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Classification of separable superintegrable systems of order four in two dimensional Euclidean space and algebras of integrals of motion in one dimension

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Classification of separable superintegrable systems of order four in two dimensional Euclidean space and algebras of integrals of motion in one dimension

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Sommaire

Cette thèse constitue une étape dans l'étude systématique des systèmes superintégrables, tant classiques que quantiques. Nous présentons les résultats de deux articles. Dans le premier, nous considérons tous les hamiltoniens de l'espace euclidien de dimension deux qui admettent une intégrale de deuxième ordre et une de quatrième ordre. La présence d'une intégrale de deuxième ordre rend les fonctions potentielles séparables. Nous classifions aussi tous les potentiels quantiques qui sont des solutions d'EDO non linéaires et donnons les intégrales correspondantes. Nous obtenons de nouveaux potentiels, exprimés en termes de troisièmes et cinquièmes fonctions transcendantes de Painlevé.
Dans le second article, nous donnons de nouvelles constructions d'hamiltoniens superintégrables en dimension deux, tant classiques que quantiques, et dont les potentiels sont séparables en coordonnées cartésiennes. Nous construisons quatre types de systèmes hamiltoniens algébriques en dimension un. Nous étudions deux copies d'algèbres d'opérateurs en dimension un et les combinons pour former des systèmes superintégrables dans E_2. Nous prouvons que tous les systèmes superintégrables d'ordre au plus cinq qui sont séparables en coordonnées cartésiennes, sont réductibles.

Mots-clés: superintégrabilité, séparation de variables, mécanique quantique, propriété de Painlevé, fonctions transcendantes de Painlevé, opérateur d'échelle.

Summary

The purpose of this thesis is to continue a systematic research on classical and quantum superintegrable systems. We present the results from two articles. In the first article, we consider a general Hamiltonian in two-dimensional Euclidean space admitting a second order and a fourth order integral of motion. The second order integral imposes the separation of variable in the potentials. We present a complete classification of all quantum potentials that are solutions of nonlinear ODEs and the corresponding integrals. New potentials expressed in terms of the third and fifth Painlevé transcendents are obtained.

In the second article, we develop new constructions of two-dimensional classical and quantum superintegrable Hamiltonians with separation of variables in Cartesian coordinates. Here we construct four types of algebraic Hamiltonian systems in one dimension. We study two copies of operator algebras in one dimension and combine these two to form superintegrable systems in E_2. We demonstrate that all quantum and classical superintegrable systems separable in Cartesian coordinates up to order 5 are in fact reducible.

keywords: superintegrability, separation of variables, quantum mechanics, Painlevé property, Painlevé transcendent, ladder operator.

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Dédicaces

To my parents

به پدر و مادرم

.

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Introduction

In order to study behaviour of a physical system we can look at its mathematical model. Quantum and classical systems are among the ones which their models are of special interests. Solving ordinary and partial differential equations analytically or numerically plays a role in studying these models. In the case we are able to solve the system with explicit analytic expressions, and as a result predict the future behaviour, we call it classical or quantum integrable Hamiltonian system. Moreover, when a system admits the maximum possible symmetry, it can be solved algebraically as well as analytically. It is called a superintegrable system.

In this thesis, we present recent results obtained in classical and quantum integrability and superintegrability of a Hamiltonian system that allows the separation of variables in coordinates in a two-dimensional Euclidean space. The results presented are in the form of two articles. The first article has been published in the Journal of Physics A: Mathematical and Theoretical [**77**]. The second one will be submitted soon in the same journal.

In the first chapter, we detail the formalism and definition of the integrable and superintegrable systems in classical and quantum mechanics.

In chapter 2, we present partial results on the classification of the superintegrable systems in two-dimensional Cartesian coordinates with integrals of motion of order two, three and four [**40, 63, 47, 48, 91**].

In chapters 3, we discuss superintegrability of order four. We consider a separable Hamiltonian system in quantum mechanics that admits a fourth order integrals of motion. Here we present the quantum systems written in terms of functions that do not satisfy any linear equations. These functions are expressed in terms of elliptic functions or Painlevé transcendents.

Finally, in chapter 4, we present different structures and algebraic methods. These are necessary, since the direct approach of classifying superintegrable Hamiltonians based on solving PDEs becomes more difficult as the dimension of the underlying space or the order of the integrals increase. Here we point out how different types of construction can be used to build two-dimensional superintegrable systems with higher order integrals of motion.

Chapitre 1

Preliminaries: Definitions and theorems

1.1. Classical Mechanics

1.1.1. Equations

A Hamiltonian system describes the dynamics of a physical system. A physical system describing the position of a particle at the time t involves n position coordinates $q_j(t)$, and n momentum coordinates $p_j(t)$. The phase space of the system is the $2n$ dimensional space with coordinates $p_1,\dots,p_n;q_1,\dots,q_n$ where $p_j,q_j \in \mathbb{R}$. The simplest form of it is the total energy of the system

$$H = T + V = E$$

where T is the kinetic energy and V the potential energy. The Hamiltonian we are going to consider in this thesis is a function on the phase space of the form

$$H = \frac{1}{2}\sum_{j,k=1}^{n} g_{jk}(\mathbf{q})p_j p_k + V(\mathbf{q}) \tag{1.1.1}$$

where $(\mathbf{q},\mathbf{p}) \in \mathbb{R}^{2n}$.

The dynamics of the system are given by Hamilton's equations

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j},\ j = 1,2,\dots,n \tag{1.1.2}$$

A solution $(\mathbf{q}(t),\mathbf{p}(t))$ of Hamilton's equations represents the trajectory of the particle submitted to the external force V.

Definition 1.1.1.

- *A physical quantity X characterized by a differentiable function on the phase space, $X = X(\mathbf{q},\mathbf{p})$, is called an* ***integral*** *or a* ***constant of the motion*** *if it remains* *constant along the trajectory* $(\mathbf{q}(t),\mathbf{p}(t))$*, which means it satisfies*

$$0=\frac{dX}{dt}=\sum_{j=1}^{n}(\frac{\partial X}{\partial q_j}\frac{dq_j}{dt}+\frac{\partial X}{\partial p_j}\frac{dp_j}{dt}). \tag{1.1.3}$$

- *An integral of motion is of order* N *if it is a polynomial of order* N *in the momenta.*

Definition 1.1.2. *The* ***Poisson bracket*** *of two functions* $X(\mathbf{q},\mathbf{p},t)$ *and* $Y(\mathbf{q},\mathbf{p},t)$ *on the phase space is the function*

$$\{X,Y\}(\mathbf{p},\mathbf{q})=\sum_{j=1}^{n}(\frac{\partial X}{\partial p_j}\frac{\partial Y}{\partial q_j}-\frac{\partial X}{\partial q_j}\frac{\partial Y}{\partial p_j}). \tag{1.1.4}$$

The Poisson bracket of the canonical coordinates $(\mathbf{q},\mathbf{p})$ are

$$\{q_j,p_k\}=\delta_{j,k},\quad \{q_j,q_k\}=\{p_j,p_k\}=0. \tag{1.1.5}$$

where $\delta_{j,k}$ is the Kronecker delta, i.e.

$$\delta_{j,k}=\begin{cases}1 & \text{if } j=k\\ 0 & \text{if } j\neq k\end{cases}$$

The Poisson bracket satisfies the following properties. Given X,Y,Z functions on the phase space and a,b constants:

- $\{X,Y\}=-\{Y,X\}$, (anti-symmetry)
- $\{X,aY+bZ\}=a\{X,Y\}+b\{X,Z\}$, (bilinearity)
- $\{X,\{Y,Z\}\}+\{Y,\{Z,X\}\}+\{Z,\{X,Y\}\}=0$, (Jacobi identity)
- $\{X,YZ\}=\{X,Y\}Z+Y\{X,Z\}$, (Leibniz rule)
- If $\{H,G\}(\mathbf{q}_0,\mathbf{p}_0)=0$ for all G, then $(\mathbf{q}_0,\mathbf{p}_0)$ is a critical point of H, (non-degeneracy)

Definition 1.1.3. *A* ***Poisson algebra*** *over* $\mathbb{R}$ *is a triple* $(A,\cdot,\{\})$*, where* $(A,\cdot)$ *is an associative* $\mathbb{R}$*-algebra and* $(A,\{\})$ *is a real Lie algebra, such that the identity*

$$\{X\cdot Y,Z\}=X\cdot\{Y,Z\}+\{X,Z\}\cdot Y$$

is satisfied for each $X,Y,Z\in A$*.*

In terms of Poisson bracket, we can rewrite Hamilton's equations as

$$\frac{dq_j}{dt}=\{H,q_j\},\quad \frac{dp_j}{dt}=\{H,p_j\},\ j=1,2,\dots,n \tag{1.1.6}$$

and the dynamics of any function $X(\mathbf{q},\mathbf{p})$ along the motion is

$$\frac{dX}{dt}=\{H,X\}. \tag{1.1.7}$$

Thus if $\{H,X\} = 0$, $X(\mathbf{q},\mathbf{p})$ is a constant of motion.

Definition 1.1.4. *Two functions X_1 and X_2 are in **involution** if their Poisson bracket is equal to zero.*

Definition 1.1.5. *Let $\mathbb{F} = \{f_1(\mathbf{q},\mathbf{p}), \ldots, f_N(\mathbf{q},\mathbf{p})\}$ be a set of N functions defined and locally analytic in some region of a $2n$-dimensional phase space. We say that $\mathbb{F}$ is **functionally independent** if the $N \times 2n$ Jacobian matrix $(\frac{\partial f_l}{\partial q_j}, \frac{\partial f_l}{\partial p_k})$ has rank N through the region.*

1.1.2. Integrable and superintegrable Classical systems

Definition 1.1.6. *In Classical mechanics an n-dimensional system with Hamiltonian H is **integrable** if it admits n integrals of motion $H, X_1, \ldots, X_{n-1}$ that are well defined functions on phase space, in involution*

$$\{H,X_k\} = 0, \{X_j,X_k\} = 0,\ j,k = 1, \ldots, n-1, \tag{1.1.8}$$

and that are functionally independent, meaning that

$$rang \frac{\partial(H,X_1, \ldots, X_{n-1})}{\partial(q_1, \ldots, q_n, p_1, \ldots, p_n)} = n \tag{1.1.9}$$

By the non-degeneracy of the Poisson bracket, there can be at most n functionally independent integrals of motion in involution [**4**].

The interest of integrable systems lies in Liouville's theorem. It states that such systems are integrable by quadratures and that motion is conditionally periodic, since the level sets of the system are tori whenever they are compact [**4**, Section 49].

Theorem 1.1.7 (Liouville)**.** *Let $X_1, \ldots, X_n$ be functions satisfying (1.1.8) and (1.1.9) on a $2n$ dimension phase space. Considering a level set of the functions X_i defined as*

$$M_{\mathbf{c}} = \{\mathbf{q} : X_i(\mathbf{q}) = c_i, i = 1, \ldots, n\}$$

Then

- *$M_{\mathbf{c}}$ is a smooth manifold, invariant under the phase flow with the Hamiltonian H.*
- *If the manifold $M_{\mathbf{c}}$ is compact and connected, then it is diffeomorphic to the n dimensional torus*

$$T^n = \{(\phi_1, \ldots, \phi_n) mod 2\pi\}$$

- *The phase flow with Hamiltonian H determines a conditionally periodic motion on $M_{\mathbf{c}}$, i.e. in angular coordinates $\phi = (\phi_1, \ldots, \phi_n)$ we have*

$$\frac{d\phi}{dt} = \omega, \, \omega = \omega(\mathbf{c})$$

- *The canonical equations with Hamiltonian H can be integrated by quadratures.*

Remark 1.1.8.

- *By **conditionally periodic motion**, we mean that there exists a dense subset of initial conditions for which the trajectories are periodic.*
- *In action-angle coordinates, ϕ can be integrated to*

$$\phi(t) = \phi(0) + \omega t,$$

therefore motion is periodic whenever $\omega \in (\mathbb{Q}/\mathbb{Z})^n$, which is a dense subset of T^n.

In this thesis we focus on classical systems that are polynomial in the momenta.

Definition 1.1.9. *An n-dimensional classical Hamiltonian system is **superintegrable** if it admits more integrals of motion than degrees of freedom. more specifically, if it admits $n+k$ integrals of motion $\{H, X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_k\}$ that are functionally independent and*

$$\{H, X_j\} = \{H, Y_l\} = 0, \, j = 1, \ldots, n, \; l = 1, \ldots, k. \tag{1.1.10}$$

It is minimally superintegrable if $k = 1$, and maximally superintegrable if $k = n - 1$.

By non-degeneracy of the Poisson bracket, in a $2n$-dimensional phase space, there can be at most $2n - 1$ functionally independent integrals of motions, including H.

The best known Superintegrable systems in 3-dimensional Euclidean space E_3 the harmonic oscillator $V(r) = \alpha r^2$, for $r = \sqrt{x^2 + y^2 + z^2}$. and the Kepler-Coulomb potential $V(r) = \frac{\alpha}{r}$. Here is a theorem concerning these two systems.

Theorem 1.1.10 (Bertrand's theorem)**.** *In 3-dimensional Euclidean space E_3, the only spherically symmetric potentials for which all bounded trajectories are closed are Kepler-Coulomb potential and harmonic oscillator.*

Lemma 1.1.11. *Let H be a Hamiltonian. Then the set of integrals of H is a Poisson sub-algebra of the Poisson algebra of smooth functions on phase space.*

1.2. Quantum Mechanics

In this section, we give basic concepts and definitions in Quantum mechanics and adapt the definitions of integrability and superintegrability from Classical mechanics.

1.2.1. Equations

In Quantum mechanics, physical states are represented as one dimensional subspaces in a complex Hilbert space. The dynamics of any time dependent non-relativistic physical system in Euclidean space is described by Schrödinger equation

$$H\Psi(\mathbf{x},t) = i\hbar\frac{d}{dt}\Psi(\mathbf{x},t) \tag{1.2.1}$$

where i is the imaginary unit, $\hbar$ is the reduced Planck constant and Ψ is the time-dependent wave function.

When one is interested in stationary states (quantum states with all observables independent of time) of a quantum system, (1.2.1) is reduced to the eigenvalue equation

$$H\Psi(\mathbf{x}) = E\Psi(\mathbf{x})$$

E is the eigenvalue of H and characterizes the energy spectrum of the Quantum system.

In analogy to classical Hamiltonian (1.1.1), we have the quantum Hamiltonian on n-dimensional Euclidean space as

$$H : L_2(\mathbb{R}^n,\mathbb{C}) \to L_2(\mathbb{R}^n,\mathbb{C})$$

In this model $q_j \to x_j, \quad p_j \to -i\hbar\partial_{x_j}$ and

$$H = -\frac{\hbar^2}{2m}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + V(\mathbf{x}) = -\frac{\hbar^2}{2m}\Delta_n + V(\mathbf{x}).$$

As in classical mechanics, we create most of our observables out of the quantities position and momentum. However, in quantum mechanics these quantities correspond to self-adjoint operators.

The Lie bracket of two operators X and Y is defined by $[X,Y] = XY - YX$. The position and momentum operators satisfy the commutation relations,

$$[x_j, x_k] = [p_j, p_k] = 0, [x_j, p_k] = i\hbar\delta_{jk}, \tag{1.2.2}$$

These relations define a Heisenberg algebra $\mathcal{H}_n$ in an n-dimensional space.

The expected value of an observable X in a state $\Psi(t)$ obeys the relation

$$\frac{d}{dt}\langle \Psi, X\Psi\rangle = \frac{i}{\hbar}\langle \Psi, [H,X]\Psi\rangle \tag{1.2.3}$$

where $\langle \cdot,\cdot\rangle$ is the inner product $\langle \Psi, \Phi\rangle = \int_{\mathbb{R}^n} \Psi(\mathbf{x},t)\overline{\Phi(\mathbf{x},t)}d\mathbf{x}$.

This is in analogy with the classical relation (1.1.7). Here the Lie bracket plays the role of the Poisson bracket.

Definition 1.2.1.

- *An observable quantity X is called an* ***integral of the motion*** *if it does not depend explicitly on time and commutes with H.*
- *An integral of motion is of order N if it is a hermitian operator of order N in the momenta.*

If $[H, X] = 0$, then we can choose a basis for the Hilbert space which is a set of simultaneous eigenfunctions of X and H [**83**]. It is important to determine a complete set of commuting observables to specify the system. This idea leads naturally to quantum integrability.

1.2.2. Integrable and superintegrable Quantum systems

Definition 1.2.2. *A quantum mechanical system in n dimensions is* ***integrable*** *if there exist n integrals of motion (including H) that are Hermitian operators in the enveloping algebra of the Heisenberg algebra with the basis $\{x_j, p_j, \hbar\}, j = 1,2,\ldots,n$, in involution w.r.t the Lie bracket*

$$[X_a,X_b] = X_aX_b - X_bX_a = 0 \quad \textit{where} \quad a,b = 1,\ldots,n$$

and algebraically independent in the sense that no Jordan polynomial formed entirely out of anti-commutators in X_a vanishes identically (i.e the operators are not zero set of any polynomials).

Definition 1.2.3. *A Quantum Hamiltonian system in n dimensions is* ***superintegrable****, if it admits $n + k$ with $k = 1,2,\ldots,n-1$; algebraically independent integrals of motion $Y_1 = H, Y_2, \ldots, Y_{n+k}$.*

Unlike in the case of classical superintegrability, there is no proof that $2n - 1$ is indeed the maximal number of possible algebraically independent symmetry operators however there are no known counter examples.

The set of integrals for a quantum Hamiltonian H forms a Poisson sub-algebra of the algebra of operators. In other words:

Lemma 1.2.4. *Let H be a Hamiltonian with integrals of motion L,K, and α, β be scalars. Then $\alpha L + \beta K$, LK and $[L,K]$ are also integrals of motion.*

For the $2n-1$ generators of a superintegrable system the Lie brackets cannot all vanish and the symmetry algebra has a non-Abelian structure [**83**].

Tempesta, Turbiner and Winternitz in 2011 conjectured that all maximally superintegrable systems are exactly solvable [**98**]. If this holds the bound state spectra can be calculated algebraically, without solving the Schrödinger equation, and wave functions can be expressed as polynomials in some appropriate variables. Another aspect of studying superintegrable quantum systems is that the non-Abelian polynomial algebra of integrals of motion can be used to calculate energy levels and gives information on wave functions.

Chapitre 2

Superintegrable systems in two dimensional Euclidean space

Following the results obtained for the harmonic and Coulomb potentials, a systematic study of integrable and superintegrable systems in a 2-dimensional and 3-dimensional Euclidean space was initiated in [**40, 63**]. These works study the classifications of superintegrable systems that have particular integrals of motion. Following these results there have been more studies and examples of superintegrable systems.

Here, our focus is on Hamiltonian systems in two dimensional Euclidean space. The Hamiltonian considered is

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(x,y) \tag{2.0.1}$$

where V is a real function on E_2.

If such a system admits an Nth order integral of motion X that is polynomial in momenta, then X can be written as

$$X = \sum_{j+k=0}^{N} \{f_{jk}(x,y), p_1^j p_2^k\} \tag{2.0.2}$$

where $p_1 = -i\hbar\partial_x$, $p_2 = -i\hbar\partial_y$ and $\{A,B\} = AB + BA$ is the anti-commutator. In quantum mechanics, the anti-commutator preserves the integral of motion as an Hermitian operator and in classical mechanics it is reduced to $2f_{jk}p_jp_k$ and therefore does not really have any influence on the value of the integrals of motion.

In this section we present the known results on the quadratic and cubic superintegrable systems in a $2D$ Euclidean space. In each cases, the integrals of motions are second and third order polynomials in the momenta, respectively.

2.1. Second order integrability and superintegrability

In this section we present the results from [**40, 63**]. First we consider the case in which the Hamiltonian (2.0.1) is integrable and it admits a second order integral of motion. Then we consider the case when this system has two integrals of motion of order two.

Let us consider the most general form of integral of motion in (2.0.2) when $N = 2$. The condition $[H,X] = 0$ implies that even order terms and odd terms in X commute independently with (2.0.1), so we will consider only the terms of even order in (2.0.2)

$$X = \sum_{j+k=2} \{f_{jk}, p_1^j p_2^k\} + g(x,y) \tag{2.1.1}$$

Further, the authors of [**40, 63**] found that the leading (second order) term in X lies in the enveloping algebra of the group $e(2)$. This group is generated by rotations and translations of the plane.

All this leads to the following form for X,

$$X = aL_3^2 + b\{L_3, p_1\} + c\{L_3, p_2\} + d(p_1^2 - p_2^2) + 2ep_1p_2 + g(x,y), \tag{2.1.2}$$

where a,b,c,d,e are arbitrary constants, and $L_3 = yp_1 - xp_2$. The function $g(x,y)$ satisfies the following determining equations

$$\begin{aligned} g_x &= 2(ay^2 + 2by + d)V_x - 2(axy + bx - cy - e)V_y, \\ g_y &= -2(axy + bx - cy - e)V_x + 2(ax^2 - 2cx - d)V_y \end{aligned} \tag{2.1.3}$$

The compatibility condition $g_{xy} = g_{yx}$ implies

$$\begin{aligned} &(axy + bx - cy - e)(V_{xx} - V_{yy}) + (a(y^2 - x^2) + 2by + 2cx + 2d)V_{xy} \\ &+ 3(ay + b)V_x - 3(ax - c)V_y = 0 \end{aligned} \tag{2.1.4}$$

Since none of the equations (2.1.3) and (2.1.4) involve the Planck constant $\hbar$, we get the same results by considering classical Hamiltonian and integral of motion in (2.0.1 & 2.1.1). Thus, for quadratic integrability and superintegrability the potential and integrals of motion coincide in classical and quantum mechanics.

The Hamiltonian (1.1.1) is form invariant under Euclidean transformations, so we can classify the integrals X into equivalence classes under rotations, translations and linear combinations with H. In [**40**] it is shown that we have four forms for integral (2.1.2). The family of potentials for those integrals are the following:

$$\begin{aligned} &V_C = f(x) + g(y), \qquad X_C = p_1^2 - p_2^2 - 2f(x) + 2g(y) \\ &V_S = f(r) + \frac{g(\theta)}{r^2}, \qquad X_S = L_3^2 + 2g(\theta), \quad (x,y) = (r\cos\theta, r\sin\theta) \\ &V_P = \frac{f(\xi) + g(\eta)}{\xi^2 + \eta^2}, \qquad X_P = \{L_3, p_1\} + \frac{f(\xi)\xi^2 - g(\eta)\eta^2}{\xi^2 + \eta^2}, \quad (x,y) = (\frac{\xi^2 - \eta^2}{2}, \xi\eta) \\ &V_E = \frac{f(\alpha) + g(\beta)}{\cos^2(\alpha) - \cosh^2(\beta)}, \qquad (x,y) = (l\cosh\beta\cos\alpha, l\sinh\beta\sin\beta); l > 0 \\ &X_E = L_3^2 + \frac{l^2}{2}(p_1^2 - p_2^2) - l^2\frac{(\cosh^2(\beta) + \sinh^2(\beta))f(\alpha) + (\cos^2(\alpha) - \sin^2(\alpha))g(\beta)}{\cos^2(\alpha) - \cosh^2(\beta)}. \end{aligned} \tag{2.1.5}$$

Hence, as we see, second order integrability in E_2 implies separation of variables in either Cartesian, polar, parabolic, or elliptic coordinates. A complete classification of systems whose second-order operators commute with the Hamiltonian has been done in [**63, 40**]. They show that two independent integrals of motion exist if we have potentials separating in at least two coordinate systems [**40**]. Four families of superintegrable potentials are as follows:

Separation in Cartesian, polar and elliptic coordinates,

$$V = \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}. \tag{2.1.6}$$

Separation in Cartesian and parabolic coordinates,

$$V = \alpha(4x^2 + y^2) + \beta x + \frac{\gamma}{y^2}. \tag{2.1.7}$$

Separation in polar and parabolic coordinates,

$$V = \frac{\alpha}{r} + \frac{1}{r^2}(\frac{\beta}{\cos^2\frac{\theta}{2}} + \frac{\gamma}{\sin^2\frac{\theta}{2}}). \tag{2.1.8}$$

Separation in two mutually orthogonal parabolic systems,

$$V = \frac{\alpha}{r} + \frac{1}{\sqrt{r}}(\beta\cos\frac{\theta}{2} + \gamma\sin\frac{\theta}{2}) = \frac{2\alpha + \sqrt{2(\beta\xi + \gamma\eta)}}{\xi^2 + \eta^2}. \tag{2.1.9}$$

A similar study was done in a 3-dimensional Euclidean space ([**34, 36, 63**]). In these works, a maximally superintegrable Hamiltonian which has 5 functionally independent integrals of motion is multiseparable, that is, separable in more than one coordinate system. More specifically, there exist 11 coordinate systems in which the system is separable, by considering different possibilities of quadratic integrals of motion.

Some work has linked superintegrability and exact solvability of a quantum system [**53, 96, 98**]. Everything leads to believe, of all known cases of superintegrable systems with scalar potential, that the superintegrability implies the solvability of the system. The same situation seems to occur in the case of non-scalar potentials [**16, 78, 105, 106, 95**], despite the fact that in these cases the existence of an integral of second order does not induce the separation of variables in the Hamiltonian.

In what follows, we will consider systems with integrals of motion order three. In this case, the differences between classical and quantum integrability are significant. As opposed to the quadratic integrability, cubic integrability does not impose separation of variables in the Hamiltonian.

2.2. Third order Integrability and superintegrability

In 1935, Jules Drach studied Hamiltonian systems admitting third order integrals of motion [**30**]. For further details we refer to the introduction section in our first article [**77**].

In this section we recall some results for superintegrable systems with third order integrals of motion [**66, 48**]. We consider the Hamiltonian as in (2.0.1) and the integral of motion of the form (2.0.2) for $N = 3$.

First, we characterize in a precise way the general form that an integral of motion must take according to the parity of its order.

From ([**48**]), in quantum mechanics if we have

$$\{f, p_1^j p_2^k\}^+ + i\{f, p_1^j p_2^k\}^- = \{\Re[f], p_1^j p_2^k\} + i\{Im[f], p_1^j p_2^k\},$$

the operator X in (2.0.2) can be written as the form

$$X = X^+ + iX^-.$$

where X^+ and X^- are self-adjoint operators. Since the Hamiltonian is self-adjoint and $[H,X] = 0$, then $X^\dagger = X^+ - iX^-$. Therefore both X^+, and X^- should commute with H. Since H is a real differential operator, the real and imaginary parts of X must commute with H separately [**91**]. This is summarized in the following proposition, from [**48**, proposition

3.1]).

Proposition 2.2.1. *For each self-adjoint integral of motion of order N, there exists one integral of order N with definite parity, i.e.,*

$$X = \sum_{j=0}^{[\frac{N}{2}]} \sum_{k=0}^{N-2j} \{F_{N-2j,k}(x,y), p_1^k p_2^{N-2j-k}\}, \tag{2.2.1}$$

where F is a real function.

