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Complex structures

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Université de Montréal

Faculté des études supérieures

Ce mémoire intitulé

Complex structures

présenté par

Leila Ezeddin

a été évalué par un jury composé des personnes suivantes :

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Mémoire accepté le:

SOMMAIRE

Initialement, notre but était de considérer la question principale au sujet des structures complexes; La 6-sphère a-t-elle une structure complexe? Après avoir construit une structure presque complexe sur la 6-sphère, nous avons rapidement constaté que ce problème était beaucoup trop compliqué. Nous nous sommes alors résignés à simplement nous familiariser avec quelques notions élémentaires en rapport avec les structures complexes, en préparation pour un retour éventuel à la 6-sphère.

Nous dirons quelques mots au sujet des structures complexes en général, mais notre intention est de mettre l'accent sur les structures complexes sur les surfaces de Riemann. En cours de route, nous toucherons aussi à quelques notions voisines, telles que les structures presque complexes et les structures de Cauchy-Riemann.

Au chapitre 1, nous parlons d'espaces vectoriels réels et complexes. Plus exactement, nous alons induire une structure complexe sur un espace vectoriel réel de dimension paire, pour en faire un espace vectoriel complexe.

Nous continuons cette approche au chapitre 2 pour imposer des structures complexes sur des variétés. Nous étudierons aussi les structures presque complexes et les structures de Cauchy-Riemann. Une variété presque complexe est une variété lisse munie d'une structure linéaire complexe sur chaque espace tangent, qui varie de façon lisse. Une variété de Cauchy-Riemann est définie par un sousfibré du fibré tangent complexifié.

Au chapitre 3, nous construisons l'espace de Riemann des modules d'une surface. Pour une surface de Riemann donnée S, l'espace de Riemann des modules consiste des classes d'équivalence conforme (biholomorphe) de surfaces de Riemann qui sont homéomorphes à S. Dans ce chapitre, nous montrons aussi que toute surface de Riemann, sauf quelques exceptions, est représentée comme quotient du demi-plan supérieur par un groupe Fuchsien. Nous parlerons aussi des fonctions automorphes. Une fonction automorphe sur un domaine est une fonction méromorphe qui est invariante par rapport à un certain genre de groupe d'automorphismes du domain.

Au chapitre 4, on introduit les transformations quasiconformes, utilisant une méthode géométrique et une procedure analytique. Ensuite, on construit l'espace de Teichmüler en utilisant les transformation quasiconformes. On montre aussi que l'espace de Teichmüler est complet pour la distance de Teichmüler. Ensuite on explique que les automorphismes d'une surface forment un groupe discret, le groupe modulaire, qui agit sur l'espace de Teichmüler de cette surface. Le quotient de l'espace de Teichmüler par cette action est précisément l'espace de modules de cette surface.

Au chapitre 5, on utilise les produits dans les quaternions et les octonions pour construire une structure presque complexe sur la 2-sphère et la 6-sphère. On montre aussi que cette approche ne marche pas sur les sphères de dimension autre que 2 ou 6. Enfin, on montre que la structure presque complexe que nous avons construite sur la 2-sphère provient d'une structure complexe.

Mots Clés: Espace de Teichmüller, espace des modules, surfaces de Riemann, structure complexe, structure CR, structure presque complexe.

SUMMARY

Our initial project was to consider the most outstanding problem concerning complex structures, namely, whether the 6-sphere $\mathbf{S}^6$ admits a complex structure. We constructed an almost complex structure on 6-sphere, however it quickly became apparent that the original problem was far beyond our reach and we resigned ourselves to merely familiarizing ourselves with some preliminary notions regarding complex structures, in the hope of eventually returning to the 6-sphere. We shall say a few words about complex structures in general, but we intend to emphasize complex structures on Riemann surfaces. Along the way we shall also touch upon related notions such as almost complex structures and Cauchy-Riemann structures.

In Chapter 1, we deal with real vector spaces and complex vector spaces. Precisely, we will implement a linear complex structure on even dimensional real vector spaces to become complex vector spaces.

We continue this approach to Chapter 2, to establish complex structures on manifolds. We shall also study Almost complex structures and CR structures. An almost complex manifold is a smooth manifold equipped with smooth linear complex structure on each tangent space and a CR manifold is defined by a subbundle of the complexified tangent bundle.

In Chapter 3, we construct Riemann's moduli space. For a given Riemann surface S, Riemann's Moduli space consists of the conformal (biholomorphic) equivalence classes of Riemann surfaces which are homeomorphic to S. In this chapter we also show that every Riemann surface, except for a few types, is represented as a quotient space of the upper half plane by a Fuchsian group. We shall also give the definition of automorphic functions. An automorphic function is one which is meromorphic in its domain and is invariant under a certain type of group of automorhisms of the domain.

In Chapter 4 we define quasiconformal mappings, using a geometric method and an analytic procedure. Then the Teichmüller space is constructed by using quasiconformal mappings. We also show that Teichmüller space is complete with respect to the Teichmüller distance. We also explain that the automorphisms of a surface form a discrete group, the modular group, that acts on the Teichmüller space of that surface. The quotient of the Teichmüller space by this action is precisely the moduli space of that surface.

In Chapter 5, we use the product in quaternions and octonions to construct an almost complex structure in the 2-sphere and 6-sphere. We also show that this approach cannot be applied to spheres in other dimensions. Finally, we show that the almost complex structure which we constructed on the 2-sphere derives from a complex structure.

Key words:

Teichmüller space, Moduli space, Riemann surface, complex structure, CR structure, almost complex structure.

CONTENTS

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INTRODUCTION

Around 500 BC, the Greek mathematicians led by Pythagoras realized the need for irrational numbers in particular the irrationality of the square root of two. Negative numbers were invented by Indian mathematicians around 600 AD, and then possibly invented independently in China shortly after. In the 18th and 19th centuries there was much work on irrational and transcendental numbers.

The earliest fleeting reference to square roots of negative numbers perhaps occurred in the work of the Greek mathematician and inventor Heron of Alexandria in the 1st century CE, when he considered the volume of an impossible frustum of a pyramid, though negative numbers were not conceived in the Hellenistic world. Complex numbers became more prominent in the 16th century, when closed formulas for the roots of cubic and quartic polynomials were discovered by Italian mathematicians.

After introducing real and complex vector spaces, we shall define complex structures on manifolds. The name manifold comes from Riemann's original German term "Mannigfaltigkeit" which William Kingdon Clifford translates as "manifoldness". Hermann Weyl gave an intrinsic definition for differentiable manifolds in 1912. The foundational aspects of the subject were clarified during the 1930s by Hassler Whitney and others, making precise intuitions dating back to the later half of the 19th century.

The notion of a complex manifold is a natural outgrowth of that of a differentiable manifold. Its importance lies to a large extent in the fact that the complex manifolds include the Riemann surfaces as special cases, and furnish the geometric basis for functions of several complex variables.

The problem of how to parametrize the variation of complex structures on a fixed base surface originated with G.F.Bernhard Riemann. This problem has spurred extensive investigations, and progress has been considerable in the area of the theory of Riemann surfaces. Riemann's Moduli space consists of the conformal (biholomorphic) equivalence classes of Riemann surfaces.

One of Oswald Teichmüller's great contributions to the moduli problem was to recognize that it becomes more accessible if we consider not only conformal mappings but also quasiconformal mappings. In the end of the 1950s, Lars V. Ahlfors and Lipman Bers developed the fundamentals of the theory of Teichmüller spaces.

The theory of Teichmüller space gives a parametrization of all the complex structures on a given surface. This subject lies in the intersection of many important areas of mathematics. These include complex manifolds, holomorphic functions, Riemann surfaces, Fuchsian groups and complex analysis. Recently, the theory of Teichmüller spaces has begun to play an important role in string theory.

String theory is a fundamental model of physics whose building blocks are one dimensional extended objects (strings) rather than the zero dimensional points (particles) that are the basis of the standard model of particle physics. String theorists are attempting to adjust the Standard Model by removing the assumption in quantum mechanics that particles are point-like. By removing this assumption and replacing the point-like particles with strings, it is hoped that string theory will develop into a sensible quantum theory of gravity. Moreover, string theory appears to be able to "unify" the known natural forces (gravitational, electromagnetic, weak and strong) by describing them with the same set of equations. How do surfaces enter the picture? A string is 1-dimensional. As time varies, its world-orbit is hence 2-dimensional, thus a surface. Teichmüler theory is useful in studying the physics of 2-dimensional space time. The 2-dimensional model of space time is of interest to physicists, because it gives them a simpler context in which to study complicated phenomena. The understanding so obtained will hopefully yield insights into higher dimensional space-time.

I hope to have whetted the reader's appetite for more of this subject (Teichmüller theory), a subject that Lipman Bers has called "the higher theory of Riemann surface."

Chapter 1

COMPLEX STRUCTURE ON VECTOR SPACES

1.1. Complex structures on vector spaces

Definition 1.1.1. *If V is a real vector space, a linear map $J : V \to V$ such that $J^2 = -I$ is called a complex structure on V.*

Example 1.1.2. *Let V be a complex vector space, then i is a complex structure on V.*

Remark 1.1.3. *A complex structure on a vector space is an automorphism.*

If J is a complex structure so $J^2 = -I$. The function I is a bijection so $J^2 = J \circ J$ is a bijection. If the composition of two functions is a bijection, then it can be concluded that the first applied is injective and the second applied is surjective. Here we have $J \circ J$, so J is injective and surjective.

The simplest example of a vector space over R is the trivial one: $\{0\}$ which contains only one element, 0, of R. Both vector addition and scalar multiplication are trivial. A basis for this vector space is the empty set, so that $\{0\}$ is 0-dimensional vector space over R. The linear map $J(0) = 0$ is a complex structure on R^0.

R is a vector space over itself. Vector addition is just field addition and scalar multiplication is just field multiplication. The identity element, 1, of R serves as a basis so that R is a 1-dimensional vector space over itself.

Assume that R has a complex structure. So there is a linear operator

$$J : R \longrightarrow R \text{ such that } J^2 = -I.$$

In particular, this implies that:

$$J(J(1)) = -I(1) = -1. \tag{1.1.1}$$

Let a denote $J(1)$. Then from (1.1.1) and linearity, we have:

$$-1 = J(J(1)) = J(a) = aJ(1) = a^2.$$

Thus $a^2 = -1$ and this is a contradiction. Hence R has no complex structure.

In linear algebra, linear transformations can be represented by matrices. If T is a linear transformation mapping R^n to R^m and x is a column vector with n entries, then:

$$T(\overrightarrow{x}) = A\overrightarrow{x}$$

for some $m \times n$ matrix A, called the transformation matrix of T.

There is a linear transformation J from $R^2 \rightarrow R^2$ by $(x, y) \mapsto (-y, x)$. The matrix representation of J is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and this is a complex structure on R^2.

Theorem 1.1.4. *There exists a complex structure on a real vector space if and only if it is not of finite odd dimension.*

PROOF. Let J be a complex structure on a real finite dimensional vector space. So $J^2 = -I$. Consider that $(-1)^n$ is the determinant of $-I$ which is the determinant of J^2 that is $(|J|^2)$. Thus $(-1)^n = (|J|)^2$ and so n is even. So R^n has no complex structure when n is odd.

Every member of a vector space is a linear combination of the basis elements and a complex structure is a linear map, so it is sufficient to define a complex structure on the basis of a vector space.

Let $e_1, e_2, e_3, \cdots, e_{2n}$ be a basis of the vector space R^{2n}. In order to define a linear transformation $J : R^{2n} \rightarrow R^{2n}$, it is sufficient to define the effect of J on the basis. Set $J(e_k) = e_{n+k}$, $J(e_{k+n}) = -e_k$, $k = 1, \cdots, n$. The matrix representation of J is $\begin{pmatrix} 0_{n\times n} & -I_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{pmatrix}$.

A consequence of the axiom of choice is that every vector space has a basis and conversely, if B is an arbitrary set, a vector space with dimension $|B|$ over a field F can be constructed as follows: Take the set $F(B)$ of all functions $f : B \to F$. These functions can be added and multiplied by elements of F, and we obtain the desired F-vector space. A basis for $F(B)$ would be given by the set of all functions $f_b(a) = \begin{cases} 0 & \text{if } a \neq b, \\ 1 & \text{if } a = b \end{cases}$.

These functions generate $F(B)$, because $\forall f \in F(B)$

$$\begin{aligned} f :& B \to F \\ b_i &\mapsto f_i \end{aligned}$$

we have $f = \sum_{finite}^{b_i \in B} f_i f_{b_i}$. They are also linearly independent: Let $\sum_{finite}^{b_i \in B} \alpha_i f_{b_i} = 0$, so for every $x = b_j \in B$, we have $\sum_{finite}^{b_i \in B} \alpha_i f_{b_i}(x) = \alpha_j f_{b_j}(b_j) = 0$, so $\alpha_j = 0$. Hence there exists a vector space having arbitrary cardinality.

Let V be a real vector space of infinite dimension m with basis B. We can see the infinite number m as a $m = m + m$. There exist vector spaces V_1 and V_2, both of dimension m, with bases C and D respectively. So the vector space V can be represented as $V = V_1 \oplus V_2$ and its basis B can be considered as a decomposition $\{C, D\}$. Since C and D have the same cardinality, there is a bijection $f : C \to D$, $d_\alpha = f(c_\alpha)$ where $C = \{c_\alpha\}$, $\alpha \in \mathfrak{a} = |B|$. Now we define a complex structure J on the real infinite dimensional vector space V as $J(c_\alpha) = d_\alpha$ and $J(d_\alpha) = -c_\alpha$ where $\alpha \in \mathfrak{a}$. □

1.2. All complex structures on vector spaces

If a vector space V has a complex structure J, then $-J$ is also a complex structure and if the dimension of V is greater than 0, then J is not equal to $-J$. Thus the complex structure is always non-unique in a vector space of non zero dimension. This does not depend on the dimension of V. It works also for infinite dimension. So the complex structure on a vector space is not unique.

Definition 1.2.1. *The complexification V^c of a real vector space V is the complex vector space V^c that is obtained from the real vector space V by extending the field of scalars.*

The space V^c is the set of expressions $X + iY$, where $X, Y \in V$, with the operations of addition and multiplication by complex numbers are defined as follows:
$(X_1 + iY_1) + (X_2 + iY_2) := (X_1 + X_2) + i(Y_1 + Y_2)$, $X_1 + iY_1, X_2 + iY_2 \in V^c$
$(a + ib)(X + iY) := (aX - bY) + i(aY + bX)$, $a + ib \in C, X + iY \in V^c$.
And since we just extend the field of scalars of a vector space,

$$\begin{aligned} V &= \{\sum_{finite} \alpha_i b_i : \alpha_i \in R, b_i \in \text{basis of } V\} \\ V^c &= \{\sum_{finite} (\alpha_i + i\beta_i) b_i : \alpha_i + i\beta_i \in C, b_i \in \text{basis of } V\} \\ &= \{\sum_{finite} \alpha_i b_i + i \sum_{finite} \beta_i b_i : \alpha_i, \beta_i \in R, b_i \in \text{basis of } V\} \end{aligned}$$

so the dimension of V^c over C is equal to the dimension of V over R, since every basis of V is a basis of V^c over C.

Definition 1.2.2. *A group G is said to act on a set X i.e. X is a G-set, when there is a map $\phi : G \times X \to X$ such that the following conditions hold for all elements $x \in X$.*

(1) $\phi(e, x) = x$ where e is the identity element of G.

(2) $\phi(g, \phi(h, x)) = \phi(gh, x)$ for all $g, h \in G$.

