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ON THE GAME-THEORETIC STRUCTURE
OF PUBLIC-GOOD ECONOMIES

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Observe that $\tilde{y}(1) = 0$. Moreover, $\tilde{y}(\cdot)$ is nondecreasing: for $s = 1, \dots, n-1$, $\tilde{y}(s+1) - \tilde{y}(s) = s\tilde{v}(s+1) - 2\tilde{v}(s) + \tilde{v}(s-1)$, which is non-negative by virtue of (4). We now construct a benefit function U_0 as follows. First we define the function at the points $\tilde{y}(s)$ by

$$U_0(\tilde{y}(s)) = \tilde{v}(s) \quad \text{for } s = 1, \dots, n.$$

Then we extend U_0 to the whole of \mathbb{R}_+ by linear interpolation: $U_0(\lambda\tilde{y}(s) + (1-\lambda)\tilde{y}(s+1)) = \lambda U_0(\tilde{y}(s)) + (1-\lambda) U_0(\tilde{y}(s+1))$ for $s = 1, \dots, n-1$ and $\lambda \in [0, 1]$, and $U_0(\tilde{y}(n) + z) = U_0(\tilde{y}(n)) + k z$ for all $z \geq 0$, where k is any number in $(0, 1/n)$. To see that U_0 meets the criteria for a benefit function, note first that $U_0(0) = \tilde{v}(1) \geq 0$. By convexity - as expressed in (3) -, $U_0(\tilde{y}(s)) \leq U_0(\tilde{y}(s+1))$ for $s = 1, \dots, n-1$. Since \tilde{y} is nondecreasing, it follows that U_0 is nondecreasing (on the whole of \mathbb{R}_+). It is also obviously continuous.

Next, define C to be the identity mapping on \mathbb{R}_+ . This function is continuous, nondecreasing, and $C(0) = 0$. Moreover there exists $\bar{y} > 0$ such that $nU_0(y) < C(y)$ for all $y > \bar{y}$ because $k < 1/n$. For each nonempty $S \subset \mathbb{N}$, we have

$$s U_0(\tilde{y}(s)) - C(\tilde{y}(s)) = s \tilde{v}(s) - (s-1) \tilde{v}(s) + s \tilde{v}(s-1) = \tilde{v}(s).$$

To establish that (U_0, \dots, U_0, C) generates v , it only remains to be checked that

$$s U_0(\tilde{y}(s)) - C(\tilde{y}(s)) \geq s U_0(y) - C(y) \quad \text{for } s = 1, \dots, n,$$

for all $y \geq 0$. Since U_0 is piecewise linear, it suffices to prove the above inequality for $y = \tilde{y}(t)$, where $t = 0, \dots, n$. When $t = 0$, the inequality reduces to $\tilde{v}(s) \geq s \tilde{v}(1)$, which holds true by convexity. It remains to be shown that

$$\tilde{v}(s) \geq (s-t+1) \tilde{v}(t) - (s-t) \tilde{v}(t-1) \tag{5}$$

for $s = 1, \dots, n$ and $t = 1, \dots, n$. Clearly (5) holds whenever $1 \leq t = s \leq n$. To show that (5) holds true when $1 \leq t \leq s \leq n$, we proceed by induction. Suppose (5) holds for all s, t such that $1 \leq t = s - k \leq s \leq n$. Fix s, t such that $1 \leq t = s - k - 1 \leq s \leq n$. We must prove that

$$\tilde{v}(s) \geq (k + 2) \tilde{v}(s - k - 1) - (k + 1) \tilde{v}(s - k - 2).$$

Rewrite this inequality as

$$\tilde{v}(s) \geq [\tilde{v}(s - k - 1) - \tilde{v}(s - k - 2)] + [(k + 1) \tilde{v}(s - k - 1) - k \tilde{v}(s - k - 2)].$$

By the induction hypothesis, the second bracket on the right-hand side is not greater than $\tilde{v}(s - 1)$. Therefore we need only show that

$$\tilde{v}(s) - \tilde{v}(s - 1) \geq \tilde{v}(s - k - 1) - \tilde{v}(s - k - 2).$$

