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EXACT INFERENCE METHODS FOR FIRST-ORDER AUTOREGRESSIVE DISTRIBUTED LAG MODELS

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RÉSUMÉ

Nous proposons des tests et régions de confiance exacts applicables à des modèles autorégressifs d'ordre un avec régresseurs exogènes et perturbations i.i.d. Pour des hypothèses linéaires générales sur les coefficients de régression, nous obtenons des procédures d'inférence dont le niveau est connu. Les tests proposés sont α-semblables (i.e., la probabilité de rejet est la même sous tous les processus générateurs compatibles avec l'hypothèse nulle) ou sont basés sur des bornes qui ne dépendent pas de paramètres de nuisance. Par conséquent, les régions de confiance correspondantes sont soit α-semblables (i.e., la probabilité de couverture est constante), soit conservatrices. Nous développons aussi des tests et régions de confiance pour des transformations non linéaires des paramètres du modèle, tels que des multiplicateurs de long terme et des délais moyens. Nous établissons aussi la validité asymptotique des méthodes exactes proposées sous des conditions de régularité usuelles et illustrons leur utilité par des applications à un modèle de tendance pour la vélocité de la monnaie aux É.-U. ainsi qu'un modèle de demande de monnaie au Canada.

Mots clés : autorégression, test à bornes, inférence exacte, restrictions linéaires, multiplicateur de long terme, délai moyen, test de Monte Carlo, randomisation, test α-semblable

ABSTRACT

Methods are proposed to build exact tests and confidence sets in the linear first-order autoregressive distributed lag model with i.i.d. disturbances. For general linear hypotheses concerning the regression coefficients, inference procedures are obtained which have known level. The tests proposed are either similar (i.e., they have constant rejection probability for all data generating processes consistent with the null hypothesis) or involve bounding distributions which are free of nuisance parameters. Correspondingly, the confidence sets are either similar with known size (i.e., they have constant coverage probability) or conservative. We also develop exact tests and confidence sets for various nonlinear transformations of the model parameters, such as long-run multipliers and mean lags. The asymptotic validity of the finite sample inference methods proposed is also established under usual regularity conditions, and their practical usefulness is illustrated with applications to a dynamic trend model of money velocity and to a model of Canadian money demand.

Key words : autoregression, bounds test, exact inference, linear restrictions, long-run multiplier, mean lag, Monte Carlo test, randomization, similar test

1. Introduction

Since most economic relationships are determined by temporal factors much attention has been paid in econometric theory over the last half century to the specification and analysis of dynamic models. A very early reference in the statistical literature is Yule (1926), and many studies followed; see Hendry *et al.* (1984) for an overview. A serious problem in dynamic models and econometrics generally is that statistical procedures which yield finite-sample inference in simple static linear models have an indeterminate distribution in more general and realistic models, because of the presence of unknown nuisance parameters. Rothenberg (1984) discusses procedures to approximate the unknown exact finite sample distributions of estimators and test statistics. When a dynamic relationship can be modelled in the form of the well-known normal classical linear model with finite distributed lags, no serious small sample problems are met. However, if the appropriate model has infinite distributed lags and has to be parametrized in the popular form of an autoregressive distributed lag model, then the nuisance parameter problem arises. Since invariance of standard inference techniques with respect to nuisance parameters is often regained asymptotically as the sample size increases, it is common practice to employ asymptotic approximations in dynamic models. However, by simply relying on asymptotic results when analyzing a finite (and often rather small) sample, one just accepts to commit approximation errors of a largely unknown nature and magnitude. This objectionable practice is due to the fact that hardly any operational exact inference procedures are available for dynamic models to date.

In the literature some solutions to the nuisance parameter problem have been suggested for particular simple dynamic models. These solutions often involve the derivation of bounds which do not depend on nuisance parameters, resulting in bounds tests and conservative confidence sets; see Dufour (1990), Hillier and King (1987), Kiviet (1980, 1991), Krämer, Kiviet and Breitung (1991), Vinod (1976), Vinod and Ullah (1981, Ch.4) and Zinde-Walsh and Ullah (1987). For very special cases also alternative inference procedures have been developed which are invariant with respect to nuisance parameters and allow exact inference; see Hillier (1987) and Kiviet and Phillips (1990, 1992). Here we combine these two approaches to develop exact inference techniques for a much more general and popular class of dynamic models.

The very special case where the lags are characterized by the presence

of one common factor (see Sargan, 1980) and where all the explanatory variables are fixed, has recently been examined in Dufour (1990). This linear regression model with AR(1) errors can be stated as:

$$(1.1) \qquad y_t = x_t'\beta + u_t \,, \qquad t = 1,\ldots,T \,,$$

$$u_t = \rho\, u_{t-1} + \varepsilon_t \,, \qquad \varepsilon_t \sim \text{IIN}(0,\sigma^2)$$

where y_t is the dependent variable (at time t), x_t is a kx1 vector of fixed (or strongly exogenous) regressors, β is a kx1 vector of fixed coefficients, and u_t is a random disturbance; the parameters β, ρ and σ^2 are unknown. Various assumptions regarding the range of the serial correlation coefficient (ρ) and the distribution of the initial disturbance term (u_1) are considered. In Dufour (1990) the literature on exact inference procedures for hypotheses on ρ and β is reviewed and several shortcomings of these earlier procedures are discussed. Then an alternative method to obtain exact tests and confidence sets in model (1.1) is proposed which proves to be more effective. After showing how an exact confidence set for ρ can be constructed from an exact autocorrelation test, the latter is first used to obtain a simultaneous confidence set for ρ and any relevant linear transformation of β. A corresponding exact similar test easily follows. By a projection method exact bounding confidence sets are then constructed for linear transformations of β only, and so an exact bounds-type test procedure for those functions of β follows. This procedure avoids the problems of previous bounds methods which did not exploit exact inference on ρ. See Kiviet (1991) for a recent application of the latter approach, and Krämer, Kiviet and Breitung (1991) for an illustration of one of its major drawbacks.

It is well-known that model (1.1), which forms one of the simplest dynamic alternatives to a fully static model (where $\rho = 0$ and x_t does not contain any lagged explanatory variables), has serious shortcomings when the relationship is genuinely dynamic; see Hendry and Mizon (1978). In this paper we develop exact inference procedures for the coefficients of the model

$$(1.2) \qquad y_t = \lambda\, y_{t-1} + x_t'\beta + \varepsilon_t \,, \qquad t = 1,\ldots,T \,,$$

$$\varepsilon_t \sim \text{IID}(0,\sigma^2)$$

where y_t, x_t, ε_t and β are as in (1.1). Note that we do not make special

assumptions on the domain of λ or the distribution of y_0. The exact inference procedures to be developed are based on test statistics whose distributions, under the null hypothesis, do not depend on the nuisance parameters (λ, β or σ) or the distribution of y_0; hence y_0 may either be fixed or random. Normality of the disturbances is not required for these invariance properties to hold. However, the actual distribution function has to be known for the calculation of exact significance points and for the construction of exact confidence sets, and this is often relatively easy under normality. The class of models (1.2) does not only represent relevant simple dynamic econometric relationships of the autoregressive-distributed lag form; it also covers the simple Dickey-Fuller type models for inference on unit roots in the presence of an intercept, drift, polynomial trends, seasonal dummies and any interactions. We shall illustrate the exact procedures in both types of dynamic models.

The basic building block that is exploited in Dufour (1990) to obtain exact inference for any subvector of β in model (1.1) is an exact confidence interval for the parameter ρ. In the same way an exact confidence interval for the coefficient λ enables exact inference in the more general dynamic model (1.2). Exact inference on λ can be obtained by applying least-squares to an augmented regression model, as set out in Kiviet and Phillips (1990, 1992), where in fact two different procedures are proposed. Here we shall develop a closely related third procedure for producing exact inference on λ from an augmented regression model. The latter procedure has a neat likelihood-ratio interpretation. Therefore, from that point of view, it appears more attractive than the relatively ad-hoc procedures developed earlier. Next we shall show how this type of approach also yields an exact similar test for any joint hypothesis on the value of λ and some subvector of β as well as corresponding joint confidence sets. However, we also employ the two-stage procedure proposed in Dufour (1990) to obtain joint confidence sets, since it is found (in a third stage) that this easily yields exact inference for any vector linear transformation (or subvector) of β. In this approach an exact confidence interval for λ is combined with the corresponding family of "conditional" confidence sets for the relevant linear transform of β, and then in the third stage, a test on just the β vector follows by applying a union-intersection method. Such a test takes the form of a "generalized bounds test", which is based on two test statistics with nested critical regions: the smaller critical region yields a conservative test and the larger one gives a liberal test, while the difference between

the two regions may be viewed as an "inconclusive" region.

The exact critical values or p-values of the test statistics mentioned above are obtainable from Monte Carlo experiments on particular pivotal statistics, which are determined by the observed data, the parameters specified by the null hypothesis, and also by the adopted type of distribution function of the disturbances. They are not determined by any unknown parameters. These simulated p- or critical values can, in principle, be calculated to any required degree of precision and thus yield exact inference. However, the number of required simulation replications may get extremely large. Therefore, we propose another method for producing simulation based genuinely exact tests and confidence sets from a finite number of replications which proves to be operational.

Apart from inference on linear transformations of the coefficients of model (1.2), practitioners are usually interested in inference on particular nonlinear transformations of the coefficients, such as long-run multipliers, (interim) impact multipliers, and mean and median lags. Exact procedures for testing nonlinear hypotheses in the static linear model are examined in Dufour (1989). We show here that analogous procedures can be applied in the dynamic model (1.2) in such a way that exact inference on any nonlinear hypothesis can be obtained.

A drawback of the exact inference techniques may seem to be that they require relatively stronger (distributional) assumptions, in contrast with the usual approach which capitalizes on asymptotic arguments. However, we will show that the exact techniques are also valid asymptotically under usual weaker assumptions, such as weakly exogenous regressors and reasonably regular non-normal disturbances. Thus, having an equivalent asymptotic justification, the main advantage of the techniques proposed here is that they are also exact for a reference case, and hence are probably more accurate in many other situations. The feasibility of the exact inference techniques will be illustrated in a number of empirical examples, where the exact results are set against the usual asymptotic approximations.

The structure of this paper is as follows. In Section 2, we discuss various ways to produce exact inference on the value of λ (the coefficient of the lagged dependent variable) by putting the test problem into a form such that the likelihood-ratio-type test statistic does not depend on nuisance parameters. Then, in Section 3, we extend this approach in various ways and develop procedures for exact inference on both the value of λ and linear transformations of β (the coefficients of the exogenous regressors). These

results are used in Section 4 for the derivation of exact inference on hypotheses which do not involve λ and only restrict the components of β, as well as restrictions involving both λ and elements of β. In Section 5, we develop exact tests for nonlinear transformations of the coefficients such as long-run multipliers. In Section 6, we prove the asymptotic validity of the procedures. Section 7 illustrates the practical usefulness of the exact inference techniques by applying them to a dynamic trend model of money velocity in the U.S. and to a model of Canadian money demand. Section 8 concludes.

2. Exact Inference on λ

Due to its notation, model (1.2) has the appearance of a simple partial adjustment model. However, the dynamic relationship between the explanatory variables and the dependent variable may involve dynamics of a more sophisticated nature. The underlying relationship could in fact be

$$(2.1) \qquad y_t = \lambda y_{t-1} + \sum_{j=1}^{J} \sum_{i=0}^{L(j)} \delta_{ji} z_{t-i}^{(j)} + \varepsilon_t , \qquad t = 1,\ldots,T$$

where the regressors are the one-period lagged dependent variable and finite (hence $L(j) \leq L < \infty$) distributed lags of J linearly independent fixed or strongly exogenous explanatory variables $z^{(j)}$, $j = 1,\ldots,J$. For regressors $z^{(j)}$ such as the constant or the linear trend $L(j) = 0$. Particular coefficients δ_{ji} may be zero, but $\delta_{j,L(j)} \neq 0$. All non-zero coefficients δ_{ji} can be stacked in a k×1 coefficient vector β, and the corresponding values of the regressor variables $z_{t-i}^{(j)}$ can be collected in a corresponding k×1 vector x_t such that (2.1) can be written as (1.2) or as

$$(2.2) \qquad y = \lambda y_{-1} + X\beta + \varepsilon ,$$

where $X = [x_1,\ldots,x_T]'$ is a $T \times k$ matrix, $y = (y_1,\ldots,y_T)'$, $y_{-1} = (y_0,\ldots,y_{T-1})'$, and $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_T)'$.

For our finite sample results we make use of the following assumption.

ASSUMPTION A: *The T×k matrix X and the T×1 vector ε are independent, the value y_0 is independent of ε, and rank(X) = k with probability 1. The coefficients λ and β are constant but unknown, with $\beta \in \mathbb{R}^k$ and $\lambda \in \mathcal{D}_\lambda$ where $\mathcal{D}_\lambda = \{\lambda | -\infty \leq \lambda_L \leq \lambda \leq \lambda_R \leq \infty\}$.*

ASSUMPTION B: *The distribution of the vector ε/σ (given X and y_0) is known, where σ is an unknown positive constant.*

The assumption that X and ε are independent (*strong exogeneity*) entails that X can be treated as fixed for inference purposes, i.e. all inferences can be made conditionally on X, which we shall do from now on. Assumption B means that the distribution of ε is known up to a scale factor. Often we will simply suppose that the elements of ε are i.i.d. normally distributed with mean zero, but it will be straightforward to use other distributions, provided they are known up to a scale factor. This also includes cases where the elements of ε show heteroscedasticity and/or serial correlation of a given form, but below we shall not pay special attention to these cases and assume that the model has already been transformed in such a way that the application of maximum-likelihood (conditional on X and y_0) is equivalent with ordinary least-squares.

When dealing with asymptotic results we will make different (in some respects weaker and in other respects more specific) assumptions. Several of the invariance results given below hold without Assumption B.

From (2.2), one easily obtains

$$(2.3) \qquad y_{-1} = y_0 \iota(\lambda) + C(\lambda)X\beta + C(\lambda)\varepsilon \ ,$$

where

$$\iota(\lambda) = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ . \\ . \\ . \\ \lambda^{T-1} \end{bmatrix} \ , \qquad C(\lambda) = \begin{bmatrix} 0 & . & . & . & . & . & 0 \\ 1 & 0 & & & & & . \\ \lambda & 1 & 0 & & & & . \\ \lambda^2 & . & . & . & & & . \\ . & & . & . & . & & . \\ . & & & . & . & . & . \\ \lambda^{T-2} & & . & . & \lambda & 1 & 0 \end{bmatrix} .$$

So, apart from dependence on y_0, X, β and λ, the elements of y_{-1} are a weighted sum of present and previous elements of the disturbance vector ε. This all complicates the distribution of the least-squares estimators of the coefficients of (2.2) considerably. These estimators are

$$(2.4) \qquad \hat{\lambda} = (y'_{-1}M[X]y_{-1})^{-1}y'_{-1}M[X]y = \lambda + (y'_{-1}M[X]y_{-1})^{-1}y'_{-1}M[X]\varepsilon ,$$

where $M[X] = I - X(X'X)^{-1}X'$ and

$$(2.5) \qquad \hat{\beta} = (X'X)^{-1}X'(y - \hat{\lambda}y_{-1})$$
$$= \beta + (X'X)^{-1}X'\varepsilon - (X'X)^{-1}X'y_{-1}(y'_{-1}M[X]y_{-1})^{-1}y'_{-1}M[X]\varepsilon .$$

Testing $H(\lambda_0)$: $\lambda = \lambda_0$ by the standard t-ratio, which we denote by $t_\lambda(\lambda_0)$, and using the tabulated critical values, does not provide an exact test, even when ε has a standard normal distribution. This will be shown in a more general context in Section 3.1 below.