For simplicity, we write $\phi(g, x) = gx$.

A group action $G \times X \to X$ is transitive if it possesses only a single group orbit, i.e. for every pair of elements x and y, there is a group element g such that $gx = y$.

Some elements of a group G acting on a set X may fix a point x. These group elements form a subgroup called the isotropy group of x, defined by:

$$G_x = \{g \in G : gx = x\}.$$

When two points x and y are in the same group orbit, say $y = gx$, then the isotropy groups are conjugate subgroups. More precisely, $G_y = gG_xg^{-1}$.

$$h \in G_y \Leftrightarrow hy = y \Leftrightarrow h(gx) = gx \Leftrightarrow g^{-1}hg(x) = x \Leftrightarrow g^{-1}hg \in G_x \Leftrightarrow h \in gG_xg^{-1}.$$

For any subgroup $H \subseteq G$, consider $G/H = \{aH : a \in G\}$ as a G-set, where $\forall g \in G$, $g(aH) = (ga)H \in G/H$.

Definition 1.2.3. *Let S_1 and S_2 be two G-sets. A map $T : S_1 \longrightarrow S_2$ is a G-morphism if*

$$T(hs) = h(Ts), \forall h \in G, \forall s \in S_1.$$

If x and y are in the same orbit, $G/G_x \simeq G/G_y$: Define a map

$$\begin{aligned} T : G/G_x &\longrightarrow G/G_y \\ aG_x &\mapsto agG_xg^{-1} \end{aligned}$$

where $G/G_x = \{aG_x : a \in G\}$ and $y = gx$ or $G_y = gG_xg^{-1}$. We can easily check that this map is a G-morphism because $T(h(aG_x)) = T((ha)G_x) = (ha)gG_xg^{-1} = h(agG_xg^{-1}) = h(T(aG_x))$.

Because a transitive group action implies that there is only one group orbit, if G acts transitively on X, then X is G-isomorphic to the quotient space G/G_x by

$$\begin{aligned} G/G_x &\longrightarrow X \\ gG_x &\longmapsto gx \end{aligned}$$

where G_x is the isotropy group of x. The choice of $x \in X$ does not affect the isomorphism type of G/G_x because all of the isotropy groups are conjugate.

Theorem 1.2.4. *An R-linear map $L : V_1 \longrightarrow V_2$ is a C-linear map $L : (V_1, J_1) \longrightarrow (V_2, J_2)$ if and only if $LJ_1 = J_2L$.*

PROOF. Suppose $L : V_1 \longrightarrow V_2$ is R-linear and $LJ_1 = J_2L$. Since L is R-linear, then

$$L(v+w) = L(v) + L(w), \forall v, w \in V_1,$$
$$L((a+ib)v) = L(av + bJ_1v) = L(av) + L(bJ_1v) = aL(v) + bL(J_1v)$$
$$= aL(v) + bJ_2(L(v)) = (a+ib)L(v)$$

, so L is C-linear. Conversely, suppose L is C-linear. So $L(J_1v) = L((i)v) = (i)L(v) = J_2L(v)$, that means $LJ_1 = J_2L$. □

Corollary 1.2.5. *$L \in Aut_R(V)$ is in $Aut_C(V, J)$ if and only if $LJ = JL$ or $L = JLJ^{-1}$ or $J = L^{-1}JL$.*

Corollary 1.2.6. *Let J_1 and J_2 be complex structures on V. An R-linear map $L : V \to V$ is a C-linear map $L : (V, J_1) \to (V, J_2)$ if and only if $LJ_1 = J_2L$.*

Corollary 1.2.7. *$L \in Aut_R(V)$ is a C-isomorphism $L : (V, J_1) \to (V, J_2)$ if and only if $LJ_1 = J_2L$.*

Theorem 1.2.8. *The space $\Delta(R^{2n})$ of all complex structures on a 2n-dimensional real vector space is a G-set for $GL(2n, R)$ which is G-isomorphic with the G-set $GL(2n, R)/GL(n, C)$.*

PROOF. Define a map

$$\begin{array}{ccc} GL(2n,R) & \longrightarrow & \Delta(R^{2n}) \\ A & \longmapsto & A^{-1}J_0A \end{array}$$

where $J_0 = \begin{pmatrix} 0_{nxn} & -I_{nxn} \\ I_{nxn} & 0_{nxn} \end{pmatrix}$ is the standard complex structure on R^{2n}. This map is obviously well defined and it is surjective also: Let V^c denote the complexification of V. Let J be a complex structure on V. Since V and V^c have the same basis, the complex structure J on V extends its definition on V^c by

$$J(v) = Jx + iJy, \text{where } \ v = x + iy, v \in V^c, x, y \in V.$$

We can check that J remains linear and $J^2 = -I$:

$$J^2(v) = J(J(x+iy)) = J(Jx+iJy) = J(Jx)+iJ(Jy) = -x+i(-y) = -v.$$

$$\begin{aligned} J(v_1+v_2) &= J((x_1+iy_1)+(x_2+iy_2)) = J((x_1+x_2)+i(y_1+y_2)) \\ &= J(x_1+x_2)+iJ(y_1+y_2) = (J(x_1)+iJ(y_1))+(J(x_2)+iJ(y_2)) \\ &= J(v_1)+J(v_2). \end{aligned}$$

$$\begin{aligned} J(\alpha v) &= J((a+ib)(x+iy)) = J((ax-by)+i(ay+bx)) \\ &= J(ax-by)+iJ(ay+bx) = aJx - bJy + iaJy + ibJx \\ &= (a+ib)(Jx+iJy) = \alpha J(v) \text{ where } \alpha = a+ib \in C. \end{aligned}$$

Since $J^2 = -I$, J has eigenvalues $\lambda = \mp i$:

$$Jv = \lambda v \Longrightarrow -v = J^2 v = J(Jv) = J(\lambda v) = \lambda^2 v \Longrightarrow \lambda^2 = -1.$$

Denote by $E^{\mp}$ the eigenspaces of J for $\mp i$ respectively, and $E^{\mp} = \text{Ker }(I \mp iJ)$:

$$x \in \text{Ker }(I-iJ) \Rightarrow (I-iJ)(x) = 0 \Rightarrow iJx = x \Rightarrow Jx = -ix \Rightarrow x \in E^-,$$

$$x \in \text{Ker }(I+iJ) \Rightarrow (I+iJ)(x) = 0 \Rightarrow iJx = -x \Rightarrow Jx = ix \Rightarrow x \in E^+.$$

And also Ker $(I \mp iJ) =$ Range $(I \pm iJ)$:

$$(I+iJ)(I-iJ) = I + iJ - iJ - i^2J^2 = 0, \;\; i^2 = -1, \;\; J^2 = -I,$$

so Range $(I-iJ) \subseteq$ Ker $(I+iJ) = E^+$. On the other hand, if $v \in E^+$, then:

$$(I-iJ)(\frac{1}{2}v) = \frac{1}{2}v - iJ(\frac{1}{2}v) = \frac{1}{2}v - \frac{1}{2}iJv = \frac{1}{2}v - \frac{1}{2}i(iv) = v,$$

so Ker $(I+iJ) \subseteq$ Range $(I-iJ)$. Hence Ker $(I+iJ) =$ Range $(I-iJ)$. The other equality is similar.

We now have $V^c =$ Range $(I-iJ)$ + Range $(I+iJ) = E^+ \bigoplus E^-$.

We define the complex conjugation as the bijection

$$\begin{array}{cccc} \varphi: & V^c & \longrightarrow & V^c \\ & v & \longmapsto & \overline{v} \\ & X+iY & \longmapsto & X-iY. \end{array}$$

This bijection is a group homomorphism:

$$\varphi(v+w) = \overline{v+w} = \overline{v} + \overline{w} = \varphi(v) + \varphi(w),$$

$$\varphi(\lambda v) = \overline{\lambda v} = \overline{(a+ib)(x+iy)} = \overline{(ax-by)+i(ay+bx)} = (ax-by) - i(ay+bx)$$
$$= (a-ib)(x-iy) = \overline{\lambda}\overline{v} = \overline{\lambda}\varphi(v).$$

The complex conjugation φ interchange E^+ and E^-, i.e. $E^- = \varphi(E^+)$ and $E^+ = \varphi(E^-)$: Since V and V^c have the same basis and J inherits its definition on V^c from its definition on V, so $\overline{Jv} = \overline{J(x+iy)} = Jx - iJy = J\overline{v}$. Let $v \in E^+$ be an eigenvector, i.e. $Jv = iv$. So, if $v \in E^+$ then $J\overline{v} = \overline{Jv} = \overline{iv} = -i\overline{v}$ so $\overline{v} \in E^-$.

Similarly, if $v \in E^-$ is an eigenvector, i.e. $Jv = -iv$ then $J\overline{v} = \overline{Jv} = \overline{-iv} = i\overline{v}$ so $\overline{v} \in E^+$.

Let $w_j = u_j + iv_j$, $j = 1, \cdots, m$ be a basis of E^+. Since $E^- = \varphi(E^+) = \overline{E^+}$, so $\overline{w_j}, j = 1, \cdots, m$ generate E^-. They are linearly independent too:

Let $\sum_{j=1}^m a_j \overline{w_j} = 0$. So $0 = \sum_{j=1}^m a_j \overline{w_j} = \sum_{j=1}^m \varphi(\overline{a_j} w_j) = \varphi(\sum_{j=1}^m \overline{a_j} w_j)$.
>From the definition of φ, if $v = x+iy$ and $\varphi(v) = 0$, then, since $\varphi(x+iy) = x-iy$, we have $x = y = 0$ and so $v = 0$.
So $\sum_{j=1}^m \overline{a_j} w_j = 0$, and because $w_j, j = 1, \cdots, m$ are linearly independent $\overline{a_j} = 0, j = 1, \cdots, m$, and so $a_j = 0, j = 1, \cdots, m$.

Hence $\overline{w_j}, j = 1, \cdots, m$ form a basis of E^- and dimE^-= dimE^+. We have already shown that $V^c = E^+ \bigoplus E^-$ so dimV^c=dimE^-+dimE^+=2n. Hence dim$E^\pm$= n.

The vectors $u_1, \cdots, u_n, v_1, \cdots, v_n$, where $w_j = u_j + iv_j$ $j = 1, \cdots, n$, are linearly independent:

$\sum_{i=1}^n a_i u_i + \sum_{i=1}^n b_i v_i = 0$, $u_i = \dfrac{w_i + \overline{w_i}}{2}$, $v_i = \dfrac{w_i - \overline{w_i}}{2i}$
$\Rightarrow \sum_{i=1}^n a_i u_i + \sum_{i=1}^n b_i v_i = \sum_{i=1}^n (\frac{a_i}{2} + \frac{b_i}{2i}) w_i + \sum_{i=1}^n (\frac{a_i}{2} - \frac{b_i}{2i}) \overline{w_i} = 0$
$\Rightarrow a_i = 0, b_i = 0, i = 1, \cdots, n.$

Also the vectors $u_1, \cdots, u_n, v_1, \cdots, v_n$ generate V^c:

If $v \in E^+$ so $v = \sum_{i=1}^n a_i w_i = \sum_{i=1}^n a_i(u_i + iv_i) = \sum_{i=1}^n a_i u_i + \sum_{i=1}^n (ia_i)v_i$.

If $v \in E^-$ so $v = \sum_{i=1}^n a_i \overline{w_i} = \sum_{i=1}^n a_i(u_i - iv_i) = \sum_{i=1}^n a_i u_i + \sum_{i=1}^n (-ia_i)v_i$.

If $v \in V^c$, since $V^c = E^+ \bigoplus E^-$, so v can be generated by the vectors $u_1, \cdots, u_n, v_1, \cdots, v_n$ too. Hence $u_1, \cdots, u_n, v_1, \cdots, v_n$ form a basis of V^c. By the result of the definition 1.2.1, the basis of V^c over C is the basis of V over R, so $u_1, \cdots, u_n, v_1, \cdots, v_n$ is the basis of V too.

Since $w_j = u_j + iv_j$, $j = 1, \cdots, n$ is a basis of E^+ so $w_j = u_j + iv_j \in E^+$ is an eigenvector and $Jw_j = iw_j$ and $Jw_j = J(u_j + iv_j) = -v_j + iu_j$. This shows that:

$$Ju_j = -v_j, Jv_j = u_j.$$

Let the linear transformation $A : R^{2n} \longrightarrow V$ be given by:

$$A\zeta = \sum_{j=1}^{n} (\xi_j u_j - \eta_j v_j)$$

for $\zeta = (\xi, \eta)$. It is easy to check that:

$$JA\zeta = J(\sum_{j=1}^{n} (\xi_j u_j - \eta_j v_j)) = \sum_{j=1}^{n} (\xi_j Ju_j - \eta_j Jv_j) = \sum_{j=1}^{n} (-\xi_j v_j - \eta_j u_j),$$

$$AJ_0\zeta = A(-\eta, \xi) = \sum_{j=1}^{n} (-\xi_j v_j - \eta_j u_j).$$

Hence $JA = AJ_0$ and so $J = AJ_0A^{-1}$.

By defining action $(A, J) \mapsto A^{-1}JA$, $GL(2n, R)$ acts on $\Delta(R^{2n})$. So $\Delta(R^{2n})$ is a G-set for $G = GL(2n, R)$ and the surjectivity of

$$\begin{array}{ccc} GL(2n, R) & \longrightarrow & \Delta(R^{2n}) \\ A & \longmapsto & A^{-1}J_0A \end{array}$$

shows that $GL(2n, R)$ acts on the set $\Delta(R^{2n})$ transitively. So $\Delta(R^{2n})$, the set of all complex structures on R^{2n}, is isomorphic to the set $GL(2n, R)/H$ where H is the isotropy group $GL(2n, R)_J$. The choice of J does not affect the isomorphism because there is only one group orbit and so all of isotropy groups are conjugate. The isotropy group of J_0,$H_{J_0} = \{A \in GL(2n, R) : J_0 = A^{-1}J_0A\}$, consists of all matrices that commute with J_0. If we identify R^{2n} with C^n with $z = (x, y)$ corresponding to $x + iy$ for $x, y \in R^n$, we can have another form of corollary 1.2.5

in finite dimensional as: $A \in Aut_R(R^{2n}) = GL(2n, R)$ is in $Aut_C(C^n) = GL(n, C)$ if and only if $AJ = JA$ or $A = JAJ^{-1}$ or $J = A^{-1}JA$. So, H_{J_0}, the isotropy group of J_0 is exactly $GL(n, C)$.

□

Let J be an arbitrary complex structure on a real vector space V. With such a structure, V becomes a complex vector space:

$$(a + ib)v = av + bJv, (a + ib) \in C, v \in V. \qquad (1.2.1)$$

Let B_1 be a basis of V over C then B_1 is linearly independent i.e. if $\sum_{finite}^{b_i \in B_1} \alpha_i b_i = 0$ the $\alpha_i = 0, \forall i$. Let $\sum_{finite}^{b_i \in B_1} \alpha_i J b_i = 0$. By 1.2.1 $\sum_{finite}^{b_i \in B_1} (i\alpha_i) b_i = 0$. And because B_1 is linearly independent, so $i\alpha_i = 0$ and $\alpha_i = 0, \forall i$. So $J(B_1)$ is linearly independent. The set $\{B_1, J(B_1)\}$ is linearly independent too, because if $\sum_{finite}^{b_i \in B_1} (\alpha_i b_i + \beta_i J b_i) = 0$, by 1.2.1 $\sum_{finite}^{b_i \in B_1} (\alpha_i + i\beta_i) b_i = 0$. And because B_1 is linearly independent, so $\alpha_i + i\beta_i = 0$ and so $\alpha_i = 0, \beta_i = 0 \forall i$.