But this inequality holds true because of (3). We still have to check that (5) holds also when $1 \leq s \leq t \leq n$. For such values of s, t , writing $k = t - s$ allows us to rewrite (5) under the form

$$\tilde{v}(s + k - 1) - \tilde{v}(s) \leq (k - 1) (\tilde{v}(s + k) - \tilde{v}(s + k - 1)). \quad (6)$$

Writing the left-hand side as $[\tilde{v}(s + k - 1) - \tilde{v}(s + k - 2)] + \dots + [\tilde{v}(s + 1) - \tilde{v}(s)]$ and using (3) shows that (6) is true. The proof is now complete. ■

A second condition ensuring that a convex game is a public-good game is that the number of agents does not exceed three:

Proposition 2. Suppose $n \leq 3$. Then a game is convex if and only if it is a public-good game.

Proof. Since the result is obvious if $n \leq 2$, let us assume $n = 3$. As the "if" part needs no proof, fix a convex game v . For all (not necessarily distinct) $i, j \in N$, define the numbers $u_i(\emptyset) = 0$, $u_i(\{j\}) = v(\{i\})$, $u_i(N \setminus \{j\}) = v(N) - v(N \setminus \{i\})$, and $u_i(N) = v(N)$.

For each nonempty $S \subset N$, define $y(S) = \sum_{i \in S} u_i(S) - v(S)$. For each $i \in N$, let $U_i(y(S)) = u_i(S)$ for each S and extend the function U_i to \mathbb{R}_+ by linear interpolation as in the proof of Proposition 1. Let C be the identity mapping. It is a simple matter to check that (U_1, U_2, U_3, C) indeed generates v . ■

3. NECESSARY AND SUFFICIENT CONDITIONS

Let v be a game. Suppose that v is generated by a public-good economy (U, C) . For each nonempty coalition S , let $y(S)$ denote the smallest maximizer of $\sum_{i \in S} U_i(\cdot) - C(\cdot)$ over \mathbb{R}_+ and set $y(\emptyset) = 0$. For each $i \in N$ and $S \subset N$, define the numbers $u_i(S) := U_i(y(S))$ and $c(S) = C(y(S))$, and define the preordering¹ \leq on \mathcal{N} by

$$S \leq S' \Leftrightarrow y(S) \leq y(S'). \quad (7)$$

Notice that this preordering is compatible with the inclusion relation: for any $S, S' \subset N$, $S \subset S' \Rightarrow S \leq S'$. Also, the following conditions hold:

$$\left\{ \begin{array}{ll} v(S) = \sum_{i \in S} u_i(S) - c(S) & \forall S \subset N, \\ v(T) \geq \sum_{i \in T} u_i(S) - c(S) & \forall S, T \subset N, \\ c(S) \leq c(T) \text{ and } u_i(S) \leq u_i(T) & \forall i \in N, \forall S, T \subset N \text{ s.t. } S \leq T. \end{array} \right. \quad (8)$$

Conversely, suppose there exist an inclusion-compatible preordering \leq on \mathcal{N} and numbers $u_i(S), c(S)$ satisfying (8). (Note that this implies $c(\emptyset) = 0$, $c(S) \geq 0$ for all $S \subset N$ and $u_i(S) \geq 0$ for all $i \in N$ and $S \subset N$). Then it is straightforward to define a public-good economy that generates v . Select for each $S \subset N$ a number $y(S)$ in any way compatible with (7) and $y(\emptyset) = 0$. (Note that these numbers will be non-negative). For each agent i , let U_i assume the value $u_i(S)$ at the production level $y(S)$ and let the cost function C assume the value $c(S)$ at that level. Then extend these mappings to \mathbb{R}_+ by linear

¹ A preordering is a complete and transitive binary relation. The binary relations $<$ and \sim are defined from the preordering \leq in the usual way.

interpolation as in the proof of Proposition 1. We conclude that v is a public-good game if and only if there exist an inclusion-compatible preordering \leq on \mathcal{N} and numbers $u_i(S)$ and $c(S)$ for all $i \in N$ and $S \subset N$ that satisfy (8).

