Z) = &-\frac{1}{4r^2}T''(\phi)M(Z) - \frac{1}{4}T(\phi)M''(Z) - \frac{r^2}{8}M'(Z)^2 \\ &- \frac{1}{8r^4}T'(\phi)^2 + W_1(r) + \frac{1}{r^2}W_2(\phi) + W_3(Z), \end{aligned} \tag{2.9.3}$$

We are left with a single equation (2.9.1d) to solve, which simplifies to

$$T(\phi)T'(\phi)M'''(Z) - M(Z)M'(Z)T'''(\phi) = 4\left(T'(\phi)W_3'(Z) - M'(Z)W_2'(\phi)\right). \tag{2.9.4}$$

If we assume that $\tau(\phi) \neq 0$, it is possible to separate the variables ϕ and Z by dividing the above expression by $M'(Z)T'(\phi)$ and then differentiating with respect to ϕ and Z. This leads to the following condition:

$$\frac{T''''(\phi)T'(\phi) - T'''(\phi)T''(\phi)}{T'(\phi)^3} = -3C = \frac{M''''(Z)M'(Z) - M'''(Z)M''(Z)}{M'(Z)^3}, \tag{2.9.5}$$

for some separation constant C. Reducing the order of the separated equations, we find that

$$M'(Z)^2 = CM(Z)^3 + C_1M(Z)^2 + C_2M(Z) + C_3, \tag{2.9.6}$$

$$T'(\phi)^2 = CT(\phi)^3 + \tilde{C}_1T(\phi)^2 + \tilde{C}_2T(\phi) + \tilde{C}_3$$

where C_1,C_2,C_3,$\tilde{C}_1$,$\tilde{C}_2$,$\tilde{C}_3$ are constants of integration. Thus $T(\phi)$ and $M(Z)$ functions take the form of Weierstrass's elliptic functions when $C \neq 0$ and the roots of the third order polynomials in (2.9.6) are mutually distinct. The functions of interest $\tau(\phi)$ and $\mu(Z)$ then become:

$$\tau(\phi) = K\frac{\mathrm{d}}{\mathrm{d}\phi}\wp\Big(\frac{\phi + c_1}{K}; -2Kc_1, c_2\Big), \tag{2.9.7}$$
$$\mu(Z) = K\frac{\mathrm{d}}{\mathrm{d}Z}\wp\Big(\frac{Z + c_3}{K}; -2Kc_3, c_4\Big),$$

where $K \neq 0$ is a redefinition of C, and c_1, c_2, c_3 and c_4 are integration constants corresponding to elliptic invariants which are related to the periods of the elliptic functions.

If $C = 0$, we find solutions expressed in terms of exponentials and trigonometric functions,

$$\tau(\phi) = k_1 \cos(k_0\phi) + k_2 \sin(k_0\phi),$$
$$\mu(Z) = \tilde{k}_1 \exp(\tilde{k}_0 Z) + \tilde{k}_2 \exp(-\tilde{k}_0 Z), \tag{2.9.8}$$

where $k_0,\ k_1,\ k_2,\ \tilde{k}_0,\ \tilde{k}_1$ and $\tilde{k}_2$ are arbitrary parameters. Solving $W_2(\phi)$ and $W_3(Z)$ we arrive at the magnetic field and the potential

$$B_r = -\frac{\tilde{k}_0\left(\tilde{k}_1\exp\left(2\tilde{k}_0 Z\right)-\tilde{k}_2\right)r^2}{2\exp\left(\tilde{k}_0 Z\right)} - \frac{k_0\left(k_1\sin\left(k_0\phi\right)-k_2\cos\left(k_0\phi\right)\right)}{2r^2},$$

$$B_\phi = \frac{\cos\left(k_0\phi\right)k_1+\sin\left(k_0\phi\right)k_2}{r^3},\quad B_Z = \frac{\left(\tilde{k}_1\exp\left(2\tilde{k}_0 Z\right)+\tilde{k}_2\right)r}{\exp\left(\tilde{k}_0 Z\right)}, \tag{2.9.9}$$

