Université de Montréal
Faculté des arts et des sciences
Département de sciences économiques

CAHIER 9624

ORDINAL COST SHARING

Yves SPRUMONT[1]

[1] Centre de recherche et développement en économique (C.R.D.E.) and Département de sciences économiques, Université de Montréal

October 1996

Many thanks to H. Moulin for very helpful discussions. Financial support from the Fonds pour la formation de chercheurs et l'aide à la recherche (FCAR) of Québec and the Social Sciences and Humanities Research Council (SSHRC) of Canada is gratefully acknowledged.

C.P. 6128, succursale Centre-ville
Montréal (Québec) H3C 3J7

Télécopieur (FAX) : (514) 343-~~5801~~ 7221
Courrier électronique (E-Mail) : econo@ere.umontreal.ca

Ce cahier a également été publié au Centre de recherche et développement en économique (C.R.D.E.) (publication no 2796).

Dépôt légal - 1996
Bibliothèque nationale du Québec
Bibliothèque nationale du Canada

ISSN 0821-4441

RÉSUMÉ

Nous cherchons à généraliser les mécanismes de partage de coût les plus populaires au cas où les fonctions de coût ne sont pas homogènes. Nous proposons l'axiome d'Ordinalité qui exige l'invariance de la solution proposée à toute transformation préservant la nature du problème de partage considéré. Suivant d'abord l'approche de la *valeur*, nous utilisons l'axiome d'Ordinalité pour caractériser la règle de Shapley-Shubik. Nous proposons une extension du mécanisme *proportionnel* qui, contrairement à la règle d'Aumann-Shapley, est ordinale. Enfin, nous définissons et caractérisons deux extensions ordinales du mécanisme *sériel*.

Mots clés : partage des coûts, valeur, proportionnel, sériel, invariance ordinale

ABSTRACT

We ask how the three best known mechanisms for solving cost sharing problems with homogeneous cost functions - the value, the proportional, and the serial mechanisms - should be extended to arbitrary problems. We propose the Ordinality axiom, which requires that cost shares be invariant under all transformations preserving the nature of a cost sharing problem. Following the *value* approach first, we present a characterization of the Shapley-Shubik rule based on Ordinality. Next, we note that the Aumann-Shapley extension of the *proportional* mechanism is not ordinal; we propose an alternative proportional extension satisfying Ordinality. Finally, we define and characterize two extensions of the *serial* mechanism which, contrary to the Friedman-Moulin rule, are ordinal.

Key words : cost sharing, value, proportional, serial, ordinal invariance

1 INTRODUCTION

This paper reconsiders the problem of allocating the cost of a jointly used productive facility among its users. Each agent demands a quantity of a personalized good, and we wish to divide the total cost equitably among all agents. The entire cost function is known but the agents' preferences are not : the cost sharing rule must rely exclusively on the cost function and the reported demands.

Even in the so-called homogeneous case, i.e., when all goods enter additively in the cost function and may therefore be regarded as just a single good, at least three radically different mechanisms deserve our attention. Under the *value mechanism* advocated by Shubik (1962), the vector of cost shares is the Shapley value of the so-called stand-alone cost game (in which each coalition is assigned the cost of meeting the demands of its members). Under the *proportional mechanism*, cost shares are simply proportional to demands. Finally, under the *serial mechanism* of Moulin and Shenker (1992), demands are ranked, say, $q_1 \leq q_2 \leq \ldots \leq q_n$, and the successive increments in production and cost along the sequence $0, nq_1, q_1 + (n-1)\, q_2, \ldots, q_1 + \ldots + q_n$ are split equally among the agents who are not fully served.

We address the issue of extending the three above-mentioned mechanisms to arbitrary cost functions. The problem is not new. The so-called *Shapley-Shubik (cost sharing) rule* proposed by Shubik (1962) generalizes the value mechanism in the obvious way. The two other mechanisms are harder to extend because they use the sizes of the agents' demands to compute the cost shares : while such comparisons make sense in the homogeneous case, they are not meaningful in general. The *Aumann-Shapley rule* (1974) generalizes the proportional mechanism by charging each agent the integral of his marginal cost along the ray to the demand vector $(q_1, q_2, \ldots, q_n)$. Finally, the *Friedman-Moulin rule* (1995) extends the serial mechanism by charging each agent the integral of his marginal cost along the "piecewise diagonal curve" linking 0 to $(q_1, q_1, ..., q_1)$ to $(q_1, q_2, ..., q_2)$ to .... to $(q_1, q_2, ..., q_n)$.

This paper argues that cost shares should not depend on the conventions used to measure the agents' demands. The formal expression of this principle is the *Ordinality* axiom defined in the next section : it strengthens the classic axiom of Scale Invariance by imposing that the cost shares be invariant under essentially all increasing transformations of the measuring scales rather than just the linear ones.

By its very definition, the Shapley-Shubik rule is ordinal. We prove in Section 3 that it is in fact the only ordinal rule satisfying the classic axioms of Additivity, Dummy, and Symmetry. The Aumann-Shapley rule, though scale-invariant, is not ordinal. Section 5 proposes and defends an alternative extension of the proportional mechanism, called the *ordinally proportional rule*, which is ordinal. Contrary to the Aumann-Shapley rule, it also satisfies the important axiom of Demand Monotonicity introduced in Moulin (1995) : when an agent's demand increases while all other demands stay put, that agent's cost share does not decrease. Turning to the Friedman-Moulin rule, we note that it violates not only Ordinality but even Scale Invariance (as remarked by the authors themselves). Section 4 analyzes two ordinal extensions of the serial mechanism : the *Moulin-Shenker serial rule* (due to these authors but never formally studied) and a new method called the *axial serial rule*. Both rules are axiomatized.

2 A FRAMEWORK AND THE MAIN AXIOM

2.1 Cost Sharing

Let $N = \{1, ..., n\}$ be a nonempty finite set of agents, or *coalition*. A *demand vector (for* N*)* is a vector q in $\mathbb{R}_+^N$. Vector inequalities are written $\leq, <, \ll$.

Let $\mathcal{C}_0(N)$ be the set of functions $C : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$ which are nondecreasing ($p \leq q \Rightarrow C(p) \leq C(q)$ for all p, q) and satisfy $C(0) = 0$. A *cost function (for N)* is an element of some domain $\mathcal{C}(N) \subset \mathcal{C}_0(N)$. Two different domains are used in this paper. If the first-order partial derivative of $C \in \mathcal{C}_0(N)$ with respect to its ith argument exists at $q \in \mathbb{R}_+^N$, we denote it by $\partial_i C(q)$.[1] The domain $\mathcal{C}_1(N)$ is made up of all continuously differentiable functions in $\mathcal{C}_0(N)$. The domain $\mathcal{C}_2(N)$ consists of all twice continuously differentiable functions C in $\mathcal{C}_0(N)$ which have bounded derivatives : there exist real numbers $a(C)$ and $b(C)$ such that $0 < a(C) \leq \partial_i C \leq b(C)$ for all $i \in N$. A ***(cost-sharing) problem*** is a list $(N; q; C)$ where N is a coalition, q is a demand vector for N, and C is a cost function for N. A *(cost sharing) rule* x assigns to each problem $(N; q; C)$ a vector of cost shares $x_N(q; C)$ in $\mathbb{R}_+^N$ satisfying the budget balance condition $\sum_{i \in N} x_{iN}(q; C) = C(q)$. Section 3 assumes $\mathcal{C} = \mathcal{C}_1$ (i.e., $\mathcal{C}(N) = \mathcal{C}_1(N)$ for all N) while Sections 4 and 5 assume $\mathcal{C} = \mathcal{C}_2$.[2]

A cost function $C \in \mathcal{C}(N)$ is *homogeneous* if there is a mapping $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $C(q) = c\left(\sum_{i \in N} q_i\right)$ for all $q \in \mathbb{R}_+^N$. We call a problem $(N; q; C)$ homogeneous if the cost function C is homogeneous. A *(cost sharing) mechanism* is the restriction of a cost sharing rule to the homogeneous problems.

2.2 Ordinality

Fix a coalition N and a domain $\mathcal{C}(N)$. Let $f = (f_1, ..., f_n)$ be a bijection from $\mathbb{R}_+^N$ onto itself. For each cost function C in $\mathcal{C}(N)$, define the function $C^f : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$ by

$$C^f(t) = C(f(t)) \text{ for all } t \in \mathbb{R}_+^N .$$

1 If $q_i = 0$, it is understood that $\partial_i C(q)$ stands for the right-hand derivative.

2 The reason for choosing different domains is the following. The main result of Section 3, namely, Theorem 1, relies on the well-known Dummy axiom which has absolutely no bite if only increasing cost functions are admissible, as in Sections 4 and 5. On the other hand, two of the rules in Sections 4 and 5 are based on systems of differential equations whose solutions would be ill-defined under the domain assumption of Section 3.

We call f an *ordinal transformation* if $\mathcal{C}(N)$ is closed under it, i.e.,

$$C^f \in \mathcal{C}(N) \quad \text{for all } C \in \mathcal{C}(N).$$

Which bijection qualifies as an ordinal transformation depends on the domain of cost functions under consideration. When $\mathcal{C}(N) = \mathcal{C}_1(N)$, a bijection f is an ordinal transformation if and only if it is increasing and continuously differentiable. When $\mathcal{C}(N) = \mathcal{C}_2(N)$, f must be twice continuously differentiable and each derivative f_i' must be positive and bounded.

Consider now two problems $(q; C)$ and $(q'; C')$ for N. We call these problems *ordinally equivalent* if there exists an ordinal transformation f such that

$$C' = C^f \text{ and } q = f(q'),$$

in which case, of course, $f^{-1} := \left(f_1^{-1}, ..., f_n^{-1}\right)$ is also an ordinal transformation under which $C = (C')^{f^{-1}}$ and $q' = f^{-1}(q)$. (The suggestive notation ${}^f q$ will be used instead of $f^{-1}(q)$ in the sequel.) The central axiom of this paper reads as follows :

Ordinality. Let N be an arbitrary coalition. If $(q; C)$ and $(q'; C')$ are two ordinally equivalent problems for N, then $x_N(q; C) = x_N(q'; C')$.

In words : cost shares should be invariant under all transformations that preserve the nature of the problem under consideration. Compared with the standard requirement of Scale Invariance (formally defined in Samet and Tauman (1982) or Friedman and Moulin (1995), for instance), Ordinality may seem exceedingly demanding. We hope to convince the reader in the following three sections that it does leave us a lot of flexibility. Let us just emphasize at this point that Ordinality allows us to use more information than just the "stand-alone cost data".

Definition 1. The *stand-alone (cost) game* generated by a problem $(N; q; C)$ is the cooperative game C_q given by

$$C_q(S) = C\left(q_S, 0_{N\setminus S}\right) \quad \text{for all } S \subset N.$$

(If $q, t \in \mathbb{R}_+^N$ and $S \subset N$, $\left(q_S, t_{N \backslash S}\right)$ is the vector in $\mathbb{R}_+^N$ whose ith coordinate is q_i if $i \in S$ and t_i if $i \in N \backslash S$.)

Let us call *Simplicity* the requirement that all problems generating the same stand-alone game receive the same solution. Simplicity obviously implies Ordinality. To see that the converse is false - and to appreciate how wide is the gap between the two principles - fix N and assume that $\mathcal{C}(N) = \mathcal{C}_2(N)$. Observe that the relation "being ordinally equivalent to" is indeed an equivalence relation on the set of all problems for N. It is easily seen that each equivalence class that it generates contains exactly one problem $(q; C)$ for N which is *normalized* in the following sense :

$$C\left(t_i, 0_{N \backslash i}\right) = t_i \quad \text{for each } t_i \geq 0 \quad \text{and } i \in N.$$

(Throughout the paper, we write i instead of $\{i\}$ whenever there is no risk of confusion.) Since two different normalized problems for N cannot be ordinally equivalent, Ordinality allows us to solve them differently. By contrast, Simplicity commands that all the (uncountably many!) normalized problems generating a same stand-alone game receive the same solution.

3 ORDINALITY AND THE VALUE APPROACH

Throughout this section, we let $\mathcal{C} = \mathcal{C}_1$. We combine Ordinality with the following three axioms.

Additivity. If N is a coalition, q a demand vector for N, and C, C' two cost functions for N, then $x_N(q; C + C') = x_N(q; C) + x_N(q; C')$.

Dummy. Let $(N; q; C)$ be a problem and $i \in N$. If $\partial_i C = 0$, then $x_{iN}(q; C) = 0$.

Symmetry. Let $(N; q; C)$ be a problem and $i, j \in N$. If C is symmetrical in the demands of i and j and $q_i = q_j$, then $x_{iN}(q; C) = x_{jN}(q; C)$.

These conditions are among the most classic requirements in the literature and need no introduction. They constitute in the cost sharing model the natural counterparts of Shapley's (1953) axioms in the cooperative game model. However, while Shapley's axioms characterize the Shapley value, a wide variety of cost sharing rules satisfy Additivity, Dummy, and Symmetry : see Friedman and Moulin (1995). Among them is the so-called Shapley-Shubik rule.

Definition 2. The *Shapley-Shubik rule* x^{SS} assigns to each problem $(N;q;C)$ the Shapley value of the stand-alone game that it generates :

$$x_{iN}^{SS}(q;C) = \sum_{S \subset N} \frac{(s-1)!\,(n-s)!}{n!}\left(C_q(S) - C_q(S\backslash i)\right),$$

where s denotes the cardinality of S.