Since B_1 is a basis of V over C then $Span(B_1) = V$ over C so
$V = \{\sum_{finite} (x_i + iy_i) b_i = \sum_{finite} x_i b_i + y_i J b_i : x_i + iy_i \in C$ and $b_i \in B_1\}$
in other words:
$V = \{\sum_{finite} x_i b_i + y_i J b_i : x_i, y_i \in R, b_i \in B_1$ and $Jb_i \in J(B_1)\}$,
so Span$(\{B_1, J(B_1)\}) = V$ over R. Therefore if B_1 is a basis of V over C then $\{B_1, J(B_1) = B_2\}$ forms a basis of V over R.

Let V_1 and V_2 be real vector spaces with respective complex structures J_1 and J_2. Then (V_1, J_1) and (V_2, J_2) are complex vector spaces. The morphisms between real vector spaces are R-linear maps and the morphisms between complex vector spaces are C-linear maps. And also every C-linear map is R-linear.

Theorem 1.2.9. *The set of all complex structures on an infinite dimensional real vector spaces $\Delta(V)$ can be identified with the $Aut_R(V)$-set $Aut_R(V)/Aut_C(V, J)$.*

PROOF. Let J be an arbitrary complex structure on an infinite dimensional real vector space V. So (V, J) is a complex vector space.

Let $B = \{b_\alpha\}, \alpha \in \mathfrak{a} = |B|$, be a basis of the vector space V over C. Then $\{B, JB = C\}$ is a basis of V over R.

We know by the end of theorem 1.1.4 that if there was a decomposition of the basis of a vector space into two set with the same cardinality, so we could consider a bijection $f : B \to C$ by $b_\alpha \mapsto c_\alpha$ where $C = \{c_\alpha\}, \alpha \in \mathfrak{a} = |C| = |B|$, and we could define a complex structure J_0 on the vector space V:

$$J_0(b_\alpha) = f(b_\alpha) = c_\alpha, J_0(c_\alpha) = -f^{-1}(c_\alpha) = -b_\alpha, \alpha \in \mathfrak{a}.$$

If we take a bijection between B and C as f then $J(b_\alpha) = f(b_\alpha) = c_\alpha = J_0(b_\alpha)$ and $J(c_\alpha) = J(J(b_\alpha)) = -b_\alpha = J_0(c_\alpha)$ so $J = J_0$. If we take another bijection between B and C as $f \neq g : B \to C$, there is a map $\phi : C \to C$ such that $f = \phi g$. We define $\phi = JJ_0^{-1} \in Aut_R(V)$. So $J = \phi J_0$.

So $Aut_R(V)$ acts on $\Delta(V)$ transitively. By 1.2.1, we see that J corresponds to multiplication by i, so the isotropy group $Aut_R(V)_J$, that is all R-linear map that commute with J, is all Complex linear maps $Aut_C(V, J)$. Hence the set of all complex structures on an infinite dimensional real vector spaces $\Delta(V)$ can be identified with the $Aut_R(V)$-set $Aut_R(V)/Aut_C(V, J)$. □

1.3. THE EQUIVALENCE CLASS OF COMPLEX STRUCTURES ON VECTOR SPACES

Definition 1.3.1. *Two complex structures J_1 and J_2 on a real vector spaces V are equivalent if there exist a C-isomorphism $L : (V, J_1) \to (V, J_2)$.*

Corollary 1.3.2. *$J_1 \sim J_2 \iff \exists L \in Aut_R(V)$ such that $J_2 = LJ_1L^{-1}$.*

The following theorem results easily from combining the previous theorems.

Theorem 1.3.3. *All complex structures on a real vector space are equivalent.*

Chapter 2

COMPLEX MANIFOLDS

2.1. Differentiable manifolds

Let M be a Hausdorff topological space such that any point of M admits a neighborhood homeomorphic to an open set in R^n: roughly, such a space is obtained by gluing together open subsets of R^n. We decide to carry the differentiable structure from the open subset of R^n onto these neighborhoods in M.

Definition 2.1.1. *A C^p atlas on a Hausdorff topological space M is given by an open cover U_i, $i \in I$ of M and a family of homeomorphism $\phi_i \colon U_i \to \Omega_i$ where the Ω_i are open subsets of R^n such that for any i and j in I, the homeomorphism $\phi_j \circ \phi_i^{-1}$ (transition functions) is in fact a C^p diffeomorphism from $\phi_i(U_i \cap U_j)$ onto $\phi_j(U_i \cap U_j)$.*

Two C^p atlases for M, (U_i, ϕ_i) and (V_j, ψ_j), are C^p equivalent if their union is still a C^p atlas, that is if the $\phi_i \circ \psi_j^{-1}$ are C^p diffeomorphisms from $\psi_j(U_i \cap V_j)$ onto $\phi_i(U_i \cap V_j)$.

A differentiable structure of class C^p on M is an equivalence class of consistent C^p atlases. A differentiable manifold will be a connected Hausdorff topological space, together with a differentiable structure.

2.2. Complex manifolds

A real (differentiable) manifold is a topological space which is locally real Euclidean. We shall now introduce complex manifolds, which are locally complex Euclidean. Indeed, to define a complex manifold of complex dimension n, we copy the definition of a real manifold of real dimension n. The only difference is that, instead of requiring the Ω_i to be open sets in real Euclidean space R^n, we require that they be open sets in complex Euclidean space C^n. We may speak of complex coordinates, charts, atlases etc. Thus, a complex manifold of complex dimension n can be considered as a real manifold of real dimension $2n$. Thus, it would seem that the study of complex manifolds is merely the study of real manifolds in even real dimensions. However, when considering complex manifolds, we usually require a very high level of smoothness. A complex atlas A is said to be a holomorphic atlas if the changes of coordinates $\phi_j \circ \phi_i^{-1}$ are biholomorphic. A holomorphic structure on M is an equivalence class of holomorphic atlases on M. Often, we shall give a holomorphic atlas U for a manifold and think of it as the equivalence class of all structures which are biholomorphically compatible with it. Of course we shall associate the same holomorphic structure to two holomorphic atlases U and V if and only if the two atlases are compatible. Since the union of compatible holomorphic atlases is a holomorphic atlas, for any holomorphic atlas A, there is a maximal holomorphic atlas compatible with an atlas A. This is merely the union of all holomorphic atlases compatible with A. Thus we may think of holomorphic structure on M as a maximal holomorphic atlas. It seems we now defined a holomorphic structure on M in three ways: as an equivalence class of holomorphic atlases, as a holomorphic atlas which is maximal with respect to equivalence or simply as a holomorphic atlas U, meaning the equivalence class of U or the maximal holomorphic atlas equivalent with U. All that matters at this point is to be able to tell whether two holomorphic structures on M are the same or not. No matter which definition we use, we shall always come up with the same answer. That is, two structures will be considered different with respect to one of the definitions if and only if they are considered different with respect to the other definitions.

A complex holomorphic manifold is a connected, Hausdorff topological space M as above, together with a holomorphic structure. Since complex manifolds of dimension n of smoothness less than holomorphic are merely real manifolds of dimension $2n$, we shall consider only holomorphic complex manifolds. Thus, when we speak of a complex structure, we shall mean a holomorphic structure and when we speak of a complex manifold, we shall always mean a manifold endowed with a complex holomorphic structure. A Riemann surface is a complex manifold of dimension one. Thus, complex manifolds are higher dimensional analogs of Riemann surfaces.

Definition 2.2.1. *Let M and N be complex manifolds with atlases A and B respectively. A map $f : M \to N$ ia said to be holomorphic if*

$$\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(V)$$

is holomorphic for all $(U, \phi) \in A$ and $(V, \psi) \in B$. If f is a homeomorphism and both f and f^{-1} are holomorphic, we say that f is biholomorphic and that M and N are biholomorphic or biholomorphically equivalent.

2.3. Tangent space

2.3.1. Tangent space to a real manifold

Loosely, the tangent space $T(X)$ of a real manifold X of dimension n is the set of formal expressions

$$T(X) = \{a_1 \frac{\partial}{\partial x_1} + \ldots + a_n \frac{\partial}{\partial x_n} : a_j \in C^1(X)\},$$

which is the space of smooth vector fields on X. We shall define the tangent space $T_p(X)$ of X at a point $p \in X$ and we shall set

$$T(X) = \bigcup_{p \in X} T_p(X).$$

More precisely, let X be a smooth (real) manifold. If U is an open subset of X, we denote by $\varepsilon(U)$ the set of smooth functions on U. If $p \in X$, let us say that f is a smooth function at p if $f \in \varepsilon(U)$ for some open neighborhood U of p. Two smooth functions f and g at p are said to be equivalent if $f = g$ in some neighborhood of p. This is an equivalence relation and the equivalence classes are called germs of smooth functions at p. For simplicity, we shall denote the germ of a smooth function f at p also by f. Denote by ε_p the set of germs of smooth functions at p. The set ε_p is an R-algebra.

A derivation of the algebra ε_p is a vector space homomorphism

$$D : \varepsilon_p \longrightarrow R$$

such that

$$D(fg) = D(f) \cdot g(p) + f(p) \cdot D(g),$$

where $g(p)$ and $f(p)$ are the evaluations at p of the germs g and f at p.

The tangent space of X at p, denoted by $T_p(X)$, is the vector space of derivations of the algebra ε_p.

Since X is a smooth manifold, there is a diffeomorphism h of an open neighborhood U of p onto an open set $U\prime \subset R^n$:

$$h : U \longrightarrow U\prime,$$

and if we set $h^*f(x) = f \circ h(x)$, then h has the property that, for open $V \subset U\prime$,

$$h^* : \varepsilon(V) \longrightarrow \varepsilon(h^{-1}(V))$$

is an algebra isomorphism. Thus h^* induces an algebra isomorphism on germs:

$$h^* : \varepsilon_{h(p)} \longrightarrow \varepsilon_p,$$

and hence induces an isomorphism on derivations:

$$h_* : T_p(X) \longrightarrow T_{h(p)}(R^n).$$

Indeed, if $D \in T_p(X)$ we define $h_*(D) \in T_{h(p)}(R^n)$ as follows: if $f \in \varepsilon_{h(p)}$, we set $h_*(D)f = D(h^*f)$.

Fix $a \in R^n$. Then $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ are derivation of $\varepsilon_a(R^n)$ and form a basis of $T_a(R^n)$.

2.3.2. Tangent space to a complex manifold

Having discussed the tangent space to a smooth manifold, we now introduce the (complex) tangent space to a complex manifold. Let p be a point of a complex manifold M and let O_p be the C-algebra of germs of holomorphic functions at p. The complex or holomorphic tangent space $T_p(M)$ to M at p is the complex vector space of all derivations of the C-algebra O_p, hence the complex vector space homomorphisms $D : O_p \to C$ such that

$$D(fg) = f(p) \cdot D(g) + D(f) \cdot g(p).$$

In local coordinates, we note that $T_p(M) = T_z(C^n)$ and that partial derivatives $\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\}$ form a basis of $T_z(C^n)$. Having defined the complex tangent space $T_p(M)$ to a complex manifold at a point $p \in M$, we define the complex tangent space $T(M)$ of M:

$$T(M) = \bigcup_{p \in M} T_p(M).$$

2.4. Almost complex structure

2.4.1. Almost complex structure

Let M be a differentiable manifold of dimension $2n$. Suppose that J associates to each x, a complex structure $J_x : T_x(M) \to T_x(M)$ for $T_x(M)$, i.e. $J_x^2 = -I_x$, where I_x is the identity isomorphism acting on $T_x(M)$.

We also suppose that J_x varies differentially with x. This means that if x is a local coordinate in R^n and A_x is matrix representing J_x with respect to the basis $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$, then the coefficients of A_x vary smoothly with x.

Then J is called an almost complex structure for the differentiable manifold M. If M is equipped with an almost complex structure J, then (X, J) is called an almost complex manifold.

An almost complex structure J on M defines a complex structure in each tangent space $T_x(M)$. As we have shown, dimension of $T_x(M)$ as a vector space is even and dimension of M is equal to dimension of $T_x(M)$. So every almost complex manifold is of even real dimension.

To show that every complex manifold carries a natural almost complex structure, we consider the space C^n of n-tuples of complex numbers $(z_1, \ldots, z_n)$ with $z_j = x_j + iy_j, j = 1, \ldots, n$. With respect to the coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ we define an almost complex structure J on C^n by

$$J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}, J(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j}, j = 1, \ldots, n.$$

2.4.2. Differential forms

Let M be a one dimensional complex manifold. A $0-form$ on M is a function on M. A $1-form\,\omega$ on M is an ordered assignment of two continuous functions f and g to each local coordinate $z = (x+iy)$ on an open set U in M such that

$$fdx + gdy$$

is invariant under coordinate changes; that is, if z' is another local coordinate on an open set V in M and the domain of z' intersects non trivially the domain of z, and if ω assigns the functions f' and g' to z', then using matrix notation

$$\begin{pmatrix} f'(z') \\ g'(z') \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} \end{pmatrix} \begin{pmatrix} f(z(z')) \\ g(z(z')) \end{pmatrix} \tag{2.4.1}$$

on $z(U \cap V)$. The 2×2 matrix appearing in 2.4.1 is, of course, the Jacobian matrix of the mapping $z' \longrightarrow z$.

A $2-form\,\Omega$ on M is an assignment of a continuous function f to each local coordinate z such that

$$fdx \wedge dy$$

is invariant under coordinate changes; that is, in terms of the local coordinate z' we have

$$f'(z') = f(z(z'))\frac{\partial(x,y)}{\partial(x',y')} \tag{2.4.2}$$

where $\frac{\partial(x,y)}{\partial(x',y')}$ is the determinant of the Jacobian. Since we consider only holomorphic coordinate change 2.4.2 has the simple form

$$f'(z') = f(z(z')) \mid \frac{dz}{dz'} \mid^2 .$$

Many times it is more convenient to use complex notation for differential forms. Using the complex analytic coordinate z, a $1-form$ may be written as

$$u(z)dz + v(z)d\overline{z},$$

where

$$dz = dx + idy,\ d\overline{z} = dx - idy \tag{2.4.3}$$

and $f = u + v$ and $g = i(u - v)$. It follows from 2.4.3 that

$$dz \wedge d\overline{z} = -2i dx \wedge dy.$$

Similarly, a $2 - form$ can be written as

$$h(z) dz \wedge d\overline{z}.$$

Remark 2.4.1. *We have made use of the exterior multiplication of forms. This multiplication satisfies the conditions:* $dx \wedge dx = 0 = dy \wedge dy$, $dx \wedge dy = -dy \wedge dx$. *The product of a* $k - form$ *and an* $l - form$ *is a* $k + l - form$ *provided* $k + l \leq 2$ *and is the zero form for* $k + l > 2$.

For C^1 forms, that is, forms whose coefficients are C^1 functions, we introduce the differential operator d. Define

$$df = f_x dx + f_y dy$$

for C^1 functions f. For the C^1 $1 - form\, \omega$, we have by definition

$$\begin{aligned} d\omega &= d(f dx) + d(g dy) = df \wedge dx + dg \wedge dy \\ &= (f_x dx + f_y dy) \wedge dx + (g_x dx + g_y dy) \wedge dy \\ &= (g_x - f_y) dx \wedge dy. \end{aligned}$$

For a $2 - form\, \Omega$ we, of course, have by definition

$$d\Omega = 0.$$

The most important fact concerning this operator is that

$$d^2 = 0,$$

whenever d^2 is defined.