$$\begin{aligned}W = &-\frac{\left(\tilde{k}_1\left(\exp\left(\tilde{k}_0 Z\right)\right)^2+\tilde{k}_2\right)^2 r^2}{8\left(\exp\left(\tilde{k}_0 Z\right)\right)^2}-\\ &-\frac{\tilde{k}_0}{4k_0}\left(\tilde{k}_1\exp\left(\tilde{k}_0 Z\right)-\tilde{k}_2\exp\left(-\tilde{k}_0 Z\right)\right)\left(k_1\sin\left(k_0\phi\right)-k_2\cos\left(k_0\phi\right)\right)+\\ &+W_1\left(r\right)+\frac{\tilde{k}_4}{\tilde{k}_0}\left(\tilde{k}_1\exp\left(\tilde{k}_0 Z\right)-\tilde{k}_2\exp\left(-\tilde{k}_0 Z\right)\right)+\\ &+\frac{k_0^2}{8\tilde{k}_0^2}\left(\tilde{k}_1^2\exp\left(\tilde{k}_0 Z\right)^2+\tilde{k}_2^2\exp\left(-\tilde{k}_0 Z\right)^2\right)-\\ &-\frac{\tilde{k}_0 k_3}{4k_0}\left(\tilde{k}_1\exp\left(\tilde{k}_0 Z\right)\tilde{k}_2\exp\left(-\tilde{k}_0 Z\right)\right)+\tilde{k}_5+...\end{aligned}$$

In the case $\tau(\phi)=0$ the solution is much more straightforward and we immediately arrive at the potential and the magnetic field

$$W = W_1(r) - \frac{r^2}{8}\mu(Z)^2 + W_3(Z),$$
$$B^r = -\frac{r^2}{2}\mu'(Z),\quad B^\phi = 0,\quad B^Z = r\mu(Z), \tag{2.9.10}$$

where $\mu(Z)\neq 0$ is arbitrary. Transforming into cartesian coordinates, we find

$$W = W_1\big(\sqrt{x^2+y^2}\big) - \frac{x^2+y^2}{8}\mu(z)^2 + W_3(z),$$
$$\vec{B} = \left(-\frac{x}{2}\mu'(z), -\frac{y}{2}\mu'(z), \mu(z)\right). \tag{2.9.11}$$

Obviously, for this system the integral X_1 reduces to a first order one since the magnetic field and the potential are invariant with respect to rotations around z-axis.

(b) On the other hand if we have $\rho(r) \neq 0$, thus $W_\phi = 0$, $\tau(\phi) = 0$ and there is only one remaining equation to be solved, namely

$$W_{rZ} = -\frac{r}{2}\mu'(Z)\mu(Z) + \frac{1}{4}\mu'(Z)\rho'(r). \tag{2.9.12}$$

Solving for the potential, we conclude that both $\mu(Z)$ and $\rho(r)$ remain arbitrary non-zero functions, and the potential and magnetic field read

$$\begin{aligned} W &= W_1(r) - \frac{r^2}{8}\mu(Z)^2 + \frac{1}{4}\rho(r)\mu(Z) + W_3(Z), \\ B^r &= -\frac{r^2}{2}\mu'(Z), \qquad B^\phi = 0, \qquad B^Z = r\mu(Z) - \frac{1}{2}\rho'(r). \end{aligned} \tag{2.9.13}$$

In cartesian coordinates they become

$$\begin{aligned} W &= W_1\big(\sqrt{x^2+y^2}\big) - \frac{x^2+y^2}{8}\mu(z)^2 + \frac{1}{4}\rho\big(\sqrt{x^2+y^2}\big)\mu(z) + W_3(z), \\ \vec{B} &= \left(-\frac{x}{2}\mu'(z), -\frac{y}{2}\mu'(z), \mu(z) - P\big(\sqrt{x^2+y^2}\big)\right), \end{aligned}$$

where $P(r) = \frac{\rho'(r)}{2r}$. Also for this system the integral X_1 reduces to a first order one since the magnetic field and the potential are invariant with respect to rotations around z-axis.

(2) $\mu(Z) = 0$, $\tau(\phi) \neq 0$, $\rho(r) = 0$

In this case we have $\sigma(r)W_Z = 0$, so a priori there are two possible subcases. However, $\sigma(r) = 0$ implies equations of the form (2.9.1) but with $\mu(Z) = 0$. Equation (2.9.1d) together with our assumptions imposes W_Z. Thus we must have $W_Z = 0$ and the only remaining equation reads

$$r^3\tau'(\phi)\sigma'(r) + 2\tau'(\phi)\tau(\phi) - 4r^5 W_{r\phi} - 8r^4 W_\phi = 0, \tag{2.9.14}$$

