

2006-21

## Self-Selective Social Choice Functions

*SLINKO, Arkadii*  
*KORAY, Semih*

**Département de sciences économiques**

Université de Montréal

Faculté des arts et des sciences

C.P. 6128, succursale Centre-Ville

Montréal (Québec) H3C 3J7

Canada

<http://www.sceco.umontreal.ca>

[SCECO-information@UMontreal.CA](mailto:SCECO-information@UMontreal.CA)

Téléphone : (514) 343-6539

Télécopieur : (514) 343-7221

Ce cahier a également été publié par le Centre interuniversitaire de recherche en économie quantitative (CIREQ) sous le numéro 18-2006.

*This working paper was also published by the Center for Interuniversity Research in Quantitative Economics (CIREQ), under number 18-2006.*

ISSN 0709-9231

# Self-Selective Social Choice Functions

Semih Koray, Arkadii Slinko\*

November 7, 2006

Prof. Semih Koray  
Department of Economics  
Bilkent University  
06533 Bilkent Ankara  
TURKEY  
k.semih@bilkent.edu.tr

Dr. Arkadii Slinko  
Department of Mathematics  
University of Auckland  
Private Bag 92019 Auckland  
NEW ZEALAND  
a.slinko@auckland.ac.nz  
and  
CIREQ

**Abstract:** It is not uncommon that a society facing a choice problem has also to choose the choice rule itself. In such situation voters' preferences on alternatives induce preferences over the voting rules. Such a setting immediately gives rise to a natural question concerning consistency between these two levels of choice. If a choice rule employed to resolve the society's original choice problem does not choose itself when it is also used in choosing the choice rule, then this phenomenon can be regarded as inconsistency of this choice rule as it rejects itself according to its own rationale.

Koray (2000) proved that the only neutral, unanimous universally self-selective social choice functions are the dictatorial ones. Here we introduce to our society a constitution, which rules out inefficient social choice rules. When inefficient social choice rules become unavailable for comparison, the property of self-selectivity becomes weaker and we show that some non-trivial self-selective social choice functions do exist. Under certain assumptions on the constitution we describe all of them.

**Key words:** social choice function, social choice correspondence, self-selectivity, resistance to cloning

**JEL Classification:** D7

---

\*This project was initiated in 1999 during a visit of Arkadii Slinko to Boğaziçi University of Istanbul. A support from TÜBİTAK NATO PC-B Programme for that visit is gratefully acknowledged. Both authors are grateful to Murat Sertel for helpful discussions and encouragement. Arkadii Slinko is grateful to CIREQ and the Département de sciences économiques, Université de Montréal, for their hospitality and financial support.

# 1 Introduction

It is not uncommon that a society facing a choice problem has also to choose the choice rule itself. Such a setting immediately gives rise to a natural question concerning consistency between these two levels of choice. If a choice rule employed to resolve the society's original choice problem does not choose itself when it is also used in choosing the choice rule, then this phenomenon can be regarded as inconsistency of this choice rule as it rejects itself according to its own rationale.

This idea of self-selectivity for social choice functions was first analyzed by Koray (2000). Barberà and Beviá (2002) and Barberà and Jackson (2004) also consider it but from a different perspective. Jackson (2001) in his survey "A crash course on implementation theory" underlined the importance of the idea.

The difficulty of defining such a concept lies in the necessity to construct a profile on the set of available social choice functions starting from the profile on the existing alternatives. Koray (2000) resolved this difficulty by a clever use of duality which will be described below.

Let  $A$  stand for the set of alternatives, from which the society will be eventually choosing, and let  $\mathcal{A}$  stand for the finite nonempty set of social choice functions (SCFs) available to this society at the moment of choice, then Koray showed that the society's preference profile  $R$  on  $A$  will induce a "dual" preference profile  $R^{\mathcal{A}}$  on  $\mathcal{A}$ . According to him, it is natural to expect that the agents will rank the SCFs in  $\mathcal{A}$  in accordance with what they will choose from  $A$ . If each agent's preferences on  $A$  are represented by a linear order, then the dual preferences on  $\mathcal{A}$  will be complete preorders since an agent will be indifferent between two SCFs choosing the same alternative from  $A$ . This framework now allows to apply the consistency test introduced above. If an SCF in  $\mathcal{A}$  passes this test, i.e. if it selects itself from  $\mathcal{A}$  at the dual preference profile  $R^{\mathcal{A}}$ , then it is called self-selective at the preference profile  $R$  on  $A$  relative to  $\mathcal{A}$ . Moreover, an SCF  $F$  is said to be universally self-selective if it is self-selective at each preference profile on any finite nonempty set  $A$  relative to any set finite set  $\mathcal{A}$  of available SCFs containing  $F$ . Koray (2000) confined itself to neutral SCFs only, so that it was only the size of the alternative set  $A$  that mattered rather than the names of the alternatives in  $A$ . The main result in Koray (2000) is the impossibility theorem stating that a unanimous and neutral SCF is universally self-selective if and only if it is dictatorial. Koray and Unel (2003) showed also that impossibility still survives in the tops-only domain. Allowing social choice rules to be multi-valued also does not lead to any interesting examples and ends up with a rediscovery of the Condorcet rule as the maximal neutral and self-selective social choice rule (Koray, 1998).

These theorems showed that the concept of self-selectivity was made too strong to be useful. In particular, according to the definition of self-selectivity given above, a self-selective rule must select itself even when grouped together with most ridiculous rules which no society will ever contemplate using. Moreover some voting rules are unavailable to the society on legitimacy grounds. Also, it would be very difficult to argue against the decision of a society to rule out the usage of inefficient social choice rules.

Since the use of inefficient rules was essential for the proof of Koray's impossibility

theorem it has become gradually clear that the sets  $\mathcal{A}$  cannot be arbitrary. In the present paper we make an initial attempt to pursue this idea. The main model example that we have in mind throughout the paper restricts rival SCFs to singleton-valued refinements of the Pareto correspondence. This special case is of course important in itself, however we explore self-selectivity in a much broader framework. We introduce a rather large family of suitable restrictions that yield an interesting class of non-dictatorial self-selective SCFs (which are not universally self-selective, of course, as the self-selectivity test is not universal any more). In particular, we show that non-dictatorial self-selectivity is achievable under efficiency.

Each restriction of rival SCFs against which self-selectivity is to be tested in the present study corresponds to a particular set of norms on the part of the society. We start with a social choice correspondence  $\pi$  and confine our test functions to singleton-valued refinements of  $\pi$ . Thus  $\pi$  is to be thought of as a constitutional rule reflecting the norms that the society wishes to adhere to. We assume that the correspondence  $\pi$  is neutral, tops-inclusive and hereditary. These properties that our constitutional correspondence is required to possess are all consistent with our conception of social desirability as will be seen later in the paper.

Moreover, the family of restrictions of test functions via constitutional correspondences is sufficiently wide to also include the unrestricted domain as well as the tops-only domain as its special cases. Thus, we obtain the main results of Koray (2000) and Koray and Unel (2003) as corollaries to our main result, hence also providing alternative proofs to those results.

Both Koray (2000) and Koray and Unel (2003) dealt exclusively with neutral SCFs. Here we show that neutrality is not crucial for self-selectivity results. The notion of self-selectivity can be extended to the non-neutral case in an easy and natural manner. For the simplicity of exposition, however, we delegate the non-neutral case to Section 6.

An alternative approach to the “choosing how to choose” problem is pursued by Houy (2003,2006). He assumes that individuals do not pay attention to immediate consequences of the choice but form their preferences on the basis of the intrinsic values of the rules alone: for example some voters might have ethical objections to dictatorship despite the benefit that it can bring to them personally. This is an important point and in the future a combined approach might appear.

## 2 Basic Notions and Examples

Let  $\mathcal{N}$  stand for a finite nonempty society of cardinality  $n$  which will be fixed throughout the paper. For each finite nonempty set  $A$ , we denote the set of all linear orders on  $A$  by  $\mathcal{L}(A)$ . Any  $n$ -tuple  $R = (R_1, \dots, R_n)$  of linear orders  $R_i$  will be called a *profile*, and the set of all profiles will be denoted by  $\mathcal{L}(A)^n$ . Denoting, as usual, the set of all positive integers by  $\mathbb{N}$ , we set  $I_m = \{1, 2, \dots, m\}$  for each  $m \in \mathbb{N}$ . We call a mapping

$$F: \bigcup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n \rightarrow 2^{\mathbb{N}}$$

a *social choice rule* (SCR), if and only if, for each  $m \in \mathbb{N}$  and  $R \in \mathcal{L}(I_m)^n$ , one has  $F(R) \subseteq I_m$ . If  $F(R)$  is a singleton for each  $m$  and  $R$ , we refer to the SCR  $F$  as a *social choice function* (SCF) and write  $F(R) = a$  instead of  $F(R) = \{a\}$ . An SCR which is not an SCF is called a *social choice correspondence* (SCC).

