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Mean values and correlations of multiplicative functions: The “pretentious" approach

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SOMMAIRE

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Le sujet principal de cette thèse est l'étude des valeurs moyennes et corrélations de fonctions multiplicatives. Les résultats portant sur ces derniers sont subséquemment appliqués à la résolution de plusieurs problèmes.

Dans le premier chapitre, on rappelle certains résultats classiques concernant les valeurs moyennes des fonctions multiplicatives. On y énonce également les théorèmes principaux de la thèse.

Le deuxième chapitre consiste de l'article *"Mean values of multiplicative functions over the function fields"*. En se basant sur des résultats classiques de Wirsing, de Hall et de Tenenbaum concernant les fonctions multiplicatives arithmétiques, on énonce et on démontre des théorèmes qui y correspondent pour les fonctions multiplicatives sur les corps des fonctions $\mathbb{F}_q[x]$. Ainsi, on résoud un problème posé dans un travail récent de Granville, Harper et Soundararajan. On décrit dans notre thése certaines caractéristiques du comportement des fonctions multiplicatives sur les corps de fonctions qui ne sont pas présentes dans le contexte des corps de nombres. Entre autres, on introduit pour la première fois une notion de "simulation" pour les fonctions multiplicatives sur les corps de fonctions $\mathbb{F}_q[x]$.

Les chapitres 3 et 4 comprennent plusieurs résultats de l'article *"Correlations of multiplicative functions and applications"*. Dans cet article, on détermine une formule asymptotique pour les corrélations

$$\sum_{n\leqslant x} f_1(P_1(n))\cdots f_m(P_m(n)),$$

où $f_1,\ldots,f_m$ sont des fonctions multiplicatives de module au plus ou égal à 1 "simulatrices" qui satisfont certaines hypothèses naturelles, et $P_1,\ldots,P_m$ sont des polynomes ayant des coefficients positifs. On déduit de cette formule plusieurs conséquences intéressantes. D'abord, on donne une classification des fonctions multiplicatives $f:\mathbb{N}\to\{-1,+1\}$ ayant des sommes partielles uniformément bornées. Ainsi, on résoud un problème d'Erdős datant de 1957 (dans la forme conjecturée par Tao). Ensuite, on démontre que si la valeur moyenne des écarts $|f(n+1)-f(n)|$ est zéro, alors soit $|f|$ a une valeur moyenne de zéro, soit $f(n)=n^s$ avec

$\mathrm{Re}(s) < 1$. Ce résultat affirme une ancienne conjecture de Kátai. Enfin, notre théorème principal est utilisé pour compter le nombre de représentations d'un entier n en tant que somme $a+b$, où a et b proviennent de sous-ensembles multiplicatifs fixés de $\mathbb{N}$. Notre démonstration de ce résultat, dû à l'origine à Brüdern, évite l'usage de la "méthode du cercle".

Les chapitres 5 et 6 sont basés sur les résultats obtenus dans l'article *"Effective asymptotic formulae for multilinear averages and sign patterns of multiplicative functions,"* un travail conjoint avec Alexander Mangerel. D'après une méthode analytique dans l'esprit du théorème des valeurs moyennes de Halász, on détermine une formule asymptotique pour les moyennes multidimensionelles

$$x^{-l} \sum_{\boldsymbol{n} \in [x]^l} \prod_{1 \leqslant j \leqslant k} f_j(L_j(\boldsymbol{n})),$$

lorsque $x \to \infty$, où $[x] := [1,x]$ et $L_1, \dots, L_k$ sont des applications linéaires affines qui satisfont certaines hypothèses naturelles. Notre méthode rend ainsi une démonstration neuve d'un résultat de Frantzikinakis et Host avec, également, un terme principal explicite et un terme d'erreur quantitatif. On applique nos formules à la démonstration d'un phénomène *local-global* pour les normes de Gowers des fonctions multiplicatives. De plus, on découvre et explique certaines irrégularités dans la distribution des suites de signes de fonctions multiplicatives $f : \mathbb{N} \to \{-1, +1\}$. Visant de tels résultats, on détermine les densités asymptotiques des ensembles d'entiers n tels que la fonction f rend une suite fixée de 3 ou 4 signes dans presque toutes les progressions arithmétiques de 3 ou 4 termes, respectivement, ayant n comme premier terme. Ceci mène à une généralisation et amélioration du travail de Buttkewitz et Elsholtz, et donne un complément à un travail récent de Matomäki, Radziwiłł et Tao sur les suites de signes de la fonction de Liouville.

Mots clés: Théorie analytique des nombres, Fonctions multiplicatives, Corrélations de fonctions multiplicatives, Problème d'Erdős, Les corps des fonctions.

SUMMARY

The main theme of this thesis is to study mean values and correlations of multiplicative functions and apply the corresponding results to tackle some open problems.

The first chapter contains discussion of several classical facts about mean values of multiplicative functions and statement of the main results of the thesis.

The second chapter consists of the article *"Mean values of multiplicative functions over the function fields"*. The main purpose of this chapter is to formulate and prove analog of several classical results due to Wirsing, Hall and Tenenbaum over the function field $\mathbb{F}_q[x]$, thus answering questions raised in the recent work of Granville, Harper and Soundararajan. We explain some features of the behaviour of multiplicative functions that are not present in the number field settings. This is accomplished by, among other things, introducing the notion of "pretentiousness" over the function fields.

Chapter 3 and Chapter 4 include results of the article *"Correlations of multiplicative functions and applications"*. Here, we give an asymptotic formula for correlations

$$\sum_{n\leq x} f_1(P_1(n))f_2(P_2(n))\cdot\dots\cdot f_m(P_m(n))$$

where $f\dots,f_m$ are bounded "pretentious" multiplicative functions, under certain natural hypotheses. We then deduce several desirable consequences. First, we characterize all multiplicative functions $f:\mathbb{N}\to\{-1,+1\}$ with bounded partial sums. This answers a question of Erdős from 1957 in the form conjectured by Tao. Second, we show that if the average of the first divided difference of multiplicative function is zero, then either $f(n) = n^s$ for $\mathrm{Re}(s) < 1$ or $|f(n)|$ is small on average. This settles an old conjecture of Kátai. Third, we apply our theorem to count the number of representations of $n = a + b$ where a,b belong to some multiplicative subsets of $\mathbb{N}$. This gives a new "circle method-free" proof of the result of Brüdern.

Chapters 5 and Chapter 6 are based on the results obtained in the article *"Effective asymptotic formulae for multilinear averages and sign patterns of multiplicative functions,"* joint with Alexander Mangerel. Using an analytic approach in the spirit of Halász' mean value theorem, we compute multidimensional averages

$$x^{-l}\sum_{\boldsymbol{n}\in[x]^l}\prod_{1\leqslant j\leqslant k}f_j(L_j(\boldsymbol{n}))$$

as $x\to\infty$, where $[x]:=[1,x]$ and $L_1,\ldots,L_k$ are affine linear forms that satisfy some natural conditions. Our approach gives a new proof of a result of Frantzikinakis and Host that is distinct from theirs, with *explicit* main and error terms.

As an application of our formulae, we establish a *local-to-global* principle for Gowers norms of multiplicative functions. We reveal and explain irregularities in the distribution of the sign patterns of multiplicative functions by computing the asymptotic densities of the sets of integers n such that a given multiplicative function $f:\mathbb{N}\to\{-1,1\}$ yields a fixed sign pattern of length 3 or 4 on almost all 3- and 4-term arithmetic progressions, respectively, with first term n. The latter generalizes and refines the work of Buttkewitz and Elsholtz and complements the recent work of Matomaki, Radziwiłł and Tao.

We conclude this thesis by discussing some work in progress.

Key words: Analytic number theory, Multiplicative functions, Correlations of multiplicative functions, Erdős discrepancy problem, Function fields.

CONTENTS

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Chapter 1

INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

1.1. Various ways to count primes

By studying tables of primes, Gauss realized, as a boy of fifteen, that the primes occur with density $\frac{1}{\log x}$ around x. In other words,

$$\pi(x) = \{p \leq x | p \ prime\} \sim \frac{x}{\log x}.$$

It took another half a century, until Riemann in 1859, wrote a seven page memoir outlining a profound plan to estimate $\pi(x)$. It was rather surprising that in order to tackle a seemingly elementary question of counting prime numbers, Riemann brought in deep ideas from the complex analysis and the theory of analytic continuation that took others forty more years to bring to fruition. Riemann's approach is based upon considering what is now called *Riemann zeta-function*

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \ prime} \left(1 - \frac{1}{p^s}\right)^{-1}$$

when $\Re(s) > 1$. One can extend ζ analytically to the whole complex plane except for a simple pole at $s = 1$. The prime number theorem would follow if one could show that $\zeta(s) \neq 0$ for $\Re(s) = 1$, and in a quantitative form by bounding zeros away from the $1-$ line. This program was carried out, independently, by Hadamard and de la Vallée-Poussin in 1896.

A natural question to ask: is it really necessary to use complex analysis to count primes? Can one come up with an "elementary" proof that does not use zeros of the analytic functions? This is discussed in the introduction to the Ingham's book. In particular a famous quote of Hardy says: "No elementary proof of the PNT is known, and one may ask whether it is reasonable to expect one. Now we know that this theorem is roughly equivalent to a theorem about an analytic function, the theorem that Riemann's zeta function has no roots on a certain line. A proof of such a theorem, not fundamentally dependent on the theory of functions seems extraordinarily unlikely... If anyone produces an elementary proof of the

PNT, he/she will show that these views are wrong, that the subject does not hang together in the way we have supposed and that it is time for the books to be cast aside and for the theory to be rewritten."

It thus came as a huge surprise to the mathematical community, when in 1948 Selberg found an elementary proof of

$$\sum_{p\le x}\log^2 p+\sum_{p,q\le x}\log p\log q=2x\log x+O(x), \tag{1.1.1}$$

from which himself and independently Erdős quickly deduced the proof of the prime number theorem.

Since then, several other elementary proofs have appeared, most using formulas like (1.1.1).

One way to proceed to prove the prime number theorem is via the identity valid for $\Re(s)>1$

$$\frac{1}{\zeta(s)}=\prod_p\left(1-\frac{1}{p^s}\right)=\sum_{n\ge 1}\frac{\mu(n)}{n^s},$$

where μ is the Mőbius function. It turns out that absence of zeros of $\zeta(s)$ on the line $\Re(s)=1$ and thus the prime number theorem is easily seen to be equivalent to the cancelation

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\mu(n)=0$$

or in other words

$$\sum_{n\le x}\mu(n)=o(x)$$

when $x\to\infty$. The Möbius function is a particular example of multiplicative function $f:\mathbb{N}\to\mathbb{C}$, such that $f(mn)=f(m)f(n)$ for all pairs $(m,n)=1$. Throughout this thesis, we confine ourselves to the study of multiplicative functions $f(n)$ such that $|f(n)|\le 1$. Naturally, this brings us to the question of establishing under which conditions the mean value

$$M_f(x):=\frac{1}{x}\sum_{n\le x}f(n)$$

is "large", namely $M_f(x)\gg 1$ for all x. There are some obvious examples, such as $f(n)=1$. Less obvious, but crucial for our study is an example $f(n)=n^{it}$, in which case

$$\frac{1}{x}\sum_{n\le x}n^{it}\sim\frac{1}{x}\int_0^x u^{it}du=\frac{x^{it}}{1+it}.$$

Note, that in the last example the limit $|M_f(x)|$ when $x\to\infty$ does exist while the limit of $M_f(x)$ does not.

Do other examples with large mean values exist? An obvious class of examples come from minor perturbations of the function n^{it}, for instance one may take $f(3)=-1$ and $f(p)=1$ for other primes p. It is easy to check that in this case $M_f(x)\to\frac{1}{2}$ when $x\to\infty$. In general, the mean value can be large if $f(n)$ is "close" to n^{it} for some value $t\in\mathbb{R}$. This, roughly speaking, is a content of a celebrated theorem of Halász which we are going to discuss in the following section.

1.2. Mean values of multiplicative functions. Classical results

Given a multiplicative function f with $|f(n)| \leq 1$ our main objective is to understand $M_f(x)$. The basic heuristic suggests that when $x \to \infty$, we must have

$$M_f(x) \to M_f$$

where

$$M_f = \prod_{p\geq 2}\left(1-\frac{1}{p}\right)\left(\sum_{k\geq 0}\frac{f(p^k)}{p^k}\right).$$

Erdős and Wintner conjectured that is true when $f : \mathbb{N} \to [-1,1]$. The easier case, $M_f \neq 0$, has been established by Wintner in 1944. It was only in 1967, when Wirsing brought in the ideas from the theory of delay differential equations to settle the conjecture completely.

Theorem 1.2.1. *[Wirsing, 1967] Let $f : \mathbb{N} \to [-1,1]$ be multiplicative. Then,*

$$\sum_{n\leq x} f(n) = o(x),$$

unless

$$\sum_{p\geq 2}\frac{1-f(p)}{p} < \infty,$$

in which case

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leq x} f(n) = M_f = \prod_{p\geq 2}\left(1-\frac{1}{p}\right)\left(\sum_{k\geq 0}\frac{f(p^k)}{p^k}\right).$$

Since

$$\sum_{p\geq 2}\frac{1-\mu(p)}{p} = \sum_{p\geq 2}\frac{2}{p} = \infty,$$

applied to the Mőbius function $f(n) = \mu(n)$, Theorem 1.2.1 yields $M_\mu = 0$, which implies the prime number theorem in a non quantitative form.

Following Granville and Soundararajan [GS07a], we define the "distance" between two multiplicative functions $f,g : \mathbb{N} \to \mathbb{U}$

$$\mathbb{D}(f,g;y;x) = \left(\sum_{y\leq p\leq x}\frac{1-\mathrm{Re}\left(f(p)\overline{g(p)}\right)}{p}\right)^{\frac{1}{2}},$$

and $\mathbb{D}(f,g;x) := \mathbb{D}(f,g;1;x)$. We remark, that $\mathbb{D}(f,f;\infty) = 0$ if and only if $|f(p)| = 1$ for all primes $p \geq 2$. The crucial feature of this "distance" is that it satisfies the triangle inequality

$$\mathbb{D}(f,g;y;x) + \mathbb{D}(g,h;y;x) \geq \mathbb{D}(f,h;y;x)$$

for any multiplicative functions f,g,h bounded by 1.

Usually, the distance $\mathbb{D}(f,g;\infty)$ is infinite. However, in the case $\mathbb{D}(f,g;\infty) < \infty$ we say that f "pretends" to be g. In this way, Wirsing's theorem says that the mean value of real-valued f is zero unless f "pretends" to be 1, in which case it is easily computable. Wirsing's theorem was vastly generalized by Halász who proved, among other things

Theorem 1.2.2. *[Halász, 1971] Let $f : \mathbb{N} \to \mathbb{U}$ be multiplicative. Then,*

$$\sum_{n \leq x} f(n) = o(x)$$

unless there exists $t \in \mathbb{R}$ such that $\mathbb{D}(f, n^{it}; \infty) < \infty$ in which case

$$M_f(x) = \frac{x^{it}}{1+it} \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(\sum_{k \geq 0} \frac{f(p^k)p^{-kit}}{p^k}\right) + o_{x\to\infty}(1).$$

During the past fifty years, quantitative improvements of Halász's and Wirsing's results have been obtained by several authors [MV01], [HT91], [GS03].

1.3. Multiplicative functions over function fields

Consider the polynomial ring $\mathbb{F}_q[x]$ over a field with q elements. Let $\mathcal{M}$ denote the set of monic polynomials and $\mathcal{M}_n$ denote the set of monic polynomials of degree n, so that $|\mathcal{M}_n| = q^n$. Let $\mathcal{P}$ denote the set of irreducible monic polynomials and $\mathcal{P}_n$ be the corresponding set of degree n. One of the fruitful analogies in number theory is the one between the integers $\mathbb{Z}$ and the polynomial ring $\mathbb{F}_q[x]$. Thus, for instance prime numbers correspond to the irreducible polynomials over $\mathbb{F}_q[x]$ and the fundamental theorem of arithmetic applies.

Let $\mathbb{U}$ denote the unit disc. In the recent paper [GHS15], Granville, Harper, and Soundararajan initiated the study of mean values of multiplicative functions over the function field $\mathbb{F}_q[x]$ by proving a quantitative analog of the celebrated theorem of Halasz [Hal71]. We begin by introducing the objects of study, borrowing the notations from [GHS15]. See also the book [Ros02] for a general introduction of the function fields.

We define a multiplicative function $f : \mathcal{M} \to \mathbb{C}$ such that $f(FG) = f(F)f(G)$ for any two coprime monic polynomials F and G. By analogy with the number field setting, we define the associated Dirichlet series to be

$$\mathcal{F}(z) = \sum_{F \in \mathcal{M}} f(F) z^{\deg(F)} = \prod_{P \textit{ monic irreducible}} \left(1 + f(P)z^{\deg P} + f(P^2)z^{2\deg P} + \dots\right),$$

where the corresponding Euler product converges uniformly for $|z| < 1/q$ whenever, say, $|f(F)| \leq 1$ for all $F \in \mathcal{M}$.

For the moment, we consider the function which takes value 1 for all $F \in \mathcal{M}$. The associated Dirichlet series is then equal to

$$\mathcal{F}(z) = \sum_{n=0}^{\infty} |\mathcal{M}_n| z^n = (1-qz)^{-1} = \prod_{P \textit{ monic irreducible}} \left(1 - z^{\deg P}\right)^{-1},$$

converges in the domain $|z| < 1/q$, and is a direct analog of the *Riemann zeta-function* $\zeta(s)$. Taking logarithms, we end up with

$$\sum_{m=1}^{\infty} \frac{(qz)^m}{m} = \sum_{P} \sum_{k=1}^{\infty} \frac{z^{k \deg P}}{k} = \sum_{P} \sum_{k=1}^{\infty} \Lambda(P^k) \frac{z^{\deg(P^k)}}{\deg(P^k)} = \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{F \in \mathcal{M}_n} \Lambda(F).$$

Equating the corresponding coefficients yields

$$\sum_{F \in \mathcal{M}_n} \Lambda(F) = q^n,$$

which is the form of the prime number theorem over the function field $\mathbb{F}_q[x]$. Using Möbius inversion gives

$$|\mathcal{P}_n| = \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d} = \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right).$$

The last formula counts primes over $\mathbb{F}_q[x]$ with an error term as strong as predicted by the Riemann hypothesis in the number field setting.

For a general multiplicative function $f : \mathcal{M} \to \mathbb{C}$ we proceed in a similar fashion and write

$$\log \mathcal{F}(z) = \sum_{F \in \mathcal{M}} \frac{\Lambda_f(F)}{\deg(F)} z^{\deg(F)}$$

for certain coefficients $\Lambda_f(F)$. By analogy with the number field case, $\Lambda_f(F)$ is supported on the powers of monic irreducible polynomials. Upon differentiating, we have

$$z \frac{\mathcal{F}'(z)}{\mathcal{F}(z)} = \sum_{F \in M} \Lambda_f(F) z^{\deg(F)}.$$

Our main goal is to study for a general multiplicative function $f : \mathcal{M} \to \mathbb{C}$, the mean values of the form

$$\sigma(n) = \sigma(n; f) := \frac{1}{q^n} \sum_{F \in \mathcal{M}_n} f(F).$$

In order to do so, we introduce the mean value over prime powers, that is

$$\chi(n) = \chi(n; f) := \frac{1}{q^n} \sum_{F \in \mathcal{M}_n} \Lambda_f(F).$$

It is easy to see that $\sigma(0) = 1$, $\chi(0) = 0$ and $\sigma(1) = \chi(1)$. It was observed in [GHS15], that the corresponding mean values satisfy the recursion relation

$$n\sigma(n) = \sum_{k=1}^{n} \chi(k) \sigma(n-k), \tag{1.3.1}$$

which is a discrete analog of the delay differential equation

$$u\sigma(u) = \int_0^u \chi(t) \sigma(u-t) dt$$

that arises while studying mean values of multiplicative functions in the number field case. In what follows, we will work with a more general class of functions with the following definition.

Definition 1.3.1. For any $\kappa \geq 1$, we define class of functions $\mathcal{C}(\kappa)$, such that $f \in \mathcal{C}(\kappa)$ if and only if $|\chi(n)| \leq \kappa$ for all $n \geq 1$.

We remark, that this class of functions is more general than the one considered by Granville, Harper and Soundararajan in the number field setting where they impose the stronger condition $|\Lambda_f(n)| \leq \kappa\Lambda(n)$ for all $n \geq 1$.

1.4. Wirsing and Halász type theorems over $\mathbb{F}_q[x]$

Our first result is related to the celebrated theorem of Wirsing [Wir67], which asserts that any multiplicative function $f : \mathbb{N} \to [-1,1]$ has a mean value, that is

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} f(n) := M_f.$$

As pointed out in [GHS15], a direct analog of Wirsing's theorem is however false in the function field setting. Indeed, consider the function $f(F) = (-1)^{\deg(F)}$ for which $\sigma(n) = (-1)^n$ clearly oscillates. One may expect that this type of functions is the "only" counterexample to the potential Wirsing type result over the function field $\mathbb{F}_q[x]$. The next corollary confirms this guess.

Corollary 1.4.1. *For any real valued multiplicative $f \in \mathcal{C}(1)$, either $f(P)$ or $(-1)^{\deg P} f(P)$ has a mean value.*

Since any multiplicative function $f : \mathcal{M} \to [-1,1]$ belongs to $\mathcal{C}(1)$, Corollary 1.4.1 provides the function field analog of Wirsing's result. We will deduce this from a more general result.

Theorem 1.4.2. *Let $\kappa \geq 1$. For every real valued multiplicative function $f \in \mathcal{C}(\kappa)$, either $\frac{f(P)}{(\deg P)^{\kappa-1}}$ or $\frac{(-1)^{\deg P} f(P)}{(\deg P)^{\kappa-1}}$ has a mean value.*

As was mentioned in the first chapter, Halász showed that for any complex valued $f : \mathbb{N} \to \mathbb{U}$,

$$\sum_{n\leq x} f(n) = o(x)$$

unless there exists $t \in \mathbb{R}$, such that $\mathbb{D}(f(n),n^{it};\infty) < \infty$, in which case Delange [Del67] earlier computed the asymptotic

$$\frac{1}{x}\sum_{n\leq x} f(n) = \frac{x^{it}}{1+it} \prod_{p\leq x} \left(1 - \frac{1}{p}\right) \left(\sum_{k\geq 0} \frac{f(p^k)p^{-kit}}{p^k}\right) + o(1).$$

Let $e(\alpha) = \exp(2\pi i\alpha)$. A natural function field analog of the function $h_t(n) = n^{it}$, $t \in \mathbb{R}$ is the function $h_\theta(Q) = \exp(\theta \deg Q)$, $Q \in \mathcal{M}$ and $\theta \in [0,1)$. For any two multiplicative functions $f,g : \mathcal{M} \to \mathbb{U}$ we define the "distance" to be

$$\mathbb{D}^2(f,g;m,n) = \sum_{\substack{m \le \deg P \le n, \\ P \text{ irreducible}}} \frac{1 - \mathrm{Re}(f(P)\overline{g(P)})}{q^{\deg P}}$$

and $\mathbb{D}(f,g;n) := \mathbb{D}(f,g;1,n)$. For any given multiplicative function $f : \mathcal{M} \to \mathbb{U}$ we define the corresponding Euler product

$$\mathcal{P}(f,n) = \prod_{\substack{\deg P \le n, \\ P \textit{ irreducible}}} \left(1 - \frac{1}{q^{\deg P}}\right) \left(\sum_{k=0}^{\infty} \frac{f(P^k)}{q^{k \deg P}}\right).$$

We establish the following explicit version of Halász and Delange's results:

Theorem 1.4.1. *For a given multiplicative function $f : \mathcal{M} \to \mathbb{U}$, one of the following holds:*

- *If $\mathbb{D}(f(P),\ e(\theta \deg P); \infty) = \infty$ for all $\theta \in [0,1)$, then*

$$\lim_{n \to \infty} \frac{1}{q^n} \sum_{F \in \mathcal{M}_n} f(F) = 0.$$

- *There exists $\theta_0 \in [0,1)$ such that $\mathbb{D}(f(P),\ e(\theta_0 \deg P); \infty) < \infty$. For any given $\varepsilon > 0$, let $m = \lceil (1-\varepsilon) \frac{\log n}{\log q} \rceil$. Then*

$$\frac{1}{q^n} \sum_{F \in \mathcal{M}_n} f(F) = e(n\theta_0) \cdot \mathcal{P}(f(P)e(-\theta_0 \deg P), n) + O_\varepsilon \left(\mathbb{D}(f(P),\ e(-\theta_0 \deg P); m,n) + \frac{1}{n^{1-\varepsilon}} \right).$$

1.5. Spectrum of multiplicative functions over $\mathbb{F}_q[x]$

In 1995, Hall [Hal96] proved the conjecture of Heath-Brown that there exists $\delta > -1$ such that for any completely multiplicative function $f : \mathbb{N} \to [-1,1]$

$$\sum_{n \le x} f(n) \ge x(\delta + o(1)).$$

Despite being short, Hall's proof relies on two deep theorems about mean values of multiplicative functions. In 1999, in their seminal work [GS01], by carefully analyzing extremal solutions of the delay differential equations, Granville and Soundararajan found the sharp threshold $\delta = \delta_0$:

$$\delta_0 = 1 - 2\log(1 + \sqrt{e}) + 4 \int_1^{\sqrt{e}} \frac{\log t}{1+t} dt = -0.656999\ldots$$

One arithmetic consequence of their results is that if x is sufficiently large, then for any prime p, there is at least 17.15% of quadratic residues up to x modulo p.

The same example of the function $f(F) = (-1)^{\deg(F)}$ with $\sigma(n) = (-1)^n$ shows that the direct analog of this result is false in the function field setting. In [GHS15] the authors suggested that parity might be the only obstruction to such a result. We offer two possible answers to this question.

Corollary 1.5.1. Let $f \in \mathcal{C}(1)$ be a real valued multiplicative function. Then there exists an absolute constant $\delta > 0$ such that if $\chi(1) \geq 0$ then $\sigma(n) \geq -1+\delta$ for all $n \geq 1$. Otherwise, if $\chi(1) < 0$, then $(-1)^n \sigma(n) \geq -1+\delta$ for all $n \geq 1$.

One might view the conclusion of Corollary 1.5.1 to be somewhat unsatisfactory, since the answer should not, in principle, depend on the value of $\chi(1)$. In the future chapters we give a more conceptual explanation of this phenomena by invoking the notion of "pretentiousness" over the function field $\mathbb{F}_q[x]$ and argue that the "correct" result must be the following. Let

$$D_n = \min_{\theta \in \{0,1/2\}} \sum_{1 \leq k \leq n} \frac{1-\chi(k)\cos(2\pi k\theta)}{k}.$$

Theorem 1.5.1. *Let $f \in \mathcal{C}(1)$ be a real valued multiplicative function and let $n_0 \geq 1$ be given. Then there exists $\delta(n_0) > 0$, such that:*

- *If $D_{n_0} \gg \log(\frac{1}{1-\delta(n_0)})$, then $\sigma(n) \geq -1+\delta(n_0)$ and $(-1)^n\sigma(n) \geq -1+\delta(n_0)$ for all $n \geq n_0$.*

- *Let $\theta_{n_0} = argmin_{\theta\in\{0,1/2\}}\{D_{n_0}\}$. Then $e(-\theta_{n_0} n)\sigma(n) \geq -1+\delta(n_0)$ for all $n \geq n_0$.*

In fact, we will show that one can choose $\delta(n_0)$, such that $\delta(n_0) \to \delta_0 > 0$ when $n_0 \to \infty$. Corollary 1.5.1 is thus the special case of Theorem 1.5.1 for $n_0 = 1$. On the other hand, the analog of Granville and Soundararajan's result [GS01] in the number field case would correspond to the case $n_0 \to \infty$.

As a byproduct of our results we establish

Proposition 1.5.2. *Let $f \in \mathcal{C}(1)$ be a real valued multiplicative function and define*

$$D_n = \min\left(\sum_{1\leq k\leq n} \frac{1-(-1)^k\chi(k)}{k}, \sum_{1\leq k\leq n}\frac{1-\chi(k)}{k}\right).$$

Then

$$|\sigma(n)| \ll (1+D_n)\exp\left(-\frac{1}{42}D_n\right).$$

In a number field setting, the result analogous to Proposition 1.5.2 has been proved by Hall and Tenenbaum [HT91]. Namely, they proved that for any multiplicative $f : \mathbb{N} \to [-1,1]$

$$\frac{1}{x}\sum_{n\leq x} f(n) \ll \left(1+\sum_{p\leq x}\frac{1-f(p)}{p}\right)\exp\left(-C\sum_{p\leq x}\frac{1-f(p)}{p}\right)$$

where the sharp constant is given by $C = -\cos\beta$ and β is the solution of $\sin\beta - \beta\cos\beta = \frac{1}{2}\pi$. With more effort, following the lines of the proof of Proposition 1.5.2, one can find the sharp constant in the function field setting as well.

1.6. Correlations of multiplicative functions and applications

1.6.1. Introduction and motivation

Let $\mathbb{U}$ denote the unit disc, and let $\mathbb{T}$ be the unit circle. It is generally believed that the multiplicative structure of an object (a set of integers, say) should not, in principal, interfere with its additive structure, and thus the values of $f(n)$ and $f(n+a)$, where $f : \mathbb{N} \to \mathbb{U}$ is multiplicative, should be roughly independent unless f is "exceptional" in some sense. One measure of this "independence" is cancellation in the binary correlations. In fact, we expect that

$$\sum_{n \leq x} f(n)\overline{f(n+h)} = o(x)$$

unless $\mathbb{D}(f,\chi n^{it};x)$ is small for some Dirichlet character χ and $t \in \mathbb{R}$. It is therefore of a current interest in analytic number theory to understand the correlations

$$\sum_{n \leq x} f_1(P_1(n))f_2(P_2(n)) \cdot \dots \cdot f_m(P_m(n))$$

for arbitrary multiplicative functions $f_1, \dots, f_m : \mathbb{N} \to \mathbb{U}$, and arbitrary polynomials $P_1, \dots, P_m \in \mathbb{Z}[x]$. For example, Chowla's conjecture asserts that for any distinct natural numbers $h_1, \dots h_k$

$$\sum_{n \leq x} \lambda(n+h_1) \dots \lambda(n+h_k) = o(x),$$

where $\lambda(n)$ is a Liouville function. These problems are still widely open in general, though spectacular progress has been made recently due to the breakthrough of Matomäki and Radziwiłł [MR] and subsequent work of Matomäki, Radziwiłł, and Tao [MRT]. In particular, this led Tao [Taob] to establish a weighted version of Chowla's conjecture in the form

$$\sum_{n \leq x} \frac{\lambda(n)\lambda(n+h)}{n} = o(\log x)$$

for all $h \geq 1$. Combining this with ideas from the Polymath5 project, and a new "entropy decrement argument", led to the resolution of the Erdős Discrepancy Problem which we are going to discuss in more details in the forthcoming section.

Halász's theorem [Hal71], [Hal75] implies Wirsing's Theorem that for multiplicative $f : \mathbb{N} \to [-1,1]$, the mean value satisfies a decomposition into local factors,

$$\frac{1}{x}\sum_{n \leq x} f(n) = \prod_p M_p(f) + o(1) \tag{1.6.1}$$

when $x \to \infty$, where we define the multiplicative function f_p for each prime p to be

$$f_p(q^k) = \begin{cases} f(q^k), & \text{if } q = p \\ 1, & \text{if } q \neq p, \end{cases} \tag{1.6.2}$$

for all $k \geqslant 1$, and

$$M_p(f) := \lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} f_p(n) = \left(1-\frac{1}{p}\right)\sum_{k\ge 0}\frac{f(p^k)}{p^k}.$$

This last equality, evaluating $M_p(f)$, is an easy exercise. Substituting this into (1.6.1) one finds that the mean value there is $\asymp \exp(-\mathbb{D}(f,1;\infty))^2$, and so is non-zero if and only if $\mathbb{D}(f,1;\infty) < \infty$ and each $M_p(f) \neq 0$. Moreover, using our explicit evaluation of $M_p(f)$, we see that $M_p(f) = 0$ if and only if $p = 2$ and $f(2^k) = -1$ for all $k \geqslant 1$. We also note that one can truncate the product in (1.6.1) to the primes $p \leqslant x$, and retain the same qualitative result.

1.6.2. Mean values of multiplicative functions acting on polynomials

Our first goal is to prove the analog of (1.6.1) for the mean value of $f(P(n))$ for any given polynomial $P(x) \in \mathbb{Z}[x]$. This is not difficult for linear polynomials P but, as the following example shows, it is not so straightforward for higher degree polynomials:

Proposition 1.6.1. *There exists a multiplicative function* $f : \mathbb{N} \to [-1,1]$ *such that* $\mathbb{D}^2(1,f;x) = 2\log\log x + O(1)$ *for all* $x \geq 2$ *and*

$$\limsup_{x\to\infty}\left|\frac{1}{x}\sum_{n\le x} f(n^2+1)\right| \geq \frac{1}{2} + o(1).$$

As we shall see, in the proof of Proposition 1.6.1, the choice of $f(p)$ for certain primes $p \geq x$ have a significant impact on the mean value of $f(n^2+1)$ up to x. In order to tame this effect we introduce the set

$$N_P(x) = \{p^k, p \geq x \mid \exists n \leq x,\, p^k || P(n)\}$$

for any given $P \in \mathbb{Z}[x]$, and modify the “distance" to

$$\mathbb{D}_P(f,g;y;x) = \left(\sum_{y\le p\le x}\frac{1-\mathrm{Re}\,(f(p)\overline{g(p)})}{p} + \sum_{p^k\in N_P(x)}\frac{1-\mathrm{Re}\,(f(p^k)\overline{g(p^k)})}{x}\right)^{\frac{1}{2}}.$$

and $\mathbb{D}_P(f,g;x) := \mathbb{D}_P(f,g;1;x)$. Moreover, we define

$$M_p(f(P)) = \lim_{x\to\infty}\frac{1}{x}\sum_{n\le x} f_p(P(n)),$$

and one easily shows that

$$M_p(f(P)) = \sum_{k\ge 0} f(p^k)\left(\frac{\omega_P(p^k)}{p^k} - \frac{\omega_P(p^{k+1})}{p^{k+1}}\right),$$

where $\omega_P(m) := \#\{n \pmod m : \;\; P(n) \equiv 0 \pmod m\}$ for every integer m (and note that $\omega_P(.)$ is a multiplicative function by the Chinese Remainder Theorem). We establish the following analog of (1.6.1):

Corollary 1.6.2. *Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function and let $P(x) \in \mathbb{Z}[x]$ be a polynomial. Then*

$$\frac{1}{x}\sum_{n\leq x} f(P(n)) = \prod_{p\leq x} M_p(f(P)) + O\left(\mathbb{D}_P(1,f;\log x;x) + \frac{1}{\log\log x}\right).$$

This implies that if $\mathbb{D}(1,f;\infty) < \infty$ and

$$\sum_{p^k\in N_P(x)} 1 - \mathrm{Re}(f(p^k)) = o(x)$$

then

$$\frac{1}{x}\sum_{n\leq x} f(P(n)) = \prod_{p\leq x} M_p(f(P)) + o(1) = \prod_{p\geq 1} M_p(f(P)) + o(1)$$

when $x \to \infty$.

1.6.3. Mean values of correlations of multiplicative functions

We now move on to correlations. For $P,Q \in \mathbb{Z}[x]$, we define the local correlation

$$M_p(f(P),g(Q)) = \lim_{x\to\infty} \frac{1}{x}\sum_{n\leq x} f_p(P(n))g_p(Q(n)). \tag{1.6.3}$$

Evaluating these local factors is also easy yet can be technically complicated, as we shall see below in the case that P and Q are both linear.

More generally we establish the following

Theorem 1.6.3. *Let $f,g : \mathbb{N} \to \mathbb{U}$ be multiplicative functions. Let $P,Q \in \mathbb{Z}[x]$ be two polynomials, such that $res(P,Q) \neq 0$. Then,*

$$\frac{1}{x}\sum_{n\leq x} f(P(n))g(Q(n)) = \prod_{p\leq x} M_p(f(P),g(Q)) + \mathrm{Error}(f(P),g(Q),x)$$

where

$$\mathrm{Error}(f(P),g(Q),x) \ll \mathbb{D}_P(1,f;\log x;x) + \mathbb{D}_Q(1,g;\log x;x) + \frac{1}{\log\log x}.$$

Theorem 1.6.3 implies that if $\mathbb{D}(1,f;x),\mathbb{D}(1,g;x) < \infty$ and $\sum_{p\in N_P(x)}(1 - \mathrm{Re}(f(p^k))) = o(x)$, $\sum_{p\in N_Q(x)} 1 - \mathrm{Re}(g(p^k)) = o(x)$, then

$$\frac{1}{x}\sum_{n\leq x} f(P(n))g(Q(n)) = \prod_{p\leq x} M_p(f(P),g(Q)) + o(1) = \prod_{p\geq 1} M_p(f(P),g(Q)) + o(1).$$

If $\mathbb{D}_P(f,n^{it};\infty),\mathbb{D}_P(g,n^{iu};\infty) < \infty$ then we let $f_0(n) = f(n)/n^{it}$ and $g_0(n) = g(n)/n^{iu}$ so that $\mathbb{D}_P(1,f_0;\infty),\mathbb{D}_P(1,g_0;\infty) < \infty$. We apply Theorem 1.6.3 to the mean value of $f_0(P(n))g_0(Q(n))$, and then proceed by partial summation to obtain

$$\frac{1}{x}\sum_{n\leq x} f(P(n))g(Q(n)) = M_i(f(P),g(Q),x)\prod_{p\leq x} M_p(f_0(P),g_0(Q)) + \mathrm{Error}(f_0(P),g_0(Q),x)$$

where, if $P(x) = ax^D + \dots$ and $Q(x) = bx^d + \dots$ then we define $T = Dt + du$ and

$$M_i(f(P),g(Q),x) := \frac{1}{x}\sum_{n\le x} P(n)^{it}Q(n)^{iu} = a^{it}b^{iu}\frac{x^{iT}}{1+iT} + o(1).$$

Here, the $o(1)$ term depends on the polynomials $P,Q \in \mathbb{Z}[x]$ and

$$\text{Error}(f_0(P),g_0(Q),x) \ll_{t,u} \mathbb{D}_P(1,f_0;\log x;x) + \mathbb{D}_Q(1,g_0;\log x;x) + \frac{1}{\log\log x}$$

where the implied constant depends on t,u. The same method works for m-point correlations

$$\sum_{n\le x} f_1(P_1(n))f_2(P_2(n))\cdot\dots\cdot f_m(P_m(n))$$

for multiplicative functions $f_j : \mathbb{N} \to \mathbb{U}$ and polynomials P_j with each $\mathbb{D}_{P_j}(n^{it_j},f_j,\infty) < \infty$. We give a more explicit version of our results in the case that P and Q are linear polynomials:

Corollary 1.6.4. *Let $f,g : \mathbb{N} \to \mathbb{U}$ be multiplicative functions with $\mathbb{D}(f,n^{it},\infty)$, $\mathbb{D}(g,n^{iu},\infty) < \infty$, and write $f_0(n) = f(n)/n^{it}$ and $g_0(n) = g(n)/n^{iu}$. Let $a,b \ge 1$, c,d be integers with $(a,c) = (b,d) = 1$ and $ad \ne bc$. As above we have*

$$\frac{1}{x}\sum_{n\le x} f(an+c)g(bn+d)) = M_i(f(P),g(Q),x)\prod_{p\le x} M_p(f_0(P),g_0(Q)) + o(1)$$

when $x \to \infty$ and $o(1)$ term depends on the variables a,b,c,d,t,u.

We have

$$M_i(f(P),g(Q),x) = \frac{a^{it}b^{iu}x^{i(t+u)}}{1+i(t+u)} + o(1)$$

when $x \to \infty$ and $o(1)$ term and $o(1)$ depends on a,b,t,u.

If $p|(a,b)$ then $M_p(f_0(P),g_0(Q)) = 1$. If $p \nmid ab(ad-bc)$, then

$$M_p(f_0(P),g_0(Q)) = M_p(f_0(P)) + M_p(g_0(Q)) - 1 = 1 + \left(1-\frac{1}{p}\right)\left(\sum_{j\ge1}\frac{f_0(p^j)}{p^j} + \sum_{j\ge1}\frac{g_0(p^j)}{p^j}\right).$$

In general, if $p \nmid (a,b)$ we have a more complicated formula

$$M_p(f_0(P),g_0(Q)) = \sum_{\substack{0\le i\le k,\\ k\ge0,\\ p^k||ad-bc}}\left(\frac{\theta(p^i)\gamma(p^i)}{p^i} + \delta_b\sum_{j>i}\frac{\theta(p^i)\gamma(p^j)}{p^j} + \delta_a\sum_{j>i}\frac{\gamma(p^i)\theta(p^j)}{p^j}\right)$$

and $\delta_\natural = 0$ when $p|\natural$ and $\delta_\natural = 1$ otherwise. Here $f_0 = 1 * \theta$ and $g_0 = 1 * \gamma$.

For $t = u = 0$, some version of Corollary 1.6.4 also appeared in Hildebrand [Hil88a], Elliot [Ell92], Stepanauskas [Ste02].

1.6.4. Correlations with characters

In the future sections, we shall apply Theorem 1.6.3 to obtain a number of consequences. The key idea for our applications is that one expands

$$\frac{1}{x}\sum_{n\le x}\left|\sum_{k=n+1}^{n+H+1} f(k)\right|^2 = \sum_{|h|\le H}(H-|h|)\sum_{n\le x} f(n)\overline{f(n+h)} + O\left(\frac{H^2}{x}\right)$$

and then one observes that the $h=0$ term equals H if each $|f(n)|=1$. Therefore if the above sum is small then

$$\frac{1}{x}\sum_{n\le x} f(n)\overline{f(n+h)} \gg 1$$

for some h, $1\le |h|\le H$. As Tao showed, if some weighted version of this is true, then $\mathbb{D}(f(n),\chi(n)n^{it};x)\ll 1$ for some primitive character χ. Therefore, to understand the above better, we need to give a version of Theorem 1.6.3 for functions f with $\mathbb{D}(f(n),\chi(n)n^{it};x)\ll 1$.

Now we will suppose that $\mathbb{D}(f(n),n^{it}\chi(n),\infty)<\infty$ for some $t\in\mathbb{R}$ where χ is a primitive character of conductor q. We define F to be the multiplicative function such that

$$F(p^k) = \begin{cases} f(p^k)\overline{\chi(p^k)}p^{-ikt}, & \text{if } p\nmid q \\ 1, & \text{if } p\mid q, \end{cases}$$

and

$$M_p(F,\overline{F};d) = \lim_{x\to\infty}\frac{1}{x}\sum_{n\le x} F_p(n)\overline{F_p(n+d)}.$$

We will prove

Theorem 1.6.5. *Let $f:\mathbb{N}\to\mathbb{U}$ be a multiplicative function such that $\mathbb{D}(f(n),n^{it}\chi(n);\infty)<\infty$ for some $t\in\mathbb{R}$ and χ is a primitive character of conductor q. Then for any non-zero integer d we have*

$$\frac{1}{x}\sum_{n\le x} f(n)\overline{f(n+d)} = \prod_{\substack{p\le x\\ p\nmid q}} M_p(F,\overline{F};d)\prod_{p^{\text{ł}}\|q} M_{p^{\text{ł}}}(f,\overline{f},d) + o(1)$$

when $x\to\infty$. Here, $o(1)$ term depends on d,χ,t and

$$M_{p^{\text{ł}}}(f,\overline{f},d) = \begin{cases} 0, & \text{if } p^{\text{ł}-1}\nmid d \\ 1-\frac{1}{p}, & \text{if } p^{\text{ł}-1}\| d \\ \left(1-\frac{1}{p}\right)\sum_{j=0}^{k}\frac{|f(p^j)|^2}{p^j} - \frac{|f(p^k)|^2}{p^k}, & \text{if } p^{\text{ł}+k}\| d \end{cases}$$

for any $k\ge 0$ and if $p^n\| d$ for some $n\ge 0$, then

$$M_p(F,\overline{F},d) = 1-\frac{2}{p^{n+1}} + \left(1-\frac{1}{p}\right)\sum_{j>n}\left(\frac{F(p^n)\overline{F(p^j)}}{p^j} + \frac{\overline{F(p^n)}F(p^j)}{p^j}\right).$$

In particular, the mean value is $o(1)$ *if* $q \nmid d\prod_{p|q} p$.

The same method works for correlations

$$\sum_{n\leq x} f(n)g(n+m),$$

where $\mathbb{D}(f(n),n^{it}\chi(n);\infty)$, $\mathbb{D}(g(n),n^{iu}\psi(n);\infty) < \infty$.

1.6.5. The Erdős discrepancy problem for multiplicative functions

In 1957, Erdős made his famous conjecture, known as the Erdős discrepancy problem, which asserts that given any sequence $\{a_n\}_{n\geq 1}$ with each element being either 1 or -1, the discrepancy along homogeneous arithmetic progressions is infinite, namely

$$\sup_{n,d\geq 1}\left|\sum_{k=1}^{n} a_{kd}\right| = \infty.$$

In September 2015, this conjecture was finally settled by Tao [Taoa]. Somewhat surprisingly, previously the Polymath5 project showed, using Fourier analysis, that the Erdős discrepancy problem can be reduced to a statement about completely multiplicative functions. In particular, Tao [Taoa] established that for any completely multiplicative $f : \mathbb{N} \to \{-1,1\}$,

$$\limsup_{x\to\infty}\left|\sum_{n\leq x} f(n)\right| = \infty.$$

In [Erd57], [Erd85a], [Erd85b], Erdős along with the Erdős discrepancy problem, asked to classify all multiplicative $f : \mathbb{N} \to \{-1,1\}$ such that

$$\limsup_{x\to\infty}\left|\sum_{n\leq x} f(n)\right| < \infty. \tag{1.6.4}$$

There are examples known with bounded sums, such as the multiplicative function f for which $f(n) = +1$ when n is odd and $f(n) = -1$ when n is even. In [Taoa], Tao, partially answering this question, proved that if for a multiplicative $f : \mathbb{N} \to \{-1,1\}$, (1.6.4) holds, then $f(2^j) = -1$ for all j, and

$$\sum_{p} \frac{1-f(p)}{p} < \infty.$$

In this thesis, we resolve this question completely by proving

Theorem 1.6.6. [Erdős-Coons-Tao conjecture] *Let* $f : \mathbb{N} \to \{-1,1\}$ *be a multiplicative function. Then* (1.6.4) *holds if and only if there exists an integer* $m \geq 1$ *such that* $f(n+m) = f(n)$ *for all* $n \geq 1$ *and* $\sum_{n=1}^{m} f(n) = 0$.