which is easily integrated. We find

$$\begin{aligned} W &= W_1(r) - \frac{1}{8r^4}\tau(\phi)^2 + \frac{1}{4r^2}\tau(\phi)\sigma(r) + \frac{1}{r^2}W_2(\phi), \\ B^r &= \frac{1}{2r^2}\tau'(\phi), \quad B^\phi = \frac{1}{r^3}\tau(\phi) + \frac{1}{2}\sigma'(r), \quad B^Z = 0, \end{aligned} \tag{2.9.15}$$

where $\tau(\phi)$ and $\sigma(r)$ are arbitrary functions, $\tau(\phi)$ not vanishing identically. In cartesian coordinates we have

$$\begin{aligned}
W &= W_1\left(\sqrt{x^2+y^2}\right) - \frac{\tau(\phi)^2}{8\left(x^2+y^2\right)^2} + \frac{\tau(\phi)\sigma\left(\sqrt{x^2+y^2}\right)}{4(x^2+y^2)} + \frac{W_2(\phi)}{x^2+y^2}, \\
B^x &= \frac{x\tau'(\phi)}{2\left(x^2+y^2\right)^2} - y\left(\frac{\tau(\phi)}{\left(x^2+y^2\right)^2} + S\left(\sqrt{x^2+y^2}\right)\right), \\
B^y &= \frac{y\tau'(\phi)}{2\left(x^2+y^2\right)^2} + x\left(\frac{\tau(\phi)}{\left(x^2+y^2\right)^2} + S\left(\sqrt{x^2+y^2}\right)\right), \\
B^z &= 0,
\end{aligned} \tag{2.9.16}$$

where $S(r) = \frac{\sigma'(r)}{2r}$, and $\phi = \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right)$.

(3) $\mu(Z) = 0$, $\tau(\phi) = 0$, $\sigma(r) = 0$ and $\rho(r) \neq 0$

We have $\rho(r)W_\phi = 0$, which implies that $W_\phi = 0$. Again there is only one equation left to solve

$$W_{rZ} = 0, \tag{2.9.17}$$

thus we have

$$W = W_1(r) + W_3(Z),\ B^r = 0,\ B^\phi = 0,\ B^Z = -\frac{1}{2}\rho'(r), \tag{2.9.18}$$

thus this class of systems is equivalent to the polar case in two dimensions, which was explored in Ref. [**6**], complemented by one-dimensional independent motion in the z-direction, governed by the potential $W^3(z)$.

(4) $\mu(Z) = 0$, $\tau(\phi) = 0$, $\rho(r) = 0$, $\sigma(r) \neq 0$.

We see that $\sigma(r)W_Z = 0$ thus $W_Z = 0$. There is one remaining equation

$$4r^5 W_{r\phi} - 8r^4 W_\phi = 0 \tag{2.9.19}$$

which is identical to (2.9.14) with $\tau(\phi) = 0$. Thus the solution is

$$W = W_1(r) + \frac{1}{r^2}W_2(\phi), \quad B^r = 0, \quad B^\phi = \frac{1}{2}\sigma'(r), \quad B^Z = 0. \tag{2.9.20}$$

2.9.2. Case 3b: $\mu(Z) = 0$, $\psi'(\phi) \neq 0$

Let's recall the reduced row echelon form of M for this case reads (2.8.6). For its rank to be 1, the only possibility is that both $\sigma(r)$ and $\tau(\phi)$ vanish. Equations (2.6.2) imply that the potential separates as

$$W(r,\phi,Z) = W^{12}(r,\phi) + W^{3}(Z). \tag{2.9.21}$$

Equations (2.6.1-2.6.3) reduce to the two following equations which are identical to the ones considered in (2.8.10):

$$\begin{aligned} r\psi'(\phi)W_r + \big(r\rho(r) - \psi(\phi)\big)W_\phi = 0, \\ \psi'(\phi)\big(- 3\psi''(\phi) + r^3\rho''(r) - r^2\rho'(r) + r\rho(r) - 4\psi(\phi)\big) \\ + \psi'''(\phi)\big(r\rho(r) - \psi(\phi)\big) - 4r^5 W_{r\phi} - 8r^4 W_\phi = 0. \end{aligned} \tag{2.9.22}$$

The magnetic field reads

$$B_r = 0, \quad B_\phi = 0, \quad B_Z = -\frac{1}{2r^2}\left(\rho'(r)r^2 + \psi''(\phi + \psi(\phi)\right). \tag{2.9.23}$$

Thus this class of systems is equivalent to the polar case in two dimensions, which was explored in previous work [**6**], complemented by one-dimensional independent motion in the z-direction, governed by the potential $W^3(z)$.