Clearly, this rule is ordinal. As it turns out, no other ordinal rule satisfies Additivity, Dummy and Symmetry.

Theorem 1 : *The Shapley-Shubik rule is the only rule satisfying Additivity, Dummy, Symmetry and Ordinality.*

Proof : It is clear that the Shapley-Shubik rule satisfies our four axioms. Conversely, let x be a rule satisfying these axioms. The proof that $x = x^{SS}$ is conveniently divided into five steps.[3] Throughout Steps 1 to 4 we fix an arbitrary coalition N. If $p, q \in \mathbb{R}_+^N$, we let $[p,q]$ denote the rectangle $\left\{t \in \mathbb{R}_+^N \mid p \leq t \leq q\right\}$.

Step 1 : We derive the implications of Dummy, Symmetry, and Ordinality on a class of particularly simple problems.

Fix a nonempty $S \subset N$ and $q = \left(q_S, 0_{N\backslash S}\right) \in \mathbb{R}^N$ such that $q_S \gg 0$. Let $p = \left(p_S, 0_{N\backslash S}\right) \in \mathbb{R}^N$ and $\varepsilon \in \mathbb{R}$ be such that $0 \leq p_S \ll q_S$ and $0 < \varepsilon < q_i - p_i$ for all

3 The proof below shows explicitly how our axioms lead to the Shapley-Shubik formula. While this may be of some interest, it is not necessary, since we need only show that at most one rule satisfies the axioms. Following that alternative approach would make Step 4 a bit shorter.

$i \in S$. Define the mapping $C_{\varepsilon pS} : \mathbb{R}_+^N \to \mathbb{R}_+$ by

$$C_{\varepsilon pS}(t) = \sum_{T \subset S} (-1)^{|T|+1} \prod_{i \in T} c_{\varepsilon p_i}(t_i), \tag{1}$$

where $c_{\varepsilon p_i}(t_i) = \min\left\{1, \frac{1}{\varepsilon} \max\{0, t_i - p_i\}\right\}$. It is obvious that $C_{\varepsilon pS}(0) = 0$ and we claim that $C_{\varepsilon pS}$ is nondecreasing. To prove this point, let us consider the real-valued function C_S defined on the unit square $[0, e]$ in $\mathbb{R}_+^N$ by $C_S(y) = \sum_{T \subset S} (-1)^{|T|+1} \prod_{i \in T} y_i$ and let us show that

$$C_S\left(e_S, y_{N \setminus S}\right) = 1 \text{ and } \partial_i C_S(y) \geq 0 \text{ for all } i \in N \text{ and all } y \in [0, e]. \tag{2}$$

Clearly, (2) is true if $|S| = 1$. Fix now $s_0 > 1$ and suppose that (2) is true for each $S \subset N$ with $|S| \leq s_0 - 1$. Fix S_0 with $|S_0| = s_0$. If $i \notin S_0$, then $\partial_i C_{S_0} = 0$. If $i \in S_0$, observe that

$$\partial_i C_{S_0}(y) = 1 - C_{S_0 \setminus i}(y) \text{ for all } y \in [0, e]. \tag{3}$$

By the induction hypothesis, $C_{S_0 \setminus i}\left(e_{S_0 \setminus i}, y_{N \setminus (S_0 \setminus i)}\right) = 1$ and $\partial_j C_{S_0 \setminus i}(y) \geq 0$ for all $j \in N$ and $y \in [0, e]$. Therefore $C_{S_0 \setminus i}(y) \leq 1$, hence $\partial_i C_{S_0}(y) \geq 0$, for all $y \in [0, e]$. Moreover,

$$C_{S_0}\left(e_{S_0}, y_{N \setminus S_0}\right) = \sum_{k=1}^{s_0} \sum_{\substack{T \subset S \\ |T| = k}} (-1)^{k+1} = \sum_{k=1}^{s_0} (-1)^{k+1} \binom{s_0}{k} = -\sum_{k=0}^{s_0} (-1)^k \binom{s_0}{k} + 1 = 1,$$

proving (2). Since each $c_{\varepsilon p_i}$ is nondecreasing in t_i, it follows that $C_{\varepsilon pS}$ is nondecreasing.

Unfortunately, $C_{\varepsilon pS}$ is not quite a cost function because it is not differentiable. But we can approximate it arbitrarily well by a continuously differentiable function. For each $\alpha = 1, 2, \ldots$, and $t \in \mathbb{R}_+^N$, define

$$C_{\varepsilon pS}^{\alpha}(t) = \sum_{T \subset S} (-1)^{|T|+1} \prod_{i \in T} c_{\varepsilon p_i}^{\alpha}(t_i), \tag{4}$$

where $c_{\varepsilon p_i}^{\alpha}(t_i)$ is worth $\frac{1}{2}\left(\frac{2}{\varepsilon} \max\{0, t_i - p_i\}\right)^{1+\frac{1}{\alpha}}$ if $0 \leq t_i \leq p_i + \frac{\varepsilon}{2}$ and $1 - \frac{1}{2}\left(\frac{2}{\varepsilon} \max\{0, \varepsilon - t_i + p_i\}\right)^{1+\frac{1}{\alpha}}$ otherwise. Check that each $C_{\varepsilon pS}^{\alpha}$ is in $\mathcal{C}_1(N)$

and that their sequence converges uniformly on $[0,q]$ to $C_{\varepsilon pS}$. We claim that

$$\lim_{\alpha\to\infty} x_{iN}\left(q;C^{\alpha}_{\varepsilon pS}\right)=\left\{\begin{array}{ll}\frac{1}{|S|} & \text{if } i\in S,\\ 0 & \text{otherwise.}\end{array}\right\}. \tag{5}$$

The reason is simply that every problem of the form $\left(q;C^{\alpha}_{\varepsilon pS}\right)$ (where we recall that $q_S \gg 0$ and $q_{N\backslash S}=0_{N\backslash S}$) is ordinally equivalent to a problem in which all members of S express the same demand, all members of $N\backslash S$ demand zero, and the cost function is symmetrical in the demands of the members of S. Indeed, let $\delta := \min_{j\in N}(q_j-p_j)$. For each $i\in N$, construct a strictly increasing differentiable mapping f_i such that

$$\begin{aligned} f_i(0) &= 0,\\ f_i(t_i) &= 1+t_i-p_i \text{ when } p_i\le t_i\le p_i+\varepsilon,\\ f_i(q_i) &= 1+\delta. \end{aligned} \tag{6}$$

This is obviously feasible. The mapping $f=(f_1,\dots,f_n)$ is an ordinal transformation that makes the problem $\left(q;C^{\alpha}_{\varepsilon pS}\right)$ ordinally equivalent to

$$\left(\left((1+\delta)\,e_S,0_{N\backslash S}\right);C^{\alpha}_{\varepsilon\left(e_S,0_{N\backslash S}\right)S}\right),$$

where e denotes the unit vector in $\mathbb{R}^N$. By Dummy and Symmetry, $i's$ cost share in the latter problem is $1/|S|$ if $i\in S$ and zero otherwise. By Ordinality, the cost shares are the same in the former problem. Since this is true for each α, (5) follows.

Step 2 : Since x satisfies Additivity and Dummy, we may invoke the integral representation result of Friedman and Moulin (1995, Lemma 3). For all $q\in\mathbb{R}^N_+$ and all $i\in N$, there is a nonnegative measure μ_{iq} on $[0,q]$ such that, for every $C\in\mathcal{C}(N)$,

$$x_{iN}(q;C)=\int_{[0,q]}\partial_i C\,d\mu_{iq}. \tag{7}$$

Moreover, the projection of μ_{iq} on the one-dimensional interval $0\le t_i\le q_i$ is the one-dimensional Lebesgue measure.

Step 3 : We derive the key restriction imposed by Symmetry and Ordinality on the measures μ_{iq}.

Fix a demand vector $q \gg 0$ and $i \in N$. If $i \in S \subset N$, define the set $Q_i(S) = \left[\left(q_{S\setminus i}, 0_{N\setminus(S\setminus i)}\right), \left(q_S, 0_{N\setminus S}\right)\right]$ and let $Q_i = \cup_{S: i\in S\subset N} Q_i(S)$. We claim that the latter set has full measure in $[0,q]$:

$$\mu_{iq}(Q_i) = \mu_{iq}([0,q]). \tag{8}$$

To prove this claim, fix a coalition S containing i, and fix $p = \left(p_S, 0_{N\setminus S}\right)$ and ε such that $0 \le p \ll q$ and $\varepsilon < q_j - p_j$ for all $j \in N$. Consider the mapping $C_{\varepsilon pS}$ defined in (1). Note that $\partial_i C_{\varepsilon pS}$ does not exist on the set $Z_i(\varepsilon, p) := \{t \in [0,q] \mid t_i \in \{p_i, p_i + \varepsilon\}\}$ but

$$\mu_{iq}(Z_i(\varepsilon, p)) = 0 \tag{9}$$

because the projection of μ_{iq} on the interval $0 \le t_i \le q_i$ is the Lebesgue measure. Elsewhere in $[0,q]$, $\partial_i C_{\varepsilon pS}(t)$ exists and is easily computed. Letting $c_{\varepsilon p}(t) = (c_{\varepsilon p_1}(t_1), \ldots, c_{\varepsilon p_n}(t_n))$ and recalling (3),

$$\partial_i C_{\varepsilon pS}(t) = \left(1 - C_{S\setminus i}(c_{\varepsilon p}(t))\right) c'_{\varepsilon p_i}(t_i).$$

Note that this expression can be positive only when t belongs to

$$A_i(\varepsilon, p, S) := \{t \in [0,q] \mid p_i < t_i < p_i + \varepsilon \text{ and } t_j < p_j + \varepsilon \ \ \forall j \in S\setminus i\}$$

and is worth $1/\varepsilon$ on the set

$$B_i(\varepsilon, p, S) := \{t \in [0,q] \mid p_i < t_i < p_i + \varepsilon \text{ and } t_j < p_j \ \ \forall j \in S\setminus i\}.$$

Now, for each α, $\partial_i C^{\alpha}_{\varepsilon pS}$ exists and is continuous everywhere in $[0,q]$. Moreover, the sequence $\left\{\partial_i C^{\alpha}_{\varepsilon pS}(t)\right\}$ is uniformly bounded and converges to $\partial_i C_{\varepsilon pS}(t)$ whenever $t \notin Z_i(\varepsilon, p)$. Using (5), (7), (9), and Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}
\frac{1}{|S|} &= \lim_{\alpha\to\infty} \int_{[0,q]\setminus Z_i(\varepsilon,p)} \partial_i C^{\alpha}_{\varepsilon pS}\, d\mu_{iq} \\
&= \int_{[0,q]\setminus Z_i(\varepsilon,p)} \partial_i C_{\varepsilon pS}\, d\mu_{iq} \\
&= \frac{1}{\varepsilon}\mu_{iq}(B_i(\varepsilon, p, S)) + \frac{1}{\varepsilon} \int_{A_i(\varepsilon,p,S)\setminus B_i(\varepsilon,p,S)} \left(1 - C_{S\setminus i}(c_{\varepsilon p}(t))\right) d\mu_{iq}(t)
\end{aligned}$$

and therefore,

$$\mu_{iq}(B_i(\varepsilon,p,S)) \leq \frac{\varepsilon}{|S|} \leq \mu_{iq}(A_i(\varepsilon,p,S)). \tag{10}$$

But if $p' = \left(p'_S, 0_{N\setminus S}\right)$ is defined by $p'_i = p_i$ and $p'_j = p_j + \varepsilon$ for all $j \in S\setminus i$, we see that $A_i(\varepsilon,p,S) = B_i(\varepsilon,p',S)$. As long as $\varepsilon < q_j - p'_j$ for all $j \in S$, it follows from the first inequality in (10) that $\mu_{iq}(A_i(\varepsilon,p,S)) \leq \varepsilon/|S|$. Therefore,

$$\mu_{iq}(A_i(\varepsilon,p,S)) = \frac{\varepsilon}{|S|} \tag{11}$$

for all $p = \left(p_S, 0_{N\setminus S}\right)$ and ε such that $0 \leq p_S \ll q_S$ and $0 < \varepsilon < (q_j - p_j)/2$ for all $j \in S$. Letting S and p vary, it follows that

$$\mu_{iq}(Q_i(\varepsilon)) = \mu_{iq}([0,q])$$

where $Q_i(\varepsilon) = \cup_{S\ni i} Q_i(\varepsilon,S)$, $Q_i(\varepsilon,S) = \{t \in [0,q] \mid t_j \leq \varepsilon$ if $j \in N\setminus S$ and $t_j \geq q_j - \varepsilon$ if $j \in S\setminus i\}$. Letting ε go to zero yields (8).

Step 4 : We find the exact expression of the measures μ_{iq}.