One can easily show that f satisfies the above hypotheses if and only if m is even, $f(2^k) = -1$ for all $k \geqslant 1$, and $f(p^k) = f((p^k,m))$ for all odd prime powers p^k. In particular if p does not divide m, then $f(p^k) = 1$.

It would be interesting to classify all complex valued multiplicative $f : \mathbb{N} \to \mathbb{T}$ for which (1.6.4) holds. Using Theorem 1.6.5 it is easy to prove

Theorem 1.6.7. *Suppose for a multiplicative $f : \mathbb{N} \to \mathbb{T}$, (1.6.4) holds. Then there exists a primitive character χ of an odd conductor q and $t \in \mathbb{R}$, such that $\mathbb{D}(f(n),\chi(n)n^{it};\infty) < \infty$ and $f(2^k) = -\chi^k(2)2^{-ikt}$ for all $k \geq 1$.*

1.6.6. Distribution of $(f(n), f(n+1))$

Let $\triangle(n) = f(n+1) - f(n)$. The archetype for the problems we shall consider is the famous theorem of Erdős [Erd46a] that states that if $f : \mathbb{N} \to \mathbb{N}$ is a *non-decreasing* multiplicative function, that is $\triangle f(n) \geq 0$ for all $n \geq 1$, then $f(n) = n^k$ for some non-negative integer k. Another example of such a rigidity result, first conjectured by Kátai and solved by Wirsing (and independently by Shao and Tang, see [WTS96]), is that if $f : \mathbb{N} \to \mathbb{T}$ is multiplicative and $|f(n+1) - f(n)| \to 0$ as $n \to \infty$ then $f(n) := n^{it}$ for some $t \in \mathbb{R}$ (see also a nice paper of Wirsing and Zagier [WZ01] for a simpler proof). One would naturally expect that if $\triangle f(n) \to 0$ in some averaged sense, than the similar conclusion must hold. Kátai [Kát83] made the following conjecture which we prove in Chapter 4 :

Theorem 1.6.8. [Kátai's Conjecture, 1983] *If $f : \mathbb{N} \to \mathbb{C}$ is a multiplicative function and*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \leq x} |\triangle f(n)| = 0,$$

then either

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \leq x} |f(n)| = 0$$

or $f(n) = n^s$ for some $\mathrm{Re}(s) < 1$.

We note that the content of the last theorem is rather deep. In particular, applied to the functions $f : \mathbb{N} \to \{-1,1\}$, Theorem 1.6.8 immediately implies that any such f has a positive proportion of sign changes, which is the main consequence of the recent breakthrough work [MR]. Since $f(n) = \mathrm{e}^{h(n)}$ is multiplicative, where $h(n) : \mathbb{N} \to \mathbb{R}$ is an additive function, one may compare Theorem 1.6.8 with the following statement about additive functions, first conjectured by Erdős [Erd46b] and proved later by Kátai [Kát70] (and independently by Wirsing): if $h : \mathbb{N} \to \mathbb{C}$ is an additive function and

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \leq x} |h(n+1) - h(n)| = 0,$$

then $h(n) = c \log n$.

The conjecture attracted considerable attention of several authors including Kátai, Hildebrand, Phong and others. See, for example [Hil88b], [Pho14], [Pho00], [Kát91] for some of the results and the survey paper [Kát00] with an extensive list of the related references.

1.6.7. Application to the binary additive problems

A sequence A of positive integers is called multiplicative, if its characteristic function, 1_A, is multiplicative. We define

$$\rho_A(d) = \lim_{x\to\infty} \frac{1}{x/d} \sum_{k\le x/d} \mathrm{I}_A(kd),$$

with $\rho_A = \rho_A(1)$, which is the density of A. Note that these constants all exist by Wirsing's Theorem.

Binary additive problems, which involve estimating quantities like

$$r(n) = |\{(a,b) \in A \times B : \ a+b=n\}|$$

are considered difficult. However, using a variant of a circle method Brüdern [Brü09], among other things, established the following theorem, which we will deduce from Theorem 1.6.3.

Theorem 1.6.9. [Brüdern, 2008] *Suppose A and B are multiplicative sequences of positive density ρ_A and ρ_B respectively. For $k \ge 1$, let*

$$a(p^k) = \rho_A(p^k)/p^k - \rho_A(p^{k-1})/p^{k-1}$$

Define $b(p^k)$ in the same fashion. Then,

$$r(n) = \rho_A\rho_B\sigma(n)n + o(n)$$

when $n \to \infty$, where

$$\sigma(n) = \prod_{p^m||n} \left(1 + \sum_{k=1}^{m} \frac{p^{k-1}a(p^k)b(p^k)}{p-1} - \frac{p^m a(p^{m+1})b(p^{m+1})}{(p-1)^2}\right).$$

1.7. Multilinear correlations of multiplicative functions

Let $f_1, \ldots, f_k : \mathbb{N} \to \mathbb{C}$ be multiplicative functions taking values in the closed unit disc. Our first result is to compute multidimensional averages

$$x^{-l} \sum_{\boldsymbol{n} \in [x]^l} \prod_{1 \leqslant j \leqslant k} f_j(L_j(\boldsymbol{n}))$$

as $x \to \infty$, where $[x] := [1,x]$ and $L_1, \ldots, L_k$ are affine linear forms that satisfy some natural conditions, generalizing correlation formulas discussed in the previous chapter.
As an application of our formulae, we establish a *local-to-global* principle for Gowers norms of multiplicative functions. We also compute the asymptotic densities of the sets of integers n such that a given multiplicative function $f : \mathbb{N} \to \{-1, 1\}$ yields a fixed sign pattern of length 3 or 4 on almost all 3- and 4-term arithmetic progressions, respectively, with first term n.

1.7.1. Previous work and new results

For $k,l \geqslant 2$, let $\boldsymbol{L} := (\boldsymbol{L}_1, \ldots, \boldsymbol{L}_k)$ be a vector of k (affine) linear forms $L_j : \mathbb{R}^l \to \mathbb{R}$ with non-negative integer coefficients, i.e.,

$$L_j(\boldsymbol{n}) = \alpha_{j,0} + \sum_{1 \leqslant r \leqslant l} \alpha_{j,r} n_r,$$

where $(\alpha_{j,r})_{0 \leqslant r \leqslant l} \in \mathbb{N}_0^{l+1}$. We will call such a vector an *integral system.* Assume moreover that $(\alpha_{j,1}, \ldots, \alpha_{j,l}) = 1$, for each j, and that the forms are pairwise linearly independent. We will say that a system of forms that satisfies these properties is *primitive.* We will concern ourselves throughout this thesis with *primitive integral systems* of affine linear forms. We remark that this primitivity assumption is merely technical and can be removed with more effort.
Let $\mathbb{U}$ denote the closed unit disc. We say that a function $f : \mathbb{N} \to \mathbb{C}$ is *1-bounded* if $f(n) \in \mathbb{U}$ for all n. For a vector $\boldsymbol{f} := (f_1, \ldots, f_k)$ of 1-bounded multiplicative functions, a vector $\boldsymbol{x} := (x_1, \ldots, x_l) \in (0,\infty)^l$ and a system of primitive integral affine linear forms $\boldsymbol{L}$, put

$$M(\boldsymbol{x}; \boldsymbol{f}, \boldsymbol{L}) := \langle \boldsymbol{x} \rangle^{-1} \sum_{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x})} \prod_{1 \leqslant j \leqslant k} f_j(L_j(\boldsymbol{n})),$$

where $\mathcal{B}(\boldsymbol{x})$ denotes the box $\prod_{1 \leqslant j \leqslant k}(0,x_j]$, and $\langle \boldsymbol{x} \rangle = x_1 \cdots x_l$ is its volume. When $\boldsymbol{x} = (x, \ldots, x)$ for some $x \geqslant 1$ then we will write $M(x; \boldsymbol{f}, \boldsymbol{L})$ instead.
The main purpose of the present investigation is to establish an asymptotic formula for $M(\boldsymbol{x}; \boldsymbol{f}, \boldsymbol{L})$ with *explicit* main and error terms using analytic techniques in the spirit of Halász' mean value theorem. In contrast, results in this direction have thus far been obtained by either using ergodic theoretic machinery, as in the works of Frantzikinakis and Host [FH16],[HF16], or, more recently, by using the nilpotent Hardy-Littlewood method of Green

and Tao (see the recent paper of Matthiesen [Mat] for details). Neither of these papers give quantitative error terms.

Recall, that for multiplicative functions $f,g : \mathbb{N} \to \mathbb{U}$, we set

$$\mathbb{D}(f,g;y,x) := \left(\sum_{y<p\leqslant x} \frac{1-\mathrm{Re}(f(p)\overline{g(p)})}{p}\right)^{\frac{1}{2}}$$

for $1 \leqslant y \leqslant x$, as well as $\mathbb{D}(f,g;x) := \mathbb{D}(f,g;1,x)$. We then define $\mathbb{D}(f,g;\infty) := \lim_{x\to\infty} \mathbb{D}(f,g;x)$. We also put

$$\mathbb{D}^*(f,g;y,x) := \left(\sum_{y<p^k\leqslant x} \frac{1-\mathrm{Re}(f(p^k)\overline{g(p^k)})}{p^k}\right)^{\frac{1}{2}}.$$

For $Q,X \geqslant 1$, we shall write

$$\mathcal{D}(g;X,Q) := \inf_{|t|\leqslant X;q\leqslant Q,\chi(q)} \mathbb{D}(f,\chi n^{it};X)^2,$$

where the infimum in q is over all Dirichlet characters χ modulo q, for all $q \leqslant Q$.

Recently, using their deep structural theorem for multiplicative functions (see Theorem 2.1 in [HF16]), Frantzikinakis and Host proved that for a vector of 1-bounded *multiplicative* functions $\boldsymbol{f}$ and a system of integral, affine linear forms,

$$M(x;\boldsymbol{f},\boldsymbol{L}) = cx^{iT}e(\omega(x)) + o_{x\to\infty}(1), \tag{1.7.1}$$

where $\omega : \mathbb{R} \to \mathbb{R}$ is a slowly-oscillating function and $c = 0$ unless all of the functions f_j are *pretentious* in the sense that for each $1 \leqslant j \leqslant k$ there is a primitive Dirichlet character χ_j with modulus q_j, and $t_j \in \mathbb{R}$ such that $\mathbb{D}(f_j,\chi_j n^{it_j};\infty) < \infty$. In the latter case, they show that the parameter T in (1.7.1) depends in some way on $t_1,\dots,t_k$ (for instance, when the system is primitive they prove that $T = t_1 + \cdots + t_k$). However, they do not give an explicit expression for c.

Our first result is a quantitative version of (1.7.1), with explicit main and error terms, in the case that *all* of the functions f_j are pretentious in the above sense. To state it, we need to introduce some notation and conventions.

Given a vector $\boldsymbol{x} \in (0,\infty)^l$ we write

$$\ell(\boldsymbol{x}) := \sum_{1\leqslant j\leqslant l} |x_j|.$$

We will also write x_- and x_+ to denote, respectively, the minimum and maximum components of $\boldsymbol{x}$. Given $A \geqslant 1$ and $B > 0$, we will say that a vector $\boldsymbol{x} \in (0,\infty)^l$ is (A,B)-*appropriate* if $x_- \geqslant 3$ and

$$x_- > l\log_2((l+1)Ax_+)^2(\log x_+)^B.$$

This condition ensures that $\boldsymbol{x}$ is not too skew.

For a system of linear forms $\boldsymbol{L}$, we write $\boldsymbol{L}(\boldsymbol{0})$ to be the vector with components $L_j(\boldsymbol{0})$, for

$1 \leqslant j \leqslant k$. We also say that the *height* of the system $\boldsymbol{L}$ of affine linear forms is the maximum of the coefficients of all linear forms in $\boldsymbol{L}$.

For any multiplicative function $f : \mathbb{N} \to \mathbb{U}$ and any prime p, we define the multiplicative function f_p by

$$f_p(q^\nu) := \begin{cases} f(p^\nu) & \text{if } q = p \\ 1 & \text{if } q \neq p, \end{cases}. \tag{1.7.2}$$

We then define the p-adic local average of $\boldsymbol{f}$ on $\boldsymbol{L}$ by

$$M_p(\boldsymbol{f},\boldsymbol{L}) := \lim_{x\to\infty} x^{-l} \sum_{\boldsymbol{n}\in[x]^l} \prod_{1\leqslant j\leqslant k} f_{j,p}(L_j(\boldsymbol{n})).$$

For an integral vector $\boldsymbol{a} = (a_1, \ldots, a_k)$ and primitive Dirichlet characters $\chi_1, \ldots, \chi_k$ to respective moduli $q_1, \ldots, q_k$, we set

$$\mathcal{I}(\boldsymbol{x},\boldsymbol{L},\boldsymbol{t}) := \int_{[0,1]^l} \prod_{1\leqslant j\leqslant k} L_j((u_1x_1, \ldots, u_lx_l))^{it_j} d\boldsymbol{u};$$

$$\Xi_{\boldsymbol{a}}(\boldsymbol{\chi},\boldsymbol{L}) := \sum_{\substack{b_1(q_1) \\ \exists \boldsymbol{n}: L_j(\boldsymbol{n})/a_j\equiv b_j(q_j)\forall j}} \cdots \sum_{b_k(q_j)} \prod_{1\leqslant j\leqslant k} \chi_j(b_j);$$

$$R(m_1, \cdots, m_k) := \lim_{x\to\infty} x^{-l} \sum_{\substack{\boldsymbol{n}\in[x]^l \\ m_j|L_j(\boldsymbol{n})\forall j}} 1,$$

and

$$C_{\boldsymbol{a}}(\boldsymbol{x},\boldsymbol{\chi},\boldsymbol{t},\boldsymbol{L}) := R(q_1a_1, \ldots, q_ka_k)\Xi_{\boldsymbol{a}}(\boldsymbol{\chi},\boldsymbol{L})\mathcal{I}(\boldsymbol{x},\boldsymbol{L},\boldsymbol{t}).$$

Finally, we recall that the *radical* of a positive integer n is $\mathrm{rad}(n) := \prod_{p|n} p$.

We begin by stating one corollary of our main theorem.

Corollary 1.7.1. *Let $A, q \geqslant 2$, $B > 0$, and let $\boldsymbol{x} \in (0,\infty)^l$ be (A,B)-appropriate. Let $\boldsymbol{f} = (f_1, \ldots, f_k)$ be a vector of 1-bounded multiplicative functions. Let $\boldsymbol{L}$ be a primitive integral system of k affine linear forms in l variables with height at most A.*

Suppose that there are primitive Dirichlet characters $\chi_1, \ldots, \chi_k$ modulo q and $t_1, \ldots, t_k \in \mathbb{R}$ such that $\mathbb{D}(f_j(n), \chi_j n^{it_j}) < \infty$ for all $1 \leq j \leq k$. Let $F_j(n) := f_j(n)\overline{\chi}_j(n)n^{-it_j}$. Put $X := \ell(\boldsymbol{x}) + 1$. Then

$$M(\boldsymbol{x}; \boldsymbol{f},\boldsymbol{L}) = \left(\sum_{\substack{rad(a_j)|q \\ \forall 1\leqslant j\leqslant k}} \prod_{1\leqslant j\leqslant k} \frac{f_j(a_j)}{a_j^{it_j}} C_{\boldsymbol{a}}(\boldsymbol{x},\boldsymbol{\chi},\boldsymbol{t},\boldsymbol{L}) \right) \prod_{\substack{p\leqslant AX \\ p\nmid q}} M_p(\boldsymbol{F},\boldsymbol{L}) + o(1).$$

More generally we have the following fully explicit result.

Theorem 1.7.1. *Let $A \geqslant 2$, $B > 0$, and let $\boldsymbol{x} \in (0,\infty)^l$ be (A,B)-appropriate. Let $\boldsymbol{f} = (f_1, \ldots, f_k)$ be a vector of 1-bounded multiplicative functions. Let $\boldsymbol{L}$ be a primitive integral system of k affine linear forms in l variables with height at most A.*

Fix a set of primitive Dirichlet characters $\chi_1,\ldots,\chi_k$ *to respective moduli* $q_1,\ldots,q_k$, *and* $t_1,\ldots,t_k \in \mathbb{R}$. *Let* $F_j(n) := f_j(n)\overline{\chi}_j(n)n^{-it_j}$. *Put* $X := \ell(\boldsymbol{x})+1$ *and let* $\max_{1\leqslant j\leqslant k} q_j < y \leqslant X$. *If* $q_j = q$ *for all* j *then*

$$M(\boldsymbol{x};\boldsymbol{f},\boldsymbol{L}) = \left(1+O_{k,l}\left(\frac{1}{\log y}\right)\right)\left(\sum_{\substack{rad(a_j)|q\\ \forall 1\leqslant j\leqslant k}}\prod_{1\leqslant j\leqslant k}\frac{f_j(a_j)}{a_j^{it_j}}C_{\boldsymbol{a}}(\boldsymbol{x},\boldsymbol{\chi},\boldsymbol{t},\boldsymbol{L})\right)\prod_{\substack{p\leqslant AX\\ p\nmid q}} M_p(\boldsymbol{F},\boldsymbol{L})$$
$$+O_{k,l}\left(\prod_{p|q}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}\left(\sum_{1\leqslant j\leqslant k}\mathbb{D}^*(f_j,\chi_j n^{it_j};y,AX)+\frac{1}{(\log X)^{B'}}\right)\right)$$
$$+O_{k,l}\left(\frac{1}{x_-}\left(A+q^k e^{\frac{3ky}{\log y}}\left(\sum_{\substack{rad(a_j)|q\\ \forall 1\leqslant j\leqslant k}}[a_1,\ldots,a_k]^{-1}\right)\prod_{1\leqslant j\leqslant k}\max\{1,|t_j|\}\right)+\frac{(\log y)^2}{\sqrt{y}}\right),$$

where $B' := \min\{1,B/2\}$. *More generally, for any collection of moduli* q_j,

$$M(\boldsymbol{x};\boldsymbol{f},\boldsymbol{L}) = \left(1+O_{k,l}\left(\frac{1}{\log y}\right)\right)\left(\sum_{\substack{rad(a_j)|q_j\\ \forall 1\leqslant j\leqslant k}}\prod_{1\leqslant j\leqslant k}\frac{f_j(a_j)}{a_j^{it_j}}C_{\boldsymbol{a}}(\boldsymbol{x},\boldsymbol{\chi},\boldsymbol{t},\boldsymbol{L})\right)\mathcal{S}_{\boldsymbol{a}}(y;\boldsymbol{f},\boldsymbol{L})\prod_{y<p\leqslant AX} M_p(\boldsymbol{F},\boldsymbol{L}) \tag{1.7.3}$$

$$+O_{k,l}\left(\sum_{1\leqslant j\leqslant k}\prod_{p|q_j}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}\left(\mathbb{D}^*(f_j,\chi_j n^{it_j};y,AX)+\frac{1}{(\log X)^{B'}}\right)\right)$$
$$+O_{k,l}\left(\frac{1}{x_-}\left(A+e^{\frac{3ky}{\log y}}\left(\sum_{\substack{rad(a_j)|q_j\\ \forall 1\leqslant j\leqslant k}}[a_1,\ldots,a_k]^{-1}\right)\prod_{1\leqslant j\leqslant k}q_j\max\{1,|t_j|\}\right)+\frac{(\log y)^2}{\sqrt{y}}\right), \tag{1.7.4}$$

where, for $\boldsymbol{a},\boldsymbol{d}\in\mathbb{N}^k$,

$$R_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L};\boldsymbol{u},\boldsymbol{v}) := \lim_{x\to\infty} x^{-l}\sum_{\substack{\boldsymbol{n}\in[x]^l\\ L_j(\boldsymbol{n})/a_j\equiv u_j(q_j),L_j(\boldsymbol{n})\equiv v_j(a_jd_j)\forall j}} 1$$

and

$$\mathcal{S}_{\boldsymbol{a}}(y;\boldsymbol{f},\boldsymbol{L}) := R(q_1a_1,\ldots,a_kq_k)^{-1}\sum_{\substack{P^+(d_j)\leqslant y\\ (d_j,q_j)=1\forall j}} R_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L}-\boldsymbol{L}(\boldsymbol{0}),\boldsymbol{0},\boldsymbol{0})\prod_{1\leqslant j\leqslant k}(\mu * F_j)(d_j).$$

Theorem 1.7.1 shows that a *local-to-global* phenomenon occurs for correlations of multiplicative functions, i.e., the global average correlation is the product of the local average correlations, determined by the functions $f_{j,p}$ and the characters χ_j and $n\mapsto n^{it_j}$. Our proof of Theorem 1.7.1 generalizes and extends the ideas from [Klu].

Remark 1.7.2. Note that when $x_j = x$ for all j, we have $\mathcal{I}(\boldsymbol{x},\boldsymbol{L},\boldsymbol{t}) = x^{iT}\mathcal{I}(\boldsymbol{L},\boldsymbol{t})$, where $T := \sum_{1 \leqslant j \leqslant k} t_j$ and

$$\mathcal{I}(\boldsymbol{L},\boldsymbol{t}) := \int_{[0,1]^l} \prod_{1 \leqslant j \leqslant k} L_j(\boldsymbol{u})^{it_j} d\boldsymbol{u}.$$

This is consistent with the result in [FH16] mentioned above.

Remark 1.7.3. The distinction between the case in which the q_j are all equal and the case in which they are not stems from the fact that the Chinese Remainder Theorem implies that $R(m_1, \ldots ,m_k)$ is only multiplicative, and not *firmly multiplicative* (see Section 3.2 of [T́14]). That is, R satisfies the identity

$$R(m_1 n_1, \ldots ,m_k n_k) = R(m_1, \ldots ,m_k)R(n_1, \ldots ,n_k)$$

whenever $(m_1 \cdots m_k,n_1 \cdots n_k) = 1$, but in general it is *not* sufficient that $(m_j,n_j) = 1$ for all j. This nuance concerning multiplicative functions in several variables (which is manifest in (5.3.18) below) prevents us from getting a conclusion that is uniform over all fixed moduli q_j.

Remark 1.7.4. The error term in 1.7.1 can be improved in a number of ways when the functions f_j satisfy certain natural restrictions. For example, it follows from the proof of Theorem 1.7.1 that the term $\frac{(\log y)^2}{\sqrt{y}}$ can be replaced by $y^{-1+o(1)}$ when each f_j is supported on squarefree integers, and when the f_j are all completely multiplicative we can replace $(\log X)^{-B'}$ by $(\log X)^{-B/2}$.

When at least one of the functions f_j is *non-pretentious*, we are able to recover quantitative versions of the results from [FH16] whenever $k \leqslant 3$ or the linear forms L_j, $1 \leq j \leq k$ are *sufficiently* linearly independent. This independence is measured by *Cauchy-Schwarz complexity* (see the end of Section 2 for a definition).

Proposition 1.7.5. *Let $A \geqslant 1$. Let $\boldsymbol{f} = (f_1, \ldots ,f_k)$ be a vector of multiplicative functions $f_j : \mathbb{N} \to \mathbb{U}$. Let $\boldsymbol{L}$ be a primitive integral system of affine linear forms in l variables with height at most A and Cauchy-Schwarz complexity at most 1. Then there are absolute constants $c_1,c_2 > 0$ such that if, for some $1 \leqslant j_0 \leqslant k$, we have $\mathcal{D}_{j_0}(x) := \mathcal{D}(f_{j_0}; 10Ax, (\log x)^{1/125}) \to \infty$ as $x \to \infty$,*

$$M(x; \boldsymbol{f},\boldsymbol{L}) \ll_{k,l,A} e^{-c_1 \mathcal{D}_{j_0}(x)} + (\log x)^{-c_2}.$$

This result is a consequence of the recent work of Matomaki, Radziwiłł and Tao on the averaged Elliott conjecture (see Theorem 1.6 of [MRT15])..

1.7.2. Gowers norms of 1-bounded multiplicative functions

One motivation for investigations regarding affine linear averages of multiplicative functions comes from the study of Gowers norms. Let $(G,+)$ be a finite Abelian group, and let

$f : G \to \mathbb{C}$ be a map. Write

$$\mathbb{E}_{x\in G}(f) := |G|^{-1} \sum_{x\in G} f(x),$$

and $\mathbb{E}_{x_1,\dots,x_{k+1}\in G}(f) = \mathbb{E}_{x_{k+1}\in G}\mathbb{E}_{x_1,\dots,x_k\in G}(f)$. For each $k \geqslant 1$ we define the $U^k(G)$-*Gowers norm* of f via

$$\|f\|_{U^k(G)}^{2^k} := \mathbb{E}_{x,h_1,\dots,h_k\in G} \prod_{\boldsymbol{s}\in\{0,1\}^k} \mathcal{C}^{|\boldsymbol{s}|} f(x + \boldsymbol{s}\cdot\boldsymbol{h}),$$

where, given a vector $\boldsymbol{s} \in \{0,1\}^k$ we write $|\boldsymbol{s}| = \sum_{1\leqslant j\leqslant k} s_j$, and $\mathcal{C} : \mathbb{C}^{|G|} \to \mathbb{C}^{|G|}$ is the conjugation operator $\mathcal{C}(g) = \overline{g}$. Gowers norms are fundamental in Additive Combinatorics as they provide a Fourier analytic framework for counting arithmetic progressions in groups. For background information regarding Gowers norms, see [Tao12].

We can extend Gowers norms to maps on intervals $[1,x] \subset \mathbb{N}$ as follows: let $N > x$ be a sufficiently large prime and let $G = \mathbb{Z}/N\mathbb{Z}$. Then the Gowers norm of a map $f : \mathbb{N} \to \mathbb{C}$ on $[1,x]$ is given by

$$\|f\|_{U^k(x)} := \|f1_{[1,x]}\|_{U^k(\mathbb{Z}/N\mathbb{Z})}/\|1_{[1,x]}\|_{U^k(\mathbb{Z}/N\mathbb{Z})},$$

where $1_{[1,x]}$ is the characteristic function of the interval $[1,x]$ as a subset of $\mathbb{Z}/N\mathbb{Z}$.

Definition 1.7.6. Let $k \geqslant 2$ and $K := 2^k$. The *Gowers system (of order k)* is the system $\boldsymbol{L}_k := \{L_j\}_{1\leqslant j\leqslant K}$ of k-ary homogeneous linear forms such that if the binary expansion of j is $\sum_{0\leqslant l\leqslant k-1} \alpha_l 2^l \leqslant K$ then

$$L_j(n_1,\dots,n_{k+1}) = n_{k+1} + \sum_{0\leqslant l\leqslant k-1} \alpha_j n_{j+1}.$$

Note that for f multiplicative, if $j = \sum_{0\leqslant l\leqslant k-1} \alpha_l 2^l$ and $d_j := \sum_{0\leqslant l\leqslant k-1} \alpha_l$ then with $f_j := \mathcal{C}^{d_j} f$, we have $\|f\|_{U^k(x)}^{2^k} = M(x; \boldsymbol{f},\boldsymbol{L}_k)$. Theorem 1.7.1 thus indeed furnishes estimates for Gowers norms of multiplicative functions.

Consider the $U^k(x)$ norm of a multiplicative function f such that for some primitive character χ with conductor q and a real number t we have $\mathbb{D}(f,\chi n^{it};\infty) < \infty$. With the notation above, our correlation has the form

$$\|f\|_{U^k(x)}^{2^k} := x^{-(k+1)} \sum_{\boldsymbol{n}\in[x]^{k+1}} \left(\prod_{\substack{1\leqslant j\leqslant K\\ d_j \text{ even}}} f(L_j(\boldsymbol{n})) \right) \overline{\left(\prod_{\substack{1\leqslant j\leqslant K\\ d_j \text{ odd}}} f(L_j(\boldsymbol{n})) \right)},$$

and Theorem 1.7.1 applies. The Dirichlet character factor takes the form

$$\Xi_{k,\boldsymbol{a}}(\chi) := \Xi_{\boldsymbol{a}}((\mathcal{C}^{d_1}\chi,\dots,\mathcal{C}^{d_K}\chi),\boldsymbol{L}_k) = \sum_{\substack{b_1(q)\\ \exists \boldsymbol{n}: L_j(\boldsymbol{n})/a_j \equiv b_j(q)\forall j}} \cdots \sum_{b_K(q)} \chi\left(\prod_{\substack{1\leqslant j\leqslant K\\ d_j \text{ even}}} b_j \right) \overline{\chi\left(\prod_{\substack{1\leqslant j\leqslant K\\ d_j \text{ odd}}} b_j \right)},$$

while the Archimedean character factor is

$$I_k(t) := I(\boldsymbol{L}_k,((-1)^{d_1}t,\ldots,(-1)^{d_K}t)) = \int_{[0,1]^{k+1}} \left(\prod_{\substack{1\leqslant j\leqslant K\\ d_j \text{ even}}} L_j(\boldsymbol{u})\right)^{it} \left(\prod_{\substack{1\leqslant j\leqslant K\\ d_j \text{ odd}}} L_j(\boldsymbol{u})\right)^{-it} d\boldsymbol{u},$$

for Archimedean characters.

The local-to-global principle for Gowers norms is thus as follows.

Corollary 1.7.7. *Let $f : \mathbb{N} \to \mathbb{C}$ be a 1-bounded multiplicative function. Let $k \geq 2$ and put $K := 2^k$.*

i) If $\mathbb{D}(f,n^{it}\chi,\infty) = \infty$ for all Dirichlet characters χ and all $t \in \mathbb{R}$, then

$$\|f\|_{U^2(x)} = o_{x\to\infty}(1).$$

ii) If there exists a primitive Dirichlet character χ of conductor q and $t \in \mathbb{R}$ such that $\mathbb{D}(f,n^{it}\chi,\infty) < \infty$, then

$$\|f\|_{U^k(x)}^{2^k} = I_k(t) \left(\sum_{\substack{rad(a_j)|q\\ \forall 1\leqslant j\leqslant K}} \prod_{1\leqslant j\leqslant K} \mathcal{C}^{d_j}\left(\frac{f(a_j)}{a_j^{1+it}}\right) R(qa_1,\cdots,qa_K)\Xi_{k,\boldsymbol{a}}(\chi)\right) \prod_{p\leq x,\ p\nmid q} \|f_p(n)\overline{\chi(n)}n^{-it}\|_{U^k(x)}^{2^k}$$
$$+ O\left(\mathbb{D}(1,f(n)\overline{\chi(n)}n^{-it};\log x;x)\right),$$

where d_j is the sum of the binary digits of j.

1.7.3. Sign patterns of multiplicative f in 3- and 4-term AP's

Let λ denote the Liouville function $\lambda(n) := (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime factors of n, counted with multiplicity. Chowla [Cho65] conjectured the following regarding sign patterns of λ.

Conjecture 1.7.8 (Chowla for Sign Patterns)**.** *Let $k \geqslant 2$, let $\{h_1,\ldots,h_k\}$ be a sequence of distinct non-negative integers, and let $\boldsymbol{\epsilon} \in \{-1,1\}^k$ be a vector of signs. Then*

$$|\{n \leqslant x : \lambda(n+h_j) = \epsilon_j \text{ for all } 1 \leqslant j \leqslant k\}| = \left(\frac{1}{2^k} + o(1)\right)x.$$

In other words, it is expected that the vectors $(\lambda(n+h_1),\ldots,\lambda(n+h_k))$ are uniformly distributed among the 2^k possible patterns of $+$ and $-$ signs. Of particular interest is the case in which the forms $n \mapsto n+h_j$ constitute an arithmetic progression. This case requires that one understands the behaviour of a function sensitive to multiplicative structure on sets with additive structure.

Recently, lower density estimates for sign patterns of λ of length 3 were given by Matomäki, Radziwiłł and Tao [MRT16], and the exact logarithmic density of the set of n yielding any fixed sign pattern of length 2 for λ was obtained by Tao [Taoc].

One may ask about the frequency of sign patterns for arbitrary multiplicative functions $f : \mathbb{N} \to \{-1,1\}$ in place of λ, and on arithmetic progressions of length 3 or more. Such questions have interest, for instance, because of their relationship with the distribution of quadratic non-residues modulo primes.

As an example of such investigations, Buttkewitz and Elsholtz [BE11] recently classified those multiplicative sign functions that only yield a fixed length four sign pattern finitely often on certain 4-term APs.

We shall study several questions in this direction. We first consider fixed arithmetic progressions, giving explicit lower bounds for the upper logarithmic density of sign patterns of length 3 and 4 when f is non-pretentious. In particular, we show the following.

Proposition 1.7.9. *Let $d \geqslant 1$. Let $f : \mathbb{N} \to \{-1,1\}$ be a non-pretentious multiplicative function.*

i) Given $\boldsymbol{\epsilon} \in \{-1,1\}^3$, the upper logarithmic density of the set of integers n such that

$$(f(n),f(n+d),f(n+2d)) = \boldsymbol{\epsilon}$$

is at least $\frac{1}{28}$.

ii) Given $\boldsymbol{\epsilon} \in \{-1,1\}^4$, the upper logarithmic density of the set of integers n such that

$$(f(n),f(n+d),f(n+2d),f(n+3d)) = \boldsymbol{\epsilon} \text{ or } -\boldsymbol{\epsilon}$$

is at least $\frac{1}{28}$.

We remark that the bound in [MRT16] on the lower density of length 3 sign patterns of λ was *inexplicit*, due to the use of nonstandard analysis there.

We also consider corresponding questions about the natural density of sign patterns in *almost all* progressions. We will establish the following equidistribution-type results for sign patterns of non-pretentious functions on *almost* all 3-term APs in a suitable sense. In particular, we have an averaged analogue of Conjecture 1.7.8 in this context.

For $c_1,c_2 > 0$ the constants in Proposition 1.7.5, define

$$\mathcal{R}_f(x) := e^{-c_1 \mathcal{D}(f;x,(\log x)^{\frac{1}{125}})} + (\log x)^{-c_2}. \tag{1.7.5}$$

Theorem 1.7.10. *Let $f : \mathbb{N} \to \{-1,1\}$ be multiplicative. Let $\boldsymbol{\epsilon} \in \{-1,1\}^3$. Except for $O\left(x\mathcal{R}_f(x)^{\frac{1}{3}}\right)$ choices of $d \leqslant x$, we have*

$$|\{n \leqslant x : f(n+jd) = \epsilon_j \text{ for all } 0 \leqslant j \leqslant 2\}| = x\left(\frac{1}{8} + O\left(\mathcal{R}_f(x)^{\frac{1}{3}}\right)\right).$$

Finally, we establish an analogue of Theorem 1.7.10 when f is pretentious and investigate to what extent this average density can be biased away from the density predicted by equidistribution. In so doing, we establish a quantitative refinement of the results of Buttkewitz and Elsholtz [BE11]. See Remark 1.7.16 for a discussion of the connection between our results and those of [BE11].

It turns out that when f is pretentious to a real primitive character χ with conductor q, f behaves well on arithmetic progressions with difference d not divisible by q.

Theorem 1.7.11. *Let $\delta > 0$ and let $2 \leqslant (\log x)^\delta \leqslant z \leqslant x$, with $z = o(x)$. Let χ be a real primitive character with conductor q, where q is coprime to 6. Let $f : \mathbb{N} \to \{-1,1\}$ be a multiplicative function with $\mathbb{D}(f,\chi;\infty) < \infty$, and $\boldsymbol{\epsilon} \in \{-1,1\}^4$. Then for all but $o(z)$ integers $d \leqslant z$ not divisible by q, we have*

$$|\{n \leqslant x : f(n+jd) = \epsilon_j \text{ for each } 0 \leqslant j \leqslant 3\}| = \left(\frac{1}{16} + o(1)\right) x. \tag{1.7.6}$$

In particular, if $q \geqslant 5$ and coprime to 6 then a positive proportion of the length 4 arithmetic progressions in $[1,z] \times [1,x]$ exhibit the sign pattern $\boldsymbol{\epsilon}$.

We note that the restriction $(q,6) = 1$ is merely technical, and could be removed with more effort. With additional effort we could also quantify the size of the exceptional set in Theorem 1.7.11; we have chosen not to do this in order to avoid making the computations even more tedious.

On the other hand, when the shifts d are divisible by q, the behaviour is much more erratic. In fact, for such arithmetic progressions there can be a bias, as is evident from the following theorems. To state them, we require additional notation.

Given $r \in \mathbb{N}$, set $[r] := \{0, \dots ,r\}$. For $S \subseteq [r]$, we write

$$\boldsymbol{L}_S := \{(n,d) \mapsto n + jd : j \in S\},$$

and for each pair of sets $S,T \subseteq [r]$ we associate the system of forms

$$\boldsymbol{L}_{S,T} := \{(n,n',d) \mapsto n + jd : j \in S\} \cup \{(n,n',d) \mapsto n' + j'd : j' \in T\}.$$

For each $\lambda \in \mathbb{Z}$ and $p|q$, define $\mathbb{E}_\lambda/\mathbb{F}_p$ to be the elliptic curve over $\mathbb{F}_p$ with Legendre model

$$E_\lambda : y^2 \equiv x(x-1)(x-\lambda) \ (p).$$

Let b denote a reduced element of the residue class inverse to 2 modulo q, and set $\Delta_p := p + 1 - \#E_{3b^2}(\mathbb{F}_p)$. Finally, put

$$A_{\boldsymbol{\epsilon}}(f;q) := \epsilon_0\epsilon_1\epsilon_2\epsilon_3 \prod_{p|q} \frac{\mu(p)\Delta_p}{p+1} \prod_{p\nmid q} M_p(f\chi\mathbf{1}_4,\boldsymbol{L}_{[3]}). \tag{1.7.7}$$

Theorem 1.7.12. *Let $\delta > 0$ and let $2 \leqslant (\log x)^\delta \leqslant z \leqslant x$, and $z = o(x)$. Let χ be a real primitive character with modulus q, with q coprime to 6. Let $f : \mathbb{N} \to \{-1,1\}$ be a multiplicative function with $\mathbb{D}(f,\chi;\infty) < \infty$. For any $\boldsymbol{\epsilon} \in \{-1,1\}^4$,*

$$\frac{1}{xz} \sum_{d \leqslant z} |\{n \leqslant x : f(n+jd) = \epsilon_j \forall j\}| = \frac{1}{16}\left(1 + A_{\boldsymbol{\epsilon}}(f;q)\right) + o(1). \tag{1.7.8}$$

Remark 1.7.13. Put $\lambda = 3b^2$, where b is as above. The role that the elliptic curve E_λ plays in this problem stems from the complete character sum yielded by the character local factor

in Corollary 1.7.1, taking account of the compatibility conditions imposed on its summands. Note that q is necessarily squarefree, being the odd conductor of a real character. By the Chinese Remainder Theorem, the complete sum over $\mathbb{Z}/q\mathbb{Z}$ splits as a product of complete sums of Legendre symbols of cubic polynomials over $\mathbb{Z}/p\mathbb{Z}$, which is then easily related to point counts for elliptic curves over $\mathbb{F}_p$.
The quantity Δ_p, which is the trace of the Frobenius element of E_λ over $\mathbb{F}_p$, is non-zero if, and only if, the curve E_λ is not supersingular over $\mathbb{F}_p$ (see Exercise V.5.10 of [Sil08]). Since the set of primes at which an elliptic curve is supersingular is typically small (e.g., for non-CM elliptic curves, see Theorem V.4.7) we expect that if E_λ is generic with respect to each of the primes dividing q then the product in (1.7.8) is non-vanishing, and a bias exists according to the sign of $\epsilon_0\epsilon_1\epsilon_2\epsilon_3$.
For concreteness, we may note that if q is composed solely of primes $p \equiv 1(4)$ (i.e., q is an odd sum of two squares) then the product is non-zero. Indeed, note that the points (1,0) and (λ,0) are both trivially 2-torsion on $E_\lambda(\mathbb{F}_p)$. As such, $E_\lambda(\mathbb{F}_p)$ contains a subgroup of order 4. Hence,

$$\Delta_p \equiv p+1\ (4) \equiv 2\ (4),$$

so that $|\Delta_p| \geqslant 2$ for all $p|q$.

We also compute the mean-squared deviation. For a discussion regarding the size of the deviation in (1.7.9), including an heuristic for why it should generally be $\Omega(1)$, see Remark 6.4.3 below.

Theorem 1.7.14. *With the hypotheses in Theorem 1.7.12,*

$$\frac{1}{z}\sum_{d\leqslant z}\left(x^{-1}|\{n\leqslant x : f(n+jd)=\epsilon_j \forall\ 0\leqslant j\leqslant 3\}| - \frac{1}{16}\left(1+A_{\boldsymbol{\epsilon}}(f;q)\right)\right)^2$$
$$= \frac{1}{256}\left((T_{4,4}-A_{\boldsymbol{\epsilon}}(f;q)^2) + 2\epsilon_0\epsilon_1\epsilon_2\epsilon_3\left(\sum_{0\leqslant i<j\leqslant 3}\epsilon_i\epsilon_j\right)T_{4,2} + \left(\sum_{0\leqslant i<j\leqslant 3}\epsilon_i\epsilon_j\right)^2 T_{2,2}\right) + o(1), \tag{1.7.9}$$

where we have set

$$T_{2,2} := \prod_{p\nmid q} M_p(f\chi\mathbf{1}_4, \boldsymbol{L}_{[1],[1]})\prod_{p|q}\frac{p}{p^2+p+1}$$
$$T_{4,2} := \prod_{p\nmid q} M_p(f\chi\mathbf{1}_6, \boldsymbol{L}_{[3],[1]})\prod_{p|q}\frac{(p-\Delta_p)(p+1)-\Delta_p}{p^2(p+1)}$$
$$T_{4,4} := A_{\boldsymbol{\epsilon}}(f;q)^2\prod_{p|q}\frac{(1+1/\Delta_p)^2+1/p+p/\Delta_p^2}{1+1/p(p+1)}.$$

Remark 1.7.15. When d is a multiple of q, the contribution to the sign given by χ on $(n,n+d,n+2d,n+3d)$ is completely determined by n, and since $f\chi$ is 1-pretentious this means that $f\chi$ *should* only change sign infrequently on 4-term arithmetic progressions with difference d. As such, we heuristically expect that certain sign patterns (depending on χ) occur more often than others among the vectors $(f(n),f(n+d),f(n+2d),f(n+3d))$, an intuition that is confirmed by Theorems 1.7.12 and 1.7.14.

Remark 1.7.16. It is worthwhile mentioning how the results of this paper relate to the results in [BE11]. In the latter paper, it is shown that, except for two explicit collections of multiplicative functions $f : \mathbb{N} \to \{-1,1\}$, any f takes on each length 4 sign pattern on infinitely many 4-term arithmetic progressions. The counterexamples are of one of the following two types:

(1) there is a prime p such for all $\nu \geqslant 1$, $f(q^\nu) = 1$ if $q \neq p$, while $f(p^\nu) = (-1)^\nu$;

(2) $f(n) = \chi_3(n)$ for all $(n,3) = 1$, where χ_3 is the primitive real character modulo 3.

In each of these two cases, certain sign patterns can never be exhibited on length 4 arithmetic progressions. For functions of the first type, for example, the sign patterns $(1,1,1,-1)$ and $(1,-1,-1,-1)$ only occur on finitely many length 4 arithmetic progressions.

Note that the functions of both of these types are necessarily pretentious. In the first case, they are pretentious to the trivial character, with conductor $q = 1$, while in the second they are pretentious to χ_3. This latter example is excluded from the above analysis, so consider instead the 1-pretentious examples.

In this case, all common differences d are divisible by the conductor. Hence, Theorem 1.7.11 does not apply to any d, and the irregularity of distribution in Theorem 1.7.14 is necessary (notice that the examples of sign patterns given above are both such that $\epsilon_0\epsilon_1\epsilon_2\epsilon_3 = -1$, as we should expect from Theorem 1.7.12). Conversely, if a function is pretentious to a real character with conductor strictly greater than 1 then it follows from Theorem 1.7.11 that for any given sign pattern we can find infinitely many arithmetic progressions giving an instance of this sign pattern. Thus, the theorems of this section are consistent with, and quantitatively refine, the results of Buttkewitz and Elsholtz.

Chapter 2

MEAN VALUES OF MULTIPLICATIVE FUNCTIONS OVER THE FUNCTION FIELDS

2.1. Preparatory lemmas and "pretentiousness" over $\mathbb{F}_q[x]$

We start by introducing some basic definitions of "pretentiousness" over the function field $\mathbb{F}_q[x]$. For any two multiplicative functions $f,g : \mathcal{M} \to \mathbb{U}$ we define the "distance" to be

$$\mathbb{D}^2(f,g;m,n) = \sum_{\substack{m \le \deg P \le n, \\ P \text{ irreducible}}} \frac{1 - \mathrm{Re}(f(P)\overline{g(P)})}{q^{\deg P}}$$

and $\mathbb{D}(f,g;n) := \mathbb{D}(f,g;1,n)$. The crucial feature is that $\mathbb{D}(f,g;n)$ satisfies the triangle inequality, which is in a number field case due to Granville and Soundararajan [GS07a]:

$$\mathbb{D}(g,f;n) + \mathbb{D}(f,h;n) \ge \mathbb{D}(g,h;n).$$

Here we present a short proof of this claim which also works in a number field setting.

