Oligarchies in Spatial Environments

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Abstract

In spatial environments we consider social welfare functions satisfying Arrow's requirements, i.e. *weak Pareto* and *independence of irrelevant alternatives*. Individual preferences measure distances between alternatives according to the L_p-norm (for a fixed $p \geq 1$). When the policy space is multi-dimensional and the set of alternatives has a non-empty interior and it is compact and convex, any quasi-transitive welfare function must be oligarchic. As a corollary we obtain that for transitive welfare functions *weak Pareto*, *independence of irrelevant alternatives*, and *non-dictatorship* are inconsistent if the set of alternatives has a non-empty interior and it is compact and convex.

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1 Introduction

A social welfare function is a procedure for aggregating profiles of individual preferences into social orderings. Arrow's theorem shows that it is impossible for a social welfare function to satisfy *weak Pareto* (if all individuals strictly prefer one alternative

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to another, then so does society), *independence of irrelevant alternatives* (the social ranking of two alternatives only depends on the individual rankings of these alternatives), and *non-dictatorship* when the preference domain is unrestricted. When the set of alternatives is structured, the assumption of unrestricted domain might be unreasonable. One important exception of this kind are spatial environments. In spatial environments, alternatives are points in a multi-dimensional space of issue positions and individual preferences are continuous, quasi-concave, and have bliss points.[1]

There is one well-known spatial environment in which Arrow's requirements are consistent. If the number of voters is odd and the policy space is one-dimensional, then simple majority voting is transitive and satisfies *weak Pareto*, *independence of irrelevant alternatives*, and *anonymity* (Black ,1948; Arrow, 1951, 1963). Simple majority voting is an example of an Arrovian welfare function belonging to the following class: if the number of voters is n, fix $n-1$ preferences of additional constant voters, and apply to each profile of individual preferences majority voting over this profile and the $n-1$ fixed voters.[2,3]

If the policy space is multi-dimensional and unbounded, and preferences are Euclidean, then Arrow's requirements are inconsistent (Border, 1984). Other proofs show that the domain of Euclidean preferences is "saturating" and apply a result due to Kalai, Muller, and Satterthwaite (1979) to deduce Arrow's impossibility (Le Breton and Weymark, 1996, 2000, 2002; Campbell, 1993[4]). However, the proofs of these results rely importantly on the unboundedness of the policy space. As far as we

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[1]An excellent review of the literature is Le Breton and Weymark (2000).

[2]Ehlers and Storcken (2002) provide a characterization of all welfare functions satisfying *weak Pareto* and *independence of irrelevant alternatives*.

[3]For finite sets of alternatives Sethuraman, Teo, and Vohra (2003,2006) introduce an integer linear programming formulation of anonymous and transitive Arrovian social welfare functions (ruling out social indifferences). They show that any such welfare function is monotonic if and only if it is simple majority voting with $n+1$ fixed voters for any domain with no Condorcet triples and which contains an ordering and its inversion.

[4]Campbell (1993) drops *weak Pareto* and imposes instead continuity of social preference.

know, Arrow's theorem has not been established in spatial environments when the set of alternatives is compact and convex. It is natural to assume that a government faces budget constraints and therefore the set of policy alternatives becomes bounded. We derive Arrow's theorem in such spatial environments. If individual preferences are measure distances according to a given L_p-norm and the set of alternatives has a non-empty interior and it is compact and convex, then *weak Pareto*, *independence of irrelevant alternatives*, and *non-dictatorship* are inconsistent, i.e. again decisiveness of a coalition spreads from one pair of alternatives to all pairs of alternatives and dictatorship results.

Instead of showing Arrow's theorem directly, we will establish a stronger result. Gibbard (1969) and Mas-Colell and Sonnenschein (1972) require social orderings only to be quasi-transitive and show that for a finite set of alternatives with an unrestricted domain any quasi-transitive welfare functions satisfying *weak Pareto* and *independence of irrelevant alternatives* must be oligarchic. This means that there exists a non-empty coalition S (called the oligarchy) such that an alternative a is socially strictly preferred to another alternative b if and only if all members of the oligarchy agree that a is strictly preferred to b. Our main result shows the following: if individual preferences measure distances according to the L_p-norm and the set of alternatives has a non-empty interior and it is compact and convex, then any quasi-transitive welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives* must be oligarchic.

The paper is organized as follows. In Section 2 we introduce our notation and the main definitions. In Section 3 we state that *weak Pareto*, *independence of irrelevant alternatives*, and *non-dictatorship* are inconsistent if the policy space is compact, convex, and at least two-dimensional. In Section 4 we introduce oligarchic welfare functions and give our main result: any quasi-transitive welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives* must be oligarchic. Section 5 contains the proof of our main result. Section 6 concludes.