2.10. Conclusions

Let us first of all sum up the results of this study. The problem stated in the title and Introduction was formulated mathematically in Section 2.4 and lead to the determining equations (2.4.9)–(2.4.16) for the scalar potential W, the magnetic field $\vec{B}$ and the coefficients $\vec{s}_1$, $\vec{s}_2$, m_1 and m_2 of two second order integrals of motion X_1 and X_2 (2.4.3). All of the above functions are assumed to be smooth functions of 3 variables, the cylindrical coordinates r,ϕ,Z in $\mathbb{E}_3$, with $0 \leq r < \infty$, $0 \leq \phi \leq 2\pi$, $-\infty < Z < \infty$. In Section 2.5 we partially solve this overdetermined system of 28 PDEs for 12 functions. We express the vector functions $\vec{B}$, $\vec{s}_1$, $\vec{s}_2$ in terms of 5 scalar auxiliary functions of one variable $\rho(r)$,$\sigma(r)$,$\tau(\phi)$,$\psi(\phi)$ and $\mu(Z)$ (2.5.1)–(2.5.2). We also derive a system of 12 equations (2.5.3)–(2.5.4) for the remaining scalar functions m_1, m_2 and W and the auxiliary functions. Some compatibility equations are presented in (2.5.5).

The reduced system of the determining equations is presented in Section 2.6. It consists of 3 PDEs for the scalar potential $W(r,\phi,Z)$ (2.6.2), 2 ODEs (2.6.1) for the auxiliary functions and 3 algebraic equations (2.6.3) for the first derivatives W_r, W_ϕ and W_z. Equation (2.6.3) involves a matrix M depending only on the auxiliary functions. The rank of M, $r(M) = r$ satisfies $0 \leq r \leq 3$. The case $r = 0$ can be discarded since it implies that the magnetic field is absent, $\vec{B} = 0$. in Section 2.7 we show that the reduced determining system has no solutions for $r = 3$, i.e. the system is inconsistent.

The main results of this paper are obtained for $r = 2$ and $r = 3$, presented in Sections 2.8 and 2.9. The obtained integrable magnetic fields $\vec{B}(r,\phi,Z)$ and $W(r,\phi,Z)$ are as follows:

(1) $r = 2$

The matrix M depends on all 5 auxiliary functions. The rank condition $r(M) = 2$ forces at least one of them to vanish. Three subcases can occur and in all of them the scalar potential splits into two parts as in (2.8.5).

(a) $\psi(\phi) = 0$

The magnetic field and potential are given in (2.8.2), so W does not depend on Z. The second order integrals X_1 and X_2 are actually squares of first order ones given in (2.8.4). They were already found and analysed in an earlier article [**3**].

(b) $\psi'(\phi) \neq 0$, $\mu(Z) = 0$, $(\frac{\tau'(\phi)}{\psi'(\phi)})' = 0$

We again find $W_Z = 0$ and $\vec{B}$ is given in (2.8.12). One of the integrals of motion can be reduced to $X_2 = p_2$. We obtain a two–dimensional case in $\mathbb{E}_2$, analyzed earlier in [**6**] and [**32**]. In the perpendicular direction Z we have free motion.

(c) $\psi'(\phi) \neq 0$, $\mu(Z) = 0$, $(\frac{\tau'(\phi)}{\psi'(\phi)})' \neq 0$

Our analysis leads to the magnetic field (2.8.25) and the potential (2.8.26). Both are expressed in terms of one function $\beta(\phi) = \sqrt{\gamma(\phi)}$ where $\gamma(\phi)$ satisfies the nonlinear ODE (2.8.24).

(2) $r = 1$

All 5 auxiliary functions are *a priori* present in M but the rank condition forces at least 2 of them to vanish. Again we obtain several cases:

(a) $\psi(\phi) = \sigma(r) = \rho(r) = 0$, $\mu(Z) \neq 0$

The field $\vec{B}$ and the potential W are expressed in terms of elliptic functions (2.9.7). In special cases this simplifies to elementary functions as in (2.9.9) and (2.9.10).

(b) $\psi(\phi) = \sigma(r) = \tau(\phi) = 0$, $\mu(Z) \neq 0$, $\rho(r) \neq 0$

The result is given in (2.9.13).

(c) $\psi(\phi) = \mu(Z) = \rho(r) = 0$, $\tau(\phi) \neq 0$

We obtain (2.9.15).

(d) $\psi(\phi) = \mu(Z) = \sigma(r) = \tau(\phi) = 0$, $\rho(r) \neq 0$

This leads to (2.9.18), a case already treated in [**6**] for $\mathbb{E}_2$. The motion in the perpendicular Z direction depends on an arbitrary potential $W_3(Z)$.