In order to do so, we first note the following combinatorial result : for any $n = 0, 1, \ldots$ and $t = 0, 1, \ldots,$

$$\sum_{k=0}^{n} \frac{(k+t)!}{k!} = \frac{(n+t+1)!}{n!\,(t+1)}. \tag{12}$$

(The proof is by induction on n. If $n = n_0 = 0$, (12) reduces to $t! = (t+1)!/(t+1)$, which is true for all t. Letting $n_0 \geq 1$ and assuming that (12) holds for $n \leq n_0 - 1$ and for all t, we get

$$\begin{aligned}
\sum_{k=0}^{n_0} \frac{(k+t)!}{k!} &= \sum_{k=0}^{n_0-1} \frac{(k+t)!}{k!} + \frac{(n_0+t)!}{n_0!} \\
&= \frac{(n_0+t)!}{(n_0-1)!\,(t+1)} + \frac{(n_0+t)!}{n_0!} \\
&= \frac{(n_0+t+1)!}{n_0!\,(t+1)},
\end{aligned}$$

as desired.)

Let now $i \in S \subset N$, $0 \leq p = \left(p_S, 0_{N \setminus S}\right) \ll q$, and $0 < \varepsilon < (q_j - p_j)/2$ for all $j \in N$. Defining $E_i(\varepsilon, p) := \{t \in [0, q] \mid p_i \leq t_i \leq p_i + \varepsilon\}$, we note that

$$A_i(\varepsilon, p, S) \cap Q_i = E_i(\varepsilon, p) \cap \left(\bigcup_{T: T \cap S = \{i\}} Q_i(T) \right).$$

(Indeed, if $t \in A_i(\varepsilon, p, S) \cap Q_i(T)$, we know that $t_j < q_j$ for all $j \in S$ by definition of $A_i(\varepsilon, p, S)$ and $t_j = q_j$ for $j \in T \setminus i$ and $t_j < q_j$ for $j \in N \setminus T$ by definition of $Q_i(T)$: therefore $T \cap S = \{i\}$ and the claim follows.) In view of (8) and (11), this observation implies that

$$\sum_{T: T \cap S = \{i\}} \mu_{iq}(E_i(\varepsilon, p) \cap Q_i(T)) = \frac{\varepsilon}{|S|}. \tag{13}$$

Using (12), we may now conclude from (13) that

$$\mu_{iq}(E_i(\varepsilon, p) \cap Q_i(T)) = \frac{(t-1)!\,(n-t)!}{n!}\varepsilon \tag{14}$$

whenever $i \in T \subset N$ and $|T| = t$. The proof is again by induction. If $S = N$, (13) becomes $\mu_{iq}(E_i(\varepsilon, p) \cap Q_i\{i\}) = \varepsilon$: this establishes (14) for $T = \{i\}$. Now, let $2 \leq t_0 \leq n$ and suppose that (14) holds whenever $i \in T \subset N$ and $1 \leq |T| = t \leq t_0 - 1$. Fix T_0 such that $i \in T_0 \subset N$ and $|T_0| = t_0$, and define $S_0 = (N \setminus T_0) \cup \{i\}$. Note that $|S_0| = n - t_0 + 1$. By (13),

$$\begin{aligned} \frac{\varepsilon}{n - t_0 + 1} &= \sum_{T: T \cap S_0 = \{i\}} \mu_{iq}(E_i(\varepsilon, p) \cap Q_i(T)) \\ &= \sum_{\substack{T: i \in T \subset T_0 \\ \neq}} \mu_{iq}(E_i(\varepsilon, p) \cap Q_i(T)) + \mu_{iq}(E_i(\varepsilon, p) \cap Q_i(T_0)) \end{aligned}$$

and therefore,

$$\mu_{iq}(E_i(\varepsilon, p) \cap Q_i(T_0)) = \frac{\varepsilon}{n - t_0 + 1} - \sum_{t=1}^{t_0 - 1} \sum_{\substack{T: |T| = t, \\ i \in T \subset T_0}} \mu_{iq}(E_i(\varepsilon, p) \cap Q_i(T)).$$

Using the induction hypothesis and the fact that there are $(t_0 - 1)!/(t-1)!\,(t_0 - t)!$ coalitions T of cardinality t such that $i \in T \subset T_0$, the right-hand side of the above equality is worth

$$\left(\frac{1}{n - t_0 + 1} - \frac{(t_0 - 1)!}{n!} \sum_{t=1}^{t_0 - 1} \frac{(n-t)!}{(t_0 - t)!} \right) \varepsilon.$$

Defining $k = t_0 - t$ and applying (12), the sum in the above expression equals $(n!/(t_0-1)!(n-t_0+1)) - (n-t_0)!$. We obtain that (14) holds true for $t = t_0$, as was to be proved.

Step 5 : We conclude the proof.

From (7) and (14) follows that $x_N(q;C) = x_N^{SS}(q;C)$ for all C and $q \gg 0$. If q is an arbitrary demand vector in $\mathbb{R}_+^N$, let $N^+ = \{i \in N \mid q_i > 0\}$ and define C^+ on $\mathbb{R}_+^{N^+}$ by $C^+(t_{N^+}) = C\left(t_{N^+}, 0_{N \setminus N^+}\right)$. For each $i \in N^+$, we know that $x_{iN^+}(q_{N^+};C^+)$ is $i's$ Shapley value in the stand-alone game with player set N^+ generated by $(q_{N^+};C^+)$. A familiar combinatorial argument shows that this is also $i's$ Shapley value in the stand-alone game with player set N generated by $(q;C)$. Hence $x_N(q;C) = x_N^{SS}(q;C)$. ■

A few remarks are in order. First, Theorem 1 is tight : the reader should have no difficulty finding examples of rules other than Shapley-Shubik's satisfying any combination of three of our four axioms.

It should also be noted that the variable population assumption is used only in the very last step of the proof, and in a very inessential way. Theorem 1 remains true for a fixed coalition of agents whose demands are constrained to be positive. That proviso can be dropped if the rule is required to be continuous in the demands.

Theorem 1 is related to several known results. McLean and Sharkey (1992) characterized the Shapley-Shubik rule by combining Additivity, Dummy, Symmetry, and the Simplicity axiom discussed in Subsection 2.2. As already mentioned, Simplicity is stronger than Ordinality. In Friedman and Moulin's (1995) Theorem 2, Scale Invariance and Demand Monotonicity replace our Ordinality axiom. In their Theorem 4, a lower bound condition replaces the combination of Symmetry and Ordinality in our result.

Since Dummy and Symmetry are hardly disputable conditions of equity[4], Theorem 1 indicates that equitable ordinal rules must violate Additivity. The three main rules discussed in the following two sections are nonadditive.

4 ORDINALITY AND THE SERIAL APPROACH

4.1 Serial Extensions and the Serial Principle

Throughout this section, we let $C = C_2$. We search for ordinal generalizations of the serial mechanism introduced by Moulin and Shenker (1992) for the case of homogeneous problems. Moulin and Shenker's mechanism works as follows. Agent 1, with the lowest demand of output q_1, pays $(1/n)$th of the cost of nq_1. Agent 2, with the next lowest demand q_2, pays agent 1's cost share, plus $1/(n-1)$th of the incremental cost from nq_1 to $q_1 + (n-1)\,q_2$. Agent 3, with the next lowest demand q_3, pays agent 2's cost share, plus $1/(n-2)$th of the incremental cost from $q_1 + (n-1)\,q_2$ to $q_1 + q_2 + (n-2)\,q_3$. And so on. Formally, let us call a demand vector q in $\mathbb{R}_+^N$, or a problem $(q; C)$, naturally ordered if $q_1 \leq ... \leq q_n$. For any demand vector q in $\mathbb{R}_+^N$ and $i \in N$, define the demand vector $q(i)$ by $q_j(i) = \min\{q_i, q_j\}$ for all $j \in N$.

Definition 3. For any coalition N, any naturally ordered vector q in $\mathbb{R}_+^N$, and any *homogeneous* cost function C for N, the *serial mechanism s* computes the cost shares for $(N; q; C)$ according to the formula

$$s_{iN}(q; C) = \frac{C(q(i))}{(n-i+1)} - \sum_{j<i} \frac{C(q(j))}{(n-j+1)(n-j)} \text{ for all } i \in N. \qquad (15)$$

Demand vectors that are not naturally ordered are reordered before formula (15) is applied.

Moulin and Shenker (1992, 1994), and others, have shown that the serial

4 We have in mind the standard interpretation of the cost-sharing model [see, e.g., Shubik (1962)]. The Dummy axiom can be criticized if the agents are not held responsible for their marginal cost functions.

mechanism enjoys remarkable strategic and ethical properties. It is therefore natural to investigate how it could be "extended" to arbitrary cost sharing problems.

Definition 4. A cost sharing rule is a *serial extension* if it coincides with the serial mechanism for every homogeneous problem.

We want our extension to preserve the spririt of the serial mechanism as well as possible. Perhaps the most essential feature of that mechanism is the protection it offers to smaller demanders against larger demanders. As Moulin and Shenker (1992) point out, their mechanism is in fact directly characterized by combining *Equal Treatment of Equals* (which requires that agents with equal demands pay equal cost shares) with the requirement that an agent's cost share be independent of demands larger than his own. The difficulty is that the latter condition, which we call the *Serial Property*, does not make much sense in the context of heterogeneous goods because the quantities of different goods are not comparable. Only cost shares remain comparable. The natural extension of the Serial Property, therefore, requires that an agent's cost share be unaffected by changes in the demands of those who *pay* more than him. The precise condition is as follows.

Serial Principle. Fix a coalition N and a cost function C for N. A rule x satisfies the *Serial Principle for* (N, C) if for all $q, q' \in \mathbb{R}_+^N$ and all $i \in N$,

$$\left\{q'_j = q_j \text{ for } j = i \text{ and for all } j \in N \backslash i \text{ such that } x_{jN}(q; C) < x_{iN}(q; C)\right\}$$

and

$$\left\{q'_j \geq q_j \text{ for all } j \in N \backslash i \text{ such that } x_{jN}(q; C) \geq x_{iN}(q; C)\right\}$$

imply that $x_{iN}(q'; C) = x_{iN}(q; C)$. The rule satisfies the *Serial Principle* if it satisfies the Serial Principle for every (N, C).

An immediate consequence of the Serial Principle, which will be useful in the proof of Lemma 1 below, is the following classic property.

No Exploitation. Let $(N; q; C)$ be a problem and $i \in N$. If $q_i = 0$, then $x_{iN}(q; C) =$

0.

Clearly, every rule which satisfies the Serial Principle and meets Equal Treatment of Equals on the homogeneous problems (or the stronger property of Symmetry defined in Section 3) is a serial extension. Not all serial extensions satisfy the Serial Principle, however. It is easily seen, for instance, that the Friedman-Moulin rule violates it. The principle nevertheless defines a rich class of rules that we now describe.

For any coalition S, a *path* $\left(in\ \mathbb{R}_+^S\right)$ is a continuous map $\pi : \mathbb{R}_+ \to \mathbb{R}_+^S$ such that $\pi(0) = 0$. It is convenient to think of the argument of π as being time. The path π is *increasing* if $\pi(r) \ll \pi(r')$ whenever $r < r'$.

Definition 5. Fix (N, C). *A path function for* (N, C) is a mapping Π which assigns to each $(t; S)$ in $\mathbb{R}_+^N \times \mathcal{N}$ an increasing path $\Pi(t; S)$ in $\mathbb{R}_+^S$. (Here, $\mathcal{N}$ denotes the set of nonempty subsets of N). A *path function* is made up of a collection of path functions for all (N, C).

For any demand vector q in $\mathbb{R}_+^N$, the path function Π for (N, C) generates a path π in $\mathbb{R}_+^N$ in the following natural way. We first follow the path $\Pi(0; N)$ until we meet, say, the first coordinate of q at a point that we call $q^1 = \Pi(0; N)(r^1)$. We define

$$\pi(r) = \Pi(0; N)(r) \text{ for } 0 \le r \le r^1.$$

We then follow the path $q^1 + (0, \Pi(q^1; N\backslash 1))$ in $\mathbb{R}_+^N$ until we meet, say, the second coordinate of q at $q^2 = q^1 + (0, \Pi(q^1; N\backslash 1)(r^2 - r^1))$. We define

$$\pi(r) = q^1 + \left(0, \Pi\left(q^1; N\backslash 1\right)\left(r - r^1\right)\right) \text{ for } r^1 \le r \le r^2.$$

We continue in this way until we get $\pi(r^n) = q$ and set $\pi(r) = q$ for every $r > r^n$. This completes the definition for the case where the coordinates of q are met at distinct times. The definition is easily extended to handle the case where several coordinates are met simultaneously. For instance, if $\Pi(0; N)$ meets the demands of all members of $M \subset N$ simultaneously at q^M, we subsequently follow the path $q^M + \left(0_M, \Pi\left(q^M; N\backslash M\right)\right)$.

Definition 6. Fix (N, C). *The cost sharing rule for* (N, C) *generated by the path function* Π for (N, C) is the mapping $x_N^{\Pi}(.;C)$ defined on $\mathbb{R}_+^N$ as follows. Let $q \in \mathbb{R}_+^N$. Assume, without loss, that the path π generated by Π at q meets the coordinates of q in natural order, i.e., $q_j^i = q_j$ whenever $j \leq i$ (where $q^i := \pi(r^i)$ for each i). Compute the cost shares by splitting the successive cost increments along the sequence $q^1, ..., q^n$ equally among the agents who are not fully served. This yields the formula :

$$x_{iN}^{\Pi}(q;C) = \frac{C(q^i)}{n-i+1} - \sum_{j<i} \frac{C(q^j)}{(n-j+1)(n-j)} \text{ for all } i \in N. \tag{16}$$

The *cost sharing rule generated by a path function* is defined in the obvious way.