2   Notation and Definitions

We use the same terminology and notation as Le Breton and Weymark (2000). Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}^m$ denote the m-dimensional Euclidean space. Let $N \equiv \{1, 2, \ldots, n\}$ denote a finite set of agents with $n \geq 2$, and let $A \subseteq \mathbb{R}^m$ denote a set of alternatives. Each point in $\mathbb{R}^m$ identifies the changes in the level of m different policies, for example public spending on police, health care, and so on.

Let $\mathcal{W}$ denote the set of all complete, reflexive, and transitive relations over A. An element of $\mathcal{W}$ is called a *weak ordering* over A. Let $\mathcal{Q}$ denote the set of all complete, reflexive, and quasi-transitive relations over A. An element of $\mathcal{Q}$ is called a *quasi ordering* over A.

Given $R_i \in \mathcal{Q}$, the corresponding strict relation, P_i, and the indifference relation, I_i, are defined as follows: for all $a, b \in A$, (i) $aP_ib \Leftrightarrow \neg bR_ia$, and (ii) $aI_ib \Leftrightarrow aR_ib$ and bR_ia. Note that if $\neg aR_ib$, then by completeness of R_i, aR_ib. Hence, $\neg bR_ia$ is enough to describe the strict relation P_i. Recall that R_i is transitive if for all $a, b, c \in A$, aR_ib and $bR_ic \Rightarrow aR_ic$; and R_i is quasi-transitive if for all $a, b, c \in A$, $\neg bR_ia$ and $\neg cR_ib \Rightarrow \neg cR_ia$.

Let $\mathcal{R} \subseteq \mathcal{W}$ denote an individual's preference domain. A *(preference) profile* is a list $R \equiv (R_i)_{i \in N} \in \mathcal{R}^N$. A *(social) welfare function* assigns to each profile a quasi ordering over A. Any welfare function f is a mapping $f : \mathcal{R}^N \to \mathcal{Q}$ such that for all $R \in \mathcal{R}^N$, $f(R) \in \mathcal{Q}$. We call $f(R)$ the *social ordering* (that f assigns to R). Any welfare function f is *quasi-transitive* because its chosen social orderings are quasi-transitive.

We call a welfare function f *transitive* if it is a mapping $f : \mathcal{R}^N \to \mathcal{W}$ such that for all $R \in \mathcal{R}^N$, $f(R) \in \mathcal{W}$

Note that social orderings need not belong to the individual preference domain $\mathcal{R}$. Other authors impose the restriction that any social ordering belongs to each individual's preference domain (for example, Peters, van der Stel, and Storcken, 1992, and Bossert and Weymark, 1993).

Arrow's requirements are as follows. The first axiom says that if all agents strictly prefer a to b, then a should also be socially strictly preferred to b.

Weak Pareto: For all $R \in \mathcal{R}^N$ and all $a, b \in A$, if for all $i \in N$, aP_ib, then $\neg b f(R) a$.

Given $R \in \mathcal{R}^N$, $X \subseteq A$, and $j \in N$, let $R_j | X$ denote the restriction of R_j to X, and $R|X \equiv (R_i|X)_{i \in N}$. The second axiom says that the social ordering of two alternatives only depends on the profile of individual preferences restricted to these two alternatives.

Independence of Irrelevant Alternatives: For all $R, \bar{R} \in \mathcal{R}^N$ and all $a, b \in A$, if $R|\{a, b\} = \bar{R}|\{a, b\}$, then $f(R)|\{a, b\} = f(\bar{R})|\{a, b\}$.

A welfare function is *Arrovian* if it satisfies *weak Pareto* and *independence of irrelevant alternatives*. A welfare function is *dictatorial* if there exists some agent such that for each profile the social strict preference relation respects the strict preference relation of this agent.

Non-Dictatorship: There exists no $i \in N$ such that for all $R \in \mathcal{R}^N$ and all $a, b \in A$, if aP_ib, then $\neg b f(R) a$.

Arrow's Theorem shows that for transitive welfare functions, *weak Pareto, independence of irrelevant alternatives*, and *non-dictatorship* are incompatible if the set of alternatives is finite and the individual preference domain $\mathcal{R}$ contains all strict orderings over the set of alternatives.