Before proceeding any further, let us note two aspects in which our definition of an SCR differs from the standard one. Firstly, unlike the more common framework, when the set of alternatives is fixed but the set of voters can vary, we have a fixed set of voters and variable set of alternatives. This reflects the fact that we study how a society chooses a voting rule. During this process the society is fixed but the exact set of alternatives at this stage is unknown and cannot be known because the voting rule must be applicable to all voting situations that might emerge in the future. Thus we consider a sequence of finite sets of alternatives  $I_1, I_2, \dots, I_k, \dots$ , rather than a single fixed one. When the choice problem with  $m$  alternatives is defined, the component of the rule that maps  $\mathcal{L}(I_m)^n$  into  $2^{I_m}$  is used. Most common SCFs can be used for sets of alternatives of variable sizes. Secondly, the common domain of our SCRs consists of profiles on representative sets  $I_m$ , one for each cardinality  $m \in \mathbb{N}$ , rather than on arbitrary finite sets. In the case of a neutral SCR, this is nothing but a more compact way of describing how the SCR acts on the profiles composed of linear orders on an arbitrary finite set  $A$ . In the first four sections we will restrict ourselves to considering only neutral SCRs. In the last section we show how the case of non-neutral SCRs can be handled. Finally, the last but not the least, the concept of self-selectivity can be defined only for a SCF which is capable of choosing an alternative from sets of alternatives of different sizes. Below are several examples of SCFs and SCCs that satisfy our criteria.

**Example 1.** *Dictatorial and anti-dictatorial SCFs play an important role. They are defined as follows. For a given profile  $R$ ,*

$$\begin{aligned} D_i(R) &= \max R_i, \\ AD_i(R) &= \min R_i, \end{aligned}$$

*$i = 1, 2, \dots, n$ . An important SCC, which will later be defined as  $P$ , is defined as follows: for any profile  $R$  the set  $P(R)$  consists of all Pareto optimal alternatives. Later we will generalize this example.*

**Example 2.** *Let  $R$  be a profile. By  $P(R)$  we denote the set of all Pareto optimal alternatives and by  $T(R)$  we the set of all alternatives which are top ranked by at least one agent.*

These two SCCs, denoted  $P$  and  $T$ , will be important later. Obviously  $T(R) \subseteq P(R)$  for every profile  $R$ .

Let us recap what does it mean for SCR to be neutral. For each  $m \in \mathbb{N}$ , let  $S_m$  stand for the symmetric group of all permutations on  $I_m$ . Given  $R = (R_1, \dots, R_n) \in \mathcal{L}(I_m)^n$  and  $\sigma \in S_m$ , we define a new profile  $R^\sigma = (R_1^\sigma, \dots, R_n^\sigma)$  such that  $k R_i^\sigma \ell$  if and only if

$\sigma^{-1}(k) R_i \sigma^{-1}(\ell)$ , where  $i \in N$  and  $k, \ell \in I_m$ . An SCR  $F$  is said to be *neutral at a profile*  $R$  if, for any  $m \in \mathbb{N}$ ,  $R \in \mathcal{L}(I_m)^n$  and  $\sigma \in S_m$

$$\sigma(F(R^\sigma)) = F(R). \quad (1)$$

An SCR  $F$  is said to be *neutral* if it is neutral at any profile.

In the definition of a social choice rule it was convenient to use a generic set of alternatives  $I_m$ . However, in practice we may have to deal with various sets of alternatives, thus we have to show how to use a SCR  $F$  to select from an arbitrary finite set of alternatives  $A$  given a preference profile on  $A$ . The natural way of doing this is, of course, by indexing the elements of  $A$  using the initial segment  $I_m$  of  $\mathbb{N}$  with  $m = |A|$  and then paying attention to indices only. This indexation is given by any bijection  $\mu: A \rightarrow I_m$  and in practice it corresponds, to assigning to each candidate their order on a ballot. Given this bijection, any profile  $Q = (Q_1, \dots, Q_n) \in \mathcal{L}(A)^n$  will induce a profile  $Q^\mu = (Q_1^\mu, \dots, Q_n^\mu) \in \mathcal{L}(I_m)^n$  such that, for any  $i \in N$  and  $k, \ell \in I_m$

$$k Q_i^\mu \ell \iff \mu^{-1}(k) Q_i \mu^{-1}(\ell). \quad (2)$$

We may now define

$$F^\mu(Q) = \mu^{-1}(F(Q^\mu)). \quad (3)$$

If  $F$  is neutral, then it is straightforward to see that  $F^\mu = F^\nu$  for any two bijections from  $A$  to  $I_m$ . This means that  $F$  treats all candidates equally, regardless of their position on the ballot. Thus in the neutral case we may assume that  $F$  is defined on any finite set of alternatives  $A$ . It will cause no confusion to write  $F$  instead of  $F^\mu$ , when  $F$  is neutral. If we abandon the neutrality assumption, then such a transfer of an SCF  $F$  is no longer uniquely determined,  $F^\mu$  will depend on  $\mu$ . Hence in this case the set of alternatives must be indexed. Our main results concerning self-selective SCRs, with appropriate definition of self-selectivity, will still hold in the non-neutral case but for the clarity of exposition and convenience of the reader, we delegate this case to Section 5.

In the sequel we will use the concept of isomorphism for profiles which we give in the following definition.

**Definition 1.** *Let  $A$  and  $B$  be two equinumerous sets of alternatives,  $R = (R_1, \dots, R_n)$  and  $Q = (Q_1, \dots, Q_n)$  be profiles on  $A$  and  $B$ , respectively. Then  $R$  and  $Q$  are called *isomorphic* if there is a bijection  $\sigma: A \rightarrow B$  such that a  $R_i a'$  if and only if  $\sigma(a) Q_i \sigma(a')$ .*

The following proposition can now be proved as an easy exercise.

**Proposition 1.** *Let  $A$  and  $B$  be two equinumerous sets of alternatives,  $R = (R_1, \dots, R_n)$  and  $Q = (Q_1, \dots, Q_n)$  be two isomorphic profiles on  $A$  and  $B$ , respectively with  $\sigma: A \rightarrow B$  be the corresponding bijection. Then for any neutral SCF  $F$  we have  $\sigma(F(R)) = F(Q)$ .*

Now suppose that the society  $\mathcal{N}$ , endowed with a preference profile on an  $m$ -element set of alternatives  $A$ , from which the choice is to be made, is also to choose an SCF that will be employed to make its choice from  $A$ . Suppose that a nonempty finite set  $\mathcal{A}$  of SCFs

is available to  $\mathcal{N}$  for this purpose. We assume that the agents in  $\mathcal{N}$  are only interested in the outcomes that the SCFs from  $\mathcal{A}$  will produce and thus rank SCFs accordingly. Therefore any agent  $i \in \mathcal{N}$  will also have a preference relation  $R_i^A$  on  $\mathcal{A}$  such that for any  $F, G \in \mathcal{A}$

$$F R_i^A G \iff F(R) R_i G(R). \quad (4)$$

This preference relation  $R_i^A$  will be a complete preorder and may not be, in general, antisymmetric. Indeed, two different SCFs  $F, G \in \mathcal{A}$  may well choose the same alternative  $a \in A$ , in which case the  $i$ th agent will be indifferent between  $F$  and  $G$ . By breaking ties and introducing linear orders on indifference classes we may obtain a number of linear orders compatible with  $R_i^A$ . When we do it for all  $i \in \mathcal{N}$ , we obtain a profile from  $\mathcal{L}(\mathcal{A})^n$ . Any profile, so obtained, will be called a *profile dual to  $R$*  on the set of SCFs  $\mathcal{A}$ . Let us denote the set of all such profiles as  $\mathcal{L}(\mathcal{A}, R)$ .

We have now apparatus to formalize the concept of self-selectivity. If  $\mathcal{A}$  is a finite set of SCFs, then we say that  $F$  is *self-selective at a profile  $R$  relative to  $\mathcal{A}$*  if and only if there exists a dual profile  $R^* \in \mathcal{L}(\mathcal{A} \cup \{F\}, R)$  such that  $F(R^*) = F$ . We say that  $F$  is *self-selective at a profile  $R$*  if it is self-selective at  $R$  relative to every finite set of SCFs  $\mathcal{A}$ . Finally  $F$  is said to be *universally self-selective* if and only if  $F$  is self-selective at each profile  $R \in \mathcal{L}(\mathcal{A})$ .

It may be worthwhile to emphasize that in the definition of self-selectivity of  $F$  we only require that  $F$  chooses itself at just one (not all) dual profile. A natural question arises, what will happen if we require that  $F$  selects itself at all dual profiles. It is not difficult to see that this leads to a vacuous concept. Indeed, if we compare  $F$  with SCFs  $F_1, \dots, F_n$ , which at some profile (unanimous, for example), all select the same winner, then the set of dual profiles will consist of all possible profiles and  $F$  selects itself at all of them if and only if it is constant.

Another important thing to note that we are talking about set and not multisets here. This means in particular that we just cannot repeat  $F$  or any other SCF in  $\mathcal{A}$  several times. The importance of this will become clear in Section 3.

In other words, universal self-selectivity requires that  $F$  passes the self-selectivity test at each preference profile and against any finite set of test functions. From Koray (2000) we know that the only unanimous neutral universally self selective SCFs are the dictatorial ones. There are two kinds of natural restrictions that one can resort to in an attempt to avoid this impossibility result. The first one is to restrict the domain of preference profiles at which self-selectivity is required. The second is to restrict the class of SCFs against which the self-selectivity is to be tested. In this study we will be interested in the latter approach. This interest does not only stem from our intention to escape from impossibility results but we also believe that this approach is actually consistent with the realities of a modern society.