Lemma 2.1.1. *Let $f,g,h : \mathcal{M} \to \mathbb{U}$ be multiplicative. Then*

$$\mathbb{D}(g,f;n) + \mathbb{D}(f,h;n) \ge \mathbb{D}(g,h;n). \tag{2.1.1}$$

Proof. We first note that for any complex vectors u,v,w such that $|u|,|v|,|w| \le 1$ we have

$$\sqrt{1 - \mathrm{Re}(u\overline{v})} + \sqrt{1 - \mathrm{Re}(v\overline{w})} \ge \sqrt{1 - \mathrm{Re}(w\overline{u})}. \tag{2.1.2}$$

Indeed, let $\triangle(x) = \sqrt{1-|x|^2}$. Then, since $2(1 - \mathrm{Re}(x\overline{y})) = |x-y|^2 + \triangle^2(x) + \triangle^2(y)$ the result follows by applying the triangle inequality to the vector addition in $\mathbb{R}^4$:

$$(u-v, \triangle(u),\triangle(v),0) + (v-w,0,-\triangle(v),\triangle(w)) = (u-w,\triangle(u),0,\triangle(w)).$$

Now (2.1.1) follows by applying triangle inequality in $\mathbb{R}^m$

$$\sqrt{\sum_{i\le n} |a_i|^2} + \sqrt{\sum_{i \le n} |b_i|^2} \ge \sqrt{\sum_{i\le n} |a_i + b_i|^2}$$

for $a_i^2 = \frac{1-\mathrm{Re}(f(F)\overline{g(F)})}{q^{\deg F}}$ and $b_i^2 = \frac{1-\mathrm{Re}(f(F)\overline{h(F)})}{q^{\deg F}}$ together with (2.1.2). □

We begin by proving the analog of the aforementioned Delange's result [Del67] over the function field $\mathbb{F}_q[x]$. Our starting point is the following analog of the Turán-Kubilius inequality for function the field $\mathbb{F}_q[x]$ established in [Zha96]. Let

$$\mu_h = \sum_{\deg P^k \leq n} \frac{h(P^k)}{q^{k \deg P}} \left(1 - \frac{1}{q^{\deg P}}\right)$$

and

$$\mathbb{D}^*(f,g;n)^2 = \sum_{\substack{k \deg P \leq n, \\ P \text{ irreducible}}} \frac{1 - \mathrm{Re}(f(P^k)\overline{g(P^k)})}{q^{k \deg P}}.$$

Lemma 2.1.2. *If $h(F) : \mathcal{M} \to \mathbb{C}$ is an additive function, then*

$$\sum_{\deg F \leq n} |h(F) - \mu_h|^2 \ll q^n \sum_{\substack{\deg P^k \leq n, \\ P \text{ irreducible}}} \frac{|h(P^k)|^2}{q^{k \deg P}}.$$

In the next Lemma we establish the concentration inequality for the values of multiplicative function $f : \mathcal{M} \to \mathbb{U}$.

Lemma 2.1.3. *Let $f : \mathcal{M} \to \mathbb{U}$ be a multiplicative function. Then*

$$\sum_{\deg F \leq n} |f(F) - \mathcal{P}(f,n)|^2 \ll q^n \mathbb{D}^*(1,f,n)^2 + \frac{q^n}{n^2}.$$

Proof. By repeatedly applying the triangle inequality we have that for all $|z_i|,|w_i| \leq 1$

$$\left| \prod_{1 \leq i \leq n} z_i - \prod_{1 \leq i \leq n} w_i \right| \leq \sum_{1 \leq i \leq n} |z_i - w_i|.$$

Since $e^{z-1} = z + O(|z-1|^2)$ for $|z| \leq 1$ we conclude

$$f(Q) = \prod_{P^k || Q} f(P^k) = \prod_{P^k || Q} e^{f(P^k)-1} + O\left(\sum_{P^k || Q} |f(P^k) - 1|^2 \right).$$

We now introduce an additive function h, such that $h(P^k) = f(P^k) - 1$. Clearly,

$$\sum_{\deg Q \leq n} |f(Q) - e^{h(Q)}|^2 \ll \sum_{\deg Q \leq n} |f(Q) - e^{h(Q)}| \ll \sum_{\deg Q \leq n} \sum_{P^k || Q} |f(P^k) - 1|^2$$
$$\ll q^n \sum_{\deg P^k \leq n} \frac{|f(P^k) - 1|^2}{q^{k \deg P}} \ll q^n \mathbb{D}^*(f,1,n)^2.$$

Note, that $|e^a - e^b| \leq |a - b|$ for $\mathrm{Re}(a),\mathrm{Re}(b) \leq 0$. Moreover, Lemma 2.1.2 implies that

$$\sum_{\deg Q \leq n} |e^{h(Q)} - e^{\mu_h}|^2 \ll \sum_{\deg Q \leq n} |h(Q) - \mu_h|^2 \ll q^n \mathbb{D}^*(f,1,n)^2.$$

For a fixed monic irreducible polynomial $P \in \mathbb{F}_q[x]$, we introduce $\mu_h = \sum_{\deg P \leq n} \mu_{h,P}$ where

$$\mu_{h,P} = \sum_{\deg P^k \leq n} \frac{h(P^k)}{q^{k \deg P}} \left(1 - \frac{1}{q^{\deg P}}\right)$$

and observe

$$e^{\mu_{h,P}} = 1 + \mu_{h,P} + O(\mu_{h,P}^2) = \left(1 - \frac{1}{q^{\deg P}}\right) \sum_{k \deg P \leq n} \frac{f(P^k)}{q^{k \deg P}} + O\left(\frac{1}{q^n} + \frac{1}{q^{\deg P}} \sum_{k \deg P \leq n} \frac{h(P^k)}{q^{k \deg P}}\right).$$

Using Cauchy-Schwarz and the prime number theorem over $\mathbb{F}_q[x]$ yields

$$|e^{\mu_h} - \mathcal{P}(f,n)|^2 \ll \left(\sum_{k \deg P \leq n} \frac{1}{q^{\deg P}} \frac{|f(P^k) - 1|}{q^{k \deg P}} + \frac{1}{q^n}\right)^2 \ll \mathbb{D}^*(f,1,n)^2 + \frac{1}{n^2}.$$

Combining all of the above together with the triangle inequality completes the proof of the lemma. □

The following lemma allows to conveniently decompose the averages with a good error term when the function f is 1-pretentious.

Lemma 2.1.4. *Let $f : \mathcal{M} \to \mathbb{U}$ be a multiplicative function and $g : \mathcal{M} \to \mathbb{U}$ be any function. Then*

$$\sum_{\deg F \leq n} f(F)g(F) = \mathcal{P}(f,n) \sum_{\deg F \leq n} g(F) + O\left(q^n \mathbb{D}^*(1,f,n) + \frac{q^n}{n}\right).$$

Proof. Using Lemma 2.1.3, the triangle inequality and the Cauchy-Schwartz inequality we obtain

$$\begin{aligned}
\sum_{\deg F \leq n} f(F)g(F) - \mathcal{P}(f,n) \sum_{\deg F \leq n} g(F) &\ll \sum_{\deg F \leq n} |f(F)) - \mathcal{P}(f,n)| \\
&\ll \left(q^n \sum_{\deg F \leq n} |f(F) - \mathcal{P}(f,n)|^2\right)^{\frac{1}{2}} \\
&\ll q^n \mathbb{D}^*(1,f,n) + \frac{q^n}{n}.
\end{aligned}$$

□

2.2. Proof of Theorem 1.4.1.

As was pointed out in the introduction the role of multiplicative characters of $\mathcal{M}$ is played by the functions $h_\theta(Q) = e(\theta \deg Q)$ for $\theta \in [0,1)$.

Theorem 1.4.1. *For a given multiplicative function $f : \mathcal{M} \to \mathbb{U}$ one of the following holds:*

- *If $\mathbb{D}(f(P), e(\theta \deg P); \infty) = \infty$ for all $\theta \in [0,1)$, then*

$$\lim_{n \to \infty} \frac{1}{q^n} \sum_{F \in \mathcal{M}_n} f(F) = 0.$$

- *There exists $\theta_0 \in [0,1)$ such that $\mathbb{D}(f(P),e(\theta_0 \deg P);\infty) < \infty$. For any given $\varepsilon > 0$, let $m = \lceil (1-\varepsilon)\frac{\log n}{\log q} \rceil$. Then*

$$\frac{1}{q^n}\sum_{F\in\mathcal{M}_n} f(F) = e(n\theta)\cdot \mathcal{P}(f(P)e(-\theta_0 \deg P),n) + O_\varepsilon\left(\mathbb{D}(f(P),e(\theta_0 \deg P);m,n) + \frac{1}{n^{1-\varepsilon}}\right).$$

Proof. Consider the function

$$F\left(\frac{z}{q}\right) = \exp\left(\sum_{k\geq 1}\frac{\chi(k)z^k}{k}\right).$$

Since $|\chi(k)| \leq 1$, the function F is analytic in the open unit disc $|z| < 1$. Applying the main result of [GHS15], we have that for any multiplicative $f \in \mathcal{C}(1)$

$$|\sigma(n)| \leq 2(M+1)e^{-M} \tag{2.2.1}$$

where $\max_{|z|=\frac{1}{q}} |F^{\perp}(z)| := e^{-M}(2n)$ and

$$F^{\perp}\left(\frac{z}{q}\right) = \exp\left(\sum_{k=1}^{n}\frac{\chi(k)z^k}{k}\right).$$

We note that for any multiplicative $f : \mathcal{M} \to \mathbb{U}$,

$$\sum_{\substack{\deg P\leq n,\\ P \text{ irreducible}}} \frac{1-\mathrm{Re}(f(P))}{q^{\deg P}} = \sum_{k\leq n}\frac{1-\mathrm{Re}(\chi(k))}{k} + O(1)$$

since prime powers P^k with $k \geq 2$ and $k \deg P \leq n$ give $O(1)$ contribution. Consequently,

$$\begin{aligned}\max_{|z|=\frac{1}{q}} |F^{\perp}(z)| &= \exp\left(\max_{\theta\in[0,1)}\sum_{k=1}^{n}\frac{\mathrm{Re}(\chi(k)e(k\theta))}{k}\right)\\ &= e^{O(1)}n\cdot\exp\left(-\min_{\theta\in[0,1)}\sum_{k=1}^{n}\frac{1-\mathrm{Re}(\chi(k)e(k\theta))}{k}\right)\\ &= e^{O(1)}n\cdot\exp\left(-\min_{\theta\in[0,1)}\mathbb{D}^2(f(P),e(\theta\deg P),n)\right).\end{aligned}$$

If $\sigma(n) \neq o_{n\to\infty}(1)$, then there exists an increasing sequence $\{n_k\}_{k\geq 1}$ such that $\sigma(n_k) \gg 1$. Therefore, (2.2.1) implies that there exists a corresponding sequence $\{\theta_k\}_{k\geq 1}$ such that

$$\mathbb{D}^2(f(P),e(\theta_k \deg P),n_k) = O(1)$$

uniformly for all $k \geq 1$. Since $\{\theta_k\}_{k\geq 1}$ is bounded, passing to a subsequence, if necessary, we may assume that $\theta_k \to \theta_0$ whenever $k \to \infty$. For a fixed $n \geq 1$, by monotonicity uniformly for all $n_k \geq n$, we have

$$\mathbb{D}(f(P),e(\theta_k \deg P),n) \leq \mathbb{D}(f(P),e(\theta_k \deg P),n_k) = O(1).$$

Taking first limit $k \to \infty$ and then $n \to \infty$ yields $\mathbb{D}(f(P),e(\theta_0 \deg P),\infty) = O(1)$. We rescale $f(P) \to f(P)e(-\theta_0 \deg P)$ to assume that $\theta_0 = 0$. In this case we have that $\mathbb{D}(1,f,\infty) < \infty$. Select $m = \lceil (1-\varepsilon)\frac{\log n}{\log q} \rceil$, and decompose $f(Q) = f_s(Q)f_l(Q)$ where

$$f_s(P^k) = \begin{cases} f(P^k), & \text{if } k \deg P \leq m; \\ 1, & \text{if } k \deg P > m, \end{cases}$$

and

$$f_l(P^k) = \begin{cases} 1, & \text{if } k \deg P \leq m; \\ f(P^k), & \text{if } k \deg P > m. \end{cases}$$

Lemma 2.1.4 gives

$$\sum_{\deg Q \leq n} f(Q) = \sum_{\deg Q \leq n} f_s(Q) f_{\mathfrak{l}}(Q) = \mathcal{P}(f_l,n) \sum_{\deg Q \leq n} f_s(Q) + O\left(q^n \mathbb{D}^*(1,f_l,n) + \frac{q^n}{n}\right).$$

Let $f_s = 1 * g_s$, $f_{\mathfrak{l}} = 1 * g_{\mathfrak{l}}$. Note, $g_s(P^k) = f_s(P^k) - f_s(P^{k-1}) = 0$ provided $(k-1)\deg P > m$. Moreover, by the prime polynomial theorem over $\mathbb{F}_q[x]$

$$\sum_{\substack{k \deg P \leq m, \\ P \ irreducible}} k \deg P = (1 + o_{m\to\infty}(1)) \sum_{\deg F \leq m} \Lambda(F) = (1 + o_{m\to\infty}(1)) \frac{q^{m+1} - 1}{q - 1} \leq n$$

and therefore the following sums are supported on the polynomials of $\deg Q \leq n$:

$$\sum_{\deg Q \leq n} f_s(Q) = q^n \sum_{\deg Q \leq n} \frac{g_s(Q)}{q^{\deg Q}} = q^n \prod_{(k-1)\deg P \leq m} \left(\sum_{k \geq 0} \frac{g_s(P^k)}{q^{k \deg P}} \right) = q^n \mathcal{P}(f_s,n).$$

We now turn to the estimation of the Euler products. In what follows we assume that all products are taken over irreducible polynomials $P \in \mathcal{M}$. Since $g_{\mathfrak{l}}(P^k) = 0$ whenever $k \deg P \leq m$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{g_s(P^k)}{q^{k \deg P}} \sum_{k=1}^{\infty} \frac{g_{\mathfrak{l}}(P^k)}{q^{k \deg P}} &= \sum_{(k-1)\deg P \leq m} \frac{g_s(P^k)}{q^{k \deg P}} \sum_{k \deg P > m} \frac{g_{\mathfrak{l}}(P^k)}{q^{k \deg P}} \\ &= O\left(\frac{1}{q^{\deg P}} \sum_{k \deg P \geq m} \frac{1}{q^{k \deg P}} \right) = O\left(\frac{1}{q^{m + \deg P}} \right). \end{aligned}$$

By the prime polynomial theorem over $\mathbb{F}_q[x]$, we have $|\mathcal{P}_k| = \frac{q^k}{k} + O\left(\frac{q^{k/2}}{k}\right)$ and so

$$\begin{aligned} \mathcal{P}(f_s,n)\mathcal{P}(f_{\mathfrak{l}},n) &= \prod_{\deg P \leq n} \left(1 + \sum_{k=1}^{\infty} \frac{g(P^k)}{q^{k \deg P}} + \sum_{k=1}^{\infty} \frac{g_s(P^k)}{q^{k \deg P}} \sum_{k=1}^{\infty} \frac{g_{\mathfrak{l}}(P^k)}{q^{k \deg P}} \right) \\ &= \prod_{\deg P \leq n} \left(1 + \sum_{k=1}^{\infty} \frac{g(P^k)}{q^{k \deg P}} + O\left(\frac{1}{q^{m + \deg P}} \right) \right) \\ &= \mathcal{P}(f,n) + O\left(\sum_{1 \leq k \leq n} \frac{1}{q^{k+m}} \cdot \frac{q^k}{k} \right) = \mathcal{P}(f,n) + O\left(\frac{\log n}{q^m} \right). \end{aligned}$$

Combining all of the above, we arrive at the result claimed. $\square$

2.3. Proof of Theorem 1.4.2

We now prove the analog of Wirsing's theorem over the function field $\mathbb{F}_q[x]$ in a more general setting. Instead of requiring $-1 \leq f(P) \leq 1$ for all $P \in \mathcal{M}$, we shall only require the average over primes to be bounded. For the rest of the paper, we confine ourselves with the functions f that belong to the class $\mathcal{C}(\kappa)$ for $\kappa \geq 1$. We will require the following "Halász type" estimate established in [GHS15].

Lemma 2.3.1. *Let* $f \in \mathcal{C}(\kappa)$. *Then*

$$|\sigma(n)| \leq 2\kappa(\kappa + M + 1)e^{-M}(2n)^{\kappa-1}$$

where $\max_{|z|=\frac{1}{q}} |F^{\perp}(z)| := e^{-M}(2n^{\kappa})$ *and*

$$F^{\perp}\left(\frac{z}{q}\right) = \exp\left(\sum_{k=1}^{n} \frac{\chi(k)z^k}{k}\right).$$

Our first step is to get an asymptotic formula for the magnitude of the corresponding mean value.

Lemma 2.3.2. *Let* $\kappa \geq 1$. *For any multiplicative function* $f \in \mathcal{C}(\kappa)$, *there exists constant* c_f *such that* $|\sigma(n)| = c_f n^{\kappa-1} + o_{n\to\infty}(n^{\kappa-1})$.

Proof. Using the same arguments as the beginning of the proof of Theorem 1.4.1 with Lemma 2.3.1 in place of (2.2.1), we have that $\sigma(n) = o_{n\to\infty}(1)$ unless there exists $\theta \in [0,1)$, such that

$$\sum_{k\geq 1} \frac{1 - \mathrm{Re}(\chi(k)e(-k\theta))}{k} = O(1).$$

Changing $\chi(k) \to \chi(k)e(-k\theta)$ we may assume that $\theta = 0$. Since $|\chi(n)| \leq \kappa$, we have

$$(n+1)S_{n+1} - nS_n = |\sigma(n)| \leq \frac{\sum_{k=0}^{n-1} |\chi(n-k)||\sigma(k)|}{n} \leq \kappa\frac{\sum_{i=0}^{n-1} |\sigma(i)|}{n} = \kappa S_n$$

and therefore $(n+1)S_{n+1} \leq (n+\kappa)S_n$. From an obvious inequality $(1+\kappa/n) \leq (1+1/n)^{\kappa}$, we infer

$$\frac{S_{n+1}}{(n+1)^{\kappa-1}} \leq \frac{n+\kappa}{n+1} \cdot \frac{S_n}{n^{\kappa-1}} = \frac{\left(1+\frac{\kappa}{n}\right)}{\left(1+\frac{1}{n}\right)^{\kappa}} \cdot \frac{S_n}{n^{\kappa-1}} \leq \frac{S_n}{n^{\kappa-1}}.$$

Hence, the sequence $S_n/n^{\kappa-1}$ is decreasing and therefore converges. This implies that $S_n = \beta_f n^{\kappa-1} + o_{n\to\infty}(n^{\kappa-1})$. Fix $n \geq 1$ and $1 \leq m \leq n$. Lemma 2.3.1 implies that for all ł ≥ 1, $|\sigma(ł)| \ll ł^{\kappa-1}$ and therefore by the triangle inequality

$$|\sigma(n+m) - \sigma(n)|$$

$$= \left|\frac{\sum_{k=0}^{m+n-1} \sigma(k)\chi(m+n-k)}{n+m} - \frac{\sum_{k=0}^{n-1} \sigma(k)\chi(n-k)}{n}\right|$$

$$\leq \left| \frac{\sum_{k=0}^{m+n-1} \sigma(k)\chi(m+n-k)}{n} - \frac{\sum_{k=0}^{n-1} \sigma(k)\chi(n-k)}{n} \right| + \left| \sum_{k=0}^{m+n-1} |\sigma(k)\chi(n-k)| \left(\frac{1}{n} - \frac{1}{n+m} \right) \right|$$

$$\ll \left| \frac{\sum_{k=0}^{n-1} \sigma(k)(\chi(m+n-k) - \chi(n-k))}{n} \right| + \sum_{k=n}^{n+m-1} \frac{|\sigma(k)\chi(n+m-k)|}{n} + \frac{m(m+m)}{n(n+m)}(m+n)^{\kappa-1}$$

$$\leq n^{\kappa-1} \cdot \frac{\sum_{k=0}^{n-1} |(\chi(m+n-k) - \chi(n-k)|}{n} + \frac{m}{n} \cdot n^{\kappa-1}$$

$$\ll n^{\kappa-1} \sum_{k=0}^{n-1} \frac{|\kappa - \chi(m+n-k)|}{n} + n^{\kappa-1} \sum_{k=0}^{n-1} \frac{|\kappa - \chi(n-k)|}{n} + \frac{m}{n} \cdot n^{\kappa-1}.$$

We now truncate the last sum at $\log n$ at a cost of an acceptable error and use the Cauchy-Schwartz inequality together with the obvious bound $|\kappa - \chi(k)|^2 \leq \kappa(\kappa - \Re(\chi(k)))$ to arrive at

$$|\sigma(n+m) - \sigma(n)| \ll n^{\kappa-1} \frac{\sum_{\log n \leq k \leq n+m-1} |\kappa - \chi(k)|}{n} + \frac{\log n}{n} \cdot n^{\kappa-1} + \frac{m}{n} \cdot n^{\kappa-1}$$

$$\ll n^{\kappa-1} \left(\sum_{k=\log n}^{n+m-1} \frac{|\kappa - \chi(k)|^2}{k} \sum_{k=\log n}^{n+m-1} \frac{k}{n^2} \right)^{\frac{1}{2}} + \left(\frac{m}{n} + \frac{\log n}{n} \right) n^{\kappa-1}$$

$$\ll n^{\kappa-1} \left(\sum_{k=\log n}^{n+m-1} \frac{\kappa - \Re(\chi(k))}{k} \right)^{\frac{1}{2}} + \left(\frac{m}{n} + \frac{\log n}{n} \right) n^{\kappa-1}.$$

Consequently, for any $1 \leq m \leq n$, we have

$$|\sigma(m+n)| = |\sigma(n)| + O\left(n^{\kappa-1} \left(\sum_{k=\log n}^{n+m-1} \frac{\kappa - \Re(\chi(k))}{k} \right)^{\frac{1}{2}} + \left(\frac{m}{n} + \frac{\log n}{n} \right) n^{\kappa-1} \right)$$

We can now estimate

$$S_{n+m} = \frac{\sum_{k=0}^{n+m-1} |\sigma(k)|}{n+m} = \frac{\sum_{k=0}^{n-1} |\sigma(k)|}{n+m} + O\left(\frac{\log n}{n+m} \cdot n^{\kappa-1} \right)$$

$$+ \frac{1}{n+m} \sum_{k=n+\log n}^{n+m-1} \left(|\sigma(n)| + O\left(n^{\kappa-1} \left(\sum_{k=\log n}^{n+m-1} \frac{\kappa - \Re(\chi(k))}{k} \right)^{\frac{1}{2}} + \frac{m}{n} \cdot n^{\kappa-1} \right) \right)$$

$$= \frac{n}{n+m} S_n + \frac{m}{n+m} |\sigma(n)| + O\left(n^{\kappa-1} \left(\sum_{k=\log n}^{n+m-1} \frac{\kappa - \Re(\chi(k))}{k} \right)^{\frac{1}{2}} + \frac{m}{n} \cdot n^{\kappa-1} \right).$$

Select $m = \alpha n$ and note that

$$\sum_{k=\log n}^{n+m-1} \frac{\kappa - \Re(\chi(k))}{k} = o_{n\to\infty}(1).$$

This, together with the asymptotic for S_n implies that

$$\beta_f n^{\kappa-1}(1+\alpha)^{\kappa-1} + o(n^{\kappa-1}) = \frac{1}{1+\alpha} \beta_f n^{\kappa-1} + \frac{\alpha}{1+\alpha} n^{\kappa-1} + O(\alpha n^{\kappa-1}).$$

Multiplying both sides by $1+\alpha \leq 2$ and rewriting the last asymptotic, yields

$$\beta_f((1+\alpha)^\kappa - 1)n^{\kappa-1} = \alpha|\sigma(n)| + o(\alpha n^{\kappa-1}).$$

Using Taylor expansion $(1+\alpha)^\kappa = 1+\kappa\alpha+o_{\alpha\to 0}(\alpha)$ and taking $\alpha = 1/\log n$ and substituting this into the last asymptotic gives

$$\kappa b_f n^{\kappa-1} = |\sigma(n)| + o(n^{\kappa-1}),$$

which proves the claim. □

Equipped with the last lemma we proceed to proving Wirsing's type result over $F_q[x]$.

Theorem 1.4.2. *Let $\kappa > 1$. For every real valued multiplicative function $f \in \mathcal{C}(\kappa)$, either $\frac{f(P)}{(\deg P)^{\kappa-1}}$ or $\frac{(-1)^{\deg P} f(P)}{(\deg P)^{\kappa-1}}$ has a mean value.*

Proof. Lemma 2.3.1 together with compactness arguments as in the proof of Theorem 1.4.1 yield $\sigma(n) = o_{n\to\infty}(n^{\kappa-1})$ unless there exists $\theta_0 \in [0,1)$ such that

$$\sum_{k=1}^{\infty} \frac{\kappa - \mathrm{Re}(\chi(k)e(k\theta_0))}{k} < \infty.$$

Since $\{k\theta_0\}_{k\geq 1}$ is either uniformly distributed or periodic mod1, we have

$$\sum_{k=1}^{n} \frac{\kappa - \chi(k)\cos(2k\pi\theta_0)}{k} \geq \kappa \sum_{k=1}^{n} \frac{1 - |\cos(2k\pi\theta_0)|}{k} \to \infty$$

unless $\theta_0 = 0$ or $\theta_0 = \frac{1}{2}$. Changing $f(P) \to (-1)^{\deg P} f(P)$ if necessary we may assume that $\theta_0 = 0$. By the previous Lemma $|\sigma(n)| = c_f n^{\kappa-1} + o(n^{\kappa-1})$. If $c_f = 0$, then the result follows. If $c_f > 0$, we have

$$\begin{aligned} |\sigma(n+1) - \sigma(n)|^2 &= 2c_f^2 n^{2\kappa-2} + o(n^{2\kappa-2}) - 2\sigma(n+1)\sigma(n) \\ &= 2c_f^2(1 - \mathrm{sign}(\sigma(n)\sigma(n+1)))n^{2\kappa-2} + o(n^{2\kappa-2}) \end{aligned}$$

Since the sequence $\{\mathrm{sign}(\sigma(n)\sigma(n+1))\}_{n\geq 1}$ is discrete it must stabilize. Hence, $\mathrm{sign}(\sigma(n)$ is constant for all sufficiently large $n \in \mathbb{N}$. This concludes the proof.

□

Remark 2.3.3. The difference between the behaviour of mean values of *real* valued multiplicative functions in the number field and the function field settings can now easily be explained using the notion of "pretentiousness". More precisely, in contrast with the number field case, where the only real character n^{it} is the function $h(n) = 1$ corresponding to the value $t = 0$, in the function field setting there exists an extra real character $h_{\frac{1}{2}}(F) = (-1)^{\deg F}$. The behaviour of each real multiplicative $f \in \mathcal{C}(1)$ is then modelled by the "closest" real $h_\theta(F)$ for $\theta = \{0, \frac{1}{2}\}$.

2.4. Hall type theorem over $\mathbb{F}_q[x]$. Preparatory lemmas and the proof of Proposition 1.5.2

This section is devoted to the preparation of the proof of Theorem 1.5.1. As was mentioned in the introduction, a direct analog of Hall's result does not hold in the function field setting due to the existence of another oscillating real character $h_{\frac{1}{2}}(P) = (-1)^{\deg P}$. One way to think about the conclusion of Theorem 1.5.1 is given a multiplicative function $f \in \mathcal{C}(1)$, Hall type result holds for either function $f(F)$ or the twist $(-1)^{\deg F} f(F)$. More precisely, if f is not "pretentious", then both mean values of $f(F)$ and $(-1)^{\deg F} f(F)$ are bounded away from -1 and the explicit bound is supplied by Corollary 1.5.2. Alternatively, f is "close" to one of the functions 1 or $(-1)^{\deg F}$ and the result holds for the appropriate twist. We now collect some technical lemmas required for the proof of the main result.

Lemma 2.4.1. *Suppose $n \geq 2$ and $\alpha \in (0,1)$ are given. Let $R := \lceil \log n \rceil$ and select $m \leq 2R$ such that $|\alpha - b/m| \leq 1/(2mR)$ for some $(b,m) = 1$. Then*

$$\sum_{k \leq n} \frac{1 - |\cos(k\alpha\pi)|}{k} = \left(1 - \frac{2}{\pi}\right) \log n + \left(\frac{2}{\pi} - c_m\right) \log\left(\min\{n, 1/||m\alpha||\}\right) + O(\log em)$$

where

$$c_m := \frac{1}{m} \sum_{a=0}^{m-1} |\cos(\pi a/m)| = \begin{cases} \frac{\csc(\pi/2m)}{m}, & \text{if } m \text{ is odd,} \\ \frac{\cot(\pi/2m)}{m}, & \text{if } m \text{ is even.} \end{cases}$$

Proof. This essentially follows from Lemma 5.2 of [GHS15]. We give the proof for completeness. Using Fourier series expansion for $|\cos(\pi\alpha)|$, a simple computation shows that

$$\sum_{k=1}^{n-1} \frac{|\cos(\pi k\alpha)|}{k} = \frac{2}{\pi} \left(\sum_{k=1}^{n-1} \frac{1}{k} - 2 \sum_{r \geq 1} \frac{(-1)^r}{4r^2 - 1} \sum_{k=1}^{n-1} \frac{\cos(2\pi k r\alpha)}{k} \right).$$

Clearly,

$$\sum_{r \geq \log n} \frac{(-1)^r}{4r^2 - 1} \sum_{k=1}^{n-1} \frac{\cos(2\pi k r\alpha)}{k} \ll \sum_{r \geq \log n} \frac{\log n}{r^2} \ll 1.$$

We note that if $r\alpha$ is not an integer, then

$$S(x) := \sum_{k \leq x} \cos(2\pi k r\alpha) \ll \frac{1}{||r\alpha||},$$

and therefore integration by parts yields

$$\sum_{1/||r\alpha|| \leq k \leq M} \frac{\cos(2\pi k r\alpha)}{k} = \int_{1/||r\alpha||}^{M} \frac{dS(t)}{t} = \frac{S(M)}{M} - \frac{S(1/||r\alpha||)}{1/||r\alpha||} + \int_{1/||r\alpha||}^{M} \frac{S(t)dt}{t^2} \ll 1.$$

Consequently

$$\sum_{k=1}^{n-1} \frac{\cos(2\pi k r\alpha)}{k} = \sum_{k=1}^{\min\{1/||r\alpha||, n\}} \frac{\cos(2\pi k r\alpha)}{k} + O(1) = \sum_{k=1}^{\min\{1/||r\alpha||, n\}} \frac{1}{k}$$

$$- 2 \sum_{k=1}^{\min\{1/||r\alpha||,n\}} \frac{\sin^2(\pi k r\alpha)}{k} + O(1) = \log\left(\min\{n,1/||m\alpha||\}\right) + O(1).$$

For each $r \le R$ we have $|r\alpha - rb/m| \le 1/2m$. Therefore if $m \nmid r$ we have $||r\alpha|| \ge 1/2m$, and so $\log(\min\{1/||r\alpha||,n\}) = \log(1/||rb/m||) + O(1) \ll \log m$. Hence,

$$\sum_{\substack{r\ge 1\\ m\nmid r}} \frac{(-1)^r}{4r^2-1} \sum_{k=1}^{n-1} \frac{\cos(2\pi k r\alpha)}{k} \ll \sum_{\substack{1\le r\le \log n\\ m\nmid r}} \frac{(-1)^r}{4r^2-1} \sum_{k=1}^{n-1} \frac{\cos(2\pi k r\alpha)}{k} + O(1)$$

$$\ll \sum_{\substack{1\le r\le \log m\\ m\nmid r}} \frac{(-1)^r}{4r^2-1} \log \frac{1}{||rb/m||} + \sum_{\log m\le r\le \log n} \frac{1}{r^2\log m} \ll \log(em).$$

Moreover, for all $r \le \log n$ and $m \mid r$ we have $||r\alpha|| = (r/m)||m\alpha||$ and thus

$$\sum_{\substack{r\ge 1\\ m\mid r}} \frac{(-1)^r}{4r^2-1} \sum_{k=1}^{n-1} \frac{\cos(2\pi k r\alpha)}{k} = \sum_{\substack{1\le r\ge \log n\\ m\mid r}} \frac{(-1)^r}{4r^2-1} \log\min\left(n,\frac{1}{r/m||m\alpha||}\right) + O(1)$$

$$= \left(c_m - \frac{2}{\pi}\right) \log\left(\min\{n,1/||m\alpha||\}\right) + O(1).$$

Combining all of the above yields the result. □

Lemma 2.4.2. *For any real valued sequence $\{\chi(k)\}_{k\ge 1}$ such that $|\chi(k)| \le 1$ and for all $k \ge 1$ and any $\theta \in [0,1)$ we have*

$$\sum_{k\le n} \frac{1-\chi(k)\cos(2\pi k\theta)}{k} \ge \frac{1}{3}\log(n||2\theta||) + O(1),$$

and

$$\sum_{k\le n} \frac{1-\chi(k)\cos(2\pi k\theta)}{k} \ge \frac{1}{6}\min\left\{\sum_{k\le n}\frac{1-(-1)^k\chi(k)}{k}, \sum_{k\le n}\frac{1-\chi(k)}{k}\right\} - 2\log(n||2\theta||) + O(1).$$

Proof. It is easy to check that the subsequence $\{c_{2k}\}_{k\ge 1}$ is strictly increasing and $1/2 \le c_{2k} < 2/\pi$ and the subsequence $\{c_{2k+1}\}_{k\ge 1}$ is strictly decreasing with $2/\pi < c_{2k+1} \le 2/3$. If $m = 2k \ge 2$, then $2/\pi - c_m > 0$ and since $m \ll \log n$, Lemma 2.4.1 implies

$$\sum_{k\le n} \frac{1-\chi(k)\cos(2\pi k\theta)}{k} \ge \sum_{k\le n} \frac{1-|\cos(2\theta k\pi)|}{k} \ge \left(1-\frac{2}{\pi}\right)\log n + O(\log m) \ge \frac{1}{3}\log n + O(1).$$

If $m = 2k+1 > 2$, then $\log\left(\min\{n,1/||m\alpha||\}\right) \le \log n$, $\log m \ll \log\log n$ and Lemma 2.4.1 implies

$$\sum_{k\le n} \frac{1-\chi(k)\cos(2\pi k\theta)}{k} \ge \sum_{k\le n} \frac{1-|\cos(2\theta k\pi)|}{k}$$

$$\ge \frac{1}{3}\log n + \left(\frac{2}{3} - c_m\right)\log n + O(\log m) \ge \frac{1}{3}\log n + O(1)$$

$$\geq \frac{1}{6}\min\left\{\sum_{k\leq n}\frac{1-(-1)^k\chi(k)}{k},\sum_{k\leq n}\frac{1-\chi(k)}{k}\right\}+O(1),$$

and

$$\sum_{k\leq n}\frac{1-\chi(k)\cos(2\theta k\pi)}{k}\geq \frac{1}{3}\log(n||2\theta||)+O(1).$$

We now suppose that $m=1$ which implies that $b=0$ or $b=1$ and $c_1=1$. Note, that by Lemma 2.4.1

$$\begin{aligned}\sum_{k\leq n}\frac{1-\chi(k)\cos(2\pi k\theta)}{k}&\geq \sum_{k\leq n}\frac{1-|\cos(2\theta k\pi)|}{k}\\ &\geq \left(1-\frac{2}{\pi}\right)\log n+\left(\frac{2}{\pi}-1\right)\log\left(\frac{1}{||2\theta||}\right)+O(1)\\ &=\left(1-\frac{2}{\pi}\right)\log\left(n||2\theta||\right)+O(1).\end{aligned}$$

Replacing θ with $1/2-\theta$ and $\chi(k)$ with $(-1)^k\chi(k)$ if necessary, we may assume that $|\theta|\leq 1/4$. In this range $2\theta=||2\theta||$ and $|\sin(\pi\theta k)|\leq k\pi||\theta||$ and consequently

$$\begin{aligned}\left|\sum_{k\leq n}\frac{1-\chi(k)\cos(2\pi\theta k)}{k}-\sum_{k\leq n}\frac{1-\chi(k)}{k}\right|&=2\cdot\left|\sum_{k\leq n}\frac{\chi(k)\sin^2(\pi\theta k)}{k}\right|\\ &\leq 1+2\cdot\sum_{k\leq\frac{1}{||\theta||}}\frac{k^2\pi^2||\theta||^2}{k}+2\cdot\sum_{\frac{1}{||\theta||}\leq k\leq n}\frac{1}{k}\\ &=2\log(n||2\theta||)+O(1).\end{aligned}$$

Thus we arrive at the result. The other case $|\theta|>\frac{1}{4}$ is completely analogous. □

Combining Lemma 2.4.2 with (2.2.1) leads to the following corollary.

Proposition 1.5.2. *Let $f\in\mathcal{C}(1)$ be a real valued multiplicative function and define*

$$D_n=\min\left(\sum_{k\leq n}\frac{1-(-1)^k\chi(k)}{k},\sum_{k\leq n}\frac{1-\chi(k)}{k}\right).$$

Then,

$$|\sigma(n)|\ll(1+D_n)\exp\left(-\frac{1}{42}D_n\right).$$

In a number field setting, the result analogous to Corollary 1.5.2 has been established by Hall and Tenenbaum [HT91]. Namely, they proved that for any multiplicative $f:\mathbb{N}\to[-1,1]$

$$\frac{1}{x}\sum_{n\leq x}f(n)\ll\left(1+\sum_{p\leq x}\frac{1-f(p)}{p}\right)\exp\left(-C\sum_{p\leq x}\frac{1-f(p)}{p}\right)$$

where the sharp constant is given by $C=-\cos\beta$ and β is the solution of $\sin\beta-\beta\cos\beta=\frac{1}{2}\pi$.

2.5. Proof of Theorem 1.5.1

We now proceed to the proof of Theorem 1.5.1. We distinguish two cases depending on whether $\chi(1) \geq 0$ or $\chi(1) < 0$. The following proposition takes care of the positive case.

Proposition 2.5.1. *Let $f \in \mathcal{C}(1)$ be real valued multiplicative function and $\chi(1) \geq 0$. Then there exists an absolute constant $\delta > 0$ such that $\sigma(n) \geq -1+\delta$ for all $n \geq 1$.*

Proof. Let $\delta > 0$ be a small parameter to be determined later. Recall that for every $i \geq 1$, $|\chi(i)| \leq 1$ with the boundary conditions $\sigma(1) = \chi(1)$, $\sigma(0) = 1$, $\chi(0) = 0$ and

$$\sigma(n) = \frac{\sum_{i=0}^{n} \sigma(i)\chi(n-i)}{n} \qquad (n \geqslant 2). \tag{2.5.1}$$

Suppose that $\sigma(n_0) \leq -1+\delta$ for some $n_0 \geq 2$. We observe that

$$\sigma(2) = \frac{\chi(2) + \sigma(1)^2}{2}\cdot$$

If $\chi(2) \leqslant 0$, then $|\sigma(2)| \leqslant 1/2$ and for $n_0 \geq 3$ we have $|\sigma(n_0)| \leq S_{n_0} \leq S_3 \leq \frac{5}{6} < 1-\delta$ for sufficiently small δ. Hence, $\chi(2) > 0$ and consequently $\sigma(2) > 0$. Since $\sigma(1) = \chi(1) \geq 0$, we have $\chi(3) \geqslant 0$, $\sigma(3) \geq 0$, since otherwise $|\sigma(n_0)| \leq |S_4| \leqslant 11/12 < 1-\delta$. Now let

$$k = \min\{n : \ \sigma(n+1) < 0\}.$$

If such k does not exist, we are clearly done. By the definition of k, we have $\sigma(1), \sigma(2), \ldots \sigma(k) \geqslant 0$. Our first observation is that we cannot have too many negative $\chi(i)$'s for $i \leq k+1$. More precisely, suppose

$$m = \#\{n \leq k+1 : \ \chi(n) < 0\}.$$

Then for any $r \geq 0$, we have

$$m - r \leq \#\{n \leq k+1-r : \ \chi(n) < 0\} \leq m$$

and trivially for all $1 \leq ł \leq k+1$,

$$-\frac{1}{ł} \sum_{\substack{n \leq ł, \\ \chi(n) < 0}} 1 \leq \sigma(ł) \leq \frac{1}{ł} \sum_{\substack{n \leq ł, \\ \chi(n) > 0}} 1.$$

Consequently,

$$\frac{-m}{k+1} \leq \sigma(k+1) \leq 0.$$

Moreover, for all $1 \leq r \leq m$ we have

$$0 \leq \sigma(k+1-r) \leq \frac{k+1-r-(m-r)}{k+1-r} \leq 1 - \frac{m-r}{k}\cdot$$

By the triangle inequality we have

$$(1-\delta) \leq |\sigma(n_0)| \leq S_k = \frac{\sum_{0 \leq r < k} |\sigma(r)|}{k} \leq 1 - \sum_{r=1}^{m} \frac{m-r}{k^2} = 1 - \frac{m(m-1)}{2k^2},$$

which implies that $m \leq 2\sqrt{\delta}k + 1$.

By our assumption $\sigma(k+1) < 0$ and thus

$$\sum_{i=1}^{k} \sigma(i)\chi(k+1-i) < -\chi(k+1) \leq 1.$$

Since we have at most $m \leq 2\sqrt{\delta}k + 1$ negative $\chi(i)'$s, this yields the estimate

$$\sum_{\substack{i \leq k, \\ \chi(i)>0}} \chi(i)\sigma(k+1-i) \leq -\sum_{\substack{j \leq k, \\ \chi(j)<0}} \chi(j)\sigma(k+1-j) + 1 \leq 2k\sqrt{\delta} + 2. \tag{2.5.2}$$

By monotonicity $S_{k+1} \geq S_{n_0} \geq |\sigma(n_0)| \geq 1 - \delta$ and therefore

$$\frac{\sum_{m \leq k,\ \sigma(m) \geq 1/2} 1 + \sum_{m \leq k,\ \sigma(m) \leq 1/2} \frac{1}{2}}{k} \geq \frac{\sum_{i=1}^{k} \sigma(i)}{k} = \frac{(k+1)S_{k+1} - 1}{k} \geq 1 - \delta\left(1 + \frac{1}{k}\right),$$

which gives the estimate

$$\#\{n \leq k :\ \sigma(n) \geq 1/2\} \geq (1-2\delta)k - 2. \tag{2.5.3}$$

Let $(x)_+ = \max(x{,}0)$. We introduce $B(j) = \sum_{1 \leq i \leq j} (\chi(i))_+$ and claim that

$$B(k) = \sum_{1 \leq i \leq k} (\chi(i))_+ \geq \frac{1}{2}k. \tag{2.5.4}$$

Inserting for each $\sigma(ł)$ the expression (2.5.1) and rearranging yields

$$\sum_{i=1}^{k} \sigma(i) = \sum_{i=1}^{k}\sum_{j=0}^{i} \frac{\sigma(j)\chi(i-j)}{i} = \sum_{n=0}^{k} \sigma(n) \sum_{j=1}^{k-n} \frac{\chi(j)}{n+j} \geqslant k(1-\delta) - \delta,$$

and since $0 \leq \sigma(i) \leq 1$ for all $0 \leq i \leq k$, we have

$$\begin{aligned}
(1-\delta)k - \delta \leqslant \sum_{n=0}^{k}\sum_{j=1}^{k-n} \frac{(\chi(j))_+}{n+j} &= \sum_{1 \leq j \leq k} \frac{B(j)}{j} \\
&\leq \sum_{1 \leq j \leq k/2} \frac{j}{j} + B(k) \sum_{k/2 < j \leq k} \frac{1}{j} \\
&\leq \frac{k}{2} + B(k)\log 2.
\end{aligned}$$

Comparing both sides of the last inequality implies the claim for sufficiently small $\delta > 0$. Using (2.5.4) together with (2.5.3) gives

$$\sum_{\substack{i \leq k, \\ \chi(i)>0}} \chi(i)\sigma(k+1-i) \geq \frac{1}{2} \sum_{\substack{1 \leq i \leq k, \\ \sigma(k+1-i) \geq \frac{1}{2}}} (\chi(i))_+ \geq \frac{1}{2}\left(\frac{k}{2} - 2k\delta - 2\right).$$

The last estimate clearly violates (2.5.2) for sufficiently small δ and $k \geq 13$. For $k < 13$, we can perform the same arguments as at the beginning of the proof to conclude that $\sigma(n)$, $n \leq 13$ is uniformly bounded away from -1. This completes the proof. $\square$

To handle the remaining case $\chi(1) < 0$ and prove the main result of this section we require the following simple lemma.

Lemma 2.5.2. *Either for all $1 \le j \le n$ we have $(-1)^j\chi(j) \ge 0$ and $(-1)^j\sigma(j) \ge 0$ or $|\sigma(m)| \le 1 - 1/n(n+1)$ for all $m \ge n+1$.*

Proof. We proceed by induction. Clearly, if $(-1)^j\chi(j) \ge 0$ and $(-1)^j\sigma(j) \ge 0$ for all $1 \le j \le m < n$ and $(-1)^{m+1}\chi(m+1) \ge 0$ then

$$(-1)^{m+1}\sigma(m+1) = \frac{\sum_{i=0}^{m+1}(-1)^j\sigma(i)\cdot(-1)^{m+1-j}\chi(m+1-i)}{n} \ge 0.$$

Thus if the first possibility does not hold we have

$$k_0 = \min\{k : \ (-1)^k\chi(k) < 0\}$$

and $k_0 \le n$. In this case

$$(-1)^{k_0}k_0\sigma(k_0) = (-1)^{k_0}\chi(k_0) + \sum_{i=1}^{k_0-1}(-1)^i\chi(i)\sigma(k_0-i)(-1)^{k_0-i} \le k_0 - 1$$

and therefore $|\sigma(k_0)| \le 1 - 1/k_0$. Furthermore, for all $m \ge k_0+1$ we have

$$|\sigma(m)| \le S_m \le S_{k_0} = \frac{|\sigma(k_0)| + \sum_{i\le k_0}|\sigma(i)|}{k_0+1} \le \frac{1-\frac{1}{k_0}+k_0}{k_0+1} = 1 - \frac{1}{k_0(k_0+1)}.$$

This completes the proof of the lemma. $\square$

We recall the notation from Corollary 1.5.2:

$$D_n = \min_{\theta\in\{0,1/2\}}\sum_{k\le n}\frac{1-\chi(k)\cos(2\pi k\theta)}{k}.$$

Theorem 1.5.1. *Let $f \in \mathcal{C}(1)$ be real valued multiplicative function and let $n_0 \ge 1$ be given. Then there exists $\delta(n_0) > 0$, such that:*

- *If $D_{n_0} \gg \log(\frac{1}{1-\delta(n_0)})$, then $\sigma(n) \ge -1+\delta(n_0)$ and $(-1)^n\sigma(n) \ge -1+\delta(n_0)$ for all $n \ge n_0$.*
- *Let $\theta_{n_0} = argmin_{\theta\in\{0,1/2\}}\{D_{n_0}\}$. Then $e(-\theta_{n_0}n)\sigma(n) \ge -1+\delta(n_0)$ for all $n \ge n_0$.*

Proof. We fix small $\delta > 0$ to be determined later and suppose that $\sigma(m) < -1+\delta$ for some $m \ge n_0$. Applying Corollary 1.5.2 gives

$$1-\delta < |\sigma(m)| \ll (1+D_m)\exp\left(-\frac{1}{42}D_m\right).$$

Consequently,

$$\min\left\{\sum_{k\le n_0}\frac{1-(-1)^k\chi(k)}{k}, \sum_{k\le n_0}\frac{1-\chi(k)}{k}\right\} \le D_m \ll -\log(1-\delta) + O(1)$$

which clearly violates our condition with appropriately chosen constants.