We must now investigate under which conditions on v and \leq system (8) admits a solution $(u, c) = (u_i(S), c(S))_{i \in N, S \subset N}$. In fact, such a solution exists if and only if there exists a vector $u = (u_i(S))_{i \in N, S \subset N}$ satisfying

$$\left\{ \begin{array}{l} \sum_{i \in T} u_i(S) - \sum_{i \in S} u_i(S) \leq v(T) - v(S) \quad \forall S, T \subset N, \\ u_i(S) \leq u_i(T) \quad \forall i \in N, \forall S, T \subset N \text{ s.t. } S \leq T. \end{array} \right\} \quad (9)$$

Necessity is obvious. To check sufficiency, fix u satisfying (9) and define $c(S) = \sum_{i \in S} u_i(S) - v(S)$: it is then apparent that (u, c) satisfies (8). It will be convenient to rewrite (9) in vector form. For any $T \subset N$, denote by $e(T)$ the n -dimensional vector defined by $e_i(T) = 1$ if $i \in T$ and $e_i(T) = 0$ if $i \in N \setminus T$. For any $S \subset N$ and any n -dimensional vector x , denote by x^S the $n \cdot 2^n$ -dimensional vector $(x^S(T))_{T \subset N}$ defined by $x^S(S) = x$ and $x^S(T) = 0$ if $T \neq S$. If $S \neq T$ and x, y are n -dimensional vectors, (x^S, y^T) denotes the $n \cdot 2^n$ -dimensional vector such that $(x^S, y^T)(S) = x$, $(x^S, y^T)(T) = y$, and $(x^S, y^T)(R) = 0$ for $R \neq S, T$. Using these notations, (9) is equivalent to

$$\left\{ \begin{array}{l} (e(T) - e(S))^S \cdot u \leq v(T) - v(S) \quad \forall S, T \subset N, S \neq T, \\ (e(\{i\})^S, -e(\{i\})^T) \cdot u \leq 0 \quad \forall i \in N, \forall S, T \text{ s.t. } S \leq T, S \neq T. \end{array} \right\} \quad (10)$$

Now, by a standard separation result (e.g., Rockafellar (1970), Theorem 22.1), system (10) admits a solution u if and only if the following conditions are satisfied: for all vectors $\alpha = (\alpha_{ST})_{S, T \subset N, S \neq T}$ and $\beta = (\beta_{iST})_{i \in N; S, T \subset N; S \leq T, S \neq T}$ of non-negative numbers such that

$$\sum_{\substack{S, T: \\ S \neq T}} \alpha_{ST} (e(T) - e(S))^S + \sum_{\substack{i, S, T: \\ S \leq T \\ S \neq T}} \beta_{iST} (e(\{i\})^S, -e(\{i\})^T) = 0, \quad (11)$$

we have

$$\sum_{\substack{S, T: \\ S \neq T}} \alpha_{ST}(v(T) - v(S)) \geq 0. \quad (12)$$

System (11) consists of n^2 equations, each one corresponding to a pair (i, S) in $N \times \mathcal{N}$. The equation corresponding to a given pair (i, S) reads

$$\sum_{T \neq S} \alpha_{ST}(e(T) - e(S))_i - \sum_{\substack{T: T \leq S \\ \neq}} \beta_{ITS} + \sum_{\substack{T: S \leq T \\ \neq}} \beta_{IST} = 0.$$

Since the first sum in this expression is worth $-\sum_{T \neq i} \alpha_{ST}$ when $i \in S$ and $\sum_{T \ni i} \alpha_{ST}$ when $i \in N \setminus S$, we obtain:

Lemma. A game v is a public-good game if and only if there exists an inclusion-compatible preordering \leq on \mathcal{N} such that

$$\sum_{\substack{S, T: \\ S \neq T}} \alpha_{ST}(v(T) - v(S)) \geq 0 \quad (12)$$

for all non-negative vectors α, β satisfying the conditions

$$v(i, S) \in N \times \mathcal{N}, \quad \sum_{\substack{T \leq S \\ \neq}} \beta_{ITS} - \sum_{\substack{T \geq S \\ \neq}} \beta_{IST} = \begin{cases} -\sum \alpha_{ST} & \text{if } S \ni i \\ T \ni i \\ \sum \alpha_{ST} & \text{if } S \not\ni i \\ T \ni i \end{cases}. \quad (13)$$

Inequality (12) does not involve the numbers β_{IST} . In order to obtain a clearer characterization result, we shall now replace system (13) by a system that does not involve the numbers β_{IST} either. To do this, we must find conditions on the numbers α_{ST} that are implied by (13) and conversely imply the existence of non-negative numbers β_{IST} such that (13) holds.