Indeed, every society has certain normative criteria according to which the notion of social acceptability is reflected at the constitutional level. This is naturally confines the set of SCFs that may be used by that society, from the very outset, to a certain subclass of all SCFs, ruling out all other SCFs as socially unacceptable. We would find it very difficult on our part to argue, for example, against the decision of a society to adopt Pareto



efficiency as a constitutional principle and thus restrict itself to using efficient SCFs only. In this case the set of acceptable SCFs would consist of all singleton-valued refinements of the Pareto correspondence. For such a society it will be natural to test self-selectivity of an SCF against Paretian SCFs only.

Let  $\mathcal{F}$  be a nonempty set of neutral SCFs which will be used to denote the set of test functions for self-selectivity. We say that  $F$  is  $\mathcal{F}$ -self-selective at a profile  $R \in \mathcal{L}(A)^n$  if and only if  $F$  is self-selective at  $R$  relative to  $\mathcal{A} \cup \{F\}$  for any finite subset  $\mathcal{A}$  of  $\mathcal{F}$ . We say that  $F$  is  $\mathcal{F}$ -self-selective if it is  $\mathcal{F}$ -self-selective at any profile  $R$ . We illustrate the concept with the following three examples.

**Example 3.** Let  $Q = (Q_1, \dots, Q_{19})$  be the following profile:

$Q_1 - Q_4$	$Q_5 - Q_8$	$Q_9 - Q_{13}$	$Q_{14} - Q_{19}$
$a$	$b$	$c$	$d$
$c$	$a$	$a$	$b$
$b$	$c$	$b$	$a$
$d$	$d$	$d$	$c$

Let  $B$  be the Borda rule,  $C$  be any Condorcet consistent rule,  $E$  be the Plurality rule, and  $R$  be the Runn-off rule. Then  $B(Q) = a$ ,  $C(Q) = b$ ,  $R(Q) = c$ ,  $E(Q) = d$ . The same voters will rank the rules in the dual profile  $Q^*$  as follows:

$Q_1^* - Q_4^*$	$Q_5^* - Q_8^*$	$Q_9^* - Q_{13}^*$	$Q_{14}^* - Q_{19}^*$
$B$	$C$	$R$	$E$
$R$	$B$	$B$	$C$
$C$	$R$	$C$	$B$
$E$	$E$	$E$	$R$

We see that  $B(Q^*) = B$ ,  $C(Q^*) = C$ ,  $R(Q^*) = R$ ,  $P(Q^*) = P$ . Each rule is self-selective at  $Q$ .

**Example 4.** Any dictatorial or anti-dictatorial SCF is universally self-selective.

Finally, we give an example when the Borda rule is not self-selective.

**Example 5.** Let  $\mathcal{D} = \{D_1, \dots, D_n\}$  be the class of all dictatorial SCFs. Then it is easy to see that the Borda count is not  $\mathcal{D}$ -self-selective. To illustrate this let us denote the Borda count as  $B$  and check that  $B$  does not choose itself from  $\{B, D_1\}$  at the profile

$R_1$	$R_2$	$R_3$
$a$	$a$	$e$
$b$	$b$	$b$
$c$	$c$	$c$
$d$	$d$	$d$
$e$	$e$	$a$

Here  $D_1$  chooses  $a$  and the Borda count chooses  $b$ . The unique dual profile on  $\{B, D_1\}$  will be

$$\begin{array}{ccc} R_1^* & R_2^* & R_3^* \\ \hline D_1 & D_1 & B \\ B & B & D_1 \end{array}$$

where Borda will choose  $D_1$ .

In this study the notion of social acceptability at the constitutional level will be represented via a neutral SCC  $\pi$ . Once the society chooses such a constitutional correspondence, the set  $\mathcal{F}$  of its admissible SCFs will be restricted to singleton-valued refinements of  $\pi$ . We will refer to such SCFs as to *selections* of  $\pi$ . We also wish to secure that  $\mathcal{F}$  fully reflects  $\pi$  in the sense that there is no smaller constitution that  $\mathcal{F}$  is consistent with. Formally, we require that for every profile  $R$

$$\bigcup_{F \in \mathcal{F}} F(R) = \pi(R) \quad (5)$$

at each profile  $R$ , in which case we say that  $\mathcal{F}$  is  $\pi$ -complete.

To illustrate this concept with the following example.

**Example 6.** Let  $\mathcal{D} = \{D_1, \dots, D_n\}$  be the class of all dictatorial SCFs. Then  $\mathcal{D}$  is  $T$ -complete.  $\mathcal{D}$  will not be  $P$ -complete though since it is possible that a Pareto-optimal alternative is not anybody's first preference.

Now let us turn our attention to the properties that the correspondence  $\pi$  is expected to possess. This correspondence should be both sufficiently restrictive and sufficiently flexible. It is to be restrictive to reflect certain normative criteria. If  $\pi$  is the universal correspondence, which we denote  $\Omega$ , i.e.  $\pi(R) = A$  for every set of alternatives  $A$  and every profile  $R \in \mathcal{L}(A)^n$ , then  $\pi$  is vacuous from the normative viewpoint. On the other hand,  $\pi$  should be sufficiently flexible as it is meant to be a rule at the constitutional level. If  $\pi$  itself is always singleton-valued, then  $\pi$  itself would be the only admissible SCF available to the society to resolve any choice problem whatsoever. Moreover, if a constitution is to respect preferences of individuals, then it does not seem too-far-fetched to require that there should be no agent whose best outcome is constitutionally ruled out at some preference profile. This means that  $\pi(R)$  should contain all top-ranked outcomes at any profile  $R$ . This leads to the requirement that  $\pi$  must be *tops-inclusive* i.e., the inclusion  $T(R) \subseteq \pi(R)$  must hold. We also require one additional condition which will be discussed in Section 4.

We will also require that our constitutional correspondence behave consistently under restrictions of preference profiles to subsets of alternatives chosen by it. More specifically, we will say that an SCC  $\pi$  is *hereditary* if and only if for every profile  $R$  and every nonempty subset  $\emptyset \neq X \subseteq \pi(R)$  there holds  $\pi(R|_X) = X$ , where  $R|_X$  is the restriction of the profile  $R$  to the set of alternatives  $X$ . This requirement is very natural. Indeed, if an alternative was eligible for choice for the society at an early stages of selection, and then other alternatives were eliminated, then that particular alternative should remain eligible for choice.

In the sequel, we assume that our constitutional correspondence  $\pi$  is tops-inclusive and hereditary. We note that our “role models”  $P$ ,  $T$ , and  $\Omega$  all satisfy these requirements. The collection  $\mathcal{F}$  of all admissible under  $\pi$  social choice test-functions will be always assumed  $\pi$ -complete.

We note that, when  $\pi = \Omega$  and  $\mathcal{F}$  is the set of all selections of  $\Omega$  we obtain the framework studied by Koray (2000) and his main result as a corollary. Similarly taking  $\pi = T$  and  $\mathcal{F}$  to be the set of all selections of  $T$ , we obtain the framework of the paper by Koray and Unel (2003) and their main result as a corollary too.

Let us also define some more SCCs which will play some role in the rest of the paper. Firstly, we remind to the reader that the upper contour set  $U(a, L)$  of an alternative  $a$  relative to a linear order  $L$  is defined as  $U(a, L) = \{x \in A \mid x L a\}$ .

Let  $q \geq 1$  be a positive integer and  $R = (R_1, \dots, R_n) \in \mathcal{L}(A)^n$  be a profile. An alternative  $a \in A$  is said to be  $q$ -Pareto optimal if

$$\text{card} \left( \bigcap_{i=1}^n U(a, R_i) \right) \leq q.$$

In particular, for  $q = 1$  we note that 1-Pareto optimal elements are the classical Pareto optimal ones, i.e.,  $P_1(R) = P(R)$ . Let  $P_q(R)$  be the set of all  $q$ -Pareto optimal elements of  $R$ . An alternative  $a \in A$  is said to get at least one  $q$ th degree approval if

$$\min_{i=1}^n \text{card} (U(a, R_i)) \leq q.$$

Let  $T_q(R)$  be the set of all alternatives which got at least one  $q$ th degree approval. In particular,  $T_1(R)$  is the set of elements who ranked first by at least one agent, thus  $T_1(R) = T(R)$ . Obviously,

$$T_1(R) \subseteq T_2(R) \subseteq \dots \subseteq T_k(R) \subseteq \dots$$

We note also that  $T_q(R) \subseteq P_q(R)$ .

### 3 Self-selectivity and resistance to cloning

Here we will show that self-selectivity is closely related with two other properties of SCFs that often appear in the literature: resistance to cloning alternatives and Arrow’s choice axiom. The resistance to cloning alternatives is one of the many forms of manipulation that exists [12, 14, 16]. For example, producing a clone of a leading candidate in the race splits her vote and may allow the second candidate in the race to win the election. We treat cloning in generalised terms. In particular, withdrawal of a candidate from the race may also change the outcome of the election and this move can also be manipulative (see, e.g. [13]). We treat withdrawals as a particular type of cloning when an alternative is replaced with zero clones.