By this proposition, it is sufficient to look for integrals of motion as

$$X = \sum_{j+k=3} \{f_{jk}(x,y), p_1^j p_2^k\} + \{g_1(x,y), p_1\} + \{g_2(x,y), p_2\}, \tag{2.2.2}$$

where f_{jk} are real functions.

This expression follows from the fact that the Hamiltonian and momentum operator are respectively real and imaginary, so the even and odd terms of an integral of motion commute separately with Hamiltonian. Thus, for any integral of order N in E_2, when N is even (resp. odd), lower order terms are also even (resp. odd). In the case of a classical integral, the result remains valid [**48**].

Imposing the condition $[H,X] = 0$, (2.2.2) is reduced to

$$X = \sum_{j+k+l=3} A_{jkl}\{L_3^j, p_1^k p_2^l\} + \{g_1(x,y), p_1\} + \{g_2(x,y), p_2\}, \tag{2.2.3}$$

for arbitrary real constants A_{jkl}. Also for g_1,g_2 and V we get

$$\begin{aligned} g_1 V_x + g_2 V_y =& \frac{\hbar^2}{4}\Big(f_1 V_{xxx} + f_2 V_{xxy} + f_3 V_{xyy} + f_4 V_{yyy} \\ & + 8A_{300}(xV_y - yV_x) + 2(A_{210}V_x + A_{201}V_y)\Big), \end{aligned} \tag{2.2.4}$$

$$\begin{aligned} (g_1)_x &= 3f_1(y)V_x + f_2(x,y)V_y \\ (g_2)_y &= f_3(x,y)V_x + 3f_4(x)V_y \\ (g_1)_y + (g_2)_x &= 2\Big(f_2(x,y)V_x + f_3(x,y)V_y\Big) \end{aligned} \tag{2.2.5}$$

where the real functions f_i are defined as

$$
\begin{aligned}
f_1(y) &= A_{300}y^3 - A_{210}y^2 + A_{120}y - A_{030} \\
f_2(x,y) &= -3A_{300}xy^2 + 2A_{210}xy - A_{210}y^2 - A_{120}x + A_{111}y - A_{021} \\
f_3(x,y) &= 3A_{300}x^2y - A_{210}x^2 + 2A_{201}xy - A_{111}x + A_{102}y - A_{012} \\
f_4(x) &= -A_{300}x^3 - A_{201}x^2 - A_{102}x - A_{003} \qquad (2.2.6)
\end{aligned}
$$

From(2.2.5), we obtain the following third order linear equation for the potential

$$
\begin{aligned}
0 = &- f_3V_{xxx} + (-2f_2 - 3f_4)V_{xxy} + (-3f_1 + 2f_3)V_{xyy} - f_2V_{yyy} + 2(f_{2y} - f_{3x})V_{xx} \\
&+ 2(-3f_{1y} + f_{2x} + f_{3y} - 3f_{4x})V_{xy} + 2(-f_{2y} + f_{3x})V_{yy} + (-3f_{1yy} + 2f_{2xy} - f_{3xx})V_x \\
&+ (-f_{2yy} + 2f_{3xy} - 3f_{4xx})V_y \qquad (2.2.7)
\end{aligned}
$$

The absence of the Planck constant $\hbar$ in equations (2.2.5) implies that in the classical case, one compatibility condition for the potential is (2.2.7). On the other hand, equation (2.2.4) involves $\hbar$, which by considering the $\lim \hbar \to 0$ will reduce to $g_1V_x + g_2V_y = 0$. Also, we get three more compatibility conditions for the potentials from equations (2.2.4 & 2.2.5), which are nonlinear and difficult to solve. In any case, we see that when the right hand side of (2.2.4) does not vanish, classical and quantum superintegrable systems with third and higher order integrals can be very different as has been presented in [**48**].

2.2.1. Superintegrable systems with one third order and one first order integral

It has been shown that a potential $V(x,y)$ in the Hamiltonian (2.0.1) allows an integral of first order in momenta if and only if it is invariant under the rotations and translations [**63**]. The potential and first order integrals are as follows:

$$V(x,y) = V(r); \quad r = \sqrt{x^2 + y^2}, \quad X = L_3,$$

and

$$V(x,y) = V(x), \quad X = p_2.$$

In this section, we briefly recall the obtained results in both cases [**48, 66**].
For the system with a potential invariant under rotations in both classical and quantum mechanics, from (2.2.4 & 2.2.5), we have $V(r) = \frac{\alpha}{r}$, and $V(r) = \alpha r^2$, the Coulomb potential and the harmonic oscillator respectively. This has been explained as well by Bertrand theorem

[**6**].

The third-order integrals of motion in the case with the Coulomb potential in E_2 can be obtained by commutation of the two following second-order integrals

$$X_1 = \{L_3,p_1\} - \frac{2\alpha y}{r},$$
$$X_2 = \{L_3,p_2\} + \frac{2\alpha x}{r}. \tag{2.2.8}$$

Also, the harmonic oscillator potential allows two second order integrals of motion of the form

$$X_1 = -\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \alpha x^2 - \alpha y^2,$$
$$X_2 = -p_1 p_2 + 2\alpha xy. \tag{2.2.9}$$

The Lie bracket (or Poisson bracket) of X_1 and X_2 gives the third-order integral.

In the case with the potential invariant under translations, setting $V_y = 0$ simplifies equations (2.2.4 & 2.2.5). The solutions are

$$V(x) = ax, \quad V(x) = \frac{a}{x^2}.$$

In classical mechanics, the Poisson bracket of the square of the first order and the second order integrals of motion associated with these potentials will be the third-order integrals. Thus, these systems are second order superintegrable.

In quantum mechanics, the third order integral of motion for the potential $V(x) = \frac{a}{x^2}$ is

$$X_1 = \{L_3^2,p_2\} + a\{2\frac{y^2}{x^2},p_2\},$$

and by commuting X_1^2 with the first order integral $X = p_2$ there will be two more as

$$X_2 = \{L_3,p_1p_2\} - a\{2\frac{y}{x^2},p_2\},$$

$$X_3 = 2p_1^2p_2 + a\{\frac{2}{x^2},p_2\}.$$

Setting $A_{ijk}=0$ for all A_{ijk} figuring in the following equations

$$(A_{210}x^2+A_{111}x+A_{012})V_{xxx}+4(2A_{210}x+A_{111})V_{xx}+12A_{210}V_x=0,$$

$$(3A_{300}x^2+2A_{201}x+A_{102})V_{xxx}+4(6A_{300}x+2A_{201})V_{xx}+36A_{300}V_x=0.$$

The other potential is obtained by solving the following equation

$$\hbar^2V'(x)^2=4V(x)^3+\alpha V(x)^2+\beta V(x)+\gamma\equiv 4(V-a_1)(V-a_2)(V-a_3), \qquad (2.2.10)$$

where $\alpha,\beta,\gamma,$ are real integration constants, and a_1,a_2,a_3 are either all real or one of them is real and the other two are complex conjugates. Depending on these constants, where $a_1\neq a_2\neq a_3$, and real, the finite and singular potentials are

$$V_1=(\hbar\omega)^2k^2sn^2(\omega x,k)\quad,\quad V_2=\frac{(\hbar\omega)^2}{sn^2(\omega x,k)}.$$

If we have, e.g., $a_3=a_2^*$ and $\Im a_2\neq 0$, the obtained singular potential is

$$V_3=\frac{(\hbar\omega)^2}{2\Big(cn^2(\omega x,k)+1\Big)}.$$

where $sn(\omega x,k)$ and $cn(\omega x,k)$ are Jacobi elliptic functions, and $0\leq k\leq 1$.
If some of the roots a_i coincide, e.g., $k=1$ in V_1 we get the soliton potential

$$V=\frac{(\hbar\omega)^2}{\cosh^2(\omega x)}.$$

Also, for $k=0$ or $k=1$ in V_2 we get a singular nonperiodic or a periodic potential in the form:

$$V=\frac{(\hbar\omega)^2}{\sinh^2(\omega x)}\quad,\quad V=\frac{(\hbar\omega)^2}{\sin^2(\omega x)}.$$

The two algebraically independent integrals for all solutions of (2.2.10) are

$$Y=p_2,\quad X=\{L_3,p_1^2\}+\{(\sigma-3V)y,p_1\}+\{-\sigma x+2xV+\int V(x)dx,p_2\}$$

with $\sigma=a_1+a_2+a_3$.

2.2.2. Superintegrable systems with one third order and one second order integral

In this case, in the Hamiltonian $H=\frac{1}{2}(p_1^2+p_2^2)+V(x,y)$, the potential $V(x,y)$ allows the separation of variables in Cartesian, polar, parabolic, or elliptic coordinates [**63, 40**].

2.2.3. Third order superintegrable systems separable in Cartesian coordinates

The Hamiltonians that allow the separation of variables in Cartesian coordinates will be of the form

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V_1(x) + V_2(y).$$

In (2.2.4) and (2.2.5), if we set $V = V_1(x) + V_2(y)$, the determining equations will be the following:

$$\begin{aligned}
g_1 V_{1x} + g_2 V_{2y} &= \frac{\hbar^2}{4}\Big(f_1 V_{1xxx} + f_4 V_{2yyy} + 8A_{300}(xV_{2y} - yV_{1x}) + 2(A_{210}V_{1x} + A_{201}V_{2y})\Big) \\
(g_1)_x &= 3f_1(y)V_{1x} + f_2(x,y)V_{2y} \\
(g_2)_y &= f_3(x,y)V_{1x} + 3f_4(x)V_{2y} \\
(g_1)_y + (g_2)_x &= 2\Big(f_2(x,y)V_{1x} + f_3(x,y)V_{2y}\Big)
\end{aligned}$$

We have

$$X = \frac{1}{2}(p_1^2 - p_2^2) + V_1(x) - V_2(y)$$

as the second order integral of motion. The classical potentials which do not admit enough first or second order integrals to make them superintegrable are

$$V = a(9x^2 + y^2), \quad V = c_1\sqrt{x} + c_2\sqrt{y}, \quad V = ay + b\sqrt{x},$$

$$V = ay^2 + g(x), \quad V = ay + f(x)$$

where $g(x)$ and $f(x)$ satisfy

$$(9g - ax^2)(g - ax^2)^3 - 2d(3g + ax^2)(g - ax^2) - cx^2 - d^2 = 0$$

and

$$(f - bx)^2 f = d$$

with a, b, c, d, c_1, c_2 constants.

In the quantum case there are more such potentials. There are six potentials which are expressed in terms of rational function, two by elliptic functions and five in terms of the first, second and fourth Painlevé transcendents. The superintegrable potentials and their third order integrals obtained by the above equations are presented in [**47**].

2.3. Fourth order superintegrability

In 2011, S.Post and P.Winternitz published an article on superintegrable systems in E_2 that does not allow separation of variables neither in the quantum case nor in its classical limit [**91**]. In this article, they consider a Hamiltonian with an integral of third order and an integral of fourth order and construct a superintgerable nonseparable system.

Admitting a third order integral lead to the third order integral of the form (2.2.3) and the determining equation in (2.2.4& 2.2.5 & 2.2.7). Since the potential does not allow separation of variables the determining equations are difficult to solve. Here they make an Ansatz that the potential is linear in y. In this case the potential and the third order integral in the quantum case are

$$\begin{aligned} V(x,y) &= \alpha\frac{y}{x^{\frac{2}{3}}} - \frac{5\hbar^2}{72x^2}, \\ X &= 3p_1^2p_2 + 2p_2^3 + \{\frac{9}{2}\alpha x^{\frac{1}{3}},p_1\} + \{\frac{3\alpha y}{x^{\frac{2}{3}}} - \frac{5\hbar^2}{24x^2},p_2\}, \end{aligned} \tag{2.3.1}$$

with $\alpha \in \mathbb{R}$. The classical potential and integral are obtained in the limit $\hbar \to 0$.

Now we impose a fourth order integral to the system. Using the results of Proposition (2.2.1) and considering that the leading term of the integral lies in the enveloping algebra of Euclidean Lie algebra $e(2)$ leads to the following form for integral [**91**, theorem3]

$$Y = \sum_{j+k+l=4} \frac{A_{jkl}}{2}\{L_3^j,p_1^kp_2^l\} + \frac{1}{2}(\{g_1(x,y),p_1^2\} + \{g_2(x,y),p_1p_2\} + \{g_3(x,y),p_2^2\}) + l(x,y), \tag{2.3.2}$$

where A_{jkl} are real constants and $g_i,i = 1,2,3, l$ are real functions of (x,y).

The relation $[H,Y] = 0$ provides 7 determining equations that are given in Section 2 of our first article [**77**].

The results for a nonseparable superintegrable system is presented in two Theorems [**91**, Theorem1,Theorem2]. The system with the potential and the third order integral as in (2.3.1) admits the fourth order integral of the form

$$Y = p_1^4 + \{\frac{2\alpha y}{x^{\frac{2}{3}}} - \frac{5\hbar^2}{36x^2},p_1^2\} + \{6\alpha x^{\frac{1}{3}},p_1p_2\} - 2\alpha^2\frac{(9x^2-2y^2)}{x^{\frac{4}{3}}} - \frac{5\alpha\hbar^2 y}{9x^{\frac{8}{3}}} + \frac{25\hbar^4}{1296x^4}. \tag{2.3.3}$$

The classical analog of this system would be as follows

$$\begin{aligned} H &= \frac{1}{2}(p_1^2+p_2^2)+\alpha\frac{y}{x^{\frac{2}{3}}},\\ X &= 3p_1^2p_2+2p_2^3+9\alpha x^{\frac{1}{3}}p_1+\frac{6\alpha y}{x^{\frac{2}{3}}}p_2,\\ Y &= p_1^4+\frac{4\alpha y}{x^{\frac{2}{3}}}p_1^2+12\alpha x^{\frac{1}{3}}p_1p_2-2\alpha^2\frac{(9x^2-2y^2)}{x^{\frac{4}{3}}}. \end{aligned} \tag{2.3.4}$$

The system in quantum case is the first known fourth order nonseparable superintegrable system. However the classical case is already known.

It is a special case of

$$V=\frac{1}{x^{\frac{2}{3}}}(a+by+c(4x^2+3y^2)). \tag{2.3.5}$$

which belongs to the family of potentials obtained by Drach in 1935 [**30**] (the constant a can be translated away for $b\neq 0$). For $c\neq 0$ (2.3.5) does not allow a fourth-order integral, though it still might be superintegrable.

Chapitre 3

Fourth order Superintegrable systems separating in Cartesian coordinates. I. Exotic quantum potentials

par

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Mes contribution et le rôle des coauteurs: Le sujet de la recherche a été suggérée par Pavel Winternitz. J'ai effectué les calculs nécessaires liés aux résultats obtenus. La rédaction se fait à parts égales entre les coauteurs.

Abstract. A study is presented of two-dimensional superintegrable systems separating in Cartesian coordinates and allowing an integral of motion that is a fourth order polynomial in the momenta. All quantum mechanical potentials that do not satisfy any linear differential equation are found. They do however satisfy nonlinear ODEs. We show that these equations always have the Painlevé property and integrate them in terms of known Painlevé transcendents or elliptic functions.

Keywords: Fourth order integral of motion, Painlevé transcendent, superintegrable system

3.1. Introduction

This article is part of a general program the aim of which is to derive, classify, and solve the equations of motion of superintegrable systems with integrals of motion that are polynomials of finite order N in the components of linear momentum. So far, we are concentrating on superintegrable systems with Hamiltonians of the form

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(x,y), \tag{3.1.1}$$

in two dimensional Euclidean space E_2. In classical mechanics, p_1 and p_2 are the canonical momenta conjugate to the Cartesian coordinates x and y. In quantum mechanics, we have

$$p_1 = -i\hbar\partial_x, \quad p_2 = -i\hbar\partial_y, \quad L_3 = xp_2 - yp_1. \tag{3.1.2}$$

The angular momentum L_3 is introduced because it will be needed below.

We recall that a superintegrable system has more integrals of motion than degrees of freedom (see [**79**] for a recent review with an extensive list of references). More precisely, a classical Hamiltonian system with n degrees of freedom is integrable if it allows n integrals of motion $\{X_1,X_2,...X_n\}$ (including the Hamiltonian) that are in involution, are well defined functions on the phase space and are functionally independent. It is superintegrable if further functionally independent integrals exist, $\{Y_1, Y_2, ...,Y_k\}$ with $1 \leq k \leq n-1$. The value $k = 1$ corresponds to "minimal superintegrability," $k = n-1$ to "maximal superintegrability." In quantum mechanics, the integrals are operators in the enveloping algebra of the Heisenberg algebra (or in some generalization of the enveloping algebra). In this article we assume that all integrals are polynomials in the momenta of the order $1 \leq j \leq N$, and at least one of them is of order N. We require the integrals to be algebraically independent, i.e no Jordan polynomial (completely symmetric) formed out of the $n+k$ integrals of motion can vanish identically.

In classical mechanics, all bounded trajectories in a maximally superintegrable system are closed [**82**], and the motion is periodic. In quantum mechanics, it has been conjectured by Tempesta, Turbiner and Winternitz [**98**] that all maximally superintegrable systems are exactly solvable. This means that the bound states spectra can be calculated algebraically and their wave functions expressed as polynomials in some appropriate variables (multiplied by an overall gauge factor).

The best known superintegrable systems in E_n, $n \geq 2$, correspond to the Kepler-Coulomb potential $V = \frac{\alpha}{r}$ (see [**39, 5**]) and the isotropic harmonic oscillator $V = \alpha r^2$ (see [**55, 81**]). A sizable recent literature on superintegrable systems has been published. It includes theoretical studies of such systems in Riemannian and pseudo-Riemannian spaces of arbitrary dimensions and with integrals of arbitrary order. The potentials are either scalar ones, or may involve vector potentials, or particles with spin [**14, 13, 28, 51, 52, 86, 85, 94, 64, 99**]. For recent applications of superintegrable systems in such diverse fields as particle physics, general relativity, statistical physics and the theory of orthogonal polynomials see [**27, 37, 38, 44, 57, 62, 80, 89**].

According to Bertrand's theorem, (see [**6, 45**]), the only spherically symmetric potentials (in E_3) for which all bounded trajectories are closed are precisely $\frac{1}{r}$ and $\omega^2 r^2$. Hence when searching for further superintegrable systems, we must go beyond spherically symmetrical potentials.

A systematic search for second order superintegrable systems in E_2 was started by Friš, Mandrosov, Smorodinsky, Uhlíř and Winternitz [**40**] and in E_3 by Makarov, Smorodinsky, Valiev and Winternitz [**63**] , and Evans [**34, 36**]. A relation between second order superintegrability and multiseparability of the Schrödinger or Hamilton-Jacobi equation was also established in these articles.

Most of the subsequent work was devoted to second order superintegrability (X and Y polynomials of order 2 in the momenta) and is reviewed in an article by Miller, Post and Winternitz [**79**]. The study of third order superintegrability (X of order 1 or 2, Y of order 3) started in

2002 by Gravel and Winternitz [**48, 47**], and new features were discovered. Third order integrals in classical mechanics in a complex plane were studied earlier by Drach and he found 10 such integrable systems [**30**]. The Drach systems were more recently studied by Rañada [**93**] and Tsiganov [**101**] who showed that 7 of the 10 systems are actually reducible. These 7 systems are second order superintegrable and the third order integral is a commutator (or Poisson commutator) of two second order ones.

The determining equations for the existence of an Nth order integral of motion in two-dimensional Euclidean space were derived by Post and Winternitz in [**92**]. The Planck constant $\hbar$ enters explicitly in the quantum case. The classical determining equations are obtained in the limit $\hbar \to 0$. The classical and quantum cases differ for $N \geq 3$ and in the classical case the determining equations are much simpler. The determining equations constitute a system of partial differential equations (PDE) for the potential $V(x,y)$ and for the functions $f_{ab}(x,y)$ multiplying the monomials $p_1^a p_2^b$ in the integral of motion. If $V(x,y)$ is given, the PDEs for $f_{ab}(x,y)$ are linear. If we are searching for potentials that allow an integral of order N the set of PDEs is nonlinear. A linear compatibility condition for the potential $V(x,y)$ alone was derived in [**92**]. It is an Nth order PDE with polynomial coefficients also of order up to N.

An interesting phenomenon was observed when studying third order superintegrable quantum systems in E_2. Namely, when the potential allows a third order integral and in addition a second order one (that leads to separation of variables in either Cartesian or polar coordinates) "exotic potentials" arise, (see [**48, 47, 100**]). These are potentials that do not satisfy any linear differential equation but instead satisfy nonlinear ordinary differential equations (ODEs). It turned out that all the ODEs obtained in the quantum case have the Painlevé property. That means that the general solution of these equations has no movable critical singularities (see [**58, 87, 43, 19, 20**]). It can hence be expanded into a Laurent series with a finite number of negative powers. The separable potentials were then expressed in terms of elliptic functions, or known (second order) Painlevé transcendents (i.e. the solutions of the Painlevé equations [**58**, page 345]).

We conjecture that this is a general feature of quantum superintegrable systems in two-dimensional Euclidean spaces. Namely, that if they allow an integral of motion of order $N \geq 3$ and also allow the separation of variables in Cartesian or polar coordinates, they will involve potentials that are solutions of ordinary differential equations that have the Painlevé property. All linear equations have this property by default, they have no movable singularities at all. Exotic potentials, on the other hand, are solutions of a genuinely nonlinear ODEs that have the Painlevé property.

The specific aim of this article is to test the above conjecture for superintegrable systems allowing one fourth order integral of motion and one second order one that leads to the separation of variables in Cartesian coordinates. We will determine all such exotic potentials and obtain their explicit expressions.

In Section 2, we present the set of 6 determining equations for the fourth order integral Y_L as well as a linear compatibility condition for 4 of these equations. This is a fourth order linear PDE for the potential $V(x,y)$. In Section 3, we impose the existence of an additional second order "Cartesian" integral that restricts the form of the potential to $V(x,y) = V_1(x) + V_2(y)$. The linear compatibility condition then reduces to two linear ODEs for $V_1(x)$ and two for $V_2(y)$. Section 4 is an auxiliary one. In it we review same basic facts about nonlinear equations with the Painlevé property that will be needed below (they come mainly from the references [**9, 10, 11, 12, 17, 15, 23, 25, 41**]). The main original results of this paper are contained in Section 5. We impose that the linear equation for at least one of the functions $V_1(x)$ or $V_2(y)$ be satisfied trivially (otherwise the potential would not be exotic.) This greatly simplifies the form of the integral Y (6 out of 10 free constants must vanish). The remaining linear and nonlinear determining equations can be solved exactly and completely. As expected, we find that the potentials satisfy nonlinear equations that pass the Painlevé test introduced by Ablowitz, Ramani, and Segur [**1**] (see also Kowalevski [**60**] and Gambier [**43**]). Using the results of [**17, 12, 25, 23**], we integrate these 4th order ODEs in terms of the original 6 Painlevé transcendents, elliptic functions, or solutions of linear equations. In Section 6, we study the classical analogs of exotic potentials. They satisfy first order ODEs that are polynomials of second degree in the derivative. Section 7 is devoted to conclusions and future outlook.

3.2. Determining equations and linear compatibility condition for a fourth order integral

The determining equations for fourth-order classical and quantum integrals of motion were derived earlier by Post and Winternitz [**91**] and they are a special case of Nth order ones given in [**92**]. In the quantum case, the integral is $Y^{(4)} = Y$:

$$Y = \sum_{j+k+l=4} \frac{A_{jkl}}{2}\{L_3^j,p_1^k p_2^l\} + \frac{1}{2}(\{g_1(x,y),p_1^2\} + \{g_2(x,y),p_1p_2\} + \{g_3(x,y),p_2^2\}) + l(x,y), \tag{3.2.1}$$

where A_{jkl} are real constants, the brackets $\{.,.\}$ denote anti-commutators and the Hermitian operators p_1,p_2 and L_3 are given in (3.1.2). The functions $g_1(x,y)$, $g_2(x,y)$, $g_3(x,y)$, and $l(x,y)$ are real and the operator Y is self adjoint. Equation (3.2.1) is also valid in classical mechanics where p_1, p_2 are the canonical momenta conjugate to x and y, respectively (and the symmetrization becomes irrelevant).

The commutation relation $[H,Y] = 0$ with H in (3.1.1) provides the determining equations

$$g_{1,x} = 4f_1V_x + f_2V_y \tag{3.2.2a}$$

$$g_{2,x} + g_{1,y} = 3f_2V_x + 2f_3V_y \tag{3.2.2b}$$

$$g_{3,x} + g_{2,y} = 2f_3V_x + 3f_4V_y \tag{3.2.2c}$$

$$g_{3,y} = f_4V_x + 4f_5V_y, \tag{3.2.2d}$$

and

$$\ell_x = 2g_1 V_x + g_2 V_y + \frac{\hbar^2}{4}\Big((f_2+f_4)V_{xxy} - 4(f_1-f_5)V_{xyy} - (f_2+f_4)V_{yyy}$$

$$+ (3f_{2,y} - f_{5,x})V_{xx} - (13 f_{1,y} + f_{4,x})V_{xy} - 4(f_{2,y} - f_{5,x})V_{yy}$$

$$- 2(6A_{400}x^2 + 62A_{400}y^2 + 3A_{301}x - 29A_{310}y + 9A_{220} + 3A_{202})V_x$$

$$+ 2(56A_{400}xy - 13A_{310}x + 13A_{301}y - 3A_{211})V_y\Big), \qquad (3.2.3a)$$

$$\ell_y = g_2 V_x + 2g_3 V_y + \frac{\hbar^2}{4}\Big(-(f_2+f_4)V_{xxx} + 4(f_1-f_5)V_{xxy} + (f_2+f_4)V_{xyy}$$

$$+ 4(f_{1,y} - f_{4,x})V_{xx} - (f_{2,y} + 13f_{5,x})V_{xy} - (f_{1,y} - 3f_{4,x})V_{yy}$$

$$+ 2(56A_{400}xy - 13A_{310}x + 13A_{301}y - 3A_{211})V_x$$

$$- 2(62A_{400}x^2 + 6A_{400}y^2 + 29A_{301}x - 3A_{310}y + 9A_{202} + 3A_{220})V_y\Big). \qquad (3.2.3b)$$

The quantities f_i, $i = 1,2,..,5$ are polynomials, obtained from the highest order term in the condition $[H,Y] = 0$, and explicitly we have

$$f_1 = A_{400}y^4 - A_{310}y^3 + A_{220}y^2 - A_{130}y + A_{040}$$

$$f_2 = -4A_{400}xy^3 - A_{301}y^3 + 3A_{310}xy^2 + A_{211}y^2 - 2A_{220}xy - A_{121}y + A_{130}x + A_{031}$$

$$f_3 = 6A_{400}x^2y^2 + 3A_{301}xy^2 - 3A_{310}x^2y + A_{202}y^2 - 2A_{211}xy + A_{220}x^2 - A_{112}y + A_{121}x + A_{022}$$

$$f_4 = -4A_{400}yx^3 + A_{310}x^3 - 3A_{301}x^2y + A_{211}x^2 - 2A_{202}xy + A_{112}x - A_{103}y + A_{013}$$

$$f_5 = A_{400}x^4 + A_{301}x^3 + A_{202}x^2 + A_{103}x + A_{004}.$$

(3.2.4)

For a known potential the determining equations (3.2.2) and (3.2.3) form a set of 6 linear PDEs for the functions g_1,g_2,g_3, and l. If V is not known, we have a system of 6 nonlinear PDEs for g_i,l and V. In any case the four equations (3.2.2) are a priori incompatible. The compatibility equation is a fourth-order linear PDE for the potential $V(x,y)$ alone, namely

$$\partial_{yyy}(4f_1V_x + f_2V_y) - \partial_{xyy}(3f_2V_x + 2f_3V_y) + \partial_{xxy}(2f_3V_x + 3f_4V_y) - \partial_{xxx}(f_4V_x + 4f_5V_y) = 0. \quad (3.2.5)$$

This is a special case of the Nth order linear compatibility equation obtained in [**92**]. We see that the equation (3.2.5) does not contain the Planck constant and is hence the same in quantum and classical mechanics (this is true for any N [**92**]). The difference between classical and quantum mechanics manifests itself in the two equations (3.2.3). They greatly simplify in the classical limit $\hbar \to 0$. Further compatibility conditions on the potential $V(x,y)$ can be derived for the systems (3.2.2) and (3.2.3), they will however be nonlinear. We will not go further into the problem of the fourth order integrability of the Hamiltonian (3.1.1). Instead, we turn to the problem of superintegrability formulated in the Introduction.

