Université de Montréal

Groupes de cobordisme lagrangien immergé et structure des polygones pseudo-holomorphes

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Résumé

Cette thèse s'intéresse à la théorie de Floer pour les immersions lagrangiennes. On commence par montrer un théorème de décomposition des disques pseudo-holomorphes à bord dans une immersion générique. On donne ensuite une application au calcul du complexe de Floer. On conclut par une esquisse d'un travail en cours sur le calcul de l'obstruction de la chirurgie de deux lagrangiennes plongées et transverses.

Dans un deuxième temps, on se restreint au cas des surfaces. On montre qu'un groupe de cobordisme dont les relations sont données par certains cobordismes lagrangien immergés est isomorphe au groupe de Grothendieck de la catégorie de Fukaya. Au passage, on calcule le groupe de cobordisme lagrangien immergé.

Mots-clés : Sous-variétés lagrangiennes, Immersions lagrangiennes, Polygones holomorphes, Cobordismes Lagrangiens, Groupes de cobordisme, Homologie de Floer, Catégories de Fukaya.

Summary

In this thesis, we shall study Floer theory for Lagrangian immersions. In the first chapter, we prove a decomposition theorem for pseudo-holomorphic disks with boundary on a given generic Lagrangian immersion. We apply this result to the computation of certain Floer complexes. We conclude with work in progress on the computation of the obstruction of the surgery of two transverse Lagrangian submanifolds.

In the second chapter, we consider surfaces. We show that a cobordism group, whose relations are given by unobstructed immersed lagrangian cobordisms, is isomorphic to the Grothendieck group of the derived Fukaya category. We also compute the immersed Lagrangian cobordism group.

Key words: Lagrangian submanifolds, Lagrangian immersions, Holomorphic polygons, Lagrangian cobordisms, Cobordism groups, Floer homology, Fukaya categories.

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Introduction

Quelques bases de topologie symplectique

Variétés symplectiques

Dans cette thèse, on étudiera les *variétés symplectiques*. Il s'agit des couples (M, ω) avec M une variété et $\omega \in \Omega^2(M)$ une forme différentielle fermée et non dégénérée. Ces objets apparaissent naturellement dans de nombreux champs de la géométrie. Voici quelques exemples.

- L'espace complexe $\mathbb{C}^n$ de coordonnées $z_k = x_k + iy_k$ peut être muni de la forme symplectique
$$\omega = \sum_{k=1}^{n} dx_k \wedge dy_k.$$
- Soit V une variété, son cotangent T^*V est naturellement muni d'une forme différentielle $\lambda \in \Omega^1(T^*V)$, sa *forme de Liouville*. Sa différentielle
$$\omega = d\lambda,$$
est une forme symplectique. Ces variétés permettent de donner une formulation géométrique de la mécanique hamiltonienne[1].
- En géométrie complexe, toute variété kählérienne (M, J, ω) est symplectique. Cela inclut, entre autres, les variétés projectives et affines lisses.

Notons dès maintenant que la dimension d'une variété symplectique est nécessairement paire, disons $2n$ avec $n \in \mathbb{N}$.

Pour classifier ces objets, on a besoin d'une notion de morphisme. On appelle *symplectomorphisme* un difféomorphisme
$$\psi : (M_1, \omega_1) \to (M_2, \omega_2)$$

[1]Voir [**Arn89**] pour une très belle introduction.

entre deux variétés symplectiques (M_1, ω_1) et (M_2, ω_2) tel que

$$\psi^* \omega_2 = \omega_1.$$

Une des spécificités du sujet est qu'à la différence du cas riemannien il n'y a pas d'invariants locaux. En effet, soit $x \in M$. Le *théorème de Darboux* affirme qu'il existe des coordonnées locales $x_1, \ldots, x_n, y_1, \ldots, y_n$ autour de x dans lesquelles

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

On conclut que toutes les variétés symplectiques sont localement équivalentes !

Heureusement il existe des invariants, nécessairement globaux, permettant de distinguer ces objets. Avant d'expliquer cela, rappelons qu'une *structure presque complexe* $J \in \Gamma(\mathrm{End}(TM))$ est une section du fibré des endomorphismes de l'espace tangent satisfaisant $J^2 = -\mathrm{Id}$. La section J est dite *compatible* avec ω si l'application bilinéaire $g_J = \omega(\cdot, J\cdot)$ est symétrique, définie et positive. On note $\mathcal{J}(M, \omega)$ l'ensemble des structures presque complexes compatibles avec ω. Muni de la topologie $\mathcal{C}^\infty$, c'est un espace non vide et contractile (voir par exemple [**ALP94**]).

En 1985, Gromov introduit les courbes *pseudo-holomorphes* dans l'article [**Gro85**]. Soient $J \in \mathcal{J}(M, \omega)$ une structure presque complexe compatible avec ω et (Σ, j) une surface de Riemann fermée. Une *courbe pseudo-holomorphe* est une application lisse

$$u : (\Sigma, j) \to (M, J)$$

qui satisfait l'équation de Cauchy-Riemann

$$du \circ j = J \circ du.$$

En étudiant ces applications, Gromov montre de nombreux résultats fondateurs et étonnants. Par exemple, fixons $J \in \mathcal{J}(\mathbb{C}P^2, \omega_{\mathrm{FS}})$ une structure presque complexe générique sur le plan projectif complexe et deux points $z_1 \neq z_2 \in \mathbb{C}P^2$. À paramétrisation près, il existe une unique sphère pseudo-holomorphe $u : \mathbb{C}P^1 \to \mathbb{C}P^n$ homologue à $\mathbb{C}P^1 \subset \mathbb{C}P^2$ passant par z_1 et z_2. Dans ce cas, le comptage de courbes pseudo-holomorphes ne dépend donc pas de la structure complexe et fournit un invariant symplectique !

Ces techniques ont donné naissance au sujet et les résultats sont donc trop nombreux pour pouvoir tous les citer. Cependant, le livre de McDuff et Salamon [**MS12**] contient une très belle introduction ainsi que des références plus complètes.

Systèmes hamiltoniens

La variété (M, ω) possède une famille importante d'automorphismes. On appelle *hamiltonien* (dépendant du temps) toute fonction

$$H \in \mathcal{C}^\infty([0,1] \times M, \mathbb{R}).$$

On associe à H le champ de vecteurs X_H défini par la formule

$$\iota_{X_H}\omega = -dH.$$

Si X_H est complet, par intégration, on obtient un flot de transformations symplectiques $\phi_H^t : M \to M$. Il s'agit du *flot hamiltonien* associé à H.

Cette terminologie provient de la physique. En effet, considérons un système physique dont l'espace de configuration est une variété V. L'espace dans lequel vit sa position et son impulsion généralisée s'identifie alors naturellement au cotangent T^*V tandis que son comportement est décrit par une fonction $H : V \to \mathbb{R}$ qu'on appelle son *hamiltonien*. L'évolution du système suit alors une trajectoire de T^*V tangente au champ de vecteur X_H.

L'étude de ces systèmes dynamiques est le sujet d'une littérature très riche qui remonte à Poincaré. Souvent, on s'attache à montrer l'existence d'orbites périodiques. Comme ce n'est pas le sujet de cette thèse, je renvoie les personnes intéressées au chapitre 11 de [MS17] pour un historique très complet.

Sous-variétés lagrangiennes

Une immersion $i : L^n \looparrowright M^{2n}$ est dite lagrangienne si son domaine est de dimension n et la forme $i^*\omega$ est identiquement nulle. Quand i est un plongement, on parle de *sous-variétés lagrangiennes*. Donnons quelques exemples.

- Le sous-espace vectoriel $\mathbb{R}^n \subset \mathbb{C}^n$.
- Le *tore de Clifford* est le produit $T_{\text{Cliff}} = S^1 \times \ldots \times S^1 \subset \mathbb{C}^n$ de n copies de S^1.
- Pour $x \in V$, la fibre T_x^*V au dessus de x.
- La section nulle $V \subset T^*V$.
- Soit $H = \{P^{-1}(0)\} \subset \mathbb{C}P^n$ l'ensemble des zéros d'un polynôme homogène à coefficients réels $P \in \mathbb{R}[X_1, \ldots X_{n+1}]$. On suppose l'hypersurface H lisse. Alors, sa partie réelle $H_\mathbb{R} := H \cap \mathbb{R}^n$ est lagrangienne.

Il y a de nombreuses raisons de s'intéresser à cette classe de sous-variétés et immersions.

- Si $\psi : (M_1, \omega_1) \to (M_2, \omega_2)$ est un symplectomorphisme, son graphe $gr(\psi)$ est lagrangien dans la variété produit $M_1 \times M_2$ munie de la forme $-\omega_1 \oplus \omega_2$. Ainsi,

toute information sur les lagrangiennes de $M_1 \times M_2$ permet de caractériser les symplectomorphismes $\psi : M_1 \to M_2$.
– Les sous-variétés (et immersions) lagrangiennes donnent des invariants en étudiant leur topologie et leur comportement sous les automorphismes de (M, ω).

Soient $i_0 : L \looparrowright M$ et $i_1 : L \looparrowright M$ deux immersions lagrangiennes, on appelle *isotopie lagrangienne* entre i_0 et i_1, une famille lisse d'immersions lagrangiennes

$$i_t : L \looparrowright M, \ t \in [0,1].$$

Une immersion lagrangienne $i : L^n \looparrowright M$ se relève naturellement en une application de fibrés :

$$\begin{array}{cccc} F: & TL & \to & TM \\ & (x,v) & \mapsto & di_x(v) \end{array}.$$

qui est (i) linéaire, (ii) injective sur les fibres et (iii) telle que l'image de chaque espace tangent

$$di_x(T_x(L))$$

est lagrangienne. Gromov [**Gro85**] et Lees [**Lee76**] ont montré que deux immersions lagrangiennes $i_0 : L \to M$ et $i_1 : L \to M$ sont isotopes si et seulement si
– Il existe une famille continue $(i_t)_{t\in[0,1]} : L \to M$,
– Il existe une famille continue d'applications de fibrés $F_t : TL \to TM$ relevant les (i_t) telle que F_t satisfait $(i), (ii), (iii)$ pour tout $t \in [0,1]$.

On voit donc que décider si deux immersions lagrangiennes sont isotopes revient à un calcul d'homotopie ! On dit que les immersions lagrangiennes satisfont le *h-principe* (pour principe d'homotopie). Suivant Gromov ([**Gro87**]), on dit que les problèmes de topologie symplectique qui se ramènent à des calculs d'homotopie sont *flexibles*.

En revanche, les plongements lagrangiens n'obéissent pas à ces phénomènes de flexibilité. Par exemple, Chekanov ([**Che96**]) introduit un tore $T_{\text{Chek}} \subset \mathbb{C}^n$ lagrangien qui est lagrangien isotope au tore de Clifford. En revanche, il n'existe pas de symplectomorphisme $\psi : \mathbb{C}^n \to \mathbb{C}^n$ tel que

$$\psi(T_{\text{Chek}}) = T_{\text{Cliff}}.$$

Cela peut se démontrer en comptant des disques pseudo-holomorphes dont le bord est inclus dans T_{Chek} et T_{Cliff} ([**Che97**],[**CS10**]). Les phénomènes qui ne se ramènent pas à des problèmes d'homotopie sont dits *rigides*.

Cette dichotomie entre phénomènes rigides et flexibles est un des thèmes majeurs de la topologie symplectique. En particulier, c'est un des thèmes majeurs de mon travail de thèse. Au passage, on notera que dans le cas du tore de Chekanov, la rigidité a été détectée au moyen de courbes pseudo-holomorphes : cela semble être un principe général ([**Eli15**, 6.1]).

Quelques outils techniques

Théorie de Floer et rigidité

Homologie de Floer. Un exemple de technique permettant d'étudier les phénomènes de rigidité est donné par l'*homologie de Floer*.

Le problème initial est le suivant. Étant données L_0 et L_1 deux sous-variétés lagrangiennes de (M, ω). Peut-on modifier le nombre de points d'intersection entre L_0 et L_1 en déplaçant une des lagrangiennes par un flot hamiltonien ? Quel est le nombre minimal de points d'intersection ?

Cette question est en fait liée au comptage des orbites périodiques d'un hamiltonien $H_t : M \times [0,1] \to \mathbb{R}$. En effet, on a vu que la diagonale

$$\Delta = \{(z,z) | z \in M\}$$

est lagrangienne dans le produit $M \times M$. Les points d'intersections de Δ et du graphe de flot au temps 1 de H_t sont en bijection avec les orbites périodiques de H_t. Compter les intersections entre ces deux lagrangiennes donne donc des informations sur la dynamique du flot de H_t !

Pour caractériser ces intersections Floer propose, durant les années 80, une construction assez remarquable que je vais maintenant expliquer (les articles fondateurs sont [**Flo88a**],[**Flo88b**],[**Flo88c**]).

Pour la suite de cette section, supposons que M soit fermée et asphérique (i.e. $\pi_2(M) = 0$). Soient L_0 et L_1 deux sous-variétés lagrangiennes fermées de M telles que $\pi_2(M, L_i) = 0$ pour $i = 0, 1$. Soit $H_t : [0,1] \times M \to \mathbb{R}$ un hamiltonien. Notons $\mathcal{P}_H$ l'ensemble des trajectoires hamiltoniennes allant de L_0 à L_1, c'est à dire

$$\mathcal{P}_H = \left\{ \gamma : [0,1] \to M \middle| \gamma(0) \in L_0,\ \gamma(1) \in L_1,\ \frac{d\gamma}{dt} = \phi_H^t(\gamma(t)) \right\}.$$

Ses éléments sont aussi appelés des *cordes hamiltoniennes*. L'ensemble $\mathcal{P}_H$ est naturellement en bijection avec $L_0 \cap \phi_H^{-1}(L_1)$. On suppose que cette intersection est transverse de sorte que $\mathcal{P}_H$ est fini.

On fixe aussi une structure presque complexe

$$J \in \mathcal{C}^\infty([0,1], \mathcal{J}(M, \omega))$$

qui dépend du temps $t \in [0,1]$.

Soient γ_- et γ_+ deux éléments de $\mathcal{P}_H$, Floer considère l'ensemble

$$\widetilde{\mathcal{M}}(\gamma_-, \gamma_+, J, H)$$

des solutions $u : \mathbb{R} \times [0,1] \to M$ de *l'équation de Floer*

$$\partial_s u + J_t(u)(\partial_t u - X_{H_t}(u(s,t))) = 0. \tag{0.1}$$

satisfaisant les conditions aux limites

$$\lim_{s\to\pm\infty} u(s,t) = \gamma_\pm(t),\ u(\mathbb{R}\times\{0\}) \subset L_0,\ u(\mathbb{R}\times\{1\}) \subset L_1 \tag{0.2}$$

et d'énergie finie

$$\int_{\mathbb{R}\times[0,1]} |\partial_t u - X_{H_t}(u(s,t))|^2 < +\infty. \tag{0.3}$$

Le groupe additif $\mathbb{R}$ agit sur l'ensemble $\widetilde{\mathcal{M}}(\gamma_-,\gamma_+,J,H)$ par la formule

$$(s_0 \cdot u)(s,t) = u(s+s_0,t).$$

On note $\mathcal{M}(\gamma_-,\gamma_+,J,H)$ l'ensemble des orbites de cette action. Ses éléments sont appelés *trajectoires de Floer*. Quand $H=0$, un élément de $\mathcal{M}(\gamma_-,\gamma_+,J,H)$ sera aussi appelé un *bande pseudo-holomorphe* puisqu'il satisfait l'équation de Cauchy-Riemann. Floer commence par remarquer, suivant les idées de Gromov, que l'espace

$$\mathcal{M}(x,y,J,H) \coprod \left(\bigcup_{z\in\mathcal{P}_H} \mathcal{M}(x,z,J,H) \times \mathcal{M}(z,y,J,H) \right)$$

admet une topologie naturelle qui en fait un espace métrisable compact. Rappelons qu'un sous-ensemble d'un espace de Baire est dit *générique* s'il contient une intersection dénombrable d'ouverts denses. Il existe un ensemble générique

$$\mathcal{J}_{\mathrm{reg}}(L_0,L_1) \subset \mathcal{C}^\infty([0,1],\mathcal{J}(M,\omega)),$$

tel que pour tout $J \in \mathcal{J}_{\mathrm{reg}}(L_0,L_1)$, l'ensemble $\mathcal{M}(\gamma_-,\gamma_+,J,H)$ est une variété de dimension finie. Appelons $\mathcal{M}^k(\gamma_-,\gamma_+,J,H)$ l'union des composantes connexes de dimension k de $\mathcal{M}(\gamma_-,\gamma_+,J,H)$.

Si $J \in \mathcal{J}_{\mathrm{reg}}(L_0,L_1)$, on montre que l'espace topologique

$$\mathcal{M}^0(\gamma_-,\gamma_+,J,H)$$

est compact. Des résultats difficiles d'analyse montrent que l'espace topologique

$$\mathcal{M}^1(\gamma_-,\gamma_+,J,H) \coprod \left(\bigcup_{z\in\mathcal{P}_H} \mathcal{M}^0(\gamma_-,z,J,H) \times \mathcal{M}^0(z,\gamma_+,J,H) \right), \tag{0.4}$$

est une variété compacte de dimension un dont le bord est donné par

$$\bigcup_{z\in\mathcal{P}_H} \mathcal{M}^0(\gamma_-,z,J,H) \times \mathcal{M}^0(z,\gamma_+,J,H).$$

Le *complexe de Floer* est l'espace vectoriel sur $\mathbb{Z}/2\mathbb{Z}$ engendré par les éléments de $\mathcal{P}_H$

$$CF(L_1, L_2, J, H) = \bigoplus_{\gamma \in \mathcal{P}_H} \mathbb{Z}/2\mathbb{Z} \cdot \gamma.$$

On le munit d'une différentielle $d : CF(L_1, L_2, J, H) \to CF(L_1, L_2, J, H)$ qui compte le nombre de trajectoires entre deux cordes hamiltoniennes modulo 2,

$$d(\gamma_+) = \sum_{\gamma_-} \#\mathcal{M}^0(\gamma_-, \gamma_+, J, H) \cdot \gamma_-.$$

L'union

$$\bigcup_{z \in \mathcal{P}_H} \mathcal{M}^0(\gamma_-, z, J, H) \times \mathcal{M}^0(z, \gamma_+, J, H)$$

est de cardinal pair puisque c'est le bord d'une variété de dimension 1. On déduit facilement que l'application d satisfait $d^2 = 0$.

L'*homologie de Floer* entre L_1 et L_2 est l'homologie du complexe $(CF(L_1, L_2, J, H), d)$

$$HF(L_1, L_2, J, H) = H(CF(L_1, L_2, J, H), d).$$

On peut montrer que cet espace vectoriel ne dépend ni du choix de $J \in \mathcal{J}_{reg}(L_1, L_2)$, ni du choix du hamiltonien H. Ainsi, on peut oublier J et H dans la notation.

Floer ([**Flo88a**]) montre, en choisissant bien le hamiltonien H et la structure complexe J, qu'il existe un isomorphisme

$$HF(L, L) \simeq H^*(L, \mathbb{Z}/2\mathbb{Z}).$$

En particulier, le nombre d'intersections d'une déformation hamiltonienne $\phi_H^1(L)$ avec L est minoré par la somme des *nombres de Betti* de L

$$\#\big(L \cap \phi_H^1(L)\big) \geqslant \sum_k \beta_k(L, \mathbb{Z}/2\mathbb{Z}).$$

On peut appliquer cela à la diagonale Δ pour obtenir un minorant du nombre de points fixes d'un difféomorphisme hamiltonien ; cela prouve une forme faible d'une conjecture d'Arnold.

Structure produit. Il est possible de définir des structures algébriques sur les groupes d'homologie de Floer. Expliquons premièrement comment définir un produit.

Soient L_0, L_1, L_2 trois sous-variétés lagrangiennes. On peut définir une application

$$m_2 : HF(L_1, L_2) \otimes HF(L_0, L_1) \to HF(L_0, L_2), \tag{0.5}$$

qui compte des triangles pseudo-holomorphes dont les conditions de bord sont données par L_1, L_2 et L_3 (voir la figure 1). Comme la construction est compliquée, je n'en dirai

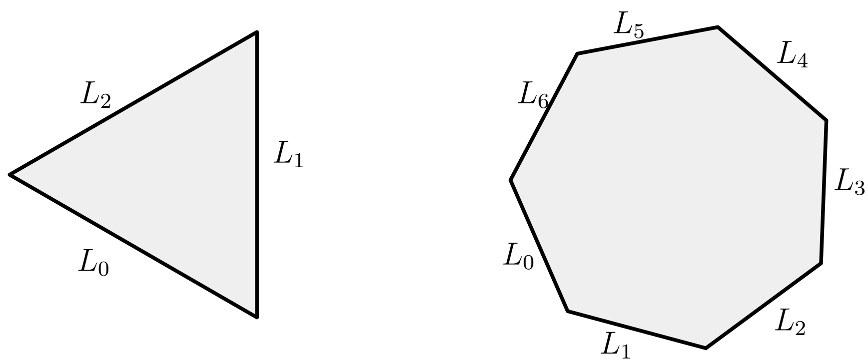

FIGURE 1 – *À gauche, un polygone qui contribue à m_2.*
À droite, un polygone qui contribue à m_6.

pas plus sur le sujet, mais on pourra consulter la section 2.3 du chapitre 2 pour un résumé plus complet.

Il se trouve que le produit ainsi construit est associatif :

$$m_2 \circ (m_2 \otimes \mathrm{Id}) = m_2 \circ (\mathrm{Id} \otimes m_2). \tag{0.6}$$

Suivant une idée de Donaldson on peut définir une catégorie $\mathrm{Don}(M, \omega)$, dite de *Donaldson-Fukaya*, dont

– Les objets sont les lagrangiennes compactes de (M, ω),
– Les morphismes entre deux objets L_1 et L_2 sont donnés par le groupe d'homologie de Floer $HF(L_1, L_2)$,
– La composition entre deux morphismes est donnée par l'application m_2 (0.5).

Malheureusement la catégorie de Donaldson-Fukaya est un invariant peu pratique pour plusieurs raisons.

(A) Il n'est pas clair que la catégorie $\mathrm{Don}(M, \omega)$ possède assez d'objets pour être calculée et manipulée algébriquement. En effet, elle n'est *a priori* même pas additive (elle n'admet pas, *a priori*, de produits finis) ! On aimerait donc travailler avec un invariant qui admet plus de structure.

(B) Les groupes de morphismes de $\mathrm{Don}(M, \omega)$ sont des groupes d'homologie. Il existe en réalité beaucoup d'information au niveau des chaînes qui est perdue par ce passage au quotient[2].

Pour régler le problème (B), Fukaya propose de construire une « catégorie » $\mathrm{Fuk}(M, \omega)$ dont les objets sont toujours des lagrangiennes mais dont les morphismes sont donnés

[2]Une bonne analogie dans le cas d'une algèbre différentielle graduée (A, d) est l'existence de *produits de Massey*. Ceux-ci sont un invariant du *complexe* A et ne peuvent pas être retrouvés par la donnée de l'algèbre $H(A, d)$.

par les complexes de Floer

$$\mathrm{Hom}_{\mathrm{Fuk}(M,\omega)}(L_1, L_2) = CF(L_1, L_2).$$

La composition est donnée par l'application m_2 dont j'ai esquissé la définition plus haut. Malheureusement cette application n'est *pas* associative. Fukaya remarque qu'il existe des applications

$$m_3 : CF(L_3, L_4) \otimes CF(L_2, L_3) \otimes CF(L_1, L_2) \to CF(L_1, L_4)$$

qui satisfont la relation

$$m_2 \circ (m_2 \otimes \mathrm{Id}) - m_2 \circ (\mathrm{Id} \otimes m_2) = d \circ m_3 + m_3 \circ (d \otimes \mathrm{Id}^{\otimes 2}) + m_3 \circ (\mathrm{Id} \otimes d \otimes \mathrm{Id}) + m_3 \circ (\mathrm{Id}^{\otimes 2} \otimes d).$$

Le produit m_2 est donc associatif à une homotopie m_3 près ! L'application m_3 est définie par un comptage de polygones pseudo-holomorphes à condition de bord dans quatre lagrangiennes (voir la figure 1).

Plus généralement, pour tout $d \geqslant 1$, il existe des applications

$$m_d : CF(L_{d-1}, L_d) \otimes \ldots \otimes CF(L_0, L_2) \to CF(L_0, L_d)$$

données par des comptages de polygones pseudo-holomorphes comme en figure 1 qui satisfont les *relations* A_∞

$$\sum_{\substack{j=1\ldots d \\ i=0\ldots d-j}} m_{d-j+1} \circ \left(\mathrm{Id}^{\otimes i} \otimes m_j \otimes \mathrm{Id}^{\otimes d-(i+j)} \right) = 0.$$

Cela munit $\mathrm{Fuk}(M,\omega)$ d'une structure de A_∞-*catégorie* .

Pour ce qui est du problème (A), la bonne notion pour faire de l'algèbre homologique est celle de *catégorie triangulée*, due à Grothendieck et Verdier. Il s'agit d'une catégorie additive munie d'une famille de triangles de la forme

$$A \xrightarrow{f} B \xrightarrow{g} C \to A[1],$$

dits distingués qui satisfait un certain nombre d'axiomes. Dans un tel triangle, l'objet C est l'analogue du cone du morphisme f en algèbre homologique.

En 1994 Kontsevich ([**Kon95**]) propose de généraliser une construction de Bondal et Kapranov ([**BK90**]) pour construire une catégorie triangulée à partir de $\mathrm{Fuk}(M,\omega)$. Concrètement, la structure A_∞ de $\mathrm{Fuk}(M,\omega)$ permet de définir une famille de triangles distingués. Cette catégorie n'est cependant pas triangulée. On ajoute alors formellement les triangles distingués manquants pour obtenir *la catégorie dérivée de Fukaya* $\mathrm{DFuk}(M,\omega)$.

La construction de $\mathrm{Fuk}(M,\omega)$ est difficile et demande beaucoup d'analyse. Le problème principal est que les espaces de courbes qui interviennent dans la définition ne sont génériquement des variétés que pour *les courbes simples*, c'est-à-dire les courbes qui n'ont pas de multiplicité[3]. Pour résoudre cela, on compte des polygones qui satisfont une équation de Cauchy-Riemann perturbée. Tout cela est expliqué dans le livre de Seidel [**Sei08**] dans le cas des variétés symplectiques exactes[4] dont le bord est convexe. Je donne aussi un (court) résumé des points essentiels de la construction dans la section 2.3 du Chapitre 2.

Bien que les catégories de Fukaya soient des objets algébriquement compliqués, elles ont l'avantage de contenir beaucoup d'information géométrique. Par exemple, soit V une variété compacte. Désignons par $p : T^*V \to V$ la projection sur la section nulle. En étudiant $\mathrm{Fuk}(T^*V)$, Fukaya-Seidel-Smith ([**FSS08**]) puis Abouzaid ([**Abo12**]) et Abouzaid-Kragh ([**AK18**]) ont montré que que si $L \subset T^*V$ est une sous-variété lagrangienne exacte et compacte, la projection de L sur la section nulle V est une équivalence d'homotopie simple ! C'est un pas vers la preuve d'une conjecture d'Arnold qui affirme que L est hamiltonienne isotope à la section nulle.

Théorie de Floer et immersions lagrangiennes

Homologie de Floer et lagrangiennes immergées. Des travaux récents montre qu'une certaine classe d'immersions lagrangiennes, strictement plus grande que les plongements, satisfait des propriétés de rigidité, malgré les h-principes. J'explique maintenant d'où provient cette classe : l'idée est de rechercher les immersions auxquelles on peut appliquer la théorie de Floer.

Plus précisément, soient $i_0 : L_0 \looparrowright M$ et $i_1 : L_1 \looparrowright M$ deux immersions lagrangiennes. On suppose que leurs points doubles sont transverses et qu'elles n'admettent pas de points triples. On construit comme précédemment un complexe différentiel

$$CF(L_0, L_1, J, H)$$

dont les générateurs sont les cordes hamiltoniennes de L_1 à L_2. La différentielle d compte des applications $u : \mathbb{R} \times [0,1] \to M$ solutions de l'équation de Floer 0.1, d'énergie finie 0.3 et de bord satisfaisant 0.2. On exige de plus qu'il existe des applications

$$\gamma_0 : \mathbb{R} \to L_0,\ \gamma_1 : \mathbb{R} \to L_1$$

qui relèvent la condition de bord

$$i_0 \circ \gamma_0 = u(\cdot, 0),\ i_1 \circ \gamma_1 = u(\cdot, 1).$$

[3]La définition exacte est dans l'introduction du chapitre 1.

[4]Ce sont les variétés dont la forme symplectique satisfait $\omega = d\lambda$ pour une forme $\lambda \in \Omega^1(M)$.

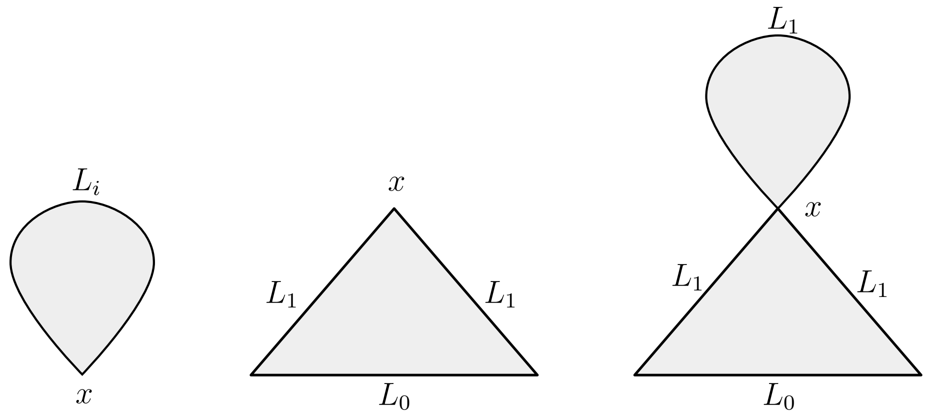

FIGURE 2 – *À gauche, une larme,*
Au centre, une trajectoire de Floer avec un coin,
À droite, les configurations comptées par d^2*.*

Cette hypothèse élimine l'existence de « coins » (comme dessinés sur la figure 2).

Malheureusement, l'application d ne satisfait pas $d^2 = 0$. Un théorème de compacité dû à Ivashkovich et Shevchishin ([**IS02**]) montre que la compactification des espaces de solutions n'est pas aussi simple que 0.4. Expliquons rapidement quel est le terme d'obstruction.

On appelle *larme* un polygone pseudo-holomorphe dont le bord est contenu dans L et qui a un unique sommet (voir la figure 2). Si x est un point double de l'immersion i_k (avec $k \in \{0, 1\}$), on appelle

$$\mathcal{M}(x, L_k, J)$$

l'ensemble des larmes pseudo-holomorphes dont le coin est x. Appelons de plus

$$\mathcal{M}(\gamma_-, \gamma_+, x, L_i, J)$$

l'ensemble des triangles pseudo-holomorphes dont les conditions de bord sont données dans la figure 2. Si tous les espaces de courbes sont des variétés, le carré de la différentielle est une somme formelle d'orbites

$$d^2(\gamma_+) = \sum_{\gamma_- \in \mathcal{P}_H} \left(\sum_x A_{\gamma_-,\gamma_+,x} \right) \gamma_-. \tag{0.7}$$

Le coefficient $A_{\gamma_-,\gamma+,x}$ est donné par un comptage de configurations représentées en figure 2

$$A_{\gamma_-,\gamma_+,x} = \#\mathcal{M}^0(\gamma_-, \gamma_+, x, L_0, J) \cdot \#\mathcal{M}^0(x, L_1, J) + \#\mathcal{M}^0(\gamma_-, \gamma_+, x, L_1, J) \cdot \#\mathcal{M}^0(x, L_0, J).$$

Le complexe $(CF(L_0, L_1), d)$ est donc différentiel quand les immersions L_0 et L_1 sont telles que ce comptage algébrique s'annule. On dit alors que L_0 et L_1 *sont non obstruées*. Cela arrive par exemple quand les immersions n'admettent pas de larmes.

J'insiste : tout cela est valable lorsque tout les espaces considérés sont des variétés. Il faut pour cela régler des problèmes d'analyse importants ou analyser finement la structure des courbes pseudo-holomorphes à bord dans une immersion. J'énoncerai, au chapitre 1, un théorème permettant de montrer que cette description est vraie pour une structure complexe générique quand la dimension de la variété (M, ω) est plus grande que six.

Comme je l'ai dit plus haut, l'homologie de Floer pour les immersions a été étudiée par de nombreux auteurs sous des hypothèses permettant d'éliminer ces larmes pseudo-holomorphes. Donnons-en une liste, forcément non exhaustive...

- En 2005, Akaho [**Aka05**] est le premier à remarquer qu'on peut définir une variante de l'homologie de Floer pour une sous-variété lagrangienne immergée L. Il suppose alors que (M, ω) est compacte et que le groupe d'homotopie relatif $\pi_2(M, L)$ est nul, ce qui implique en particulier qu'il n'existe pas de larmes pseudo-holomorphes. Il en déduit une minoration du nombre de points d'intersection entre L et une déformation hamiltonienne $\phi_H^1(L)$
$$\#\big(L \cap \phi_H^1(L)\big) \geqslant \sum_k \beta_k(L, \mathbb{Z}/2\mathbb{Z}) + 2N,$$
où N est le nombre de points doubles de l'immersion L.
- Plus tard, Akaho et Joyce [**AJ10**] associent une algèbre A_∞ *avec courbure* à toute immersion lagrangienne générique $i : L \looparrowright M$ d'une variété compacte. Il s'agit d'une généralisation de la notion d'algèbre A_∞ qui tient compte de l'obstruction 0.7. La construction utilise les structures de Kuranishi et des techniques de perturbations virtuelles.
- Abouzaid ([**Abo08**]) construit une A_∞-(pré-)catégorie A_∞ dont les objets sont des courbes immergées d'une surface S_g de genre $g \geqslant 2$. Celles-ci n'admettent pas de larmes pour des raisons topologiques. Je renvoie au chapitre 2 (2.3) pour une construction plus précise.
- Dans [**She11**], Sheridan calcule l'algèbre A_∞ associée à une immersion de la sphère S^n dans une paire de pantalons généralisée $\mathcal{P}^{2n}$. Ici, l'immersion est aussi non obstruée pour des raisons topologiques. Dans [**She15**] et [**She16**], ce calcul permet d'obtenir les catégories de Fukaya d'une hypersurface projective de degré d compris entre 1 et $n+1$.
- En adaptant la définition de Seidel [**Sei08**], Alston et Bao définissent une catégorie de Fukaya dont les objets sont une certaine classe d'immersions. Celles-ci sont non

obstruées pour des raisons analytiques. Pour faire simple, les configurations de la figure 1 apparaissent en familles de dimension trop grande pour contribuer à la différentielle.

Lagrangiennes immergées et catégorie dérivée de Fukaya. Un des problèmes de la catégorie $\mathrm{DFuk}(M,\omega)$ est que ses objets sont construits par un procédé algébrique formel et n'ont donc pas d'interprétation géométrique évidente. Cornea et indépendamment Kontsevich ont proposé la conjecture suivante.

Conjecture 0.1 (Cornea, Kontsevich)**.** Tout les objets de $\mathrm{DFuk}(M,\omega)$ sont représentés par des immersions non obstruées.

Biran et Cornea ont de plus proposé un programme pour montrer cette conjecture que j'esquisse rapidement.

La catégorie $\mathrm{DFuk}(M,\omega)$ est construite à partir de $\mathrm{Fuk}(M,\omega)$ en ajoutant formellement pour chaque morphisme

$$L_0 \xrightarrow{c} L_1$$

le cône du morphisme c

$$C = \mathrm{Cone}(L_0 \xrightarrow{c} L_1).$$

qui fait partie d'un triangle distingué

$$L_0 \xrightarrow{c} L_1 \to C \to L_0[1].$$

On itère alors cette construction pour obtenir une catégorie triangulée. Le morphisme c est un élément de $CF(L_0,L_1)$ et est donc représenté par une somme formelle de points d'intersection entre L_0 et L_1 à coefficients dans $\{0,1\}$

$$c = \sum_{x\in A\subset L_0\cap L_1} x,$$

où A est un sous-ensemble fini de $L_0\cap L_1$. Il existe une procédure qui permet de résoudre les points d'intersection de l'ensemble A pour obtenir une immersion lagrangienne. Biran et Cornea conjecturent alors

Conjecture 0.2 (Cornea)**.** Cette immersion est non obstruée et représente $\mathrm{Cone}(L_0 \xrightarrow{c} L_1)$.

La conjecture de Cornea est en réalité plus forte, elle affirme que $\mathrm{DFuk}(M,\omega)$ est équivalente à une catégorie dont les morphismes sont donnés par des *cobordismes lagrangiens*. Ce qui nous amène à la prochaine sous-section.

Cobordismes lagrangiens. . .

. . . et flexibilité. Soient $L_1, \dots, L_m$ et $N_1, \dots, N_k$ des sous-variétés lagrangiennes immergées de (M, ω). Une sous-variété lagrangienne

$$V \subset \mathbb{C} \times M,$$

est un *cobordisme lagrangien* de $L_1, \dots, L_m$ vers $N_1, \dots, N_k$ si en dehors d'un compact $K \subset \mathbb{C} \times M$, V coincide avec une union de rayons horizontaux

$$\coprod_{i=1}^{m} (]-\infty, -1] \times \{i\} \times L_i) \bigcup \coprod_{j=1}^{k} ([1, +\infty[\times \{j\} \times N_j).$$

Dans ce cas, on note

$$V : (L_1, \dots, L_m) \rightsquigarrow (N_1, \dots, N_k).$$

En particulier, V est un cobordisme lagrangien dont les bouts sont les variétés $L_1, \dots L_m$ et $N_1, \dots, N_k$.

Remarque 0.1. On peut, bien sûr, parler de cobordismes immergés quand la sous-variété V est immergée ; ou encore de cobordismes orientés quand la sous-variété V est orientée et respecte l'orientation de ses bouts.

Les cobordismes lagrangiens ont été introduits par Arnold dans [**Arn80**] qui avait en vue l'étude de la propagation de certains fronts d'ondes. Dans cet article, Arnold introduit le *groupe de cobordisme lagrangien immergé* $\Omega_{\mathrm{cob}}^{\mathrm{imm,or}}(M, \omega)$. C'est un groupe abélien dont les générateurs sont les immersions lagrangiennes orientées de (M, ω) et dont les relations sont données par

$$L_1 + \ldots + L_m = 0$$

dès qu'il existe un cobordisme lagrangien immergé et orienté

$$V : (L_1, \dots, L_m) \rightsquigarrow \emptyset.$$

Les groupes $\Omega_{\mathrm{cob}}^{\mathrm{imm,or}}(M, \omega)$ ont été étrangement peu calculés. Ils sont souvent déterminés par des invariants d'isotopie, à cause du h-principe de Gromov (et sont donc dits flexibles). Par exemple, dans [**Arn80**], Arnold calcule

$$\Omega_{\mathrm{cob}}^{\mathrm{imm,or}}\left(T^* S^1, \omega\right) \simeq \mathbb{Z}$$

où l'isomorphisme est donné par le nombre de rotation du vecteur tangent. D'autres calculs ont été faits par Eliashberg ([**Eli84**]) et Audin ([**Aud85**]). Je présenterai, au chapitre 2, un calcul du groupe de cobordisme immergé

$$\Omega_{\mathrm{cob}}^{\mathrm{imm,or}}(S_g, \omega)$$

d'une surface S_g de genre $g \geqslant 1$.

...et rigidité. Supposons (M, ω) fermée. Du côté de la rigidité, Biran et Cornea remarquent dans [**BC13**] que certains cobordismes lagrangiens plongés permettent d'obtenir des relations algébriques dans les catégories de Fukaya.

Avant d'énoncer leur résultat, fixons une lagrangienne plongée L. En intégrant la forme symplectique, on obtient un morphisme

$$[\omega] : \pi_2(M, L) \to \mathbb{R}.$$

On dispose aussi d'un autre morphisme, *l'indice de Maslov*

$$\mu_L : \pi_2(M, L) \to \mathbb{R}$$

obtenu en calculant un analogue du nombre de rotation de l'espace tangent à L le long du bord d'un disque. La variété L est dite *monotone* s'il existe $\lambda > 0$ tel que

$$[\omega] = \lambda \cdot \mu_L.$$

Cette condition permet de définir une catégorie de Fukaya dont les objets sont des lagrangiennes monotones L telles que l'image de $\pi_1(L)$ dans $\pi_1(M)$ est triviale (voir [**BC14**],[**She16**]).

Biran et Cornea montrent donc le

Théorème 0.1 ([**BC14**]). *Soient $L_1, \ldots, L_m$ et L des sous-variétés lagrangiennes, plongées et monotones. On suppose qu'il existe un cobordisme lagrangien V plongé, monotone et tel que l'image de $\pi_1(V) \to \pi_1(\mathbb{C} \times M)$ est nulle*

$$V : (L_1, \ldots, L_n) \rightsquigarrow L.$$

Alors, dans $\mathrm{DFuk}(M, \omega)$*, L est isomorphe à un cône itéré de la forme*

$$L \simeq \mathrm{Cone}(L_m \to \ldots \mathrm{Cone}(L_2 \to L_1) \ldots).$$

L'énoncé algébrique de ce théorème est un peu difficile. Expliquons donc ce qu'il signifie pour $m = 2$. On suppose qu'il existe un cobordisme lagrangien $V : (L_1, L_2) \rightsquigarrow L$ satisfaisant les hypothèses précédentes. Les axiomes des catégories triangulées impliquent que pour toute lagrangienne monotone N, il existe un suite exacte longue périodique[5]

$$HF(N, L_1) \to HF(N, L_2) \to HF(N, L) \to HF(N, L_1).$$

Les morphismes sont donnés par un compte de courbes pseudo-holomorphes dans (M, ω) et ce, même s'ils dépendent de V. Cela permet donc de calculer l'homologie de Floer de L à l'aide de celles de L_1 et L_2 quand les deux dernières sont plus simples.

[5]Rappelons que nos complexes ne sont pas gradués !

Plus généralement, la conclusion du théorème 0.1 permet de calculer l'homologie $HF(L, N)$ à l'aide d'une suite spectrale (voir [Sei08, (51)]) dont les différentielles sont données par un comptage de courbes pseudo-holomorphes dans (M, ω).

Groupes de cobordisme plongé. Pour étudier la relation de cobordisme monotone, Biran et Cornea proposent de construire une variante du groupe introduit par Arnold. Le groupe de cobordisme lagrangien *plongé et monotone* est le groupe abélien engendré par les lagrangiennes monotones satisfaisant $\mathrm{Im}(\pi_1(L) \to \pi_1(M)) = 0$ et avec relations

$$L_1 + \ldots + L_m = 0$$

pour chaque cobordisme lagrangien monotone

$$V : (L_1, \ldots, L_m) \rightsquigarrow \emptyset,$$

satisfaisant $\mathrm{Im}(\pi_1(V) \to \pi_1(\mathbb{C} \times M)) = 0$. On le note $\Omega^{\mathrm{mon}}_{\mathrm{cob}}(M, \omega)$.

Il existe un groupe semblable du côté algébrique. Le *groupe de Grothendieck* de $\mathrm{DFuk}(M, \omega)$ est le groupe abélien engendré par les objets de $\mathrm{DFuk}(M, \omega)$ et dont les relations sont

$$L_3 = L_2 - L_1,$$

pour chaque quasi-isomorphisme

$$L_3 \simeq \mathrm{Cone}(L_1 \to L_2).$$

On le note $K_0(\mathrm{DFuk}(M, \omega))$.

On vérifie sans peine que le théorème de Biran et Cornea 0.1 implique qu'il existe un morphisme de groupes naturel

$$\Theta_{\mathrm{BC}} : \begin{array}{ccc} \Omega^{\mathrm{mon}}_{\mathrm{cob}}(M, \omega) & \to & K_0(\mathrm{DFuk}(M, \omega)) \\ [L] & \mapsto & [L] \end{array}.$$

On voit facilement que Θ_{BC} est surjectif.

Question 0.1. L'application Θ_{BC} est-elle un isomorphisme ? Si non, quel est son noyau ?

Dans [Hau15], Haug montre que cette application est un isomorphisme dans le cas du tore T^2 lorsque les relations sont données par des cobordismes lagrangiens orientés.

Contenu de la thèse

Contenu du chapitre 1

On a vu que l'étude des phénomènes de rigidité pour les immersions lagrangiennes fait intervenir un comptage de polygones holomorphes à bord dans celles-ci. Donnons une définition plus précise. Soit $i : L \looparrowright M$ un immersion générique, c'est-à-dire sans points triples et dont tout les points doubles sont transverses. Fixons aussi une structure presque complexe $J \in \mathcal{J}(M, \omega)$.

Au chapitre 1, je définis un polygone J-holomorphe avec bord dans L et des coins (cf 1.1.1). Il s'agit d'une application pseudo-holomorphe

$$u : (\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$$

qui admet un certain nombre de coins $x_1, \ldots, x_N$.

Un point injectif de u est un point $z \in \mathbb{D}$ tel que

$$du_z \neq 0, \;\; u^{-1}(u(z)) = \{z\}.$$

On dit que u est *simple* si l'ensemble de ses points injectifs est dense.

Le résultat principal du chapitre 1 est un résultat de décomposition en pièces simples pour ces polygones.

Théorème (1.1.3)**.** *On suppose que le disque u est d'énergie finie (i.e. $\int u^*\omega < +\infty$).*

Il existe des polygones pseudo-holomorphes simples $v_1, \ldots, v_N$ d'énergies finies et des entiers naturels $m_1, \ldots, m_N \in \mathbb{N}$ tels que

(i) $\mathrm{Im}(u) = \bigcup_{k=1\ldots N} \mathrm{Im}(v_k)$,

(ii) On a dans $H_2(M, i(L))$

$$[u] = \sum_{k=1}^{N} m_k [v_k].$$

Il s'agit d'une généralisation d'un résultat de Lazzarini ([**Laz00**], [**Laz11**]) dans le cas des disques à bord dans une lagrangienne plongée.

La démonstration du théorème repose sur l'introduction d'un *patron* $\mathcal{W}(u)$ qui est un graphe plongé dans le domaine $\mathbb{D}$ du polygone u (définition 1.2.2 et proposition 1.2.23). La restriction de u à chacune des composantes connexes du complémentaire est un revêtement multiple d'un polygone simple. Autrement dit, on retrouve les morceaux simples de la décomposition en « découpant le polygone le long du patron ».

L'intérêt du théorème 1.1.3 est qu'il permet de montrer que certains espaces de courbes pseudo-holomorphes sont des variétés pour un choix générique de $J \in \mathcal{J}(M, \omega)$. Comme on l'a vu plus tôt, cela permet de définir des objets algébriques en comptant des polygones.

Je donne ensuite des applications du Théorème 1.1.3. La première est une généralisation d'un résultat de Lazzarini. Soient L_1 et L_2 deux sous-variétés lagrangiennes transverses. On suppose que la dimension de M est *plus grande que six.*

Proposition (1.3.4 et1.3.7)**.** *Il existe des ensembles génériques*

$$\mathcal{J}_{reg}(M, L, \omega) \subset \mathcal{J}(M, \omega) \;\; et \; \mathcal{J}_{reg}(M, L_1, L_2, \omega) \subset \mathcal{J}(M, \omega)$$

tels que

(i) Pour tout $J \in \mathcal{J}_{reg}(M, L, \omega)$, toute larme à bord sur L est simple,

(ii) Pour tout $J \in \mathcal{J}_{reg}(M, L_1, L_2, \omega)$, toute bande pseudo-holomorphe est simple.

Je donne ensuite un exemple d'application de cette proposition à la définition d'un objet algébrique, en l'occurence le complexe de Floer entre L_1 et L_2.

Théorème (1.3.12). *Supposons (M, ω) fermée et monotone. Il existe un ensemble générique*

$$\mathcal{J}_{reg}(M, \omega, L_1, L_2) \subset \mathcal{J}(M, \omega)$$

tel que le complexe

$$(CF(L_1, L_2, J), d)$$

est bien défini.

En d'autres mots, l'homologie de Floer peut se calculer en utilisant des structures complexes *indépendantes* du temps !

J'ai montré le théorème 1.1.3 avec en vue une preuve de la conjecture 0.2 quand le morphisme c coïncide avec un point d'intersection $x \in L_1 \cap L_2$. Malheureusement, je n'ai pour l'instant pu qu'obtenir des résultats partiels que je détaille dans la sous-section 1.3.3.

Contenu du chapitre 2

Dans le chapitre 2, on se restreint au cas d'une surface de genre g, S_g munie d'une forme d'aire ω. On note $\chi(S_g)$ sa caractéristique d'Euler.

Mon premier résultat est le calcul du groupe de cobordisme immergé $\Omega^{\text{imm}}_{\text{cob}}(S_g)$ quand le genre g est plus grand que un.

Théorème (2.1.5). *Supposons le genre g plus grand que un. Il existe un isomorphisme*

$$\Omega^{imm}_{cob}(S_g) \simeq H_1(S_g, \mathbb{Z}) \oplus \mathbb{Z}/\chi(S_g)\mathbb{Z}.$$

Ce théorème est encore un exemple de *flexibilité*. En effet, l'application

$$\Omega^{\text{imm}}_{\text{cob}}(S_g) \to H_1(S_g, \mathbb{Z})$$

est donnée par la classe d'homologie. L'application

$$\Omega^{\text{imm}}_{\text{cob}}(S_g) \to \mathbb{Z}/\chi(S_g)\mathbb{Z}$$

est une variante du nombre de rotation originellement due à Chilllingworth ([**Chi72b**]). Ces deux quantités sont invariantes par isotopies lagrangiennes !

Je passe ensuite à l'étude des phénomènes de rigidité. On doit alors supposer le genre g plus grand que deux. Je commence par introduire une variante du groupe de cobordisme

lagrangien
$$\Omega_{\text{cob}}^{\text{imm,unob}}(S_g).$$
Celui-ci est engendré par les plongements $S^1 \hookrightarrow S_g$ et ses relations sont données par des cobordismes lagrangiens immergés, non obstrués et orientés (voir la définition 2.4.1). Ici les cobordismes sont non obstrués si et seulement s'ils n'admettent pas de larmes continues.

Il se trouve que les cobordismes non obstrués donnent aussi des relations dans les catégories de Fukaya de la surface S_g (Théorème 2.4.2). On en déduit l'existence d'un morphisme de groupe
$$\Theta_{\text{BC}} : \Omega_{\text{cob}}^{\text{imm,unob}}(S_g) \to K_0(\text{DFuk}(S_g)).$$
Je montre alors le

Théorème (2.5.1)**.** *L'application Θ_{BC} est un isomorphisme.*

Abouzaid ([**Abo08**]) a calculé le groupe $\mathbb{K}_0(\text{DFuk}(S_g))$. Avec son résultat, on obtient donc le

Corollaire (2.1.7)**.** *Il existe un isomorphisme*
$$\Omega_{cob}^{imm,unob}(S_g) \simeq \mathbb{R} \oplus H_1(S_g, \mathbb{Z}) \oplus \mathbb{Z}/\chi(S_g)\mathbb{Z}.$$

Le Théorème 2.5.1 montre que les relations données par les cobordismes non obstrués suffisent à définir $K_0(\text{DFuk}(S_g))$. Il s'agit donc d'un indice très fort de la validité de la conjecture 0.1 !

Les preuves des Théorèmes 2.1.5 et 2.5.1 sont toutes les deux très géométriques. Elles reposent sur une caractérisation de l'action du *groupe modulaire* (ou Mapping Class Group) :
$$\text{Mod}(S_g) = \text{Diff}^+(S_g)/\operatorname{Diff}_0(S_g),$$
sur les groupes $\Omega_{\text{cob}}^{\text{imm}}(S_g)$ et $\Omega_{\text{cob}}^{\text{imm,unob}}(S_g)$. L'idée est initialement due à Abouzaid (voir [**Abo08**]).

Il est connu que ce groupe est engendré par les twists de Dehn T_α autour d'une courbe plongée α. La contribution principale de mon travail est donc de montrer une formule qui caractérise l'action d'une telle transformation.

Pour les groupes $\Omega_{\text{cob}}^{\text{imm}}(S_g)$, cela repose sur la description de l'image $T_\alpha\beta$ d'une courbe β au moyen d'une suite de résolutions successives de points doubles (qui est représentée dans la figure 5).

Pour adapter cet argument au cas de $\Omega_{\text{cob}}^{\text{imm,unob}}(S_g)$, il faut montrer que toutes les courbes et cobordismes qui interviennent dans cette procédure sont non obstrués. Pour cela, on découpe des disques holomorphes en utilisant (sans le dire) le patron défini ci-dessus. On utilise de plus les propriétés des groupes d'homotopie des surfaces de genre plus grand que deux. C'est le contenu des preuves des propositions 2.5.7 et 2.5.13.

On montre donc, en particulier, une forme très faible de la conjecture 0.2 dans le cas des surfaces de genre plus grand que deux.

Chapitre 1

Structure des courbes pseudo-holomorphes à bord sur une immersion Lagrangienne

Ce chapitre reproduit la prépublication *Structure of J-holomorphic disks with immersed Lagrangian boundary conditions*, [**Per18**].

Résumé. Nous généralisons un théorème de structure dû à Lazzarini ([**Laz11**]) au cas des courbes à bord sur une immersion lagrangienne. En appliquant ce résultat, on montre que l'homologie de Floer entre deux lagrangiennes peut être calculée à l'aide de structures complexes indépendantes du temps. Nous donnons aussi quelques autres applications et expliquons des projets futurs.

Abstract. We explain how to generalize Lazzarini's structural Theorem from [**Laz11**] to the case of curves with boundary on a given Lagrangian immersion. As a consequence of this result, we show that we can compute Floer homology with time-independent almost complex structures. We also give some applications as well as topics for future work.

1.1. Introduction

1.1.1. Setting

Let (M, ω) be a symplectic manifold and $J \in \mathcal{J}(M, \omega)$ be a compatible almost complex structure. It is well known that any J-holomorphic curve $u : \Sigma \to M$ with Σ a closed Riemann surface factors through a simple curve (see [**MS12**, Proposition 2.5.1]).

Let $L \subset M$ be an embedded Lagrangian submanifold and

$$u : (\mathbb{D}, \partial\mathbb{D}) \to (M, L)$$

be a J-holomorphic disk satisfying $u(\partial\mathbb{D}) \subset L$. In general, it is not true that such a map factors through a branched cover to a simple curve. However there are results of Kwon-Oh ([Oh97],[KO00]) and Lazzarini ([Laz00], [Laz11]) about the structure of such disks.

Moduli spaces of disks with Lagrangian boundaries appear in the definitions of several differential complexes associated to Lagrangian embeddings such as the pearl complex (due to Biran-Cornea [BC07], [BC09]) or Lagrangian intersection Floer homology in the monotone case (due to Oh, [Oh93a], [Oh93b]). The results of Lazzarini and Kwon-Oh are essential to study the generic regularity of such moduli spaces.

1.1.2. Main theorem

In this paper, we shall explain how to adapt Lazzarini's result ([Laz11]) to disks with corners whose boundaries lie in the image of a Lagrangian immersion. In this section, we provide the basic definitions of the objects we will consider.

From now on, we fix a connected symplectic manifold (M^{2n}, ω) and a Lagrangian immersion $i : L^n \looparrowright M$, with L a closed (not necessarily connected) manifold such that

(1) i does not have triple points,

(2) the double points of i are transverse.

If this is satisfied, we say that i is *generic*.

Let

$$R = \{(p,q) \in L \times L | i(p) = i(q)\}$$

be the set of ordered double points of i and $i(R)$ be the set of their images. The hypotheses on i imply that this is a finite subset.

Moreover, we fix a (smooth) compatible almost complex structure $J \in \mathcal{J}(M, \omega)$. We now explain what we mean by an almost complex curve with corners and boundary on L.

Definition 1.1.1. *Let S be a compact Riemann surface with boundary ∂S.*

*A J-*holomorphic curve with corners and boundary on *L is a continuous map*

$$u : (S, \partial S) \to (M, i(L))$$

which satisfies the following assumptions.

(i) There are $x_1, \dots, x_N \in \partial S$ with

$$\forall 1 \leqslant k \leqslant N,\ u(x_k) \in i(R).$$

(ii) There is a continuous map $\gamma : \partial S \backslash \{x_1, \dots, x_N\} \to L$ such that

$$u_{|\partial S \backslash \{x_1, \dots, x_N\}} = i \circ \gamma.$$

(iii) *The map γ does not extend to a continuous map $\partial S \to L$ at any of the punctures.*

(iv) *The map u is a smooth J-holomorphic curve on $S \backslash \{x_1, \ldots, x_N\}$.*

Remark 1.1.2. (1) Keeping the notations of Definition 1.1.1, we call $x_1, \ldots, x_N$ the *corner points* of the curve.

(2) We also consider maps $u : S \to M$ which satisfy the hypotheses $(i), (ii), (iii)$ without (iv). We call such a map a *topological* curve with corners.

For a J-holomorphic curve $u : (S, \partial S) \to (M, i(L))$ with corners and boundary on L, a point $z \in \mathrm{Int}(S)$ is an *injective point* if it satisfies

$$du_z \neq 0, u^{-1}(u(z)) = \{z\}.$$

We say that such a curve is *simple* if the set of its injective points is dense.

We can now state the main theorem of this paper.

Theorem 1.1.3. *Let $u : (\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$ be a non-constant J-holomorphic disk with corners, boundary on L and finite energy (meaning $\int u^*\omega < +\infty$).*

There are simple finite-energy J-holomorphic disks $v_1, \ldots, v_N$ with corners, boundary on L and natural integers $m_1, \ldots, m_N \in \mathbb{N}$ such that

(i) $\mathrm{Im}(u) = \cup_{k=1\ldots N} \mathrm{Im}(v_k)$

(ii) *In $H_2(M, i(L))$ we have*

$$[u] = \sum_{k=1}^{N} m_k [v_k].$$

The proof of this is an adaptation of Lazzarini's proof to the case of immersed Lagrangians.

1.1.3. Applications

Assume that the complex dimension n is greater than 3. For a generic almost complex structure J, any finite-energy J-holomorphic disk with corners and boundary on L is either simple or multiply covered. The proof is an adaptation of [Laz11, Proposition 5.15].

Corollary 1.1.4. *Assume that the complex dimension of M satisfies $n \geqslant 3$. There is a second category subset*

$$\mathcal{J}_{reg}(M, \omega, L) \subset \mathcal{J}(M, \omega)$$

satisfying the following property.