Using complex analytic coordinates we introduce two differential operators ∂ and $\overline{\partial}$ by setting for a C^1 function f,

$$\partial f = f_z dz, \overline{\partial} f = f_{\overline{z}} d\overline{z},$$

and setting for a C^1 $1 - form$ $\omega = u dz + v d\overline{z}$,

$$\partial \omega = \partial u \wedge dz + \partial v \wedge d\overline{z} = v_z dz \wedge d\overline{z},$$

$$\overline{\partial}\omega = \overline{\partial}u \wedge dz + \overline{\partial}v \wedge d\overline{z} = u_{\overline{z}} d\overline{z} \wedge dz = -u_{\overline{z}} dz \wedge d\overline{z},$$

where

$$f_z = \frac{1}{2}(f_x - i f_y), f_{\overline{z}} = \frac{1}{2}(f_x + i f_y).$$

For $2 - forms$, the operators ∂ and $\overline{\partial}$ are defined as the zero operators.

Remark 2.4.2. *The equation $f_{\overline{z}} = 0$ is equivalent to the Cauchy-Riemann equations for Ref and Imf, that is, $f_{\overline{z}} = 0$ if and only if f is holomorphic.*

On a complex manifold, we have defined operator ∂ and $\overline{\partial}$ which act on forms. These operators satisfy:

$$d = \partial + \overline{\partial},$$

$$\partial^2 = \partial\overline{\partial} + \overline{\partial}\partial = \overline{\partial}^2 = 0.$$

In the next section, we characterize integrable almost complex structures in this way. Namely, for a given almost complex structure J, we define operators ∂_J and $\overline{\partial_J}$. We shall present the Newlander-Nirenbeg theorem which asserts that J is integrable if and only if these operators satisfy the preceding relations.

2.4.3. Complex and almost complex structure

On an arbitrary almost complex manifold, one can always find coordinates for which the almost complex structure takes the above canonical form at any given point p. In general, however, it is not possible to find coordinates, so that J takes the canonical form on an entire neighborhood of p. Such coordinates, if they exist, are called local holomorphic coordinates for J. If around every point M admits local holomorphic coordinates which induce J, then J is said to be integrable. The local holomorphic coordinates patch together to form a holomorphic atlas for M giving it the structure of a complex manifold. A complex structure can then be defined as an integrable almost complex structure.

The existence of an almost complex structure is a topological question and is relatively easy to answer. The existence of an integrable almost complex structure, on the other hand, is a much more difficult analytic question. For example, it has long been known that S^6 admits an almost complex structure, but it is still an open question as to whether or not it admits an integrable complex structure.

Given an almost complex structure there are several ways for determining whether or not that structure is integrable. Let J be an almost complex structure on a manifold M, one can associate to J certain operators ∂_J and $\overline{\partial_J}$ on forms. In case, J is integrable, then these operators are just the usual operators ∂ and $\overline{\partial}$ arising from the complex structure associated to J. We may now state the Newlander-Nirenberg theorem which characterizes integrability of almost complex structures. The Newlander-Nirenberg theorem states that the following are equivalent:

- J is integrable. (i.e. M is a complex manifold.)
- The Nijenhuis tensor, defined by

$$N_J = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY]$$

vanishes for all smooth vector fields X and Y. (i.e. An almost complex structure is said to be integrable if it has no torsion.)

The Lie bracket of vector fields X and Y, $[X,Y]$, is also a vector field, defined by the equation $[X,Y](f) = X(Y(f)) - Y(X(f))$.

- We can decompose the exterior derivative as $d = \partial_J + \overline{\partial_J}$.
- $\overline{\partial_J}^2 = 0$.

2.5. THE EQUIVALENCE CLASS OF COMPLEX STRUCTURES ON MANIFOLDS

Proposition 2.5.1. *A mapping f of an open subset of C^n into C^m preserves the almost complex structures of C^n and C^m, i.e. $f_* \circ J = J \circ f_*$, if and only if f is holomorphic.*

PROOF. Let $(w_1, \ldots, w_n)$ with $w_k = u_k + iv_k, k = 1, \ldots, m$, be the natural coordinate system in C^m. If we express f in terms of these coordinate systems in C^n and C^m:

$$u_k = u_k(x_1, \ldots, x_n, y_1, \ldots, y_n), v_k = v_k(x_1, \ldots, x_n, y_1, \ldots, y_n),$$

where $k = 1, \ldots, m$, then f if holomorphic when and only when the following Cauchy-Riemann equations holds:

$$\frac{\partial u_k}{\partial x_j} - \frac{\partial v_k}{\partial y_j} = 0, \frac{\partial u_k}{\partial y_j} + \frac{\partial v_k}{\partial x_j} = 0,$$

where $k = 1, \ldots, m$ and $j = 1, \ldots, n$.

On the other hand, we have always (whether f is holomorphic or not):

$$f_*(\frac{\partial}{\partial x_j}) = \sum_{k=1}^{m}(\frac{\partial u_k}{\partial x_j})(\frac{\partial}{\partial u_k}) + \sum_{k=1}^{m}(\frac{\partial v_k}{\partial x_j})(\frac{\partial}{\partial v_k}),$$

$$f_*(\frac{\partial}{\partial y_j}) = \sum_{k=1}^{m}(\frac{\partial u_k}{\partial y_j})(\frac{\partial}{\partial u_k}) + \sum_{k=1}^{m}(\frac{\partial v_k}{\partial y_j})(\frac{\partial}{\partial v_k}),$$

for $j = 1, \ldots, n$. From these formulas and the definition of J in C^n and C^m given above, we see that $f_* \circ J = J \circ f_*$ if and only if f satisfies the Cauchy-Riemann equations.

□

To define an almost complex structure on a complex manifold M, we transfer the almost complex structure of C^n to M by means of charts. Proposition 2.5.1 implies that an almost complex structure can be thus defined on M independently of the choice of charts.

Definition 2.5.2. *An almost complex structure J on a manifold M is called a complex structure if M is an underlying differentiable manifold of a complex manifold which induces J in the way just described.*

Let M and M' be almost complex manifolds with almost complex structures J and $J\prime$, respectively. A mapping $f : M \to M'$ is said to be almost complex or complex linear with respect to the given complex structures on the tangent spaces, if $J' \circ f_* = f_* \circ J$. In particular, two almost complex structure on the same complex manifold coincide if the identity mapping is almost complex. From proposition 2.5.1 we obtain:

Proposition 2.5.3. *Let M and $M\prime$ be complex manifolds. A mapping $f : M \to M\prime$ is holomorphic if and only if f is almost complex with respect to the complex structures of M and $M\prime$.*

In particular, two complex manifolds with the same underlying differentiable manifold are identical if the corresponding almost complex structures coincide.

2.5.1. Pseudoholomorphic curves

We can specify a plane curve in two different ways: either as the set of solution of an equation $f(x,y)=0$ or via a parametrisation $x=x(t)$, $y=y(t)$. For example, we can specify a circle by the equation $x^2+y^2=1$ or by the parametrisation $x=\cos t$, $y=\sin t$.

We replace the real variables x, y above by complex variables z, w and consider complex or holomorphic curves in C^2. Thus the same equation $z^2+w^2=1$, for example, describes such a holomorphic curve. We can consider parametrised holomorphic curves $z=z(\tau)$, $w=w(\tau)$ where $z(\tau)$ and $w(\tau)$ are holomorphic functions of a complex variable τ. More generally we may consider holomorphic curves in complex manifolds, parametrised by holomorphic maps from Riemann surfaces. A Riemann surface is a complex manifold of dimension one. The details will come in chapter 3.

A pseudoholomorphic curve is just the natural modification of the notion of holomorphic curve to the case when the ambient manifold is almost complex. That is, we consider a Riemann surface Σ with complex structure j, an almost complex manifold (M,J), and a differentiable map $f\colon\Sigma\to M$ such that for each $\sigma\in\Sigma$ the derivative

$$df_\sigma\colon T_\sigma\Sigma\longrightarrow T_{f(\sigma)}M$$

is complex linear or almost complex with respect to the given complex structures on the tangent spaces.

The classical case occurs when M and Σ are both simply the complex number plane. In real coordinates

$$j=J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

and

$$df=\begin{pmatrix}\frac{du}{dx} & \frac{du}{dy}\\ \frac{dv}{dx} & \frac{dv}{dy}\end{pmatrix}$$

where $f(x,y) = (u(x,y), v(x,y))$. After multiplying these matrices in two different orders, one sees immediately that the equation

$$J \circ df = df \circ j$$

is equivalent to the classical Cauchy-Riemann equations

$$\frac{du}{dx} = \frac{dv}{dy}, \quad \frac{dv}{dx} = -\frac{du}{dy}.$$

2.6. CR STRUCTURE

2.6.1. The observation of Poincaré

Any two real analytic curves in C^1 are locally equivalent: Given points p and q on the curves Γ_1 and Γ_2 there are open subsets of C^1, U_1 containing p and U_2 containing q, and a biholomorphism $\Phi\colon U_1 \to U_2$ with $\Phi(U_1 \cap \Gamma_1) = U_2 \cap \Gamma_2$. This may be taken as a very weak form of the Riemann mapping theorem. Poincaré showed that the analogous result does not hold in C^2. Namely, let s and S be real analytic surfaces of real dimension three in C^2. In general, there will not be a local biholomorphism taking one to the other. The Poincaré proof uses the fact that a function on a hypersurface is the restriction of a holomorphic function only if it satisfies a certain partial differential equation.

Let the hypersurface be written as a graph

$$s = \{(x_1 + iy_1, x_2 + iy_2) \,:\, x_1 = \phi(y_1, x_2, y_2)\}$$

and let $F(y_1, x_2, y_2)$ be the function on s. If there exists a holomorphic function $f(z_1, z_2)$ of $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ such that

$$F(y_1, x_2, y_2) = f(\phi(y_1, x_2, y_2) + iy_1, x_2 + iy_2)$$

so

$$\frac{\partial F}{\partial y_1} = \frac{\partial f}{\partial z_1}(\phi_{y_1} + i), \quad \frac{\partial F}{\partial \overline{z_2}} = \frac{\partial f}{\partial z_1}\phi_{\overline{z_2}}.$$

Let

$$L = \phi_{\overline{z_2}}\frac{\partial}{\partial y_1} - (\phi_{y_1} + i)\frac{\partial}{\partial \overline{z_2}}.$$

Thus $LF = 0$.

We consider two surfaces

$$s = \{x_1 = \phi(y_1, x_2, y_2)\}, \quad S = \{X_1 = \Phi(Y_1, X_2, Y_2)\}.$$

For s and S to be locally equivalent we need to find three real functions Y_1, X_2, Y_2 of the variables (y_1, x_2, y_2) such that both $f_1 = \Phi(Y_1, X_2, Y_2) + iY_1$ and $f_2 = X_2 + iY_2$ are the restriction of holomorphic functions. That is, we must solve for $j = 1, 2$

$$\left(\phi_{\overline{z_2}}(y_1, x_2, y_2)\frac{\partial}{\partial y_1} - (i + \phi_{y_1}(y_1, x_2, y_2))\frac{\partial}{\partial \overline{z_2}}\right) f_j = 0.$$

We have four real equation for three real unknowns. Thus solutions usually do not exist.

Since not all hypersurfaces are locally equivalent, it is natural to seek invariants which allows us to distinguish one from another.

2.6.2. CR manifolds

Let $(z_1, \ldots, z_n)$ be the usual coordinates for C^n and $(x_1, y_1, \ldots, x_n, y_n)$ the corresponding coordinates for R^{2n}. We define the first-order partial differential operator $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$ and its conjugate operator $\frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$.

A function $F(z) = f(x, y)$ of one complex variable is holomorphic if and only if $\frac{\partial}{\partial \overline{z}}f = 0$. We may consider $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \overline{z_j}}$ as complex vector fields.

If V is a vector space over the real numbers, then $C \otimes V$ is the corresponding vector space over the complex numbers. If $\{v_1, \ldots, v_n\}$ is a basis for V then

$$C \otimes V = \{\sum_{j=i}^{N} \alpha_j v_j \ : \ \alpha_j \in C\}$$

$C \otimes V$ is called the complexification of V.

Consider a manifold M and the tangent space T_pM to M at a point p. The tangent bundle is given by $T\,M = \bigcup_p T_pM$ and the complexified tangent space by $C \otimes T\,M = \bigcup_p C \otimes T_pM$.

When M is R^{2n}, we have as a basis for $T\,M$ and $C \otimes T\,M$

$$\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\}$$

which yields a second basis for $C \otimes T\,M$

$$\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z_1}}, \ldots, \frac{\partial}{\partial \overline{z_n}}\}.$$

We have seen that there is a differential operator L defined on $M^3 \subset C^2$ with the property that $Lf = 0$ if f is the restriction to M of a holomorphic function. So L may be considered as the induced Cauchy-Riemann operator. We shall soon see that for $M^{2n+1} \subset C^{n+1}$ there are such induced Cauchy-Riemann operators. This is what is abstracted as the definition of the CR manifolds. To explain this, let us see another definition of the almost complex structure of C^{n+1}.

Definition 2.6.1. *Let M be a manifold and let V be a subspace of $C \otimes T\,M$. Then (M, V) is an almost complex manifold if*

$$V \cap \overline{V} = \{0\}, \quad V \oplus \overline{V} = C \otimes T\,M.$$

Set $dim_C\, V = n$; it follows that $dim_R\, M = 2n$.

If M is a complex manifold then the underlying almost complex manifold is given locally by choosing complex coordinates and setting

$$V = Linear\; span_C\{\frac{\partial}{\partial \overline{z_1}}, \ldots, \frac{\partial}{\partial \overline{z_n}}\}.$$

This subspace is independent of the choice of coordinates since it is preserved by a holomorphic transformation. We use $L \in V$ as an abbreviation for "L is a section of V over an appropriate set".

Note that a function f is holomorphic if $Lf = 0$ for all $L \in V$. So V is the space of Cauchy-Riemann operators.

We had another definition of an almost complex manifold, using an isomorphism $J\colon T\,M \to T\,M$ where $J^2 = -Id$. If M is a complex manifold, then the underlying almost complex manifold is given by

$$J(\frac{\partial}{\partial \overline{z_k}}) = -i\frac{\partial}{\partial \overline{z_k}}, \quad J(\frac{\partial}{\partial z_k}) = i\frac{\partial}{\partial z_k}.$$

Again, J is independent of the choice of coordinates. For if $z_k = \phi(w_1, \ldots, w_n)$ is a change of coordinates, and if J is defined using z, then

$$J(\frac{\partial}{\partial w_k}) = \sum J(\frac{\partial \phi_j}{\partial w_k}\frac{\partial}{\partial z_j}) = \sum \frac{\partial \phi_j}{\partial w_k}(i\frac{\partial}{\partial z_j}) = i\frac{\partial}{\partial w_k}$$

and in the same way

$$J(\frac{\partial}{\partial \overline{w_k}}) = -i\frac{\partial}{\partial \overline{w_k}}.$$

So J coincides with the operator defined using w.

Note that J is real in the sense that it provides a map of $T\,M$ to itself, Since, if X is a real tangent vector then:

$$X = \sum(\alpha_j\frac{\partial}{\partial z_j} + \overline{\alpha_j}\frac{\partial}{\partial \overline{z_j}})$$

and

$$JX = \sum(i\alpha_j\frac{\partial}{\partial z_j} - \overline{\alpha_j}\frac{\partial}{\partial \overline{z_j}}).$$

So JX is also a real tangent vector.

The two definition of almost complex manifolds are equivalent.