3 Arrow's Theorem for Multi-Dimensional Policy Spaces

Throughout we consider environments where the set of alternatives is multi-dimensional and connected with respect to the standard topology on $\mathbb{R}^m$. Let $A \subseteq \mathbb{R}^m$ be a set of alternatives such that the relative interior of A in $\mathbb{R}^m$ is non-empty. A preference relation $R_i \in \mathcal{W}$ is L_p-*single-peaked* if there exists a *bliss point* $p(R_i) \in A$ such that for all $a, b \in A$, $a R_i b \Leftrightarrow \|a - p(R_i)\|_p \leq \|b - p(R_i)\|_p$ (here $\|\cdot\|_p$ denotes the L_p norm in $\mathbb{R}^m$ for a given $p \geqslant 1$)[5]. In case $p = 2$ the L_2-single-peaked preferences are also called *Euclidean*. Let $\mathcal{R}_p$ denote the set of all L_p-single-peaked preferences over A. Note that p remains fixed throughout.

For the special case of $A = \mathbb{R}^m$ and the Euclidean norm, Border (1984) and Le Breton and Weymark (1996) establish Arrow's theorem for transitive welfare functions. For $A = \mathbb{R}_+^m$, Le Breton and Weymark (2000, 2002) show Arrow's theorem for transitive welfare functions. However, all these results rely importantly on the unboundedness of the set of alternatives. In real life, policy makers face budget constraints and these results do not apply.

A corollary of our main result will be the following theorem: Arrow's requirements are inconsistent if the set of alternatives has a non-empty and connected interior and it is contained in the closure of its interior (i.e. its boundary has no "tails").

Let $\operatorname{int}(A)$ denote the interior of A in $\mathbb{R}^m$ and $\operatorname{cl}(A)$ the closure of A in $\mathbb{R}^m$.

Theorem 1 *Let $A \subseteq \mathbb{R}^m$ be such that $int(A) \neq \emptyset$, $int(A)$ is connected, and $A \subseteq cl(int(A))$. If $m \geq 2$, then for transitive welfare functions weak Pareto, independence of irrelevant alternatives, and non-dictatorship are inconsistent on the domain $\mathcal{R}_p^N$.*

In particular, Theorem 1 yields Arrow's theorem for compact and convex sets of policy alternatives with non-empty interior.

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[5]For any $x \in \mathbb{R}^m$ we have $\|x\|_p = (x_1^p + x_2^p + \cdots + x_m^p)^{1/p}$

If the interior of the set of alternatives is non-empty and not connected, then an Arrovian social welfare function may not be *dictatorial*.

Example 1 *Let $m \equiv 2$, $p \equiv 2$, $A_1 \equiv [0,1] \times [0,1]$, $A_2 \equiv [0,1] \times [9,10]$, and $A \equiv A_1 \cup A_2$. For all $R \in \mathcal{R}_2^N$, let $f(R)|A_1 = R_1|A_1$, $f(R)|A_2 = R_2|A_2$, and for all $a \in A_1$ and all $b \in A_2$, $f(R)|\{a,b\} = R_1|\{a,b\}$. Then f is a transitive Arrovian welfare function satisfying* non-dictatorship.

The conclusion of Example 1 does not remain true if the sets A_1 and A_2 are connected through a line segment, for example if $A = A_1 \cup A_2 \cup [(0,0),(0,10)]$. For this set of alternatives an Arrovian welfare function must be dictatorial.[6]

If the boundary of A contains tails, then again an Arrovian welfare function may not be *dictatorial*.

Example 2 *Let $m \equiv 2$, $p \equiv 2$, $N \equiv \{1,2\}$, $B \equiv [0,1] \times [0,1]$, and $L \equiv](0,1),(0,2)]$. Let $A \equiv B \cup L$. For all $R \in \mathcal{R}_2^N$ and all $a,b \in A$, (i) if $a,b \in L$, $a_2 < b_2$, and for some $i \in N$, aP_ib, then $\neg bf(R)a$, and (ii) otherwise, $f(R)|\{a,b\} = R_1|\{a,b\}$. We show that f is a transitive welfare function.*

Claim: For all $R \in \mathcal{R}_2^N$, $f(R)$ is transitive.