Let $J \in \mathcal{J}_{reg}(M, \omega)$ and $u : (\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$ be a non-constant finite-energy J-holomorphic disk with corners and boundary on L. Then there exist

(i) *a holomorphic map $p : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{D}, \partial\mathbb{D})$ with branch points in $\mathrm{Int}(\mathbb{D})$ (in particular the map p restricts to a cover $\partial\mathbb{D} \to \partial\mathbb{D}$).*

(ii) a simple J-holomorphic disk with corners and boundary on L,

$$u' : (\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$$

such that

$$u = u' \circ p.$$

Recall that there are two morphisms $\omega : \pi_2(M, L) \to \mathbb{R}$ and $\mu : \pi_2(M, L) \to \mathbb{Z}$ induced respectively by the symplectic area and the Maslov class. A Lagrangian submanifold $L \subset M$ is *monotone* if there is a $\lambda > 0$ such that

$$\omega = \lambda\mu.$$

Denote by N_L the minimal Maslov number of a Lagrangian submanifold L. Consider two transverse Lagrangian submanifolds L_1 and L_2 satisfying $N_{L_1} \geqslant 3$ and $N_{L_2} \geqslant 3$. As a direct application of Corollary 1.1.4, we will see that for a generic *time-independent* $J \in \mathcal{J}(M, \omega)$, there is a well-defined Floer complex between these two objects. This differs from the usual situation where one usually considers time-dependent almost complex structures to achieve transversality (as considered in [**Oh93a**] or [**FHS95**]).

1.1.4. Outline of the proof of the Main Theorem

We prove the Main Theorem 1.1.3 in several steps which follow Lazzarini's approach. We will emphasize along the argument the differences with [**Laz11**].

First, we define a set $\mathcal{W}(u) \subset \mathbb{D}$ called the *frame* of the disk which contains $\partial\mathbb{D}$. This is roughly the set of points where u "overlaps" with its boundary. We then prove that this is a $\mathcal{C}^1$-embedded graph. We do this by providing an asymptotic expansion of the J-holomorphic curve around its corners.

The simple or multiply covered pieces are found by cutting the curve along the graph $\mathcal{W}(u)$. More precisely, we pick for each connected component Ω of $\mathbb{D}\backslash\mathcal{W}(u)$ a holomorphic embedding $h_\Omega : (\mathbb{D}, \partial\mathbb{D}) \to (\Omega, \mathcal{W}(u))$. The curve $u \circ h_\Omega$ satisfies $\mathcal{W}(u \circ h_\Omega) = \partial\mathbb{D}$ and is therefore either simple or multiply covered. The pieces of the decomposition are the simple curves underlying $u \circ h_\Omega$ for Ω a connected component.

Notice that a connected component Ω of $\mathbb{D}\backslash\mathcal{W}(u)$ is not necessarily simply connected, so we cannot immediately conclude that $u \circ h_\Omega$ factors through a simple disk. It turns out that if such a component exists, there is a simple holomorphic sphere $v : \mathbb{C}P^1 \to M$ such that $u(\mathbb{D}) = v(\mathbb{C}P^1)$. From this, we conclude that each piece is a disk. Here the details do not differ much from Lazzarini's paper ([**Laz11**]).

1.1.5. Outline of the paper

The first section of the paper explains how to adapt Lazzarini's proof ([**Laz00**], [**Laz11**]) to finite-energy curves with boundary on a given Lagrangian immersion $i : L \looparrowright M$. We cut the curve into multiply covered pieces along a frame. This frame is a $\mathcal{C}^1$-embedded graph. The proof of this fact is the main technical part of the argument. Second, we explain how to get the decomposition from this.

The second section of the paper gives the proof of Corollary 1.1.4. In a second subsection, we will explain why this implies that the Floer complex is well-defined for a generic time-independent almost complex structure.

Lastly, we give some expected applications of the main theorem to a count of holomorphic curves with boundary on the surgery of two Lagrangian embeddings. These results fit in a more general program of Biran-Cornea and is the subject of work in progress.

1.1.6. Acknowledgements:

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1.2. The Frame of a J-holomorphic curve

Fix $u_1 : (S_1, \partial S_1) \to (M, i(L))$ and $u_2 : (S_2, \partial S_2) \to (M, i(L))$ two finite-energy J-holomorphic curves with corners and boundaries on L.

We define the set of "bad points" of u_1 with respect to u_2 :

$$\begin{aligned}\mathcal{C}(u_1, u_2) := u_1^{-1}(u_1(\{z \in \mathrm{Int}(S_1) | du_1(z) = 0\}))\\ \cup\, u_1^{-1}(u_2(\{z \in \mathrm{Int}(S_2) | du_2(z) = 0\})) \cup u_1^{-1}(i(R))\end{aligned}$$

The following definition is due to Lazzarini ([**Laz11**]).

Definition 1.2.1. *We keep the above notations. Let $z_1 \in \mathrm{Int}(S_1) \backslash \mathcal{C}(u_1, u_2)$ and $z_2 \in \mathrm{Int}(S_2) \backslash \mathcal{C}(u_2, u_1)$.*

We let $z_1 \mathcal{R}_{u_1}^{u_2} z_2$ if and only if for any open neighborhoods $V_1 \ni z_1$ (resp. $V_2 \ni z_2$), there are open neighborhoods $\Omega_1 \ni z_1$ (resp. $\Omega_2 \ni z_2$) in V_1 (resp. V_2) such that

$$u_1(\Omega_1) = u_2(\Omega_2).$$

Now if $z_1 \in S_1$ and $z_2 \in S_2$, we let $z_1 \mathcal{R}_{u_1}^{u_2} z_2$ if and only if there are sequences $(z_1^\nu)_{\nu \geqslant 0}$ (resp. $(z_2^\nu)_{\nu \geqslant 0}$) such that $z_1^\nu \in \mathrm{Int}(S_1) \backslash \mathcal{C}(u_1, u_2)$ (resp. $z_2^\nu \in \mathrm{Int}(S_2) \backslash \mathcal{C}(u_2, u_1)$), $z_1^\nu \to z_1$

(resp. $z_2^\nu \to z_2$) and

$$\forall \nu \geqslant 0,\ z_1^\nu \mathcal{R}_{u_1}^{u_2} z_2^\nu.$$

We now define the graph along which we will cut to get the simple pieces of the curve.

Definition 1.2.2. *We let u_1 and u_2 be two finite-energy J-holomorphic curves with corners and boundaries on L.*

The frame *of u_1 with respect to u_2 is the set of points related to the boundary of S_2:*

$$\mathcal{W}(u_1, u_2) := \mathcal{R}_{u_1}^{u_2}(\partial S_2).$$

The completed frame *of u_1 with respect to u_2 is the union of this with ∂S_1:*

$$\overline{\mathcal{W}}(u_1, u_2) := \mathcal{R}_{u_1}^{u_2}(\partial S_2) \cup \partial S_1.$$

Remark 1.2.3. Let u be a finite-energy J-holomorphic curve with corners and boundary on L. We readily check

$$\partial S \subset \mathcal{W}(u, u),$$

so

$$\overline{\mathcal{W}}(u, u) = \mathcal{W}(u, u).$$

From now on, we will abbreviate

$$\mathcal{W}(u) := \mathcal{W}(u, u).$$

In this section, we shall prove that the completed frame $\overline{\mathcal{W}}(u_1, u_2)$ is a $\mathcal{C}^1$ embedded graph in S_1. This is however not the case for $\mathcal{W}(u_1, u_2)$. Along the way, we will prove important properties of the relation $\mathcal{R}_{u_1}^{u_2}$, always following Lazzarini's proof.

Examples of frames and the decomposition

As explained in the introduction, the simple pieces of the curve are found among the connected components of $\mathbb{D}\backslash\mathcal{W}(u)$.

The decomposition may introduce corner points which do not appear in the original curve.

Example 1.2.4. Consider the 2-dimensional torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ equipped with the standard area form $dx \wedge dy$ and the standard complex structure.

We let i be the immersion of two copies of S^1 drawn in Figure 1. Moreover, we let u be a J-holomorphic polygon with corners and boundary on L, whose image is represented in Figure 1. The parameterization of u is chosen so that $u(-1) = x_1$ and $u(1) = x_4$. These are the only corner points of u.

It is an easy exercise to check that the frame of u is a graph with four vertices which map to the double points of i. The restriction of u to each connected component of

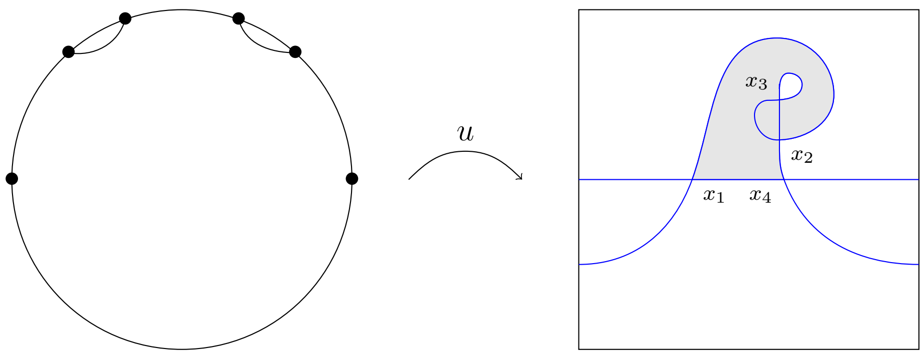

FIGURE 1 – *The immersion i (blue, on the right), the disk u (shaded, on the right) and its frame $\mathcal{W}(u)$ on the left*

$\mathbb{D}\backslash\mathcal{W}(u)$ is a simple J-holomorphic curve with corners. Notice that each piece now has corners which map to x_2 and x_3. These corners did not appear in u.

Moreover, the frame need not be connected, as shown by the following example.

Example 1.2.5. Consider $\mathbb{C}P^1 = \mathbb{C}\cup\{\infty\}$ equipped with its standard complex structure and let $L \subset \mathbb{C}$ be the ellipse with semi-major axis $\frac{5}{2}$ and semi-minor axis $\frac{3}{2}$. We consider a map with domain the disk of radius 2

$$\begin{array}{cccc} u: & \mathbb{D}(0,2) & \to & \mathbb{C}\cup\{\infty\} \\ & z & \mapsto & z+\frac{1}{z} \end{array}.$$

We claim that the frame of u is given by $\partial\mathbb{D}(0,2)\cup\partial\mathbb{D}\big(0,\frac{1}{2}\big)$. Notice first that

$$\mathcal{W}(u) \subset u^{-1}(L) = \partial\mathbb{D}(0,2)\cup\partial\mathbb{D}\left(0,\frac{1}{2}\right).$$

To prove the other inclusion, let $z \in \partial\mathbb{D}\big(0,\frac{1}{2}\big)$. Let $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ be a sequence of positive real numbers converging to 0. Put $z_\nu = (1+\varepsilon_\nu)z$. Then the sequences $(z_\nu)_{\nu\in\mathbb{N}}$ and $\left(\frac{1}{z_\nu}\right)_{\nu\in\mathbb{N}}$ satisfy

$$\forall \nu \in \mathbb{N},\ z_\nu \mathcal{R}_u^u \frac{1}{z_\nu}$$

since

$$\forall \nu \in \mathbb{N},\ u(z_\nu) = u\left(\frac{1}{z_\nu}\right)$$

and the derivatives

$$u'(z_\nu),\ u'\left(\frac{1}{z_\nu}\right)$$

are non-zero. Hence $z\mathcal{R}_u^u\frac{1}{z} \in \partial\mathbb{D}$, so $z \in \mathcal{W}(u)$.

1.2.1. Local coordinates around double points of the immersion

In this subsection, we construct several local charts $\phi : U \subset \mathbb{C}^n \to M$ around a double point $x = i(p) = i(q)$ of a generic Lagrangian immersion $i : L \to M$. These charts map the image of the immersion i to the union of two transverse linear subspaces of $\mathbb{C}^n$.

In order to characterize the analytic behavior of a pseudo-holomorphic polygon around a double point, we need these charts to preserve complex and symplectic structures at the origin. Therefore, we start by classifying the pairs of *linear* Lagrangian subspaces of a symplectic vector space under the action of the unitary group. This is done in 1.2.1. Then, we will exponentiate this local model in order to find local coordinates around a given double point. This is done in 1.2.1.

Some linear symplectic geometry

We start by the linear case, i.e. the classification of pairs of transverse Lagrangian subspaces under the action of the unitary group. It turns out that the orbits of this action are classified by a n-uple of real numbers called the *Kähler angles* ([**FOOO06**, Definition 54.11]).

In what follows, we equip the standard symplectic space

$$\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$$

with the standard symplectic form, scalar product and complex structures:

$$\begin{aligned}\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^n \times \mathbb{R}^n,\ \omega_{\mathrm{std}}((x_1, y_1), (x_2, y_2)) &= x_1 \cdot y_2 - y_1 \cdot x_2,\\ g_{\mathrm{std}}((x_1, y_1), (x_2, y_2)) &= x_1 \cdot x_2 + y_1 \cdot y_2,\\ J_{\mathrm{std}}(x_1, y_1) &= (-y_1, x_1).\end{aligned}$$

Fix a symplectic vector space (V^{2n}, ω) of complex dimension n. Let $L_1, L_2 \subset V$ be two transverse Lagrangian subspaces. Choose a compatible almost complex structure $J \in \mathcal{J}(V, \omega)$.

There is a complex linear symplectomorphism

$$f : (\mathbb{C}^n, g_{\mathrm{std}}, J_{\mathrm{std}}) \to (V, g_J, \omega) \text{ with } f(L_1) = \mathbb{R}^n. \tag{1.1}$$

To see this, we let $f_1, \dots, f_n$ be a basis of L_1, orthonormal with respect to the scalar product $g_J := \omega(\cdot, J\cdot)$. Moreover, denote by $e_1, \dots, e_n$ the canonical complex linear basis of $\mathbb{C}^n$. Then, $f : \mathbb{C}^n \to V$ is the unique complex linear map such that $f(e_i) = f_i$ for $i \in \{1, \dots, n\}$.

The linear subspace $f^{-1}(L_2)$ is transverse to L_1. Hence, there is a unique linear map $A:\mathbb{R}^n\to\mathbb{R}^n$ such that

$$f^{-1}(L_2)=\{(Ax,x)|x\in\mathbb{R}^n\}. \tag{1.2}$$

Since $f^{-1}(L_2)$ is Lagrangian, the linear map A is autoadjoint. Therefore, by the spectral theorem, there are eigenvalues $\lambda_n\leqslant\ldots\leqslant\lambda_1\in\mathbb{R}$ and an orthonormal basis of eigenvectors $v_1,\ldots,v_n\in\mathbb{R}^n$ such that

$$\forall k\in\{1,\ldots,n\},\ Av_k=\lambda_k v_k.$$

We let

$$0<\alpha_1\leqslant\ldots\leqslant\alpha_n<\pi$$

be real numbers such that

$$\forall k\in\{1,\ldots,n\},\ e^{i\alpha_k}=\frac{\lambda_k+i}{|\lambda_k+i|}. \tag{1.3}$$

Notice, that with respect to the complex basis $v_1,\ldots,v_n$, we have

$$f^{-1}(L_2)=e^{i\alpha_1}\cdot\mathbb{R}\times\ldots\times e^{i\alpha_n}\cdot\mathbb{R}.$$

Definition 1.2.6. *Following the above discussion, we let L_1,L_2 be transverse Lagrangian subspaces of the symplectic vector space (V,ω) and $f:\mathbb{C}^n\to V$ as in equation 1.1.*

We call the real numbers $\alpha_1\leqslant\ldots\leqslant\alpha_n$ defined in equation 1.3, the Kähler angles *of the pair (L_1,L_2).*

These real numbers do not depend on the choice of f as in equation 1.1.

Proof. We let $g:\mathbb{C}^n\to V$ be another complex linear symplectomorphism as in equation 1.1. We call $0<\beta_1\ldots\leqslant\beta_n<\pi$ the angles associated to the map g by the procedure described above. Notice that

$$g^{-1}(L_2)=(g^{-1}\circ f)\big(f^{-1}(L_2)\big).$$

Since the map $g^{-1}\circ f$ is unitary and satisfies $g^{-1}\circ f(\mathbb{R}^n)=\mathbb{R}^n$, there is $B\in O(n)$ such that

$$\forall(x,y)\in\mathbb{R}^n\times\mathbb{R}^n,\ (g^{-1}\circ f)(x,y)=(Bx,By).$$

Keeping the notation of equation 1.2, we obtain

$$\begin{aligned}g^{-1}(L_2)&=\{(BAx,Bx)|x\in\mathbb{R}^n\}\\&=\big\{(BAB^{-1}x,x)\big|x\in\mathbb{R}^n\big\}\end{aligned}$$

Since BAB^{-1} is conjugate to A, their eigenvalues are the same. Therefore, we conclude

$$\forall k\in\{1,\ldots,n\},\ \alpha_k=\beta_k.$$

□

Further, we also need to introduce some additional notation. We fix real numbers $0 < \alpha_1 \leqslant \ldots \leqslant \alpha_n < \pi$. We define the following vector space

$$V_\alpha := \left\{ v \in \mathbb{R}^n \middle| e^{i\alpha} v \in e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R} \right\}. \tag{1.4}$$

It is straightforward to check that there is a direct sum decomposition

$$\mathbb{R}^n = \bigoplus_{\alpha \in \{\alpha_1, \ldots, \alpha_n\}} V_\alpha. \tag{1.5}$$

Moreover, we let

$$\pi_\alpha : \mathbb{R}^n \to V_\alpha \tag{1.6}$$

be the linear projection onto V_α with respect to the decomposition 1.5. We abuse notation slightly and denote its complexification $\pi_\alpha : \mathbb{C}^n \to \mathbb{C}^n$ as well.

A bit of vocabulary

Recall that we fixed a generic Lagrangian immersion $i : L \to (M, \omega)$. Let $(p, q) \in R$, since di_p (resp. di_q) is an immersion, there is an open neighborhood $U_p \ni p$ (resp. $U_q \ni q$) such that $i_{|U_p}$ (resp. $i_{|U_q}$) is an embedding. We call the submanifold $i(U_p)$ (resp. $i(U_q)$) the *branch of* i *at* p (resp. at q) and denote it by L_p (resp. L_q).

In what follows, we will sometimes forget about U_p and denote by L_p the image of any neighborhood of p on which i is an embedding.

Some local charts

We can now state

Proposition 1.2.7. *Let $i : L \to (M, \omega)$ be a generic Lagrangian immersion. Fix a double point $(p, q) \in R$ and denote $x = i(p) = i(q)$.*

Then there are open neighborhoods U of 0 in $\mathbb{C}^n$, V of x in M, U_p (resp. U_q) of p (resp. q) in L together with a smooth chart $\phi : U \to V$ satisfying the following properties.

(i) Let $g_J := \omega(\cdot, J\cdot)$ be the metric induced by the almost complex structure J, and g_{std} be the standard scalar product on $\mathbb{C}^n$. We have

$$\phi^* g_J(0) = g_{\mathrm{std}},\ \phi^* J(0) = i.$$

(ii) The chart maps the branches of i at x to linear subspaces :

$$\phi(U \cap \mathbb{R}^n) = i(U_p),\ \phi(U \cap e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R}) = i(U_q).$$

Here $\alpha_1 \leqslant \ldots \leqslant \alpha_n \in (0, \pi)$ are the Kähler angles of the pair $(T_x L_p, T_x L_q)$ with respect to the complex structure J.

Proof. There is a smooth chart $\tilde{\phi} : U \subset \mathbb{C}^n \to V \subset M$ such that $\tilde{\phi}(\mathbb{R}^n \cap U) = L_p \cap V$ and $\tilde{\phi}(i \cdot \mathbb{R}^n \cap U) = L_q \cap V$.

We now modify $\tilde{\phi}$ so that it satisfies the assertions of the proposition. For this pick an orthonormal basis (with respect to the metric g_J) $\mathcal{B} = (e_1, \ldots, e_n)$ of $T_x L_p$. We assume that, with respect to the complex coordinates given by $\mathcal{B}$, we have

$$T_x L_q = e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R}.$$

We put $(f_1, \ldots, f_n) := (d\tilde{\phi}^{-1}(e_1), \ldots, d\tilde{\phi}^{-1}(e_n))$. Pick a real linear isomorphism $A : \mathbb{C}^n \to \mathbb{C}^n$ such that the image of the canonical basis of $\mathbb{R}^{2n}$ is the basis $(f_1, \ldots, f_n)$. The sought-after local chart is $\tilde{\phi} \circ A$ (it is defined on a small enough ball). $\square$

We will modify this chart to get a more precise behavior along $\mathbb{R}^n$.

Proposition 1.2.8. *Recall that we fixed a generic Lagrangian immersion $i : L \looparrowright M$.*

There is a smooth local chart $\phi : U \subset \mathbb{C}^n \to V \subset M$ such that

(i) we have $\phi^* J_{|\mathbb{R}^n} = i$ and $(\phi^* g_J)_0 = g_{\mathrm{std}}$,

(ii) the preimages of the branches at x are linear subspaces

$$\phi^{-1}(L_p \cap V) = \mathbb{R}^n \cap U,\ \phi^{-1}(L_q \cap V) = \left(e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R}\right) \cap U.$$

Proof. By the preceding Proposition 1.2.7, one can assume that the two branches of the immersion are the given linear Lagrangian subspaces, that the almost complex structure J satisfies $J(0) = i$ and that the metric g_J satisfies $g_J(0) = g_{\mathrm{std}}$.

Now choose $\psi : W \to U \subset \mathbb{C}^n$ a local chart such that $\psi^* J_{|\mathbb{R}^n} = i$ and $d\psi(0) = \mathrm{Id}$ (such a chart always exists, see the construction in [**Laz11**, lemma 3.7]).

Notice that $\psi^{-1}(L_q)$ is an embedded submanifold whose tangent space at 0 is transverse to $\mathbb{R}^n$ (it is given by $e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R}$). Therefore the implicit function theorem implies that there is a smooth map $f : W \cap \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\psi^{-1}(L_q) = \{f(y) + iy | y \in \mathbb{R}^n\}.$$

Consider the map $\phi(x + iy) = f(y) - df(0) \cdot y + x + iy$. Its differential is given by the matrix

$$\begin{pmatrix} \mathrm{Id} & df_y - df_0 \\ 0 & \mathrm{Id} \end{pmatrix},$$

so $d\phi_0 = \mathrm{Id}$ and ϕ is a local diffeomorphism.

A small computation shows that df_0 is given by a diagonal matrix

$$\begin{pmatrix} \cot \alpha_1 & & \\ & \ddots & \\ & & \cot \alpha_n \end{pmatrix}.$$

Hence for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we have $\phi(x) = (f(0) + 0 + x, 0) \in \mathbb{R}^n$. Moreover, notice that

$$df_0(x_1 \sin \alpha_1, \ldots, x_n \sin \alpha_n) = (x_1 \cos \alpha_1, \ldots, x_n \cos \alpha_n),$$

hence

$$\phi\big(x_1 e^{i\alpha_1}, \ldots, x_n e^{i\alpha_n}\big) = f(x_1 \sin \alpha_1, \ldots, x_n \sin \alpha_n) + i(x_1 \sin \alpha_1, \ldots, x_n \sin \alpha_n),$$

so $\phi(\mathbb{R}^n) = e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R}$. Moreover, for $x \in \mathbb{R}^n$, we have $d\phi_x = \mathrm{Id}$ so $\phi^* J_x = i$ and $\phi^* g_J(0) = g_{\mathrm{std}}$. □

Behavior of a J-holomorphic curve around an interior point

We describe precisely the asymptotic behavior of a J-holomorphic curve around an interior point. For instance, the following is proved in Lazzarini's paper [**Laz11**, Proposition 3.3].

Proposition 1.2.9. *Assume that $J : \mathbb{D} \to GL(2n, \mathbb{R})$ is a $\mathcal{C}^1$ map such that $J^2 = -\mathrm{Id}$ and $J(0) = J_{std}$ is the standard complex structure. Let $u : S \to \mathbb{C}^n$ be a J-holomorphic curve with $u(0) = 0$. Then there are*

(1) an integer $k \geqslant 1$,

(2) a $\mathcal{C}^1$-local chart $\phi : \Omega \to \mathbb{D}$ with Ω and open neighborhood of 0 in $\mathbb{D}$ and $\phi(0) = 0$,

(3) a positive real number $\lambda_u > 0$ and a matrix $A \in U(n)$,

such that

$$u \circ \phi(z) = \lambda_u A\big(z^k, U(z)\big),$$

with $U(z) = O\big(z^{k+1}\big)$.

Behavior of a J-holomorphic curve around a double point

In this subsection, we describe, following [**Laz11**, section 3.2], the local form of a curve around the corner points.

For this, let us fix $(p, q) \in R$ and put $x = i(p) = i(q)$. As usual, we call

$$0 < \alpha_1 \leqslant \ldots \leqslant \alpha_n < \pi$$

the Kähler angles of the pair $(T_x L_p, T_x L_q)$ with respect to J. We also choose a chart $\phi : U \to V$ such as the one given in Proposition 1.2.7.

Proposition 1.2.10. *Let $\mathbb{D}^+ = \{x + iy \mid |x + iy| < 1, y \geqslant 0\}$ be the unit upper half-disk and $\mathbb{D}^+_{\mathbb{R}} = \mathbb{D}^+ \cap \mathbb{R}$ be its real part.*

Let $u : (\mathbb{D}^+, \mathbb{D}^+_{\mathbb{R}}) \to (M, i(L))$ be a non-constant J-holomorphic half-disk with boundary on L and finite energy (i.e. $\int u^*\omega < +\infty$). Assume that the lift $\gamma_{[0,1)}$ (resp. $\gamma_{(-1,0]}$) of $u_{|[0,1)}$ (resp. $u_{|(-1,0]}$) to L satisfies $\gamma_{[0,1)}(0) = p$ (resp. $\gamma_{[0,1)}(0) = q$)[1].

Then there are integers $k \in \{1, \ldots, n\}$ and $m \geqslant 0$, together with a positive real number $\delta > 0$ and a vector $a_k \in V_{\alpha_k}$[2] such that

$$\phi^{-1} \circ u(z) = a_k z^{\frac{\alpha_k}{\pi}+m} + o\Big(z^{\frac{\alpha_k}{\pi}+m+\delta}\Big).$$

Moreover, we have

$$d(\phi^{-1} \circ u)(z) = \Big(\frac{\alpha_k}{\pi} + m\Big) a_k z^{\frac{\alpha_k}{\pi}+m-1} + o\Big(z^{\frac{\alpha_k}{\pi}+m+\delta-1}\Big).$$

Note that this implies that there are no critical points in a sufficiently small punctured neighborhood of a corner point.

Remark 1.2.11. We call the integer $m+1$ the *multiplicity* of the curve u at 0.

Proof. The proof is an application of a theorem of Robbin and Salamon ([**RS01**, Theorem B]) on the asymptotics of a finite-energy J-holomorphic strip.

To see this, fix $r > 0$, and define the strip-like end

$$\begin{array}{rccc} \varepsilon_r : & S := [0, +\infty) \times [0, 1] & \to & (\mathbb{D}^+, \mathbb{D}^+_{\mathbb{R}}) \\ & (s, t) & \mapsto & -re^{-\pi(s+it)} \end{array}.$$

For $r \ll 1$ consider the map

$$\tilde{u} := \phi^{-1} \circ u \circ \varepsilon_r : S \to \mathbb{C}^n.$$

Then $\tilde{u}$ is pseudo-holomorphic with respect to the almost complex structure $\phi^* J$, has finite energy with respect to the metric $g_{\phi^* J}$ and satisfies the boundary condition

$$\tilde{u}([0, +\infty) \times \{0\}) \subset e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R},\ \tilde{u}([0, +\infty) \times \{1\}) \subset \mathbb{R}^n.$$

Moreover, since u is continuous, for r small enough the map $\phi^{-1} \circ u \circ \varepsilon_r$ has relatively compact image in $\mathbb{C}^n$. Hence by [**RS01**, Theorem A], $\tilde{u}(s, \cdot)$ converges uniformly to x as $s \to +\infty$ and its derivative $\partial_s u$ decays exponentially with respect to the usual $\mathcal{C}^\infty$ pseudo-distance.

We can now apply [**RS01**, Theorem B]. There exist a $\lambda > 0$ and a map $v : [0, 1] \to \mathbb{C}^n$ such that

$$i\partial_t v = \lambda v,\ v(0) \in e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R},\ v(1) \in \mathbb{R}^n,$$

[1]Geometrically this means that the curve has right boundary condition along the branch L_p and left boundary condition along the branch L_q.

[2]see 1.2.1 for the the definition of V_α

and a $\delta > 0$ such that

$$u(s,t) = \exp_0\left(-\frac{1}{\lambda}e^{-\lambda s}v(t) + w(s,t)\right),\ |w|_{\mathcal{C}^k} \leqslant c_k e^{-(\lambda+\delta)s}.$$

A small computation shows that there exist an integer $m \geqslant 0$, an α_k and a vector $v_k \in V_{\alpha_k}$ such that

$$\lambda = \alpha_k + m\pi,\ v(t) = e^{i\alpha_k}e^{-i(\alpha_k+m\pi)t}.$$

Now notice that if $z = -e^{-\pi(s+it)}$, then $z^{\frac{\alpha_k}{\pi}+m} = e^{-s(\alpha_k+m\pi)}e^{i(\alpha_k+m\pi)(1-t)}$. Hence,

$$u(z) = \exp_0\left(-\frac{(-1)^m}{\lambda}e^{-(\alpha_k+m\pi)s}e^{i(\alpha_k+m\pi)(1-t)}\right),$$

and so

$$u(z) = \exp_0\left(-\frac{(-1)^m}{\lambda}z^{\alpha_k+m\pi} + w(z)\right).$$

This gives the relevant estimate.

The estimate on the derivative follows from the chain rule applied to $\phi \circ u \circ \varepsilon$. $\square$

From now on, we will work locally in M with the help of the chart given by Proposition 1.2.8. Therefore, we shall consider J-holomorphic curves with values in $\mathbb{C}^n$ equipped with an almost complex structure J such that $J_{|\mathbb{R}^n} = J_{\text{std}}$. We assume these curves have boundaries on the union of the branches

$$L_p = \mathbb{R}^n \text{ and } L_q = e^{i\alpha_1}\cdot\mathbb{R}\times\ldots\times e^{i\alpha_n}\cdot\mathbb{R}. \tag{1.7}$$

We shall describe their behavior around the double point 0.

Proposition 1.2.12. *Recall that the branches L_p and L_1 are given in equation 1.7. Assume that $u : (\mathbb{D}^+, \mathbb{D}^+_{\mathbb{R}}) \to (\mathbb{C}^n, L_p \cup L_q)$ is a pseudo-holomorphic curves which satisfies the hypothesis of Proposition 1.2.10.*

Then there exist

(1) an open neighborhood Ω of 0 in $\mathbb{D}^+$,

(2) a $\mathcal{C}^1$ chart

$$\psi : (\Omega, \Omega\cap\mathbb{R}) \to (\mathbb{D}^+, \mathbb{D}^+_{\mathbb{R}}),$$

(3) a linear isometry $A_u \in \mathcal{L}(\mathbb{R}^{\dim V_{\alpha_k}}, V_{\alpha_k})$ and a $\lambda_u \in \mathbb{R}^+$ such that

$$\pi_{\alpha_k}(u\circ\psi(z)) = \lambda_u A_u\left(z^{\frac{\alpha_k}{\pi}+m}, \tilde{U}(z)\right)$$

with $\tilde{U}(z) = o\left(z^{\frac{\alpha_k}{\pi}+m+\delta}\right)$.

Moreover, if

$$U(z) = \sum_{\alpha\in\{\alpha_1,\ldots,\alpha_n\}\setminus\{\alpha_k\}} \pi_\alpha(u\circ\psi(z)),$$

we have

$$U(z) = o\Big(z^{\frac{\alpha_k}{\pi}+m+\delta}\Big),\ dU(z) = o\Big(z^{\frac{\alpha_k}{\pi}+m+\delta-1}\Big).$$

Proof. Replacing u by $\phi \circ u$ we can assume that u has values in $\mathbb{C}^n$. Using Proposition 1.2.10, there are k and $a_k \in V_{\alpha_k}$ such that

$$u(z) = a_k z^{\frac{\alpha_k}{\pi}+m} + o\Big(z^{\frac{\alpha_k}{\pi}+m+\delta}\Big).$$

Choose an isometry $A \in \mathcal{L}(\mathbb{R}^n, V_{\alpha_k})$ and a $\lambda_u > 0$ such that $A(1,0) = \lambda_u a_k$, then we have

$$\pi_{\alpha_k}(u(z)) = \lambda_u z^{\frac{\alpha_k}{\pi}+m} A(1 + a(z), U_1(z))$$

with $a(z) \in \mathbb{C}$, $a(z) = o(z^\delta)$ and $U_1(z) = o(z^\delta)$.

Now if $r > 0$ is small enough, define ϕ on $\mathbb{D}(0,r)$ by

$$\phi(z) = z(1 + a(z))^{\frac{1}{\frac{\alpha_k}{\pi}+m}}.$$

The map ϕ is $\mathcal{C}^1$ on $\mathbb{D}^+(0,r)\backslash\{0\}$, and if $z \neq 0$ we have

$$\phi'(z) = (1 + a(z))^{\frac{1}{\frac{\alpha_k}{\pi}+m}} dz + \frac{z a'(z)}{\frac{\alpha_k}{\pi} + m}(1 + a(z))^{\frac{1}{\frac{\alpha_k}{\pi}+m}-1}.$$

Therefore $\phi'(z) \to 1$ as $z \to 0$. Hence, ϕ extends to a $\mathcal{C}^1$ map on $\mathbb{D}^+(0,r)$.

Now if $z \in \mathbb{R}_+$, we have $\pi_{\alpha_k}(u(z)) \in \mathbb{R}^{\dim V_{\alpha_k}}$, so $\lambda_u z^{\frac{\alpha_k}{\pi}+m}(1+a(z)) \in \mathbb{R}$ and $1+a(z) \in \mathbb{R}$. If $z \in \mathbb{R}_-$, since $\pi_{\alpha_k}(u(z)) \in e^{i\alpha_k} \cdot \mathbb{R}^{\dim V_{\alpha_k}}$ we similarly obtain $1 + a(z) \in \mathbb{R}$.

Since $a(z) \to 0$ as $z \to 0$, we can assume that for $z \in \mathbb{D}(0,r) \cap \mathbb{R}$ we have $1+a(z) \in \mathbb{R}^+$. Hence $(1 + a(z))^{\frac{1}{\frac{\alpha_k}{\pi}+m}} \in \mathbb{R}$ and $\phi(z) \in \mathbb{R}$.

We can now use the Schwarz reflection principle to see that ϕ extends to a map defined on $\mathbb{D}(0,r)$ with invertible differential at the origin. Therefore it admits a local inverse. We will now assume that $r > 0$ is small enough so that ϕ is invertible.

The image of $\mathbb{D}(0,r)$ by ϕ is an open subset of $\mathbb{C}$ with boundary a $\mathcal{C}^1$ simple closed curve. By the Jordan curve theorem, this image is cut by the real line $\mathbb{R}$ into two connected components. These are necessarily the images of the connected components of $\mathbb{D}(0,r)\backslash\mathbb{R}$ by ϕ. We conclude that $\phi(\mathbb{D}^+(0,r))$ is a subset of $\mathbb{H}$.

Now

$$\pi_{\alpha_k}(u(z)) = \lambda_u A\Big(\phi(z)^{\frac{\alpha_k}{\pi}+m}, U_1(z)\Big),$$

and so

$$\pi_{\alpha_k}(u(\phi^{-1}(z))) = \lambda_u A\Big(z^{\frac{\alpha_k}{\pi}+m}, U_1 \circ \phi^{-1}(z)\Big).$$

□

Let us recall the analog of this (Proposition 1.2.12) in the case of a curve with boundary along a single branch of the immersion. This is [**Laz11**, Lemma 3.5]. It is proved in the same manner as above (with a bit less trouble).

Proposition 1.2.13. *Recall that the branch L_p is given in equation 1.7.*

Assume that

$$u : (\mathbb{D}^+, \mathbb{D}^+_{\mathbb{R}}) \to (\mathbb{C}^n, L_p)$$

is a finite-energy, J-holomorphic curve with $u(0) = 0$.

There are a matrix $A_u \in O_n(\mathbb{R})$, a $\lambda_u > 0$, a natural $m \in \mathbb{N}$ and ψ a $\mathcal{C}^1$ local chart around 0 such that

$$u \circ \phi(z) = \lambda_u A_u(z^m, U(z)),$$

with $U(z) = o(z^m)$ and $dU(z) = o(z^{m-1})$.

These two propositions allow us to give the local behavior of these curves when they have boundary conditions along L_q rather than L_p.

For this let us introduce $D_{\alpha_1,\dots,\alpha_n}$ the $n \times n$ diagonal matrix with successive entries $e^{i\alpha_1}, \dots, e^{i\alpha_n}$.

Proposition 1.2.14. *Recall that the branch L_q is defined in equation 1.7.*

Assume that

$$u : \left(\mathbb{D}^+, \mathbb{D}^+_{\mathbb{R}}\right) \to (\mathbb{C}^n, L_q)$$

is a finite-energy, J-holomorphic curve with $u(0) = 0$.

There are a matrix $B_u \in O_n(\mathbb{R})$, a positive real number $\lambda_u > 0$ such that

$$u(z) = \lambda_u D_{\alpha_1,\dots,\alpha_n} B_u(z^m, U(z)),$$

with $U(z) = o(z^m)$ and $dU(z) = o(z^{m-1})$.

Proof. This follows directly from Proposition 1.2.13. To see this, consider the curve

$$v = D_{-\alpha_1,\dots,-\alpha_n} u.$$

Then v satisfies the hypotheses of 1.2.13 with the complex structure

$$D_{-\alpha_1,\dots,-\alpha_n} J D_{\alpha_1,\dots,\alpha_n}.$$

This immediately gives the conclusion. $\square$

The same trick allows us to give a local form around a corner point.

Proposition 1.2.15. *Recall that the branches L_p and L_q are given in equation 1.7.*

Assume that

$$u : \left(\mathbb{D}^+, \mathbb{D}^+_{\mathbb{R}}\right) \to (\mathbb{R}^n, L_p \cup L_q)$$

is a finite-energy J-holomorphic curve such that $u(0) = 0$ and $u([0,1)) \subset L_q$ and $u((-1,0]) \subset L_p$ (this implies in particular that there is a corner point at 0).

Then there exist

(1) an open neighborhood Ω of 0 in $\mathbb{D}^+$,

(2) a $\mathcal{C}^1$ chart

$$\psi : (\Omega, \Omega \cap \mathbb{R}) \to (\mathbb{D}^+, \mathbb{D}^+_{\mathbb{R}}),$$

(3) an linear isometry $B_u \in \mathcal{L}(\mathbb{R}^{\dim V_{\alpha_k}}, V_{\alpha_k})$ and a $\lambda_u \in \mathbb{R}^+$ such that

$$\pi_{\alpha_k}(u \circ \psi(z)) = \lambda_u e^{i\alpha_k} B_u \Big(z^{\frac{\pi - \alpha_k}{\pi} + m}, \tilde{U}(z) \Big)$$

with $\tilde{U}(z) = o\Big(z^{\frac{\pi-\alpha_k}{\pi} + m + \delta} \Big)$.

Moreover, if

$$U(z) = \sum_{\alpha \in \{\alpha_1, \dots, \alpha_n\} \setminus \{\alpha_k\}} \pi_\alpha(u \circ \psi(z))$$

we have

$$U(z) = o\Big(z^{\frac{\pi-\alpha_k}{\pi} + m + \delta} \Big), \ dU(z) = o\Big(z^{\frac{\pi-\alpha_k}{\pi} + m + \delta - 1} \Big).$$

Proof. As before, consider the curve $v = D_{-\alpha_1, \dots, -\alpha_n} u$. It satisfies the boundary condition $v([0,1)) \subset \mathbb{R}^n$ and $v((-1,0]) \subset e^{-i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{-i\alpha_n} \cdot \mathbb{R}$. Notice that the Kähler angles of the second boundary condition are $\pi - \alpha_1, \dots, \pi - \alpha_n$.

We now apply Proposition 1.2.12 to obtain an $\alpha_k \in (0, \pi)$, a $\lambda_u > 0$, a linear map B_u and a local $\mathcal{C}^\infty$ diffeomorphism ψ such that

$$\pi_{\alpha_k}(v(\psi(z))) = \lambda_u B_u \Big(z^{\frac{\pi-\alpha_k}{\pi} + m}, U(z) \Big)$$

with $U(z) = o(^{\frac{\pi-\alpha_k}{\pi} + m})$ and $\delta > 0$. Notice that $\pi_{\alpha_k}(D_{-\alpha_1, \dots, -\alpha_n} u) = e^{-i\alpha_k} \pi_{\alpha_k}(u)$, so

$$\pi_{\alpha_k}(u \circ \psi(z)) = \lambda_u e^{i\alpha_k} B_u \Big(z^{\frac{\pi-\alpha_k}{\pi} + m}, \tilde{U}(z) \Big).$$

□

1.2.2. The relative frame of the curve is a graph

In this subsection, we will explain how to adapt the argument of [Laz11] to show that given two finite-energy J-holomorphic curves with boundary on L, their relative frame is a $\mathcal{C}^1$-embedded graph.

First let us state [Laz11, Lemma 3.10]. The proof adapts without difficulty.

Proposition 1.2.16. *Recall that u_1 and u_2 are two finite-energy J-holomorphic curves with corners and boundary on L.*

Let

$$p_1 : S_1 \times S_2 \to S_1$$

be the projection onto the first factor. Then if $V_1 \subset S_1$ and $V_2 \subset S_2$, the map $p_1 : (V_1 \times V_2) \cap \mathcal{R}_{u_1}^{u_2} \to V_1$ is open if

(1) either $V_2 \subset \mathrm{Int}(S_2)$ is open,

(2) or V_1 is an open set such that $V_1 \cap \mathcal{W}(u_1, u_2) \subset \partial S_1$.

PROOF. Let $q_1 \mathcal{R}_{u_1}^{u_2} q_2$ and let $V_i \ni q_i$ be two open neighborhoods satisfying $V_2 \subset \mathrm{Int}(S_2)$ and $V_1 \cap \mathcal{W}(u_1, u_2) \subset \partial S_1$. Assume that the V_i are open half-disks or open disks and that $V_1 \backslash \{0\} \cap \mathcal{C}(u_1, u_2) = \emptyset$.

Up to reparameterization by $z \to u_1(\lambda z)$ with $\lambda > 0$ small enough, we can assume that if $(z_1, z_2) \in V_1 \times V_2$ is such that $u_1(z_1) = u_2(z_2)$, then $|z_2| \leqslant \frac{1}{2}$.

First, assume that z_1 is a corner point and z_2 is not. There are constants such that

$$|u_1(\lambda z)| \leqslant C_1 \lambda^{\frac{\alpha_i}{\pi}+m} |z|^{\frac{\alpha}{\pi}+m}, \ |u_2(z)| \geqslant C_2 |z|^k.$$

So if $u_2(z_2) = u_1(\lambda z_1)$, we have

$$C_2 |z|^{k_2} \leqslant C_1 |\lambda|^{\frac{\alpha_i}{\pi}+m} |z_1|^{\frac{\alpha_i}{\pi}+m}.$$

Hence

$$|z_2| \leqslant \left(\frac{C_1}{C_2} \lambda^{\frac{\alpha_i}{\pi}+m} \right)^{\frac{1}{k_2}}.$$

The right term goes to zero as $\lambda \to 0^+$. Therefore, the result is true for λ small enough.

Second, assume that z_2 is a corner point and z_1 isn't. Then there are constants such that

$$|u_1(\lambda z)| \leqslant C_1 \lambda^k |z|^k, \ |u_2(z)| \geqslant C_2 |z|^{\frac{\alpha}{\pi}+m}.$$

So if $u_2(z_2) = u_1(\lambda z_1)$, we have

$$C_2 |z_2|^{\frac{\alpha}{\pi}+m} \leqslant |\lambda|^k |z_1|^k,$$

hence

$$|z_2| \leqslant \left(\frac{C_1}{C_2} \lambda^k \right)^{\frac{1}{\frac{\alpha}{\pi}+m}}.$$

The right term goes to zero as $\lambda \to 0^+$. Therefore the result is true for $\lambda > 0$ small enough.

Last assume that both z_1 and z_2 are corner points. Then there are constants such that

$$|u_1(\lambda z)| \leqslant C_1 \lambda^{\frac{\alpha_1}{\pi}+m_1} |z|^{\frac{\alpha_1}{\pi}+m_1}, \ |u_2(z)| \geqslant C_2 |z|^{\frac{\alpha_2}{\pi}+m_2}.$$

So if $u_2(z_2) = u_1(\lambda z_1)$, we have

$$C_2 |z_2|^{\frac{\alpha_2}{\pi}+m_2} \leqslant |\lambda|^{\frac{\alpha_1}{\pi}+m_1} |z_1|^{\frac{\alpha_1}{\pi}+m_1}.$$

Hence
$$|z_2| \leqslant \left(\frac{C_1}{C_2}\lambda^{\frac{\alpha_1}{\pi}+m_1}\right)^{\frac{1}{\frac{\alpha_2}{\pi}+m_2}}.$$
The right term goes to zero as $\lambda \to 0^+$. Therefore the result is true for $\lambda > 0$ small enough.

Now, let $\Omega = \mathcal{R}_{u_1}^{u_2}(V_2) \cap (\mathrm{Int}(V_1)\backslash\{0\}) \subset \mathrm{Int}(V_1\backslash\{0\}) \cup \partial S_1$.

- We have that $\Omega \neq \emptyset$. Indeed, there are sequences $(q_{1,\nu})$ and $(q_{2,\nu})$ with values in $\mathrm{Int}(S_1)\backslash\mathcal{C}(u_1,u_2)$ and $\mathrm{Int}(S_2)\backslash\mathcal{C}(u_2,u_1)$ such that $q_{1,\nu} \to q_1$, $q_{2,\nu} \to q_2$ and $q_{1,\nu} \neq q_1$. Now for ν large enough, $q_{1,\nu} \in V_1\backslash\{q_1\}$ and $q_{2,\nu} \in V_2$.
- The set Ω is open in $\mathrm{Int}(V_1)$: if $z_1 \in \Omega$, then $z_1 \notin \mathcal{C}(u_1,u_2)$. Let $z_2 \in V_2$ be such that $z_1\mathcal{R}_{u_1}^{u_2}z_2$. Then $z_2 \in \mathrm{Int}(V_2)$ since if $z_2 \in \partial S_2$ we have $z_1 \in \partial S_2$ which is a contradiction. Moreover, $du_1(z_1) \neq 0$ and $du_2(z_2) \neq 0$. So the restrictions of the two curves to small enough open neighborhoods of z_1 and z_2 are reparameterizations of each other.
- The set Ω is closed in $\mathrm{Int}(V_1)\backslash\{0\}$ since if $z_{1,\nu} \to z \in \mathrm{Int}(V_1)\backslash\{0\}$, there is $z_{2,\nu} \in V_2$ such that $z_{1,\nu}\mathcal{R}_{u_1}^{u_2}z_{2,\nu}$. One can assume that the sequence $(z_{2,\nu})$ converges to z_2. Since $|z_2| \leqslant \frac{1}{2}$, we get $z_2 \in V_2$.

Hence $\Omega = \mathrm{Int}(V_1)\backslash\{0\}$ and the result follows by taking the closure of this in V_1 and V_2 since $\mathcal{R}_{u_1}^{u_2}$ is closed. □

Let us also recall a characterization of simple curves with corners and boundary on L.

Proposition 1.2.17. *Let $u : (S, \partial S) \to (M, L)$ be a finite-energy J-holomorphic curve with corners and boundary in L.*

The curve u is simple if and only if $\mathcal{R}_u^u$ is the trivial relation.

Proof. If $\mathcal{R}_u^u$ is non-trivial, it is easy to show that u is not simple : see [Laz11, Corollary 3.16].

Assume that $\mathcal{R}_u^u = \Delta$ and let $\mathcal{N} = \{z \in \mathrm{Int}(S)\backslash\mathcal{C}(u,u) | \#u^{-1}(u(z)) \geqslant 2\}$. Suppose that $z_{1,\nu} \to z_1 \in \mathcal{N}$ and $u(z_{1,\nu}) = u(z_{2,\nu})$ with $z_{2,\nu} \to z_2 \in S$ and $z_{1,\nu} \neq z_{2,\nu}$. Then since $z_1 \notin \mathcal{C}_{u,u}$, we have that $u_1(z_1) \notin i(R)$, hence by [Laz11], $z_1\mathcal{R}_u^u z_2$ and so $z_1 = z_2$. This is a contradiction since $du(z_1) \neq 0$ and u is locally injective around z_1. □

For the remaining part of this subsection, we fix u_1 and u_2 two finite-energy J-holomorphic curves with corners and boundary on L. We now explain how to prove that the set $\mathcal{W}(u_1,u_2)$ is a $\mathcal{C}^1$ embedded graph. The proof is still an adaptation of [Laz00] and [Laz11] with special care given to corner points.

Let us fix $z_1 \in S_1$ and $z_2 \in \partial S_2$ such that $z_1\mathcal{R}_{u_1}^{u_2}z_2$. We will show that the desired result holds locally around z_1.

There are several cases to consider depending on the type of the points z_1 and z_2. The proofs of all of these follow a variation of the same scheme (and are therefore interchangeable). Namely

(1) For $i = 1, 2$ we find an expression of u_i around z_i of the type

$$u_i(z) = \lambda_i A_i(1,0) z^{c_i} + A_i(0, U_i(z))$$

with c_i a positive real number and $U_i(z) = o(z^{c_i})$ (for this we apply one of the Propositions 1.2.14, 1.2.13, 1.2.15, 1.2.12 according to the type of z_i).

(2) Since there are sequences $z_{1,\nu} \to z_2$ and $z_{2,\nu} \to z_2$ which satisfy $u_1(z_{1,\nu}) = u_2(z_{2,\nu})$, we deduce that $A_1(1,0)$ and $A_2(1,0)$ are dependent over $\mathbb{C}$ and lie in the complexification of V_{α_p} for some p.

(3) We then use the complexification of the standard scalar product to conclude that $\mathcal{W}(u_1, u_2)$ is included in a union of rays.

Lemma 1.2.18. *In the above setting, assume that $z_1 \in \partial S_1$ and that z_1 and z_2 are not corner points.*

Moreover, we suppose that $u_1(z_1) = u_2(z_2) = i(p) = i(q)$ is a double point, that u_1 has boundary condition along the branch L_p around z_1 and u_2 has boundary condition along the branch L_q around z_2.

Then there is an open neighborhood Ω of z_1 such that $\mathcal{W}(u_1, u_2) \cap \Omega$ is a $\mathcal{C}^1$-embedded graph in Ω.

Proof. Using Propositions 1.2.8, 1.2.13, and 1.2.14, we can assume that

(1) u_1 and u_2 have values in $\mathbb{C}^n$,

(2) L_p is given by $\mathbb{R}^n$ and L_q is given by $e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R}$,

(3) there are local $\mathcal{C}^1$ diffeomorphisms ψ_1 and ψ_2 around z_1 and z_2 respectively with images Ω_1 and Ω_2 such that

$$\begin{aligned} u_1(\psi_1(z)) &= \lambda_1 A\big(z^k, U(z)\big) \\ u_2(\psi_2(z)) &= \lambda_2 D_{\alpha_1,\ldots,\alpha_n} B\Big(z^m, \tilde{U}(z)\Big). \end{aligned}$$

Replacing Ω_1 and Ω_2 by smaller neighborhoods if necessary, we can assume that

$$\mathcal{C}(u_1, u_2) \cap \Omega_1 \subset \{0\}.$$

We claim that there is a complex $\mu \in \mathbb{C} \setminus \{0\}$ such that $\mu A(1,0) = D_{\alpha_1,\ldots,\alpha_n} B(1,0)$ and that there is an α_k such that $A(1,0) \in V_{\alpha_k}$.

Since $z_1 \mathcal{R}_{u_1}^{u_2} z_2$, there are sequences $(z_{1,\nu})$ and $(z_{2,\nu})$ of points distinct from z_1 and z_2 such that $z_{1,\nu} \to z_1$ and $z_{2,\nu} \to z_2$ and $u_1(z_{1,\nu}) = u_2(z_{2,\nu})$. There are $\mu \in \mathbb{C}$ and $v \in \mathbb{R}^{n-1}$ such that $D_{\alpha_1,\ldots,\alpha_n} B(1,0) = \mu A(1,0) + A(0, v)$.

If by contradiction $\mu = 0$, from the equality

$$\lambda_1 z_{1,\nu}^k A(1,0) + \lambda_1 A(0, U(z_{1,\nu})) = \lambda_2 z_{2,\nu}^m A(0,v) + \lambda_2 D_{\alpha_1,\ldots,\alpha_n} B(0,\tilde{U}),$$

we get $z_{1,\nu}^k = o(z_{2,\nu}^m)$. Therefore, we would have $u_1(z_{1,\nu}) = o(z_{2,\nu}^m) = o(u_2(z_{2,\nu^m}))$. This is of course a contradiction.

From this, we deduce $\lambda_1 z_{1,\nu}^k \sim \mu\lambda_2 z_{2,\nu}^m$. Denote by $\pi : \mathbb{R}^n \to \mathbb{R}^n$ the real orthogonal projection onto $A(1,0)^\perp$. Since A is orthogonal, we get

$$o(z_{1,\nu}^k) = \pi(u_1(z_{1,\nu})) = \pi(u_2(z_{2,\nu})) = z_{2,\nu}^m \pi(D_{\alpha_1,\ldots,\alpha_n} B(1,0) + o(z_{2,\nu})).$$

Hence, $\pi(D_{\alpha_1,\ldots,\alpha_n} B(1,0)) = 0$.

Moreover, we have $\mu A(1,0) = D_{\alpha_1,\ldots,\alpha_n} B(1,0) \in e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R}$ and $A(1,0) \in \mathbb{R}^n$, so $A(1,0) \in V_{\alpha_p}$ for some p. We conclude that μ has argument α_k mod π.

Assume that $z \in \Omega_1 \cap \mathcal{W}(u_1,u_2)$. Then $u(z) \in L_q$. Denoting by $\langle \cdot, \cdot \rangle$ the complexification of the usual scalar product on $\mathbb{R}^n$, we have

$$\lambda_1 z^k = \langle u_1(z), A(1,0) \rangle.$$

Since $u_1(z) \in L_q$ and $A(1,0) \in V_{\alpha_k}$, we have $\langle u_1(z), A(1,0) \rangle \in e^{i\alpha_p} \cdot \mathbb{R}$, so $z^k \in e^{i\alpha_p} \cdot \mathbb{R}$.

We conclude that $\mathcal{W}(u_1,u_2) \subset A$ where A is the union of rays given by

$$A := \left(\bigcup_q e^{i\frac{\alpha_p}{k} + i\frac{2\pi q}{p}} \cdot \mathbb{R}_+ \right) \cup \left(\bigcup_q e^{i\frac{\alpha_p}{k} + i\frac{(2q+1)\pi}{p}} \cdot \mathbb{R}_+ \right).$$

We claim that the frame $\mathcal{W}(u_1,u_2) \backslash \{0\}$ is a (possibly empty) union of connected components of $A \backslash \{0\}$. We prove this by showing that it is an open and closed subset of $A \backslash \{0\}$.

Notice that $\mathcal{W}(u_1,u_2) = \mathcal{R}(\partial S_2)$ is closed, since $\mathcal{R}_{u_1}^{u_2}$ and ∂S_2 are both closed. We conclude that $\mathcal{W}(u_1,u_2) \backslash \{0\}$ is closed in $A \backslash \{0\}$.

Since $\Omega_2 \cap \mathcal{C}(u_1,u_2) \subset \{0\}$ any point of $\mathcal{W}(u_1,u_2) \backslash \{0\}$ is not in $\mathcal{C}(u_1,u_2)$. Therefore, we can apply the proof of [**Laz11**, Theorem 3.18] to conclude that $\mathcal{W}(u_1,u_2) \backslash \{0\}$ is open in $A \backslash \{0\}$. □

Lemma 1.2.19. *In the above setting, assume that $z_1 \in \partial S_1$ and that z_1 is not a corner point but z_2 is.*

Moreover we suppose that $u_1(z_1) = u_2(z_2) = i(p) = i(q)$ and that u_1 has boundary condition along the branch L_p around z_1.

Then there is an open neighborhood Ω of z_1 such that $\mathcal{W}(u_1,u_2) \cap \Omega$ is a $\mathcal{C}^1$-embedded graph in Ω.

Proof. The curve u_2 can have two different types of boundary conditions. Accordingly, we will consider two different cases.

First Case: By the Propositions 1.2.8, 1.2.12, and 1.2.13 we can assume that

(1) the maps u_1 and u_2 have values in $\mathbb{C}^n$,
(2) the branch L_p is given by $\mathbb{R}^n$ and L_q is given by $e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R}$,
(3) there are local $\mathcal{C}^1$ diffeomorphisms ψ_1 and ψ_2 around z_1 and z_2 respectively with images Ω_1 and Ω_2 such that

$$u_1(\psi_1(z)) = \lambda_1 A_1(z^p, U_1(z))$$

with $U_1(z) = o(z^p)$,

$$\pi_{\alpha_k}(u_2 \circ \psi_2(z)) = \lambda_2 A_2\Big(z^{\frac{\alpha_k}{\pi}+m}, \tilde{U}_2(z)\Big)$$

with $\tilde{U}_2(z) = o\Big(z^{\frac{\alpha_k}{\pi}+m+\delta}\Big)$ and

$$\sum_{\alpha \in \{\alpha_1,\ldots,\alpha_n\}\setminus\{\alpha_k\}} \pi_\alpha(u_2 \circ \psi_2(z)) = o\Big(z^{\frac{\alpha_k}{\pi}+m+\delta}\Big).$$

Moreover, we can assume that $\mathcal{C}(u_1, u_2) \cap \Omega_1 \subset \{0\}$.

We claim that $A_1(1,0) \in V_{\alpha_k}$ and that there is a $\mu \neq 0$ such that $A_1(1,0) = \mu A_2(1,0)$.