Since dependence and invariance, especially under the null, are primary issues in this study, the notation of exact critical values of test statistics (indicated by a "$\bar{\ }$") is deliberately chosen such that it is made clear what is tested, by which test statistic, and what the determining factors of these critical values are. Hence, the right-hand and left-hand critical values of the exact level $\alpha = \alpha_R + \alpha_L$ test $t_\lambda(\lambda_0)$ are denoted by $\bar{t}_\lambda(\alpha_R;\lambda_0,X,y_0,\beta,\sigma)$ and $\bar{t}_\lambda(1-\alpha_L;\lambda_0,X,y_0,\beta,\sigma)$ respectively. These are such that under Assumptions A and B (for $\alpha_L \geq 0$, $\alpha_R \geq 0$ and $0 < \alpha_L + \alpha_R = \alpha < 1$):

$$P[t_\lambda(\lambda_0) \leq \bar{t}_\lambda(\alpha_R;\lambda_0,X,y_0,\beta,\sigma) \mid \lambda = \lambda_0] = 1 - \alpha_R$$

and

$$P[t_\lambda(\lambda_0) \leq \bar{t}_\lambda(1-\alpha_L;\lambda_0,X,y_0,\beta,\sigma) \mid \lambda = \lambda_0] = \alpha_L .$$

These critical values are different from the Student-t critical values $\bar{t}(\alpha_R;T-k-1) = -\bar{t}(1-\alpha_R;T-k-1)$. Hence, even under normality the standard classical least-squares based inference procedures, i.e. t- and $\mathcal{F}$- tests, do not yield exact results in model (2.1); this is due to the (weakly exogenous) stochastic nature of the regressor y_{-1}.

Of course practitioners can take the inexactness of standard tests for granted and, although a finite sample is analyzed, rely simply on asymptotic theory assuming that particular regularity conditions are fulfilled and

asymptotic properties do reasonably well hold for the actual finite sample under study. Here, however, we want to show how such speculations and sometimes quite inaccurate approximations [see Nankervis and Savin (1985, 1987)] can be avoided by devising exact inference on λ and/or a subset of β in model (2.1) under Assumptions A and B.

2.1 Various Similar Tests on λ

In Kiviet and Phillips (1990, 1992) two procedures are developed for testing the null hypothesis $\lambda = \lambda_0$ exactly in model (2.2). The null distributions of these test statistics are free of the nuisance parameters β and σ; moreover, they are invariant with respect to the value and the (stochastic) nature of y_0. However, the null distribution of these tests does depend on both λ_0 and X, so it is not feasible to produce general tables of exact critical values, but it is possible to compute the relevant p-values (or critical values) by using numerical or simulation methods. Exact confidence sets for λ may also be constructed by "inverting" these test procedures. These two particular exact tests of $\lambda = \lambda_0$ are based on straightforward least-squares results in a regression model which corresponds to (2.2) augmented by a number of redundant strongly exogenous regressors:

$$y = \lambda y_{-1} + X(\lambda_0)\beta_* + \varepsilon \ , \tag{2.6}$$

where the matrix $X(\lambda_0)$ has full column rank and is constructed such that the linear space spanned by its columns contains the space spanned by the columns of the matrix $Z(\lambda_0) = [X \vdots \varepsilon(\lambda_0) \vdots C(\lambda_0)X]$. When $Z(\lambda_0)$ has full column rank, we can take $X(\lambda_0) = Z(\lambda_0)$; otherwise $X(\lambda_0)$ can be obtained from $Z(\lambda_0)$ by deleting appropriate columns from $Z(\lambda_0)$. Extending a model by including particular redundant regressors in order to achieve invariance of tests has also been suggested in Dufour and Dagenais (1985) and Dufour and King (1991,p.125) with respect to inference problems in model (1.1).

The least-squares estimator of λ in (2.6) is

$$\hat{\lambda}(\lambda_0) = (y'_{-1}M[X(\lambda_0)]y_{-1})^{-1}y'_{-1}M[X(\lambda_0)]y \ , \tag{2.7}$$

where

$$M[X(\lambda_0)] = I - X(\lambda_0)[X(\lambda_0)'X(\lambda_0)]^{-1}X(\lambda_0)' \ .$$

The two statistics proposed in Kiviet and Phillips (1990, 1992) for testing $\lambda = \lambda_0$ are

$$(2.8) \qquad c^*_\lambda(\lambda_0) = \hat{\lambda}(\lambda_0) - \lambda_0$$

and the t-ratio

$$(2.9) \qquad t^*_\lambda(\lambda_0) = c^*_\lambda(\lambda_0) \; / \; [\hat{V}[\hat{\lambda}(\lambda_0)]]^{1/2} \; ,$$

where $\hat{V}[\hat{\lambda}(\lambda_0)]$ is the estimated variance of $\hat{\lambda}(\lambda_0)$:

$$(2.10) \qquad \hat{V}[\hat{\lambda}(\lambda_0)] = \frac{1}{T\text{-}1\text{-}\mathrm{rank}[X(\lambda_0)]} \frac{[y - \hat{\lambda}(\lambda_0)y_{-1}]' M[X(\lambda_0)][y - \hat{\lambda}(\lambda_0)y_{-1}]}{y'_{-1} M[X(\lambda_0)] y_{-1}} \; .$$

We also suggest here alternative similar test statistics for $\lambda = \lambda_0$, which are based on applying the likelihood ratio principle to (2.6) while acting as if normality holds. For the moment we only derive and investigate such tests informally. In the next section, we shall formulate and prove a proposition concerning similar tests on hypotheses involving both λ and linear transformations of β of which the present procedures are special cases.

Regression (2.6) can be obtained by rewriting model (2.2) as

$$(2.11) \qquad y = \lambda[y_{-1} - y_0{}'(\lambda_0) - C(\lambda_0)X\beta] + \lambda[y_0{}'(\lambda_0) + C(\lambda_0)X\beta] + X\beta + \varepsilon \; ,$$

which is a special case (for $\lambda_* = \lambda$) of the model

$$y = \lambda[y_{-1} - y_0{}'(\lambda_0) - C(\lambda_0)X\beta] + \lambda_*[y_0{}'(\lambda_0) + C(\lambda_0)X\beta] + X\beta + \varepsilon \; ,$$

or

$$(2.12) \qquad y = \lambda y_{-1} + X\beta + \lambda_{**}{}'(\lambda_0) + C(\lambda_0)X\beta_{**} + \varepsilon \; ,$$

where $\lambda_{**} = (\lambda_* - \lambda)y_0$ and $\beta_{**} = (\lambda_* - \lambda)\beta$. Clearly, we can test

$$H_*(\lambda_0): \; \lambda = \lambda_0$$

by testing the extended restricted model

$$(2.12a) \qquad y = \lambda_0 y_{-1} + X\beta + \lambda_{**} s(\lambda_0) + C(\lambda_0) X\beta_{**} + \varepsilon \ ,$$

where β, λ_{**} and β_{**} are taken as free parameters, against (2.12).

For any T×m matrix Z let $SS_0(Z|\lambda_0)$ and $SS_1(Z)$ be the following residual sums of squares:

$$(2.13) \qquad SS_0(Z|\lambda_0) = \min_{\theta} (y - \lambda_0 y_{-1} - Z\theta)'(y - \lambda_0 y_{-1} - Z\theta)$$

$$(2.14) \qquad SS_1(Z) = \min_{\theta} (y - Z\theta)'(y - Z\theta) \ .$$

Then the likelihood ratio (LR) statistic for testing $H_*(\lambda_0)$ [i.e. (2.12a)] against (2.12) is a simple monotonic transformation of the statistic

$$(2.15) \qquad \mathcal{L}^*_\lambda(\lambda_0) = \frac{SS_0[X(\lambda_0)|\lambda_0]}{SS_1[W(\lambda_0)]} = \frac{(y-\lambda_0 y_{-1})' M[X(\lambda_0)](y-\lambda_0 y_{-1})}{y' M[W(\lambda_0)] y} \ ,$$

where $W(\lambda_0) = [y_{-1} | X(\lambda_0)]$, with $X(\lambda_0)$ as defined below (2.6). It is easily verified that

$$(T - 1 - \text{rank}[X(\lambda_0)]) \cdot [\mathcal{L}^*_\lambda(\lambda_0) - 1] = [t^*_\lambda(\lambda_0)]^2 \ ,$$

which shows clearly that the statistics defined in (2.8) and (2.9) can be interpreted as tests of $\lambda = \lambda_0$ in the context of the extended model (2.12). Further, as we shall show below, the null distribution of $\mathcal{L}^*_\lambda(\lambda_0)$, like that of $t^*_\lambda(\lambda_0)$, does not depend on β, σ or y_0.

It is clear, however, that these tests do not take into account all the restrictions entailed by $\lambda = \lambda_0$ on (2.12). Model (2.12) reduces to model (2.2) with $\lambda = \lambda_0$ only when the following restrictions are imposed on (2.12):

$$H_{**}(\lambda_0): \ \lambda = \lambda_0 \ , \ \lambda_{**} = 0 \ , \ \beta_{**} = 0 \ .$$

This suggests one to consider the LR statistic for testing $H_{**}(\lambda_0)$ in the context of model (2.12), which is a monotonic transformation of

$$(2.16) \qquad \mathcal{L}_{\lambda}^{**}(\lambda_0) = \frac{SS_0[X|\lambda_0]}{SS_1[W(\lambda_0)]} = \frac{(y-\lambda_0 y_{-1})'M[X](y-\lambda_0 y_{-1})}{y'M[W(\lambda_0)]y} .$$

Exploiting the partitioning of the matrix $W(\lambda_0)$, a decomposed expression for $M[W(\lambda_0)]$ can be obtained by use of the following Lemma, which is also known as the Frisch-Waugh theorem (see, Gouriéroux and Monfort, 1989, Vol.2, p.478). The proofs of the lemmas and propositions are given in the Appendix.

LEMMA 1: *Let the partitioned matrix* $C = [A|B]$ *have full column rank and let* $M[C] = I - P[C]$, *where* $P[C] = C(C'C)^{-1}C'$ *denotes the orthogonal projection matrix on* C . *Then* $M[C] = M[B] - P[M[B]A]$.

Applying this Lemma, the denominator of (2.15) and (2.16) can be written as

$$(2.17) \qquad SS_1[W(\lambda_0)] = y'M[W(\lambda_0)]y = \varepsilon'M[W(\lambda_0)]\varepsilon$$

$$= \varepsilon'M[X(\lambda_0)]\varepsilon - \frac{(\varepsilon'M[X(\lambda_0)]y_{-1})^2}{y'_{-1}M[X(\lambda_0)]y_{-1}} .$$

Under $H_{**}(\lambda_0)$, we have $SS_0[X|\lambda_0] = \varepsilon'M[X]\varepsilon$, whereas it follows from (2.3) that $M[X(\lambda_0)]y_{-1} = M[X(\lambda_0)]C(\lambda_0)\varepsilon$ under $\lambda = \lambda_0$, so that

$$(2.18) \qquad \mathcal{L}_{\lambda}^{**}(\lambda_0) = \frac{\varepsilon'M[X]\varepsilon}{\varepsilon'M[X(\lambda_0)]\varepsilon - \dfrac{(\varepsilon'M[X(\lambda_0)]C(\lambda_0)\varepsilon)^2}{\varepsilon'C(\lambda_0)'M[X(\lambda_0]C(\lambda_0)\varepsilon}}$$

under $H_{**}(\lambda_0)$. The distribution of $\mathcal{L}_{\lambda}^{*}(\lambda_0)$ under $H_{*}(\lambda_0)$ is analogous with $\varepsilon'M[X(\lambda_0)]\varepsilon$ in the numerator. Clearly, the scale factor σ cancels in (2.18) and the expression for the null distribution of $\mathcal{L}_{\lambda}^{*}(\lambda_0)$ and for $\mathcal{L}_{\lambda}^{**}(\lambda_0)$ only involve: ε/σ, λ_0 and X. Hence, like for $c_{\lambda}^{*}(\lambda_0)$ and for $t_{\lambda}^{*}(\lambda_0)$, the statistics $\mathcal{L}_{\lambda}^{*}(\lambda_0)$ and $\mathcal{L}_{\lambda}^{**}(\lambda_0)$ are pivotal under the null. Their null distributions can be assessed by simulation, provided the distribution of ε/σ is entirely specified [e.g., when $\varepsilon_1,\ldots,\varepsilon_T$ are i.i.d. $N(0,\sigma^2)$]. Note that the $\mathcal{L}_{\lambda}^{**}(\lambda_0)$

procedure is a generalization of the test denoted Φ_3 and presented in Dickey and Fuller (1981, p.1059) for the special case where $\lambda_0 = 1$ and $X = \iota(1)$.

2.2 Exact Confidence Sets for λ

Let us now consider the construction of confidence sets for λ. We focus first on sets derived from the $t^*_\lambda(\lambda_0)$ test procedure. The $c^*_\lambda(\lambda_0)$ test, whose null distribution can be obtained by the Imhof (1961) procedure (see Kiviet and Phillips, 1990), can be used in the same way. We reject $\lambda = \lambda_0$ when the $t^*_\lambda(\lambda_0)$ test statistic is either too small or too large. So let $\bar{t}^*_\lambda(\alpha;\lambda_0,X)$ be the point such that

$$(2.19) \qquad G_{t*}[\bar{t}^*_\lambda(\alpha;\lambda_0,X)|\lambda_0,X] = 1 - \alpha$$

where

$$(2.20) \qquad G_{t*}[x|\lambda_0,X] = P[t^*_\lambda(\lambda_0) \leq x \mid \lambda_0,X]$$

is the cumulative distribution function of $t^*_\lambda(\lambda_0)$ when $\lambda = \lambda_0$ (for a given regressor matrix X). It is easy to see that $G_{t*}[x|\lambda_0,X]$ is continuous in x. If we choose α_L and α_R such that $\alpha = \alpha_L + \alpha_R$ $(0 < \alpha < 1\ ,\ \alpha_L \geq 0\ ,\ \alpha_R \geq 0)$, then the set

$$(2.21) \qquad \bar{\mathcal{C}}_\lambda(\alpha_L,\alpha_R) = \left\{ \lambda_0 \in \mathcal{D}_\lambda : \bar{t}^*_\lambda(\alpha_L;\lambda_0,X) \leq t^*_\lambda(\lambda_0) \leq \bar{t}^*_\lambda(1-\alpha_R;\lambda_0,X) \right\},$$

is a confidence set for λ with level $1-\alpha$, i.e. $P[\lambda \in \bar{\mathcal{C}}_\lambda(\alpha_L,\alpha_R)] \geq 1-\alpha$, with the equality holding when $t^*_\lambda(\lambda_0)$ has a continuous distribution. Correspondingly, the set

$$(2.22) \qquad \mathcal{C}_\lambda(\alpha_L,\alpha_R) = \left\{ \lambda_0 \in \mathcal{D}_\lambda : \alpha_L \leq G_{t*}[t^*_\lambda(\lambda_0)|\lambda_0,X] \leq 1-\alpha_R \right\},$$

contains $\bar{\mathcal{C}}_\lambda(\alpha_L,\alpha_R)$ and must also have level $1-\alpha$, with $\mathcal{C}_\lambda(\alpha_L,\alpha_R) = \bar{\mathcal{C}}_\lambda(\alpha_L,\alpha_R)$ when $G_{t*}[x|\lambda_0,X]$ is strictly monotonic in x for all $\lambda_0 \in \mathcal{D}_\lambda$. So, by checking if $\alpha_L \leq G_{t*}[t^*_\lambda(\lambda_0)|\lambda_0,X] \leq 1-\alpha_R$, we can decide whether a given λ_0 belongs to $\mathcal{C}_\lambda(\alpha_L,\alpha_R)$. Note that the distribution of the random variable $G_{t*}[t^*_\lambda(\lambda_0)|\lambda_0,X]$ is uniform over the interval [0,1] when $t^*_\lambda(\lambda_0)$ has a continuous distribution.

Even if $\alpha_L = \alpha_R$, the set $\mathcal{C}_\lambda(\alpha_L,\alpha_R)$ is generally not a symmetric

interval around a least-squares estimate of λ, as is the case for standard confidence intervals based on t-ratio's. Also, one cannot simply state that an interval $[\lambda_0^L , \lambda_0^R]$ is a tight $(1-\alpha_L-\alpha_R)$ level confidence interval for λ if

$$t_\lambda^*(\lambda_0^L) = \tau_\lambda^*(1-\alpha_L;\lambda_0^L,X) \quad , \quad t_\lambda^*(\lambda_0^R) = \tau_\lambda^*(\alpha_R;\lambda_0^R,X) \ ,$$

because nothing guarantees that the function $G_{t*}[t_\lambda^*(\lambda_0)|\lambda_0,X]$ is a monotonic function of λ_0.