For given (M, V), we define $J\colon C\otimes T\,M \rightarrow C\otimes T\,M$ by letting V and $\overline{V}$ be its eigenspaces corresponding to the eigenvalues $-i$ and i respectively. Such a J restricts to a map of $T\,M$ to itself and satisfies $J^2 = -Id$. Conversely, given $J\colon T\,M \rightarrow T\,M$ with $J^2 = -Id$, we extend J linearly to a map of $C\otimes T\,M$ to itself and let V be the eigenspace corresponding to the eigenvalue $-i$.

Let X be any real tangent vector space. Note that

$$J(X + iJX) = -i(X + iJX).$$

Thus $V = \{X + iJX\ :\ X \in T\,M\}$.

We have seen that not every almost complex manifold comes from some complex manifold. A necessary condition for an almost complex manifold to be complex is that

$$[V, V] \subset V.$$

Consider X, $Y \in V$ where $X = a_k\frac{\partial}{\partial \overline{z_k}}$ and $Y = b_k\frac{\partial}{\partial \overline{z_k}}$, so by definition of Lie bracket

$$[X, Y] = (X(b_k) - Y(a_k))\frac{\partial}{\partial \overline{z_k}} = \left(a_j\frac{\partial b_k}{\partial \overline{z_j}} - b_j\frac{\partial a_k}{\partial \overline{z_j}}\right)\frac{\partial}{\partial \overline{z_k}}.$$

In terms of J this means

$$[X, JY] + [JX, Y] = J\{[X, Y] - [JX, JY]\}.$$

This is equivalent to

$$J([X, JY] + [JX, Y]) = [JX, JY] - [X, Y]$$

that is the Nijenhuis tensor, that we have already seen.

Let M^{2n+1} be a submanifold of C^{n+1}. We show how each of our definitions of an almost complex structure on C^{n+1} leads to an induced structure on M and then that these two included structures are essentially the same.

We set

$$H(C^{n+1}) = Linear\ span_C\{\frac{\partial}{\partial \overline{z_1}}, \dots, \frac{\partial}{\partial \overline{z_{n+1}}}\} \subset C \otimes TC^{n+1}$$

and define $H(M)$ by

$$H(M) = H(C^{n+1}) \cap C \otimes T\,M.$$

We denote $H(M)$ by V. So

$$V \cap \overline{V} = \{0\}.$$

Note that $w \in V$ if $w = \sum \alpha_k \frac{\partial}{\partial \overline{z_k}}$ and in addition w is tangent to M.

Definition 2.6.2. *A complex vector W is tangent to M if the real vectors $Re\,W$ and $Im\,W$ are tangent to M.*

We also could use the first definition of an almost complex manifold to define the structure induced on $M^{2n+1} \subset C^{n+1}$. To do this we let $H = T\,M \cap JT\,M$ and consider the restriction to H of J. We have

$$H \subset T\,M, \quad dim_R H = 2n, \quad J\colon H \to H, \quad J^2 = -Id.$$

Since the almost complex structure on C^{n+1} is complex, the induced structure on M defined by $H(M)$ satisfies

$$[V, V] \subset V.$$

Note that a vector W is in H if both W and JW are tangent to M. In the structure on M defined using $H = T\,M \cap JT\,M$, X and Y in H satisfy

$$[JX, Y] + [X, JY] \in H$$

and

$$J\{[JX, Y] + [X, JY]\} = [JX, JY] - [X, Y].$$

Definition 2.6.3. *(M, V) is a CR manifold if $dim_R M = 2n+1$, V is a subspace of $C \otimes T\,M$ with $dim_C V = n$, $V \cap \overline{V} = \{0\}$ and $[V, V] \subset V$.*

Definition 2.6.4. *(M, H, J) is a CR manifold if $dim_R M = 2n + 1$, H is a subspace of $T\,M$ with $dim_R H = 2n$, $J \colon H \longrightarrow H$ and $J^2 = -Id$.*

If X and Y are in H, then so is $[JX, Y]+[X, JY]$ and $J\{[JX, Y]+[X, JY]\} = [JX, JY] - [X, Y]$.

Remark 2.6.5. *The CR refers to Cauchy-Riemann because for $M \subset C^{n+1}$, V consists of the induced Cauchy-Riemann operators. That is, a function f on M can be the restriction of a holomorphic function on an open subset of C^{n+1} only if $Lf = 0$ for all sections L of V.*

Let us show that the two definitions of CR structures are equivalent. Given V we choose some basis $\{L_1, \ldots, L_n\}$ and note that $V \cap \overline{V} = \{0\}$ implies that $\{Re\, L_1, \ldots, Im\, L_n\}$ are linearly independent. Set H equal to the linear span over R of this set and define J by

$$J(Re\, L_k) = Im\, L_k, \quad J(Im\, L_k) = -Re\, L_k.$$

H does not depend on the choice of basis. The map J extends to a complex linear map of $C \otimes H$ to itself with V as its $-i$ eigenspace and $\overline{V}$ as its $+i$ eigenspace. So J also is independent of the choice of basis. The integrability condition for J follows from that for V.

Given H and J, extend J to a complex linear map of $C \otimes H$ to itself and let V be the $-i$ eigenspace. So $V \cap \overline{V} = \{0\}$. And the integrability condition for J implies the one for V.

Chapter 3

MODULI SPACE

3.1. RIEMANN SURFACES

A Riemann surface is a one complex dimensional connected complex analytic manifold, that is, a two real dimensional connected manifold M with a maximal set of charts $\{U_\alpha, z_\alpha\}_{\alpha \in A}$ on M, that is, the $\{U_\alpha\}_{\alpha \in A}$ constitute an open cover of M and

$$z_\alpha \colon U_\alpha \to C$$

is a homeomorphism onto an open subset of the complex plane C such that the transition functions

$$f_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} \colon z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic whenever $U_\alpha \cap U_\beta \neq \emptyset$.

Example 3.1.1. *The simplest example of a Riemann surface is the complex plane C. The single coordinate chart (C, id) defines the Riemann surface structure on C.*

Given any Riemann surface M, then a domain D (connected open subset) on M is also a Riemann surface. The coordinate charts on D are obtained by restricting the coordinate charts of M to D. Thus, every domain in C is again a Riemann surface.

The one point compactification, $C \cup \{\infty\}$, of C (known as the extended complex plane or Riemann sphere) is the simplest example of a closed (= compact)

Riemann surface. The charts we use are $\{U_j, z_j\}_{j=1,2}$ *with*

$$U_1 = C$$

$$U_2 = (C\backslash\{0\}) \cup \{\infty\}$$

and

$$z_1(z) = z, \quad z \in U_1$$

$$z_2(z) = \frac{1}{z}, \quad z \in U_2.$$

The two non trivial transition functions involved are

$$f_{kj}\colon C\backslash\{0\} \to C\backslash\{0\}, \quad k \neq j, j = 1, 2$$

with

$$f_{kj}(z) = \frac{1}{z}.$$

3.1.0.1. *Conformal mapping*

Suppose that an arc γ with the equation $z = z(t)$, $\alpha \leq t \leq \beta$, is contained in a region Ω, and let $f(z)$ be defined and continuous in Ω. Then the equation $\omega = \omega(t) = f(z(t))$ defines an arc γ' in the ω-plane which may be called the image of γ.

Consider the case of an $f(z)$ which is holomorphic in Ω. If $z'(t)$ exists, we find that $\omega'(t)$ also exists and is determined by

$$\omega'(t) = f'(z(t))z'(t).$$

We will investigate the meaning of this equation at a point $z_0 = z(t_0)$ with $z'(t_0) \neq 0$ and $f'(z_0) \neq 0$.

The first conclusion is that $\omega'(t_0) \neq 0$. Hence γ' has a tangent at $\omega_0 = f(z_0)$, and its direction is determined by

$$arg\, \omega'(t_0) = arg\, f'(z_0) + arg\, z'(t_0).$$

This relation asserts that the angle between the directed tangents to γ at z_0 and to γ' at ω_0 is equal to $arg\, f'(z_0)$. It is hence independent of the curve γ. For this reason curves through z_0 which are tangent to each other are mapped onto curves with a common tangent at ω_0. Moreover, two curves which form an angle at z_0 are mapped upon curves forming the same angle, in sense as well as in size. In view of this property the mapping by $\omega = f(z)$ is said to be conformal at all points with $f'(z) \neq 0$.

A map

$$f\colon M \to N$$

between Riemann surfaces is called conformal if for every local coordinate (U, z) on M and every local coordinate (V, ζ) on N with $U \cap f^{-1}(V) \neq \emptyset$, the mapping

$$\zeta \circ f \circ z^{-1}\colon z(U \cap f^{-1}(V)) \to \zeta(V)$$

is conformal (as a mapping from C to C).

3.1.0.2. *Riemannian surfaces and conformal structures*

Suppose that a Riemannian metric ds is given on a real two-dimensional real manifold M. This metric is represented as

$$ds^2 = E\,dx^2 + 2F\,dxdy + G\,dy^2$$

on a coordinate neighborhood $(U, (x, y))$ of M. Setting $z = x + iy$, we see that it is written in the form

$$ds^2 = \lambda \mid dz + \mu d\bar{z} \mid^2, \tag{3.1.1}$$

where λ is a positive smooth function on U and μ is a complex valued smooth function with $\mid \mu \mid < 1$ on U. Actually, λ and μ are given by

$$\lambda = \frac{1}{4}(E + G + 2\sqrt{EG - F^2}),$$

$$\mu = \frac{E - G + 2iF}{E + G + 2\sqrt{EG - F^2}}.$$

Local coordinates (u, v) on U are said to be isothermal coordinates for ds^2 if ds^2 is represented as

$$ds^2 = \rho(du^2 + dv^2)$$

on U, where ρ is a positive smooth function on U. The complex coordinate $w = u + iv$ is also called an isothermal coordinate for ds^2.

Since an isothermal coordinate w for ds^2 satisfies

$$\rho \mid dw \mid^2 = \rho \mid w_z \mid^2 \mid dz + \frac{w_{\overline{z}}}{w_z} d\overline{z} \mid^2,$$

comparing with 3.1.1, we conclude that an isothermal coordinate w for ds^2 exists if the partial differential equation

$$\frac{\partial w}{\partial \overline{z}} = \mu \frac{\partial w}{\partial z}$$

has a diffeomorphic solution w. This equation is called a Beltrami equation. We shall see in chapter 4 that such a solution w always exists provided that $\| \mu \|_\infty < 1$. Hence for a system of coordinate neighborhoods $\{(U_j, (x_j, y_j))\}_{j \in J}$ on M, there exists an isothermal coordinate w_j on each U_j. $\{(U_j, w_j)\}_{j \in J}$ define a complex structure on M. Denote by R the Riemann surface obtained in this way. The complex structure on R may be called the conformal structure induced by the Riemannian metric ds^2.

For oriented 2-dimensional Riemannian manifolds (M, ds^2) and (N, ds_1^2), an orientation preserving diffeomorphism $f\colon M \to N$ is a conformal mapping if the pull back of ds_1^2 by f is equal to $exp(\varphi) ds^2$ on M, where φ is a real valued smooth function on M. Intuitively, it means that the angle, measured by ds^2, between any smooth curves c_1 and c_2 on M equals the angle, measured by ds_1^2, between $f(c_1)$ and $f(c_2)$ on N. We say that (M, ds^2) and (N, ds_1^2) are conformally equivalent or have the same conformal structure if there exists a conformal mapping between them.

In the case of dimension 2, the uniqueness of the representation 3.1.1 leads to the following theorem.

Theorem 3.1.2. *Let R and S be Riemann surfaces induced by oriented 2-dimensional Riemannian manifolds (M, ds^2) and (N, ds_1^2), respectively. Then $f\colon (M, ds^2) \to (N, ds_1^2)$ is conformal if and only if $f\colon R \to S$ is biholomorphic.*

This theorem shows that, in the two dimensional case, concepts of complex structure and of conformal structure are equivalent. This is the reason that a biholomorphic mapping is called a conformal mapping. This assertion is a remarkable property for one dimensional complex manifolds, i.e. two dimensional real manifolds, which is not true for the higher dimensional case.

3.1.1. Classification of Riemann surfaces

The classification of Riemann surfaces is given by the Uniformization theorem which we shall discuss in this section.

3.1.1.1. *The Riemann mapping theorem*

We shall prove that the unit disk can be mapped conformally onto any simple connected region in the plane, other than the plane itself. This will imply that any two such regions can be mapped conformally onto each other, for we can use the unit disk as an intermediary step.

Although the mapping theorem was formulated by Riemann, its first successful proof was due to Koebe.

Theorem 3.1.3. *Given any simply connected region Ω which is not the whole plane, and a point $z_0 \in \Omega$, there exists a unique holomorphic function $f(z)$ in Ω, normalized by the conditions $f(z_0) = 0$, $f'(z_0) > 0$, such that $f(z)$ defines a one-to-one mapping of Ω onto the disk $\mid w \mid < 1$.*

PROOF. The uniqueness is easily proved, for if f_1 and f_2 are two such functions, then $f_1(f_2^{-1}(w))$ defines a one-to-one mapping of $\mid w \mid < 1$ onto itself. We know that such a mapping is given by a linear transformation S. The conditions $S(0) = 0$, $S'(0) > 0$ imply $S(w) = w$, hence $f_1 = f_2$.

A holomorphic function $g(z)$ in Ω is said to be univalent if $g(z_1) = g(z_2)$ only for $z_1 = z_2$, in other words, if the mapping by g is one-to-one. For the existence proof we consider the family Θ formed by all functions g with the following properties: (i) g is holomorphic and univalent in Ω, (ii) $\mid g(z) \mid \leq 1$ in Ω, (iii) $g(z_0) = 0$ and $g'(z_0) > 0$. We contend that f is the function in Θ for which the derivative $f'(z_0)$ is a maximum. The proof will consist of three parts: (1) it is shown that the family Θ is not empty, (2) there exists f with maximal derivative, (3) this f has the desired properties.

To prove that Θ is not empty we note that there exists, by assumption, a point $a \neq \infty$ not in Ω. Since Ω is simply connected, it is possible to define a single-valued branch of $\sqrt{z-a}$ in Ω, denote it by $h(z)$. This function does not take the same value twice, nor does it take opposite values. The image of Ω under the mapping h covers a disk $\mid (w - h(z_0)) \mid < \rho$, and therefore it does not meet the disk $\mid (w + h(z_0)) \mid < \rho$. In other words, $\mid h(z) + h(z_0) \mid \geq \rho$ for $z \in \Omega$, and in particular $2 \mid h(z_0) \mid \geq \rho$. It can now be verified that the function

$$g_0(z) = \frac{\rho \mid h'(z_0) \mid}{4 \mid h(z_0) \mid^2} \cdot \frac{h(z_0)}{h'(z_0)} \cdot \frac{h(z) - h(z_0)}{h(z) + h(z_0)}$$

belongs to the family Θ. Indeed, because it is obtained from the univalent function h by means of a linear transformation, it is itself univalent. Moreover, $g_0(z_0) = 0$ and $g_0'(z_0) = (\frac{\rho}{8})\frac{|h'(z_0)|}{|h(z_0)|^2} > 0$. Finally, the estimate

$$\left|\frac{h(z) - h(z_0)}{h(z) + h(z_0)}\right| = \mid h(z_0) \mid \cdot \left|\frac{1}{h(z_0)} - \frac{2}{h(z) + h(z_0)}\right| \leq \frac{4 \mid h(z_0) \mid}{\rho}$$

shows that $\mid g_0(z) \mid \leq 1$ in Ω.