Since $z_1 \mathcal{R}_{u_1}^{u_2} z_2$, there are two sequences $z_{1,\nu} \neq z_1$ and $z_{2,\nu} \neq z_2$ such that $z_{1,\nu} \to z_1$, $z_{2,\nu} \to z_2$ and $u_1(z_{1,\nu}) = u_2(z_{2,\nu})$. Moreover, we let $\mu \in \mathbb{R}$ and v be a real vector such that $\pi_{\alpha_k}(A_1(1,0)) = \mu A_2(1,0) + A_2(0,v)$. If, by contradiction, $\mu = 0$, from the equality

$$\lambda_1 z_{1,\nu}^p \pi_{\alpha_k} A_1(1,0) + \lambda_1 \pi_{\alpha_k} A_1(0, U_1(z_{1,\nu})) = \lambda_2 A_2(z_{2,\nu}^{\frac{\alpha_k}{\pi}+m}, \tilde{U}_2(z_{2,\nu})),$$

we get

$$\lambda_1 z_{1,\nu}^p A_2(0,v) + \lambda_1 \pi_{\alpha_k} A_1(0, U_1(z_{1,\nu})) = \lambda_2 A_2(z_{2,\nu}^{\frac{\alpha_k}{\pi}+m}, \tilde{U}_2(z_{2,\nu})),$$

so $z_{1,\nu}^p = o\Big(z_{2,\nu}^{\frac{\alpha_k}{\pi}+m}\Big)$. Hence, $u_1(z_{1,\nu}) = o\Big(z_{2,\nu}^{\frac{\alpha_k}{\pi}+m}\Big) = o(u_2(z_{2,\nu}))$, which is a contradiction.

In particular, we can deduce that $\mu\lambda_1 z_{1,\nu}^p \sim \lambda_2 z_{2,\nu}^{\frac{\alpha_k}{\pi}+m}$. Denote by $\pi : V_{\alpha_k} \to V_{\alpha_k}$ the (real) orthogonal projection onto $A_2(1,0)^\perp$. Since A_2 is orthogonal, we get

$$\lambda_1 z_{1,\nu}^p A_2(0,v) + \lambda_1 \pi \circ \pi_{\alpha_k} A_1(0, U_1(z_{1,\nu})) = \lambda_2\Big(0, \tilde{U}_2(z_{2,\nu})\Big).$$

Hence, if $v \neq 0$, we have $z_{1,\nu}^p = o\Big(z_{2,\nu}^{\frac{\alpha_k}{\pi}+m+\delta}\Big)$, which is a contradiction.

Assume that $z \in \Omega_1 \cap \mathcal{W}(u_1, u_2)$, then we have $u_1(z) \in \mathbb{R}^n \cup e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R}$. Hence, from

$$\langle A_1(1,0), u_1(z)\rangle = z^p,$$

and the fact that $A_1(1,0) \in V_{\alpha_k}$, we deduce that $z^p \in \mathbb{R} \cup e^{i\alpha_k} \cdot \mathbb{R}$.

So $z \in A$ where A is the union of arcs given by

$$A = \bigcup_{q=0}^{p} e^{i\frac{q\pi}{p}} \cdot \mathbb{R}_+ \cup \bigcup_{q=0}^{E\left(p-\frac{\alpha_k}{\pi}\right)} e^{i\frac{\alpha_k+q\pi}{p}} \cdot \mathbb{R}_+.$$

Now we show that the frame $(\mathcal{W}(u_1,u_2)\cap\Omega_1)\backslash\{0\}$ is a (possibly empty) union of connected components of $A\backslash\{0\}$.

Indeed it is closed in $A\backslash\{0\}$ and $\mathcal{W}(u_1,u_2)=\mathcal{R}(\partial S_2)$ is closed.

Since $\Omega_2\cap\mathcal{C}(u_1,u_2)\subset\{0\}$ any point of $\mathcal{W}(u_1,u_2)\backslash\{0\}$ is not in $\mathcal{C}(u_1,u_2)$. Therefore, we can apply the proof of [**Laz11**, Theorem 3.18] to conclude that $\mathcal{W}(u_1,u_2)\backslash\{0\}$ is open in $A\backslash\{0\}$.

Second Case: This is practically the same as the first case. We explain the differences. This time Proposition 1.2.15 implies that we can assume

$$\pi_{\alpha_k}(u_2\circ\psi_2(z)) = \lambda_2 e^{i\alpha_k} A_2\left(z^{\frac{\pi-\alpha_k}{\pi}+m}, U_2(z)\right)$$

with $U_2(z) = o\left(z^{\frac{\pi-\alpha_k}{\pi}+m\delta}\right)$ and

$$\sum_{\alpha\in\{\alpha_1,\ldots,\alpha_n\}\backslash\{\alpha_k\}} \pi_\alpha(u_2\circ\psi_2(z)) = o\left(z^{\frac{\pi-\alpha_k}{\pi}+m+\delta}\right).$$

The same argument as in case 1 shows that $A_1(1,0)\in V_{\alpha_k}$ and that there is a real $\mu\neq 0$ such that $A_1(1,0)=\mu A_2(1,0)$.

Assume that $z\in\Omega_1\cap\mathcal{W}(u_1,u_2)$. Then from $u_1(z)\in\mathbb{R}^n\cup e^{i\alpha_1}\cdot\mathbb{R}\times\ldots\times e^{i\alpha_n}\cdot\mathbb{R}$ and $\langle A_1(1,0),u_1(z)\rangle = z^p$ we deduce $z^p\in\mathbb{R}\cup e^{i\alpha_k}\mathbb{R}$. Hence $z\in A$.

Now the proof is the same as in the first case. □

Lemma 1.2.20. *In the above setting, assume that $z_1\in\partial S_1$ and that z_1 is a corner point but z_2 is not.*

Moreover, we suppose that u_2 has boundary condition along the branch L_p around z_2.

Then there is an open neighborhood Ω of z_1 such that $\mathcal{W}(u_1,u_2)\cap\Omega$ is a $\mathcal{C}^1$-embedded graph in Ω.

Proof. Exchanging the roles of u_1 and u_2, the proof is the same as in Lemma 1.2.19. □

Lemma 1.2.21. *In the above setting, assume that $z_1\in\partial S_1$ and that both z_1 and z_2 are corner points.*

Moreover, we suppose that u_1 and u_2 have boundary condition along L_p followed by L_q around z_1 and z_2 respectively.

Then there is an open neighborhood Ω of z_1 such that $\mathcal{W}(u_1,u_2)\cap\Omega$ is a $\mathcal{C}^1$-embedded graph in Ω.

PROOF. We can assume by Propositions 1.2.8 and 1.2.12, that

(1) the maps u_1 and u_2 have values in $\mathbb{C}^n$,
(2) the branch L_p is given by $\mathbb{R}^n$ and L_q is given by $e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R}$,
(3) there are local $\mathcal{C}^1$ diffeomorphisms ψ_1 and ψ_2 around z_1 and z_2 respectively with images Ω_1 and Ω_2 such that

$$\pi_{\alpha_k}(u_1(z)) = \lambda_1 A_1\Big(z^{\frac{\alpha_k}{\pi}+m}, U_1(z)\Big)$$

with $U_1(z) = o\Big(z^{\frac{\alpha_k}{\pi}+m+\delta}\Big)$ and

$$\sum_{\alpha \in \{\alpha_1,\ldots,\alpha_n\}\backslash\{\alpha_k\}} \pi_\alpha(u_1(z)) = o\Big(z^{\frac{\alpha_k}{\pi}+m+\delta}\Big).$$

Moreover,

$$\pi_{\alpha_l}(u_2(z)) = \lambda_2 A_2\Big(z^{\frac{\alpha_l}{\pi}+p}, U_2(z)\Big)$$

with $U_2(z) = o\Big(z^{\frac{\alpha_l}{\pi}+p+\delta}\Big)$ and

$$\sum_{\alpha \in \{\alpha_1,\ldots,\alpha_n\}\backslash\{\alpha_l\}} \pi_\alpha(u_2(z)) = o\Big(z^{\frac{\alpha_l}{\pi}+p+\delta}\Big).$$

Furthermore, there are two non-zero sequences $(z_{1,\nu})$ and $(z_{2,\nu})$ which converge to 0 such that $u_1(z_{1,\nu}) = u_2(z_{2,\nu})$.

First we can easily see that $\alpha_k = \alpha_p$. Assume the opposite. Then

$$o\Big(z_{2,\nu}^{\frac{\alpha_l}{\pi}+p+\delta}\Big) = \pi_{\alpha_k}(u_2(z_{2,\nu})) = \pi_{\alpha_k}(u_1(z_{1,\nu})) = \lambda_1 A_1\Big(z_{1,\nu}^{\frac{\alpha_k}{\pi}+m}, U_1(z_{1,\nu})\Big).$$

So $z_{1,\nu}^{\frac{\alpha_k}{\pi}+m} = o\Big(z_{2,\nu}^{\frac{\alpha_l}{\pi}+p+\delta}\Big)$. Exchanging the roles of α_k and α_l, we get that $z_{2,\nu}^{\frac{\alpha_l}{\pi}+p} = o\Big(z_{1,\nu}^{\frac{\alpha_k}{\pi}+m+\delta}\Big)$, a contradiction.

As usual, we claim that there is $\mu \in \mathbb{R}\backslash\{0\}$ such that $A_1(1,0) = \mu A_2(1,0)$.

Let $\mu \in \mathbb{R}$ and v be a vector such that $A_1(1,0) = \mu A_2(1,0) + A_2(0,v)$. Assume by contradiction that $\mu = 0$. Then since

$$\lambda_1 z_{1,\nu}^{\frac{\alpha_k}{\pi}+m} A_2(0,v) + \lambda_1 A_1(0, U_1(z_{1,\nu})) = \lambda_2 A_2\Big(z_{2,\nu}^{\frac{\alpha_l}{\pi}+p}, U_2(z_{2,\nu})\Big),$$

and since A_2 is an isometry, we get $z_{1,\nu}^{\frac{\alpha_k}{\pi}+m} = o\Big(z_{2,\nu}^{\frac{\alpha_l}{\pi}+p}\Big)$. Hence, $u_1(z_{1,\nu}) = o(u_2(z_{2,\nu}))$, a contradiction.

In particular $\mu\lambda_1 z_{1,\nu}^{\frac{\alpha_k}{\pi}+m} \sim \lambda_2 z_{2,\nu}^{\frac{\alpha_l}{\pi}+p}$. Moreover, applying the (complexified) orthogonal projection onto $A_2(1,0)^\perp$, we get

$$\lambda_1 z_{1,\nu}^{\frac{\alpha_k}{\pi}+m} A_2(0,v) + \lambda_1 A_1(0, U_1(z_{1,\nu})) = o\Big(z_{2,\nu}^{\frac{\alpha_l}{\pi}+p+\delta}\Big).$$

This implies $A_2(0,v) = 0$.

Now assume that $z \in \mathcal{W}(u_1, u_2)$. Then $u_1(z) \in \mathbb{R}^n \cup e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R}$. As usual, we take the scalar product of $\pi_{\alpha_k} u_1(z)$ with $A_1(1,0)$ to obtain that $z^{\frac{\alpha_k}{\pi}+m} \in \mathbb{R} \cup e^{i\alpha_k} \cdot \mathbb{R}$. Therefore $z \in A$, where A is the set

$$A := \left(\bigcup_{q=0}^{m} e^{i\frac{\frac{\alpha_k}{\pi}+q\pi}{\frac{\alpha_k}{\pi}+m}} \cdot \mathbb{R}_+ \right) \cup \left(\bigcup_{q=0}^{E(\frac{\alpha_k}{\pi}+m)} e^{i\frac{q\pi}{\frac{\alpha_k}{\pi}+m}} \cdot \mathbb{R}_+ \right).$$

We now show that the frame $(\mathcal{W}(u_1, u_2) \cap \Omega_1) \backslash \{0\}$ is a (possibly empty) union of connected components of $A \backslash \{0\}$.

It is closed in $A \backslash \{0\}$ since $\mathcal{W}(u_1, u_2)$ is closed.

Since $\Omega_2 \cap \mathcal{C}(u_1, u_2) \subset \{0\}$ any point of $\mathcal{W}(u_1, u_2) \backslash \{0\}$ is not in $\mathcal{C}(u_1, u_2)$. Therefore, we can apply the proof of [**Laz11**, Theorem 3.18] to conclude that $\mathcal{W}(u_1, u_2) \backslash \{0\}$ is open in $A \backslash \{0\}$. $\square$

Lemma 1.2.22. *In the above setting, assume that $z_1 \in \mathrm{Int}(S_1)$ and that z_2 is a corner point.*

Then there is an open neighborhood Ω of z_1 such that $\mathcal{W}(u_1, u_2) \cap \Omega$ is a $\mathcal{C}^1$-embedded graph in Ω.

Proof. Using Propositions 1.2.8, 1.2.12 and 1.2.9, we can assume that

(1) the maps u_1 and u_2 have values in $\mathbb{C}^n$,
(2) the branch L_p is given by $\mathbb{R}^n$ and the branch L_q is given by $e^{i\alpha_1} \cdot \mathbb{R} \times \ldots \times e^{i\alpha_n} \cdot \mathbb{R}$,
(3) there are local $\mathcal{C}^1$-diffeomorphisms ψ_1 and ψ_2 around z_1 and z_2 respectively with images Ω_1 and Ω_2 such that

$$u_1(\psi_1(z)) = \lambda_1 A_1\big(z^k, U_1(z)\big),$$

with $U_1(z) = O\big(z^{k+1}\big)$ and

$$\pi_{\alpha_l} \circ u_2(\psi_2(z)) = \lambda_2 A_2\Big(z^{\frac{\alpha_l}{\pi}+m}, U_2(z)\Big),$$

with $U_2(z) = o\Big(z^{\frac{\alpha_l}{\pi}+m+\delta}\Big)$ and

$$\sum_{\alpha \in \{\alpha_1, \ldots, \alpha_n\} \backslash \{\alpha_l\}} \pi_\alpha \circ u_2(\psi_2(z)) = o\Big(z^{\frac{\alpha_l}{\pi}+m+\delta}\Big).$$

Replacing Ω_1 and Ω_2 by smaller neighborhoods if necessary, we can assume that

$$\mathcal{C}(u_1, u_2) \cap \Omega_1 \subset \{0\}.$$

Moreover, since $z_1 \mathcal{R}_{u_1}^{u_2} z_2$, there are non-zero sequences $(z_{1,\nu})$ and $(z_{2,\nu})$ converging to 0 such that $u_1(\psi_1(z_{1,\nu})) = u_2(\psi_2(z_{2,\nu}))$.

First, we easily see that for $p \neq l$, $\pi_{\alpha_p} A_1(1,0) = 0$. Indeed, assume by contradiction that there is p such that $\pi_{\alpha_p} A_1(1,0) \neq 0$. Apply the projection π_{α_p} to the equality above to get

$$o\Big(z_{2,\nu}^{\frac{\alpha_l}{\pi}+m}\Big) = \lambda_1 z_{1,\nu}^k \pi_{\alpha_p} A_1(1,0) + o\big(z_{1,\nu}^k\big).$$

So necessarily $z_{1,\nu}^k = o\Big(z_{2,\nu}^{\frac{\alpha_l}{\pi}+m}\Big)$ and $u_2(z_{2,\nu}) = u_1(z_{1,\nu}) = o\Big(z_{2,\nu}^{\frac{\alpha_l}{\pi}+m}\Big)$, a contradiction.

Now, we claim that there is a $\lambda \in \mathbb{C}\backslash\{0\}$ such that $A_1(1,0) = \lambda A_2(1,0)$. To see this, let $\lambda \in \mathbb{C}$ and v be a complex vector such that

$$\pi_{\alpha_l}(A_1(1,0)) = \lambda A_2(1,0) + A_2(0,v).$$

If $\lambda = 0$, we apply the complexification of the real scalar product with $A_2(1,0)$ to get

$$o\big(z_{1,\nu}^k\big) = \lambda_2 z_{2,\nu}^{\frac{\alpha_l}{\pi}+m} + o\Big(z_{2,\nu}^{\frac{\alpha_l}{\pi}+m}\Big).$$

so $z_{2,\nu}^{\frac{\alpha_l}{\pi}+m} = o\big(z_{1,\nu}^k\big)$ and $u_2(z_{2,\nu}) = o(u_1(z_{1,\nu}))$, a contradiction.

In particular, we have $\lambda\lambda_1 z_{1,\nu}^k \sim \lambda_2 z_{2,\nu}^{\frac{\alpha_l}{\pi}+m}$.

Assume that π is the real orthogonal projection onto $A_2(1,0)^{\perp}$. We apply its complexification to get

$$z_1^k \lambda\lambda_1 A_2(0,v) = o\Big(z_{2,\nu}^{\frac{\alpha_k}{\pi}+m}\Big).$$

This shows that $A_2(0,v) = 0$.

Now assume that $z \in \mathcal{W}(u_1,u_2) \cap \Omega_1$, so $u_1(z) \in (e^{i\alpha_1} \cdot \mathbb{R} \times \ldots e^{i\alpha_n} \cdot \mathbb{R}) \cup \mathbb{R}^n$. Denote by h_{std} the standard hermitian scalar product (which is complex linear in the first variable). Then, since $A_1 \in U(n)$,

$$\begin{aligned} h_{\mathrm{std}}(u_1(z), A_1(1,0)) &= h_{\mathrm{std}}\big(\lambda_1 z^k A_1(1,0) + A_1(0, U_1(z)), A_1(1,0)\big) \\ &= \lambda_1 z^k. \end{aligned}$$

Recall that $\lambda^{-1} A_1(1,0) \in V_{\alpha_l}$. For $v \in e^{i\alpha_1} \cdot \mathbb{R} \times \ldots e^{i\alpha_n} \cdot \mathbb{R}$, we have $h_{\mathrm{std}}\Big(v, \frac{A_1(1,0)}{\lambda}\Big) \in e^{i\alpha_l} \cdot \mathbb{R}$. For $v \in \mathbb{R}^n$, $h_{\mathrm{std}}\Big(v, \frac{A_1(1,0)}{\lambda}\Big) \in \mathbb{R}$.

In the end, we conclude that $z^k \in \overline{\lambda} \cdot \mathbb{R} \cup \overline{\lambda} e^{i\alpha_l} \cdot \mathbb{R}$. Call $\theta_\lambda \in [0,\pi]$ an argument of $\overline{\lambda}$ (resp. $-\overline{\lambda}$) if $Im(\overline{\lambda}) > 0$ (resp. $Im(\overline{\lambda}) < 0$). Then $z \in A$ where A is a union of half-rays with extremities at 0,

$$A := \bigcup_{p=0}^{2k} e^{i\frac{p\pi}{k}} \cdot \mathbb{R}_+ \cup \bigcup_{p=E\left(-\frac{\theta_\lambda+\alpha_l}{\pi}\right)}^{E\left(2k-\frac{\theta_\lambda+\alpha_l}{\pi}\right)} e^{i\frac{p\pi+\theta_\lambda+\alpha_l}{k}} \cdot \mathbb{R}_+.$$

Now we show that the frame $(\mathcal{W}(u_1,u_2) \cap \Omega_1)\backslash\{0\}$ is a (possibly empty) union of connected components of $A\backslash\{0\}$.

Indeed it is closed in $A\backslash\{0\}$ since $\mathcal{W}(u_1,u_2) = \mathcal{R}_{u_1}^{u_2}(\partial S_2)$ is closed.

Since $\Omega_2 \cap \mathcal{C}(u_1,u_2) \subset \{0\}$ any point of $\mathcal{W}(u_1,u_2)\backslash\{0\}$ is not in $\mathcal{C}(u_1,u_2)$. Therefore, we can apply the proof of [**Laz11**, Theorem 3.18] to conclude that $\mathcal{W}(u_1,u_2)\backslash\{0\}$ is open in $A\backslash\{0\}$. □

With the cases already proved in [**Laz11**], we readily conclude that the following proposition holds.

Proposition 1.2.23. *Recall that $u_1 : (S_1, \partial S_1) \to (M, i(L))$ and $u_2 : (S_2, \partial S_2) \to (M, i(L))$ are two finite energy J-holomorphic curves with corners and boundary in L.*

Let $\mathcal{D}(u_1,u_2)$ be the set of isolated points of $\mathcal{W}(u_1,u_2)$. The following is true

(i) $\mathcal{D}(u_1,u_2) \subset \mathcal{C}(u_1,u_2) \cap \partial S_1$,

(ii) $\mathcal{W}(u_1,u_2)\backslash\mathcal{D}(u_1,u_2)$ is a $\mathcal{C}^1$-embedded graph in S_1, its vertices are in $\mathcal{C}(u_1,u_2)$.

(iii) $\overline{\mathcal{W}}(u_1,u_2)$ is a $\mathcal{C}^1$-embedded graph.

Proof. First, note that *(iii)* follows immediately from *(i)* and *(ii)*. We shall see that (ii) follows from the lemmas proved above.

Let $z \in \mathcal{W}(u_1,u_2) \cap \mathrm{Int}(S_1)$ so there is $z_2 \in \partial S_2$ such that $z\mathcal{R}_{u_1}^{u_2}z_2$. Assume first that z_2 is not a corner point. It follows from the proof of [**Laz11**, Theorem 3.18] that $\mathcal{W}(u_1,u_2)$ is a $\mathcal{C}^1$-graph around z. The same results holds if z_2 is a corner point, this is the content of Lemma 1.2.22.

We prove, using that the frame relation is open in some cases (Proposition 1.2.16), that z is not isolated in $\mathcal{W}(u_1,u_2)$. To see this, pick an open neighborhood V_2 of z_2 such that $V_2 \cap u_2^{-1}(z_2) = \{z_2\}$ and a decreasing sequence of neighborhoods $V_{1,\nu} \subset \mathrm{Int}(S_1)$ such that $\{z\} = \cap_\nu V_{1,\nu}$. For $\nu \in \mathbb{N}$, the projection $(V_{1,\nu} \times V_2) \cap \mathcal{R}_{u_1}^{u_2} \to V_2$ is open (since $V_{1,\nu} \subset \mathrm{Int}(S_1)$). Hence, there are $z_{2,\nu} \neq z_2 \in \partial S_2$ and $z_{1,\nu} \in V_{1,\nu}$ with $z_{1,\nu}\mathcal{R}_{u_1}u_2 z_{2,\nu}$. Necessarily $z_{1,\nu} \neq z_1$ (otherwise $u_1(z_1) = u_2(z_{2,\nu})$ which would yield $z_2 \in u_2^{-1}(u_2(z_2)))$. We conclude that z is an accumulation point of $\mathcal{W}(u_1,u_2)$.

Now assume that $z \in \mathcal{W}(u_1,u_2) \cap \partial S_1$ and and that $z \notin \mathcal{C}(u_1,u_2)$. Pick z_2 such that $z\mathcal{R}_{u_1}^{u_2}z_2$. In particular z_2 is not a corner point, $du_1(z) \neq 0$ and $du_2(z_2) \neq 0$. We can apply [**Laz11**, Proposition 3.13] : there are open neighborhoods ω_1 and ω_2 of z and z_2 respectively such that $\phi(\omega_2 \cap \partial S_2) = \omega_1 \cap \partial S_1$ and $z\mathcal{R}_{u_1}^{u_2}z'$ if and only if $z = \phi(z')$. Therefore $\omega_1 \cap \mathcal{W}(u_1,u_2) = \partial S_1 \cap \omega_1$: the frame is a local $\mathcal{C}^1$-graph around z and z is not isolated in $\mathcal{W}(u_1,u_2)$.

Assume that z is not a corner point, then Lemma 1.2.19 if z_2 is a corner point, and Lemma 1.2.18 if z_2 is not, show that $\mathcal{W}(u_1,u_2)$ is a graph around z.

If z is a corner point, Lemma 1.2.20 if z_2 is not a corner point, and Lemma 1.2.21 if z_2 is, show again that $\mathcal{W}(u_1,u_2)$ is a graph around z. □

1.2.3. Frame and simple curves

Lifts of curves

Two simple curves which have the same images are reparameterizations of each other through a biholomorphism ([**Laz11**, Section 4]). In this section, we shall recall the statement of these results as well as provide proofs when needed.

The relation that we just defined is not quite transitive. However, there still are a few cases where transitivity holds. Let us consider three J-holomorphic curves with boundary in L, $u_i : (S_i, \partial S_i) \to (M, i(L))$ with $i = 1 \dots 3$.

Proposition 1.2.24 (Proposition 4.1, [**Laz11**]). *If $(z_1, z_2, z_3) \in S_1 \times S_2 \times S_3$ satisfy $z_1 \mathcal{R}_{u_1}^{u_2} z_2$ and $z_2 \mathcal{R}_{u_2}^{u_3} z_3$, and one of the following holds*

(1) $z_1 \in \mathrm{Int}(S_1)$ or $z_3 \in \mathrm{Int}(S_3)$,

(2) $z_2 \in \partial S_2$ and there is a neighborhood $\omega_2 \subset S_2$ of z_2 such that $\mathcal{W}(u_1, u_2) \cap \omega_2 \subset \partial S_2$ or $\mathcal{W}(u_2, u_3) \cap \omega_2 \subset \partial S_2$,

then $z_1 \mathcal{R}_{u_1}^{u_3} z_3$.

Moreover, it turns out that the relation $\mathcal{R}$ has the lifting property with respect to the projection on the second factor.

Proposition 1.2.25 (Lemma 4.3, [**Laz11**]). *Let $z_1 \in S_1 \backslash (\mathcal{W}(u_1, u_2) \cup \mathcal{C}(u_1, u_2))$ and $z_2 \in S_2$ such that $z_1 \mathcal{R}_{u_1}^{u_2} z_2$.*

Assume that $\gamma_1 : [0, 1] \to S_1$ is a continuous map such that $\gamma_1(0) = z_1$ and for $t \in [0, 1[$, $\gamma_1(t) \notin \mathcal{W}(u_1, u_2) \cup \mathcal{C}(u_1, u_2)$.

There exists a unique continuous map $\gamma_2 : [0, 1] \to S_2$ such that $\gamma_2(0) = z_2$ and for $t \in [0, 1]$, $\gamma_1(t) \mathcal{R}_{u_1}^{u_2} \gamma_2(t)$.

Proposition 1.2.26 (Lemma 4.4, [**Laz11**]). *Let $\gamma_1 : [0.1] \to \overline{\mathcal{W}}(u_1, u_2)$ be a continuous path such that $\gamma_1(t) \notin \mathcal{C}(u_1, u_2)$ for $0 < t < 1$. If $z_1 = \gamma_1(0)$ and $z_1 \mathcal{R}_{u_1}^{u_2} z_2$ with $z_2 \in \mathrm{Int}(S_2)$.*

There is $\gamma_2 : [0, 1] \to \mathcal{W}(u_2; u_1, u_2)$ such that $\gamma_2(0) = z_2$ and $\gamma_1(t) \mathcal{R}_{u_1}^{u_2} \gamma_2(t)$ for $0 \leqslant t \leqslant 1$.

Moreover, if $\gamma_1([0, 1]) \subset \mathcal{W}(u_1, u_2)$, then $\gamma_2([0, 1]) \subset \mathcal{W}(u_2, u_2)$. Otherwise, $\gamma_2([0, 1]) \subset \mathcal{W}(u_2, u_1)$.

Proposition 1.2.27 (Lemma 4.5, [**Laz11**]). *Assume ∂S_2 is connected. If C is a connected component of $\mathcal{W}(u_1, u_2)$ with $C \subset \mathrm{Int}(S_1)$, then for $z \in \partial S_2$ there is $w \in C$ such that $w \mathcal{R}_{u_1}^{u_2} z$.*

We also introduce the following notion.

Definition 1.2.28. *The J-holomorphic curve $u : (S, \partial S) \to (M, i(L))$ is* properly bordered *if ∂S is open in $\mathcal{W}(u, u)$.*

Simple curves are determined by their frames

Definition 1.2.29. *Two J-holomorphic curves with boundaries on L defined on connected surfaces u_1 and u_2 have relatively simple frames if $\mathcal{R}_{u_1}^{u_2} \neq \emptyset$, $\mathcal{W}(u_1, u_2) \subset \partial S_1$ and $\mathcal{W}(u_2, u_1) \subset S_2$.*

The argument of [**Laz11**, section 4] holds without modifications to yield

Theorem 1.2.30 (Theorem 4.13, [**Laz11**])**.** *If u_1 and u_2 are two simple curves with boundary in L, the following assertions are equivalent*

(1) $u_1(S_1) = u_2(S_2)$ and $u_1(\partial S_1) = u_2(\partial S_2)$,

(2) $\mathcal{W}(u_1, u_2) = \partial S_1$ and $\mathcal{W}(u_2, u_1) = \partial S_2$,

(3) u_1 and u_2 have relatively simple frames,

(4) $\mathcal{R}_{u_1}^{u_2} \neq \emptyset$ and $u_1(\partial S_1) = u_2(\partial S_2)$,

(5) There exist biholomorphism $\phi_{12} : (S_2, \partial S_2) \to (S_1, \partial S_1)$ such that

$$u_2 = u_1 \circ \phi_{12}.$$

1.2.4. Factorizations of J-holomorphic disks

Factorization of curves

Proposition 1.2.31. *Let $u : (S, \partial S) \to (M, i(L))$ be a non-constant, finite-energy, J-holomorphic curve with corners and boundary on L. We assume*

$$\mathcal{W}(u) = \partial S.$$

Let $\{z_1, \dots, z_N\} \subset \partial S$ be the set of point related to z_1:

$$\mathcal{R}_u^u\{z_1\} = \{z_1, \dots, z_N\}.$$

There are simply connected open neighborhoods of z_i, $\Omega_i \ni z_i$ for $i = 1, \dots, N$ such that

$$\Omega_i \cap \mathcal{C}(u, u) \subset \{z_i\}$$

and applications

$$\psi_{ij} : \overline{\Omega_i} \to \overline{\Omega_j}$$

nonsuch that

(1) ψ_{ij} is the unique biholomorphism such that $u \circ \psi_{ij} = u$,

(2) If $(z, w) \in \overline{\Omega_i} \times \overline{\Omega_j}$, we have $z \mathcal{R}_u^u w$ if and only if $w = \psi_{ij}(z)$.

Proof. For $i = 1 \dots N$, choose V_i a neighborhood of z_i and $\mathcal{C}^1$ charts such that, in these charts,

$$u(z) = a_i(z - z_i)^{k_i} + o(|z - z_i|^{k_i})$$

if z_i is not a corner point, or

$$u(z) = a_i(z - z_i)^{\frac{\alpha_i}{\pi} + m_i} + o(|z - z_i|^{\frac{\alpha_i}{\pi} + m_i})$$

if z_i is.

Moreover, we can assume that $d|u|_z$ is non-zero on V_i. We choose an $\alpha \in$ such that $0 < \alpha < \inf_{\partial(\cup V_i)\setminus\partial\mathbb{D}}$. Then α is a regular value of $|u|$. Denote by Ω_i the connected component of $\mathbb{D}^+\setminus|u|^{-1}\{\alpha\}$ such that $\Omega_i \ni z_i$. Since

$$d|u|(z - z_i) = |a_i|^2\left(\frac{\alpha_i}{\pi} + m_i\right) + o(1)$$

is positive for z close enough to z_i, we can assume (choosing α smaller if necessary) that Ω_i is simply connected. It implies that it is biholomorphic to a disk. Furthermore, its boundary is the union of an embedded arc in ∂S and an embedded arc in the interior.

Let us show that the ψ_{ij} exist. Choose $\tilde{z}_1 \in \Omega_1 \cap \mathrm{Int}(S)$ and $\tilde{z}_2 \in \Omega_2 \cap \mathrm{Int}(S)$. We build $\psi_{12} : \Omega_1\setminus\partial\mathbb{D} \to \Omega_2\setminus\partial\mathbb{D}$.

For this, choose $z \in \Omega_1\setminus\partial\mathbb{D}$. There is a continuous path $\gamma : [0,1] \to \Omega_1\setminus\partial\mathbb{D}$ from $\tilde{z}_1$ to z. This path lifts to a unique continuous $\gamma_2 : [0,1] \to S$ such that $\gamma_2(0) = \tilde{z}_2$ and $\gamma(t)\mathcal{R}_{u_1}^{u_2}\gamma_2(t)$. Notice that since $\mathcal{W}(u) = \partial S$, we have $\gamma_2(t) \notin \partial S$. Moreover, since $|u(\gamma(t))| < \alpha$ for all $t \in [0,1]$ and $u(\gamma_2(t)) = u(\gamma(t))$, we get $\gamma_2(t) \in \Omega_2\setminus S$. We put $\psi_{12}(z) = \gamma_2(1)$.

It remains to see that $\psi_{12}(z)$ does not depend on the choice of γ. For this, suppose that there is a homotopy $H : [0,1] \times [0,1] \to \Omega_1\setminus\partial\mathbb{D}$ such that $H(0,\cdot) = \gamma$. By the same argument as before, there is a unique lift $\tilde{H} : [0,1] \times [0,1] \to \Omega_2\setminus\partial\mathbb{D}$ such that $H(s,t)\mathcal{R}_u^u\tilde{H}(s,t)$. It is easy to see, using that there is no critical points of u in $\Omega_2\setminus\partial S$, that H is actually smooth. Thus the existence of ψ_{12} is proved.

Recall that there are no critical points of u in $\Omega_2\setminus\{z_2\}$. It is easy to see, using the same argument, that ψ_{12} extends to a holomorphic map $\Omega_1\setminus\{z_1\} \to \Omega_2\setminus\{z_2\}$. In a local chart for S, this is a bounded holomorphic map from $\mathbb{D}_{\mathbb{R}}^+\setminus\{0\}$ to $\mathbb{D}_{\mathbb{R}}^+\setminus\{0\}$ sending the real line to the real line. Hence, it extends to a holomorphic map $\Omega_1 \to \Omega_2$.

To see that this is a biholomorphism, notice that the same argument allows us to build a holomorphic map $\psi_{21} : \Omega_2 \to \Omega_1$ such that $\psi_{21}(\tilde{z}_2) = \tilde{z}_1$. Now $\Phi = \psi_{12} \circ \pi_{21} : \Omega_1 \to \Omega_1$ satisfies $u \circ \Phi = u$ and $\Phi(\tilde{z}_1) = \tilde{z}_1$. Hence by the unicity of lifts of paths, it is the identity. Exchanging the order of the composition, we get $\psi_{12} \circ \psi_{21} = \mathrm{Id}$.

The unicity follows from a beautiful argument given in [**Laz00**, Proposition 5.9]. Assume that ψ is a biholomorphism $\Omega_1 \to \Omega_1$ such that $u \circ \psi = u$. Since Ω_1 is simply connected, we can assume that it is actually equal to $\mathbb{D}$ by the Riemann mapping theorem. Then ψ has a fixed point, say z_0. If $z_0 \in \partial\mathbb{D}$, then $\psi^n(z) \to z_0$ for all z, so u is constant. Hence $z_0 \in \mathring{\mathbb{D}}$ and we can assume $z_0 = 0$ so $\psi(z) = \zeta z$ for some $\zeta \in \partial\mathbb{D}$. Either ζ has infinite order, in which case u is constant, or it has finite order so u factors through $z \mapsto z^k$. But u is an immersion on $\mathbb{D}$, so $k = 1$. $\square$

From this, we immediately get the following corollary.

Corollary 1.2.32. *Recall that $u : (S, \partial S) \to (M, i(L))$ is a finite energy J-holomorphic curve with corners and boundary on L.*

If $\mathcal{W}(u) = \partial S$ and $z_1, \ldots, z_N \in \partial S$ are such that

$$\mathcal{R}_u^u\{z_1\} = \{z_1, \ldots, z_n\}.$$

There are holomorphic charts $h_i : (\mathbb{D}^+, \mathbb{D}^+_{\mathbb{R}}) \to (S, \partial S)$ with $h_i(0) = z_i$ such that $h_i(z) = h_j(z')$ if and only if $z\mathcal{R}_u^u z'$.

We also have the following.

Corollary 1.2.33. *Recall that $u : (S, \partial S) \to (M, i(L))$ is a finite energy J-holomorphic curve with corners and boundary on L.*

Assume that its frame satisfies $\mathcal{W}(u) = \partial S$. Then, there is a simple, finite-energy J-holomorphic curve $v : (S', \partial S') \to (M, i(L))$ and a finite branched cover $p : (S, \partial S) \to (S', \partial S')$ which restricts to an actual cover $p : \partial S \to \partial S'$ such that

$$u = v \circ p.$$

Proof. The relation $\mathcal{R}_u^u$ is transitive by Proposition 1.2.24. Therefore, the quotient $S' = S/\mathcal{R}_u^u$ is well-defined.

It remains to define holomorphic charts on S' such that the quotient map $p : S \to S'$ is holomorphic.

Let $z' \in S'$, let

$$p^{-1}(z) = \{z_1, \ldots, z_N\} \subset \partial S$$

be the preimages of z'. We consider the biholomorphisms $h_1, \ldots, h_N$ given by Corollary 1.2.32. Then the restriction of p to each Ω_i is a bijection. Remark that, up to taking smaller neighborhoods, we can assume tahtthe Ω_i are relatively compact. Therefore, we can assume that p restricted to each Ω_i is a homeomorphism onto its image. The chart around z' is given by $h_i \circ p^{-1}_{|\Omega_i}$.

If the preimages of z' are contained in $\mathrm{Int}(S)$, the charts are constructed in [**MS12**, Proposition 2.5.1].

Notice that this immediately implies that $p_{|\partial S} : \partial S \to \partial S'$ is a finite cover.

The map u goes through the quotient to induce a holomorphic map $v : (S', \partial S') \to (M, i(L))$ which is simple. Call $E = \{y_1, \ldots, y_m\}$ the corner points of u and let $\gamma : \partial S \backslash E \to L$ be the boundary condition of u. Corollary 1.2.32 immediately implies that if $z \in \partial S$ is not a corner point and z' is such that $z\mathcal{R}_u^u z'$, then z' is not a corner point.

We deduce that in a neighborhood of z, we have $\gamma \circ h_z = \gamma \circ h_{z'}$. Hence, we can quotient out γ to obtain a continuous map $\gamma' : \partial S' \backslash p(E)$ which satisfies $i \circ \gamma' = v_{|S' \backslash p(E)}$.

It is now immediate from the definition of γ' that each point of $p(E)$ is a corner point.

Since p is a branched cover of finite degree, say $d \geqslant 1$, we have

$$\int u^* \omega = d \int v^* \omega,$$

so $\int v^* \omega < +\infty$.

The fact that v is simple is an easy consequence of the definition of S'. If $z \mathcal{R}_v^v z'$, there are two sequences $z_\nu \to z$ and $z'_\nu \to z'$ such that $z_\nu \notin \mathcal{C}(v)$, $z'_\nu \notin \mathcal{C}(v)$ and $z_\nu \mathcal{R}_v^v z'_\nu$. Now pick two sequences of lifts of these $\tilde{z}_\nu, \tilde{z}'_\nu$ which converge (up to a subsequence) to two points say $\tilde{z}$ and $\tilde{z}'$. Notice that p is a branched cover, hence a local embedding outside the critical points. With this in mind, $z_\nu \mathcal{R}_v^v z'_\nu$ implies $\tilde{z}_\nu \mathcal{R}_u^u \tilde{z}'_\nu$. This implies in turn $\tilde{z} \mathcal{R}_u^u \tilde{z}'$. So by definition $z = p(\tilde{z}) = p(\tilde{z}')$. Hence, the relation $\mathcal{R}_v^v$ is trivial. Now apply Proposition 1.2.17. ☐

We conclude with the following Theorem.

Theorem 1.2.34. *Let $u : (S, \partial S) \to (M, i(L))$ be a finite-energy J-holomorphic curve with corners and boundary in L.*

There are finite-energy simple J-holomorphic curves with corners

$$v_i : (S_i, \partial S_i) \to (M, i(L))$$

for $i = 1 \dots N$ such that

$$\mathrm{Im}(u) = \bigcup_{i=1}^{N} \mathrm{Im}(v_i).$$

Further, there are natural integers $m_1, \dots, m_N \geqslant 1$ such that

$$[u] = \sum_{i=1}^{N} m_i [v_i] \text{ in } H_2(M, i(L)).$$

Proof. The proof of the first point proceeds as in Lazzarini's paper. For each connected component Ω of $S \backslash \mathcal{W}(u)$, choose a complex embedding $h_\Omega : (S_\Omega, \partial S_\Omega) \to (\Omega, \partial\Omega)$ ([**Laz11**, Lemma 2.6]) and consider the map $u \circ h_\Omega$.

We have

$$E(u \circ h_\Omega) \leqslant E(u_{|\Omega}) \leqslant E(u) < +\infty.$$

The set of the preimages of double points $u^{-1}(i(R))$ is finite, hence by [**Laz11**, Lemma 2.4], the set $(u \circ h_\Omega)^{-1}(i(R))$ is also finite. Therefore $u \circ h_\Omega$ has a finite number of corner points.

Now we claim that $\mathcal{W}(u \circ h_\Omega) = \partial S_\Omega$. To see this, let $z \in \mathcal{W}(u \circ h_\Omega)$. There is $z' \in \partial S_\Omega$ such that $z' \mathcal{R}_{u \circ h_\Omega}^{u \circ h_\Omega} z$. From this, it follows that $h_\Omega(z) \mathcal{R}_u^u h_\Omega(z')$. If $h_\omega(z') \in \mathrm{Int}(S)$, from $h_\Omega(z') \in \mathcal{W}(u)$ it follows that $h_\Omega(z) \in \mathcal{W}(u)$ by transitivity. Hence, $z \in \partial S_\Omega$. Therefore, $\mathcal{W}(u \circ h_\Omega) \subset \partial S_\Omega$ and there is equality since the other inclusion holds by definition.

By Corollary 1.2.33, there is a Riemann surface with boundary S'_Ω, a map $p_\Omega : S_\Omega \to S'_\Omega$ and a simple curve $v_\Omega : S'_\Omega \to (M, i(L))$ such that

$$u \circ h_\Omega = v_\Omega \circ p_\Omega.$$

Moreover, we see immediately that

$$\mathrm{Im}(u) = \bigcup_\Omega \mathrm{Im}(v_\Omega),$$

where the union is taken over the set of connected components of $\mathbb{D} \backslash \mathcal{W}(u)$. □

Now the conclusion of the main Theorem 1.1.3 follows immediately from the following proposition.

Proposition 1.2.35. *Assume that $u : (\mathbb{D}, \partial\mathbb{D}) \to (M, L)$ is a finite-energy J-holomorphic disk with corners and boundary on L. Keeping the notations of the proof of Theorem 1.2.34, each of the surfaces S'_Ω is biholomorphic to a disk.*

Given what we have already shown, the proof of Proposition 1.2.35 does not differ much from the proof of the corresponding proposition in [**Laz11**, Proposition 5.5]. For the convenience of the reader, we shall recall the proof in the next subsection.

Connectedness of the frame and holomorphic spheres

The main result is the following proposition.

Proposition 1.2.36. *Assume that $u : (\mathbb{D}, \partial\mathbb{D}) \to (M, L)$ is a finite-energy J-holomorphic disk with corners with $\mathcal{W}(u)$ not connected. There is a simple J-holomorphic sphere $v : \mathbb{C}P^1 \to M$ such that $\mathrm{Im}(u) = \mathrm{Im}(v)$ and $\mathcal{R}_v^u \neq \emptyset$.*

Proof of Proposition 1.2.35 assuming 1.2.36 . First, assume that $\mathcal{W}(u)$ is connected. Let Ω be a connected component of $\mathbb{D} \backslash \mathcal{W}(u)$. It is simply connected, hence S_Ω is a disk. Keeping the notation of 1.2.34 in mind, let g'_Ω be the genus of S'_Ω. The Riemann-Hurwitz formula applied to the cover p_Ω yields :

$$1 = \deg(p_\Omega)(1 - g'_\Omega) - m$$

with $m \geqslant 0$ an integer. From $m + 1 > 0$ and $1 - g'_\Omega \leqslant 1$, we deduce $g'_\Omega = 0$.

Now assume that $\mathcal{W}(u)$ is not connected. Therefore, there is a simple J-holomorphic curve $v : \mathbb{C}P^1 \to M$ such that $\mathrm{Im}(u) = \mathrm{Im}(v)$ and $\mathcal{R}_v^u(\mathbb{C}P^1) = \mathbb{D}$. Notice that if $z \in \mathbb{D}$,

$z_1 \in \mathbb{C}P^1$ and $z_2 \in \mathbb{C}P^1$ are such that $z\mathcal{R}_v^u z_1$ and $z\mathcal{R}_v^u z_2$, then $z_1 = z_2$ (in other words every element of $\mathbb{D}$ lifts to a unique point in $\mathbb{C}P^1$). Indeed by transitivity (Proposition 1.2.24), we get $z_1\mathcal{R}_v^v z_2$ and since v is simple $z_1 = z_2$ (see Proposition 1.2.17).

The points of $\mathcal{C}(u, v)$ are isolated. Therefore, Proposition 1.2.26 implies that the boundary of u lifts to a continuous curve $\gamma : \partial\mathbb{D} \to \mathbb{C}P^1$ such that $u(z)\mathcal{R}\gamma(z)$ for $z \in \partial D$ and whose image is $\mathcal{W}(v, u)$. Hence $\mathcal{W}(v, u)$ is connected, so each connected component Ω of $\mathbb{C}P^1\backslash\mathcal{W}(v, u)$ is simply connected and gives rise to a simple J-holomorphic disk $v_{|\Omega} : \Omega \to M$.

Consider Ω a connected component of $\mathbb{D}\backslash\mathcal{W}(u)$, and S_Ω, S'_Ω as in the proof of 1.2.34. If z is in the interior of S'_Ω, there is a point $\tilde{z} \in \mathbb{C}P^1$ such that $z\mathcal{R}^v_{v'_\Omega}\tilde{z}$. Let $\tilde{\Omega}$ be the connected component of $\mathbb{C}P^1\backslash\mathcal{W}(v, u)$ containing $\tilde{z}$.

Then one checks that $\mathcal{R}(\partial\tilde{\Omega}) = \partial\Omega$ and $\mathcal{R}(\partial\Omega) = \partial\tilde{\Omega}$, so v'_Ω is a J-holomorphic disk by Theorem 1.2.30. $\square$

We will give the proof of Propositon 1.2.36 at the end of the next subsection after some preliminary results.

Cutpoints and holomorphic spheres

In this subsection, we state some results whose proofs are in [**Laz11**]. An exception is point (2) of Proposition 1.2.38 which is specific to our own situation. Nevertheless, for the convenience of the reader, we shall sum up the main arguments.

First, we need to define the notion of *cutpoint*. Those are the points at the boundary where the disk closes on itself.

Definition 1.2.37. *Let $u : (S, \partial S) \to (M, i(L))$ be a finite-energy J-holomorphic curve with corners and $z \in \partial S$.*

The point z is a cutpoint *if there is a complex embedding*

$$h : (\mathbb{D}^+, \partial\mathbb{D}^+) \to (S, \partial S) \text{ with } h(0) = z$$

and a J-holomorphic disk

$$v : \mathbb{D} \to M$$

such that 0 is a dead-end of $\mathcal{W}(v, u \circ h)$.

We denote by $\mathrm{Cut}(u) \subset \partial S$ *the set of cutpoints of u.*

Here are some properties of the set of cutpoints.

Proposition 1.2.38. *Let $u : (S, \partial S) \to (M, i(L))$ be a finite-energy J-holomorphic curve with corners and boundary in L.*

(1) If $\mathrm{Cut}(u) = \emptyset$, then $\mathcal{W}(u)$ has no dead-ends.

(2) If $z \in \mathrm{Cut}(u)$, then z is not a corner point.

(3) If $z \in \mathrm{Cut}(u)$, there is a neighborhood ω of z in ∂S and a continuous involution $\sigma : \omega \to \omega$ such that $\sigma(z) = z$ and $z\mathcal{R}^u_u\sigma(z)$ for $z \in \omega$.

Proof. The proof of (1) is clear. Assume z_0 is a dead-end, then there is a point $z \in \partial S$ such that $z_0\mathcal{R}^u_u z_0$. Choose an embedding $\phi : \mathbb{D} \to S$ such that $\phi(0) = z_0$ and $\phi^{-1}(\mathcal{W}(u))$ is an embedded Jordan arc. By definition $\mathcal{W}(u \circ \phi, u) = \phi^{-1}(\mathcal{W}(u))$, so 0 is a dead end of $\mathcal{W}(u \circ \phi, u)$. Now choose an embedding $h : (\mathbb{D}^+, \partial\mathbb{D}^+) \to (S, \partial S)$ with $h(0) = z$. Then $\mathcal{W}(u \circ \phi, u \circ h) \subset \mathcal{W}(u \circ \phi, u)$ and the former is open in the latter by Proposition 1.2.16. Hence 0 is a dead-end of $\mathcal{W}(u \circ \phi, u \circ h)$ and $z \in \mathrm{Cut}(u)$.

Let z be a cutpoint. Assume by contradiction that z is a corner point mapping to $x = i(p) = i(1)$. There are a disk $v : \mathbb{D} \to M$ and an embedding $h : (\mathbb{D}^+, \partial\mathbb{D}^+) \to (S, \partial S)$ with $h(0) = z$ such that $0\mathcal{R}^{u\circ h}_v z$ and 0 is a dead end of $W(v, u \circ h)$. Without loss of generality, we can assume that $u \circ h(\mathbb{R}_+) \subset L_p$ and $u \circ h(\mathbb{R}_-) \subset L_q$. The paths $\gamma_+ : [0, 1) \to \mathbb{D}^+$ and $\gamma_- : (-1, 0] \to \mathbb{D}^+$ defined by $\gamma_\pm(t) = t$ lift to continuous paths $\tilde{\gamma}_\pm$ with values in $\mathcal{W}(v, u \circ h)$ such that $\tilde{\gamma}_\pm(t)\mathcal{R}^{u\circ h}_v\gamma_\pm(t)$. Since the image of $\gamma_\pm$ is not contained in $\mathcal{C}(u \circ h, v)$, the paths $\tilde{\gamma}_\pm$ are not constant. Hence, since the frame is locally path-connected, there is a small neighborhood ω of 0 in $\mathcal{W}(v, u \circ h)$ such that $v(\omega) \subset L_p$ and $v(\omega) \subset L_q$, so $v(\omega) \subset \{0\}$. This implies that v is constant. This contradiction proves (2).

As before, assume that $z \in \mathrm{Cut}(u)$ and keep the notations of the proof of (2). By Proposition 1.2.13 and (2), one can assume that h is such that in a suitable local chart $u\circ h(z) = A(z^k, U(z))$ with $U(z) = o(z^k)$. We conclude that the paths $\tilde{\gamma}_\pm$ are embeddings with values in $\mathcal{W}(v, u \circ h)$ which is one-dimensional. Hence $v \circ \gamma_+(t) = v \circ \gamma_-(t)$ (and k is even). The involution σ maps $h(\gamma_+(t))$ to $h(\gamma_-(t))$. □

The next proposition gives a sufficient condition for a holomorphic disk to be a sphere.

Proposition 1.2.39. *Let u be a J-holomorphic disk. Assume $\mathcal{W}(u)$ is open in ∂S. If there is a J-holomorphic disk $v : \mathbb{D} \to M$ such that $\mathcal{W}(v, u)$ is an embedded Jordan curve, then there is a simple J-holomorphic sphere $w : \mathbb{C}P^1 \to M$ such that*

$$\mathrm{Im}(u) = \mathrm{Im}(w),$$

and $\mathcal{R}^w_u(\mathbb{C}P^1) \neq \emptyset$.

Proof. The idea of the proof is that the boundary of the disk u closes itself on the image of $\mathcal{W}(v, u)$ by v.

First, one can assume without loss of generality that v is a simple disk.

Let z_0 be an extremity of $\mathcal{W}(v, u)$ and choose a point $z \in \partial\mathbb{D}$ such that $z_0\mathcal{R}^u_v z$ (in particular z_0 is a cutpoint of u). One can prove as in the preceding proposition that there are two distinct paths $\gamma_\pm : \mathbb{R}_+ \to \partial\mathbb{D}$, and a path $\tilde{\gamma} : \mathbb{R}_+ \to \mathcal{W}(v, u)$ satisfying

(1) $\gamma_\pm(0) = z$,
(2) $\tilde{\gamma}(0) = z_0$,
(3) $\gamma_\pm(t)\mathcal{R}^v_u\tilde{\gamma}(t)$.

Let $N > 0$ be the first number such that $\gamma_+(t) = \gamma_-(t)$. The surface $S := \mathbb{D}/\sim$ with $\gamma_+(t) \sim \gamma_=(t)$ for $t \in [0, N]$ is a topological sphere. The map u factors through the quotient projection $\pi : \mathbb{D} \to S$ to give a map $w : S \to M$. It remains to see that S admits a structure of Riemann surface such that π is holomorphic.

To construct charts, consider a point $\gamma_+(t_0)\mathcal{R}^u_u\gamma_-(t_0)$ with $t_0 \in (0, N)$ which is not a cutpoint. The proof of Proposition 1.2.31 shows that there are local charts $h_\pm : (\mathbb{D}^+, \partial\mathbb{D}^+) \to (\mathbb{D}, \partial\mathbb{D})$ and $\tilde{h}_\pm(\mathbb{D}^+, \partial\mathbb{D}+) \to (\mathbb{D}, \mathcal{W}(v, u))$ such that $h_\pm(z_1)\mathcal{R}^v_u\tilde{h}_\pm(z_2)$ if and only if $z_1 = z_2$. Since v is simple, there is a unique map ϕ_- such that $\tilde{h}_+(t) = \tilde{h}_-(\phi_-(t))$. The surface $\mathbb{D}^+ \sqcup \mathbb{D}^+/\sim$ where $t \sim \phi_-(t)$ has a structure of a Riemann surface with the charts given by the union of the maps $\tilde{h}_+$ and $\tilde{h}_-$. The chart for the surface S is then given by $h_+ \sqcup h_-$. The map w restricted to this chart is holomorphic since equal to the restriction of v to the images of $\tilde{h}_+$ and $\tilde{h}_-$.

If we consider a point $\gamma_+(t_0)$ which is a cutpoint, one can check that $\tilde{\gamma}(t_0)$ is an endpoint of $\mathcal{W}(v, u)$. □

Proposition 1.2.40. *Let $u : (\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$ be a finite energy J-holomorphic disk such that $\partial\mathbb{D}$ is open in $\mathcal{W}(u)$. Moreover, assume that the frame $\mathcal{W}(u)$ is not connected and that u is not a J-holomorphic sphere.*

Pick $z_0, z_1 \in \mathrm{Cut}(u)$ such that $z_0\mathcal{R}^u_u z_1$ and let $\gamma : [0, 1] \to \partial\mathbb{D}$ be an embedded path with $\gamma(0) = z_0$ and $\gamma(1) = z_1$.

There are $0 < t_0 < t_1 < 1$ such that $\gamma(t_i) \in \mathcal{C}(u, u)\setminus \mathrm{Cut}(u)$ with $i \in \{0, 1\}$.

Proof. The point t_1 is the smallest $t \in (0, 1]$ such that $\gamma(t) \in \mathcal{C}(u, u)$. It is enough to show that $\gamma(t_1) \notin \mathrm{Cut}(u)$ since this implies $t_1 \neq 1$.

The idea is as follows. Assume that $\gamma(t_1)$ is a cutpoint. Pick a connected component $C \subset \mathcal{W}(u) \cap \mathrm{Int}(\mathbb{D})$ and a lift $\tilde{\gamma} : [0, 1] \to C$ such that $\tilde{\gamma}(t)\mathcal{R}^u_u\gamma(t)$. Since $\gamma(0) \in \mathrm{Cut}(u)$ (resp. $\gamma(1) \in \mathrm{Cut}(1)$), there is a J-holomorphic disk $w_0 : \mathrm{Int}(\mathbb{D}) \to M$ (resp. $w_1 : \mathrm{Int}(\mathbb{D}) \to M$) and an embedding $h_0 : (\mathbb{D}^+, \partial\mathbb{D}^+) \to (M, i(L))$ (resp. $h_1 : (\mathbb{D}^+, \partial\mathbb{D}^+) \to (M, i(L)))$ such that 0 is a dead-end of $\mathcal{W}(w_0, u \circ h_0)$ (resp. $\mathcal{W}(w_1, u \circ h_1))$. We also pick an embedding $h_2 : \mathrm{Int}(\mathbb{D}) \to \mathrm{Int}(\mathbb{D})$ such that $h_2(-1, 1) = \tilde{\gamma}(\varepsilon, t_0 - \varepsilon)$ and $0 \notin \mathcal{R}^{w_2}_{w_i} w_2(\mathbb{D})$. Now we attach the three disks w_0, w_1 and w_2 using the relation $\mathcal{R}$ to obtain a disk w such that $\mathcal{W}(w, u)$ is a Jordan arc. Therefore, u is a J-holomorphic sphere, a contradiction.

Let t_0 be the largest $t \in (0, t_1]$ such that $\gamma(t) \in \mathcal{C}(u, u)$. We are done if we show that $t_0 \neq t_1$.

Assume $t_0 = t_1$. Since $\gamma(0)\mathcal{R}_u^u\gamma(1)$, one can choose γ such that $\gamma(t)\mathcal{R}_u^u\gamma(1-t)$. One can then check that this implies $t_0 \in \mathrm{Cut}(u)$, a contradiction. $\square$

All of this allows us to show that some disks are equivalent to disks whose frame does not possess dead-ends.

Proposition 1.2.41. *Let $u : (\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$ be a finite-energy J-holomorphic disk with corners and boundary in L such that ∂S is open in $\mathcal{W}(u)$ and the frame $\mathcal{W}(u)$ is not connected. We assume that u is not a J-holomorphic sphere.*

There is a finite-energy J-holomorphic disk with corners and boundary in L, $\tilde{u} : (\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$ such that

(1) $\mathrm{Im}(u) = \mathrm{Im}(\tilde{u})$,

(2) we have $\mathrm{Cut}(\tilde{u}) = \emptyset$,

(3) there is a surjection $\pi_0(\mathcal{W}(u)) \to \pi_0(\mathcal{W}(\tilde{u}))$.

Proof. The idea is to fold the boundary along the cutpoints.

More precisely, choose a a point $z_1 \in \mathrm{Cut}(u)$ and let $\{z_1, \ldots, z_N\}$ be the set $\mathcal{R}_u^u\{z_1\} \cap \partial\mathbb{D}$. Let $\tilde{z} \in \mathrm{Int}(\mathbb{D})$ be a point such that $z_1\mathcal{R}_u^u\tilde{z}$. There are injective paths $\gamma_2 : [0,1] \to \mathcal{W}(u)$ and $\gamma_{i,\pm} : [0,1] \to \partial\mathbb{D}$ such that

(1) $\tilde{\gamma}(t)\mathcal{R}_u^u\gamma_{i,\pm}(t)$ for $i \in \{1, \ldots, N\}$,

(2) $\tilde{\gamma}(0) = \tilde{z}$, $\tilde{\gamma}(1) \in \mathcal{C}(u,u)$,

Notice that the preceding proposition shows that the points $\gamma_{\pm,i}(1)$ are distinct for $i = 1 \ldots N$.

We let $S := \mathbb{D}/\sim$ be the quotient of $\mathbb{D}$ identifying $\gamma_{\pm,i}(t)$ with $\gamma_{\pm,j}(t)$ for $i,j \in \{1, \ldots, N\}$. Topologically, the surface S is a disk. Then u factors through a map $v : S \to M$. It remains to show that there is a complex structure on S such that the quotient map $\pi : \mathbb{D} \to S$ is holomorphic.