Confidence sets based on the $\mathcal{L}_\lambda^*(\lambda_0)$ and $\mathcal{L}_\lambda^{**}(\lambda_0)$ tests are somewhat less complicated to describe. For example, if we define the critical value $\mathcal{L}_\lambda^*(\alpha;\lambda_0,X)$ such that

$$(2.23) \qquad G_{\mathcal{L}*}[\mathcal{L}_\lambda^*(\alpha;\lambda_0,X)|\lambda_0,X] = 1 - \alpha \ ,$$

where

$$(2.24) \qquad G_{\mathcal{L}*}[x|\lambda_0,X] = P[\mathcal{L}_\lambda^*(\lambda_0) \le x \mid \lambda_0,X]$$

is the cumulative distribution function of $\mathcal{L}_\lambda^*(\lambda_0)$ when $\lambda = \lambda_0$ (for a given regressor matrix X), then the set

$$(2.25) \qquad \mathcal{C}_\lambda^*(\alpha) = \left\{ \lambda_0 \in \mathcal{D}_\lambda : \mathcal{L}_\lambda^*(\lambda_0) \le \mathcal{L}_\lambda^*(\alpha;\lambda_0,X) \right\}$$

$$= \left\{ \lambda_0 \in \mathcal{D}_\lambda : G_{\mathcal{L}*}[\mathcal{L}_\lambda^*(\lambda_0)|\lambda_0,X] \le 1 - \alpha \right\}$$

is a confidence set for λ with level $1-\alpha$. Confidence sets based on $\mathcal{L}_\lambda^{**}(\lambda_0)$ are built in the same way.

In practice the explicit assessment of the critical value for each iteration of λ_0 can be avoided by constructing the set from the second formulation in (2.25) as follows. Generating in a Monte Carlo simulation N independent realizations of the $\mathcal{L}_\lambda^*(\lambda_0)$ statistic under $\lambda = \lambda_0$, which can be written as

$$(2.26) \qquad \mathcal{L}_\lambda^*(\lambda_0) = \left[1 - \frac{(\varepsilon' M[X(\lambda_0)]C(\lambda_0)\varepsilon)^2}{(\varepsilon' C(\lambda_0)' M[X(\lambda_0)]C(\lambda_0)\varepsilon)(\varepsilon' M[X(\lambda_0)]\varepsilon)} \right]^{-1} ,$$

the frequency of the occurrence of $\mathfrak{L}^*_\lambda(\lambda_0) \leq \hat{\mathfrak{L}}^*_\lambda(\lambda_0)$ gives an estimate of $G_{\mathfrak{L}^*}[\hat{\mathfrak{L}}^*_\lambda(\lambda_0)|\lambda_0,X]$, where $\hat{\mathfrak{L}}^*_\lambda(\lambda_0)$ is the test test statistic computed from the observed data. If this exceeds $1-\alpha$ then λ_0 is not in the $\mathfrak{C}^*_\lambda(\alpha)$ set. By searching over $\mathfrak{D}_\lambda$ the full set (2.25) can be established to a certain degree of precision, depending on the number of Monte Carlo replications N and the intricacy of the (grid-)search over the λ domain. As it happens the number of replications has in fact to be extremely high. Using a normal approximation, the standard deviation of the $G_{\mathfrak{L}^*}[\mathfrak{L}^*_\lambda(\lambda_0)|\lambda_0,X]$ estimate in the neighbourhood of $1-\alpha$ is $[\alpha(1-\alpha)/N]^{1/2}$, and keeping this below 1% of α thus requires $N > 10^4(1-\alpha)/\alpha$; i.e. at $\alpha = 0.05$ this requires $N = 190{,}000$ and that seems prohibitively large.

To avoid this difficulty, we shall use instead "randomized" or "Monte Carlo" versions of the tests based on $\mathfrak{L}^*_\lambda(\lambda_0)$ and $\mathfrak{L}^{**}_\lambda(\lambda_0)$ which yield genuinely exact and much cheaper procedures. Such tests were originally suggested by Dwass (1957), Barnard (1963) and Birnbaum (1974). They are based on a general property stated in the following lemma.

LEMMA 2: *Let Z_j, $j = 1,\ldots,N$, be independent and identically distributed (i.i.d.) real random variables with continuous distribution, and let $R_j(N)$ be the rank of Z_j when $Z_1,\ldots,Z_N$ are ranked in nondecreasing order ($j = 1,\ldots,N$), i.e.,*

$$(2.27) \qquad R_j(N) = \sum_{i=1}^{N} U(Z_j - Z_i) ,$$

where $U(x) = 1$ if $x \geq 0$, and $U(x) = 0$ if $x < 0$. Then, for $j = 1,\ldots,N$,

$$(2.28) \qquad P[R_j(N)/N \geq x] = \begin{cases} 1 & , \text{ if } x \leq 0 \\ (1 + I[N(1-x)])/N & , \text{ if } 0 < x \leq 1 \\ 0 & , \text{ if } x > 1 , \end{cases}$$

where $I[x]$ is the largest integer less than or equal to x.

This lemma can be used in the following way. Let Z_1 be the value of a test statistic computed from an observed sample, and $Z_2,\ldots,Z_N$ i.i.d. random variables with the same distribution as Z_1 under a null hypothesis of

interest. For example, to test $\lambda = \lambda_0$, we may take $Z_1 = \hat{\ell}^*_\lambda(\lambda_0) = \ell^*_{\lambda 1}(\lambda_0)$ the observed test statistic $\ell^*_\lambda(\lambda_0)$, and $Z_j = \ell^*_{\lambda j}(\lambda_0)$, $j = 2,\ldots,N$, independent replications of the variable defined in (2.26). Clearly $Z_2,\ldots,Z_N$ can easily be generated by Monte Carlo methods. Then for $0 < \alpha < 1$ and by choosing $c_N(\alpha)$ to be a positive real number such that $I[(1-c_N(\alpha))N] + 1 \leq N\alpha$, we can conclude from Lemma 2 that the critical region

$$(2.29) \qquad R_1(N)/N \geq c_N(\alpha)$$

has size not larger than α, provided Z_1 has a continuous distribution [which is the case for $\ell^*_\lambda(\lambda_0)$ whenever the vector ε has a continuous distribution]. In particular, for

$$(2.30) \qquad c_N(\alpha) = 1 - \frac{I[N\alpha]}{N} + \frac{1}{N},$$

we have

$$(2.31) \qquad \alpha - (1/N) \leq P[R_1(N)/N \geq c_N(\alpha)] \leq \alpha$$

with $P[R_1(N)/N \geq c_N(\alpha)] = \alpha$ when $N\alpha$ is a positive integer. Thus by using a sufficiently large number of replications (N), we can make the size of the critical region $R_1(N)/N \geq c_N(\alpha)$ as close as we wish to α, and equal to α (also for relatively small values of N) by setting $N\alpha$ to be an integer. Note also that the critical region defined by (2.29)-(2.30) is equivalent to

$$(2.32) \qquad p_N = \frac{N + 1 - R_1(N)}{N} \leq \frac{I[N\alpha]}{N},$$

so that p_N can be interpreted as a "randomized" or "Monte Carlo" p-value for testing the null hypothesis.

It is straightforward to obtain a confidence set for λ from such a Monte Carlo test. If we generate N-1 i.i.d. replications $\eta_{(2)},\ldots,\eta_{(N)}$ of the vector $\eta = \varepsilon/\sigma$ and compute the corresponding values $\ell^*_{\lambda j}(\lambda_0)$, $j = 2,\ldots,N$, of $\ell^*_\lambda(\lambda_0)$ in (2.26), the following confidence set has level $1-\alpha$ for λ:

$$(2.33) \qquad \mathcal{C}^*_\lambda(\alpha) = \{ \lambda_0 \in \mathcal{D}_\lambda : R_1(N,\lambda_0)/N \leq c_N(\alpha) \}$$

where $R_1(N,\lambda_0)$ is the rank of the observed test statistic $\hat{\ell}^*_\lambda(\lambda_0) = \ell^*_{\lambda 1}(\lambda_0)$

among $\xi^*_{\lambda j}(\lambda_0)$, $j = 1,\ldots,N$, and $c_N(\alpha)$ is defined by (2.30). Note that the same disturbance vectors $\eta_{(2)},\ldots,\eta_{(N)}$ are used for all values λ_0.

For finite N this confidence set is not equivalent with (2.25), nor are the corresponding test procedures. In the simulation test an extra random element has been brought in which yields genuine exactness for finite values of N. Given the fact that we cannot assess the null distribution of (2.26) analytically, the simulation test procedure is the only really exact way to proceed. There is typically a power loss associated with the simulation. However, as $N \to \infty$, the Monte Carlo procedure becomes equivalent to the corresponding non-randomized procedure under weak regularity conditions; for further discussion, see Birnbaum (1974), Dwass (1957), Edgington (1980), Foutz (1980) and Jöckel (1986). For other applications of Monte Carlo tests in a time series context, see Dufour and Hallin (1987) and Theil and Shonkweiler (1986).

Since it is not clear which one of the various tests for $\lambda = \lambda_0$ given above is the most powerful in a particular model, it is tempting (and perhaps sensible) to construct exact confidence intervals for λ based on various procedures, and to compare their width and location before a choice is made. However, this will affect the significance level and so leaves room for further study of the relative power and interdependence of these similar tests.

3. Exact Inference on λ and β

The above results will be used now in order to enable exact inference on λ and β simultaneously. Let R be a known rxk matrix with $\text{rank}(R) = r$. Since R may be void we have $0 \leq r \leq k$. Let

$$\gamma = R\beta \quad , \tag{3.1}$$

hence γ is an rx1 vector of linearly independent linear transformations of β. We now construct exact tests for the null hypothesis $\lambda = \lambda_0$, $\gamma = \gamma_0$ and corresponding joint confidence intervals for (λ,γ) with exact size $1-\alpha$. In Section 2 we already considered the case $r = 0$. In Section 4 we shall develop exact inference procedures on γ only, and also for the case where we have just one restriction involving both λ and some elements of β.

To keep the notation relatively simple, we reparametrize the model and transform the regressors. Let $\bar{R}$ be a (k-r)×k matrix, such that $Q = [R' \vdots \bar{R}']'$ is non-singular, and let $Q\beta = (\gamma', \bar{\gamma}')'$; of course, $\bar{R}$ and $\bar{\gamma} = \bar{R}\beta$ are not unique. Model (2.2) can be rewritten now as

$$(3.2) \qquad y = \lambda\, y_{-1} + X(Q)^{-1}Q\beta + \varepsilon = \lambda\, y_{-1} + Z\gamma + \bar{Z}\bar{\gamma} + \varepsilon \ ,$$

where $[Z \vdots \bar{Z}] = XQ^{-1}$ is of order T×k and has full column rank.

3.1 A Comprehensive Procedure

First we consider testing $H(\lambda_0, \gamma_0)$, i.e. using a non-extended model, and show that the usual procedure yields a non-similar test. We base our test statistic again on the LR principle, and act – in order to obtain the test statistic – as if we have normally distributed disturbances, whereas in fact the true distribution of the similar test statistics will be established by simulation (instead of relying on the asymptotic null distribution of LR statistics under regularity). Proceeding in the standard way and conditioning upon y_0, the LR statistic is $T.\ln(SS_0[\bar{Z}|\lambda_0,\gamma_0]/SS_1[W])$, where $SS_0[\bar{Z}|\lambda_0,\gamma_0]$ is the restricted sum of squared residuals

$$(3.3) \qquad SS_0[\bar{Z}|\lambda_0,\gamma_0] = (y - \lambda_0 y_{-1} - Z\gamma_0)'M(\bar{Z})(y - \lambda_0 y_{-1} - Z\gamma_0)$$

and $SS_1[W]$ is the unrestricted sum of squared residuals defined by

$$(3.4) \qquad SS_1[W] = y'M(W)y = \varepsilon'M(W)\varepsilon \ , \quad W = [y_{-1} \vdots X] \ .$$

Note that $M[y_{-1} \vdots X] = M[y_{-1} \vdots XQ^{-1}]$.

We examine the monotonic transformation of the LR statistic

$$(3.5) \qquad \ell_{\lambda,\gamma}(\lambda_0,\gamma_0) = \frac{SS_0[\bar{Z}|\lambda_0,\gamma_0]}{SS_1[W]} \ .$$

Under $H(\lambda_0,\gamma_0)$ we have $SS_0[\bar{Z}|\lambda_0,\gamma_0] = \varepsilon'M(\bar{Z})\varepsilon$. Using Lemma 1, (3.4) can be written as

$$(3.6) \qquad SS_1[W] = \varepsilon' M(W)\varepsilon = \varepsilon' M[X]\varepsilon - \frac{(\varepsilon' M[X] y_{-1})^2}{y'_{-1} M[X] y_{-1}} .$$

Note that, under the null hypothesis, the statistic $\mathcal{L}_{\lambda,\gamma}(\lambda_0,\gamma_0)$ is not pivotal, i.e. its distribution depends on unknown nuisance parameters. This is easily seen when both the numerator $SS_0[\bar{Z}|\lambda_0,\gamma_0]$ and the denominator $SS_1[W]$ are divided by σ^2 and it is recognized that the test statistic consists of a number of (inner) products of three stochastic vectors. The vectors $M(\bar{Z})\varepsilon/\sigma$ and $M[X]\varepsilon/\sigma$ are pivotal, but $M[X]y_{-1}/\sigma$ is not pivotal, which is seen as follows. Using (2.3), we have

$$(3.7) \qquad y_{-1} = y_0^{\iota}(\lambda) + C(\lambda)Z\gamma + C(\lambda)\bar{Z}\tilde{\gamma} + C(\lambda)\varepsilon$$

and hence, under $H(\lambda_0,\gamma_0)$, we find

$$(3.8) \qquad y_{-1}/\sigma = y_0^{\iota}(\lambda_0)/\sigma + C(\lambda_0)Z\gamma_0/\sigma + C(\lambda_0)\bar{Z}\tilde{\gamma}/\sigma + C(\lambda_0)\varepsilon/\sigma ,$$

where σ and $\tilde{\gamma}$ are unknown. Premultiplication by $M[X]$ does not lead to simplification, leaving only the fourth term of (3.8) pivotal; in the first two the factor $1/\sigma$ causes problems, and in the third it is $\tilde{\gamma}/\sigma$.

To get a pivotal statistic for testing $\lambda = \lambda_0$, $\gamma = \gamma_0$ we operate in a comparable way as we did in (2.11). We rewrite model (3.2) as

$$(3.9) \qquad \begin{aligned} y = {} & \lambda[y_{-1} - y_0^{\iota}(\lambda_0) - C(\lambda_0)Z\gamma_0 - C(\lambda_0)\bar{Z}\tilde{\gamma}] \\ & + \lambda[y_0^{\iota}(\lambda_0) + C(\lambda_0)Z\gamma_0 + C(\lambda_0)\bar{Z}\tilde{\gamma}] + Z\gamma + \bar{Z}\tilde{\gamma} + \varepsilon \end{aligned}$$

and next consider a more general model by relaxing some of the coefficient restrictions in (3.9). Consider

$$\begin{aligned} y = {} & \lambda[y_{-1} - y_0^{\iota}(\lambda_0) - C(\lambda_0)Z\gamma_0 - C(\lambda_0)\bar{Z}\tilde{\gamma}] \\ & + \lambda_*[y_0^{\iota}(\lambda_0) + C(\lambda_0)Z\gamma_0 + C(\lambda_0)\bar{Z}\tilde{\gamma}] + Z\gamma + \bar{Z}\tilde{\gamma} + \varepsilon \end{aligned}$$

or

$$(3.10) \qquad y = \lambda\, y_{-1} + Z\gamma + \bar{Z}\tilde{\gamma} + \lambda_{**}\iota(\lambda_0) + \dot{\lambda}_{**}[C(\lambda_0)Z\gamma_0] + C(\lambda_0)\bar{Z}\tilde{\gamma}_{**} + \varepsilon ,$$

which specializes to model (3.2) if $\lambda_{**} = (\lambda_* - \lambda)y_0 = 0$, $\dot{\lambda}_{**} = \lambda_* - \lambda = 0$ and $\dot{\gamma}_{**} = (\lambda_* - \lambda)\tilde{\gamma} = 0$.