It is easily seen that every rule generated by a path function satisfies the Serial Principle. The converse is also true if the rule is continuous (in the demands).

Lemma 1. *A continuous rule satisfies the Serial Principle if and only if it is generated by a path function.*

Proof. The "if" part is clear. To prove the "only if" part, we fix (N, C), a rule x for (N, C) satisfying the Serial Principle for (N, C), and we construct a path function Π for (N, C) that generates x. (It will be clear that many such path functions exist, but that is irrelevant.) Throughout the proof, N and C are dropped from the notations whenever this causes no confusion. The reference to (N, C) is implicit in all concepts involved in the argument.

Step 1 : Defining $\Pi(.;N)$.

Let $P(0;N) := \left\{q | x(q) = \left(\frac{C(q)}{n}, ..., \frac{C(q)}{n}\right)\right\}$. We claim that

(i) for any distinct q, q' in $P(0;N)$, either $q \ll q'$ or $q \gg q'$;

(ii) for each $k \geq 0$, there is some q in $P(0;N)$ such that $C(q) = k$.

These two facts mean that $P(0;N)$ is the image of some increasing path $\Pi(0;N)$ in $\mathbb{R}_+^N$. We then complete the definition of $\Pi(.;N)$ by letting $\Pi(t;N)$ be any arbitrary increasing path in $\mathbb{R}_+^N$ when $t \neq 0$.

To prove (i), let $q, q' \in P(0;N)$. Contrary to the claim, suppose that for some nonempty strict subset M of N, we have $q_i < q'_i$ for all $i \in M$ and $q_j \geq q'_j$ for all $j \in N\backslash M$. Consider the demand vector $q \vee q'$.

From q to $q \vee q'$, the demands from the agents in $N\backslash M$ remain fixed. Since $q \in P(0;N)$, the Serial Principle implies that

$$\forall j \in N\backslash M, x_j(q \vee q') = x_j(q). \tag{17}$$

Since $q < q \vee q', C(q) < C(q \vee q')$ and budget balance implies that

$$\exists i \in M : x_i(q \vee q') > x_i(q). \tag{18}$$

Since $q \in P(0;N)$, (17) and (18) imply that $\exists i \in M : \forall j \in N\backslash M, x_i(q \vee q') > x_j(q \vee q')$.

$$\exists i \in M : \forall j \in N\backslash M, x_i(q \vee q') > x_j(q \vee q'). \tag{19}$$

Now, from q' to $q \vee q'$, the demands from the members of M remain fixed. By the same argument as above (*mutantis mutandis*),

$$\exists j \in N\backslash M : \forall i \in M, x_j(q \vee q') \geq x_i(q \vee q'),$$

which contradicts (19). This proves (i).

The proof of (ii) relies on a fairly standard continuity argument. Fix $k \geq 0$. Since the case $k = 0$ is trivial, assume $k > 0$. Our assumptions on C guarantee that the isocost surface $\{q|C(q) = k\}$ is homeomorphic to the closed n-dimensional simplex Δ. Since x satisfies the Serial Principle, it also satisfies No Exploitation. Moreover, recall that x is continuous in q. Therefore, we need only show that if a continuous mapping $\xi : \Delta \rightarrow \Delta$ meets the condition

$$\forall q \in \Delta \text{ and } i \in N, q_i = 0 \Rightarrow \xi_i(q) = 0, \tag{20}$$

there must exist some $q \in \Delta$ such that $\xi(q) = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$. Define $F : \Delta \rightarrow \Delta$ by

$$F_i(q) = \frac{q_i + \max\left\{\frac{1}{n} - \xi_i(q), 0\right\}}{1 + \sum_{j \in N} \max\left\{\frac{1}{n} - \xi_j(q), 0\right\}}$$

for all $i \in N$. Observe that (20) guarantees that $F_i(q) > 0$ even if $q_i = 0$. We claim that $\xi(q) = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$ if $F(q) = q$. Indeed, if $\xi(q) \neq \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$, there must exist two agents, say 1 and 2, such that $\xi_1(q) < \frac{1}{n} < \xi_2(q)$. But then $\sum_{j \in N} \max\left\{\frac{1}{n} - \xi_j(q), 0\right\} > 0$ and therefore $F_i(q) < q_i + \max\left\{\frac{1}{n} - \xi_i(q), 0\right\}$ for all $i \in N$. In particular, $F_2(q) < q_2$, the desired contradiction. Since the continuity of ξ implies that of F, Brouwer's theorem ensures the existence of some q such that $F(q) = q$ and we are done.

Step 2 : Completing the definition of Π.

We complete the definition of Π by applying the same argument as in Step 1 to coalitions of decreasing sizes. Thus, to construct $\Pi(., N\backslash 1)$, we proceed as follows. For any $q^1 \in P(0; N)$, define

$$\begin{aligned} P\left(q^1; N\backslash 1\right) &= \Big\{d_{N\backslash 1} | \forall i \in N\backslash 1, x_i\left(q^1 + \left(0_1, d_{N\backslash 1}\right)\right) - x_i\left(q^1\right) \\ &= \frac{1}{n-1}\left(C\left(q^1 + \left(0_1, d_{N\backslash 1}\right)\right) - C\left(q^1\right)\right)\Big\}. \end{aligned}$$

By essentially the same argument as before, this set is the image of some increasing path $\Pi(q^1; N\backslash 1)$ in $\mathbb{R}_+^{N\backslash 1}$. We then complete the definition of $\Pi(.; N\backslash 1)$ by letting $\Pi(t; N\backslash 1)$ be an arbitrary increasing path if $t \notin P(0; N)$. Continuing in this way completes the definition of a path function Π.

Step 3 : Checking that Π generates x.

Let x^{Π} be the rule for (N, C) generated by Π and fix an arbitrary demand vector q. We must check that $x^{\Pi}(q) = x(q)$. If $q \in P(0; N)$, then $x^{\Pi}(q) = \left(\frac{C(q)}{n}, \dots, \frac{C(q)}{n}\right) = x(q)$ and we are done. If $q \notin P(0; N)$, we move up along the curve $P(0; N)$ and observe that there exists a unique $q^1 \in P(0; N)$ such that

$$q = q^1 + \left(0_i, d_{N\backslash i}\right)$$

for some agent $i \in N$ and some vector $d_{N\backslash i} \in \mathbb{R}_+^{N\backslash i}$. Without loss, let us assume $i = 1$. If $d_{N\backslash 1} \in P(q^1; N\backslash 1)$, then $x_1^{\Pi}(q) = \frac{C(q^1)}{n} = x_1(q)$ and $x_i^{\Pi}(q) = \frac{C(q^1)}{n} + \frac{1}{n-1}(C(q) - C(q^1)) = x_i(q)$ for all $i \in N\backslash 1$, and we are done. If $d_{N\backslash 1} \notin P(q^1; N\backslash 1)$, we repeat the above argument. ■

An important consequence of Lemma 1, which reinforces the appeal of the Serial Principle, should be mentioned at this stage. Recall the following definition.

Demand Monotonicity. Let $(N; q; C)$ and $(N; q'; C)$ be two problems and let i$\in N$. If $q_i \leq q'_i$ and $q_j = q'_j$ for all $j \in N\backslash i$, then $x_{iN}(q; C) \leq x_{iN}(q'; C)$.

Then, we clearly have the following result.

Corollary 1. *Every continuous rule satisfying the Serial Principle satisfies Demand Monotonicity.*

Proof. This follows directly from Lemma 1 and Definition 6.

The remaining subsections are devoted to three rules satisfying the Serial Principle. The rule described in Subsection 4.2 is the crudest of all. It is not ordinal but will be useful to define and/or understand the ordinal rules proposed in Subsections 4.3 and 4.4.

4.2 The Direct Serial Rule

The simplest rule satisfying the Serial Principle just applies the serial formula to all problems without any modification.

Definition 7. For any coalition N, any naturally ordered demand vector q in $\mathbb{R}_+^N$, and *any* cost function C for N, the *direct serial rule*, which we also denote by s, computes the cost shares for $(N; q; C)$ according to formula (15). Again, demand vectors that are not naturally ordered are reordered before applying the formula.

This rule is generated by any path function which assigns to every (N, C) and every $(t; S)$ in $\mathbb{R}_+^N \times \mathcal{N}$ a path sending $\mathbb{R}_+$ onto the diagonal of $\mathbb{R}_+^S$. We hasten to repeat that it violates Ordinality. In fact, it is not even scale-invariant. Moreover, an agent whose marginal cost function is arbitrarily small may have to bear a substantial part of the total cost : if weakly increasing cost functions were allowed, Dummy would in fact be violated. Finally, the direct serial rule also violates the following basic axiom.

Separation. Let $(N; q; C)$ be a problem. If $C(t) = \sum_{i \in N} C_i(t_i)$ for all $t \in \mathbb{R}_+^N$, then $x_{iN}(q; C) = C_i(q_i)$ for each $i \in N$.

In spite of all its shortcomings, the direct serial rule enjoys several interesting properties. Besides Additivity, Symmetry (and, of course, No Exploitation and Demand Monotonicity), we mention for future reference two properties that have not been studied so far in the cost sharing context. The first property says that an agent who demands nothing may safely be ignored : counting him or not does not affect the cost shares of those with a positive demand. Formally, if C is a cost function for some coalition N and S is a subset of N, define $C_S : \mathbb{R}_+^S \to \mathbb{R}$ by $C_S(q_S) = C\left(q_S, 0_{N \setminus S}\right)$. The axiom reads :

Independence of Null Agents. For every problem $(N; q; C)$ and every $i \in N$, $\{q_i = 0\} \Rightarrow \left\{x_{jN}(q; C) = x_{jN\setminus i}\left(q_{N\setminus i}; C_{N\setminus i}\right) \text{ for all } j \in N \setminus i\right\}$.

This axiom really contains two distinct requirements : it implies No Exploitation and, in addition, embodies a limited form of consistency. Haller (1994) studied a variant of this property in the abstract framework of cooperative games under the name of "null player out property".

The second property states that a change in an agent's demand does not affect the ranking of the other agents' cost shares : whether agent i pays more than j depends on these two agents' demands only.

Rank Independence of Irrelevant Agents. If $(N; q; C)$ and $(N; q'; C)$ are two problems and i, j are two agents in N for whom $q_i = q'_i$ and $q_j = q'_j$, then $\{x_{iN}(q;C) \leq x_{jN}(q;C)\} \Leftrightarrow \{x_{iN}(q';C) \leq x_{jN}(q';C)\}$.

This powerful axiom is violated by most well-known rules, including the Shapley-Shubik, Aumann-Shapley, and Friedman-Moulin rules. The fundamental idea that justifies the axiom, however, is extremely simple and fairly compelling : if an agent i pays more than another agent j, it must be because we *judge* that i's demand is larger than j's. While any information *about the cost function* is meaningful when comparing i and j's demands, the demands of the other agents are irrelevant and should not be used.

4.3 The Axial Serial Rule

We are now ready to describe our first ordinal rule satisfying the Serial Principle. The idea is simply to apply the direct serial rule after a suitable normalization of the problem under consideration.

If N is a coalition and $(q;C)$ an arbitrary problem for N, there is a unique problem $\left({}^A q; C^A\right)$ for N which is ordinally equivalent to $(q;C)$ and satisfies the following *axial normalization condition*

$$C_i^A(t_i) = t_i \text{ for all } t_i \in \mathbb{R}_+, \text{ all } i \in N.$$

In fact, the *axially normalized* problem $\left({}^A q; C^A\right)$ is explicitly given by

$${}^A q_i = C_i(q_i) \text{ for all } i \in N \text{ and } C^A(t) = C\left(C_1^{-1}(t_1), \ldots, C_n^{-1}(t_n)\right) \text{ for all } t \in \mathbb{R}_+^N.$$

(Here and below, ${}^A q_i$ must be read $\left({}^A q\right)_i$.)

Definition 8. The *axial (serial) rule* s^A computes the cost shares for any given problem $(N;q;C)$ by applying the direct serial rule to the axially normalized form of that problem :

$$s_N^A(q;C) = s_N\left({}^A q; C^A\right). \tag{21}$$

For instance, if ${}^Aq$ is naturally ordered (which does not amount to q being naturally ordered), we obtain

$$s^A_{iN}(q;C) = \frac{C^A\left({}^Aq(i)\right)}{(n-i+1)} - \sum_{j<i} \frac{C^A\left({}^Aq(j)\right)}{(n-j+1)(n-j)} \text{ for all } i \in N, \quad (22)$$

where ${}^Aq(i)$ means $\left({}^Aq\right)(i)$. These cost shares do not generally coincide with those recommended by the direct serial rule. They do, however, if the cost function C satisfies the axial normalization condition and, more generally, whenever $C_1 = C_2 = \ldots = C_n$. To see this, suppose without loss that q is naturally ordered. Since $C_1 = \ldots = C_n$, ${}^Aq$ is naturally ordered as well and $C^A\left({}^Aq(i)\right) = C^A\left({}^Aq_1, \ldots, {}^Aq_{i-1}, {}^Aq_i, \ldots, {}^Aq_i\right) = C\left(q_1, \ldots, q_{i-1}, q_i, \ldots, q_i\right) = C\left(q\left(i\right)\right)$ for every $i \in N$, as desired. This implies in particular that s^A is a serial extension (which also follows from the fact that it satisfies the Serial Principle and Symmetry).