As in Proposition 1.2.39, the idea is to build a chart around $\gamma_{i,+}(t)$ using as a chart a quotient of a small disk around the corresponding point $\tilde{\gamma}(t)$. For $\gamma_{i,+}(t)$ with $t \in [0,1)$ it is the same process as in Proposition 1.2.39.

Consider the points $\gamma_{i,+}(1)$ and $\gamma_{i,-}(1)$ and assume that $\tilde{\gamma}(1)$ is not a vertex of the graph $\mathcal{W}(u)$. Choose a small enough holomorphic embedding $\phi : \mathrm{Int}\,\mathbb{D} \to \mathbb{D}$ such that $\phi(0) = \tilde{\gamma}(1)$ and $\mathrm{Im}\,\phi \cap \mathcal{C}(u,u) = \widetilde{\gamma(1)}$. The set $\phi^{-1}(\mathcal{W}(u,u))$ is divided in two arcs, one is simply $\phi^{-1}(\tilde{\gamma})$ and we call the other γ'. Choose an embedding $h' : (\mathbb{D}^+, \partial\mathbb{D}^+) \to (\mathbb{D}, \gamma')$. The graph $\mathcal{W}(u \circ \phi \circ h, u)$ consists of the boundary$(-1,1)$ and an arc going from 0 to the outer boundary of the half-disk. We call this arc γ''.

Using the proof of Proposition 1.2.31, we show that there are 4 embeddings

$$\tilde{h}_\pm : (\mathbb{D}^+, \partial\mathbb{D}^+) \to (\mathbb{D}^+, \partial\mathbb{D}^+ \cup \gamma''),\ h_\pm : (\mathbb{D}^+, \partial\mathbb{D}^+) \to (\mathbb{D}, \partial D)$$

with $\tilde{h}_\pm(0) = 0$ and $h_\pm(0) = \gamma_\pm(1)$. These satisfy $\tilde{h}_\pm(z_1)\mathcal{R}h_\pm(z_2)$ if and only if $z_1 = z_2$. As before one can attach the two half-disks along their boundaries and identify the resulting surface with $\mathbb{D}^+$ through the disjoint union of the maps $\tilde{h}_\pm$.

Suppose that $\tilde{\gamma}(1)$ is a vertex of the graph $\mathcal{W}(u)$. Let $\phi : \operatorname{Int}\mathbb{D} \to \mathbb{D}$ be a small enough holomorphic embedding so that $\phi(0) = \tilde{\gamma}(1)$ and $\operatorname{Im}\phi \cap \mathcal{C}(u,u) = \tilde{\gamma(1)}$. The paths $\gamma_\pm(1-t)$ lift to two (not necessarily distinct) arcs γ_1 and γ_2 in $\mathcal{W}(u,u)$. Let $\tilde{h} : (\mathbb{D}^+, \partial\mathbb{D}^+) \to (\operatorname{Int}\mathbb{D}, \phi^{-1}(\gamma_1 \cup \gamma_2))$ be a holomorphic embedding with $\phi^{-1}(\tilde{\gamma}) \subset \operatorname{Im}(\tilde{h})$. We then proceed just as before!

The end product is a finite-energy J-holomorphic disk with corners v such that

$$\#\operatorname{Cut}(v) \leqslant \#\operatorname{Cut}(u) - 1.$$

We then repeat the process by induction to get the desired disk $\tilde{u}$. ☐

After these results, we now return to the proof of Proposition 1.2.36.

Proof of Proposition 1.2.36. Assume by contradiction that the map u is not a J-holomorphic sphere. Then by Propositions (1.2.41), there is a finite-energy J-holomorphic disk $\tilde{u}$ with corners and boundary on L which satisfies the following.

(1) The set $\partial\mathbb{D}$ is open in $\mathcal{W}(u)$.
(2) The connected component Ω of $\mathbb{D}\mathcal{W}(\tilde{u})$ which contains $\partial\mathbb{D}$ is not simply connected. It is the unique connected connected component with this property.
(3) The set of cutpoints is empty.
(4) The map $\tilde{u}$ is not a J-holomorphic sphere.

Call Ω_1 the connected component of $\mathbb{D}\backslash\mathcal{W}(u)$ which contains $\partial\mathbb{D}$ and choose a map $h : (S, \partial S) \to (\Omega_1, \partial\Omega_1)$ which is a biholomorphism from $\operatorname{Int}(S)$ to $\operatorname{Int}(\Omega_1)$. The map $u_{\Omega_1} := u \circ h$ factors through a simple J-holomorphic map $v_{\Omega_1} : S'_{\Omega_1} \to M$. We show that this is a disk.

For C a connected component of $\mathcal{W}(u)$, denote by Ω_C the connected component of $\mathbb{D}\backslash\mathcal{W}(u)$ with boundary C. It is simply connected, hence biholomorphic to a disk. Hence, the map $u_{\Omega_C} := u_{|\Omega_C}$ factors through a simple disk v_{Ω_C}.

Let $z \in \partial S'_{\Omega_1}$ and pick a point $\tilde{z} \in p_{\Omega_1}^{-1}(z)$. There is $s \in C$ such that $h(\tilde{z})\mathcal{R}_u^u s$. Then, either $h(\tilde{z})\mathcal{R}_{u_{\Omega_1}}^{u_{\Omega_C}} s$ or $h(\tilde{z})\mathcal{R}_{u_{\Omega_1}}^{u_{\Omega_1}} s$. In the first case, the J-holomorphic maps v_{Ω_1} and v_{Ω_C} satisfy $\mathcal{W}(v_{\Omega_1}, v_{\Omega_C}) = \partial\Omega_1$ and $\mathcal{W}(v_{\Omega_C}, v_{\Omega_1}) = \partial\Omega_C$. Hence, v_{Ω_1} and v_{Ω_C} are conjugate.

If there is no connected component such that $h(\tilde{z})\mathcal{R}_{u_{\Omega_1}}^{u_{\Omega_C}} s$, the surface $S_{\Omega_1}/\mathcal{R}_{u_{\Omega_1}}^{u_{\Omega_1}}$ has a unique connected component. Therefore, it is a disk

Now choose a connected component C. We glue the disks v_{Ω_C} and v_Ω along their boundaries to get a J-holomorphic sphere $v : \mathbb{C}P^1 \to M$ such $\mathcal{R}_{\tilde{u}}^v(\mathbb{C}P^1) \neq \emptyset$. We readily conclude that $\tilde{u}$ is a sphere. This is a contradiction. ☐

1.3. Consequences of the main theorem

1.3.1. Simplicity of curves for generic almost complex structures

In this subsection, we give the proof of Corollary 1.1.4. The proof is basically contained in [**Laz11**, Theorem B] and [**BC07**]. Here, we sum up the main arguments involved in the proof. Recall that we fixed a generic Lagrangian immersion $i : L \to M$ whose set of double points is $R = \{(p, q) \in L \times L | p \neq q, i(p) = i(q)\}$.

Intersection points and indices of curves.

For each (ordered) double point $(p, q) \in R$ (with as usual $x = i(p) = i(q)$), denote by $\mathcal{G}(T_xM)$ the Lagrangian Grassmannian of T_xM. We choose once and for all a smooth path

$$\lambda_{(p,q)} : [0, 1] \to \mathcal{G}(T_xM)$$

such that

$$\lambda_{(p,q)}(0) = di_p(T_xL) \text{ and } \lambda_{(p,q)}(1) = di_q(T_xL).$$

Moreover, we may assume that $\lambda_{(q,p)}$ is $\lambda_{(p,q)}$ parameterized in the reverse direction.

Now define a *Maslov pair* (E, F) (we use the terminology of [**MS12**, Appendix C.3]) as follows. We let

$$E = \mathbb{H} \times T_xM$$

be the trivial symplectic vector bundle over the closed Poincaré half-plane $\mathbb{H}$ with fiber T_xM equipped with the symplectic form ω_x.

Consider a strictly increasing smooth function $f : \mathbb{R} \to [0, 1]$ such that $f(t) = 0$ for $t << 0$ and $f(t) = 1$ for $t >> 0$. Then the Lagrangian boundary condition is given by

$$\forall t \in \mathbb{R},\ F_t = \lambda_{(p,q)}(f(t)).$$

We endow $\mathbb{H}$ with the following strip-like end

$$\begin{array}{cccc} \varepsilon &]-\infty, 0] \times [0, 1] & \to & \mathbb{H} \\ & (s, t) & \mapsto & e^{-\pi(s+it)} \end{array}$$

and endow the bundle $\mathbb{H} \times T_xM$ with the trivial symplectic connection ∇. This satisfies the hypotheses of [**Sei08**, section 8h], hence admits an associated Fredholm Cauchy-Riemann operator $\overline{\partial}_\Delta$ between suitable Sobolev completions of the spaces of sections. We denote by $\mathrm{Ind}(p, q)$ the index of this operator :

$$\mathrm{Ind}(p, q) = \mathrm{Ind}(\overline{\partial}_\Delta).$$

Now, choose a compatible almost complex structure $J \in \mathcal{J}(M, \omega)$ and let

$$(p_0, q_0), \ldots, (p_d, q_d) \in \mathbb{R}$$

be $d \in \mathbb{N}^*$ ordered self-intersection points of i.

Let $u_0 : (\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$ and $u_1 : (\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$ be two topological disks with corners[3] at $(p_0, q_0), \ldots, (p_d, q_d)$ in cyclic order. Let $\tilde{u}_0 : \mathbb{D}\backslash\{z_1, \ldots, z_d\} \to L$ and $\tilde{u}_1 : \mathbb{D}\backslash\{z_1, \ldots, z_d\}$ be their boundary lifts to L.

We say that u_0 and u_1 are *homotopic* (as topological disks with corners) if there are

- a continuous family $(v_t)_{t\in[0,1]}$ of maps $(\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$,
- a continuous family $(\tilde{v}_t)_{t\in[0,1]}$ of maps $\mathbb{D}\backslash\{z_1, \ldots, z_d\} \to L$,

such that

- for each $t \in [0, 1]$, the map v_t is a topological disk with corners and with lift $\tilde{v}_t$,
- we have $(v_0, \tilde{v}_0) = (u_0, \tilde{u}_0)$ and $(v_1, \tilde{v}_1) = (u_1, \tilde{u}_1)$.

Let A be a homotopy class of topological disks with corners and corner points given in cyclic order by $(p_0, q_0), \ldots, (p_d, q_d)$.

Assume first $d \geqslant 2$. Recall that there is a universal family $\mathcal{S}^{d+1} \xrightarrow{\pi} \mathcal{R}^{d+1}$ of disks with $d + 1$ marked points. Fix a universal choice of positive strip-like ends [4].

We let $\mathcal{M}(A, (p_0, q_0), \ldots, (p_d, q_d), J)$ be the space of maps $u : \pi^{-1}(r) \to M$ for some $r \in \mathcal{R}^{d+1}$ satisfying the following conditions,

(1) u is a finite-energy J-holomorphic disk with corners and boundary on L,
(2) the corner points of u coincide with the limits of the strip-like ends and the switch condition at the i-th marked point is given by (p_i, q_i),
(3) the homotopy class of u is A.

Each $u \in \mathcal{M}(A, (p_0, q_0), \ldots, (p_d, q_d), J)$ gives rise to the bundle pair (u^*TM, u^*TL). The linearization of the Cauchy-Riemann equation at u yields a Cauchy-Riemann operator between suitable Sobolev completions of the space of sections of this bundle pair

$$D_u : W^{k,p}(u^*TM, u^*TL) \to W^{k-1,p}(\Lambda^{0,1}u^*TM).$$

Fix such a $u : \pi^{-1}(r) \to M$ and denote by $x_0, \ldots, x_d$ the marked points in the domain ordered counterclockwise. There is a natural compactification of $\pi^{-1}(r)$ given by the union of $\pi^{-1}(r)$ and $d+1$ copies of the interval $[0, 1]$ topologized so that the positive (resp. negative) strip-like ends $\varepsilon_i : (0, +\infty[\times[0, 1] \to \pi^{-1}(r)$ (resp. $\varepsilon_i :]-\infty, 0)\times[0, 1] \to \pi^{-1}(r)$) extend to homeomorphisms $\varepsilon_i : (0, +\infty] \times [0, 1] \to \pi^{-1}(r)$ (resp. $\varepsilon_i : [-\infty, 0) \times [0, 1] \to \pi^{-1}(r)$). We denote it $\overline{\pi^{-1}(r)}$.

[3]See Remark 1.1.2 for the definition
[4]See [Sei08, Section (9)] for the relevant definitions

The map u admits a unique extension to a continuous map $\overline{u} : \overline{\pi^{-1}(r)} \to M$. This gives rise to a Maslov pair $(E, F) \to (\overline{\pi^{-1}(r)}, \partial\overline{\pi^{-1}(r)})$ as follows :

- for $z \in \partial\pi^{-1}(r)$, we put

$$F_z = T_{u(z)}L,$$

- over each interval $\varepsilon_i(\pm\infty \times \ [0, 1])$,

$$\forall t \in [0, 1],\ T_{\overline{u}(\varepsilon_i(\pm\infty,t))} = \lambda_{p_i,q_i}(t).$$

The index of this Maslov pair only depends on the homotopy class A of the map u. Therefore, we define *the Maslov index of the class* A by

$$\mu_A = \mu(E, F).$$

The index of the operator D_u is

$$\mathrm{Ind}(D_u) = n + \mu_A - \sum_{i=0}^{d} \mathrm{Ind}(p_i, q_i). \tag{1.8}$$

See the paper of Akaho-Joyce [**AJ10**, Section 4.3, Proposition 4.6], or the exposition in Seidel's book [**Sei08**, Section (11)].

For the case $d = 1$, we consider the space of J-holomorphic strips with corners at (p_0, q_0) and (p_1, q_1). More precisely, define $Z = \mathbb{R} \times [0, 1]$. We denote by

$$\widetilde{\mathcal{M}}(A, (p_0, q_0), (p_1, q_1), J)$$

the space of finite-energy J-holomorphic maps $u : Z \to M$ such that $u(0, \cdot)$ (resp. $u(1, \cdot)$) lifts to a map $\gamma_- : \mathbb{R} \to L$ (resp. $\gamma_+ : \mathbb{R} \to L$) with

$$\lim_{s\to+\infty} (\gamma_-(s), \gamma_+(s)) = (p_1, q_1),\ \lim_{s\to-\infty} (\gamma_-(s), \gamma_+(s)) = (p_0, q_0).$$

The index of such a curve is given by

$$\mathrm{Ind}(D_u) = \mu_A + \mathrm{Ind}(p_0, q_0) - \mathrm{Ind}(p_1, q_1). \tag{1.9}$$

For the case $d = 0$, we consider the space of J-holomorphic teardrops with corner at (p_0, q_0). More precisely, we denote by

$$\widetilde{\mathcal{M}}(A, (p_0, q_0), J)$$

the space of finite-energy J-holomorphic maps $u : \mathbb{H} \to M$ such that $u_{|\mathbb{R}}$ lifts to a map $\gamma : \mathbb{R} \to L$ with

$$\lim_{s\to-\infty} \gamma(s) = p_0 \text{ and } \lim_{s\to+\infty} \gamma(s) = q_0.$$

The index of an element $u \in \widetilde{\mathcal{M}}(A,(p_0,q_0),J)$ of this moduli space is given by

$$\operatorname{Ind}(D_u) = \mu_A + \operatorname{Ind}(p_0,q_0). \tag{1.10}$$

Generic transversality of moduli spaces

Assume that $d \geqslant 2$ and denote by

$$\mathcal{M}^*(A,(p_0,q_0),\dots,(p_d,q_d),J) \subset \mathcal{M}(A,(p_0,q_0),\dots,(p_d,q_d),J)$$

the moduli space of simple J-holomorphic curves with corners at the (p_i,q_i). Standard regularity arguments (such as in [MS12] or [FHS95]) imply that there is a second category subset

$$\mathcal{J}_{\mathrm{reg}}(M,\omega) \subset \mathcal{J}(M,\omega)$$

such that for $J \in \mathcal{J}_{\mathrm{reg}}(M,\omega)$ the space

$$\mathcal{M}^*(A,(p_0,q_0),\dots,(p_d,q_d),J)$$

is either empty or a manifold of dimension $\operatorname{Ind}(D_u)+d-2$.

If $d \in \{0,1\}$, we quotient $\widetilde{\mathcal{M}}^*(A,(p_0,q_0),\dots,(p_d,q_d),J)$ by the space of conformal reparameterizations leaving the marked points fixed and denote the resulting space by

$$\mathcal{M}^*(A,(p_0,q_0),\dots,(p_d,q_d),J).$$

There is a second category subset

$$\mathcal{J}_{\mathrm{reg}}(M,\omega) \subset \mathcal{J}(M,\omega)$$

such that the space $\mathcal{M}^*(A,(p_0,q_0),\dots,(p_d,q_d),J)$ is either empty or a manifold of dimension $\operatorname{Ind}(D_u)+d-2$.

We let A_1 and A_2 be two homotopy classes of topological disks with corners at $(p_0,q_0),\dots,(p_d,q_d)$ and $(\tilde{p}_0,\tilde{q}_0),\dots,(\tilde{p}_m,\tilde{q}_m)$ respectively. We define

$$\mathcal{M}^*(A_1,A_2,(p_0,q_0),\dots,(p_d,q_d),(\tilde{p}_0,\tilde{q}_0),\dots,(\tilde{p}_m,\tilde{q}_m),J)$$

to be the set of pairs of simple disks (u_1,u_2) such that $u_1(\mathbb{D}) \not\subset u_2(\mathbb{D})$ and $u_2(\mathbb{D}) \not\subset u_1(\mathbb{D})$. There is a second category subset $\mathcal{J}_{\mathrm{reg}}(M,\omega)$ such that for each $J \in \mathcal{J}_{\mathrm{reg}}(M,\omega)$ the space

$$\mathcal{M}^*(A_1,A_2,(p_0,q_0),\dots(\tilde{p}_m,\tilde{q}_m),J)$$

is either empty or a smooth manifold of dimension $2n+\mu_{A_1}+\mu_{A_2}-\sum_i \operatorname{Ind}(p_i,q_i)-\sum_i \operatorname{Ind}(\tilde{p}_i,\tilde{q}_i)$.

Now for $k \geqslant 0$ consider the moduli space of (parameterized) pairs of disks with marked points at the boundary

$$\mathcal{M}^*_k(A_1, A_2, (p_0, q_0), \dots (\tilde{p}_m, \tilde{q}_m), J) := \mathcal{M}^*(A_1, A_2, (p_0, q_0), \dots (\tilde{p}_m, \tilde{q}_m), J) \times (\partial D)^{2k}.$$

There is a smooth evaluation map

$$\begin{array}{cccc} \mathrm{ev}_k : & \mathcal{M}^*_k(A_1, A_2, (p_0, q_0), \dots (\tilde{p}_m, \tilde{q}_m), J) & \to & L^{2k} \\ & (u_1, u_2, z_1, \dots z_k, x_1, \dots, x_k) & \mapsto & (u_1(z_1), u_2(x_1), \dots, u_1(z_k), u_2(z_k)) \end{array}.$$

Denote by $\Delta = \{(x, x) | x \in L\} \subset L \times L$ the diagonal. There is a second category subset $\mathcal{J}_{\mathrm{reg}}(M, \omega)$ such that for every $J \in \mathcal{J}_{\mathrm{reg}}(M, \omega)$ and $k \geqslant 1$, the evaluation map ev_k is transversal to the product Δ^k. Hence, if not empty, the set $\mathrm{ev}_k^{-1}(\Delta^k)$ has the structure of a smooth manifold of dimension $2n + \mu_{A_1} + \mu_{A_2} - \sum \mathrm{Ind}(p_i, q_i) - \sum \mathrm{Ind}(\tilde{p}_i, \tilde{q}_i) + (2-n)k$.

Assume that $n \geqslant 3$, then for k large enough, we have $2n + \mu_{A_1} + \mu_{A_2} - \sum \mathrm{Ind}(p_i, q_i) - \sum \mathrm{Ind}(\tilde{p}_i, \tilde{q}_i) + (2-n)k \leqslant 0$, so $\mathrm{ev}_k^{-1}(\Delta^k)$ is empty. We conclude that the following proposition holds.

Proposition 1.3.1. *There is a second category subset*

$$\mathcal{J}_{reg}(M, L, \omega) \subset \mathcal{J}(M, \omega)$$

such that if

- $J \in \mathcal{J}_{reg}(M, L, \omega)$,
- $u_1 \in \mathcal{M}^*(A_1, (p_0, q_0), \dots, (p_d, q_d), J)$,
- $u_2 \in \mathcal{M}^*(A_2, (\tilde{p}_0, \tilde{q}_0), \dots, (\tilde{p}_m, \tilde{q}_m), J)$,

satisfy

$$u_1(\mathbb{D}) \not\subset u_2(\mathbb{D}) \text{ and } u_2(\mathbb{D}) \not\subset u_1(\mathbb{D}),$$

then the set

$$\{z_1, z_2 \in \partial\mathbb{D} | u_1(z_1) = u_2(z_2)\}$$

is finite.

The same argument for self intersections yields

Proposition 1.3.2. *There is a second category subset*

$$\mathcal{J}_{reg}(M, L, \omega) \subset \mathcal{J}(M, \omega)$$

such that if $u \in \mathcal{M}^*(A, (p_0, q_0), \dots, (p_d, q_d), J)$, then the set

$$\{(z_1, z_2) \in \partial\mathbb{D} \times \partial\mathbb{D} | u(z_1) = u(z_2)\}$$

is finite.

Proof of Corollary 1.1.4. The rest of the proof follows closely the proof of [**Laz11**, Theorem B].

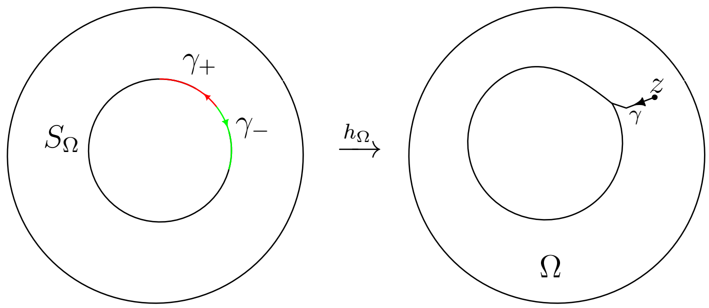

FIGURE 2 – *The surface S_Ω, the paths γ and $\gamma_\pm$.*

Since the space of continuous maps $(\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$ is separable and locally connected, the set of homotopy classes of polygons with corners and boundary on $i(L)$ is either finite or countable.

Therefore, by Propositions 1.3.1 and 1.3.2, there is a second category subset

$$\mathcal{J}_{\mathrm{reg}}(M, L, \omega) \subset \mathcal{J}_{\mathrm{reg}}(M, \omega)$$

such that for every almost complex structure $J \in \mathcal{J}_{\mathrm{reg}}(M, L, \omega)$ the following holds.

(1) If $u_1, u_2 : (\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$ are two finite energy, simple J-holomorphic polygons then
   - either $\mathrm{Im}(u_1) \subset \mathrm{Im}(u_2)$ or $\mathrm{Im}(u_2) \subset \mathrm{Im}(u_1)$,
   - or the set $\{z_1, z_2 \in \partial\mathbb{D} | u_1(z_1) = u_2(z_2)\}$ is finite.

(2) If $u : (\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$ is a finite energy, simple, J-holomorphic polygon with corners, then the set $\{(z_1, z_2) \in \partial\mathbb{D} \times \partial\mathbb{D} | u(z_1) = u(z_2)\}$ is finite.

We fix $J \in \mathcal{J}_{\mathrm{reg}}(M, L, \omega)$. Let u be a finite energy, J-holomorphic polygon with corners. We will show that u is multiply covered by showing that its frame satisfies $\mathcal{W}(u) = \partial\mathbb{D}$.

First, we claim that $\mathcal{W}(u)$ has no dead-ends. Assume the opposite, i.e. that there is $z \in \mathrm{Int}(\mathbb{D})$ which is a dead-end for $\mathcal{W}(u)$. Call Ω the connected component adjacent to this dead-end (see Figure 2). Recall that there are

- a Riemann surface S_Ω as well as a biholomorphism $h_\Omega : (S_\Omega, \partial S_\Omega) \to (\Omega, \partial\Omega)$,
- a holomorphic map $p_\Omega : (S_\Omega, \partial S_\Omega) \to (\mathbb{D}, \partial\mathbb{D})$, which restricts to a cover on the boundary,
- and a simple, finite energy, J-holomorphic polygon with v_Ω,

such that $u \circ h_\Omega = v_\Omega \circ p_\Omega$. Let $\gamma : [0, \varepsilon) \to \mathcal{W}(u)$ be a $\mathcal{C}^1$-embedded path such that $\gamma(0) = z$. By definition of S_Ω, this lifts to two $\mathcal{C}^1$ paths $\gamma_\pm : [0, \varepsilon) \to \partial S_\Omega$ with

- $\gamma_+(0) = \gamma_-(0)$,
- γ_+ orientation preserving and γ_- orientation reversing,
- $h_\Omega \circ \gamma_\pm = \gamma$.

Again, we refer to Figure 2. Therefore, we have for all $t \in [0, \varepsilon)$, $v_\Omega \circ p_\Omega(\gamma_\pm(t)) = u \circ \gamma(t)$. So v_Ω is a simple pseudo-holomorphic polygon with infinitely many intersection points at its boundary. This is contradiction, by property (1) of the definition of $\mathcal{J}_{\text{reg}}(M, L, \omega)$. Therefore, $\mathcal{W}(u)$ has no dead-ends.

Assume that $\mathbb{D} \backslash \partial \mathcal{W}(u)$ has several distinct connected components. Pick two distinct, adjacent, connected components Ω_1 and Ω_2 of $\partial\mathbb{D} \backslash \mathcal{W}(u)$. Then, there are two simple, finite energy, J-holomorphic polygons v_1 and v_2 and two holomorphic maps $p_1 : (\overline{\Omega}_1, \partial\overline{\Omega}_1) \to (\mathbb{D}, \partial\mathbb{D})$ and $p_2 : (\overline{\Omega}_2, \partial\overline{\Omega}_2) \to (\mathbb{D}, \partial\mathbb{D})$ such that

$$v_i = u_{|\Omega_i} \circ p_i,\ \ i = 1, 2.$$

Let $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{W}(u)$ be an embedded $\mathcal{C}^1$ path whose image is contained in the intersection $\partial\overline{\Omega}_1 \cap \partial\overline{\Omega}_2$. Since $u_{|\Omega_1}(\gamma(t)) = u_{|\Omega_2}(\gamma(t))$, we get

$$\forall t \in (-\varepsilon, \varepsilon) v_1 \circ p_1 \circ \gamma(t) = v_2 \circ p_2(\gamma(t)).$$

So, by property (1) of $\mathcal{J}_{\text{reg}}(M, L, \omega)$, we can assume $\operatorname{Im}(v_1) \subset \operatorname{Im}(v_2)$.

Then, $\mathcal{R}_{v_1}^{v_2} \neq \emptyset$. To see this, let $t \in (-\varepsilon, \varepsilon)$ be such that $p_1 \circ \gamma(t) \notin \mathcal{C}(v_1, v_2)$ and $p_2\gamma(t) \notin \mathcal{C}(v_2, v_1)$. Since $du_1(p_1 \circ \gamma(t)) \neq 0$, the map v_1 is a local embedding around $p_1 \circ \gamma(t)$ with image contained in v_2. Moreover, v_2 is a local embedding around $p_2 \circ \gamma(t)$. So, we conclude that v_1 is a local reparameterization of v_2 near $p_1 \circ \gamma(t)$. Hence $p_1 \circ \gamma(t)\ \mathcal{R}_{v_1}^{v_2}\ p_2 \circ \gamma(t)$.

Moreover, we have $\mathcal{W}(v_1, v_2) \subset \mathcal{W}(v_1, u) \subset \partial\mathbb{D}$ and $\mathcal{W}(v_2, v_1) \subset \mathcal{W}(v_2, u) \subset \partial\mathbb{D}$. So v_1 and v_2 have relatively simple frame. Therefore v_1 is a reparameterization of v_2 (and vice-versa) by Theorem 1.2.30.

However, we also have $v_1 \circ p_1 \circ \gamma = v_2 \circ p_2 \circ \gamma$. Moreover, (up to a switch between v_1 and v_2) the map $p_1 \circ \gamma$ has the same orientation as $\partial\mathbb{D}$ while $p_2 \circ \gamma$ does not. Therefore, the polygon v_1 has infinitely many self-intersection point, a contradiction with property (2) of the definition of $\mathcal{J}_{\text{reg}}(M, L, \omega)$.

Hence, it must be that $\mathbb{D} \backslash \mathcal{W}(u)$ is connected. Since $\mathcal{W}(u)$ has no dead ends, we have $\mathcal{W}(u) = \partial\mathbb{D}$. This finishes the proof. $\square$

Generically teardrops and strips are simple

Definition 1.3.3. *A finite-energy J-holomorphic disk $u : (\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$ with corners and boundary on L is a* teardrop *if it has* a unique *corner point*

An elementary argument gives the following consequence of Corollary 1.1.4.

Corollary 1.3.4. *Assume that the complex dimension of (M, ω) satisfies $n \geqslant 3$. There is a second category subset*

$$\mathcal{J}_{reg}(M, L, \omega) \subset \mathcal{J}(M, \omega)$$

such that the following holds.

If $J \in \mathcal{J}_{reg}(M, L, \omega)$, every finite-energy J-holomorphic teardrop u with boundary in L is simple.

Proof. We assume that $\mathcal{J}_{\mathrm{reg}}(M, L, \omega)$ is the second category subset given in Corollary 1.1.4 (such that all the J-holomorphic curves are either simple or multiply covered for $J \in \mathcal{J}_{\mathrm{reg}}(M, L, \omega)$).

Assume that $u : (\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$ is a teardrop with finite energy. Denote by z_1 its corner point. There is a simple disk v as well as a branched cover $p : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{D}, \partial\mathbb{D})$ which restricts to a cover on the boundary such that $u = v \circ p$. Notice that v has a corner point at $p(z_1)$. Now if the the degree of p is more than 2, we see that u has corner points at $p^{-1}\{z_1\}$. This set is of cardinality greater or equal to 2, a contradiction. $\square$

Now, let $i_1 : L_1 \to M$ and $i_2 : L_2 \to M$ be two Lagrangian immersions. The disjoint union give rise to a Lagrangian immersion

$$i : L_1 \sqcup L_2 \to M. \tag{1.11}$$

We assume that i is generic (transverse double points and no triple points).

Definition 1.3.5. *We assume L_1 and L_2 are two generic Lagrangian immersions, and that the immersion i is the one defined in 1.11.*

Let $J \in \mathcal{J}(M, \omega)$ be a compatible almost complex structure, x_- and x_+ be two intersection points between L_1 and L_2. A J-holomorphic strip between L_1 and L_2 from x_- to x_+ is a J holomorphic map $u : \mathbb{R} \times [0, 1] \to M$ such that

(1) $\lim_{s\to+\infty} u(s, t) = x_-, \lim_{s\to+\infty} = x_+$,

(2) $u(\cdot, 0)$ (resp. $u(\cdot, 1)$) admits a continuous lift γ_1 to L_1 (resp. a continuous lift γ_2 to L_2).

Remark 1.3.6. Precompose a J-holomorphic strip u with a biholomorphism

$$\psi : \mathbb{D}\backslash\{-1, 1\} \to \mathbb{R} \times [0, 1].$$

The resulting map $u \circ \psi$ is a J-holomorphic disk with two corner points. The boundary lifts are

$$\tilde{\gamma}_1 = \gamma_1 \circ \psi \text{ and } \tilde{\gamma}_2 = \gamma_2 \circ \psi.$$

We now have the following proposition about the structure of such strips.

Proposition 1.3.7. *We assume that we are in the setting of Definition 1.3.5.*

There is a second category subset

$$\mathcal{J}(M, L_1, L_2, \omega) \subset \mathcal{J}(M, \omega)$$

such that the following holds.

If $J \in \mathcal{J}(M, L_1, L_2, \omega)$, every finite-energy J-holomorphic strip[5] *between i_1 and i_2 is simple.*

Proof. Recall, from Corollary 1.1.4, that there is a second category subset

$$\mathcal{J}(M, L_1, L_2, \omega) \subset \mathcal{J}(M, \omega)$$

such that for every almost complex structure $J \in \mathcal{J}(M, L_1, L_2, \omega)$, every finite-energy J-holomorphic polygon with boundary on $i : L_1 \sqcup L_2 \to M$ is simple or multiply covered. Let us fix one such J.

Let $u : \mathbb{R} \times [0, 1] \to M$ be a finite-energy J-holomorphic strip from x_- to x_+. Let $\psi : \mathbb{D} \backslash \{-1, 1\} \to \mathbb{R} \times [0, 1]$ be a biholomorphism. Then, the map, $u \circ \psi$ is a finite-energy, J-holomorphic polygon with two corner points at -1 and 1. It is therefore multiply covered. There is a simple pseudo-holomorphic polygon $v : \mathbb{D} \to M$ and a covering $p : \mathbb{D} \to \mathbb{D}$ such that $u = v \circ p$. Assume by contradiction that the degree of p is greater or equal than 2.

Then, the set $p^{-1}(p(1))$ has a cardinal greater or equal than 2. Moreover, it is easy to see that each $z \in p^{-1}(p(1))$ is a corner point. So $p^{-1}(p(1))$ is the pair $\{-1, 1\}$.

However, we claim that $-1 \notin p^{-1}(p(1))$. Indeed, for each $z \in p^{-1}(p(1))$, any conformal embedding $h : (\mathbb{D}^+, \mathbb{D}^+_{\mathbb{R}}) \to (\mathbb{D}, \partial\mathbb{D})$ such that $h(0) = z$ satisfies $u \circ h(\mathbb{R}^+) \subset L_2$ and $u \circ h(\mathbb{R}^-) \subset L_1$. This can be seen, for instance, by using local charts. However, a conformal local chart $h : (\mathbb{D}^+, \mathbb{D}^+_{\mathbb{R}}) \to (\mathbb{D}, \partial\mathbb{D})$ such that $h(0) = -1$ satisfies $u \circ h(\mathbb{R}^-) \subset L_2$ and $u \circ h(\mathbb{R}^+) \subset L_1$. □

We need one last statement. For this, let $d \geqslant 2$ be a natural integer and

$$L_0, \dots, L_d \subset M$$

be $d + 1$ embedded Lagrangian submanifolds in *general position*. This means that the induced immersion

$$\coprod_{i=0}^{d} L_i \to M$$

is generic.

Definition 1.3.8. *Assume that we are in the above setting.*

For $i \in \{0, \dots, d\}$, fix $x_i \in L_i \cap L_{i+1}$. A J-holomorphic polygon *with boundary condition $L_1, \dots, L_d$ is a J-holomorphic map*

$$u : \pi^{-1}(r) \to M$$

[5] see Definition 1.3.5.

for some $r \in \mathcal{R}^{d+1}$ *such that*

$$\lim_{s \to +\infty} u \circ \varepsilon_i(s,t) = x_i,$$

and the image by u *of the arc between the* i *and* $i+1$ *end is included in* L_i*.*

We now have the following proposition.

Proposition 1.3.9. *Assume that* $L_0, \ldots, L_d$ *are Lagrangian submanifolds as above (in general position).*

There is a second category subset

$$\mathcal{J}(M, L_0, \ldots, L_d, \omega) \subset \mathcal{J}(M, \omega)$$

satisfying the following property.

If $J \in \mathcal{J}(M, L_0, \ldots, L_d, \omega)$*, every* J*-holomorphic polygon* u *of finite energy with boundary condition*[6] $L_0, \ldots, L_d$ *is simple.*

Proof. As before, call $i : \bigsqcup L_i \to M$ the natural Lagrangian immersion. Recall from Corollary 1.1.4, that there is a second category subset

$$\mathcal{J}(M, L_0, \ldots, L_d, \omega) \subset \mathcal{J}(M, \omega)$$

such that any finite-energy J-holomorphic disk with boundary on i is simple or multiply covered. Fix an almost complex structure J in the set $\mathcal{J}(M, L_0, \ldots, L_d, \omega)$.

Now let u be a J-holomorphic polygon of finite energy and fix a biholomorphism $\psi : \mathbb{D} \to \pi^{-1}(r)$. Call $y_0, \ldots, y_d$ the preimages of the marked points of $\pi^{-1}(r)$ by ψ. Then $u \circ \psi$ is a finite-energy disk with boundary on i. It is therefore multiply covered. So there is a simple pseudo-holomorphic polygon $v : (\mathbb{D}, \partial\mathbb{D}) \to (M, i(L))$ and a covering $p : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{D}, \partial\mathbb{D})$ such that $u \circ \psi = p \circ v$.

As before, we see that for $i \neq j$, $y_i \notin p^{-1}(p(y_j))$ since the image of any neighborhood of $z \in p^{-1}(p(y_j))$ in $\partial\mathbb{D}$ by u intersects both L_j and L_{j+1}.

Assume, by contradiction that the degree of p is greater or equal than 2. Then the set $p^{-1}(p(y_j))$ has more than two elements. It therefore contains a corner point y_i with $i \neq j$. This is in clear contradiction with the above claim. Hence, the covering p is of degree 1. $\square$

Remark 1.3.10. One should be aware that the set $\mathcal{J}(M, L_1, \ldots, L_d, \omega)$ *depends on the Lagrangian submanifolds* $L_0, \ldots, L_d$. In particular, the author does not know if for a generic J, any J-holomorphic polygon (without restriction on the boundaries) is simple.

[6] See Definition 1.3.8

1.3.2. Time-independent Floer homology

As an application of the Main Theorem 1.1.3, we show that we can compute Lagrangian intersection Floer homology with time-independent complex structures. This is (as far as the author knows) new.

To do this, let us now assume that (M, ω) is *closed and monotone*. Let

$$[\omega] : H_2(M) \to \mathbb{R}$$

be the morphism induced by symplectic area. There is $\lambda > 0$ such that

$$[\omega] = \lambda c_1(TM).$$

Let $L_1 \subset M$ and $L_2 \subset M$ be two embedded compact Lagrangian submanifolds and denote by $N_1 \geqslant 1$ and $N_2 \geqslant 1$ their minimal Maslov number. We assume that $N_p \geqslant 3$ for $p \in \{1, 2\}$. Moreover, symplectic area and Maslov class induce two morphisms

$$[\omega] : H_2(M, L) \to \mathbb{R}, \mu_L : H_2(M, L) \to \mathbb{Z}.$$

We assume that L_1 and L_2 are *monotone* :

$$[\omega] = \frac{\lambda}{2}\mu_L.$$

Further, we assume that L_1 and L_2 are *transverse*.

For x and y two distinct intersection points in $L_1 \cap L_2$, and A a homotopy class of finite-energy strips from x to y, we denote

- by $\widetilde{\mathcal{M}}(x, y, L_1, L_2, A, J)$ the set of J-holomorphic strips from x to y in the homotopy class A,
- by $\mathcal{M}(x, y, L_1, L_2, A, J)$ its quotient by the natural $\mathbb{R}$-action,
- by $\widetilde{\mathcal{M}}^*(x, y, A, L_1, L_2, J) \subset \widetilde{\mathcal{M}}(x, y, A, L_1, L_2, J)$ the set of simple J-holomorphic strips,
- by $\mathcal{M}^*(x, y, A, L_1, L_2, J)$ its quotient by the $\mathbb{R}$-action.

From Proposition 1.3.7 and the standard transversality arguments (cf [**FHS95**]), we immediately deduce the following.

Proposition 1.3.11. *In the above setting, there is a second category subset*

$$\mathcal{J}_{reg}(M, L_1, L_2, \omega) \subset \mathcal{J}(M, \omega)$$

such that

(1) If $J \in \mathcal{J}_{reg}(M, L_1, L_2, \omega)$, for each homotopy class of strips A, all J-holomorphic strips are simple :

$$\forall x, y \in L_1 \cap L_2,\ \mathcal{M}^*(x, y, L_1, L_2, A, J) = \mathcal{M}(x, y, L_1, L_2, A, J).$$

(2) If $J \in \mathcal{J}_{reg}(M, L_1, L_2, \omega)$, then for all the intersection points $x, y \in L_1 \cap L_2$, the moduli space $\mathcal{M}(x, y, L_1, L_2, A, J)$ is either empty or a finite-dimensional manifold.

Now for such a generic $J \in J \in \mathcal{J}_{\mathrm{reg}}(M, L_1, L_2, \omega)$ and a natural integer $k \in \mathbb{N}$, denote by

$$\mathcal{M}(x, y, L_1, L_2, A, J)^k$$

the k dimensional component of $\mathcal{M}(x, y, L_1, L_2, A, J)$.

By a standard Gromov compactness argument, the set $\mathcal{M}^0(x, y, A, J)$ is compact. Furthermore, the set $\mathcal{M}^1(x, y, A, J)$ admits a compactification

$$\overline{\mathcal{M}}^1(x, y, A, J)$$

which is a compact 1-dimensional manifold with boundary

$$\partial\overline{\mathcal{M}}^1(x, y, A, J) = \bigsqcup_{z \in L_1 \cap L_2} \mathcal{M}^0(x, z, A, J) \times \mathcal{M}^0(z, y, A, J).$$

The coefficients are over the Novikov ring of formal power series with coefficients in $\mathbb{Z}_2$:

$$\Lambda_{\mathbb{Z}_2} = \left\{ \sum_{\substack{\lambda_i \to +\infty, \\ \lambda_i \geqslant 0}} a_i T^{\lambda_i} \middle| a_i \in \mathbb{Z}_2 \right\}.$$

As usual we define the Floer complex between L_1 and L_2 to be the $\Lambda_{\mathbb{Z}_2}$-module generated by the intersection points

$$CF(L_1, L_2, J) = \bigoplus_{x \in L_1 \cap L_2} \mathbb{Z}_2 \cdot x.$$

The differential on this complex is given by a count of rigid J-holomorphic strips in $\mathbb{Z}_2$

$$\begin{array}{rccc} d: & CF(L_1, L_2, J) & \to & CF(L_1, L_2, J) \\ & y & \mapsto & \sum_{y \in L_1 \cap L_2} \#_{\mathbb{Z}_2} \mathcal{M}^0(x, y, A, J)\ T^{\omega(A)} y. \end{array}$$

It is immediate to see from the usual Gromov compactness argument that $\mathrm{d}^2 = 0$, so $(CF(L_1, L_2), \mathrm{d})$ is a well-defined differential complex.

Since any generic almost complex structure is, in particular, a generic time-dependent almost complex structure, the homology of this complex computes the usual Lagrangian intersection Floer homology. Hence we can conclude that the following theorem is true.

Theorem 1.3.12. *In the above setting, there is a second category subset*

$$\mathcal{J}_{reg}(M, \omega, L_1, L_2) \subset \mathcal{J}(M, \omega)$$

such that the Floer complex

$$(CF(L_1, L_2, J), d)$$

is well-defined (as a differential complex). Moreover, its homology computes the usual Lagrangian intersection Floer homology.

1.3.3. Work in progress

Since Akaho's ([**Aka05**]) and Akaho-Joyce's ([**AJ10**]) work, it is well-known that pseudo-holomorphic teardrops yield obstructions to the definition of Floer homology for (generic) Lagrangian immersions. More precisely, let L_1 and L_2 be two transverse generic Lagrangian immersions. Assume that J is a compatible almost complex structure. Using virtual perturbation techniques, Akaho and Joyce ([**AJ10**]) expressed the square of the differential of the Floer complex between L_1 and L_2 with an algebraic count of J-holomorphic teardrops with boundaries on on of the immersions L_1 or L_2.

On the other hand, an ongoing research program of Biran and Cornea aims to show that some Lagrangian immersions are geometric representatives of distinguished cones in the Fukaya category. I will informally describe the main idea of their research plan. Assume that L_1 and L_2 are two transverse, embedded, Lagrangians. Further, assume that we are in a setting where Floer homology is well-defined (without virtual fundamental cycle techniques). For instance, one can assume

- (M, ω) is exact with convex boundary and L_1, L_2 are exact, graded, closed Lagrangians,
- or M, L_1, L_2 are closed, weakly monotone with Maslov number greater or equal than two.

Fix a cycle $c \in CF(L_1, L_2)$ which is a formal sum of intersection points. Then, the surgery of L_1 and L_2 along these intersection points is expected to be an immersed Lagrangian which represents, in a sense, the surgery between L_1 and L_2. Note that a similar idea was used in [**CDRGG17**] to find generators of the wrapped Fukaya category of a Weinstein sector.

Let us restrict to an exact symplectic manifold with convex boundary. We assume that L_1 and L_2 are closed, exact, graded Lagrangian submanifolds. We expect that we can use our work to prove Theorem 1.3.16. It asserts that, for a generic almost complex structure, the algebraic count of pseudo-holomorphic teardrops with boundary on a surgery equals an algebraic count of pseudo-holomorphic strips between the embeddings L_1 and L_2. This is the first step of Biran and Cornea's program. Furthermore, we outline the proof of this Theorem. We will highlight, along the way, the points that are not yet proven.

Framework

In this subsection, we describe some expected applications of the main theorems to Floer theoretic properties of the surgery of two immersed Lagrangian submanifolds.

We assume that (M, ω) is an exact symplectic manifold with Liouville form λ, convex boundary and complex dimension $n \geqslant 3$. We denote by $\widehat{M}$ its completion.

Gradings: We assume that the first Chern class of (M, ω) satisfies

$$2c_1(TM) = 0$$

in $H^2(M, \mathbb{Z})$. This implies that the complex line bundle $\Lambda^n T^*M \otimes \Lambda^n T^*M$ is trivial. Hence, it admits a non-vanishing section Ω.

For each Lagrangian subspace $\Lambda \in \mathcal{G}(T_xM)$, choose a real basis $v_1, \ldots, v_n$ of Λ and define

$$\det{}^2_\Omega(\Lambda) = \frac{\Omega(v_1 \wedge \ldots \wedge v_n,\ v_1 \wedge \ldots \wedge v_n)}{|\Omega(v_1 \wedge \ldots \wedge v_n,\ v_1 \wedge \ldots \wedge v_n)|}.$$

If $w_1, \ldots, w_n$ is another basis of Λ, denote by A the transition matrix from the basis $(v_1, \ldots, v_n)$ to the basis $(w_1, \ldots, w_n)$. Then,

$$\begin{aligned}\frac{\Omega(v_1 \wedge \ldots \wedge v_n,\ v_1 \wedge \ldots \wedge v_n)}{|\Omega(v_1 \wedge \ldots \wedge v_n,\ v_1 \wedge \ldots \wedge v_n)|} &= \frac{\det(A)^2\Omega(w_1 \wedge \ldots \wedge w_n,\ w_1 \wedge \ldots \wedge w_n)}{\det(A)^2|\Omega(w_1 \wedge \ldots \wedge w_n,\ w_1 \wedge \ldots \wedge w_n)|} \\ &= \frac{\Omega(w_1 \wedge \ldots \wedge w_n,\ w_1 \wedge \ldots \wedge w_n)}{|\Omega(w_1 \wedge \ldots \wedge w_n,\ w_1 \wedge \ldots \wedge w_n)|}\end{aligned}$$

Hence, the real number $\det^2_\Omega(\Lambda)$ does not depend on the choice of basis of the real vector space Λ. Therefore, this defines a smooth function

$$\det{}^2_\Omega : \mathcal{G}(TM) \to \mathbb{R}.$$

Definition 1.3.13. *In the above setting, an* exact graded Lagrangian immersion *is a tuple (L, i, f_L, θ_L) with*

- *L a compact manifold,*
- *$i : L \looparrowright M$ a generic Lagrangian immersion,*
- *$f_L : L \to \mathbb{R}$ a smooth function such that $i^*\theta = df_L$,*
- *$\theta_L : L \to \mathbb{R}$ a smooth function such that*

$$e^{2i\pi\theta_L} = \det{}^2_\Omega \circ \iota$$

where $\iota : L \to \mathcal{G}(TM)$ is the map $x \mapsto \mathrm{Im}(di_x)$.

Now, let L_1 and L_2 be two transverse exact, graded, immersions which intersect transversally. Let $x \in L_1 \cap L_2$ be an intersection point. Fix an adapted almost complex structure J and denote by $\alpha_1, \ldots, \alpha_n$ the Kähler angles of the pair (T_xL_1, T_xL_2). The *index* of x as an element of $CF(L_1, L_2)$ is the number

$$|x| = n + \theta_{L_2}(x) - \theta_{L_1}(x) - \frac{\alpha_1 + \ldots + \alpha_n}{\pi}.$$

Similarly, we can define the *index* of a self intersection point $(p, q) \in R$ of L_i for $i = 1, 2$:

$$|(p,q)| = n + \theta_{L_2}(q) - \theta_{L_1}(p) - \frac{\alpha_1 + \ldots + \alpha_n}{\pi},$$

where $\alpha_1, \ldots, \alpha_n$ are the Kähler angles of the pair $(di_p(T_pL), di_q(T_qL))$ with respect to J.

Remark 1.3.14. There is a choice of path $\lambda_{p,q}$ such that the index $\mathrm{Ind}(p, q)$ (defined in subsubsection 1.3.1) agrees with the index $|(p, q)|$. We refer to [**Sei08**, (11g)] for a complete proof.

Lagrangian surgery: We describe the surgery of L_1 and L_2 at an intersection point following the presentation of Biran and Cornea ([**BC13**, 6.1]).

For each intersection point y between L_1 and L_2, fix a Darboux chart

$$\phi_y : B(0, r_y) \subset \mathbb{C}^n \to (M, \omega) \tag{1.12}$$

such that

$$\phi_y(0) = y,\ \phi_y(\mathbb{R}^n) \subset L_1,\ \phi_y(i\mathbb{R}^n) \subset L_2$$

and whose image does not contain any other intersection point.

Now consider a smooth path

$$\gamma(t) := (a(t), b(t)) \in \mathbb{C},\ t \in \mathbb{R},$$

such that

- $\gamma(t) = (t, 0)$ for $t < -1$,
- $\gamma(t) = (0, t)$ for $t > 1$,
- $a'(t) > 0$ and $b'(t) > 0$ for $t \in (-1, 1)$.

For $\varepsilon > 0$, the set

$$H_\varepsilon := \left\{(x_1\gamma(t), \ldots, x_n\gamma(t)) \middle| t \in \mathbb{R},\ (x_1, \ldots, x_n) \in S^{n-1}\right\}$$

is a smooth Lagrangian submanifold of $\mathbb{C}^n$ diffeomorphic to $\mathbb{R} \times S^{n-1}$.

Let $x \in L_1 \cap L_2$ be an intersection point. For $\varepsilon > 0$ small enough, there is a generic immersion $L_1\#_{x,\varepsilon}L_2$ obtained by removing the set $\phi_x(\mathbb{R}^n \cup i\mathbb{R}^n)$ and replacing it by $\phi_x(H_\varepsilon)$. This has domain the connected sum $L_1\#L_2$ of L_1 and L_2 and turns out to be exact. Moreover, if the index of x satisfies $|x| = 1$, the gradings of L_1 and L_2 canonically induce a grading

$$\theta : L_1\#L_2 \to \mathbb{R}$$

on $L_1 \#_{x,\varepsilon} L_2$ which satisfies

$$\theta_{|L_1 \setminus \phi_x(B(0,r_x))} = \theta_{L_1}$$
$$\theta_{|L_2 \setminus \phi_x(B(0,r_x))} = \theta_{L_2}$$

This is a result of Seidel, see [**Sei00**, Lemma 2.13]).

For the remainder of this section, we assume that the submaniolds L_1 and L_2 are *embedded* and *compact*. Let $x \in L_1 \cup L_2$ be an intersection point of degree 1.

We somewhat restrict the space of almost complex structures we consider. Recall that J_{std} is the standard complex structure on $\mathbb{C}^n$ and that we fixed Darboux charts ϕ_y near each intersection point $y \in L_1 \cap L_2$ (see 1.12).

We let $\mathcal{J}_\phi(M, \omega)$ be the set of adapted almost complex structures such that

$$\forall y \in L_1 \cap L_2,\ (\phi_y)_* J_{\mathrm{std}} = J \text{ on } \phi_y(B(0, r_y)).$$

In the proof of Corollary 1.1.4, we proved the following Lemma.

Lemma 1.3.15. *We let L_1 and L_2 be two exact, graded and compact Lagrangian submanifolds as in the above setting.*

There is a second category subset

$$\mathcal{J}_{\phi,reg}(M, \omega) \subset \mathcal{J}_\phi(M, \omega) \tag{1.13}$$

such that

(1) any non-constant J-holomorphic disk with corners and boundary on the immersion $L_1 \sqcup L_2 \looparrowright M$ is either simple or multiply covered,

(2) any simple J-holomorphic disk with corners and boundary on the immersion $L_1 \sqcup L_2 \looparrowright M$ is regular (meaning that the linearization of the Cauchy-Riemann operator is surjective).

Surgery and count of holomorphic disks

We expect that we can apply our work to prove the following.

Theorem 1.3.16 ($\star$)**.** [7] *We assume that L_1 and L_2 are two transverse graded Lagrangian submanifolds as in the above setting. Let $(\varepsilon_\nu)_{\nu \in \mathbb{N}}$ be a sequence of positive real numbers which converges to 0 : $\varepsilon_\nu \to 0$. There is a second category subset of 1.13*

$$\mathcal{J}_{\phi,reg,2}(M, \omega) \subset \mathcal{J}_{\phi,reg}(M, \omega) \tag{1.14}$$

and a natural integer $\nu_0 \in \mathbb{N}$ such that the following holds.

[7] We use the symbol $\star$ to indicate the results that are work in progress.

Let $\nu \geqslant \nu_0$, $J \in \mathcal{J}_{\phi,reg,2}(M,\omega)$ and $y \in L_1 \cap L_2$ with $|y| = 2$. There is a bijection

$$\mathcal{M}(y, J, L_1 \#_{x,\varepsilon_\nu} L_2) \to \mathcal{M}(y, x, J, L_1, L_2)$$

between the set of J-holomorphic teardrops with boundary on $L_1 \#_{x,\varepsilon_\nu} L_2$ and the set of J-holomorphic strips between L_1 and L_2 from y to x.

Below, we list the main steps expected to lead to this result.

Surgery and holomorphic disks

The proof of theorem 1.3.16 relies on three results. The first is a result about the multiplicity [8] of an isolated generic J-holomorphic strip at its corners.

Proposition 1.3.17. *The Lagrangian submanifolds L_1 and L_2 are as in the above setting. There is a second category subset of 1.13*

$$\mathcal{J}_{\phi,reg,3}(M, L_1, L_2, \omega) \subset \mathcal{J}_{\phi,reg}(M,\omega)$$

such that if $J \in \mathcal{J}_{\phi,reg,3}(M, L_1, L_2, \omega)$, then every J-holomorphic strip of Fredholm index 1 has multiplicity 1 at its corners.

Sketch of the proof. Fix x and y such that $|x| - |y| = 1$. Consider the universal moduli space $\mathcal{M}^*\big(L_1, L_2, x, y, \mathcal{J}^l\big)$ of pairs (u, J) with

- J a $\mathcal{C}^l$ almost complex structure in $\mathcal{J}_\phi(M,\omega)$,
- u a J-holomorphic strip between L_1 and L_2 from x to y.

The usual arguments (as in [MS12, Chapter 3]) show that $\mathcal{M}^*\big(L_1, L_2, x, y, \mathcal{J}^l\big)$ admits the structure of a smooth separable Banach manifold.

Assume that $(u, J) \in \mathcal{M}^*\big(L_1, L_2, x, y, \mathcal{J}^l\big)$, then (see Proposition 1.2.10)[9] the limits

$$\mathrm{ev}_{\mathrm{x,jet}}(u, J) := \lim_{s \to -\infty} e^{-\frac{\pi}{2}s} u(s,t),$$

and

$$\mathrm{ev}_{\mathrm{y,jet}}(u, J) := \lim_{s \to +\infty} e^{+\frac{\pi}{2}s} u(s,t)$$

exist. This defines two smooth maps

$$\mathrm{ev}_{\mathrm{x,jet}} : \mathcal{M}^*(x, y, L_1, L_2, \mathcal{J}) \to \mathbb{R}^n$$

and

$$\mathrm{ev}_{\mathrm{y,jet}} : \mathcal{M}^*(x, y, L_1, L_2, \mathcal{J}) \to \mathbb{R}^n.$$

Notice that $\mathrm{ev}_{\mathrm{x,jet}}^{-1}(0)$ (resp. $\mathrm{ev}_{\mathrm{y,jet}}^{-1}(0)$) is the set of J-holomorphic strip with multiplicity greater than 1 at x (resp. y).

[8] See Remark 1.2.11 for the definition of multiplicity.

[9] However, in this setting there is no real necessity to use [RS01]. Since the curves are holomorphic near the double points, we can use the Schwarz reflection principle twice.

A variation of the arguments of [MS12, 3.4] show that these are submersions. Hence, the sets $\mathrm{ev}^{-1}_{\mathrm{x,jet}}(0)$ and $\mathrm{ev}^{-1}_{\mathrm{y,jet}}(0)$ are smooth submanifolds of codimension n.

Now, one can see from the Sard-Smale theorem and an argument due to Taubes (see [MS12, 3.2] or [FHS95, Section 5]) that there is a generic subset $\widetilde{\mathcal{J}} \subset \mathcal{J},\phi(M,\omega)$ satisfying the following. For each $J \in \widetilde{\mathcal{J}}$, $\mathrm{ev}^{-1}_{\mathrm{x,jet}}(0) \cap \mathcal{M}^*(x,y,L_1,L_2,J)$ is a submanifold of codimension n in $\mathcal{M}^*(x,y,L_1,L_2,J)$ which has dimension 1. It is therefore empty (since the complex dimension satisfies $n \geqslant 3$).

The conclusion follows since any J-holomorphic strip is simple (cf Proposition 1.3.7). $\square$

In complex dimension greater than 3, the same conclusion holds for teardrops.

Proposition 1.3.18. *Here, L_1 and L_2 are graded, exact and compact Lagrangian submanifolds as in the above setting.*

There is a second category subset of 1.13

$$\mathcal{J}_{\phi,reg,4}(M,L_1,L_2,\omega) \subset \mathcal{J}_{\phi,reg}(M,L_1,L_2,\omega)$$

such that the following holds.

Let $J \in \mathcal{J}_{\phi,reg,4}(M,L_1,L_2,\omega)$ and $\nu \geqslant 0$ be a natural integer. Every J-holomorphic teardrop of Fredholm index 2 and boundary on the immersion $L_1\#_{x,\varepsilon_\nu}L_2$ has multiplicity 1 at its corner.