Below we consider testing

$$H_{**}(\lambda_0,\gamma_0):\ \lambda = \lambda_0\ ,\ \gamma = \gamma_0\ ,\ \lambda_{**} = 0\ ,\ \dot{\lambda}_{**} = 0\ ,\ \tilde{\gamma}_{**} = 0$$

in model (3.10) with the criterion

$$(3.11)\qquad \mathcal{L}^{**}_{\lambda,\gamma}(\lambda_0,\gamma_0) = \frac{SS_0[\bar{Z}|\lambda_0,\gamma_0]}{SS_1[W(\lambda_0,\gamma_0)]} = \frac{(y-\lambda_0 y_{-1}-Z\gamma_0)'M(\bar{Z})(y-\lambda_0 y_{-1}-Z\gamma_0)}{y'M[W(\lambda_0,\gamma_0)]y}\ ,$$

where $W(\lambda_0,\gamma_0) = [\ y_{-1}|Z|\bar{Z}|\dot{y}(\lambda_0)|C(\lambda_0)Z\gamma_0|C(\lambda_0)\bar{Z}\ ] - [y_{-1}|Z|\bar{Z}(\lambda_0,\gamma_0)] - [y_{-1}|Z(\lambda_0,\gamma_0)]$. We also consider testing

$$H_{*}(\lambda_0,\gamma_0):\ \lambda = \lambda_0\ ,\ \gamma = \gamma_0$$

in model (3.10) by

$$(3.12)\qquad \mathcal{L}^{*}_{\lambda,\gamma}(\lambda_0,\gamma_0) = \frac{SS_0[\bar{Z}(\lambda_0,\gamma_0)|\lambda_0,\gamma_0]}{SS_1[W(\lambda_0,\gamma_0)]} - \frac{(y-\lambda_0 y_{-1}-Z\gamma_0)'M[\bar{Z}(\lambda_0,\gamma_0)](y-\lambda_0 y_{-1}-Z\gamma_0)}{y'M[W(\lambda_0,\gamma_0)]y}\ .$$

As we shall see it is the case that if $r > 0$ and y_0 is supposed to be fixed, we can also obtain similar inference from an extended regression with one redundant regressor less. Therefore we also consider testing

$$H_{\dagger *}(\lambda_0,\gamma_0):\ \lambda = \lambda_0\ ,\ \gamma = \gamma_0\ ,\ \lambda_{*} = 0\ ,\ \tilde{\gamma}_{**} = 0$$

and

$$H_{\dagger}(\lambda_0,\gamma_0):\ \lambda = \lambda_0\ ,\ \gamma = \gamma_0$$

in

$$(3.13)\qquad y = \lambda\, y_{-1} + Z\gamma + \bar{Z}\tilde{\gamma} + \lambda_{*}[y_0\dot{y}(\lambda_0) + C(\lambda_0)Z\gamma_0] + C(\lambda_0)\bar{Z}\tilde{\gamma}_{**} + \varepsilon\ ,$$

which also is a generalization of (3.9), by

$$(3.14)\qquad \mathcal{L}^{\dagger *}_{\lambda,\gamma}(\lambda_0,\gamma_0) = \frac{SS_0[\bar{Z}|\lambda_0,\gamma_0]}{SS_1[W^{\dagger}(\lambda_0,\gamma_0)]} - \frac{(y-\lambda_0 y_{-1}-Z\gamma_0)'M(\bar{Z})(y-\lambda_0 y_{-1}-Z\gamma_0)}{y'M[W^{\dagger}(\lambda_0,\gamma_0)]y},$$

where $W^{\dagger}(\lambda_0,\gamma_0) = [\, y_{-1}|Z|\bar{Z}|[y_0{}'(\lambda_0)+C(\lambda_0)Z\gamma_0]|C(\lambda_0)\bar{Z}\,] = [y_{-1}|Z|\bar{Z}^{\dagger}(\lambda_0,\gamma_0)] = [y_{-1}|Z^{\dagger}(\lambda_0,\gamma_0)]$, and by

$$(3.15)\qquad \mathcal{L}^{\dagger}_{\lambda,\gamma}(\lambda_0,\gamma_0) = \frac{SS_0[\bar{Z}^{\dagger}(\lambda_0,\gamma_0)|\lambda_0,\gamma_0]}{SS_1[W^{\dagger}(\lambda_0,\gamma_0)]} - \frac{(y-\lambda_0 y_{-1}-Z\gamma_0)'M[\bar{Z}^{\dagger}(\lambda_0,\gamma_0)](y-\lambda_0 y_{-1}-Z\gamma_0)}{y'M[W^{\dagger}(\lambda_0,\gamma_0)]y}$$

respectively. Note that for $r = 0$ (a test on λ only) the statistics (2.15), (3.12) and (3.15) are equivalent, and so are (2.16), (3.11) and (3.14).

The procedures for testing joint restrictions on λ and β based on the above statistics are based on the following proposition.

PROPOSITION 1: *Suppose that model (3.2) and Assumption A hold. Then the statistics* $\mathcal{L}^{**}_{\lambda,\gamma}(\lambda_0,\gamma_0)$, $\mathcal{L}^{*}_{\lambda,\gamma}(\lambda_0,\gamma_0)$, $\mathcal{L}^{\dagger *}_{\lambda,\gamma}(\lambda_0,\gamma_0)$ *and* $\mathcal{L}^{\dagger}_{\lambda,\gamma}(\lambda_0,\gamma_0)$ *given in (3.11), (3.12), (3.14) and (3.15) respectively can be written as follows when* $\lambda = \lambda_0$ *and* $\gamma = \gamma_0$:

$$(3.16)\qquad \mathcal{L}^{**}_{\lambda,\gamma}(\lambda_0,\gamma_0) = \frac{\eta' M[\bar{Z}]\eta}{S_1[\eta,Z(\lambda_0,\gamma_0)]}, \qquad \mathcal{L}^{*}_{\lambda,\gamma}(\lambda_0,\gamma_0) = \frac{\eta' M[\bar{Z}(\lambda_0,\gamma_0)]\eta}{S_1[\eta,Z(\lambda_0,\gamma_0)]}$$

$$(3.17)\qquad \mathcal{L}^{\dagger *}_{\lambda,\gamma}(\lambda_0,\gamma_0) = \frac{\eta' M[\bar{Z}]\eta}{S_1[\eta,Z^{\dagger}(\lambda_0,\gamma_0)]}, \qquad \mathcal{L}^{\dagger}_{\lambda,\gamma}(\lambda_0,\gamma_0) = \frac{\eta' M[\bar{Z}^{\dagger}(\lambda_0,\gamma_0)]\eta}{S_1[\eta,Z^{\dagger}(\lambda_0,\gamma_0)]}$$

where $\eta = \varepsilon/\sigma$ *and*

$$(3.18)\qquad S_1(\eta,A) = \eta' M[A]\eta - \frac{(\eta' M[A]C(\lambda_0)\eta)^2}{\eta' C(\lambda_0)' M[A]C(\lambda_0)\eta}$$

for $A = Z(\lambda_0,\gamma_0)$ *or* $A = Z^{\dagger}(\lambda_0,\gamma_0)$ *as defined in (3.11) and (3.14).*

The main usefulness of Proposition 1 comes from the fact that the null distribution of the test statistics considered do not involve the nuisance parameters $\bar{\gamma}$ and σ, provided an appropriate assumption is put on the distribution of ε/σ. The null distributions of $\mathcal{L}^{**}_{\lambda,\gamma}(\lambda_0,\gamma_0)$ and $\mathcal{L}^{*}_{\lambda,\gamma}(\lambda_0,\gamma_0)$ only depend on the known quantities λ_0, γ_0, R, and the regressor matrix X, as well as on the distribution of ε/σ; those of $\mathcal{L}^{\dagger *}_{\lambda,\gamma}(\lambda_0,\gamma_0)$ and $\mathcal{L}^{\dagger}_{\lambda,\gamma}(\lambda_0,\gamma_0)$ also depend on y_0.

For the moment let us concentrate on applying the $\mathcal{L}^{*}_{\lambda,\gamma}(\lambda_0,\gamma_0)$ statistic; the others work similarly. If $\mathcal{L}^{*}_{\lambda,\gamma}(\alpha;\lambda_0,\gamma_0,X,R)$ is the point such that

$$(3.19)\qquad P\left[\, \mathcal{L}^{*}_{\lambda,\gamma}(\lambda_0,\gamma_0) > \mathcal{L}^{*}_{\lambda,\gamma}(\alpha;\lambda_0,\gamma_0,X,R) \;\middle|\; \lambda = \lambda_0 \,,\, \gamma = \gamma_0 \,\right] = \alpha \,,$$

an exact confidence set for (λ,γ) with size $1-\alpha$ is then given by the set

$$(3.20)\qquad \mathcal{C}_{\lambda,\gamma}(\alpha) = \left\{ (\lambda_0,\gamma_0) : \mathcal{L}^{*}_{\lambda,\gamma}(\lambda_0,\gamma_0) \le \mathcal{L}^{*}_{\lambda,\gamma}(\alpha;\lambda_0,\gamma_0,X,R) \right\} .$$

The exact critical values denoted by $\mathcal{L}^{*}_{\lambda,\gamma}(\alpha;\lambda_0,\gamma_0,X,R)$ can in principle be obtained to any reasonable degree of precision from Monte Carlo experiments. Exact Monte Carlo tests and confidence sets can also be obtained by using the simulation argument given in Lemma 2.

For normally distributed disturbances ε, it is also interesting to note that a liberal critical value for statistic $\mathcal{L}^{*}_{\lambda,\gamma}(\lambda_0,\gamma_0)$ can be found without performing any simulations at all as follows. From (A.1) we obtain the inequality $SS_1[W(\lambda_0,\gamma_0)] \le \varepsilon' M[Z(\lambda_0,\gamma_0)]\varepsilon$. Hence, under $H_*(\lambda_0,\gamma_0)$,

$$(3.21)\qquad \mathcal{L}^{*}_{\lambda,\gamma}(\lambda_0,\gamma_0) = \frac{SS_0[\bar{Z}(\lambda_0,\gamma_0)|\lambda_0,\gamma_0]}{SS_1[W(\lambda_0,\gamma_0)]} \ge \frac{\varepsilon' M[\bar{Z}(\lambda_0,\gamma_0)]\varepsilon}{\varepsilon' M[Z(\lambda_0,\gamma_0)]\varepsilon}$$

$$= 1 + \frac{\varepsilon' M[\bar{Z}(\lambda_0,\gamma_0)]\varepsilon - \varepsilon' M[Z(\lambda_0,\gamma_0)]\varepsilon}{\varepsilon' M[Z(\lambda_0,\gamma_0)]\varepsilon}$$

$$= 1 + \frac{\operatorname{rank}[Z(\lambda_0,\gamma_0)] - \operatorname{rank}[\bar{Z}(\lambda_0,\gamma_0)]}{T - \operatorname{rank}[Z(\lambda_0,\gamma_0)]} F = F^* .$$

Here $F = \mathcal{F}(\nu_1,\nu_2)$, i.e. F is Fisher distributed with degrees of freedom $\nu_1 = \operatorname{rank}[Z(\lambda_0,\gamma_0)] - \operatorname{rank}[\bar{Z}(\lambda_0,\gamma_0)] = r$ and $\nu_2 = T - \operatorname{rank}[Z(\lambda_0,\gamma_0)] \ge T-2k+r-2$. Thus

$$(3.22)\quad P\left[\mathcal{L}^{*}_{\lambda,\gamma}(\lambda_0,\gamma_0) \geq 1 + \frac{\nu_1}{\nu_2}\mathcal{F}(\alpha;\nu_1,\nu_2)\right] \geq P\left[F^{*} \geq 1 + \frac{\nu_1}{\nu_2}\mathcal{F}(\alpha;\nu_1,\nu_2)\right] = \alpha\ ,$$

where $\mathcal{F}(\alpha;\nu_1,\nu_2)$ denotes the α level critical value of the Fisher distribution with ν_1 and ν_2 degrees of freedom respectively. Therefore, if we find $\mathcal{L}^{*}_{\lambda,\gamma}(\lambda_0,\gamma_0) \leq 1+(\nu_1/\nu_2)\mathcal{F}(\alpha;\nu_1,\nu_2)$, we can infer that the test statistic is certainly not significant at level α. Such liberal critical values can also be derived for the other three tests proposed in this subsection.

3.2 A Two-Stage Procedure

Above we developed direct procedures for joint inference on λ and any linear transformation of β. Such inference can also be obtained from a two-stage procedure, i.e. an exact simultaneous confidence set for (λ,γ) can be constructed by combining an exact confidence set for λ [see (2.22), (2.25) or (2.33)], denoted (neutrally) now by $\mathcal{C}_{\lambda}(\alpha)$, with the corresponding family of "conditional" confidence sets for γ; see Dufour (1990). The duality between tests and confidence sets then again leads to an exact test for any joint null hypothesis on λ and γ. We proceed as follows.

First assume that the true value of λ is given. An exact similar test for $\gamma = \gamma_0$ is then obtained easily (without extending the model) by the statistic

$$(3.23)\quad \mathcal{L}_{\gamma|\lambda}(\gamma_0) = \frac{SS_0[\bar{Z}|\lambda,\gamma_0]}{SS_1[Z|\lambda]} = \frac{(y-\lambda y_{-1}-Z\gamma_0)'M[\bar{Z}](y-\lambda y_{-1}-Z\gamma_0)}{(y-\lambda y_{-1})'M[X](y-\lambda y_{-1})}\ ,$$

which reduces to

$$(3.24)\quad \mathcal{L}_{\gamma|\lambda}(\gamma_0) = \frac{\varepsilon' M[\bar{Z}]\varepsilon}{\varepsilon' M[X]\varepsilon}$$

under $\gamma = \gamma_0$. Given Assumption B, it can yield a similar test, and its critical values will be denoted by $\mathcal{L}_{\gamma|\lambda}(\alpha;X,R)$, i.e. $\mathcal{L}_{\gamma|\lambda}(\alpha;X,R)$ is the smallest point such that $P[\mathcal{L}_{\gamma|\lambda}(\gamma_0) > \mathcal{L}_{\gamma|\lambda}(\alpha;X,R)] \leq \alpha$; these do not depend on γ_0 or λ. Then the set

$$(3.25)\qquad \mathcal{C}_{\gamma|\lambda}(\alpha) = \left\{ \gamma_0 : \mathcal{L}_{\gamma|\lambda}(\gamma_0) \le \mathcal{L}_{\gamma|\lambda}(\alpha;X,R) \right\}$$

is an exact confidence set for γ (given λ). Note that the (monotonically transformed) statistic

$$(3.26)\qquad \mathcal{F}_{\gamma|\lambda}(\gamma_0) = \frac{T-k}{r}\left[\mathcal{L}_{\gamma|\lambda}(\gamma_0) - 1\right]$$

produces equivalent inference. When $\varepsilon/\sigma \sim N[0,I_T]$, $\mathcal{F}_{\gamma|\lambda}(\gamma_0)$ follows (under the null hypothesis) a Fisher distribution with (r,T-k) degrees of freedom. So, under normality, $\mathcal{L}_{\gamma|\lambda}(\alpha;X,R) = 1 + r(T-k)^{-1}\mathcal{F}(\alpha;r,T-k)$.

We now suppose that λ is unknown. Let

$$(3.27)\qquad 0 < \alpha = \alpha_1 + \alpha_2 < 1 \quad \text{with} \quad 0 \le \alpha_i < 1 \quad (i = 1,2) .$$

A size $1-\alpha_1$ confidence set $\mathcal{C}_{\lambda}(\alpha_1)$ for λ can be constructed following one of the procedures set out in Section 2.2. Now consider the set

$$(3.28)\qquad \mathcal{C}_{\lambda,\gamma}(\alpha_1,\alpha_2) = \left\{ (\lambda_0,\gamma_0) : \lambda_0 \in \mathcal{C}_{\lambda}(\alpha_1) \text{ and } \gamma_0 \in \mathcal{C}_{\gamma|\lambda_0}(\alpha_2) \right\} .$$

Note the difference between the set $\mathcal{C}_{\gamma|\lambda}(\alpha)$ of (3.25), which has size $1-\alpha$, and

$$(3.29)\qquad \mathcal{C}_{\gamma|\lambda_0}(\alpha_2) = \left\{ \gamma_0 : \mathcal{L}_{\gamma|\lambda_0}(\lambda_0,\gamma_0) \le \mathcal{L}_{\gamma|\lambda_0}(\alpha_2;X,R) \right\} .$$

The latter set is a "conditional" confidence set for γ; it is conditional on λ_0 and has level α_2. The critical value in (3.25) is obtained from $\mathcal{L}_{\gamma|\lambda}(\gamma_0)$ under the assumption $(\lambda_0,\gamma_0) = (\lambda,\gamma)$. Hence, it is simply a quantile of (3.24) and does not depend on nuisance parameters nor on λ_0 and γ_0. Its invariance is very convenient when (3.29) has to be assessed for various λ_0 values, which – as we shall see – is required when these conditional sets are used to construct unconditional inference for (λ,γ) or γ only.