Proof of Claim. If $p(R_1) \in B$, then $f(R) = R_1$. Let $p(R_1) \in L$ and $a,b,c \in A$. If $a,b,c \in B$ or $a,b,c \in L$, then it is easy to check that $f(R)|\{a,b,c\}$ is transitive. If $a \in L$ and $b,c \in B$, then $f(R)|\{a,b,c\} = R_1|\{a,b,c\}$ and $f(R)|\{a,b,c\}$ is transitive. Let $a,b \in L$ and $c \in B$. Then $f(R)|\{a,c\} = R_1|\{a,c\}$ and $f(R)|\{b,c\} = R_1|\{b,c\}$. Obviously, if $f(R)|\{a,b\} = R_1|\{a,b\}$, then $f(R)|\{a,b,c\}$ is transitive. Let $f(R)|\{a,b\} \neq R_1|\{a,b\}$. Without loss of generality, suppose that $a_2 < b_2$. Thus, by (i), $\neg bf(R)a$, and $a_2 \leq p_2(R_1)$. Then aP_1c and $\neg cf(R)a$. Hence, $f(R)|\{a,b,c\}$ is transitive. $\square$

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[6]By Theorem 2, $f|A_1$ and $f|A_2$ are dictatorial. Because A_1 and A_2 are connected through the line segment $[(0,0),(0,10)]$ it follows from Theorem 1 that the dictator of $f|A_1$ and $f|A_2$ is the same and f is dictatorial.

Now it is easy to check that f is a transitive welfare function satisfying weak Pareto, independence of irrelevant alternatives, *and non-dictatorship.*

4   Oligarchies

Above any welfare function satisfied transitivity. Below social orderings are only required to be quasi-transitive. In order to formulate our main result, we need to introduce the notion of decisiveness and oligarchies.

Given $\emptyset \neq S \subseteq N$ and $a, b \in A$, we say that S is *decisive* over "a preferred to b" if for all $R \in \mathcal{R}_p^N$ we have $\neg b f(R) a$ whenever $a P_i b$ for all $i \in S$. Let $D(a, b)$ denote the set of all coalitions that are decisive over "a preferred to b". We say that S blocks "a preferred to b" if for all $R \in \mathcal{R}_p^N$ we have $b f(R) a$ whenever $b P_i a$ for all $i \in S$. Let $B(a, b)$ denote the set of all coalitions that block "a preferred to b".

Let f be a welfare function and S be a non-empty coalition. Then f is called *oligarchic with oligarchy S* if for all $a, b \in A$ such that $a \neq b$ we have $S \in D(a, b)$ and $\{i\} \in B(a, b)$ for all $i \in S$. In other words, for oligarchic welfare functions a is socially strictly preferred to b if and only if all members of the oligarchy strictly prefer a to b. Note that f is dictatorial if there exists $i \in N$ such that f is oligarchic with oligarchy $\{i\}$. It is easy to check that oligarchic welfare functions satisfy *weak Pareto* and *independence of irrelevant alternatives* and that the chosen social orderings are quasi-transitive.

Our main result establishes that oligarchic welfare functions are the only quasi-transitive welfare functions satisfying *weak Pareto* and *independence of irrelevant alternatives* if the set of alternatives has a non-empty and connected interior and it is contained in the closure of its interior.

Theorem 2 *Let $A \subseteq \mathbb{R}^m$ be such that $int(A) \neq \emptyset$, $int(A)$ is connected, and $A \subseteq cl(int(A))$. If $m \geq 2$, then oligarchic welfare functions are the only quasi-transitive welfare functions satisfying* weak Pareto *and* independence of irrelevant alternatives

on the domain $\mathcal{R}_p^N$.

We show Theorem 2 in the next section. Theorem 1 will be an immediate consequence of Theorem 2.

5 Proof of Theorem 2

Let f be a welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives* on the domain $\mathcal{R}_p^N$. Given $X \subseteq A$, let f^X denote the restriction of f to the set of alternatives X. Formally, for all $R \in \mathcal{R}_p^N$ such that for all $i \in N$, $p(R_i) \in X$, let $f^X(R|X) \equiv f(R)|X$. It is straightforward that f^X inherits *weak Pareto* and *independence of irrelevant alternatives* from f.