Proof. Fix $\nu \in \mathbb{N}$, as in the proof of Proposition 1.3.17, there is a second category subset $\mathcal{J}^{\nu}_{\phi,\mathrm{reg}}(M,\omega)$ such that every J-holomorphic teardrop with boundary on $L_1\#_{x,\varepsilon_\nu}L_2$ has multiplicity 1 at its corner.

Now the countable intersection

$$\mathcal{J}_{\phi,\mathrm{reg},3}(M,L_1,L_2,\omega) := \bigcap_{\nu\geqslant 0} \mathcal{J}^{\nu}_{\phi,\mathrm{reg}}(M,\omega)$$

is of second category and satisfies the conclusion of the theorem. $\square$

Consider an $\alpha > 0$, a complex structure

$$J \in \mathcal{J}_{\phi,\mathrm{reg},3}(M,L_1,L_2,\omega) \cap \mathcal{J}_{\phi,\mathrm{reg},4}(M,L_1,L_2,\omega)$$

and an intersection point $y \in L_1 \cap L_2$ with $|y| = 2$. We let

$$\mathcal{M}(y, L_1\#_{\varepsilon_\nu,x}L_2, J, \alpha) \subset \mathcal{M}(y, L_1\#_{\varepsilon_\nu,x}L_2, J)$$

be the set of elements of $\mathcal{M}(y, L_1\#_{\varepsilon_\nu,x}L_2, J)$ represented by a $u \in \widetilde{\mathcal{M}}(y, L_1\#_{\varepsilon_\nu,x}L_2, J)$ such that there is a strip $w \in \widetilde{\mathcal{M}}(y,x,L_1,L_2,J)$ with

$$\sup_z d_J(u(z), w(z)) < \alpha.$$

Here, d_J is the distance induced by the metric g_J.

By the propositions above, there is a second category subset $\mathcal{J}_{\mathrm{reg},5}(M, \omega, L_1, L_2)$ such that for $J \in \mathcal{J}_{\mathrm{reg},5}(M, \omega, L_1, L_2)$

- Every J-holomorphic strip in $\mathcal{M}(y, x, L_1, L_2, J)$ for $y \in L_1 \cap L_2$ with $|y| = 2$ is regular, simple and has corners of multiplicity 1,
- every J-holomorphic teardrop in $\mathcal{M}(y, L_1 \#_{\varepsilon,\nu} L_2, J)$ for $\nu \geqslant 0$ is regular, simple and has a corner of multiplicity 1.

Since the Lagrangians L_1 and L_2 are exact, Gromov compactness for J-holomorphic strips and regularity imply that the space $\mathcal{M}(y, x, L_1, L_2, J)$ is compact.

Therefore, we can apply a result stated in [**FOOO06**, Theorem 5.11] to obtain the following corollary.

Corollary 1.3.19 ($\star$)**.** *Here, L_1 and L_2 are compact, exact, graded submanifolds as in the above setting.*

There is a second category subset $\mathcal{J}_{\phi,reg,5}(M, L_1, L_2, \omega)$ such that the following holds.

Let $J \in \mathcal{J}_{\phi,reg,5}(M, \omega, L_1, L_2)$. There exist $\alpha > 0$ and a natural integer $\nu_0 \geqslant 0$ such that for any $\nu \geqslant \nu_0$ there is a bijection

$$\mathcal{M}(y, L_1 \#_{\varepsilon_\nu, x} L_2, J, \alpha) \to \mathcal{M}(y, x, L_1, L_2, J).$$

Gromov compactness

Last we need a version of Gromov compactness for J-holomorphic curves as the surgery parameter ε_ν goes to 0. We emphasize that it is not (to our knowledge) proved in the literature and that it is the subject of future work.

A *d-leafed tree* is a planar tree $T \subset \mathbb{R}^2$ with a choice of vertex α called the root, oriented so that the root has no incoming edge and with d leaves (beware that it is *not* the definition of [**Sei08**, (9d)]). For each vertex v of T, we denote by $|v|$ its valency.

Definition 1.3.20. *Let T be a d-leafed tree. A* labeled domain *consists of*

(i) a d-leafed tree T,

(ii) for each vertex v, an element $r_v \in \mathcal{R}^{|v|}$,

(iii) for each vertex v, $k_v \in \mathbb{N}$ cyclically ordered marked points at the boundary $z_1, \dots, z_{k_v} \in \partial r_v$,

(iv) for each connected component C of $\partial r_v \backslash \{z_1, \dots, z_{k_v}\}$, an element $L_C \in \{L_1, L_2\}$,

which satisfy the following conditions.

(1) If C_1 and C_2 are two adjacent connected components, then the labels L_{C_1} and L_{C_2} should be different,

(2) for every leaf v, $k_v \geqslant 1$.

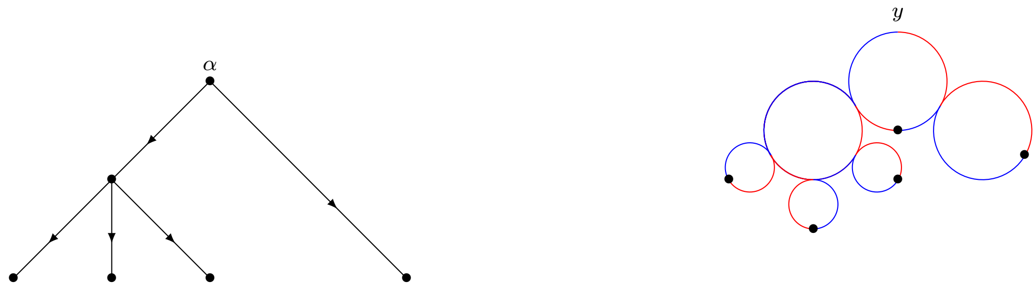

FIGURE 3 – *A labeled domain and its underlying tree*
Red corresponds to a label L_1 and blue to L_2
The dots are mapped to x

Remark 1.3.21. For each vertex v one can number the outgoing edges as counterclockwise (remember that T is embedded in $\mathbb{R}^2$). The number of an edge e going from v_1 to v_2 will be denoted n_e.

We now define the limit curves when the handle parameter goes to 0.

Definition 1.3.22. *A broken strip from y to x modeled on the labeled domain T consists of*

(i) for each vertex v, $|v|+1$ intersection points $y_0^v, \ldots, y_{|v|}^v \in L_1 \cap L_2$ such that if e is an edge from v_1 to v_2 then $y_{n_e}^{v_1} = y_0^{v_2}$,

(ii) for each vertex v a J-holomorphic curve with corners $u_v : (r_v, \partial r_v) \to (M, L_1 \sqcup L_2)$, such that

(1) for each vertex v and connected component C of $\partial r_v \backslash \{z_1, \ldots, z_{k_v}\}$, we have

$$u_v(C) \subset L_C,$$

(2) for each vertex v and $i \in \{1, \ldots, k_v\}$, we have $v(z_i) = x$,

(3) for the root vertex α, u_α converges to y on the 0-th strip-like end of r_α,

(4) for each vertex v, the curve u converges to y_i^v on the i-th strip-like end of r_v.

We expect that the following proposition is an adaptation of the neck-stretching procedure (as it appears in [BEH+03] and [CM05]) for curves with boundary on a Lagrangian manifold.

Proposition 1.3.23 ($\star$)**.** *Recall that we fixed two transverse, compact, exact, graded Lagrangian submanifolds L_1 and L_2 Fix an almost complex compatible structure $J \in \mathcal{J}_\phi(M, \omega)$.*

Let $(u_\nu)_{\nu \in \mathbb{N}}$ be a sequence of J-holomorphic teardrops such that

$$\forall \nu \geqslant 0,\ u_\nu \in \mathcal{M}(y, L_1 \#_{x,\varepsilon_\nu} L_2, J).$$

There is a subsequence $(u_{\nu_k})_{k \geqslant 0}$ which Gromov converges to a broken strip v from y to x.

Proof of Theorem 1.3.16 ($\star$)

We now prove Theorem 1.3.16 assuming that Proposition 1.3.23 is true. For this fix $J \in \mathcal{J}_{\phi,\mathrm{reg},5}(M,\omega)$.

Given the conclusion of Corollary 1.3.19, it only remains to check that there is ν_0 such that for all $\nu \geqslant \nu_0$, any teardrop $v \in \widetilde{\mathcal{M}}(y, J, L_1 \#_{x,\varepsilon_\nu} L_2)$ is α close to a J-holomorphic strip $v \in \widetilde{\mathcal{M}}(y, x, J, L_1, L_2)$.

Assume by contradiction that there is a strictly increasing sequence (ν_k), and a sequence of teardrops $u_{\nu_k} \in \widetilde{\mathcal{M}}(y, J, L_1 \#_{x,\varepsilon_{\nu_k}} L_2)$ such that

$$\forall v \in \widetilde{\mathcal{M}}(y, x, J, L_1, L_2),\ \sup_{z \in \mathbb{D}} d_J(u(z), v(z)) \geqslant \alpha.$$

By Proposition 1.3.23, there is a subsequence of (u_{ν_k}) which converges in the sense of Gromov to a broken strip w.

To conclude, it remains to see that v is an actual teardrop.

Lemma 1.3.24. *In the above setting, let $w = (u_v)_{v \in T}$ be a broken strip with underlying tree T such that all u_v are simple. Then the tree T consists of one vertex and w is a strip from y to x.*

Proof. The conclusion follows from a simple combinatorial argument which uses regularity and simplicity of the underlying holomorphic curves.

First, notice that the index of x as an element of $CF(L_2, L_1)$ is $n-1$ which is greater than 1 since $n \geqslant 3$.

For $v \in T$ different from the root, call y_v the incoming limit point and $x_1, \dots x_p$ the outgoing limit points. Moreover, assume that there are $k_{1,v}$ marked points mapping to x going from L_1 to L_2 and $k_{2,v}$ marked points mapping to x going from L_2 to L_1. Since the curve u_v is regular, we have

$$|y_v| - \sum_{i=1}^{p} |x_i| - k_{1,v} - k_{2,v}(n-1) + k_{1,v} + k_{2,v} + |v| - 3 \geqslant 0,$$

so

$$|y_v| - \sum_{i=1}^{p} |x_i| + |v| - 3 \geqslant 0.$$

Similarly, if v is the root, we get

$$|y| - \sum_{i=1}^{p} |x_i| - k_{1,v} - k_{2,v}(n-1) + k_{1,v} + k_{2,v} + |v| + 1 - 3 \geqslant 0,$$

so

$$|y| - \sum_{i=1}^{p} |x_i| + |v| - 2 \geqslant 0.$$

Adding these equalities for $v \in T$, we obtain

$$|y| + \sum_{v \in T} |v| - 3V(T) + 1 \geqslant 0,$$

where $V(T)$ is the number of vertices of T. Now notice that $\sum_{v \in T} |v|$ is twice the number of edges of T and therefore equal to $2V(T) - 2$. So

$$2 = |y| \geqslant 1 + V(T).$$

Hence $V(T) = 1$. Therefore we have a single curve w with one corner at y and the others at x.

Now since y is an incoming point from L_1 to L_2, there are $2k - 1$ other corners mapping to x (with k an integer greater than 1). Among them, k are outgoing points from L_1 to L_2 and $k - 1$ are outgoing points from L_2 to L_1. Since w is regular, we get

$$|y| - k - (n-1)(k-1) + 2k - 3 \geqslant 0,$$

so

$$|y| - (n-2)k + n - 4 \geqslant 0,$$

hence (since $|y| = 2$)

$$n - 2 \geqslant (n-2)k.$$

Since $n - 2 \geqslant 1$, we readily conclude that $1 \geqslant k$ hence $k = 1$. □

Lemma 1.3.25. *In the above setting, let $w = (u_v)_{v \in T}$ be a broken strip with underlying tree T. There is a tree T_1 and a broken strip $w_1 = (u_{v,1})_{v \in T}$ with underlying tree T_1 such that the following assertions hold.*

(1) For each $v \in T_1$, the curve $u_{v,1}$ is simple.

(2) There is an injective tree morphism $f : T_1 \to T$ mapping the root of T_1 to the root of T satisfying the following. If $v \in T_1$, the underlying simple curve of $u_{f(v)}$ is $u_{v,1}$.

(3) If $V(T) \geqslant 2$, then $V(T_1) \geqslant 2$.

Proof. We build the simple curve by induction.

Start with the root v_0. The curve u_{v_0} is multiply covered by the choice of almost complex structure J. Let $u_{v_0,1}$ be the underlying simple curve : there is a branched cover π such that $u_{v_0} = u_{v_0,1} \circ \pi$. We associate the curve $u_{v_0,1}$ to the root of T_1.

The domain of $u_{v_0,1}$ has one incoming strip-like end (the image of the incoming strip-like end of r_{v_0} by π) and $m_v \in \mathbb{N}$ outgoing strip-like ends. Call $\zeta_1, \ldots, \zeta_{m_v}$ their asymptotic points. For each ζ_i we put an outgoing edge e_{ζ_i}. Call v_{ζ_i} the outgoing end of e_{ζ_i}.

For each $i \in \{1, \ldots, m_v\}$, choose a point $\tilde{\zeta}_i \in r_{v_0}$ such that $\pi(\tilde{\zeta}_i) = \zeta_i$. Each $\tilde{\zeta}_i$ is the limit of an outgoing strip-like ends and corresponds to an edge in T with endpoint $v_{\tilde{\zeta}_i}$. The curve $u_{v_{\zeta_i}}$ is the simple curve underlying $u_{v_{\tilde{\zeta}_i}}$.

If we repeat this process by induction, we see that the end-product is a broken strip satisfying the hypotheses. $\square$

Chapitre 2

Groupes de cobordisme lagrangien

Ce chapitre reproduit la prépublication *Lagrangian cobordism groups of higher genus surfaces*, [**Per19**].

Résumé. Nous étudions les groupes de cobordisme lagrangien de surfaces orientées de genre plus grand ou égal à deux. Nous calculons le groupe de cobordisme lagrangien immergé. On montre ensuite qu'une variante de ce groupe dont les relations sont données par des cobordismes lagrangiens *non-obstrués* calcule le groupe de Grothendieck de la catégorie dérivée de Fukaya. La démonstration repose sur un argument d'Abouzaid.

Abstract. We study Lagrangian cobordism groups of oriented surfaces of genus greater than two. We compute the immersed oriented Lagrangian cobordism group of these surfaces. We show that a variant of this group, with relations given by *unobstructed* immersed Lagrangian cobordisms computes the Grothendieck group of the derived Fukaya category. The proof relies on an argument of Abouzaid [**Abo08**].

2.1. Introduction

2.1.1. Immersed Lagrangians and cobordisms

In this paper, we consider a (Riemann) surface S_g of genus $g \geqslant 1$ equipped with an area form ω.

We recall the definition of a Lagrangian cobordism following Biran-Cornea ([**BC13**]).

Definition 2.1.1. *Let $\gamma_0, \ldots, \gamma_N : S^1 \looparrowright S_g$ and $\tilde{\gamma}_0, \ldots \tilde{\gamma}_M : S^1 \looparrowright S_g$ be immersed curves. Let $F : V \looparrowright \mathbb{C} \times S_g$ be a Lagrangian immersion.*

We say that F is an immersed Lagrangian cobordism *from $\gamma_1, \ldots, \gamma_N$ to $\tilde{\gamma}_1, \ldots \tilde{\gamma}_M$ if* *(i) there is $\varepsilon > 0$ such that outside $[-\varepsilon, \varepsilon] \times \mathbb{R}$, F is an embedding with image*

$$\coprod_{i=1\ldots N} (-\infty, -\varepsilon] \times \gamma_i \cup \coprod_{j=1\ldots M} [\varepsilon, \infty) \times \tilde{\gamma}_j,$$

(ii) the set $F^{-1}([-\varepsilon, \varepsilon] \times \mathbb{R})$ is compact.

Such a cobordism will be denoted by $V : (\gamma_1, \ldots, \gamma_N) \rightsquigarrow (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_N)$.

Remark 2.1.2. (1) If V is oriented and its orientation agrees with the natural orientations of $(-\infty, -\varepsilon] \times \gamma_i$ and $[\varepsilon, \infty) \times \tilde{\gamma}_j$, we say that V is an *oriented* immersed Lagrangian cobordism.

(2) When the curves γ_i, $\tilde{\gamma}_i$ and the surface F are embedded, we will say that V is a *Lagrangian cobordism.*

(3) The definition goes back to Arnold ([**Arn80**]). The reader should be aware that the definition in [**Arn80**] is slightly different from ours although equivalent (see Lemma 2.2.6).

Now, consider the set $\mathcal{L}_{Imm}$ of Lagrangian immersions from an arbitrary number of copies of S^1 to S_g. Define an equivalence relation $\sim$ on $\mathcal{L}_{Imm}$ by

$$\gamma_1 \sim \gamma_2$$

if and only if there is an immersed Lagrangian cobordism from γ_1 to γ_2.

Here γ_1 and γ_2 are two elements of $\mathcal{L}_{Imm}$.

Definition 2.1.3. *The* immersed Lagrangian cobordism group *of S_g is the quotient*

$$\mathcal{L}_{Imm}/\sim .$$

We will denote it by $\Omega^{imm}_{cob}(S_g)$

The set $\Omega^{\mathrm{imm}}_{\mathrm{cob}}(S_g)$ an abelian group whose sum is given by disjoint union. The neutral element is the void set. The inverse of a generator $\gamma : S^1 \to S_g$ is the curve γ^{-1} obtained by reversing the orientation of γ.

The following Lemma shows that this group effectively detects the cobordism class of a curve.

Lemma 2.1.4. *Let $\gamma_1, \ldots, \gamma_n : S^1 \looparrowright S_g$ be immersed curve in S_g. Their classes in $\Omega^{imm}_{cob}(S_g)$ satisfy*

$$[\gamma_1] + \ldots + [\gamma_n] = 0$$

if and only if there is an oriented immersed Lagrangian cobordism

$$V : (\gamma_1, \ldots, \gamma_n) \rightsquigarrow \emptyset.$$

Due to Gromov's h-principle for Lagrangian immersions, topological invariants determine the Lagrangian cobordism group of the surface S_g.

Theorem 2.1.5. *We denote by $\chi(S_g)$ the Euler characteristic of S_g. There is an isomorphism*
$$\Omega^{imm}_{cob}(S_g) \to H_1(S_g, \mathbb{Z}) \oplus \mathbb{Z}/\chi(S_g)\mathbb{Z}.$$

Here, the map $\Omega^{\mathrm{imm}}_{\mathrm{cob}}(S_g) \to H_1(S_g, \mathbb{Z})$ is the homology class. Meanwhile, the map $\Omega^{\mathrm{imm}}_{\mathrm{cob}}(S_g) \to \mathbb{Z}/\chi(S_g)$ is a variant of a topological index defined by Chillingworth (see [**Chi72a**], see also [**Abo08**, Appendix A]). Along the way, we give an alternate definition of this index in line with the usual definition of the Maslov index in symplectic topology.

Remark 2.1.6. We can find many computations of Lagrangian cobordism groups in the literature. In [**Arn80**], Arnold computed the Lagrangian cobordism groups of $\mathbb{R}^2$ and of the cotangent bundle T^*S^1.

Eliashberg showed in [**Eli84**] that some of these groups are isomorphic to fundamental groups of some Thom spaces. Audin used these results to compute the generators of some other cobordism groups ([**Aud85**] and [**Aud87**]).

2.1.2. Floer theory and cobordism groups

Let L be a Lagrangian submanifold of a symplectic manifold (M, ω). The Maslov index induces a morphism
$$\mu_L : \pi_2(M, L) \to \mathbb{Z}.$$
On the other hand, symplectic area induces a morphism
$$\omega : \pi_2(M, L) \to \mathbb{R}.$$
The lagrangian submanifold L is *monotone* if there is $\lambda > 0$ such that
$$\omega_L = \lambda \mu_L.$$

In this case, there is a well-defined Fukaya category $\mathrm{Fuk}(M, \omega)$ whose objects are monotone Lagrangian submanifolds satisfying a topological condition (see [**BC14**] or [**She16**]). The A_∞-category $Fuk(M, \omega)$ has a *derived category* $DFuk(M, \omega)$ (defined in [**Sei08**]). Note that this category is *not* the split-completion of $\mathrm{DFuk}(M, \omega)$. The category $\mathrm{DFuk}(M, \omega)$ is triangulated, so one can speak of its *Grothendieck group*
$$K_0(\mathrm{DFuk}(M, \omega)).$$
Recall that this is the abelian group generated by the objects of $\mathrm{DFuk}(M, \omega)$ with relations given by
$$Y = Z + X$$
whenever there is an exact triangle
$$X \to Y \to Z \to X[1].$$

Now, Biran and Cornea proved that there is a natural surjective group morphism ([BC14, Corollary 1.2.1])

$$\Theta_{BC} : \Omega_{\text{cob}}^{\text{emb}}(M,\omega) \twoheadrightarrow K_0(\text{DFuk}(M,\omega)). \tag{2.1}$$

There are several results on this map. In [Hau15], Haug shows that the map 2.1 is an isomorphism when (M,ω) is a torus of dimension 2 and the Lagrangians are equipped with local systems.

In [Hen17], Hensel gives algebraic conditions under which the map 2.1 is an isomorphism. These are, in particular, verified for the torus.

More recently, in [SS18b], Sheridan and Smith use Mirror symmetry to prove the existence of certain Maslov 0 Lagrangian tori in $K3$ surfaces. In [SS18a], they study Lagrangian cobordism group in Lagrangian torus fibrations over tropical affine manifolds.

The main Theorem of our paper is a generalization of Haug's result to surfaces of genus $g \geqslant 2$. For this, we use a version of the cobordism group which takes a broader class of cobordisms into account.

Theorem 2.1.7. *There is an isomorphism*

$$\Omega_{cob}^{imm,unob}(S_g) \to \mathbb{R} \oplus H_1(S_g,\mathbb{Z}) \oplus \mathbb{Z}/\chi(S_g)\mathbb{Z}.$$

In [Abo08], Abouzaid showed that the abelian group $K_0(\text{DFuk}(S_g))$ is isomorphic to $\mathbb{R} \oplus H_1(S_g,\mathbb{Z}) \oplus \mathbb{Z}/\chi(S_g)\mathbb{Z}$ as well.

Therefore, the two groups are isomorphic. We show a slight improvement of this: a version of the map 2.1 (see Corollary 2.4.4) is an isomorphism as well.

Theorem 2.1.8. *Assume that the genus g of S_g is greater or equal than 2. There is a natural isomorphism*

$$\Theta_{BC} : \Omega_{cob}^{imm,unob}(S_g) \xrightarrow{\sim} K_0(\text{DFuk}(S_g)).$$

Unobstructed Lagrangian cobordisms give the relations of $\Omega_{\text{cob}}^{\text{imm,unob}}(S_g)$. These are immersed Lagrangian cobordism which satisfy a technical condition. We postpone the actual definition to section 2.4.

We shall consider a variant of the Fukaya category whose objects are defined below.

Definition 2.1.9. *An immersion $\gamma : S^1 \looparrowright S_g$ is* unobstructed *if it satisfies the following assumptions.*

(i) It has no triple points and all its double points are transverse.

(ii) Let $\tilde{S}_g$ be the universal cover of S_g, the curve γ lifts to a curve $\tilde{\gamma} : \mathbb{R} \to \tilde{S}_g$. We assume that $\tilde{\gamma}$ is properly embedded.

Remark 2.1.10. When (i) holds, we say that γ is *generic*.

At this point, there is only one thing the reader needs to keep in mind. Unobstructed objects do not bound *teardrops* which are polygons with a unique corner (see figure 1).

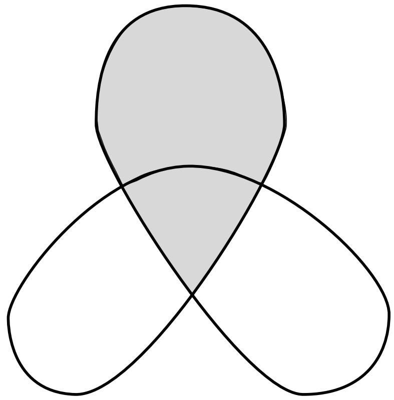

FIGURE 1 – *An obstructed immersed curve and a teardrop (shaded)*

It is indeed well-known that these give an obstruction to the definition of Floer theory of immersed objects. See the work of Akaho and Joyce ([**AJ10**]), Abouzaid ([**Abo08**]), Alston and Bao ([**AB18**]).

2.1.3. Relation with [Hau15]

In [**Hau15**], Haug actually showed that there is a (split) exact sequence

$$0 \to \mathbb{R}/\mathbb{Z} \xrightarrow{i} \Omega_{\text{cob}}^{\text{emb}}(M,\omega) \to H_1(T^2,\mathbb{Z}) \to 0. \tag{2.2}$$

The proof requires *Mirror symmetry*. More precisely, using geometric arguments, Haug proves that the kernel of $\Omega_{\text{cob}}^{\text{emb}}(M,\omega) \to H_1(T^2,\mathbb{Z})$ is the image of i. On the other hand, Mirror Symmetry for the torus yields an equivalence $\text{DFuk}(T^2) \simeq D^b(X)$ between the derived Fukaya category of the torus, whose objects are curves equipped with local systems, and the bounded derived category of Coherent sheaves of an elliptic curve X over the Novikov field Λ. Haug uses this to show that the application i is injective.

In our paper, we show that there is an analog of the exact sequence 2.2 for the group $\Omega_{\text{cob}}^{\text{imm,unob}}(S_g)$ (see Theorem 2.5.1 for the precise statement). However, the main difference is as follows. We *do not use* Mirror symmetry for the proof. Moreover, we do not take local systems into account. Therefore, our main result is purely geometric.

This is in contrast with all the results above which use ideas coming from mirror symmetry to study Lagrangian cobordism groups.

2.1.4. Outline of the paper

The proofs of both Theorems 2.1.5 and 2.5.1 use the action of the Mapping Class Group of S_g to find generators of $\Omega_{\text{cob}}^{\text{imm}}(S_g)$ and $\Omega_{\text{cob}}^{\text{imm,unob}}(S_g)$. This idea is due to Abouzaid [**Abo08**].

In the first section, we study the immersed Lagrangian cobordism group $\Omega_{\text{cob}}^{\text{imm}}(S_g)$ and give the proof of Theorem 2.1.5. Most of the results and definitions are not new (some even date back to Arnold). However, we tried to give details which we did not find in the literature.

In the second section, we define the Fukaya category of unobstructed curves following Seidel's book [**Sei08**] and Alston and Bao's paper [**AB18**]. We also give a combinatorial description of this category.

In the third section, we give the definition of an unobstructed Lagrangian cobordism. We explain why Biran-Cornea's map 2.1 extends to this setting.

In the fourth section, we prove Theorem 2.1.8 and 2.1.7. To do this, we describe the action of the Mapping Class Group on $\Omega_{\text{cob}}^{\text{imm,unob}}(S_g)$ using unobstructed Lagrangian cobordisms.

2.1.5. Acknowledgements

This work is part of my doctoral thesis at Université de Montréal under the direction of Octav Cornea. I wish to thank him for proposing me this project and his very thoughtful guidance over these years!

I also wish to thank Jordan Payette for explaining the proof of Lemma 2.5.3 to me as well as Jean-Philippe Chassé for reading the text and pointing out some mistakes.

At last, many thanks to Alexandru Oancea for his careful relecture of the text and his remarks about the exposition.

2.2. Computation of the immersed cobordism group

In this section, we give the proof of Theorem 2.1.5. In the first subsection, we show that embedded curves generate $\Omega_{\text{cob}}^{\text{imm}}(S_g)$. In the second subsection, we define a map

$$\Omega_{\text{cob}}^{\text{imm}}(S_g) \to H_1(S_g, \mathbb{Z}) \oplus \mathbb{Z}/\chi(S_g)\mathbb{Z}.$$

We check that it is well-defined and surjective. At last, we modify a geometric argument of Abouzaid ([**Abo08**]) to show that this map is injective.

2.2.1. Properties of the immersed cobordism group

Lagrangian cobordism and isotopy

We will use the following Lemma.

Lemma 2.2.1. *Assume that $\gamma_- : S^1 \looparrowright S_g$ and $\gamma_+ : S^1 \looparrowright S_g$ are two isotopic immersed curves, then there is an immersed Lagrangian cobordism from γ_- to γ_+.*

Proof. Choose an isotopy $(\gamma_s(t))_{s\in\mathbb{R}}$ such that

$$\begin{cases} \gamma_s(t) = \gamma_-(t) \text{ for s} < 0 \\ \gamma_s(t) = \gamma_+(t) \text{ for s} > 1 \end{cases}$$

Now, consider the following immersion

$$\begin{array}{rccc} f: & \mathbb{R}\times S^1 & \mapsto & \mathbb{C}\times S_g \\ & (s,t) & \mapsto & (s,\gamma_s(t)). \end{array}$$

This map is covered by an isotropic bundle map

$$F : T\mathbb{R}\times TS^1 \to T\mathbb{C}\times TS_g$$

defined by

$$F(\partial_s) = (1,0),\ F(\partial_t) = \left(0, \frac{d\gamma_s(t)}{dt}\right).$$

Moreover, since $H^2(\mathbb{R}\times S^1,\mathbb{R}) = 0$, we have $f^*(dx\wedge dy+\omega) = 0$ in $H^2(\mathbb{R}\times S^1,\mathbb{R})$.

We can now apply [**EM02**, 16.3.2] to find an immersion $\tilde{f} : S^1\times\mathbb{R}\to\mathbb{C}\times S_g$ whose ends coincide with f. The map $\tilde{f}$ is the relevant Lagrangian cobordism. □

Resolution of double points

We will use a variant of the Weinstein neighborhood Theorem for Lagrangian immersions constantly throughout this section.

Recall that there is a canonical identification between the fiber bundle $\pi : T^*S^1\to S^1$ and the product bundle $S^1\times\mathbb{R}\to S^1$. We denote by $T^*_\varepsilon S^1$ the set $\{(q,p)\in T^*S^1 | |p| < \varepsilon\}$.

Lemma 2.2.2. *Let $\gamma : S^1\to S_g$ be an immersed curve. There are $\varepsilon > 0$ and a local embedding*

$$\psi : T^*_\varepsilon S^1 \to S_g$$

such that

- *ψ restricted to the zero section is equal to γ,*
- *$\psi^*\omega$ is the standard symplectic form ω_{std} on T^*S^1.*

The proof is a simple exercise.

Lemma 2.2.3. *Let $\gamma : S^1 \looparrowright S_g$ be an immersed curve. Then γ is Lagrangian cobordant to a generic*[1] *immersed curve $\tilde{\gamma} : S^1 \looparrowright S_g$.*

Remark 2.2.4. The proof uses a variant of the *Lagrangian suspension* ([**ALP94**, 2.1.2]). If $(\phi^t_H)_{t\in[0,1]}$ is a Hamiltonian isotopy of S_g and γ an immersed curve, then the immersion

$$(t,x)\in[0,1]\times S^1 \mapsto \big(t, H_t\circ\phi^t_H\circ\gamma(x), \phi^t_H\circ\gamma(x)\big)$$

[1] See 2.1.10 for the definition

is a Lagrangian cobordism between γ et $\phi_H^1(\gamma)$. Notice that this cobordism is embedded if γ is.

PROOF. We call ψ the local embedding given by Lemma 2.2.2. We let $x \in S^1$. Choose a disk neighborhood $U_x \subset S^1$ containing x such that $\psi_{|\pi^{-1}(U_x)}$ is an embedding which we denote by Φ. Moreover we let $U = \gamma^{-1}(\psi(U_x)) \backslash U_x$

Let $\eta > 0$. We claim that there is a function $f_x : S^1 \to \mathbb{R}$ such that the following holds.

- The derivatives of f_x satisfy $|f_x'| < \eta$ and $|f_x''| < \eta$.
- Denote by γ_x the immersion $t \mapsto \psi(t, -f_x'(t))$. If $\gamma_x(t_1) = \gamma_x(t_2)$ with $t_1 \in U_x$, then $\gamma_x'(t_1)$ and $\gamma_x'(t_2)$ are transverse.

The proof is an application of Sard's Theorem. Consider the map $F = \pi_{\mathbb{R}} \circ \Phi^{-1} \circ \gamma_{|U}$. For every $\alpha > 0$, there is a regular value z of F satisfying $|z| < \alpha$. Now we let f_x be a smooth function such that $-f'$ is constant equal to a regular value over U_x and $|f_x'|, |f_x''| < \eta$. The reader may easily check that this is the desired function.

The curves γ and γ_x are cobordant. Indeed, choose a smooth cutoff function $\beta : \mathbb{R} \to \mathbb{R}$ such that $\beta(t) = 0$ for $t \leqslant 0$ and $\beta(t) = 1$ for $t \geqslant 1$. The relevant cobordism is the image of the map

$$\begin{array}{ccc} \mathbb{R} \times S^1 & \to & \mathbb{C} \times S_g \\ (t, x) & \mapsto & (t, \beta(t) f(x), \psi(x, -\beta(t) f'(x))) \end{array}.$$

Now choose $x_1, \ldots, x_N$ such that $U_{x_1}, \ldots, U_{x_N}$ is a covering of S^1. We use the construction above iteratively to get an immersed curve which is cobordant to γ and with transverse double points. $\square$

We can solve any double point of a generic immersion through a cobordism ([**Arn80**, page 9], see also [**ALP94**, 1.4]). For the convenience of the reader, we will summarize the proof of this fact and fill in some details.

First, we recall the (standard) procedure for solving the double point of a generic immersion. This is a particular instance of the Lagrangian surgery (see [**LS91**] and [**Pol91**]). Let $\gamma : S^1 \looparrowright S_g$ be a generic immersed curve and $x = \gamma(p) = \gamma(q)$ a double point with $p \neq q \in S^1$. There are

- an open neighborhood U of x,
- a real number $r > 0$,
- a symplectomorphism $\phi : U \to B(0, r) \subset \mathbb{C}$

such that $\phi \circ \gamma$ parameterizes the real axis near p and parameterizes the imaginary axis near q.

Pick a smooth path $c : \mathbb{R} \to \mathbb{C}$ such that

(1) for $t \leqslant -1$, $c(t) = t$,
(2) for $t \geqslant 1$, $c(t) = it$,
(3) the derivatives x' and y' satisfy $x' > 0$ and $y' > 0$,
(4) for all $t \in \mathbb{R}$, $(x(-t), y(-t)) = (-y(t), -x(t))$.

The *surgery* of γ at the point (p, q) with parameter $\varepsilon > 0$ is the curve obtained by replacing the image of γ by the images of the curves εc and $-\varepsilon c$ in the chart ϕ. We denote it by $\gamma_{(p,q),\phi,\varepsilon}$. Note that this curve depends on the *ordered* pair (p, q).

Notice in particular that all surgeries at a given double point are isotopic to one another, hence Lagrangian cobordant by Lemma 2.2.3.

We shall prove the following Proposition.

Proposition 2.2.5. *Let $\gamma : S^1 \looparrowright S_g$ be a generic immersed curve and $x = \gamma(p) = \gamma(q)$ a double point with $p \neq q \in S^1$.*

There are a chart ϕ and a real number $\varepsilon > 0$ such that γ is cobordant to $\gamma_{(p,q),\phi,\varepsilon}$.

Proof. First, we need the following.

Lemma 2.2.6. *Let Σ a compact surface with boundary and let*

$$F : (\Sigma, \partial\Sigma) \looparrowright ([-1, 1] \times \mathbb{R} \times S_g, \partial[-1, 1] \times \mathbb{R} \times S_g)$$

be a Lagrangian immersion transverse to $\partial[-1, 1] \times \mathbb{R} \times \mathbb{C}$ along $\partial\Sigma$.

Then, the projection of $F_{|\partial\Sigma}$ to S_g is the union of two immersions γ_- and γ_+ lying over $\{-1\} \times \mathbb{R} \times S_g$ and $\{1\} \times \mathbb{R} \times S_g$ respectively.

There is an immersed Lagrangian cobordism from γ_- to γ_+.

Remark 2.2.7. Such immersions are what Arnold called Lagrangian cobordisms in his original paper([**Arn80**]). Therefore, we will call these objects *Lagrangian cobordisms in Arnold's sense.*

Proof of Lemma 2.2.6. Denote by $\partial^+\Sigma$ the union of connected components of $\partial\Sigma$ which projects to $\{1\} \times \mathbb{R}$ in the $\mathbb{C}$ factor. We identify $\partial^+\Sigma$ with a disjoint union of copies of S^1 : $\partial^+\Sigma = \sqcup_{i=1\ldots N} S^1$. On $\partial^+\Sigma$, F is of the form $t \mapsto (1, f(t), \gamma_+(t))$ with $f : \partial^+\Sigma \to \mathbb{R}$ a smooth function.

By Lemma 2.2.2, we can extend γ_+ to a local symplectomorphism

$$\psi : \coprod_{i=1\ldots N} S^1 \times (-\varepsilon, \varepsilon) \to S_g.$$

We choose a smooth function $\tilde{f}$ with

$$\begin{array}{cccc} \tilde{f} : & \coprod_{i=1\ldots N} S^1 \times (-\varepsilon, \varepsilon) & \to & S_g \\ & (s, t) & \mapsto & \left\{ \begin{array}{c} 0 \text{ if } |t| > \frac{2\varepsilon}{3}, \\ f(s,t) if |t| < \frac{\varepsilon}{3} \end{array} \right. \end{array}.$$

Notice that its hamiltonian flow $\left(\phi^t_{\tilde{f}}\right)_{t\in[0,1]}$ satisfies $\phi^t_{\tilde{f}}(S^1 \times (-\varepsilon,\varepsilon)) = S^1 \times (-\varepsilon,\varepsilon)$.

We define a Lagrangian cobordism in Arnold sense as follows

$$\begin{array}{rccc} G: & [0,1]\times S^1 & \to & \mathbb{C}\times S_g \\ & (t,z) & \to & \left(t, \psi\circ\phi^t_{t\tilde{f}}(\gamma_+(z))\right) \end{array}.$$

Now, we consider the union of the maps F and $G+(1,0)$. A Lagrangian smoothing of the resulting cobordism is the desired Lagrangian cobordism. □

We build a local model for the resolution of the double point. The quartic

$$\Sigma = \left\{(t,x,y)\in[-1,1]\times\mathbb{R}^2 \middle| y^2 - x^2 + t = 0\right\}.$$

is the set of critical points of the generating families

$$\begin{array}{rccc} f_{t,x}: & \mathbb{R} & \to & \mathbb{R} \\ & y & \mapsto & \frac{y^3}{3} - yx^2 + ty \end{array}.$$

We rotate the Lagrangian immersion associated to this by an angle of $\frac{\pi}{4}$ to obtain

$$\begin{array}{rccc} F: & \Sigma & \to & \mathbb{C}\times\mathbb{C} \\ & (t,x,y) & \mapsto & \left(t, y, \frac{x+2xy}{\sqrt{2}}, \frac{x-2xy}{\sqrt{2}}\right) \end{array}.$$

Notice also that the map F a Lagrangian cobordism in Arnold sense between a double point and its resolution.

We now modify F so that it is equal to $\mathbb{R}\times\mathbb{R}\sqcup\mathbb{R}\times i\mathbb{R}$ outside a neighborhood of $\mathbb{R}\times\{0\}$.

The set $\mathrm{Im}(F)\backslash([-1,1]\times B(0,1))$ is an embedded manifold with four connected components. Two of them are contained in the quadrant

$$[-1,1]\times\left\{e^{i\theta}\middle|\theta\in[-\frac{\pi}{4},\frac{\pi}{4}] \mod \pi\right\}.$$

We denote them by L_1. Two of them are contained in the quadrant

$$[-1,1]\times\left\{e^{i\theta}\middle|\theta\in[-\frac{\pi}{4},\frac{\pi}{4}] \mod \pi\right\}.$$

We denote them by L_2.

Notice that the linear projection $\pi_{\mathbb{R}}:[-1,1]\times\mathbb{C}\to\mathbb{R}\times\mathbb{R}$ which maps (t,s,u,v) to (t,u) restricts to a diffeomorphism from L_1 to the band $[-1,1]\times(\mathbb{R}\backslash[-1,1])$. From this, we deduce that L_1 is of the form

$$\{(t,f(x,t),x,g(x,t))|(t,x)\in[-1,1]\times(\mathbb{R}\backslash[-1,1])\}.$$

Since L_1 is Lagrangian, the form $fdt + gdx$ is closed. Furthermore, the set $[-1,1] \times (\mathbb{R}\backslash[-1,1])$ is homotopy equivalent to two points. Hence, there is a smooth function h such that $f = \partial_t h$ and $g = \partial_t h$.

Now, choose a bump function $\beta : [-1,1] \times (\mathbb{R}\backslash[-1,1]) \to \mathbb{R}$ such that $\beta(t,x) = 1$ on $[-1,1] \times [-\frac{5}{4}, \frac{5}{4}]$ and $\beta(t,x) = 0$ outside $[-1,1] \times [-2,2]$. Define $\tilde{L}_1$ to be the following embedded Lagrangian

$$\tilde{L}_1 := \{(t, \partial_t(\beta h), x, \partial_x(\beta h)) | (t,x) \in [-1,1] \times (\mathbb{R}\backslash[-1,1])\}.$$

We define $\tilde{L}_2$ in the same manner. The projection $\pi_{i\mathbb{R}} : [-1,1] \times \mathbb{C} \to \mathbb{R} \times \mathbb{R}$ which maps (t,s,u,v) to (t,v) restricts to a diffeomorphism $L_2 \to [-1,1] \times (\mathbb{R}\backslash[-1,1])$. We deduce that L_2 is of the form

$$\{(t, \partial_t h, \partial_y h, y\ ) | (t,y) \in [-1,1] \times (\mathbb{R}\backslash[-1,1])\}.$$

and we put

$$\tilde{L}_2 := \{(t, \partial_t(\beta h), \partial_y(\beta h), y) | (t,y) \in [-1,1] \times (\mathbb{R}\backslash[-1,1])\}.$$

Now, the map F restricted to the set $\Sigma \cap [-1,1]^3$ and the two embedded submanifolds $\tilde{L_1}, \tilde{L_2}$ yield an immersion

$$H : \Sigma \to \mathbb{C} \times \mathbb{C}$$

equal to $[-1,1] \times \mathbb{R} \cup [-1,1] \times i\mathbb{R}$ outside $\Sigma \cap [-1,1]^3$.

Consider the immersion $i := [-1,1] \times \gamma$. Recall that we chose a chart $\phi : U \to B(0,r)$ around x. In the chart $\mathrm{Id} \times \phi$, the immersion i is equal to $[-1,1] \times \mathbb{R} \cup [-1,1] \times i\mathbb{R}$. We replace this by εH for ε small enough and smooth the resulting immersion. The result is a Lagrangian cobordism in Arnold sense (see Remark 2.2.7) between γ and its surgery at x. By Lemma 2.2.6, we obtain that γ and its surgery are cobordant. ☐

All of this allows us to deduce the following result due to Arnold ([**Arn80**]).

Proposition 2.2.8. *The classes of Lagrangian embeddings generate the immersed Lagrangian cobordism group $\Omega^{imm}_{cob}(S_g)$.*

Proof. Let γ be a Lagrangian immersion. By Lemma 2.2.3, γ is Lagrangian cobordant to a generic curve $\tilde{\gamma}$. Repeated applications of Lemma 2.2.5 show that $\tilde{\gamma}$ is Lagrangian cobordant to an embedding. ☐

Resolution of intersection points

Let $\gamma_1 : S^1 \to S_g$ and $\gamma_2 : S^1 \to S_g$ be two transverse generic immersed curves. Let $x = \gamma_1(p) = \gamma_2(q)$ be an intersection point of γ_1 and γ_2. We can perform the Lagrangian surgery of γ_1 and γ_2 at x (as defined in [**Pol91**] and [**LS91**]) to obtain a curve $\gamma_1 \#_x \gamma_2$.

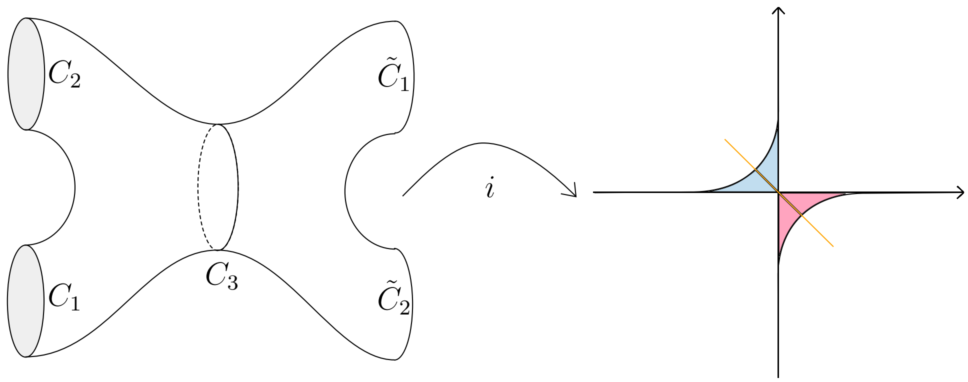

FIGURE 2 – *The doubled pair of pants S and the projection of the immersions i^+ (blue), i^- (red) and j (yellow)*

It is an observation of Biran and Cornea that in the embedded case, the curves γ_1 and γ_2 are cobordant to their surgery $\gamma_1 \#_{x,\varepsilon} \gamma_2$ ([**BC13**, Lemma 6.1.1]). We explain how to adapt their argument to the case of oriented immersed curves.

We say that x is of degree $1 \in \mathbb{Z}/2\mathbb{Z}$ if the oriented basis $(\gamma_1'(p), \gamma_2'(q))$ is positive with respect to the orientation of $T_x S_g$ and that the degree is 0 otherwise.

Proposition 2.2.9. *In the above setting, assume that the intersection point x is of degree 1. Then, there is an immersed oriented Lagrangian cobordism*

$$V : (\gamma_1, \gamma_2) \rightsquigarrow \gamma_1 \#_{x,\varepsilon} \gamma_2.$$

PROOF. First, we introduce some notations. We choose a Darboux chart $\phi : U \ni x \to B(0,r)$ (with $0 < r < \frac{1}{2}$) such that $\phi \circ \gamma_1$ parameterizes $\mathbb{R} \cap B(0,r)$ from left to right and such that $\phi \circ \gamma_2$ parameterizes $i\mathbb{R} \cap B(0,r)$ from bottom to top. We let $\psi : B(0,r) \times S_g \to B(0,r) \times B(0,r)$ be the Darboux chart given by $\mathrm{Id} \times \phi$.

Moreover, let $\alpha : \mathbb{R} \to \mathbb{C}$ be the path given by $\alpha(t) = t$ and $\beta = (x,y) : \mathbb{R} \to \mathbb{C}$ be a smooth path satisfying the following conditions

- $\beta(t) = t + i$ for $t < -1$,
- $\beta(t) = t - i$ for $t > 1$,
- $\beta(t) = -it$ for $t \in (-r, r)$,
- $x'(t) \geqslant 0$ and $y'(t) \leqslant 0$ for all t.

We also define P to be a smooth oriented pair of pants with boundary components labeled by C_1, C_2 and C_3 (see Figure 2).

We give the handle that is used to resolve the intersection point x. Let $c : \mathbb{R} \to \mathbb{C}$ be a path such as in subsection 2.2.1. For $\varepsilon > 0$, put

$$H_\varepsilon^+ = \left\{ \varepsilon c(t) z \middle| t \in \mathbb{R}, z = (x,y) \in S^1, \Re(c(t)x) \leqslant 0 \right\}.$$

Define an new immersion as follows. Consider the immersion given by $(\alpha_{|\mathbb{R}_-} \times \gamma_1) \coprod (\beta_{|\mathbb{R}_-} \times \gamma_2)$. Remove its intersection with $B(0,r) \times U$ and replace it with $\psi^{-1}(H_\varepsilon^+)$. This yields an oriented Lagrangian immersion $i^- : P \to \mathbb{C} \times S_g$ such that i^- coincides with $\alpha \times \gamma_1$ on a neighborhood of C_1, i^{-1} coincides with $\beta \times \gamma_2$ on a neighborhood of C_2 and i^- is the immersion $\{0\} \times \gamma_1 \#_{x,\varepsilon} \gamma_2$ over C_3. (The immersion $i-$ is oriented because of the assumption on the degree of x). Moreover, the outward pointing direction to C_3 maps through di^- to a vector pointing into fourth quadrant.

Notice that the double points of the immersion i are of three types,

- those given by the cartesian product of $\alpha_{|\mathbb{R}_-}$ and the double points of γ_1,
- those given by the cartesian product of $\beta_{|\mathbb{R}_-}$ and the double points of γ_2,
- the intersection points between γ_1 and γ_2 different from x at the point 0.

We now extend this immersion so that it becomes an actual Lagrangian cobordism. We explain this following the procedure of [**BC13**].

We consider the genus 0 surface with four boundary components S obtained by the gluing of two copies of P along the boundary C_3 and call its two new boundary components $\tilde{C}_1, \tilde{C}_2$ (see Figure 2). Moreover, we let i^+ be the immersion $P \to \mathbb{C} \times S_g$ given by the composition of i^- with the reflexion $(z, x) \in \mathbb{C} \times S_g \mapsto (-z, x)$. Their union yields a Lagrangian immersion $i : S \to \mathbb{C} \times S_g$ which is a Lagrangian cobordism $(\gamma_1, \gamma_2) \rightsquigarrow (\gamma_2, \gamma_1)$. We extend this to a local embedding $\iota : T^*_\varepsilon S \to \mathbb{C} \times S_g$ such that its restriction to the zero section coincides with i and the pullback of the symplectic form coincides with standard one.

For $\alpha > 0$ small enough, the immersion $j : (-\alpha, \alpha) \times S^1 \to \mathbb{C} \times S_g$ given by $j(s,t) = (s(1-i), \gamma_1 \#_{x,\varepsilon} \gamma_2(t))$ lifts to a Lagrangian embedding $\tilde{j} : (-\alpha, \alpha) \times S^1 \to T^*_\varepsilon S$ through ι. This is a consequence of the homotopy lift Theorem for covers. Indeed, j coincides with $\{0\} \times \gamma_1 \# \gamma_2$ on $\{0\} \times S^1$.

Reducing α if necessary, we can assume that $\tilde{j}$ is the graph of a closed one-form λ (because the tangent space of $\tilde{j}$ at a point $(0, x)$ is transversal to the fiber). Since $(-\alpha, \alpha) \times S^1$ is homotopy equivalent to $\{0\} \times S^1$ and $\lambda_{|\{0\} \times S^1}$ is zero, there is a smooth function $F : (-\alpha, \alpha) \times S^1 \to \mathbb{R}$ such that $\lambda = dF$ and $F_{|\{0\} \times S^1} = 0$. Let $\beta : (-\alpha, \alpha)$ be a smooth function such that $\beta(t) = 0$ for $t < 0$, $\beta(t) = 1$ for $t > \frac{\alpha}{2}$ and $\beta'(t) > 0$. We replace the graph of λ by the graph of $d(\beta F)$. Composing with ι, we get an immersion of the pair of pants with coincides with i and with j at its ends. $\square$

We give a precise description of the double points of the cobordism $(\gamma_1, \gamma_2) \rightsquigarrow \gamma_1 \#_x \gamma_2$. In order to do this, we first describe some relevant charts near the double points of $i^+ \sqcup i^-$.

In what follows, we identify the restriction of $i^+ \sqcup i^-$ to C (see Figure 2) with the immersion $\{0\} \times (\gamma_1 \#_x \gamma_2)$. We let ε be a positive real smaller than $\frac{2}{3}$ that we may reduce if necessary.

Chart near a self-intersection point of γ_1. Let y be a self-intersection point of γ_1. Call $s \neq t \in C$ its pre-images by $\{0\} \times (\gamma_1 \#_x \gamma_2)$.

We choose a Darboux chart $\phi_{1,y} : U_{1,y} \to B(0, r_{1,y}) \subset \mathbb{C}$ near y such that $\phi_{1,y}(\gamma_1)$ parameterizes the real line $\mathbb{R}$ (resp. $i\mathbb{R}$) near s (resp. near t).

We can consider the following maps,

$$\psi_s : (x, y, a, b) \in (-\varepsilon, \varepsilon)^4 \mapsto (x, y, a, b) \in \mathbb{C} \times U_{i,y}, \tag{2.3}$$

$$\psi_t : (x, y, a, b) \in (-\varepsilon, \varepsilon)^4 \mapsto (x, y, -b, a) \in \mathbb{C} \times U_{i,y}, \tag{2.4}$$

which is expressed in the chart $\mathrm{Id} \times \phi_{1,y}$. These are Darboux embedding (the domain is equipped with the symplectic form $dx \wedge dy + da \wedge db$).

Since $i^+ \sqcup i^-$ coincides with $\mathbb{R} \times \mathbb{R}$ near s, ψ_s restricted to $(-\varepsilon, \varepsilon) \times \{0\} \times (-\varepsilon, \varepsilon) \times \{0\}$ yields coordinates of S near s. So the map ψ_s is actually an embedding of a neighborhood of s in $T^*_\varepsilon S$.

Similarly, the map ψ_t is an embedding of a neighborhood of t in $T^*_\varepsilon S$.

Chart near a self-intersection point of γ_2. We let y be a self-intersection point of γ_2 and call $s \neq t$ its pre-images by $\{0\} \times \gamma_1 \#_x \gamma_2$. We choose a Darboux chart $\phi_{2,y} : U_{2,y} \to B(0, r_{2,y}) \subset \mathbb{C}$ such that $\phi_{2,y}(\gamma_2)$ parameterizes the line $\mathbb{R}$ (resp. $i\mathbb{R}$) near s (resp. near t). We consider the following maps,

$$\psi_s : (x, y, a, b) \in (-\varepsilon, \varepsilon)^4 \mapsto (-y, x, a, b) \in \mathbb{C} \times U_{i,y}, \tag{2.5}$$

$$\psi_t : (x, y, a, b) \in (-\varepsilon, \varepsilon)^4 \mapsto (-y, x, -b, a) \in \mathbb{C} \times U_{i,y}, \tag{2.6}$$

which is read in the chart $\mathrm{Id} \times \phi_{1,y}$. These are Darboux embedding (the domain is equipped with the symplectic form $dx \wedge dy + da \wedge db$).

Since $i^+ \sqcup i^-$ coincides with $\mathbb{R} \times \mathbb{R}$ near s, ψ_s restricted to $(-\varepsilon, \varepsilon) \times \{0\} \times (-\varepsilon, \varepsilon) \times \{0\}$ yields coordinates of S near s. So the map ψ_s is actually an embedding of a neighborhood of s in $T^*_\varepsilon S$.

Similarly, the map ψ_t is an embedding of a neighborhood of t in $T^*_\varepsilon S$.

Chart near an intersection point of γ_1 and γ_2. We let $y \neq x$ be an intersection point of γ_1 and γ_2 different from the surgered point above. We choose a Darboux chart $\phi_y : U_y \to B(0, r_y) \subset \mathbb{C}$ such that $\phi_y(\gamma_1) \subset \mathbb{R}$ and $\phi_y(\gamma_2) \subset i\mathbb{R}$.

We consider the following maps

$$\psi_s : (x, y, a, b) \in (-\varepsilon, \varepsilon)^4 \mapsto (x, y, a, b) \in \mathbb{C} \times U_y \tag{2.7}$$

$$\psi_t : (x, y, a, b) \in (-\varepsilon, \varepsilon)^4 \mapsto (-y, x, -b, a) \in \mathbb{C} \times U_y, \tag{2.8}$$

which is read in the chart ϕ_y. These are Darboux embedding when the domain is equipped with the symplectic form $dx \wedge dy + da \wedge db$.

Call $s \in C$ (resp. $t \in C$) the preimage of $(0, y)$ by $i^+ \sqcup i^-$ such that a small neighborhood of s (resp. t) is mapped to $\mathbb{R} \times \mathbb{R}$ (resp. $i\mathbb{R} \times i\mathbb{R}$). The map ψ_s (resp. ψ_t) yields local coordinates of S near (resp. t). Hence, the map ψ_s (resp. ψ_t) is a local embedding of a neighborhood of s (resp. t) in $T^*_\varepsilon S$.

An easy Moser argument shows that the maps above extends to a local Weinstein embedding $\Psi : V \subset T^*_\varepsilon \to \mathbb{C} \times S_g$. Here V is a neighborhood of C in T^*_ε.

Recall that near $\{0\} \times \gamma_1 \#_x \gamma_2$ the immersion $\{y = -x\} \times \gamma_1 \#_x \gamma_2$ is the image of $Gr(dF)$ by Ψ.

- Choose a self-intersection point $y = \gamma_1(s) = \gamma_1(t)$ of γ_1. We see that, in the coordinates (x, a) near s, the function F is given by $(x, a) \mapsto -\frac{x^2}{2}$. In the coordinates (x, a) near t, F is given by $(x, a) \mapsto -\frac{x^2}{2}$
- Similarly, consider a self-intersection point $y = \gamma_2(s) = \gamma_2(t)$ of γ_2. In the coordinates (x, a) near s, the function F is given by $(x, a) \mapsto \frac{x^2}{2}$. Near t, F coincides with $(x, a) \mapsto \frac{x^2}{2}$.
- Lastly, choose an intersection point $y = \gamma_1(s) = \gamma_2(s)$ between γ_1 and γ_2. In the coordinates (x, a) near s, the function F is given by $(x, a) \mapsto -\frac{x^2}{2}$. In the coordinates (x, a) near t, the function F is given by $(x, a) \mapsto \frac{x^2}{2}$.

We choose the function cutoff function β so that it depends only on x in each of the coordinate patches above. Moreover β satisfies the following hypotheses with $\frac{1}{2} > \alpha > 0$ and $0 < \eta < \alpha$.

- $\beta = 0$ for $t \leqslant \frac{\alpha}{2}$,
- $\beta = 1$ for $t \geqslant \frac{\alpha}{2}$,
- $\beta' \geqslant 0$ and $\beta' \leqslant \frac{1}{\varepsilon - \alpha + \eta}$.

In particular this implies, for $x \in (-\varepsilon, \varepsilon)$, $x\beta + \frac{x^2}{2}\beta' \leqslant \frac{3\varepsilon}{2} < 1$.

We deduce the following.