Now we examine the level of $\mathcal{C}_{\lambda,\gamma}(\alpha_1,\alpha_2)$ given in (3.28). Upon using Bonferroni's inequality, we find

$$(3.30)\qquad P\left[(\lambda,\gamma) \in \mathcal{C}_{\lambda,\gamma}(\alpha_1,\alpha_2) \right] = P\left[\lambda \in \mathcal{C}_{\lambda}(\alpha_1) \text{ and } \gamma \in \mathcal{C}_{\gamma|\lambda}(\alpha_2) \right]$$
$$\ge 1 - P\left[\lambda \notin \mathcal{C}_{\lambda}(\alpha_1) \right] - P\left[\gamma \notin \mathcal{C}_{\gamma|\lambda}(\alpha_2) \right] = 1 - \alpha_1 - \alpha_2 = 1 - \alpha .$$

Hence $\mathcal{C}_{\lambda,\gamma}(\alpha_1,\alpha_2)$ is a conservative $1 - \alpha$ level confidence set and, of course, this implies a conservative test. This test is of minor interest since we have already obtained an exact test of size (precisely) α in the foregoing subsection. The two-stage joint confidence set, however, provides a basis for making inference on individual elements of β and cross-restrictions between λ and β.

4. Inference on β only and cross-restrictions between λ and β

Below we shall "marginalize" the conservative two-stage joint confidence set $\mathcal{C}_{\lambda,\gamma}(\alpha_1,\alpha_2)$ defined in (3.28) with respect to λ, by exploiting the exact interval $\mathcal{C}_{\lambda}(\alpha_1)$ in such a way that exact inference on γ only, hence on a linear transform of β, results. First bounding confidence sets for $\gamma = R\beta$ will be constructed and then an exact generalized bounds test for the hypothesis $\gamma = \gamma_0$ easily follows.

We define the r dimensional random sets

$$(4.1) \qquad \mathcal{C}^U_{\gamma}(\alpha_1,\alpha_2) = \left\{ \gamma_0 : \exists\, \lambda_0 \in \mathcal{C}_{\lambda}(\alpha_1) \text{ such that } (\lambda_0,\gamma_0) \in \mathcal{C}_{\lambda,\gamma}(\alpha_1,\alpha_2) \right\}$$

and

$$(4.2) \qquad \mathcal{C}^L_{\gamma}(\alpha_1,\alpha_2) = \left\{ \gamma_0 : \forall\, \lambda_0 \in \mathcal{C}_{\lambda}(\alpha_1) \;,\; (\lambda_0,\gamma_0) \in \mathcal{C}_{\lambda,\gamma}(\alpha_1,\alpha_2) \right\},$$

with the convention $\mathcal{C}^U_{\gamma}(\alpha_1,\alpha_2) = \mathcal{C}^L_{\gamma}(\alpha_1,\alpha_2) = \emptyset$ when $\mathcal{C}_{\lambda}(\alpha_1) = \emptyset$, where $\emptyset$ is the empty set. The sets (4.1) and (4.2) are nested, i.e. $\emptyset \subseteq \mathcal{C}^L_{\gamma}(\alpha_1,\alpha_2) \subseteq \mathcal{C}^U_{\gamma}(\alpha_1,\alpha_2)$, and they can be used as bounding confidence sets as follows.

PROPOSITION 2: *Suppose that model (3.2) holds jointly with the Assumptions A and B. Let $\mathcal{C}_{\lambda}(\alpha_1)$ be a confidence set for λ such that $P[\lambda \in \mathcal{C}_{\lambda}(\alpha_1)] \geq 1-\alpha_1$. If $0 \leq \alpha_1 < 1$, $0 < \alpha_1 + \alpha_2 = \alpha \leq 1 - \alpha_1$ and $\alpha'_2 = \alpha + \alpha_1 < 1$, then for the random sets defined in (4.1) and (4.2) we have $\mathcal{C}^L_{\gamma}(\alpha_1,\alpha'_2) \subseteq \mathcal{C}^U_{\gamma}(\alpha_1,\alpha_2)$ and*

$$P[\gamma \in \mathcal{C}^L_{\gamma}(\alpha_1,\alpha'_2)] \leq 1-\alpha \leq P[\gamma \in \mathcal{C}^U_{\gamma}(\alpha_1,\alpha_2)] \,.$$

From the above results, conservative and liberal tests for $\gamma = \gamma_0$ can be derived. We define

$$(4.3) \qquad \mathcal{L}^L_\gamma(\gamma_0) = \text{Inf}\left\{ \mathcal{L}_{\gamma|\lambda_0}(\gamma_0) : \lambda_0 \in \mathcal{C}_\lambda(\alpha_1) \right\}$$

$$(4.4) \qquad \mathcal{L}^U_\gamma(\gamma_0) = \text{Sup}\left\{ \mathcal{L}_{\gamma|\lambda_0}(\gamma_0) : \lambda_0 \in \mathcal{C}_\lambda(\alpha_1) \right\} .$$

Note that $\gamma_0 \in \mathcal{C}^U_\gamma(\alpha_1,\alpha_2)$ is equivalent with $\mathcal{L}^U_\gamma(\gamma_0) \leq \mathcal{L}_{\gamma|\lambda}(\alpha_2;X,R)$. With respect to the true γ this implies for the statistic $\mathcal{L}^U_\gamma(\gamma_0)$ that

$$(4.5) \qquad P\left[\mathcal{L}^U_\gamma(\gamma_0) > \mathcal{L}_{\gamma|\lambda}(\alpha_2;X,R) \right] = P\left[\gamma \notin \mathcal{C}^U_\gamma(\alpha_1,\alpha_2) \right] \leq \alpha ,$$

when $\gamma = \gamma_0$. Similarly, the event $\mathcal{L}^L_\gamma(\gamma_0) > \mathcal{L}_{\gamma|\lambda}(\alpha'_2;X,R)$ implies $\gamma_0 \notin \mathcal{C}^L_\gamma(\alpha_1,\alpha'_2)$, hence

$$(4.6) \qquad P\left[\mathcal{L}^L_\gamma(\gamma_0) > \mathcal{L}_{\gamma|\lambda}(\alpha'_2;X,R) \right] \leq P\left[\gamma \notin \mathcal{C}^L_\gamma(\alpha_1,\alpha'_2) \right] \leq \alpha ,$$

when $\gamma = \gamma_0$. Because $\alpha'_2 \geq \alpha_2$ implies $\mathcal{L}_{\gamma|\lambda}(\alpha_2;X,R) \geq \mathcal{L}_{\gamma|\lambda}(\alpha'_2;X,R)$, the generalized bounds principle can be applied, yielding the test:

$$(4.7) \qquad \begin{aligned} &\text{reject } \gamma = \gamma_0 \text{ when } \mathcal{L}^L_\gamma(\gamma_0) > \mathcal{L}_{\gamma|\lambda}(\alpha_2;X,R) , \\ &\text{accept when } \mathcal{L}^U_\gamma(\gamma_0) \leq \mathcal{L}_{\gamma|\lambda}(\alpha'_2;X,R) , \\ &\text{whereas the test is inconclusive otherwise.} \end{aligned}$$

In practice, simulation procedures have to be used first to obtain a $\mathcal{C}_\lambda(\alpha_1)$ interval, and then $\mathcal{L}^L_\gamma(\gamma_0)$ and $\mathcal{L}^U_\gamma(\gamma_0)$ have to be assessed. Under normality of ε the $\mathcal{L}_{\gamma|\lambda}$ critical values can easily be obtained from tabulated critical F-values. In general, however, this test procedure should be brought fully into the simulation context of Lemma 2 and then the decision rules of (4.7) involve the assessment of the *p*-values of the rank statistics based on $\mathcal{L}^L_\gamma(\gamma_0)$ and $\mathcal{L}^U_\gamma(\gamma_0)$. These *p*-values are to be compared with α_2 and α'_2 respectively.

As far as linear restrictions on λ and β are concerned, the only situation for which we haven't yet obtained an exact inference procedure is the case where the restrictions involve both λ and elements of β, but do not explicitly (or implicitly) specify $\lambda = \lambda_0$. We consider the one restriction case:

(4.8) $\quad H_0(\kappa_0)\colon\ \lambda + \kappa_1'\beta = \kappa_0\ ,$

where κ_1 is a kx1 vector and κ_0 a scalar; κ_0 and κ_1 both known. This can be handled as follows. Let

(4.9) $\quad \gamma = \kappa_1'\beta\ ,\quad \kappa = \lambda + \gamma = \lambda + \kappa_1'\beta\ .$

Hence, both γ and κ are scalar unknown parameters here. We define the set

(4.10) $\quad \mathcal{C}_\kappa(\alpha) = \left\{ \kappa_0 : \exists\ (\lambda_0,\gamma_0) \in \mathcal{C}_{\lambda,\gamma}(\alpha) \text{ such that } \kappa_0 = \lambda_0 + \gamma_0 \right\}\ ,$

where $\mathcal{C}_{\lambda,\gamma}(\alpha)$ can be obtained as in (3.20). Obviously,

(4.11) $\quad P[\ (\lambda,\gamma) \in \mathcal{C}_{\lambda,\gamma}(\alpha)\ ] = 1 - \alpha \le P[\ \kappa \in \mathcal{C}_\kappa(\alpha)\ ]\ ,$

and the test of (4.8), which can also be expressed as $H_0(\kappa_0)\colon\ \kappa = \kappa_0$, corresponds to the confidence set (4.10). It is performed by simply rejecting $H_0(\kappa_0)$ when $\kappa_0 \notin \mathcal{C}_\kappa(\alpha)$, and it has level α (though usually not size α). We shall not pay separate attention to the case where we have various restrictions, including at least one of the form (4.8), since this, like all other forms of linear restrictions, are covered by the general results in the next section.

5. Tests for Non-Regular Hypotheses

We now generalize the foregoing results on linear restrictions and consider cases where two arbitrary hypotheses on the regressor coefficients are compared, viz.

(5.1) $\quad h_0(\lambda,\gamma) = 0 \iff (\lambda,\gamma')' \in \Gamma_0 \subseteq \mathbb{R}^{r+1}\ ,\quad \Gamma_0 \neq \emptyset$

and

(5.2) $\quad h_1(\lambda,\beta) = 0 \iff (\lambda,\beta')' \in \Omega_1 \subseteq \mathbb{R}^{k+1}\ ,\quad \Omega_1 \neq \emptyset\ .$

These cover special cases such as nonlinear restrictions, inequality

restrictions and non-nested hypotheses.

When λ is known, the model $y - \lambda y_{-1} = y(\lambda) = X\beta + \varepsilon$ satisfies the assumptions of the standard linear regression model, except that ε/σ may be non-normal. Dufour (1989) deals with exact solutions to inference problems on non-regular hypotheses in that model under normality. We shall stick to our more general Assumptions A and B and extend these results to dynamic models.

Let

$$(5.3) \qquad S(\lambda_0,\beta_0) = [y(\lambda_0) - X\beta_0]'[y(\lambda_0) - X\beta_0]$$

denote the residual sum of squares for arbitrary coefficient values $(\lambda_0,\beta_0')'$, where $(\lambda_0,\beta_0')' \in \mathbb{R}^{k+1}$ with $\lambda_0 \in \mathcal{D}_\lambda$. We first consider the case where the true value of λ is known, and define:

$$(5.4) \qquad SS_0(\lambda) = \text{Inf}\left\{ S(\lambda,\beta_0) : \beta_0 \in \mathbb{R}^k \text{ and } h_0(\lambda,R\beta_0) = 0 \right\}$$

and

$$(5.5) \qquad SS_1(\lambda) = \text{Inf}\left\{ S(\lambda,\beta_0) : \beta_0 \in \mathbb{R}^k \text{ and } h_1(\lambda,\beta_0) = 0 \right\}.$$

The LR type test for the two hypotheses can be written (after a monotonic transformation, and still assuming λ to be known) as:

$$(5.6) \qquad \mathcal{L}_{r|\lambda}(\Gamma_0,\Omega_1) = \frac{SS_0(\lambda)}{SS_1(\lambda)}.$$

We also define:

$$(5.7) \qquad \overline{SS}_0(\lambda,\gamma_0) = \text{Inf}\left\{ S(\lambda,\beta_0) : \beta_0 \in \mathbb{R}^k \text{ and } R\beta_0 - \gamma_0 \in \mathbb{R}^r \right\}$$

for some given value γ_0, and

$$(5.8) \qquad SS(\lambda) = \text{Inf}\left\{ S(\lambda,\beta_0) : \beta_0 \in \mathbb{R}^k \right\}.$$

Note that the statistic (3.23) for testing $\gamma = \gamma_0$ now can be expressed as

$$(5.9) \qquad \mathcal{L}_{\gamma|\lambda}(\gamma_0) = \frac{\overline{SS}_0(\lambda,\gamma_0)}{SS(\lambda)}.$$

In the proposition below it is shown that the critical value $\mathcal{Z}_{\gamma|\lambda}(\alpha;X,R)$ of test statistic (5.9) is a conservative critical value for testing the non-regular hypotheses (5.1) and (5.2) by the statistic $\mathcal{L}_{\Gamma|\lambda}(\Gamma_0,\Omega_1)$.

PROPOSITION 3: *Suppose that model (3.2) holds jointly with the Assumptions A and B. Then, if* $(\lambda,\gamma')' \in \Gamma_0$, *we have*

$$P[\ \mathcal{L}_{\Gamma|\lambda}(\Gamma_0,\Omega_1) > \mathcal{Z}_{\gamma|\lambda}(\alpha;X,R)\ ] \leq \alpha\ .$$

It is also possible to obtain liberal critical values. These are, however, of less interest. Proposition 3 concerns the unrealistic case where λ is known. We shall show now how we can exploit an exact confidence interval $\mathcal{C}_\lambda(\alpha)$ in order to obtain unconditional inference on the hypotheses (5.1) and (5.2). We define:

$$(5.10)\qquad \mathcal{L}^L_\Gamma(\Gamma_0,\Omega_1) = \mathrm{Inf}\left\{\ \mathcal{L}_{\Gamma|\lambda_0}(\Gamma_0,\Omega_1) : \lambda_0 \in \mathcal{C}_\lambda(\alpha_1)\ \right\}\ .$$

$$(5.11)\qquad \mathcal{L}^U_\Gamma(\Gamma_0,\Omega_1) = \mathrm{Sup}\left\{\ \mathcal{L}_{\Gamma|\lambda_0}(\Gamma_0,\Omega_1) : \lambda_0 \in \mathcal{C}_\lambda(\alpha_1)\ \right\}\ .$$

PROPOSITION 4: *Under the assumptions of Proposition 3, let* $\mathcal{C}_\lambda(\alpha_1)$ *be a confidence set for* λ *such that* $P[\lambda \in \mathcal{C}_\lambda(\alpha_1)] \geq 1 - \alpha_1$. *If* $0 \leq \alpha_1 < 1$, $0 < \alpha_1 + \alpha_2 = \alpha \leq 1 - \alpha_1$ *and* $\alpha'_2 = \alpha + \alpha_1$, *then for the statistics defined in (5.10) and (5.11) we have under* $(\lambda,\gamma')' \in \Gamma_0$:

$$P[\ \mathcal{L}^L_\Gamma(\Gamma_0,\Omega_1) > \mathcal{Z}_{\gamma|\lambda}(\alpha_2;X,R)\ ] \leq \alpha \leq P[\ \mathcal{L}^U_\Gamma(\Gamma_0,\Omega_1) > \mathcal{Z}_{\gamma|\lambda}(\alpha'_2;X,R)\ ]\ .$$

From this proposition we find the following generalized bounds test procedure:

$$(5.12)\qquad \begin{aligned} &\text{reject } (\lambda,\gamma')' \in \Gamma_0 \text{ when } \mathcal{L}^L_\Gamma(\Gamma_0,\Omega_1) > \mathcal{Z}_{\gamma|\lambda}(\alpha_2;X,R)\ ,\\ &\text{accept } (\lambda,\gamma')' \in \Gamma_0 \text{ when } \mathcal{L}^U_\Gamma(\Gamma_0,\Omega_1) \leq \mathcal{Z}_{\gamma|\lambda}(\alpha'_2;X,R)\ ,\\ &\text{whereas the test is inconclusive otherwise.}\end{aligned}$$

Rather than obtaining a test for $(\lambda,\gamma')' \in \Gamma_0$ from the above procedure, which is inspired by the two-stage approach of Section 3.2, we can also make use of the comprehensive procedures of Section 3.1. We shall illustrate this for a particular example.