3.3. Potentials separable in Cartesian coordinates

We shall now assume that the potential in the Hamiltonian (3.1.1) has the form

$$V(x,y) = V_1(x) + V_2(y). \tag{3.3.1}$$

This is equivalent to saying that a second order integral exists which can be taken in the form

$$X = \frac{1}{2}(p_1^2 - p_2^2) + V_1(x) - V_2(y). \tag{3.3.2}$$

Equivalently, we have two one dimensional Hamiltonians

$$H_1 = \frac{p_1^2}{2} + V_1(x), \quad H_2 = \frac{p_2^2}{2} + V_2(y). \tag{3.3.3}$$

We are looking for a third integral of the form (3.2.1) satisfying the determining equations (3.2.2) and (3.2.3). This means that we wish to find all potentials of the form (3.3.1) that satisfy the linear compatibility condition (3.2.5). Once (3.3.1) is substituted, (3.2.5) is no longer a PDE and will split into a set of ODEs which we will solve for $V_1(x)$ and $V_2(y)$.
The task thus is to determine and classify all potentials of the considered form that allow the existence of at least one fourth order integral of motion. As in every classification we must avoid triviality and redundancy. Since H_1 and H_2 of (3.3.3) are integrals, we immediately obtain 3 "trivial" fourth order integrals, namely H_1^2, H_2^2, and $H_1 H_2$. The fourth order integral Y of equation (3.2.1) can be simplified by taking linear combination with polynomials in the second order integrals H_1 and H_2 of (3.3.3):

$$Y \to Y' = Y + a_1 H_1^2 + a_2 H_2^2 + a_3 H_1 H_2 + b_1 H_1 + b_2 H_2 + b_0, \quad a_i, b_i \in \mathbb{R}. \tag{3.3.4}$$

Using the constants a_1,a_2 and a_3 we set

$$A_{004} = A_{040} = A_{022} = 0, \tag{3.3.5}$$

in the integral Y we are searching for. At a later stage we will use the constants b_0,b_1 and b_2 to eliminate certain terms in g_1, g_3 and l.

Other trivial fourth order integrals are more difficult to identify. They arise whenever the potential (3.3.1) is lower order superintegrable i.e. in addition to (3.3.2), allows another second or third order integral. In such a case, the fourth order integral may be a commutator (or Poisson commutator) of two lower order ones. Such cases must be weeded out a posteriori. In our case this is actually quite simple. The exotic potentials separating in Cartesian coordinates and allowing an additional third order integral are listed as $Q16-Q20$ in [**47**]. For 4 of them the leading terms in the third order integral $Y^{(3)}$ has the form $ap_1^3+bp_2^3$ or $ap_1^3+bp_1^2p_2$. Hence commuting $Y^{(3)}$ with a second order integral H_1 can not give rise to a fourth order integral.

The remaining case is $Q18$ with

$$V(x,y)=a(y^2+x^2)-2\sqrt[4]{\frac{a^3}{2}\hbar^2}xP_4(-\sqrt[4]{\frac{2a}{\hbar^2}}x)+\sqrt{\frac{a}{2}}\hbar(\epsilon P_4'(-\sqrt[4]{\frac{2a}{\hbar^2}}x)+P_4^2(-\sqrt[4]{\frac{2a}{\hbar^2}}x)),\quad \epsilon=\pm1, \tag{3.3.6}$$

and integral

$$Y^{(3)}=\{L_3,p_1^2\}+\{ax^2y-3yV_1,p_1\}-\frac{1}{2a}\{\frac{\hbar^2}{4}V_{1xxx}+(ax^2-3V_1)V_{1x},p_2\}.$$

Commuting $Y^{(3)}$ with H_1 we obtain a fourth order integral

$$Y^{(4)}=2p_1^3p_2+... \tag{3.3.7}$$

Hence the potential (3.3.6) must appear (and does appear) in our present study, but the existence of (3.3.7) is a "trivial" consequence of third order superintegrability. However, an integral of the type (3.3.7) may appear for more general potentials than (3.3.6).

Two potentials will be considered equivalent if and only if they differ at most by translations of x and y.

Substituting (3.3.1) into the compatibility condition (3.2.5), we obtain a linear condition, relating the functions $V_1(x)$ and $V_2(y)$

$$\begin{aligned}
&(-60A_{310}+240yA_{400})V_1'(x)+(-20A_{211}+60yA_{301}-60xA_{310}+240xyA_{400})V_1''(x)+\\
&(-5A_{112}+10yA_{202}-10xA_{211}+30xyA_{301}-15x^2A_{310}+60x^2yA_{400})V_1^{(3)}(x)+\\
&(-A_{013}+yA_{103}-xA_{112}+2xyA_{202}-x^2A_{211}+3x^2yA_{301}-x^3A_{310}+4x^3yA_{400})V_1^{(4)}(x)+\\
&(-60A_{301}-2140xA_{400})V_2'(y)+(20A_{211}-60yA_{301}+60xA_{310}-240xyA_{400})V_2''(y)+\\
&(-5A_{121}++10yA_{211}-10xA_{220}-15y^2A_{301}+30xyA_{310}-60xy^2A_{400})V_2^{(3)}(y)+\\
&(A_{031}-yA_{121}+xA_{130}+y^2A_{211}-2xyA_{220}-y^3A_{301}+3xy^2A_{310}-4xy^3A_{400})V_2^{(4)}(y)=0.
\end{aligned} \tag{3.3.8}$$

It should be stressed that this is no longer a PDE, since the unknown functions $V_1(x)$ and $V_2(y)$ both depend on one variable only.
We differentiate (3.3.8) twice with respect to x and thus eliminate $V_2(y)$ from the equation. The resulting equation for $V_1(x)$ splits into two linear ODEs (since the coefficients contain terms proportional to y^0, and y^1), namely

$$\begin{aligned}
&210A_{310}V_1^{(3)}(x)+42(A_{211}+3A_{310}x)V_1^{(4)}(x)+7(A_{112}+2A_{211}x+3A_{310}x^2)V_1^{(5)}(x)\\
&+(A_{013}+A_{112}x+A_{211}x^2+A_{310}x^3)V_1^{(6)}(x)=0,
\end{aligned} \tag{3.3.9a}$$

$$\begin{aligned}
&840A_{400}V_1^{(3)}(x)+(126A_{301}+504A_{400}x)V_1^{(4)}(x)+14(A_{202}+3A_{301}x+6A_{400}x^2)V_1^{(5)}(x)\\
&+(A_{103}+2A_{202}x+3A_{301}x^2+4A_{400}x^3)V_1^{(6)}(x)=0.
\end{aligned} \tag{3.3.9b}$$

Similarly, differentiating (3.3.8) with respect to y we obtain two linear ODEs for $V_2(y)$,

$$\begin{aligned}
&210A_{301}V_2^{(3)}(y)-42(A_{211}-3A_{301}y)V_2^{(4)}(y)+7(A_{121}-2A_{211}y+3A_{301}y^2)V_2^{(5)}(y)\\
&-(A_{031}-A_{121}y+A_{211}y^2-A_{301}y^3)V_2^{(6)}(y)=0,
\end{aligned} \tag{3.3.10a}$$

$$\begin{aligned}
&840A_{400}V_2^{(3)}(y)-(126A_{310}-504A_{400}y)V_2^{(4)}(y)+14(A_{220}-3A_{310}y+6A_{400}y^2)V_2^{(5)}(y)\\
&-(A_{130}-2A_{220}y+3A_{310}y^2-4A_{400}y^3)V_2^{(6)}(y)=0.
\end{aligned} \tag{3.3.10b}$$

The compatibility condition $\ell_{xy} = \ell_{yx}$, for (3.2.3a) and (3.2.3b) implies

$$\begin{aligned}
&- g_2 V_1''(x) + g_2 V_2''(y) + (2g_{1y} - g_{2x})V_1'(x) + (g_{2y} - 2g_{3x})V_2'(y) + \\
&\frac{\hbar^2}{4}\Bigg((f_2 + f_4)(V_1^{(4)} - V_2^{(4)}) + (f_{2x} - 4f_1'(y))V_1^{(3)} + (4f_5'(x) - 5f_{2y} - f_{4y})V_2^{(3)} \\
&+ (3f_{2yy} + 4f_{4xx} + 6A_{211} - 26A_{301}y + 26A_{310}x - 112A_{400}xy)V_1'' \\
&- (4f_{2yy} + 3f_{4xx} + 6A_{211} - 26A_{301}y + 26A_{310}x - 112A_{400}xy)V_2'' \\
&+ (84A_{310} - 360A_{400}y)V_1' + (84A_{310} + 360A_{400}y)V_2' \Bigg) = 0. \qquad (3.3.11)
\end{aligned}$$

This equation, contrary to (3.3.9) and (3.3.10), is nonlinear since it still involves the unknown functions g_1, g_2, and g_3, (in addition to $V_1(x)$ and $V_2(y)$).
Our next task is to solve equations (3.3.9) and (3.3.10) and ultimately also (3.3.11) and the other determining equations. The starting point is given by the linear compatibility conditions (3.3.9) and (3.3.10) for $V_1(x)$ and $V_2(y)$. These are third order linear ODEs for the functions $W_1(x) = V_1^{(3)}(x)$, and $W_2(y) = V_2^{(3)}(y)$. They have polynomial coefficients and are easy to solve. Once the potentials are known, the whole problem becomes linear. However, the coefficients A_{jkl} (in the integral (3.2.1) and in (3.3.9) and (3.3.10)) may be such that the equations (3.3.9) or (3.3.10) vanish identically. Then the equations provide no information. This may lead to exotic potentials not satisfying any linear equation at all. In a previous study [**48, 47**] involving third order integrals, it was shown that all exotic potentials can be expressed in terms of elliptic functions or Painlevé transcendents. Here we will show that the same is true for integrals of order 4.

3.4. ODEs with the Painlevé property

In order to study exotic potentials $V(x,y) = V_1(x) + V_2(y)$, allowing fourth order integrals of motion in quantum mechanics we must first recall some known results on Painlevé type equations.

3.4.1. The Painlevé property, Painlevé test, and the classification of Painlevé type equations

An ODE has the Painlevé property if its general solution has no movable branch points, (i.e. branch points whose location depends on one or more constants of integration). We shall use the Painlevé test in the form introduced in [**1**]. For a review and further developments see Conte, Fordy, and Pickering [**18**], Conte [**19**], Conte and Musette [**20, 21**], Grammaticos and Ramani [**46**], Hone [**56**], Kruskal and Clarkson [**61**]. Passing the test is a necessary condition for having the Painlevé property. We shall need it only for equations of the form

$$W^{(n)} = F(y,W,W',W'',...,W^{(n-1)}), \tag{3.4.1}$$

where F is polynomial in $W,W',W'',...,W^{(n-1)}$ and rational in y. The general solution must have the form of a Laurent series with a finite number of negative power terms

$$W = \Sigma_{k=0}^{\infty} d_k (y - y_0)^{k+p},\ d_0 \neq 0, \tag{3.4.2}$$

satisfying the requirements

(1) The constant p is a negative integer.

(2) The coefficients d_k satisfy a recursion relation of the form

$$P(k)d_k = \phi_k(y_0,d_0,d_1,...,d_{k-1}),$$

where $P(k)$ is a polynomial that has $n-1$ distinct nonnegative integer zeros. The values of k_j for which we have $P(k_j) = 0$ are called resonances and the values of d_k for $k = k_j$ are free parameters. Together with the position y_0 at the singularity we thus have n free parameters in the general solution (3.4.2).

(3) A compatibility condition, also called the resonance condition:

$$\phi_k(y_0,d_0,d_1,...,d_{k-1}) = 0,$$

must be satisfied identically in y_0 and in the values of d_{k_j} for all $k_j; j = 1,2,...,n-1$. This test is a generalization of the Frobenius method used to study fixed singularities of linear ODEs (for the Frobenius method see e.g. the book by Boyce and Diprima [**7**]). Passing the Painlevé test is a necessary condition only. To make it sufficient one would have to prove that the series (3.4.2) has a nonzero radius of convergence and that the n free parameters can be used to satisfy arbitrary initial conditions. A more practical procedure that we shall adopt is the following. Once a nonlinear ODE passes the Painlevé test one can try to integrate it explicitly. The Riccati equation is the only first order and first degree equation which has the Painlevé property. A first order algebraic differential equation of degree n ≥ 1 has the form

$$A_0(W,y)W'^n + A_1(W,y)W'^{n-1} + ... + A_n(W,y) = 0, \tag{3.4.3}$$

where A_i are polynomials in W. When all solutions of such equation are free of movable branch points, the degree of polynomials A_i must satisfy $deg(A_i) \leq 2i$ for $i = 0,1,2,...,n$. The necessary and sufficient conditions for such equation to have the Painlevé property is given by the Fuchs' theorem (Theorem1.1,[**15**, page 80],proof in [**58**, page 304-311]). Painlevé type differential equations of the first order and nth degree have been studied in [**41**], [**58**, chapter 13]. All such equations are either reducible to linear equations or solvable in terms of elliptic functions. Painlevé type second order first degree equation are of the from

$$W'' = F(W',W,y),$$

where F is a polynomial of degree at most 2 in W', with coefficients that are rational in W, and analytic in y. They were classified by Painlevé and Gambier, (see [**58, 26**]). They can be solved in terms of solutions of linear equations, elliptic functions or in terms of the 6 irreducible Painlevé transcendents $P_1, P_2,...,P_6$.

Bureau initiated a study of ODEs of the form

$$A(W',W,y)W''^2 + B(W',W,y)W'' + C(W',W,y) = 0,$$

where A, B and C are polynomials in W, and W' with coefficients analytic in y, [**12**]. This work was continued by Cosgrove and Scoufis [**25**] who constructed all Painlevé type ODEs of the form

$$W''^2 = F(W',W,y),$$

where F is rational in W', and W and analytic in y. They also succeeded in integrating all of these equations in terms of known functions (including the six original Painlevé transcendents).

We will need to integrate equations of the form (3.4.1) for $n = 3$. Chazy in [**17**] studied the

Painlevé type third order differential equations in the polynomial class and proved that they have the form

$$W''' = aWW''+bW'^2+cW^2W'+dW^4+A(y)W''+B(y)WW'+C(y)W'+D(y)W^3+E(y)W^2+F(y)W+G(y), \tag{3.4.4}$$

where a,b,c, and d are certain rational or algebraic numbers, and the remaining coefficients are locally analytic functions of y.

Chazy and Bureau have determined all cases for the reduced equation, obtained by using the α-test, $(y,W) \to (y_0 + \alpha y, \frac{W}{\alpha})$ when $\alpha \to 0$, [**17**]. Chazy classified the reduced equations into 13 classes, denoted by Chazy class I-XIII. The list of these equations is in [**23**, page 181]. Each Chazy class is a conjugacy class of differential equations under transformations of the form

$$U(Y) = \lambda(y)W + \mu(y), \ Y = \phi(y).$$

In Section 5, we will encounter some fourth order differential equations, but we always succeed in integrating them to third order ones. We then transform to a Chazy-I equation. Cosgrove in [**23**] introduces the canonical form for Chazy-I equation as

$$\begin{aligned} W''' = &- \frac{f'(y)}{f(y)}W'' - \frac{2}{f^2(y)}\Big(3k_1y(yW'-W)^2 + k_2(yW'-W)(3yW'-W) + k_3W'(3yW'-2W) \\ &+ k_4(W')^2 + 2k_5y(yW'-W) + k_6(2yW'-W) + 2k_7W' + k_8y + k_9\Big), \end{aligned} \tag{3.4.5}$$

where $f(y) = k_1y^3 + k_2y^2 + k_3y + k_4$; Equation (3.4.5) admits the first integral,

$$\begin{aligned} (W'')^2 = &- \frac{4}{f^2(y)}\Big(k_1(yW'-W)^3 + k_2W'(yW'-W)^2 + k_3(W')^2(yW'-W) + k_4(W')^3 + k_5(yW'-W)^2 \\ &+ k_6W'(yW'-W) + k_7(W')^2 + k_8(yW'-W) + k_9W' + k_{10}\Big), \end{aligned} \tag{3.4.6}$$

where k_{10} is the constant of integration. In [**25**], Cosgrove and Scoufis give a complete classification of Painlevé type equations of second order and second degree. There are six classes of them, denoted by SD-I, SD-II,...,SD-VI. The equation (3.4.6), which is introduced as SD-I equation, splits into six canonical subcases (SD-Ia, SD-Ib, SD-Ic, SD-Id, SD-Ie, and SD-If). The solution of SD-Ia is expressed in terms of the sixth Painlevé transcendent. Here, we do not get any equation of this form. The solutions for the SD-Ib is expressed in terms of either the third or fifth Painlevé transcendent. The solutions of SD-1c, SD-Id, SD-Ie, and SD-If are, respectively, expressed in terms of P_4, P_2, P_1 and elliptic function [**25**, page 66]. These equations and their solutions appear in Section 5.

3.5. Search for exotic potentials in the quantum case

3.5.1. General comments

Let us first investigate the cases that may lead to "exotic potentials", that is potentials which do not satisfy any linear differential equations. That means that either (3.3.9) or (3.3.10)(or both) must be satisfied trivially. The linear ODEs (3.3.9) are satisfied identically if we have

$$A_{400} = A_{310} = A_{301} = A_{211} = A_{202} = A_{112} = A_{103} = A_{013} = 0. \tag{3.5.1}$$

The linear ODEs (3.3.10) are satisfied identically if we have

$$A_{400} = A_{310} = A_{301} = A_{211} = A_{220} = A_{121} = A_{130} = A_{031} = 0. \tag{3.5.2}$$

If (3.5.1) and (3.5.2) both hold then the only fourth order integrals are the trivial ones H_1^2, H_2^2 and $H_1 H_2$. Their existence does not assure superintegrability, it is simply a consequence of second order integrability. In other words, no fourth order superintegrable systems, satisfying (3.5.1) and (3.5.2) simultaneously, exist. This means that at most one of the functions $V_1(x)$ or $V_2(y)$ can be "exotic". The other one will be a solution of a linear ODE. For third order integrals both $V_1(x)$ and $V_2(y)$ could be exotic [**47**].

3.5.2. Linear equations for $V_2(y)$ satisfied trivially

3.5.2.1. *General setting and the three possible forms of* $V_1(x)$

In this case, (3.5.2) is valid and (3.5.1) not. The leading-order term for the nontrivial fourth order integral has the form

$$Y_L = A_{202}\{L_3^2, p_2^2\} + A_{112}\{L_3, p_1 p_2^2\} + A_{103}\{L_3, p_2^3\} + 2A_{013} p_1 p_2^3. \tag{3.5.3}$$

Let us classify the integrals (3.5.3) under translations. The three classes are:

$$
\begin{aligned}
&I. A_{202} \neq 0, A_{112} = A_{103} = 0.\\
&II. A_{202} = 0, A_{112}^2 + A_{103}^2 \neq 0, A_{013} = 0,\\
&\quad IIa. A_{103} \neq 0,\\
&\quad IIb. A_{103} = 0, A_{112} \neq 0.\\
&III. A_{202} = A_{112} = A_{103} = 0, A_{013} \neq 0.
\end{aligned}
\tag{3.5.4}
$$

The functions f_i in (3.2.4) reduce to

$$
\begin{aligned}
&f_1 = f_2 = 0,\\
&f_3(y) = A_{202}y^2 - A_{112}y,\\
&f_4(x,y) = -2A_{202}xy + A_{112}x - A_{103}y + A_{013},\\
&f_5(x) = A_{202}x^2 + A_{103}x.
\end{aligned}
\tag{3.5.5}
$$

Let us now extract all possible consequences from the determining equations (3.2.2). Using separability (3.3.1) we obtain

$$
\begin{aligned}
g_1(x,y) =& G_1(y),\\
g_2(x,y) =& \Big(- G_1'(y) + 2(A_{202}y^2 - A_{112}y)V_2'(y)\Big)x + G_2(y),\\
g_3(x,y) =& 2(A_{202}y^2 - A_{112}y)V_1(x) + \frac{1}{2}x(-10A_{202}xy + 5A_{112}x - 6A_{103}y + 6A_{013})V_2'(y)\\
&- x^2(A_{202}y^2 - A_{112}y)V_2''(y) + \frac{1}{2}x^2G_1''(y) - xG_2'(y) + G_3(y).
\end{aligned}
\tag{3.5.6}
$$

The functions that remain to be determined are $V_1(x)$,$V_2(y)$,$G_1(y)$,$G_2(y)$,$G_3(y)$, and $l(x,y)$. So far we have no information on $V_2(y)$, since equations (3.3.10) are satisfied trivially. The potential $V_1(x)$ must satisfy (3.3.9a) and (3.3.9b).

Let us substitute (3.5.5) and (3.5.6) into (3.2.2d). We obtain

$$
\begin{aligned}
&(4A_{202}y - 2A_{112})V_1 + (2A_{202}xy - A_{112}x + A_{103}y - A_{013})V_1' - (9A_{202}x^2 + 7A_{103}x)V_2'\\
&- (7A_{202}x^2y - \frac{7}{2}A_{112}x^2 + 3A_{103}xy - 3A_{013}x)V_2'' - (A_{202}x^2y^2 - A_{112}x^2y)V_2^{(3)}\\
&+ G_3'(y) - xG_2''(y) + \frac{1}{2}x^2G_1^{(3)}(y) = 0.
\end{aligned}
\tag{3.5.7}
$$

Differentiating (3.5.7) three times with respect to x and requiring that terms proportional to y and independent of y vanish separately, we obtain two equations for $V_1(x)$ namely

$$5A_{112}V_1^{(3)}(x) + (A_{013} + A_{112}x)V_1^{(4)}(x) = 0, \tag{3.5.8a}$$

$$10A_{202}V_1^{(3)}(x) + (A_{103} + 2A_{202}x)V_1^{(4)}(x) = 0. \tag{3.5.8b}$$

(They replace equations (3.3.9)). These two equations imply $V_1^{(3)} = V_1^{(4)} = 0$ unless we have

$$A_{112}A_{103} - 2A_{202}A_{013} = 0. \tag{3.5.9}$$

If (3.5.9) is not satisfied, the only solution of (3.5.8) is $V_1(x) = c_0 + c_1x + c_2x^2$. We can always put $c_0 = 0$. If $c_2 \neq 0$ we can translate x to set $c_1 = 0$. Thus, with no loss of generality we can in this case put

$$V_1^{(a)}(x) = c_1x + c_2x^2; \quad c_1c_2 = 0. \tag{3.5.10}$$

This case will be investigated separately below in the section (3.5.2.3).

Now let us assume that (3.5.9) is satisfied and consider the 3 cases in (3.5.4) separately.

I. $A_{202} \neq 0, A_{112} = A_{103} = 0, Y_L = A_{202}\{L_3^2,p_2^2\}$.

The condition (3.5.9) implies $A_{013} = 0$ and from (3.5.8) we obtain

$$V_1^{(b)}(x) = \frac{c_{-2}}{x^2} + c_1x + c_2x^2;\ c_{-2} \neq 0. \tag{3.5.11}$$

For $c_{-2} = 0$, $V_1^{(b)}$ reduces to the case $V_1^{(a)}$ of (3.5.10) .

II. $A_{202} = 0, A_{112}^2 + A_{103}^2 \neq 0, A_{013} = 0$.

The condition (3.5.9) implies $A_{112}A_{103} = 0$, so we have 2 subcases

IIa. $A_{103} \neq 0, A_{112} = 0,\ Y_L = A_{103}\{L_3,p_2^3\}$.

The solution for (3.5.8) is

$$V_1^{(c)}(x) = c_1x + c_2x^2 + c_3x^3,$$

however (3.5.7) implies $c_3 = 0$. So in this case $V_1^{(c)}$ is reduced to $V_1^{(a)}$.

IIb. $A_{103} = A_{013} = 0, A_{112} \neq 0, Y_L = A_{112}\{L_3,p_1p_2^2\}$.

The potential $V_1(x) = V_1^{(b)}(x)$ and satisfies (3.5.11).

III. $A_{202} = A_{112} = A_{103} = 0, A_{013} \neq 0, Y_L = 2A_{013}p_1p_2^3$.

The potential is $V_1(x) = V_1^{(a)}(x)$ of (3.5.10).

Let us now return to the determining equations (3.2.3) and their compatibility condition (3.3.11). We substitute (3.5.5) and (3.5.6) into (3.3.11) and obtain

$$\begin{aligned}
&3G_1'(y)V_1' + 3(G_2'(y) - xG_1''(y))V_2' + 6(A_{112}y - A_{202}y^2)V_1'V_2' \\
&+ (xG_1'(y) - G_2(y))(V_1'' - V_2'') + 6(-A_{013} + A_{103}y - 2A_{112}x + 4A_{202}xy)(V_2')^2 \\
&+ 2x(A_{112}y - A_{202}y^2)V_2'V_1'' + 8(A_{202}xy^2 - A_{112}xy)V_2'V_2'' \\
&+ \frac{\hbar^2}{4}\Big((5A_{112} - 10A_{202}y)V_1^{(3)} + (5A_{103} + 10A_{202}x)V_2^{(3)} \\
&+ (2A_{202}xy - A_{112}x + A_{103}y - A_{013})(V_2^{(4)} - V_1^{(4)})\Big) = 0.
\end{aligned} \tag{3.5.12}$$

So far we have identified possible forms of the potential $V_1(x)$ in the case when the linear equations (3.3.10) for $V_2(y)$ are satisfied trivially. Now we shall consider the two classes of potentials V_1^a, and V_1^b separately and obtain nonlinear ODEs for $V_2(y)$. Our main tool for solving these nonlinear ODEs will be singularity analysis. More precisely, we will show that these equations always pass the Painlevé test. The same was true in the case of third order integrals of motion. It was shown that the ODEs actually have the Painlevé property and they were solved in terms of known Painlevé transcendents, or elliptic functions [**47, 48**]. We will now show that the same is true in this case.