- Near $y = \gamma_1(s) = \gamma_1(t)$, the immersion is given by the two embeddings

$$(x, a) \mapsto \left(x, -x\beta(x) - \frac{x^2}{2}\beta'(x), a, 0\right)$$

$$(x, a) \mapsto \left(x, -x\beta(x) - \frac{x^2}{2}\beta'(x), 0, a\right).$$

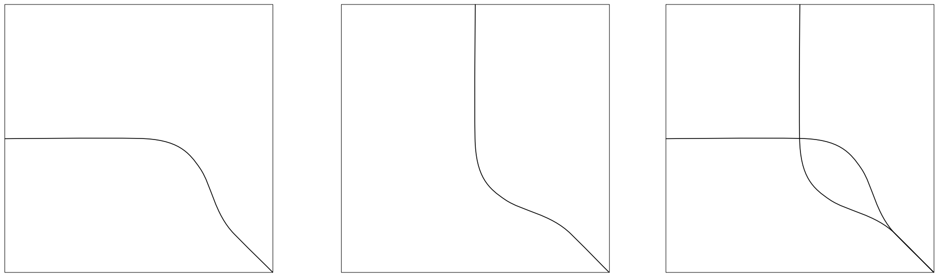

FIGURE 3 – *The projections of the surgery cobordism near the double points.*

There is a segment of double point which projects to the line $x \mapsto (x, -x\beta - \frac{x^2}{2}\beta')$(see Figure 3).

- Similarly, near $y = \gamma_2(s) = \gamma_2(t)$, the immersion is given by the two embeddings

$$(x,a) \mapsto \left(-x\beta(x) - \frac{x^2}{2}\beta'(x), x, a, 0\right)$$
$$(x,a) \mapsto \left(-x\beta(x) - \frac{x^2}{2}\beta'(x), x, 0, a\right).$$

There is a segment of double points which projects to the line $x \mapsto (-x\beta(x) - \frac{x^2}{2}\beta'(x), x)$ (see Figure 3).

- Near $y = \gamma_1(s) = \gamma_2(t)$, the immersion is given by the two embeddings

$$(x,a) \mapsto \left(x, -x\beta(x) - \frac{x^2}{2}\beta'(x), a, 0\right)$$
$$(x,a) \mapsto \left(-x\beta(x) - \frac{x^2}{2}\beta'(x), x, a, 0\right).$$

There is a double point at $(0,0)$ and a segment of double points which projects to the line $y = -x$ (see Figure 3).

In what follows, we will call DP_1 (resp. DP_2) the set of double points coming from the double points of γ_1 (resp.γ_2). We will call DP the set of double points along the line $y = -x$ and DP_0 the set of double points which project to $(0,0) \in \mathbb{C}$.

Lemma 2.2.10. *There is a family $(i_\lambda)_{\lambda\in[0,1]}$ of immersions $S \looparrowright \mathbb{C} \times S_g$ such that*

(i) we have $i_1 = i$, i_0 is the piecewise smooth immersion $S \to S_g$ given by $i^+ \sqcup j$,

(ii) for any compact set $K \subset S\backslash C_3$, the map i_λ is constant for λ small enough,

(iii) i_λ converges uniformly to i_0 as λ goes to 0.

(iv) The maps i_λ are constant in a neighborhood of DP_1, DP_2 and DP.

PROOF. We consider the family of cutoff functions $\beta_\lambda(x) = \beta(\frac{x}{\lambda})$. The family $(i_\lambda)_{\lambda\in[0,1]}$ is obtained by replacing $\Phi(Gr(d(\beta F)))$ by $\Phi(Gr(d(\beta_\lambda F)))$ for $\lambda \in [0,1]$. $\square$

2.2.2. Computation of the cobordism group

The applications π and μ

Let $\gamma_1, \dots, \gamma_N$ be immersed curves. We assume that there is an oriented immersed Lagrangian cobordism $V : (\gamma_1, \dots, \gamma_n) \rightsquigarrow \emptyset$. In this case, we say that $\gamma_1, \dots, \gamma_N$ are *immersed Lagrangian cobordant.*

It is easy to see that the classes of theses curves in $H_1(S_g, \mathbb{Z})$ must satisfy

$$\sum_{i=1}^{N} [\gamma_i] = 0.$$

Therefore, the map which associates to an immersed curve γ its homology class $[\gamma]$ induces a well-defined group morphism

$$\pi : \Omega^{\text{imm}}_{\text{cob}}(S_g) \to H_1(S_g, \mathbb{Z}).$$

Note that a variant of this map was used by Abouzaid in the case of the Grothendieck group of the derived category (see [Abo08]).

As stated in the introduction, there is a morphism

$$\mu : \Omega^{\text{imm}}_{\text{cob}}(S_g) \to \mathbb{Z}/\chi(S_g)\mathbb{Z},$$

which is a variant of the Maslov index. We define it following Seidel's paper ([Sei00, 2.b.]). An alternate definition as a winding number appears in a paper of Chillingworth ([Chi72a]). Moreover, a variant of this morphism for the Grothendieck group $K_0(\text{DFuk}(S_g)$ was considered by Abouzaid in [Abo08].

We fix a complex structure j on S_g, so that TS_g is a complex line bundle. Choose another line bundle $Z \to S_g$ of degree 1 over S_g and a complex isomorphism

$$\Phi : TS_g \xrightarrow{\sim} Z^{\otimes \chi(S_g)}. \tag{2.9}$$

Denote by $TS_g \backslash \{0\}$ the tangent bundle of S_g minus the zero section. We let $(\gamma, \tilde{\gamma}) : S^1 \to TS_g \backslash \{0\}$ be a nowhere vanishing curve in TS_g. We also let $v : S^1 \to Z$ be a nowhere vanishing section of the fiber bundle $\gamma^* Z$. There is a function $\lambda_v : S^1 \to \mathbb{C}^*$ such that for all $t \in S^1$

$$\tilde{\gamma}(t) = \lambda_v(t)\Phi(v(t))^{\otimes \chi(S_g)}.$$

If w is another nowhere vanishing section of $\gamma^* Z$, denote by $\lambda_w : S^1 \to \mathbb{C}^*$ the function such that

$$\forall t \in S^1, \ \tilde{\gamma}(t) = \lambda_w(t)\Phi(w(t))^{\otimes \chi(S_g)}.$$

There is a function $\mu : S^1 \to \mathbb{C}^*$ such that

$$\forall t \in S^1,\ v(t) = \mu(t)w(t).$$

So

$$\forall t \in S^1,\ \tilde{\gamma}(t) = \lambda_v(t)\mu(t)^{\chi(S_g)}\phi(w(t))^{\otimes\chi(S_g)}.$$

Therefore, we have $\deg(\lambda_v) = \deg(\lambda_w)$ modulo $\chi(S_g)$. So it makes sense to define the Maslov index $\mu_\Phi(\tilde{\gamma}) \in \mathbb{Z}/\chi(S_g)$ by

$$\mu_\Phi(\tilde{\gamma}) = \deg(\lambda_v) \mod \chi(S_g).$$

Let $(\gamma_1, \tilde{\gamma}_1) : S^1 \to TS_g\backslash\{0\}$ be another nowhere vanishing curve. We assume that $\tilde{\gamma}$ and $\tilde{\gamma}_1$ are homotopic. Then, it is easy to check that

$$\mu_\Phi(\tilde{\gamma}) = \mu_\Phi(\tilde{\gamma}_1).$$

We conclude that there is a well defined morphism

$$\mu_\Phi \in \mathrm{Hom}(\pi_1(TS_g\backslash\{0\}), \mathbb{Z}/\chi(S_g)\mathbb{Z}) = H^1(TS_g\backslash\{0\}, \mathbb{Z}/\chi(S_g)\mathbb{Z}).$$

An immersed curve $\gamma : S^1 \to S_g$ has a canonical lift $\tilde{\gamma}$ to $TS_g\backslash\{0\}$ given by

$$\begin{array}{rccc} \tilde{\gamma}: & S^1 & \to & TS_g\backslash\{0\} \\ & t & \mapsto & (\gamma(t), \gamma'(t)) \end{array}.$$

We put,

$$\mu_\Phi(\gamma) := \mu_\Phi([\tilde{\gamma}]) \in \mathbb{Z}/\chi(S_g)\mathbb{Z},$$

where $[\tilde{\gamma}]$ is the class of $\tilde{\gamma}$ in the homology group $H_1(TS_g\backslash\{0\}, \mathbb{Z}/\chi(S_g)\mathbb{Z})$.

Proposition 2.2.11. *Let*

$$\gamma_1, \ldots, \gamma_N : S^1 \looparrowright S_g$$

be immersed Lagrangian cobordant curves. In $\mathbb{Z}/\chi(S_g)\mathbb{Z}$, we have the relation

$$\sum_{i=1}^{N} \mu_\Phi(\gamma_i) = 0.$$

Proof. First, we generalize the above construction of μ. Denote by $\pi_\mathbb{C} : \mathbb{C} \times S_g \to \mathbb{C}$ and $\pi_{S_g} : \mathbb{C} \times S_g \to S_g$ the projection on the first and second factor respectively. There is a canonical isomorphism

$$\pi_{S_g}^* \Lambda^1 TS_g \tilde{\to} \Lambda^2 T(\mathbb{C} \times S_g).$$

We compose this with the map 2.9 to obtain an isomorphism

$$\Psi : (\pi_{S_g}^* Z)^{\otimes\chi(S_g)} \xrightarrow{\sim} \Lambda^2 T(\mathbb{C} \times S_g).$$

Let $\gamma : S^1 \to \mathbb{C} \times S_g$ be a smooth loop. We let

$$\Lambda(t) \subset T_{\gamma(t)}(\mathbb{C} \times S_g),\ t \in S^1$$

be a smooth loop of oriented lagrangian subspaces over γ. For each $t \in S^1$, we let $(e_1(t), e_2(t))$ be a (real) basis of the vector space $\Lambda(t)$. We assume that the family (e_1, e_2) is smooth. We let v be a trivialization of the complex line bundle $\big(\pi_{S_g} \circ \gamma\big)^* Z$.

The function

$$\begin{array}{ccc} S^1 & \mapsto & \Lambda^2 T(\mathbb{C} \times S_g) \\ t & \mapsto & e_1(t) \wedge e_2(t) \end{array},$$

is nowhere vanishing. So there is a smooth function $\lambda : S^1 \to \mathbb{C}^*$ such that

$$e_1 \wedge e_2 = \lambda(t)\Psi(v(t)).$$

Now, we put

$$\mu_\Phi(\Lambda) = \deg(\lambda).$$

As before, one can easily check that this does not depend on the homotopy class of Λ and does not depend on the choice of the section v. Thus, this induces a well-defined class

$$\mu_\Phi \in H^1(\mathcal{GL}^{or}(T(\mathbb{C} \times S_g)), \mathbb{Z}/\chi(S_g)\mathbb{Z}),$$

in the first cohomology group of the oriented Lagrangian Grassmannian.

Let $\gamma : S^1 \to S_g$ be an immersed curve and $x \in \mathbb{R}$. For $t \in S^1$, we let

$$\Lambda(t) = \mathrm{Span}((1,0), (0, \gamma'(t))) \subset T_{(x,\gamma(t))}(\mathbb{C} \times S_g).$$

Then, for any $t \in S^1$, the space $\Lambda(t)$ is Lagrangian. It is an easy exercise to check

$$\mu_\Phi(\Lambda) = \mu(\gamma).$$

Let $i : W \looparrowright \mathbb{C} \times S_g$ be an oriented immersed Lagrangian cobordism between the immersed curves $\gamma_1, \dots, \gamma_N$. Then by the discussion above

$$\mu_\Phi(\gamma_1) + \ldots + \mu_\Phi(\gamma_N) = \langle i_V^* \mu, [\partial W] \rangle.$$

The class ∂W is a boundary in $H_1(W, \mathbb{Z}/\chi(S_g)\mathbb{Z})$, so the left term is 0. □

We conclude that there is a well-defined morphism

$$\mu_\Phi : \Omega^{\mathrm{imm}}_{\mathrm{cob}}(S_g) \to \mathbb{Z}/\chi(S_g)\mathbb{Z}.$$

Remark 2.2.12. The map μ_Φ *depends* on the choice of the isomorphism Φ. In fact, two such maps differ by the morphism induced by a cohomology class $p^*\alpha$ with $\alpha \in H^1(S_g, \mathbb{Z}/\chi(S_g)\mathbb{Z})$ and p the projection $TS_g \backslash \{0\} \to S_g$.

From now on, we fix once and for all one such Φ. We will, therefore, denote μ_Φ by μ.

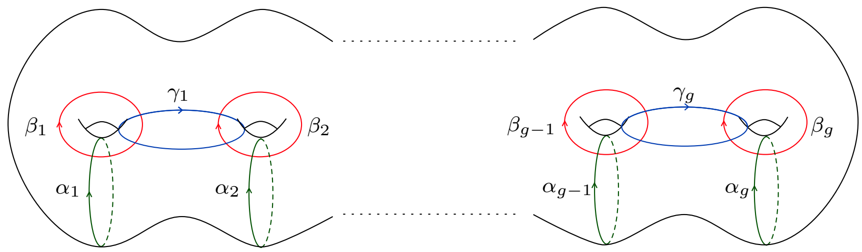

FIGURE 4 – *The Lickorish generators of the Mapping Class Group*

Action of the mapping class group on $\Omega_{cob}^{imm}(S_g)$

As usual, the *Mapping Class Group* of the surface S_g is the quotient of the group of orientation preserving diffeomorphisms by its identity component,

$$\mathrm{Mod}(S_g) = \mathrm{Diff}^+(S_g)/\,\mathrm{Diff}_0(S_g).$$

The Mapping Class Group has a natural left action on $\Omega_{\text{cob}}^{\text{imm}}(S_g)$. Given two classes $[\phi] \in \mathrm{Mod}(S_g)$ and $[\gamma] \in \Omega_{\text{cob}}^{\text{imm}}(S_g)$, the action is given by $[\phi] \cdot [\gamma] = [\phi \circ \gamma]$.

We recollect a few well-known facts on the Mapping Class Group. The reader may find proofs and statements in the book by Farb and Margalit [**FM12**].

A particular class of elements of the Mapping Class groups are given by *Dehn twists*, which we now define. Let $\alpha : S^1 = \mathbb{R}/\mathbb{Z} \to S_g$ be an embedded curve. Choose a Weinstein embedding $\psi : [0,1] \times S^1 \to S_g$ such that $\psi_{|\{\frac{1}{2}\}\times S^1} = \alpha$. We let $f : [0,1] \to \mathbb{R}$ be an increasing smooth function equal to 1 in a neighborhood of 1 and equal to 0 in a neighborhood of 0. The map

$$\begin{array}{ccc} [0,1] \times S^1 & \to & [0,1] \times S^1 \\ (t,\theta) & \mapsto & (t, \theta + f(t)) \end{array}$$

extends by the identity to a symplectomorphism

$$T_\alpha : S_g \to S_g$$

which is called the *Dehn twist about* α. Notice that its class in $\mathrm{Mod}(S_g)$ does not depend on the choice of ϕ and f.

It is a well-known fact that these transformations generate the Mapping Class Group. More precisely, we let $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ and $\gamma_1, \ldots, \gamma_{g-1}$ be the embedded curves represented in Figure 4.

Theorem 2.2.13 (Lickorish,1964, [**Lic64**])**.** *The Dehn twists about the curves* $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ *and* $\gamma_1, \ldots, \gamma_{g-1}$ *generate the Mapping Class Group.*

In particular, any orientation-preserving diffeomorphism ϕ *is the product of a symplectomorphism* ψ *and a diffeomorphism* χ *isotopic to the identity.*

In particular, any positive homeomorphism is isotopic to a symplectomorphism As a corollary of this and Lemma 2.2.1, we obtain the following.

Lemma 2.2.14. *The map*

$$\begin{array}{rcc} \mathrm{Mod}(S_g) \times \Omega_{cob}^{imm}(S_g) & \to & \Omega_{cob}^{imm}(S_g) \\ ([\phi], [\gamma_1] + \ldots + [\gamma_N]) & \mapsto & [\phi \circ \gamma_1] + \ldots + [\phi \circ \gamma_N] \end{array}$$

is well-defined and is a group action on $\Omega_{cob}^{imm}(S_g)$.

Proof. First we check that if $[\gamma_1]+\ldots+[\gamma_N] = 0$ in $\Omega^{\mathrm{imm}}_{\mathrm{cob}}(S_g)$, then $[\phi\circ\gamma_1]+\ldots+[\phi\circ\gamma_N] = 0$. For this, write $\phi = \psi \circ \chi$ with ψ symplectic and χ isotopic to the identity. There is an immersed oriented Lagrangian cobordism $V : (\gamma_1, \ldots, \gamma_N) \rightsquigarrow \emptyset$. Then $\psi(V)$ is a Lagrangian cobordism between the curves $\psi(\gamma_1), \ldots, \psi(\gamma_N)$ which are isotopic (hence Lagrangian cobordant by 2.2.1) to $\phi(\gamma_1), \ldots, \phi(\gamma_N)$.

Similarly, Lemma 2.2.1 implies that if ϕ is isotopic to ψ, then $[\phi\circ\gamma_1]+\ldots+[\phi\circ\gamma_N] = [\psi \circ \gamma_1] + \ldots + [\psi \circ \gamma_N]$ in $\Omega^{\mathrm{imm}}_{\mathrm{cob}}(S_g)$. $\square$

We also have the following proposition:

Proposition 2.2.15. *Let β be an embedded curve in S_g. Then in $\Omega_{cob}^{imm}(S_g)$*

$$[T_\alpha(\beta)] = (\beta \cdot \alpha)[\alpha] + [\beta].$$

Here, $\beta \cdot \alpha$ is the homological intersection number of β and α.

Proof. Up to isotopy, we assume that α and β are in *minimal position* (i.e. the number of intersection points is minimal in their respective isotopy class).

There is a geometric procedure which produces a curve isotopic to $T_\alpha\beta$ after a sequence of surgeries such as in 2.2.1. The whole process is represented in Figure 5.

Call $x_1, \ldots, x_N$ the intersection points of α and β ordered according to the orientation of α (here N is the number of intersection points between α and β). For $k \in \{1, \ldots, N\}$, we fix a Darboux chart $\phi_k : B(0, r) \to S_g$ around x_k such that

- in this chart, β is the oriented line $\mathbb{R}$,
- in this chart, α has image $i\mathbb{R}$.

The first step consists of the surgery between α and β if x_1 is of degree 1 and of the surgery between α^{-1} and β if x_1 is of degree 0. This yields a curve c_1

In the second step, we perturb α to a curve $\tilde{\alpha}_2$ as in the second row of Figure 5. The main features of $\tilde{\alpha}$ are as follows

- there are $x_2^2, \ldots x_N^2$ intersection points lying close to $x_2, \ldots, x_N$.
- there is one other intersection point y_1 *above* β in the Darboux chart ϕ_2.

Now, we perform the surgery between c_1 and $\tilde{\alpha_2}$ at x_2^2 if it is of degree 1 and between c_1 and $\tilde{\alpha_2}^{-1}$ otherwise.

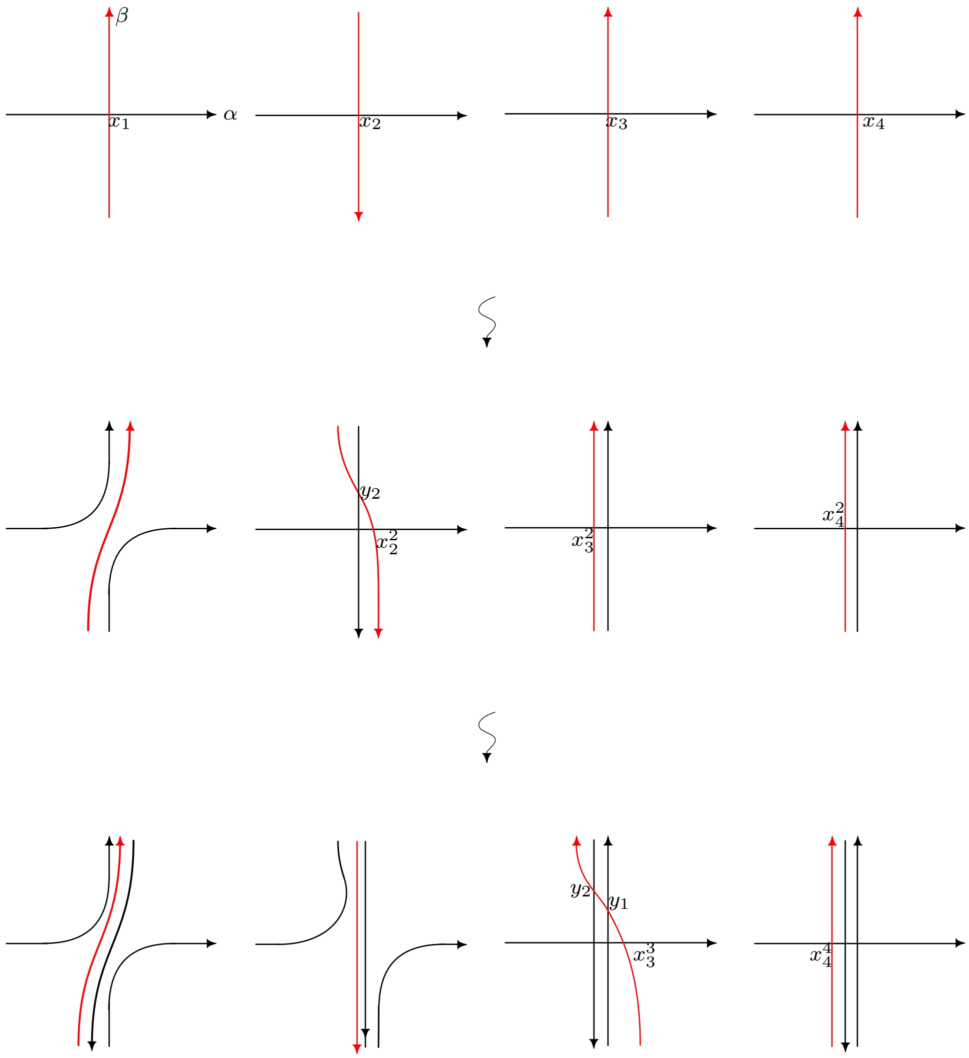

FIGURE 5 – *The surgery procedure to obtain a curve isotopic to a Dehn Twist,*
The successive c_k are represented in red.

Assume that we performed the surgery of β with k curves $\alpha, \tilde{\alpha}_2, \dots, \tilde{\alpha}_k$ isotopic to α to obtain a curve c_k. We perturb α to a curve $\tilde{\alpha}_{k+1}$ such as in Figure 5. It satisfies the following assumptions.

- There are $x_k^k, \dots, x_k^N$ intersection points between c_k and $\tilde{\alpha}_{k+1}$ close to $x_k, \dots, x_N$.
- There are intersections points $y_1, \dots, y_k$ which lie above β in the chart ϕ_k.

Now, we perform the surgery between c_k and $\tilde{\alpha}_{k+1}$ at x_k according to the orientation of $\tilde{\alpha}_{k+1}$. The handle is big enough to delete the intersection points $y_1, \dots, y_k$.

Notice that each surgery produces an oriented immersed Lagrangian cobordism by Propositions 2.2.9. Composing these cobordisms and using Lemma 2.2.1 about isotopic curves, we obtain an immersed Lagrangian cobordism

$$\left(\alpha, \beta, \dots, \beta, \beta^{-1}, \dots, \beta^{-1}\right) \rightsquigarrow \gamma,$$

with as many copies of α as there are intersection points of degree 0 and as many copies of α^{-1} as there are intersection points of degree 1. Hence in the Lagrangian cobordism group $\Omega^{\text{imm}}_{\text{cob}}(S_g)$

$$[\gamma] = [\beta] + (\alpha \cdot \beta)[\alpha].$$

This concludes the proof since γ is Lagrangian cobordant to $T_\alpha(\beta)$ (Lemma 2.2.1). $\square$

Tori and pairs of pants

We describe a set of generators for $\Omega^{\text{imm}}_{\text{cob}}(S_g)$ using the action of the mapping class group described above.

First, let γ_1 and γ_2 be two embedded curves in S_g. We suppose that each of these is the oriented boundary of an embedded torus. By the change of coordinates principle ([**FM12**, 1.3]), there is a product of Dehn twists ϕ which maps γ_1 to a curve isotopic (hence immersed Lagrangian cobordant) to γ_2. By Proposition 2.2.15, we have $[\gamma_1] = [\gamma_2]$ in $\Omega^{\text{imm}}_{\text{cob}}(S_g)$. We conclude that there is a well-defined element

$$T \in \Omega^{\text{imm}}_{\text{cob}}(S_g) \tag{2.10}$$

which represent any oriented boundary of a torus in S_g.

First, we compute the Maslov index of the class T.

Lemma 2.2.16. *For any choice of isomorphism*

$$\Phi : Z^{\otimes \chi(S_g)} \xrightarrow{\sim} TS_g,$$

we have

$$\mu_\Phi(T) = -1 \in \mathbb{Z}/\chi(S_g)\mathbb{Z},$$

(T is the class defined in 2.10).

Proof. First, the index of T does not depend on Φ. To see this, let γ be a representative of T and v a trivialization of Z along γ. Let $\Psi : TS_g \xrightarrow{\sim} Z^{\otimes \chi(S_g)}$ be an another complex isomorphism. Then $\Psi \circ \Phi^{-1}$ has the form $(z, v) \mapsto (z, \mu(z)v)$ where $\mu : S_g \to \mathbb{C}^*$ is a nowhere vanishing function. If

$$\gamma'(t) = \lambda(t)\Phi^{-1}(v \otimes \dots \otimes v),$$

then

$$\begin{aligned} \gamma'(t) &= \Psi^{-1} \circ \Psi \circ \Phi^{-1}(v \otimes \dots \otimes v) \\ &= \lambda(t)\mu(\gamma(t))\Psi^{-1}(v \otimes \dots \otimes v). \end{aligned}$$

But since μ extends to S_g and γ is homologically trivial, we have $\deg(\mu \circ \gamma) = 0$.

Let us turn to the computation of $\mu(T)$. It is a quick application of the Poincaré-Hopf theorem. Let $\tilde{T}$ be the torus bounded by γ and D be a disk embedded in $\tilde{T}$.

We choose trivializations of TS_g over $S_g \backslash D$ and over D so that TS_g is identified with the fiber bundle obtained by gluing $(S_g \backslash D) \times \mathbb{C}$ on $D \times \mathbb{C}$ along the map

$$\begin{array}{rccc} f: & (S_g \backslash D) \times \mathbb{C} & \to & D \times \mathbb{C} \\ & \big(\phi(e^{i\theta}), z\big) & \mapsto & \big(e^{i\theta}, e^{i\chi(S_g)\theta} z\big). \end{array}$$

Here $\phi : \partial D \to \partial(S_g \backslash D)$ is an orientation reversing diffeomorphism.

Similarly, we define the line bundle Z as the gluing of $(S_g \backslash D \times \mathbb{C})$ on $(D \times \mathbb{C})$ along the map

$$\begin{array}{rccc} g: & (S_g \backslash D) \times \mathbb{C} & \to & D \times \mathbb{C} \\ & \big(\phi(e^{i\theta}), z\big) & \mapsto & \big(e^{i\theta}, e^{i\theta} z\big). \end{array}$$

The isomorphism $\Phi : Z^{\otimes \chi(S_g)} \to TS_g$ is given by $(a, \lambda_1 \otimes \ldots \otimes \lambda_n) \mapsto (a, \lambda_1 \ldots \lambda_n)$. Moreover, a non-zero section of Z over γ is given by $z \in \mathrm{Im}(\gamma) \mapsto (z, 1)$. So the Maslov index of γ is just the index of γ' read in the trivialization above.

Choose a vector field X on $\tilde{T}$ which coincides with γ' over γ and has a unique zero in D. This zero has degree -1 since the Euler characteristic of $\tilde{T}$ is -1. The degree of γ' in the trivialization above is equal to the degree of X over the boundary of S_g D since this is a homological invariant. Given the expression of -1, this degree is also the degree of X in the trivialization over D plus $\chi(S_g)$. Hence it is $-1 + \chi(S_g)$ since X has a zero of degree -1 on D. $\square$

Remark 2.2.17. The same proof also shows that if γ is the oriented boundary of an embedded surface S_1 of genus $\tilde{g}$, then its index satisfies

$$\mu_\Phi(\gamma) = \chi(S_1) \mod \chi(S_g),$$

for any trivialization Φ.

We now express the class of any separating curve with T. The proof uses the surgeries of [**Abo08**, Lemma 7.6].

Lemma 2.2.18. *Let γ be the oriented boundary of an embedded surface S_1. Then in $\Omega^{imm}_{cob}(S_g)$*

$$[\gamma] = \chi(S_1) \cdot T.$$

Proof. The proof follows from induction over the genus of the surface bounded by γ. If γ bounds a torus, there is nothing to prove.

We assume that the formula is true for any curve which bounds a surface of genus less than $g_1 - 1$. We assume that γ is the oriented boundary a surface S_1 of genus $g_1 \geqslant 2$.

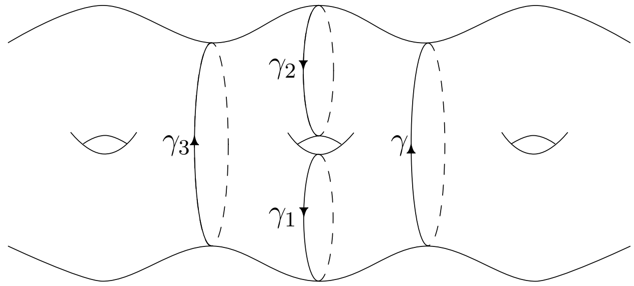

FIGURE 6 – *Two pair of pants*

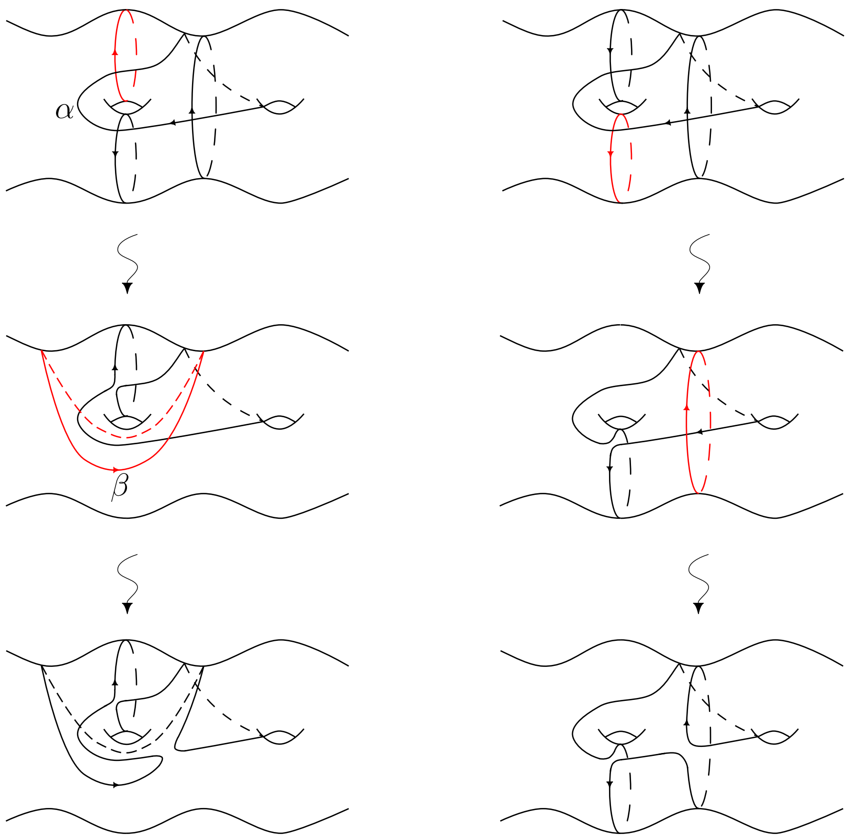

FIGURE 7 – *The successive surgeries to compute the class of a pair of pants*

Choose three curves γ_1, γ_2 and γ_3 such that the following hold (see Figure 6).

- The curves γ_1 and γ_2 are non-separating and γ, γ_1 and γ_2 form the oriented boundary of a pair of pants.
- The curve γ_3 is separating and bounds a surface S_3 of genus $g_1 - 1$.
- The curves γ_1, γ_2 and γ_3 bound another pair of pants.

Now, choose two embedded curves α and β as in Figure 7. We first perform surgeries of α with γ_2 and β as indicated in the left-hand side of Figure 7. This yields an immersed curve c. If we perform surgeries of α with γ_1 and γ as in the right-hand side of Figure 7, we obtain an immersed curve isotopic to c. Therefore,

$$-\gamma_2 + \alpha + T = \gamma_1 + \alpha + \gamma,$$

so

$$\gamma + \gamma_1 + \gamma_2 = T.$$

The same argument yields

$$T = -\gamma_3 - \gamma_1 - \gamma_2.$$

Hence,

$$\gamma = \gamma_3 + 2T.$$

But γ_3 is the oriented boundary of a surface of genus $g_1 - 1$, so

$$[\gamma_3] = \chi(S_3) \cdot T.$$

Hence,

$$\gamma = (\chi(S_3) + 2) \cdot T = \chi(S_1)T.$$

□

We now have the following Lemma.

Lemma 2.2.19. *The restriction of μ to the subgroup H of $\Omega^{imm}_{cob}(S_g)$ generated by separating curves is an isomorphism.*

Proof. Lemma 2.2.18 implies that T generates the group H. Moreover, $\mu(T) = -1$ implies that the order of T is either infinite or a multiple of $\chi(S_g)$.

On the other hand, let γ be the oriented boundary of a torus. Then, γ^{-1} is the oriented boundary of a surface of genus $g-1$. Hence by Lemma 2.2.18, we have

$$\begin{aligned} T &= -(1 - 2(g-1))T \\ &= (-3 + 2g)T. \end{aligned}$$

So $\chi(S_g)T = 0$. This concludes the proof. □

Finally, we compute the class of the Lickorish generator γ_i (see 2.2.13) in $\Omega^{\mathrm{imm}}_{\mathrm{cob}}(S_g)$.

Lemma 2.2.20. *Recall that we denoted by $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ and $\gamma_1, \ldots, \gamma_{g-1}$.*

Let $i \in \{1, \ldots, g-1\}$. Then we have

$$[\gamma_i] = [\alpha_{i+1}] - [\alpha_i] - T.$$

Proof. By the change of coordinates principle, we can assume that α_i, γ_i and α_{i+1}^{-1} are as in Figure 8. We also fix two curves α and β as in Figure 8.

The proof proceeds as in the proof of Lemma 2.2.18. We perform the surgeries indicated on the left to obtain a curve c and the surgeries on the right to obtain a curve

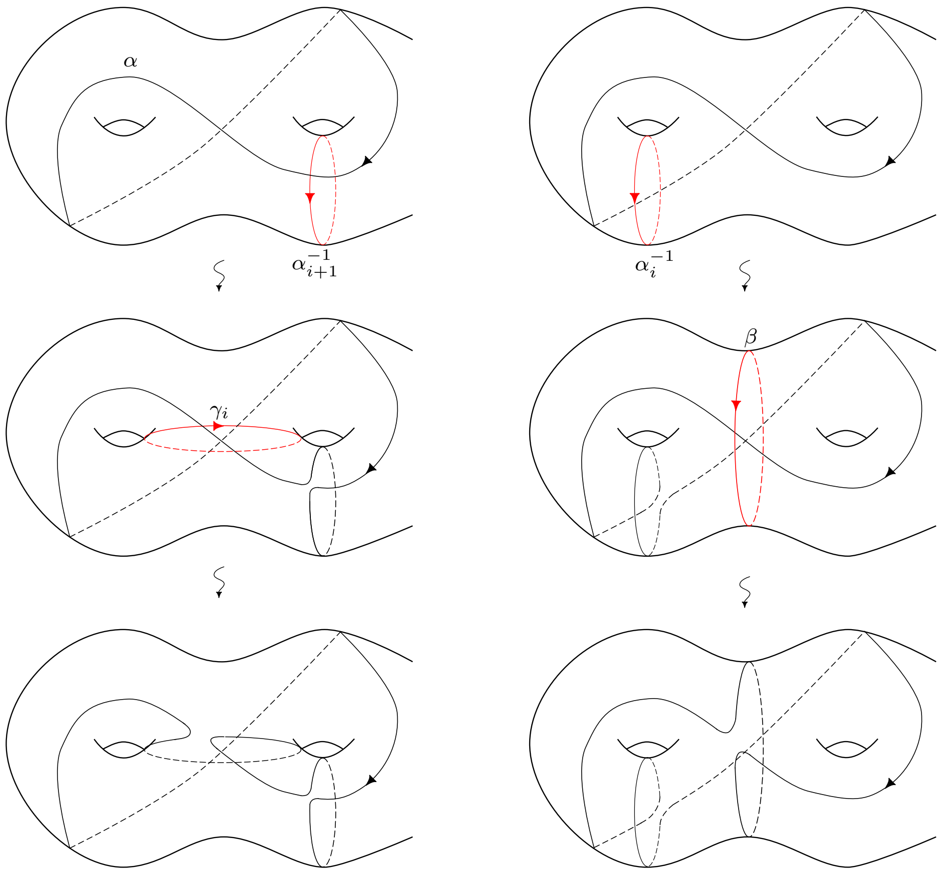

FIGURE 8 – *The successive surgeries to compute the class of a pair of pants in the non-separating case.*

isotopic to c. So

$$\alpha + \gamma_i - \alpha_{i+1} = \alpha + \beta - \alpha_i.$$

Moreover, we have $\beta = -T$, so

$$\alpha_{i+1} - \alpha_i - \gamma_i = T.$$

□

With this preparation, we can pass to the

Proof of Theorem 2.1.5

PROOF. We use the action of the Mapping Class group of S_g to conclude. Let γ be a non-separating curve. By the change of coordinates principle ([**FM12**, 1.3]), there is a product of Dehn twists about the α_i, γ_i and β_i which maps γ to α_1. Hence, γ lies in the group generated by the α_i, β_i and γ_i by Proposition 2.2.15. By Lemma 2.2.20, γ lies in the subgroup generated by the α_i, β_i and T.

Since any separating embedded curve is a multiple of T, we conclude that the group $\Omega^{\text{imm}}_{\text{cob}}(S_g)$ is generated by the α_i, β_i and T.

The map $\pi : \Omega^{\text{imm}}_{\text{cob}}(S_g) \to H_1(S_g, \mathbb{Z})$ is surjective. Furthermore, we have $\pi \oplus \mu(T) = (0, -1)$. So the map $\pi \oplus \mu$ is surjective.

On the other hand, $\pi \oplus \mu$ is injective. To see this, let

$$x = \sum_{i=1}^{g} n_i\alpha_i + \sum_{i=1}^{g} m_i\beta_i + kT.$$

such that $\pi(x) = 0$ and $\mu(x) = 0$. Take the image under π to obtain

$$\sum_{i=1}^{g} n_i\alpha_i + \sum_{i=1}^{g} m_i\beta_i$$

in $H_1(S_g, \mathbb{Z})$. So the n_i and the m_i are zero. Furthermore, $0 = \mu(x) = -k$, so $\chi(S_g)|k$. Hence, $x = kT = 0$. □

2.3. Fukaya categories of surfaces

There is a well-defined Fukaya category whose objects are the unobstructed immersed curves. Its construction follows Seidel's perturbation scheme [**Sei08**]. Such a category has already been studied for exact manifolds with convex boundary by Alston and Bao [**AB18**]. Abouzaid ([**Abo08**]) also constructed a pre-category with immersed objects using combinatorial methods.

In this section, we recall the main steps of the construction of $\text{Fuk}(S_g)$, highlighting the parts that need special care due to the immersed setting. We do this in Subsections 2.3.2 and 2.3.1.

In Subsection 2.3.3, we shall briefly explain why $\text{Fuk}(S_g)$ recovers the pre-A_∞ category defined in Abouzaid's paper.

2.3.1. Preliminaries

Floer datum

Recall from definition 2.1.9 that an unobstructed curve is an immersed curve with no triple points, transverse double points and which lifts to an embedding in the universal cover. For each ordered pair of unobstructed immersions (γ_1, γ_2), we fix a *Floer datum* H_{γ_1,γ_2}. This is a smooth, time-independent, hamiltonian $H_{\gamma_1,\gamma_2} : S_g \to \mathbb{R}$ such that γ_1 and $\phi^{-1}_{H_{\gamma_1,\gamma_2}}(\gamma_2)$ are in general position. If γ_1 and γ_2 already are in general position, we make the choice $H_{\gamma_1,\gamma_2} = 0$.

We also fix a smooth complex structure j on S_g which is compatible with ω.

Coherent perturbations

Recall from [Sei08, (9f)] that for $d \geqslant 2$, there is a compactified universal family of pointed disks

$$\mathcal{S}^{d+1} \to \mathcal{R}^{d+1}.$$

We fix a coherent universal choice of strip-like ends (see [Sei08, 9g]) for these families. We denote these ends by $(\varepsilon_r^i)_{0\leqslant i\leqslant d, r\in\mathcal{S}^{d+1}}$.

Let $(\gamma_0, \ldots, \gamma_d)$ be a $d+1$ tuple of unobstructed curves. We fix a *perturbation datum* for the family $\mathcal{S}^{d+1} \to \mathcal{R}^{d+1}$ labeled by the tuple $(\gamma_0, \ldots, \gamma_d)$. This is a family of one form

$$K_{\gamma_0,\ldots,\gamma_d} \in \Omega^1(\mathcal{S}, \mathcal{H})$$

with values in the space $\mathcal{H}$ of hamiltonians on S_g. Additionally, we assume that for any $r \in \mathcal{R}^{d+1}$,

$$K_{|T\partial\pi^{-1}(r)} = 0.$$

We require that this family satisfies the following hypothesis.

(H) : If the curves $\gamma_0, \ldots, \gamma_d$ are in general position, then

$$K_{\gamma_0,\ldots,\gamma_d} = 0.$$

We have the following proposition.

Proposition 2.3.1. *There is a coherent choice of perturbation datum*[2] *which satisfies the hypothesis (H).*

Proof. We start with $K_{\gamma_0,\gamma_1,\gamma_2} = 0$ for every 3-uple of two by two transverse lagrangians. Then we use the induction process described in [Sei08, (9i)] to obtain a coherent pertubation perturbation datum.

For every $d+1$-tuple $(\gamma_0, \ldots, \gamma_d)$ of two by two transverse unobstructed curves, any m-tuple of the form $(\gamma_{i_1}, \ldots, \gamma_{i_m})$ (with $m < d$) consists of two by two transverse unobstructed curves. Therefore the gluing induction process provides a perturbation with $K_{\gamma_0,\ldots,\gamma_d} = 0$. ☐

Definition and regularity of the relevant moduli spaces

We introduce the moduli spaces of (perturbed) holomorphic curves that we consider throughout this section.

First, let $Z = \mathbb{R} \times [0, 1]$ be the standard strip with coordinates s and t and equipped with the standard complex structure.

[2]See [Sei08, section (9i)] for the definition

Let γ_0 and γ_1 be two unobstructed curves. Then the set $\mathcal{P}_{H_{\gamma_0,\gamma_1}}$ of hamiltonian chords from γ_0 to γ_1 is finite. We choose two such chords c_- and c_+.

We consider continuous strips $u : Z \to S_g$ from c_- to c_+ satisfying the following conditions.

(i) There is a continuous lift $u^- : \mathbb{R} \times \{0\} \to S^1$ such that $u(s,0) = \gamma_0 \circ u_0$.

(ii) There is a continuous lift $u^+ : \mathbb{R} \times \{0\} \to S^1$ such that $u(s,1) = \gamma_1 \circ u_1$.

(iii) The map u converges uniformly to c_- and c_+ when s goes to infinity:

$$\lim_{s\to-\infty} u(s,\cdot) = c_-, \ \lim_{s\to+\infty} u(s,\cdot) = c_+.$$

Let u_0 and u_1 be two continuous strips with lifts given by u_0^+, u_0^- and u_1^+, u_1^- respectively. We say that u_0 and u_1 are *homotopic* if there are

- a continuous family $(v_t)_{t\in[0,1]}$ of maps $Z \to S_g$,
- continuous families $\left(v_t^{\pm}\right)_{t\in[0,1]}$ of maps $\mathbb{R} \to S^1$,

such that

- for each $t \in [0,1]$, v_t is a continuous strip from c_- to c_+ with continuous lifts given by $v_t^{\pm}$,
- we have $\left(u_0, u_0^-, u_0^+\right) = \left(v_0, v_0^-, v_0^+\right)$ and $\left(u_1, u_1^-, u_1^+\right) = \left(v_1, v_1^-, v_1^+\right)$.

We fix such a homotopy class A.

Definition 2.3.2. *We let $\widetilde{\mathcal{M}}(c_-, c_+, A)$ be the set of* Floer strips *from c_- to c_+ in the homotopy class A.*

A map $u : Z \to S_g$ is an element of $\widetilde{\mathcal{M}}(c_-, c_+, A)$ if it satisfies the conditions (i), (ii), (iii) above and the Floer equation

$$\frac{\partial u}{\partial s} + j\left(\frac{\partial u}{\partial t} - X_{H_{\gamma_0,\gamma_1}}(u)\right) = 0.$$

This set admits a natural $\mathbb{R}$-action and we let

$$\mathcal{M}(c_-, c_+, A) = \widetilde{\mathcal{M}}(c_-, c_+, A)/\mathbb{R}.$$

Let $\gamma_0, \ldots, \gamma_d$ be a $d+1$-uple of unobstructed curves, $c_0 \in \mathcal{P}_{H_{\gamma_0,\gamma_d}}$ and $c_i \in \mathcal{P}_{H_{\gamma_{i-1},\gamma_i}}$ for $1 \leqslant i \leqslant d$. We also consider *continuous polygons* with boundary conditions at $\gamma_0, \ldots, \gamma_d$.

To define these, fix a disk with $d+1$-marked points $s = \pi^{-1}(r)$ with $r \in \mathcal{R}^{d+1}$. A *continuous polygon* is a continuous map $u : s \to S_g$ such that

(i) For each arc $C_i \subset \partial s$ between the i-th and $i+1$-th punctures, there is a continuous map $u_i : C_i \to S^1$ such that $u_{|C_i} = \gamma_i \circ u_i$.

(ii) The map u converges uniformly to c_i on the i-th strip-like end,

$$\lim_{s\to-\infty} u \circ \varepsilon_r^0(s,\cdot) = c_0, \ \lim_{s\to+\infty} u \circ \varepsilon_r^i(s,\cdot) = c_i.$$

Let u^0 and u^1 be two continuous polygons with lifts given by u_i^0 and u_i^1 for $i = 0 \dots d$ respectively. We say that u^0 and u^1 are *homotopic* if there are

- continuous families $(v^t)_{t\in[0,1]}$ of maps $s \to S_g$ for a fixed $s \in \pi^{-1}(r)$ with $r \in \mathcal{R}^{d+1}$,
- continuous families $(v_i^t)_{t\in[0,1]}$ of maps $C_i \to S^1$ for $i \in \{0, \dots, d\}$,

such that

- for each $t \in [0,1]$, v_t is continuous polygon with boundary lifts given by the v_t^i for $i = 0 \dots d$,
- we have $v_0 = u_0$ (resp. $v_1 = u_1$) and $v_0^i = u_0^i$ (resp. $v_0^1 = u_0^1$) for $i = 0 \dots d$.

We fix such a homotopy class B.

Definition 2.3.3. *We let $\mathcal{M}(c_0, \dots, c_d, B)$ be the set of* holomorphic polygons *in the homotopy class A.*

A map $u : s \to S_g$, with $s \in \pi^{-1}(r)$ for some $r \in \mathcal{R}^{d+1}$, is an element of the moduli space

$$\mathcal{M}(c_0, \dots, c_d, B)$$

if and only if it satisfies the conditions (i),(ii),(iii) above and the following equation

$$\big(du - X_{K_{\gamma_0,\dots,\gamma_d}}\big)^{(0,1)} = 0.$$

Standard regularity arguments imply that for a generic choice of Floer and perturbation datum, these moduli spaces are regular. See for instance [**AB18**, Section 5] for a write up in the case of immersions.

However, we would like to keep perturbation data which verify the hypothesis (H). The following lemma, due to Seidel, makes this possible. The proof contained in [**Sei08**] goes through as stated in the book.

Let $u : (s, \partial s) \to (M, i(L))$ be a Floer strip or a holomorphic polygon. One can linearize the Cauchy-Riemann equation at u to obtain an *extended Cauchy-Riemann operator*

$$D_{s,u} : T_s\mathcal{R} \times W^{1,p}(u^*TM, u^*TL) \to L^p\big(\Lambda^{0,1}T^*s \otimes u^*TM\big),$$

from suitable Sobolev completions of the space of sections of u^*TM. (cf [**Sei08**, (8i)]). We say that u is *regular* if this operator is surjective. In particular, this implies that the set of solutions is a manifold near u.

In particular notice that $D_{s,u}$ takes into account the *variations* of the domain.

Lemma 2.3.4 (Automatic regularity, [**Sei08**], Lemma 13.2)**.** *In the above setting, assume that u is either a non-constant Floer strip or a non-constant holomorphic polygon. Then it is automatically regular, meaning that the extended Cauchy-Riemann operator*

$$D_{s,u} : T_s\mathcal{R} \times W^{1,p}(u^*TM, u^*TL) \to L^p\big(\Lambda^{0,1}T^*s \otimes u^*TM\big),$$

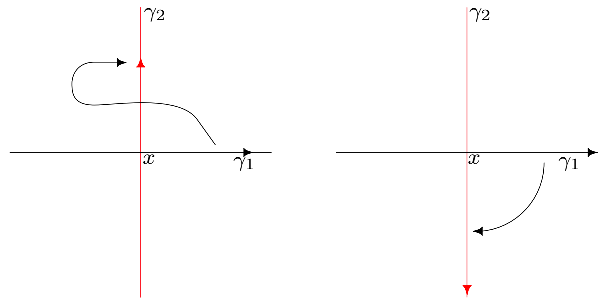

FIGURE 9 – *The path $\lambda_{\gamma_1,\gamma_2,x}$ when x is of degree 1 (left) and of degree 0 (right)*

is surjective.

From now on, we will fix a choice of Floer and perturbation datum which satisfy hypothesis (H).

Indices

Let u be an element of one of the moduli spaces $\mathcal{M}(c_-, c_+, A)$ or $\mathcal{M}(c_0, \dots, c_d, A)$. We call $\mathrm{Ind}(u)$ the virtual-dimension of its moduli space. Using (for instance) the index formula of [**AJ10**], it is easily seen to depend only on the homotopy class A of u. Therefore, in what follows, we will denote this index by $\mathrm{Ind}(A)$.

Spin structures and signs

To take care of signs issues, we need some additional datum which we explain now.

We need to equip each unobstructed curve $\gamma : S^1 \to S_g$ with a *Spin structure.* Since the space of oriented orthonormal frames of a tangent space T_xS^1 has a unique point, a choice of Spin structure is a choice of a double covering of S^1. We choose the Spin structure given by the nontrivial double covering. This is called the *bounding Spin structure* (see [**LM89**, II.1]).

Let γ_1 and γ_2 be two transverse unobstructed curves and x an intersection point. We let

$$\lambda_{\gamma_1,\gamma_2,x} : [0,1] \to \mathcal{G}(TS_g),$$

be the path represented in Figure 9.

If γ_1 and γ_2 are not transverse, we choose a hamiltonian chord $c \in \mathcal{P}_{H_{\gamma_1,\gamma_2}}$. This corresponds to a unique intersection point $x \in \gamma_1 \cap \phi^{-1}_{H_{\gamma_1,\gamma_2}}(\gamma_2)$. We define, for $t \in [0,1]$, the vector space $\lambda_{\gamma_1,\gamma_2,c}(t) \in \mathcal{G}(T_c(t)S_g)$ by

$$\lambda_{\gamma_1,\gamma_2,c}(t) = d\phi^t_{H_{\gamma_1,\gamma_2}}(\lambda_{x,\gamma_1,\phi^{-1}_{H_{\gamma_1,\gamma_2}}(\gamma_2)}(t)).$$

For each Hamiltonian chord $c \in \mathcal{P}_{H_{\gamma_1,\gamma_2}}$, define a one dimensional real vector space $o(c)$ as follows. Consider the Poincaré half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} | y \leqslant 0\}$. Equip this with the incoming strip-like end

$$\begin{array}{rccc} \varepsilon : & \mathbb{R}\times[0,1] & \to & \mathbb{H} \\ & (s,t) & \mapsto & -e^{-\pi(s+it)} \end{array}.$$

As explained, for instance, in [Sei08], there is a complex bundle pair (E, F) associated to $\lambda_{\gamma_1,\gamma_2,c}$. In particular, there is a Cauchy-Riemann operator D associated to it.

The *orientation line* of c is the real one-dimensional vector space

$$o(c) := \det(D).$$

We choose once and for all an orientation of each vector space $o(c)$.

Further, assume that γ_1 and γ_2 are unobstructed curves. Let $u \in \mathcal{M}(c_-, c_+, A)$ be a holomorphic strip which satisfies $\mathrm{Ind}(A) = 0$. Since γ_1 and γ_2 are equipped with Spin structures, gluing induces an isomorphism

$$\Lambda T_u\widetilde{\mathcal{M}}(c_-, c_+, A)\tilde{\to}o(c_-)^\vee \otimes o(c_+).$$

We orient the left side by the vector field generated by the $\mathbb{R}$ action. On the other hand, the right hand sign inherits an orientation from $o(c_-)$ and $o(c_+)$. The difference between these two orientations yields a sign

$$\mathrm{Sign}(u) \in \{-1, 1\}.$$

Similarly let $v \in \mathcal{M}(c_0, \ldots, c_d, A)$ be a holomorphic polygon which satisfies $\mathrm{Ind}(A) = 0$. Now there is an isomorphism

$$\Lambda T_u\mathcal{M}(c_0, \ldots, c_d, A)\tilde{\to}o(c_0)^\vee \otimes o(c_1) \otimes \ldots \otimes o(c_d).$$

But the left-hand side is naturally oriented as the determinant of a 0-dimensional vector space. Comparing orientations yields a sign

$$\mathrm{Sign}(v) \in \{-1, 1\}.$$

Gromov compactness

In order to define the A_∞ operations for our category, we need to describe a Gromov-type compactification of the moduli spaces introduced above. Gromov compactness for holomorphic curves with immersed boundaries has already been considered in Ivashkovich and Shevchishin's paper [IS02].

Here, our situation is slightly different since we considered the solutions of a perturbed Cauchy-Riemann equation. However, the relevant analysis is worked-out in Alston and Bao's article [AB18, Proposition 4.4]. We can summarize their results in our setting as follows.

As is usual by now, we let γ_0, γ_1 be unobstructed curves in S_g and c_-, c_+ be Hamiltonian chords from γ_0 to γ_1.

Proposition 2.3.5 (Gromov compactness for strips)**.** *(1) Let A be a homotopy class of strips satisfying* $\operatorname{Ind}(A) = 1$*. Then the topological space*

$$\mathcal{M}(c_-, c_+, A)$$

is a compact, 0-dimensional manifold.

(2) Assume that A is a homotopy class of strips satisfying $\operatorname{Ind}(A) = 2$*. Then the topological space*

$$\overline{\mathcal{M}}(c_-, c_+, A)$$

admits a natural compactification $\overline{\mathcal{M}}(c_-, c_+, A)$ given by

$$\overline{\mathcal{M}}(c_-, c_+, A) := \coprod_{c \in \mathcal{P}_{H_{\gamma_-,\gamma_+}}} \mathcal{M}(c_-, c, A) \times \mathcal{M}(c, c_+, A) \bigcup \mathcal{M}(c_-, c_+, A).$$

The space $\overline{\mathcal{M}}(c_-, c_+, A)$ has a natural structure of 1-dimensional manifold with boundary

$$\partial \overline{\mathcal{M}}(c_-, c_+, A) = \coprod_{c \in \mathcal{P}_{H_{\gamma_-,\gamma_+}}} \mathcal{M}(c_-, c, A) \times \mathcal{M}(c, c_+, A).$$

Proof. Let (u_n) be a sequence of holomorphic strips in the homotopy class A. It is easy to check that their energy is finite so that we can apply [**AB18**, Proposition 4.4]. Hence, there is a subsequence (u_{n_k}) which converges in Gromov's sense to a set of broken holomorphic strips, polygons, spheres, and disks. Moreover, if there is one polygon in this decomposition, there must be a polygon with one corner.

Notice that there cannot be holomorphic disks or holomorphic polygon with one corner with boundary condition on γ_+ or γ_- since both of these are unobstructed. Moreover, there are no holomorphic spheres. Hence the above set can only consist of strips which are all regular. A dimension counting argument finishes the proof. ☐

Similarly, let $\gamma_0, \ldots, \gamma_d$ (with $d \geqslant 2$) be unobstructed curves, c_0 be in $\mathcal{P}_{H_{\gamma_0,\gamma_d}}$ and c_i be in $\mathcal{P}_{H_{\gamma_i,\gamma_{i+1}}}$ for $1 \leqslant i \leqslant d$. We fix a homotopy class B of holomorphic polygons with corners at the γ_i.

Proposition 2.3.6. *(1) Assume that* $\operatorname{Ind}(B) = 0$*, then the space $\mathcal{M}(c_0, \ldots, c_d, B)$ is compact.*

(2) Assume that $\operatorname{Ind}(B) = 1$*, then the topological space*

$$\mathcal{M}(c_0, \ldots, c_d, B)$$

admits a natural compactification $\overline{\mathcal{M}}(c_0, \ldots, c_d, B)$

$$\begin{aligned} &\overline{\mathcal{M}}(c_0, \ldots, c_d, B) := \\ &\mathcal{M}(c_0, \ldots, c_d, B) \coprod \\ &\coprod_{\substack{1 \leqslant i \leqslant d, \tilde{c}_{i+1} \in \mathcal{P}_{H_{\gamma_{i+1}, \gamma_{i+2}}}, \\ \mathrm{Ind}(B_k)=0}} \mathcal{M}(c_0, \ldots, \tilde{c}_{i+1}, \ldots c_d, B_1) \times \mathcal{M}(\tilde{c}_{i+1}, c_{i+1}, \ldots, c_d, B_2). \end{aligned} \tag{2.11}$$

The space $\overline{\mathcal{M}}(c_0, \ldots, c_d, B)$ *has a natural structure of* 1*-dimensional manifold with boundary*

$$\begin{aligned} &\partial\overline{\mathcal{M}}(c_0, \ldots, c_d, B) := \\ &\coprod_{\substack{1 \leqslant i \leqslant d, \tilde{c}_{i+1} \in \mathcal{P}_{H_{\gamma_{i+1}, \gamma_{i+2}}}, \\ \mathrm{Ind}(B_k)=0}} \mathcal{M}(c_0, \ldots, \tilde{c}_{i+1}, \ldots c_d, B_1) \times \mathcal{M}(\tilde{c}_{i+1}, c_{i+1}, \ldots, c_d, B_2). \end{aligned}$$

2.3.2. Definition

We now have all the ingredients to define a A_∞ category $\mathrm{Fuk}(S_g)$. The coefficients are taken over the *Novikov field*

$$\Lambda := \left\{ \sum_{i=0}^{+\infty} a_i T^{\lambda_i} \middle| a_i \in \mathbb{R}, \lambda_i \in \mathbb{R}, \lambda_i \to +\infty \right\}.$$

The objects of $\mathrm{Fuk}(S_g)$ are unobstructed curves. Given two unobstructed curves γ_1, γ_2, their morphism space is the $\mathbb{Z}/2$-graded Λ-vector space generated by Hamiltonian chords between these

$$\mathrm{Hom}^i_{\mathrm{Fuk}(S_g)}(\gamma_1, \gamma_2) = \bigoplus_{\substack{c \in \mathcal{P}_H \\ |c|=i}} \Lambda \cdot c.$$

The A_∞ operations are defined as follows

$$\mu^d(c_1, \ldots, c_d) = (-1)^\bullet \sum_{c_0, A} \sum_{u \in \mathcal{M}(c_0, \ldots, c_d, A)} \mathrm{Sign}(u) \cdot T^{\omega(A)} \cdot c_0.$$

Here the sum is over the Hamiltonian chords $c_0 \in \mathcal{P}_{H_{\gamma_1, \gamma_2}}$ and over the homotopy classes A of polygons such that $\mathrm{Ind}(A) = 0$. The sign $\bullet$ is given by

$$\bullet = \sum_{k=1}^{d} k|c_k|.$$

Proposition 2.3.6 implies that the operations $(\mu^d)_{d \geqslant 2}$ satisfy the A_∞ relation modulo 2. To see that these are satisfied over $\mathbb{R}$, one has to use that gluing is compatible with the isomorphisms. This is done in [Sei08, Sections (12b) and (12g)].