Some further definitions will be useful to analyze the axial serial rule. If x is a cost sharing rule, $(N;q;C)$ a problem, and i, j two members of N, we write $iR_x(N;q;C)\,j$ if and only if $x_{iN}(q;C) \leq x_{jN}(q;C)$. We refer to $R_x(N;q;C)$ as the *(cost-share) ranking* prescribed by the rule x for the problem $(N;q;C)$. If $S \subset N$, $R_x(N;q;C) \mid S$ denotes the restriction of that ranking to S. We call R_x the *ranking function* of x.

Lemma 2. *(i) The axial serial rule is a serial extension satisfying Ordinality, Independence of Null Agents, and Rank Independence of Irrelevant Agents.*

(ii) The ranking function of any serial extension satisfying these three axioms is precisely that of the axial serial rule.

Proof. (i) We already know that s^A is a serial extension. Since the solution to two ordinally equivalent problems is computed by applying the direct serial rule to their common axially normalized form, it is obvious that the axial rule is ordinal.

Independence of Null Agents and Rank Independence of Irrelevant Agents are easily proved by invoking the fact that the *direct* serial rule enjoys these properties.

(ii) Let x be a serial extension satisfying Ordinality, Independence of Null Agents, and Rank Independence of Irrelevant Agents. Let $(N;q;C)$ be a problem with $n \geq 3$ and let i,j be two distinct agents in N. By Rank Independence of Irrelevant Agents, $R_x(N;q;C) \mid \{ij\} = R_x\left(N;\left(q_{\{ij\}},0_{N\setminus\{ij\}}\right);C\right) \mid \{ij\}$. On the other hand, by repeated application of Independence of Null Agents, $x_{kN}\left(\left(q_{\{ij\}},0_{N\setminus\{ij\}}\right);C\right) = x_{k\{ij\}}\left((q_i,q_j);C_{\{ij\}}\right)$ for $k=i,j$. From these two facts,

$$R_x(N;q;C) \mid \{ij\} = R_x\left(\{ij\};(q_i,q_j);C_{\{ij\}}\right). \tag{23}$$

In words : the ranking of agents i and j's cost shares in the problem $(N;q;C)$ coincides with that for the two-agent problem $\left(\{ij\};(q_i,q_j);C_{\{ij\}}\right)$. Since that must be true for all pairs of agents, the rankings of cost shares in the two-agent problems must be consistent with each other : for any three i,j,k in N, the statements

$$x_{i\{ij\}}\left((q_i,q_j);C_{\{ij\}}\right) \leq x_{j\{ij\}}\left((q_i,q_j);C_{\{ij\}}\right)$$

and

$$x_{j\{jk\}}\left((q_j,q_k);C_{\{jk\}}\right) \leq x_{k\{jk\}}\left((q_j,q_k);C_{\{jk\}}\right)$$

imply that

$$x_{i\{ik\}}\left((q_i,q_k);C_{\{ik\}}\right) \leq x_{k\{ik\}}\left((q_i,q_k);C_{\{ik\}}\right).$$

This consistency, we claim, implies that the ranking of i and k's cost shares in the two-agent problem $\left(\{ik\};(q_i,q_k);C_{\{ik\}}\right)$ depends upon $C_{\{ik\}}$ only through the behavior of that mapping on the *ith* and *kth* axes : if $C'_{\{ik\}}$ satisfies $C'_i = C_i$ and $C'_k = C_k$, then $R_x\left(\{ik\};(q_i,q_k);C'_{\{ik\}}\right) = R_x\left(\{ik\};(q_i,q_k);C_{\{ik\}}\right)$. To see why, it suffices to observe that $C'_{\{ik\}}$ can be extended to a cost function C' for N that satisfies $C'_{\{ij\}} = C_{\{ij\}}$ and $C'_{\{jk\}} = C_{\{jk\}}$.

Recalling (23), we thus reach the conclusion that $R_x(N;q;C)$ depends on C through $C_1,...,C_n$ only. The proof is now easily concluded. If C satisfies the axial normalization $C_i(t_i) = t_i$ for all t_i and i, we obtain $R_x(N;q;C) = R_x(N;q;C_0)$, where C_0 is the cost function $C_0(t) = \sum_{i \in N} t_i$. But $R_x(N;q;C_0) = R_s(N;q;C_0) = R_s(N;q;C) = R_{s^A}(N;q;C)$ (by the assumption that x is a serial extension, the definition of the direct serial rule, and the fact that the axial and serial rules coincide on axially normalized problems). Therefore, $R_x(N;q;C) = R_{s^A}(N;q;C)$. We now invoke Ordinality to extend this conclusion to the case when C does not satisfy the axial normalization. This proves that the ranking function of x coincides with that of the axial rule on every problem involving at least three agents. This conclusion is extended to two-agent problems by recalling (23) and noting that the latter expression is valid in particular for $x = s^A$. The proof is now complete. ■

The reader may have noticed that the full force of Ordinality and Independence of Null Agents was not exploited in the proof of (ii). The argument carries over, *mutatis mutandis*, if the following "rank" versions replace the original axioms.

Rank Ordinality. For each coalition N and any two ordinally equivalent problems $(q;C)$ and $(q';C')$ for N, $R_x(N;q;C) = R_x(N;q';C')$.

Rank Independence of Null Agents. For every problem $(N;q;C)$ and every $i \in N$, $\{q_i = 0\} \Rightarrow \left\{R_x(N;q;C) \mid N \setminus i = R_x\left(N \setminus i; q_{N\setminus i}; C_{N\setminus i}\right)\right\}$.

More importantly, the assumption that x is a serial extension is also unnecessarily strong. It can be replaced with Separation. In fact, the following *extremely* mild condition suffices :

Weak Rank Separation.[5] Let $(N;q;C)$ be a problem and $i,j \in N$. If $C(t) = \sum_{i \in N} t_i$ for all $t \in \mathbb{R}_+^N$, then $\{x_i(N;q;C) \leq x_j(N;q;C)\} \Leftrightarrow \{q_i \leq q_j\}$.

Summing up the foregoing discussion, we obtain :

5 The term "weak" stresses the fact that the condition is imposed only on the cost function $C(t) = \sum_{i \in N} t_i$ rather than on all separable cost functions. The term "rank" emphasizes that only the ranking – not the magnitude – of the cost shares is at stake.

Lemma 3. *The ranking function of any cost sharing rule satisfying Weak Rank Separation, Rank Ordinality, Rank Independence of Null Agents, and Rank Independence of Irrelevant Agents is precisely that of the axial serial rule.*

Notice that only "rank" axioms are used in this result : this makes sense since the conclusion also bears on the ranking of cost shares. Lemma 3 is an important result in itself because it does not assume that the rule under consideration meets the Serial Principle or is a serial extension. It can also be used to help single out the axial rule from the class of cost sharing rules that do satisfy the Serial Principle.

Theorem 2. *The axial serial rule is the only rule satisfying the Serial Principle, Symmetry, Ordinality, Rank Independence of Null Agents, and Rank Independence of Irrelevant Agents.*

Proof. In view of Lemma 2, we need only prove uniqueness. Fix a rule x satisfying the axioms of Theorem 2.

Step 1 : x is a serial extension.

This follows directly from the Serial Principle and Symmetry by essentially the same argument as in Moulin and Shenker (1992), as already mentioned in Subsection 4.1.

Step 2 : The ranking function of x is that of the axial serial rule.

This follows at once from Step 1 and Lemma 3.

Step 3 : Fix a problem $(N; q; C)$ and $i, j \in N$. If C satisfies the axial normalization condition and $q_i = q_j$, then $x_{iN}(q; C) = x_{jN}(q; C)$.

The argument is as follows. If C is axially normalized, the cost shares prescribed by the axial serial rule coincide with those prescribed by the direct serial rule. Thus $s_{iN}^A(q; C) = s_{jN}^A(q; C)$ if $q_i = q_j$. The claim follows now from Step 2.

Step 4 : Fix a problem $(N; q; C)$. If C satisfies the axial normalization condition, then $x_N(q; C) = s_N^A(q; C)$.

Assume without loss that q is naturally ordered. By Step 3, $x_{iN}(q(1); C) = C(q(1))/n$ for all $i \in N$. By the Serial Principle, $x_{1N}(q(2); C) = x_{1N}(q(1); C) = C(q(1))/n$ and by Step 3, $x_{iN}(q(2)) = C(q(1))/n + (C(q(2)) - C(q(1)))/(n-1)$ for all $i \in N \setminus 1$. Continuing in this way shows that $x_N(q; C) = x_N(q(n); C) = s_N(q; C)$. But $s_N(q; C) = s_N^A(q; C)$ since C is normalized; thus, we are done.

Step 5 : x is the axial serial rule.

This follows immediately from Step 4 and Ordinality. ■

4.4 The Moulin-Shenker Serial Rule

We turn now to a second ordinal serial extension, which emerged from discussions between Moulin and Shenker but was never formally analyzed. By contrast with the previous subsection, the variable population framework will not be of any use in this subsection. We therefore fix an arbitrary coalition N once and for all and drop it from our notations whenever we can.

We begin by taking a second look at the Friedman-Moulin serial rule. Recall that if q is naturally ordered, this rule computes each agent's cost share by integrating his marginal cost along the piecewise-diagonal curve to q. Crucially, this curve does not depend upon the cost function : this ensures Additivity at the cost of violating Ordinality. By letting the curve vary with the cost function in an appropriate manner, we will generate a (nonadditive) ordinal rule.

Let $(q; C)$ be a given problem. Consider the following system of differential equations. For all $i \in N$ and $r \in \mathbb{R}_+$,

$$\dot{a}_i^q(r) = \begin{cases} 1 \,/\, \partial_i C(a^q(r)) & \text{if } a_i^q(r) < q_i, \\ 0 & \text{otherwise.} \end{cases}$$

Think of r as measuring time. Together with the initial condition $a^q(0) = 0$, this system defines a unique nondecreasing path $a^q : \mathbb{R}_+ \to [0, q]$ which reaches q for some value of r and remains there afterwards. Even though our notation does not emphasize the fact that this path depends on C, it clearly does.

Definition 9. The *Moulin-Shenker (serial) rule* s^{MS} computes agent i's cost share in $(q; C)$ by integrating his marginal cost along the path a^q :

$$s_i^{MS}(q; C) = \int_0^\infty \partial_i C(a^q(r))\, \dot{a}_i^q(r)\; dr \text{ for all } i \in N. \tag{24}$$

By its very definition, this rule satisfies the Ordinality axiom. It is important to note that at any point on the path a^q, the incremental cost generated by a small move along the path is shared equally among the agents not yet fully served : indeed, $\partial_i C(a^q(r))\, \dot{a}_i^q(r)\, dr = \partial_j C(a^q(r))\, \dot{a}_j^q(r)\, dr$ whenever $a_i^q(r) < q_i$ and $a_j^q(r) < q_j$. For that reason, letting $r_i = \inf\{r \mid a_i^q(r) = q_i\}$ and assuming, without loss, that $r_1 \leq r_2 \leq \ldots \leq r_n$, we can rewrite (24) as follows :

$$s_i^{MS}(q; C) = \frac{C(a^q(r_i))}{(n-i+1)} - \sum_{j<i} \frac{C(a^q(r_j))}{(n-j+1)(n-j)} \text{ for all } i \in N. \tag{25}$$

This is a particular form of formula (16) : the Moulin-Shenker rule satisfies the Serial Principle.

To better understand this rule and relate it with the direct and axial rules, assume that q is naturally ordered and that the marginal costs of all agents are equal along the piecewise diagonal curve to q. This would be the case if (but not only if) C was homogeneous. Then $a^q(r_i) = q(i)$ for each i and the cost shares are those prescribed by the direct serial rule.

In fact, just like the axial rule, the Moulin-Shenker rule computes the cost shares for any given problem by applying the direct serial rule to a suitably normalized version of that problem. If q is naturally ordered, define the *piecewise diagonal curve through* q, denoted $D(q)$, to be the union of the line segments in $\mathbb{R}_+^N$ which link 0 to $q(1)$ to $q(2)$... to $q(n) = q$ to $q(n+1)$, where $q_i(n+1)$ is q_i if $q_i < q_n$ and $+\infty$ if $q_i = q_n$. Call a problem $(q; C)$ *piecewise-diagonally normalized* if (i) q is naturally ordered, and (ii) agent i's marginal cost is one along the piecewise diagonal curve through $q(i)$:

$$\partial_i C(t) = 1 \text{ for each } t \in D(q(i)) \text{ and each } i \in N.$$

It can be shown that each problem $(q; C)$ is ordinally equivalent to a uniquely defined piecewise-diagonally normalized problem $\left({}^P q; C^P\right)$. The Moulin-Shenker rule solves $(q; C)$ by applying the direct serial rule to $\left({}^P q; C^P\right)$: $s^{MS}(q; C) = s\left({}^P q; C^P\right)$.

The Moulin-Shenker rule admits a surprisingly compact characterization which does not even use Ordinality but relies on a technical property.