We define the function

$$W(y) = \int V_2 dy, \tag{3.5.13}$$

and derive ODEs for $W(y)$. Since the potential $V_2(y)$ is defined up to a constant, two integrals $W_1(y)$ and $W_2(y)$ will be considered equivalent if they satisfy

$$W_2(y) = W_1(y) + \alpha y + \beta;\ \alpha, \beta \in \mathbb{R} \tag{3.5.14}$$

The ODEs for $W(y)$ will a priori be fourth order nonlinear ones but we will always be able to integrate them once.

3.5.2.2. *The potential* $V_1^{(b)}(x) = \frac{c_{-2}}{x^2} + c_1 x + c_2 x^2,\ c_{-2} \neq 0$

The potential $V_1^{(b)}$ provides interesting results. It occurs in cases I, and IIb of (3.5.4). Solving (3.5.7) and (3.5.12) and using (3.3.4) we obtain

$$G_1(y) = 2y(yA_{202} - A_{112})W' + (2yA_{202} - A_{112})W - \frac{2}{3}c_2 A_{202} y^4 + \frac{4}{3}c_2 A_{112} y^3 + a_2 y^2 + a_1 y, \tag{3.5.15}$$

where $W(y)$ is defined in (3.5.13) and moreover we obtain $c_1 = G_2(y) = G_3(y) = 0$. The function $W(y)$ satisfies the ODE

$$\begin{aligned}
&\frac{1}{4}\hbar^2(2A_{202}y - A_{112})W^{(4)} + 2\hbar^2 A_{202} W^{(3)} - 3(2A_{202}y - A_{112})W'W'' \\
&- 2A_{202}WW'' + (\frac{8}{3}c_2 A_{202} y^3 - 4c_2 A_{112} y^2 - 2a_2 y - a_1)W'' - 8A_{202}W'^2 \\
&+ 4(4c_2 A_{202} y^2 - 4c_2 A_{112} y - a_2)W' + 8c_2(2A_{202}y - A_{112})W \\
&- \frac{16}{3}c_2^2 A_{202} y^4 + \frac{32}{3}c_2^2 A_{112} y^3 + 8a_2 c_2 y^2 + 8a_1 c_2 y + k = 0,
\end{aligned} \tag{3.5.16}$$

where k is an integration constant.

Case I. $A_{202} \neq 0, A_{112} = 0; Y_L = A_{202}\{L_3^2, p_2^2\}$.

Let $A_{202} = 1$. From (3.5.16) and (3.5.14) we obtain

$$\begin{aligned}
&\frac{1}{2}\hbar^2 y W^{(4)} + 2\hbar^2 W^{(3)} - 6yW'W'' - 2WW'' + \frac{8}{3}c_2 y^3 W'' - 8W'^2 + 16c_2 y^2 W' \\
&+ 16c_2 y W - \frac{16}{3}c_2^2 y^4 + k_1 = 0,
\end{aligned} \tag{3.5.17}$$

integrating once we get

$$\begin{aligned}
&\hbar^2 y^2 W^{(3)} + 2\hbar^2 y W'' - 6y^2 W'^2 - 4yWW' + (\frac{16}{3}c_2 y^4 - 2\hbar^2)W' + 2W^2 + \frac{32}{3}c_2 y^3 W \\
&- \frac{16}{9}c_2^2 y^6 + k_1 y^2 + k_2 = 0.
\end{aligned} \tag{3.5.18}$$

The equation (3.5.18) passes the Painlevé test. Substituting the Laurent series (3.4.2) into (3.5.18), we find $p = -1$. The resonances are $r = 1$, and $r = 6$, and we obtain $d_0 = -\hbar^2$. The constants d_1 and d_6 are arbitrary, as they should be. We now proceed to integrate (3.5.18). By the following transformation

$$Y = y^2, U(Y) = -\frac{y}{2\hbar^2}W(y) + \frac{c_2}{6\hbar^2}y^4 + \frac{1}{16},$$

we transform (3.5.18) to

$$Y^2U^{(3)} = -2(U'(3YU' - 2U) - \frac{c_2}{\hbar^2}Y(YU' - U) + k_3Y + k_4) - YU'', \tag{3.5.19}$$

where $k_3 = \frac{-2k_1-12c_2\hbar^2}{64\hbar^4}$, $k_4 = \frac{-k_2}{32\hbar^4}$. The equation (3.5.19) is a special case of the Chazy class I equation. It admits the first integral

$$Y^2U''^2 = -4(U'^2(YU' - U) - \frac{c_2}{2\hbar^2}(YU' - U)^2 + k_3(YU' - U) + k_4U' + k_5), \tag{3.5.20}$$

where k_5 is the integration constant. The equation is the canonical form SD-I.b in [**25**, page 65-73]. When c_2 and k_3 are both nonzero the solution is

$$\begin{aligned} U =&\frac{1}{4}(\frac{1}{P_5}(\frac{YP_5'}{P_5-1} - P_5)^2 - (1-\sqrt{2\alpha})^2(P_5-1) - 2\beta\frac{P_5-1}{P_5} + \gamma Y\frac{P_5+1}{P_5-1} + 2\delta\frac{Y^2P_5}{(P_5-1)^2}), \\ U' =&-\frac{Y}{4P_5(P_5-1)}(P_5' - \sqrt{2\alpha}\frac{P_5(P_5-1)}{Y})^2 - \frac{\beta}{2Y}\frac{P_5-1}{P_5} - \frac{1}{2}\delta Y\frac{P_5}{P_5-1} - \frac{1}{4}\gamma, \end{aligned} \tag{3.5.21}$$

where $P_5 = P_5(Y); Y = y^2$, satisfies the fifth Painlevé equation

$$P_5'' = (\frac{1}{2P_5} + \frac{1}{P_5-1})P_5'^2 - \frac{1}{Y}P_5' + \frac{(P_5-1)^2}{Y^2}(\alpha P_5 + \frac{\beta}{P_5}) + \gamma\frac{P_5}{Y} + \delta\frac{P_5(P_5+1)}{P_5-1},$$

with

$$c_2 = -\hbar^2\delta,\ k_3 = -\frac{1}{4}(\frac{1}{4}\gamma^2 + 2\beta\delta - \delta(1-\sqrt{2\alpha})^2),\ k_4 = -\frac{1}{4}(\beta\gamma + \frac{1}{2}\gamma(1-\sqrt{2\alpha})^2),$$

$$k_5 = -\frac{1}{32}(\gamma^2((1-\sqrt{2\alpha})^2 - 2\beta) - \delta((1-\sqrt{2\alpha})^2 + 2\beta)^2).$$

The solution for the potential up to a constant is

$$\begin{aligned} V(x,y) =&\frac{c_{-2}}{x^2} - \delta\hbar^2(x^2+y^2) + \hbar^2\Big(\frac{\gamma}{P_5-1} + \frac{1}{y^2}(P_5-1)(\sqrt{2\alpha} + \alpha(2P_5-1) + \frac{\beta}{P_5}) \\ &+ y^2(\frac{P_5'^2}{2P_5} + \delta P_5)\frac{(2P_5-1)}{(P_5-1)^2} - \frac{P_5'}{P_5-1} - 2\sqrt{2\alpha}P_5'\Big) + \frac{3\hbar^2}{8y^2}. \end{aligned} \tag{3.5.22}$$

And we have

$$
\begin{aligned}
g_1(x,y) =& 2y^2W' + 2yW + \frac{2}{3}\hbar^2\delta y^4,\ g_2(x,y) = -x(6yW' + 2W + \frac{8}{3}\hbar^2\delta y^3),\\
g_3(x,y) =& 4x^2W' + 2\hbar^2\delta x^2y^2 + 2c_{-2}\frac{y^2}{x^2},\\
l(x,y) =& \hbar^2x^2(\frac{1}{4}yW^{(4)} + W^{(3)}) - x^2(3yW' + W)W'' - (\frac{4}{3}\hbar^2\delta x^2y^3 + \frac{3\hbar^2}{2}y)W''\\
&+ (4(\frac{c_{-2}}{x^2} - \hbar^2\delta x^2)y^2 - 3\hbar^2)W' + 4y(\frac{c_{-2}}{x^2} - \hbar^2\delta x^2)W + \frac{4c_{-2}}{3x^2}\hbar^2\delta y^4\\
&- 2\hbar^2\delta x^2(\frac{2}{3}\hbar^2\delta y^4 - \hbar^2) - 2\hbar^4\delta y^2.
\end{aligned}
\tag{3.5.23}
$$

The solution of (3.5.20) when $c_2 = 0$ is

$$
\begin{aligned}
U =& \frac{1}{4}(\frac{1}{P^2}(YP' - P)^2 - \frac{1}{16}\alpha P^2 - \frac{1}{8}(\beta + 2\sqrt{\alpha})P + \frac{1}{8P}\gamma Y + \frac{1}{16P^2}\delta Y^2),\\
U' =& -\frac{1}{4}\sqrt{\alpha}P' - \frac{1}{8Y}(\alpha P^2 + \beta P),\\
k_3 =& \frac{1}{64}\alpha\delta,\ k_4 = -\frac{1}{64}\gamma(\beta + 2\sqrt{\alpha}),\ k_5 = -\frac{1}{1024}(\alpha\gamma^2 - \delta(\beta + 2\sqrt{\alpha})^2),
\end{aligned}
\tag{3.5.24}
$$

where $P(Y) = yP_3(y)$, and P_3 satisfies the third Painlevé equation

$$
P_3'' = \frac{P_3'^2}{P_3} - \frac{P_3'}{y} + \alpha P_3^3 + \frac{\beta P_3^2 + \gamma}{y} + \frac{\delta}{P_3}.
$$

The solution for the potential is

$$
V(x,y) = \frac{c_{-2}}{x^2} + \frac{\hbar^2}{2}(\sqrt{\alpha}P_3' + \frac{3}{4}\alpha P_3^2 + \frac{\delta}{4P_3^2} + \frac{\beta P_3}{2y} + \frac{\gamma}{2P_3y} - \frac{P_3'}{2yP_3} + \frac{P_3'^2}{4P_3^2}). \tag{3.5.25}
$$

And we have

$$
\begin{aligned}
g_1(x,y) =& 2y^2W' + 2yW,\ g_2(x,y) = -6xyW' - 2xW,\\
g_3(x,y) =& 4x^2W' + 2c_{-2}\frac{y^2}{x^2},\\
l(x,y) =& \hbar^2x^2(\frac{1}{4}yW^{(4)} + W^{(3)}) - x^2(3yW' + W)W'' - \frac{3}{2}\hbar^2yW'' + (4\frac{c_{-2}}{x^2}y^2 - 3\hbar^2)W' + 4\frac{c_{-2}}{x^2}yW.
\end{aligned}
\tag{3.5.26}
$$

Case IIb. $A_{202} = 0, A_{112} \neq 0; Y_L = A_{112}\{L_3,p_1p_2^2\}$.

Let $A_{112} = 1$. In this case,

$$
\begin{aligned}
g_1(x,y) = &- 2yW' - W + \frac{4}{3}c_2y^3 + a_2y^2,\ g_2(x,y) = 3xW' - 4c_2xy^2 - 2a_2xy,\\
g_3(x,y) = &\ 2c_2x^2y + a_2x^2 - 2c_{-2}\frac{y}{x^2},\\
l(x,y) = &-\frac{1}{8}\hbar^2x^2W^{(4)} + \frac{3}{2}x^2W'W'' - (2c_2x^2y^2 + a_2x^2y - \frac{3}{4}\hbar^2)W'' - 2(2c_{-2}\frac{y}{x^2} + 2c_2x^2y)W'\\
&- 2(\frac{c_{-2}}{x^2} + c_2x^2)W + 2x^2(\frac{4}{3}c_2^2y^3 + a_2c_2y^2) + 2\frac{c_{-2}}{x^2}(\frac{4}{3}c_2y^3 + a_2y^2) - 2c_2\hbar^2y.
\end{aligned}
\tag{3.5.27}
$$

Integrating the equation (3.5.16) we get

$$
\hbar^2W^{(3)} - 6W'^2 + 8(2c_2y^2 + a_2y)W' + 8(4c_2y + a_2)W - \frac{32}{3}c_2^2y^4 - \frac{32}{3}c_2a_2y^3 + k_1y + k_2 = 0. \tag{3.5.28}
$$

The equation (3.5.28) passes the Painlevé test. Substituting the Laurent series (3.4.2) into (3.5.28), we obtain $p = -1$. The resonances are $r = 1$, and $r = 6$, and $d_0 = -\hbar^2$. The constants d_1 and d_6 are arbitrary. By an appropriate linear transformation of the form

$$
Y = \lambda_1y + \lambda_2,\ U(Y) = \lambda_3W(y) + \mu(y),
$$

we transform the equation (3.5.28) into a special case of the canonical form for the Chazy class I. The general form of the equation is

$$
U^{(3)} = -2(3U'^2 + 2k_3Y(YU' - U) + k_4(2YU' - U) + 2k_5U' + k_6Y + k_7), \tag{3.5.29}
$$

Depending on the choice of λ_1,λ_2, λ_3 and μ, the parameters k_3,k_4,k_5,k_6, and k_7 get different values, and the first integral of the equation (3.5.29) with respect to Y corresponds to one of the four canonical subcases, listed below.

For $c_2 \neq 0, k_3 = -1, k_4 = k_5 = k_6 = 0$, we get equation $SD - I.c$:

$$
U''^2 = -4(U'^3 - (YU' - U)^2 + k_7U' + k_8), \tag{3.5.30}
$$

where k_8 is the integration constant. The solution for the equation $SD-I.c$ is

$$\begin{aligned}U =&\frac{1}{8P_4}P_4'^2-\frac{1}{8}P_4^3-\frac{1}{2}YP_4^2-\frac{1}{2}(Y^2-\alpha+\epsilon)P_4+\frac{1}{3}(\alpha-\epsilon)Y+\frac{\beta}{4P_4},\\ U' =&-\frac{1}{2}\epsilon P_4'-\frac{1}{2}P_4^2-YP_4+\frac{1}{3}(\alpha-\epsilon),\end{aligned} \tag{3.5.31}$$

where

$$\epsilon=\pm 1, k_7=-\frac{1}{3}(\alpha-\epsilon)^2-2\beta, k_8=\frac{1}{3}(\alpha-\epsilon)(\beta+\frac{2}{9}(\alpha-\epsilon)^2),$$

and $P_4=P_4(-\sqrt[4]{\frac{8c_2}{\hbar^2}}y-\frac{a_2}{2\sqrt[4]{2c_2^3\hbar^2}})$, satisfies the fourth Painlevé equation (for arbitrary α and β)

$$P_4''=\frac{P_4'^2}{2P_4}+\frac{3}{2}P_4^3+4YP_4^2+2(Y^2-\alpha)P_4+\frac{\beta}{P_4}.$$

Therefore, the solution for potential is

$$V(x,y) =2a_2y+c_2(x^2+4y^2)+\frac{c_{-2}}{x^2}-\frac{\sqrt[4]{2}a_2\sqrt{\hbar}P_4}{\sqrt[4]{c_2}}-4\sqrt[4]{2c_2^3}\sqrt{\hbar}yP_4+\sqrt{2c_2}\hbar(\epsilon P_4'+P_4^2). \tag{3.5.32}$$

For $a_2\neq 0, c_2=k_3=k_5=k_6=k_7=0, k_4=\frac{1}{2}$ we obtain equation $SD-I.d$:

$$U''^2=-4U'^3-2U'(YU'-U)+k_8. \tag{3.5.33}$$

The solution for the equation $SD-I.d$ is

$$\begin{aligned}U =&\frac{1}{2}(P_2')^2-\frac{1}{2}(P_2^2+\frac{1}{2}Y)^2-(\alpha+\frac{1}{2}\epsilon)P_2,\\ U' =&-\frac{1}{2}(\epsilon P_2'+P_2^2+\frac{1}{2}Y),\end{aligned} \tag{3.5.34}$$

where $k_8=\frac{1}{4}(\alpha+\frac{1}{2}\epsilon)^2$, and $P_2(Y)=P_2(-2\sqrt[3]{\frac{a_2}{\hbar^2}}y-\frac{3k_1}{16\sqrt[3]{a_2^5\hbar^2}})$, satisfies the second Painlevé equation

$$P_2''=2P_2^3+YP_2+\alpha.$$

Therefore, the solution for potential is

$$V(x,y) =\frac{c_{-2}}{x^2}+2\sqrt[3]{a_2^2\hbar^2}(\epsilon P_2'+P_2^2). \tag{3.5.35}$$

For $c_2 = a_2 = k_3 = k_4 = k_5 = k_7 = 0, k_6 = \frac{1}{2}$, we get equation $SD - I.e$:

$$U''^2 = -4U'^3 - 2(YU' - U). \tag{3.5.36}$$

The solution for the equation $SD - I.e$ is

$$U = \frac{1}{2}(P_1')^2 - 2P_1^3 - YP_1, \quad U' = -P_1.$$

The function $P_1(Y) = P_1(-\sqrt[5]{\frac{k_1}{\hbar^4}}y - \frac{k_2}{\sqrt[5]{k_1^4\hbar^4}})$, satisfies the first Painlevé equation

$$P_1'' = 6P_1^2 + Y,$$

and we have

$$V(x,y) = \frac{c_{-2}}{x^2} + \sqrt[5]{k_1^2\hbar^2}P_1. \tag{3.5.37}$$

For $c_2 = a_2 = k_3 = k_4 = k_5 = k_6 = 0$, we obtain equation $SD - I.f$:

$$U''^2 = -4(U'^3 + k_7U' + k_8). \tag{3.5.38}$$

The solution for the equation $SD - I.f$ is

$$U = -\int u dy + \alpha_1, \quad u = \wp(y - \alpha_2, -4k_7, 4k_8)$$

where α_1, α_2 are integration constants, and $\wp$ is the Weierstrass elliptic function. Thus

$$V(x,y) = \frac{c_{-2}}{x^2} + \hbar^2\wp. \tag{3.5.39}$$

3.5.2.3. *The potential* $V_1^{(a)}(x) = c_1 x + c_2 x^2;\ c_1 c_2 = 0$

We again define $W(y)$ as in (3.5.13). From (3.5.7) and (3.3.4) we obtain

$$
\begin{aligned}
G_1(y) &= 2(A_{202}y^2 - A_{112}y)W' + (2A_{202}y - A_{112})W - \frac{2}{3}c_2 A_{202}y^4 + \frac{4}{3}c_2 A_{112}y^3 + a_2 y^2 + a_1 y,\\
G_2(y) &= -3(A_{103}y - A_{013})W' - A_{103}W - c_1 A_{202}y^3 + \frac{1}{3}c_2 A_{103}y^3 - \frac{3}{2}c_1 A_{112}y^2 - c_2 A_{013}y^2 + b_1 y + b_0,\\
G_3(y) &= -c_1 y(\frac{1}{2}A_{103}y - A_{013}).
\end{aligned}
\tag{3.5.40}
$$

Substituting G_1,G_2,G_3 in (3.5.12) and integrating it with respect to y, we get

$$K_1 x + K_2 = 0,$$

where

$$
\begin{aligned}
K_1 &= \hbar^2(A_{112} - 2A_{202}y)W^{(4)} - 8\hbar^2 A_{202}W^{(3)} + 12(2A_{202}y - A_{112})W'W'' + 8A_{202}WW''\\
&- 4(\frac{8}{3}c_2 A_{202}y^3 - 4c_2 A_{112}y^2 - 2a_2 y - a_1)W'' + 32A_{202}W'^2 - 16(4c_2 A_{202}y^2 - 4c_2 A_{112}y - a_2)W'\\
&- 32c_2(2A_{202}y - A_{112})W + \frac{64}{3}c_2^2 A_{202}y^4 - \frac{128}{3}c_2^2 A_{112}y^3 - 32a_2 c_2 y^2 - 32a_1 c_2 y + k_1,\\
K_2 &= \hbar^2(A_{013} - A_{103}y)W^{(4)} - 4\hbar^2 A_{103}W^{(3)} + 12(A_{103}y - A_{013})W'W'' + 4A_{103}WW''\\
&- (4c_1 A_{202}y^3 + \frac{4}{3}c_2 A_{103}y^3 - 6c_1 A_{112}y^2 - 4c_2 A_{013}y^2 + 4b_1 y + 4b_0)W'' + 16A_{103}W'^2\\
&- 8(3c_1 A_{202}y^2 + c_2 A_{103}y^2 - 2c_2 A_{013}y - 3c_1 A_{112}y + b_1)W'\\
&- 4(6c_1 A_{202}y + 2c_2 A_{103}y - 3c_1 A_{112} - 2c_2 A_{013})W + \frac{2}{3}c_2^2 A_{103}y^4 - \frac{8}{3}c_2^2 A_{013}y^3 + 4c_2 b_1 y^2\\
&- 12a_2 c_1 y^2 + 8c_2 b_0 y - 12a_1 c_1 y + k_2,
\end{aligned}
\tag{3.5.41}
$$

and we must have $K_1 = 0$, $K_2 = 0$. In general the two ODEs in (3.5.41) are not compatible and we will analyze their compatibility conditions. A crucial role is played by the matrix

$$A = \begin{pmatrix} A_{202} & A_{112} \\ A_{103} & A_{013} \end{pmatrix}.$$

For the integral (3.5.3) to exist the rank of A must be 1, or 2. Let us analyze different possibilities.

1. rank$(A) = 1,\ A_{112} = A_{202} = 0$. In this case $K_1 = 0$ reduces to a linear second order ODE for $W(y)$;

$$(a_1 + 2a_2 y)W'' + 4a_2 W' - 8a_2 c_2 y^2 - 8a_1 c_2 y + \frac{k_1}{4} = 0. \tag{3.5.42}$$

For $(a_2,a_1) \neq (0,0)$, equation (3.5.42) together with $K_2 = 0$ leads to the elementary potentials that allow second order integrals of motion. They were already discussed in [**40**]. Of more interest is the case when we also have $a_1 = a_2 = 0$, so (3.5.42) is satisfied identically, and $K_2 = 0$ reduces to

$$\begin{aligned}&\hbar^2(A_{013} - yA_{103})W^{(4)} - 4\hbar^2 A_{103}W^{(3)} + 4(A_{103}W - \frac{1}{3}c_2 y^3 A_{103} + c_2 y^2 A_{013} - b_1 y - b_0)W''+\\ &12(yA_{103} - A_{013})W'W'' + 16A_{103}W'^2 - 8(c_2 y^2 A_{103} - 2c_2 y A_{013} + b_1)W' + 8c_2(A_{013} - yA_{103})W+\\ &4b_1 c_2 y^2 + 8b_0 c_2 y + \frac{2}{3}c_2^2 y^4 A_{103} - \frac{8}{3}c_2^2 y^3 A_{013} + k_2 = 0.\end{aligned} \tag{3.5.43}$$

Thus, we have one 4th order nonlinear ODE to solve and we must distinguish two cases, according to (3.5.4).
Case IIa. $A_{103} \neq 0, A_{202} = A_{112} = A_{013} = 0; Y_L = A_{103}\{L_3,p_2^3\}$.
Setting $A_{103} = 1$, we obtain

$$\begin{aligned}&g_1(x,y) = 0,\ g_2(x,y) = -3yW' - W + \frac{1}{3}c_2 y^3,\ g_3(x,y) = 4xW' - c_2 x y^2 - \frac{1}{2}c_1 y^2,\\ &l(x,y) = \frac{1}{4}\hbar^2 x(yW^{(4)} + 4W^{(3)}) - 3xyW'^2 - xWW' + \frac{1}{3}c_2 x y^3 W'' - c_1 y^2 W' - c_1 y W - \frac{1}{2}\hbar^2 c_2 x.\end{aligned} \tag{3.5.44}$$

From (3.5.43) we have

$$\hbar^2 y W^{(4)} + 4\hbar^2 W^{(3)} - 12yW'W'' - 4WW'' + \frac{4}{3}c_2 y^3 W'' - 16W'^2 + 8c_2 y^2 W' + 8c_2 y W - \frac{2}{3}c_2^2 y^4 + k = 0. \tag{3.5.45}$$

This equation is the same type of equation as (3.5.17), (with slightly different parameters, and c_2 in (3.5.17) is replaced by $\frac{c_2}{4}$), and has solutions expressed in terms of the fifth and third Painlevé transcendents. For $c_2 \neq 0$, we have

$$V(x,y) = -\delta\hbar^2(4x^2+y^2) + \hbar^2\Big(\frac{\gamma}{P_5(y^2)-1} + \frac{1}{y^2}(P_5(y^2)-1)(\sqrt{2\alpha} + \alpha(2P_5(y^2)-1) + \frac{\beta}{P_5(y^2)}) + y^2(\frac{P_5'^2(y^2)}{2P_5(y^2)} + \delta P_5(y^2))\frac{(2P_5(y^2)-1)}{(P_5(y^2)-1)^2} - \frac{P_5'(y^2)}{P_5(y^2)-1} - 2\sqrt{2\alpha}P_5'(y^2)\Big) + \frac{3\hbar^2}{8y^2}, \tag{3.5.46}$$

and for $c_2 = 0$,

$$V(x,y) = c_1x + \frac{\hbar^2}{2}(\sqrt{\alpha}P_3'(y) + \frac{3}{4}\alpha(P_3(y))^2 + \frac{\delta}{4P_3^2(y)} + \frac{\beta P_3(y)}{2y} + \frac{\gamma}{2yP_3(y)} - \frac{P_3'(y)}{2yP_3(y)} + \frac{P_3'^2(y)}{4P_3^2(y)}). \tag{3.5.47}$$

Case III. $A_{202} = A_{112} = A_{103} = 0, A_{013} \neq 0; Y_L = 2A_{013}p_1p_2^3$.

We set $A_{013} = 1$.

$$\begin{aligned} g_1(x,y) &= 0,\ g_2(x,y) = 3W' - c_2y^2 + b_1y,\ g_3(x,y) = 2c_2xy + c_1y - b_1x, \\ l(x,y) &= -\frac{1}{4}\hbar^2xW^{(4)} + 3xW'W'' + (b_1xy - c_2xy^2)W'' + 2c_1yW' + c_1W + \frac{1}{2}b_1c_1y^2. \end{aligned} \tag{3.5.48}$$

Integrating (3.5.43), we get

$$\hbar^2W^{(3)} - 6W'^2 + 4(c_2y^2 - b_1y)W' + (8c_2y - 4b_1)W - \frac{2}{3}c_2^2y^4 + \frac{4}{3}b_1c_2y^3 + k_2y + k_3 = 0, \tag{3.5.49}$$

which is the same type of equation as (3.5.28), (with slightly different parameters, and c_2 in (3.5.28) is replaced by $\frac{c_2}{4}$) and can be solved in terms of the fourth, second and first Painlevé transcendents and elliptic functions. Depending on the values of the parameters in (3.5.49) and following the procedure after (3.5.29), we obtain the following potentials.