We introduce additional notation. Let $a, b, c \in \mathbb{R}^m$ be three distinct alternatives. Let $[a, b]$ denote the line segment between a and b, i.e. $[a, b] \equiv \{\lambda a + (1 - \lambda)b \,|\, \lambda \in [0, 1]\}$. Let $H(a, b)$ denote the surface of points with equal distance to a and b. This surface bisects $[a, b]$. In the non-degenerate case, when a, b, and c are non-collinear, the surfaces $H(a, b)$, $H(a, c)$, and $H(b, c)$ determine six disjoint open subspaces, say space(abc), space(acb), space(bac), space(bca), space(cab), and space(cba) (see Figure 1 for Euclidean preferences in $\mathbb{R}^2$). We named these spaces such that for all $R_i \in \mathcal{R}_p$, $p(R_i) \in$ space(acb) $\Leftrightarrow aP_icP_ib$ and likewise for the other spaces. Note that the collective boundary of space(abc) and space(acb) is a subset of $H(b, c)$. Let space($a(bc)$) = space(abc)$\cap$ space(acb) that is the set of all points $x \in H(b, c)$ such that for $R_i \in \mathcal{R}_p$, if $p(R_i) = x$, then aP_ibI_ic. Note that $H(a, b) \cap H(a, c) \cap H(b, c) \neq \emptyset$ and $\{a, b, c\}$ is a free triple in the domain $\mathcal{R}_p$ if $H(a, b) \cap H(a, c) \cap H(b, c) \cap \text{int}(A) \neq \emptyset$. Then the domain $\mathcal{R}_p$ contains all conceivable rankings of a, b, and c. Let d and u be two points in $\mathbb{R}^m$ such that for all coordinates $j \in \{1, 2, ..., m\}$ $d_j \leq u_j$. Let box(d, u) = $\{x \in \mathbb{R}^m|$ for all coordinates $j \in \{1, 2, ..., m\}$, $d_j \leq x_j \leq u_j\}$ denote the hyper box between d and u. Let cpbox(d, u) = $\{x \in \mathbb{R}^m|$ for all coordinates $j \in \{1, 2, ..., m\}$, $d_j = x_j$ or $x_j = u_j\}$ denote the set of corner points of box(d, u). Now for every triple of these

corner points, say a, b and c, $m \in H(a,b) \cap H(a,c) \cap H(b,c)$, where $m = \frac{1}{2}d + \frac{1}{2}u$ the middle of $[d,u]$. If $\text{box}(d,u) \subseteq \text{int}(A)$, then any triple of its corner points is a free triple.

Given $a \in \mathbb{R}^m$ and $\epsilon > 0$, let $\odot(a,\epsilon)$ denote the open ball with center a and radius ϵ, i.e. $\odot(a,\epsilon) \equiv \{x \in \mathbb{R}^m \mid \|a - x\|_p < \epsilon\}$. Given $a, b, c \in \mathbb{R}^m$, let $\text{plane}(a,b,c)$ denote the plane spanned by a, b, and c, i.e. $\text{plane}(a,b,c) \equiv \{a + \lambda(b-a) + \mu(c-a) \mid \lambda, \mu \in \mathbb{R}\}$. Let v and w be two points belonging to the boundary of $\odot(a,\epsilon)$, i.e. $\|a - v\|_p = \epsilon$ and $\|a - w\|_p = \epsilon$. The circular arc between v and w on $\odot(a,\epsilon)$ in $\text{plane}(a,v,w)$ is denoted by $\text{arc}(a;v,w)$.

Lemma 1 *Let $a \in int(A)$ and $b \in A$ such that $a \neq b$. Let $\epsilon > 0$ be such that $\odot(a,\epsilon) \subseteq int(A)$. Then there exist $c \in \odot(a,\epsilon)$ and $d \in int(A)$ such that for all $c' \in arc(d;a,c)$, $\{a,b,c'\}$ is a free triple in $\mathcal{R}_p$.*

Proof. Because $\text{int}(A)$ is connected and $A \subseteq \text{cl}(\text{int } A)$, we may choose $d \in H(a,b) \cap \text{int}(A)$ such that d belongs to a path from a to b in A. Further, as $d \in \text{int}(A)$ we may choose d such that it does not belong to $[a,b]$. Then a, b and d are not collinear. Because $d \in \text{int}(A)$, we have for some $\epsilon' > 0$, $\odot(d,\epsilon') \subseteq \text{int}(A)$. Now choose $c \in \odot(a,\epsilon) \cap \text{plane}(a,b,d)$ such that c belongs to the boundary of $\odot(d, \|a - d\|)$. For $c' \in \text{arc}(d;c,a)$ we have that $H(a,b) \cap H(a,c') \cap \text{plane}(a,b,d) = \{d\}$. Thus, $H(a,b) \cap H(a,c') \cap H(b,c') \cap \text{plane}(a,b,d) = \{d\}$. Since $d \in \text{int} A$, the two previous facts imply that for all $c' \in \text{arc}(d;c,a)$, $\{a,b,c'\}$ is a free triple in $\mathcal{R}_p$. $\square$

The following is a direct consequence of Blair and Pollack (1979).