Theorem 3. *The Moulin-Shenker rule is the only continuous rule that satisfies the Serial Principle and has all partial first-order derivatives.*

Proof. The theorem remains in fact valid if we restrict our attention to any subset of $\mathcal{C}$. Throughout the proof, we fix a given C in $\mathcal{C}$ and drop it from our notations whenever possible. We give the argument for $n = 2$; the general proof is heavier on notations but does not bring substantial new insights.

Let x be a continuous rule satisfying the Serial Principle. By Lemma 1, there is an increasing path $\pi : \mathbb{R}_+ \to \mathbb{R}_+^2$ that generates x in the following sense. For each q, let q^* be the largest vector on $\pi(\mathbb{R}_+)$ such that $q^* \leq q$, let $r^* = \pi^{-1}(q^*)$, and assume without loss that $q_1^* \leq q_2^*$. Then,

$$x_1(q) = \frac{C(q_1^*)}{2}$$

and $x_2(q) = C(q) - C(q_1^*)/2$.

Suppose now that x has all partial first-order derivatives. We may then assume, without loss of generality, that π is differentiable : if it is not, there is some other path which is differentiable and generates x as well.

Assume now, contrary to the claim, that x is not the Moulin-Shenker rule. There must exist some $r \in \mathbb{R}_+$ such that, say,

$$\partial_1 C(\pi(r))\,\dot{\pi}_1(r) < \partial_2 C(\pi(r))\,\dot{\pi}_2(r)\ . \tag{26}$$

Let $q = \pi(r)$. We claim that the partial derivative of agent 2's cost share with respect to agent 1's demand does not exist at q. To see this, note first that the right-hand derivative

$$\partial_1^+ x_2(q) = \lim_{\substack{\varepsilon \to 0 \\ >}} \frac{x_2(q_1+\varepsilon, q_2) - x_2(q)}{\varepsilon}$$

exists and is zero since $x_2(q_1+\varepsilon, q_2) = x_2(q) = C(q)/2$ for all $\varepsilon > 0$. We claim next that the left-hand derivative

$$\partial_1^- x_2(q) = \lim_{\substack{\varepsilon \to 0 \\ <}} \frac{x_2(q_1+\varepsilon, q_2) - x_2(q)}{\varepsilon}$$

also exists but is not zero. Define the mapping $p : \pi_1(\mathbb{R}_+) \to \mathbb{R}_+$ by $p(t_1) = t_2 \Leftrightarrow (t_1, t_2) \in \pi(\mathbb{R}_+)$. For each $\varepsilon < 0$, we have

$$\begin{aligned} & x_2(q_1+\varepsilon, q_2) - x_2(q) \\ = \;& \left[C(q_1+\varepsilon, q_2) - C(q_1+\varepsilon, p(q_1+\varepsilon))\right] - \tfrac{1}{2}\left[C(q) - C(q_1+\varepsilon, p(q_1+\varepsilon))\right] \\ = \;& \left[C(q_1+\varepsilon, q_2) - C(q)\right] + \tfrac{1}{2}\left[C(q) - C(q_1+\varepsilon, p(q_1+\varepsilon))\right]\ . \end{aligned}$$

Therefore,

$$\partial_1^- x_2(q) = \partial_1 C(q) - \tfrac{1}{2} \lim_{\substack{\varepsilon \to 0 \\ <}} \frac{C(q_1+\varepsilon, p(q_1+\varepsilon)) - C(q)}{\varepsilon}. \tag{27}$$

The limit on the right-hand side exists and equals the directional derivative of C at q in the direction $(1, \dot{\pi}_2(r)/\dot{\pi}_1(r))$, i.e.,

$$\lim_{\substack{\varepsilon \to 0 \\ <}} \frac{C(q_1+\varepsilon, p(q_1+\varepsilon)) - C(q)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{C\left(q_1+\varepsilon, q_2 + \frac{\dot{\pi}_2(r)}{\dot{\pi}_1(r)}\varepsilon\right) - C(q)}{\varepsilon}$$

$$=: D_{(1,\dot{\pi}_2(r)/\dot{\pi}_1(r))} C(q) \; .$$

But this directional derivative is just a linear combination of the partial derivatives at q, namely,

$$D_{(1,\dot{\pi}_2(r)/\dot{\pi}_1(r))} C(q) = \partial_1 C(q) + \frac{\dot{\pi}_2(r)}{\dot{\pi}_1(r)} \partial_2 C(q) \; . \tag{28}$$

Because of (26), this expression is strictly greater than $2\partial_1 C(q)$. Returning to (27), we conclude that $\partial_1^- x_2(q) < 0$, as desired.

We have proved that if a continuous rule satisfies the Serial Principle and has all first-order partial derivatives, it can only be the Moulin-Shenker rule s^{MS}. The s^{MS} rule is continuous and satisfies the Serial Principle. That it has all first-order derivatives is clear from the above argument : if π generates s^{MS}, the inequality sign in (26) must be replaced with an equality. At any q, (27) and (28) hold true for $x = s^{MS}$; now, however, expression (28) is exactly equal to $2\partial_1 C(q)$, and it follows that $\partial_1^- s_2^{MS}(q) = 0 = \partial_1^+ s_2^{MS}(q)$. Of course, the first-order derivatives exist at any demand vector not on the path. ■

The reader will check that the Moulin-Shenker rule is in fact differentiable, and even continuously so. Because the main interest of Theorem 3 lies in the uniqueness part of the result, we found it preferable to use continuity and the existence of the partial derivatives rather than the stronger property of differentiability.

5 ORDINALITY AND THE PROPORTIONAL APPROACH

As in the previous section, we assume that $C = \mathcal{C}_2$. For simplicity, we restrict our attention to the two-agent case (with strictly positive demands). Throughout the present section, any problem $(q; C)$ must be understood as a problem for $N = \{1,2\}$, with $q \gg 0$. The reference to N will be dropped. Our purpose is to generalize the proportional mechanism. Recall that this mechanism solves every homogeneous problem $(q; C)$ by splitting the cost in proportion to demands :

$x_i(q; C) = q_i C(q) / (q_1 + q_2)$ for $i = 1, 2$.

Definition 10. A cost sharing rule is a *proportional extension* if it coincides with the proportional mechanism for every homogeneous problem.

We impose on our proportional extension the Ordinality requirement. As explained in Section 1, this forces us to discard the Aumann-Shapley cost sharing rule. To construct an ordinal proportional extension, it seems natural to try to apply the proportional formula to some normalized version of the problem under consideration. Indeed, this type of approach proved fruitful when constructing ordinal *serial* extensions in the previous section. Finding a suitable normalization procedure is not straightforward, however. To understand the difficulty, consider the axial normalization procedure of Subsection 4.3. Applying the *proportional* formula after axial normalization yields the rule

$$x_i(q; C) = \frac{C_i(q_i)}{C_1(q_1) + C_2(q_2)} C(q) \text{ for } i = 1, 2 \text{ and all } (q; C).$$

This rule makes good sense but is *not* a proportional extension. For instance, if $C(t) = (t_1 + t_2)^2$ and $q = (1, 2)$, it recommends the cost shares (9/5, 36/5) while the proportional mechanism yields the solution (3, 6).

Instead of normalizing the problem along the axes, we suggest to normalize it along the ray to the demand vector.

Definition 11. A problem $(q; C)$ is *proportionally normalized* if

$$\partial_i C(rq) = 1 \text{ for } i = 1, 2 \text{ and all } r \geq 0. \tag{29}$$

The *ordinally proportional rule* applies the proportional formula to the proportionally normalized problem which is ordinally equivalent to the problem at hand. A formal definition will be given shortly. The purpose of the current section is twofold. First, we show (in Lemmata 4 and 5) that the ordinally proportional rule is a well-defined proportional extension : Ordinality is compatible with the "proportional approach". Secondly, we establish a number of interesting properties of the ordinally proportional rule.

Lemma 4. *To each problem $(q;C)$ corresponds a unique proportionally normalized problem which is ordinally equivalent to $(q;C)$.*

Proof. Fix a problem $(q;C)$. We must check that there exists a unique ordinal transformation f such that $\left({}^f q; C^f\right)$ is proportionally normalized. Since $\partial_i C^f(t) = \partial_i C(f(t)) f_i'(t_i)$ for $i = 1,2$ and every $t \in \mathbb{R}^2_+$, the condition that $\left({}^f q; C^f\right)$ is proportionally normalized reads

$$\partial_i C\left(f\left(r\,{}^f q\right)\right) f_i'\left(r f_i^{-1}(q_i)\right) = 1 \quad \text{for } i = 1,2, r \geq 0. \tag{30}$$

Let us assume for a moment that we know the value of ${}^f q$, say, ${}^f q = q^*$. For $i = 1,2$ and $r \geq 0$, define

$$\phi_i(r) = f_i(r q_i^*). \tag{31}$$

Then, $\dot{\phi}_i(r) = q_i^* f_i'(r q_i^*)$ and if we write $\phi(r) := (\phi_1(r), \phi_2(r))$, (30) becomes

$$\dot{\phi}_i(r) = \frac{q_i^*}{\partial_i C(\phi(r))} \quad \text{for } i = 1,2, r \geq 0. \tag{32}$$

Since f is an ordinal transformation, we also have

$$\phi(0) = 0. \tag{33}$$

For each choice of q^*, (32) and (33) form a so-called initial value problem. The unique solution to this problem is an increasing path ϕ in $\mathbb{R}^2_+$: to indicate that this path depends on q^*, we also denote it by $\phi(q^*, \cdot)$. Associated with this path is an ordinal transformation $f(q^*, \cdot)$ given implicitly by (31). Explicitly,

$$f_i(q^*; r) = \phi_i\left(q^*; \frac{r}{q_i^*}\right) \quad \text{for } i = 1,2, r \geq 0.$$

What must be shown now, is that a unique choice of q^* is "consistent" in the sense that the ordinal transformation $f(q^*; \cdot)$ that it generates does satisfy ${}^{f(q^*;\cdot)}q = q^*$.

To prove that point, note first that the condition ${}^{f}q = q^*$ is equivalent to

$$\phi(1) = q\,. \tag{34}$$

Given the problem $(q; C)$, we must therefore show that there is a unique q^* for which the system (32), (33), (34) has a solution. Three observations will be useful. First,

(i) If $q_1^*/q_2^* < q_1^{**}/q_2^{**}$, the curve $\phi(q^*; \mathbb{R}_{++})$ is strictly above $\phi(q^{**}; \mathbb{R}_{++})$ in the following sense : for all $(t_1, t_2) \in \phi(q^*; \mathbb{R}_{++})$ and $\left(t_1, t_2'\right) \in \phi(q^{**}; \mathbb{R}_{++})$ we have $t_2 > t_2'$.

The reason is as follows. From (32), we know that $\dot{\phi}_1(q^*; 0)/\dot{\phi}_2(q^*; 0) < \dot{\phi}_1\ (q^{**}; 0)/\dot{\phi}_2(q^{**}; 0)$: hence, the curve $\phi(q^*; \mathbb{R}_{++})$ is strictly above the curve $\phi(q^{**}; \mathbb{R}_{++})$ in a neighborhood of the origin. Suppose the former does not remain strictly above the latter : let $\hat{q}$ be the smallest nonzero vector belonging to both curves, say, $\hat{q} = \phi(q^*; r^*) = \phi(q^{**}; r^{**})$. Then, if $(t_1, t_2) \in \phi(q^*; \mathbb{R}_{++})$, $\left(t_1, t_2'\right) \in \phi(q^{**}; \mathbb{R}_{++})$, and $t_1 < \hat{q}_1$, we have $t_2 > t_2'$ and therefore

$$\dot{\phi}_1(q^*; r^*)/\dot{\phi}_2(q^*; r^*) \geq \ \dot{\phi}(q^{**}; r^{**})/\dot{\phi}_2(q^{**}; r^{**})\ .$$

From (32), however, the opposite strict inequality holds. This is the desired contradiction.

Next, we note :

(ii) There exists q^* such that $\phi(q^*; \mathbb{R}_{++})$ lies entirely above the ray through q, i.e., $t_1 > q_2 t_2/q_1$ for all $(t_1, t_2) \in \phi(q^*; \mathbb{R}_{++})$. Likewise, there exists q^{**} such that $\phi(q^{**}; \mathbb{R}_{++})$ lies entirely below the ray through q.

This observation follows directly from (32) and our assumption that marginal costs are bounded away from zero and infinity.

Finally, we have :

(iii) For each r, $\phi(q^*; r)$ varies continuously with q^*.

This is a standard property of solutions to ordinary differential equations : see, e.g., Coddington and Levinson (1955), Chapter 1, Section 7.

Using observations (i) to (iii), a simple intermediate value argument shows that there exists a number λ, which must be unique, such that $\phi(q^*; \mathbb{R}_{++}) \ni q$ if and only if $q_2^*/q_1^* = \lambda$. The curve $\phi(q^*; \mathbb{R}_{++})$ is the same for all q^* such that $q_2^*/q_1^* = \lambda$: denote it Λ. Define $a : \mathbb{R}_+ \rightarrow \mathbb{R}_+^2$ by

$$a(q_1^*) = \phi((q_1^*, \lambda q_1^*); 1) .$$

From (32) and our assumptions, a is strictly increasing, continuous and onto Λ. It follows that there exists a unique q_1^* such that $a(q_1^*) = q$. The vector $(q_1^*; \lambda q_1^*)$ is the unique q^* for which system (32)-(33)-(34) has a solution. ■

We are now in a position to define the ordinally proportional rule.