(e) $\psi(\phi) = \tau(\phi) = \mu(Z) = \rho(r) = 0$, $\sigma(r) \neq 0$

See (2.9.20).

(f) $\psi(\phi) \neq 0$ $\tau(\phi) = \mu(Z) = \sigma(r) = 0$

See (2.9.21) and (2.9.23). This case again decomposes into integrable motion in $\mathbb{E}_2$ plus a motion governed by $W_3(Z)$ in the perpendicular direction.

This sums up the results on integrable systems of the considered type. Some of the potentials and magnetic fields depend on arbitrary functions as well as constants. This leaves us with the freedom to impose further restrictions, in particular to request that the system be superintegrable. Thus we can request that 1 or 2 more integrals exist.

Let us review the differences and similarities between the cases with and without magnetic fields:

(1) In both cases the leading part of the integral of motion lies in the enveloping algebra of the Euclidean Lie algebra $\mathfrak{e}_3$.

(2) For $B = 0$ a second order integral contains no first order terms. For $B \neq 0$ first order terms can be present.

(3) Second order integrability in $\mathbb{E}_n$ implies separation of variables in the Hamiton–Jacobi and Schrödinger equations for $B = 0$, but not for $B \neq 0$.

(4) For $B = 0$ second order integrable and superintegrable systems are the same in quantum and classical mechanics. For $B \neq 0$ this is not necessarily true.

Thus second order integrable and superintegrable systems in magnetic fields are similar to systems without magnetic fields but with integrals of order N, $N > 2$ [**42**–**49**].

Our future plans include the following. To find all superintegrable systems among the integrable ones in this article. To solve the classical equations of motion and verify that in maximally superintegrable systems all bounded trajectories are closed [**50**–**52**]. To determine the cylindrical type quantum integrable and superintegrable systems in a magnetic field. We expect the quantum maximally superintegrable systems to be exactly solvable [**53**–**57**].

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Conclusion

Le présent mémoire fournit une approche systématique de classification de systèmes quadratiquement intégrables, à partir d'une paire d'intégrales de mouvement donnée en (2.4.8) qui est associée au type cylindrique, en trois dimensions.

La section 2.10 fournit une analyse détaillée des résultats de cette recherche. En résumé, on en conclut qu'il existe plusieurs familles d'intégrales de mouvement correspondant au type cylindrique. Les champs magnétiques associés s'expriment toujours en termes d'au plus cinq fonctions d'une seule variable (voir (2.5.1)–(2.5.2)), et le potentiel scalaire admet parfois une séparation de la variable Z des variables du plan perpendiculaire r et ϕ, tel qu'à la section 2.8.2. L'un de ces cas se réduit au cas à deux dimensions en coordonnées polaires, lorsque le champ magnétique est uniquement dirigé en Z, tel que vu à l'équation (2.9.18). Plusieurs résultats sont exprimés en termes de fonctions et de constantes arbitraires, donc il est encore possible d'imposer la condition supplémentaire de superintégrabilité et ainsi trouver une ou deux nouvelles intégrales de mouvement dans plusieurs cas.

La présence du champ magnétique impose généralement l'existence de termes de premier ordre dans l'expression des intégrales de mouvement. Elle enlève également la certitude que le cas quantique soit complètement équivalent au cas classique. La disparition complète du champ magnétique, dans chacun des sous-cas, est cohérente avec les résultats trouvés en 1967 et la séparation des variables en coordonnées cylindriques est retrouvée [**4**], ainsi que la disparition des termes de premier ordre dans les intégrales de mouvement. On voit aussi, notamment, que les systèmes intégrables et superintégrables sont alors les mêmes dans les cas classique et quantique.

Il existe quelques cas traités dans cet ouvrage qui ne sont pas complètement résolus. Une classification complète et exhaustive nécessiterait de résoudre des équations différentielles ordinaires non linéaires, notamment l'équation (2.8.24), dont on ne connait présentement aucune solution générale dans la littérature. L'étude approfondie de la superintégrabilité s'inscrit également dans les projets futurs du domaine. Il sera nécessaire, dans le cas cylindrique comme dans les autres, d'établir une classification exhaustive des systèmes superintégrables et de montrer que, pour tout système maximalement superintégrable, toute trajectoire classique bornée est fermée sur elle-même. On s'attend également à ce que tout système quantique maximalement superintégrable soit complètement résoluble.

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