The derivatives $g'(z_0)$, $g \in \Theta$, have a least upper bound B which a priori could be infinite. There is a sequence of functions $g_n \in \Theta$ such that $g_n'(z_0) \to B$. The family Θ is normal, that is, every sequence of functions in Θ contains a subsequence which converges uniformly on every compact subset of Ω. Hence there exists a subsequence $\{g_{n_k}\}$ which tends to a holomorphic limit function f, uniformly on compact sets. It is clear that $\mid f(z) \mid \leq 1$ in Ω, $f(z_0) = 0$ and $f'(z_0) = B$ (this proves that $B < \infty$). If we can show that f is univalent, it will follow that f is in Θ and has a maximal derivative at z_0.

In the first place f is not a constant, for $f'(z_0) = B > 0$. Choose a point $z_1 \in \Omega$, and consider the functions $g_1(z) = g(z) - g(z_1)$, $g \in \Theta$. They are all $\neq 0$ in the region obtained by omitting z_1 from Ω. Every limit function is either identically zero or never zero. But $f(z) - f(z_1)$ is a limit function, and it is not identically zero. Hence $f(z) \neq f(z_1)$ for $z \neq z_1$, and since z_1 was arbitrary we have proved that f is univalent.

It remains to show that f takes every value w with $\mid w \mid < 1$. Suppose it were true that $f(z) \neq w_0$ for some w_0, $\mid w_0 \mid < 1$. Then, since Ω is simply connected, it is possible to define a single valued branch of

$$F(z) = \sqrt{\frac{f(z) - w_0}{1 - \overline{w_0} f(z)}}.$$

It is clear that F is univalent and that $| F | \leq 1$. To normalize it we form

$$G(z) = \frac{| F'(z_0) |}{F'(z_0)} \cdot \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)} F(z)}$$

which vanishes and has a positive derivative at z_0. For its value we find, after brief computation and using $f(z_0) = 0$ and $f'(z_0) = B$,

$$G'(z_0) = \frac{| F'(z_0) |}{1 - | F(z_0) |^2} = \frac{\dfrac{1 - | w_0 |^2}{2 | w_0 |}}{1 - | w_0 |} = \frac{1 + | w_0 |}{2 | w_0 |} B > B.$$

This is a contradiction, and we conclude that $f(z)$ assumes all values w, $| w | <$ 1. □

3.1.1.2. *Uniformization of simply-connected Riemann surfaces*

Up to conformal equivalence, there exist three simply connected Riemann surfaces,

- $\hat{C} \equiv C \cup \{\infty\}$ the Riemann sphere,
- C the complex plane,
- $\Delta = \{z \in C : |z| < 1\}$ the unit disk.

Its proof is based on the use of subharmonic functions. Subharmonic functions are defined on Riemann surfaces with the aid of local parameters. This is possible because subharmonicity is a local and conformally invariant property.

First of all, classification of Riemann surfaces into compact, parabolic, and hyperbolic surfaces in needed. A non compact Riemann surface S is parabolic if every negative subharmonic function on S is constant, otherwise S is hyperbolic.

Using subharmonic functions and Perron families, we can define Green's functions for Riemann surfaces just as it is done for the case of plane domains. The Green's function g_p of a Riemann surface S with singularity at the point $p \in S$ is a function which is positive and harmonic on $S - \{p\}$. To describe its singularity, we consider a local parameter z mapping a neighborhood of p onto the unit disc such that $z(p) = 0$. Then it is required that $g_p + \log \mid z \mid$ be harmonic at p. This is an invariant definition not depending on the choice of the local parameter. The Green's function is characterized by the property that among all functions positive and harmonic on $S - \{p\}$ and possessing the same singularity at p as g_p, the function g_p is the smallest. If a Green's function exists for some $p \in S$, then it exists for every $p \in S$. By a theorem of Ohtsuka, the Green's function exists if and only if S is hyperbolic.

If S is parabolic or compact, Green's functions do not exist but it is possible to prove the existence of a function $u_{p,q}$ with the following properties: $u_{p,q}$ is harmonic in $S - \{p\} - \{q\}$, if $z(p) = 0$, then $u_{p,q} - \log \mid z \mid$ is harmonic at p, and if $z(q) = 0$, then $u_{p,q} + \log \mid z \mid$ is harmonic at q, outside parameter discs (pre images of discs under z) containing p and q, the function $u_{p,q}$ is bounded ([**25**]).

Remark 3.1.4. *The möbius transformation* $w = \dfrac{z-i}{z+i}$ *maps biholomorphically, the upper half-plane* H *onto the unit disc* Δ*, and hence we often use the unit disc* Δ *instead of the upper half-plane* H.

3.1.1.3. *Uniformization of arbitrary Riemann surfaces*

We provide the uniformization theorem for Riemann surfaces, based on universal covering surfaces, Möbius transformations and Fuchsian groups.

In order to formulate the Uniformization theorem for an arbitrary Riemann surface R, we shall consider the universal covering surface $\tilde{R}$ of R and its covering transformation group Γ will be constructed. By the uniformization theorem for simply connected Riemann surfaces, $\tilde{R}$ is biholomorphically equivalent to $\hat{C}$, C or H, and Γ acts properly discontinuously on $\tilde{R}$ as a group consisting of Möbius transformations. In particular, when $\tilde{R} = H$, we call Γ a Fuchsian group. In this way, we conclude that every Riemann surface for which $\tilde{R} = H$ is represented by a quotient space H/Γ of H by a Fuchsian group Γ.

3.1.1.4. *Universal covering*

Let R and $\tilde{R}$ be Riemann surfaces. A surjective holomorphic mapping $\pi\colon \tilde{R} \to R$ is said to be a covering map if every point p of R has a neighborhood U such that for each connected component V of the inverse image $\pi^{-1}(U)$ of U, the restricted map $\pi\colon V \to U$ is biholomorphic. We call $\tilde{R}$ a covering surface of R. The covering map π is also called the projection of $\tilde{R}$ onto R. When $\tilde{R}$ is simply connected, we call $\tilde{R}$ a universal covering surface of R.

Example 3.1.5. *We give a few examples of covering surfaces.*

(1) *Let* $\pi\colon C \to C - \{0\}$ *be given by* $\pi(z) = e^z$. *Then* C *is a universal covering surface of* $C - \{0\}$.

(2) *Let* $\pi\colon H \to \Delta - \{0\}$ *be given by* $\pi(z) = e^{2\pi i z}$. *Then* H *is a universal covering surface of* $\Delta - \{0\}$.

(3) *Let* $\pi\colon C - \{0\} \to C - \{0\}$ *be given by* $\pi(z) = z^n$, *where* n *is a positive integer. Then* $C - \{0\}$ *is a covering surface of itself, but it is not a universal covering surface.*

(4) *For a given* λ (> 1), *set* $r = \exp(\frac{-2\pi^2}{\log \lambda})$ *and* $A = \{w \in C : r <| w |< 1\}$. *Define* $\pi\colon H \to A$ *by* $\pi(z) = \exp(2\pi i \frac{\log z}{\log \lambda})$. *Then* H *becomes a universal covering surface of the annulus* A.

(5) *Let* Γ_τ *be a lattice group generated by* 1 *and a point* $\tau \in H$, *and let* π *be the projection of* C *onto the quotient space* C/Γ_τ. *Then* C *is a universal covering surface of the torus* C/Γ_τ.

Any biholomorphic mapping $\gamma\colon \tilde{R} \to \tilde{R}$ with $\pi \circ \gamma = \pi$ is called a covering transformation. We denote by Γ the set of all its covering transformations. By the composition of mappings, Γ forms a group, which is called the covering transformation group. In particular, we call Γ the universal covering transformation group if $\tilde{R}$ is a universal covering surface of R.

Example 3.1.6. *We give the covering transformation groups of the coverings in the previous examples. The notation* $< \gamma_1, \gamma_2, \cdots, \gamma_n >$ *expresses the group generated by* $\gamma_1, \gamma_2, \cdots, \gamma_n$.

(1) $\Gamma = < \gamma_1 >$ *with* $\gamma_1(z) = z + 2\pi i$.

(2) $\Gamma = <\gamma_1>$ *with* $\gamma_1(z) = z+1$.

(3) $\Gamma = <\gamma_1>$ *with* $\gamma_1(z) = z\exp(\frac{2\pi i}{n})$.

(4) $\Gamma = <\gamma_1>$ *with* $\gamma_1(z) = \lambda z$.

(5) $\Gamma = <\gamma_1, \gamma_2>$ *where* $\gamma_1(z) = z+1$ *and* $\gamma_2(z) = z+\tau$.

Theorem 3.1.7. *(The pull-back structure theorem). If $f: X \to X_1$ is a homeomorphism and X_1 is a complex manifold, then f induces a complex structure on X.*

PROOF. X_1 is said to have the structure of an n-dimensional complex manifold if there exists an atlas $A = \{(U_i, \phi_i) : i \in I\}$ of charts on X_1 such that:

1. $\{U_i\}$ is an open cover of X_1.
2. ϕ_i is a homeomorphism of U_i onto the open subset $\phi_i(U_i)$ of C^n for all $i \in I$.
3. For all $i, j \in I$, $\phi_i\phi_j^{-1}$ is a biholomorphic map of $\phi_j(U_i \cap U_j)$ onto $\phi_i(U_i \cap U_j)$.

Now we are going to construct an atlas on X.

When $\{U_i\}$ is an open cover of X_1 and f is a homeomorphism, so $\{f^{-1}(U_i)\}$ is an open cover of X.

We know that the composition of two homeomorphism is a homeomorphism, so $\phi_i f$ is a homeomorphism and it is a map of $f^{-1}(U_i)$ onto the open subset $\phi_i(U_i)$ of C^n for all $i \in I$.

For all $i, j \in I$, $(\phi_i f)(\phi_j f)^{-1}$ is a map from $\phi_j(U_i \cap U_j)$ onto $\phi_i(U_i \cap U_j)$ and $(\phi_i f)(\phi_j f)^{-1} = \phi_i f f^{-1}\phi_j^{-1} = \phi_i\phi_j^{-1}$ where $\phi_i\phi_j^{-1}$ is a biholomorphic map from $\phi_j(U_i \cap U_j)$ onto $\phi_i(U_i \cap U_j)$. So $(\phi_i f)(\phi_j f)^{-1}$ is biholomorphic.

Hence, $B = \{(f^{-1}(U_i), \phi_i f) : i \in I\}$ is a complex holomorphic atlas on X, and X together with the atlas B is a complex manifold. □

Theorem 3.1.8. *For every Riemann surface R, there exists a universal covering surface $\tilde{R}$ of R, which is biholomorphic to one of the three Riemann surfaces $\hat{C}$, C or H.*

Theorem 3.1.9. *(Uniqueness of the universal covering) For any two universal covering surfaces $\tilde{R}$ and $\tilde{R}_1$ where $\pi: \tilde{R} \to R$ and $\pi_1: \tilde{R}_1 \to R$, there exist a biholomorphic mapping φ of $\tilde{R}$ to $\tilde{R}_1$ with $\pi_1 \circ \varphi = \pi$.*

3.1.1.5. *Construction of the universal covering*

A path on a Riemann surface R is a continuous curve $c\colon I \to R$, where I is the interval $[0,1]$. The points $c(0)$ and $c(1)$ are said to be the initial and terminal points of c, respectively. We also say that c is a path from $c(0)$ to $c(1)$. Its image is also denoted by the same letter c.

For two paths c and c' on R such that $c(1) = c'(0)$, by connecting the terminal point of c with the initial point of c', we get a path $c \cdot c'$ on R with the initial point $c(0)$ and terminal point $c'(1)$.

Let $\tilde{R}$ be a covering surface of a Riemann surface R. A point $\tilde{p}$ in $\tilde{R}$ is said to lie over a point p in R if $\pi(\tilde{p}) = p$. A lift of a path c on R is a path $\tilde{c}$ on $\tilde{R}$ with $\pi \circ \tilde{c} = c$.

Fix a point p_0 on a given Riemann surface R. Let (c,p) be a pair consisting of a point p on R and a path c on R from p_0 to p. Two pairs (c,p) and (c',p') are equivalent if $p = p'$ and c is homotopic to c' on R. Denote by $[c,p]$ the equivalence class of (c,p). Let $\tilde{R}$ be the set of all the equivalence classes $[c,p]$, and $\pi\colon \tilde{R} \to R$ be the projection given by $\pi([c,p]) = p$.

Lemma 3.1.10. *(Existence and uniqueness of a lift of a path) For any path c on R with initial point p, and for any point $\tilde{p}$ of $\tilde{R}$ over p, there exists a unique lift $\tilde{c}$ of c with initial point $\tilde{p}$.*

Theorem 3.1.11. *For Riemann surfaces R and S, let $\tilde{R}$ and $\tilde{S}$ be their universal covering surfaces where $\pi_R\colon \tilde{R} \to R$ and $\pi_S\colon \tilde{S} \to S$. Then given an arbitrary continuous mapping $f\colon R \to S$, there exists a continuous mapping $\tilde{f}\colon \tilde{R} \to \tilde{S}$ with $f \circ \pi_R = \pi_S \circ \tilde{f}$. This mapping $\tilde{f}$ is uniquely determined under the condition that $\tilde{f}(\tilde{p_1}) = \tilde{q_1}$, where $\tilde{p_1} \in \tilde{R}$ and $\tilde{q_1} \in \tilde{S}$ are such that $\pi_S(\tilde{q_1}) = f(\pi_R(\tilde{p_1}))$.*

Moreover, if f is differentiable or holomorphic, then $\tilde{f}$ is also differentiable or holomorphic.

Proof. Setting $\tilde{p_1} = [c_1, p_1]$ and $\tilde{q_1} = [d_1, f(p_1)]$, we get a mapping defined by $f([c,p]) = [d_1 \cdot f(c_1)^{-1} \cdot f(c), f(p)]$ for all points $[c,p]$ in $\tilde{R}$. Then it is obvious that $\tilde{f}(\tilde{p_1}) = \tilde{q_1}$ and $f \circ \pi_R = \pi_S \circ \tilde{f}$. Since π_R and π_S are locally biholomorphic and

f is continuous, $\tilde{f}$ must be continuous and if f is differentiable or holomorphic, then so is $\tilde{f}$. The uniqueness assertion follows from lemma 3.1.10. □

3.1.1.6. *Universal covering transformation group*

Theorem 3.1.12. *For a given universal covering surface $\tilde{R}$ of a Riemann surface R, its universal covering group Γ is isomorphic to the fundamental group $\pi_1(R, p_0)$ of R.*

Proof. For any element $[c_0] \in \pi_1(R, p_0)$, we define the action $[c_0]_*$ on $\tilde{R}$ by

$$[c_0]_*([c,p]) = [c_0 \cdot c, p], \quad [c,p] \in \tilde{R}.$$

This $[c_0]_*$ belongs to Γ, that is, it is a covering transformation.

The above correspondence $[c_0] \longrightarrow [c_0]_*$ yields an isomorphism of the fundamental group $\pi_1(R, p_0)$ of R onto the universal covering transformation group Γ.

This correspondence is a homomorphism of $\pi_1(R, p_0)$ to Γ. To prove that it is injective, suppose that $[c_0]_*$ is the unit element of Γ. Then we have

$$[c_0]_*([I_0, p_0]) = [c_0, p_0] = [I_0, p_0],$$

where I_0 is the path on R such that $I_0(t) = p_0$ for any $t \in I = [0,1]$. Thus, c_0 is homotopic to I_0, and hence $[c_0]$ is the unit element of $\pi_1(R, p_0)$. It follows that this correspondence is injective.