Definition 12. Let us denote by $(q^*; C^*)$ the unique proportionally normalized problem which is ordinally equivalent to $(q; C)$ (see Lemma 4). The *ordinally proportional rule* x^* computes agent i's cost share in any given problem by applying the proportional formula to its proportionally normalized version :

$$x_i^*(q; C) = \frac{q_i^*}{q_1^* + q_2^*} C^*(q^*) = \frac{q_i^*}{q_1^* + q_2^*} C(q) , \ i = 1, 2 .$$

By its very definition, this rule is ordinal. Our next task is to check that it yields the standard proportional formula when the cost function C is homogeneous.

Lemma 5. *The ordinally proportional rule is a proportional extension.*

Proof. Let $(q; C)$ be a homogeneous problem with $C(t) = c(t_1 + t_2)$ for all $t \in \mathbb{R}_+^2$. Consider the initial value problem (32)-(33) with

$$q_i^* = \frac{q_i}{c^{-1}(1)} \quad \text{for } i = 1, 2 .$$

It is easily seen that the solution to that problem is

$$\phi_i(r) = \frac{q_i c^{-1}(r)}{c^{-1}(1)} \quad \text{for } i = 1,2 \text{ and all } r \geq 0 .$$

The corresponding ordinal transformation f is given by

$$f_i(r) = \frac{q_i c^{-1}\left(\frac{rc^{-1}(1)}{q_i}\right)}{c^{-1}(1)} \quad \text{for } i = 1,2 \text{ and all } r \geq 0 . \tag{35}$$

Obviously $f_i^{-1}(q_i) = q_i / c^{-1}(1)$ for $i = 1, 2$, meaning that our choice of q^* is consistent. Therefore,

$$x_i^*(q; C) = \frac{q_i^*}{q_1^* + q_2^*} C(q) = \frac{q_i / c^{-1}(1)}{(q_1 + q_2) / c^{-1}(1)} C(q) = \frac{q_i}{q_1 + q_2} C(q)$$

for $i = 1, 2$, as was to be shown. ■

The above proof shows that the ratio of the individual demands in a homogeneous problem is preserved after proportional normalization. The proportionally normalized form of a homogeneous problem, however, need not be homogeneous. This is clear from (35). For instance, if $C(t) = (t_1 + t_2)^2$ and $q = (1, 2)$, the unique proportionally normalized problem $(q^*; C^*)$ which is ordinally equivalent to $(q; C)$ is given by $q^* = (1, 2)$ and $C^*(t) = \left(\sqrt{t_1} + 2\sqrt{t_2/2}\right)^2$.

Our next result establishes two elementary but important properties of the ordinally proportional rule. To put this result in perspective, recall that the Aumann-Shapley rule is *not* demand-monotonic.

Lemma 6. *The ordinally proportional rule satisfies Separation and Demand Monotonicity.*

Proof. (i) We prove Separation first. Suppose $C(t) = C_1(t_1) + C_2(t_2)$ for all $t \in \mathbb{R}_+^2$ and fix a demand vector q. We claim that $x_i^*(q; C) = C_i(q_i)$ for $i = 1, 2$. To see this, consider problem (32)-(33) with

$$q_i^* = C_i(q_i) \quad \text{for } i = 1, 2 .$$

Its solution is

$$\phi_i(r) = C_i^{-1}(rC_i(q_i)) \text{ for } i = 1,2 \text{ and all } r \geq 0 \, .$$

The corresponding ordinal transformation is given by

$$f_i(r) = \phi_i\left(\frac{r}{C_i(q_i)}\right) = C_i^{-1}(r) \text{ for } i = 1,2 \text{ and all } r \geq 0 \, .$$

Since $f_i^{-1}(q_i) = C_i(q_i) = q_i^*$, our choice of q^* is consistent. Therefore,

$$x_i^*(q;C) = \left(\frac{C_i(q_i)}{C_1(q_1) + C_2(q_2)}\right) C(q) = C_i(q_i) \text{ for } i = 1,2,$$

as claimed.

(ii) To prove Demand Monotonicity, let q, y be two demand vectors such that $q_1 > y_1$ and $q_2 = y_2$, and let C be an arbitrary cost function. It is enough to show that

$$q_1^*/q_2^* \geq y_1^*/y_2^* \, .$$

Suppose the opposite strict inequality holds. From observation (i) in the proof of Lemma 4, the curve $\phi(q^*, \mathbb{R}_{++})$ solving (32)-(33) must lie strictly above the curve $\phi(y^*; \mathbb{R}_{++})$. But the former contains q and the latter y. This contradicts the fact that $q_1/q_2 > y_1/y_2$. ■

It appears from the proof of Demand Monotonicity that the *proportion* of the total cost borne by an agent increases with his demand. This property, which is stronger than Demand Monotonicity, is very much in the spirit of the proportional approach.

We conclude with two further properties of the ordinally proportional rule. A few definitions are in order. A cost function C - or, by extension, a problem $(q;C)$ - is *supermodular* if $\partial_{12}C(t) = (\partial_{21}C(t)) \geq 0$ for all t; it is *submodular* if the reverse weak inequality holds for all t. Note that these restrictions are ordinal : if f is an ordinal transformation, we have, for each t,

$$C^f(t) \quad = \quad C(f(t))$$

$$\Longrightarrow \quad \left(\partial_1 C^f\right)(t) = \partial_1 C\left(f(t)\right) f_1^{'}(t_1)$$
$$\Longrightarrow \quad \left(\partial_{12} C^f\right)(t) = \partial_{12} C\left(f(t)\right) f_1^{'}(t_1)\, f_2^{'}(t_2) \ ;$$

hence, C^f is supermodular (or submodular) if and only if C is.

When a problem is supermodular, it makes sense to ask that an increase in an agent's demand does not decrease the cost share paid by the other; the opposite requirement is meaningful for a submodular problem. Formally,

Cross Demand Monotonicity. If i, j are distinct and $q_i \leq q_i^{'}$ and $q_j = q_j^{'}$, then $x_j(q;C) \leq x_j\left(q^{'};C\right)$ whenever C is supermodular and $x_j(q;C) \geq x_j(q';C)$ whenever C is submodular.

Interestingly, this axiom is satisfied by the Aumann-Shapley rule.[6] As it turns out, the ordinally proportional rule passes the test too :

Lemma 7. *The ordinally proportional rule satisfies Cross Demand Monotonicity.*

Proof. We only give a sketch. Let $(q;C)$ be a submodular problem (the supermodular case is similar). It can be shown that the ordinally proportional rule x^* is differentiable. We will prove that $\partial_2 x_1^*(q;C) \leq 0$. Since $x_1^*(q;C) = q_1^* C(q) / (q_1^* + q_2^*)$,

$$\partial_2 x_1^*(q;C) = \frac{C(q)\left[(q_1^* + q_2^*)\frac{dq_1^*}{dq_2} - q_1^*\left(\frac{dq_1^*}{dq_2} + \frac{dq_2^*}{dq_2}\right)\right]}{(q_1^* + q_2^*)^2} + \frac{q_1^* \partial_2 C(q)}{q_1^* + q_2^*} .$$

But $C(q) = C^*(q^*)$ and $\partial_2 C(q) = \partial_2 C^*(q^*)\frac{dq_2^*}{dq_2} + \partial_1 C^*(q^*)\frac{dq_1^*}{dq_2}$. Since $(q^*;C^*)$ is proportionally normalized, we know that $\partial_1 C^*(q^*) = \partial_2 C^*(q^*) = 1$. Therefore,

$$\partial_2 x_1^*(q;C) = \frac{q_1^*}{q_1^* + q_2^*}\left[\left(1 - \frac{C^*(q^*)}{q_1^* + q_2^*} + \frac{C^*(q^*)}{q_1^*}\right)\frac{dq_1^*}{dq_2} + \left(1 - \frac{C^*(q^*)}{q_1^* + q_2^*}\right)\frac{dq_2^*}{dq_2}\right] .$$

But since $(q^*;C^*)$ is proportionally normalized, we also know that

$$\frac{C^*(q^*)}{q_1^* + q_2^*} = \int_0^1 \partial_2 C^*(rq^*)\, dr = 1$$

[6] This fact may come as a surprise since the Aumann-Shapley rule is not demand-monotonic. Yet, it follows directly from the very definition of the rule. It implies that the Aumann-Shapley rule is demand-monotonic on the submodular problems. This remains true with more than two agents.

and therefore,

$$\partial_2 x_1^* (q; C) = \frac{C^* (q^*)}{q_1^* + q_2^*} \frac{dq_1^*}{dq_2} .$$

We claim now that the submodularity of C implies that $dq_1^*/dq_2 \leq 0$. Suppose the opposite strict inequality holds. Consider the path $\phi (q^*; \mathbb{R}_{++})$ solving the complete system (32), (33), (34). Let the demand vector y satisfy $y_1 = q_1$ and $y_2 > q_2$, and denote by $\phi (y^*; \mathbb{R}_{++})$ the unique solution to the complete system associated with y, i.e.,

$$\begin{aligned} \dot{\phi}_i (r) &= \frac{y_i^*}{\partial_i C (\phi (r))} \text{ for } i = 1, 2, \text{ all } r \geq 0, \\ \phi (0) &= 0, \\ \phi (1) &= y . \end{aligned}$$

If y is sufficiently close to q, $dq_1^*/dq_2 > 0$ implies $y_1^* > q_1^*$. Comparing the systems associated with q and y and using the submodularity of C, we conclude that $\phi_1 (y^*; r) > \phi_1 (q^*; r)$ for each $r > 0$. (The proof is similar to that of observation (i) in the proof of Lemma 4.) But $\phi_1 (y^*; 1) = y_1 = q_1 = \phi_1 (q^*; 1)$, which contradicts this inequality. ■

The last property that we mention is a limited form of solidarity with respect to changes in the cost function.

Cost Solidarity. Let C^1 and C^2 be two cost functions. Suppose there exists a mapping $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that $C^2 = \gamma \circ C^1$. Then, $x (q; C^1) \ll x (q; C^2)$, or $x (q; C^1) = x (q; C^2)$, or $x (q; C^1) \gg x (q; C^2)$.

Observe that the isocost surfaces of C^1 and C^2 are the same. A simple argument shows that every rule satisfying Cost Solidarity and Separation is a proportional extension. The stronger solidarity property obtained by asking that cost shares be affected in a common direction by *any* change in the cost function is incompatible with Separation, as the reader will easily check. This incompatibility motivates our weaker version.

Lemma 8. *The ordinally proportional rule satisfies Cost Solidarity.*

Proof. Let C^1 and C^2 be two cost functions such that $C^2 = \gamma \circ C^1$, and let q be a demand vector. From the initial value problems defining the proportionally normalized forms of $(q; C^1)$ and $(q; C^2)$, it is clear that the cost share ratio is the same under both cost functions : $x_1^*(q; C^1)/x_2^*(q; C^1) = x_1^*(q; C^2)/x_2^*(q; C^2)$. Cost Solidarity follows. ■

The results of this section show that the ordinally proportional rule is well-defined and very well-behaved. Characterizing it within the class of ordinal proportional extensions remains an open problem.

BIBLIOGRAPHY

Aumann, R.J. and L. Shapley (1974), *Values of Nonatomic Games*, Princeton University Press.

Coddington, E.A. and N. Levinson (1955), *Theory of Ordinary Differential Equations*, McGraw-Hill.

Friedman, E. and H. Moulin (1995), "Three Methods to Share Joint Costs (or Surplus)", mimeo, Duke University.

Haller, H. (1994), "Collusion Properties of Values", *International Journal of Game Theory* 23, 261-281.

McLean, R. and W. Sharkey (1992), "Alternative Methods for Cost Allocation in Stochastic Service Systems", mimeo, Rutgers University.

Moulin, H. (1995), "On Additive Methods to Share Joint Costs", *Japanese Economic Review* 46, 303-332.

Moulin, H. and S. Shenker (1992), "Serial Cost Sharing", *Econometrica* 60, 1009-1037.

Moulin, H. and S. Shenker (1994), "Average Cost Pricing versus Serial Cost Sharing : An Axiomatic Comparison", *Journal of Economic Theory* 64, 178-201.

Samet, D. and Y. Tauman (1982), "The Determination of Marginal Cost Prices under a Set of Axioms", *Econometrica* 50, 895-909.

Shapley, L. (1953), "A Value for n-Person Games", in *Contributions to the Theory of Games II*, H.W. Kuhn and W. Tucker (eds.), Annals of Mathematical Studies 28, Princeton University Press.

Shubik, M. (1962), "Incentives, Decentralized Control, the Assignment of Joint Costs, and Internal Pricing", *Management Science* 8, 325-343.

Université de Montréal
Département de sciences économiques
Centre de documentation
C.P. 6128, succursale Centre-ville
Montréal (Québec)
H3C 3J7

Cahiers de recherche (Discussion papers)
1995 à aujourd'hui (1995 to date)

Si vous désirez obtenir un exemplaire, vous n'avez qu'à faire parvenir votre demande et votre paiement (5 $ l'unité) à l'adresse ci-haut mentionnée. / To obtain a copy ($ 5 each), please send your request and prepayment to the above-mentioned address.