Let us describe the cloning procedure formally. Let  $R$  be a profile on a set of alternatives  $A = \{a_1, \dots, a_k\}$ . For each  $1 \leq i \leq k$  we introduce the set of alternatives

$A'_i$  which is either empty or  $A'_i = \{a_{i1}, a_{i2}, \dots, a_{ik_i}\}$  with  $k_i \geq 1$  and  $a_i = a_{i1}$ . We set  $A' = A'_1 \cup \dots \cup A'_k$ . In the profile  $R$  we drop all alternatives  $a_i$ , for which  $A'_i = \emptyset$  and replace each alternative  $a_i$ , for which  $A'_i$  is non-empty, with a linear order on  $A'_i$  (not necessarily the same for different occurrences of  $a_i$ ) and this gives us a profile  $R'$  on the set of alternatives  $A'$  which we will call a cloned profile. We emphasise the following two features of any cloned profile: in each linear order of  $R'$  all clones of the same alternative are standing “together” but the order on these clones may be different from one linear order of  $R'$  to another. Another important thing to note is that a subset of  $A$  is contained in  $A'$ ; this is the set of alternatives which have not been “withdrawn”. The possibility to withdraw an alternative is absent in the definitions of cloning used in [12, 14, 16].

**Definition 2.** *Let  $R$  be a profile on a set of alternatives  $A = \{a_1, \dots, a_k\}$  and  $C$  be an SCC. We say that  $C$  is resistant to cloning of essential alternatives at a profile  $R$  if for any cloned set of alternatives  $A' = A'_1 \cup \dots \cup A'_k$ , where  $a_i \notin C(R)$  implies  $A'_i = \emptyset$ , there exists a cloned profile  $R'$  on  $A'$  at which*

$$C(R) \cap A' = C(R'). \quad (6)$$

*We say that an SCC  $C$  is resistant to cloning of essential alternatives if it is resistant to cloning of essential alternatives at any profile  $R$ .*

As in the case of self-selectivity, it is important to note that we require the existence of just one cloned profile  $R'$  with the property (6). Asking for all profiles to satisfy this condition makes the concept vacuous again.

It may be worthwhile to note that (6) represents a weak version of Arrow’s choice axiom [2] which he proved to be equivalent to a rationalisability of the SCC  $C$  by a social welfare function.

**Example 7.** *Already mentioned SCCs  $T$ ,  $P$  and  $\Omega$  are resistant to cloning of essential alternatives.*

*Proof.* Suppose that  $a \in T(R)$ . Then  $a = \max R_i$  for some  $1 \leq i \leq n$ . Let  $a = a_j$ . Suppose  $A'_j$  is not empty and therefore includes  $a$ . We order all elements of  $A'_j$  so that  $a = a_j = a_{j1}$  is on the top of  $A'_j$ . Let  $R'$  be any cloned profile where this order on  $A'_j$  is chosen. Then  $a$  will be on the top of  $R'_i$ , hence  $a \in T(R')$ . On the other hand no other element of  $A'_j$  will be on the top of any  $R'_j$  since it  $a$  majorises it in every  $R'_j$ . Thus (6) is satisfied for  $T$ .

The proof for  $P$  is similar and the resistant to cloning of essential alternatives for  $\Omega$  is obvious.  $\square$

**Proposition 2.** *Any SCC which is resistant to cloning of essential alternatives is hereditary.*

*Proof.* Let  $C$  be a clone resistant SCC,  $R$  be a profile on  $A = \{a_1, \dots, a_k\}$ , and let  $X \subseteq C(R)$ . Suppose without loss of generality that  $X = \{a_1, \dots, a_q\}$  for  $q \leq k$ . Then the restriction  $R|_X$  of  $R$  onto the set of alternatives  $X$  is a cloned profile on  $A'$ , where  $|A'_1| = \dots = |A'_q| = 1$  and  $|A'_{q+1}| = \dots = |A'_k| = 0$ , and the proposition follows.  $\square$

An important link between self-selectivity and resistant to cloning of essential alternatives is presented in the following theorem which will give us a non-trivial example of self-selective SCFs.

**Theorem 1.** *Let  $\pi$  be any neutral SCC which is resistant to cloning of essential alternatives and  $\mathcal{F}$  be any class of SCFs, each of which is a selection of  $\pi$ . Then for each  $1 \leq i \leq n$  the two SCF given by*

$$F(R) = \min R_i |_{\pi(R)}, \quad G(R) = \max R_i |_{\pi(R)} \quad (7)$$

are  $\mathcal{F}$ -self-selective.

*Proof.* We will prove the statement only for the first function. The proof for the second function is similar. Let  $R$  be a profile on a set of alternatives  $A = \{a_1, \dots, a_m\}$  and  $\mathcal{A} = \{F_1, \dots, F_k\} \subseteq \mathcal{F}$  be any finite subset of  $\mathcal{F}$  not containing  $F$ . Let us also denote  $F_0 = F$ . Suppose  $F_j(R) = a_j$ , where  $j = 0, 1, \dots, k$  and some  $a_j$ 's may coincide. Without loss of generality we may assume that  $a_0, a_1, \dots, a_p$  are distinct and that  $a_q \in \{a_0, a_1, \dots, a_p\}$  for all  $q > p$ . Let  $\mathcal{F}_j$  be the set of all SCFs from  $\{F_0\} \cup \mathcal{A}$  which select  $a_j$  for all  $j = 1, \dots, p$ . Note that  $F = F_0 \in \mathcal{F}_0$ . By their definition all  $\mathcal{F}_j$ 's for  $j = 1, \dots, p$  are non-empty, let us denote the elements of  $\mathcal{F}_j$  as  $F_{j1}, F_{j2}, \dots, F_{jk_j}$  with  $F_j = F_{j1}$ .

Let  $B = \{a_0, a_1, \dots, a_p\}$ . Since every SCF from  $\mathcal{A}$  is a selection of  $\pi$ , we note that  $B \subseteq \pi(R)$ . To construct a dual profile  $R^*$ , firstly, we have to restrict  $R$  to the set  $B$ , then to change  $a_j \in B$  into  $\mathcal{F}_j$ , treating  $\mathcal{F}_j$  as equivalence classes, and then to break ties selecting linear orders on each  $\mathcal{F}_j$  (which may be different for different linear orders of  $R^*$ ). Instead, we will consider a cloned profile isomorphic to  $R^*$ . We set  $A'_j = \{a_{j1}, a_{j2}, \dots, a_{jk_j}\}$  for  $j = 0, 1, \dots, p$  and  $A'_q = \emptyset$  for  $p < q \leq m$ . Let  $R'$  be the resulting cloned profile on  $A' = A'_1 \cup \dots \cup A'_m$ . By resistance to cloning  $B \subseteq \pi(R')$  and hence  $\{F_0\} \cup \mathcal{A} \subseteq \pi(R^*)$ .

Let  $\bar{R}$  be the restriction of  $R$  onto  $B$ . We note that by its definition  $a_0 = \min \bar{R}_i$ . As  $B \subseteq \pi(R)$ , by resistance to cloning of essential alternatives  $a_0 = a_{01}$  is the only element of  $\pi(R')$  among elements of  $A'_0 = \{a_{01}, a_{02}, \dots, a_{0k_0}\}$ . Hence in the cloned profile  $R'$  the element  $a_0$  is the worst element of  $\pi(R')$  in  $R'_i$ . Hence  $F(R') = a_0$  and  $F(R^*) = F_0 = F$ . Hence  $F$  chooses itself at  $R^*$ .  $\square$

The SCFs introduced in Theorem 1 will be called  $\pi$ -antidictatorship and  $\pi$ -dictatorship of the  $i$ th voter, respectively. A certain degree of clone resistance of  $\pi$  is necessary for this theorem to be true. We illustrate this in the following example.

**Example 8.** *Let us consider the following profile  $R = (R_1, R_2, R_3)$ :*

$R_1$	$R_2$	$R_3$
$a$	$b$	$b$
$c$	$a$	$a$
$d$	$d$	$d$
$b$	$c$	$c$

and let  $\pi = T_2$ . Then  $a = B(R)$  is the Borda winner and  $b = E(R)$  is the Plurality winner, and  $\pi(R) = \{a, b, c\}$ . Let  $D_1$  be the dictatorship of the first voter and  $F$  be the

$\pi$ -antidictatorship of the second voter. Then there are eight dual profiles (one of which  $R^*$  is shown below)

$R_1^*$	$R_2^*$	$R_3^*$
$B$	$E$	$E$
$D_1$	$D_1$	$D_1$
$F$	$B$	$B$
$E$	$F$	$F$

None of the eight dual profiles have  $F \in T_2(R^*)$ , hence  $F$  is not self-selective at  $R$  relative to  $\mathcal{F} = \{B, D_1, E\}$  or any larger set of SCFs.

We see that it is exactly the failure of clone resistance that leads to the failure of  $F$  to be self-selective. It is easy to construct examples which show that the  $\pi$ -antidictatorship is not self-selective for  $\pi = T_q$ , when  $q > 2$ , and for  $\pi = P_q$ , when  $q \geq 2$ .