Changing $\chi(k)$ to $(-1)^k\chi(k)$ if necessary, we may assume that $\theta_0 = 0$. If $\chi(1) \geq 0$, then the result follows from Proposition 2.5.1 for sufficiently small δ. Otherwise we have $\chi(1) = \sigma(1) < 0$. Let

$$k_0 = \min\{k : \ (-1)^k\chi(k) < 0\}.$$

Since

$$\sum_{k \leq n_0} \frac{1-\chi(k)}{k} \leq \frac{1-\chi(k)(-1)^k}{k},$$

we have

$$\sum_{k \leq n_0/2} \frac{\chi(2k+1)}{2k+1} \geq 0$$

and so $k_0 \leq n_0$. Applying Lemma 2.5.2 gives $\sigma(m) \geq -1 + 1/k_0(k_0+1)$ for all $m \geq k_0 + 1$, and thus we can adjust the constants δ and $n_0(\delta)$ to conclude the result. $\square$

Applying Theorem 1.5.1 for $n_0 = 1$ immediately implies the following corollary.

Corollary 1.5.1. *Let $f \in \mathcal{C}(1)$ be a real valued multiplicative function. Then there exists an absolute constant $\delta > 0$ such that if $\chi(1) \geq 0$ then $\sigma(n) \geq -1 + \delta$ for all $n \geq 1$. Otherwise, if $\chi(1) < 0$, then $(-1)^n\sigma(n) \geq -1 + \delta$ for all $n \geq 1$.*

From the proof of Theorem 1.5.1 it follows that if

$$\tilde{\delta}(n_0) = \sup_{\delta(n_0)>0} \{\sigma(n)e(-n\theta_{n_0}) \geq -1 + \delta(n_0),\, n \geq n_0\},$$

then there exists $\delta_0 > 0$, such that $\delta(\tilde{n_0}) \geq \delta_0 > 0$. Modifying the proof of Theorem 1.5.1 slightly, one can establish quantitative version with $\delta_0 = 0.12$.

One of the consequences of the work of Granville and Soundararajan [GS01] in the number field setting is that for any prime $p \geq 2$ and all sufficiently large $n \geq 1$, the interval $[1,n]$ contains at least 17.5% of quadratic residues modulo p.

For any irreducible polynomial $D \in \mathcal{M}$, let χ_D be the quadratic character modulo D, defined by the Kronecker symbol $\chi_D(f) = (\frac{D}{f})$ with $f \in \mathcal{M}$.

Corollary 2.5.3. *Le $D \in \mathbb{F}_q[x]$ be any monic irreducible polynomial. Then for any sufficiently large $k \geq 1$, there exists at least 6% of quadratic residues modulo D in $\mathcal{M}_{2k}$.*

Proof. We note that by Theorem 1.5.1 with $\delta_0 = 0.12$ we have

$$\sum_{F \in \mathcal{M}_{2k}} \frac{1 + \left(\frac{D}{F}\right)}{2} \geq \frac{q^{2k}}{2} + \frac{-1+\delta_0}{2}q^{2k} \geq 0.06q^{2k}$$

and the results follows. $\square$

Chapter 3

CORRELATIONS OF MULTIPLICATIVE FUNCTIONS OF POLYNOMIALS

3.1. Multiplicative functions of polynomials. Proof of Proposition 1.6.1.

For any given polynomial $P(x) \in \mathbb{Z}[x]$ we define $\omega_P(p^k)$ to be the number of solutions of $P(x) = 0(\mathrm{mod}(p^k))$. Clearly, $\omega_P(p^k) \leq \deg P$ for all but finitely many primes p. We begin by showing that the mean value of $f(P(n))$ in general significantly depends on the large primes. We restrict ourselves to the case $P(x) = x^2 + 1$ but the same arguments work for all polynomials $P(x) \in \mathbb{Z}[x]$ that are not product of linear factors.

Lemma 3.1.1. *Let $P(x) = x^2 + 1$. For any $x \geq 2$, and any complex numbers $g(p^k) \in \mathbb{T}$, $p \leq 2x$, $k \geq 1$, there exists a multiplicative function $f : \mathbb{N} \to \mathbb{T}$ such that $f(p^k) = g(p^k)$ for all $p \leq 2x$ and*

$$\left| \frac{1}{x} \sum_{n \leq x} f(P(n)) \right| \geq \frac{1}{2} + o(1).$$

Proof. Let

$$\mathrm{M}(x) = \{n_p \leq x \mid \exists p \in N_P(x), p|P(n_p)\}.$$

We note that for each $p \geq 2x$, there exists at most one element $n_p \in \mathrm{M}(x)$ such that $p|P(n_p)$ and moreover all prime factors of $P(n_p)/p$ are smaller than x. We have

$$\begin{aligned} 2x\log x + O(x) &= \sum_{n \leq x} \log P(n) = \sum_{n \leq x} \sum_{d|P(n)} \Lambda(d) \\ &\leq 2 \sum_{\substack{p \leq x, \\ p=1 \bmod(4)}} \log p \cdot \frac{x}{p} + \sum_{\substack{p>2x, \\ p|P(n_p), \\ n_p \leq x}} \log p + O(x) \\ &\leq x\log x + 2\log x \cdot |\mathrm{M}(x)| + O(x) \end{aligned}$$

and therefore

$$|M(x)| \geq x\left(\frac{1}{2}+o(1)\right).$$

Consider the multiplicative function f defined as follows: $f(p^k) = g(p^k)$ for all primes $p \leq 2x$ and

$$f(p) = e^{i\phi}\overline{f\left(\frac{P(n_p)}{p}\right)}$$

if $p > 2x$ and there exists $n_p \in \mathrm{M}(x)$ such that $p|P(n_p)$, where

$$\phi = \arg\left(\sum_{\substack{n\in\overline{\mathrm{M}(x)}\\ n\leq x}} f(P(n))\right).$$

Define $f(p^k) = 1$ for all other primes and all $k \geq 1$. Clearly,

$$\sum_{n\leq x} f(P(n)) = \sum_{\substack{n\in\overline{\mathrm{M}(x)}\\ n\leq x}} f(P(n)) + \sum_{n_p\in \mathrm{M}(x)} f(P(n_p)) = \sum_{\substack{n\in\overline{\mathrm{M}(x)},\\ n\leq x}} f(P(n)) + e^{i\phi}|\mathrm{M}(x)|.$$

Selecting ϕ so that the two sums point in the same direction, we deduce that

$$\left|\frac{1}{x}\sum_{n\leq x} f(P(n))\right| \geq \frac{|\mathrm{M}(x)|}{x} \geq \frac{1}{2}+o(1).$$

$\square$

Proposition 1.6.1. *There exists a multiplicative function $f : \mathbb{N} \to [-1,1]$ such that $\mathbb{D}^2(1,f;x) = 2\log\log x + O(1)$ for all $x \geq 2$ and*

$$\limsup_{x\to\infty}\left|\frac{1}{x}\sum_{n\leq x} f(n^2+1)\right| \geq \frac{1}{2}+o(1).$$

Proof. Take the sequence $x_k = 2^{2^k}$ for $k \geq 1$ and define a completely multiplicative function f inductively: $f(p) = -1$ for all primes in $p \in (x_k,x_{k+1}]$ unless $p \in N_P(x_k)$, in which case we define the function as in the proof of Lemma 3.1.1. This guarantees that for all $k \geq 1$,

$$\left|\frac{1}{x_k}\sum_{n\leq x_k} f(n^2+1)\right| \geq \frac{1}{2}+o(1).$$

Since $N_P(x)$ contains at most x elements, we have

$$\sum_{p\in N_P(x)} 1/p \leqslant \sum_{x<p\leqslant 2x\log x} 1/p \ll (\log\log x)/\log x,$$

so that $\sum_{k\geqslant 1}\sum_{p\in N_P(x_k)} 1/p \ll \sum_{k\geqslant 1} k/2^k \ll 1$. Therefore

$$\mathbb{D}^2(1,f;x) \geq \sum_{\substack{p\leq x\\ p\notin\cup_{k\geq 1}N_P(x_k)}} \frac{2}{p} \geq 2\log\log x - O(1).$$

$\square$

3.2. Preparatory lemmas for the proof of Theorem 1.6.3

For technical reasons, we define an auxiliary distance

$$\mathbb{D}^*(f,g;x) = \left(\sum_{p^k \leq x} \frac{1 - \mathrm{Re}(f(p^k)\overline{g(p^k)})}{p^k} \right)^{\frac{1}{2}}.$$

We thus focus on the class of functions such that $f(p)$ is close to 1 on large primes $p \geq x$ where the distance is given by $\mathbb{D}_P(1,f;x)$ where

$$\mathbb{D}_P^2(1,f;x) \asymp \sum_p (1 - \mathrm{Re}\, f(p^k)) \cdot \frac{1}{x} \sum_{\substack{n \leq x, \\ p^k || P(n)}} 1,$$

which generalizes $\mathbb{D}(1,f;x)$ where

$$\mathbb{D}^2(1,f;x) \asymp \mathbb{D}^{*2}(1,f;x) \asymp \sum_p (1 - \mathrm{Re}\, f(p^k)) \cdot \frac{1}{x} \sum_{\substack{n \leq x, \\ p^k || n}} 1.$$

In order to prove Theorem 1.6.3, we begin by proving a few auxiliary results. The following lemma is a simple consequence of the Turán-Kubilius type inequality for the polynomial sequences.

Lemma 3.2.1. *Let $h : \mathbb{N} \to \mathbb{C}$ be an additive function such that $h(p^k) = 0$ for $p^k \geq x$ and $|h(p^k)| \leq 2$ for all p and $k \geq 1$. Suppose $P(x) \in \mathbb{Z}[x]$ is irreducible. Define*

$$\mu_{h,P} = \sum_{p^k \leq x} \frac{h(p^k)}{p^k} \left(\omega_P(p^k) - \frac{\omega_P(p^{k+1})}{p} \right)$$

and

$$\sigma_{h,P}^2 = \sum_{p^k \leq x} \frac{|h(p^k)|^2}{p^k} \left(\omega_P(p^k) - \frac{\omega_P(p^{k+1})}{p} \right).$$

Then

$$\sum_{n \leq x} |h(P(n)) - \mu_{h,P}|^2 \ll x \sum_{p^k \leq x} \frac{|h(p^k)|^2}{p^k} + x \frac{(\log\log x)^3}{\log x}. \tag{3.2.1}$$

Proof. By multiplicativity, we have

$$|\{n \leq x \mid d | P(n)\}| = \frac{\omega_P(d)}{d} x + r_d,$$

where $r_d = O(\omega_P(d))$. Furthermore, by Proposition 4 from [GS07b] applied to the additive functions in place of strongly additive

$$\sum_{n \leq x} |h(P(n)) - \mu_{h,P}|^2 \leq C_2 x \sigma_{h,P}^2 + O\left(\left(\max_{p \leq y} |h(p^k)|^2 \right) \left(\sum_{p \leq x} \frac{\omega_P(p)}{p} \right)^2 \sum_{\substack{d = p_1 p_2, \\ p_i \leq x}} |r_d| \right).$$

The error term is bounded by

$$\left(\max_{p\le x}|h(p^k)|^2\right)\left(\sum_{p\le x}\frac{\omega_P(p)}{p}\right)^2\sum_{\substack{d=p_1p_2,\\ p_i\le x}}|r_d|\ll\max_{p\le x}|h(p^k)|^2(\log\log x)^2\cdot\frac{x\log\log x}{\log x}.$$

Combining this observation with the estimate

$$\sigma_{h,P}^2\ll\sum_{p^k\le x}\frac{|h(p^k)|^2}{p^k}$$

we conclude the proof of (3.2.1). □

In what follows, we are going to focus on two-point correlations but the same method actually works for $m-$ point correlations with mostly notational modifications. Let

$$\mu_{h,P}=\sum_{p^k\le x}h(p^k)\left(\frac{\omega_P(p^k)}{p^k}-\frac{\omega_P(p^{k+1})}{p^{k+1}}\right)$$

and

$$\mathrm{P}(f;P;x)=\prod_{p\le x}\left(\sum_{k\ge0}f(p^k)\left(\frac{\omega_P(p^k)}{p^k}-\frac{\omega_P(p^{k+1})}{p^{k+1}}\right)\right).$$

We also introduce an auxiliary distance

$$\mathbb{D}_P^*(f,g;y;x)=\left(\sum_{y\le p^k\le x}\frac{1-\mathrm{Re}\left(f(p^k)\overline{g(p^k)}\right)}{p^k}+\sum_{p^k\in N_P(x)}\frac{1-\mathrm{Re}\left(f(p^k)\overline{g(p^k)}\right)}{x}\right)^{\frac{1}{2}}.$$

We begin by proving the concentration inequality for the values of a multiplicative function $f:\mathbb{N}\to\mathbb{U}$, which is in other form appeared earlier in the works of Maustavicious, Elliott and Hildebrand.

Proposition 3.2.2. *Let $f:\mathbb{N}\to\mathbb{U}$ be a multiplicative function. Let $P(n)\in\mathbb{Z}[x]$. Then*

$$\sum_{n\le x}|f(P(n))-P(f;P;x)|^2\ll x{\mathbb{D}^*}_P^2(1,f;x)+\frac{x(\log\log x)^3}{\log x}.$$

Proof. We begin by proving the proposition for the multiplicative function f such that $f(p^k)=1$ for all $p^k\ge x$. Note $e^{z-1}=z+O(|z-1|^2)$ for $|z|\le1$. By repeatedly applying the triangle inequality we have that for all $|z_i|,|w_i|\le1$

$$\left|\prod_{1\le i\le n}z_i-\prod_{1\le i\le n}w_i\right|\le\sum_{1\le i\le n}|z_i-w_i|. \tag{3.2.2}$$

Therefore,

$$\prod_{p^k||P(n)}e^{f(p^k)-1}=\prod_{p^k||P(n)}\left(f(p^k)+O(|f(p^k)-1|^2)\right)=\prod_{p^k||P(n)}f(p^k)+O\left(\sum_{p^k||P(n)}|f(p^k)-1|^2\right)$$

and

$$f(P(n)) = \prod_{p^k||P(n)} f(p^k) = \prod_{p^k||P(n)} e^{f(p^k)-1} + O\left(\sum_{p^k||P(n)} |f(p^k)-1|^2\right).$$

We now introduce an additive function h, such that $h(p^k) = f(p^k) - 1$. Clearly,

$$\sum_{n\le x} |f(P(n)) - e^{h(P(n))}|^2 \ll \sum_{n\le x} |f(P(n)) - e^{h(P(n))}|$$
$$\ll \sum_{n\le x} \sum_{\substack{p^k||P(n),\\ p^k\le x}} |f(p^k)-1|^2 \ll x \sum_{p^k\le x} \frac{|f(p^k)-1|^2}{p^k} \ll x\mathbb{D}^{*2}(f{,}1;x).$$

Since $|e^a - e^b| \ll |a-b|$ for $\mathrm{Re}\,(a), \mathrm{Re}\,(b) \le 0$, Lemma 5.1.2 implies

$$\sum_{n\le x} |e^{h(P(n))} - e^{\mu_{h,P}}|^2 \ll \sum_{n\le x} |h(P(n)) - \mu_{h,P}|^2 \le x\mathbb{D}^{*2}(f{,}1;x) + \frac{x(\log\log x)^3}{\log x}\cdot$$

We introduce $\mu_{h,P} = \sum_{p\le x} \mu_{h,p}$, where

$$\mu_{h,p} = \sum_{p^k\le x} h(p^k)\left(\frac{\omega_P(p^k)}{p^k} - \frac{\omega_P(p^{k+1})}{p^{k+1}}\right)$$

and observe

$$e^{\mu_{h,p}} = 1 + \mu_{h,p} + O(\mu_{h,p}^2) = \sum_{1\le p^k\le x} f(p^k)\left(\frac{\omega_P(p^k)}{p^k} - \frac{\omega_P(p^{k+1})}{p^{k+1}}\right) + O\left(\frac{1}{x} + \frac{1}{p}\sum_{p^k\le x}\frac{|h(p^k)|}{p^k}\right).$$

Note that $|e^{\mu_{h,p}}| \le 1$. Using (5.1.6) and the Cauchy-Schwarz inequality once again yields

$$|e^{\mu_{h,P}} - \mathrm{P}(f;P;x)|^2 \le \left(\sum_{p\le x}\left|e^{\mu_{h,p}} - \sum_{1\le p^k\le x} f(p^k)\left(\frac{\omega_P(p^k)}{p^k} - \frac{\omega_P(p^{k+1})}{p^{k+1}}\right) + O\left(\frac{1}{x}\right)\right|\right)^2$$
$$\ll \left(\sum_{p^k\le x}\frac{1}{p}\frac{|f(p^k)-1|}{p^k} + \sum_{p\le x}\frac{1}{x}\right)^2 \ll \mathbb{D}^{*2}(f{,}1;x) + \frac{1}{\log^2 x}$$

which together with the triangle inequality completes the proof of the lemma in the special case when $f(p^k) = 1$ for $p^k \ge x$.

We now consider any multiplicative function f and decompose $f(n) = f_s(n)f_l(n)$ where

$$f_s(p^k) = \begin{cases} f(p^k), & \text{if } p^k \le x \\ 1, & \text{if } p^k > x \end{cases}$$

and

$$f_l(p^k) = \begin{cases} 1, & \text{if } p^k \le x \\ f(p^k), & \text{if } p^k > x. \end{cases}.$$

Note that for a fixed prime power $p^k \in N_P(x)$,

$$|\{n \le x \mid p^k|P(n)\}| \le \omega_P(p^k)$$

and each $P(n)$ is divisible by $\ll \deg P$ elements of $N_P(x)$. Using the Cauchy-Schwarz inequality yields

$$\sum_{n\le x} |f(P(n)) - f_s(P(n))|^2 \ll \sum_{n\le x} \left(\sum_{\substack{p^k || P(n), \\ p^k \ge x}} |f(p^k) - 1| \right)^2 \ll x \cdot \sum_{p^k \in N_P(x)} \frac{|f(p^k) - 1|^2}{x}.$$

We are left to collect the error terms and note that

$$\mathbb{D}^{*2}(1,f;x) + \sum_{p^k \in N_P(x)} \frac{1 - \operatorname{Re} f(p^k)}{x} = \mathbb{D}_P^{*\,2}(1,f;x).$$

$\square$

Proposition 5.1.3 immediately implies the following corollary which will be used in the proof of Theorem 1.6.3.

Corollary 3.2.3. *Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function and let $g : \mathbb{N} \to \mathbb{U}$ be any function. Let $P(n) \in \mathbb{Z}[x]$. Then*

$$\sum_{n\le x} f(P(n))g(n) = P(f;P;x) \sum_{n\le x} g(n) + O\left(x\mathbb{D}_P^*(1,f;x) + \frac{x(\log\log x)^{\frac{3}{2}}}{\sqrt{\log x}} \right).$$

Proof. Using Proposition 5.1.3, the triangle inequality and the Cauchy-Schwarz inequality give

$$\begin{aligned} \sum_{n\le x} f(P(n))g(n) - \mathrm{P}(f;P;x) \sum_{n\le x} g(n) &\ll \sum_{n\le x} |f(P(n)) - \mathrm{P}(f;P;x)| \\ &\ll \left(x \sum_{n\le x} |f(P(n)) - \mathrm{P}(f;P;x)|^2 \right)^{\frac{1}{2}} \\ &\ll x\mathbb{D}_P^*(1,f;x) + \frac{x(\log\log x)^{\frac{3}{2}}}{\sqrt{\log x}}. \end{aligned}$$

$\square$

Let $f,g : \mathbb{N} \to \mathbb{U}$ be multiplicative functions. For any two irreducible polynomials $P,Q \in \mathbb{Z}[x]$ we define

$$M(f,g;x) = \frac{1}{x} \sum_{n\le x} f(P(n))g(Q(n)).$$

We define $\omega(p^k,p^{\text{ł}})$ to be the quantity such that

$$\{n \le x \mid p^k || P(n), p^{\text{ł}} || Q(n)\} = x\omega(p^k,p^{\text{ł}}) + O(1).$$

We note that if $p \nmid \operatorname{res}(P,Q)$ then $\omega(p^k,p^{\text{ł}}) = 0$ unless $k = 0$ or $\text{ł} = 0$. In the latter case,

$$\omega(p^k,1) = \frac{\omega_P(p^k)}{p^k} - \frac{\omega_P(p^{k+1})}{p^{k+1}}$$

and

$$\omega(1,p^{\ell}) = \frac{\omega_Q(p^{\ell})}{p^{\ell}} - \frac{\omega_Q(p^{\ell+1})}{p^{\ell+1}}.$$

Furthermore, by the Chinese Remainder Theorem we have

$$\{n \le x \mid d_1|P(n), d_2|Q(n)\} = xF(d_1,d_2) + O(\omega_P(d_1)\omega_Q(d_2)) = xF(d_1,d_2) + O_\varepsilon(x^\varepsilon)$$

for some multiplicative function $F(d_1,d_2)$ and any $\varepsilon > 0$. Our main goal in this section is to prove that the mean value $M(f,g;x)$ satisfies the "local-to-global" principle. We first evaluate the local correlations.

Lemma 3.2.4. *Let $f,g : \mathbb{N} \to \mathbb{U}$ be multiplicative functions. Define f_p,g_p as in (1.7.2). Let $P,Q \in \mathbb{Z}[x]$ and $res(P,Q) \ne 0$. Then,*

$$\frac{1}{x}\sum_{n\le x} f_p(P(n))g_p(Q(n)) = \sum_{p^k,p^{\ell}\ge 1} f(p^k)g(p^{\ell})\omega(p^k,p^{\ell}) + O\left(\frac{\log x}{x\log p}\right).$$

In particular, if $p \nmid res(P,Q)$, then

$$\begin{aligned}&\frac{1}{x}\sum_{n\le x} f_p(P(n))g_p(Q(n))\\ &= \left(\sum_{k\ge 0} f(p^k)\left(\frac{\omega_P(p^k)}{p^k} - \frac{\omega_P(p^{k+1})}{p^{k+1}}\right) + \sum_{k\ge 0} g(p^k)\left(\frac{\omega_Q(p^k)}{p^k} - \frac{\omega_Q(p^{k+1})}{p^{k+1}}\right) - 1\right)\\ &+ O\left(\frac{\log x}{x\log p}\right).\end{aligned}$$

Proof. We first suppose that $p \nmid \mathrm{res}(P,Q)$. In this case we have

$$\begin{aligned}\frac{1}{x}\sum_{n\le x} f_p(P(n))g_p(Q(n)) &= \frac{1}{x}\left(\sum_{\substack{p^k\le x,\\ p^k||P(n)}} f(p^k) + \sum_{\substack{p^{\ell}\le x,\\ p^{\ell}||Q(n)}} g(p^{\ell}) + \sum_{\substack{n\le x,\\ p^0||P(n)Q(n)}} 1\right)\\ &= \left(\sum_{k\ge 0} f(p^k)\left(\frac{\omega_P(p^k)}{p^k} - \frac{\omega_P(p^{k+1})}{p^{k+1}}\right) + \sum_{k\ge 0} g(p^k)\left(\frac{\omega_Q(p^k)}{p^k} - \frac{\omega_Q(p^{k+1})}{p^{k+1}}\right) - 1\right)\\ &+ O\left(\frac{\log x}{x\log p}\right).\end{aligned}$$

More generally,

$$\frac{1}{x}\sum_{n\le x} f_p(P(n))g_p(Q(n)) = \frac{1}{x}\sum_{\substack{p^k,p^{\ell}\le x,\\ p^k||P(n),\\ p^{\ell}||Q(n)}} f(p^k)g(p^{\ell}) = \sum_{p^k,p^{\ell}\ge 1} f(p^k)g(p^{\ell})\omega(p^k,p^{\ell}) + O\left(\frac{\log x}{x\log p}\right).$$

This completes the proof of the lemma. $\square$

3.3. Proof of Theorem 1.6.3

We are going to prove

Theorem 3.3.1 (1.6.3)**.** *Let $f,g : \mathbb{N} \to \mathbb{U}$ be multiplicative functions. Let $P,Q \in \mathbb{Z}[x]$ be two polynomials, such that $res(P,Q) \neq 0$. Then,*

$$\frac{1}{x}\sum_{n\le x} f(P(n))g(Q(n)) = \prod_{p\le x} M_p(f(P),g(Q)) + \mathrm{Error}(f(P),g(Q),x)$$

where

$$\mathrm{Error}(f(P),g(Q),x) \ll \mathbb{D}_P(1,f;\log x;x) + \mathbb{D}_Q(1,g;\log x;x) + \frac{1}{\log\log x}.$$

Proof. Choose $y = (1-\varepsilon)\log x$. We begin by decomposing $f(n) = f_s(n)f_\natural(n)$ where

$$f_s(p^k) = \begin{cases} f(p^k), & \text{if } p^k \le y \\ 1, & \text{if } p^k > y \end{cases}$$

and

$$f_\natural(p^k) = \begin{cases} 1, & \text{if } p^k \le y \\ f(p^k), & \text{if } p^k > y. \end{cases}$$

By analogy, we write $g(n) = g_s(n)g_\natural(n)$. We apply Corollary 3.2.3 to get

$$\sum_{n\ge 1} f_\natural(P(n))f_s(P(n))g(Q(n)) = \mathrm{P}(f_\natural;P;x)\sum_{n\le x} f_s(P(n))g(Q(n)) + O\left(x\mathbb{D}^*_P(1,f_\natural;y;x) + \frac{x(\log\log x)^{\frac{3}{2}}}{\sqrt{\log x}}\right).$$

We now apply Corollary 3.2.3 to the inner sum to arrive at

$$\sum_{n\le x} g_\natural(Q(n))g_s(Q(n))f_s(P(n)) = \mathrm{P}(g_\natural;Q;x)\sum_{n\le x} f_s(P(n))g_s(Q(n)) + O\left(x\mathbb{D}^*_P(1,f_\natural;y;x) + x\mathbb{D}^*_Q(1,g_\natural;y;x) + \frac{x(\log\log x)^{\frac{3}{2}}}{\sqrt{\log x}}\right).$$

Combining the last two identities we conclude

$$\sum_{n\le x} f(P(n))g(Q(n)) = \mathrm{P}(f_\natural;P;x)\mathrm{P}(g_\natural;Q;x)\sum_{n\le x} f_s(P(n))g_s(Q(n)) + O\left(x\mathbb{D}^*_P(1,f_\natural;y;x) + x\mathbb{D}^*_Q(1,g_\natural;y;x) + \frac{x(\log\log x)^{\frac{3}{2}}}{\sqrt{\log x}}\right).$$

Let $f_s = 1 * \theta_s$, $g_s = 1 * \gamma_s$. Then $\theta_s(p^k) = 0$ and $\gamma_s(p^k) = 0$ whenever $p^k \ge y$. Since $\prod_{p^k\le y} p = e^{y+o(y)} \le x$ as long as $y \le (1-\varepsilon)\log x$ the following sums are supported on the integers $d_1,d_2 \le x$. Hence,

$$\sum_{n\le x} f_s(P(n))g_s(Q(n)) = \sum_{\substack{d_1,d_2\le x,\\ p|d_i\Rightarrow p\le y}} \theta_s(d_1)\gamma_s(d_2) \sum_{\substack{n\le x,\\ d_1|P(n),\\ d_2|Q(n)}} 1$$

$$= \sum_{\substack{d\le x,\\ d|\mathrm{res}(P,Q)}} \sum_{\substack{d_1,d_2\le x,\\ (d_1,d_2)=d,\\ p|d_i\Rightarrow p\le y}} \theta_s(d_1)\gamma_s(d_2)F(d_1,d_2)x + O\left(x^{\varepsilon} \sum_{d_1,d_2\le x} |\theta_s(d_1)\gamma_s(d_2)|\right)$$

$$= \sum_{\substack{d\le x,\\ d|\mathrm{res}(P,Q)}} \sum_{\substack{d_1,d_2\ge 1,\\ (d_1,d_2)=d,\\ p|d_i\Rightarrow p\le y}} \theta_s(d_1)\gamma_s(d_2)F(d_1,d_2)x + O\left(x^{\varepsilon} \sum_{d_1,d_2\le x} |\theta_s(d_1)\gamma_s(d_2)|\right).$$

To estimate the error term we observe

$$\sum_{d_1,d_2\le x} |\theta_s(d_1)\gamma_s(d_2)| \le x^{\frac{1}{2}} \left(\sum_{d\ge 1} \frac{|\theta_s(d)|}{d^{\frac{1}{4}}}\right)\left(\sum_{d\ge 1} \frac{|\gamma_s(d)|}{d^{\frac{1}{4}}}\right) \tag{3.3.1}$$

$$\le x^{\frac{1}{2}} \left(\prod_{p\le y}\left(\sum_{k\ge 0} \frac{|\theta_s(p^k)|}{p^{\frac{k}{4}}}\right)\right)\left(\prod_{p\le y}\left(\sum_{k\ge 0} \frac{|\gamma_s(p^k)|}{p^{\frac{k}{4}}}\right)\right)$$

$$\ll x^{\frac{1}{2}} \left(\prod_{p\le y}\left(1+\frac{2}{p^{\frac{1}{4}}}\right)\right)^2 \ll x^{\frac{1}{2}} \exp\left(\frac{3y^{3/4}}{\log y}\right).$$

The last sum is $O(x^{\frac{1}{2}+\varepsilon})$ for $y \ll \log x$ and $y \to \infty$. It easy to see that for $p \le y$, Lemma 3.2.4 implies

$$M_p(f,g) = \sum_{p^k,p^{\text{ł}}\ge 1} \theta(p^k)\gamma(p^{\text{ł}})F(p^k,p^{\text{ł}}),$$

where $M_p(f,g)$ defined as in (1.6.3). By multiplicativity the contribution of small primes is

$$\sum_{d|\mathrm{res}(P,Q)} \sum_{\substack{d_1,d_2\ge 1,\\ (d_1,d_2)=d,\\ p|d_i\Rightarrow p\le y}} \theta_s(d_1)\gamma_s(d_2)F(d_1,d_2) = \prod_{p\le y} M_p(f,g). \tag{3.3.2}$$

We are left to estimate $\mathrm{P}(f_{\text{ł}};P;x)\mathrm{P}(g_{\text{ł}};Q;x)$. The contribution of primes $p^k > y$ and $p \le y$ is

$$\prod_{\substack{p^k\ge y,\\ p<y}} \left(1+\sum_{i\ge k} \frac{\theta_{\text{ł}}(p^k)\omega_P(p^k)}{p^k}\right) \prod_{\substack{p^k\ge y,\\ p<y}} \left(1+\sum_{i\ge k} \frac{\gamma_{\text{ł}}(p^k)\omega_Q(p^k)}{p^k}\right) = 1 + O\left(\sum_{\substack{p^k\ge y\\ p<y}} \frac{1}{p^k}\right)$$

$$= 1 + O\left(\frac{1}{y}\cdot\frac{y}{\log y}\right) = 1 + O\left(\frac{1}{\log y}\right).$$

Furthermore, for $p \ge y$ we clearly have $(p,\mathrm{res}(P,Q)) = 1$ and

$\mathrm{P}(f_{\text{ł}};P;x)\mathrm{P}(g_{\text{ł}};Q;x)$

$$= \left(1+O\left(\frac{1}{\log y}\right)\right) \cdot \prod_{y<p\le x}\left(1+\sum_{k\ge 1}\frac{\theta_{\natural}(p^k)\omega_P(p^k)}{p^k}\right)\prod_{y<p\le x}\left(1+\sum_{k\ge 1}\frac{\gamma_{\natural}(p^k)\omega_Q(p^k)}{p^k}\right)$$
$$= \left(1+O\left(\frac{1}{\log y}\right)\right)$$
$$\times \prod_{y<p\le x}\left(1+\sum_{k\ge 1}\frac{\theta(p^k)\omega_P(p^k)}{p^k}+\sum_{k\ge 1}\frac{\gamma(p^k)\omega_Q(p^k)}{p^k}+\sum_{k\ge 1}\frac{\theta(p^k)\omega_P(p^k)}{p^k}\sum_{k\ge 1}\frac{\gamma(p^k)\omega_Q(p^k)}{p^k}\right)$$
$$= \left(1+O\left(\frac{1}{\log y}\right)\right)\exp\left(O\left(\sum_{y\le p\le x}\frac{1}{p^2}\right)\right)\prod_{y<p\le x}\left(1+\sum_{k\ge 1}\frac{\theta(p^k)\omega_P(p^k)}{p^k}+\sum_{k\ge 1}\frac{\gamma(p^k)\omega_Q(p^k)}{p^k}\right)$$
$$= \left(1+O\left(\frac{1}{\log y}\right)\right)\prod_{y<p\le x}\left(1+\sum_{k\ge 1}\frac{\theta(p^k)\omega_P(p^k)}{p^k}+\sum_{k\ge 1}\frac{\gamma(p^k)\omega_Q(p^k)}{p^k}\right)$$

and thus
$$\mathrm{P}(f_{\natural};P;x)\mathrm{P}(g_{\natural};Q;x) = \prod_{p\ge y} M_p(f,g) + O\left(\frac{1}{\log y}\right).$$

We note that $D^*_P(1,f;\log x;x)$ can be replaced with $D_P(1,f;\log x;x)$ at a cost $O(\frac{\log\log x}{\log x})$. Combining all of the above we arrive at the result claimed. □

Applying Theorem 1.6.3 and Lemma 3.2.4 with $g=1$, we deduce the following corollary.

Corollary 1.6.2. *Let $f:\mathbb{N}\to\mathbb{U}$ be a multiplicative function and $P\in\mathbb{Z}[x]$ Then*
$$\frac{1}{x}\sum_{n\le x} f(P(n)) = \prod_{p\le x} M_p(f(P)) + O\left(\mathbb{D}_P(1,f;\log x;x)+\frac{1}{\log\log x}\right).$$

3.4. Corollaries required for further applications. Proof of Corollary 1.6.4.

To state some corollaries required for our future applications we introduce some notation. We fix two integer numbers $a,b\ge 1$. For multiplicative functions $f,g:\mathbb{N}\to\mathbb{C}$ such that $\mathbb{D}(1,f;\infty)<\infty$, $\mathbb{D}(1,g;\infty)<\infty$, we set $f=1*\theta$, $g=1*\gamma$. For $(r,(a,b))=1$ we define
$$G(f;g;r;x)=G(r,x) := \prod_{p^k\| r,\ p\le x}\left(\theta(p^k)\gamma(p^k)+\delta_b\sum_{i>k}\frac{\theta(p^k)\gamma(p^i)}{p^{i-k}}+\delta_a\sum_{i>k}\frac{\gamma(p^k)\theta(p^i)}{p^{i-k}}\right) \tag{3.4.1}$$
and $\delta_{\natural}=0$ when $p|\natural$ and $\delta_{\natural}=1$ otherwise. We remark that in (3.4.1) we allow $k=0$ if $p\nmid r$. For $(r,(a,b))>1$ we set
$$G(r,x) := 0.$$
We can now deduce the following corollary.

Corollary 3.4.1. *Let $f,g : \mathbb{N} \to \mathbb{U}$ be multiplicative functions. Suppose that $\mathbb{D}(1,f;\infty) < \infty$, $\mathbb{D}(1,g;\infty) < \infty$. Let $a,b \geq 1$, c,d be integers with $(a,c) = (b,d) = 1$ and $ad \neq bc$. Then,*

$$\frac{1}{x}\sum_{n\leq x} f(an+c)g(bn+d) = \sum_{r|ad-bc} \frac{G(f;g;r;x)}{r} + o(1)$$

as $x \to \infty$ and the error term $o(1)$ depends on the coefficients a,b,c,d.

Proof. We note that

$$\Big|\{n \leq x \mid \exists p^k \geq x, p^k|an+c\}\Big| \ll \frac{x}{\log x}$$

and thus the contribution of terms with large prime power factors can be absorbed into the error term. We can now apply Theorem 1.6.3 (using the same notation) with $P(n) = an+c$ and $Q(n) = bn+d$ and note that $\mathrm{res}(P,Q) = ad-bc$, $\omega_P(p^k) = 1$ for $p \nmid a$ and $\omega_P(p^k) = 0$ for $p|a$, $\omega_Q(p^k) = 1$ for $p \nmid b$ and $\omega_Q(p^k) = 0$ for $p|b$, $p^k \leq x$. We are left to note that

$$F(d_1,d_2) = \frac{1}{[d_1,d_2]}$$

and the terms coming from small primes $p \leq y$, such that $(r,(a,b)) = 1$

$$G_s(r) = \sum_{\substack{d_1,d_2\geq 1\\(d_1,d_2)=r\\(d_1,a)=1\\(d_2,b)=1\\p|rd_i\Rightarrow p\leq y}} \frac{\theta_s(d_1)\overline{\gamma_s(d_2)}}{[d_1,d_2]}$$

each has an Euler product

$$G_s(a) := \prod_{p^k||a,\ p\leq y}\left(\theta(p^k)\gamma(p^k) + \delta_b\sum_{i>k}\frac{\theta(p^k)\gamma(p^i)}{p^{i-k}} + \delta_a\sum_{i>k}\frac{\gamma(p^k)\theta(p^i)}{p^{i-k}}\right)$$

and $\delta_{\mathfrak{l}} = 0$ when $p|\mathfrak{l}$ and $\delta_{\mathfrak{l}} = 1$ otherwise. ∎

We will require the following extension of Corollary 3.4.1 to all "pretentious" functions.
Corollary 1.6.4. *Let $f,g : \mathbb{N} \to \mathbb{U}$ be multiplicative functions with $\mathbb{D}(f,n^{it},\infty)$, $\mathbb{D}(g,n^{iu},\infty) < \infty$, and write $f_0(n) = f(n)/n^{it}$ and $g_0(n) = g(n)/n^{iu}$. Let $a,b \geq 1$, c,d be integers with $(a,c) = (b,d) = 1$ and $ad \neq bc$. As above we have*

$$\frac{1}{x}\sum_{n\leq x} f(an+c)g(bn+d)) = M_i(f(P),g(Q),x)\prod_{p\leq x} M_p(f_0(P),g_0(Q)) + o(1)$$

as $x \to \infty$ and $o(1)$ term depends on the variables a,b,c,d,t,u.

We have

$$M_i(f(P),g(Q),x) = \frac{a^{it}b^{iu}x^{i(t+u)}}{1+i(t+u)} + o(1)$$

when $x \to \infty$ *and* $o(1)$ *depends on* a,b,t,u. *If* $p|(a,b)$ *then* $M_p(f_0(P),g_0(Q)) = 1$. *If* $p \nmid ab(ad-bc)$, *then*

$$M_p(f_0(P),g_0(Q)) = M_p(f_0(P)) + M_p(g_0(Q)) - 1 = 1 + \left(1 - \frac{1}{p}\right)\left(\sum_{j\geq 1}\frac{f_0(p^j)}{p^j} + \sum_{j\geq 1}\frac{g_0(p^j)}{p^j}\right).$$

In general, if $p \nmid (a,b)$ *we have a more complicated formula*

$$M_p(f_0(P),g_0(Q)) = \sum_{\substack{0\leq i\leq k,\\ k\geq 0,\\ p^k||ad-bc}}\left(\frac{\theta(p^i)\gamma(p^i)}{p^i} + \delta_b\sum_{j>i}\frac{\theta(p^i)\gamma(p^j)}{p^j} + \delta_a\sum_{j>i}\frac{\gamma(p^i)\theta(p^j)}{p^j}\right)$$

and $\delta_ł = 0$ *when* $p|ł$ *and* $\delta_ł = 1$ *otherwise. Here* $f_0 = 1 * \theta$ *and* $g_0 = 1 * \gamma$.

Proof. We observe $\mathbb{D}(f_0,1,\infty) < \infty$ and $\mathbb{D}(g_0,1,\infty) < \infty$ and let

$$M(x) = \sum_{n\leq x} f_0(an+c)g_0(bn+d).$$

Corollary 3.4.1 implies

$$M(y) = y\sum_{r|ad-bc}\frac{G(f_0;g_0;r;y)}{d} + o(y).$$

Recall that for any $r \geq 1$, $(r,(a,b)) = 1$

$$G(f_0;g_0;r;x) = G(r,x) := \prod_{p^k||r,\ p\leq x}\left(\theta(p^k)\gamma(p^k) + \delta_b\sum_{i>k}\frac{\theta(p^k)\gamma(p^i)}{p^{i-k}} + \delta_a\sum_{i>k}\frac{\gamma(p^k)\theta(p^i)}{p^{i-k}}\right).$$

Note that $\mathbb{D}(1,f_0,\infty) < \infty$ together with the fact that $\mathrm{Re}\,(\theta(p)) \leq 0$ imply

$$-\sum_{p\geq 1}\frac{\mathrm{Re}\,(\theta(p))}{p} < \infty$$

and thus for $y \gg r$ we have

$$G(r,y) \ll \exp\left(\sum_{p\geq 1}\frac{\mathrm{Re}\,(\theta(p))}{p} + \frac{\mathrm{Re}\,(\gamma(p))}{p}\right) = O(1).$$

Furthermore, since $\frac{\mathrm{Re}\,(\theta(p))}{p} \leq 0$ and $\frac{\mathrm{Re}\,(\gamma(p))}{p} \leq 0$ we use (5.1.6) to estimate

$$\begin{aligned}G(r,x) - G(r,y) &= G(r,y)\left[\prod_{y<p\leq x}\left(1 + \sum_{k\geq 1}\frac{\theta(p^k)}{p^k} + \sum_{k\geq 1}\frac{\gamma(p^k)}{p^k}\right) - 1\right] \qquad (3.4.2)\\ &= G(r,y)\left[\exp\left(\log\sum_{y<p\leq x}\left(1 + \sum_{k\geq 1}\frac{\theta(p^k)}{p^k} + \sum_{k\geq 1}\frac{\gamma(p^k)}{p^k}\right)\right) - 1\right]\\ &\ll \left|\exp\left(\sum_{y\leq p\leq x}\frac{\mathrm{Re}\,(\theta(p))}{p} + \frac{\mathrm{Re}\,(\gamma(p))}{p}\right)\left(1 + O\left(\frac{1}{y}\right)\right) - 1\right|\end{aligned}$$

$$\ll \left(\sum_{y<p\le x}\frac{1}{p}\right) \ll \log\left(\frac{\log x}{\log y}\right).$$

For $(r,(a,b)) > 1$ we have $G(r,x) = G(r,y) = 0$ and (3.4.2) holds. Hence,

$$\sum_{r|ad-bc}\frac{G(r,y)}{r} = \sum_{r|ad-bc}\frac{G(r,x)}{r} + O\left(\log\left(\frac{\log x}{\log y}\right)\right).$$

Since

$$M(y) = y\sum_{r|ad-bc}\frac{G(r,y)}{r} + o(y),$$

we have

$$\frac{M(y)}{y} = \frac{M(x)}{x} + O\left(\log\left(\frac{\log x}{\log y}\right)\right).$$

Summation by parts yields

$$\begin{aligned}
\sum_{n\le x} f(an+c)g(bn+d) &= \sum_{n\ge 1}(an+c)^{it}(bn+d)^{iu}f_0(an+c)g_0(bn+d)\\
&= \int_1^x (ay+c)^{it}(by+d)^{iu}d(M(y))\\
&= M(x)(ax+c)^{it}(bx+d)^{iu} - \int_1^x M(y)\left[(ay+c)^{it}(by+d)^{iu}\right]' dy\\
&= M(x)(ax+c)^{it}(bx+d)^{iu} - \frac{1}{x}\int_1^x M(x)y\left[(ay+c)^{it}(by+d)^{iu}\right]' dy\\
&+ O\left(\int_2^x y\log\left(\frac{\log x}{\log y}\right)\left|\left[(ay+c)^{it}(by+d)^{iu}\right]'\right| dy\right)\\
&= \frac{M(x)}{x}\int_2^x (ay+c)^{it}(by+d)^{iu}dy\\
&+ O\left(\int_2^x y\log\left(\frac{\log x}{\log y}\right)\left|\left[(ay+c)^{it}(by+d)^{itu}\right]'\right| dy\right).
\end{aligned}$$

Note,

$$y\left|\left[(ay+c)^{it}(by+d)^{iu}\right]'\right| \ll \frac{y}{ay+c} + \frac{y}{by+d} = O(1),$$

and so the error term is bounded by

$$\int_2^x \log\left(\frac{\log x}{\log y}\right) dy \ll \frac{x}{\log x} = o(x).$$

Since $|(ay+c)^{it} - (ay)^{it}| = O\left(\frac{t}{y}\right)$, we have

$$\int_2^x (ay+c)^{it}(by+d)^{iu}dy = \int_2^x (ay)^{it}(by)^{iu}dy + o(x).$$

Evaluating the last integral and performing simple manipulations with the Euler factors we conclude

$$\sum_{r|ad-bc}\frac{G(f_0;g_0;r;x)}{r} = \prod_{p\le x} M_p(f_0(P),g_0(Q)) + o(1)$$

and the result follows. ◻

Remark 3.4.2. Let $f_k(n)$, $k=\overline{1,m}$ be multiplicative functions such that $|f_k(n)|\le 1$ and $\mathbb{D}(f_k(n),n^{it_k};\infty)<\infty$ for all $n\in\mathbb{N}$. Following the lines of the proof one can generalize Corollary 1.6.4 to compute correlations of the form

$$\sum_{n\le x} f_1(a_1n+b_1)f_2(a_2n+b_2)\cdot\dots\cdot f_m(a_mn+b_m).$$

Finally, we will require the following special case of Corollary 3.4.1.