Fix $i \in N$ and some $S_0 \subset N$. Adding up over all coalitions $S \geq S_0$ the equations in (13) that correspond to i , we obtain

$$\sum_{\substack{S \geq S_0 \\ S \neq \emptyset}} \sum_{\substack{T \subset S_0 \\ T \neq \emptyset}} \beta_{iTS} = \sum_{\substack{S \geq S_0 \\ S \neq \emptyset}} \sum_{T \ni i} \alpha_{ST} - \sum_{\substack{S \geq S_0 \\ S \neq \emptyset}} \sum_{\substack{T \ni i \\ T \subset S_0}} \alpha_{ST}.$$

Since each β_{iTS} is non-negative and since both i and S_0 are arbitrary, we get the conditions

$$\sum_{\substack{S \geq S_0 \\ S \neq \emptyset}} \sum_{T \ni i} \alpha_{ST} \leq \sum_{\substack{S \geq S_0 \\ S \neq \emptyset}} \sum_{T \ni i} \alpha_{ST} \quad \forall i \in N, \forall S_0 \subset N. \quad (14)$$

In a similar manner, adding up the equations in (13) corresponding to i over all $S \leq S_0$ yields

$$-\sum_{\substack{S \leq S_0 \\ S \neq \emptyset}} \sum_{\substack{T \supset S_0 \\ T \neq \emptyset}} \beta_{iST} = \sum_{\substack{S \leq S_0 \\ S \neq \emptyset}} \sum_{T \ni i} \alpha_{ST} - \sum_{\substack{S \leq S_0 \\ S \neq \emptyset}} \sum_{\substack{T \ni i \\ T \supset S_0}} \alpha_{ST},$$

whence the conditions

$$\sum_{\substack{S \leq S_0 \\ S \neq \emptyset}} \sum_{T \ni i} \alpha_{ST} \leq \sum_{\substack{S \leq S_0 \\ S \neq \emptyset}} \sum_{T \ni i} \alpha_{ST} \quad \forall i \in N, \forall S_0 \subset N. \quad (15)$$

Having derived (14) and (15) from (13), let us consider the converse direction. Let α be a non-negative vector meeting conditions (14) and (15). Note that

$$\sum_{S \ni i} \sum_{T \ni i} \alpha_{ST} = \sum_{S \ni i} \sum_{T \ni i} \alpha_{ST} \quad \forall i \in N. \quad (16)$$

[Just choose $S_0 = \emptyset$ in (14) and $S_0 = N$ in (15)]. In order to find a non-negative vector β such that (α, β) satisfies (13), we will use the additional assumption that the preordering \leq is in fact an ordering (i.e., $S \leq S'$ implies $S = S'$). In view of (7), this amounts to assuming that different coalitions desire different levels of the public good.

More precisely, the smallest maximizer of $\sum_{i \in S} U_i(\cdot) - C(\cdot)$ differs from the smallest maximizer of $\sum_{i \in S'} U_i(\cdot) - C(\cdot)$ whenever $S \neq S'$. We call *regular* a public-good economy satisfying this condition. If \leq is an ordering, we can assign to each coalition an index $m = 1, \dots, M = 2^n$ so that $\emptyset = S_1 < \dots < S_m < S_{m+1} < \dots < S_M = N$. Define β as follows: for all $i \in N$ and $m = 1, \dots, M - 1$, let

$$\beta_{iS_m S_{m+1}} = \sum_{\substack{S < S_m \\ S \ni i}} \sum_{T \ni i} \alpha_{ST} - \sum_{\substack{S < S_{m+1} \\ S \ni i}} \sum_{T \ni i} \alpha_{ST}, \quad (17)$$

and let $\beta_{iSS'} = 0$ for all remaining pairs (S, S') . Expression (17) is non-negative because of (15). Moreover, (16) allows us to write

$$\beta_{iS_m S_{m+1}} = \sum_{\substack{S > S_{m+1} \\ S \ni i}} \sum_{T \ni i} \alpha_{ST} - \sum_{\substack{S > S_m \\ S \ni i}} \sum_{T \ni i} \alpha_{ST}. \quad (18)$$