In the context of our dynamic model (1.2), which originates from the underlying relationship (2.1), nonlinear transformations of the regression coefficients which are particularly relevant are expressions such as:

$\sum_{i=0}^{L(j)} \delta_{ji}/(1-\lambda)$, i.e. the total multiplier for regressor $z^{(j)}$, $j = 1,\ldots,J$;

$1 - (\lambda)^{i+1}$, the standardized interim multiplier after i time periods;

$\lambda/(1-\lambda)$, the mean lag;

$\text{Max}\left\{\frac{\ln(\frac{1}{2})}{\ln(\lambda)} - 1 \ , \ 0\right\}$, the median lag.

Note that the above nonlinear characterizations of particular aspects of the dynamic adjustment process are only meaningful if $\mathcal{D}_\lambda \subseteq (-1,+1)$.

We shall focus now on the problem of producing exact inference on the parameter $\varphi = \gamma/(1 - \lambda)$, where again $\gamma = R\beta$; hence, φ may represent a vector of total or long-run multipliers. From the confidence set $\mathcal{C}_{\lambda,\gamma}(\alpha)$ for (λ,γ) given in (3.20) we construct another set, viz.:

$$(5.13) \qquad \mathcal{C}_\varphi(\alpha) = \left\{ \varphi_0 : \exists\ (\lambda_0,\gamma_0) \in \mathcal{C}_{\lambda,\gamma}(\alpha) \text{ such that } \varphi_0 = \gamma_0/(1 - \lambda_0) \right\} .$$

Note that this set is not necessarily bounded. It is clear that

$$(5.14) \qquad P[\ \varphi \in \mathcal{C}_\varphi(\alpha)\ ] \geq 1 - \alpha \ ,$$

and that an exact test of $\varphi = \varphi_0$ corresponds to this confidence set. In case we start off from the two-stage confidence set $\mathcal{C}_{\lambda,\gamma}(\alpha_1,\alpha_2)$ of (3.28) we can obtain the set, which corresponds to test procedure (5.12):

$$(5.15)\quad \mathfrak{C}_{\varphi}(\alpha_1,\alpha_2) = \Big\{ \varphi_0 : \varphi_0 = \gamma_0/(1-\lambda_0) \text{ and } \mathfrak{L}_{\gamma|\lambda_0}(\gamma_0,\lambda_0) \le \mathfrak{L}_{\gamma|\lambda}(\alpha_2';X,R) \text{ for some } \lambda_0 \in \mathfrak{C}_{\lambda}(\alpha_1) \Big\}.$$

6. Pointwise Asymptotic Validity

The procedures proposed above are exact under the Assumptions A and B, i.e. when X may be treated as fixed and ε has a distribution which is completely known up to a scale factor. In this section, we show that these procedures also yield "asymptotically valid" tests under considerably weaker distributional assumptions on X and ε, which allow both stationary and non-stationary regressors. More precisely, we wish to show that proceeding as if Assumptions A and B hold jointly with a specific distributional assumption on ε/σ (e.g., $\varepsilon/\sigma \sim N[0,I_T]$) yields tests whose probability of type I error converges to the nominal level of the test as $T \to \infty$ under any parameter configuration compatible with the null hypothesis (pointwise asymptotic validity).

All our results up to now have been established for a given sample size T. To formulate asymptotic properties, we need to consider a sequence of models indexed by T so that it will be useful to rewrite model (3.2) in the form

$$(6.1)\quad y(T) = \lambda y_{-1}(T) + Z(T)\gamma + \bar{Z}(T)\bar{\gamma} + \varepsilon(T) ,$$

where $y(T) = y = (y_1,\ldots,y_T)'$, $y_{-1}(T) = y_{-1}$, $Z(T) = Z$, $\bar{Z}(T) = \bar{Z}$ and $\varepsilon(T) = \varepsilon$. All the tests proposed in Sections 2 and 3.1 are based on considering a regression of the form

$$(6.2)\quad y_T(\lambda_0,\gamma_0) = \lambda_* y_{-1}(T) + \dot{Q}_T(\lambda_0,\gamma_0)\gamma_* + \bar{Q}_T(\lambda_0,\gamma_0)\bar{\gamma} + \varepsilon(T)$$

$$= \lambda_* y_{-1}(T) + Q_T(\lambda_0,\gamma_0)\delta + \varepsilon(T) ,$$

where $y_T(\lambda_0,\gamma_0) = y(T) - \lambda_0 y_{-1}(T) - Z(T)\gamma_0$. The matrix $Q_T(\lambda_0,\gamma_0) = [\dot{Q}_T(\lambda_0,\gamma_0) \vdots \bar{Q}_T(\lambda_0,\gamma_0)]$ is a full column rank $T \times m$ matrix and $\bar{Q}_T(\lambda_0,\gamma_0)$ has $\bar{m}$ columns ($0 \le \bar{m} \le m$). Note that either $\dot{Q}_T(\lambda_0,\gamma_0)$ or $\bar{Q}_T(\lambda_0,\gamma_0)$ may be void.

The null hypothesis of interest is

$$(6.3) \qquad H_0: \lambda_* = 0 \ , \ \gamma_* = 0$$

which is tested with the LR-type statistic

$$(6.4) \qquad \mathcal{L}_T = \frac{S_{0T}(\lambda_0,\gamma_0)}{S_{1T}(\lambda_0,\gamma_0)} \ ,$$

where

$$S_{0T}(\lambda_0,\gamma_0) = y_T(\lambda_0,\gamma_0)'M[\bar{Q}_T(\lambda_0,\gamma_0)]y_T(\lambda_0,\gamma_0)$$

and

$$S_{1T}(\lambda_0,\gamma_0) = y_T(\lambda_0,\gamma_0)'M[y_{-1}(T)\,|\,Q_T(\lambda_0,\gamma_0)]y_T(\lambda_0,\gamma_0) \ ,$$

or equivalently by the Fisher-type statistic

$$(6.5) \qquad \mathcal{F}_T = \left(\frac{T-m-1}{\dot{m}+1}\right)\frac{S_{0T}(\lambda_0,\gamma_0) - S_{1T}(\lambda_0,\gamma_0)}{S_{1T}(\lambda_0,\gamma_0)} = \frac{T-m-1}{\dot{m}+1}(\mathcal{L}_T - 1) \ ,$$

where $\dot{m} = m - \bar{m}$ (for the case $\bar{m} = 0$ we define $M[\bar{Q}_T(\lambda_0,\gamma_0)] = I_T$). For example, the statistic $\mathcal{L}^*_\lambda(\lambda_0)$ of (2.15) is obtained by taking $Q_T(\lambda_0,\gamma_0) = \bar{Q}_T(\lambda_0,\gamma_0) = X(\lambda_0)$ and statistic $\mathcal{L}^{**}_\lambda(\lambda_0)$ of (2.16) is obtained by taking $Q_T(\lambda_0,\gamma_0) = X(\lambda_0)$ and $\bar{Q}_T(\lambda_0,\gamma_0) = X$. Similarly, $\mathcal{L}^{**}_{\lambda,\gamma}(\lambda_0,\gamma_0)$ of (3.10) can be obtained by taking $Q_T(\lambda_0,\gamma_0) = [Z\,|\,\bar{Z}\,|\,\varepsilon(\lambda_0)\,|\,C(\lambda_0)Z\gamma_0\,|\,C(\lambda_0)\bar{Z}]$ (assuming that this has full column rank, otherwise redundant columns should be omitted) and $\bar{Q}_T(\lambda_0,\gamma_0) = \bar{Z}$. Similarly all the other $\mathcal{L}$-type statistics introduced in the foregoing sections are special cases of (6.4) with particular $Q_T(\lambda_0,\gamma_0)$ and $\bar{Q}_T(\lambda_0,\gamma_0)$ matrices.

Applying Lemma 1, the statistic $\mathcal{F}_T$ under the null hypothesis has the form:

$$(6.6) \qquad \mathcal{F}_T = K(T)\left[\frac{A_1(T) - A_2(T) + A_3(T)}{\frac{1}{T}A_2(T) - \frac{1}{T}A_3(T)}\right]$$

where $K(T) = (T - m - 1)/[(\dot{m} + 1)T]$,

$$(6.7) \qquad A_1(T) = \varepsilon(T)'M[\bar{Q}_T(\lambda_0,\gamma_0)]\varepsilon(T) \ , \quad A_2(T) = \varepsilon(T)'M[Q_T(\lambda_0,\gamma_0)]\varepsilon(T) \ ,$$

$$A_3(T) = \frac{[\varepsilon(T)'M[Q_T(\lambda_0,\gamma_0)]C_T(\lambda_0)\varepsilon(T)/c_T]^2}{\varepsilon(T)'C_T(\lambda_0)'M[Q_T(\lambda_0,\gamma_0)]C_T(\lambda_0)\varepsilon(T)/c_T^2} ,$$

and c_T can be any positive number. We shall now examine the asymptotic distribution of $\mathcal{F}_T$ under the following "generic" assumptions (where $\Rightarrow$ refers to weak convergence as $T \to \infty$).

ASSUMPTION C: *In model (2.2), the matrix* $X = X(T)$ *has rank* k *with probability one, and the coefficients* λ *and* β *are constant but unknown, with* $\beta \in \mathbb{R}^k$ *and* $\lambda \in \mathcal{D}_\lambda \subseteq \mathbb{R}$ *. Furthermore for* $\lambda = \lambda_0$, $\gamma = \gamma_0$, *where* $\lambda_0 \in \mathcal{D}_\lambda$ *and* γ *is defined in (3.2), the matrix* $[y_{-1}(T) \vdots Q_T(\lambda_0,\gamma_0)]$, *as defined in (6.2) has full column rank with probability one for* $T \geq T_0 > m$ *and the sequence* $([Q_T(\lambda_0,\gamma_0),\ \varepsilon(T)]\ ,\ T \geq T_0)$ *belongs to a class* $\mathcal{Z}_0$ *of stochastic processes such that for each process in* $\mathcal{Z}_0$ *the following properties hold:*

(i) $\underset{T\to\infty}{plim}\ \frac{1}{T}\varepsilon(T)'\varepsilon(T) = \sigma^2 > 0$,

where σ^2 *is a positive constant which is the same for all processes in* $\mathcal{Z}_0$;

(ii) *there exists a sequence of non-singular* $m \times m$ *matrices* D_T *and a sequence of positive constants* c_T *such that:*

(a) $c_T \xrightarrow[T\to\infty]{} \infty$;

(b) $\underset{T\to\infty}{plim}\ D_T'Q_T(\lambda_0,\gamma_0)'Q_T(\lambda_0,\gamma_0)D_T = \Sigma_{QQ} = \begin{bmatrix} \Sigma_{\dot{Q}\dot{Q}} & \Sigma_{\dot{Q}\bar{Q}} \\ \Sigma_{\bar{Q}\dot{Q}} & \Sigma_{\bar{Q}\bar{Q}} \end{bmatrix}$,

where Σ_{QQ} *is the same for all processes in* $\mathcal{Z}_0$, $|\Sigma_{QQ}| \neq 0$, *and* $\Sigma_{\bar{Q}\bar{Q}}$ *is a* $\bar{m} \times \bar{m}$ *nonsingular matrix;*

(c) $D_T'Q_T(\lambda_0,\gamma_0)'\varepsilon(T) \Rightarrow q_1$,

$$\frac{1}{c_T} D_T'Q_T(\lambda_0,\gamma_0)'C_T(\lambda_0)\varepsilon(T) \Rightarrow q_2 ,$$

$$\frac{\varepsilon(T)'C_T(\lambda_0)\varepsilon(T)}{c_T} \Rightarrow q_3 ,$$

$$\frac{\varepsilon(T)' C_T(\lambda_0)' C_T(\lambda_0) \varepsilon(T)}{c_T^2} \Rightarrow q_4 ,$$

where q_1, q_2, q_3 *and* q_4 *are random vectors such that* q_1 *and* q_3^2/q_4 *have absolutely continuous (non degenerate) distributions on* $\mathbb{R}^m$ *and* $\mathbb{R}$ *respectively, and the distribution of* $(q_1' , q_3^2/q_4)'$ *is the same for all processes in* $\mathcal{Z}_0$.

In the above assumption, the sequence (c_t, D_T) and the joint distribution of $(q_2', q_3, q_4)'$ may depend on λ_0 and γ_0 as well as on each particular process in $\mathcal{Z}_0$. The limiting constant σ, the matrix Σ_{QQ} and the distribution of $(q_1' , q_3^2/q_4)'$ may also depend on (λ_0, γ_0) but should be the same for all processes in $\mathcal{Z}_0$. The latter invariance is the crucial property which will assure the invariance of the limiting null distribution of $\mathcal{F}_T$.

Note that none of the Assumptions A, B or C do exclude non-normal disturbances. In contrast with A, however, Assumption C allows stochastic and weakly exogenous regressors (although at the cost of introducing additional assumptions on the asymptotic behaviour of the model). Before we investigate the test statistic $\mathcal{F}_T$, we shall discuss a few of the particulars of assumption C.

The scaling factor c_T and the matrix D_T are typically of the form

$$c_T = (T)^{\delta_0} \quad \text{and} \quad D_T = \text{diag}\left[(T)^{-\delta_1}, \ \ldots \ , (T)^{-\delta_m} \right] ,$$

where $\delta_1 > 0$, e.g. $\delta_1 = 0.5$ or $\delta_1 = 1$, depending on the degree of nonstationarity of the variables involved. For example, for $\lambda_0 = 0$, we have

$$\varepsilon(T)' C_T(\lambda_0) \varepsilon(T) = \sum_{t=2}^{T} \varepsilon_t \varepsilon_{t-1}$$

$$\varepsilon(T)' C_T(\lambda_0)' C_T(\lambda_0) \varepsilon(T) = \sum_{t=2}^{T} \varepsilon_t^2 ,$$

so that $c_T = T^{1/2}$ should be an appropriate scaling factor which will assure the convergence of $T^{-1/2} \sum_{t=2}^{T} \varepsilon_t \varepsilon_{t-1}$ and $T^{-1} \sum_{t=2}^{T} \varepsilon_t^2$ (under quite general regularity conditions on the disturbances ε_t).

For $\lambda_0 = 1$, we have

$$T^{-1}\varepsilon(T)'C_T(1)\varepsilon(T) = T^{-1}\varepsilon(T)'(0 \ , \ \varepsilon_1 \ , \ \varepsilon_1 + \varepsilon_2 \ , \ \cdots \ , \ \sum_{t=1}^{T-1}\varepsilon_t)'$$

$$- \frac{1}{2}\left[(T^{-1/2}\sum_{t=1}^{T}\varepsilon_t)^2 - T^{-1}\sum_{t=1}^{T}\varepsilon_t^2\right] .$$

This has an asymptotic distribution under standard conditions, while

$$E[\varepsilon(T)'C_T(1)'C_T(1)\varepsilon(T)] = E\sum_{t=1}^{T-1}(\sum_{i=1}^{t}\varepsilon_i)^2$$

$$= \sigma^2[(T-1) + (T-2) + \ldots + 1] = O(T^2) ,$$

which suggests that the appropriate scaling factor is $c_T = T$.

For the case $\lambda_0 > 1$, let $w(\lambda_0) = [w_1(\lambda_0), \ \ldots \ , w_T(\lambda_0)]' = C_T(\lambda_0)\varepsilon(T)$. We have $w_1(\lambda_0) = 0$ and

$$w_t(\lambda_0) = \sum_{i=1}^{t-1}(\lambda_0)^{t-1-i}\varepsilon_i \ , \quad t = 2, \ \ldots \ , T \ ,$$

hence

$$E\ [w_t(\lambda_0)]^2 = \sigma^2 \sum_{i=1}^{t-1}(\lambda_0^2)^{t-1-i} = \sigma^2[1 - (\lambda_0^2)^{t-1}]/[1 - \lambda_0^2]$$

$$E[\varepsilon' C(\lambda_0)' C(\lambda_0)\varepsilon] = E\sum_{t=1}^{T}\ [w_t(\lambda_0)]^2 = \sigma^2(\lambda_0^2)^T/[1 - \lambda_0^2]^2 .$$

In this case, one would typically take $c_T = |\lambda_0|^T$. Similar arguments can be applied to the determination of the matrix D_T.

Under Assumption C, we can establish the following result on the asymptotic null distribution of $\mathcal{F}_T$.

Proposition 5: *Under Assumption C, the statistic* $\mathcal{F}_T$ *in (6.6) has an asymptotic distribution as* $T \to \infty$ *which is the same for all processes in the class* $\mathcal{Z}_0$ *defined in C.*

An important special case of the latter proposition is the one where $\mathcal{Z}$ includes cases (among others) where $\varepsilon(T)/\sigma \sim N[0, I_T]$ and $X(T)$ is independent of $\varepsilon(T)$. Then the inferences drawn using the latter assumptions remain valid asymptotically under the more general assumptions represented by the class $\mathcal{Z}$.