When $c_2 \neq 0$, $c_1 = 0$, and the potential is

$$V(x,y) = -b_1y + c_2(x^2+y^2) - \frac{b_1\sqrt{\hbar}P_4}{\sqrt[4]{2c_2}} - \sqrt[4]{8c_2^3\hbar^2}yP_4 + \sqrt{\frac{c_2}{2}}\hbar(\epsilon P_4' + P_4^2), \tag{3.5.50}$$

where $\epsilon = \pm 1$, and $P_4 = P_4(-\sqrt[4]{\frac{2c_2}{\hbar^2}}y + \frac{b_1}{\sqrt[4]{2^3c_2^3\hbar^2}})$, satisfies the fourth Painlevé equation.

When $c_2 = 0, b_1 \neq 0$, the solutions are

$$V(x,y) = c_1x + \sqrt[3]{2b_1^2\hbar^2}(\epsilon P_2' + P_2^2), \tag{3.5.51}$$

where $P_2 = P_2(\sqrt[3]{\frac{4b_1}{\hbar^2}}y + \frac{3k_2}{2\sqrt[3]{4b_1^5\hbar^2}})$, satisfies the second Painlevé equation.
For $c_2 = b_1 = 0, k_2 \neq 0$, the potential is

$$V(x,y) = c_1 x + \sqrt[5]{k_2^2\hbar^2}P_1, \tag{3.5.52}$$

for $P_1 = P_1(-\sqrt[5]{\frac{k_2}{\hbar^4}}y - \frac{k_3}{\sqrt[5]{k_2^4\hbar^4}})$, satisfying the first Painlevé equation. and finally, for $c_2 = b_1 = k_2 = 0$, we are left with

$$V(x,y) = c_1 x + \hbar^2\wp, \tag{3.5.53}$$

where $\wp$ is the Weierstrass elliptic function.

2. rank$(A) = 1$, $A_{013} = A_{103} = 0$. In this case $K_2 = 0$ reduces to a linear second order ODE

$$(4c_1A_{202}y^3 - 6c_1A_{112}y^2 + 4b_1y + 4b_0)W'' + 8(3c_1A_{202}y^2 - 3c_1A_{112}y + b_1)W' + 4(6c_1A_{202}y - 3c_1A_{112})W$$
$$- 4c_2b_1y^2 + 12a_2c_1y^2 - 8c_2b_0y + 12a_1c_1y - k_2 = 0. \tag{3.5.54}$$

Since at least one of A_{112} and A_{202} must be nonvanishing, (3.5.54) leads to elementary potentials (unless it satisfied trivially). Equation (3.5.54) is satisfied trivially if $c_1 = b_1 = b_0 = 0$. We are left with one fourth order nonlinear ODE, $K_1 = 0$. In view of (3.5.4) two cases must be considered.

Case I. $A_{202} \neq 0, A_{112} = A_{103} = A_{013} = 0; Y_L = A_{202}\{L_3^2, p_2^2\}$.
In this case, we have

$$g_1(x,y) = 2y^2W' + 2yW - \frac{2}{3}c_2y^4,\ g_2(x,y) = -6xyW' - 2xW + \frac{8}{3}c_2xy^3,\ g_3(x,y) = x^2W' - 2c_2x^2y^2,$$
$$l(x,y) = \hbar^2x^2(\frac{1}{4}yW^{(4)} + W^{(3)}) - x^2(3yW' + W)W'' + (\frac{4}{3}c_2x^2y^3 - \frac{3\hbar^2}{2}y)W'' + (4c_2x^2y^2 - 3\hbar^2)W'$$
$$+ 4c_2x^2yW - c_2^2(\frac{4}{3}x^2 + \frac{1}{4})y^4 + 2c_2\hbar^2(y^2 - x^2). \tag{3.5.55}$$

and

$$\hbar^2 yW^{(4)} + 4\hbar^2 W^{(3)} - 12yW'W'' - 4WW'' + \frac{16}{3}c_2 y^3 W'' - 16W'^2 + 32c_2 y^2 W' + 32c_2 yW - \frac{32}{3}c_2^2 y^4 + k = 0, \tag{3.5.56}$$

which is exactly the same equation as (3.5.17) and hence has the same solutions expressed in terms of the fifth and third Painlevé transcendents.

Case IIb. $A_{112} \neq 0, A_{202} = A_{103} = A_{013} = 0; Y_L = A_{112}\{L_3,p_1p_2^2\}$.

$$g_1(x,y) = -2yW' - W + \frac{4}{3}c_2 y^3 + a_2 y^2,\ g_2(x,y) = 3xW' - 4c_2 xy^2 - 2a_2 xy,\ g_3(x,y) = 2c_2 x^2 y + a_2 x^2,$$
$$l(x,y) = -\frac{1}{8}\hbar^2 x^2 W^{(4)} + \frac{3}{2}x^2 W'W'' + (-2c_2 x^2 y^2 + \frac{3\hbar^2}{4})W'' - 4c_2 x^2 yW' - 2c_2 x^2 W + \frac{8}{3}c_2^2 x^2 y^3 - 2c_2\hbar^2 y. \tag{3.5.57}$$

Integrating $K_1 = 0$ once we obtain

$$\hbar^2 W^{(3)} - 6W'^2 + 8(2c_2 y^2 + a_2 y)W' + 8(4c_2 y + a_2)W - \frac{32}{3}c_2^2 y^4 - \frac{32}{3}c_2 a_2 y^3 + k_3 y + k_4 = 0, \tag{3.5.58}$$

which is the same equation as (3.5.28) and is solved in terms of the fourth, second and first Painlevé transcendents and elliptic function.

3. rank$(A) = 2$. Both $K_1 = 0$, and $K_2 = 0$, are satisfied nontrivially.

Case I. $A_{202} \neq 0, A_{013} \neq 0; A_{112} = A_{103} = 0; Y = A_{202}\{L_3^2,p_2^2\} + 2A_{013}p_1p_2^3$.

Let us set $A_{202} = 1, A_{013} = \alpha$, with $\alpha \neq 0$. In this case, both equations in (3.5.41) can be integrated once and we obtain two third order equations

$$\begin{aligned}&\hbar^2 y^2 W^{(3)} + 2\hbar^2 yW'' - 6y^2 W'^2 + (\frac{16c_2}{3}y^4 - 4a_2 y^2 - 2a_1 y - 2\hbar^2 - 4yW)W' + 2W^2\\&+ (2a_1 + \frac{32c_2}{3}y^3)W - (\frac{16}{9}c_2^2 y^4 - 4a_2 c_2 y^2 - \frac{16}{3}a_1 c_2 y + \frac{k_1}{4})y^2 + k_3 = 0,\end{aligned} \tag{3.5.59}$$

$$\begin{aligned}&\alpha\hbar^2 W^{(3)} - 6\alpha W'^2 - 4(c_1 y^3 - \alpha c_2 y^2 + b_1 y + b_0)W' - 4(3c_1 y^2 - 2c_2\alpha y + b_1)W\\&- \frac{2}{3}c_2^2\alpha y^4 + 4(\frac{1}{3}c_2 b_1 - c_1 a_2)y^3 - 2(3c_1 a_1 - 2c_2 b_0)y^2 + k_2 y + k_4 = 0,\end{aligned} \tag{3.5.60}$$

where k_3 and k_4 are integration constants. Eliminating third order derivatives between (3.5.59) and (3.5.60), we obtain a second order ODE. This equation admits a first integral,

$$\alpha\hbar^2 W' - \alpha W^2 - (\alpha a_1 - 2(b_0 - \alpha a_2)y - 2b_1 y^2 - \frac{2}{3}\alpha c_2 y^3 - 2c_1 y^4)W - \frac{1}{9}\alpha c_2^2 y^6 + \frac{1}{6}(3a_2 c_1 - b_1 c_2)y^5$$
$$+ (\frac{2}{3}(\alpha a_2 c_2 - b_0 c_2) + a_1 c_1)y^4 + (\frac{4}{3}\alpha a_1 c_2 - \frac{k_2}{4})y^3 - \frac{1}{8}(\alpha k_1 + 4k_4)y^2 + k_5 y - \alpha\frac{k_3}{2} = 0, \tag{3.5.61}$$

where k_5 is an integration constant. Equation (3.5.61) is a Riccati equation and can be linearized by a Cole-Hopf transformation. Setting $W = -\hbar^2 \frac{U'}{U}$, we get the following linear ODE

$$\alpha\hbar^4 U''(y) + \hbar^2(2c_1 y^4 + \frac{2}{3}\alpha c_2 y^3 + 2b_1 y^2 - 2\alpha a_2 y + 2b_0 y - \alpha a_1)U'(y) + (\frac{1}{9}\alpha c_2^2 y^6 - (\frac{a_2 c_1}{2} - \frac{b_1 c_2}{6})y^5$$
$$- (\frac{2}{3}\alpha a_2 c_2 + a_1 c_1 - \frac{2b_0 c_2}{3})y^4 - (\frac{4}{3}\alpha a_1 c_2 - \frac{k_2}{4})y^3 + \frac{1}{8}(4k_3 + \alpha k_1)y^2 + \frac{k_5}{2\hbar^2}y + \frac{\alpha k_3}{2})U(y) = 0. \tag{3.5.62}$$

Consequently, in this case we do not obtain any exotic potential.

Case II. $A_{202} = 0, A_{112} \neq 0, A_{103} \neq 0; Y_L = A_{112}\{L_3, p_1 p_2^2\} + A_{103}\{L_3, p_2^3\}$.

Same as the previous case, we can again integrate the equations in (3.5.41), and if we apply the same procedure we generate another Riccati equation

$$\alpha\hbar^2 W' - \alpha W^2 + (2b_0 + 2(b_1 - \alpha a_1)y - (3c_1 + 4\alpha a_2)y^2 - \frac{22}{3}\alpha c_2 y^3)W + \frac{19}{18}\alpha c_2^2 y^6 + (\frac{4}{3}\alpha a_2 c_2 + \frac{1}{2}c_1 c_2)y^5$$
$$+ (\frac{8}{3}\alpha a_1 c_2 - \frac{1}{6}b_1 c_2 + \frac{1}{2}a_2 c_1)y^4 + (a_1 c_1 - \frac{2}{3}b_0 c_2 - \frac{1}{4}\alpha k_1)y^3 - (\frac{1}{4}k_2 + k_3)y^2 + k_4 = 0, \tag{3.5.63}$$

where k_3, and k_4 are constants of integration. Again it can be linearized by a Cole-Hopf transformation.

3.5.3. Linear equations for V_1 satisfied trivially

In this case, (3.5.2) are valid, and (3.5.1) not. The leading-order term for the nontrivial fourth order integral has the form

$$Y_L = A_{220}\{L_3^2, p_1^2\} + A_{130}\{L_3, p_1^3\} + A_{121}\{L_3, p_1^2 p_2\} + 2A_{031} p_1^3 p_2. \tag{3.5.64}$$

Let us classify the integrals (3.5.64) under translations. The three classes are

$$(i) A_{220} \neq 0, A_{121} = A_{130} = 0.$$
$$(ii) A_{220} = 0, A_{121}^2 + A_{130}^2 \neq 0, A_{031} = 0.$$
$$(iii) A_{220} = A_{121} = A_{130} = 0, A_{031} \neq 0. \tag{3.5.65}$$

Since we can just adapt the results from the section 4.2 to this case, we will not consider it separately. The results are obtained by interchanging $x \leftrightarrow y$, $(A_{202}, A_{112}, A_{103}, A_{013}) \leftrightarrow (A_{220}, A_{121}, A_{130}, A_{031})$.

3.6. Classical analogs of the quantum exotic potentials

In the classical case, we are dealing with the classical limit ($\hbar \to 0$) of the determining equations (3.2.2) and (3.2.3) and therefore the compatibility condition (3.2.5) and (3.3.11). The equations (3.2.2) and (3.2.5) are actually the same in the classical and quantum case. We continue our investigation for the classical potentials followed by the classifications of the integrals in (3.5.4). Here we present the results briefly for each cases.
Integrating the classical analog of the equations (3.5.17), (3.5.45) and (3.5.56), we get

$$3y^2 W'^2 + (2yW - \frac{2}{3}\lambda y^4)W' - W^2 - \frac{4}{3}\lambda y^3 W + \frac{1}{18}\lambda^2 y^6 + k_1 y^2 + k_2 = 0, \tag{3.6.1}$$

where $\lambda = c_2, 4c_2$ respectively for $Y_L = L_3 p_2^3$, and $Y_L = L_3^2 p_2^2$.
The classical analog of the equations (3.5.28), (3.5.49), and (3.5.58) is

$$W'^2 + \frac{2}{3}y(k_1 - \lambda y)W' + \frac{2}{3}(k_1 - 2\lambda y)W + \frac{1}{9}\lambda^2 y^4 - \frac{2}{9}\lambda k_1 y^3 + k_2 y + k_3 = 0, \tag{3.6.2}$$

where $\lambda = c_2, 4c_2$ respectively for $Y_L = p_1 p_2^3$, and $Y_L = L_3 p_1 p_2^2$.
Equations (3.6.1) and (3.6.2) are special cases of equation (3.4.3). They do not satisfy the conditions in the Fuchs' theorem, (Theorem1.1, [**15**, page 80], proof in [**58**, page 304-311]), hence do not have the Painlevé property. They will be further investigated in Part II of this project.

3.7. Summary of results and future outlook

3.7.1. Quantum potentials

The list of exotic superintegrable quantum potentials in quantum case that admit one second order and one fourth order integral is given below. We also give their fourth order integrals by listing the leading terms Y_L and the functions $g_i(x,y); i = 1,2,3$; and $l(x,y)$. Each of the exotic potentials has a non-exotic part that comes from $V_1(x)$. By construction $V_2(y)$ is exotic, however in 4 cases a non-exotic part proportional to y^2 splits off from $V_2(y)$ and can be combined with an x^2 term in $V_1(x)$. We order the final list below in such a manner that the first two potentials are isotropic harmonic oscillators (possibly with an additional $\frac{1}{x^2}$ term) with an added exotic part. The next two are $2:1$ anisotropic harmonic oscillators, plus an exotic part (in y).

Based on previous experience (see Marquette [**67, 68, 66**]) we expect these harmonic terms to determine the bound state spectrum. The remaining 8 cases have either $\frac{a}{x^2}$ or $c_1 x$ as their non-exotic terms and we expect the energy spectrum to be continuous.

I. Isotropic harmonic oscillator:

Q_1^1 :

$$\begin{aligned} V(x,y) =& -\delta\hbar^2(x^2+y^2) + \frac{a}{x^2} + \hbar^2\Big(\frac{\gamma}{P_5-1} + \frac{1}{y^2}(P_5-1)(\sqrt{2\alpha} + \alpha(2P_5-1) + \frac{\beta}{P_5}) \\ &+ y^2(\frac{P_5'^2}{2P_5} + \delta P_5)\frac{(2P_5-1)}{(P_5-1)^2} - \frac{P_5'}{P_5-1} - 2\sqrt{2\alpha}P_5'\Big) + \frac{3\hbar^2}{8y^2}. \end{aligned}$$

$$Y_L = \{L_3^2, p_2^2\},$$
$$g_1(x,y) = 2y(yW' + W + \frac{1}{3}\hbar^2\delta y^3), \quad g_2(x,y) = -2x(3yW' + W + \frac{4}{3}\hbar^2\delta y^3),$$
$$g_3(x,y) = x^2(4W' + 2\hbar^2\delta y^2) + \frac{2a}{x^2}y^2,$$

$$\begin{aligned} l(x,y) =& \hbar^2x^2(\frac{1}{4}yW^{(4)} + W^{(3)}) - x^2(3yW' + W)W'' - \hbar^2 y(\frac{4}{3}\delta x^2y^2 + \frac{3}{2})W'' + (4(\frac{a}{x^2} - \hbar^2\delta x^2)y^2 - 3\hbar^2)W' \\ &+ 4y(\frac{a}{x^2} - \hbar^2\delta x^2)W + \frac{4a}{3x^2}\hbar^2\delta y^4 - 2\hbar^2\delta x^2(\frac{2}{3}\hbar^2\delta y^4 - \hbar^2) - 2\hbar^4\delta y^2. \end{aligned}$$

For

$$W(y) = -\frac{\hbar^2}{2y}\Big(\frac{1}{P_5}(\frac{YP_5'}{P_5-1} - P_5)^2 - (1-\sqrt{2\alpha})^2(P_5-1) - 2\beta\frac{P_5-1}{P_5} + \gamma Y\frac{P_5+1}{P_5-1} + 2\delta\frac{Y^2P_5}{(P_5-1)^2}\Big)$$
$$+\frac{\hbar^2}{8y} - \frac{\delta\hbar^2}{3}y^3,$$

where $P_5 = P_5(Y); Y = y^2$.

Q_1^2 :

$$V(x,y) = c_2(x^2+y^2) - \sqrt[4]{8c_2^3\hbar^2}yP_4(-\sqrt[4]{\frac{2c_2}{\hbar^2}}y) + \sqrt{\frac{c_2}{2}}\hbar(\epsilon P_4'(-\sqrt[4]{\frac{2c_2}{\hbar^2}}y) + P_4^2(-\sqrt[4]{\frac{2c_2}{\hbar^2}}y)); \quad \epsilon = \pm 1.$$

$$Y_L = 2p_1p_2^3,$$
$$g_1(x,y) = 0,\ g_2(x,y) = 3V - c_2(3x^2+y^2),\ g_3(x,y) = 2c_2xy,$$
$$l(x,y) = -\frac{1}{4}\hbar^2 xV_{yyy} + 3xVV_y - c_2x(3x^2+y^2)V_y.$$

II. Anisotropic harmonic oscillator:

Q_2^1 :

$$V(x,y) = c_2(x^2+4y^2) + \frac{a}{x^2} - 4\sqrt[4]{2c_2^3\hbar^2}yP_4 + \sqrt{2c_2}\hbar(\epsilon P_4' + P_4^2); \quad \epsilon = \pm 1.$$

$$Y_L = \{L_3, p_1p_2^2\},$$
$$g_1(x,y) = -2yW' - W + \frac{4}{3}c_2y^3,\ g_2(x,y) = 3xW' - 4c_2xy^2,\ g_3(x,y) = 2c_2x^2y - 2a\frac{y}{x^2},$$

$$l(x,y) = -\frac{1}{8}\hbar^2x^2W^{(4)} + \frac{3}{2}x^2W'W'' - (2c_2x^2y^2 - \frac{3}{4}\hbar^2)W'' - 2(2a\frac{y}{x^2} + 2c_2x^2y)W' - 2(\frac{a}{x^2} + c_2x^2)W$$
$$+\frac{8}{3}c_2y^3(c_2x^2 + \frac{a}{x^2}) - 2c_2\hbar^2y.$$

For

$$W(y) = \sqrt[4]{8c_2\hbar^6}\Big(\frac{1}{8P_4}P_4'^2 - \frac{1}{8}P_4^3 - \frac{1}{2}YP_4^2 - \frac{1}{2}(Y^2-\alpha+\epsilon)P_4 + \frac{1}{3}(\alpha-\epsilon)Y + \frac{\beta}{4P_4}\Big) + \frac{4c_2}{3}y^3,$$

where $P_4 = P_4(Y); Y = -\sqrt[4]{\frac{8c_2}{\hbar^2}}y$.

Q_2^2 :

$$V(x,y) = -\delta\hbar^2(4x^2+y^2) + \hbar^2\Big(\frac{\gamma}{P_5-1} + \frac{1}{y^2}(P_5-1)(\sqrt{2\alpha}+\alpha(2P_5-1)+\frac{\beta}{P_5})$$
$$+ y^2(\frac{P_5'^2}{2P_5}+\delta P_5)\frac{(2P_5-1)}{(P_5-1)^2} - \frac{P_5'}{P_5-1} - 2\sqrt{2\alpha}P_5'\Big) + \frac{3\hbar^2}{8y^2}.$$

$$Y_L = \{L_3,p_2^3\},$$

$$g_1(x,y) = 0,\ g_2(x,y) = -3yW' - W - \frac{4}{3}\hbar^2\delta y^3,\ g_3(x,y) = 4xW' + 4\hbar^2\delta xy^2,$$

$$l(x,y) = \frac{1}{4}\hbar^2x(yW^{(4)} + 4W^{(3)}) - 3xyW'^2 - xWW' - \frac{4}{3}\hbar^2\delta xy^3W'' + 2\hbar^4\delta x.$$

For

$$W(y) = -\frac{\hbar^2}{2y}\Big(\frac{1}{P_5}(\frac{YP_5'}{P_5-1} - P_5)^2 - (1-\sqrt{2\alpha})^2(P_5-1) - 2\beta\frac{P_5-1}{P_5} + \gamma Y\frac{P_5+1}{P_5-1} + 2\delta\frac{Y^2P_5}{(P_5-1)^2}\Big)$$
$$+\frac{\hbar^2}{8y} - \frac{4\delta\hbar^2}{3}y^3,$$

where $P_5 = P_5(Y); Y = y^2$.

III. Potentials with no confining (harmonic oscillator) term:

Q_3^1 :

$$V(x,y) = \frac{a}{x^2} + \frac{\hbar^2}{2}(\sqrt{\alpha}P_3' + \frac{3}{4}\alpha(P_3)^2 + \frac{\delta}{4P_3^2} + \frac{\beta P_3}{2y} + \frac{\gamma}{2yP_3} - \frac{P_3'}{2yP_3} + \frac{P_3'^2}{4P_3^2}).$$

$$Y_L = \{L_3^2,p_2^2\},$$

$$g_1(x,y) = 2y^2W' + 2yW,\ g_2(x,y) = -6xyW' - 2xW,\ g_3(x,y) = 4x^2W' + 2a\frac{y^2}{x^2},$$

$$l(x,y) = \hbar^2 x^2 (\frac{1}{4} y W^{(4)} + W^{(3)}) - x^2 (3yW' + W) W'' - \frac{3}{2}\hbar^2 y W'' + (4\frac{a}{x^2} y^2 - 3\hbar^2) W' + 4\frac{a}{x^2} y W.$$

For

$$W(y) = -\frac{\hbar^2}{2y}\Big(\frac{1}{4}(y\frac{P_3'}{P_3} - 1)^2 - \frac{1}{16}\alpha y^2 P_3^2 - \frac{1}{8}(\beta + 2\sqrt{\alpha}) y P_3 + \frac{\gamma}{8P_3} y + \frac{\delta}{16P_3^2} y^2\Big) + \frac{\hbar^2}{8y}.$$

Q_3^2 :

$$V(x,y) = \frac{a}{x^2} + \frac{b^2\hbar^2}{2}(\epsilon P_2' + P_2^2); \quad \epsilon = \pm 1.$$

$$Y_L = \{L_3, p_1 p_2^2\},$$

$$g_1(x,y) = -2yW' - W - \frac{b^3\hbar^2}{8} y^2,\ g_2(x,y) = 3xW' + \frac{b^3\hbar^2}{4} xy,\ g_3(x,y) = -\frac{b^3\hbar^2}{8} x^2 - 2a\frac{y}{x^2},$$

$$l(x,y) = -\frac{1}{8}\hbar^2 x^2 W^{(4)} + \frac{3}{2} x^2 W' W'' + (\frac{b^3\hbar^2}{8} x^2 y + \frac{3}{4}\hbar^2) W'' - 4a\frac{y}{x^2} W' - 2\frac{a}{x^2} W - ab^3\hbar^2 \frac{y^2}{4x^2}.$$

For

$$W(y) = \frac{-b\hbar^2}{2}\Big((P_2')^2 - (P_2^2 + \frac{b}{2} y)^2 - 2(\alpha + \frac{\epsilon}{2}) P_2\Big) - \frac{b^3}{8}\hbar^2 y^2,$$

where $P_2 = P_2(by)$.

Q_3^3 :

$$V(x,y) = \frac{a}{x^2} + \hbar^2 b^2 P_1.$$

$$Y_L = \{L_3^2, p_2^2\},$$

$$g_1(x,y) = -2yW' - W,\ g_2(x,y) = 3xW',\ g_3(x,y) = -2a\frac{y}{x^2},$$

$$l(x,y) = -\frac{1}{8}\hbar^2 x^2 W^{(4)} + \frac{3}{2} x^2 W' W'' + \frac{3}{4}\hbar^2 W'' - 4a\frac{y}{x^2} W' - 2\frac{a}{x^2} W$$

For

$$W(y) = -b\hbar^2(\frac{1}{2}(P_1')^2 - 2P_1^3 - byP_1),$$

where $P_1 = P_1(by)$.

Q_3^4 :

$$V(x,y) = \frac{a}{x^2} + \hbar^2\wp.$$

$$Y_L = \{L_3^2,p_2^2\},$$

$$g_1(x,y) = -2yW' - W,\ g_2(x,y) = 3xW',\ g_3(x,y) = -2a\frac{y}{x^2},$$

$$l(x,y) = -\frac{1}{8}\hbar^2x^2W^{(4)} + \frac{3}{2}x^2W'W'' + \frac{3}{4}\hbar^2W'' - 4a\frac{y}{x^2}W' - 2\frac{a}{x^2}W$$

For

$$W(y) = \hbar^2\int u dy, \quad u = \wp(y).$$

Q_3^5 :

$$V(x,y) = c_1x + \frac{\hbar^2}{2}(\sqrt{\alpha}P_3'(y) + \frac{3}{4}\alpha(P_3(y))^2 + \frac{\delta}{4P_3^2(y)} + \frac{\beta P_3(y)}{2y} + \frac{\gamma}{2yP_3(y)} - \frac{P_3'(y)}{2yP_3(y)} + \frac{P_3'^2(y)}{4P_3^2(y)}).$$

$$Y_L = \{L_3,p_2^3\},$$

$$g_1(x,y) = 0,\ g_2(x,y) = -3yW' - W,\ g_3(x,y) = 4xW' - \frac{1}{2}c_1y^2,$$

$$l(x,y) = \frac{1}{4}\hbar^2x(yW^{(4)} + 4W^{(3)}) - 3xyW'^2 - xWW' - c_1y^2W' - c_1yW.$$

For

$$W(y) = -\frac{\hbar^2}{2y}\Big(\frac{1}{4}(y\frac{P_3'}{P_3} - 1)^2 - \frac{1}{16}\alpha y^2P_3^2 - \frac{1}{8}(\beta + 2\sqrt{\alpha})yP_3 + \frac{\gamma}{8P_3}y + \frac{\delta}{16P_3^2}y^2\Big) + \frac{\hbar^2}{8y}.$$

Q_3^6 :

$$V(x,y) = c_1 x + \frac{b^2\hbar^2}{2}(\epsilon P_2' + P_2^2); \quad \epsilon = \pm 1.$$

$$Y_L = 2p_1 p_2^3,$$

$$g_1(x,y) = 0,\, g_2(x,y) = 3W' + \frac{b^3\hbar^2}{4}y,\, g_3(x,y) = c_1 y - \frac{b^3\hbar^2}{4}x,$$

$$l(x,y) = -\frac{1}{4}\hbar^2 x W^{(4)} + 3xW'W'' + \frac{b^3\hbar^2}{4}xyW'' + 2c_1 yW' + c_1 W + \frac{b^3\hbar^2}{8}c_1 y^2.$$

For

$$W(y) = \frac{-b\hbar^2}{2}\Big((P_2')^2 - (P_2^2 + \frac{b}{2}y)^2 - 2(\alpha + \frac{\epsilon}{2})P_2\Big) - \frac{b^3}{8}\hbar^2 y^2,$$

where $P_2 = P_2(by)$.