Lemma 2 *Let $a,b,c \in A$ be such that $\{a,b,c\}$ is a free triple in the domain $\mathcal{R}_p$. Then $f^{\{a,b,c\}}$ is oligarchic.*

Now if $\{a,b,c\}$ is a free triple in the domain $\mathcal{R}_p$, then let $\emptyset \neq S_{\{a,b,c\}} \subseteq N$ be such that $f^{\{a,b,c\}}$ is oligarchic with oligarchy $S_{\{a,b,c\}}$. Note that any oligarchy is uniquely determined, i.e. if $f^{\{a,b,c\}}$ is oligarchic with oligarchies S_1 and S_2, then $S_1 = S_2$.

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The next lemma establishes that f is oligarchic on any open ball, which is contained in the interior of A.

Lemma 3 *Let $a \in int(A)$ and $\epsilon > 0$ be such that $\odot(a, \epsilon) \subseteq int(A)$. Then there exists $S \subseteq N$ such that , $f^{\odot(a,\epsilon)}$ is oligarchic with oligarchy S.*

Proof. Let $d, u \in (a, \epsilon)$ be such that $d_j \leq u_j$ for all $j \in \{1, \ldots, m\}$ and $box(d, u) \subseteq \odot(a, \epsilon)$. Every triple $a', b, c \in cpbox(d, u)$ is a free triple in $\mathcal{R}_p$. Thus, by Lemma 2, $f^{\{a',b,c\}}$ is oligarchic, say with oligarchy $S_{\{a',b,c\}}$. The same holds for any other triple of corner points $\{a'', b, c\}$ of $box(d, u)$. But then $S_{\{a',b,c\}} = S_{\{a'',b,c\}}$. Now it is immediate that $f^{cpbox(d,u)}$ is oligarchic with oligarchy $S_{\{a',b,c\}}$. Since this is true for any hyper box in $\odot(a, \epsilon)$ and the triples of corner points of those are connected, i.e. there are sequences of triples of corner points of boxes starting with one triple and ending with another triple, whose subsequent members have at least two points in common, it follows that $f^{\odot(a,\epsilon)}$ is oligarchic with oligarchy $S_{\{a,b,c\}}$. $\qquad\square$

Lemma 4 *There exists a non-empty $S \subseteq N$ such that for all $a \in int(A)$ and all $b \in A$, with $a \neq b$, $f^{\{a,b\}}$ is oligarchic with oligarchy S.*

Proof. First we proof the following: for all $a \in int(A)$, there exists a non-empty $S_a \subseteq N$ such that for all $b \in A \backslash \{a\}$,

$$f^{\{a,b\}} \text{ is oligarchic with oligarchy } S_a. \tag{1}$$

In order to proof this claim, let $a \in int(A)$ and $b, b' \in A$ be such that a, b and b' are distinct. It is sufficient to prove that $f^{\{a,b\}}$ and $f^{\{a,b'\}}$ are both oligarchic with the same oligarchy.

Lemma 1 implies that there exist $c, c' \in \odot(a, \epsilon) \subseteq int(A)$ such that $\|a - c\| = \|a - c'\|$ and both $\{a, b, c\}$ and $\{a, b', c'\}$ are free triples. Thus, by Lemma 2, both $f^{\{a,b,c\}}$ and $f^{\{a,b',c'\}}$, respectively, are oligarchic with oligarchy $S_{\{a,b,c\}}$ and $S_{\{a,b',c'\}}$, respectively. By Lemma 3 and $a, c, c' \in \odot(a, \epsilon)$, $S_{\{a,b,c\}} = S_{\{a,b',c'\}}$, which implies (1).

In order to finish the proof of Lemma 4, let $a, a' \in \text{int}(A)$. By (1), it is sufficient to show $S_a = S_{a'}$. If $a = a'$, then this is trivial. If $a \neq a'$, then by (1) we have $f^{\{a,a'\}}$ is oligarchic with oligarchies S_a and $S_{a'}$. Therefore, $S_a = S_{a'}$, the desired conclusion. $\square$

Theorem 2 will follow from our final lemma.

Lemma 5 *There exists a non-empty $S \subseteq N$ such that f is oligarchic with oligarchy S.*

Proof. By Lemma 4, $f^{\text{int}A}$ is oligarchic, say with oligarchy S. Let $a, b \in A \backslash \text{int}(A)$.