## 4 The Main Theorem

In Theorem 1 we introduced the  $\pi$ -antidictatorship and  $\pi$ -dictatorship, respectively. The  $\pi$ -dictatorship is not very interesting since for any tops-inclusive SCC  $\pi$  it will give us the ordinary dictatorship. Not so with the  $\pi$ -antidictatorship. One of the most interesting SCFs of this kind is the SCF given by

$$F(R) = \min R_i |_{P(R)},$$

i.e. Pareto antidictatorship. This SCF chooses the worst Pareto optimal alternative for the  $i$ th voter. So the  $i$ th voter is a *Pareto anti-dictator*.

Unlike the standard anti-dictatorial SCFs, the  $\pi$ -antidictatorship constitutes a rather complex arrangement made by the society in such a way that the choice is always efficient, depends on the opinion of all agents, not just one, and does not give anybody an unfair advantage.

Now we discuss the condition of tops-inclusiveness in detail. We say that  $\pi$  is *tops-inclusive* if and only if the following two conditions hold:

- (i)  $T(R) \subseteq \pi(R)$  for every profile  $R$ .
- (ii) If  $\pi(R) \subseteq P(R)$  does not hold for at least one profile  $R \in \mathcal{L}(A)^n$ , then  $\pi(R) \supseteq T_2(R)$  for every profile  $R \in \mathcal{L}(A)^n$ .

We have already discussed condition (i) above. As for condition (ii), it looks like a technical condition that we need for our results to hold. However, it does have a simple meaning based on the notions of efficiency and fairness at the constitutional level. Before the discussion of its meaning, let us note that in this study we regard the Pareto correspondence as our primary “role model” for the constitutional correspondence  $\pi$ . And, as long as  $\pi$  is Paretian, condition (ii) is vacuous and may be forgotten. However, if  $\pi$  includes not only all Pareto optimal alternatives but also at least one  $a \in \pi(R)$ , which is

not Pareto optimal, then it is not top-ranked by any of the agents at  $R$ . If some agent's  $k$ th ranked alternative, with  $k \geq 2$ , is included in  $\pi(R)$  although it is not efficient, then one could argue on the grounds of fairness that no alternative that is ranked  $k$ th or higher should be excluded from  $\pi(R)$ , i.e. the inclusion  $T_k(R) \subseteq \pi(R)$  must hold. Condition (ii) is the weakest of this kind and, as long as the correspondence  $\pi$  satisfies the two conditions we do not want to complicate the matter any further.

The main result of this paper presented in a theorem below states that  $\pi$ -antidictatorships are effectively the only non-trivial examples of self-selective SCFs if we restrict the set of rival SCFs to selections of  $\pi$ .

**Theorem 2.** *Suppose  $n \geq 3$ . Let  $\pi$  be any neutral, hereditary and tops-inclusive SCC and  $F$  be a selection of  $\pi$  which is  $\mathcal{F}$ -self-selective for some  $\pi$ -complete set  $\mathcal{F}$  of SCFs. Then either  $F$  is dictatorial or  $\pi$ -antidictatorial.*

We will give a proof in the next section. Now we are going to single out some interesting cases which fall under this general result.

**Corollary 1.** *Let  $F$  be a universally self-selective SCF. Then it is dictatorial or antidictatorial.*

This generalises the main result of Koray (2000), where unanimity was assumed which precluded antidictatorial SCFs from being a possibility. We obtain this corollary assuming  $\pi = \Omega$ . Another consequence of the main result follows from Theorem 1:

**Corollary 2.** *Let  $F$  be a clone resistant SCF. Then it is dictatorial or antidictatorial.*

**Corollary 3.** *Let  $F$  be a selection of  $T$ . Let  $\mathcal{D} = \{D_1, \dots, D_n\}$  be the set of all dictatorial SCFs and  $\mathcal{F}$  is any set of SCFs containing  $\mathcal{D}$ . Then  $F$  is  $\mathcal{F}$ -self-selective if and only if it is dictatorial or  $T$ -antidictatorial.*

This generalises the main result of Koray and Unel (2003) in several directions. We obtain their result by setting  $\pi = T$ .

**Corollary 4.** *Let  $F$  be a selection of  $P$ . Let  $\mathcal{F}$  be any set of SCFs which is  $P$ -complete. Then  $F$  is  $\mathcal{F}$ -self-selective if and only if it is dictatorial or  $P$ -antidictatorial.*

Finally, we will mention several of SCCs for which dictatorial SCFs are still the only self-selective SCFs even if unanimity is not postulated.

**Corollary 5.** *Let  $n \geq 3$  and let  $\pi$  be either  $T_q$  or  $P_q$ , where  $q \geq 2$ , and  $F$  be a selection of  $\pi$  which is  $\mathcal{F}$ -self-selective for some  $\pi$ -complete set  $\mathcal{F}$  of SCFs. Then  $F$  is dictatorial.*

*Proof.* Since  $\pi$  is neutral, tops-inclusive and hereditary, by Theorem 2  $F$  is either dictatorial or  $\pi$ -antidictatorial. But as we have seen for  $T_q$  or  $P_q$ , where  $q \geq 2$ ,  $\pi$ -antidictatorial SCFs are not self-selective.  $\square$

## 5 Proof of Theorem 2

In this section we assume that all conditions of Theorem 2 hold. Let  $R$  be a profile. The alternatives in  $\pi(R)$  will be called  $\pi$ -optimal relative to  $R$ . By  $\pi^-(R)$  we will denote the set of all remaining alternatives (which are not thus  $\pi$ -optimal). The following key lemma relates the condition of  $\mathcal{F}$ -self-selectivity with the more familiar conceptual framework of Independence of Irrelevant Alternatives.

**Lemma 1.** *Let  $R \in \mathcal{L}(I_m)^n$  be a profile, and  $F$  be an SCF which is  $\mathcal{F}$ -self-selective at  $R$ . Let  $B$  be a subset of  $I_m$  such that  $\pi^-(R) \subseteq B \subseteq I_m$ , and  $C = I_m \setminus B$ . Then*

$$(F(R) \in C) \implies (F(R) = F(R|_C)). \quad (8)$$

*Proof.* Let the cardinality of  $C$  be  $k$ . Suppose  $F(R) \in C$ . Note that all elements in  $C$  are  $\pi$ -optimal, hence  $k \geq 1$  as  $\pi$  is tops-inclusive. Since  $\mathcal{F}$  is  $\pi$ -complete, there exists a subset  $\mathcal{G} \subseteq \mathcal{F}$  of cardinality  $k$  such that  $F \in \mathcal{G}$  and for every  $a \in C$  there exists an SCF  $G \in \mathcal{G}$  such that  $G(R) = a$ . Let  $\mu: \mathcal{G} \rightarrow C$  be a bijection such that  $\mu(G) = G(R)$ .

Let  $S = R|_C$  be the restriction of  $R$  onto  $C$ . Then, using the mapping  $\mu^{-1}$ , as in (2) we can induce a profile  $S^{\mu^{-1}}$  on  $\mathcal{G}$ . Note that  $S^{\mu^{-1}}$  coincides with the unique dual profile  $S^{\mathcal{G}}$  as defined in (4). Thus, by  $\mathcal{F}$ -self-selectivity of  $F$ , we have

$$F(S^{\mu^{-1}}) = F(S^{\mathcal{G}}) = F.$$

Having the definition of  $\mu$  in mind, and (3) we obtain

$$F(R) = \mu(F) = \mu F(S^{\mu^{-1}}) = F^\mu(S).$$

Due to neutrality of  $F$  we have  $F(R) = F(S)$ , as required.  $\square$

We will call the condition (8) the Independence of Irrelevant Alternatives with respect to  $\pi$ . We will omit  $\pi$ , if this invites no confusion.

**Corollary 6.** *Let  $R \in \mathcal{L}(I_m)^n$  be a profile and  $F$  be a selection of  $\pi$  which satisfies the Independence of Irrelevant Alternatives. Then*

$$F(R) = F(R|_{\pi(R)}).$$

*Proof.* Since  $F$  is a selection of  $\pi$ ,  $F(R) \notin \pi^-(R)$ . By the Independence of Irrelevant Alternatives

$$F(R) = F(R|_{I_m \setminus \pi^-(R)}) = F(R|_{\pi(R)}), \quad (9)$$

as required.  $\square$

Let  $F$  be an SCF and let  $R$  be a profile. Then for  $X \subseteq \pi(R)$  we define

$$c_R(X) \stackrel{\text{def}}{=} F(R|_X),$$

and for every  $x, y \in \pi(R)$

$$x \succ_R y \stackrel{\text{def}}{\iff} c_R(\{x, y\}) = x.$$

By doing this, we attach to every SCF  $F$  and every profile  $R$  a binary relation  $\succ_R$  on  $\pi(R)$ .