The A_∞-category $\mathrm{Fuk}(S_g)$ admits a triangulated envelope : this is the smallest triangulated A_∞-category generated by $\mathrm{Fuk}(S_g)$. We call its 0-th degree cohomology the *derived category* of $\mathrm{Fuk}(S_g)$ and denote it by $\mathrm{DFuk}(S_g)$. We refer to Seidel's book [**Sei08**, (3j)] for more details.

2.3.3. Properties of the Fukaya category

The following theorem also immediately follows from the recipe presented in Seidel's book.

Proposition 2.3.7. *The A_∞-category $\mathrm{Fuk}(S_g)$ is homologically unital and independent of the choice of perturbation datum and almost compatible complex structure.*

Moreover, two Hamiltonian isotopic curves are quasi-isomorphic.

Since we chose our perturbation datum (H), there is a combinatorial description of the operations of the category $\mathrm{Fuk}(S_g)$ with boundaries in a tuple $(\gamma_0, \dots, \gamma_d)$ in general position.

We let $c_0 \in \mathcal{P}_{H_{\gamma_0,\gamma_d}}$, $c_i \in \mathcal{P}_{H_{\gamma_i,\gamma_{i+1}}}$ for $1 \leqslant i \leqslant d$. We choose $s \in \mathcal{S}^{d+1}$ and label it by $(\gamma_0, \dots, \gamma_d)$ and call r its pre-image by $\pi : \mathcal{S}^{d+1} \to \mathcal{R}^{d+1}$. We fix a homotopy class A of polygons with corners at $c_0, \dots, c_d$ such that $\mathrm{Ind}(A) = 0$.

We recall the following definition.

Definition 2.3.8. *We keep the above notations. Let $s \in \pi^{-1}(r)$ be a $d+1$ pointed disk with ordered marked points $x_0, \dots, x_d$. Let $f : s \to S_g$ be an orientation preserving immersion which is also a polygon with corners at $c_0, \dots, c_d$.*

For $i \in \{0, \dots, d\}$, fix two smooth embedded arcs $\gamma_\pm : [0, \varepsilon) \to \partial s$ such that

- $\gamma_+(0) = \gamma_-(0) = x_i$,
- γ_+ *is orientation preserving,*
- γ_- *is orientation reversing.*

We say that x_i is a convex *corner if*

(1) for any open neighborhood U of x_i in s, the set $f(U)$ is not *an open neighborhood of $f(x_i)$,*

(2) the oriented angle from $\gamma'_-(0)$ to $\gamma'_+(0)$ satisfies

$$0 < \big(\gamma'_-(0), \gamma'_+(0)\big) < \pi.$$

Remark 2.3.9. (1) Geometrically, this implies that the image of a small neighborhood of x_i by u is convex. See Figure 10 for a picture and an unauthorized configuration where the corner is not convex. The terminology comes from [**Abo08**].

(2) Consider the symplectic manifold $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ with its standard area form. We let L be the generic immersion whose image is the union of $\mathbb{R}$ and $i\mathbb{R}$. The

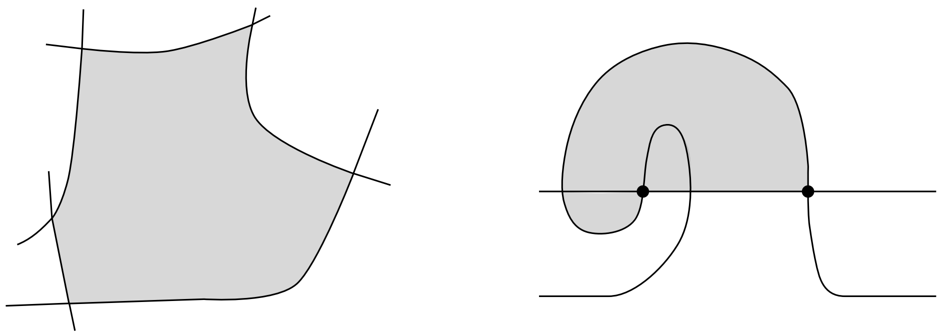

FIGURE 10 – *Left, an immersed polygon with convex corners, Right, an immersed polygon with two corners indicated by $\bullet$, one of which is non-convex.*

following map, whose domain is the Poincaré half plane,

$$\begin{array}{ccc} \mathbb{H}\cup\{\infty\} & \to & \mathbb{C}P^1 \\ z & \mapsto & z^{\frac{5}{2}} \end{array}.$$

is an immersed polygon with two corners at $0 \in \mathbb{C}P^1$ and $\infty \in \mathbb{C}P^1$. These corners satisfy condition (2) of Definition 2.3.8 but not condtion (1).

We let $\tilde{\Delta}(c_0, \dots, c_d, A)$ be the set of orientation preserving immersions (up to the boundary) $f : s \to S_g$ with $s \in \pi^{-1}(r)$ such that

- the map f is a polygon with corners at $c_0, \dots, c_d$,
- each corner of f is convex (see Figure 10).

We let $\Delta(c_0, \dots, c_d, A)$ be the quotient of $\tilde{\Delta}(c_0, \dots, c_d, A)$ by the group of diffeomorphisms of r which preserve the marked points.

Proposition 2.3.10. *Each holomorphic polygon $u \in \overline{\mathcal{M}}(c_0, \dots, c_d, A)$ is (up to reparameterization) an element of $\tilde{\Delta}(c_0, \dots, c_d)$.*

Moreover, the inclusion

$$\mathcal{M}(c_0, \dots, c_d, A) \hookrightarrow \Delta(c_0, \dots, c_d, A)$$

is bijective.

PROOF. This result is well-known (see [**Sei08**, (13b)], [**ENS02**], [**dSRS14**]). Let us quickly recall the idea of the proof.

Let $u \in \overline{\mathcal{M}}(c_0, \dots, c_d, A)$. If u has an interior branch point, there is a two-dimensional continuous family in $\overline{\mathcal{M}}(c_0, \dots, c_d, A)$ (see the proof of [**ENS02**, Proposition 7.8]). But $\dim(\overline{\mathcal{M}}(c_0, \dots, c_d, A)) \leqslant 1$, a contradiction. Similarly, if there is a branch point on the boundary, there is a contribution of 1 to the dimension. So u is an immersion up to the boundary. Moreover, it is easy to see that u cannot have non convex corners at the boundary.

Now if $u \in \tilde{\Delta}(c_0, \ldots, c_d, A)$, an easy application of the uniformization theorem shows that u can be reparameterized to a holomorphic curve. Hence, the inclusion $\mathcal{M}(c_0, \ldots, c_d, A) \hookrightarrow \Delta(c_0, \ldots, c_d, A)$ is surjective.

If u_1 and u_2 are such that $u_1 \circ \phi = u_2$ with ϕ a diffeomorphism $r \to r$, ϕ is holomorphic since u_1 and u_2 are immersions. Hence, the inclusion $\mathcal{M}(c_0, \ldots, c_d, A) \hookrightarrow \Delta(c_0, \ldots, c_d, A)$ is injective. □

In particular, one can define a A_∞ pre-category $\mathrm{Fuk}_{\mathrm{comb}}(S_g)$ whose objects are unobstructed immersed curves, whose morphisms spaces are given by the Floer complexes and whose higher operations are given by a count of elements of $\Delta(c_0, \ldots, c_d, A)$. This is done, using combinatorial arguments, in Abouzaid's paper ([Abo08]). We conclude that there is a pre-A_∞ quasi-isomorphism

$$\mathrm{Fuk}_{\mathrm{comb}}(S_g) \hookrightarrow \mathrm{Fuk}(S_g).$$

2.4. Immersed Lagrangian cobordisms and iterated cones

In this section, we study immersed Lagrangian cobordisms which are well-behaved for Floer theory. We have already seen that the main obstruction to this is the existence of teardrops, that is polygons with one corner points. Our objects of interest are Lagrangian cobordisms which do not bound topological teardrops.

In what follows, we will consider only compatible almost complex structures on $\mathbb{C} \times S_g$ such that the projection on the first factor $\pi_{\mathbb{C}} : \mathbb{C} \times S_g$ is holomorphic. If this holds, we say that the almost complex structure is *adapted*.

Definition 2.4.1. *Let* $(\gamma_1, \ldots, \gamma_n)$ *and* $(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m)$ *be* embedded *curves in* S_g. *An* unobstructed Lagrangian cobordism *from* $(\gamma_1, \ldots, \gamma_N)$ *to* $(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m)$ *is an oriented immersed Lagrangian cobordism*

$$V : (\gamma_1, \ldots, \gamma_N) \rightsquigarrow (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m)$$

which satisfies the following conditions.

(i) The immersion V has no triple points, and all its double points are transverse.

(ii) There are no topological teardrops with boundary on V.

Biran and Cornea proved that, in the monotone setting, an embedded, monotone Lagrangian cobordism induces a cone relation between its end in the Fukaya category ([BC14]). This result still holds for unobstructed Lagrangian cobordisms.

Theorem 2.4.2. *Let*

$$V : (\gamma_1, \ldots, \gamma_n) \rightsquigarrow \gamma$$

be an unobstructed Lagrangian cobordism. Then, there is an isomorphism in $\mathrm{DFuk}(S_g)$

$$\gamma \simeq \mathrm{Cone}(\gamma_1[1] \to \mathrm{Cone}(\gamma_2[1] \ldots \to \mathrm{Cone}(\gamma_{n-1}[1] \to \gamma_n))).$$

In our setting, the proof is the same as [**Hau15**, Theorem 4.1].

There are two natural corollaries to this result, which we now give.

Corollary 2.4.3. *Let γ be an unobstructed curve. In* $\mathrm{DFuk}(S_g)$ *we have the isomorphism*

$$\gamma[1] \simeq \gamma^{-1}.$$

PROOF. Consider a properly embedded path $\begin{array}{cccc} \alpha : & \mathbb{R} & \to & \mathbb{C} \\ & t & \mapsto & (x(t), y(t)) \end{array}$ such that

- for $s < 0$, we have $\alpha(s) = (s, 0)$,
- for $s > 1$, we have $\alpha(s) = (1 - s, 1)$,
- for $s \in [0, 1]$, the derivative y' satisfies $y'(s) > 0$.

The immersed manifold

$$\begin{array}{ccc} \mathbb{R} \times S^1 & \to & \mathbb{C} \times S_g \\ (s, t) & \mapsto & (\alpha(s), \gamma(t)) \end{array}.$$

is a Lagrangian cobordism $(\gamma, \gamma^{-1}) \rightsquigarrow \emptyset$. Therefore, Theorem 2.4.2 gives the desired isomorphism. $\square$

Corollary 2.4.4. *There is a natural group morphism*

$$\Theta_{BC} : \Omega_{cob}^{imm,unob}(S_g) \to K_0(\mathrm{DFuk}(S_g)),$$

which maps the class of an embedded curve γ to its representative in $K_0(\mathrm{DFuk}(S_g))$. *Further, this morphism is surjective.*

PROOF. The existence of the morphism $\Theta_{BC} : \Omega_{\mathrm{cob}}^{\mathrm{imm,unob}}(S_g) \to K_0(\mathrm{DFuk}(S_g))$ is immediate from Theorem 2.4.2.

Moreover, recall from [**Abo08**] that the group $K_0(\mathrm{DFuk}(S_g))$ is generated by embedded curves. Hence, the image of Θ_{BC} is the whole group $K_0(\mathrm{DFuk}(S_g))$. $\square$

2.5. Computation of the unobstructed Lagrangian Cobordism Group

In this section, we compute the unobstructed Lagrangian cobordism group $\Omega_{\mathrm{cob}}^{\mathrm{imm,unob}}(S_g)$.

First, notice that by the results of subsection 2.2.2, the homology class and the Maslov class yield maps

$$\pi : \Omega_{\mathrm{cob}}^{\mathrm{imm,unob}}(S_g) \to H_1(S_g, \mathbb{Z}), \;\; \mu : \Omega_{\mathrm{cob}}^{\mathrm{imm,unob}}(S_g) \to \mathbb{Z}/\chi(S_g)\mathbb{Z}.$$

Our main tool is the following

Theorem 2.5.1. *There is a long exact sequence*

$$0 \to \mathbb{R} \xrightarrow{i} \Omega_{cob}^{imm,unob}(S_g) \xrightarrow{\pi \oplus \mu} H_1(S_g, \mathbb{Z}) \oplus \mathbb{Z}/\chi(S_g)\mathbb{Z} \to 0.$$

Furthermore, this exact sequence is split.

2.5.1. Holonomy and the map i

In this subsection, we define an injection

$$i : \mathbb{R} \to \Omega_{\text{cob}}^{\text{imm,unob}}(S_g).$$

This map has a simple geometric interpretation: to a real number x, it associates the oriented boundary of a cylinder of area x. We have to check that the map is indeed well-defined.

Let $p : S(TS_g) \to S_g$ be the unit tangent bundle with respect to the metric $g_j = \omega(\cdot, j\cdot)$. We choose a one-form $A \in \Omega^1(S(TS_g))$ such that

$$p^*\omega = dA.$$

An immersed curve $\gamma : S^1 \to S_g$ admits a canonical lift to $S(TS_g)$:

$$\begin{array}{cccc} \tilde{\gamma} : & S^1 & \to & S(TS_g) \\ & t & \mapsto & \left(\gamma(t), \frac{\gamma'(t)}{|\gamma'(t)|}\right) \end{array}$$

We define the *holonomy* of γ with respect to A by

$$\mathrm{Hol}_A(\gamma) := \int_{S^1} \tilde{\gamma}^* A.$$

We shall use the following properties of this number.

Proposition 2.5.2. *The following assertions are true.*

(i) Let $F : [0,1] \times S^1 \to S_g$ be an isotopy between the two immersed curves γ_0 and γ_1. Then

$$\mathrm{Hol}_A(\gamma_1) - \mathrm{Hol}_A(\gamma_0) = \int_{[0,1]\times S^1} F^*\omega.$$

(ii) There is a well-defined group morphism

$$\mathrm{Hol}_A : \Omega_{cob}^{imm,unob}(S_g) \to \mathbb{R}$$

whose value on the class of an embedded curve γ is $\mathrm{Hol}_A(\gamma)$*.*

Proof. There is a natural lift of F to $S(TS_g)$:

$$\begin{array}{cccc} \tilde{F} : & [0,1] \times S^1 & \to & S(TS_g) \\ & (t,s) & \mapsto & \left(F(t,s), \frac{\partial_t F(t,s)}{|\partial_t F(t,s)|}\right) \end{array}.$$

We now apply Stokes Theorem:

$$\begin{aligned}\mathrm{Hol}_A(\gamma_1) - \mathrm{Hol}_A(\gamma_0) &= \int_{S^1\times\{1\}} \tilde{F}^*A - \int_{S^1\times\{0\}} \tilde{F}^*A \\ &= \int_{S^1\times[0,1]} \tilde{F}^*dA \\ &= \int_{S^1\times[0,1]} \tilde{F}^*p^*\omega \\ &= \int_{S^1\times[0,1]} F^*\omega.\end{aligned}$$

Recall from Corollary 2.4.4 that there is a natural group morphism

$$\Theta_{BC} : \Omega^{\text{imm,unob}}_{\text{cob}}(S_g) \to K_0(\mathrm{DFuk}(S_g)).$$

On the other hand, it is a result of Abouzaid ([**Abo08**, Proposition 6.1]) that the holonomy induces a group morphism

$$K_0(\mathrm{DFuk}(S_g)) \to \mathbb{R}.$$

Therefore, the composition of these two is a group morphism. This proves (ii). □

The following lemma seems to be a well-known fact ([**Sei11**, Section 6]). I learned its proof from Jordan Payette.

Lemma 2.5.3. *Let γ_0 and γ_1 be two isotopic embedded curves with*

$$\mathrm{Hol}_A(\gamma_1) = \mathrm{Hol}_A(\gamma_0).$$

Then the curves γ_0 and γ_1 are Hamiltonian isotopic to each other.

Proof. We fix an isotopy $(\gamma_t)_{t\in[0,1]}$ from γ_0 to γ_1.

Let $\phi^t : S_g \to S_g$ be a global isotopy of diffeomorphisms such that $\phi^t \circ \gamma_0 = \gamma_t$. Choose a symplectic embedding $\psi : S^1 \times (-\varepsilon,\varepsilon) \to S_g$ such that $\psi(s,0) = \gamma_0(s)$. In the coordinates $(s,u) \in S^1 \times (-\varepsilon,\varepsilon)$ there is a smooth function $f > 0$ such that $(\phi^t)^*\omega = f(s,u,t) ds \wedge du$. We let $\beta : S^1 \times (-\varepsilon,\varepsilon) \times [0,1] \to \mathbb{R}$ be a smooth function such that

- we have $\beta(s,u,t) = \frac{1}{f(s,u,t)}$ for $|u| < \frac{\varepsilon}{3}$,
- we have $\beta(s,u,t) = 1$ for $|u| > \frac{2\varepsilon}{3}$.

Define, for $t \in [0,1]$

$$\begin{array}{rccc}\psi^t : & S^1 \times (-\varepsilon,\varepsilon) & \to & S^1 \times (-\varepsilon,\varepsilon) \\ & (s,u) & \mapsto & (s, u\beta(u,s,t))\end{array}$$

Since this map coincides with the identity on the open set $\left\{(s,u) \middle| |u| > \frac{2\varepsilon}{3}\right\}$, it extends to a diffeomorphism $\psi^t : S_g \to S_g$.

Consider $\chi^t = \phi^t \circ \psi^t$ and let $\omega_t = (\chi^t)^*\omega$. This isotopy satisfies $\chi^t \circ \gamma_0 = \gamma_t$. Moreover, the family $(\omega_t)_{t\in[0,1]}$ is constant along γ_0. One can easily apply Moser's trick to find a family of transformation Φ^t such that $(\Phi^t)^*\omega_t = \omega$ and $\Phi^t(\gamma_0) = \gamma_0$.

The composition $\chi^t \circ \Phi^t$ is a symplectic isotopy $\Psi^t : S_g \to S_g$ such that

$$\Psi^t \circ \gamma_0 = \gamma_t.$$

Moreover, this has zero flux along the path γ_0. We call X_t the symplectic vector field generated by Ψ^t.

We adapt the construction of [MS17, Theorem 10.2.5] to obtain a Hamiltonian isotopy between γ_0 and γ_1.

The difference of holonomy between γ_1 and γ_0 is

$$\begin{aligned}
\mathrm{Hol}_A(\gamma_1) - \mathrm{Hol}_A(\gamma_0) &= \int_0^1 \int_0^1 \omega\left(\frac{d}{dt}(\Psi^t \circ \gamma_0)(s), \frac{d}{ds}(\Psi^t \circ \gamma_0)(s)\right) dtds \\
&= \int_0^1 \int_0^1 \omega\big(X_t \circ \Psi^t \circ \gamma_0(s), d\Psi^t(\gamma_0'(s))\big) dtds \\
&= \int_0^1 \int_0^1 \omega\big(\big(\Psi^t\big)^* X_t(\gamma_0(s)), \gamma_0'(s)\big) dtds \\
&= \int_{S^1} \gamma_0^*\left[\int_0^1 \omega\big(\big(\Psi^t\big)^* X_t, \cdot\big) dt\right]
\end{aligned}$$

So the one-form

$$\gamma_0^*\left[\int_0^1 \omega\big(\big(\Psi^t\big)^* X_t, \cdot\big) dt\right]$$

is exact on S^1. Hence, there is a smooth function

$$F : \mathrm{Im}(\gamma_1) \to \mathbb{R}$$

such that

$$\forall v \in T(\mathrm{Im}(\gamma_0)), \ \int_0^1 \omega\big(\big(\Psi^t\big)^* X_t, v\big) dt = -dF \cdot d\Psi^1(v).$$

We extend F to a smooth function

$$F : S_g \to \mathbb{R}.$$

We define a new isotopy $(\Phi^t)_{t\in[0,1]}$ by

$$\Phi^t = \begin{cases} \psi^{2t} & \text{for } t \in \left[0, \frac{1}{2}\right] \\ \phi_F^{1-2t} \circ \psi^1 & \text{for } t \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Call Y_t its associated vector field. We compute for $v \in \mathrm{Im}(\gamma_0)$,

$$\begin{aligned}\int_0^1 \omega\big((\Phi^t)^* Y_t, v\big) &= \int_0^{\frac{1}{2}} \omega\big((\Psi^{2t})^* X_{2t}, v\big) 2dt - \int_{\frac{1}{2}}^1 \omega\big((\phi_F^{1-2t} \circ \Psi^1)^* X_F, v\big) 2dt \\ &= \int_0^1 \omega\big((\Psi^t)^* X_t, v\big) dt - \int_{\frac{1}{2}}^1 \omega(X_{F\circ\Psi^1}, v) 2dt \\ &= -d\big(F \circ \Psi^1\big)(v) + d\big(F \circ \Psi^1\big)(v).\end{aligned}$$

So

$$\int_0^1 \omega\big((\Phi^t)^* Y_t, v dt\big) = 0 \tag{2.12}$$

We let

$$Z_t = -\int_0^t \big(\Phi^\lambda\big)^* Y_\lambda d\lambda$$

and θ_t^s be the flow associated to Z_t. Then the isotopy $\mu^t = \Phi^t \circ \theta_t^1$ is Hamiltonian (see the proof of [**MS17**, Theorem 10.2.5]).

Moreover, from 2.12, we have for all $v \in T(\mathrm{Im}(\gamma_0))$

$$\omega(Z_1, v) = 0.$$

Hence, Z_1 is tangent to $\mathrm{Im}(\gamma_0)$. Therefore, $\theta_1^1(\gamma_0) \subset \mathrm{Im}(\gamma_0)$. We deduce

$$\begin{aligned}\mu^1(\mathrm{Im}(\gamma_0)) &= \Phi^1(\theta_1^1(\mathrm{Im}(\gamma_0))) \\ &= \phi_F^{-1}(\mathrm{Im}(\gamma_1)).\end{aligned}$$

So the Hamiltonian isotopy $(\phi_F^t \circ \mu^t)_{t\in[0,1]}$ maps the image of the curve γ_0 to the image of γ_1. □

Here is the main result of this section.

Proposition 2.5.4. *Let γ_1, γ_2 (resp. $\tilde{\gamma}_1, \tilde{\gamma}_2$) be two isotopic non-separating embedded curves such that*

$$\mathrm{Hol}_A(\gamma_2) - \mathrm{Hol}_A(\gamma_1) = \mathrm{Hol}_A(\tilde{\gamma}_2) - \mathrm{Hol}_A(\tilde{\gamma}_1).$$

Then, in $\Omega_{cob}^{imm,unob}(S_g)$, we have

$$[\gamma_2] - [\gamma_1] = [\tilde{\gamma}_1] - [\tilde{\gamma}_2].$$

Proof. **First case.** We assume that the curves γ_1, γ_2, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ are pairwise disjoint. Furthermore, we assume that the pair (γ_1, γ_2) does not bound any oriented surface. Then, by the change of coordinates principle ([**FM12**, 1.3]), we are in the situation of Figure 11.

We do the successive surgeries indicated in Figure 11. This process produces a curve γ_7 isotopic to γ_2. Here are the steps in detail.

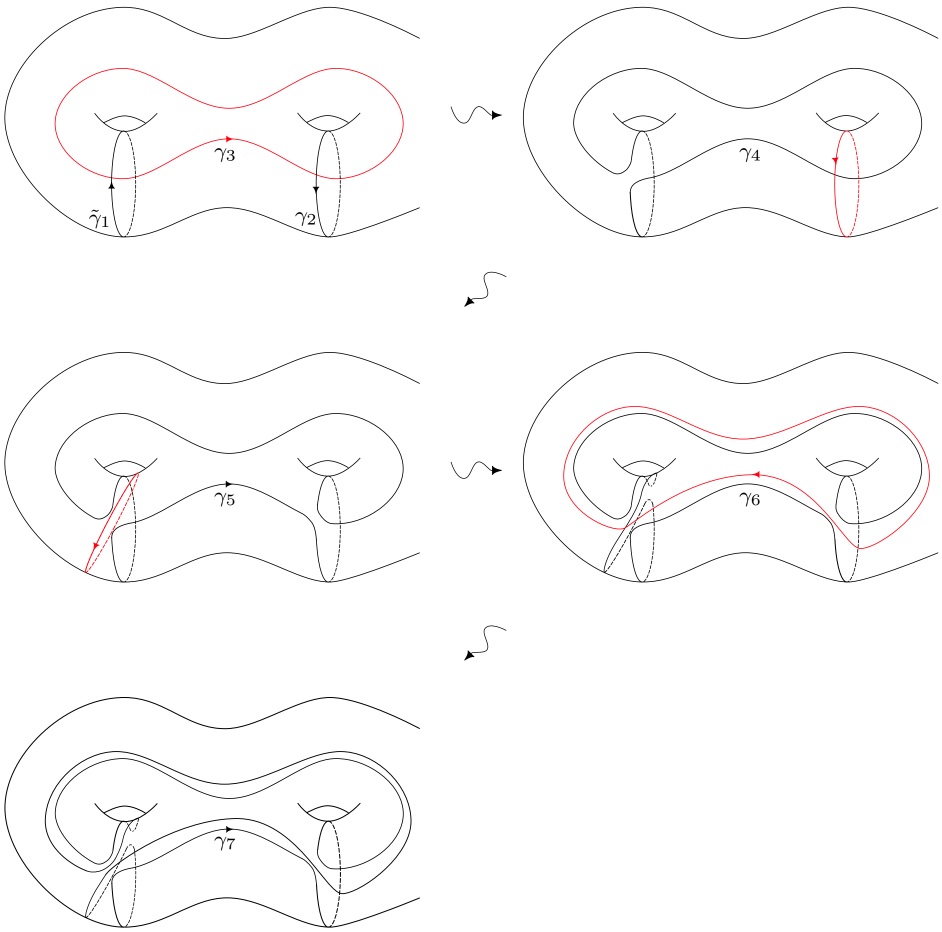

FIGURE 11 – *The successive surgeries which link two non separating curves*

(1) The first step is the surgery of $\tilde{\gamma}_1$ with the curve γ_3 represented at the left of the first row of Figure 11. This yields a curve γ_4 whose holonomy is

$$\mathrm{Hol}_A(\gamma_4) = \mathrm{Hol}_A(\tilde{\gamma}_1) + \mathrm{Hol}_A(\gamma_3).$$

(2) The second step is the surgery of γ_2 with the curve γ_4 at their unique intersection point. This yields a curve γ_5 represented in the left of the second row of Figure 11. Its holonomy is

$$\mathrm{Hol}_A(\gamma_5) = \mathrm{Hol}_A(\gamma_4) + \mathrm{Hol}_A(\gamma_2).$$

(3) We pick a curve Hamiltonian isotopic to γ_1 which intersects γ_5 in a *unique point*. It is represented in red in the left of the second row of Figure 11 We perform the surgery of this curve with γ_5. This yields a curve γ_6 represented in the right of the second row. Its holonomy is

$$\mathrm{Hol}_A(\gamma_6) = \mathrm{Hol}_A(\gamma_5) - \mathrm{Hol}_A(\gamma_1).$$

(4) We pick a curve Hamiltonian isotopic to γ_3^{-1} which intersects γ_6 in a *unique point*. It is represented in red in the right of the second row of Figure 11 We perform the surgery of this curve with γ_6. This yields the curve γ_7 represented in the third row. Its holonomy is

$$\mathrm{Hol}_A(\gamma_7) = \mathrm{Hol}_A(\gamma_6) - \mathrm{Hol}_A(\gamma_3).$$

Therefore, the holonomy of γ_7 is given by

$$\begin{aligned}
\mathrm{Hol}_A(\gamma_7) &= \mathrm{Hol}_A(\gamma_6) - \mathrm{Hol}_A(\gamma_3) \\
&= \mathrm{Hol}_A(\gamma_5) - \mathrm{Hol}_A(\gamma_1) - \mathrm{Hol}_A(\gamma_3) \\
&= \mathrm{Hol}_A(\gamma_4) + \mathrm{Hol}_A(\gamma_2) - \mathrm{Hol}_A(\gamma_1) - \mathrm{Hol}_A(\gamma_3) \\
&= \mathrm{Hol}_A(\tilde{\gamma}_1) + \mathrm{Hol}_A(\gamma_3) + \mathrm{Hol}_A(\gamma_2) - \mathrm{Hol}_A(\gamma_1) - \mathrm{Hol}_A(\gamma_3) \\
&= \mathrm{Hol}_A(\tilde{\gamma}_1) - \mathrm{Hol}_A(\gamma_1) + \mathrm{Hol}_A(\gamma_2) \\
&= \mathrm{Hol}_A(\tilde{\gamma}_2)
\end{aligned}$$

since $\mathrm{Hol}_A(\gamma_2) - \mathrm{Hol}_A(\gamma_1) = \mathrm{Hol}_A(\tilde{\gamma}_2) - \mathrm{Hol}_A(\tilde{\gamma}_1)$. Hence, the curve γ_7 is Hamiltonian isotopic to $\tilde{\gamma}_2$ by Proposition 2.5.2. Since two Hamiltonian isotopic curves are embbedded Lagrangian cobordant (see Remark 2.2.4), we have

$$\tilde{\gamma}_2 = \gamma_7.$$

in $\Omega_{\mathrm{cob}}^{\mathrm{imm,unob}}(S_g)$.

The steps described above produce several Lagrangian cobordisms that are all embedded and oriented. We can glue these together to obtain an embedded oriented Lagrangian cobordism $(\gamma_3^{-1}, \tilde{\gamma}_1, \gamma_3, \gamma_2, \gamma_1^{-1}) \to \gamma_7$. Hence, in $\Omega_{\mathrm{cob}}^{\mathrm{imm,unob}}(S_g)$

$$\begin{aligned}
\tilde{\gamma}_2 &= \gamma_7 \\
&= -\gamma_3 + \tilde{\gamma}_1 + \gamma_3 + \gamma_2 - \gamma_1 \\
&= \tilde{\gamma}_1.
\end{aligned}$$

This finishes the proof of the first case.

General case. [FM12, Theorem 4.3] there exists a sequence of non-separating embedded curves $\gamma_1 = \alpha_1, \ldots, \alpha_k = \gamma_2$ such that for each $i \in \{1, \ldots, k\}$, α_i and α_{i+1} have no intersection points. Now, apply the first case iteratively to conclude. ☐

There is a direct definition for the application i. Let γ be a non-separating embedded curve. There is $\varepsilon > 0$ such that if $|x| < \varepsilon$, there exists an embedded curve $\tilde{\gamma}$ isotopic to γ such that

$$\mathrm{Hol}_A(\tilde{\gamma}) - \mathrm{Hol}_A(\gamma) = x.$$

Now, let $x \in \mathbb{R}$ and $x_1, \ldots, x_m \in \mathbb{R}$ with $\begin{cases} x_1 + \ldots x_m = x \\ \forall i \in \{1 \ldots m\},\ |x_i| < \varepsilon \end{cases}$.

For $i = 1 \ldots m$, we choose an embedded curve γ_i isotopic to γ such that

$$\mathrm{Hol}_A(\gamma_i) - \mathrm{Hol}_A(\gamma) = x_i.$$

We put

$$i(x) = \sum_{i=1}^{m} ([\gamma_i] - [\gamma]).$$

Corollary 2.5.5. *In the setting above, this defines an injective group morphism*

$$i : \mathbb{R} \to \Omega_{cob}^{imm,unob}(S_g).$$

Proof. It is easy to see, using Proposition 2.5.4, that $i(x)$ does not depend on the choice of $x_1, \ldots, x_m$, nor on the choice of γ. Therefore, it defines a group morphism $i : \mathbb{R} \to \Omega_{\mathrm{cob}}^{\mathrm{imm,unob}}(S_g)$.

Moreover, notice that by definition

$$\mathrm{Hol}_A(i(x)) = x,$$

so i is injective. □

2.5.2. Surgery of immersed curves and obstruction

Throughout this subsection we let $\alpha : S^1 \to S_g$ and $\gamma : S^1 \to S_g$ be immersed curves such that

(i) α is embedded,
(ii) γ is unobstructed,
(iii) α and γ are transverse.

A *bigon* is an immersed polygon with one boundary arc on α and one boundary arc on γ. Notice that a bigon is *not necessarily injective*. We say that α and γ are in *minimal position* if there are no bigons between α and γ. There is a useful criterion to detect minimal position for transverse curves.

Lemma 2.5.6. *In the above setting, if there are $s_0 \neq \overline{s_0}$ and $t_0 \neq \overline{t_0}$ such that the loop $\gamma_{|[t_0,\overline{t_0}]} \cdot \alpha^{-1}_{|[s_0,\overline{s_0}]}$ is homotopic to a constant, then γ and α are not in minimal position.*

Proof. The hypothesis implies that the loop $\gamma_{|[t_0,\overline{t_0}]} \cdot \alpha^{-1}_{|[s_0,\overline{s_0}]}$ lifts to the universal cover $\tilde{S}_g$ of S_g. Denote this loop by $f : S^1 \to \tilde{S}_g$.

So there are two lifts $\tilde{\alpha} : \mathbb{R} \to \tilde{S}_g$ and $\tilde{\gamma} : \mathbb{R} \to \tilde{S}_g$ of α and γ respectively such that

$$\tilde{\gamma}_{|[t_0,\overline{t_0}]} \cdot \tilde{\alpha}^{-1}_{|[s_0,\overline{s_0}]} = f.$$

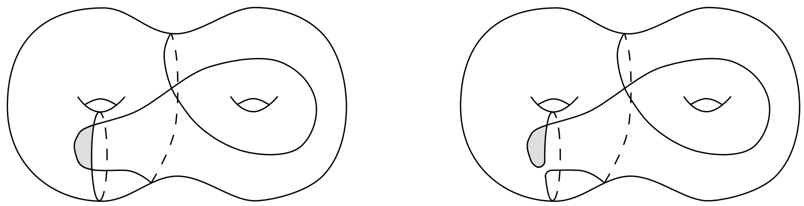

FIGURE 12 – *Rounding off a corner yields a teardrop.*

Assume there are

- two increasing sequences $(s_n)_{n\in\mathbb{N}}$, $(t_n)_{n\in\mathbb{N}}$,
- two decreasing sequences $(\overline{s}_n)_{n\in\mathbb{N}}$, $(\overline{t}_n)_{n\in\mathbb{N}}$,

such that

$$\forall n \in \mathbb{N},\ s_0 < s_n < \overline{s}_n < \overline{s}_0,$$

$$\forall n \in \mathbb{N},\ t_0 < t_n < \overline{t}_n < \overline{t}_0,$$

and such that the path $\tilde{\gamma}_{|[t_n,\overline{t_n}]} \cdot \tilde{\alpha}^{-1}_{|[s_n,\overline{s_n}]}$ is a loop.

If $\overline{s}_n - s_n \to 0$, then the adjacent sequences (s_n) and $(\overline{s}_n)$ converge to a common limit, say l. So there is a sequence of intersection points between $\tilde{\alpha}$ and $\tilde{\gamma}$ which accumulates at $\tilde{\alpha}(l)$. This is absurd since α and γ are transverse. We conclude that the sequence $(\overline{s}_n - s_n)$ has a positive limit. The same argument shows that $(\overline{t}_n - t_n)$ has a positive limit.

From this we infer that there are $s < \overline{s}$ and $t < \overline{t}$ such that the path

$$\tilde{\gamma}_{|[t,\overline{t}]} \cdot \tilde{\alpha}^{-1}_{|[s,\overline{s}]}$$

is an embedded loop. Hence, this bounds an embedded bigon, say u. Now since the projection $p : \tilde{S}_g \to S_g$ is an immersion, the map $p \circ u$ is an immersed bigon with boundary arcs on γ and α. □

Let x be an intersection point between α and γ. In what follows, we will need to find situations where the surgery $\gamma\#_x\alpha$ is unobstructed. Unfortunately, if the curves α and γ are not in minimal position, the surgery can be obstructed. I will now sketch what can go wrong in this case.

Assume that there is a bigon between the curves α and γ with

- convex corners[3],
- and one corner at x.

If the surgery at the point x is done the wrong way, the corner at x can be rounded off to produce a teardrop with boundary on $\gamma\#_x\alpha$. See Figure12.

[3]See Definition 2.3.8

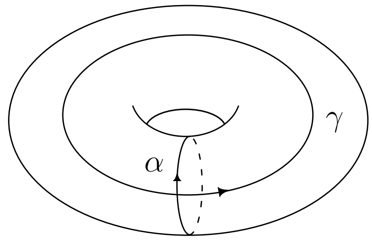

FIGURE 13 – *The curves α and γ bound a square with four corners.*

At first, it seems that the reverse procedure shows that if α and γ are in minimal position, then the surgery is unobstructed. If there is a teardrop u with boundary on $\gamma\#_x\alpha$, we should be able to modify it to produce a bigon between α and γ. Unfortunately, this is not true. Indeed, assume that there is a teardrop u with boundary on the surgery $\gamma\#_x\alpha$. Following the boundary of u, we see that it must switch several times between α and γ. Since this number may be greater or equal than two, the procedure will in general produce a polygon with boundary on α and γ and more than two corners. These can certainly exist even when α and γ are in minimal position, see Figure 13.

However, we can bypass this difficulty. It turns out that, by a topological argument, the number of boundary switches of u is at most two. So the resulting polygon has at most three corners. With this bound in hand, it is possible to cut the polygon to obtain an actual bigon on α and γ. This is the content of the following proposition.

Proposition 2.5.7. *We let α and γ be as in the beginning of Subsection 2.5.2. Let x be an intersection point of degree* 1 *between γ and α.*

Further, we assume

(i) the curves γ and α are in minimal position,

(ii) the curve $\gamma\#_x\alpha$ is not homotopic to a constant.

Then, the surgery $\gamma\#_x\alpha$ is unobstructed.

Remark 2.5.8. As mentioned earlier, throughout the proof of Proposition 2.5.7, we cut a polygon u with boundary on the immersion $\alpha \cup \gamma$ along the set $u^{-1}(\alpha)$.

This is a very natural technique to find bigons with certain properties (in particular it is used in [**FM12**, Proposition 1.7]). In symplectic topology, a variant of this was introduced by Lazzarini ([**Laz00**],[**Laz11**]). If $u : \mathbb{D} \to M$ is a J-holomorphic disk with boundary on an embedded Lagrangian $L \subset M$, he introduced a graph $\mathcal{W}(u) \subset u^{-1}(L)$ called the *frame* of u. Cutting along $\mathcal{W}(u)$ produces several multiply covered or simple pieces of the original disk u. See also [**KO00**] for a detailed analysis of the set $u^{-1}(L)$ and [**Per18**] for the case of pseudo-holomorphic polygons with boundary on a Lagrangian immersion.

Here, our goal is slightly different (even though the set $u^{-1}(\alpha)$ is a subgraph of Lazzarini's frame). We will cut the polygon u in order to find pseudo-holomorphic curves with the *least* number of corners.

Proof of Proposition 2.5.7. The proof will proceed by contradiction. We assume that the surgery is obstructed.

Let us start with a quick outline of the proof of the Proposition.

(1) In Lemma 2.5.9, we show that there is an *immersed* teardrop u on $\gamma\#_x\alpha$.
(2) In Lemma 2.5.10 and Figure 16, we describe precisely the behavior of the teardrop around the surgered point. From this we construct an immersed polygon v with boundary on α and γ in Lemma 2.5.12.
(3) Using algebraic properties of the fundamental group of S_g, we bound the number of corners of the polygon v. Then by considering the connected components of $\mathbb{D}\backslash v^{-1}(\alpha)$, we conclude that there is a bigon between γ and α. This is a contradiction since γ and α are in minimal position.

We start with the following lemma.

Lemma 2.5.9. *Let $\gamma : S^1 \to S_g$ be a generic curve which is obstructed and non-homotopic to zero. Then there exists an immersed holomorphic teardrop with boundary on γ and corner at a double point of γ.*

Moreover, the corner of this teardrop covers one or three quadrants.

Proof. Let $\tilde{\gamma} : \mathbb{R} \to \tilde{S}_g$ be a lift of γ to the universal cover $p : \tilde{S}_g \to S_g$ of S_g. Since γ is obstructed, there are two reals $s_0 < t_0$ such that $\tilde{\gamma}(s_0) = \tilde{\gamma}(t_0)$.

We claim that there are $s < t$ such that $\tilde{\gamma}(s) = \tilde{\gamma}(t)$ and such that $\tilde{\gamma}_{|(s,t)}$ is injective. Assume by contradiction that there are sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ with

- $\forall n \in \mathbb{N},\ s_0 < s_n < t_n < t_0$,
- the sequence $(s_n)_{n\in\mathbb{N}}$ is increasing, the sequence $(t_n)_{n\in\mathbb{N}}$ is decreasing and the sequence $(t_n - s_n)_{n\in\mathbb{N}}$ converges to 0,
- $\forall n \in \mathbb{N},\ \tilde{\gamma}(s_n) = \tilde{\gamma}(t_n)$.

Call s_∞ the limit of the adjacent sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$. Since $\tilde{\gamma}$ is an immersion, it is in particular a local embedding around s_∞. But there is a sequence of distinct double points converging to S_∞.

Now, since S^1 is compact and $\tilde{\gamma}_{|(s,t)}$ is injective, $\tilde{\gamma}_{|(s,t)}$ is an embedded curve. Call Ω the bounded connected component of $\tilde{S}_g\backslash \operatorname{Im}\big(\tilde{\gamma}_{|(s,t)}\big)$. By the Riemann mapping Theorem, there is a biholomorphism $u : \mathbb{D} \to \Omega$ which extends to a homeomorphism $\tilde{u} : \overline{\mathbb{D}} \to \Omega$. Moreover, we assume that $\tilde{u}(-1) = \tilde{\gamma}(s) = \tilde{\gamma}(t)$. Since its image is of finite area, $\tilde{u}$ is of finite energy.

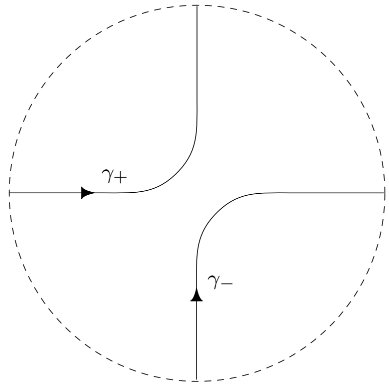

FIGURE 14 – *The curves γ_+ and γ_- near the intersection point x.*

The map $\tilde{u}$ is a holomorphic teardrop. Let us call $\tilde{y} = \tilde{\gamma}(s) = \tilde{\gamma}(t)$ its corner. Choose $\varepsilon \ll 1$ and denote by $\alpha \in (0, \pi)$ the angle between $\tilde{\gamma}_{|(s-\varepsilon,s+\varepsilon)}$ and $\tilde{\gamma}_{|(t-\varepsilon,t+\varepsilon)}$. Choose a local chart ϕ around y such that $\phi(\tilde{\gamma}_{|(s-\varepsilon,s+\varepsilon)}) = \mathbb{R}$ and $\phi(\tilde{\gamma}_{|(t-\varepsilon,t+\varepsilon)}) = e^{i\alpha}\mathbb{R}$. Then, by [Per18, Proposition 5] , there is a local chart $\psi : \Omega \to \mathbb{D}$ around -1 with domain a neighborhood of 0 in the Poincaré half-plane such that

$$\phi \circ \tilde{u} \circ \psi(z) = z^{\alpha+m-1}.$$

Here, m is the number of quadrants covered by $\tilde{u}$ at its corner. Hence, if $m \geqslant 4$, $\tilde{u}$ is not injective.

Now, since the projection p is an immersion, the map $p \circ \tilde{u}$ satisfies the conclusion of the lemma. □

We now return to the setting of Proposition 2.5.7. By Lemma 2.5.9 above, there is an immersed holomorphic teardrop u with boundary on $\gamma \#_x \alpha$. We denote its corner by y.

Recall from 2.2.1 that we denoted by U_x the Darboux neighborhood in which we perform the surgery. We call γ_+ (resp. γ_-) the upper (resp. lower) connected component of $\mathrm{Im}(\gamma \#_x \alpha) \cap U_x$. See Figure 14.

The set $u^{-1}_{|\partial\mathbb{D}}(U_x)$ is a finite union of arcs. We label them clockwise by $A_1, \dots, A_N$. Moreover, for $i = 1 \dots N$, we call C_i the connected component of $u^{-1}(U_x)$ which contains A_i. See Figure 15.

Lemma 2.5.10. *Let $i \in \{1, \dots, N\}$. With the notations above, the open set C_i is an embedded half-disk. Moreover, the map u restricts to a biholomorphism from C_i to one of the quadrants represented in Figure 16.*

PROOF. There are four possibilities for the image of $u_{|C_i}$.

(1) The image of the map $u_{|A_i}$ is a subset of $\mathrm{Im}(\gamma_+)$ and the orientations of the arcs $u_{|A_i}$ and $\mathrm{Im}(\gamma_+)$ coincide. This is represented as Type $A+$ in the upper left

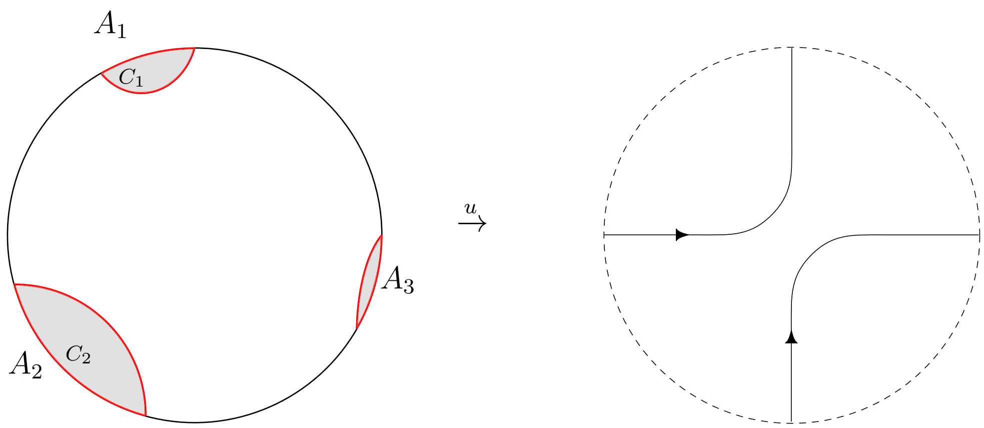

FIGURE 15 – *The arcs A_i map to the connected components of $U_x \cap \gamma \#_x \alpha$*

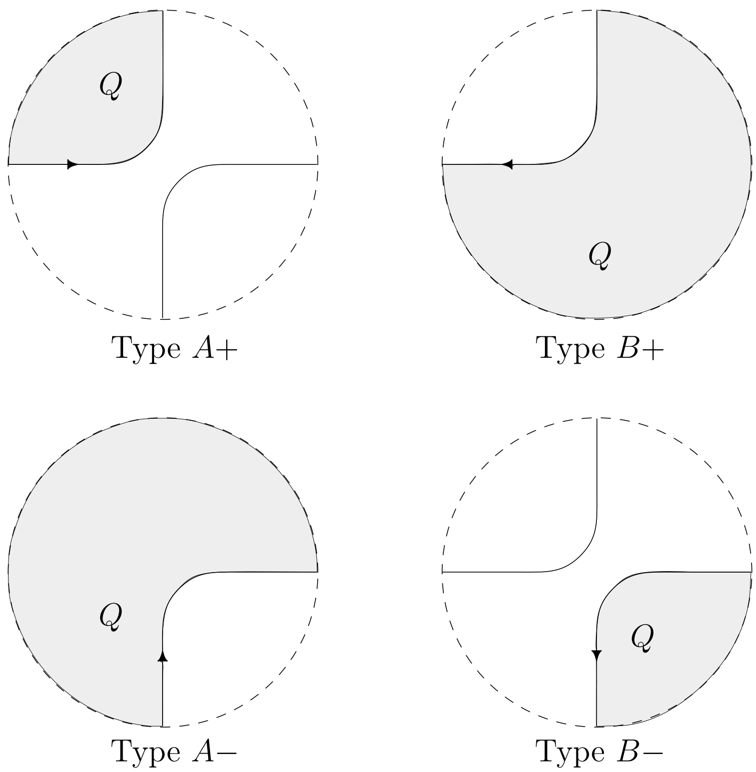

FIGURE 16 – *The four possibilities for the image of u around the surgered point.*
The arrows correspond to the orientation of the boundary of u.

of Figure 16. In this case, we call Q the closure (in S_g) of the shaded region represented in the upper left of Figure 16. Moreover, we put $\varepsilon = +$.

(2) The image of the map $u_{|A_i}$ is a subset of $\mathrm{Im}(\gamma_+)$ and the arcs $u_{|A_i}$ and $\mathrm{Im}(\gamma_+)$ have opposite orientation. This is represented as Type $B+$ in the upper right

of Figure 16. In this case, we call Q the closure (in S_g) of the shaded region represented in the upper right of Figure 16. Moreover, we put $\varepsilon = +$.

(3) The image of the map $u_{|A_i}$ is a subset of $\mathrm{Im}(\gamma_-)$ and the orientations of the arcs $u_{|A_i}$ and $\mathrm{Im}(\gamma_-)$ coincide. This is represented as Type $A-$ in the bottom left of Figure 16. In this case, we call Q the closure (in S_g) of the shaded region represented in the bottom left of Figure 16. Moreover, we put $\varepsilon = -$.

(4) The image of the map $u_{|A_i}$ is a subset of $\mathrm{Im}(\gamma_-)$ and the orientations of the arcs $u_{|A_i}$ and $\mathrm{Im}(\gamma_+)$ differ. This is represented as Type $B-$ in the bottom right of Figure 16. In this case, we call Q the closure (in S_g) of the shaded region represented in the bottom right of Figure 16. Moreover, we put $\varepsilon = -$.

We will also denote the closure of C_i in $\mathbb{D}$ by $\overline{C_i}$. We also put $\mathrm{Int}(C_i) = C_i \backslash \partial\mathbb{D}$.

Notice that, since $u_{|C_i}$ is immersed and holomorphic, there is a $z \in \mathrm{Int}(C_i)$ such that $u(z) \in \mathrm{Int}(Q)$. We will also denote the closure of C_i in $\mathbb{D}$ by $\overline{C_i}$.

First step: We claim that $\mathrm{Im}(u_{|\overline{C_i}}) \subset Q$. Assume, by contradiction, that there is a $z_1 \in \overline{C_i}$ such that $u(z_1) \notin Q$.

We claim that $u(\mathrm{Int}(C_i)) \cap \mathrm{Im}(\gamma_\varepsilon) \neq \emptyset$. Indeed, pick a continuous path $\mu : [0,1] \to C_i$ such that $\mu(0) = z$ and $\mu(1) = z_1$. One can assume that for all $t \in [0,1)$, $\mu(t) \in \mathrm{Int}(C_i)$. If for all $t \in [0,1], u(\mu(t)) \notin \mathrm{Im}(\gamma_\varepsilon)$, then $u \circ \mu$ is path in $U_x \backslash \mathrm{Im}(\gamma_\varepsilon)$ whose endpoints are in two distinct connected components, a contradiction.

Now, since u is immersed and γ_ε is embedded, the set $\overline{C_i} \cap u^{-1}(\gamma_\varepsilon)$ is a union of disjoint embedded arcs. We let Ω be the connected component of $\overline{C_i} \backslash u^{-1}(\gamma_\varepsilon)$ which is adjacent to A_i. Its closure $\overline{\Omega}$ is a polygon with corners.

Notice that $u(\overline{\Omega}) \subset Q$. Otherwise, by a connexity argument, there would exist $z_2 \in \Omega$ such that $u(z_2) \in \mathrm{Im}(\gamma_\varepsilon)$. This is in clear contradiction with the definition of Ω. See Figure 17 for a picture.

The map $u_{|\overline{\Omega}}$ is

- proper, since the sets $\overline{C_i}$ and Q are compact,
- a local homeomorphism since u is immersed.

So $u_{|\overline{\Omega}}$ is a connected cover of disk. Hence, it is a homeomorphism. By definition of Ω, two arcs of $\partial\overline{\Omega}$ have image contained in γ_+. So $u_{|\overline{\Omega}}$ can not be injective, a contradiction.

Therefore, $\mathrm{Im}(u_{|C_i}) \subset Q$.

Second step: The map $u_{|C_i} \to Q$ is

- proper, since the sets C_i and Q are compact,
- a local homeomorphism since u is immersed.

Therefore, it is a connected cover of a disk. We conclude that it is a homeomorphism. In particular C_i is an embedded half-disk and u a biholomorphism. □

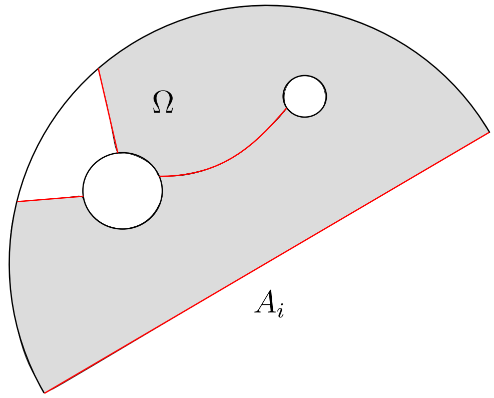

FIGURE 17 – *The connected component C_i, the set Ω (shaded) and $u^{-1}(\gamma_\pm)$ (in red).*
Note that this is hypothetical : the proof of Lemma 2.5.10 shows that this situation cannot happen.

Remark 2.5.11. As explained in Remark 2.5.8, in the proof of Lemma 2.5.10, the set $u^{-1}(\gamma_\varepsilon)$ is a subset of the *frame* $\mathcal{W}(u)$ of the curve u in the sense of Lazzarini ([**Laz00**], [**Laz11**]), see also [**Per18**]. This is a $\mathcal{C}^1$-embedded graph along which we can cut the map u to obtain simple or multiply covered polygons or disks.

The next step of the proof is the construction of an immersed holomorphic polygon v with boundary on γ and α by "filling the corners".

Lemma 2.5.12. *We use the notations above. There is an immersed holomorphic polygon*

$$v : (\mathbb{D}, \partial\mathbb{D}) \to (S_g, \mathrm{Im}(\alpha) \cup \mathrm{Im}(\gamma))$$

such that

- *the map $v_{|v^{-1}(U_x)}$ is a reparameterization of $u_{|u^{-1}(U_x)}$,*
- *each connected component of $v^{-1}(U_x)$ is a disk,*
- *v has a unique corner which maps to x within each connected component of $v^{-1}(U_x)$,*
- *v has one corner at y and an odd number of corners at x.*

PROOF. The construction of v is represented in Figure 18. Let $i \in \{1 \dots N\}$.

First case: The map u is of Type $A+$ or of Type $B-$ near A_i (see Figure 16 for the definition of Type). Consider the closed region R_i at x indicated in Figure 18. Choose a biholomorphism

$$v_i : \mathbb{D} \to R_i.$$

This extends, by Caratheodory Theorem, to a homeomorphism

$$v_i : \overline{\mathbb{D}} \to \overline{R}_i.$$

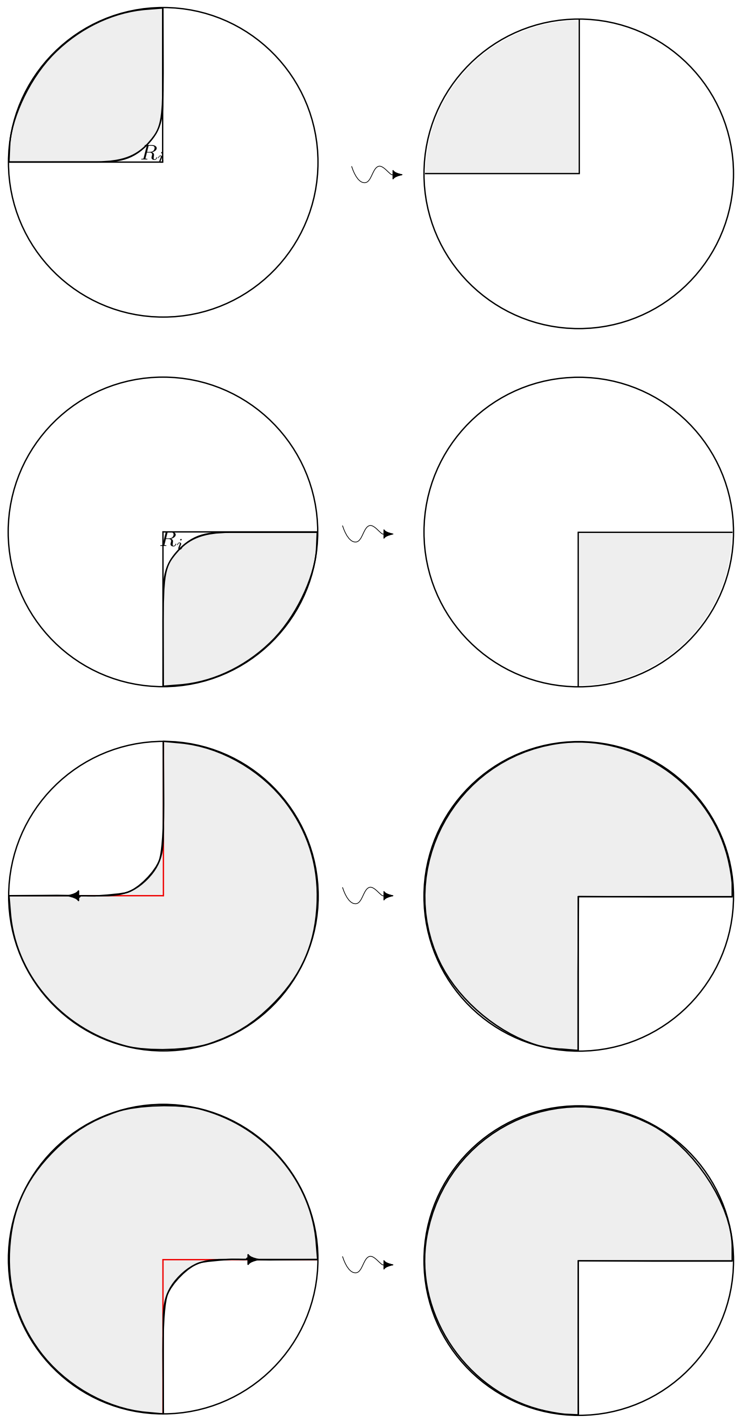

FIGURE 18 – *The procedure to obtain the polygon v*
.