Corollary 3.4.3. *Let $f:\mathbb{N}\to\mathbb{U}$ be a multiplicative function such that $\mathbb{D}(1,f;\infty)<\infty$, $m\in\mathbb{N}$. Then,*

$$\frac{1}{x}\sum_{n\ge 1} f(n)\overline{f(n+m)}=\sum_{r|m}\frac{G_0(r)}{r}+o(1)$$

when $x\to\infty$ and $o(1)$ depends on m, where $f=1*\theta$ and

$$G_0(r):=\prod_{p^k||r}\left(|\theta(p^k)|^2+2\sum_{i>k}\frac{\mathrm{Re}\,(\theta(p^k)\overline{\theta(p^i)})}{p^{i-k}}\right).$$

Proof. We apply Corollary 3.4.1 with $g=\overline{f}$, $a=b=1$, $d=0$, $c=m$ and observe

$$\prod_{p>x}\left(|\theta(p^k)|^2+2\sum_{i>k}\frac{\mathrm{Re}\,(\theta(p^k)\overline{\theta(p^i)})}{p^{i-k}}\right)=\prod_{p>x}\left(1+2\sum_{i\ge 1}\frac{\mathrm{Re}\,(\overline{\theta(p^i)})}{p^i}\right)\to 1.$$

Hence, the Euler factors

$$G(a):=\prod_{p^k||a,\ p\le x}\left(|\theta(p^k)|^2+2\sum_{i>k}\frac{\mathrm{Re}\,(\theta(p^k)\overline{\theta(p^i)})}{p^{i-k}}\right)$$

converge to

$$G_0(a):=\prod_{p^k||a}\left(|\theta(p^k)|^2+2\sum_{i>k}\frac{\mathrm{Re}\,(\theta(p^k)\overline{\theta(p^i)})}{p^{i-k}}\right).$$

◻

3.5. Correlations with modulated characters. Proof of Theorem 1.6.5

Let f be a multiplicative function such that $|f(n)|\le 1$ and $\mathbb{D}(f(n),n^{it}\chi(n);\infty)<\infty$ for some $t\in\mathbb{R}$ where χ is a primitive character of conductor q. We define F to be the multiplicative function such that

$$F(p^k)=\begin{cases} f(p^k)\overline{\chi(p^k)}p^{-ikt}, & \text{if } p\nmid q\\ 1, & \text{if } p\mid q,\end{cases} \tag{3.5.1}$$

and

$$M_p(F,\overline{F};d) = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} F_p(n)\overline{F_p(n+d)}.$$

We are now ready to establish the formula for correlations when f "pretends" to be a modulated character.

Theorem 1.6.5. *Let $f:\mathbb{N}\to\mathbb{U}$ be a multiplicative function such that $\mathbb{D}(f(n),n^{it}\chi(n);\infty) < \infty$ for some $t\in\mathbb{R}$ and χ is a primitive character of conductor q. Then, for any non-zero integer d we have*

$$\frac{1}{x}\sum_{n\le x} f(n)\overline{f(n+d)} = \prod_{\substack{p\le x\\ p\nmid q}} M_p(F,\overline{F};d) \prod_{p^{\mathfrak{l}}||q} M_{p^{\mathfrak{l}}}(f,\overline{f},d) + o(1)$$

when $x\to\infty$. Here, $o(1)$ term depends on d,χ,t and

$$M_{p^{\mathfrak{l}}}(f,\overline{f},d) = \begin{cases} 0, & \text{if } p^{\mathfrak{l}-1}\nmid d,\\ 1-\frac{1}{p}, & \text{if } p^{\mathfrak{l}-1}||d,\\ \left(1-\frac{1}{p}\right)\sum_{j=0}^{k}\frac{|f(p^j)|^2}{p^j} - \frac{|f(p^k)|^2}{p^k}, & \text{if } p^{\mathfrak{l}+k}||d, \end{cases}$$

for any $k\ge 0$ and if $p^n||d$ for some $n\ge 0$, then

$$M_p(F,\overline{F},d) = 1 - \frac{2}{p^{n+1}} + \left(1-\frac{1}{p}\right)\sum_{j>n}\left(\frac{F(p^n)\overline{F(p^j)}}{p^j} + \frac{\overline{F(p^n)}F(p^j)}{p^j}\right).$$

In particular, the mean value is $o(1)$ if $q\nmid d\prod_{p|q} p$.

Proof. We partition the sum according to $r,s\ge 1$ such that $r|n$ and $\mathrm{rad}(r)|q$, $(n/r,q)=1$ and $s|(n+d)$ and $\mathrm{rad}(s)|q$, $((n+d)/s,q)=1$. Note that $(r,s)|d$. We write

$$n = m\cdot \mathrm{lcm}(r,s) + rb(r)$$

such that $sb(s)-rb(r)=d$ for some integers $b(r),b(s)$. The sum can now be rewritten as

$$\sum_{n\le x} f(n)\overline{f(n+d)} = \sum_{r,s} f(r)\overline{f(s)} \sum_{m^*\le \frac{x}{\mathrm{lcm}(r,s)}} f\left(m^*\frac{s}{(r,s)}+b(r)\right)\overline{f\left(m^*\frac{r}{(r,s)}+b(s)\right)}$$

where the inner sum runs over m^* such that

$$\left(m^*\frac{s}{(r,s)}+b(r),q\right)=1$$

and

$$\left(m^*\frac{r}{(r,s)}+b(s),q\right)=1.$$

We can therefore define the function f_1 such that $f_1(p^k) = f(p^k)$ for all primes $p \nmid q$ and $f_1(p^k) = 0$ otherwise. In this case, Corollary 1.6.4 implies

$$\sum_{m^* \leq \frac{x}{\mathrm{lcm}(r,s)}} f\left(m^* \frac{s}{(r,s)} + b(r)\right) \overline{f\left(m^* \frac{r}{(r,s)} + b(s)\right)} \tag{3.5.2}$$

$$= \sum_{m \leq \frac{x}{\mathrm{lcm}(r,s)}} f_1\left(m \frac{s}{(r,s)} + b(r)\right) \overline{f_1\left(m \frac{r}{(r,s)} + b(s)\right)}$$

where now m runs over all integers up to $\frac{x}{\mathrm{lcm}(r,s)}$. We can now factor $f_1(n) = \chi(n)F(n)$. Note $\mathbb{D}(F,1,\infty) < \infty$. Let $m = kq+a$ where a runs over residue classes mod(q). The sum in (3.5.2) can be rewritten as

$$\sum_{r,s} f(r)f(s) \sum_{a \ \mathrm{mod}(q)} \chi\left(a \frac{s}{(r,s)} + b(r)\right) \overline{\chi\left(a \frac{r}{(s,r)} + b(s)\right)}$$

$$\times \sum_{k \leq \frac{x}{q\mathrm{lcm}(r,s)}} F\left(kq \frac{s}{(r,s)} + a \frac{s}{(r,s)} + b(r)\right) \overline{F\left(kq \frac{r}{(r,s)} + a \frac{r}{(r,s)} + b(s)\right)}.$$

We apply Corollary 1.6.4 to the inner sum and observe that

$$a_2 b_1 - a_1 b_2 = \frac{dq}{(r,s)}$$

and the asymptotic in Corollary 1.6.4 does not depend on b_1,b_2 and consequently on the residue class $a(\mathrm{mod}(q))$. Hence, up to a small error the innermost sum is equal to

$$\sum_{m \leq \frac{x}{q[s,r]}} F\left(m \frac{s}{(r,s)} + b(r)\right) \overline{F\left(m \frac{r}{(r,s)} + b(s)\right)}.$$

We now focus on the sum

$$\sum_{a \ \mathrm{mod}(q)} \chi\left(a \frac{s}{(r,s)} + b(r)\right) \overline{\chi\left(a \frac{r}{(s,r)} + b(s)\right)}. \tag{3.5.3}$$

Let $q = p_1^{a_1} p_2^{a_2} ... p_k^{a_k}$ and $\chi = \chi_{p_1^{a_1}} \chi_{p^{a_2}} \cdot ... \cdot \chi_{p_k^{a_k}}$, where each $\chi_{p_i^{a_i}}$ is a primitive character of conductor $p_i^{a_i}$. By the Chinese Remainder Theorem the sum (3.5.3) equals

$$\sum_{a \ \mathrm{mod}(q)} \chi\left(a \frac{s}{(r,s)} + b(r)\right) \overline{\chi\left(a \frac{r}{(s,r)} + b(s)\right)}$$

$$= \prod_{p^k || q} \sum_{a \ \mathrm{mod}(p^k)} \chi_{p^k}\left(a \frac{s}{(r,s)} + b(r)\right) \overline{\chi_{p^k}\left(a \frac{r}{(s,r)} + b(s)\right)}.$$

We claim that the last sum is zero unless $r = s$. Indeed, if $r \neq s$, then there exists prime p such that $p^i || r$ and $p^j || s$ for $j > i$. Since $(r/(r,s),p) = 1$ we can make the change of variables

$$a \to \frac{ar}{(r,s)} (mod(p^k))$$

and the $p-$th factor can be rewritten as

$$\sum_{a \mod(p^k)} \chi_{p^k}(ap^{j-i}t + b_1(r))\overline{\chi_{p^k}(a + b_1(s))},$$

where $(t,p) = 1$. If $j - i \geq k$, then the first term is fixed and the second runs over all residues modulo p^k. So the sum is zero. If $j - i < k$, we write $a = A + p^{k-(j-i)}L$ where A runs over residues $\mathrm{mod}(p^{k-(j-l)})$ and L runs over residues modulo p^{j-i}. Then, our sum becomes

$$\sum_{A \mod(p^{k-(j-l)})} \chi_{p^k}(Ap^{j-i}t + b_1(r)) \sum_{L \mod p^{j-i}} \overline{\chi_{p^k}(A + b_1(s) + p^{k-j+i}L)}.$$

It is easy to show that the inner sum

$$\sum_{L \mod p^{j-i}} \overline{\chi(A + b_1(s) + p^{k-j+i}L)} = 0.$$

Thus, the main contribution comes from the terms $r = s = R$. In this case we have $R(b(s) - b(r)) = d = bR$ and we can take $b(r) = 0$, $b(s) = b$. Our character sum can be rewritten as

$$\sum_{a \mod(q)} \chi(a)\overline{\chi(a + b)}.$$

To evaluate the last sum, we split it into prime powers. Now if $p^k||q$ and $p^i||b$ (possibly $i = 0$) then we have a nonzero contribution if and only if $i \geq k - 1$. Indeed, let $b = p^i b_1$, $(b_1,p) = 1$. We note

$$\sum_{a \mod(p^k)} \chi_{p^k}(a)\overline{\chi_{p^k}(a + b)} = \sum_{\substack{c \mod(p^k),\\ (c,p)=1}} \chi_{p^k}(p^i c + 1).$$

This sum is 0 if $i \leq k-2$ and equals to $-p^{k-1}$ whenever $i = k-1$ and $\phi(p^k)$ whenever $i \geq k$. We thus have

$$\sum_{a \mod(q)} \chi(a)\overline{\chi(a + b)} = \prod_{\substack{p^k||q\\ p^i||b\\ i\leq k-1}} \mu(p^{k-i})p^i \prod_{\substack{p^k||q\\ p^k|b}} \phi(p^k)$$

and the result follows by combining this with Corollary 1.6.4 and easy manipulations with the Euler products. $\square$

Combining the last proposition with Corollary 3.4.3 we deduce

Corollary 3.5.1. *Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function with $\mathbb{D}(f(n),n^{it}\chi(n);\infty) < \infty$ for some primitive character χ of conductor q. Then*

$$\frac{1}{x}\sum_{n\leq x} f(n)\overline{f(n+1)} = \frac{\mu(q)}{q} \prod_{\substack{p\geq 1\\ p\nmid q}} \left(2\,\mathrm{Re}\left(1 - \frac{1}{p}\right)\left(\sum_{k\geq 0} \frac{f(p^k)\overline{\chi(p^k)}p^{-ikt}}{p^k}\right) - 1\right) + o(1)$$

when $x \to \infty$ and $o(1)$ depends on χ,t.

We remark that using the same arguments one may establish the formula for the correlations

$$\sum_{n\leq x} f(n)g(n+m)$$

for $\mathbb{D}(f(n),n^{it_1}\chi(n),\infty) < \infty$ and $\mathbb{D}(g(n),n^{it_2}\psi(n),\infty) < \infty$. We state here one particular case when $m = 1$.

Proposition 3.5.2. *Let $f,g : \mathbb{N} \to \mathbb{U}$ be two multiplicative functions with $\mathbb{D}(f(n),n^{it_1}\chi(n),\infty) < \infty$ and $\mathbb{D}(g(n),n^{it_2}\psi(n),\infty) < \infty$ for some primitive characters χ,ψ. Let $R = \frac{q_\psi}{(q_\chi,q_\psi)}$ and $S = \frac{q_\chi}{(q_\chi,q_\psi)}$, $Q = [q_\chi,q_\psi]$. Then*

$$\frac{1}{x}\sum_{n\leq x} f(n)g(n+1) = \frac{R^{it_1}S^{it_2}}{i(t_1+t_2)+1} f(R)g(S) \sum_{a \ mod(Q)} \chi(aS+b(R))\psi(aR+b(S))$$

$$\times \prod_{\substack{p\leq x\\ p\nmid Q}} \left(\left(1-\frac{1}{p}\right)\left(\sum_{k\geq 0}\frac{f(p^k)p^{-ikt_1}}{p^k}\right) + \left(1-\frac{1}{p}\right)\left(\sum_{k\geq 0}\frac{g(p^k)p^{-ikt_2}}{p^k}\right) - 1\right) + o(1)$$

when $x \to \infty$ and $o(1)$ depends on parameters t_1,t_2,χ,ψ.

Proof. We follow the lines of the proof of Proposition 1.6.5 and note that in this case $(r,s) = 1$ and the only term that contributes is

$$r = R = \frac{q_\psi}{(q_\chi,q_\psi)}$$

and

$$s = S = \frac{q_\chi}{(q_\chi,q_\psi)}.$$

$\square$

Chapter 4

APPLICATIONS OF THE CORRELATION FORMULAS

4.1. Application to the Erdős-Coons-Tao conjecture

In this sections we are going to study multiplicative functions $f : \mathbb{N} \to \mathbb{T}$, such that

$$\limsup_{x\to\infty} |\sum_{n\le x} f(n)| < \infty. \tag{4.1.1}$$

We first focus on the complex valued case and the proof of Theorem 1.6.7. The key tool is the following recent result by Tao [Taob].

Theorem 4.1.1. [Tao] *Let a_1,a_2 be natural numbers, and let b_1,b_2 be integers such that $a_1b_2 - a_2b_1 \neq 0$. Let $\varepsilon > 0$, and suppose that A is sufficiently large depending on $\varepsilon,a_1,a_2,b_1,b_2$. Let $x \ge \omega \ge A$, and let $g_1,g_2 : \mathbb{N} \to \mathbb{U}$ be multiplicative functions with g_1 non-pretentious in the sense that*

$$\sum_{p\le x} \frac{1 - \mathrm{Re}(g_1(p)\chi(p)p^{it})}{p} \ge A$$

for all Dirichlet character χ of period at most A, and all real numbers $|t| \le Ax$. Then

$$\left| \sum_{x/\omega<n\le x} \frac{g_1(a_1n + b_1)g_2(a_2n + b_2)}{n} \right| \le \varepsilon \log \omega.$$

We will require the following technical lemma.

Lemma 4.1.2. *Let $a > 1$ be given and let x_n be an increasing sequence such that $x_n < x_{n+1} \le x_n^a$. Suppose that for each x_m, there exists a primitive character χ_m of conductor $O(1)$ and a real t_m with $|t_m| \ll x_m$ such that $\mathbb{D}(f(n),n^{it_m}\chi_m(n), x_m) = O(1)$. Then, there exists $t \in \mathbb{R}$ and a primitive character χ such that $\mathbb{D}(f(n),n^{it}\chi(n),\infty) < \infty$.*

Proof. Without loss of generality, we may assume that $x_{n+1} = x_n^a$ (otherwise we can choose a suitable subsequence and modify the values of a if necessary). We note that there exists $k = O(1)$ such that for all $n \ge 1$, $\chi_n^k(p) = 1$ for all but finitely many primes p. The triangle inequality now implies that

$$\mathbb{D}(f^k(n),n^{ikt_m}, x_m) = \mathbb{D}(f^k(n),n^{ikt_m}\chi_m^k(n), x_m) + O(1) \geq k\mathbb{D}(f(n),n^{it_m}\chi_m(n), x_m) = O(1).$$

Moreover,

$$\mathbb{D}^2(f^k(n),n^{ikt_m},x_{m+1}) \leq O(1) + \sum_{x_m \leq p \leq x_{m+1}} \frac{2}{p} \leq O(1) + 2\log\frac{\log x_{m+1}}{\log x_m} = O(1)$$

and therefore applying the triangle inequality once again we end up with

$$O(1) \geq \mathbb{D}(f^k(n),n^{ikt_m}, x_{m+1}) + \mathbb{D}(f^k(n),n^{ikt_{m+1}},x_{m+1}) \geq \mathbb{D}(1,n^{ik(t_{m+1}-t_m)},x_{m+1}).$$

Clearly $k|t_{m+1} - t_m| \ll x_{m+1}$ and therefore by the classical zero-free region we get

$$|t_{m+1} - t_m| \ll \frac{1}{\log x_{m+1}}.$$

Iterating last inequality we conclude that there exists t such that

$$|t_m - t| \ll \frac{1}{\log x_{m+1}}$$

for all $m \geq 1$. Since there are only finitely many options of characters χ_m, we can pass to the subsequence and assume that $\chi_m = \chi$ is fixed. The triangle inequality now implies

$$\mathbb{D}(f(n),n^{it_m}\chi(n), x_m) + \mathbb{D}(1,n^{i(t-t_m)}, x_m) \geq \mathbb{D}(f(n),n^{it}\chi(n), x_m) + O(1).$$

We are left to note that

$$\mathbb{D}(1,n^{i(t-t_m)},x_m) = O(1)$$

as long as $|t_m - t| \ll 1/\log x_m$ and we can replace t_m with t at a cost of $O(1)$. This completes the proof of the lemma. □

Lemma 4.1.3. *Suppose that for a multiplicative $f : \mathbb{N} \to \mathbb{T}$, (4.1.1) holds. Then there exists a primitive character χ and $t \in \mathbb{R}$, such that $\mathbb{D}(f(n),\chi(n)n^{it},\infty) < \infty$.*

Proof. Let $H \in \mathbb{N}$. Suppose that for each $1 \leq h \leq H$ we have

$$\frac{1}{\log x}\sum_{n \leq x} \frac{f(n)\overline{f(n+h)}}{n} \leq \frac{1}{2H}.$$

Consider

$$T(x) := \frac{1}{\log x}\sum_{n \leq x} \frac{1}{n}\left|\sum_{k=n+1}^{n+H+1} f(k)\right|^2.$$

Expanding the square, we get

$$T(x) = \sum_{1 \leq h_1 \neq h_2 \leq H} \frac{1}{\log x}\sum_{n \leq x} \frac{f(n+h_1)\overline{f(n+h_2)}}{n}.$$

The diagonal contribution $h_1 = h_2$ is $1 + o(1)$. For $h_2 > h_1$ we introduce $h = h_2 - h_1$ and replace n in the denominator by $N = n + h_1$ at a cost $\ll H/\log x$. We change the range for N from $1 + h_1 \le N \le x + h_1$ to $1 \le n \le x$ at a cost of $\ll \log H/\log x$. Therefore

$$\begin{aligned} T(x) &= H + o(1) - \sum_{|h|\le H} (H - |h|) \cdot \frac{1}{\log x} \sum_{N\le x} \frac{f(N)\overline{f(N+h)}}{N} \\ &\ge H - (H^2 - H) \cdot \frac{1}{2H} + o(1) = \frac{H}{2} + O(1) \end{aligned}$$

for $x \to \infty$. This contradicts (4.1.1) for sufficiently large $H \ge 1$. Thus, for a fixed $H \ge 1$, and every large $x \gg 1$, there exists $1 \le h_x \le H$ such that

$$\frac{1}{\log x} \sum_{n\le x} \frac{f(n)\overline{f(n+h_x)}}{n} \gg 1.$$

Since $h_x \le H$, we can apply Theorem 4.1.1 to conclude that there exists $A = A(H) \ge 0$ such that for any sufficiently large x, there exists $t_x \in \mathbb{R}$, $|t_x| \le Ax$ and a primitive character χ of modulus $D \le A$, such that $\mathbb{D}(f(n), n^{it_x}\chi(n); x) \le A$, namely

$$\sum_{p\le x} \frac{1 - \mathrm{Re}(f(p)p^{-it_x}\overline{\chi(p)})}{p} \le A^2.$$

Since the latter holds uniformly for all large x, Lemma 4.1.2 implies the result. $\square$

We now refine the result of Lemma 4.1.3.

Theorem 1.6.7. *Suppose for a multiplicative $f : \mathbb{N} \to \mathbb{T}$, (4.1.1) holds. Then there exists a primitive character χ of an odd conductor q and $t \in \mathbb{R}$, such that $\mathbb{D}(f(n), \chi(n)n^{it}; \infty) < \infty$ and $f(2^k) = -\chi^k(2)2^{-ikt}$ for all $k \ge 1$.*

Proof. Applying Lemma 4.1.3, we can find a primitive character χ of conductor q and $t \in \mathbb{R}$ such that $\mathbb{D}(f(n), \chi(n)n^{it}; \infty) < \infty$. Theorem 1.6.5 implies that for any $d \ge 0$, we have

$$S_d = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(x)\overline{f(x+d)} = \prod_{\substack{p\le x \\ p\nmid q}} M_p(F, \overline{F}; d) \prod_{p^{\mathfrak{k}}||q} M_{p^{\mathfrak{k}}}(f, \overline{f}, d).$$

For fixed $H \ge 1$, we can now write

$$\begin{aligned} &\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \left| \sum_{k=n+1}^{n+H+1} f(k) \right|^2 \\ &= \lim_{x\to\infty} \frac{1}{x} \left[\sum_{h=0,\ n\le x} Hf(n)\overline{f(n+h)} + 2 \sum_{1\le h\le H} (H-h) \sum_{n\le x} f(n)\overline{f(n+h)} \right] \\ &= HS_0 + 2\sum_{h=1}^{H} (H-h)S_h = H + 2\sum_{N=1}^{H-1} \sum_{n=1}^{N} S_m. \end{aligned}$$

We note that all $S_m \leq 1$ and Theorem 1.6.5 implies that each S_m behaves like a scaled multiplicative function, since it is given by the Euler product. We are going to show that there exists $\lim_{N\to\infty} \frac{1}{N}\sum_{n\leq N} S_n = c$ and so

$$H + 2\sum_{N=1}^{H-1}\sum_{n=1}^{N} S_m = O(1) \sim H + 2\sum_{N=1}^{H} cn = cH^2 + O(H).$$

The latter would imply that $c = 0$. We turn to the computations of the corresponding mean values. Clearly

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n\leq N} S_n = \prod_{p\leq N} S(p)$$

where $S(p)$ denotes the local factor that corresponds to prime p. If $p \nmid q$, then using Theorem 1.6.5 and simple computations

$$S_p = \sum_{k\geq 0}\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right) M_p(F,\overline{F},p^k) = \left|\left(1 - \frac{1}{p}\right)\sum_{k\geq 0}\frac{F(p^k)}{p^k}\right|^2.$$

If $p^{\mathfrak{l}}||q$, then again using Theorem 1.6.5 we get

$$S_p = \sum_{k\geq 0}\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right) M_{p^{\mathfrak{l}}}(f,\overline{f},p^k) = \frac{1}{p^{\mathfrak{l}-1}}\left(1 - \frac{1}{p}\right)^2.$$

Since $c = 0$, one of the Euler factors has to be 0. The only possibility then is $S_2 = 0$ and $2 \nmid q$ and $F(2^k) = -1$ for all $k \geq 1$. This completes the proof. □

4.2. Proof of the Erdős-Coons-Tao conjecture

We now move on to the proof of Theorem 1.6.6. It turns out that periodic multiplicative functions with zero mean have the following equivalent characterization that we will use throughout the proof.

Proposition 4.2.1. *Suppose that f is multiplicative with $|f(n)| \leq 1$ for all $n \in \mathbb{N}$. Then there exists an integer m such that $f(n+m) = f(n)$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{m} f(n) = 0$ if and only if $f(2^k) = -1$ for all $k \geq 1$ and there exists an integer M such that if prime power $p^k \geq M$ then $f(p^k) = f(p^{k-1})$.*

Proof. Suppose that $f(n+m) = f(n)$ for all $n \geq 1$ and $\sum_{n=1}^{m} f(n) = 0$. From periodicity we have $f(km) = f(m)$ for all $k \geq 1$, and so if $p^a||m$ then $f(p^b) = f(p^a)$ for all $b \geq a$. In particular if p does not divide m then $f(p^b) = 1$. Hence,

$$\sum_{n=1}^{m} f(n) = \sum_{d|m} f(d)\phi\left(\frac{m}{d}\right) = \prod_{p^a||m}\left(p^a\left(1 - \frac{1}{p}\right)\left(\sum_{1\leq k\leq a-1}\frac{f(p^k)}{p^k}\right) + f(p^a)\right).$$

Consequently, some factor has to be 0. The only possibility is then $p = 2$ and $f(2^k) = -1$ for all $k \geq 1$. The other direction immediately follows from the Chinese Remainder Theorem. □

Our starting point is the following result:

Theorem 4.2.2. [Tao, 2015] *If for a multiplicative* $f : \mathbb{N} \to \{-1,1\}$

$$\limsup_{x\to\infty} |\sum_{n\le x} f(n)| < \infty,$$

then $f(2^j) = -1$ *for all* $j \ge 1$ *and*

$$\sum_p \frac{1-f(p)}{p} < \infty.$$

In what follows we restrict ourselves to the multiplicative functions $f : \mathbb{N} \to \{-1,1\}$ such that $\mathbb{D}(1,f,\infty) < \infty$, $f = 1 * g$ and $f(2^j) = -1$ for all $j \ge 1$. For such functions we are going to drop the subscript and set

$$G_0(a) = G(a) := \prod_{p^k || a} \left(|g(p^k)|^2 + 2 \sum_{i \ge k+1} \frac{g(p^k)g(p^i)}{p^{i-k}} \right). \tag{4.2.1}$$

Here, we allow $k = 0$ if $p \nmid a$. The following lemma summarizes properties of $G(a)$ that we will use throughout the proof.

Lemma 4.2.3. *Let* $G(a)$ *be as above. Then*

(1) $G(4a) = 0$, $a \in \mathbb{N}$;

(2) $G(2a) = -4G(a)$ *for odd* a*;*

(3) $\sum_{a\ge1} \frac{G(a)}{a^2} = 0$;

(4) If $f(3) = 1$, *then* $G(a) \le 0$ *for all odd* a;

(5) $\sum_{a\ge1} \frac{G(a)}{a} = 1$.

Proof. Note that $g(2) = -2$ and $g(2^i) = f(2^i) - f(2^{i-1}) = 0$ for $i \ge 2$. Thus $G(4a) = 0$ and $G(2a) = -4G(a)$ for odd a. The third part immediately follows from

$$\sum_{a\ge1} \frac{G(a)}{a^2} = \sum_{a\ge1,\ a \text{ odd}} \frac{G(a)}{a^2} + \sum_{a\ge1,\ a \text{ odd}} \frac{G(2a)}{(2a)^2} = 0.$$

To prove (4), fix p and suppose $p^k || a$. We note that for $k = 0$, the Euler factor

$$E_p(a) = 1 + 2\sum_{i\ge1} \frac{g(p^i)}{p^i} \ge 1 - \frac{4}{p-1} \ge 0$$

for $p \geq 5$. Note $E_2(a) = 1-2 = -1$. If $3^0||a$, then $g(3) = f(3)-1 = 0$ and $E_3(a) \geq 1-\frac{4}{9}\cdot\frac{3}{2} = \frac{1}{3} > 0$. Suppose that $p^k||a$ and $k \geq 1$. Then,

$$E_p(a) = |g(p^k)|^2 + 2\sum_{i\geq k+1} \frac{g(p^k)g(p^i)}{p^{i-k}} \geq 4 - \frac{8}{p-1} \geq 0$$

for $p \geq 3$. Hence the only negative Euler factor is E_2 and (4) follows. To prove (5), we take $m = 0$ in Corollary 3.4.3 to arrive at

$$\lim_{x\to\infty} \frac{1}{x}\sum_{n\leq x} f(n)\overline{f(n+0)} = 1 = \sum_{a|0} \frac{G(a)}{a} = \sum_{a\geq 1} \frac{G(a)}{a}\cdot$$

□

Lemma 4.2.4. *Suppose $G(a) \neq 0$. Then,*

$$|G(a)| \gg \left(\frac{5}{4}\right)^{\omega(a)-1} \cdot \frac{2}{5} \cdot |G(1)|.$$

PROOF. Recall,

$$G(a) = \prod_{p^k||a} \left(|g(p^k)|^2 + 2\sum_{i\geq k+1} \frac{g(p^k)g(p^i)}{p^{i-k}} \right).$$

Note $g(p^k)g(p^{k+1}) \leq 0$ and so if $p^k||a$ and $k \geq 1$ we have

$$E_p(a) = |g(p^k)|^2 + 2\sum_{i\geq k+1} \frac{g(p^k)g(p^i)}{p^{i-k}} \geq 4 - \frac{8}{p}\cdot\frac{1}{1-\frac{1}{p^2}} = 4 - \frac{8p}{p^2-1}\cdot$$

For $p = 3$ the last bound reduces to $E_3(a) \geq 1$ and for $p \geq 5$ we clearly have $E_p(a) \geq 2$. For $k = 0$, we have

$$E_p(1) = 1 + 2\sum_{i\geq 1} \frac{g(p^i)}{p^i} \leq 1 + \frac{4}{p}\cdot\frac{1}{1-\frac{1}{p^2}} = 1 + \frac{4p}{p^2-1}.$$

Consequently, for $k \geq 1$ and $p > 3$

$$E_p(p^k) \geq \frac{5}{4}E_p(1).$$

Taking into account $p = 3$ we conclude

$$|G(a)| = \left| \prod_{p^k||a,\ k\geq 1} \left(|g(p^k)|^2 + 2\sum_{i\geq k+1} \frac{g(p^k)g(p^i)}{p^{i-k}} \right) \right| \geq \left(\frac{5}{4}\right)^{\omega(a)-1} \cdot \frac{2}{5} \cdot |G(1)|.$$

□

In fact, it is easy to check that $G(1) \neq 0$ and thus the last lemma provides nontrivial lower bound for $G(a)$. In the next lemma we compute the second moment of the partial sums over the interval of fixed length.

Lemma 4.2.5. *Let $H \in \mathbb{N}$. Then*

$$\frac{1}{x}\sum_{n\le x}\left(\sum_{k=n+1}^{n+H+1} f(k)\right)^2 = -2\sum_{a\ge 1,\ a\ odd} G(a)\left\|\frac{H}{2a}\right\| + o_{x\to\infty}(1).$$

Proof. Note

$$\frac{1}{x}\sum_{n\le x}\left(\sum_{k=n+1}^{n+H+1} f(k)\right)^2 = \frac{1}{x}\left[\sum_{h=0,\ n\le x} Hf(n)f(n+h) + 2\sum_{1\le h\le H}(H-h)\sum_{n\le x} f(n)f(n+h)\right] + o(1)$$
$$= \sum_{a\ge 1}\frac{G(a)}{a}\left(H + 2\sum_{\substack{1\le h\le H,\\ a|h}}(H-h)\right) + o_{x\to\infty}(1)$$

To compute the corresponding coefficient we write $H = ra+s$, $0\le s<a$ to arrive at

$$ra+s+2\sum_{1\le m\le r}(ra+s-ma) = ra+s+ar(r-1)+2rs$$
$$= \frac{(ra+s)^2}{a} + a\left(\frac{s}{a} - \left(\frac{s}{a}\right)^2\right).$$

Plugging this into our formula and using (3), (1), (2) from the Lemma 4.2.3 we get

$$H^2\sum_{a\ge 1}\frac{G(a)}{a^2} + \sum_{a\ge 1} G(a)\left(\left\{\frac{H}{a}\right\} - \left\{\frac{H}{a}\right\}^2\right) = \sum_{a\ge 1} G(a)\left(\left\{\frac{H}{a}\right\} - \left\{\frac{H}{a}\right\}^2\right)$$
$$= \sum_{a\ge 1,\ a\ odd} G(a)\left[\left(\left\{\frac{H}{a}\right\} - \left\{\frac{H}{a}\right\}^2\right) - 4\left(\left\{\frac{H}{2a}\right\} - \left\{\frac{H}{2a}\right\}^2\right)\right]$$
$$= -2\sum_{a\ge 1,\ a\ odd} G(a)\left\|\frac{H}{2a}\right\|,$$

since

$$\left(\left\{\frac{H}{a}\right\} - \left\{\frac{H}{a}\right\}^2\right) - 4\left(\left\{\frac{H}{2a}\right\} - \left\{\frac{H}{2a}\right\}^2\right) = -2\left\|\frac{H}{2a}\right\|,$$

where $\|x\|$ denotes the distance from x to the nearest integer. $\square$

We are now ready to prove Theorem 1.6.6.

Theorem 1.6.6. *Let $f : \mathbb{N} \to \{-1,1\}$ be a multiplicative function. Then*

$$\limsup_{x\to\infty}\left|\sum_{n\le x} f(n)\right| < \infty,$$

if and only if there exists an integer $m \ge 1$ such that $f(n+m) = f(n)$ for all $n \ge 1$ and $\sum_{n=1}^{m} f(n) = 0$.

Proof. If f satisfies $\sum_{i=1}^m f(i) = 0$ and $f(n) = f(n+m)$ for some $m \geq 1$, then for all $x \geq 1$,

$$\left|\sum_{n\leq x} f(n)\right| \leq m$$

and the claim follows. In the other direction, we assume $|\sum_{n\leq x} f(n)| = O_{x\to\infty}(1)$. By Theorem 4.2.2 we must have $f(2^i) = -1$ for all $i \geq 1$ and $\mathbb{D}(1,f,\infty) < \infty$. By the Lemma 4.2.5 we must have that for all $H \geq 1$,

$$\frac{1}{x}\sum_{n\leq x}\left(\sum_{k=n+1}^{n+H+1} f(k)\right)^2 = -2\sum_{a\geq 1,\ a\ odd} G(a)\left\|\frac{H}{2a}\right\| + o_{x\to\infty}(1) = O_{x\to\infty}(1).$$

Suppose that there is an infinite sequence of odd numbers $\{a_n\}_{n\geq 1}$ such that $g(a_n) \neq 0$. Observe, $|G(a_n)| \gg 1$. Choose $H = \mathrm{lcm}[a_1, \ldots a_M]$. If $f(3) = 1$, then by Lemma 4.2.3, part (4) we have

$$-2\sum_{a\geq 1,\ a\ odd} G(a)\left\|\frac{H}{2a}\right\| \geq -2\sum_{1\leq n\leq M} G(a_n)\left\|\frac{H}{2a_n}\right\| \gg M.$$

This is clearly impossible if M is sufficiently large.

Suppose $f(3) = -1$. Let

$$G^*(a) = \prod_{p^k||a,\ p>3}\left(|g(p^k)|^2 + 2\sum_{i\geq k+1}\frac{g(p^k)g(p^i)}{p^{i-k}}\right)$$

and

$$S(H) = -2\sum_{a\geq 1,\ (a,6)=1} G^*(a)\left\|\frac{H}{2a}\right\|.$$

Note that

$$-2\sum_{a\geq 1,\ a\ odd} G(a)\left\|\frac{H}{2a}\right\| = \sum_{i\geq 0} E_3\left(3^i\right) S\left(\frac{H}{3^i}\right) = O(1). \tag{4.2.2}$$

If $E_3(1) \geq 0$ then we proceed as in the previous case. If $E_3(1) < 0$, then $g(3) = f(3)-1 = -2$. Since $g(p^k)g(p^{k+1}) \leq 0$ for all $k \geq 0$ we get

$$E_3(3) \geq 4 - \frac{8}{9}\cdot\frac{1}{1-\frac{1}{9}} \geq 3$$

and

$$0 > E_3(1) = 1 + 2\sum_{i\geq 1}\frac{g(3^i)}{3^i} \geq 1 - \frac{4}{3}\cdot\frac{1}{1-\frac{1}{9}} = -\frac{1}{2}.$$

Since $E_3(3^k) \geq 0$ for all $k \geq 1$, applying triangle inequality in (4.2.2) yields

$$S(H) \geq \frac{E_3(3)S\left(\frac{H}{3}\right)}{-E_3(1)} + O(1) \geq 6S\left(\frac{H}{3}\right) - M. \tag{4.2.3}$$

If there is an infinite sequence $\{b_n\}_{n\geq 1}$ such that $g(b_n) \neq 0$ and $(b_n,6) = 1$, then we select H_0 as before such that $S(H_0) \geq M$ and $S(3H_0) \geq M$. Then (4.2.3) yields $S(3H_0) \geq 5S(H_0)$. By induction one easily gets that for all $n \geq 1$,

$$S(3^n H_0) \geq 5^n S(H_0).$$

This implies, that for the sequence $H_n = 3^n H_0$ we have $S(H_n) \gg H_n^{1+c}$. From the other hand

$$\sum_{a \geq H,\ (a,6)=1} \frac{G^*(a)}{a} = o_{H\to\infty}(1)$$

and so

$$\begin{aligned} -S(H) = 2 \sum_{a\geq 1,\ (a,6)=1} G^*(a) \left\| \frac{H}{2a} \right\| &\ll \sum_{a \leq H,\ (a,6)=1} G^*(a) + H \sum_{a \geq H,\ (a,6)=1} \frac{G^*(a)}{a} \\ &\ll \sqrt{H} \sum_{a \leq \sqrt{H},\ (a,6)=1} \frac{G^*(a)}{a} + H \sum_{\sqrt{H} \leq a \leq H,\ (a,6)=1} \frac{G^*(a)}{a} + o(H) \end{aligned}$$

and so $S(H) = o(H)$.

To finish the proof we are left to handle the case $g(3^k) \neq 0$ for infinitely many $k \geq 1$ and there exists finitely many b_1,$b_2 \ldots$,b_m $(b_i,6) = 1$, $i \geq 1$ and $g(b_i) \neq 0$. In this case we have

$$S(H) \leq \sum_{i=1}^{m} G^*(b_i) := M.$$

Choose $H_0 = \mathrm{lcm}[b_1, \ldots, b_m]$ and observe that $S(3^k H_0) \geq M/2$ for $k = 1, \ldots K$. Then,

$$\begin{aligned} -2 \sum_{a \geq 1, a\ odd} G(a) \left\| \frac{3^K H_0}{2a} \right\| &= \sum_{i \geq 0} E_3\left(3^i\right) S\left(\frac{3^K H_0}{3^i}\right) \\ &\geq \sum_{1 \leq i \leq K} E_3\left(3^i\right) S\left(\frac{3^K H_0}{3^i}\right) - E_3(1) S(H_0) \\ &\geq \frac{M}{2} \sum_{1 \leq i \leq K} E_3(3^k) - M. \end{aligned}$$

The last sum is bounded if $E_3(3^k) = 0$ for all $k \geq K_0$. Consequently, $f(3^k) = f(3^{k+1})$ for $k \geq K_0$ and the result follows. $\square$

4.3. Application to the conjecture of Kátai. Proof of Theorem 1.6.8

Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function and $\triangle f(n) = f(n+1) - f(n)$. In this section we focus on proving

Theorem 1.6.8. *If $f : \mathbb{N} \to \mathbb{C}$ is a multiplicative function and*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \leq x} |\triangle f(n)| = 0 \tag{4.3.1}$$

then either

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} |f(n)| = 0$$

or $f(n) = n^s$ *for some* $\mathrm{Re}(s) < 1$.

In [Kát00], Kátai, building on the ideas of Maclauire and Murata [MM80], showed that in order to prove Theorem 1.6.8, it is enough to consider a multiplicative f with $|f(n)| = 1$ for all $n \ge 1$. Observe, that if we denote

$$S(x) = \frac{1}{x} \sum_{n\le x} |\triangle(n)|$$

then (4.3.1) implies

$$\sum_{n\le x} \frac{|\triangle f(n)|^2}{n} \le \sum_{n\le x} \frac{2|\triangle f(n)|}{n} \ll \int_1^x \frac{S(t)}{t^2} dt = o(\log x).$$

We begin by proving the following lemma.

Lemma 4.3.1. *Suppose that* $f : \mathbb{N} \to \mathbb{T}$ *is multiplicative and*

$$\sum_{n\le x} \frac{|\triangle f(n)|^2}{n} \le 2(1-\varepsilon)\log x$$

for x *sufficiently large and some* $0 < \varepsilon < 1$*. Then, there exists a primitive character* $\chi_1(n)$ *and* $t_{\chi_1} \in \mathbb{R}$ *such that* $\mathbb{D}(f(n), \chi_1(n) n^{it_{\chi_1}}; \infty) < \infty$.

Proof. We note that

$$\mathrm{Re}\, f(n)\overline{f(n+1)} = 1 - \frac{|\triangle f(n)|^2}{2}$$

and therefore

$$\sum_{n\le x} \frac{\mathrm{Re}\, f(n)\overline{f(n+1)}}{n} \ge \varepsilon \log x + O(1).$$

We can now apply Lemma 4.1.3, since the only fact that was used in the proof is that the logarithmic correlation is large to conclude the result. $\square$

Remark 4.3.2. The conclusion of the lemma also holds if $f : \mathbb{N} \to \mathbb{T}$ satisfies

$$\sum_{n\le x} \frac{|\triangle f(n)|^2}{n} \ge 2(1+\varepsilon)\log x$$

for some $\varepsilon > 0$. In other words, if $\sum_{n\le x} \frac{|\triangle f(n)|^2}{n}$ is bounded away from $2\log x$, then

$$\mathbb{D}(f(n), \chi_1(n) n^{it_{\chi_1}}; \infty) < \infty.$$

Proposition 4.3.3. *Let* $f : \mathbb{N} \to \mathbb{T}$ *be a multiplicative function and* $\mathbb{D}(f, n^{it}\chi(n); \infty) < \infty$ *for some* $t \in \mathbb{R}$ *and a primitive character* χ *of conductor* q*. Then*

$$\sum_{n\le x} \frac{|\triangle f(n)|^2}{n} = 2(1 - E(f) + o(1))\log x$$

where

$$E(f) = \frac{\mu(q)}{q} \prod_{\substack{p \geq 1 \\ p \nmid q}} \left(2 \operatorname{Re} \left(1 - \frac{1}{p}\right) \left(\sum_{k \geq 0} \frac{f(p^k)\overline{\chi(p^k)} p^{-ikt}}{p^k} \right) - 1 \right).$$

Proof. Applying Corollary 3.5.1 we have that

$$M(y) = \sum_{n \leq y} f(n)\overline{f(n+1)} = y\frac{\mu(q)}{q} \prod_{\substack{p \geq 1 \\ p \nmid q}} \left(2 \operatorname{Re} \left(1 - \frac{1}{p}\right) \left(\sum_{k \geq 0} \frac{f(p^k)\overline{\chi(p^k)} p^{-ikt}}{p^k} \right) - 1 \right) + o(y).$$

Consequently,

$$\sum_{n \leq x} \frac{\operatorname{Re} f(n)\overline{f(n+1)})}{n} = \frac{M(x)}{x} + \int_1^x \frac{M(y)}{y^2} dy = \log x \cdot E(f) + o(\log x)$$

and

$$\sum_{n \leq x} \frac{|\triangle f(n)|^2}{n} = 2 \log x - 2 \sum_{n \leq x} \frac{\operatorname{Re} f(n)\overline{f(n+1)})}{n} + O(1) = 2(1 - E(f) + o(1)) \log x.$$

□

Corollary 4.3.4. *Let $f : \mathbb{N} \to \mathbb{T}$ be a multiplicative function such that $\mathbb{D}(f, n^{it}\chi(n); \infty) < \infty$ for some $t \in \mathbb{R}$ and a primitive character χ of conductor q. Suppose that*

$$\sum_{n \leq x} \frac{|\triangle f(n)|^2}{n} = o(\log x).$$

Then, $f(n) = n^{it}$.

Proof. Proposition 4.3.3 implies that $E(f) = 1$. We have that for all $p \geq 2$, $p \nmid q$, each Euler factor

$$E_p(f) = 2\left(1 - \frac{1}{p}\right) \sum_{k \geq 0} \frac{\operatorname{Re} f(p^k)\overline{\chi(p^k)} p^{-ikt}}{p^k} - 1 \geq 2\left(1 - \frac{1}{p}\right)\left(1 - \sum_{k \geq 1} \frac{1}{p^k}\right) - 1 = \frac{p-4}{p} \geq -1$$

with the possible equality only at $p = 2$. From the other hand,

$$E_p(f) \leq 2\left(1 - \frac{1}{p}\right)\left(\sum_{k \geq 0} \frac{1}{p^k}\right) - 1 = 1.$$

Consequently, we must have $q = 1$ and $|E_p(f)| = 1$ for all $p \geq 2$. Since $E(f) = 1 > 0$, we have $E_2(f) \neq -1$ and

$$2\left(1 - \frac{1}{p}\right) \sum_{k \geq 0} \frac{\operatorname{Re} f(p^k) p^{-ikt}}{p^k} - 1 = 1.$$

This is possible if only if $f(p^k) = p^{kit}$ for all $p \geq 2$ and $k \geq 1$. The result follows. □

Theorem 1.6.8 now follows from the following

Proposition 4.3.5. *Let $f : \mathbb{N} \to \mathbb{T}$ be a multiplicative function such that*

$$\sum_{n \le x} \frac{|\triangle f(n)|^2}{n} = o(\log x).$$

Then, $f(n) = n^{it}$ for some $t \in \mathbb{R}$.

Proof. Applying Lemma 4.3.1 we can find a primitive character χ and $t \in \mathbb{R}$ such that

$$\mathbb{D}(f(n),\chi(n)n^{it};\infty) < \infty.$$

We now apply Corollary 4.3.4 to conclude that $f(n) = n^{it}$. ☐

4.4. Applications to the binary additive problems

As was mentioned in the introduction Brüdern established the following result.