Fixing $1 \leq m \leq M$, we get for each $i \in N$

$$\sum_{\substack{T < S_m \\ T \ni i}} \beta_{iTS_m} - \sum_{\substack{T > S_m \\ T \ni i}} \beta_{iS_m T} = \beta_{iS_{m-1} S_m} - \beta_{iS_m S_{m+1}}, \quad (19)$$

where $\beta_{iS_0 S_1} = \beta_{iS_M S_{M+1}} := 0$. Using (18) to compute $\beta_{iS_{m-1} S_m}$ and (17) to compute

$\beta_{iS_m S_{m+1}}$, (19) becomes

$$\sum_{\substack{S > S_m \\ S \ni i}} \sum_{T \ni i} \alpha_{ST} - \sum_{\substack{S > S_m \\ S \ni i}} \sum_{T \ni i} \alpha_{ST} - \sum_{\substack{S < S_m \\ S \ni i}} \sum_{T \ni i} \alpha_{ST} + \sum_{\substack{S < S_m \\ S \ni i}} \sum_{T \ni i} \alpha_{ST}.$$

If $i \in S_m$, this expression reduces to

$$\sum_{T \ni i} \sum_{S \ni i} \alpha_{ST} - \sum_{S \ni i} \sum_{T \ni i} \alpha_{ST} - \sum_{T \ni i} \alpha_{S_m T} = - \sum_{T \ni i} \alpha_{S_m T}.$$

while if $i \in S_m$ it equals

$$\sum_{S \ni i} \sum_{T \ni i} \alpha_{ST} + \sum_{T \ni i} \alpha_{S_m T} - \sum_{S \ni i} \sum_{T \ni i} \alpha_{ST} = \sum_{T \ni i} \alpha_{S_m T},$$

in accordance with (13). We have proved:

Theorem. A game v is generated by a regular public-good economy if and only if there exists an inclusion-compatible ordering \leq on \mathcal{N} such that

$$\sum_{\substack{S, T: \\ S \neq T}} \alpha_{ST} (v(T) - v(S)) \geq 0$$

for all non-negative vectors α satisfying the conditions

$$\sum_{\substack{S \geq S_0 \\ S \ni i}} \sum_{T \ni i} \alpha_{ST} \leq \sum_{\substack{S \geq S \\ S \ni i}} \sum_{T \ni i} \alpha_{ST} \quad \text{and} \quad \sum_{\substack{S \leq S_0 \\ S \ni i}} \sum_{T \ni i} \alpha_{ST} \leq \sum_{\substack{S \leq S \\ S \ni i}} \sum_{T \ni i} \alpha_{ST}$$

for all $i \in N$ and $S_0 \subset N$.

The necessary and sufficient conditions provided in the above Lemma and Theorem are difficult to interpret. They do indicate, however, that a highly structured pattern of *differences* between the worths of the various coalitions is characteristic of the public-good games. This pattern must be compatible with the classical convexity condition (as will be shown below) but it implies many more restrictions.

4. SOME NECESSARY CONDITIONS

This section extracts from our characterization results some fairly simple properties of public-good games and regular public-good games. We start off by deriving from the Lemma some "two-sided" necessary conditions. These are conditions obtained by choosing two coalitions S, S' such that $S \leq S'$ and choosing the non-negative vectors α, β in the Lemma such that

$$\alpha_{TT'} = 0 \quad \forall T \in \{S, S'\}, \forall T' \subset N, \quad \text{and} \quad \beta_{iTT'} = 0 \quad \forall i \in N, \forall (T, T') \neq (S, S').$$