To prove that this correspondence is surjective, take any element $\gamma \in \Gamma$. Let $\tilde{c}$ be a path on $\tilde{R}$ from $[I_0, p_0]$ to $\gamma([I_0, p_0])$. Then the relation $\pi \circ \gamma = \pi$ implies that $c = \pi \circ \tilde{c}$ is a closed path on R with base point p_0. Hence $\tilde{c}$ is a lift of c, and $[I_0, p_0]$ and $[c, p_0]$ are the initial and terminal points of $\tilde{c}$, respectively. Thus lemma 3.1.10 shows that

$$\gamma([I_0, p_0]) = [c, p_0] = [c]_*([I_0, p_0]).$$

Since $[c]$ is an element of $\pi_1(R, p_0)$, theorem 3.1.11 implies that $\gamma = [c]_*$, and hence this correspondence is surjective. □

Lemma 3.1.13. *The universal covering transformation group Γ of a Riemann surface R satisfies the following properties:*

- *For any $\tilde{p}, \tilde{q} \in \tilde{R}$ with $\pi(\tilde{p}) = \pi(\tilde{q})$, there exists an element $\gamma \in \Gamma$ with $\tilde{q} = \gamma(\tilde{p})$.*
- *For every $\tilde{p} \in \tilde{R}$, there is a suitable neighborhood $\tilde{U}$ of $\tilde{p}$ in $\tilde{R}$ such that $\gamma(\tilde{U}) \cap \tilde{U} = \emptyset$ for every $\gamma \in \Gamma - \{id\}$. In particular, each element of Γ except for the identity has no fixed points.*
- *Γ acts properly discontinuously on $\tilde{R}$, that is, for any compact subset K of $\tilde{R}$, there are at most finitely many elements $\gamma \in \Gamma$ such that $\gamma(K) \cap K \neq \emptyset$.*

PROOF. To prove the first property, suppose that $\pi(\tilde{p}) = \pi(\tilde{q}) = p$. Then we have $\tilde{p} = [c_1, p]$ and $\tilde{q} = [c_2, p]$ for some paths c_1 and c_2 on R. Putting $c_0 = c_2 \cdot c_1^{-1}$, we see that $\gamma = [c_0]_*$ satisfies $\tilde{q} = \gamma(\tilde{p})$.

To see the second property, take a point $\tilde{p} \in \tilde{R}$, and set $p = \pi(\tilde{p})$, $\tilde{p} = [c, p]$. Choose a neighborhood U of p in R which satisfies the condition in the definition of a covering map, and denote by $\tilde{U}$ the connected component of $\pi^{-1}(U)$ containing $\tilde{p}$. Actually, it is sufficient to take a simply connected domain U containing p. If $\gamma(\tilde{U}) \cap \tilde{U} \neq \emptyset$ for some $\gamma \in \Gamma$, then there are points $\tilde{p_1}, \tilde{q_1} \in \tilde{U}$ with $\tilde{q_1} = \gamma(\tilde{p_1})$.

Since $\pi \circ \gamma = \pi$, we get $\pi(\tilde{p_1}) = \pi(\tilde{q_1})$, and hence $\tilde{q_1} = \tilde{p_1}$, for π is biholomorphic on $\tilde{U}$. Thus we have $\gamma(\tilde{p_1}) = Id(\tilde{p_1})$ where Id is the identity. By theorem 3.1.11, we conclude that γ as a lift of $Id\colon R \to R$ is uniquely determined by Id.

Finally, to verify the third property, assume that there exists a sequence $\{\gamma_n\}_{n=1}^{\infty}$ consisting of distinct elements of Γ such that $\gamma_n(K) \cap K \neq \emptyset$ for all n. Then for each n, we can take two points $\tilde{q_n}, \tilde{r_n} \in K$ with $\tilde{r_n} = \gamma_n(\tilde{q_n})$. Since K is compact, taking a subsequence if necessary, we may assume that $\{\tilde{q_n}\}_{n=1}^{\infty}$, $\{\tilde{r_n}\}_{n=1}^{\infty}$ converge to $\tilde{q_0}, \tilde{r_0} \in K$, respectively, as $n \to \infty$. Since $\pi \circ \gamma_n = \pi$, we obtain $\pi(\tilde{q_n}) = \pi(\tilde{r_n})$ and $\pi(\tilde{q_0}) = \pi(\tilde{r_0})$. Take a neighborhood U of $\pi(\tilde{q_0})$ in R satisfying the condition of the definition of a covering map, denote by $\tilde{U}$ and $\tilde{V}$ the connected components of $\pi^{-1}(U)$ containing $\tilde{q_0}$! and $\tilde{r_0}$, respectively.

Since $\{\gamma_n(\tilde{q_n})\}_{n=1}^{\infty}$ converges to $\tilde{r_0}$, we have $\gamma_n(\tilde{U}) \cap \tilde{V} \neq \emptyset$ for a sufficiently large n. Since $\pi \circ \gamma_n(\tilde{U}) = U$, it follows that $\gamma_n(\tilde{U}) = \tilde{V}$, namely, $\gamma_{n+1}^{-1} \circ$

$\gamma_n(\tilde{U}) = \tilde{U}$. By the second assertion, we conclude that $\gamma_{n+1} = \gamma_n$. This is a contradiction. □

3.1.1.7. *Uniformization theorem for arbitrary Riemann surfaces*

Now we are ready to state the uniformization theorem for arbitrary Riemann surfaces. We construct a Riemann surface Σ/G from a Riemann surface M with universal covering surface Σ and a subgroup G of the biholomorphic automorphism group $Aut(\Sigma)$, where G is assumed to satisfy the second and third properties in lemma 3.1.13, that is, every element of G except for the unit element has no fixed points in Σ, and G acts properly discontinuously on Σ.

Theorem 3.1.14. *Every Riemann surface M is conformally equivalent to some Σ/G, where Σ is $\widehat{C}$ or C or Δ, is the universal cover of M, and G is a subgroup of $Aut(\Sigma)$ acting properly discontinuously and fixed-point freely on Σ and $G \cong \pi_1(M)$.*

3.1.1.8. *Automorphisms*

Subgroups of the group of automorphisms of the three Riemann surfaces play an important role in theorem 3.1.14, so it is a good idea to know what Aut is for each surface.

To find $Aut(\hat{C})$, we use the fact that

$$\hat{C} \simeq CP^1 \equiv P^1,$$

with the isomorphism given in homogeneous coordinates on P^1 by

$$([z_1, z_2]) \mapsto \frac{z_1}{z_2}, \quad ([1,0]) \equiv \{\infty\}.$$

The action of $GL(2,C)$ on C^2 projects to an action of $PL(2,C) \equiv \{GL(2,C)/\lambda,\ \lambda \in C^*\}$ on P^1. Then $PL(2,C)$ is the group $Aut(\hat{C})$, whose action on $\hat{C}$ is

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto z' = \frac{az+b}{cz+d}.$$

Such a transformation is called a Möbius transformation.

$$Aut(\hat{C}) \simeq PL(2, C).$$

To find $Aut(C)$, we note that the conformal automorphisms of C will be those automorphisms of $\hat{C}$ which fix the point ∞. It is clear that a Möbius transformation which fixes ∞ must have $c = 0$:

$$z' = az + b, \quad a \in C^*, b \in C.$$

This group is known as $Aff(1, C)$, the affine transformations of the plane. It is isomorphic to the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a \in C^*.$$

Thus,

$$Aut(C) \simeq Aff(1, C).$$

$Aut(\Delta) \simeq Aut(H)$ and every element of $Aut(H)$ has a form $\gamma(z) = \dfrac{az + b}{cz + d}$ where $a, b, c, d \in R$ with $ad - bc = 1$. So we have

$$Aut(H) \simeq PL(2, R).$$

3.1.2. Moduli of Riemann surfaces

Two Riemann surfaces can have the same underlying topological space, and yet be conformally inequivalent (have different complex structures). The set of conformally inequivalent Riemann surfaces over the same topological space is known as a moduli space.

3.1.2.1. *Surfaces with universal cover $\hat{C}$*

As we found before, $Aut(\hat{C})$ is the group $PL(2, C)$. Recall that if we think of $\hat{C}$ as $C \cup \infty$, then the action of $Aut(\hat{C})$ is that of a Möbius transformation.

Proposition 3.1.15. *The only Riemann surface with universal cover $\hat{C}$ is $\hat{C}$ itself.*

PROOF. First we show that Möbius transformations fix at least one point of $\hat{C}$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PL(2, C)$. Then a fixed point satisfies $z = \dfrac{az+b}{cz+d}$, which simplifies to

$$z = \frac{a-d}{2c} \pm \frac{1}{2c}\sqrt{(d-a)^2 - 4cb} \quad c \neq 0$$

$$z = \frac{b}{d-a} \quad c = 0,\ a \neq d$$

$$z = \infty \quad c = 0,\ a = d,\ b \neq 0.$$

These equations clearly have solutions for any element of $PL(2, C)$. Thus, we see that every element of $Aut(\hat{C})$ fixes at least one point of $\hat{C}$, and so no element of $PL(2, C)$ can act fixed-point freely on $\hat{C}$. From theorem 3.1.14 we have the desired result. □

An obvious consequence of this proposition is that the moduli space of genus zero surfaces which have $\hat{C}$ as their universal covering space, is a one-point set. In fact, all three of the simply connected surfaces have one-point moduli space.

3.1.2.2. *Surfaces with universal cover C*

Recall that $Aut(C) = Aff(1, C)$, with $z \mapsto az + b$. We will make use of the fact that:

Theorem 3.1.16. *If the (holomorphic) universal covering space D of M is C, then M is conformally equivalent to C, C^*, or T^2, a torus.*

The respective covering groups are $\{e\}$, Z, $Z \oplus Z$. First, the covering group $G = \{e\}$, in which case M is conformally equivalent to the plane C. Second, examine the case $G = Z$. We can take $z \mapsto z + 1$ as a generator. A fundamental domain of such a group is the interior of the parallel strip bounded by straight lines through 0 and through 1 and perpendicular to the vector from 0 to 1. Topologically, D/G is an infinite cylinder. The function $z \to exp(2\pi i z)$, shows that D/G is conformally equivalent to C^*.

$G = Z \oplus Z$ is a bit more complicated. Consider a lattice in C:

$$\Lambda(\omega, \eta) = \{m\omega + n\eta : m, n \in Z,\ \omega, \eta \in C^* \text{ linearly independent } \}.$$

Clearly it is a discrete group, isomorphic to $Z \oplus Z$, and the quotient $C/\Lambda(\omega, \eta)$ is a torus.

Proposition 3.1.17. *The conjugacy class of $\Lambda(\omega, \eta)$ in $Aut(C)$ is the set of lattices of the form $\Lambda(a\omega, a\eta)$, with $a \in C^*$.*

PROOF. An element $(a, b) \in Aut(C)$ acts on a generator $h_\omega : z \mapsto z + \omega$ of the group $\Lambda(\omega, \eta)$ as $(a, b) \cdot h_\omega(z) = a(z + \omega) + b$, and thus, if $a \in C^*$

$$(a, b)h_\omega(a, b)^{-1}(z) = z + a\omega.$$

□

Now we define $\tau = \frac{\eta}{\omega}$ and without loss of generality, choose $Im(\tau) > 0$. Furthermore, we choose $a = \frac{1}{\omega}$, so that every lattice is conjugate to one of the form $\Lambda(1, \tau)$.

Theorem 3.1.18. *For any two points τ and τ' in the upper half-plane H, two tori R_τ and $R_{\tau'}$ are biholomorphically equivalent if and only if τ and τ' satisfy the relation*

$$\tau' = \frac{a\tau + b}{c\tau + d}, \tag{3.1.2}$$

where a, b, c and d are integers with $ad - bc = 1$.

PROOF. First, assume that there is a biholomorphic mapping f of $R_{\tau'}$ onto R_τ. Since C is simply connected, the theorem 3.1.11 implies that there exist a lift $\tilde{f}$ of f, that is, a holomorphic mapping $\tilde{f}\colon C \to C$ such that $\pi_\tau \circ \tilde{f} = f \circ \pi_{\tau'}$. Because f is biholomorphic, so is $\tilde{f}$. Then $\tilde{f}$ is written as $\tilde{f} = \alpha Z + \beta$, where α, β are complex numbers and $\alpha \neq 0$, because $Aut(C) = Aff(1, C)$.

Moreover, we may assume that $\tilde{f}(0) = 0$, and hence $\beta = 0$. We have

$$\tilde{f}(\tau') = \alpha\tau' = a\tau + b, \quad \tilde{f}(1) = \alpha = c\tau + d,$$

where a, b, c and d are integers. Therefore, we obtain

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

Applying the same argument to $\tilde{f}^{-1}$, we get

$$\tau = \frac{a'\tau' + b'}{c'\tau' + d'},$$

where a', b', c' and d' are integers. Furthermore, from the relations $\tilde{f}^{-1} \circ \tilde{f}(1) = 1$ and $\tilde{f}^{-1} \circ \tilde{f}(\tau') = \tau'$, we see that $ad - bc = \pm 1$. Since $Im\,\tau' = \dfrac{ad - bc}{| c\tau + d |^2} Im\,\tau >$ 0, we have $ad - bc = 1$.

Conversely, if 3.1.2 holds, then a biholomorphic mapping $f\colon R_{\tau'} \to R_\tau$ is given by $f([z]) = [(c\tau + d)z]$. □

Now, we call the group

$$PSL(2, Z) = \{\gamma(z) = \frac{a\tau + b}{c\tau + d} : a, b, c, d \in Z \text{ and } ad - bc = 1\}$$

the modular group. Every $\gamma \in PSL(2, Z)$ is a biholomorphic automorphism of the upper half-plane H.

Let M be the moduli space of tori, i.e. the set of all biholomorphic equivalence classes of tori. Theorem 3.1.18 implies that M is identified with the quotient space of H by $PSL(2, Z)$, that is,

$$M \cong H/PSL(2, Z).$$

3.1.2.3. *Fuchsian groups*

A Riemann surface which is biholomorphic to one of $\hat{C}$, C, C^* or tori is said to be of exceptional type.

Theorem 3.1.19. *A Riemann surface M has a universal covering surface Σ biholomorphic to H if and only if M is not of exceptional type.*

We define a natural topology on $Aut(H)$, i.e. the compact-open topology. This means that a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of $Aut(H)$ converges to $\gamma \in Aut(H)$ if γ_n converges uniformly to γ on compact subset of H as n tends to ∞. This topology is equivalent to the one of the group $PSL(2, R)$. The topology of $PSL(2, R)$ is induced by the topology of $SL(2,R)$. Here, the sequence $\{A_n\}_{n=1}^{\infty}$ of $SL(2,R)$ with $A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ converges to $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2,R)$ if and only if a_n, b_n, c_n and d_n converge to a, b, c and d, respectively, as n tends to ∞.

A subgroup Γ of $Aut(H)$ is said to be discrete if Γ is a discrete subset of $Aut(H)$, i.e. Γ consist of isolated points.

Definition 3.1.20. *A discrete subgroup of $Aut(H)$ is called a Fuchsian group.*

Theorem 3.1.21. *For a subgroup Γ of $Aut(H)$ the following are equivalent:*

(1) Γ is Fuchsian.

(2) Γ acts properly discontinuously on H.

PROOF. That the second condition implies the first one is by the definition. Conversely, assume that Γ does not act properly discontinuously on H. Then we have a point $z_0 \in H$ and a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of distinct elements of Γ such that $\gamma_n(z_0) \to \omega_0 \in H$ as $n \to \infty$.