We call B_i the arc $v_i^{-1}(\gamma_\pm)$. There is a unique map $\phi_i : B_i \to A_i$ such that

$$\forall x \in B_i,\ u(\phi_i(x)) = v_i(x).$$

We let $S = \mathbb{D} \cup_{\phi_i} \mathbb{D}$ be the surface obtained by gluing two copies of the disk along ϕ_i. Then, S can be endowed with a Riemann surface structure. For x in the interior of one of the copies of $\mathbb{D}$, the chart is the natural one. If $z = \phi_i(w)$ belongs to A_i, choose a disk D_x contained in U_x and such that $u(z) \in D_x$. We put $U = u^{-1}(U_x) \cup_{\phi_i} v_i^{-1}(U_x)$. A chart around x is given by the map

$$y \in U \mapsto \begin{cases} u(y) \text{ if } y \in u^{-1}(U_x) \\ v(y) \text{ if } y \in v_i^{-1}(U_x) \end{cases}.$$

Then, the applications u and v_i induce a map $v : S \to S_g$. By definition, it is indeed holomorphic. Since v is holomorphic, its energy is the area of the image. Hence, it is finite. The surface S is simply connected, hence biholomorphic to a disk. We conclude that v is a holomorphic polygon with one more corner at x.

Second case: The corner is of type $A-$ or $B+$. Denote by A the red piecewise differentiable arc represented in Figure 18. Recall that C_i is the connected component of $u^{-1}(U_x)$ which contains A_i. Then $Ar := u^{-1}(A) \cap C_i$ is an embedded piecewise differentiable arc. Therefore, $\mathbb{D} \backslash Ar$ has two distinct simply-connected components Ω_1 and Ω_2. We can assume that Ω_1 is the only one which contains x. The map v is the restriction of u to Ω_1.

We repeat this process for $i = 1 \dots N$ to obtain an immersed holomorphic polygon with boundary on α and γ. It has one corner at y and N corners at x. Since the boundary of the polygon necessarily switches from α to γ or γ to α at each corner, the number N is odd. $\square$

Notice that the concatenation $\beta = \gamma \cdot \alpha$ is a continuous loop and as such can be regarded as an continuous map $\beta : S^1 \to S_g$. We assume that it is parameterized so that

- $\beta(i) = \beta(-i) = x$,
- $\beta(-1) = \beta(1) = y$,
- β restricted to the counterclockwise arc from i to $-i$ is a reparameterization of γ,
- β restricted to the counterclockwise arc from $-i$ to i is a reparamaterization of α.

Moreover, we introduce the following notations.

- We let $x_- : [0, 1] \to S^1$ be the counterclockwise arc of S^1 from -1 to 1 and β_- be the map $\beta \circ x_-$.
- We let $x_+ : [0, 1] \to S^1$ be the counterclockwise arc of S^1 from 1 to -1 and β_+ be the map $\beta \circ x_+$.

From the construction of the polygon v, we see that we can lift its boundary to a continuous path $f : [0,1] \to S^1$ such that $\beta \circ f = v_{|\partial\mathbb{D}}$. Since v has a corner at x, we have

- either $f(0) = -1,\ f(1) = 1$,
- or $f(0) = 1,\ f(1) = -1$.

Case $f(0) = -1, f(1) = 1$. Since the concatenation $f \cdot x_-^{-1}$ is a loop based at -1, there is $k \in \mathbb{Z}$ such that $f \cdot x_-^{-1}$ is homotopic to $(x_- \cdot x_+)^k$ relative to -1.

By a Theorem of Jaco ([**Jac70**, Corollary 2]), the subgroup of $\pi_1(S_g, y)$ generated by the classes of β_- and β_+ is free. So there are three alternatives to consider.

(1) There are no relations between β_- and β_+ in $\pi_1(S_g, y)$ (so they generate a free group of rank 2).
(2) There is $m \in \mathbb{Z}$ such that $\beta_+ = \beta_-^m$ in $\pi_1(S_g, y)$.
(3) There is $m \in \mathbb{Z}$ such that $\beta_- = \beta_+^m$ in $\pi_1(S_g, y)$.

Since $\beta \circ f$ is the boundary of v, we have in $\pi_1(S_g, y)$

$$\begin{aligned} e &= (\beta \circ f) \\ &= (\beta_- \cdot \beta_+)^k \beta_-. \end{aligned}$$

So the case (1) can not hold. Therefore, we are in one of the cases (2) or (3).

Case $\beta_- = \beta_+^m$. Then, we have in $\pi_1(S_g, y)$

$$e = (\beta_- \cdot \beta_+)^k \beta_- \tag{2.13}$$

$$= \beta_+^{k(m+1)+m}. \tag{2.14}$$

If $\beta_+ = e$ in $\pi_1(S_g, y)$, then $\beta_- = e$ so that β bounds a disk. Therefore, the surgery $\gamma \#_x \alpha$ bounds a disk and is contractible. This contradicts the hypothesis on $\gamma \#_x \alpha$. Therefore,

$$k(m+1) + m = 0.$$

There are two solutions to this equation, $(m, k) = (0, 0)$ or $(m, k) = (-2, -2)$.

If $m = k = 0$, then equation 2.14 implies $\beta_- = e$. We conclude that γ and α are not in minimal position by Lemma 2.5.6.

If $m = k = -2$, the conclusion follows from a combinatorial argument. Indeed, the boundary f is homotopic to $x_+^{-1} \cdot x_-^{-1} \cdot x_+^{-1}$. We deduce that the polygon v has three corners at x of successive types $B-$, $B+$ and $B-$ (see Figure 16). We assume that these corners are counterclockwise the image of x_1, x_2 and x_3.

Since v is immersed and α is embedded and not contractible, the set $v^{-1}(\alpha)$ is a union of embedded arcs with endpoints on the boundary of $\mathbb{D}$. The corner at x_2 is of type $B+$, so there is one arc A with an endpoint at x_2. Since α is embedded and x_1 and x_2 are of

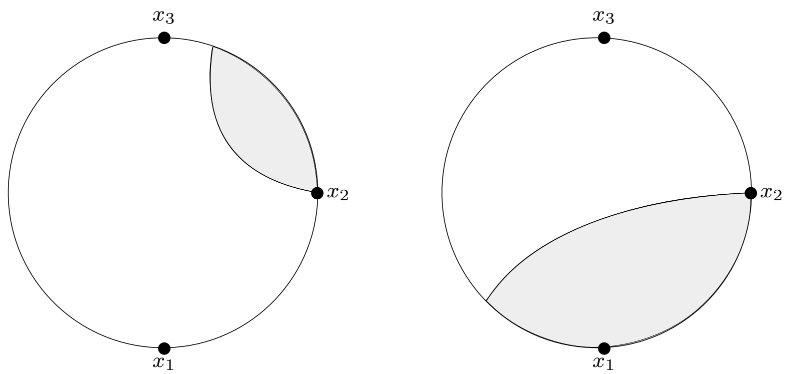

FIGURE 19 – *The arc A and the bigons delimited by A (shaded)*

type $B-$, the other endpoint of A belongs to one of the open boundary arcs (x_2, x_3) or $[-1, x_1)$ (see Figure 19). In the left case, the map v restricted to the bigon delimited by A and the boundary (represented in Figure 19) yields a strip with boundary on α and γ. In the right case, since x_2 is of type $B-$, v restricted to the shaded area is also a bigon. So α and γ are not in minimal position.

Case $\beta_+ = \beta_-^m$. Then, in $\pi_1(S_g, y)$,

$$e = (\beta_- \cdot \beta_+)^k \beta_- \tag{2.15}$$

$$= (\beta_-^{m+1})^k \beta_- \tag{2.16}$$

$$= \beta_-^{k(m+1)+1}. \tag{2.17}$$

If $\beta_- = e$, then β_+ also represents the neutral element. We deduce that the surgery $\gamma \#_x \alpha$ bounds a disk, which contradicts the hypothesis of the proposition. Hence, $k(m+1)+1 = 0$ so $(k, m) = (1, -2)$ or $(k, m) = (-1, 0)$. But if $m = 0$, then $\beta_+ = e$ so that α and γ are not in minimal position by Lemma 2.5.6. Therefore, $(k, m) = (1, -2)$. The boundary f of the polygon v is homotopic to $x_- \cdot x_+ \cdot x_-$. So the polygon v has three corners at x of successive types A_+, A_- and A_+. We call their respective pre-images x_1, x_2 and x_3

As before, v is an immersion and α is embedded and not contractible, so $v^{-1}(\alpha)$ is a union of embedded arcs with endpoints on the boundary $\partial\mathbb{D}$. Since x_2 has type A_-, there is one arc with an endpoint at x_2 which we call A. Its other end lies either in the boundary arc $[-1, x_1)$ or in the boundary arc (x_2, x_3) (see Figure 19). In each case v restricts to an immersed bigon, so that γ and α are not in minimal position, a contradiction.

Case $f(0) = 1, f(1) = -1$. Here, the concatenation $f \cdot x_-$ is a loop based at 1. So there is $k \in \mathbb{Z}$ such that f is homotopic relative endpoints to $(x_+ \cdot x_-)^k \cdot x_-^{-1}$.

Since we have, in $\pi_1(S_g, y)$

$$(\beta_+ \cdot \beta_-)^k \beta_-^{-1} = e$$

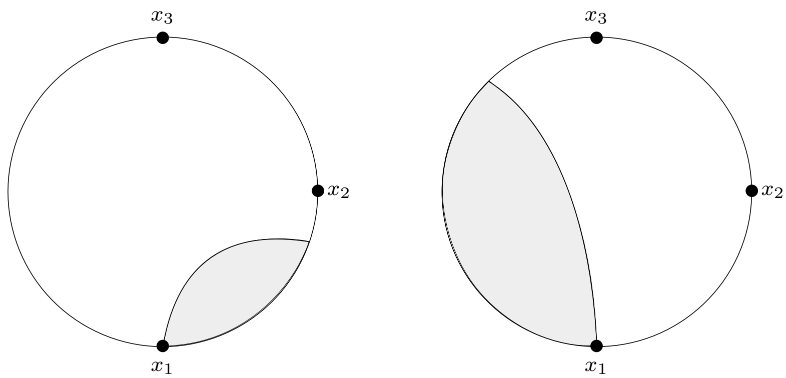

FIGURE 20 – *The arc A and the bigons delimited by A (shaded)*

We have either $\beta_- = \beta_+^m$ for some $m \in \mathbb{Z}$ or $\beta_+ = \beta_-^m$ for some $m \in \mathbb{Z}$.

Assume there is $m \in \mathbb{Z}$ such that $\beta_- = \beta_+^m$. The integer m cannot be zero, otherwise γ and α are not in minimal position. Then from $\beta \circ f = e$, we get $(\beta_+)^{k(m+1)-m} = e$. So $m = -2$ and $k = 2$. Therefore, f is homotopic to $x_+ \cdot x_- \cdot x_+$. So the polygon v has three successive corners at x of successive types A_-, A_+ and A_-. We call their respective pre-images x_1, x_2 and x_3.

The polygon v is immersed and α is embedded an not contractible, so the set $v^{-1}(\alpha)$ is a union of embedded arcs with endpoints on $\partial\mathbb{D}$. Since x_1 is of type A_-, there is an arc A with an endpoint at x_1. Since α is embedded, the other endpoint of A is either on the boundary arc $[x_3, x_1]$ or on the arc (x_1, x_2) (see Figure 20). In both cases, v restricted to the shaded area in Figure 20 is a bigon. So α and γ are not in minimal position, a contradiction.

Assume that there is $m \in \mathbb{Z}$ such that $\beta_+ = \beta_-^m$. This time, we have $(m, k) = (-2, -1)$. So the polygon v has three successive corners at x of types B_-, B_+, B_-. We call x_1, x_2 and x_3 their pre-images.

The arc $A \subset v^{-1}(\alpha)$ with an endpoint on x_1 has other endpoint either on $[x_3, x_1]$ or on the arc (x_1, x_2) (see Figure 20). In either case, v restricted to the shaded area in Figure 20 is a bigon. So α and γ are not in minimal position, a contradiction. $\square$

2.5.3. Obstruction of the surgery cobordisms

We suppose that we are in the setting of Subsection 2.5.2. There are

- an embedded curve α,
- a generic curve γ in minimal position with α,
- an intersection point x of degree 1 between α and γ.

Proposition 2.5.13. *Under the hypotheses of Subsection 2.5.2, the immersed surgery cobordism*

$$V : (\gamma, \alpha) \rightsquigarrow \gamma\#_x\alpha,$$

constructed in 2.2.1 does not bound a continuous polygon with a unique corner.

Proof. Recall from Subsection 2.5.2 that the cobordism $V : (\gamma, \alpha) \rightsquigarrow \gamma\#_x\alpha$ is an immersion $i : P \looparrowright \mathbb{C} \times S_g$ of a pair of pants P. The immersion i is the smoothing of a piecewise smooth immersion $i^+ \sqcup j : P \looparrowright \mathbb{C} \times S_g$. Moreover, by Lemma 2.2.10 there is a homotopy $(i_\lambda)_{\lambda\in[0,1]}$ which interpolates between $i^+ \sqcup j$ and i and is constant near the double points of i. See Figure 2 for the projections of these objects to the complex plane.

Assume there is a topological teardrop with boundary on i. So there are

- a continuous map $u : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C} \times S_g, i(V))$,
- a continuous map $\tilde{u} : (-\pi, \pi) \to S$ such that

  $$\forall\theta \in (-\pi, \pi),\ i \circ \tilde{u}(\theta) = u\big(e^{i\theta}\big),$$

  and

  $$\lim_{t\to-\pi^+} \gamma(t) \neq \lim_{t\to\pi^-} \gamma(t).$$

In particular $u(-1)$ is a double point of the immersion i :

$$u(-1) \in DP \cup DP_1 \cup DP_2 \cup DP_0,$$

(see the end of 2.2.1 for the notations).

First step. From Lemma 2.2.10, we know that there is a continuous homotopy $(i_\lambda)_{\lambda\in[0,1]}$ from $i_0 = i^+ \sqcup j$ to i_1 which is constant near the double points of i. For $\lambda \in [0,1]$, the path

$$i_\lambda \circ \tilde{u} : (-\pi, \pi) \to S,$$

satisfies

$$i_\lambda \circ \tilde{u}(-\pi) = i_\lambda \circ \tilde{u}(\pi).$$

We let $u_\lambda : \partial\mathbb{D} \to \mathbb{C} \times S_g$ be the path defined by

$$\forall\theta \in (-\pi, \pi),\ u_\lambda\big(e^{i\theta}\big) = i_\lambda \circ \tilde{u}(\theta).$$

Let $\overline{A}(1,2) \subset \mathbb{C}$ be the closed annulus $\{z \in \mathbb{C} | 1 \leqslant |z| \leqslant 2\}$. We now glue the map

$$\begin{array}{ccc} \overline{A}(1,2) & \to & \mathbb{C} \times S_g \\ re^{i\theta} & \mapsto & u_{2-r}\big(e^{i\theta}\big) \end{array},$$

along the boundary of u to obtain a teardrop v with boundary on i_0. Its lift to the domain of i_0 is the map $\tilde{u}$.

Second step. We let C be the union of $\mathbb{R}^- \times \{0\}$ and $\{0\} \times \mathbb{R}^+$. We let

$$\begin{array}{cccc} p: & \mathbb{R}^2 & \to & C \\ & (x,y) & \mapsto & \left\{ \begin{array}{l} (0, x+y) \text{ if } y \leqslant -x \\ (x+y, 0) \text{ if } x \leqslant -y \end{array} \right. \end{array}.$$

The map p is the continuous projection along the ray $\{x = -y\}$ onto C.

We let

$$\begin{array}{cccc} p^-: & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & (x,y) & \mapsto & \left\{ \begin{array}{c} (x,y) \text{ if } x \leqslant 0,\ y \geqslant 0 \\ p(x,y) \text{ else} \end{array} \right. \end{array}$$

The map $w = p^- \circ v$ is a continuous teardrop with boundary on $i^+ \sqcup i^-$.

Third step. Recall from 2.2.1 that, in the chart $\mathbb{C} \times U_x$, the immersion $i^+ \sqcup i^-$ coincides with the handle

$$H_\varepsilon = \left\{ \varepsilon c(t) z \middle| t \in \mathbb{R},\ z = (x,y) \in S^1 \right\},$$

where c is a smooth path interpolating between $\mathbb{R}$ and $i\mathbb{R}$.

We let

$$A = \left\{ \varepsilon c(0) z \middle|\ z = (x,y) \in S^1 \right\}$$

be the core of the handle H_ε.

Denote by $\tilde{w} : (-\pi, \pi) \to S$ the lift of w to the domain of $i^+ \sqcup i^-$ defined by

$$\forall \theta \in (0, \pi),\ w\left(e^{i\theta}\right) = i^+ \sqcup i^-(\tilde{w}(\theta)).$$

Both of the endpoints of $\tilde{w}$ do not belong to A. Therefore, we can homotope $\tilde{w}$ relative to its endpoints to a smooth path

$$\tilde{a} : (-\pi, \pi) \to S$$

which is transverse to the set A. Since it is homotopic to $\tilde{w}$, the map $i^+ \sqcup i^- \circ \tilde{a}$ bounds a smooth topological teardrop a.

Fourth step. For $t \in [0, 1]$, we let

$$\begin{array}{cccc} p_t: & \{(x,y) | x \leqslant 0,\ y \geqslant 0\} & \to & \mathbb{R}^2 \\ & (x,y) & \mapsto & \left\{ \begin{array}{l} (x + ty, (1-t)y) \text{ if } x + y \leqslant 0 \\ ((1-t)x, y + tx) \text{ if } x + y \geqslant 0 \end{array} \right. \end{array}$$

This is a continuous family which interpolates between p and the identity on the set

$$\{(x,y) | x \leqslant 0,\ y \geqslant 0\}.$$

Notice that for $t \in [0, 1)$ the paths

$$c_t := p_t \circ c$$

are smooth.

Let U be the set $\partial\mathbb{D} \cap a^{-1}(\mathbb{C} \times U_x)$. There are smooth functions

$$s : U_x \to \mathbb{R},\ (x, y) : U \to S^1$$

such that

$$\forall \theta \text{ such that } e^{i\theta} \in U_x,\ a(e^{i\theta}) = c \circ s(\theta)(x(\theta), y(\theta)).$$

Now, we attach the map

$$\begin{array}{rcc} A(1,2) & \to & \mathbb{C} \times S_g \\ re^{i\theta} & \mapsto & p_{r-1} \circ c \circ s(\theta)(x(\theta), y(\theta)) \end{array}$$

along the boundary of a to obtain a continuous polygon $\overline{b}$.

We let b be the projection of $\overline{b}$ on the surface S_g

$$b := p_{S_g} \circ \overline{b}.$$

Fifth step. We build a non-constant teardrop on $\gamma_1 \#_x \gamma_2$ from b. Notice that for $t \in \partial\mathbb{D}$, we have $a(t) \in A$ if and only if $b(t) = x$. Let t_0 be such that $a(t_0) \in A$.

There is a connected open neighborhood V of t_0 in $\partial\mathbb{D}$ such that $\forall t \in V, w(t) \in \mathbb{C} \times U_x$ and $b(t) \in U_x$. We write (in the Darboux chart for U_x that we fixed earlier) $a(t) = (w_1(t), w_2(t))$ for $t \in V$. Then, there are smooth functions $s : V \to \mathbb{R}$, $x : V \to \mathbb{R}$ and $y : V \to \mathbb{R}$ with $s(t_0) = 0$ such that

$$\forall t \in V,\ a(t) = c \circ s(t)(x(t), y(t)).$$

Up to homotopy, we can always assume that $y'(t_0) \neq 0$. So

$$\forall t \in V,\ a'(t) = s'(t)c' \circ s(t)(x(t), y(t)) + c \circ s(t)(x'(t), y'(t)).$$

At $a(t_o) = c(0)(x(0), y(0))$, the tangent space of A is generated by $c(0)(-y(0), x(0))$. Hence, since $a'(t_0)$ is transverse to A, we have $s'(0) \neq 0$.

Assume $s'(t_0) > 0$. Then, $p \circ c(t) = (c_1(t) + c_2(t), 0)$ for $t_0 - \alpha < t < t_0$ and $p \circ c(t) = (0, c_1(t) + c_2(t))$ for $t_0 + \alpha > t > t_0$ close enough to t_0. So we have

$$b(t) = \begin{cases} y(c_1 \circ s + c_2 \circ s) \text{ if } t_0 - \alpha < t < t_0 \\ iy(c_1 \circ s + c_2 \circ s) \text{ if } t_0 < t < t_0 + \alpha \end{cases}.$$

In particular, the left and right derivative at t_0 are given by

$$b'(t_0^-) = y(t_0)(c_1' + c_2')(0)s'(t_0)$$
$$b'(t_0^+) = iy(t_0)(c_1' + c_2')(0)s'(t_0)$$

So if $y(t_0) < 0$, the path b parameterizes the real line $\mathbb{R}_+$ in the opposite orientation followed by $i\mathbb{R}_-$ in the opposite orientation. If $y(t_0) > 0$, b parameterizes $\mathbb{R}_-$ according to its orientation followed by $i\mathbb{R}_+$.

If $y(t_0) = 0$, we compute the second derivatives to get

$$b^{(2)}(t_0^-) = 2y'(t_0)(c_1' + c_2')(0)s'(t_0)$$
$$b^{(2)}(t_0^+) = 2iy'(t_0)(c_1' + c_2')(0)s'(t_0)$$

So we easily see that the same conclusion holds.

Therefore, we can lift b to a continuous path $d : \partial\mathbb{D} \to \gamma\#_x\alpha$ such that $p \circ d = b$. This path is homotopic (through the applications p_t) to b. Hence, it extends to a map $d : \mathbb{D} \to S_g$ with boundary on $\gamma\#_x\alpha$. The map d is easily seen to be a teardrop. Hence, $\gamma\#_x\alpha$ is obstructed, a contradiction with the hypothesis. $\square$

Proposition 2.5.13 generalizes to the following.

Proposition 2.5.14. *Assume that $\gamma, \alpha_1, \ldots, \alpha_N$ are unobstructed curves. We, moreover, assume the following by induction.*

We assume that γ and α_1 are transverse. We let x_1 be an intersection point between γ and α_1 of degree 1.

For $k \in \{1 \ldots n-1\}$, we assume that $(\gamma\#_{x_1}\alpha_1)\ldots\#_{x_k}\alpha_k$ is transverse to α_{k+1}. We assume that these two curves are in minimal position. We let x_k be an intersection point of degree 1 between these curves.

Moreover, we assume that each of the curves $(\gamma\#_{x_1}\alpha_1)\ldots\#_{x_k}\alpha_k$ is unobstructed for $k \in \{1 \ldots n\}$.

Then, the composition of the successive cobordisms

$$((\gamma\#_{x_1}\alpha_1)\ldots\#_{x_k}\alpha_k, \alpha_{k+1}) \rightsquigarrow (\gamma\#_{x_1}\alpha_1)\ldots\#_{x_{k+1}}\alpha_{k+1}$$

does not admit any topological teardrop.

Proof. The proof is a repeated application of the proof of the preceding proposition. $\square$

We can deduce the following proposition.

Proposition 2.5.15. *Assume that γ, $\alpha_1, \ldots, \alpha_N$ are as in Proposition 2.5.14. Then there is an unobstructed Lagrangian cobordism*

$$(\gamma, \alpha_1, \ldots, \alpha_N) \rightsquigarrow (\gamma\#_{x_1}\alpha_1)\ldots\#_{x_N}\alpha_N.$$

Proof. First, we prove the following lemma which is a refinement of the construction in the proof of Lemma 2.2.3.

Lemma 2.5.16. *Assume that $i : V \looparrowright \mathbb{C} \times S_g$ is an immersed oriented Lagrangian cobordism with embedded ends such that*

(1) the set of double points

$$\{(x, y) \in V \times V | i(x) = i(y),\ x \neq y\}$$

is a finite disjoint union of embedded intervals I_k (with $k = 1 \dots N$) and of points (x_p, y_p) (with $p = 1 \dots M$),

(2) the immersion i restricts to an embedding of the intervals I_k for $k = 1, \dots, N$,

(3) if $(x, y) \in I_k$ for some k, we have

$$\dim(di_x(T_xV) \cap di_y(T_yV)) = 1,$$

(4) the space

$$di_{x_p}(T_{x_p}V) \cap di_{y_p}(T_{y_p}V)$$

is null for $p \in \{1, \dots M\}$.

Then there is a smooth family of immersions $i_t : V \looparrowright \mathbb{C} \times S_g$ for $t \in [0, 1]$ such that the following properties hold.

(i) We have $i_0 = i$.

(ii) For all $t \in [0, 1]$, the immersion i_t coincides with V outside of a compact subset.

(iii) For almost all $t \in [0, 1]$, the double points of i_t are transverse.

(iv) If $(x, y) \in V$ are such that $i_1(x) = i_1(y)$, then there are

- *a smooth path $\gamma = (x, y) : [0, 1] \to V \times V$ with $\gamma(1) = (x, y)$,*
- *a function $f : [0, 1] \to [0, 1]$ with $f(0) = 0$ and $f(1) = 1$,*

such that $i_{f(t)}(x) = i_{f(t)}(y)$ for all $t \in [0, 1]$.

Proof of Lemma 2.5.16. We extend the immersion $i : V \to \mathbb{C} \times S_g$ to a Weinstein embedding $\Phi : T^*_\varepsilon V \to \mathbb{C} \times S_g$. We let $K \subset V$ be a compact subset such that the image of $i_{|V \setminus K}$ is the disjoint union

$$\bigcup_{i=1\dots n} (\mathbb{R}^- \times \gamma_i) \cup \bigcup_{j=1\dots m} (\mathbb{R}^+ \times \gamma_j).$$

There is a smooth function $f : V \to \mathbb{R}$ such that the following holds.

(A) The function f is null outside of K.

(B) Let $\pi : T^*_\varepsilon V \to V$ be the standard projection. We let $X_{f\circ\pi}$ be the hamiltonian vector field of $f \circ \pi$. For $(x, y) \in I_k$ for some $k \in \{1, \dots N\}$, the vector

$$d\Phi_x(X_{f\circ\pi}(x)) - d\Phi_y(X_{f\circ\pi}(y))$$

is transverse to the vector space $di(T_xV) + di(T_yV)$.

To see this, choose a vector field X on each $\sqcup_k i(I_k)$ such that

$$di(T_xV) + di(T_yV) + X(i(x)) = T_{i(x)}(\mathbb{C} \times S_g)$$

for each $(x, y) \in I_k$. Choose disjoint neighborhoods D_k of I_k diffeomorphic to disks. We extend X to a vector field on $\sqcup D_k$. There is a smooth function f such that $X_{f\circ\pi} = X$ on $\sqcup D_k$. Now extend it to V using a smooth cut-off function.

We choose an increasing, smooth, cut-off function $\beta : [0, \varepsilon] \to [0, 1]$ such that $\beta(t) = 1$ for $t \in \left[0, \frac{\varepsilon}{3}\right]$ and $\beta(t) = 0$ for $t \in \left[\frac{2\varepsilon}{3}, 1\right]$. We let g be the smooth function given by

$$\begin{array}{cccc} g: & T^*_\varepsilon V & \to & \mathbb{R} \\ & (x, v) & \mapsto & \beta(|v|) f \circ \pi(x, v) \end{array}.$$

We denote by $\phi^t_g : T^*_\varepsilon V \to T^*_\varepsilon V$ the flow of X_g at the time $t \in [0, 1]$.

We claim that for $\eta > 0$ small enough, the map

$$\begin{array}{cccc} \Psi: & L \times L \times [-\eta, \eta] & \to & (\mathbb{C} \times S_g) \times (\mathbb{C} \times S_g) \\ & (x, y, t) & \mapsto & \big(\Phi\big(\phi^t_g(x)\big), \Phi\big(\phi^t_g(y)\big)\big) \end{array}.$$

is transverse to the diagonal $\Delta = \{(z, z) | z \in \mathbb{C} \times S_g\}$.

Indeed, there is $\eta > 0$ such that

(C) if $\Psi(x) = \Psi(y)$ with $x \neq y$, then $\dim[d\Phi(T_xV) \cap d\Phi(T_yV)] \leqslant 1$,

(D) if $\Psi(x) = \Psi(y)$ with $x \neq y$, then $d\Phi(X_g(\phi^t(x))) - d\Phi(X_g(\phi^t(y))) \notin d\Phi(T_xV) + d\Phi(T_yV)$.

This is seen by using assertions (A) and (B), the compactness of K and lower semi-continuity of the rank.

Let $(x, y, t) \in \Psi^{-1}(\delta)$ and $v \in T_{\Psi(x,y,t)}(\mathbb{C} \times S_g)$. Due to the preceding assumption, there are $v_1 \in T_xV, v_2 \in T_yV$ and $\lambda \in \mathbb{R}$ such that

$$d\Phi(d\phi^t_g(v_1)) - d\Phi(d\phi^t_g(v_2)) + \lambda\big[d\Phi\big(X_g \circ \phi^t_g(x)\big) - d\Phi\big(X_g \circ \phi^t_g(y)\big)\big]$$

is equal to v.

Now we conclude by the following claim (which we learned from [**MS12**]) whose proof is an easy exercise.

Claim. Let $h = (h_1, h_2) : M \to N \times N$ be smooth map. Let x be such that $h_1(x) = h_2(x)$. h is transverse to the diagonal if and only if for all $v \in T_{h_1(x)}N$, there is $w \in t_xM$ such that $d(h_1)_x(w) - d(h_2)(x)(w) = v$.

So the the set $\Psi^{-1}(\Delta)$ has a natural structure of compact 1-dimensional manifold. In particular, it has a finite number of connected components which are compact as well. Hence, for $\alpha > 0$ small enough, all the connected components of $V \times V \times [-\alpha, \alpha] \cap \Psi^{-1}(\Delta)$ have non empty intersection with $V \times V \times 0$.

The projection

$$\begin{array}{rcl}\Psi^{-1}(\Delta) & \to & \mathbb{R} \\ (x,y,t) & \mapsto & t\end{array}$$

is smooth. Hence, there is a regular value $0 < t_0 < \alpha$. The family $\Phi \circ \phi_g^t$ for $0 \leqslant t \leqslant t_0$ satisfies the conclusion of the lemma. $\square$

We let $i : V \looparrowright \mathbb{C} \times S_g$ be the immersed Lagrangian cobordism

$$(\gamma, \alpha_1, \ldots, \alpha_N) \rightsquigarrow (\gamma \#_{x_1} \alpha_1) \ldots \#_{x_N} \alpha_N,$$

given by Proposition 2.5.14. From Subsection 2.5.2, the immersion V satisfies the hypotheses of Lemma 2.5.16. Hence, there is a family $(i_t)_{t \in [0,1]}$ which satisfies the above properties (i), (ii), (iii) and (iv).

Assume there is a continuous teardrop

$$u : (\mathbb{D}, \partial\mathbb{D}) \to (M, i_1(V))$$

with boundary on the immersion i_1. In particular, there is a path

$$\tilde{u} : (-\pi, \pi) \to V$$

such that

$$\forall \theta \in (-\pi, \pi),\ u\big(e^{i\theta}\big) = i_1 \circ \tilde{u}(\theta),$$
$$x = \lim_{\theta \to \pi^-} \tilde{u}(\theta) \neq \lim_{\theta \to -\pi^+} \tilde{u}(\theta) = y.$$

We call f, $\gamma = (x, y)$ the smooth functions provided by the point (iv) of Lemma 2.5.16. There is a continuous family of paths $(\gamma_t)_{t \in [0,1]} : [-\pi, \pi] \to V$ such that

$$\forall t \in [0,1],\ \gamma_t(-\pi) = x(t),\ \gamma_t(\pi) = y(t),$$

and

$$\gamma_1 = \tilde{u}.$$

We glue the map

$$\begin{array}{rcl}A(1,2) & \to & \mathbb{C} \times S_g \\ re^{i\theta} & \mapsto & i_{f(2-r)}(\gamma_{2-r}(\theta))\end{array},$$

along the teardrop u to obtain a topological teardrop with boundary on the immersion i. This does not exist by hypothesis. Therefore, there are no topological teardrops on i_1. $\square$

2.5.4. Action of the Mapping Class Group and proof of Theorem 2.5.1

First, we need the following lemma.

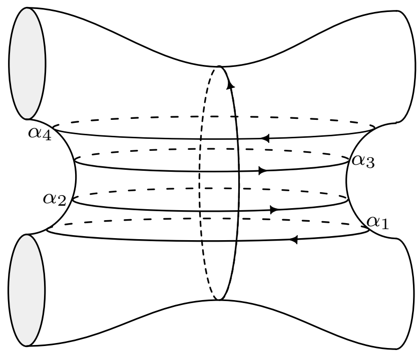

FIGURE 21 – *The curves α_1, α_2, α_3, α_4*

Lemma 2.5.17. *Let γ_1 and γ_2 be two isotopic separating curves. There is $x \in \mathbb{R}$ such that*

$$[\gamma_2] - [\gamma_1] = i(x),$$

in $\Omega_{cob}^{imm,unob}(S_g)$.

PROOF. We are in the situation of Figure 21. We perform four successive surgeries along four isotopic curves α_1, α_2, α_3 and α_4 represented in Figure 21. The surgeries with α_1 and α_3 resolve the intersection point in the front of the surface. Meanwhile, the surgeries with α_2 and α_4 resolve the intersection point in the back of the surface.

The end-product is a curve β isotopic to γ_1 whose holonomy is

$$\mathrm{Hol}_A(\gamma_1) + \mathrm{Hol}_A(\alpha_1) + \mathrm{Hol}_A(\alpha_2) + \mathrm{Hol}_A(\alpha_3) + \mathrm{Hol}_A(\alpha_4).$$

Furthermore, by Proposition 2.5.15, there is an unobstructed Lagrangian cobordism

$$V : (\alpha_3, \alpha_4, \gamma_1, \alpha_1, \alpha_2) \rightsquigarrow \beta.$$

Since α_k for $k \in \{1, \ldots, 4\}$ are non-seperating, we have by Lemma 2.2.18,

$$\alpha_1 + \alpha_2 = i[\mathrm{Hol}_A(\alpha_1) + \mathrm{Hol}_A(\alpha_2)],$$
$$\alpha_3 + \alpha_4 = i[\mathrm{Hol}_A(\alpha_3) + \mathrm{Hol}_A(\alpha_4)].$$

So

$$\beta = \gamma_1 + i[\mathrm{Hol}_A(\alpha_3 + \alpha_4) + \mathrm{Hol}_A(\alpha_2 + \alpha_1)].$$

From this and Lemma 2.5.3, we deduce that there is $\varepsilon > 0$ such that all curve γ isotopic to γ_1 with

$$|\mathrm{Hol}_A(\gamma) - \mathrm{Hol}_A(\gamma_1)| < \varepsilon$$

satisifies in $\Omega_{\mathrm{cob}}^{\mathrm{imm,unob}}(S_g)$

$$\gamma - \gamma_1 = i(\mathrm{Hol}_A(\gamma) - \mathrm{Hol}_A(\gamma_1)).$$

Let S be the set of $\varepsilon > 0$ with this property. We let ε_{γ_1} be the supremum of S.

Choose a smooth isotopy $t \mapsto \gamma_t$ between γ_0 and γ_1. Since $[0,1]$ is compact, one can build a finite sequence

$$0 = t_0 < t_1 < \ldots < t_N = 1$$

such that

$$\forall i \in \{1, \ldots, N-1\},\ \left|\mathrm{Hol}_A(\gamma_{t_i}) - \mathrm{Hol}_A(\gamma_{t_{i+1}})\right| < \varepsilon_{\gamma_{t_i}}.$$

So the conclusion follows. $\square$

The following Proposition is the analog of Proposition 2.2.15 for $\Omega_{\text{cob}}^{\text{imm,unob}}(S_g)$.

Proposition 2.5.18. *Let $\alpha : S^1 \to S_g$ and $\beta : S^1 \to S_g$ be two embedded curves. Then, there is $x \in \mathbb{R}$ such that*

$$[T_\alpha(\beta)] = [\beta] + (\beta \cdot \alpha)[\alpha] + i(x)$$

in $\Omega_{cob}^{imm,unob}(S_g)$.

Proof. Assume that α and β are in minimal position. We construct a representative γ, up to isotopy, of $T_\alpha(\beta)$ using the procedure of Proposition 2.2.15.

We show that the successive surgeries of the proof of Proposition 2.2.15 are unobstructed.

Recall that for $k \in \{1, \ldots, N\}$, the curve c_k is obtained from k surgeries along the curves $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k$. The curve $\tilde{\alpha}_{k+1}$ is a perturbation of α. We also fixed Darboux charts ϕ_m near each intersection point x_m.

Lemma 2.5.19. *In the above setting, the curves $\tilde{\alpha}_{k+1}$ and c_k are in minimal position.*

Proof. The proof proceeds by contradiction. Let us start with an outline of the proof.

(1) We first show that there is a bigon v between $\tilde{\alpha}_{k+1}$ and β whose boundary arc on $\tilde{\alpha}_{k+1}$ is embedded.
(2) This bigon has a precise behavior near the surgered points described in Figure 22.
(3) Then, we build from v a bigon between α and β following the procedure represented in 24.

So α and β are not in minimal position, this contradicts the hypothesis.

Assume there is a bigon u between $\tilde{\alpha}_{k+1}$ and c_l. Since $\tilde{\alpha}_{k+1}$ is embedded, the set $u^{-1}(\tilde{\alpha}_{k+1})$ is a union of embedded arcs. We take an innermost such arc A. We let $v : \mathbb{D} \to S_g$ be the immersed bigon delimited by this arc. We parameterize v so that the preimages of its corners are -1 and 1.

Notice that the bigon v is immersed. If one of its corners is non convex, the set $v^{-1}(\tilde{\alpha}_{k+1})$ has non-empty intersection with $\mathrm{Int}(\mathbb{D})$, so A is not innermost. Hence, v has convex corners.

We fix a (regular) parameterization $\gamma : S^1 \to S_g$ of the curve c_k. There are two by two disjoint arcs of $A_1, \ldots, A_k, B_1, \ldots, B_k \subset S^1$ satisfying the following properties

- for $m \in \{1, \ldots, k\}$, $\gamma_{|A_m}$ parameterizes an arc of $\tilde{\alpha}_m$,
- for $m \in \{1, \ldots, l\}$, $\gamma_{|B_m}$ parameterizes an arc of β

Notice that the arcs $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ are intertwined. Moreover, the complements of these in S^1 is a union of arcs which parameterize the handles of the successive surgeries. We call these arcs $C_1, \ldots, C_{2k}$.

Let A be the arc of $\partial\mathbb{D}$ which maps into c_k through v. Choose an immersed lift $\lambda : A \to S^1$ such that $\gamma \circ \lambda = v_{|A}$. Whenever the map λ parameterizes one of the arcs C_i, we say that v has a switch.

There is at least one switch. Assume otherwise. Then the image of λ is contained in one of the A_i or one of the B_i. If it is a subset of one of the A_i, then v yields an immersed strip with convex corners between $\tilde{\alpha}_i$ and $\tilde{\alpha}_{k+1}$. There are only two such strips, and both of these are not strips on the surgeries. If the image of λ is a subset of one of the B_i, then v is an immersed bigon between $\tilde{\alpha}_{k+1}$ and β. Hence these curves are not in minimal position, a contradiction.

There are at most two branch switches. If not, there is m such that $A_m \subset \mathrm{Im}(\lambda)$. So $\mathrm{Im}(c_l \circ \lambda)$ contains one of the vertical lines in the charts ϕ_k. Hence, it must intersect $\tilde{\alpha}_{k+1}$ in this chart, a contradiction.

The possible behaviors of the curve v at a switch are described in Figure 22. Using the techniques of the proof of Proposition 2.5.7, it is an easy exercise to show that these are indeed the only possible cases.

First case. One of the corners maps to a point x^k_m for some m. Moreover, if we parameterize v so that $v(-1) = x^m_k$, the lower arc is mapped to c_k and the upper boundary arc to $\tilde{\alpha}_{k+1}$.

Then there is a unique switch of type $A\pm$ (Figure 22). The other corner is one of the points y_i and must be of type $1\pm$ (Figure 23). It is then easy, following the procedure of the proof of Proposition 2.5.7, to produce a non-constant bigon with arcs on α and β (see Figure 24).

Second case. One of the corners maps to a point x^k_m for some m. Moreover, if we parameterize v so that $v(-1) = x^m_k$, the upper arc is mapped to c_k and the lower boundary arc to $\tilde{\alpha}_{k+1}$.

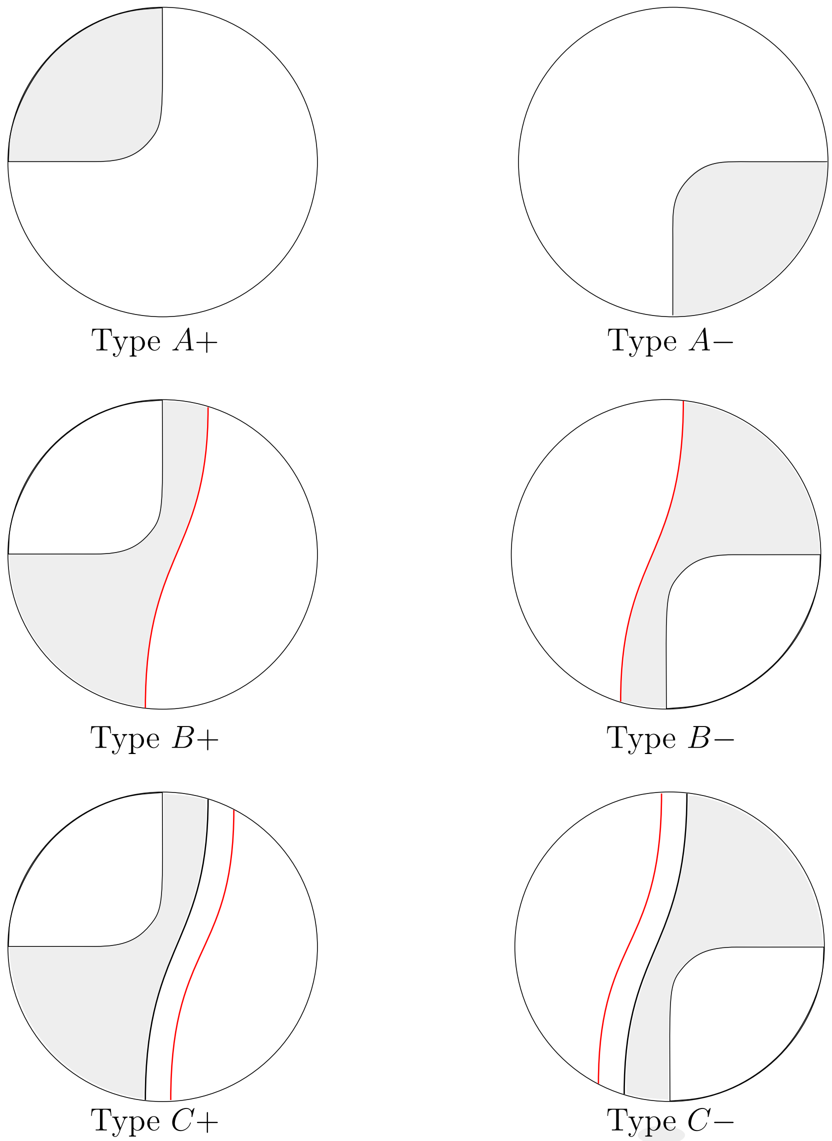

FIGURE 22 – *The different types of switch points, The curve* α_{k+1} *is in red*

Following the upper boundary arc from -1 to 1, there is a first switch of type $B\pm$ or $C\pm$. There cannot be another switch. Otherwise, the boundary condition λ parameterizes one of the vertical arcs in the chart ϕ_k. Hence, the other corner maps to one of the y_i and must be of type 2 (Figure 23). From this, we deduce that the switch was of type $B\pm$.

In the chart near the switch, β yields an arc in the image of v from the handle to $\tilde{\alpha}_{k+1}$. Cutting the disk along this arc, we obtain a bigon between $\tilde{\alpha}_{k+1}$ and β.

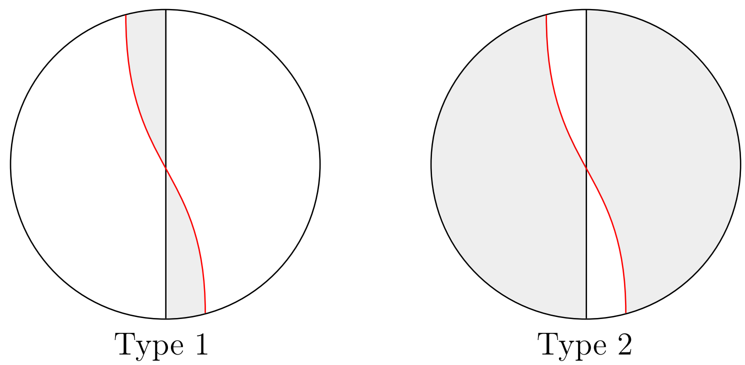

Figure 23 – *The different possibilities at a corner y_j,*
The image of the disk v is one of the four shaded areas

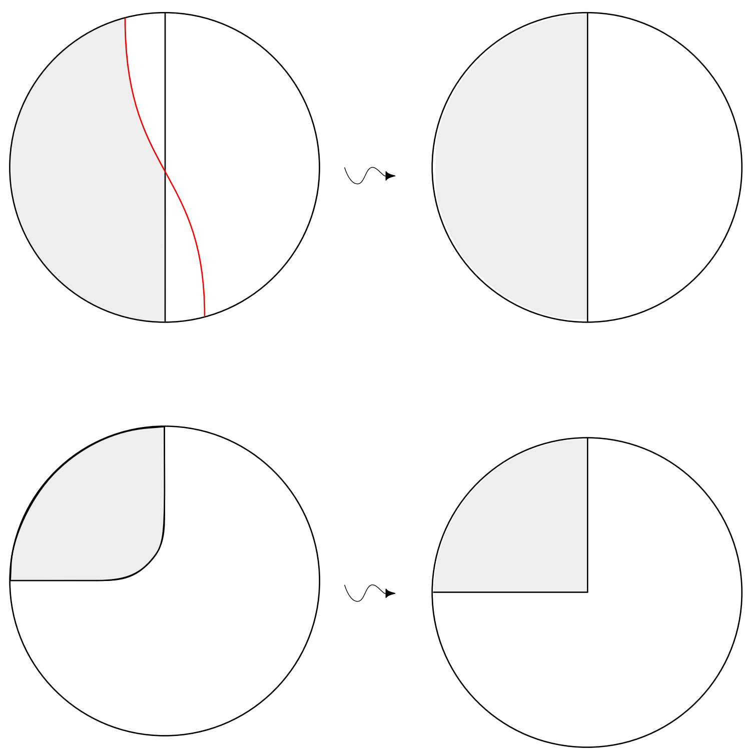

Figure 24 – *The procedure to obtain a bigon between α and β*

Third case. Both of the corners map to points y_i and y_j for some i and some j. We assume that the lower boundary arc of the bigon maps to $\tilde{\alpha}_{k+1}$ and the upper boundary arc maps to c_k.

Following the upper boundary arc from -1 to 1, there is a first switch point of type A and one second of type B or C. However, the corner at -1 is of type 2, and the corner at 1 is of type 1. So the second switch is of type B.

In the chart near the first switch point, there is an arc along β from c_k to $\tilde{\alpha}_{k+1}$ which cuts the image of v in half. We cut v along this arc and solve the corners as in Figure 24 to obtain a bigon with boundary on α and β.

In all three cases, we obtain that α and β are not in minimal position. Therefore, the lemma must hold. $\square$

Now, an induction and Proposition 2.5.7 show that each of the curve c_k is unobstructed. Recall that we denoted by γ the curve obtained by the successive surgeries of the proof of Proposition 2.2.15 and that it is isotopic to $T_\alpha(\beta)$.

Hence, by Proposition 2.5.15, there is an unobstructed Lagrangian cobordism

$$\left(\alpha, \beta, \ldots, \beta, \beta^{-1}, \ldots, \beta^{-1}\right) \rightsquigarrow \gamma,$$

with as many copies of α as there are intersection points of degree 0 and as many copies of α^{-1} as there are intersection points of degree 1. On the other hand, γ and $T_\alpha(\beta)$ are isotopic curves. By Corollary 2.5.5 when β is non-separating and Lemma 2.5.17 when β is, there is $x \in \mathbb{R}$ such that

$$\gamma = i(x) + T_\alpha(\beta).$$

Hence, in $\Omega^{\mathrm{imm,unob}}_{\mathrm{cob}}(S_g)$,

$$\begin{aligned} T_\alpha(\beta) &= \gamma + i(x) \\ &= \beta + (\alpha \cdot \beta)\alpha + i(x). \end{aligned}$$

This concludes the proof.

In general, isotope α to a curve $\tilde{\alpha}$ in minimal position with α. So there is $x \in \mathbb{R}$ such that

$$T_{\tilde{\alpha}}(\beta) = \beta + (\alpha \cdot \beta)\tilde{\alpha} + i(x).$$

The conclusion follows by Lemma 2.5.17 since $T_{\tilde{\alpha}}(\beta)$ and $\tilde{\alpha}$ are isotopic to α and $T_\alpha(\beta)$ respectively. $\square$

As a first consequence, let γ be the oriented boundary of an embedded torus. We let

$$T = [\gamma] - i(\mathrm{Hol}_A(\gamma)). \tag{2.18}$$

Lemma 2.5.20. *The class T defined in equation 2.18 does not depend on the choice of γ.*

Proof. Let γ_1 and γ_2 be two embedded curves which bound a torus. There is a sequence of Dehn Twists $T_{\delta_1}, \ldots, T_{\delta_n}$ about the curves $\delta_1, \ldots, \delta_n$ such that

$$T_{\delta_1} \ldots T_{\delta_n}(\gamma_1)$$

is isotopic to γ_2. Since γ_1 is null-homologous, by Proposition 2.5.18, there is $x \in \mathbb{R}$ such that

$$T_{\delta_1} \dots T_{\delta_n}(\gamma_1) = \gamma_1 + i(x).$$

Since γ_2 is isotopic to $T_{\delta_1} \dots T_{\delta_n}(\gamma_1)$, by Lemma 2.5.17, there is $y \in \mathbb{R}$ such that

$$\gamma_2 = T_{\delta_1} \dots T_{\delta_n}(\gamma_1) + i(y).$$

Hence,

$$\gamma_2 = \gamma_1 + i(y + x).$$

We apply the holonomy morphism to obtain

$$\mathrm{Hol}_A(\gamma_2) - \mathrm{Hol}_A(\gamma_1) = y + x.$$

So

$$\gamma_2 - \mathrm{Hol}_A(\gamma_2) = \gamma_1 - \mathrm{Hol}_A(\gamma_1).$$

$\square$

The following is the analog of Lemma 2.2.18.

Lemma 2.5.21. *Let γ be the oriented boundary of an embedded surface S_1. Then there is $x \in \mathbb{R}$ in $\Omega^{imm}_{cob}(S_g)$ such that*

$$[\gamma] = \chi(S_1) \cdot T + i(x).$$

Proof. As in the proof of 2.2.18, we choose γ_1 and γ_2 such as in Figure 7. Now, let us call c (resp. $\bar{c}$) the curve given by the successive surgeries on the left (resp. right) of Figure 7. There is a homeomorphism $h : S_g \to S_g$, isotopic to the identity, such that $h(\bar{c}) = c$.

In particular, there are curves $\tilde{\gamma}_1^{-1}$ and $\tilde{\gamma}^{-1}$ respectively isotopic to γ_1^{-1} and γ^{-1} such that the successive surgeries on the left of Figure 7 produce the curve c.

Composing these cobordisms, we obtain an immersed cobordism

$$V : \left(\gamma_2^{-1}, \alpha, \beta, \gamma_1^{-1}, \alpha^{-1}, \gamma^{-1}\right) \rightsquigarrow \emptyset.$$

By Proposition 2.5.15, there is a immersed unobstructed cobordism between these curves. Hence,

$$-\gamma_2 + \alpha + \beta - \tilde{\gamma}_1 - \tilde{\alpha} + \tilde{\gamma} = 0.$$

The curves $\tilde{\gamma}_1^{-1}$, $\tilde{\gamma}^{-1}$ and $\tilde{\alpha}$ are embedded and isotopic to $\gamma_1^{-1}, \gamma^{-1}$ and α respectively. So there is $x \in \mathbb{R}$ such that

$$-\tilde{\gamma}_1 - \tilde{\alpha} + \tilde{\gamma} = i(x) + -\alpha + \gamma.$$

So there is $y \in \mathbb{R}$ such that

$$\gamma_1 + \gamma_2 + \gamma = T + i(y).$$

Now the proof follows by induction on the genus of the surface bounded by γ. $\square$

Lemma 2.5.22. *The class $T \in \Omega_{cob}^{imm,unob}(S_g)$ defined in equation 2.18 is of order $\chi(S_g)$.*

Proof. As in the proof of Lemma 2.2.19, consider a curve γ which is the oriented boundary of a torus. By the definition of T, there is $x \in \mathbb{R}$ such that $\gamma = T + i(x)$. On the other hand, by lemma 2.5.21, there is $y \in \mathbb{R}$ such that $\gamma^{-1} = (-3+2g) \cdot T + i(y)$.

We conclude that $\chi(S_g)\gamma = i(y-x)$. Now, apply the holonomy morphism to both sides of this equation to obtain $y - x = 0$. $\square$

Lemma 2.5.23. *Recall that in our notations, $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ and $\gamma_1, \ldots, \gamma_{g-1}$ are the Lickorish generators represented in Figure 4.*

Let $i \in \{1, \ldots, g-1\}$. Then, there is $x \in \mathbb{R}$ such that

$$[\gamma_i] = [\alpha_{i+1}] - [\alpha_i] - T + i(x).$$

Proof. This follows from the sequence of surgeries in Figure 8 and from Proposition 2.5.15. $\square$

Proof of Theorem 2.5.1. It only remains to see that

$$\mathrm{Ker}(\pi \oplus \mu) \subset \mathrm{Im}(i).$$

To see this, let γ be a non-separating curve, there is a product of Dehn Twists about $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$, $\gamma_1, \ldots, \gamma_{g-1}$ which maps γ to a curve isotopic to α_1. Therefore, γ belongs to the subgroup generated by the image of i and the Lickorish generators. By Lemma 2.5.23, this is the subgroup generated by $\alpha_1, \ldots, \beta_g$ and the image of i.

Hence, by Lemma 2.5.21, the group $\Omega_{\mathrm{cob}}^{\mathrm{imm,unob}}(S_g)$ is generated by T, $\alpha_1, \ldots, \beta_g$ and the image of i.

Let $g = \sum_i n_i\alpha_i + \sum_j m_j\beta_j + i(x) + kT$ be a element of $\mathrm{Ker}(\pi \oplus \mu)$. Composing this by μ, we get $k = 0 \mod \chi(S_g)$. Moreover, taking homology classes, the n_i and m_j are zero. Hence, g is in the image of i.

Moreover, the holonomy map

$$\mathrm{Hol}_A : \Omega_{\mathrm{cob}}^{\mathrm{imm,unob}}(S_g) \to \mathbb{R},$$

is a section of the map $i : \mathbb{R} \to \Omega_{\mathrm{cob}}^{\mathrm{imm,unob}}(S_g)$. So the exact sequence is split. $\square$

Proof of Theorem 2.1.8. Recall from Corollary 2.4.4 that there is a natural group morphism

$$\Theta_{BC} : \Omega^{\text{imm,unob}}_{\text{cob}}(S_g) \to K_0(\text{DFuk}(S_g))$$

which maps an embedded curve $\gamma : S^1 \hookrightarrow S_g$ to its image in $K_0(\text{DFuk}(S_g))$.

Moreover, in [**Abo08**], Abouzaid shows that the Maslov index and homology class induce well-defined map

$$\pi : K_0(\text{DFuk}(S_g)) \to H_1(S_g, \mathbb{Z}),\ \mu : K_0(\text{DFuk}(S_g)) \to \mathbb{Z}/\chi(S_g)\mathbb{Z}.$$

Therefore, there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r, "i"] & \Omega^{\text{imm,unob}}_{\text{cob}}(S_g) \arrow[r, "\pi\oplus\mu"] \arrow[d, "\Theta_{BC}"] & H_1(S_g, \mathbb{Z}) \oplus \mathbb{Z}/\chi(S_g)\mathbb{Z} \arrow[r] \arrow[d, "\text{Id}"] & 0 \\ & & K_0(\text{DFuk}(S_g)) \arrow[r, "\pi\oplus\mu"] & H_1(S_g, \mathbb{Z}) \oplus \mathbb{Z}/\chi(S_g)\mathbb{Z} & \end{tikzcd}.$$

Let us check that the map Θ_{BC} is injective. Let $x \in \Omega^{\text{imm,unob}}_{\text{cob}}(S_g)$ such that $\Theta_{BC}(x) = 0$. Since the right square in the above diagram is commutative, we have $\pi \oplus \mu(x) = 0$. Since, by Theorem 2.5.1, the first row is exact, we have $x \in \text{Im}(i)$. Moreover, since $\Theta_{BC}(x) = 0$ and since the holonomy map factors through $K_0(\text{DFuk}(S_g))$ (cf Proposition 2.5.2), $\text{Hol}_A(x) = 0$. Since the morphism i is injective (cf Corollary 2.5.5), $x = 0$.

Moreover, by [**Abo08**], the group $K_0(\text{DFuk}(S_g))$ is generated by embedded curves. Therefore, the map Θ_{BC} is surjective. $\square$

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