Theorem 1.6.9. **[Brüdern, 2008]** *Suppose A and B are multiplicative sequences of positive density ρ_A and ρ_B respectively. For $k \ge 1$, let*

$$a(p^k) = \rho_A(p^k)/p^k - \rho_A(p^{k-1})/p^{k-1}$$

Define $b(h)$ in the same fashion. Then, $r(n) = \rho_A \rho_B \sigma(n) n + o(n)$ when $n \to \infty$, where

$$\sigma(n) = \prod_{p^m || n} \left(1 + \sum_{k=1}^{m} \frac{p^{k-1} a(p^k) b(p^k)}{p-1} - \frac{p^m a(p^{m+1}) b(p^{m+1})}{(p-1)^2}\right).$$

We now sketch how one can derive this from our main result.

Proof. Let $f(n) = \mathrm{I}_A(n)$ and $g(n) = \mathrm{I}_B(n)$. Clearly both, f and g are multiplicative taking values $\{0,1\}$. Since $\rho_A > 0$, we have

$$\limsup_x \frac{1}{x} \sum_{n \le x} f(n) > 0.$$

Theorem of Delange readily implies that $\mathbb{D}(1,f;\infty) < \infty$. By analogy, $\mathbb{D}(1,g;\infty) < \infty$. Furthermore,

$$\rho_A = \lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} f(n) = \mathrm{P}(f,1,\infty)$$

and

$$\rho_B = \lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} g(n) = \mathrm{P}(g,1,\infty).$$

Notice that

$$r(m) = \sum_{n \le m} f(n) g(m-n).$$

We note that following the proof of Corollary 1.6.4 we may let $a = 1$, $c = 0$, $b = -1$, $d = m$. Despite the fact that $d = m \to \infty$ the error term is still bounded by (3.3.1). Corollary 1.6.4

gives

$$r(m) = \sum_{\text{ł}|m} \frac{G(f; g; \text{ł}; \infty)}{\text{ł}} m + o(m).$$

A straightforward manipulation with the Euler factors shows that the latter has the Euler product described above. □

Remark 4.4.1. In case one of the sets A,B has density zero, say $\rho_A = 0$ we can apply Delange's theorem to conclude

$$r(m) = \sum_{n \leq m} f(n)g(m-n) \leq \sum_{n \leq m} f(n) = o(m).$$

Chapter 5

MULTILINEAR CORRELATIONS AND APPLICATIONS

5.1. Preparatory lemmas for the proof of Theorem 1.7.1

For the proof of Theorem 1.7.1, we will need several technical results. The first is a version of the Turán-Kubilius inequality applicable to additive functions whose arguments are integral affine linear forms in several variables. While the proof of this result is fairly routine, we could not find it in the literature. We therefore give a full proof here for completeness. We first require some definitions.

For a primitive, integral affine linear form $L : \mathbb{R}^l \to \mathbb{R}$, let

$$\omega_L(p^k) := |\{\boldsymbol{b} \in (\mathbb{Z}/p^k\mathbb{Z})^l : p^k || L(\boldsymbol{b})\}|.$$

Furthermore, given an additive function $h : \mathbb{N} \to \mathbb{C}$, put

$$\mu_{h,L}(x) := \sum_{p^k \leqslant x} h(p^k) \left(\frac{\omega_L(p^k)}{p^{kl}} - \frac{\omega_L(p^{k+1})}{p^{(k+1)l}} \right);$$

$$\sigma_{h,L}(x)^2 := \sum_{p^k \leqslant x} |h(p^k)|^2 \left(\frac{\omega_L(p^k)}{p^{kl}} - \frac{\omega_L(p^{k+1})}{p^{(k+1)l}} \right).$$

Remark 5.1.1. Note that we can lift a solution to the congruence $L(\boldsymbol{b}) \equiv 0(p^k)$ to precisely $\omega_{L-L(\mathbf{0})}(p)$ distinct solutions mod p^{k+1} via $b_j' := r_j p^k + b_j$ whenever the vector $\boldsymbol{r}$ satisfies $L(\boldsymbol{r}) - L(\mathbf{0}) \equiv 0(p)$. Moreover, since L is primitive there is some index $1 \leqslant j_0 \leqslant k$ such that the coefficient a_{j_0} satisfies $(a_{j_0}, p) = 1$. Thus, given any choice of r_j for $j \neq j_0$, there is a unique r_{j_0} mod p such that the congruence $L(\boldsymbol{r}) - L(\mathbf{0}) \equiv 0(p)$ is satisfied. Hence, $\omega_{L-L(\mathbf{0})}(p) = p^{l-1}$, and by induction, we have $\omega_L(p^\mu) = \omega_{L-L(\mathbf{0})}(p)^\mu = p^{\mu(l-1)}$. Thus, we can rewrite $\mu_{h,L}$ and $\sigma_{h,L}^2$ as

$$\mu_h(x) = \mu_{h,L}(x) = \sum_{p^k \leqslant x} \frac{h(p^k)}{p^k} \left(1 - \frac{1}{p}\right); \tag{5.1.1}$$

$$\sigma_h(x)^2 = \sigma_{h,L}(x)^2 = \sum_{p^k \leqslant x} \frac{|h(p^k)|^2}{p^k}\left(1 - \frac{1}{p}\right). \tag{5.1.2}$$

Write $X := \ell(\boldsymbol{x}) + 1$, for $\boldsymbol{x} \in (0,\infty)^l$.

Lemma 5.1.2. Let $A \geqslant 1$ and $\boldsymbol{x} \in [1,\infty)^l$. Let $h : \mathbb{N} \to \mathbb{C}$ be an additive function satisfying $|h(p^k)| \ll 1$ uniformly on prime powers p^k, and suppose that L is a primitive integral affine linear form in l variables with height at most A. Then

$$\langle \boldsymbol{x} \rangle^{-1} \sum_{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x})} |h(L(\boldsymbol{n})) - \mu_h(AX)|^2 \ll_l \sigma_h(AX)^2 + \frac{|\mu_h(AX)|}{x_-}. \tag{5.1.3}$$

Thus, if f is a 1-bounded multiplicative function and h is the additive function defined by $h(p^k) = f(p^k) - 1$ and $\boldsymbol{x}$ is (A,B)-appropriate then

$$\langle \boldsymbol{x} \rangle^{-1} \sum_{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x})} |h(L(\boldsymbol{n})) - \mu_h(AX)|^2 \ll_l \mathbb{D}^*(1,f;AX)^2 + \frac{1}{(\log X)^B}. \tag{5.1.4}$$

Proof. Observe first that

$$\begin{aligned}
\sum_{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x})} h(L(\boldsymbol{n})) &= \sum_{p^k \leqslant AX} h(p^k) \sum_{\substack{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x}) \\ p^k || L(\boldsymbol{n})}} 1 \\
&= \sum_{p^k \leqslant AX} h(p^k) \left(\sum_{\substack{\boldsymbol{b} \in (\mathbb{Z}/p^k\mathbb{Z})^l \\ L(\boldsymbol{b}) \equiv 0 (p^k)}} \sum_{\substack{\boldsymbol{n} \in \mathcal{B}(x) \\ n_j \equiv b_j (p^k) \forall j}} 1 - \sum_{\substack{\boldsymbol{b} \in (\mathbb{Z}/p^k\mathbb{Z})^l \\ L(\boldsymbol{b}) \equiv 0 (p^{k+1})}} \sum_{\substack{\boldsymbol{n} \in \mathcal{B}(x) \\ n_j \equiv b_j (p^{k+1}) \forall j}} 1 \right) \\
&= \langle \boldsymbol{x} \rangle \left(1 + O\left(x_-^{-1}\right)\right) \sum_{p^k \leqslant AX} h(p^k) \left(\frac{\omega_L(p^k)}{p^{kl}} - \frac{\omega_L(p^{k+1})}{p^{(k+1)l}} \right) \\
&= \langle \boldsymbol{x} \rangle (1 + O(x_-^{-1})) \mu_{h,L}(AX).
\end{aligned}$$

Expanding the square in (5.1.3), we thus get

$$\begin{aligned}
&\langle \boldsymbol{x} \rangle^{-1} \sum_{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x})} |h(L(\boldsymbol{n})) - \mu_{h,L}(AX)|^2 \\
&= \langle \boldsymbol{x} \rangle^{-1} \sum_{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x})} |h(L(\boldsymbol{n}))|^2 - 2\mathrm{Re}\left(\overline{\mu_{h,L}(AX)} \langle \boldsymbol{x} \rangle^{-1} \sum_{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x})} h(L(\boldsymbol{n})) \right) + |\mu_{h,L}(AX)|^2 \\
&= \langle \boldsymbol{x} \rangle^{-1} \sum_{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x})} |h(L(\boldsymbol{n}))|^2 - |\mu_{h,L}(AX)|^2 + O(|\mu_{h,L}(AX)|^2 x_-^{-1}).
\end{aligned} \tag{5.1.5}$$

The first term in (5.1.5) can be rewritten as

$$\begin{aligned}
&\langle \boldsymbol{x} \rangle^{-1} \sum_{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x})} |h(L(\boldsymbol{n}))|^2 = \langle \boldsymbol{x} \rangle^{-1} \sum_{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x})} \sum_{p^\mu, q^\nu || L(\boldsymbol{n})} h(p^\mu) \overline{h(p^\nu)} \\
&= \langle \boldsymbol{x} \rangle^{-1} \sum_{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x})} \sum_{p^\mu || L(\boldsymbol{n})} |h(p^\mu)|^2 + \langle \boldsymbol{x} \rangle^{-1} \sum_{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x})} \sum_{\substack{p^\mu, q^\nu || L(\boldsymbol{n}) \\ p \neq q}} h(p^\mu) \overline{h(q^\nu)}
\end{aligned}$$

$$= \left(1+O(x_-^{-1})\right) \sum_{p^\mu \leqslant AX} |h(p^\mu)|^2 \left(\frac{\omega_L(p^\mu)}{p^{\mu l}} - \frac{\omega_L(p^{\mu+1})}{p^{(\mu+1)l}}\right)$$

$$+ \langle \boldsymbol{x}\rangle^{-1} \sum_{\substack{p^\mu, q^\nu \leqslant AX \\ p\neq q}} h(p^\mu)\overline{h(q^\nu)} \sum_{\substack{\boldsymbol{n}\in\mathcal{B}(\boldsymbol{x}) \\ p^\mu, q^\nu || L(\boldsymbol{n})}} 1$$

$$= \left(1+O(x_-^{-1})\right)$$

$$\left(\sigma_{h,L}^2(AX) + \sum_{\substack{p^\mu, q^\nu \leqslant AX \\ p\neq q}} h(p^\mu)\overline{h(q^\nu)} \left(\frac{\omega_{h,L}(p^\mu)}{p^{\mu l}} - \frac{\omega_{h,L}(p^{\mu+1})}{p^{(\mu+1)l}}\right)\left(\frac{\omega_{h,L}(q^\nu)}{q^{\nu l}} - \frac{\omega_{h,L}(q^{\nu+1})}{q^{(\nu+1)l}}\right)\right).$$

The second term in (5.1.5) can be expressed as

$$|\mu_{h,L}(AX)|^2 = \sum_{p^\mu, q^\nu \leqslant AX} h(p^\mu)\overline{h(q^\nu)} \left(\frac{\omega_L(p^\mu)}{p^{\mu l}} - \frac{\omega_L(p^{\mu+1})}{p^{(\mu+1)l}}\right)\left(\frac{\omega_L(q^\nu)}{q^{\nu l}} - \frac{\omega_L(q^{\nu+1})}{q^{(\nu+1)l}}\right)$$

$$= \left(\sum_{\substack{p^\mu, q^\nu \leqslant AX \\ p\neq q}} + \sum_{\substack{p^\mu, q^\nu \leqslant AX \\ p= q}}\right) h(p^\mu)\overline{h(q^\nu)} \left(\frac{\omega_L(p^\mu)}{p^{\mu l}} - \frac{\omega_L(p^{\mu+1})}{p^{(\mu+1)l}}\right)\left(\frac{\omega_{h,L}(q^\nu)}{q^{\nu l}} - \frac{\omega_L(q^{\nu+1})}{q^{(\nu+1)l}}\right).$$

Subtracting these two expressions gives

$$\langle \boldsymbol{x}\rangle^{-1} \sum_{\boldsymbol{n}\in\mathcal{B}(\boldsymbol{x})} |h(L(\boldsymbol{n}))|^2 - |\mu_{h,L}(AX)|^2$$

$$\ll \sigma_{h,L}(AX)^2 + |\mu_{h,L}(AX)|^2 x{-}^{-1}$$

$$+ \left| \sum_{p^\mu \leqslant p^\nu \leqslant AX} h(p^\mu)\overline{h(p^\nu)} \left(\frac{\omega_{h,L}(p^\mu)}{p^{\mu l}} - \frac{\omega_{h,L}(p^{\mu+1})}{p^{(\mu+1)l}}\right)\left(\frac{\omega_{h,L}(p^\nu)}{p^{\nu l}} - \frac{\omega_{h,L}(p^{\nu+1})}{p^{(\nu+1)l}}\right)\right|.$$

Hence, by Cauchy-Schwarz and (5.1.1),

$$\langle \boldsymbol{x}\rangle^{-1} \sum_{\boldsymbol{n}\in\mathcal{B}(\boldsymbol{x})} |h(L(\boldsymbol{n})) - \mu_{h,L}(AX)|^2 \ll \sigma_{h,L}(AX)^2 + |\mu_{h,L}(AX)|^2 x_-^{-1}.$$

This prove (5.1.3). For (5.1.4), note that by (5.1.2)

$$\sigma_{h,L}(AX)^2 \ll \sum_{p^k \leqslant AX} \frac{|f(p^k)-1|^2}{p^k} \ll \sum_{p^k \leqslant AX} \frac{1-\mathrm{Re}(f(p^k))}{p^k} = \mathbb{D}^*(1,f;AX)^2,$$

and that (5.1.1) together with the (A,B)-appropriateness condition imply that

$$|\mu_{h,L}(AX)|^2 x_-^{-1} \ll \log_2((l+1)Ax_+)^2 x_-^{-1} \leqslant \frac{1}{(\log x_+)^B} \ll_l \frac{1}{(\log(lx_+))^B} \leqslant \frac{1}{(\log X)^B}.$$

□

For $1 \leqslant y \leqslant x$ and each $1 \leqslant j \leqslant k$, define

$$\mathfrak{P}(f_j; y,x) := \prod_{y<p\leqslant x} \sum_{\nu\geqslant 0} f_j(p^\nu) \left(\frac{\omega_{L_j}(p^\nu)}{p^{\nu l}} - \frac{\omega_{L_j}(p^{\nu+1})}{p^{(\nu+1)l}}\right) = \prod_{y<p\leqslant x}\left(1-\frac{1}{p}\right)\left(1+\sum_{k\geqslant 1}\frac{f_j(p^k)}{p^k}\right),$$

and write $\mathfrak{P}(f_j;x) := \mathfrak{P}(f_j;1,x)$. The first representation for $\mathfrak{P}(f_j;y,x)$ will be useful later; the second one follows from Remark 5.1.1.

The following lemma allows us to conveniently decompose multilinear averages of products of arithmetic functions with good error, provided that one of the sequences is multiplicative and 1-pretentious.

Lemma 5.1.3. Let $A \geqslant 2$, $q \geqslant 1$ and let $\boldsymbol{x}$ be (A,B)-appropriate. Let $g : \mathbb{N}^l \to \mathbb{U}$ be any sequence and let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function such that $f(n) = 1$ whenever $(n,q) > 1$. Also, let $L : \mathbb{R}^l \to \mathbb{R}$ be a primitive, integral form with height at most A. Then

$$\sum_{\substack{\boldsymbol{n}\in\mathcal{B}(\boldsymbol{x})\\ q|L(\boldsymbol{n})}} f(L(\boldsymbol{n}))g(\boldsymbol{n}) = \mathfrak{P}(f;AX)\left(\sum_{\substack{\boldsymbol{n}\in\mathcal{B}(\boldsymbol{x})\\ q|L(\boldsymbol{n})}} g(\boldsymbol{n})\right) + O\left(\frac{\langle\boldsymbol{x}\rangle}{\sqrt{q}}\left(\mathbb{D}^*(1,f;AX) + \frac{1}{(\log(AX))^{B'}}\right)\right),$$

where $B' := \min\{1,B/2\}$.

Proof. Since, for all $|z_j|,|w_j| \le 1$, $1 \leqslant j \leqslant n$,

$$\begin{aligned}\left|\prod_{1\le j\le n} z_j - \prod_{1\le j\le n} w_j\right| &= \left|\prod_{1\leqslant j\leqslant n-1} z_j(z_n - w_n) + w_n\left(\prod_{1\leqslant j\leqslant n-1} z_j - \prod_{1\leqslant j\leqslant n-1} w_j\right)\right| \\ &\leqslant |z_n - w_n| + \left|\prod_{1\leqslant j\leqslant n-1} z_j - \prod_{1\leqslant j\leqslant n-1} w_j\right|,\end{aligned}$$

it follows by induction that

$$\left|\prod_{1\le j\le n} z_j - \prod_{1\le j\le n} w_j\right| \leqslant \sum_{1\le j\le n} |z_j - w_j|. \tag{5.1.6}$$

Note that $e^{z-1} = z + O(|z-1|^2)$ for $|z| \le 1$. Therefore,

$$\begin{aligned}f(L(\boldsymbol{n})) &= \prod_{p^k||L(\mathbf{n})} f(p^k) = \prod_{p^k||L(\mathbf{n})} e^{f(p^k)-1} + O\left(\left|\prod_{p^k||L(\mathbf{n})} f(p^k) - \prod_{p^k||L(\mathbf{n})}\Big(f(p^k) + O(|f(p^k)-1|^2)\Big)\right|\right) \\ &= \exp\left(\sum_{p^k||L(\mathbf{n})} (f(p^k)-1)\right) + O\left(\sum_{p^k||L(\mathbf{n})} |f(p^k)-1|^2\right).\end{aligned}$$

Define $h : \mathbb{N} \to \mathbb{C}$ to be the additive function satisfying $h(p^k) = f(p^k) - 1$ for each prime p and $k \geqslant 1$. Note that $h(p^k) = 1$ whenever $p|q$. Hence,

$$\begin{aligned}\sum_{\mathbf{n}\in\mathcal{B}(\boldsymbol{x})} f(L(\mathbf{n}))g(\mathbf{n})1_{q|L(\boldsymbol{n})} - \sum_{\mathbf{n}\in\mathcal{B}(\boldsymbol{x})} g(\mathbf{n})e^{h(L(\mathbf{n}))}1_{q|L(\boldsymbol{n})} &\ll \sum_{\mathbf{n}\in\mathcal{B}(\boldsymbol{x})} 1_{q|L(\boldsymbol{n})} \sum_{\substack{p^k||L(\mathbf{n}),\\ p^k\leqslant AX}} |h(p^k)|^2 \\ &\ll \langle\boldsymbol{x}\rangle \sum_{\substack{p^k\leqslant AX\\ p|q}} \frac{|f(p^k)-1|^2}{[q,p^k]} \ll \frac{1}{q}\langle\boldsymbol{x}\rangle\mathbb{D}^*(f,1;AX)^2.\end{aligned}$$

Since $|e^a - e^b| \ll |a-b|$ for $\mathrm{Re}\,(a), \mathrm{Re}\,(b) \le 0$, Cauchy-Schwarz together with Lemma 5.1.2 imply

$$\begin{aligned}
&\sum_{\mathbf{n}\in\mathcal{B}(\boldsymbol{x})} g(\mathbf{n}) e^{h(L(\mathbf{n}))} 1_{q|L(\boldsymbol{n})} - e^{\mu_h(AX)} \sum_{\mathbf{n}\in\mathcal{B}(\boldsymbol{x})} g(\mathbf{n}) 1_{q|L(\boldsymbol{n})} \\
&\ll \sum_{\mathbf{n}\in\mathcal{B}(\boldsymbol{x})} |e^{h(L(\mathbf{n}))} - e^{\mu_h(AX)}| 1_{q|L(\boldsymbol{n})} \ll \sum_{\mathbf{n}\in\mathcal{B}(\boldsymbol{x})} |h(L(\mathbf{n})) - \mu_h(X)| 1_{q|L(\boldsymbol{n})} \\
&\le \left(\frac{\langle \boldsymbol{x} \rangle}{q} \sum_{\mathbf{n}\in\mathcal{B}(\boldsymbol{x})} |h(L(\mathbf{n})) - \mu_h(AX)|^2 \right)^{1/2} \ll \frac{\langle \boldsymbol{x} \rangle}{\sqrt{q}} \left(\mathbb{D}^*(f,1;AX) + \frac{1}{\log X} \right).
\end{aligned}$$

For each $p \leqslant AX$, put

$$\mu_{h,p} = \left(1 - \frac{1}{p}\right) \sum_{k: p^k \le AX} \frac{h(p^k)}{p^k}$$

so that in light of Remark 5.1.1, $\mu_h(AX) = \sum_{p \leqslant AX} \mu_{h,p}$. Observe that

$$\begin{aligned}
e^{\mu_{h,p}} &= 1 + \mu_{h,p} + O(\mu_{h,p}^2) \\
&= 1 + \left(1 - \frac{1}{p}\right) \sum_{k \geqslant 0} \frac{f(p^k)}{p^k} - \left(1 - \frac{1}{p}\right) \sum_{k \geqslant 1} \frac{1}{p^k} + O\left(\left| \sum_{p^k \le AX} \frac{h(p^k)}{p^k} \right|^2 + \sum_{p^k > AX} \frac{1}{p^k} \right) && (5.1.7) \\
&= \left(1 - \frac{1}{p}\right) \sum_{k \geqslant 0} \frac{f(p^k)}{p^k} + O\left(\frac{1}{p} \sum_{p^k \leqslant AX} \frac{|f(p^k) - 1|^2}{p^k} + (AX)^{-1} \right), && (5.1.8)
\end{aligned}$$

where we applied the Cauchy-Schwarz inequality to the first error term.
Since $\mathrm{Re}(h(p^k)) \leqslant 0$ for all $k \geqslant 1$, $|e^{\mu_{h,p}}| \leqslant 1$; also, $|\mathfrak{P}(f,AX)| \leqslant 1$ trivially. Thus, applying (5.1.6) and the Cauchy-Schwarz inequality once again,

$$\begin{aligned}
|e^{\mu_h(AX)} - \mathfrak{P}(f;AX)| &\leqslant \sum_{p \le AX} \left| e^{\mu_{h,p}} - \left(1 - \frac{1}{p}\right) \sum_{k \geqslant 0} \frac{f(p^k)}{p^k} \right| \\
&\ll \sum_{p^k \le AX} \frac{1}{p} \frac{|f(p^k) - 1|^2}{p^k} + (AX)^{-1} \sum_{p \le AX} 1 \\
&\ll \mathbb{D}^*(f,1;AX) + \frac{1}{\log(AX)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&e^{\mu_h(AX)} \sum_{\boldsymbol{n}\in\mathcal{B}(\boldsymbol{x})} g(\boldsymbol{n}) 1_{q|L(\boldsymbol{n})} \\
&\qquad = \mathfrak{P}(f;AX) \sum_{\boldsymbol{n}\in\mathcal{B}(\boldsymbol{x})} g(\boldsymbol{n}) 1_{q|L(\boldsymbol{n})} + O\left(\frac{\langle \boldsymbol{x} \rangle}{q} \left(\mathbb{D}^*(f,1;AX) + \frac{1}{\log(AX)} \right) \right),
\end{aligned}$$

and the claim follows. □

Next, we show how the factors $\mathfrak{P}(f_j;x)$ relate to the p-adic local factors $M_p(\boldsymbol{f},\boldsymbol{L})$.

Lemma 5.1.4. Let $X \geqslant y \geqslant 2$. Let $\boldsymbol{L}$ be a primitive integral system of size k, and let $\boldsymbol{f}$ be a vector of k 1-bounded multiplicative functions that are supported on prime powers $p^\mu > y$. Then, as $y \to \infty$,

$$\prod_{y<p\leqslant X} M_p(\boldsymbol{f};\boldsymbol{L}) = \left(1 + O\left(\frac{k}{\log y}\right)\right)\left(\prod_{1\leqslant j\leqslant k} \mathfrak{P}(f_j;X) + O\left(y^{-1+o(1)}\right)\right).$$

Proof. Let x be large positive real number. We have

$$\begin{aligned} x^{-l}\sum_{\boldsymbol{n}\in[x]^l}\prod_{1\leqslant j\leqslant k} f_{j,p}(L_j(\boldsymbol{n})) &= x^{-l}\sum_{\nu_1,\ldots,\nu_k\geqslant 0}\prod_{1\leqslant j\leqslant k} f_j(p^{\nu_j}) \sum_{\substack{\boldsymbol{b}^{(1)}\in(\mathbb{Z}/p^{\nu_1}\mathbb{Z})^l\\ p^{\nu_1}\,||\,L_1(\boldsymbol{b}^{(1)})}}\cdots\sum_{\substack{\boldsymbol{b}^{(k)}\in(\mathbb{Z}/p^{\nu_k}\mathbb{Z})^l\\ p^{\nu_k}\,||\,L_k(\boldsymbol{b}^{(k)})}}\sum_{\substack{\boldsymbol{n}\in[x]^l\\ n_j\equiv b_j^{(t)}(p^{\nu_t})\forall j,t}} 1 \\ &= x^{-l}\sum_{\nu_1,\ldots,\nu_k\geqslant 0}\prod_{1\leqslant j\leqslant k} f_j(p^{\nu_j}) \sum_{\substack{\boldsymbol{b}^{(1)}\in(\mathbb{Z}/p^{\nu_1}\mathbb{Z})^l\\ p^{\nu_1}\,||\,L_1(\boldsymbol{b}^{(1)})}}\cdots\sum_{\substack{\boldsymbol{b}^{(k)}\in(\mathbb{Z}/p^{\nu_k}\mathbb{Z})^l\\ p^{\nu_k}\,||\,L_k(\boldsymbol{b}^{(k)})}}\prod_{1\leqslant j\leqslant l}\sum_{\substack{n_j\leqslant x\\ n_j\equiv b_j^{(t)}(p^{\nu_t})\forall t}} 1. \end{aligned} \tag{5.1.9}$$

By the Chinese remainder theorem, for each j there is a unique solution modulo $p^{\max_{1\leqslant t\leqslant k}\nu_t}$ to the k simultaneous congruences in the inner sum of (5.1.9) if, and only if, $b_j^{(r)} \equiv b_j^{(s)}(p^{\min(\nu_r,\nu_s)})$ for each $1 \leqslant r < s \leqslant k$. Hence, the right side of (5.1.9) is

$$\sum_{0\leqslant\nu_1,\ldots,\nu_k\leqslant\log x/\log p}\left(\prod_{1\leqslant j\leqslant k} f_j(p^{\nu_j})\right)p^{-l\max_t\nu_t}\sum_{\substack{\boldsymbol{b}^{(1)}\in(\mathbb{Z}/p^{\nu_1}\mathbb{Z})^l\\ p^{\nu_1}\,||\,L_1(\boldsymbol{b}^{(1)})}}\cdots\sum_{\substack{\boldsymbol{b}^{(k)}\in(\mathbb{Z}/p^{\nu_k}\mathbb{Z})^l\\ p^{\nu_k}\,||\,L_k(\boldsymbol{b}^{(k)})\\ b_t^{(r)}\equiv b_t^{(s)}(p^{\min(\nu_r,\nu_s)})\forall r,s}} 1 + O\left(x^{-1}\left(\frac{\log X}{\log p}\right)^l\right).$$

Taking $x \to \infty$, we therefore have

$$\prod_{y<p\leqslant X} M_p(\boldsymbol{f},\boldsymbol{L}) = \prod_{y<p\leqslant X}\sum_{\nu_1,\ldots,\nu_k\geqslant 0}\left(\prod_{1\leqslant j\leqslant k} f_j(p^{\nu_j})\right)p^{-l\max_t\nu_t}\sum_{\substack{\boldsymbol{b}^{(1)}\in(\mathbb{Z}/p^{\nu_1}\mathbb{Z})^l\\ p^{\nu_1}\,||\,L_1(\boldsymbol{b}^{(1)})}}\cdots\sum_{\substack{\boldsymbol{b}^{(k)}\in(\mathbb{Z}/p^{\nu_k}\mathbb{Z})^l\\ p^{\nu_k}\,||\,L_k(\boldsymbol{b}^{(k)})\\ b_t^{(r)}\equiv b_t^{(s)}(p^{\min(\nu_r,\nu_s)})}} 1. \tag{5.1.10}$$

Now, consider the product of the factors $\mathfrak{P}(f_j;X)$, i.e.,

$$\prod_{1\leqslant j\leqslant k}\mathfrak{P}(f_j;X) = \prod_{1\leqslant j\leqslant k}\prod_{p\leqslant X}\sum_{\nu_j\geqslant 0} f_j(p^{\nu_j})\left(\frac{\omega_{L_j}(p^{\nu_j})}{p^{\nu_j l}} - \frac{\omega_{L_j}(p^{\nu_j+1})}{p^{(\nu_j+1)l}}\right).$$

By the Prime Number Theorem, the contribution from $p \leqslant y$ is

$$\prod_{1\leqslant j\leqslant k}\prod_{p\leqslant y}\left(1 + O\left(\sum_{\substack{p^\nu>y\\ p\leqslant y}}\frac{1}{p^\nu}\right)\right) = \prod_{1\leqslant j\leqslant k}\left(1 + O\left(y^{-1}\pi(y)\right)\right) = 1 + O\left(\frac{k}{\log y}\right), \tag{5.1.11}$$

whence

$$\prod_{1\leqslant j\leqslant k}\mathfrak{P}(f_j;X) = \left(1 + O\left(\frac{k}{\log y}\right)\right)\prod_{1\leqslant j\leqslant k}\mathfrak{P}(f_j;y,X)$$

$$= \left(1 + O\left(\frac{k}{\log y}\right)\right) \prod_{y<p\leqslant X} \sum_{\nu_1,\dots,\nu_k \geqslant 0} \prod_{1\leqslant j\leqslant k} f(p^{\nu_j}) \left(\frac{\omega_{L_j}(p^{\nu_j})}{p^{\nu_j l}} - \frac{\omega_{L_j}(p^{\nu_j+1})}{p^{(\nu_j+1)l}}\right).$$

Subtracting $\prod_{y<p\leqslant X} M_p(\boldsymbol{f},\boldsymbol{L})$ from $\prod_{1\leqslant j\leqslant k} \mathfrak{P}(f_j; y,X)$ and using the fact that $|\mathfrak{P}(f_j; y,X)| \leqslant 1$ for each j, we get

$$\left| \prod_{y<p\leqslant X} M_p(\boldsymbol{f},\boldsymbol{L}) - \prod_{1\leqslant j\leqslant k} \mathfrak{P}(X,y; f_j) \right|$$
$$\leqslant \sum_{y<p\leqslant X} \sum_{\nu_1,\dots,\nu_k \geqslant 0} \left| \prod_{1\leqslant j\leqslant k} \left(\frac{\omega_{L_j}(p^{\nu_j})}{p^{\nu_j l}} - \frac{\omega_{L_j}(p^{\nu_j+1})}{p^{(\nu_j+1)l}}\right) - p^{-l \max_t \nu_t} \sum_{\substack{\boldsymbol{b}^{(1)} \in (\mathbb{Z}/p^{\nu_1}\mathbb{Z})^l \\ p^{\nu_1} || L_1(\boldsymbol{b}^{(1)})}} \cdots \sum_{\substack{\boldsymbol{b}^{(k)} \in (\mathbb{Z}/p^{\nu_k}\mathbb{Z})^l \\ p^{\nu_k} || L_k(\boldsymbol{b}^{(k)}) \\ b_t^{(r)} \equiv b_t^{(s)} (p^{\min(\nu_r,\nu_s)})}} 1 \right|. \tag{5.1.12}$$

Observe that when at most one of the indices $1 \leqslant j \leqslant k$ satisfies $\nu_j \geqslant 1$, the compatibility condition on the vectors $\boldsymbol{b}^{(t)}$ is automatically satisfied, and can hence be dropped. Thus, for $\sum_{1\leqslant j\leqslant k} \nu_j \leqslant 1$, the k sums over vectors $\boldsymbol{b}^{(t)}$ in (5.1.10) are precisely

$$p^{-l \max_t \nu_t} \sum_{\substack{\boldsymbol{b}^{(1)} \in (\mathbb{Z}/p^{\nu_1}\mathbb{Z})^l \\ p^{\nu_1} || L_1(\boldsymbol{b}^{(1)})}} \cdots \sum_{\substack{\boldsymbol{b}^{(k)} \in (\mathbb{Z}/p^{\nu_k}\mathbb{Z})^l \\ p^{\nu_k} || L_k(\boldsymbol{b}^{(k)})}} 1 = \prod_{1\leqslant j\leqslant k} \left(\frac{\omega_{L_j}(p^{\nu_j})}{p^{\nu_j l}} - \frac{\omega_{L_j}(p^{\nu_j+1})}{p^{(\nu_j+1)l}}\right).$$

By well-known results on partitions (see, for instance [Erd42]), the number of terms in the ν_j sums with $\sum_{1\leqslant j\leqslant k} \nu_j = m \geqslant 2$ is at most $e^{C\sqrt{m}}$, where $C > 0$ is absolute. Since each of the inner terms in (5.1.12) has size $O(p^{-m})$, it follows that

$$\begin{aligned} \left| \prod_{y<p\leqslant X} M_p(\boldsymbol{f},\boldsymbol{L}) - \prod_{1\leqslant j\leqslant k} \mathfrak{P}(f_j; y,X) \right| &\ll \sum_{y<p\leqslant X} \sum_{m\geqslant 2} e^{C\sqrt{m}} p^{-m} \\ &\ll \sum_{y<p\leqslant X} p^{-2+o(1)} \ll y^{-1+o(1)}. \end{aligned}$$

Combining this with (5.1.11) completes the proof. □

Lastly, we shall require the following smooth numbers estimate due to DeBruijn [DeB51]. Recall that for $x \geqslant y \geqslant 2$, $\Psi(x,y)$ denotes the number of integers less than or equal to x, all of whose prime factors are less than or equal to y.

Lemma 5.1.5. For $x \geqslant y \geqslant 2$,

$$\log \Psi(x,y) = (1 + o(1)) \left(\frac{\log x}{\log y} \log\left(1 + \frac{y}{\log x}\right) + \frac{y}{\log y} \log\left(1 + \frac{\log x}{y}\right)\right).$$

5.2. Preparatory lemmas for the proof of Proposition 1.7.5

As mentioned in the introduction, Proposition 1.7.5 follows from Theorem 1.6 of [MRT15]. A special case of the latter, which we use in the sequel (see Section 4), is as follows.

Theorem 5.2.1 ([MRT15], Theorem 1.6)**.** *Fix $A,m \geqslant 1$ and let $x \geqslant 10$. Let $g_1,\dots,g_k$ be 1-bounded, complex-valued multiplicative functions and let $c_1,\dots,c_k,b_1,\dots,b_k \in \mathbb{N}$ be such that $c_j,b_j \leqslant A$ for each j. Then for each $1 \leqslant j_0 \leqslant x$,*

$$x^{-(k+1)} \sum_{1\leqslant h_1,\dots,h_{k-1}\leqslant mAx} \left| \sum_{1\leqslant n\leqslant x} \prod_{1\leqslant j\leqslant k} g_j(c_j n + b_j + h_j) \right| \ll m^{k-1}k^2A^k\left(e^{-\mathcal{D}_{j_0}(x)/80} + (\log x)^{-1/3000}\right).$$

Remark 5.2.1. Strictly speaking, Theorem 1.6 in [MRT15] the range of h_j is bounded above by x, rather than by mAx, as written here. However, for fixed m and A, a careful look at the proof there shows that a perturbation of H by a fixed quantity does not affect their arguments (which depend at most on $\log H$).

It turns out that we can reduce the proof of Proposition 1.7.5 to showing that a similar statement holds when the system of linear forms $\boldsymbol{L}$ is a Gowers system. This is a consequence of Lemma 5.2.3 below, which allows us to prove a quantitative refinement of Lemma 3.4 in [FH16] in Section 4. To state Lemma 5.2.3 precisely, we recall the following definition (see Definition 1.3.2 in [Tao12]).

Definition 5.2.2. A collection $\boldsymbol{L}$ of k integral linear forms in l-variables on a finite abelian group G is said to have *Cauchy-Schwarz complexity* at most s if, for each $1 \leqslant j \leqslant k$ we can partition the set of forms $\{L_1,\dots,L_k\}\setminus\{L_j\}$ into $s+1$ classes $\{C_t : 1 \leqslant t \leqslant s+1\}$ such that $L_j \notin \mathrm{Span}(C_t)$ for each $1 \leqslant t \leqslant s+1$. (If no such s exists then the collection of forms is said to have Cauchy-Schwarz complexity ∞.)

Note that if $k \geqslant 2$ then a primitive integral system of k linear forms always has Cauchy-Schwarz complexity at most $k-2$, by taking the partition of singletons. Also, if an integral system of linear forms is *linearly independent* then the Cauchy-Schwarz complexity is at most 0.

We may now state the following lemma, which is Exercise 1.3.23 in [Tao12].

Lemma 5.2.3 (Generalized von Neumann Inequality)**.** Let G be a finite Abelian group and let $\psi_1,\dots,\psi_k : G^l \to G$ be a set of integral linear forms with Cauchy-Schwarz complexity at most s. If $f_1,\dots,f_k : G \to \mathbb{C}$ are 1-bounded functions on G then

$$|G|^{-l} \sum_{\boldsymbol{g}\in G^l} \prod_{1\leqslant j\leqslant k} f_j(\psi_j(\boldsymbol{g})) \ll_{k,l} \min_{1\leqslant j\leqslant k} \|f_j\|_{U^{s+1}(G)}.$$

5.3. Proof of Theorem 1.7.1

As in the statement of Theorem 1.7.1 put $F_j(n) := f_j(n)\overline{\chi_j}(n)n^{-it_j}$ when $(n,q_j) = 1$, and $F_j(n) = 1$ otherwise. Furthermore, let $F_j = F_{j,s} \cdot F_{j,l}$, where we set

$$F_{j,s}(p^k) := \begin{cases} F_j(p^k) & : \ p^k \leqslant y \\ 1 & : \ p^k > y \end{cases}, \quad F_{j,l}(p^k) := \begin{cases} 1 & : \ p^k \leqslant y \\ F_j(p^k) & : \ p^k > y. \end{cases}$$

Given vectors $\boldsymbol{a},\boldsymbol{n} \in \mathbb{N}^k$ let

$$h_{\boldsymbol{a}}(\boldsymbol{n}) := \prod_{1\leqslant j\leqslant k} (\chi_j \cdot F_{j,s})\left(L_j(\boldsymbol{n})/a_j\right)\left(L_j(\boldsymbol{n})/a_j\right)^{it_j} 1_{a_j|L_j(\boldsymbol{n})}$$

(otherwise, set $h_{\boldsymbol{a}}(\boldsymbol{n}) = 0$). Note that $h_{\boldsymbol{a}}(\boldsymbol{n})$ is supported on vectors $\boldsymbol{n}$ such that $a_j|L_j(\boldsymbol{n})$ and $(L_j(\boldsymbol{n})/a_j,q_j) = 1$ for each j. Thus,

$$\begin{aligned}\sum_{\boldsymbol{n}\in\mathcal{B}(\boldsymbol{x})}\prod_{1\leqslant j\leqslant k} f_j(L_j(\boldsymbol{n})) &= \sum_{\mathrm{rad}(a_j)|q_j\forall j}\prod_{1\leqslant j\leqslant k} f_j(a_j)\left(\langle\boldsymbol{x}\rangle^{-1}\sum_{\substack{\boldsymbol{n}\in\mathcal{B}(\boldsymbol{x})\\ (L_j(\boldsymbol{n})/a_j,q_j)=1}}\prod_{1\leqslant j\leqslant k} f_j\left(L_j(\boldsymbol{n})/a_j\right)1_{a_j|L_j(\boldsymbol{n})}\right)\\ &= \sum_{\mathrm{rad}(a_j)|q_j\forall j}\prod_{1\leqslant j\leqslant k} f_j(a_j)\left(\sum_{\boldsymbol{n}\in\mathcal{B}(\boldsymbol{x})} h_{\boldsymbol{a}}(\boldsymbol{n})\prod_{1\leqslant j\leqslant k} F_{j,l}\left(L_j(\boldsymbol{n})/a_j\right)1_{a_j|L_j(\boldsymbol{n})}\right)\\ &=: \langle\boldsymbol{x}\rangle\sum_{\mathrm{rad}(a_j)|q_j\forall j}\prod_{1\leqslant j\leqslant k} f_j(a_j)\mathcal{M}_{\boldsymbol{a}}(\boldsymbol{x};\boldsymbol{f},\boldsymbol{L}).\end{aligned}$$

For each j and q_j, define $R_{q_j}(m) := \max\{d|m : \mathrm{rad}(d)|q_j\}$. It is easy to see that R_{q_j} is multiplicative. Thus, define $F_{j,l}^*(n) := F_{j,l}\left(\frac{n}{R_{q_j}(n)}\right)$, and note that this, too, is clearly multiplicative. Moreover, we have $h_{\boldsymbol{a}}(\boldsymbol{n}) \neq 0$ if, and only if, $a_j = R_{q_j}(L_j(\boldsymbol{n}))$ and hence $F_{j,l}\left(L_j(\boldsymbol{n})/a_j\right) = F_{j,l}^*(L_j(\boldsymbol{n}))$ in this case. Applying Lemma 5.1.3 repeatedly, we thus have

$$\sum_{\mathrm{rad}(a_j)|q_j\forall j}\prod_{1\leqslant j\leqslant k} f_j(a_j)\mathcal{M}_{\boldsymbol{a}}(\boldsymbol{x};\boldsymbol{f},\boldsymbol{L}) = \sum_{\mathrm{rad}(a_j)|q_j\forall j}\prod_{1\leqslant j\leqslant k} f_j(a_j)\mathfrak{P}(F_{j,l}^*;AX)\left(\langle\boldsymbol{x}\rangle^{-1}\sum_{\boldsymbol{n}\in\mathcal{B}(\boldsymbol{x})} h_{\boldsymbol{a}}(\boldsymbol{n})\right) \tag{5.3.1}$$

$$+O\left(\sum_{1\leqslant j\leqslant k}\left(\sum_{\mathrm{rad}(a_j)|q_j}\frac{1}{\sqrt{a_j}}\right)\left(\mathbb{D}^*(f_j,\chi_j n^{it_j};y,AX) + \frac{k}{(\log(AX))^{B'}}\right)\right)$$

$$= \left(1 + O\left(\frac{k}{\log y}\right)\right) \tag{5.3.2}$$

$$\times\left(\prod_{y<p\leqslant AX} M_p\left(\boldsymbol{F},\boldsymbol{L}\right) + O\left(y^{-1+o(1)}\right)\right)\sum_{\mathrm{rad}(a_j)|q_j\forall j}\prod_{1\leqslant j\leqslant k} f_j(a_j)\left(\langle\boldsymbol{x}\rangle^{-1}\sum_{\boldsymbol{n}\in\mathcal{B}(\boldsymbol{x})} h_{\boldsymbol{a}}(\boldsymbol{n})\right) \tag{5.3.3}$$

$$+O\left(\sum_{1\leqslant j\leqslant k}\left(\prod_{p|q_j}\left(1-\frac{1}{\sqrt{a_j}}\right)^{-1}\right)\left(\mathbb{D}^*\left(f_j,\chi_j n^{it_j};y,AX\right) + \frac{k}{(\log(AX))^{B'}}\right)\right), \tag{5.3.4}$$

where in (5.3.4) we used Lemma 5.1.4, coupled with the fact that $F^*_{j,l}(p^k) = F_{j,l}(p^k)$ except for the prime divisors of q_j which we assume are inferior to y.
For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ of the same length we will write $\boldsymbol{a} \preceq \boldsymbol{b}$ to mean that $a_j \leqslant b_j$ for each j. Now, for $\boldsymbol{z} \in \mathcal{B}(\boldsymbol{x})$ and $\boldsymbol{a}$ such that $\mathrm{rad}(a_j)|q_j$ for each j, set

$$G_{\boldsymbol{a}}(\boldsymbol{z}) := \sum_{\boldsymbol{n}\preceq\boldsymbol{z}} \prod_{1\leqslant j\leqslant k} \chi_j(L_j(\boldsymbol{n})/a_j) F_{j,s}(L_j(\boldsymbol{n})/a_j) 1_{a_j|L_j(\boldsymbol{n})},$$

so that by partial summation,

$$\sum_{\boldsymbol{n}\in\mathcal{B}(\boldsymbol{x})} h_{\boldsymbol{a}}(\boldsymbol{n}) = \left(\prod_{1\leqslant j\leqslant k} a_j^{-it_j}\right) \int_{\mathcal{B}(\boldsymbol{x})} \prod_{1\leqslant j\leqslant k} L_j(\boldsymbol{u})^{it_j} dG_{\boldsymbol{a}}(\boldsymbol{u}). \tag{5.3.5}$$

Note that we can express $G_{\boldsymbol{a}}(\boldsymbol{z})$ as

$$\begin{aligned} G_{\boldsymbol{a}}(\boldsymbol{z}) &= \sideset{}{^*}\sum_{u_1(q_1)} \cdots \sideset{}{^*}\sum_{u_k(q_k)} \left(\prod_{1\leqslant j\leqslant k} \chi_j(u_j)\right) \sum_{\substack{\boldsymbol{n}\preceq\boldsymbol{z} \\ L_j(\boldsymbol{n})/a_j\equiv u_j(q_j)\forall j}} \prod_{1\leqslant j\leqslant k} F_{j,s}(L_j(\boldsymbol{n})/a_j) \\ &=: \sideset{}{^*}\sum_{\substack{u_1(q_1) \\ \exists \boldsymbol{n}: L_j(\boldsymbol{n})/a_j\equiv u_j(q_j)\forall j}} \cdots \sideset{}{^*}\sum_{u_k(q_k)} \left(\prod_{1\leqslant j\leqslant k} \chi_j(u_j)\right) R_{\boldsymbol{a}}(\boldsymbol{z};\boldsymbol{u}). \end{aligned} \tag{5.3.6}$$