System (13) then reduces to

$$\left\{ \begin{array}{l} \sum_{T \ni i} \alpha_{ST} = \beta_{iSS'} = - \sum_{T \ni i} \alpha_{S'T} \quad \text{if } i \in S \cap S', \\ - \sum_{T \ni i} \alpha_{ST} = \beta_{iSS'} = \sum_{T \ni i} \alpha_{S'T} \quad \text{if } i \in N \setminus (S \cup S'), \\ - \sum_{T \ni i} \alpha_{ST} = \beta_{iSS'} = - \sum_{T \ni i} \alpha_{S'T} \quad \text{if } i \in S' \setminus S, \\ \sum_{T \ni i} \alpha_{ST} = \beta_{iSS'} = \sum_{T \ni i} \alpha_{S'T} \quad \text{if } i \in S \setminus S'. \end{array} \right. \quad (20)$$

Since α, β are non-negative, it follows from this system that for all i, S, S' and T ,

$$\left\{ \begin{array}{l} \beta_{iSS'} > 0 \Rightarrow i \in S \setminus S', \\ \alpha_{ST} > 0 \Rightarrow (S \cap S') \subset T \subset S, \\ \alpha_{S'T} > 0 \Rightarrow S' \subset T \subset (S \cup S'). \end{array} \right. \quad (21)$$

For each $B \subset SS'$, define $a_{SB} := \alpha_{S \setminus B}$ and $a_{S'B} := \alpha_{S' \setminus B}$. Taking account of (21), we obtain from the last condition in (20),

$$\sum_{\substack{B: \\ i \in B \subset SS'}} a_{SB} = \sum_{\substack{B: \\ i \in B \subset SS'}} a_{S'B} \quad \text{for all } i \in SS'. \quad (22)$$

Applying the Lemma yields:

Corollary 1. If v is a public-good game, then there exists an inclusion-compatible preordering \leq on \mathcal{N} such that

$$\sum_{B \subset SS'} a_{SB} (v(S) - v(S \setminus B)) \leq \sum_{B \subset SS'} a_{S'B} (v(S' \cup B) - v(S')) \quad (23)$$

for all coalitions S, S' such that $S \leq S'$ and all non-negative vectors $a_S = (a_{SB})_{B \subset SS'}$ and $a_{S'} = (a_{S'B})_{B \subset SS'}$ satisfying (22).

A simple way to satisfy (22) is to fix a set $B_0 \subset SS'$ and let $a_{SB_0} = a_{S'B_0} = 1$ and $a_{SB} = a_{S'B} = 0$ for all $B \subset SS', B \neq B_0$. This yields:

Corollary 2. If v is a public-good game, then there exists an inclusion-compatible preordering \leq on \mathcal{N} such that $v(S) - v(S \setminus B_0) \leq v(S' \cup B_0) - v(S')$ for all S, S' such that $S \leq S'$ and all $B_0 \subset S \setminus S'$.

Since \leq is a preordering, it is a complete binary relation. Given any two coalitions S, S' , we may therefore assume that, say, $S \leq S'$. Taking $B_0 = S \setminus S'$ and applying Corollary 2 yields $v(S) + v(S') \leq v(S \cap S') + v(S \cup S')$. Thus we recover the result mentioned in Section 2:

Corollary 3. If v is a public-good game, then it is convex.

Another way to satisfy (22) is as follows. Let \mathcal{A} and \mathcal{A}' be two partitions of $S \setminus S'$. For each $B \subset S \setminus S'$, set $a_{SB} = 1$ if $B \in \mathcal{A}$, $a_{SB} = 0$ otherwise, and $a_{S'B} = 1$ if $B \in \mathcal{A}'$, $a_{S'B} = 0$ otherwise. We get:

Corollary 4. If v is a public-good game, then there exists an inclusion-compatible preordering \leq on \mathcal{N} such that

$$\sum_{B \in \mathcal{A}} (v(S) - v(S \setminus B)) \leq \sum_{B' \in \mathcal{A}'} (v(S' \cup B') - v(S'))$$

for all S, S' such that $S \leq S'$ and all partitions $\mathcal{A}, \mathcal{A}'$ of $S \setminus S'$. In particular,

$$v(S) - v(S \cap S') \leq \sum_{i \in S \setminus S'} (v(S' \cup \{i\}) - v(S')), \text{ and}$$

$$\sum_{i \in S \setminus S'} (v(S) - v(S \setminus \{i\})) \leq v(S \cup S') - v(S')$$

whenever $S \leq S'$.