Q_3^7 :

$$V(x,y) = c_1 x + \hbar^2 b^2 P_1.$$

$$Y_L = 2p_1 p_2^3,$$

$$g_1(x,y) = 0,\, g_2(x,y) = 3W',\, g_3(x,y) = c_1 y,\, l(x,y) = -\frac{1}{4}\hbar^2 x W^{(4)} + 3xW'W'' + 2c_1 yW' + c_1 W.$$

For

$$W(y) = -b\hbar^2(\frac{1}{2}(P_1')^2 - 2P_1^3 - byP_1),$$

where $P_1 = P_1(by)$.

Q_3^8 :

$$V(x,y) = c_1 x + \hbar^2 \wp.$$

$$Y_L = 2p_1 p_2^3,$$

$$g_1(x,y) = 0,\, g_2(x,y) = 3W',\, g_3(x,y) = c_1 y,\, l(x,y) = -\frac{1}{4}\hbar^2 x W^{(4)} + 3xW'W'' + 2c_1 yW' + c_1 W.$$

For

$$W(y) = \hbar^2 \int u dy, \quad u = \wp(y).$$

The potentials Q_1^2, Q_3^6 and Q_3^7 are in the list of quantum potentials obtained by Gravel [**47**, (Q_{18},Q_{19}, Q_{21})]. Among the integrals of motion we have $\{L_3^2,p_2^2\}$ and $\{L_3,p_2^3\}$. These can not be obtained by commuting a third and a second order integral.

As mentioned in the Introduction, it has been conjectured [**98**] that all maximally superintegrable systems are exactly solvable. This has also been confirmed for the case of potentials allowing third order integrals of motion, in particular exotic ones. In the papers [**67, 68**], Marquette considered a superintegrable system with one second order and one third order integral of motion. He presented the polynomial algebra generated by these integrals of motion, and he showed how their representations yield the energy spectra. The ground state wave functions were also obtained by using the tools of suspersymmetric quantum mechanics. Marquette in [**72**] obtained a potential in terms of fifth Painlevé transcendent for a system admitting fourth order ladder operators which allowed a characterisation of the spectrum and wave functions in a recursive way from the zero modes and build integrals for families of 2D models.

3.7.2. Future outlook

Part II of this article will follow shortly and will be devoted to a complete analysis of the nonexotic potentials. They are obtained when the linear compatibility conditions (3.3.9) and (3.3.10) are not satisfied identically. They must then be solved as ODEs. We also hope to establish that all obtained superintegrable systems are exactly solvable, as in the case of third order integrals.

We are also currently studying whether some or possibly all exotic potentials can be generated from one-dimensional Hamiltonians using algebras of differential operators depending on one variable only.

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Chapitre 4

Two-dimensional superintegrable systems from operator algebras in one dimension

par

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Mes contribution et le rôle des coauteurs: Le sujet et les méthodes de la recherche a été suggérée par Ian Marquette. J'ai effectué les calculs nécessaires liés aux résultats obtenus. La rédaction se fait à parts égales entre les coauteurs.

Abstract. In this paper, we develop new constructions of 2D classical and quantum superintegrable Hamiltonians allowing separation of variables in Cartesian coordinates. In classical mechanics we start from two functions on a one-dimensional phase space, a natural Hamiltonian H and a polynomial of order N in the momentum p. We assume that their Poisson bracket $\{H,K\}$ vanishes, is a constant, a constant times H, or a constant times K. In the quantum case H and K are operators and their Lie bracket has one of the above properties. We use two copies of such (H,K) pairs to generate two-dimensional superintegrable systems in the Euclidean space E_2, allowing the separation of variables in Cartesian coordinates. All known separable superintegrable systems in E_2 can be obtained in this manner and we obtain new ones for $N = 4$.
Keywords: superintegrable systems, Painlevé transcendents, ladder operators

4.1. Introduction

This article is part of a general study of superintegrable systems in quantum and classical mechanics. In a nutshell a superintgerable system with n degrees of freedom is a Hamiltonian system with n integrals of motion X_1,...,X_n (including the Hamiltonian H) in involution and k further integrals $Y_k, 1 \leq k \leq 2n-1$. The additional integrals Y_k commute (or Poisson commute) with the Hamiltonian, but not necessarily with each other, nor with the integrals X_i. All the integrals are assumed to be well defined functionally independent functions on phase space in classical mechanics. In quantum mechanics they are Hermitian operators in the enveloping algebra of the Heisenberg algebra H_n (or some generalization of the enveloping algebra) and are polynomially independent. For reviews we refer to [**59, 79**].
The best known superintegrable systems (in n dimensions) are the Kepler-Coulomb system [**5, 39, 88**] and the Harmonic oscillator [**55, 81**] with the potentials $\frac{\alpha}{r}$ and ωr^2, respectively. A systematic search for superintegrable systems in Euclidean spaces E_n was started in 1965 [**40, 63**]. The integrals of motion were postulated to be second order polynomials in the momenta with coefficients that were smooth functions of the coordinates. Second order integrals were shown to be related to multiseparation of variables in the Schrödinger or Hamilton-Jacobi equation. Integrable and superintegrable systems with integrals that are higher order polynomials in the momenta were considered in [**2, 31, 32, 33, 47, 48, 64, 65, 67, 68, 69, 77, 84, 90, 91, 92, 93, 97, 100**].

A subset of the articles quoted above was devoted to a search for superintegrable systems in E_2 with 2 integrals of order 2 and one of order N with $3 \leq N \leq 5$. The order 2 ones were the Hamiltonian H, the second order one X was chosen so as to ensure separation of variables in Cartesian or polar coordinates, respectively. The third integral Y was of order $N \geq 3$. It turned out that the complexity of the calculations rapidly increased as N increased and that the obvious systematic and straight forward method became impractical for $N > 5$. On the other hand for $N \geq 3$ it turned out that quantum integrable and superintegrable systems could have different potentials than classical ones. In particular quantum superintegrable systems allowed the existence of "exotic potentials" expressed in terms of elliptic functions , Painlevé transcendents and general functions having the Painlevé property.

The purpose of this paper is to further develop and apply a different method of constructing superintegrable systems in two and more dimensions. Namely, we shall study two copies of operator algebras in one dimension, expressed in terms of the coordinates x and y, respectively and combine these two to form superintegrable systems in E_2. The generalization to n copies and to superintegrable systems in E_n is immediate.

The article is organized as follows. In Section 2 we formulate the problem and show how algebras of operators or functions in one dimension can be used to construct superintegrable systems in two dimensions. This is done both for quantum and classical mechanics. Sections 3 is devoted to the classification of operator algebras in one dimension for operator K of order $1 \leq M \leq 5$. The same problem in classical mechanics, where H and K are functions on a two dimensional phase space is solved in Section 4. The superintegrable classical and quantum systems in E_2 are presented in Section 5. Section 6 is devoted to conclusion and a summary of results.

4.2. The general method

Let us consider a Hamiltonian in a one dimensional Euclidean space E_1

$$H_1 = \frac{p_x^2}{2} + V(x) \tag{4.2.1}$$

where x is a space coordinate. In classical mechanics p_x is the momentum canonically conjugate to x and in quantum mechanics we have $p_x = -i\hbar\partial_x$.

Let us also consider the polynomial

$$K_1 = \sum_{j=0}^{M} f_j(x) p_x^j, \tag{4.2.2}$$

where $f_M(x) \neq 0$, and $f_j(x)$ are smooth functions.

Both in quantum and in classical mechanics we can consider the Lie algebra

$$[H_1,K_1] = \alpha K_1 + \beta H_1 + \gamma 1,\ [H_1,1] = [K_1,1] = 0, \tag{4.2.3}$$

where [.,.] is the Lie bracket , or the Poisson bracket, respectively and α, β and γ are constants. By change of basis we can reduce the algebra (4.2.3) into one of the 4 following forms for $\alpha = \beta = \gamma = 0;\ \alpha = \beta = 0, \gamma \neq 0;\ \alpha = \gamma = 0, \beta \neq 0$; and $\alpha \neq 0$, respectively

$$[H_1,K_1] = 0, \tag{4.2.4a}$$

$$[H_1,K_1] = \alpha_1, \tag{4.2.4b}$$

$$[H_1,K_1] = \alpha_1 H_1, \tag{4.2.4c}$$

$$[H_1,K_1] = -\alpha_1 K_1,\ \alpha_1 \in \mathbb{R}\backslash 0 \tag{4.2.4d}$$

where $\alpha_1 \neq 0$ is a constant. We shall refer to these relations as Abelian type (a), Heisenberg type (b), conformal type (c), and ladder type (d), respectively.

We shall call the systems $\{H_1,K_1\}$ in one dimension "algebraic Hamiltonian systems". The classical case (d) of ladder and the corresponding Hamiltonian and functions K_1 that are polynomials of order 3 and 4 in momentum have been studied in the references [**71, 73**]. The case of order 3 of these relations has been discussed in [**42**]. Some of these cases have been investigated e.g. in the case (c) [**29**] and case (b) [**51**]. The quantum case (d) has been studied for particular examples related with fourth and fifth Painlevé transcendents [**70, 68, 72, 104, 3, 13**]. Superintegrable deformations of the harmonic oscillator and the singular oscillator and many types of ladder operators have been studied [**74, 75, 76**]. The Heisenberg type relations have been investigated in a recent paper [**51**]. The Abelian type (a) has been studied by Hietarinta for third order operators and was refered to as pure quantum integrability [**53, 54**]. Furthermore, for the case (a) some interesting algebraic relations have been discussed [**102, 103**].

The existence of such operators K_1 will impose constraints on the potential $V(x)$ and on the coefficients $f_j(x)$ in the polynomial K_1. We shall construct such systems proceeding by order M in the following sections, both in quantum and classical physics.

We consider a second copy of E_1 with the corresponding Hamiltonian H_2 and operator K_2 satisfying one of the relations (a),(b),(c),(d) with

$$K_2 = \sum_{j=0}^{N} g_j(y) p_y^j. \tag{4.2.5}$$

Now let us consider the two-dimensional Euclidean space E_2 with the Hamiltonian

$$H = H_1 + H_2 = \frac{1}{2}(p_x^2 + p_y^2) + V_1(x) + V_2(y). \tag{4.2.6}$$

This Hamiltonian is obviously integrable because it allows the separation of variables in Cartesian coordinates, i.e. it allows an independent second order integral

$$A = K_1 - K_2, \tag{4.2.7}$$

where K_1 and K_2 are second order Abelian type operators.

We will use operators K_1, K_2 of (4.2.2) and (4.2.5) to generate integrals of motion K in E_2. We denote (.,.) where u,v = a,b,c,d and refer that in x axis $1D$ Hamiltonian allow operator type u and in the y axis the $1D$ Hamiltonian allow operator of type v. The possible combinations are

I.(a,a):

Obviously, any linear combination

$$K = c_1 K_1 + c_2 K_2 \tag{4.2.8}$$

satisfies $[H,K] = 0$ and is hence an integral of motion. The interesting point is that in the case (a) , H_1 and K_1 in E_1 can not be polynomially independent, however in E_2 the operators H, and K can be. The case of (4.2.6) and (4.2.7) is a trivial example. For higher order operators we shall produce nontrivial examples below.

II.(b,b):

The operator

$$K = \alpha_2 K_1 - \alpha_1 K_2 \tag{4.2.9}$$

will commute with H.

III.(c,b):

An integral of motion is $K = \alpha_2 K_1 - \alpha_1 H_1 K_2$.

IV.(d,d):

The case (d) is somewhat more complicated. We change notations slightly and introduce an operator $K_1^\dagger$ adjoint to K_1.

$$K_1^- \equiv K_1,\, K_1^\dagger = (K_1^-)^\dagger. \tag{4.2.10}$$

We now have

$$\begin{aligned}
&[H_1{,}K_1^-] = -\alpha_1 K_1^- \\
&[H_1{,}K_1^\dagger] = \alpha_1 K_1^\dagger \\
&K_1^\dagger K_1^- = \sum_{n=0}^{k_x} a_n H_1^n
\end{aligned} \tag{4.2.11}$$

The fact that $K_1^\dagger K_1^-$ is a polynomial in H_1 follows from the commutation relation $[H_1{,}K_1^\dagger K_1^-] = 0$ [**8**].

The same relations are introduced for H_2, K_2^- and $K_2^\dagger$. In E_2 we have

$$[H_1 + H_2, (K_1^\dagger)^m (K_2^-)^n] = (m\alpha_1 - n\alpha_2)(K_1^\dagger)^m (K_2^-)^n \tag{4.2.12}$$

To obtain an integral of motion we impose a rationality constraint on α_1 and α_2, namely

$$\frac{\alpha_1}{\alpha_2} = \frac{n}{m}. \tag{4.2.13}$$

With this constraint

$$K = (K_1^\dagger)^m (K_2^-)^n - (K_1^-)^m (K_2^\dagger)^n. \tag{4.2.14}$$

are all integrals of motion and K is the lowest order polynomial amongst them.

V.(c,c):

The integral of motion in this case is

$$K = \alpha_2 H_2 K_1 - \alpha_1 H_1 K_2 \tag{4.2.15}$$

Other possible combinations are

VI.(a,d):

An integral of motion is $K_1 + K_2^- K_2^\dagger$. However since $K_2^\dagger K_2^-$ is a polynomial in H_2, this integral is trivial (a polynomial in H_1 and H_2).

Case	Type	Integral type	K	Order of K
1	(a,a)	polynomial	$K_1 + K_2$	$max(k_1,k_2)$
2	(b,b)	polynomial	$\alpha_2 K_1 - \alpha_1 K_2$	$max(k_1,k_2)$
3	(c,b)	polynomial	$\alpha_2 K_1 - \alpha_1 H_1 K_2$	$max(k_1,k_2+2)$
4	(d,d)	polynomial	$(K_1^\dagger)^m (K_2^-)^n - (K_1^-)^m (K_2^\dagger)^n$	$(mk_1 + nk_2 - 1)$
5	(c,c)	polynomial	$\alpha_2 H_2 K_1 - \alpha_1 H_1 K_2$	$max(k_1+2,k_2+2)$
6	(a,d)	polynomial	$K_1 - K_2^- K_2^\dagger$	$max(k_1,2k_2)$
7	(b,d)	non polynomial	$e^{i\frac{\alpha_2}{\alpha_1}K_1} K_2^-$	-
8	(c,d)	non polynomial	$e^{i\frac{\alpha_2}{\alpha_1}K_1} K_2^{H_1}$	-

Tab. 4.1. Integrals of motion in E_2

In Table 1, $k_1 = order(K_1)$ and $k_2 = order(K_2)$. For the operator of type (d), setting $K_1 = K_1^-$, we have $[H_1, K_1^\pm] = \pm\alpha_1 K_1^\pm$. Also in the case 4, $m\alpha_1 = n\alpha_2 = \lambda$.
Let us consider A as the second order integral of motion introduced in (4.2.7) and B as the Mth order one. In the classical case, the polynomial Poisson algebra $\mathcal{P}_M$, generated by functions A and B has Poisson bracket given by

$$\{A,B\}_p = C, \{A,C\}_p = R(A,B,H), \{B,C\}_p = S(A,B,H). \tag{4.2.16}$$

The polynomial Lie algebra, $\mathcal{L}_M$, which is the Mth order analogue of the classical Poisson algebra $\mathcal{P}_M$, has bracket operation given by

$$[A,B] = C, [A,C] = \tilde{R}(A,B,H), [B,C] = \tilde{S}(A,B,H) \tag{4.2.17}$$

with further constraints on parameters from the Jacobi identity. Further information on the algebra is given in Table 2.
In this article we pursue the case where B is a polynomial. The cases 7 and 8 of table 1 will be treated in a separate article.

Case	Type	$R(A,B,H)$	$S(A,B,H)$	$\tilde{R}(A,B,H)$	$\tilde{S}(A,B,H)$
1	(a,a)	0	0	0	0
2	(b,b)	0	0	0	0
3	(c,b)	0	$\kappa(H+A)$	0	$\kappa(H+A)$
4	(d,d)	$-4\lambda^2 B$	$T(A,H)$	$4\lambda^2 B$	$\tilde{T}(A,H)$
5	(c,c)	0	$\frac{\kappa}{2}A(H^2-A^2)$	0	$\frac{\kappa}{2}A(H^2-A^2)$
6	(a,d)	0	0	0	0
7	(b,d)	$-4\alpha_2^2 B$	0	-	-
8	(c,d)	$-4\alpha_2 B$	0	-	-

Tab. 4.2. Polynomial algebra

In Table 2, $\kappa = \alpha_1^2\alpha_2^2$ and

$$T(A,H) = 4\lambda P(\frac{H+A}{2})^{m-1}P(\frac{H-A}{2})^{n-1}\Big(n^2Q(\frac{H+A}{2})P(\frac{H-A}{2})-m^2Q(\frac{H-A}{2})P(\frac{H+A}{2})\Big),$$

$$\tilde{T}(A,H) = -2\prod_{i=1}^{m}Q(\frac{H}{2}+\frac{A}{2}-(m-i)\alpha_1)\prod_{j=1}^{n}S(\frac{H}{2}-\frac{A}{2}+j\alpha_2)$$

with $K_1^+K_1^- = P(H_1)$, and $\{K_1^-,K_1^+\} = Q(H_1)$.

4.3. Classification of quantum algebraic systems in one dimension

We consider the one dimensional Hamiltonian (4.2.1) and the Mth order operator K_1 (4.2.2) and their Lie bracket $[H_1,K_1]$.

Once $[H_1,K_1]$ is chosen to be equal to $0, \alpha_1, \alpha_1 H_1$ or $-\alpha_1 K_1$ as in (4.2.4) this will provide us with determining equations for the potential $V(x)$ and the coefficients $f_j(x),\ 0 \leq j \leq M$ in the operator K_1.

Using $p_x = -i\hbar\partial_x = -i\hbar D$ we obtain the following operator of order $M+1$.

$$[H_1,K_1] = -\frac{\hbar^2}{2}\sum_{l=0}^{M}(-i\hbar)^l(f_l''D^l + 2f_l'D^{l+1}) - \sum_{l=1}^{M}(-i\hbar)^l f_l \sum_{j=0}^{l-1}C_j^l V^{(l-j)}D^j \tag{4.3.1}$$

where C_j^k are the Newton binomial coefficients.

In order to obtain the determining equations for arbitrary M we must reorder the double summation in the second term in (4.3.1). We obtain

$$[H_1,K_1] = \sum_{l=0}^{M+1} Z_l D^l \tag{4.3.2}$$

with

$$Z_{M+1} = (-i\hbar)^{M+2} f'_M, \tag{4.3.3a}$$

$$Z_M = -\frac{\hbar^2}{2}(-i\hbar)^{M-1}(2f'_{M-1} - i\hbar f''_M), \tag{4.3.3b}$$

$$Z_l = -\frac{\hbar^2}{2}(-i\hbar)^{l-1}(2f'_{l-1} - i\hbar f''_l) - \sum_{j=l+1}^{M}(-i\hbar)^j f_j C_l^j V^{(j-l)},\ 1 \leq l \leq M-1, \tag{4.3.3c}$$

$$Z_0 = -\frac{\hbar^2}{2} f''_0 - \sum_{j=1}^{M}(-i\hbar)^j f_j V^{(j)} \tag{4.3.3d}$$

The determining equations for arbitrary $M \geq 1$ are as follows.

Case(a):

We have

$$Z_l = 0,\ 0 \leq l \leq M+1.$$

In particular equations (4.3.3a) and (4.3.3b) imply $f'_M = 0, f'_{M-1} = 0$. Equations (4.3.3c) provide expressions for f''_l in terms of the potential $V(x)$ and its derivatives for $l = 0,...,M-1$. Substituting f_l into (4.3.3d) we obtain a nonlinear ODE for the potential $V(x)$,

Case(b):

The determining equations are

$$Z_0 = \alpha_1,\ Z_l = 0, 1 \leq l \leq M+1. \tag{4.3.4}$$

Hence the functions f_l, $0 \leq l \leq M$ are the same as in case (a) but the equation for the potential $V(x)$ is modified.

Case(c):

This case arises for $M \geq 2$. The determining equations are

$$Z_l = 0, l \neq 0{,}2 \tag{4.3.5}$$

$$Z_0 = \alpha_1 V(x), \quad Z_2 = -\alpha_1 \frac{\hbar}{2}.$$

Again, equation (4.3.3d) provides an ODE for $V(x)$.

Case(d):

The determining equations are

$$Z_{M+1} = 0, \quad Z_l = \alpha_1 f_l, \quad 0 \leq l \leq M. \tag{4.3.6}$$

In this article we concentrate on the cases $1 \leq M \leq 5$ but it is clear that one can proceed iteratively for any given M.

Let us now solve the determining equations for $1 \leq M \leq 5$.

The notation used below is V_{γ_M} where $\gamma = a,b,c,d$ refers to the four different cases in equation (4.2.4) and $M = 1,2,...,5$ refers to the order of K_1 as a differential operator.

We note that in all cases the determining equations (4.3.3a) imply $f_m = k$ a constant and we can normalize $f_M = 1$. We also note that in all cases we can add arbitrary powers of H to the operator K. We shall omit case when $V(x)$ is constant (e.g. V_{a_1}).

I. Operator of type (a):

$$V_{a_2} = V, \tag{4.3.7}$$

$$K_{a_2} = p_x^2 + \beta p_x + 2V.$$

$$V_{a_3} = \hbar^2 \wp,\ f_2 = \beta,\ f_1 = 3\hbar^2 \wp, \tag{4.3.8}$$

$$K_{a_3} = p_x^3 + \beta p_x^2 + 3\hbar^2 \wp p_x + 2\beta\hbar^2 \wp - \frac{3}{2} i\hbar^3 \wp'.$$

where $\wp(x)$ is the Weierstrass elliptic function. In the case $f_2 = 0$ the solution for V_{a_3} is

$$V(x) = \frac{\hbar^2}{x^2},\ K_{a_3} = 2p_x^3 + \{\frac{3\hbar^2}{x^2}, p_x\}. \tag{4.3.9}$$

$$V_{a_4} = \hbar^2 \wp, \tag{4.3.10}$$

$$K_{a_4} = p_x^4 + \beta p_x^3 + 4V p_x^2 + (3\beta V - 4i\hbar V')p_x^2 + (-\frac{3}{2} i\hbar\beta V' - 8V^2)$$

$$V_{a_5} = V, \tag{4.3.11}$$

$$K_{a_5} = p_x^5 + \beta p_x^4 + 5V p_x^3 + (-\frac{15}{2} i\hbar V' + 4\beta V)p_x^2 + (-\frac{25}{4}\hbar^2 V'' - 4i\beta\hbar V' + \frac{15}{2} V^2)p_x$$
$$+ \frac{15}{8} i\hbar^3 V^{(3)} - 2\beta\hbar^2 V'' - \frac{15}{2} i\hbar V V' + 4\beta V^2.$$

V satisfies in

$$\hbar^4 V^{(4)} - 20\hbar^2 V V'' - 10\hbar^2 V'^2 + 40V^3 = 0 \tag{4.3.12}$$

Setting $V = \hbar^2 U$, we get

$$U^{(4)} - 20UU'' - 10U'^2 + 40U^3 = 0 \tag{4.3.13}$$

This equation is a special autonomous case of the equation F-V, in [**22**, p42]. It has the Painlevé property and it is solvable in terms of hyperelliptic functions. The solution can be written as

$$U = \frac{1}{4}(u_1 + u_2) \tag{4.3.14}$$

where $u_1(x)$ and $u_2(x)$ are defined by inversion of the hyperelliptic integrals

$$\begin{aligned} \int_{\infty}^{u_1(x)} \frac{dt}{\sqrt{P(t)}} + \int_{\infty}^{u_2(x)} \frac{dt}{\sqrt{P(t)}} &= k_3, \\ \int_{\infty}^{u_1(x)} \frac{t\,dt}{\sqrt{P(t)}} + \int_{\infty}^{u_2(x)} \frac{t\,dt}{\sqrt{P(t)}} &= x + k_4, \end{aligned} \tag{4.3.15}$$

with $P(t) = t^5 + 32k_1 t + k_2$, where k_1 and k_2 are constants of integration.
The functions u_1 and u_2 are not meromorphic separately, each having movable quadratic branch points, however the solution U is globally meromorphic.

II. Operator of type (b):

$$V_{b_1} = \frac{\alpha_1}{\hbar} x, \tag{4.3.16}$$

$$K_{b_1} = p_x + \beta.$$

$$V_{b_2} = -\frac{\alpha_1}{\beta\hbar} x, \tag{4.3.17}$$

$$K_{b_2} = p_x^2 + \beta p_x + 2V.$$

$$V_{b_3} = V,\ f_2 = \beta, \tag{4.3.18}$$

$$K_{b_3} = p_x^3 + \beta_1 p_x^2 + 3V p_x + 2\beta_1 V - \frac{3}{2}\hbar V'.$$

where V satisfies in the following first Painlevé equation

$$V'' = \frac{6}{\hbar^2} V^2 + \frac{4\alpha_1}{\hbar^3} x, \tag{4.3.19}$$

and thus

$$V(x) = \hbar^2\omega_1^2 P_I(\omega_1 x),\, \omega_1 = \frac{\sqrt[5]{4\alpha_1}}{\hbar}. \tag{4.3.20}$$

$$V_{b_4} = V, \tag{4.3.21}$$

$$K_{b_4} = p_x^4 + \beta p_x^3 + 4Vp_x^2 + 3\beta V - 4\hbar V' - 2\hbar^2 V'' - \frac{3}{2}\hbar\beta V' + 4V^2.$$

where V satisfies the first Painlevé equation

$$V'' = \frac{6}{\hbar^2}V^2 + \frac{4\alpha_1}{\hbar^3\beta}x,$$

with the solution

$$V(x) = \hbar^2\omega_1^2 P_I(\omega_1 x),\, \omega_1 = \sqrt[5]{\frac{4\alpha_1}{\hbar^5\beta}}. \tag{4.3.22}$$

$$V_{b_5} = V, \tag{4.3.23}$$

$$K_{b_5} = p_x^5 + \beta p_x^4 + 5Vp_x^3 + (-\frac{15}{2}i\hbar V' + 4\beta V)p_x^2 + (-\frac{25}{4}\hbar^2 V'' - 4\beta\hbar V' + \frac{15}{2}V^2)p_x$$
$$+ \frac{15}{8}\hbar^3 V^{(3)} - 2\beta\hbar^2 V'' - \frac{15}{2}\hbar VV' + 4\beta V^2.$$

The potential V satisfies

$$\hbar^4 V^{(4)} - 20\hbar^2 VV'' - 10\hbar^2 V'^2 + 40V^3 + \frac{16\alpha_1 x}{\hbar} = 0. \tag{4.3.24}$$

Setting $V = \hbar^2 U$, we get

$$U^{(4)} - 20UU'' - 10U'^2 + 40U^3 + \frac{16\alpha_1 x}{h^7} = 0 \tag{4.3.25}$$

This equation is also a special case of the equation F-V in [**22**, p42]. The exact solution of it is not known and it is possible that its solution cannot be expressed in terms of classical transcendents nor one of the original Painlevé transcendents.