Define $g_{j,s} := \mu * F_{j,s}$, and let $Y := e^{3y}$. It follows by induction on ν that $g_{j,s}(p^{\nu+1}) = 0$ whenever $p^\nu > y$. By the Prime Number Theorem, $\left(\prod_{p^k\leqslant y} p\right)^2 = e^{(2+o(1))y} \leqslant Y$, and thus all divisors in the support of $g_{j,s}$ are at most Y when y is sufficiently large. Let $\mathbf{1} := (1,\ldots,1)$. By Möbius inversion,

$$R_{\boldsymbol{a}}(\boldsymbol{z};\boldsymbol{u}) = \sum_{\substack{\boldsymbol{d}\in\mathcal{B}(Y\mathbf{1}) \\ P^+(d_j)\leqslant y, (d_j,q_j)=1 \ \forall \ j}} \left(\prod_{1\leqslant j\leqslant k} g_{j,s}(d_j)\right) \sum_{\substack{\boldsymbol{n}\preceq\boldsymbol{z} \\ a_jd_j|L_j(\boldsymbol{n}), L_j(\boldsymbol{n})/a_j\equiv u_j(q_j) \ \forall j}} 1. \tag{5.3.7}$$

Let $S_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L};\boldsymbol{u},\boldsymbol{v})$ denote the set of solutions to the $2k$ simultaneous congruences $L_j(\boldsymbol{n})/a_j \equiv u_j(q_j)$, $L_j(\boldsymbol{n})/a_j \equiv v_j(d_j)$ for all $1 \leqslant j \leqslant k$, and let $R_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L};\boldsymbol{u},\boldsymbol{v})$ denote the density of this set. Then

$$R_{\boldsymbol{a}}(\boldsymbol{z};\boldsymbol{u}) = \langle \boldsymbol{z}\rangle \left(\sum_{\substack{\boldsymbol{d}\in\mathcal{B}(Y\mathbf{1}) \\ P^+(d_j)\leqslant y, (d_j,q_j)=1 \ \forall j}} \prod_{1\leqslant j\leqslant k} g_{j,s}(d_j) R_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L};\boldsymbol{u},\mathbf{0}) + O\left(\left(\sum_{1\leqslant j\leqslant k} z_j^{-1}\right) \frac{\Psi(Y,y)^k}{[a_1,\ldots,a_k]}\right)\right), \tag{5.3.8}$$

It is easy to see that $R_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L};\boldsymbol{u},\mathbf{0})$, given that it is non-zero, is independent of $\boldsymbol{u}$. Indeed, note that for any $\boldsymbol{r},\boldsymbol{s} \in \mathbb{R}^l$,

$$L_j(\boldsymbol{r}-\boldsymbol{s}) = L_j(\boldsymbol{r}) - L_j(\boldsymbol{s}) + L_j(\mathbf{0}), \tag{5.3.9}$$

$$L_j(\boldsymbol{r}+\boldsymbol{s}) = L_j(\boldsymbol{r}) + L_j(\boldsymbol{s}) - L_j(\mathbf{0}). \tag{5.3.10}$$

This implies immediately that if there exists a vector $\boldsymbol{n}$ such that $L_j(\boldsymbol{n})/a_j \equiv u_j(q_j)$ and $L_j(\boldsymbol{n})/a_j \equiv 0(d_j)$ then for any such $\boldsymbol{n}$ we have $S_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L};\boldsymbol{u},\boldsymbol{0}) = S_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L}-\boldsymbol{L}(\boldsymbol{0});\boldsymbol{0},\boldsymbol{0}) + \boldsymbol{n}$ (where, for an abelian group G and a subset S of G, we write $S+v := \{s+v : s \in S\}$ for $v \in G$). Using this remark in (5.3.8), inserting the latter into (5.3.6) and applying the bound $|g_{j,s}(d_j)| \leqslant \tau(d_j) \leqslant 2^{\pi(y)}$ for each j, it follows that

$$\begin{aligned} G_{\boldsymbol{a}}(\boldsymbol{z}) = \langle \boldsymbol{z}\rangle\, \Xi_{\boldsymbol{a}}(\boldsymbol{\chi},\boldsymbol{L}) &\sum_{\substack{\boldsymbol{d}\in\mathcal{B}(Y\boldsymbol{1})\\ P^+(d_j)\leqslant y,(d_j,q_j)=1\ \forall j}} \prod_{1\leqslant j\leqslant k} g_{j,s}(d_j) R_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L}-\boldsymbol{L}(\boldsymbol{0});\boldsymbol{0},\boldsymbol{0}) \\ &+ O\left(\langle \boldsymbol{z}\rangle\, 2^{k\pi(y)}\Psi(Y,y)^k(q_1\cdots q_k)\sum_{1\leqslant j\leqslant l} z_j^{-1}\right). \end{aligned}$$

By Lemma 5.1.5 there is a constant $C \leqslant 9/4$ such that $\Psi(Y,y) \leqslant e^{Cy/\log y}$, so we may replace the error term above by $O\left(e^{\frac{3ky}{\log y}}(q_1\cdots q_k)E(\boldsymbol{z})\right)$, where $E(\boldsymbol{z}) := \langle \boldsymbol{z}\rangle\left(\sum_{1\leqslant j\leqslant l} z_j^{-1}\right)$. The integral in (5.3.5) takes the shape

$$\begin{aligned} &\Xi_{\boldsymbol{a}}(\boldsymbol{\chi},\boldsymbol{L}) \sum_{\substack{\boldsymbol{d}\in\mathcal{B}(Y\boldsymbol{1})\\ P^+(d_j)\leqslant y\forall j}} R_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L}-\boldsymbol{L}(\boldsymbol{0});\boldsymbol{0},\boldsymbol{0}) \prod_{1\leqslant j\leqslant k} g_{j,s}(d_j) \int_{\mathcal{B}(\boldsymbol{x})} \prod_{1\leqslant j\leqslant k} L_j(\boldsymbol{u})^{it_j}\, d\boldsymbol{u} \\ &+ O\left(e^{\frac{3ky}{\log y}}(q_1\cdots q_k)\left|\int_{\mathcal{B}(\boldsymbol{x})} \prod_{1\leqslant j\leqslant k} L_j(\boldsymbol{u})^{it_j}\, dE(\boldsymbol{u})\right|\right) \\ &=: T_1 + e^{\frac{3ky}{\log y}}(q_1\cdots q_k)T_2. \end{aligned} \tag{5.3.11}$$

Now, rescaling the integral in T_1, we have

$$\begin{aligned} \int_{\mathcal{B}(\boldsymbol{x})}\left(\prod_{1\leqslant j\leqslant k} L_j(\boldsymbol{u})^{it_j}\right) d\boldsymbol{u} &= \langle \boldsymbol{x}\rangle \int_{\prod_{1\leqslant s\leqslant l}[1/x_s,1]} \left(\prod_{1\leqslant j\leqslant k} L_j((u_1x_1,\ldots,u_lx_l))^{it_j}\right) d\boldsymbol{u} \\ &= \left(1 + O\left(\frac{lA}{x_j}\right)\right)\langle \boldsymbol{x}\rangle \mathcal{I}(\boldsymbol{x};\boldsymbol{L},\boldsymbol{t}), \end{aligned} \tag{5.3.12}$$

whence that

$$T_1 = \left(1 + O(lAx_-^{-1})\right)\langle \boldsymbol{x}\rangle \Xi_{\boldsymbol{a}}(\boldsymbol{\chi},\boldsymbol{L})\mathcal{I}(\boldsymbol{x};\boldsymbol{L},\boldsymbol{t}) \sum_{\substack{\boldsymbol{d}\in\mathcal{B}(Y\boldsymbol{1})\\ P^+(d_j)\leqslant y,(d_j,q_j)=1\forall j}} R_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L}-\boldsymbol{L}(\boldsymbol{0});\boldsymbol{0},\boldsymbol{0}) \prod_{1\leqslant j\leqslant k} g_{j,s}(d_j).$$

We next consider T_2 as in (5.3.11). Applying partial summation repeatedly, we can write it as

$$\sum_{0\leqslant m\leqslant l}(-1)^{l-m} \tag{5.3.13}$$

$$\times \sum_{1\leqslant j_1<\cdots<j_m\leqslant l} \int_{\substack{u_{r_s}\leqslant x_{r_s}\forall s\\ r_s\neq j_v\forall s,v}} du_{r_1}\cdots du_{r_{l-m}} \left[E_{\boldsymbol{d}}(\boldsymbol{u})\left(\prod_{1\leqslant s\leqslant l-m}\frac{\partial}{\partial u_{r_s}}\right)\prod_{1\leqslant j\leqslant k} L_j(\boldsymbol{u})^{it_j}\right]_{\substack{u_{j_v}=1\\ \forall 1\leqslant v\leqslant m}}^{x_{j_v}}. \tag{5.3.14}$$

Observe that if L_j has a non-zero u_r coefficient, say $c_{j,r}$,

$$\left|\frac{\partial}{\partial u_r} L_j(\boldsymbol{u})^{it_j}\right| \leqslant |t_j| c_{j,r} L_j(\boldsymbol{u})^{-1} \leqslant |t_j| u_{j_r}^{-1}; \tag{5.3.15}$$

otherwise, the u_r partial derivative of $L_j^{it_j}$ is 0. Now fix $0 \leqslant m \leqslant l-1$ and a set of indices $1 \leqslant j_1 < \cdots < j_m \leqslant l$. Since the non-zero coefficients of $L_j(\boldsymbol{u})$ are positive integers, taking further derivatives as in (5.3.15) gives

$$\begin{aligned}
&\left|\int_{\substack{u_{r_s} \leqslant x_{r_s} \forall s \\ r_s \neq j_v \forall s,v}} du_{r_1} \cdots du_{r_{l-m}} \left[E_{\boldsymbol{d}}(\boldsymbol{u}) \left(\prod_{1 \leqslant s \leqslant l-m} \frac{\partial}{\partial u_{r_s}} \right) \prod_{1 \leqslant j \leqslant k} L_j(\boldsymbol{u})^{it_j} \right]_{\substack{u_{j_v}=1 \\ \forall 1 \leqslant v \leqslant m}}^{x_{j_v}} \right| \\
&\ll_l \prod_{1 \leqslant j \leqslant k} \max\{1, |t_j|\} \int_{\substack{u_{r_s} \leqslant x_{r_s} \forall s \\ r_s \neq j_v \forall s,v}} du_{r_1} \cdots du_{r_{l-m}} \left[|E_{\boldsymbol{d}}(\boldsymbol{u})| \left(\prod_{1 \leqslant s \leqslant l-m} u_{r_s}^{-1} \right) \right]_{\substack{u_{j_v}=1 \\ \forall 1 \leqslant v \leqslant m}}^{x_{j_v}}.
\end{aligned}$$

By the definition of $E_{\boldsymbol{d}}(\boldsymbol{z})$,

$$\begin{aligned}
&\int_{\substack{u_{r_s} \leqslant x_{r_s} \forall s \\ r_s \neq j_v \forall s,v}} du_{r_1} \cdots du_{r_{l-m}} \left[|E_{\boldsymbol{d}}(\boldsymbol{u})| \left(\prod_{1 \leqslant s \leqslant l-m} u_{r_s}^{-1} \right) \right]_{\substack{u_{j_v}=1 \\ \forall 1 \leqslant v \leqslant m}}^{x_{j_v}} \\
&\ll_m x_-^{-1} \left(\prod_{1 \leqslant v \leqslant m} x_{j_v} \right) \prod_{1 \leqslant s \leqslant l-m} \int_1^{x_{j_s}} \frac{du_{j_1}}{u_{j_s}} \ll_m x_-^{-1} \left(\prod_{1 \leqslant v \leqslant m} x_{j_v} \right) \prod_{1 \leqslant s \leqslant l-m} (\log x_{r_s}).
\end{aligned}$$

These contributions are all smaller than the term with $m = l$, which is bounded by $\ll |E_{\boldsymbol{d}}(\boldsymbol{x})| \ll \langle \boldsymbol{x} \rangle x_-^{-1}$. Thus, summing over all m-tuples of distinct indices j_v and all m, we get

$$T_2 \ll_l \frac{\langle \boldsymbol{x} \rangle}{x_-} \prod_{1 \leqslant j \leqslant k} \max\{1, |t_j|\}.$$

Thus, (5.3.5) gives

$$\begin{aligned}
\langle \boldsymbol{x} \rangle^{-1} \sum_{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x})} h_{\boldsymbol{a}}(\boldsymbol{n}) = &\left(1 + O\left(\frac{lA}{x_-}\right)\right) C_{\boldsymbol{a}}(\boldsymbol{x}; \boldsymbol{L}, \boldsymbol{\chi}, \boldsymbol{t}) \left(\prod_{1 \leqslant j \leqslant k} a_j^{-it_j} \right) \mathcal{S}_{\boldsymbol{a}}(Y, y; \boldsymbol{f}, \boldsymbol{L}) \\
&+ O_l\left(\frac{1}{x_-} \frac{e^{\frac{3ky}{\log y}}}{[a_1, \ldots, a_k]} \prod_{1 \leqslant j \leqslant k} q_j \max\{1, |t_j|\} \right),
\end{aligned}$$

where we put

$$\mathcal{S}_{\boldsymbol{a}}(Y, y; \boldsymbol{f}, \boldsymbol{L}) := \sum_{\substack{\boldsymbol{d} \in \mathcal{B}_k(X\mathbf{1}) \\ P^+(d_j) \leqslant y, (d_j, q_j) = 1}} R_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L} - \boldsymbol{L}(\mathbf{0}); \mathbf{0}, \mathbf{0}) \prod_{1 \leqslant j \leqslant k} g_{j,s}(d_j).$$

This coupled with (5.3.4) yields

$$\langle \boldsymbol{x} \rangle^{-1} \sum_{\boldsymbol{n} \in \mathcal{B}(\boldsymbol{x})} \prod_{1 \leqslant j \leqslant k} f_j(L_j(\boldsymbol{n}))$$

$$= \left(1 + O_{k,l}\left(\frac{1}{\log y}\right)\right)\left(\sum_{\substack{\mathrm{rad}(a_j)|q_j\\ \forall 1\leqslant j\leqslant k}}\prod_{1\leqslant j\leqslant k}\frac{f(a_j)}{a_j^{it_j}}C_{\boldsymbol{a}}(\boldsymbol{x};\boldsymbol{L},\boldsymbol{\chi},\boldsymbol{t})\mathcal{S}_{\boldsymbol{a}}(X,y;\boldsymbol{f},\boldsymbol{L})\right)$$

$$\cdot\left(\prod_{y<p\leqslant X}M_p(\boldsymbol{F},\boldsymbol{L}) + O\left(y^{-1+o(1)}\right)\right) + O(\mathcal{R}), \tag{5.3.16}$$

$$\tag{5.3.17}$$

where we have put

$$\mathcal{R} := \sum_{1\leqslant j\leqslant k}\prod_{p|q_j}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}\left(\mathbb{D}^*(f,\chi_j n^{it_j};y,AX) + \frac{1}{\log X}\right)$$
$$+\frac{1}{x_-}\left(A + e^{\frac{3ky}{\log y}}(q_1\cdots q_k)\left(\sum_{\substack{\mathrm{rad}(a_j)|q_j\\ \forall 1\leqslant j\leqslant k}}[a_1,\ldots,a_k]^{-1}\right)\prod_{1\leqslant j\leqslant k}\max\{1,|t_j|\}\right).$$

We next apply Rankin's trick with $\delta = 1/2$ to show that

$$\left|\left(\sum_{\substack{P^+(d_j)\leqslant y\\ (d_j,q_j)=1\forall j}} - \sum_{\substack{\boldsymbol{d}\in\mathcal{B}(Y\boldsymbol{1})\\ P^+(d_j)\leqslant y,(d_j,q_j)=1\forall j}}\right)R_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L}-\boldsymbol{L}(\boldsymbol{0});\boldsymbol{0},\boldsymbol{0})\prod_{1\leqslant j\leqslant k}g_{j,s}(d_j)\right|$$
$$\ll_k \sum_{\substack{d>Y\\ P^+(d)\leqslant y}}\frac{\tau(d)}{d} \ll Y^{-\delta}\prod_{p\leqslant y}\left(1+\frac{2}{p^{1-\delta}}\right) \ll Y^{-\delta}\exp\left(2\sum_{p\leqslant y}p^{-1+\delta}\right)$$
$$\ll e^{-(3\delta y - 2y^{\delta}\log_2 y)} \ll e^{-y}.$$

Thus, we have

$$S_{\boldsymbol{a}}(Y,y;\boldsymbol{f},\boldsymbol{L}) = \sum_{\substack{P^+(d_j)\leqslant y\\ (d_j,q_j)=1\forall j}}R_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L}-\boldsymbol{L}(\boldsymbol{0});\boldsymbol{0},\boldsymbol{0})\prod_{1\leqslant j\leqslant k}g_{j,s}(d_j) + O_k\left(e^{-y}\right).$$

Moreover, replacing $g_{j,s}$ by $g_j = \mu * F_j$ here produces an error

$$\left|\mathcal{S}_{\boldsymbol{a}}(y;\boldsymbol{f},\boldsymbol{L}) - \sum_{\substack{P^+(d_j)\leqslant y\\ (d_j,q_j)=1\forall j}}R_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L}-\boldsymbol{L}(\boldsymbol{0});\boldsymbol{0},\boldsymbol{0})\prod_{1\leqslant j\leqslant k}g_{j,s}(d_j)\right|$$
$$\ll_k \sum_{\substack{P^+(d)\leqslant y\\ \exists p^\nu||d,\ p^\nu>y,\nu\geqslant 2}}\frac{\tau(d)}{d} \ll \sum_{\substack{p^\nu>y\\ \nu\geqslant 2}}\frac{1}{p^\nu}\sum_{P^+(d)\leqslant y}\frac{\tau(d)}{d}$$
$$\ll y^{-\frac{1}{2}}\prod_{p\leqslant y}\left(1+\frac{2}{p}\right) \ll \frac{(\log y)^2}{\sqrt{y}}.$$

Thus, we have

$$\mathcal{S}_{\boldsymbol{a}}(Y,y;\boldsymbol{f},\boldsymbol{L}) = \mathcal{S}_{\boldsymbol{a}}(y;\boldsymbol{f},\boldsymbol{L}) + O_k\left(\frac{(\log y)^2}{\sqrt{y}}\right),$$

which, combined with (5.3.16) completes the proof of Theorem 1.7.1 in the general case.

Suppose now that $q_j = q$ for all j. We note first that by a simple calculation as in Lemma 5.1.4,

$$\prod_{\substack{p\leqslant y\\ p\nmid q}} M_p(\boldsymbol{f},\boldsymbol{L}) = \sum_{\substack{P^+(d_j)\leqslant y\\ (d_j,q)=1}} R(d_1,\ldots,d_k)\prod_{1\leqslant j\leqslant k} g_{j,s}(d_j),$$

where $R(d_1,\ldots,d_k)$ is the density of solutions in $\mathbb{N}^l$ to the simultaneous conditions $d_j|L_j(\boldsymbol{n})$ for each j. Arguing as in the remarks surrounding (5.3.9) and (5.3.10), $R(d_1,\ldots,d_k)$ is also the density corresponding to the shifted forms $L_j - L_j(\boldsymbol{0})$. Now since $(q,d_j)=1$ for all j,

$$R_{\boldsymbol{a},\boldsymbol{d}}(\boldsymbol{L}-\boldsymbol{L}(\boldsymbol{0}),\boldsymbol{0},\boldsymbol{0}) = R([qa_1,a_1d_1],\cdots,[qa_k,a_kd_k]) = R(qa_1,\ldots,qa_k)R(d_1,\ldots,d_k) \quad (5.3.18)$$

by multiplicativity. We thus have

$$S_{\boldsymbol{a}}(y;\boldsymbol{f},\boldsymbol{L}) = R(qa_1,\cdots qa_k)\prod_{\substack{p\leqslant y\\ p\nmid q}} M_p(\boldsymbol{f},\boldsymbol{L})$$

whenever $\boldsymbol{a}$ with $\mathrm{rad}(a_j)|q_j$ for each j, and Theorem 1.7.1 follows as well in the special case $q_j = q$ for all j.

5.4. Proof of Proposition 1.7.5

As mentioned in Section 2, we shall first make the following reduction, which is based on ideas of Green and Tao (see Theorem 7.1' and Appendix A of [GT10]). For convenience, we write $\mathbb{Z}_N$ to mean $\mathbb{Z}/N\mathbb{Z}$.

Lemma 5.4.1. Let $A,k,l\geqslant 1$. Let $\boldsymbol{L}$ be a primitive integral system of k linear forms in l variables and height at most A. Suppose that $f_1,\ldots,f_k:\mathbb{N}\to\mathbb{C}$ are 1-bounded arithmetic functions such that $\min_{1\leqslant j\leqslant k}\|f_j\|_{U^{k-1}(x)}\to 0$ as $x\to\infty$. Then

$$M(x;\boldsymbol{f},\boldsymbol{L}) \ll_{k,l,A} \min_{1\leqslant j\leqslant k}\|f_j\|_{U^{k-1}(x)}^{\frac{1}{2}}.$$

Moreover, if $\boldsymbol{L}$ is a system of *linearly independent* forms then we can replace the U^{k-1} norm on the right side by the U^2 norm.

Proof. Let $\rho>\rho'>lA$ and let N be a large prime satisfying $\rho' x<N\leqslant \rho x$, with ρ sufficiently large in terms of ρ' (but bounded as $x\to\infty$). Then

$$M(x;\boldsymbol{f},\boldsymbol{L}) = \left(\frac{N}{x}\right)^l N^{-l}\sum_{\boldsymbol{n}\in\mathbb{Z}_N^l}\prod_{1\leqslant j\leqslant k} f_j(L_j(\boldsymbol{n}))1_{[1,x]^l}(\boldsymbol{n}).$$

We seek to apply Lemma 5.2.3, and must hence remove the weight $1_{[1,x]^l}$. To accomplish this, we use the following harmonic analytic argument, due to Green and Tao (see Proposition

7.1' of [GT10]). Define a metric on $\mathbb{Z}_N^l$ by

$$d(\boldsymbol{m},\boldsymbol{m'}) := \left(\sum_{j\leqslant l}\left|\frac{m_j - m'_j}{N}\right|^2\right)^{\frac{1}{2}}.$$

Let $z,Z,\lambda > 0$ be parameters to be chosen. Let $\phi_N : \mathbb{Z}_N^l \to \mathbb{C}$ be a bounded (independently of N) d-Lipschitz map with Lipschitz constant λ such that $\|1_{[1,x]^l} - \phi_N\|_{L^1(\mathbb{Z}_N^l)} \ll_l N^l/Z$. It is shown in Corollary A.3 of [GT10] that $\lambda \ll Z/N$. Expanding ϕ_N as a Fourier series and convolving it with the l-dimensional Féjer kernel of length z, one can show that

$$\phi_N(\boldsymbol{n}) = \sum_{\boldsymbol{m}\in\mathbb{Z}_N^l} a_{\boldsymbol{m}} e\left(\frac{\boldsymbol{m}\cdot\boldsymbol{n}}{N}\right) = \sum_{\boldsymbol{m}\in[z]^l} a'_{\boldsymbol{m}} e\left(\frac{\boldsymbol{m}\cdot\boldsymbol{n}}{N}\right) + O_l\left(N^l\lambda\frac{\log(z+1)}{z}\right),$$

where $|a'_{\boldsymbol{m}}| \ll 1$. Inserting this expansion into our expression for $M(x;\boldsymbol{f},\boldsymbol{L})$, splitting the two contributions and bounding the main term trivially gives

$$\begin{aligned}
M(x;\boldsymbol{f},\boldsymbol{L}) &\leqslant \rho^l\left(N^{-l}\left|\sum_{\boldsymbol{n}\in\mathbb{Z}_n^l}\phi_N(\boldsymbol{n})\prod_{1\leqslant j\leqslant k} f_j(L_j(\boldsymbol{n}))\right| + Z^{-1}\right)\\
&\ll N^{-l}\left|\sum_{\boldsymbol{m}\in[z]^l} a_{\boldsymbol{m}}\sum_{\boldsymbol{n}\in\mathbb{Z}_n^l} e\left(\frac{\boldsymbol{m}\cdot\boldsymbol{n}}{N}\right)\prod_{1\leqslant j\leqslant k} f_j(L_j(\boldsymbol{n}))\right| + \lambda\frac{\log(z+1)}{z} + Z^{-1}\\
&\ll_{\rho,l}\left(\sum_{\boldsymbol{m}\in[z]^l}|a_{\boldsymbol{m}}|\right)\max_{\boldsymbol{m}\in\mathbb{Z}_N^l} N^{-l}\left|\sum_{\boldsymbol{n}\in\mathbb{Z}_N^l} e\left(\frac{\boldsymbol{m}\cdot\boldsymbol{n}}{N}\right)\prod_{1\leqslant j\leqslant k} f_j(L_j(\boldsymbol{n}))\right| + \lambda\frac{\log(z+1)}{z} + \frac{1}{Z}\\
&\ll_{\rho,l} z^l\left(N^{-l}\left|\sum_{\boldsymbol{n}\in\mathbb{Z}_N^l}\prod_{0\leqslant j\leqslant k} f_j(L_j(\boldsymbol{n}))\right|\right) + \lambda\frac{\log(z+1)}{z} + Z^{-1},
\end{aligned}$$

where, letting $\boldsymbol{m}_0$ be the index maximizing the multilinear average, we let $L_0(\boldsymbol{n}) = \boldsymbol{m_0}\cdot\boldsymbol{n}$ and $f_0(n) := e\left(\frac{n}{N}\right)$. Now if $L_0 \notin \mathrm{Span}_{\mathbb{Q}}\{L_1,\dots,L_k\}$ then $\{L_0,\dots,L_k\}$ still has Cauchy-Schwarz complexity $k-2$ so by Lemma 5.2.3,

$$M(x;\boldsymbol{f},\boldsymbol{L}) \ll_{\rho,l} z^l \min_{1\leqslant j\leqslant k}\|f_j\|_{U^{k-1}(\mathbb{Z}_N)} + \lambda\frac{\log(z+1)}{z} + Z^{-1}. \tag{5.4.1}$$

On the other hand, if $L_0 = \sum_{1\leqslant j\leqslant k}\alpha_j L_j$ with $\alpha_j \in \mathbb{Q}$ then (5.4.1) still holds with $f'_j(n) := f_j(n)e(\alpha_j n)$ in place of f_j. Since the $U^{k-1}(\mathbb{Z}_N)$ norm is invariant under multiplication by exponential phases (see (B.4) in [GT10]) we have $\|f'_j\|_{U^{k-1}(\mathbb{Z}_N)} = \|f_j\|_{U^{k-1}(\mathbb{Z}_N)}$, and thus as written (5.4.1) holds in this case as well.

By definition, $\|f_j\|_{U^{k-1}(\mathbb{Z}_N)} = \|f_j\|_{U^{k-1}(x)}\|1_{[1,x]}\|_{U^{k-1}(\mathbb{Z}_N)} \ll_{\rho,l} \|f_j\|_{U^{k-1}(x)}$. Hence

$$\begin{aligned}
M(x;\boldsymbol{f},\boldsymbol{L}) &\ll_{\rho,l} z^l\min_{1\leqslant j\leqslant k}\|f_j\|_{U^{k-1}(x)} + \lambda\frac{\log(z+1)}{z} + Z^{-1}\\
&\leqslant z^l\|f_{j_0}\|_{U^{k-1}(x)} + \lambda\frac{\log(z+1)}{z} + Z^{-1}.
\end{aligned}$$

Suppose that $1 \leqslant j_0 \leqslant k$ is the index of the function with minimal $U^{(k-1)}(x)$ norm as $x \to \infty$. The claim follows upon taking $z := \|f_{j_0}\|_{U^{k-1}(x)}^{-\frac{1}{2l}}$ and $Z = N^{1/2}$ suffices to prove the theorem. The second claim follows immediately from the fact that mutually linearly independent forms have Cauchy-Schwarz complexity at most 1 trivially. $\square$

Lemma 5.4.2. Suppose f is a 1-bounded multiplicative function such that $\mathcal{D}(x) := \mathcal{D}(f; 10x, (\log x)^{1/125}) \to \infty$ as $x \to \infty$. Then for some absolute $c_1, c_2 > 0$,

$$\|f\|_{U^2(x)} \ll e^{-c_1 \mathcal{D}(x)} + (\log x)^{-c_2}.$$

Proof. We have

$$\begin{aligned}
\|f\|_{U^2(x)}^4 &= x^{-3} \sum_{1 \leqslant n_1, n_2, n_3 \leqslant x} f(n_1) \overline{f(n_1+n_2) f(n_1+n_3)} f(n_1+n_2+n_3) \\
&\leqslant x^{-3} \sum_{1 \leqslant n_1, n_2 \leqslant x} \left| \sum_{n_3 \leqslant x} \overline{f(n_1+n_3)} f(n_1+n_2+n_3) \right| \\
&\leqslant x^{-3} \sum_{1 \leqslant h_1, h_2 \leqslant 2x} \left| \sum_{n \leqslant x} \overline{f(n+h_1)} f(n+h_2) \right|,
\end{aligned}$$

upon making the change of variables $n = n_1$, $h_1 = n_3$ and $h_2 = h_1 + n_2 \leqslant 2x$. Applying Theorem 5.2.1 with $H = 2x$ gives

$$\|f\|_{U^2(x)}^4 \ll_{k,l,A} e^{-c_1 \mathcal{D}_{j_0}(x)} + (\log x)^{-c_2},$$

and the claim follows with constants $c_1/4$ and $c_2/4$ in place of c_1, c_2. $\square$

Proof of Proposition 1.7.5. Proposition 1.7.5 follows immediately upon combining Lemmata 5.4.1 and 5.4.2. $\square$

Chapter 6

SIGN PATTERNS OF MULTIPLICATIVE FUNCTIONS

In this section, we study the frequency with which a given multiplicative function $f : \mathbb{N} \to \{-1,1\}$ yields a given sign pattern on 3- and 4-term arithmetic progressions.

6.1. Sign patterns of non-pretentious f on fixed 3- and 4-term APs. Proof of Theorem 1.7.9.

Our method relies on the remarkable result of Tao [Taoc] that establishes a logarithmically averaged version of Elliott's conjecture. A special case of his result is the following.

Theorem 6.1.1 ([Taoc], Corollary 1.5)**.** *Let b_1,b_2 be distinct, non-negative integers. Let $f_1,f_2 : \mathbb{N} \to \mathbb{C}$ be a 1-bounded multiplicative function such that for some $j \in \{1,2\}$, $\mathcal{D}(f_j; Ax,\infty) \to \infty$ as $x \to \infty$ for each $A \geqslant 1$. Then*

$$\sum_{n \leqslant x} \frac{f_1(n+b_1)f_2(n+b_2)}{n} = o(\log x).$$

We shall take advantage of this result and the unimodularity of f to establish statements about correlations of f with three or four translates of itself. We use the following basic device to this end.

Lemma 6.1.1. For $n \geqslant 1$ let $a_1,\dots,a_n,b_1,\dots,b_n \in \mathbb{C}$ have norm uniformly bounded above by X. Let $w_1,\dots,w_n \in (0,\infty)$ and put $H := \sum_{1\leqslant j\leqslant n} w_j$. Let $A := H^{-1}\left|\sum_{1\leqslant j\leqslant n} w_j a_j\right|$ and $B := H^{-1}\left|\sum_{1\leqslant j\leqslant n} w_j b_j\right|$. Then

$$\mathrm{Re}\left(\sum_{1\leqslant j\leqslant n} w_j a_j \overline{b_j}\right) \geqslant \left(\frac{1}{2}(A+B)^2 - X\right) H.$$

Proof. Rotating the sums $\sum_{1\leqslant j\leqslant n} w_j a_j$ and $\sum_{1\leqslant j\leqslant n} w_j b_j$, we may assume without loss of generality that they point in the same direction, say $e(\theta)$. As such, by Cauchy-Schwarz,

$$\mathrm{Re}\left(\sum_{1\leqslant j\leqslant n} w_j a_j \overline{b_j}\right) \geqslant \frac{1}{2}\sum_{1\leqslant j\leqslant n} w_j\left(|a_j+b_j|^2 - 2X\right) \geqslant \frac{1}{2H}\left(\left|\sum_{1\leqslant j\leqslant n} w_j(a_j+b_j)\right|^2 - 2XH\right)$$
$$= H\left(\frac{1}{2}\left|(A+B)e\left(\theta\right)\right|^2 - X\right),$$

as claimed. □

A consequence of Lemma 6.1.1 is the following, which gives us a criterion to determine whether or not a multiplicative function is pretentious based on its 4-term correlations.

Lemma 6.1.2. Let $x,d \geqslant 1$, with $d \in \mathbb{N}$ and $d = o(x)$. Let f be a unimodular multiplicative function, and let $\delta > 0$. If

$$\frac{1}{\log x}\sum_{n\leqslant x}\frac{f(n)f(n+d)f(n+2d)f(n+3d)}{n} > \frac{1}{2} + \delta$$

as $x \to \infty$ then there is a primitive Dirichlet character χ of conductor q and a real number $t \in \mathbb{R}$ such that $\mathbb{D}(f,\chi n^{it};\infty) < \infty$.

Proof. We apply Lemma 6.1.1 with $w_n := 1/n$, $a_n := f(n)f(n+d)f(n+2d)f(n+3d)$ and $b_n := \overline{a_{n+d}}$ for each $n \leqslant x$. Clearly, $a_n\overline{b_n} = f(n)\overline{f(n+4d)}$, and as $d = o(x)$,

$$A + B = (2+o(1))\frac{1}{\log x}\left|\sum_{n\leqslant x}\frac{f(n)f(n+d)f(n+2d)f(n+3d)}{n}\right|.$$

By Lemma 6.1.1,

$$\mathrm{Re}\left(\sum_{n\leqslant x}\frac{f(n)\overline{f(n+4d)}}{n}\right) \geqslant (2+o(1))\left|\sum_{n\leqslant x}\frac{f(n)f(n+d)f(n+2d)f(n+3d)}{n}\right|^2 - \log x + O(1).$$

By assumption, it follows that

$$\mathrm{Re}\left(\sum_{n\leqslant x}\frac{f(n)\overline{f(n+3d)}}{n}\right) \gg_\delta \log x.$$

The conclusion now follows from Theorem 5.1 with $f_1 = f$, $f_2 = \overline{f}$. □

Our next lemma is a trivial observation showing that the cardinality of the set of $n \leqslant x$ yielding a fixed sign pattern of a given length can be expressed as a correlation of multiplicative functions.

Lemma 6.1.3. Let $l \geqslant 1$, and let $\boldsymbol{\epsilon} \in \{-1,1\}^l$. Let $f : \mathbb{N} \to \{-1,1\}$ and $g : (0,\infty) \to \mathbb{R}$, and put

$$S_{\boldsymbol{\epsilon}} := \{n \in \mathbb{N} : f(n+jd) = \epsilon_j \text{ for all } 0 \leqslant j \leqslant l-1\}.$$

Then

$$\sum_{\substack{n\leqslant x\\ n\in S_{\boldsymbol{\epsilon}}}} g(n) = 2^{-l}\sum_{n\leqslant x} g(n)\prod_{0\leqslant j\leqslant l-1}\left(1+\epsilon_j f(n+jd)\right).$$

Proof. If $n\notin S_{\boldsymbol{\epsilon}}$ then for some $0\leqslant j\leqslant l-1$, $1+\epsilon_j f(n+jd)=0$, so such terms contribute nothing. Conversely, when $n\in S_{\boldsymbol{\epsilon}}$ then $1+\epsilon_j f(n+jd)=2$ for all $0\leqslant j\leqslant l-1$, and the product is then 2^l. This implies the claim. □

With these results in hand, we will establish Theorem 1.7.9.

Proof of Theorem 1.7.9. We will only prove ii). By a similar argument one can establish i) as well, and we leave the details of this to the reader.
Let $S_{\pm\boldsymbol{\epsilon}} := S_{\boldsymbol{\epsilon}}\cup S_{-\boldsymbol{\epsilon}}$. Write $L_{\pm\boldsymbol{\epsilon}}(x) := \sum_{\substack{n\leqslant x\\ n\in S_{\pm\boldsymbol{\epsilon}}}}\frac{1}{n}$. Applying Lemma 6.1.3 twice with $g(n):=\frac{1}{n}$ for all $n\in\mathbb{N}$,

$$\begin{aligned}
L_{\pm\boldsymbol{\epsilon}}(x) &= \frac{1}{16}\sum_{n\leqslant x}\frac{1}{n}\left(\prod_{0\leqslant j\leqslant 3}(1+\epsilon_j f(n+jd)) + \prod_{0\leqslant j\leqslant 3}(1-\epsilon_j f(n+jd))\right)\\
&= \frac{1}{16}\left(\sum_{S\subseteq\{0,1,2,3\}}\left(1+(-1)^{|S|}\right)\sum_{n\leqslant x}\frac{1}{n}\prod_{j\in S}\epsilon_j f(n+jd)\right)\\
&= \frac{1}{8}\left(\log x + \sum_{\substack{S\subseteq\{0,1,2,3\}\\ |S|=2}}\sum_{n\leqslant x}\frac{1}{n}\prod_{j\in S}\epsilon_j f(n+jd) + \epsilon_0\epsilon_1\epsilon_2\epsilon_3\sum_{n\leqslant x}\frac{\prod_{0\leqslant j\leqslant 3} f(n+jd)}{n}\right).
\end{aligned}$$

By Theorem 1.3 of [Taoc], each of the six 2-element subsets S of $\{0,1,2,3\}$ gives rise to

$$\sum_{n\leqslant x}\frac{1}{n}\prod_{j\in S} f(n+jd) = o(\log x).$$

Also, by Lemma 6.1.2, we must have

$$\liminf_{x\to\infty}\left|\frac{1}{\log x}\sum_{n\leqslant x}\frac{f(n)f(n+d)f(n+2d)f(n+3d)}{n}\right| \leqslant \frac{1}{2}.$$

As such, we have

$$\limsup_{x\to\infty}\frac{L_{\pm\boldsymbol{\epsilon}}(x)}{\log x} \geqslant \frac{1}{8}\left(1-\liminf_{x\to\infty}\left|\frac{1}{\log x}\sum_{n\leqslant x}\frac{f(n)f(n+d)f(n+2d)f(n+3d)}{n}\right|\right) \geqslant \frac{1}{16}.$$

This establishes the claim. □

By a similar argument, we can show the following. The details are left to the reader.

Proposition 6.1.4. *For any $d\geqslant 1$, any non-pretentious function $f:\mathbb{N}\to\{-1,1\}$ and any $\boldsymbol{\epsilon}\in\{-1,1\}^3$, the upper logarithmic density of the set of n such that $f(n+jd)=\epsilon_j$ for $j=0,1$ and 2 is at least $\frac{1}{16}$.*

6.2. Sign patterns of non-pretentious functions in almost all 3-term APs

By Chebyshev's inequality, in order to prove Theorem 1.7.10 it suffices to show the following variance estimate.

Proposition 6.2.1. *Let $\boldsymbol{\epsilon} \in \{-1,1\}^3$ and let $f : \mathbb{N} \to \{-1,1\}$. Then*

$$x^{-2} \sum_{d \leqslant x} \left(|\{n \leqslant x : f(n+jd) = \epsilon_j \forall j\}| - \frac{1}{8} x \right)^2 \ll \mathcal{R}_f(x), \tag{6.2.1}$$

where $\mathcal{R}_f(x)$ is as defined in (1.7.5).

The first and second moment calculations are given in the following two lemmata.

Lemma 6.2.2. Let $\boldsymbol{\epsilon} \in \{-1,1\}^3$ and let $f : \mathbb{N} \to \{-1,1\}$. Then

$$x^{-1} \sum_{d \leqslant x} |\{n \leqslant x : f(n+jd) = \epsilon_j \forall j\}| = \frac{1}{8} x + O\left(x \mathcal{R}_f(x)\right).$$

Proof. As before we have

$$\begin{aligned} x^{-1} \sum_{d \leqslant x} |\{n \leqslant x : f(n+jd) = \epsilon_j \forall\ 0 \leqslant j \leqslant 2\}| &= \frac{1}{8} \sum_{d,n \leqslant x} \prod_{0 \leqslant j \leqslant 2} (1 + \epsilon_j f(n+jd)) \\ &= \frac{1}{8} \left(x + \sum_{\substack{S \subseteq \{0,1,2\} \\ S \neq \emptyset}} \left(\prod_{j \in S} \epsilon_j \right) \sum_{d,n \leqslant x} \prod_{j \in S} f(n+jd) \right). \end{aligned} \tag{6.2.2}$$

Fix a non-empty subset S of $\{0,1,2,3\}$. The collection of forms

$$\{(n,d) \mapsto n + jd : j \in S\}$$

has Cauchy-Schwarz complexity at most that of the 3-term AP $\{n + jd : 0 \leqslant j \leqslant 2\}$, which is 1. By Theorem 1.7.5, we have

$$x^{-2} \sum_{d,n \leqslant x} \prod_{j \in S} f(n+jd) \ll \mathcal{R}_f(x).$$

As such, (6.2.2) can be transformed as

$$x^{-1} \sum_{d \leqslant x} |\{n \leqslant x : f(n+jd) = \epsilon_j \text{ for all } 0 \leqslant j \leqslant 2\}| = \frac{1}{8} x + O\left(x \mathcal{R}_f(x)\right).$$

□

Lemma 6.2.3. Let $\boldsymbol{\epsilon} \in \{-1,1\}^3$ and let $f : \mathbb{N} \to \{-1,1\}$. Then

$$x^{-1} \sum_{d \leqslant x} |\{n \leqslant x : f(n+jd) = \epsilon_j \forall j\}|^2 = \frac{x^2}{64} + O\left(x^2 \mathcal{R}_f(x)\right).$$

Proof. By Lemma 6.1.3,

$$\sum_{d\leqslant x}|\{n\leqslant x : f(n+jd)=\epsilon_j\forall j\}|^2 = \frac{1}{64}\sum_{n,n',d\leqslant x}\prod_{0\leqslant j,j'\leqslant 2}(1+\epsilon_j f(n+jd))(1+\epsilon_j f(n'+jd))$$
$$=\frac{1}{64}\left(x^3+\sum_{\substack{S,S'\subseteq\{0,1,2\}\\ S\cup S'\neq\emptyset}}\left(\prod_{j\in S}\epsilon_j\right)\left(\prod_{j'\in S'}\epsilon_{j'}\right)\sum_{n,n',d\leqslant x}\left(\prod_{j\in S}f(n+jd)\right)\left(\prod_{j\in S'}f(n'+j'd)\right)\right). \tag{6.2.3}$$

Associate to each pair of sets $S,S'\subseteq\{0,1,2\}$ with $S\cup S'\neq\emptyset$ the system of forms

$$\boldsymbol{L}_{S,S'} := \{(n,n',d)\mapsto n+jd : j\in S\}\cup\{(n,n',d)\mapsto n'+j'd : j'\in S'\}.$$

We note to that each of the sets of forms $\{n,n+d,n'+d\}$ and $\{n',n'+2d,n+2d\}$ (in the variables n,n' and d) is linearly independent. This implies that the set of forms $\{n,n',n+d,n'+d,n+2d,n'+2d\}$ and each of its subsets has Cauchy-Schwarz complexity at most 1. Applying Proposition 1.7.5 to $\boldsymbol{L}_{S,S'}$, we get

$$M(x;f\mathbf{1},\boldsymbol{L}_{S,S'})\ll\mathcal{R}_f(x).$$

Thus, the second term in the brackets in (6.2.3) can be bounded as

$$\sum_{\substack{S,S'\subseteq\{0,1,2\}\\ S\cup S'\neq\emptyset}}\left(\prod_{j\in S}\epsilon_j\right)\left(\prod_{j'\in S'}\epsilon_{j'}\right)x^3M(x;f\mathbf{1},\boldsymbol{L}_{S,S'})\ll x^3\mathcal{R}_f(x).$$

The claim of the lemma follows. □

Proof of Proposition 6.2.1. Expanding the square on the left side of (6.2.1) yields

$$x^{-2}\sum_{d\leqslant x}|\{n\leqslant x : f(n+jd)=\epsilon_j, 0\leqslant j\leqslant 2\}|^2$$
$$-\frac{1}{4}\left(x^{-1}\sum_{d\leqslant x}|\{n\leqslant x : f(n+jd)=\epsilon_j, 0\leqslant j\leqslant 2\}|\right)+\frac{1}{64}x.$$

Combining Lemmata 6.2.2 and 6.2.3 quickly establishes the proposition. □

6.3. Sign patterns of pretentious functions in almost all 4-term APs. Preliminaries and the first moment estimate

Our first quest is to understand the p-adic and character local factors M_p and $\Xi_{\boldsymbol{a}}$. In preparation for this, we introduce more notation, some of which is recalled from the introduction.

Given a set $S\subseteq\{0,1,2,3\}$, let $\boldsymbol{L}_S$ be the collection of forms $\{L_j(n,d):=n+jd : j\in S\}$. Also, write $\mathbf{1}_{|S|}$ to denote the vector in $\mathbb{R}^{|S|}$, all of whose components are 1.

Given a fixed prime p we associate to each $\lambda\in\mathbb{F}_p\backslash\{0,1\}$ a non-singular elliptic curve E_λ

defined over $\mathbb{F}_p$ given by the Legendre model $y^2 \equiv x(x-1)(x-\lambda)(p)$. Finally, we will write $\sum^*_{a(q)}$ to indicate that summation is restricted to residue classes a coprime to q. Here and throughout this section, $q \geqslant 5$ is a positive integer coprime to 6.

Lemma 6.3.1. Let $q \geqslant 2$ and let $\boldsymbol{a} := (a_0,a_1,a_2,a_3)$ be a vector of integers whose radicals divide q. Then $\Xi_{\boldsymbol{a}}(\chi\mathbf{1}_{\{0,1,2,3\}}, \boldsymbol{L}_{\{0,1,2,3\}})$ vanishes unless $a_1 = a_2 = a_3 = a_0$, in which case we have

$$\Xi_{\boldsymbol{a}}(\chi\mathbf{1}_{\{0,1,2,3\}}, \boldsymbol{L}_{\{0,1,2,3\}}) = \mu(q)\phi(q)\prod_{p||q}(p+1-\#E_{3b^2}(\mathbb{F}_p)),$$

where b is the inverse of 2 modulo q.