**Lemma 2.** *Let  $F$  be an SCF satisfying the Independence of Irrelevant Alternatives. Then for every profile  $R$  the restriction of the binary relation  $\succ_R$  to  $\pi(R)$  is a linear order on  $\pi(R)$ .*

*Proof.* Suppose  $x \succ_R y$  and  $y \succ_R z$ , where  $x, y, z \in \pi(R)$  are distinct. Then  $x, y, z \in \pi(R|_{\{x, y, z\}})$  since  $\pi$  is hereditary. Let us prove that  $c_R(\{x, y, z\}) = x$ . Indeed, if  $c_R(\{x, y, z\}) = z$ , then the Independence of Irrelevant Alternatives implies  $c_R(\{y, z\}) = z$  which contradicts to  $y \succ_R z$ . If  $c_R(\{x, y, z\}) = y$ , then the Independence of Irrelevant Alternatives implies  $c_R(\{x, y\}) = y$  which contradicts to  $x \succ_R y$ . Hence  $c_R(\{x, y, z\}) = x$  is proven and then by the Independence of Irrelevant Alternatives we get  $c_R(\{x, z\}) = x$ , i.e.  $x \succ_R z$ .  $\square$

The following proposition reveals the mechanism behind any SCF which is a selection of  $\pi$  and satisfies the Independence of Irrelevant Alternatives. It can be viewed as an extension of Corollary 6.

**Proposition 3.** *Let  $F$  be a selection of  $\pi$  satisfying the Independence of Irrelevant Alternatives. Let  $R$  be a profile. Suppose that the elements of  $\pi(R)$  are enumerated so that  $\pi(R) = \{b_1, \dots, b_r\}$  with*

$$b_1 \succ_R b_2 \succ_R \dots \succ_R b_r.$$

*Then  $F(R) = c_R(\{b_1, \dots, b_r\}) = b_1$ .*

*Proof.* The equality  $F(R) = c_R(\{b_1, \dots, b_r\})$  is implied by Corollary 6. Let us prove  $c_R(\{b_1, \dots, b_k\}) = b_1$  by induction on  $k$ . If  $k = 2$ , then  $c_R(\{b_1, b_2\}) = b_1$  is equivalent to  $b_1 \succ_R b_2$ . Suppose that  $c_R(\{b_1, \dots, b_k\}) = b_1$ , let us consider  $b_1, \dots, b_{k+1}$ . If  $c_R(\{b_1, \dots, b_{k+1}\}) = b_{k+1}$ , then the Independence of Irrelevant Alternatives implies  $b_{k+1} \succ_R b_k$ , the contradiction. Then  $c_R(\{b_1, \dots, b_{k+1}\}) \in \{b_1, \dots, b_k\}$ . Then by the Independence of Irrelevant Alternatives

$$c_R(\{b_1, \dots, b_{k+1}\}) = c_R(\{b_1, \dots, b_k\}) = b_1.$$

The proposition is proved.  $\square$

We will denote the  $i$ th voter as  $i$  so that  $\mathcal{N} = \{1, 2, \dots, n\}$ . It will not lead to a confusion. We fix  $\pi$  till the end of this section. In the rest of the proof we follow the ideas of the original proof of Arrow's Impossibility Theorem (1951,1963). The proof itself is different since here we have transitivity only on a variable set of alternatives  $\pi(R)$  which depends on the profile  $R$ . We have to be careful about that.

**Definition 3.** *Let  $F$  be an SCF. We say that a coalition  $\mathcal{D} \subseteq \mathcal{N}$  is  $\pi$ -decisive for  $F$  and a pair  $(a, b)$  of distinct alternatives  $a, b \in I_m$ , if for an arbitrary profile  $R$ , such that  $a, b \in \pi(R)$ ,  $aR_i b$  for  $i \in \mathcal{D}$ , and  $bR_j a$  for  $j \in \mathcal{N} \setminus \mathcal{D}$ , imply  $a \succ_R b$ . We say that  $\mathcal{D}$  is  $\pi$ -decisive for  $F$ , if it is  $\pi$ -decisive for every pair of distinct alternatives.*

Most of the time our  $\pi$  will be fixed and we will write decisive instead of  $\pi$ -decisive.

**Lemma 3.** *Let  $F$  be an SCF satisfying the Independence of Irrelevant Alternatives and let  $\mathcal{D}$  be a coalition. Suppose that there exists a profile  $R$ , such that for some  $a, b \in \pi(R)$ ,  $aR_i b$  for  $i \in \mathcal{D}$ , and  $bR_j a$  for  $j \in \mathcal{N} \setminus \mathcal{D}$ , and  $a \succ_R b$ . Then  $\mathcal{D}$  is decisive for  $F$  and the pair  $(a, b)$ . If the coalition  $\mathcal{D}$  is proper, i.e.  $\emptyset \neq \mathcal{D} \neq \mathcal{N}$ , then the reverse is also true.*

*Proof.* Suppose that there exists a profile  $R$ , such that  $a, b \in \pi(R)$ ,  $aR_i b$  for  $i \in \mathcal{D}$ , and  $bR_j a$  for  $j \in \mathcal{N} \setminus \mathcal{D}$ , and  $a \succ_R b$ . Let  $R'$  be any profile with  $a, b \in \pi(R')$  such that  $a, b \in \pi(R')$ ,  $aR'_i b$  for  $i \in \mathcal{D}$ , and  $bR'_j a$  for  $j \in \mathcal{N} \setminus \mathcal{D}$ . Then  $R'|_{\{a,b\}} = R|_{\{a,b\}}$ , whence  $F(R'|_{\{a,b\}}) = F(R|_{\{a,b\}}) = a$ , and  $a \succ_{R'} b$ .

Suppose now that a proper coalition  $\mathcal{D}$  is decisive for  $F$  and a pair  $(a, b)$ . Then both  $\mathcal{D}$  and  $\mathcal{N} \setminus \mathcal{D}$  are nonempty. Let us consider any profile  $R$  of the following type:

$$\begin{aligned} a \succ b \succ \dots & : \text{ agents from } \mathcal{D}, \\ b \succ a \succ \dots & : \text{ agent from } \mathcal{N} \setminus \mathcal{D}. \end{aligned}$$

Then  $a, b \in \pi(R)$ , since  $\pi$  is tops-inclusive, and hence  $a \succ_R b$  by the decisiveness of  $\mathcal{D}$ . Therefore a profile with the required properties exists.  $\square$

We illustrate this Lemma by the following

**Example 9.** *Let  $\pi = T$  and  $\mathcal{D} = \mathcal{N}$ . Then the statement  $a, b \in \pi(R)$  and  $aR_i b$  for all  $i \in \mathcal{D}$  is false and trivially implies  $a \succ_R b$ . Thus  $\mathcal{D}$  will be decisive for  $F$ .*

**Lemma 4.** *Let  $F$  be an SCF satisfying the Independence of Irrelevant Alternatives. Then a coalition  $\mathcal{D}$  is decisive for  $F$  if and only if it is decisive for  $F$  and a pair  $(a, b)$  for some distinct alternatives  $a, b \in I_m$ .*

*Proof.* Suppose  $\mathcal{D}$  is decisive for  $F$  and a pair  $(a, b)$  of distinct alternatives  $a, b \in I_m$ . First, we suppose that there exists a profile  $R$ , such that  $a, b \in \pi(R)$ ,  $aR_i b$  for  $i \in \mathcal{D}$ , and  $bR_j a$  for  $j \in \mathcal{N} \setminus \mathcal{D}$ , with  $a \succ_R b$ . By the definition the latter means that  $a = F(R|_{\{a,b\}})$ . Let us denote  $R|_{\{a,b\}} = P$ .

Let us consider any profile  $R'$  such that  $c, d \in \pi(R')$ ,  $cR'_i d$  for  $i \in \mathcal{D}$ , and  $dR'_j c$  for  $j \in \mathcal{N} \setminus \mathcal{D}$ . Let us denote  $R'|_{\{c,d\}} = Q$ . Consider the bijections  $\mu: \{a, b\} \rightarrow I_2$  and  $\nu: \{c, d\} \rightarrow I_2$  such that  $\mu(a) = \nu(c) = 1$  and  $\mu(b) = \nu(d) = 2$ . By (3)

$$a = F(P) = F^\mu(P) = \mu^{-1}F(P^\mu).$$

Since the profiles  $P^\mu$  and  $Q^\nu$  coincide, we have

$$F(Q) = F^\nu(Q) = \nu^{-1}F(Q^\nu) = c.$$

The latter means  $c \succ_{R'} d$  and by Lemma 4  $\mathcal{D}$  is decisive for  $(c, d)$ .