The solutions to (22) studied so far all share the property that a_{SB} and $a_{S'B}$ are either zero or one for all $B \subset S \setminus S'$. This is by no means necessary. Indeed, (22) is met whenever $a_S = (a_{SB})_{B \subset S \setminus S'}$ and $a_{S'} = (a_{S'B})_{B \subset S \setminus S'}$ are two collections of balanced

weights (in the Bondareva-Shapley sense) for $S \setminus S'$. As is well known, this does not imply that all coordinates of a_S and $a_{S'}$ are either zero or one.

It should be clear that all of the above corollaries remain true if the phrase "public-good game" is replaced with "regular public-good game" and the term "preordering" is replaced with "ordering". Thus, for instance,

Corollary 1'. If v is a regular public-good game, then there exists an inclusion-compatible ordering \leq on \mathcal{N} such that (23) holds for all coalitions $S \leq S'$ and all non-negative vectors $a_S, a_{S'}$, satisfying (22).

The question arises whether the two-sided necessary conditions of Corollary 1' are sufficient to ensure that a game is generated by a regular public-good economy. As we shall now see, this is unfortunately not the case. The argument may be adapted to show that the conditions of Corollary 1, do not suffice to guarantee that a game is a public-good game. Fix three coalitions $S < S' < S''$ and choose the vector α in the Theorem such that

$$\alpha_{TT} = 0 \quad \forall T \in \{S, S', S''\}, \forall T' \subset N. \quad (24)$$

Define $\Sigma = S \setminus (S' \cup S'')$, $\Sigma' = S \setminus (S \cup S'')$, $\Sigma'' = S' \setminus (S \cup S'')$, $\Sigma\Sigma' = (S \cap S') \setminus S''$, $\Sigma\Sigma'' = (S \cap S'') \setminus S'$, $\Sigma'\Sigma'' = (S' \cap S'') \setminus S$, $\Sigma\Sigma'\Sigma'' = S \cap S' \cap S''$, and $\Sigma_0 = N \setminus (S \cup S' \cup S'')$. Note that the sets just defined partition N .

Under (24), conditions (14) and (15) on α reduce to the following system:

$$\left. \begin{aligned} \sum_{T \ni i} \alpha_{ST} &\leq \sum_{T \ni i} \alpha_{S'T} + \sum_{T \ni i} \alpha_{S''T} \\ \sum_{T \ni i} \alpha_{S'T} + \sum_{T \ni i} \alpha_{S''T} &\leq \sum_{T \ni i} \alpha_{ST} \end{aligned} \right\} \text{ for all } i \in \Sigma,$$

$$\left. \begin{aligned} \sum_{T\theta i} \alpha_{S^*T} &\leq \sum_{T\exists i} \alpha_{S^*T} \\ \sum_{T\exists i} \alpha_{ST} &\leq 0 \\ \sum_{T\exists i} \alpha_{ST} + \sum_{T\exists i} \alpha_{S^*T} &\leq \sum_{T\theta i} \alpha_{S^*T} \end{aligned} \right\} \text{for all } i \in \Sigma',$$

$$\left. \begin{aligned} \sum_{T\theta i} \alpha_{S^*T} &\leq 0 \\ \sum_{T\exists i} \alpha_{ST} + \sum_{T\exists i} \alpha_{S^*T} &\leq 0 \end{aligned} \right\} \text{for all } i \in \Sigma'',$$

$$\left. \begin{aligned} \sum_{T\theta i} \alpha_{ST} + \sum_{T\theta i} \alpha_{S^*T} &\leq \sum_{T\exists i} \alpha_{S^*T} \\ \sum_{T\exists i} \alpha_{S^*T} &\leq \sum_{T\theta i} \alpha_{ST} + \sum_{T\theta i} \alpha_{S^*T} \end{aligned} \right\} \text{for all } i \in \Sigma\Sigma',$$