9501 : Boyer, Marcel et Jean-Jacques Laffont, "Environmental Risks and Bank Liability", janvier 1995, 46 pages.
9502 : Margolis, David. N., "Firm Heterogeneity and Worker Self-Selection Bias Estimated Returns to Seniority", décembre 1994, 29 pages.
9503 : Abowd, John M., Francis Kramarz et David N. Margolis, "High-Wage Workers and High-Wage Firms", janvier 1995, 73 pages
9504 : Cardia, Emanuela et Steve Ambler, "Indexation Lags and Heterodox Stabilization Programs", janvier 1995, 29 pages.
9505 : Garcia, René et Huntley Schaller, "Are the Effects of Monetary Policy Asymmetric?", février 1995, 42 pages.
9506 : Parent, Daniel, "Survol des contributions théoriques et empiriques liées au capital humain", février 1995, 70 pages.
9507 : Parent, Daniel, "Wages and Mobility : The Impact of Employer-Provided Training", février 1995, 34 pages.
9508 : Parent, Daniel, "Industry-Specific Capital and the Wage Profile : Evidence from the NLSY and the PSID", février 1995, 21 pages.
9509 : Parent, Daniel, "Matching, Human Capital, and the Covariance Structure of Earnings", février 1995, 54 pages.
9510 : Garcia, René, "Asymptotic Null Distribution of the Likelihood Ratio Test in Markov Switching Models", mars 1995, 50 pages.
9511 : Garcia, René, Annamaria Lusardi et Serena Ng, "Excess Sensitivity and Asymmetries in Consumption : An Empirical Investigation", mars 1995, 26 pages.
9512 : Sprumont, Yves, "An Axiomatization of the Pazner-Schmeidler Rules in Large Fair Division Problems", mars 1995, 26 pages.
9513 : Ghysels, Eric, Lynda Khalaf et Cosmé Vodounou, "Simulation Based Inference in Moving Average Models", mars 1995, 10 pages.
9514 : Ng, Serena, "Looking for Evidence of Speculative Stockholding in Commodity Markets", mars 1995, 25 pages.
9515 : Ng, Serena et Huntley Schaller, "The Risky Spread, Investment, and Monetary Policy Transmission : Evidence on the Role of Asymmetric Information", mars 1995, 26 pages.
9516 : Ng, Serena, "Testing for Homogeneity in Demand Systems when the Regressors are Non-Stationary", mars 1995, 26 pages.

9517 : Ghysels, Eric, Clive W.J. Granger et Pierre L. Siklos, "Is Seasonal Adjustment a Linear or Nonlinear Data Filtering Process?", mars 1995, 34 pages.

9518 : Ghysels, Eric, Alastair Hall et Hahn S. Lee, "On Periodic Structures and Testing for Seasonal Unit Roots", mars 1995, 45 pages.

9519 : Sprumont, Yves, "On the Game-Theoretic Structure of Public-Good Economies", mars 1995, 21 pages.

9520 : Charles, Sandra, François Vaillancourt et Nicolas Marceau, "The Impact of Decentralization on Growth and Democracy : A Note", mars 1995, 13 pages.

9521 : Sprumont, Yves, "Balanced Egalitarian Redistribution of Income", mars 1995, 17 pages.

9522 : Bronsard, Camille, Lise Salvas-Bronsard et Alain Trognon, "On the Residual Dynamics Implied by the Rational Expectations Hypothesis", mars 1995, 18 pages.

9523 : Campbell, Bryan et Eric Ghysels, "An Empirical Analysis of the Canadian Budget Process", mars 1995, 30 pages.

9524 : Ghysels, Eric, Alain Guay et Alastair Hall, "Predictive Tests for Structural Change with Unknown Breakpoint", avril 1995, 29 pages.

9525 : Ghysels, Eric, "On Stable Factor Structures in the Pricing of Risk", avril 1995, 37 pages.

9526 : Kollmann, Robert, "Mark Up Fluctuations in U.S. Manufacturing and Trade : Empirical Evidence Based on A Model of Optimal Storage", avril 1995, 18 pages.

9527 : Laberge-Nadeau, Claire, Georges Dionne, Urs Maag, Denise Desjardins, Charles Vanasse et J.-M. Ékoé, "Medical Conditions and the Severity of Commercial Motor Vehicle (CMV) Drivers' Road Accidents", mai 1995, 26 pages.

9528 : Dionne, Georges, Manuel Artis et Montserrat Guillén, "On the Repayment of Personal Loans Under Asymmetrical Information : A Count Data Model Approach", mai 1995, 32 pages.

9529 : Ruge-Murcia, Francisco J., "Government Expenditure and the Dynamics of High Inflation", juin 1995, 45 pages.

9530 : Pesaran, M. Hashem et Francisco J. Ruge-Murcia, "A Discrete-Time Version of Target Zone Models with Jumps", juin 1995, 50 pages.

9531 : Dionne, Georges et N. Fombaron, "Non-Convexities and Efficiency of Equilibria in Insurance Markets with Asymmetric Information", juin 1995, 22 pages.

9532 : Bronsard, C., P. Michel et L. Salvas-Bronsard, "Singular Demand Systems as an Instrument of Scientific Discovery", juillet 1995, 17 pages.

9533 : Dionne, G. et C. Fluet, "Incentives in Multi-Period Regulation and Procurement : A Graphical Analysis", juillet 1995, 46 pages.

9534 : Ng, Serena et Pierre Perron, "Estimation and Inference in Nearly Unbalanced, Nearly Cointegrated Systems", août 1995, 38 pages.

9535 : Ng, Serena et Pierre Perron, "The Exact Error in Estimating the Spectral Density at the Origin", août 1995, 33 pages.

9536 : Ghysels, Eric, Christian Gouriéroux et Joanna Jasiak, "Market Time and Asset Price Movements : Theory and Estimation", septembre 1995, 61 pages.

9537 : Dionne, Georges, Robert Gagné et Charles Vanasse, "Inferring Technological Parameters from Incomplete Panel Data", septembre 1995, 40 pages.

9538 : Dufour, Jean-Marie et Eric Renault, "Short-Run and Long-Run Causality in Time Series : Theory", septembre 1995, 40 pages.

9539 : Dufour, Jean-Marie, "Some Impossibility Theorems in Econometrics with Applications to Instrumental Variables, Dynamic Models and Cointegration", septembre 1995, 42 pages.

9540 : Mercenier, Jean et Philippe Michel, "Temporal Aggregation in a Multi-Sector Economy with Endogenous Growth", septembre 1995, 17 pages.
9541 : Loranger, Jean-Guy, "Neoliberalism and the Overwhelming Influence of Financial Markets : a Comparative Analysis between NAFTA Countries and Other G-7 Countries", septembre 1995, 34 pages.
9542 : Proulx, Pierre-Paul, "La décentralisation : facteur de développement ou d'éclatement du Québec", août 1995, 26 pages.
9543 : Arcand, Jean-Louis, Ghazi Boulila et Christian Tritten, "Intergenerational Contracts, Remittances, and Growth", octobre 1995, 57 pages.
9544 : Kichian, Maral, René Garcia et Eric Ghysels, "On the Dynamic Specification of International Asset Pricing Models", octobre 1995, 29 pages.
9545 : Arcand, Jean-Louis, "Land Ownership, Working Capital, and Agricultural Output : Egypt, 1913-1958", octobre 1995, 93 pages.
9546 : Arcand, Jean-Louis et Marcel G. Dagenais, "The Empirics of Economic Growth in a Cross Section of Countries : Do Errors in Variables Really Not Matter?", octobre 1995, 65 pages.
9547 : Dufour, Jean-Marie et Jan F. Kiviet, "Exact Inference Methods for First-Order Autoregressive Distributed Lag Models", octobre 1995, 52 pages.
9548 : Dufour, Jean-Marie et Jan F. Kiviet, "Exact Tests for Structural Change in First-Order Dynamic Models", octobre 1995, 39 pages.
9549 : Kiviet, Jan F. et Jean-Marie Dufour, "Exact Tests in Single Equation Autoregressive Distributed Lag Models", octobre 1995, 38 pages.
9550 : Briys, Eric et Pascale Viala, "Optimal Insurance Design under Background Risk", octobre 1995, 14 pages.
9551 : Ng, Serena et Joris Pinkse, "Nonparametric Two-Step Estimation of Unknown Regression Functions when the Regressors and the Regression Error Are not Independent", octobre 1995, 22 pages.
9552 : Bai, Jushan et Pierre Perron, "Estimating and Testing Linear Models with Multiple Structural Changes", octobre 1995, 58 pages.
9553 : de Lemos Grandmont, Renato, "Mexico's Financial Liberalization and Reform, A Critical Overview", octobre 1995, 66 pages.
9554 : de Lemos Grandmont, Renato, "Multivariate Cointegration in the Presence of Structural Breaks : The Case of Money Demand in Mexico", octobre 1995, 69 pages.
9555 : Boyer, Marcel, Philippe Mahenc et Michel Moreaux, "Entry Blockading Locations", octobre 1995, 37 pages.
9556 : Boyer, Marcel et Michel Moreaux, "Capacity Commitment versus Flexibility : The Technological Choice Nexus in a Strategic Context", octobre 1995, 37 pages.
9557 : Boyer, Marcel et Jean-Jacques Laffont, "Environmental Protection, Producer Insolvency and Lender Liability", octobre 1995, 38 pages.
9558 : Montmarquette, Claude, Sophie Mahseredjian et Rachel Houle, "The Determinants of University Dropouts : a Sequential Decision Model with Selectivity Bias", décembre 1995, 17 pages.
9559 : Margolis, David N., "Cohort Effects and Returns to Seniority in France", décembre 1995, 29 pages.
9560 : Dionne, Georges et Christian Gollier, "A Model of Comparative Statics for Changes in Stochastic Returns with Dependent Risky Assets", décembre 1995, 14 pages.
9601 : Deaton, Angus et Serena Ng, "Parametric and Nonparametric Approaches to Price and Tax Reform", janvier 1996, 28 pages.

9602 : Lévy-Garboua, Louis et Claude Montmarquette, "Cognition in Seemingly riskless Choices and Judgments", janvier 1996, 29 pages.

9603 : Gonzalo, Jesus et Serena Ng, "A Systematic Framework for Analyzing the Dynamic Effects of Permanent and Transitory Shocks", mars 1996, 42 pages.

9604 : Boyer, Marcel et Jean-Jacques Laffont, "Toward a Political Theory of Environmental Policy", avril 1996, 42 pages.

9605 : Ai, Chunrong, Jean-Louis Arcand et François Ethier, "Moral Hazard and Marshallian Inefficiency: Evidence from Tunisia", avril 1996, 38 pages.

9606 : Mercenier, Jean et Erinç Yeldan, "How Prescribed Policy can Mislead when Data are Defective: A Follow-up to Srinivasan (1994) Using General Equilibrium", avril 1996, 29 pages.

9607 : Fortin, Nicole M. et Thomas Lemieux, "Rank Regressions, Wage Distributions, and the Gender GAP", avril 1996, 45 pages.

9608 : Fortin, Nicole M. et Thomas Lemieux, "Labor Market Institutions and Gender Differences in Wage Inequality", avril 1996, 13 pages.

9609 : S. Hosken, Daniel et David N. Margolis, "The Efficiency of Collective Bargaining in Public Schools", mai 1996, 54 pages.

9610 : Dionne, Georges et Tahar Mounsif, "Investment Under Demand Uncertainty : the Newsboy Problem Revisited", mai 1996, 18 pages.

9611 : Perron, Pierre et Serena Ng, "An Autoregressive Spectral Density Estimator at Frequency Zero for Nonstationarity Tests", juin 1996, 44 pages.

9612 : Ghysels, Eric et Serena Ng, "A Semi-Parametric Factor Model for Interest Rates", juillet 1996, 29 pages.

9613 : Ghysels, Eric, Andrew Harvey et Eric Renault, "Stochastic Volatility", juillet 1996, 70 pages.

9614 : Kollmann, Robert, "The Exchange Rate in a Dynamic-Optimizing Current Account Model with Nominal Rigidities : A Quantitative Investigation", juillet 1996, 54 pages.

9615 : Bossaerts, Peter, Eric Ghysels et Christian Gouriéroux, "Arbitrage-Based Pricing when Volatility is Stochastic", juillet 1996, 58 pages.

9616 : Blum, U. et L. Dudley, "The Rise and Decline of the East German Economy, 1949-1989", août 1996, 33 pages.

9617 : Allard, Marie, Camille Bronsard et Christian Gouriéroux, "Actifs financiers et théorie de la consommation", juillet 1996, 50 pages.

9618 : Tremblay, Rodrigue, "La mobilité internationale des facteurs de production en situation de chômage et de libre-échange commercial : une revue de la problématique", juillet 1996, 41 pages.

9619 : Dudley, L., "The Rationality of Revolution", août 1996, 33 pages.

9620 : Dudley, L., "Communications and Economic Growth", août 1996, 34 pages.

9621 : Abdelkhalek, T. et A. Martens, "Macroclosures in Computable General Equilibrium Models : A Probabilistic Treatment with an Application to Morrocco", août 1996, 30 pages.

9622 : Proulx, Pierre-Paul., "La mondialisation de l'économie et le rôle de l'État", septembre 1996, 22 pages.

9623 : Proulx, Pierre-Paul., "Economic Integration in North America : Formal, Informal and Spatial Aspects", 22 pages

9624 : Sprumont, Yves, "Ordinal cost sharing", octobre 1996, 42 pages.