First, we show $S \in D(a, b)$. Let $R \in \mathcal{R}_p^N$ be such that aP_ib for all $i \in S$. Let $R' \in \mathcal{R}_p^N$ be such that for all $i \in S$, $p(R_i') = a$, and for all $i \in N \backslash S$, $R_i' = R_i$. Because $A \subseteq \text{cl}(\text{int}(A))$, there exists $c \in \text{int}(A)$ such that for all $i \in S$, $aP_i'cP_i'b$. By Lemma 4, we have both $S \in D(a, c)$ and $S \in D(c, b)$. Thus, both $\neg cf(R')a$ and $\neg bf(R')c$, and by quasi-transitivity of f, $\neg bf(R')a$. By $R'|\{a,b\} = R|\{a,b\}$ and *independence of irrelevant alternatives*, $\neg bf(R)a$. Because R was arbitrary, we obtain $S \in D(a, b)$.

Second, we show $\{j\} \in B(b, a)$ for all $j \in S$. Suppose to the contrary that for some $R \in \mathcal{R}_p^N$ we have both aP_jb and $\neg af(R)b$. Partition N as follows: $V = \{i \in N | bP_ia\}$, $U = \{i \in N | aI_ib\}$, and $T = \{i \in N | aP_ib\}$. Obviously, $j \in T$.

Because $A \subseteq \text{cl}(\text{int } A)$, we may choose $a' \in \text{int}(A)$ arbitrarily close to a and $b' \in \text{int}(A)$ arbitrarily close to b. Because $\text{int}(A)$ is connected, there is a path $\pi \colon [0, 1] \longrightarrow \text{int}(A)$ from a' to b' such that $\pi(0) = a'$ and $\pi(1) = b'$. Because a' is arbitrarily close to a and b' is arbitrarily close to b, we may assume that $H(a, b)$ separates a' and b'. Thus, for some $t \in [0, 1]$ $\pi(t) \in H(a, b)$. Since π is continuous, $t_1 \equiv \min\{t \in [0, 1] : \pi(t) \in H(a, b)\}$ is well-defined. Let $d \equiv \pi(t_1)$.

Next we show space$(adb) \cap A \neq \emptyset$. The surface $H(a, b)$ divides $\mathbb{R}^m$ in two open (half-)spaces $H_a \equiv \{x \in \mathbb{R}^m : \|a - x\|_p < \|b - x\|_p\}$ and $H_b \equiv \{x \in \mathbb{R}^m : \|b - x\|_p < \|a - x\|_p\}$. Note that $\pi(t) \in H_a$ for all $0 \leq t < t_1$. Because a' is arbitrarily close to a, we may assume that $H(a, d)$ separates a' and d. Now π also connects a' and d. Thus, for some $0 \leq t_0 < t_1$ we have $\pi(t_0) \in H(a, d) \cap H_a$. Since $\pi(t_0) \in \text{int}(A)$, it follows

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that $\bigodot(\pi(t_0), \epsilon) \subseteq \text{int}(A)$ for some $\epsilon > 0$. Because $\pi(t_0) \in H(a,d) \cap H_a$ it follows that $H(b,d)$ separates b and $\pi(t_0)$. Hence, $\bigodot(\pi(t_0), \epsilon) \cap \text{space}(adb) \neq \emptyset$.

Let $R' \in \mathcal{R}_p^N$ be such that for all $i \in V$, $p(R_i') \in \text{int}A$ is close to d and $\|a - p(R_i')\|_p > \|b - p(R_i)'\|_p$; for all $i \in U$, $p(R_i') = d$; and for all $i \in T$, $p(R_i') \in \text{space}(adb) \cap A$. Then for all $i \in V$, $dP_i'bP_i'a$; for all $i \in U$, $dP_i'bI_i'a$; and for all $i \in T$, $aP_i'dP_i'b$. By weak Pareto, $\neg bf(R')d$. By $R'|\{a,b\} = R|\{a,b\}$, independence of irrelevant alternatives, and $\neg af(R)b$, $\neg af(R')b$. Hence, by quasi-transitivity of f, $\neg af(R')d$. By $d \in \text{int}(A)$ and Lemma 4, $f^{\{a,d\}}$ is oligarchic with oligarchy S, a contradiction to $\{j\} \in B(d,a)$ and $aP_j'd$. $\qquad\square$

Finally, we show Theorem 1.

Proof of Theorem 1. Obviously, if f satisfies the requirements of Theorem 1, then f satisfies the requirements of Theorem 2. Hence, by Theorem 2, f is oligarchic with some non-empty oligarchy S. If $|S| = 1$, then f is dictatorial and Theorem 1 is true.

If $|S| \geq 2$, then let $j \in S$ and $a, b, c \in \text{int}(A)$ be such that $\{a, b, c\}$ is a free triple in the domain $\mathcal{R}_p$. Obviously, the existence of such a triple is guaranteed by $\text{int}(A) \neq \emptyset$ and Lemma 1.