7. Illustrations

For an application and illustration of the exact procedures in the context of pure univariate time-series analysis we used annual data published by Balke and Gordon (1986, pp. 781-786) on nominal GNP and M2 in the US. We analyze the natural logarithm of GNP/M2, i.e. the velocity of M2 (indicated by v_t) over the period 1959-1983. Estimation by OLS yields:

$$(7.1) \qquad v_t = \beta_\iota + \beta_\tau (t/T) + \lambda\, v_{t-1} + \varepsilon_t \qquad t = 1, \ldots, T$$

$$= \underset{(0.037)}{0.135} - \underset{(0.020)}{0.005}\,(t/T) + \underset{(0.090)}{0.723}\, v_{t-1} + \hat{\varepsilon}_t ,$$

$$T = 35 \; ; \; s = 0.0286 \; ; \; R^2 = 0.739 \; ; \; DF = -3.078 \; .$$

Estimated (asymptotic) standard errors are presented between parentheses; s is the degrees of freedom corrected estimator of σ and DF is the t-ratio Dickey-Fuller statistic for testing the unit root hypothesis $\lambda = 1$. Asymptotic tests for higher order (up to fourth) serial correlation, for structural change and for non-normality of the disturbances have large p-values and hence do not indicate severe misspecification. This is also the case for the more restricted model:

$$(7.2) \qquad v_t = \underset{(0.034)}{0.138} + \underset{(0.074)}{0.710}\, v_{t-1} + \hat{\varepsilon}_t , \qquad t = 1, \ldots, T;$$

$$T = 35 \; ; \; s = 0.0281 \; ; \; R^2 = 0.739 \; ; \; DF = -3.919 \; .$$

For producing exact inference we chose $\mathcal{D}_\lambda = [-1, +1]$ and used 999 replications for the simulation tests.

When assuming normality of the disturbances in specification (7.1) we found, upon using the $\mathcal{L}_\lambda^{**}$ statistic, an exact 95% confidence set for λ given by $\mathcal{C}_\lambda^{**}(0.05) = [0.64, 1.00]$. The statistic $\mathcal{L}_\lambda^{*}$ yields a much wider region, viz. $\mathcal{C}_\lambda^{*}(0.05) = [0.28, 1.00]$. Hence, we see that the corresponding unit root hypothesis tests (which are both in fact equivalent to particular Dickey-Fuller type tests) do not reject $\lambda = 1$. However, when we use our comprehensive procedures, and test in (7.1) the joint null hypothesis $\lambda = 1$ and $\beta_\tau = 0$ (we indicate the intercept coefficient by β_ι and the trend

coefficient by β_τ), we find for both the statistics $\mathcal{L}^{**}_{(\lambda,\gamma)}(\lambda_0,\gamma_0)$ and $\mathcal{L}^{\dagger *}_{(\lambda,\gamma)}(\lambda_0,\gamma_0)$ p-values of 0.042, and hence, the pure random walk with drift model is rejected at level 5%. Again the tests that do not include the zero restrictions on the redundant regressors under the null seem less powerful; they both yield a p-value of 0.136 (note that conditioning on y_0 has apparently no effect here). If we do "a test on the regression", viz. $\lambda = 0$ and $\beta_\tau = 0$, all four tests mentioned in Proposition 1 yield p-values of 0.000 . The same joint hypotheses can also be tested by a two-stage procedure. The random walk with drift hypothesis is rejected at 10% level, since the p-value of the conditional test statistic (3.23) is 0.043 . Zero restrictions on all coefficients apart from the intercept are rejected right-away as $\lambda = 0$ is not in the confidence set $\mathfrak{C}^{**}_{\lambda}(0.05)$.

Exploiting the $\mathcal{L}^{**}_{\lambda}$ interval we find (conservative) 90% confidence intervals [0.004, 0.190] and [-0.078, 0.040] for β_ι and β_τ respectively. Upon testing the significance of the trend coefficient we obtain a p-value for $\mathcal{L}^{L}_{\gamma}(\gamma_0)$ of 0.994, so a zero value of the trend coefficient should certainly not be rejected (as we already learned from the confidence set), but we also find a p-value for $\mathcal{L}^{U}_{\gamma}(\gamma_0)$ of 0.043 . As this is smaller than 0.15 this means that the bounds-test with level 10% ($\alpha_1 = 0.05 = \alpha_2$) is inconclusive. So, acceptance of a zero trend coefficient (which seems more or less self-evident following naive asymptotic reasoning) is in fact not strongly sustained by the 35 data observations either.

Upon applying the exact procedures under normality to the parsimonious parametrization (7.2) we obtained now [0.64, 0.99] for the $\mathcal{L}^{**}_{\lambda}(\gamma_0)$ based 95% confidence interval. Hence, we see that the unit root hypothesis is rejected now at the 5% level. This $\mathfrak{C}^{**}_{\lambda}(0.05)$ interval yields a 90% conservative confidence set [-0.001, 0.178] for the intercept, and hence the intercept is not significant at the 10% level.

Just for curiosity (and not primarily for its empirical relevance for these data), we also analyzed these data under the assumption of non-normal disturbances. We considered two cases viz. i.i.d. disturbances that are distributed as $(\chi^2_1 - 1)$, hence skew with zero mean, and t_1 or Cauchy distributed disturbances, i.e. symmetric but with no finite moments. For the skew disturbances we found for (7.2) the interval $\mathfrak{C}^{**}_{\lambda}(0.05) = [0.64, 0.96]$ and the intercept is now significantly positive, since $\mathcal{L}^{U}_{\gamma}(\gamma_0)$ has a p-value of 0.004 , thus the bounds test rejects. The Cauchy disturbances yield $\mathfrak{C}^{**}_{\lambda}(0.05) = [0.61, 1.00]$ in (7.2) and the p-values of $\mathcal{L}^{U}_{\gamma}(\gamma_0)$ and $\mathcal{L}^{L}_{\gamma}(\gamma_0)$ for the intercept are 0.380 and 0.001 respectively, so the bounds test is

inconclusive; the conservative confidence set for the intercept is [-0.004, 0.191].

To illustrate the inference techniques in a simple structural econometric model, we build on an empirical study on narrow money demand in Canada by Marothia and Phillips (1982), henceforth MP. They perform various regressions in order to identify the key explanatory variables for money demand and supply functions by using both single and simultaneous equation estimation techniques. In the case of the demand equation OLS, 2SLS, 3SLS and IV give virtually the same results, viz. that real cash balance (m_t) is determined only by real income (y_t), the short-term interest rate (r_t), lagged real cash balance (m_{t-1}) and an intercept. MP's study is based on logs of quarterly data from 1970(I) until 1979(IV). During this period Canada had a flexible exchange rate regime. All data, except interest rates, are seasonally adjusted. The effects of long-term interest rates and wealth are found to be insignificant. The similarity of the various estimates is interpreted here as indicating that simultaneity is not a major issue for this particular equation. Of course, strong-exogeneity (especially of y_t) is highly unlikely here, but under Assumption C we can deal with that.

We adopt the final preferred specification of MP and obtain the following OLS results (due to data revisions our findings differ slightly from the results published by MP):

$$(7.3) \qquad m_t = \beta_1 + \beta_2 y_t + \beta_3 r_t + \lambda m_{t-1} + \varepsilon_t$$

$$= \underset{(0.304)}{-0.430} + \underset{(0.039)}{0.083}\, y_t - \underset{(0.010)}{0.058}\, r_t + \underset{(0.056)}{0.913}\, m_{t-1} + \hat{\varepsilon}_t \; ;$$

$$T = 39 \; ; \quad s = 0.0136 \; ; \quad R^2 = 0.9594 \; .$$

No obvious statistical evidence is found regarding structural breaks in relationship (7.3), and tests for (higher order) serial correlation and non-normality of the disturbances are insignificant. The main conclusions of MP on Canadian narrow money demand concern the estimated values of the long-run elasticities with respect to income and interest; equation (7.3) yields the plausible estimates 0.95 and -0.67 respectively.

Before we present our exact inference results, we first examine standard (asymptotic) inference on the unknown coefficient values such as typically provided in current practice (the intercept is indicated as β_1; β_2 and β_3 are

the y and r coefficients respectively). Table 1 presents the confidence intervals for the individual coefficients obtained at a nominal confidence coefficient of 95 and 90% respectively. Such intervals can either be based on critical values of the standard Normal or the Student distribution. The former intervals follow directly from classic results on likelihood ratio's. In the latter case the estimated standard error of the disturbance and coefficient estimate should include a degrees of freedom correction. It is generally believed that using the degrees of freedom correction and the (slightly larger) Student critical values reduces the approximation errors committed in finite samples. In Table 1, the latter wider intervals are indicated by "asymptotic-t", and the (uncorrected) pure asymptotic intervals are indicated by "asymptotic-N".

<< TABLE 1 here >>

We should mention that the intervals for λ, although obtained by the standard OLS recipe, are not genuinely asymptotic intervals, since the normal asymptotic distribution for the underlying test statistic is only valid for $|\lambda| < 1$; testing $\lambda = \lambda_0$ for $\lambda_0 \geq 1$ require different asymptotic null distributions.

Table 2 presents *exact* confidence intervals for the individual coefficients. $\mathcal{C}_\lambda(\alpha_1)$ intervals are obtained from two different test statistics using the procedure described in Lemma 2. We now used $N = 2000$ (but $N = 1000$ gave virtually the same results) and assumed normality of the disturbances. In the upper part of the table we present results where $\lambda \in \mathcal{D}_\lambda = [-2, +2]$. Note that these exact intervals can be obtained for any region $\mathcal{D}_\lambda$, hence there are no objections from a statistical point of view against the values equal or greater than unity here. We see that the two intervals presented for λ are both much wider than the asymptotic intervals. Apparently the information in the data or the capabilities of the tests are very weak with respect to rejecting exorbitant (from an economic point of view) λ values. The test $\mathcal{L}_\lambda^{**}$, which restricts the coefficients of the redundant variables under the null, seems the more effective one for these data. A fair comparison with the asymptotic intervals would require the assessment of the actual confidence coefficient of the latter type of intervals, which is impossible, due to the dependence on unknown nuisance parameter values.

<< TABLE 2 here >>

The intervals for the β's have been obtained by employing (4.1) for $\alpha_1 = 0.05 = \alpha_2$, exploiting the exact interval $\mathfrak{C}_\lambda^{**}(0.05)$. Quite remarkably, the exact intervals in Table 1. The intervals for β_2 indicate that, according to these exact procedures, the income coefficient is not significantly different from zero at the 10% level. As it seems, in this relationship and for these data, an estimated coefficient of λ greater than unity can easily be accommodated by a negative income effect. From the lower part of Table 2, which presents results on the exact inference procedures obtained under the a priori restriction $|\lambda| \leq 1$, we still find that the income coefficient is not significantly positive.

The test procedures of Section 3 can easily be used to produce exact versions of "the test of the overall relation". Table 3 contains exact *p*-values on this, and on a few other hypotheses. In the first row of this Table, joint significance of the explanatory variables is found by all four comprehensive tests and also by the two-stage procedure. In the second row, we test the three values for the coefficients found by MP; these are not rejected. The exact test procedure can also be used to test the (structural) MP specification against the pure time-series simple random walk with drift specification, which involves the restrictions: $\lambda = 1$ and $\beta_2 = \beta_3 = 0$. This particular unit-root hypothesis cannot be tested by standard Dickey-Fuller procedures; the third row of Table 3 indicates that all our tests yield strong rejection at the 1% level, so the explanatory variables suggested by economic theory produce a statistically highly significant improvement over the pure descriptive unit-root model.

<< TABLE 3 here >>

The comprehensive procedures of Section 3.1 require to restrict λ under the null. The two-stage procedure, however, also enables to test the less restrictive hypotheses of the bottom two lines in Table 3. Building on the confidence interval $\mathfrak{C}_\lambda^{**}(0.05)$, procedure (4.7) yields a zero *p*-value for the hypothesis $\beta_2 = \beta_3 = 0$. Hence, it is rejected at level $\alpha = (0.05 + \alpha_2)$ for any $\alpha_2 > 0$. For the hypothesis $\beta_2 + \beta_3 = 0$ this procedure yields an insignificant $\mathfrak{L}_\gamma^L$ value. However, the test of this hypothesis is inconclusive since $\mathfrak{L}_\gamma^U$ has a zero *p*-value, which is below $\alpha_2' = 2\alpha_1 + \alpha_2 \geq 10\%$.

Tests on long-run elasticities are sensible only when we restrict the

domain of λ. If we choose $\mathcal{D}_\lambda = [0, 1)$, then the confidence set will be unbounded. Bounded sets can be obtained from (5.13) or (5.15) if we choose $\mathcal{D}_\lambda = [0, \lambda^U]$ with $\lambda^U < 1$, which is not pursued here.

8. Conclusions

By exploiting techniques to annihilate nuisance parameters from test statistics and by using these in combination with generalized bounds test procedures we are able to produce exact inference techniques on virtually any form of hypothesis on the coefficients of a first-order autoregressive distributed lag model. The resulting procedures are computer intensive, but nevertheless operational. As usual, genuine exactness is only obtained under specific conditions. In the present case the conditions are of similar nature as those for exactness of t- and $\mathcal{F}$-tests in the static linear regression model. These conditions are: strongly-exogenous regressors and i.i.d. normally distributed disturbances. In fact, the requirements for exactness of the techniques presented here for the first-order dynamic regression model are weaker, since we can deal straightaway with any (up to a scale factor known) form of distribution of the disturbances.

Our inference procedures can also be justified on basis of asymptotic arguments which are of the same type as those underlying the standard non-exact procedures. Therefore, and due to their local exactness, the accuracy of the techniques suggested here seems less vulnerable in general. After all, they are exact for a reference case, and the standard procedures are only approximate in any finite sample.

The non-similarity of the classical inference techniques in finite samples precludes the assessment of their actual significance and power in practical empirical situations, and a proper comparison of the efficiency and effectivity of the two approaches is therefore hard to establish. This seems feasible in controlled simulation experiments only, and shall be pursued in future research. Most probably, the classic procedures outperform the exact techniques on a local asymptotic power criterion, since the latter loose extra degrees of freedom. This aspect may prove to be even more important when the present procedures are generalized for the higher-order dynamics case, which requires extra redundant regressors. However, even if that really is the case, then this does not necessarily disqualify the exact procedures

for use in finite samples. If one attaches great importance to controlling significance levels, and is willing to cope with the computational burden, then the exact procedures are clearly preferable, since for that matter, they have no competitors.

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APPENDIX

APPENDIX

PROOF of Lemma 1: Making use of results for the inverse of the partitioned matrix $C'C$ and evaluating $C(C'C)^{-1}C'$ one finds the relationship $P(C) = P(B) + P[M(B)A]$, from which the Lemma immediately follows.

■

PROOF of Lemma 2: To simplify the notation set $R_j = R_j(N)$, $j = 1,\ldots,N$. Since the random variables $Z_1,\ldots,Z_N$ are i.i.d. with a continuous distribution, the rank vector $(R_1,\ldots,R_N)'$ is with probability 1 a permutation of the integers $(1,2,\ldots,N)$, and each permuation has the same probability $1/N!$; see Hájek and Šidák (1967, section 2.3). Consequently,

$$P[R_j = i] = \frac{1}{N}, \qquad i=1,2,\ldots,N,$$

and

$$P[R_j/N \le x] = \frac{I[xN]}{N}, \qquad 0 \le x \le 1,$$

from which we see that

$$P[R_j/N \le x] = \frac{I[xN] - 1}{N}, \quad \text{if } xN \in \mathbf{Z}_+,$$
$$= \frac{I[xN]}{N}, \quad \text{otherwise},$$

where $\mathbf{Z}_+$ is the set of the positive integers. Since, for any real number z,

$$I[N-z] = N - z, \quad \text{if } z \text{ is an integer},$$
$$= N - I(z) - 1, \quad \text{otherwise},$$

we then have, for $0 \le x \le 1$,

$$P[R_j/N \ge x] = 1 - P[R_j/N < x] = \frac{N - I[xN] + 1}{N}, \quad \text{if } xN \in \mathbf{Z}_+,$$
$$= \frac{N - I[xN]}{N}, \quad \text{otherwise},$$

hence

$$P[R_j/N \ge x] = \frac{I[N(1 - x)] + 1}{N}, \quad \text{if } 0 < x \le 1,$$
$$= 1, \quad \text{if } x = 0,$$

from which we get (2.28).