III. Operator of type (c):

$$V_{c_2} = \frac{\beta}{x^2}, \tag{4.3.26}$$

$$K_{c_2} = p_x^2 + \frac{\alpha_1}{2\hbar}xp_x + \frac{2\beta_1}{x^2}.$$

$$V_{c_3} = V, \tag{4.3.27}$$

$$K_{c_3} = p_x^3 + \beta p_x^2 + (3V + \frac{1}{2\hbar}x)p_x + 2\beta V - \frac{3}{2}\hbar V'.$$

Setting $V = \hbar^2 U - \frac{\alpha_1}{2\hbar}x$, $U(x)$ is the solution of the following equation

$$U^{(3)} = 12UU' - 4\frac{\alpha_1}{\hbar^3}xU' - \frac{2\alpha_1}{\hbar^3},$$

It admits the following first integral

$$2UU'' - U'^2 - 8U^3 + 4\frac{\alpha_1}{\hbar^3}xU^2 = k \tag{4.3.28}$$

where k is the integration constant. For $k = 0$, by the change of variables

$$x = \frac{1}{\lambda}X, U = \lambda^2 W^2;\ \lambda = \frac{1}{h}\sqrt[3]{\alpha_1},$$

we get a special case of the second Painlevé equation

$$W'' - 2W^3 - XW = 0 \tag{4.3.29}$$

Therefore, the solution for $V(x)$ is

$$V(x) = -\alpha_1^{\frac{2}{3}} P_2^2 - \frac{\alpha_1}{2\hbar}x.$$

with $P_2 = P_2(\frac{1}{\hbar}\sqrt[3]{\alpha_1}x)$.

For $k \neq 0$, by the following transformation

$$x = \lambda X,\ U = \sqrt{-k\lambda^2}W;\ \lambda = \sqrt[3]{\frac{\hbar^3}{2\alpha_1}}$$

we transform (4.3.28) to

$$W'' = \frac{W'^2}{2W} + 4\lambda^2\sqrt{-k\lambda^2}W^2 - XW - \frac{1}{2W} \tag{4.3.30}$$

which is Ince-XXXIV(Ince, p.340) with the solution

$$2\lambda^2\sqrt{-k\lambda^2}W = P_2' + P_2^2 + \frac{1}{2}X$$

where P_2 satisfies the second Painlevé equation

$$P_2'' = 2P_2^3 + XP_2 - 2\lambda^2\sqrt{-k\lambda^2} - \frac{1}{2}.$$

The solution for V is

$$V(x) = \frac{(2\alpha_1)^{\frac{2}{3}}}{2}(P_2' + P_2^2),$$

for $P_2 = P_2(\sqrt[3]{\frac{2\alpha_1}{\hbar^3}}x)$.

$$V_{c_4} = V, \tag{4.3.31}$$

$$K_{c_4} = p_x^4 + \beta p_x^3 + 4Vp_x^2 + (-4\hbar V' + 3\beta V + \frac{i\alpha_1}{2\hbar}x)p_x - 2\hbar^2 V'' - \frac{3}{2}\beta\hbar V' + 4V^2.$$

For $\beta = 0$, $V = \frac{k}{x^2}$, and for $\beta \neq 0$, setting $V = \hbar^2 U - \frac{\alpha_1}{6\beta\hbar}x$, $U(x)$ is the solution of the following equation

$$U^{(4)} = 12UU'' + 12U'^2 + \frac{2\alpha_1}{\beta\hbar^3}U' + \frac{2\alpha_1^2}{3\beta^2\hbar^6}$$

which is again a special case of equation F-I [**24**]. Its solution can be expressed in terms of the second Painlevé transcendent.

$$V_{c_5} = V, \tag{4.3.32}$$

$$K_{c_5} = p_x^5 + \beta p_x^4 + 5Vp_x^3 + (-\frac{15}{2}i\hbar V' + 4\beta V)p_x^2 + (-\frac{25}{4}\hbar^2 V'' - 4\beta\hbar V' + \frac{15}{2}V^2 + \frac{\alpha_1}{2\hbar}x)p_x$$
$$+ \frac{15}{8}\hbar^3 V^{(3)} - 2\beta\hbar^2 V'' - \frac{15}{2}\hbar VV' + 4\beta V^2.$$

The potential V satisfies

$$\hbar^5 V^{(5)} - 20\hbar^3 VV^{(3)} - 40\hbar^3 V'V'' + 120\hbar V^2 V' + 8\alpha_1 xV' + 16\alpha_1 V = 0 \tag{4.3.33}$$

Setting $V = \hbar^6 U(X)$, $X = \hbar^2 x$, we get

$$U^{(5)} - 20U^{(3)}U + 120U^2U' - 40U'U'' + \frac{8\alpha_1 X}{\hbar^{15}}U' + \frac{16\alpha_1}{\hbar^{15}}U = 0 \tag{4.3.34}$$

This equation is Fif-III in [**22**, p25,eq 2.71] and it has the Painlevé property. A first integral of it is

$$2uu'' - u'^2 - 8Uu^2 + k = 0$$

where $u = U'' - 6U^2 - \frac{2\alpha_1}{\hbar^{15}}X$.

When $k = 0$, a particular solution of (4.3.34) can be obtained by setting $u = 0$. This solution is $U = P_I$, where $P_I(X)$ satisfies the Painlevé first equation

$$P_I'' = 6P_I^2 + 4\frac{\alpha_1}{\hbar^{13}}X. \tag{4.3.35}$$

IV. Operator of type (d):

$$V_{d_1} = \frac{\alpha_1^2}{2\hbar^2}x^2, \tag{4.3.36}$$
$$K_{d_1} = p_x - \frac{\alpha_1}{\hbar}x.$$
$$V_{d_2} = \frac{{\alpha_1}^2}{8\hbar^2}x^2 + \frac{\beta}{x^2}, \tag{4.3.37}$$
$$K_{d_2} = p_x^2 - \frac{\alpha_1}{\hbar}xp_x - \frac{\alpha_1^2}{\hbar^2}x^2 + \frac{2\beta}{x^2}.$$
$$V_{d_3} = V(x), \tag{4.3.38}$$
$$K_{d_3} = p_x^3 - \frac{\alpha_1}{\hbar}xp_x^2 + (3V - \frac{\alpha_1^2}{2\hbar^2}x^2)p_x + (\frac{\hbar^3}{4\alpha}V^{(3)} - \frac{3\hbar}{\alpha_1}VV' - (\frac{5}{2}\hbar - \frac{\alpha_1}{2\hbar}x^2)V' + \frac{\alpha_1^2}{2\hbar}x).$$

Setting $V = \hbar^2 U(x) + \frac{{\alpha_1}^2}{6\hbar^2}x^2 - \frac{\alpha_1}{3}$, U is the solution of

$$U^{(4)} = 12UU'' + 12U'^2 - \frac{4{\alpha_1}^2}{\hbar^4}xU' - \frac{8{\alpha_1}^2}{\hbar^4}U - \frac{8{\alpha_1}^4}{3\hbar^8}x^2$$

which is a special case of equation F-I [**24**]. The solution for $V(x)$ is

$$V(x) = \epsilon\alpha_1 P_4' + \frac{2\alpha_1^2}{\hbar^2}(P_4^2 + xP_4) + \frac{\alpha_1^2}{2\hbar^2}x^2 + (\epsilon - 1)\frac{\alpha_1}{3} - \frac{\hbar^2}{6}k_1, \tag{4.3.39}$$

where $\epsilon = \pm 1$ and P_4 satisfies the fourth Painlevé equation

$$P_4'' = \frac{(P_4')^2}{2P_4} + \frac{6\alpha_1^2}{\hbar^4}P_4^3 + \frac{8\alpha_1^2}{\hbar^4}xP_4^2 + (\frac{2\alpha_1^2}{\hbar^4}x^2 - k_1)P_4 + \frac{k_2}{P_4}. \tag{4.3.40}$$

k_1 and k_2 are integration constants.

$$V_{d_4} = V, \tag{4.3.41}$$
$$K_{d_4} = p_x^4 - \frac{\alpha_1 x}{\hbar}p_x^3 + (4V - \frac{\alpha_1^2 x^2}{2\hbar^2})p_x^2 + f_1 p_x + f_0.$$

setting $u(x) = \int V dx$, we get

$$f_1 = -\frac{x\alpha_1^2}{2\hbar} + \frac{x^3\alpha_1^3}{6\hbar^3} - \frac{\alpha_1 u}{\hbar} - \frac{3x\alpha_1 u'}{\hbar} - 4\hbar u'', \tag{4.3.42}$$
$$f_0 = -\frac{\alpha_1^3}{2\hbar^2}x^2 + \frac{\alpha_1^4}{24\hbar^4}x^4 - \frac{\alpha_1^2}{\hbar^2}xu + (\alpha_x - \frac{\alpha_1^2}{\hbar^2}x^2)u' + 4u'^2 - \frac{3}{2}\alpha_1 xu'' - 2\hbar^2 u'''.$$

thus $u(x)$ is the solution of the following equation

$$
\begin{aligned}
0 =& k-\alpha_1^2x^2+\frac{3\alpha_1^3x^4}{4\hbar^2}-\frac{\alpha_1^4x^6}{36\hbar^4}+\frac{4\alpha_1^2x^3}{3\hbar^2}u+2u^2-2\hbar^2u'-6\alpha_1x^2u' \\
&+\frac{2\alpha_1^2x^4}{3\hbar^2}u'-4xuu'-6x^2u'^2+2\hbar^2xu''+\hbar^2x^2u'''.
\end{aligned} \tag{4.3.43}
$$

By the following transformation

$$
X=x^2, U=-\frac{x}{2\hbar^2}u+\frac{3\hbar^4-9\alpha_1h^2x^2+\alpha_1^2x^4}{48\hbar^4}
$$

we transform (4.3.43) to

$$
X^2U^{(3)}=-2(U'(3XU'-2U)-\frac{\alpha_1^2}{8\hbar^4}X(XU'-U)+k_1X+k_2)-XU'', \tag{4.3.44}
$$

where $k_1=-\frac{7\alpha_1^2}{256\hbar^4}$, $k_2=\frac{-4k-3\alpha_1\hbar^2}{128\hbar^4}$. The equation (4.3.44) is a special case of the Chazy class I equation. It admits the first integral

$$
X^2U''^2=-4(U'^2(XU'-U)-\frac{\alpha_1^2}{16\hbar^4}(XU'-U)^2+k_1(XU'-U)+k_2U'+k_3) \tag{4.3.45}
$$

where k_3 is the integration constant. The equation is the canonical form SD-I.b. The solution is

$$
\begin{aligned}
U =& \frac{1}{4}(\frac{1}{P_5}(\frac{XP_5'}{P_5-1}-P_5)^2-(1-\sqrt{2\lambda})^2(P_5-1)-2\beta\frac{P_5-1}{P_5}+\gamma X\frac{P_5+1}{P_5-1}+2\delta\frac{X^2P_5}{(P_5-1)^2}), \\
U' =& -\frac{X}{4P_5(P_5-1)}(P_5'-\sqrt{2\lambda}\frac{P_5(P_5-1)}{X})^2-\frac{\beta}{2X}\frac{P_5-1}{P_5}-\frac{1}{2}\delta X\frac{P_5}{P_5-1}-\frac{1}{4}\gamma,
\end{aligned} \tag{4.3.46}
$$

where $P_5=P_5(x^2)$, satisfies the fifth Painlevé equation

$$
P_5''=(\frac{1}{2P_5}+\frac{1}{P_5-1})P_5'^2-\frac{1}{X}P_5'+\frac{(P_5-1)^2}{X^2}(\lambda P_5+\frac{\beta}{P_5})+\gamma\frac{P_5}{X}+\delta\frac{P_5(P_5+1)}{P_5-1}, \tag{4.3.47}
$$

with

$$
\alpha_1^2=-8\hbar^4\delta,\ k_1=-\frac{1}{4}(\frac{1}{4}\gamma^2+2\beta\delta-\delta(1-\sqrt{2\lambda})^2),\ k_2=-\frac{1}{4}(\beta\gamma+\frac{1}{2}\gamma(1-\sqrt{2\lambda})^2),
$$

$$
k_3=-\frac{1}{32}(\gamma^2((1-\sqrt{2\lambda})^2-2\beta)-\delta((1-\sqrt{2\lambda})^2+2\beta)^2).
$$

The solution for the potential is

$$
\begin{aligned}
V(x) =& \frac{\alpha_1^2}{8\hbar^2}x^2 + \hbar^2\Big(\frac{\gamma}{P_5-1} + \frac{1}{x^2}(P_5-1)(\sqrt{2\lambda} + \lambda(2P_5-1) + \frac{\beta}{P_5})\\
&+ x^2(\frac{P_5'^2}{2P_5} - \frac{\alpha_1^2}{8\hbar^4}P_5)\frac{(2P_5-1)}{(P_5-1)^2} - \frac{P_5'}{P_5-1} - 2\sqrt{2\lambda}P_5'\Big) + \frac{3\hbar^2}{8x^2}.
\end{aligned}
\tag{4.3.48}
$$

We could also choose the values of $k_i, i = 1,2,3$ in a way to have (4.3.47) with the following parameter values:

$$
\lambda = -\beta = \frac{B^2}{8}, \gamma = 0, \delta = 2A^2 \neq 0.
$$

we can then reduce the fifth Painlevé equation with such parameters to a third Painlevé equation [**49**, Thm 34.3(p.170),Thm 41.2(p.208), Thm 41.5(p.210)]. Hence in this case we can obtain a solution in terms of third Painlevé transcendent.

$$
\begin{aligned}
&V_{d_5} = V, \\
&K_{d_5} = p_x^5 - \frac{\alpha_1}{\hbar}xp_x^4 + (-\frac{\alpha_1^2}{2\hbar^2}x^2 + 5V)p_x^3 + f_2p_x^2 + f_1p_x + f_0.
\end{aligned}
\tag{4.3.49}
$$

setting $u(x) = \int V dx$, we get

$$
\begin{aligned}
f_2 =& -\frac{\alpha_1^2}{2\hbar}x + \frac{\alpha_1^3}{6\hbar^3}x^3 - \frac{\alpha_1}{\hbar}u - \frac{4\alpha_1}{\hbar}xu' - \frac{15}{2}\hbar u'',\\
f_1 =& -\frac{\alpha_1^3x^2}{2\hbar^2} + \frac{\alpha_1^4}{24\hbar^4}x^4 - \frac{\alpha_1^2}{\hbar^2}xu + \alpha_1 u' - \frac{3\alpha_1^2}{2\hbar^2}x^2u' + \frac{15}{2}u'^2 - 4\alpha_1 xu'' - \frac{25}{4}\hbar^2u^{(3)},\\
f_0 =& \frac{1}{48\alpha_1\hbar^3}(-3\hbar^8u^{(6)} + 114\alpha_1\hbar^6u^{(4)} + 60\hbar^6u^{(4)}u' + 120\hbar^6u^{(3)}u'' - 6\alpha_1^2\hbar^4x^2u^{(4)} + 48\alpha_1^2\hbar^4xu^{(3)} - 96\alpha_1^2\hbar^4u''\\
&- 648\alpha_1\hbar^4u'u'' - 360\hbar^4u'^2u'' + 84\alpha_1^3\hbar^2x^2u'' + 48\alpha_1^2\hbar^2xuu'' + 96\alpha_1^3\hbar^2xu' + 72\alpha_1^2\hbar^2x^2u'u'' + 24\alpha_1^3\hbar^2u\\
&+ 36\alpha_1^4\hbar^2x - 2\alpha_1^4x^4u'' - 4\alpha_1^5x^3).
\end{aligned}
\tag{4.3.50}
$$

and

$$
\begin{aligned}
&9\hbar^{10}(xu^{(6)}-u^{(5)})+18\hbar^{6}x\left(-10\hbar^{2}u'+\alpha_1^2x^2-4\alpha_1\hbar^2\right)u^{(4)}\\
&+\hbar^{6}\left(-360\hbar^{2}xu''+180\hbar^{2}u'+126\alpha_1^2x^2+72\alpha_1\hbar^2\right)u^{(3)}+90\hbar^{8}u''^{2}\\
&+\left(1080\hbar^{6}xu'^{2}+864\alpha_1\hbar^{6}xu'-216\alpha_1^2\hbar^4x^3u'-144\alpha_1^2\hbar^4x^2u+6\alpha_1^4\hbar^2x^5-144\alpha_1^3\hbar^4x^3+288\alpha_1^2\hbar^6x\right)u''\\
&-360\hbar^6u'^3-432\alpha_1\hbar^6u'^2-468\alpha_1^2\hbar^4x^2u'^2-144\alpha_1^2\hbar^4xuu'-288\alpha_1^2\hbar^6u'-432\alpha_1^3\hbar^4x^2u'\\
&+42\alpha_1^4\hbar^2x^4u'+72\alpha_1^2\hbar^4u^2+48\alpha_1^4\hbar^2x^3u-90\alpha_1^4\hbar^4x^2+36\alpha_1^5\hbar^2x^4-\alpha_1^6x^6=0
\end{aligned}
\tag{4.3.51}
$$

This equation passes the Painlevé test. Substituting the Laurent series

$$u=\sum_{k=0}^{\infty}d_k(x-x_0)^{k+p},\ d_0\neq 0,$$

in (4.3.51), we find $p=-1$. The resonances are $r=1,2,5,6,8$ and we obtain $d_0=-\hbar^2$. The constants d_1,d_2,d_5,d_6 and d_8 are arbitrary, as they should be.

4.4. Classification of classical algebraic systems in one dimension

We consider the Hamiltonian (4.2.1) and polynomial K_1 (4.2.2) in classical mechanics and require that they satisfy one of the equations (4.2.4) where $[H_1,K_1]\equiv\{H_1,K_1\}_{PB}$ is now a Poisson bracket.

Instead of (4.3.2) we now have

$$\{H_1,K_1\}_{PB}=\sum_{l=0}^{M+1}Z_l(x)p_x^l \tag{4.4.1}$$

with

$$Z_0=f_1V',\ Z_l=(l+1)f_{l+1}V'-f'_{l-1}, 1\leq l\leq M-1,\ Z_M=-f'_{M-1},\ Z_{M+1}=-f'_M. \tag{4.4.2}$$

For all the algebras in (4.2.4) we obtain $f_M=k$ a constant, so we can set $f_M=1$.

I. Polynomial of type (a):

We have $Z_l=0, 0\leq l\leq M$ and the result is trivial. In the case that K_1 is of order 1,3 and 5 we get a constant potential. For K_1 of order 2 and 4, the only function of x and p_x that Poisson commutes with the Hamiltonian is a function of H_1 itself. In particular the polynomial K_1 is any polynomial in H. For all M we find that either V is constant or K is a polynomial in H.

Notice that this is quite different from the quantum case (4.3.7,...,4.3.15) where we obtain potentials expressed in terms of nonlinear special functions having the Painlevé property. The types (b), (c) and (d) are more interesting and provide specific potentials that will generate superintegrable systems in E_2.

II. Polynomial of type (b):

The determining equations in this case are

$$f_1 V' = \alpha_1,\ f'_{M-1} = 0, \qquad (4.4.3)$$
$$(l+1) f_{l+1} V' - f'_{l-1} = 0, 1 \leq l \leq M-1,$$

The determining equations in this case were already solved by Güngör et al [**51**]. For completeness we reproduce some of their results in our notations.

$$V_{b_1} = \alpha_1 x, \qquad (4.4.4)$$
$$K_{b_1} = p_x + \beta.$$
$$V_{b_2} = \frac{\alpha_1}{\beta} x, \qquad (4.4.5)$$
$$K_{b_2} = 2H_1 + \beta p_x.$$
$$V_{b_3} = \epsilon \sqrt{\frac{2\alpha_1}{3} x};\ \epsilon = \pm 1; \qquad (4.4.6)$$
$$K_{b_3} = p_x(2H_1 + V) + 2\beta H_1.$$
$$V_{b_4} = \epsilon \sqrt{\frac{2\alpha_1}{3\beta} x};\ \epsilon = \pm 1, \qquad (4.4.7)$$
$$K_{b_4} = 4H_1^2 + 2\beta p_x H_1 + \beta V p_x.$$
$$V_{b_5} = \sqrt[3]{\frac{2\alpha_1}{5} x}, \qquad (4.4.8)$$
$$K_{b_5} = 4p_x H_1^2 + 4\beta h_1^2 + 2V p_x H_1 + \frac{3}{2} V_2 p_x.$$

III. Polynomial of type (c):

The determining equations are

$$f_1 V' = \alpha_1 V, 3f_3 V' - f'_1 = \frac{\alpha_1}{2}, f'_{M-1} = 0, M \neq 2, \qquad (4.4.9)$$
$$(l+1) f_{l+1} V' - f'_{l-1} = 0, 1 \leq l \leq M-1, l \neq 2.$$

The case when K_1 is a first order polynomial does not exist. The solutions for $2 \leq M \leq 5$ are

$$V_{c_2} = \frac{c}{x^2}, \tag{4.4.10}$$
$$K_{c_2} = 2H_1 - \frac{\alpha_1}{2} x p_x.$$
$$V_{c_3} = V, \, (\alpha_1 x - 2V)^2 V = c, \tag{4.4.11}$$
$$K_{c_3} = p_x^3 + \beta p_x^2 + (3V - \frac{\alpha_1}{2} x) p_x + 2\beta V.$$
$$V_{c_4} = V; \; (\alpha_1 x - 2\beta V)^2 V = c, \tag{4.4.12}$$
$$K_{c_4} = p_x^4 + \beta p_x^3 + 4V p_x^2 + (3\beta V - \frac{\alpha_1}{2} x) p_x + 4V^2.$$
$$V_{c_5} = V; \; (\alpha_1 x - 3V^2)^2 V = c, \tag{4.4.13}$$
$$K_{c_5} = p_x^5 + \beta p_x^4 + 5V p_x^3 + 4\beta V p_x^2 + (\frac{15}{2} V^2 - \frac{\alpha_1}{2} x) p_x + 4\beta V^2.$$

IV. Polynomial of type (d):

In this case we define polynomial ladder operators as

$$K_1^{\pm} = \sum_{l=0}^{M} f_l p_x^l \tag{4.4.14}$$

where $f_l = c g_l$, with

$$c = \begin{cases} \mp i & \text{for } l \text{ even} \\ 1 & \text{for } l \text{ odd} \end{cases} \tag{4.4.15}$$

and they satisfy the algebraic relations

$$\{H_1, K_1^{\pm}\} = \pm i \alpha_1 K_1^{\pm}$$

The determining equations are

$$g_1 V' = \alpha_1 g_0, \, g'_{M-1} = (-1)^{M-1} \alpha_1 g_M, \tag{4.4.16}$$
$$(l+1) g_{l+1} V' - g'_{l-1} = -\alpha_1 g_l, 1 \leq l \leq M-1.$$

Their solutions are

$$V_{d_1} = \frac{\alpha_1^2 x^2}{2}, \tag{4.4.17}$$

$$K_{d_1} = p_x + \alpha_1 x.$$

$$V_{d_2} = \frac{\alpha_1 x^2}{8} + \frac{\gamma}{x^2}, \tag{4.4.18}$$

$$K_{d_2} = p_x^2 - \alpha_1 x p_x + 2V - \frac{\alpha_1^2}{2} x^2.$$

$$V_{d_3} = V, \tag{4.4.19}$$

$$K_{d_3} = p_x^3 + \alpha_1 x p_x^2 + (3V - \frac{\alpha_1^2}{2} x^2) p_x - (\frac{(\alpha_1^2 x^2 - 6V)}{2\alpha_1} V'.$$

The potential V satisfies

$$24xVV' - 4\alpha_1^2 x^3 V' - 12V^2 - 12\alpha_1^2 x^2 V + \alpha_1^4 x^4 + 4d = 0. \tag{4.4.20}$$

It admits the following first integral

$$9V^4 - 14\alpha_1^2 x^2 V^3 + (\frac{15}{2}\alpha_1^4 x^4 - 6d)V^2 - 2\alpha_1^2 x^2 (\frac{3}{4}\alpha_1^4 x^4 - d)V + (\frac{\alpha_1^8}{16} x^8 + \frac{1}{2} d\alpha_1^4 x^4 + d^2) = 0. \tag{4.4.21}$$

$$V_{d_4} = V, \tag{4.4.22}$$

$$K_{d_4} = p_x^4 - \alpha_1 x p_x^3 + (4u' - \frac{\alpha_1^2}{2} x^2) p_x^2 + (\frac{\alpha_1^3}{6} x^3 - \alpha_1 u - 3\alpha_1 x u') p_x + (\frac{\alpha_1^2}{6} x^3 - u - 3xu') u'',$$

where $u(x) = \int V dx$ and u satisfies

$$3x^2 u'^2 + 2xuu' - \frac{1}{3}\alpha_1^2 x^4 u' - u^2 - \frac{2}{3}\alpha_1^2 x^3 u + \frac{\alpha_1^4 x^6}{72} + k_1 x + k_2 = 0. \tag{4.4.23}$$

Also using (4.2.11) we find that $V(x)$ satisfies the following $5th$ order algebraic equation

$$
\begin{aligned}
0 &= -32\alpha_1^2x^2V^5 + (128d + 17\alpha_1^4x^4)V^4 \qquad (4.4.24)\\
&+ (-96c - \frac{128d^2}{\alpha_1^2x^2} - \frac{512e}{\alpha_1^2x^2} - 32d\alpha_1^2x^2 - \frac{7\alpha_1^6x^6}{2})V^3\\
&+ (-40d^2 + 352e + \frac{256c^2}{\alpha_1^4x^4} + \frac{256cd}{\alpha_1^2x^2} + 32c\alpha_1^2x^2 + 3d\alpha_1^4x^4 + \frac{11\alpha_1^8x^8}{32})V^2\\
&+ (-16cd - \frac{128cd^2}{\alpha_1^4x^4} - \frac{512ce}{\alpha_1^4x^4} - \frac{256c^2}{\alpha_1^2x^2} + \frac{64d^3}{\alpha_1^2x^2} + \frac{256de}{\alpha_1^2x^2} - \frac{\alpha_1^{10}x^{10}}{64})V\\
&+ 64c^2 + 4d^3 - 112de + \frac{16d^4}{\alpha_1^4x^4} + \frac{128d^2e}{\alpha_1^4x^4} + \frac{256e^2}{\alpha_1^4x^4} - \frac{64cd^2}{\alpha_1^2x^2} + \frac{256ce}{\alpha_1^2x^2}\\
&- 8cd\alpha_1^2x^2 + \frac{3}{2}d^2\alpha_1^4x^4 + \frac{17}{2}e\alpha_1^4x^4 - \frac{1}{4}c\alpha_1^6x^6 + \frac{1}{64}d\alpha_1^8x^8 + \frac{\alpha_1^{12}x^{12}}{4096}.
\end{aligned}
$$

$$
f_3 = -\alpha_1 x,\ f_2 = (-\frac{1}{2}\alpha_1^2x^2 + 4V),\ f_1 = \frac{1}{8\alpha_1 x}(16d + \alpha_1^4x^4 + 8f_0 - 16\alpha_1^2x^2V - 32V^2),
$$

$$
f_0 = (\sqrt{-16e + 16cV - 16dV^2 + 16V^4}).
$$

$$
\begin{aligned}
V_{d_5} &= V, \qquad (4.4.25)\\
K_{d_5} &= p_x^5 + \alpha_1 x p_x^4 + (5u' - \frac{\alpha_1^2}{2}x^2)p_x^3 + (4\alpha_1 x u' + \alpha_1 u - \frac{1}{6}\alpha_1^3x^3)p_x^2\\
&+ \frac{1}{24}(180u'^2 - 36\alpha_1^2x^2u' - 24\alpha_1^2xu + \alpha_1^4x^4)p_x\\
&+ \frac{1}{24\alpha_1}(\alpha_1^4x^4 - 24\alpha_1^2xu - 36\alpha_1^2x^2u' + 180u'^2)u''
\end{aligned}
$$

where $u(x) = \int V dx$ and u satisfies

$$
(\alpha_1^4x^4 - 24\alpha_1^2xu - 36\alpha_1^2x^2u' + 180u'^2)xu'' - 60u'^3 - 78\alpha_1^2x^2u'^2 - 24\alpha_1^2xuu' + 7\alpha_1^4x^4u' \qquad (4.4.26)
$$

$$
+ 12\alpha_1^2u^2 + 8\alpha_1^4x^3u - \frac{1}{6}\alpha_1^6x^6 = 0
$$

We could also use (4.2.11) to get an algebraic equation for V, but it is not very illuminating.