Then let $R \in \mathcal{R}_p^N$ be such that aP_jbP_jc and bP_icP_ia for all $i \in N\backslash\{j\}$. By weak Pareto, $\neg cf(R)b$. Because f is oligarchic with oligarchy S and $j \in S$, we have $j \in B(a,b)$. Thus, by aP_jb, $af(R)b$. Then from transitivity of $f(R)$ and $\neg cf(R)b$ we obtain $\neg cf(R)a$. Since for all $i \in N\backslash\{j\}$, we have cP_ia, this implies $\{i\} \notin B(a,c)$ for all $i \in N\backslash\{j\}$. This contradicts $|S| \geq 2$ and f is oligarchic with oligarchy S. $\qquad\square$

6 Conclusion

As the careful reader may have observed, for the proof of Theorem 2 it is crucial that for any hyperbox $\text{box}(d,u)$, any triple of its corner points is free in the domain $\mathcal{R}_p$.

This property is also true for other domains such as L_p-single-dipped preferences: a preference relation $R_i \in \mathcal{W}$ is L_p-*single-dipped* if there exists a *worst point* $d(R_i) \in A$ such that for all $a, b \in A$, $aR_ib \Leftrightarrow \|a - d(R_i)\|_p \geq \|b - d(R_i)\|_p$. Let $\mathcal{D}_p$ denote the set of all L_p-single-dipped preferences. Single-dipped preferences play an important role for the location of public bads such as waste deposits or nuclear power plants. Similarly to the domain $\mathcal{R}_p$, a, b, and c is a free triple in the domain $\mathcal{D}_p$ if and only if $H(a, b) \cap H(a, c) \cap H(b, c) \cap \text{int}(A) \neq \emptyset$. Now the proof of Theorem 2 can be easily modified to show Theorem 2 for quasi-transitive welfare functions defined over the domain $\mathcal{D}_p^N$.

Other domains for which Theorem 2 holds include the domain of L_p-separable preferences: a preference relation $R_i \in \mathcal{W}$ is L_p-*separable* if there exists a *bliss point* $p(R_i) \in A$ and $\alpha_i > 0$ for $i \in \{1, \ldots, m\}$, such that for all $a, b \in A$, $aR_ib \Leftrightarrow \sum_{i=1}^m \alpha_i |a_i - p(R_i)_i|^p \leq \sum_{i=1}^m \alpha_i |b_i - p(R_i)_i|^p$. Let $\mathcal{R}_{sp}$ denote the set of all L_p-separable preferences. For L_p-separable preferences, Theorem 2 can be either shown directly by modifying its proof or via showing that L_p-separable preferences are isomorphic to L_p-single-peaked preferences. This means there is a continuous bijective transformation φ on $\mathbb{R}^m$ such that

$$(\sum_{i=1}^m \alpha_i |a_i - b_i|^p)^{1/p} = \|\varphi(a) - \varphi(b)\|_p$$

for all $a, b \in \mathbb{R}^m$. The function φ can be simply defined by $\varphi_i(a) = \alpha_i^{1/p} a_i$ for all $i \in \{1, \ldots, m\}$. Then Theorems 1 and 2 can be rephrased for this class of isomorphic distance functions because *weak Pareto* and *independence of irrelevant alternatives* are preserved by bijective transformations.

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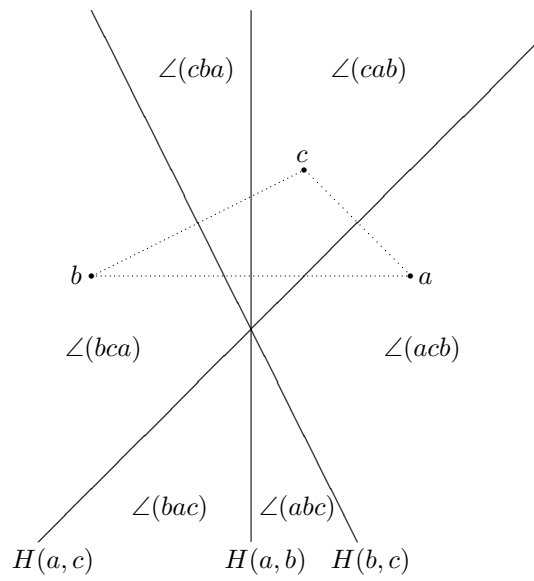
Figure 1: Illustration of the sets $\angle(abc)$, $\angle(acb)$, $\angle(bac)$, $\angle(bca)$, $\angle(cab)$, and $\angle(cba)$ for non-colinear a, b, and c in $\mathbb{R}^2$.