■

PROOF of Proposition 1: Upon dividing numerator and denominator of the various test statistics by σ^2 it is obvious that, under $\lambda = \lambda_0$ and $\gamma = \gamma_0$, the numerators are pivotal quadratic forms in ε/σ. Applying Lemma 1 the unrestricted sum of squared residuals of (3.10) can be expressed as

$$\text{(A.1)} \qquad SS_1[W(\lambda_0,\gamma_0)] = \varepsilon' M[W(\lambda_0,\gamma_0)]\varepsilon$$
$$= \varepsilon' M[Z(\lambda_0,\gamma_0)]\varepsilon - \frac{(\varepsilon' M[Z(\lambda_0,\gamma_0)]y_{-1})^2}{y'_{-1} M[Z(\lambda_0,\gamma_0)]y_{-1}} .$$

Exploiting the special composition of $Z(\lambda_0,\gamma_0)$, we find from (3.8) that under $(\lambda = \lambda_0 , \gamma = \gamma_0)$,

$$M[Z(\lambda_0,\gamma_0)]y_{-1}/\sigma = M[Z(\lambda_0,\gamma_0)]C(\lambda_0)\varepsilon/\sigma ,$$

and so both $\mathcal{L}^{*}_{\lambda,\gamma}(\lambda_0,\gamma_0)$ and $\mathcal{L}^{**}_{\lambda,\gamma}(\lambda_0,\gamma_0)$ yield similar test statistics.

Along similar lines it is found that $SS_1[W^{\dagger}(\lambda_0,\gamma_0)]/\sigma^2$ is pivotal if y_0 is fixed, since

$$M[Z^{\dagger}(\lambda_0,\gamma_0)]y_{-1}/\sigma = M[Z^{\dagger}(\lambda_0,\gamma_0)]C(\lambda_0)\varepsilon/\sigma ,$$

where $Z^{\dagger}(\lambda_0,\gamma_0)$ depends on y_0. So, if y_0 is fixed both $\mathcal{L}^{\dagger}_{\lambda,\gamma}(\lambda_0,\gamma_0)$ and $\mathcal{L}^{\dagger *}_{\lambda,\gamma}(\lambda_0,\gamma_0)$ give similar test statistics too. ∎

PROOF of Proposition 2: Since $(\lambda,\gamma) \in \mathcal{C}_{\lambda,\gamma}(\alpha_1,\alpha_2)$ implies $\gamma \in \mathcal{C}^{U}_{\gamma}(\alpha_1,\alpha_2)$, we find

$$P\left[\, \gamma \in \mathcal{C}^{U}_{\gamma}(\alpha_1,\alpha_2) \,\right] \geq P\left[\, (\lambda,\gamma) \in \mathcal{C}_{\lambda,\gamma}(\alpha_1,\alpha_2) \,\right] \geq 1 - \alpha .$$

Now define the set

$$\mathcal{C}^{L}_{\gamma}(\alpha_1,\alpha_2) = \left\{ \gamma_0 \colon \exists\, \lambda_0 \in \mathcal{C}_{\lambda}(\alpha_1) \text{ such that } (\lambda_0,\gamma_0) \notin \mathcal{C}_{\lambda,\gamma}(\alpha_1,\alpha_2) \right\} ,$$
$$\text{if } \mathcal{C}_{\lambda}(\alpha_1) \neq \emptyset ,$$
$$= R^{r} , \quad \text{if } \mathcal{C}_{\lambda}(\alpha_1) = \emptyset .$$

This is the complement of $\mathcal{C}^{L}_{\gamma}(\alpha_1,\alpha_2)$ in the space of all admissible values of

γ, hence $P[\gamma \in \mathcal{C}_\gamma^L(\alpha_1,\alpha_2)] = 1 - P[\gamma \in \overline{\mathcal{C}}_\gamma^L(\alpha_1,\alpha_2)]$. We also define

$$\mathcal{C}_{\lambda,\gamma}^{(1)}(\alpha_1,\alpha_2) = \left\{ (\lambda_0,\gamma_0) : \lambda_0 \in \mathcal{C}_\lambda(\alpha_1) \text{ and } (\lambda_0,\gamma_0) \notin \mathcal{C}_{\lambda,\gamma}(\alpha_1,\alpha_2) \right\} .$$

Then

$$P\left[(\lambda,\gamma) \in \mathcal{C}_{\lambda,\gamma}^{(1)}(\alpha_1,\alpha_2) \right] = P\left[\lambda \in \mathcal{C}_\lambda(\alpha_1) \text{ and } \gamma \notin \mathcal{C}_{\gamma|\lambda}(\alpha_2) \right]$$

$$\geq 1 - \alpha_1 - (1 - \alpha_2) = \alpha_2 - \alpha_1$$

where $\mathcal{C}_{\gamma|\lambda}(\alpha_2)$ is defined by (3.25). Note that $(\lambda,\gamma) \in \mathcal{C}_{\lambda,\gamma}^{(1)}(\alpha_1,\alpha_2)$ implies $\gamma \in \overline{\mathcal{C}}_\gamma^L(\alpha_1,\alpha_2)$, so that

$$P\left[\gamma \in \overline{\mathcal{C}}_\gamma^L(\alpha_1,\alpha_2) \right] \geq P\left[(\lambda,\gamma) \in \mathcal{C}_{\lambda,\gamma}^{(1)}(\alpha_1,\alpha_2) \right] \geq \alpha_2 - \alpha_1 .$$

Combining the above, we find

$$P\left[\gamma \in \mathcal{C}_\gamma^L(\alpha_1,\alpha_2) \right] \leq 1 - \alpha_2 + \alpha_1 .$$

Consequently, one finds a liberal confidence set for γ by replacing α_2 by $\alpha_2' = \alpha + \alpha_1$, giving

$$P\left[\gamma \in \mathcal{C}_\gamma^L(\alpha_1,\alpha_2') \right] \leq 1 - \alpha .$$

Since $\alpha_2 \leq \alpha \leq \alpha_2'$, we have $\mathcal{C}_{\lambda,\gamma}(\alpha_1,\alpha_2) \supseteq \mathcal{C}_{\lambda,\gamma}(\alpha_1,\alpha_2')$, hence

$$\mathcal{C}_\gamma^L(\alpha_1,\alpha_2') \subseteq \mathcal{C}_\gamma^L(\alpha_1,\alpha_2) \subseteq \mathcal{C}_\gamma^U(\alpha_1,\alpha_2) .$$

■

PROOF of Proposition 3: From the definitions (5.5) and (5.8), it straightforwardly follows that $SS_1(\lambda) \geq SS(\lambda)$. If $(\lambda,\beta'R')' \in \Gamma_0$ then the infimum $\overline{SS}_0(\lambda)$ is obtained over $(\lambda,\gamma_0')' \in \Gamma_0$ where $\gamma_0 = \gamma = R\beta$, whereas $SS_0(\lambda)$ is obtained over all $\gamma_0 = R\beta_0 \in \mathbb{R}^r$ obeying $(\lambda,\gamma_0')' \in \Gamma_0$, hence $\overline{SS}_0(\lambda) \geq SS_0(\lambda)$. Thus, under the null hypothesis we have $SS_0(\lambda)/SS_1(\lambda) \leq \overline{SS}_0(\lambda)/SS(\lambda)$, and so for any real x

$$P\left[\mathcal{L}_{\Gamma|\lambda}(\Gamma_0,\Omega_1) > x \right] = P\left[\frac{SS_0(\lambda)}{SS_1(\lambda)} > x \right] \leq P\left[\frac{\overline{SS}_0(\lambda)}{SS(\lambda)} > x \right]$$

$$- P\Big[\mathcal{L}_{\gamma|\lambda}(\gamma_0) > x \Big]$$

from which the conservative critical value easily follows. ∎

PROOF of Proposition 4: When $\lambda \in C_\lambda(\alpha_1)$ we have $\mathcal{L}^L_\Gamma(\Gamma_0,\Omega_1) \le \mathcal{L}_{\Gamma|\lambda}(\Gamma_0,\Omega_1)$, and thus, for any real x,

$$\begin{aligned} P\Big[\mathcal{L}^L_\Gamma(\Gamma_0,\Omega_1) > x \Big] &= P\Big[\mathcal{L}^L_\Gamma(\Gamma_0,\Omega_1) > x \text{ and } \lambda \in C_\lambda(\alpha_1) \Big] \\ &\quad + P\Big[\mathcal{L}^L_\Gamma(\Gamma_0,\Omega_1) > x \text{ and } \lambda \notin C_\lambda(\alpha_1) \Big] \\ &\le P\Big[\mathcal{L}_{\Gamma|\lambda}(\Gamma_0,\Omega_1) > x \text{ and } \lambda \in C_\lambda(\alpha_1) \Big] + P\Big[\lambda \notin C_\lambda(\alpha_1) \Big] \\ &\le P\Big[\mathcal{L}_{\Gamma|\lambda}(\Gamma_0,\Omega_1) > x \Big] + \alpha_1 \,. \end{aligned}$$

Hence,

$$P\Big[\mathcal{L}^L_\Gamma(\Gamma_0,\Omega_1) > \mathcal{Z}_{\gamma|\lambda}(\alpha_2;X,R) \Big] \le \alpha_2 + \alpha_1 - \alpha \,.$$

Similarly, $\lambda \in C_\lambda(\alpha_1)$ implies $\mathcal{L}^U_\Gamma(\Gamma_0,\Omega_1) \ge \mathcal{L}_{\Gamma|\lambda}(\Gamma_0,\Omega_1)$, and

$$\begin{aligned} P\Big[\mathcal{L}^U_\Gamma(\Gamma_0,\Omega_1) > x \Big] &\ge P\Big[\mathcal{L}^U_\Gamma(\Gamma_0,\Omega_1) > x \text{ and } \lambda \in C_\lambda(\alpha_1) \Big] \\ &\ge P\Big[\mathcal{L}_{\Gamma|\lambda}(\Gamma_0,\Omega_1) > x \text{ and } \lambda \in C_\lambda(\alpha_1) \Big] \\ &\ge 1 - P\Big[\mathcal{L}_{\Gamma|\lambda}(\Gamma_0,\Omega_1) \le x \Big] - P\Big[\lambda \notin C_\lambda(\alpha_1) \Big] \\ &\ge 1 - P\Big[\mathcal{L}_{\Gamma|\lambda}(\Gamma_0,\Omega_1) \le x \Big] - \alpha_1 \,. \end{aligned}$$

Hence,

$$P\Big[\mathcal{L}^U_\Gamma(\Gamma_0,\Omega_1) > \mathcal{Z}_{\gamma|\lambda}(\alpha'_2;X,R) \Big] \ge \alpha'_2 - \alpha_1 = \alpha \,.$$ ∎

PROOF of Proposition 5: The proof is based on looking at the asymptotic behaviour of the variables $A_1(T)$, $A_2(T)$ and $A_3(T)$ in (6.6). We have

$$A_1(T) = \varepsilon' M(Q)\varepsilon = \varepsilon'\varepsilon - \varepsilon' Q D_T (D'_T Q' Q D_T)^{-1} D'_T Q' \varepsilon$$

$$A_2(T) = \varepsilon' M(\bar{Q})\varepsilon = \varepsilon'\varepsilon - \varepsilon'\bar{Q}D_T(D_T'\bar{Q}'\bar{Q}D_T)^{-1}D_T'\bar{Q}'\varepsilon$$

where we set $\varepsilon = \varepsilon(T)$, $Q = Q_T(\lambda_0, \gamma_0)$ and $\bar{Q} = \bar{Q}_T(\lambda_0, \gamma_0)$. Using Assumption C, we then see that

$$\operatorname{plim}_{T\to\infty} \frac{1}{T} A_2(T) = \sigma^2 > 0$$

$$A_1(T) - A_2(T) \Longrightarrow q_1' \Sigma_{QQ}^{-1} q_1 - \bar{q}_1' \Sigma_{\bar{Q}\bar{Q}}^{-1} \bar{q}_1 = \tilde{q}_1$$

where $\bar{q}_1$ is the $\bar{m} \times 1$ vector such that $q_1 = (\dot{q}_1', \bar{q}_1')'$.

If $\bar{m} = m$, i.e. $\bar{Q} = Q$, then $\tilde{q}_1 = 0$, while for $0 \le \bar{m} < m$, $\tilde{q}_1$ has a continuous distribution. Similarly

$$\frac{1}{c_T} \varepsilon' M(Q) C_T(\lambda_0)\varepsilon = \frac{1}{c_T} \{\varepsilon' C_T(\lambda_0)\varepsilon - \varepsilon' Q D_T(D_T'Q'QD_T)^{-1}D_T'Q'\varepsilon\} \Longrightarrow q_3$$

$$\frac{1}{c_T^2} \varepsilon' C_T(\lambda_0)' M(Q) C_T(\lambda_0)\varepsilon \Longrightarrow q_4$$

so that

$$A_3(T) \Longrightarrow q_3^2/q_4 .$$

Consequently

$$\mathcal{F}_T \Longrightarrow \frac{\tilde{q}_1 + (q_3^2/q_4)}{\sigma^2} = \mathcal{F} ,$$

where $\mathcal{F}$ has a continuous distribution which is the same for all processes in Z_0.

■

Table 1 Asymptotic confidence intervals for the individual coefficients of the Canadian money demand equation (7.3)

coefficient	nominal confidence 95%		nominal confidence 90%	
	asymptotic-N	asymptotic-t	asymptotic-N	asymptotic-t
λ	[0.81, 1.02]	[0.80, 1.03]	[0.83, 1.00]	[0.82, 1.01]
β_1	[-0.99, 0.13]	[-1.05, 0.19]	[-0.90, 0.04]	[-0.94, 0.08]
β_2	[0.01, 0.16]	[0.00, 0.16]	[0.02, 0.14]	[0.02, 0.15]
β_3	[-0.08,-0.04]	[-0.08,-0.04]	[-0.07,-0.04]	[-0.07,-0.04]

Table 2 Exact confidence intervals for the individual coefficients of the Canadian money demand equation (7.3)

$\mathcal{D}_\lambda$	coefficient	actual confidence level	interval	inverted test statistics
[-2,+2]				
	λ	95%	[0.573, 1.363]	$\mathcal{L}_\lambda^*$
		95%	[0.716, 1.169]	$\mathcal{L}_\lambda^{**}$
	β_1	≥ 90%	[-1.365, 0.645]	$\mathcal{L}_\lambda^{**}$ and $\mathcal{L}_\gamma^L$
	β_2	≥ 90%	[-0.117, 0.246]	$\mathcal{L}_\lambda^{**}$ and $\mathcal{L}_\gamma^L$
	β_3	≥ 90%	[-0.084,-0.029]	$\mathcal{L}_\lambda^{**}$ and $\mathcal{L}_\gamma^L$
[-1,+1]				
	λ	95%	[0.573, 1]	$\mathcal{L}_\lambda^*$
		95%	[0.716, 1]	$\mathcal{L}_\lambda^{**}$
	β_1	≥ 90%	[-1.365, 0.292]	$\mathcal{L}_\lambda^{**}$ and $\mathcal{L}_\gamma^L$
	β_2	≥ 90%	[-0.017, 0.246]	$\mathcal{L}_\lambda^{**}$ and $\mathcal{L}_\gamma^L$
	β_3	≥ 90%	[-0.084,-0.036]	$\mathcal{L}_\lambda^{**}$ and $\mathcal{L}_\gamma^L$

Table 3 Exact p-values for various joint hypotheses on the coefficients of the Canadian money demand equation (7.3); $\mathcal{D}_\lambda = [-1, +1]$

null hypothesis	employed test statistic(s)				
	$\mathcal{L}^*_{\lambda,\gamma}$	$\mathcal{L}^{**}_{\lambda,\gamma}$	$\mathcal{L}^{\dagger}_{\lambda,\gamma}$	$\mathcal{L}^{\dagger *}_{\lambda,\gamma}$	$\mathcal{L}^{L}_{\gamma}$ over $\mathcal{C}^{**}_{\lambda}(0.05)$
$\lambda = 0,\ \beta_2 = \beta_3 = 0$	0.000	0.000	0.000	0.000	0.000
$\lambda = 0.86,\ \beta_2 = 0.12,\ \beta_3 = -0.05$	0.253	0.392	0.253	0.392	0.268
$\lambda = 1,\ \beta_2 = \beta_3 = 0$	0.003	0.000	0.003	0.000	0.000
$\beta_2 = \beta_3 = 0$	------not applicable------				0.000
$\beta_2 + \beta_3 = 0$	------not applicable------				0.985

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