$$\left. \begin{aligned} \sum_{T\theta i} \alpha_{ST} + \sum_{T\theta i} \alpha_{S^*T} &\leq \sum_{T\exists i} \alpha_{S^*T} \\ \sum_{T\theta i} \alpha_{S^*T} &\leq 0 \\ \sum_{T\exists i} \alpha_{S^*T} &\leq \sum_{T\theta i} \alpha_{ST} \end{aligned} \right\} \text{for all } i \in \Sigma\Sigma'',$$

$$\left. \begin{aligned} \sum_{T\theta i} \alpha_{S^*T} + \sum_{T\theta i} \alpha_{S^*T} &\leq 0 \\ \sum_{T\exists i} \alpha_{ST} &\leq 0 \end{aligned} \right\} \text{for all } i \in \Sigma'\Sigma'',$$

$$\sum_{T\theta i} \alpha_{ST} + \sum_{T\theta i} \alpha_{S^*T} + \sum_{T\theta i} \alpha_{S^*T} \leq 0 \quad \text{for all } i \in \Sigma\Sigma'\Sigma'',$$

$$\sum_{T\exists i} \alpha_{ST} + \sum_{T\exists i} \alpha_{S^*T} + \sum_{T\exists i} \alpha_{S^*T} \leq 0 \quad \text{for all } i \in \Sigma_0,$$

which in turn is equivalent to the following conditions:

$$\alpha_{ST} > 0 \Rightarrow (S \cap S' \cap S'') \subset T \subset S, \quad (25)$$

$$\alpha_{S'T} > 0 \Rightarrow (S' \cap S'') \subset T \subset (S \cup S'), \quad (26)$$

$$\alpha_{S''T} > 0 \Rightarrow S'' \subset T \subset (S \cup S' \cup S''), \quad (27)$$

$$\sum_{T \ni i} \alpha_{S'T} = \sum_{T \ni i} \alpha_{S''T} \quad \text{for all } i \in \Sigma', \quad (28)$$

$$\sum_{T \ni i} \alpha_{ST} = \sum_{T \ni i} \alpha_{S'T} \quad \text{for all } i \in \Sigma\Sigma'', \quad (29)$$

$$\sum_{T \ni i} \alpha_{ST} = \sum_{T \ni i} (\alpha_{S'T} + \alpha_{S''T}) \quad \text{for all } i \in \Sigma, \quad (30)$$

$$\sum_{T \ni i} (\alpha_{ST} + \alpha_{S'T}) = \sum_{T \ni i} \alpha_{S''T} \quad \text{for all } i \in \Sigma\Sigma'. \quad (31)$$

We shall limit ourselves to two solutions to system (24) - (31) that yield nontrivial properties of regular public-good games. The first solution is as follows: define

$$T_0 = S \setminus \Sigma\Sigma'',$$

$$T_1 = (S' \cup \Sigma\Sigma'') \setminus \Sigma\Sigma',$$

$$T_2 = S'' \cup \Sigma\Sigma',$$

and set $\alpha_{ST_0} = \alpha_{S'T_1} = \alpha_{S''T_2} = 1$, $\alpha_{TT'} = 0$ if $(T, T') \in \{(S, T_0), (S', T_1), (S'', T_2)\}$.

Check that α satisfies (24) to (31). Applying the Theorem yields the following necessary condition for v to be generated by a regular public-good economy:

$$[v(S) - v(S \setminus \Sigma\Sigma'')] + [v(S') - v((S' \cup \Sigma\Sigma'') \setminus \Sigma\Sigma')] \leq v(S'' \cup \Sigma\Sigma') - v(S'').$$

Defining $T_0 = S \setminus \Sigma$, $T_1 = (S \cup \Sigma) \setminus \Sigma$, $T_2 = S'' \cup \Sigma$ and defining α as before yields another necessary condition, namely:

$$[v(S) - v(S \setminus \Sigma)] + [v(S') - v((S \cup \Sigma) \setminus \Sigma)] \leq v(S'' \cup \Sigma) - v(S'').$$

It is not difficult to see that neither of these "three-sided" conditions is implied by the two-sided conditions of Corollary 1'.

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