Let us consider the remaining case, when no profiles exist such that  $a, b \in \pi(R)$ ,  $aR_i b$  for  $i \in \mathcal{D}$ , and  $bR_j a$  for  $j \in \mathcal{N} \setminus \mathcal{D}$ . The neutrality of  $\pi$  then implies that no profile  $Q$  can exist such that  $c, d \in \pi(Q)$ ,  $cQ_i d$  for  $i \in \mathcal{D}$ , and  $dQ_j c$  for  $j \in \mathcal{N} \setminus \mathcal{D}$ . Thus, in both cases,  $\mathcal{D}$  is decisive for  $F$ .  $\square$

**Corollary 7.** *Let  $F$  be an SCF satisfying the Independence of Irrelevant Alternatives. Let  $\mathcal{D}$  be a proper subset of  $\mathcal{N}$ . Then either  $\mathcal{D}$  is decisive or its complement  $\mathcal{N} \setminus \mathcal{D}$  is decisive.*

*Proof.* Suppose that a coalition  $\mathcal{D}$  is decisive for  $F$  and a pair  $(a, b)$ . Then  $\mathcal{D}$  is decisive by Lemma 4. If  $\mathcal{D}$  is not decisive for  $F$  and a pair  $(a, b)$ , then there exists a profile  $R$  such that  $a, b \in \pi(R)$ , and  $aR_i b$  for  $i \in \mathcal{D}$ , and  $bR_j a$  for  $j \in \mathcal{N} \setminus \mathcal{D}$ , but  $b \succ_R a$ . But now by Lemmata 3 and 4  $\mathcal{N} \setminus \mathcal{D}$  is decisive.  $\square$

The following Lemmata on the structure of the set of decisive subsets of  $\mathcal{N}$  will be proved under the assumption that  $F$  is a SCF which satisfies the Independence of Irrelevant Alternatives, where  $\pi$  is a neutral, hereditary, and tops-inclusive SCC.

**Lemma 5.** *If a decisive set  $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ , different from  $\mathcal{N}$ , is a disjoint union ( $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$ ) of two nonempty subsets  $\mathcal{D}_1$  and  $\mathcal{D}_2$ , then either  $\mathcal{D}_1$  or  $\mathcal{D}_2$  is decisive as well.*

*Proof.* Let  $\mathcal{N} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{M}$ , where  $\mathcal{M} = \mathcal{N} \setminus \mathcal{D} \neq \emptyset$ . Consider any profile  $R$  such that for some  $a, b, c \in I_m$ :

$$\begin{aligned} a \succ b \succ c \succ \dots & : \text{ agents from } \mathcal{D}_1, \\ b \succ c \succ a \succ \dots & : \text{ agents from } \mathcal{D}_2, \\ c \succ a \succ b \succ \dots & : \text{ agents from } \mathcal{M}. \end{aligned}$$

Then  $a, b, c \in \pi(R)$  as  $\pi$  is tops-inclusive. Then  $b \succ_R c$  as  $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$  is decisive. If  $b \succ_R a$  then  $\mathcal{D}_2$  is decisive and the result is proved. If not, then  $a \succ_R b$ . Since by Lemma 3 the relation  $\succ_R$  is transitive on  $\pi(R)$ ,  $a \succ_R b$  and  $b \succ_R c$  imply  $a \succ_R c$ , which means that in this case  $\mathcal{D}_1$  is decisive.  $\square$

**Lemma 6.** *There exists a singleton  $v \in \mathcal{N}$  such that  $\{v\}$  is decisive.*

*Proof.* Let  $\mathcal{N}' = \mathcal{N} \setminus \{u\}$ , where  $u \in \mathcal{N}$  is arbitrary. Then by Corollary 7 either  $\{u\}$  or  $\mathcal{N}'$  is decisive. In the first case we are done. In the second, we may repeatedly apply Lemma 5 to  $\mathcal{N}'$  and then to its decisive subsets until a decisive singleton is obtained.  $\square$

**Lemma 7.** *Let  $\mathcal{D}_1$ ,  $\mathcal{D}_2$  and  $\mathcal{D}_3$  be three nonempty disjoint subsets of  $\mathcal{N}$  such that  $\mathcal{N} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$ . Then all three subsets cannot be simultaneously decisive.*

*Proof.* If this were possible, then consider the following profile  $R$ :

$$\begin{aligned} a \succ b \succ c \succ \dots & : \text{ agents from } \mathcal{D}_1, \\ b \succ c \succ a \succ \dots & : \text{ agents from } \mathcal{D}_2, \\ c \succ a \succ b \succ \dots & : \text{ agents from } \mathcal{D}_3. \end{aligned}$$

Since  $\pi$  is tops-inclusive, the alternatives  $a, b, c$  are all  $\pi$ -optimal and, assuming that all three subsets are decisive, we will have  $a \succ_R c \succ_R b \succ_R a$ , which contradicts to the transitivity of  $\succ_R$  on  $\pi(R)$  proved in Lemma 3.  $\square$

**Lemma 8.** *Let  $\mathcal{D}_1$  and  $\mathcal{D}_2$  be two decisive subsets of  $\mathcal{N}$  such that  $\mathcal{D}_1 \cup \mathcal{D}_2 \neq \mathcal{N}$ . Then the union  $\mathcal{D}_1 \cup \mathcal{D}_2$  is decisive.*

*Proof.* Suppose first that  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are disjoint. As  $\mathcal{D}_1 \cup \mathcal{D}_2 \neq \mathcal{N}$ , then  $\mathcal{M} = \mathcal{N} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2) \neq \emptyset$ . By Lemma 7  $\mathcal{M}$  is not decisive. But then  $\mathcal{D}_1 \cup \mathcal{D}_2 = \mathcal{N} \setminus \mathcal{M}$  is decisive by Corollary 7.

Now let us assume that  $\mathcal{D}_1$  and  $\mathcal{D}_2$  have a nonzero intersection. We may also assume that this intersection is different from both of the sets because otherwise the result is trivial. Let us consider any profile such that for some alternatives  $a, b, c \in I_m$

$$\begin{aligned} a \succ b \succ c \succ \dots & : \text{agents from } \mathcal{D}_1 \cap \mathcal{D}_2, \\ a \succ c \succ b \succ \dots & : \text{agents from } \mathcal{D}_1 \setminus \mathcal{D}_2, \\ b \succ a \succ c \succ \dots & : \text{agents from } \mathcal{D}_2 \setminus \mathcal{D}_1, \\ c \succ b \succ a \succ \dots & : \text{agents from } \mathcal{M}, \end{aligned}$$

where  $\mathcal{M} = \mathcal{N} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$ . We note that  $a, b, c \in \pi(R)$  as  $\pi$  is tops-inclusive. Then  $a \succ_R b$  since  $\mathcal{D}_1$  is decisive and  $b \succ_R c$  since  $\mathcal{D}_2$  is decisive. By transitivity of  $\succ_R$  on  $\pi(R)$  we get  $a \succ_R c$  and hence  $\mathcal{D}_1 \cup \mathcal{D}_2$  is decisive.  $\square$

**Corollary 8.** *There exists a decisive subset  $\mathcal{D}$  of  $\mathcal{N}$  of cardinality  $n - 1$ .*

*Proof.* This is the same to say that one of the singletons is not decisive. Suppose to the contrary that all singletons are decisive. Then by Lemma 8 all proper subsets of  $\mathcal{N}$  are decisive. This is impossible since by Corollary 7 a subset and its complement cannot be simultaneously decisive.  $\square$

**Lemma 9.** *Let  $\emptyset \neq \mathcal{D}_1 \subseteq \mathcal{D} \subseteq \mathcal{D}_2 \neq \mathcal{N}$  with  $\mathcal{D}_1$  and  $\mathcal{D}_2$  being decisive. Then  $\mathcal{D}$  is decisive.*

*Proof.* Let us consider any profile such that for some alternatives  $a, b, c \in I_m$

$$\begin{aligned} a \succ b \succ c \succ \dots & : \text{agents from } \mathcal{D}_1, \\ b \succ a \succ c \succ \dots & : \text{agents from } \mathcal{D} \setminus \mathcal{D}_1, \\ b \succ c \succ a \succ \dots & : \text{agents from } \mathcal{D}_2 \setminus \mathcal{D}, \\ c \succ b \succ a \succ \dots & : \text{agents from } \mathcal{N} \setminus \mathcal{D}_2. \end{aligned}$$

Since  $a, b, c \in \pi(R)$ , we get  $a \succ_R b$  as  $\mathcal{D}_1$  is decisive and  $b \succ_R c$  as  $\mathcal{D}_2$  is decisive. By the transitivity of  $\succ_R$  on  $\pi(R)$  we get  $a \succ_R c$  which means that  $\mathcal{D}$  is decisive.  $\square$

**Definition 4.** *Let  $F$  be an SCF. An agent  $k \in \mathcal{N}$  will be called an  $\pi$ -dictator, if for every profile  $R$  and for every pair of two distinct alternatives  $a, b \in \pi(R)$  it is true that  $aR_k b$  implies  $a \succ_R b$ ; an agent  $k \in \mathcal{N}$  will be called an  $\pi$ -antidictator, if for every profile  $R$  and for every pair of two distinct alternatives  $a, b$  with  $a, b \in \pi(R)$  it is true that  $aR_k b$  implies  $b \succ_R a$ .*

The following two propositions are obvious.

**Proposition 4.** *An agent  $k \in \mathcal{N}$  is an  $\pi$ -dictator, if all coalitions in  $\mathcal{N}$  containing  $k$  are  $\pi$ -decisive. An agent  $k \in \mathcal{N}$  is an  $\pi$ -antidictator, if all coalitions in  $\mathcal{N}$  not containing  $k$  (including the empty one) are  $\pi$ -decisive.*

Now we are ready to prove the main results of this paper.

*Proof of Theorem 2.* We will prove that there is either a  $\pi$ -dictator or  $\pi$ -antidictator. Since  $F$  is a selection from  $\pi$  an  $\pi$ -dictator will be an ordinary dictator.