Proof. Since χ is primitive and has odd conductor, q must be squarefree, and thus χ factors as a product of Legendre symbols. Given b_0,b_1,b_2,b_3 such that $L_j(n,d)/a_j \equiv b_j(q)$ for $0 \leqslant j \leqslant 3$, we observe by the Chinese Remainder Theorem that

$$\Xi_{\boldsymbol{a}}(\boldsymbol{\chi},\boldsymbol{L}) = \prod_{p|q}\sum_{\substack{b_0,b_1,b_2,b_3(p)\\ \exists n,d:(n+jd)/a_j\equiv b_j(p)}}\left(\frac{b_0b_1b_2b_3}{p}\right).$$

We consider several cases depending on the integers a_j.

Case 1: If $p|(a_i,a_j)$ for some $0 \leqslant i < j \leqslant 3$ but $p \nmid a_k$ for $k \neq i,j$ then as $p \neq 2,3$, $p|(n,d)$. As such, $b_k \equiv 0(p)$. Hence, the sum over b_k is trivial. Thus, if p divides some two a_j's, $\Xi_{\boldsymbol{a}}$ is trivial unless $a_0 = a_1 = a_2 = a_3$.

Case 2: Suppose that $p|a_i$ but $p \nmid a_j$ for each $j \neq i$. Then $n + id = pm$ for some $m \in \mathbb{N}$, and $n + jd \equiv (j-i)d(p)$ for each $j \neq i$. Since $(p,a_j) = 1$, it follows that $(n+jd)/a_j \equiv b_j(p)$ if, and only if, $n + jd \equiv b_ja_j(p)$. Hence, as a_i is squarefree, $(a_i/p,p) = 1$ and

$$\sum_{\substack{b_0,b_1,b_2,b_3(p)\\ \exists n,d:(n+jd)/a_j\equiv b_j(p)}}\left(\frac{b_0b_1b_2b_3}{p}\right) = \sum_{d(p)}\left(d\frac{\prod_{j\neq i}(j-i)a_j}{p}\right)\sum_{m(p)}\left(\frac{m/(a_i/p)}{p}\right) = 0,$$

Case 3: We assume now that $a_0 = a_1 = a_2 = a_3$. With the above constraints on the b_j, we must have $b_3 \equiv 2b_2 - b_1$ and $b_4 \equiv 3b_2 - 2b_1$. Thus, multiplying by $\chi(2b_2)^4\chi(-1)^2 = 1$,

$$\begin{aligned}\Xi_{\boldsymbol{a}}(\chi\mathbf{1}_{\{0,1,2,3\}},\boldsymbol{L}_{\{0,1,2,3\}}) &= \sum\nolimits^*_{b_1,b_2(q)}\chi(b_1b_2(2b_2-b_1)(3b_2-2b_1))\\ &= \sum\nolimits^*_{b_1,b_2(q)}\chi(b_1\overline{b_2}(b_1\overline{b_2}-2)(2b_1\overline{b_2}-3))\\ &= \sum\nolimits^*_{b_1,b_2(q)}\chi(\overline{2}b_1\overline{b_2}(\overline{2}b_1\overline{b_2}-1)(\overline{2}b_1\overline{b_2}-3\overline{2}^2)).\end{aligned}$$

Making the change of variables $\overline{2}b_1\overline{b_2}$ in place of b_1, we get

$$\Xi_{\boldsymbol{a}}(\chi\mathbf{1}_{\{0,1,2,3\}},\boldsymbol{L}_{\{0,1,2,3\}}) = \sum\nolimits^*_{d,b_2(q)}\chi(d(d-1)(d-3b^2)) = \phi(q)\sum\nolimits^*_{d(q)}\chi(d(d-1)(d-\lambda)). \quad (6.3.1)$$

Applying the CRT again, the complete character sum factors as

$$\sum_{d(q)}^{*} \chi(d(d-1)(d-\lambda)) = \prod_{p|q} \sum_{d(p)} \left(\frac{d(d-1)(d-\lambda)}{p} \right).$$

On the other hand, we know that $1 + \sum_{d(p)} \left(1 + \left(\frac{d(d-1)(d-\lambda)}{p}\right)\right)$ is precisely the number of $\mathbb{F}_p$-rational points on E_λ (including the point at infinity). As such,

$$\sum_{d(q)}^{*} \chi(d(d-1)(d-\lambda)) = \mu(q) \prod_{p|q} (p + 1 - \#E_\lambda(\mathbb{F}_p)) . \tag{6.3.2}$$

Inserting this into (6.3.1) proves the claim. □

Lemma 6.3.2. Let $f : \mathbb{N} \to \{-1,1\}$ be pretentious to a real character χ with conductor q. If $S \subset \{0,1,2,3\}$ has size 2 or 3, then $\Xi_{\boldsymbol{a}}(\chi \mathbf{1}_{|S|}, \boldsymbol{L}_S) = 0$ for any length $|S|$ vector $\boldsymbol{a}$ of divisors of q.

Proof. When $|S| = 2$, note that the forms L_j and $L_{j'}$ are linearly independent, and thus any pair of residue classes $(b_j, b_{j'})$ can satisfy the simultaneous congruences $L_j(n,d)/a_j \equiv b_j(q)$ and $L_{j'}(n,d)/a_{j'} \equiv b_{j'}(q)$. By orthogonality,

$$\Xi_{\boldsymbol{a}}(\chi \mathbf{1}_{|S|}, \boldsymbol{L}_S) = \left(\sum_{c(q)}^{*} \chi(c) \right)^2 = 0.$$

Now suppose that $|S| = 3$, and let $0 \leqslant j_1 < j_2 < j_3 \leqslant 3$ be the elements of S. A reduction argument similar to (and, in fact, simpler than) the one in Lemma 6.3.1 allows one to assume that $a_0 = a_1 = a_2 =: a$. Then, $L_{j_t}(n,d)/a \equiv b_t(q)$ for each $1 \leqslant t \leqslant 3$ implies that $(j_2 - j_1)d \equiv b_2 - b_1$, and as $j_2 - j_1 \in \{1,2\}$ and q is odd, we have

$$b_3 \equiv b_1 + (j_3 - j_1)\overline{(j_2 - j_1)}(b_2 - b_1)(q) =: b_1(1 - J) + Jb_2(q),$$

where $J := (j_3 - j_1)\overline{(j_2 - j_1)}$. Note that since $j_3 \neq j_2$ and $j_3 - j_1 \in \{2,3\}$, $J \neq 1$ and J and $J - 1$ are both invertible. As such, we have

$$\begin{aligned}\Xi(\chi \mathbf{1}_{|S|}, \boldsymbol{L}_S) &= \sum_{b_1,b_2(q)}^{*} \chi(b_1 b_2 (b_1(1 - J) + Jb_2)) \\ &= \chi(J) \sum_{b_1,b_2(q)}^{*} \chi(b_2)^3 \sum_{b_1(q)}^{*} \chi(b_1 \overline{b_2}(1 - b_1 \overline{b_2 J}(J - 1))).\end{aligned}$$

Multiplying by $\overline{J}(J-1)$ and making the change of variables $c := b_1 \overline{b_2 J}(J - 1)$ in place of b_1 (which is a bijection onto $(\mathbb{Z}/q\mathbb{Z})^*$) yields

$$\Xi(\chi \mathbf{1}_{|S|}, \boldsymbol{L}_S) = \chi(J - 1) \left(\sum_{b_2(q)}^{*} \chi(b_2) \right) \left(\sum_{c(q)}^{*} \chi(c(1 - c)) \right) = 0.$$

□

For the remainder of the paper, we will write $[3] := \{0,1,2,3\}$, and

$$\Delta_p := p + 1 - \#E_{3b^2}(\mathbb{F}_q).$$

We can now state our first moment estimate for sign patterns of pretentious multiplicative functions in almost all 4-term arithmetic progressions.

Proposition 6.3.3. *Let $\boldsymbol{\epsilon} \in \{-1,1\}^4$. Let $\delta > 0$ be fixed and let $2 \leqslant (\log x)^\delta \leqslant z \leqslant x$ with $z = o(x)$ as $x \to \infty$. Let $f : \mathbb{N} \to \{-1,1\}$ be pretentious to a real character χ with conductor q coprime to 6. Then*

$$\frac{1}{z} \sum_{\substack{d \leqslant z \\ q \nmid d}} |\{n \leqslant x : f(n + jd) = \epsilon_j \forall j\}| = \left(1 - \frac{1}{q}\right) \frac{x}{16} + o(x), \tag{6.3.3}$$

and

$$\frac{1}{z} \sum_{d \leqslant z} |\{n \leqslant x : f(n + jd) = \epsilon_j \forall j\}| = \frac{x}{16} \left(1 + \epsilon_0\epsilon_1\epsilon_2\epsilon_3 \prod_{p \nmid q} M_p(f\chi\mathbf{1}_4, \boldsymbol{L}_{[3]}) \prod_{p | q} \frac{\mu(p)\Delta_p}{p+1}\right) + o(x). \tag{6.3.4}$$

Proof. Applying Lemma 6.1.3, we have

$$(xz)^{-1} \sum_{d \leqslant z} |\{n \leqslant x : f(n + jd) = \epsilon_j \forall j\}| = \frac{1}{16} \left(1 + \sum_{\substack{S \subseteq \{0,1,2,3\} \\ S \neq \emptyset}} \left(\prod_{0 \leqslant j \leqslant 3} \epsilon_j\right) \sum_{d \leqslant z} \sum_{n \leqslant x} \prod_{j \in S} f(n + jd)\right).$$

For each non-empty $S \in \{0,1,2,3\}$, Theorem 1.7.1 (with $\boldsymbol{t} = \mathbf{0}$) gives

$$(xz)^{-1} \sum_{d \leqslant z} \sum_{n \leqslant x} \prod_{j \in S} f(n + jd) = \sum_{\substack{\mathrm{rad}(a_j) | q \\ \forall j \in S}} \prod_{j \in S} f(a_j) R(\boldsymbol{a}, S) \Xi_{\boldsymbol{a}}(\boldsymbol{L}_S, \chi\mathbf{1}_S) \prod_{p \leqslant x} M_p(F\mathbf{1}_{|S|}, \boldsymbol{L}_S) + o(1), \tag{6.3.5}$$

where we have set

$$R(\boldsymbol{a}, S) := \lim_{x \to \infty} x^{-|S|} \sum_{\substack{\boldsymbol{n} \in [x]^{|S|} \\ a_j q | L_j(\boldsymbol{n}) \forall j \in S}} 1.$$

When $|S| \in \{2,3\}$, Lemma 6.3.1 implies that the right side of (6.3.5) is 0. Now, $\mathbb{D}(f, \chi; \infty) < \infty$ and for x sufficiently large in terms of q,

$$\begin{aligned} \mathbb{D}(1, \chi; x)^2 = \sum_{p \leqslant x} \frac{1 - \chi(p)}{p} &= \log_2 x + \log\left(\prod_{p \leqslant x} \left(1 - \frac{\chi(p)}{p}\right)\right) + O(1) \\ &= \log_2 x - \log L(1, \chi) + O(1) \gg \log_2 x. \end{aligned}$$

By the triangle inequality for $\mathbb{D}$, it follows that $\mathbb{D}(f, 1; x)^2 \gg \log_2 x$, and hence by Wirsing's theorem (for an effective version due to Hall and Tenenbaum, see Theorem 4.14 of [Ten94]),

$$\sum_{n \leqslant x} f(n + jd) = \sum_{n \leqslant x} f(n) + o(x) = o(x),$$

for each $0 \leqslant j \leqslant 3$. Thus, when $|S| = 1$, the left side of (6.3.5) is $o(1)$.
Now, when we do not sum over multiples of q, $R(\boldsymbol{a},\{0,1,2,3\}) = 0$. This implies (6.3.3). Conversely, if multiples of q are included in the sum then $R(\boldsymbol{a},\{0,1,2,3\}) = (aq)^{-2}$. Thus, for (6.3.4), the above and Lemma 6.3.1 yields

$$\begin{aligned}
&(xz)^{-1} \sum_{d \leqslant z} |\{n \leqslant x : f(n+jd) = \epsilon_j \forall j\}| \\
&= \frac{1}{16}\left(1 + \epsilon_0\epsilon_1\epsilon_2\epsilon_3 \prod_{\substack{p \leqslant x \\ p \nmid q}} M_p(F\mathbf{1}_4, \boldsymbol{L}_{[3]}) \frac{\mu(q)\phi(q)}{q^2} \sum_{\mathrm{rad}(a)|q} \frac{f(a)^4}{a^2} \prod_{p|q} \Delta_p\right) + o(1) \\
&= \frac{1}{16}\left(1 + \epsilon_0\epsilon_1\epsilon_2\epsilon_3 \prod_{\substack{p \leqslant x \\ p \nmid q}} M_p(F\mathbf{1}_4, \boldsymbol{L}_{[3]}) \frac{\mu(q)\phi(q)}{q^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{-1} \Delta_p\right) + o(1) \\
&= \frac{1}{16}\left(1 + \epsilon_0\epsilon_1\epsilon_2\epsilon_3 \prod_{\substack{p \leqslant x \\ p \nmid q}} M_p(F\mathbf{1}_4, \boldsymbol{L}_{[3]}) \frac{\mu(q)}{q} \prod_{p|q} \left(1 + \frac{1}{p}\right)^{-1} \Delta_p\right) + o(1),
\end{aligned}$$

which implies the claim. $\square$

6.4. The second moment estimate

We now establish a mean-squared deviation estimate for the cardinalities of the sets

$$S_{\boldsymbol{\epsilon}}(d) := \{n \leqslant x : f(n+jd) = \epsilon_j \ \forall\ 0 \leqslant j \leqslant 3\},$$

using the first-moment estimate in Proposition 6.3.3. In the sequel, let

$$A_{\boldsymbol{\epsilon}}(f;q) := \epsilon_0\epsilon_1\epsilon_2\epsilon_3 \prod_{p \nmid q} M_p(f\chi\mathbf{1}_4, \boldsymbol{L}_{\{0,1,2,3\}}) \prod_{p|q} \frac{\mu(p)\Delta_p}{p+1},$$

whenever f is pretentious to a real character χ modulo q.
First, we show that the product of p-adic local factors is independent of the choice of linear forms for which the factors are defined. As before, given subsets $S,T \subseteq \{0,1,2,3\}$ we let

$$\boldsymbol{L}_{S,T} := \{(n,n',d) \mapsto n + jd : j \in S\} \cup \{(n,n',d) \mapsto n' + j'd : j' \in T\}.$$

Lemma 6.4.1. Let $f : \mathbb{N} \to \{-1,1\}$ be pretentious to a real character χ with conductor q. Let $S,S',T,T' \subset \{0,1,2,3\}$ be subsets of size 2. Then

$$\prod_{p \nmid q} M_p(f\chi\mathbf{1}_4, \boldsymbol{L}_{S,T}) = \prod_{p \nmid q} M_p(f\chi\mathbf{1}_4, \boldsymbol{L}_{S',T'}).$$

Similarly,

$$\prod_{p \nmid q} M_p(f\chi\mathbf{1}_4, \boldsymbol{L}_{S,\{0,1,2,3\}}) = \prod_{p \nmid q} M_p(f\chi\mathbf{1}_4, \boldsymbol{L}_{\{0,1,2,3\},S}) = \prod_{p \nmid q} M_p(f\chi\mathbf{1}_4, \boldsymbol{L}_{S',\{0,1,2,3\}}).$$

Proof. Fix $p \nmid 6q$ for the moment. Write $S = \{j_0,j_1\}$ and $T = \{k_0,k_1\}$. We can express $M_p(f\chi\mathbf{1}_4, \boldsymbol{L}_{S,T})$ as

$$M_p(f\chi\mathbf{1}_4,\boldsymbol{L}_{S,T}) = \sum_{\nu_0,\nu_1,\nu_2,\nu_3 \geqslant 0} \prod_{0\leqslant j\leqslant 3} F(p^{\nu_j}) \left(\lim_{x\to\infty} x^{-3} \sum_{\substack{n,m,d\leqslant x \\ p^{\nu_t}||(n+j_t d), p^{\nu_{2+t}}||(m+k_t d)}} 1 \right).$$

We split the sum over n,n' and d according to the p-adic valuation of each of these variables. Given $m \in \mathbb{N}$, let $\nu_p(N)$ denote the exponent r such $p^r||N$. Given fixed n,m and d counted by the inner sum above, let $\alpha := \nu_p(n)$, $\beta := \nu_p(m)$ and $\gamma := \nu_p(d)$. By the properties of the p-adic valuation, and $p \nmid 6$, if $\alpha \neq \gamma$ then $\nu_t = \max\{\alpha,\gamma\}$ for both $t = 0,1$, and similarly, if $\beta \neq \gamma$ then $\nu_{2+t} = \max\{\beta,\gamma\}$ for both $t = 0,1$. The densities thus depend only on the p-adic valuations of n, m and d, and not on the choice of forms. Hence, these terms are independent of the choices of j_0,j_1,k_0 and k_1.
Suppose now that $\alpha = \gamma$, and write $n' = n/p^\gamma$ and $d' := d/p^\gamma$. Then it follows that $n' \equiv -j_t d' \ (p^{\nu_t-\gamma})$, and $p^{\nu_t-\gamma+1} \nmid (n' + j_t d')$. As such, given d', the density of such n' is $p^{-(\nu_t-\gamma)}(1-1/p)$, irrespective of the specific choice j_t. A similar scenario occurs when the roles of α and β are switched, and when $\alpha = \beta = \gamma$. This proves that the p-adic factors are independent of S and T when $p \nmid 6q$.
We now consider $p|6$. Given p, we define an equivalence relation $\sim_p$ among pairs of sets (S,T) with the property that

$$(S,T) \sim_p (S',T') \text{ if, and only if, } M_p(F\mathbf{1}_4,\boldsymbol{L}_{S,T}) = M_p(F\mathbf{1}_4,\boldsymbol{L}_{S',T'}).$$

We shall furthermore say that a form in $\boldsymbol{L}_{S,T}$ has *bad reduction* at p if the degree in any of the variables of the form decreases upon reduction modulo p; we say that the form has *good reduction* at p otherwise.
We make the following observations:
a) if $\boldsymbol{L}_{S,T}$ consists only of forms of good reduction at p then the arguments above still go through. For instance, $M_2(F\mathbf{1}_4,\boldsymbol{L}_{\{0,3\},\{0,1\}}) = M_2(F\mathbf{1}_4,\boldsymbol{L}_{\{0,1\},\{0,1\}})$, i.e.,

$$(\{0,3\},\{0,1\}) \sim_2 (\{0,1\},\{0,1\}).$$

In fact, if S encodes the same number of forms of good and bad reduction mod p as S' does then $(S,T) \sim_p (S',T)$;
b) if S_2 can be constructed as a translation of S_1, and T encodes forms of good reduction then $(S_1,T) \sim_p (S_2,T)$ provided that the set of primes at which forms in S_1 have bad reduction only differs by one prime from that of S_2. Indeed, this follows from Theorem 1.7.1 because when $z = o(x)$ and $z,x \to \infty$,

$$(zx^2)^{-1} \sum_{d\leqslant z} \sum_{n,n'\leqslant x} \prod_{j\in S_2} f(n+jd) \prod_{k\in T} f(n'+kd)$$

$$= (zx^2)^{-1} \sum_{d \leqslant z} \sum_{n,n' \leqslant x} \prod_{j \in S_1} f(n+jd) \prod_{k \in T} f(n'+kd) + o(1),$$

and moreover

$$(zx^2)^{-1} \sum_{d \leqslant z} \sum_{n,n' \leqslant x} \prod_{j \in S_i} f(n+jd) \prod_{k \in T} f(n'+kd)$$
$$= \left(\sum_{a_j | q \forall j} \frac{f(a_j)}{a_j} R(qa_1,qa_2,qa_3,qa_0) \Xi_{\boldsymbol{a}}(\boldsymbol{L}_{S_i,T},\boldsymbol{\chi}) \right) \prod_{p \nmid q} M_p(F\mathbf{1}_4,\boldsymbol{L}_{S_i,T}) + o(1),$$

The character factors only depend on the *gaps* between the elements of S_j, which are invariant under translation. The above remarks thus show that all but possibly the p-adic factors for $p|6$ are the same, and if, say, only one of the forms in S_2 has bad reduction at p and the other has good reduction everywhere then it also follows that $M_p(F\mathbf{1}_4,\boldsymbol{L}_{S_1,T}) = M_p(F\mathbf{1}_4,\boldsymbol{L}_{S_2,T})$.
With these remarks, we can now complete the proof. It suffices to show that

$$(S, \{0,1\}) \sim_p (S',\{0,1\})$$

for all S,S' of size two and both $p = 2$ and 3, as then the same arguments repeat (by transitivity) to fixing S and varying $\{0,1\}$ through all sets of size two. Thus, set $T := \{0,1\}$.
First, applying b) twice, we have

$$(\{0,1\},T) \sim_2 (\{1,2\},T) \sim_2 (\{2,3\},T).$$

Next, applying a), we have that $(\{1,3\},T) \sim_2 (\{0,1\},T)$, and applying b) again gives $(\{0,2\},T) \sim (\{1,3\},T)$. Finally, by a) we again have $(\{0,3\},T) \sim_2 (\{0,1\},T)$.
For $p = 3$, the same sort of arguments work. For instance, in this case we have $(\{0,3\},T) \sim_3 (\{1,3\},T)$ and $(\{0,2\},T) \sim_3 (\{0,1\},T)$ by a), and $(\{1,3\},T) \sim_2 (\{0,2\},T)$. This completes the proof in the (2,2) case.
The (2,4) and (4,2) cases follow by similar (and simpler) reasoning. ☐

Lemma 6.4.2. Let q be as above and let b be the inverse of 2 modulo q. For any $p|q$,

$$\sum_{d(p)} \left(\frac{d(d+1)(d+2)(d+3)}{p} \right) = -(\Delta_p + 1). \tag{6.4.1}$$

Proof. Let R denote the sum on the left side of (6.4.1). Since the term $d = 0$ contributes nothing, we may restrict the sum to coprime residue classes modulo p. Pulling out four factors of d and replacing d by $\overline{d}$, we get

$$R = \sideset{}{^*}\sum_{d(p)} \left(\frac{(1+d)(1+2d)(1+3d)}{p} \right) = -1 + \sum_{d(p)} \left(\frac{(1+d)(1+2d)(1+3d)}{p} \right)$$
$$= -1 + \sum_{c(p)} \left(\frac{c(2c-1)(3c-2)}{p} \right),$$

upon reinserting the term $d = 0$ and making the change of variables $c = d+1$. Removing $c = 0$ and replacing c by $\overline{2c}$, we get

$$R = -1 + \sum_{c(p)}^{*} \left(\frac{c(c-1)(c-3\overline{2}^2)}{p} \right) = -(p + 2 - \#E_\lambda(\mathbb{F}_p)),$$

as in Lemma 6.3.1. This completes the proof. □

Proof of Theorem 1.7.14. We will only consider the case that the number of + signs in $\boldsymbol{\epsilon}$ is odd. The general case is similar but involves further computations of the same type as those involved in the cases we consider.

Let $\mathcal{L}$ denote the left side of (1.7.9). Expanding the square and applying Proposition 6.3.3, we have

$$\begin{aligned}\mathcal{L} &= z^{-1} \sum_{d \leqslant z} |S_{\boldsymbol{\epsilon}}(d)|^2 - \frac{x}{8}\left(1 + A_{\boldsymbol{\epsilon}}(f;q)\right) \left(z^{-1} \sum_{d \leqslant z} |S_{\boldsymbol{\epsilon}}(d)| \right) + \frac{x^2}{256}\left(1 + A_{\boldsymbol{\epsilon}}(f;q)\right)^2 \\ &= z^{-1} \sum_{d \leqslant z} |S_{\boldsymbol{\epsilon}}(d)|^2 - \frac{x^2}{256}\left(1 + 2A_{\boldsymbol{\epsilon}}(f;q) + A_{\boldsymbol{\epsilon}}(f;q)^2\right) + o(x^2).\end{aligned}$$

We seek to evaluate the second moment of $|S_{\boldsymbol{\epsilon}}(d)|^2$ here. Using Lemma 6.1.3, we get

$$\begin{aligned} z^{-1} \sum_{d \leqslant z} |S_{\boldsymbol{\epsilon}}(d)|^2 &= \frac{1}{256z} \sum_{d \leqslant z} \sum_{n,n' \leqslant x} \prod_{0 \leqslant j \leqslant 3} \left(1 + \epsilon_j f(n+jd)\right)\left(1 + \epsilon_j f(n'+jd)\right) \\ &= \frac{1}{256} \left(x^2 + 2z^{-1} \sum_{d \leqslant z} \sum_{n \leqslant x} \prod_{j \in S} \left(1 + \epsilon_j f(n+jd)\right) \right) \\ &+ \frac{1}{256} \sum_{\substack{S,T \subseteq \{0,1,2,3\} \\ |S|,|T| \geqslant 1}} \left(\prod_{\substack{j \in S \\ k \in T}} \epsilon_j \epsilon_k \right) z^{-1} \sum_{d \leqslant z} \sum_{n,n' \leqslant x} \prod_{j \in S} f(n+jd) \prod_{k \in T} f(n'+kd) \\ &= \frac{1}{256} \left((1 + 2A_{\boldsymbol{\epsilon}}(f;q) + o(1))x^2 + \sum_{\substack{S,T \subseteq \{0,1,2,3\} \\ |S|,|T| \geqslant 1}} \left(\prod_{\substack{j \in S \\ k \in T}} \epsilon_j \epsilon_k \right) z^{-1} \sum_{d \leqslant z} \sum_{n,n' \leqslant x} \prod_{\substack{j \in S \\ k \in T}} f(n+jd) f(n'+kd) \right), \end{aligned}$$

so that

$$\mathcal{L} = \frac{1}{256} \left(\sum_{\substack{S,T \subseteq \{0,1,2,3\} \\ |S|,|T| \geqslant 1}} \left(\prod_{\substack{j \in S \\ k \in T}} \epsilon_j \epsilon_k \right) z^{-1} \sum_{d \leqslant z} \sum_{n,n' \leqslant x} \prod_{\substack{j \in S \\ k \in T}} f(n+jd) f(n'+kd) - A_{\boldsymbol{\epsilon}}(f;q)^2 x^2 \right) + o(x^2). \tag{6.4.2}$$

As above, Halász' theorem implies that when $\min\{|S|,|T|\} = 1$, the contribution here is $o(x^2)$. Similarly, by Lemma 6.3.2, when either $|S| = 3$ or $|T| = 3$, the resulting contributions are zero because the character local factor vanishes by orthogonality (because $|S|$ or $|T|$ is odd).

Hence, it remains to consider the contributions from $(|S|,|T|) \in \{(2,2),(4,2),(2,4),(4,4)\}$. For each of these contributions, we can reduce to the case in which the vector $\boldsymbol{a} = (a, \dots, a, a', \dots, a')$ upon applying Theorem 1.7.1, where the components a correspond to forms induced by S and the components a' correspond to forms induced by T. For such $\boldsymbol{a}$,

$$R(\boldsymbol{a},(S,T)) = \lim_{x\to\infty} x^{-3} \sum_{\substack{n,n',d\leqslant x \\ qa|n+jd, qa'|n'+kd \forall j\in S, k\in T}} 1 = \frac{1}{q^2 aa'[qa,qa']}. \tag{6.4.3}$$

We consider the (4,4) term first. We apply Theorem 1.7.1, rearranging the character sums as before to get

$$\begin{aligned}
&(\epsilon_0\epsilon_1\epsilon_2\epsilon_3)^2 z^{-1} \sum_{d\leqslant z} \sum_{n,n'\leqslant x} \prod_{0\leqslant j\leqslant 3} f(n+jd) f(n'+jd) \\
&= \frac{x^2}{q^2} \prod_{p\nmid q} M_p(f\chi \mathbf{1}_8, \boldsymbol{L}_{\{0,1,2,3\},\{0,1,2,3\}}) \sum_{\mathrm{rad}(a),\mathrm{rad}(a')|q} \frac{1}{aa'[qa,qa']} \\
&\cdot \sum_{\substack{b_0,b_1(q) \\ \exists d: d/a \equiv b_1 - b_0 (q), d/a'\equiv c_1-c_0(q)}} \sum_{c_0,c_1(q)} \chi(b_0b_1(2b_1-b_0)(3b_1-2b_0))\chi(c_0c_1(2c_1-c_0)(3c_1-2c_0)) + o(x^2).
\end{aligned} \tag{6.4.4}$$

A routine (and tedious) argument shows that

$$M_p(f\chi\mathbf{1}_8, L_{\{0,1,2,3\},\{0,1,2,3\}}) = M_p(f\chi\mathbf{1}_4, L_{\{0,1,2,3\}})^2$$

for each $p \nmid q$. Next, consider the character sum in (6.4.4). For those d specified by the congruence condition, write $d = [a,a']m$. Then $b_1 \equiv b_0 + ma'/(a,a')\ (q)$ and $c_1 \equiv c_0 + ma/(a,a')\ (q)$. Put $A := a/(a,a')$ and $A' := a'/(a,a')$, noting that they are coprime. We may then rewrite the sum as

$$\mathcal{S}_{A,A'}(\chi) := \sum_{b,c,m(q)} \chi(b(b+mA')(b+2mA')(b+3mA'))\chi(c(c+mA)(c+2mA)(c+3mA)).$$

For $a,b \in \mathbb{Z}/q\mathbb{Z}$ write $P_a(b) := b(b+a)(b+2a)(b+3a)$. By the CRT, we split $\mathcal{S}_{A,A'}(\chi)$ as

$$\mathcal{S}_{A,A'}(\chi) = \prod_{p|q} \sum_{m,b,c(p)} \left(\frac{P_{mA'}(b)}{p}\right)\left(\frac{P_{mA}(c)}{p}\right).$$

If $p|A'$ then the sum over b is $p-1$, and note by coprimality that $p\nmid A$ so that A is invertible modulo p. Replacing c by $c\overline{A}$, we get

$$\begin{aligned}
(p-1)\sum_{m,c(p)} \left(\frac{P_m(c)}{p}\right) &= (p-1)^2 + (p-1)\sum_{m(p)}{}^{*} \sum_{c(p)} \left(\frac{P_m(c)}{p}\right) \\
&= (p-1)^2\left(1 + \sum_{c(p)}\left(\frac{P_1(c)}{p}\right)\right) = (p-1)^2\mu(p)\Delta_p,
\end{aligned}$$

upon invoking Lemma 6.4.2. The same term occurs for primes p dividing A. Now suppose that $p \nmid AA'$. As before, we can replace (b,c) by $(b\overline{A'}, c\overline{A})$. Then, separating $m = 0$ from the remaining terms of the sum once again and then changing variables a second time, we get

$$(p-1)^2 + \sideset{}{^*}\sum_{m(p)} \sum_{b,c(p)} \left(\frac{P_m(b)}{p}\right)\left(\frac{P_m(c)}{p}\right) = (p-1)^2 + (p-1)\sum_{b,c(p)} \left(\frac{P_1(b)}{p}\right)\left(\frac{P_1(c)}{p}\right)$$
$$= (p-1)^2 + (p-1)\left(\sum_{b(p)} \left(\frac{P_1(b)}{p}\right)\right)^2 = (p-1)^2 + (p-1)(\Delta_p+1)^2.$$

As such, we have

$$\mathcal{S}_{A,A'}(\chi) = \phi(q)^2 \prod_{p|AA'} \mu(p)\Delta_p \prod_{\substack{p|q \\ p\nmid AA'}} \left(1 + \frac{(\Delta_p+1)^2}{p-1}\right). \tag{6.4.5}$$

Returning to (6.4.4), we next evaluate the expression

$$\frac{\phi(q)^2}{q^2} \prod_{p|q}\left(1+\frac{(\Delta_p+1)^2}{p-1}\right) \sum_{\mathrm{rad}(a),\mathrm{rad}(a')|q} \frac{1}{aa'[qa,qa']} \prod_{p|aa'/(a,a')^2} \frac{(p-1)\mu(p)\Delta_p}{p-1+(\Delta_p+1)^2}$$
$$= \frac{\phi(q)^2}{q^3} \prod_{p|q}\left(1+\frac{(\Delta_p+1)^2}{p-1}\right) \sum_{\mathrm{rad}(\delta)|q} \frac{1}{\delta^3} \sum_{\substack{\mathrm{rad}(a),\mathrm{rad}(a')|q \\ (a,a')=1}} \frac{1}{(aa')^2} \prod_{p|aa'} \frac{(p-1)\mu(p)\Delta_p}{p-1+(\Delta_p+1)^2}. \tag{6.4.6}$$

The inner sum can be written as the product

$$\prod_{\substack{p|q \\ p\nmid a}} \left(1 - \frac{(p-1)\Delta_p}{p-1+(\Delta_p+1)^2} \sum_{k\geqslant 1} p^{-2k}\right) = \prod_{\substack{p|q \\ p\nmid a}} \left(1 - \frac{\Delta_p}{(p+1)(p-1+(\Delta_p+1)^2)}\right),$$

so that the sum over a becomes

$$\prod_{p|q}\left(1 - \frac{\Delta_p}{(p+1)(p-1+(\Delta_p+1)^2)}\right) \prod_{p|q}\left(1 - \frac{(p^2-1)\Delta_p}{(p+1)(p-1+(\Delta_p+1)^2)-\Delta_p} \sum_{k\geqslant 1} p^{-2k}\right)$$
$$= \prod_{p|q}\left(1 - \frac{\Delta_p}{(p+1)(p-1+(\Delta_p+1)^2)}\right)\left(1 - \frac{\Delta_p}{(p+1)(p-1+(\Delta_p+1)^2)-\Delta_p}\right)$$
$$= \prod_{p|q} \frac{(p+1)(p-1+(\Delta_p+1)^2)-2\Delta_p}{(p+1)(p-1+(\Delta_p+1)^2)}$$

Inserting this expression into (6.4.6) yields

$$\frac{\phi(q)}{q^3} \prod_{p|q} \left(1-p^{-3}\right)^{-1} \frac{(p+1)(p-1+(\Delta_p+1)^2)-2\Delta_p}{p+1}$$
$$= \prod_{p|q} \frac{p^2-1+(p+1)(\Delta_p^2+2\Delta_p+1)-2\Delta_p}{(p+1)\,(p^2+p+1)}$$
$$= \prod_{p|q} \frac{\Delta_p^2+p^2+p(\Delta_p+1)^2}{(p+1)((p+1)^2-p)}$$

$$= \prod_{p|q} \left(\frac{\mu(p)\Delta_p}{p+1} \right)^2 \frac{(1+1/\Delta_p)^2 + 1/p + p/\Delta_p^2}{1+1/p(p+1)}.$$

It remains to determine the contributions from the (2,2), (4,2) and (2,4) terms. We will only consider the (2,2) case, the (4,2) (and symmetrically) the (2,4) case being similar and simpler.

The (2,2) contribution is

$$\sum_{\substack{S,T \in \{0,1,2,3\} \\ |S|=|T|=2}} \left(\prod_{j \in S} \epsilon_j \right) \left(\prod_{k \in T} \epsilon_k \right) \left(z^{-1} \sum_{d \leqslant z} \sum_{n,n' \leqslant x} \prod_{j \in S} f(n+jd) \prod_{k \in T} f(n'+kd) \right). \tag{6.4.7}$$

Fix a pair of sets S and T. We apply Theorem 1.7.1 to get

$$(zx^2)^{-1} \sum_{d \leqslant z} \sum_{n,n' \leqslant x} \prod_{j \in S} f(n+jd) \prod_{k \in T} f(n'+kd)$$
$$= \sum_{\mathrm{rad}(a_j)|q \forall j} f(a_j) R(\boldsymbol{a},(S,T)) \sum_{\substack{b_0,b_1,c_0,c_1(q) \\ \exists n,n',d:(n+j_t d)/a_t \equiv b_t(q),(n'+k_t d)/a_{2+t} \equiv b_{2+t}(q)}} \chi\left(b_0 b_1 c_0 c_1\right) \prod_{p|q} M_p(f\chi \mathbf{1}_4, \boldsymbol{L}_{S,T}).$$

By Lemma 6.4.1, the product of p-adic factors M_p is independent of S and T, and we will show that the same is true of the character sum. Indeed, arguing as in our treatment of the (4,4) case, the sums over b_j are non-zero only when $a = a_0 = a_1$ and $a' = a_2 = a_3$. Writing $d = [a,a']m$ as before, we can rewrite the character sum as

$$\sum_{b,c(q)} \chi(b(b+mA'))\chi(c(c+mA)). \tag{6.4.8}$$

We apply the CRT and consider the character sum modulo each prime p dividing q as above. When $p|A$ we get

$$(p-1) \sum_{b,m(p)} \left(\frac{b(b+m)}{p} \right) = (p-1) \left((p-1) + \left(\frac{-1}{p} \right) \sideset{}{^*}\sum_{m(p)} \sum_{b(p)} \left(\frac{b(m-b)}{p} \right) \right)$$
$$= (p-1)^2 \left(1 + \left(\frac{-1}{p} \right) J\left(\left(\frac{\cdot}{p} \right), \left(\frac{\cdot}{p} \right) \right) \right), \tag{6.4.9}$$

the last term being a Jacobi sum. It is well-known that this Jacobi sum is precisely $-\left(\frac{-1}{p}\right)$, so the right side of (6.4.9) vanishes. The same is true for $p|A'$, and hence the non-zero contributions come from $A = A' = 1$, i.e., from $a = a'$. As such, for each prime $p|q$,

$$\sum_{b,c,m(p)} \left(\frac{b(b+m)}{p} \right) \left(\frac{c(c+m)}{p} \right) = (p-1)^2 + \sideset{}{^*}\sum_{m(p)} \sum_{b,c(p)} \left(\frac{b(b+m)}{p} \right) \left(\frac{c(c+m)}{p} \right)$$
$$= (p-1)^2 + (p-1) \left(\sum_{b(p)} \left(\frac{b(b+1)}{p} \right) \right)^2$$

$$= (p-1)^2 + (p-1)\left(\frac{-1}{p}\right)^2 J\left(\left(\frac{\cdot}{p}\right), \left(\frac{\cdot}{p}\right)\right)^2$$
$$= p(p-1).$$

This expression is then clearly independent of S and T. Thus, summing over S and T, (6.4.7) becomes

$$\frac{1}{q^3}\prod_{p\nmid q} M_p(f\chi\mathbf{1}_4, \boldsymbol{L}_{\{0,1\},\{0,1\}})\left(\sum_{\mathrm{rad}(a)|q}\frac{1}{a^3}\right)\prod_{p|q} p(p-1)\sum_{\substack{S,T\subset\{0,1,2,3\}\\|S|=|T|=2}}\sum_{j\in S,k\in T}\epsilon_j\epsilon_k$$
$$= \prod_{p|q}\frac{p}{p^2+p+1}\prod_{p\nmid q} M_p(f\chi\mathbf{1}_4, \boldsymbol{L}_{\{0,1\},\{0,1\}})\left(\sum_{0\leqslant i<j\leqslant 3}\epsilon_i\epsilon_j\right)^2.$$

Now, when $\boldsymbol{\epsilon}$ has an odd number of + signs, $\sum_{0\leqslant i\leqslant 3}\epsilon_i = \pm 2$ and thus we have

$$\sum_{0\leqslant i<j\leqslant 3}\epsilon_i\epsilon_j = \frac{1}{2}\left(\left(\sum_{0\leqslant i\leqslant 3}\epsilon_i\right)^2 - \sum_{0\leqslant i\leqslant 3}\epsilon_i^2\right) = 0.$$

This implies that the (2,2) contribution is $o(x^2)$. This factor is also responsible for making the (4,2) and (2,4) contributions vanish. Thus, in the end, we have

$$\mathcal{L} = \frac{x^2}{256}\prod_{p\nmid q} M_p(f\chi\mathbf{1}_8,\boldsymbol{L}_{\{0,1,2,3\},\{0,1,2,3\}})$$
$$\cdot\prod_{p|q}\left(\frac{\mu(p)\Delta_p}{p+1}\right)^2\left(\prod_{p|q}\frac{3\Delta_p + \#E_{3b^2}(\mathbb{F}_p)}{\Delta_p^2(1-1/p(p+1))} - 1\right) + o(x^2) = \frac{x^2}{256}(T_{4,4} - A_{\boldsymbol{\epsilon}}(f;q)^2) + o(x^2)$$

which proves Theorem 1.7.14.

∎

Proof of Theorem 1.7.11. We follow the proof of Theorem 1.7.14. The summation in d is restricted such that $q\nmid d$ and normalized by $(1-1/q)z$, and 0 stands in place of $A_{\boldsymbol{\epsilon}}(f;q)$ (as in Proposition 6.3.3). Note that in (6.4.3), the restriction $q\nmid d$ there implies that the quantity $R(\boldsymbol{a},(S,T)) = 0$ for $|S|,|T| \geqslant 2$. This means that all of the contributions by $|S|,|T| \geqslant 2$ are $o(x^2)$ in (6.4.2). It thus follows that

$$z^{-1}\sum_{\substack{d\leqslant z\\ q\nmid d}}\left(|S_{\boldsymbol{\epsilon}}(d)| - \frac{x}{16}\right)^2 = o(x^2).$$

The conclusion then follows by Chebyshev's inequality. ∎

Remark 6.4.3. By Hasse's bound, we always have $|\Delta_p| \leqslant 2\sqrt{p}$. In fact, it is known [MM11] that as $p \to \infty$,

$$\pi(x)^{-1}|\{p \leqslant x : \cos^{-1}(\Delta_p/2\sqrt{p}) \in I\}| \to \mu_{ST}(I) := \begin{cases} \frac{2}{\pi}\int_I \sin^2 u du & \text{if } E_\lambda \text{ is non-CM} \\ \frac{1}{2}\left(1_{\pi/2\in I} + |I|\right) & \text{if } E_\lambda \text{ is CM.} \end{cases}$$

for all intervals $I \subset [-1,1]$. Now, if q is fixed and we choose an elliptic curve $E_\lambda/\mathbb{Q}$ such that $\lambda \equiv 3\overline{2}^2(q)$ then in general (whether or not E_λ is CM) we do not understand the behaviour of Δ_p for $p|q$. Instead, we may draw a heuristic from a result of Miller and Murty [MM11], which states that for a fixed p and a one-parameter family of elliptic curves $\{E_t/\mathbb{F}_p : t \in \mathbb{F}_p\}$, the *discrepancy*

$$\max_{I\subseteq[0,\pi]} \left|p^{-1}|\{t \in \mathbb{F}_p : \cos^{-1}(\Delta_{t,p}/2\sqrt{p}) \in I\}| - \mu_{ST}(I)\right|$$

tends to 0 as $p \to \infty$, where $\Delta_{t,p}$ is the trace of Frobenius on $E_t/\mathbb{F}_p$. This says roughly that for generic curves over $\mathbb{F}_p$ in a family, the angles $\cos^{-1}(\Delta_{t,p}/2\sqrt{p})$ behave as they should for elliptic curves over $\mathbb{Q}$ in the p-limit.

Thus, if we assume that the element $3\overline{2}^2$ modulo p yields a generic element of the one-parameter family generated by the Legendre models $E_t : y^2 = x(x-1)(x-t)$ with $t \in \mathbb{F}_p$ then for generic p, $\Delta_p \asymp \sqrt{p}$ but Δ_p is *not* close to $\sqrt{p}$. The same is true modulo q, for q a product of more than one prime. Thus, on heuristic grounds the product

$$\prod_{p|q} \frac{(1+1/\Delta_p)^2 + 1/p + p/\Delta_p^2}{1 + 1/p(p+1)}$$

is $\asymp 1$, but on the other hand, it is not asymptotically 1 as $p \to \infty$ in general.

Chapter 7

WORK IN PROGRESS

Here we briefly describe a few problems related to the those discussed in the previous chapters of this thesis.

7.0.1. Rigidity theorems for multiplicative functions

In a joint work with Alexander Mangerel, we establish several results concerning the expected general phenomenon that, given a multiplicative function $f : \mathbb{N} \to \mathbb{C}$, the values of $f(n)$ and $f(n+a)$ are "generally" independent unless f is of a "special" form.
Firstly, we prove a converse theorem that resolves the following folklore conjecture: for any completely multiplicative function $f : \mathbb{N} \to \mathbb{T}$ we have $\liminf_{n\to\infty} |f(n+1) - f(n)| = 0$.
Secondly, we settle a sixty-year-old conjecture due to N.G. Chudakov that states that any completely multiplicative function $f : \mathbb{N} \to \mathbb{C}$ that: a) takes only finitely many values, b) vanishes at only finitely many primes, and c) has uniformly bounded partial sums, is a Dirichlet character.
Finally, we show that if many of the binary correlations of a 1-bounded multiplicative function are asymptotically equal to those of a Dirichlet character χ mod q then $f(n) = \chi'(n)n^{it}$ for all n, where χ' is a Dirichlet character modulo q and $t \in \mathbb{R}$. This establishes a variant of a conjecture of H. Cohn for multiplicative arithmetic functions.

7.0.2. Chowla conjecture over the function fields. Large degree limit

Recently, big progress has been made by several people including Rudnick, Keating, BarySorocker and others proving the analogs of famous number theoretic conjectures over the function field $\mathbb{F}_q[x]$ in the limit $q \to \infty$. See extensive literature [KR16],[CR14]. Unfortunately, not much is known about the case when the ground field $\mathbb{F}_q$ is fixed.

Carmon and Rudnick [CR14] established the function field analog ($q-$limit) of the Chowla conjecture in the $q-$ limit in the by proving

$$\left|\sum_{\deg F=n}\mu(F+\alpha_1)\mu(F+\alpha_2)\ldots\mu(F+\alpha_r)\right|\leq 2rnq^{n-\frac{1}{2}}+3rn^2q^{n-1},$$

where α_j are some fixed distinct polynomials. In the forthcoming work, we develop function field analog of the result of Matomaki and Radziwiłł as well as the function field version of the "entropy decrement argument" due to Tao to prove

$$\sum_{\deg F=n}\mu(F)\mu(F+\alpha_1)=o_{n\to\infty}(q^n),$$

which establishes the $k=2$ case of the Chowla conjecture in the limit of large degree.

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