The solutions for H_{d_3} and H_{d_4} are presented in [**71, 73**].

The list of Hamiltonians reduces to $H_{b_1} \equiv H_{b_2}$, H_{d_1}, H_{c_2}, H_{d_2}, $H_{a_3} \equiv H_{a_4}$, $H_{b_3} \equiv H_{b_4}$,

$H_{c_3} \equiv H_{c_4},\ H_{d_3},\ H_{d_4},\ H_{a_5},\ H_{b_5},\ H_{c_5},\ H_{d_5}.$

4.5. Classification of superintegrable systems up to fifth order integrals:quantum and classical systems

In Section 3 we classified all quantum algebraic systems in one dimension with M satisfying $1 \leq M \leq 5$. Here we shall use 2 copies of these algebras H_1, K_1 and H_2,K_2 to construct two dimensional superintegrable systems as described in Section 2 and Table 1.

In table 2, column 1 gives the order of the operator K in the two dimensional systems. In column 2 all entries have the form (z_i,w_j). The letters z and w run through the type a,b,c,d as in (4.2.4) with z referring to the x variable and w to the y. The indices i and j correspond to the orders of the corresponding operators K_1 and K_2 and run between 1 and 5.

order of integrals	type	(α_1,α_2)
1	(b_1,b_1)	(α_1,α_2)
	(d_1,d_1)	(α,α)
2	(d_1,d_1)	$(\alpha,2\alpha)$
	(d_1,d_2)	(α,α)
3	(a_3,a_3)	$(\hbar^2,\hbar^2)$
	(b_3,b_3)	(α_1,α_2)
	(b_1,b_3)	(α_1,α_2)
	(c_2,b_1)	(α_1,α_2)
	(c_3,b_1)	(α_1,α_2)
	(d_1,d_3)	(α,α)
	(d_1,d_2)	$(\alpha,2\alpha)$
	(d_2,d_2)	(α,α)
	(d_1,d_1)	$(\alpha,3\alpha)$
4	(d_4,d_1)	(α,α)
	(d_3,d_2)	(α,α)
	(d_1,d_3)	$(\alpha,2\alpha)$
	(d_1,d_2)	$(\alpha,3\alpha)\&(2\alpha,\alpha)$
	(d_1,d_1)	$(\alpha,4\alpha)\&(2\alpha,3\alpha)$

order of integrals	type	(α_1,α_2)
5	(a_3,a_5)	....
	(a_5,a_5)	....
	(b_1,b_5)	(α_1,α_2)
	(b_3,b_5)	(α_1,α_2)
	(b_5,b_5)	(α_1,α_2)
	(c_3,c_3)	(α_1,α_2)
	(c_2,c_3)	(α_1,α_2)
	(c_2,b_3)	(α_1,α_2)
	(c_3,b_3)	(α_1,α_2)
	(c_5,b_1)	(α_1,α_2)
	(c_5,b_3)	(α_1,α_2)
	(d_5,d_1)	(α,α)
	(d_4,d_2)	(α,α)
	(d_3,d_3)	(α,α)
	(d_2,d_2)	$(\alpha,2\alpha)$
	(d_1,d_4)	$(\alpha,2\alpha)$
	(d_1,d_3)	$(\alpha,3\alpha)$
	(d_1,d_2)	$(\alpha,4\alpha)\&(\alpha,\alpha)$
	(d_1,d_1)	$(\alpha,5\alpha)\&(\alpha,2\alpha)\&(\alpha,\alpha)$

Tab. 4.3. Classification of superintegrable systems

These systems can however admit lower order integrals of motion as this construction does not necessarily provide integrals of the lowest order. As an example for one of the Smorodinsky-Winternitz potential the ladder operators lead to an integral of order 3 that is in fact the commutation of two integrals of order 2. The same phenomena occurs for some of the Gravel potentials.

The potential $(Q.10)$ in Gravel's list can be obtained by (d_2,d_2) construction of 8th order integral with $(\alpha_1,\alpha_2) = (\alpha, 3\alpha)$. The potentials Q_1^1, Q_3^2, Q_3^3 and Q_3^4 in [**77**] are the cases (d_4,d_2),(c_2,c_3),(c_2,b_3) and (a_3,a_5) respectively.

Thus all Smorodinsky-Winternitz, Gravel, fourth and fifth order ones are reducible, i.e. they can be constructed from one-dimensional algebraic Hamiltonian systems.

Let us present the list of obtained quantum superintegrable systems.

Quantum superintegrable system:

Jauch.Hill potentials:

These anisotropic harmonic oscillator potentials have the form $V = \omega^2(nx^2 + my^2)$ where n and m are two mutually prime positive integers [**55**].

(d_1,d_1) :

$$V = \frac{\alpha^2}{2\hbar^2}(x^2 + 4y^2),$$
$$K = (K_1^\dagger)^2(K_2^-) - (K_1^-)^2(K_2^\dagger) = (yp_x - xp_y)p_x.$$

$$V = \frac{\alpha^2}{2\hbar^2}(x^2 + 9y^2)$$
$$K = (K_1^\dagger)^3(K_2^-) - (K_1^-)^3(K_2^\dagger) = (xp_y - yp_x)p_x^2.$$

$$V = \frac{1}{2\hbar^2}(\alpha_x^2 x^2 + \alpha_y^2 y^2),$$

$$(\alpha_1,\alpha_2) = (\alpha,4\alpha): \; K = (K_1^\dagger)^4(K_2^-) - (K_1^-)^4(K_2^\dagger) = (xp_y - yp_x)p_x^3.$$
$$(\alpha_1,\alpha_2) = (2\alpha,3\alpha): \; K = (K_1^\dagger)^3(K_2^-)^2 - (K_1^-)^3(K_2^\dagger)^2 = (xp_y - yp_x)p_x^2 p_y.$$

$$(\alpha_1,\alpha_2) = (\alpha,\alpha): \; K = (K_1^\dagger)^3(K_2^-)^3 - (K_1^-)^3(K_2^\dagger)^3 = (xp_y - yp_x)p_x^2 p_y^2.$$
$$(\alpha_1,\alpha_2) = (\alpha,2\alpha): \; K = (K_1^\dagger)^4(K_2^-)^2 - (K_1^-)^4(K_2^\dagger)^2 = (xp_y - yp_x)p_x^3 p_y.$$
$$(\alpha_1,\alpha_2) = (\alpha,5\alpha): \; K = (K_1^\dagger)^5(K_2^-) - (K_1^-)^5(K_2^\dagger) = (xp_y - yp_x)p_x^4.$$

All values of n and m can be obtained in this manner. The $\hbar$ in the denominator of V has no meaning since we can have $\alpha_1^2 = n\hbar^2$, $\alpha_2^2 = m\hbar^2$.

Smorodinsky-Winternitz potentials:

The original multiseparable potentials in E_2 were

$$V(x,y) = \omega^2(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2} \tag{4.5.1}$$

that is separable in Cartesian, polar and elliptic coordinates, and

$$V(x,y) = \omega^2(x^2 + 4y^2) + \frac{\gamma}{y^2} \tag{4.5.2}$$

that is separable in Cartesian and parabolic coordinates.

Both allow second order integrals of motion [**40, 63**]. These, plus two further ones, not allowing separation in Cartesian coordinates, were later called Smorodinsky-Winternitz potentials [**34, 35, 36, 50**]. For $\omega = 0$ "degenerate" form of the Smorodinsky-Winternitz potentials exist, such as

$$V(x,y) = \alpha y + \frac{\beta}{x^2} + \frac{\gamma}{y^2}. \tag{4.5.3}$$

In the present approach the potentials (4.5.1) and (4.5.2) are built into infinite sets of potentials generalizing the Smorodinsky-Winternitz potentials both in classical and quantum mechanics. They occur when we consider the cases (d_1,d_2) and (d_2,d_2).

$$V = \omega^2(n^2x^2 + m^2y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}, \tag{4.5.4}$$

$$K = (K_1^\dagger)^m(K_2^-)^n - (K_1^-)^m(K_2^\dagger)^n \tag{4.5.5}$$

Taking $m = n \geq 1,\ \omega \neq 0,\ \beta \neq 0$ and $\gamma \neq 0$, we obtain the potential (4.5.1) separable in polar coordinates. Taking $m = 2n, \beta = 0$ we obtain (4.5.2). The pair (d_1,d_2) provides (4.5.4) with $\beta = 0$. The degenerate one (4.5.3) is obtained as (c_2,b_1).

Elliptic and hyperelliptic function potentials:

(a_3,a_3) :

$$V = \hbar^2(\wp(x) + \wp(y)),$$

$$K = K_1 + K_2.$$

(a_3,a_5):

$$V = \hbar^2\wp(x) + \hbar^2 g(y),\ K = K_1 + K_2$$

(a_5,a_5):

$$V = \hbar^2(f(x) + g(y)),\, K = K_1 + K_2$$

where $f(x)$ and $g(y)$ are hyperelliptic functions satisfying equation (4.3.13) and are defined in (4.3.14).

Potentials in terms of the first Painlevé transcendent:

(b_3,b_3) :

$$V = \hbar^2(\omega_1^2 P_I + \omega_2^2 P_I),$$
$$K = \alpha_2 K_1 - \alpha_1 K_2.$$

(b_1,b_3) :

$$V = \frac{\alpha_1}{\hbar}x + \hbar^2(\omega_2^2 P_I),$$
$$K = \alpha_2 K_1 - \alpha_1 K_2.$$

(c_2,b_3):

$$V = \frac{\beta}{x^2} + \hbar^2\omega_2^2 P_I(\omega_2 y),\, \omega_2 = \frac{\sqrt[5]{4i\alpha_2}}{\hbar},\, K = \alpha_2 K_1 - \alpha_1 H_1 K_2.$$

Potentials in terms of the second Painlevé transcendent:

(c_3,b_1) :

$$V = -\alpha_1^{\frac{2}{3}} P_2^2 - \frac{\alpha_1}{2\hbar}x + \frac{\alpha_1}{\hbar}y,\, P_2 = P_2(\frac{1}{\hbar}\sqrt[3]{\alpha_1}x)$$

and

$$V = \frac{\hbar^2}{2}(\epsilon P_2' + P_2^2) + \frac{\alpha_2}{i\hbar}y,\, P_2 = P_2(\sqrt[3]{\frac{2\alpha_1}{\hbar^3}}x),$$

$$K = \alpha_2 K_1 - \alpha_1 H_1 K_2 = \alpha_2 p_x^3 - \frac{\alpha_1}{2}p_x^2 p_y.$$

(c_3,c_3):

$$V = \hbar^2(f(x) + g(y)) - \frac{i}{2\hbar}(\alpha_1 x + \alpha_2 y),\, K = \alpha_2 H_2 K_1 - \alpha_1 H_1 K_2 = (\alpha_2 p_x - \alpha_1 p_y)p_x^2 p_y^2.$$

(c_2,c_3):

$$V=\frac{\beta}{x^2}+\hbar^2 g(y)-\frac{\alpha_2}{2\hbar}y,\; K=\alpha_2 H_2 K_1-\alpha_1 H_1 K_2.$$

(c_3,b_3):

$$V=f(x)+\hbar^2\omega_2^2 P_I(\omega_2 y),\; \omega_y=\frac{\sqrt[5]{4i\alpha_2}}{\hbar},\; K=\alpha_2 K_1-\alpha_1 H_1 K_2$$

where $f(x)$ and $g(y)$ satisfy equation (4.3.28).

Potentials in terms of the fourth Painlevé transcendent:

(d_1,d_3) :

$$V=\frac{\alpha^2}{2\hbar^2}(x^2+y^2)+\epsilon\alpha P_4'+\frac{2\alpha^2}{\hbar^2}(P_4^2+yP_4),\; K=(K_1^\dagger)(K_2^-)-(K_1^-)(K_2^\dagger).$$

(d_3,d_2) :

$$V=\frac{\alpha^2}{8\hbar^2}(4x^2+y^2)+\frac{\beta}{y^2}+\epsilon\alpha P_4'+\frac{2\alpha^2}{\hbar^2}(P_4^2+xP_4),$$
$$K=(K_1^\dagger)(K_2^-)-(K_1^-)(K_2^\dagger)=(xp_y-yp_x)p_x^2p_y.$$

(d_1,d_3) :

$$V=\frac{\alpha^2}{2\hbar^2}(x^2+4y^2)+2\epsilon\alpha P_4'+\frac{8\alpha^2}{\hbar^2}(P_4^2+yP_4),\; K=(K_1^\dagger)^2(K_2^-)-(K_1^-)^2(K_2^\dagger)=(xp_y-yp_x)p_xp_y^2.$$

$$V=\frac{\alpha^2}{2\hbar^2}(x^2+9y^2)+3\epsilon\alpha P_4'+\frac{18\alpha^2}{\hbar^2}(P_4^2+yP_4),\; K=(K_1^\dagger)^3(K_2^-)-(K_1^-)^3(K_2^\dagger)=(xp_y-yp_x)p_x^2p_y^2$$

(d_3,d_3):

$$V=f(x)+g(y),\; K=(K_1^\dagger)(K_2^-)-(K_1^-)(K_2^\dagger)=(xp_y-yp_x)p_x^2p_y^2$$

where $f(x)$ and $g(y)$ are given in (4.3.39) and $\alpha_1=\alpha_2=\alpha$.

Potentials in terms of the fifth Painlevé transcendent:

(d_4,d_1) :

$$V = \frac{\alpha^2}{8\hbar^2}(x^2+4y^2) + \frac{3\hbar^2}{8x^2} + \hbar^2\Big(\frac{\gamma}{P_5-1} + \frac{1}{x^2}(P_5-1)(\sqrt{2\lambda} + \lambda(2P_5-1) + \frac{\beta}{P_5})$$
$$+ x^2(\frac{P_5'^2}{2P_5} - \frac{\alpha^2}{8\hbar^4}P_5)\frac{(2P_5-1)}{(P_5-1)^2} - \frac{P_5'}{P_5-1} - 2\sqrt{2\lambda}P_5'\Big),$$
$$K = (K_1^\dagger)(K_2^-) - (K_1^-)(K_2^\dagger) = (xp_y - yp_x)p_x^3.$$

(d_4,d_2):

$$V = \frac{\alpha^2}{8\hbar^2}(x^2+y^2) + \frac{3\hbar^2}{8x^2} + \frac{\beta}{y^2} + \hbar^2\Big(\frac{\gamma}{P_5-1} + \frac{1}{x^2}(P_5-1)(\sqrt{2\lambda} + \lambda(2P_5-1) + \frac{\beta}{P_5})$$
$$+ x^2(\frac{P_5'^2}{2P_5} - \frac{\alpha^2}{8\hbar^4}P_5)\frac{(2P_5-1)}{(P_5-1)^2} - \frac{P_5'}{P_5-1} - 2\sqrt{2\lambda}P_5'\Big),$$
$$K = (K_1^\dagger)(K_2^-) - (K_1^-)(K_2^\dagger) = (xp_y - yp_x)p_x^3 p_y.$$

(d_1,d_4):

$$V = \frac{\alpha^2}{2\hbar^2}(x^2+y^2) + \frac{3\hbar^2}{8y^2} + \hbar^2\Big(\frac{\gamma}{P_5-1} + \frac{1}{y^2}(P_5-1)(\sqrt{2\lambda} + \lambda(2P_5-1) + \frac{\beta}{P_5})$$
$$+ y^2(\frac{P_5'^2}{2P_5} - \frac{\alpha^2}{2\hbar^4}P_5)\frac{(2P_5-1)}{(P_5-1)^2} - \frac{P_5'}{P_5-1} - 2\sqrt{2\lambda}P_5'\Big)$$
$$K = (K_1^\dagger)^2(K_2^-) - (K_1^-)^2(K_2^\dagger) = (xp_y - yp_x)p_x p_y^3.$$

Potentials satisfying higher order nonlinear equations passing the Painlevé test:

(b_1,b_5):

$$V = \frac{\alpha_1}{i\hbar}x + g(y),\ K = \alpha_2 K_1 - \alpha_1 K_2,$$

(b_3,b_5):

$$V = \hbar^2\omega_1^2 P_I(\omega_1 x) + g(y),\ \omega_1 = \frac{\sqrt[5]{4i\alpha_1}}{\hbar}.$$
$$K = \alpha_2 K_1 - \alpha_1 K_2.$$

(b_5,b_5):

$$V = f(x) + g(y),\ K = \alpha_2 K_1 - \alpha_1 K_2.$$

where $f(x)$ and $g(y)$ satisfy equation (4.3.24).

(d_5,d_1):

$$V = f(x) + \frac{\alpha^2}{2\hbar^2} y^2,\ K = (K_1^\dagger)(K_2^-) - (K_1^-)(K_2^\dagger) = y p_x^5$$

where $F(x) = \int f dx$ satisfies equation (4.3.51).

(c_5,b_1):

$$V = \hbar^6 f(\hbar^2 x) + \frac{\alpha_1}{\hbar} x,\ K = \alpha_2 K_1 - \alpha_1 H_1 K_2$$

(c_5,b_3):

$$V = \hbar^6 f(\hbar^2 x) + \hbar^2 \omega_2^2 P_I(\omega_2 y),\ \omega_2 = \frac{\sqrt[5]{4\alpha_2}}{\hbar},\ K = \alpha_2 K_1 - \alpha_1 H_1 K_2$$

where $f(X)$ satisfies equation (4.3.34).

Classical superintegrable system:

The systems constructed by (d_1,d_1), (d_1,d_2) and (d_2,d_2) are the same as in the quantum case. Those related to (a_i,a_j) have no classical analog. In the approach of this article we generate the following potentials and integrals.

(b_3,b_3) :

$$V = \epsilon\left(\sqrt{\frac{2\alpha_1}{3}} x + \sqrt{\frac{2\alpha_2}{3}} y\right);\ \epsilon = \pm 1,$$

$$K = \alpha_2 K_1 - \alpha_1 K_2.$$

(b_1,b_3) :

$$V = \alpha_1 x + \epsilon \sqrt{\frac{2\alpha_2}{3}} y,$$

$$K = \alpha_2 K_1 - \alpha_1 K_2.$$

(c_2,b_3):

$$V = \frac{\beta}{x^2} + \epsilon\sqrt{\frac{2\alpha_2}{3}}y,\ K = \alpha_2 K_1 - \alpha_1 H_1 K_2$$

(c_5,b_1):

$$V = f(x) + \alpha_2 y,\ K = \alpha_2 K_1 - \alpha_1 H_1 K_2$$

(c_5,b_3):

$$V = f(x) + \epsilon\sqrt{\frac{2\alpha_2}{3}}y,\ K = \alpha_2 K_1 - \alpha_1 H_1 K_2$$

where $f(x)$ satisfies equation

$$(\alpha_1 x - 3f^2)^2 f = c$$

(c_3,b_1) :

$$V = g(x) + \alpha_2 y,\ K = \alpha_2 K_1 - \alpha_1 H_1 K_2 = \alpha_2 p_x^3 - \frac{\alpha_1}{2} p_x^2 p_y.$$

(c_3,c_3):

$$V = g(x) + g(y),\ K = \alpha_2 H_2 K_1 - \alpha_1 H_1 K_2 = (\alpha_2 p_x - \alpha_1 p_y) p_x^2 p_y^2.$$

(c_2,c_3):

$$V = \frac{\beta}{x^2} + g(y),\ K = \alpha_2 H_2 K_1 - \alpha_1 H_1 K_2.$$

(c_3,b_3):

$$V = g(x) + \epsilon\sqrt{\frac{2\alpha_2}{3}}y,\ K = \alpha_2 K_1 - \alpha_1 H_1 K_2$$

where $g(x)$ satisfies equation

$$(\alpha_1 x - 2g)^2 g = c$$

$(d_1,d_3):$

$$V=\frac{\alpha^2}{2}x^2+h(y), \alpha_2=\alpha, K=(K_1^\dagger)(K_2^-)-(K_1^-)(K_2^\dagger)$$

$(d_3,d_2):$

$$V=h(x)+\frac{\alpha^2}{8}y^2+\frac{\beta}{y^2},$$
$$K=(K_1^\dagger)(K_2^-)-(K_1^-)(K_2^\dagger)=(xp_y-yp_x)p_x^2p_y.$$

$(d_1,d_3):$

$$V=\frac{\alpha^2}{2}x^2+h(y), \alpha_2=2\alpha\ K=(K_1^\dagger)^2(K_2^-)-(K_1^-)^2(K_2^\dagger)=(xp_y-yp_x)p_xp_y^2.$$

$$V=\frac{\alpha^2}{2}x^2+h(y), \alpha_2=3\alpha, K=(K_1^\dagger)^3(K_2^-)-(K_1^-)^3(K_2^\dagger)=(xp_y-yp_x)p_x^2p_y^2$$

(d_3,d_3):

$$V=h(x)+h(y), K=(K_1^\dagger)(K_2^-)-(K_1^-)(K_2^\dagger)=(xp_y-yp_x)p_x^2p_y^2$$

where h satisfies equations given in (4.4.20) and (4.4.21).

$(d_4,d_1):$

$$V=k(x)+\frac{\alpha^2}{2}y^2,$$
$$K=(K_1^\dagger)(K_2^-)-(K_1^-)(K_2^\dagger)=(xp_y-yp_x)p_x^3.$$

(d_4,d_2):

$$V=k(x)+\frac{\alpha^2}{8}y^2+\frac{\beta}{y^2},$$
$$K=(K_1^\dagger)(K_2^-)-(K_1^-)(K_2^\dagger)=(xp_y-yp_x)p_x^3p_y.$$

(d_1,d_4):

$$V=\frac{\alpha^2}{2}x^2+k(y), \alpha_y=2\alpha,$$
$$K=(K_1^\dagger)^2(K_2^-)-(K_1^-)^2(K_2^\dagger)=(xp_y-yp_x)p_xp_y^3.$$

where k satisfies equations (4.4.23) and (4.4.24).

(b_1,b_5):

$$V = \alpha_1 x + \sqrt[3]{\frac{2\alpha_2}{5}}y,\ K = \alpha_2 K_1 - \alpha_1 K_2,$$

(b_3,b_5):

$$V = \epsilon\sqrt{\frac{2\alpha_1}{3}}x + \sqrt[3]{\frac{2\alpha_2}{5}}y.$$
$$K = \alpha_2 K_1 - \alpha_1 K_2.$$

(b_5,b_5):

$$V = \sqrt[3]{\frac{2\alpha_1}{5}}x + \sqrt[3]{\frac{2\alpha_2}{5}}y,\ K = \alpha_2 K_1 - \alpha_1 K_2.$$

(d_5,d_1):

$$V = f(x) + \frac{\alpha^2}{2}y^2,\ K = (K_1^\dagger)(K_2^-) - (K_1^-)(K_2^\dagger) = yp_x^5$$

where $F(x) = \int f dx$ satisfies equation (4.4.26).

4.6. Conclusion

Our main conclusion is that the systematic use of quantum or classical algebraic systems in one dimension is an efficient method of generating superintegrable systems in a two-dimensional Euclidean space. By construction, all systems thus obtained allow the separation of variables in the Schrödinger and the Hamilton-Jacobi equation, respectively. The algebraic systems consist of a pair (H_1,K_1) where H_1 is a natural Hamiltonian as in (4.2.1) and K_1 a polynomial as in (4.2.2). The four types of algebras considered are as in (4.2.4) and all of them should be constructed in x and y spaces independently.

Let us again run through all combinations of the type (z_i, w_j) where z is in x-space and w in y-space. The subscripts give the order of the corresponding polynomial $K_l, l = 1{,}2$.

The pair (d_1,d_1) with $(\alpha_1, \alpha_2) = \omega^2(n,m)$ yields the Jauch and Hill potentials.

The pair (d_2, d_2) gives an infinite set of generalizations of the Smorodinsky-Winternitz potentials.

Pairs of the type (a_i,a_j) in quantum mechanics give potentials in terms of elliptic or hyperelliptic functions. In classical mechanics their limit is free motion (V = constant).

All other pairs lead to "exotic potentials" expressed in terms of Painlevé transcendents or their generalizations that are solutions of higher order ODEs. This is true for all examples so far considered and we conjecture that this is true for all values of i and j.

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