Firstly, we note that the existence of a decisive set of cardinality  $n - 1$  is guaranteed by Corollary 8. Without loss of generality, we assume that  $\mathcal{D} = \{1, \dots, n-1\}$  is decisive. By Lemma 6 there is a decisive singleton in  $\mathcal{D}$ ; and we may assume that it is  $\{1\}$ . By Lemma 9 all subsets of  $\mathcal{D}$ , which contain  $\{1\}$ , are decisive.

Now the key question is whether or not one of the subsets  $\mathcal{N} \setminus \{i\}$  is decisive for  $2 \leq i \leq n - 1$ . Let us assume first that there is such a subset, say  $\mathcal{N} \setminus \{i\}$ . Then every proper subset of  $\mathcal{N}$ , containing 1 is contained in either  $\mathcal{N} \setminus \{n\}$  or  $\mathcal{N} \setminus \{i\}$  and by Lemma 9 is decisive. Hence all proper subsets containing  $\{1\}$  are decisive. It remains to prove that in this case  $\mathcal{N}$  itself is also decisive, it would mean that agent 1 is an  $\pi$ -dictator.

We note first that if  $\pi(R) \subseteq P(R)$  for all profiles  $R$ , then  $\mathcal{N}$  is trivially decisive because there does not exist such  $a, b \in \pi(R)$  that  $aR_i b$  for all  $i \in \mathcal{N}$ . If this inclusion does not hold, then by the second condition of tops-inclusiveness  $\pi(R)$  contains all first and second preferences. Let us consider any profile of the following type

$$\begin{aligned} a \succ b \succ c \succ \dots & : \text{agents from } \mathcal{N} \setminus \{2, 3\} \\ b \succ a \succ c \succ \dots & : \text{agent 2,} \\ a \succ c \succ b \succ \dots & : \text{agent 3.} \end{aligned}$$

Then  $a, b, c \in \pi(R)$  as  $\pi(R)$  contains all first and second preferences. We get  $a \succ_R b$  as  $\mathcal{N} \setminus \{2\}$  is decisive and  $b \succ_R c$  as  $\mathcal{N} \setminus \{3\}$  is decisive. By transitivity we get  $a \succ_R c$  which by Lemma 4 means that  $\mathcal{N}$  is decisive. Thus agent 1 is an  $\pi$ -dictator.

Suppose now that none of the subsets  $\mathcal{N} \setminus \{i\}$  are decisive for  $2 \leq i \leq n - 1$ . This immediately implies that all agents  $2, 3, \dots, n-1$  are decisive. Now by Lemma 8 it follows that every nonempty subset of  $\mathcal{D}$  is decisive. Then  $n$  would be an  $\pi$ -antidictator if and only if an empty set is decisive.

We note first that if  $\pi(R) \subseteq P(R)$  for all profiles  $R$ , then  $\emptyset$  is trivially decisive. If not, then  $\pi(R)$  contains all second preferences. Let us consider any profile of the following type

$$\begin{aligned} a \succ b \succ c \succ \dots & : \text{agents from } \mathcal{N} \setminus \{n-1, n-2\} \\ b \succ a \succ c \succ \dots & : \text{agent } n-1, \\ a \succ c \succ b \succ \dots & : \text{agent } n-2. \end{aligned}$$

Then  $a, b, c \in \pi(R)$  as  $\pi(R)$  contains all first and second preferences. We get  $b \succ_R a$  as  $\{n-1\}$  is decisive and  $c \succ_R b$  as  $\{n-2\}$  is decisive. By transitivity we get  $c \succ_R a$  which by Lemma 4 means that  $\emptyset$  is decisive. Thus agent  $n$  is an  $\pi$ -antidictator.  $\square$

## 6 Theorem 2 without neutrality

As we mentioned above, here we have to work with indexed sets of alternatives. Instead of a set of alternative  $A$  we must consider this set  $A$  together with positions of those alternatives on the ballot. If  $m$  is the cardinality of  $A$ , then the order on the ballot is given by a bijection  $\mu: A \rightarrow I_m$ . Then, given a SCF  $F$  we may define a SCF  $F^\mu$  on  $A$  by the formula (3) but this now becomes dependent on  $\mu$ . The definition of self-selectivity has to be amended accordingly.

**Definition 5.** *An SCF  $F$  is said to be  $\mathcal{F}$ -self-selective at a profile  $R \in \mathcal{L}(I_m)^n$  if, for any finite set of SCFs  $\mathcal{F}' \subseteq \mathcal{F}$ , there exists at least one dual profile  $R^*$  on  $\mathcal{G} = \{F\} \cup \mathcal{F}'$  such that for every bijection  $\nu: \mathcal{G} \rightarrow I_k$ , where  $k$  is the cardinality of  $\mathcal{G}$ , the SCF  $F^\nu$ , being applied to  $R^*$ , chooses  $F$ , i.e.,  $F^\nu(R^*) = F$  for all  $\nu$ . An SCF  $F$  is said to be  $\mathcal{F}$ -self-selective if it is  $\mathcal{F}$ -self-selective at every profile.*

Lemma 1 has to be slightly modified too.

**Lemma 10.** *Let  $\pi$  be an SCC and let  $R \in \mathcal{L}(I_m)^n$  be a profile. Let  $F$  be an SCF which is  $\mathcal{F}$ -self-selective at  $R$ . Let  $B$  be a subset of  $I_m$  of cardinality  $m - k$  such that  $\pi^-(R) \subseteq B \subseteq I_m$ . Let  $C = I_m \setminus B$ . Then for every bijection  $\nu: C \rightarrow I_k$*

$$(F(R) \in C) \implies (F(R) = F^\nu(R|_C)). \quad (10)$$

*In particular,  $F$  is neutral at  $R|_C$ .*

*Proof.* Is very similar to the proof of Lemma 1. □

The formulation of the main theorem is not affected. Once we obtained the restricted neutrality in Lemma 10 the rest of the proof of the main theorem is without changes.

## 7 Conclusion and Further Research

In this paper the authors have made the first attempt to find a framework in which non-dictatorial self-selective SCFs may exist. To this end we relaxed the universal self-selectivity restricting the set of rival SCFs requiring them to be 'reasonable' in the sense that they are selections from a certain well-behaved constitutional correspondence. We indeed discovered some self-selective non-dictatorial SCFs. Further attempts to find interesting relaxations of universal self-selectivity are encouraged.

We showed that the property of self-selectivity is closely related to some well-known and well-studied properties of SCFs such that independence of irrelevant alternatives, resistance to cloning. But, unlike them, self-selectivity can be made rather flexible since the choice of the set of rival SCFs  $\mathcal{F}$  can be made in many different ways.

It seems that the property of self-selectivity (as well as resistance to cloning) is much more compatible with the Condorcet consistent SCFs that with point-scoring ones. It would be interesting to find out whether or not there are any self-selective SCFs in the class of Condorcet consistent rules.

Another interesting question that we left open is to characterise all neutral SCCs which are resistant to cloning of essential alternatives. In particular, we don't know if  $T$ ,  $P$  and  $\Omega$  are the only neutral SCCs with this property.

## References

- [1] Arrow, K.J. (1951, 1963) Social choice and individual values, Wiley, New York.
- [2] Arrow, K.J. (1959) Rational choice functions and orderings. *Economica* 26:121–127.
- [3] Barberà, S. and Beviá C. (2002). Self-selection consistent functions. *Journal of Economic Theory*. 105: 263–277.
- [4] Barberà, S. and Jackson M. (2004) Choosing how to choose: self-stable majority rules, *Quarterly Journal of Economics* 119: 1011-1048.
- [5] Houy, N. (2003) Dynamics of Stable Sets of Constitutions. Mimeo, Université Paris 1.
- [6] Houy, N. (2006) On the (Im)possibility of a Set of Constitutions Stable at Different Levels, to appear.
- [7] Jackson, M. (2001) A crash course in implementation theory. *Social Choice and Welfare*. 18: 655–708.
- [8] Koray, S. (1998) Consistency in electoral system design. Mimeo, Bilkent University, Ankara.
- [9] Koray, S. (2000) Self-selective social choice functions verify Arrow and Gibbard-Satterthwaite theorems. *Econometrica* 68: 981–995.
- [10] Koray, S. and Unel, B. (2003) Characterization of self-selective social choice functions on the tops-only domain. *Social Choice and Welfare*. 20: 495–507.
- [11] Laslier, J.-F. (1997), *Tournament solutions and majority voting*, Springer-Verlag, Berlin-Heidelberg.
- [12] Laffond, G., Laine, J., Laslier, J-F. (1996) *Composition consistent tournament solutions and social choice functions*, *Social Choice and Welfare*, 13: 75-93.
- [13] Moulin, H. (1988), *Axioms of cooperative decision making*. Cambridge University Press.
- [14] Tideman, T. N. (1987), "Independence of clones as a criterion for voting rules," *Social Choice and Welfare*, 4: 185–206.
- [15] Unel, B. (1999) Exploration of self-selective social choice functions. Master Thesis. Bilkent University, Ankara.

- [16] Zavist, T.M., Tideman, T. N. (1989), "Complete independence of clones in the ranked pairs rule," *Social Choice and Welfare*, 6: 167–173.