We may assume that $\{\gamma_n\}_{n=1}^{\infty}$ converges uniformly on compact subset of H to a holomorphic function γ defined in H. This γ must be an element of $Aut(H)$. Otherwise γ is a constant function. Hence Γ is not Fuchsian, because if Γ were a Fuchsian group, there would exist no sequences of distinct elements of Γ which converge in $Aut(H)$. ☐

Remark 3.1.22. *For a subgroup Γ of $Aut(\hat{C})$, the discreteness of Γ does not always imply that it acts properly discontinuously on $\hat{C}$.*

3.1.2.4. *Automorphic functions*

An automorphic function is a meromorphic function on a complex manifold M, that is invariant under some discrete group Γ of automorphisms of the given manifold:

$$f(\gamma(x)) = f(x), \quad x \in M, \quad \gamma \in \Gamma.$$

Automorphic functions are often defined so as to include only functions defined on a bounded connected domain D of the n-dimensional complex space C^n that are invariant under a discrete group Γ of automorphisms of this domain. The quotient space $X = M/\Gamma$ can be given a complex structure and automorphic functions are then meromorphic functions on X. The automorphic functions constitute a field $K(\Gamma)$ and the study of this field is one of the main tasks in the theory of automorphic functions.

Three cases are distinguished: $M = \hat{C}$ Riemann sphere, $M = C$ and $M = H$ the upper half plane. In the first case the discrete groups Γ are finite and the automorphic functions generate the field of rational functions. Examples of automorphic functions in the case $M = C$ are periodic functions and in particular, elliptic (doubly periodic) functions. Finally, for $M = H$ and a discrete group Γ, such that M/Γ is compact or has a finite volume, $K(\Gamma)$ is the field of algebraic functions on M/Γ.

Let us return to the torus constructed in the previous sections. The meromorphic functions on this torus are the elliptic functions with periods 1, τ. The canonical example here is the Weierstrass ϑ-function with periods 1, τ:

$$\vartheta(z) = \frac{1}{z^2} + \sum_{(n,m)\neq(0,0),\,(n,m)\in Z^2} \left(\frac{1}{(z-n-m\tau)^2} - \frac{1}{(n+m\tau)^2} \right).$$

The ϑ-function satisfies the differential equation

$$\vartheta'^2 = 4(\vartheta - e_1)(\vartheta - e_2)(\vartheta - e_3).$$

The points e_j can be identified as

$$e_1 = \vartheta(\frac{1}{2}), \quad e_2 = \vartheta(\frac{\tau}{2}), \quad e_3 = \vartheta(\frac{1+\tau}{2}).$$

ϑ' is again an elliptic function and hence a meromorphic function on the torus. If we now write $w = \vartheta'$, $z = \vartheta$, we obtain

$$w^2 = 4(z - e_1)(z - e_2)(z - e_3),$$

and we see that w is an algebraic function of z. The Riemann surface on which w is a single valued meromorphic function is the two sheeted branched cover of the sphere branched over $z = e_j$, $j = 1, 2, 3$ and $z = \infty$.

Consider an irreducible polynomial $P(z, w)$ and with it the set $S = \{(z, w) \in C^2 : P(z, w) = 0\}$. Most points of S are manifold points and after modifying the singular points and adding some points at infinity, S is the Riemann surface on which w is an algebraic function of z.

In the case of the torus discussed above, we started with a Riemann surface and found that the surface was the Riemann surface of an algebraic function. Another way of saying the preceding is as follows: we saw in the case of the torus that the field of elliptic functions determined the torus up to conformal equivalence. If

$$f\colon M \to N$$

is a conformal map between Riemann surface M and N, then

$$f^*\colon K(N) \to K(M)$$

defined by

$$f^*\varphi = \varphi \circ f, \quad \varphi \in K(N)$$

is an isomorphism of $K(N)$ into $K(M)$. If M and N are conformally equivalent (that is, if the function f above, has a holomorphic inverse) then the fields $K(M)$ and $K(N)$ are isomorphic.

Chapter 4

TEICHMÜLLER SPACE

4.0.3. Geometric definition of quasiconformal mappings

A quadrilateral consists of a Jordan domain Q and a sequence z_1, z_2, z_3, z_4 of boundary points of Q. The points z_i are called the vertices of the quadrilateral. In the following we shall consider only quadrilaterals $Q(z_1, z_2, z_3, z_4)$ whose sequence of vertices agrees with the positive orientation with respect to Q. The vertices of a quadrilateral $Q(z_1, z_2, z_3, z_4)$ divide its boundary into four Jordan arcs, the sides of the quadrilateral.

By a homeomorphism of the quadrilateral $Q(z_1, z_2, z_3, z_4)$ onto the quadrilateral $Q'(w_1, w_2, w_3, w_4)$ we understand a topological mapping $w\colon \overline{Q} \to \overline{Q'}$ which carries the points z_i to $w_i = w(z_i)$. If the restriction of w to Q is conformal, then w is called a conformal mapping of $Q(z_1, z_2, z_3, z_4)$ onto $Q'(w_1, w_2, w_3, w_4)$. It is not in general possible to map given quadrilaterals onto one another conformally, since the images of three boundary points determine the mapping uniquely. All quadrilaterals are therefore divided into several equivalence classes.

It follows from the Riemann mapping theorem that every quadrilateral $Q(z_1, z_2, z_3, z_4)$ can be mapped onto a quadrilateral $Q'(-\frac{1}{k}, -1, 1, \frac{1}{k})$ where $0 < k < 1$ and Q' is the upper half plane. The function

$$\omega(z) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}},$$

maps the quadrilateral $Q'(-\frac{1}{k}, -1, 1, \frac{1}{k})$ conformally onto a quadrilateral which consists of a rectangle and its corners. We call such a quadrilateral simply a rectangle. By combining the above mappings, we can map an arbitrary quadrilateral conformally onto a rectangle. Such a mapping will be called the canonical mapping of the quadrilateral and the corresponding rectangle the canonical rectangle of the quadrilateral.

Every conformal equivalence class of quadrilaterals thus contains rectangles and all similar rectangles belong to the same class. Conversely, every conformal mapping between two rectangles is a similarity transformation.

Now, suppose that $R = \{x + iy : 0 < x < a, 0 < y < b\}$ is a canonical rectangle of $Q(z_1, z_2, z_3, z_4)$ and that the first side (z_1, z_2) corresponds to the line segment $0 \leq x \leq a$. The number a/b, which does not depend on the particular choice of the canonical rectangle, is called the module of the quadrilateral Q. We shall use the notation

$$M(Q(z_1, z_2, z_3, z_4)) = a/b$$

for the module.

>From the definition it is clear that the module of a quadrilateral is conformally invariant.

Given a domain A, consider all quadrilaterals $Q(z_1, z_2, z_3, z_4)$ with $\overline{Q} \subset A$. Let $f\colon A \to A'$ be an orientation preserving homeomorphism. The number

$$\sup_{Q} \frac{M(f(Q)(f(z_1), f(z_2), f(z_3), f(z_4)))}{M(Q(z_1, z_2, z_3, z_4))}$$

is called the maximal dilatation of f.

Since the module is a conformal invariant, the maximal dilatation of a conformal mapping is 1.

Definition 4.0.23. *An orientation preserving homeomorphism with a finite maximal dilatation is quasiconformal, if the maximal dilatation is bounded by a number K, the mapping is said to be K-quasiconformal.*

By this terminology, f is 1-quasiconformal if and only if f is conformal. If f is K-quasiconformal, then

$$M(f(Q)) \geq \frac{M(Q)}{K}$$

for every quadrilateral in A. So a mapping f and its inverse f^{-1} are simultaneously K-quasiconformal.

>From the definition it also follows that if $f: A \longrightarrow B$ is K_1-quasiconformal and $g: B \longrightarrow C$ is K_2-quasiconformal, then $g \circ f$ is K_1K_2-quasiconformal.

It is possible to arrive at the notion of the module of a quadrilateral by use of the length area method. In order to arrive at this characterization of the module, we consider the canonical mapping f of the quadrilateral $Q(z_1, z_2, z_3, z_4)$ onto the rectangle $R = \{u + iv : 0 < u < a,\, 0 < v < b\}$. Then

$$\iint_Q | f'(z) |^2 \, dxdy = ab.$$

Let Γ be the family of all rectifiable arcs in Q which join the sides (z_1, z_2) and (z_3, z_4). Then

$$\int_\gamma | f'(z) | \geq b$$

for every $\gamma \in \Gamma$, with equality if γ is the inverse image of a vertical line segment of R joining its horizontal sides. Hence

$$M(Q(z_1, z_2, z_3, z_4)) = \frac{\iint_Q | f'(z) |^2 \, dxdy}{(\inf_{\gamma \in \Gamma} \int_\gamma | f'(z) || \, dz \, |)^2}. \tag{4.0.3}$$

We can get rid of the canonical mapping f if we introduce the family P whose elements ρ are non-negative measurable functions in Q and satisfy the condition $\int_\gamma \rho(z) \mid dz \mid \geq 1$ for every $\gamma \in \Gamma$. With the notation

$$m_\rho(Q) = \iint_Q \rho^2 dxdy,$$

we then have

$$M(Q(z_1, z_2, z_3, z_4)) = \inf_{\rho \in P} m_\rho(Q).$$

This basic formula can be proved by a length area reasoning. Define for every given $\rho \in P$ a function ρ_1 in the canonical rectangle R by $(\rho_1 \circ f) \mid f' \mid = \rho$. Then, by Fubini's theorem and Schwarz's inequality,

$$m_\rho(Q) = \iint_R \rho_1^2 dudv \geq \frac{1}{b} \int_0^a du (\int_0^b \rho_1(u + iv) dv)^2.$$

The last integral at the right is taken over a line segment whose pre image is in Γ. Therefore, the integral is ≥ 1, and so $m_\rho(Q) \geq \frac{a}{b} = M(Q(z_1, z_2, z_3, z_4))$.

To complete the proof we note that $\rho = \dfrac{| f' |}{b}$ belongs to P. By 4.0.3 this is a function for which $m_\rho(Q) = M(Q(z_1, z_2, z_3, z_4))$.

4.0.4. Analytic definition of quasiconformal mappings

We can generalize the characteristic property of conformal mappings that the derivative is independent of the direction.

For a diffeomorphism $f\colon A \longrightarrow f(A)$ define the complex derivatives setting

$$\partial f = \frac{1}{2}(f_x - if_y), \quad \overline{\partial} f = \frac{1}{2}(f_x + if_y).$$

Here f_x and f_y denote the partial derivatives of f with respect to x and to y, $z = x + iy$, respectively.

Let $\partial_\alpha f(z)$ denote the directional derivative of a diffeomorphic mapping $f(x, y)$ in a direction making an angle α with the positive x-direction. Thus

$$\partial_\alpha f(z) = \lim_{r \longrightarrow \infty} \frac{f(z + re^{i\alpha}) - f(z)}{r}.$$

>From calculus $\partial_\alpha f = f_x \cos(\alpha) + f_y \sin(\alpha)$, and consequently in complex notation

$$\partial_\alpha f = f_z e^{i\alpha} + f_{\overline{z}} e^{-i\alpha}.$$

We conclude that if f is an orientation preserving diffeomorphic map between planar domain, then

$$\max_\alpha | \partial_\alpha f(z) | = | f_z(z) | + | f_{\overline{z}}(z) |, \quad \min_\alpha | \partial_\alpha f(z) | = | f_z(z) | - | f_{\overline{z}}(z) | .$$

The difference $| \partial f(z) | - | \overline{\partial} f(z) |$ is positive, because the Jacobian of the function f, $J_f = | \partial f |^2 - | \overline{\partial} f |^2$ is positive for an orientation preserving diffeomorphism.

We define the dilatation quotient as

$$D_f = \frac{\max_\alpha | \partial_\alpha f |}{\min_\alpha | \partial_\alpha f |} = \frac{| \partial f | + | \overline{\partial} f |}{| \partial f | - | \overline{\partial} f |},$$

and conclude that the dilatation quotient if finite.

The mapping f is conformal if and only if $\overline{\partial} f$ vanishes identically. Then $\partial_\alpha f$ is independent of α: we have $\partial_\alpha f = \partial f = f'$. This is equivalent to the dilatation quotient being identically equal to 1.

Theorem 4.0.24. *Let $f\colon A \longrightarrow A'$ be an orientation preserving diffeomorphism with the property*

$$D_f(z) \leq K$$

for every $z \in A$. Then f is a K-quasiconformal mapping.

PROOF. We pick an arbitrary quadrilateral Q of A. Let w be the mapping which is induced from the canonical rectangle $R(0, M, M+i, i)$ of Q onto the canonical rectangle $R'(0, M', M'+i, i)$ of $f(Q)$. Because of the conformal invariance of the dilatation quotient, D_w is also majorized by K. Hence

$$| w_x |^2 \leq \max | \partial_\alpha w |^2 \leq K J_w$$

and the desired result $M' \leq KM$ follows by use of a length area reasoning:

$$\begin{aligned} M' = m(R') = \iint_R J_w(z) dx dy &\geq \frac{1}{K} \iint_R | w_x(z) |^2 \, dx dy \\ &\geq \frac{1}{MK} \int_0^1 dy \left(\int_0^M | w_x(z) | \, dx \right)^2 \geq \frac{M'^2}{MK}. \end{aligned}$$

□

Theorem 4.0.25. *An orientation preserving diffeomorphism f is K-quasiconformal if and only if the dilatation condition*

$$\max_\alpha | \partial_\alpha f(z) | \leq K \min_\alpha | \partial_\alpha f(z) |$$

holds everywhere.

A real function u is said to be absolutely continuous on lines or ACL in A, if for each closed oriented rectangle

$$R = [a, b] \times [c, d] \subset A$$

$u(x + iy)$ is absolutely continuous in x for almost all $y \in [c, d]$ and absolutely continuous in y for almost all $x \in [a, b]$. A complex valued function f is said to be ACL in A if its real and imaginary parts are ACL in A.

Remark 4.0.26. *If a homeomorphism $f\colon D \longrightarrow D'$ is ACL in A, then it has finite partial derivatives a.e. in A and hence has a differential a.e..*

Theorem 4.0.27. *An orientation preserving homeomorphism f of a domain A is K-quasiconformal if*

(1) f is ACL in A,

(2) $\max_\alpha | \partial_\alpha f(z) | \leq K \min_\alpha | \partial_\alpha f(z) |$ *a.e. in A.*

This theorem yields the analytic definition of quasiconformality.

PROOF. To prove that f is K-quasiconformal, we consider a quadrilateral Q, $\overline{Q} \subset A$, and its image $f(Q)$. We have to show that the modules $M = M(Q)$ and $M' = M(f(Q))$ satisfy the inequality

$$M' \leq KM.$$

Let f_2 be the canonical mapping of $f(Q)$ onto the rectangle $R_2 = \{u + iv : 0 < u < M', 0 < v < 1\}$, and f_1 the inverse of the canonical mapping of Q onto the rectangle $R_1 = \{\xi + i\eta : 0 < \xi < M, 0 < \eta < 1\}$. The composed mapping $f^* = f_2 \circ f \circ f_1$ is an orientation preserving homeomorphism of R_1 onto R_2, which can be extended to the boundary. Our next step will be to show that f^* satisfies condition 1 and 2 in R_1.

The mapping $f \circ f_1$ is absolutely continuous on lines in R_1, and the equation

$$\partial_\alpha (f \circ f_1)(\zeta) = f_1'(\zeta) \partial_{\alpha+\beta} f(f_1(\zeta)),$$

where $\beta = \arg f_1'(\zeta)$, holds almost everywhere in R_1 for every direction α.

Since f_2 is conformal, $f^* = f_2 \circ f \circ f_1$ is also absolutely continuous on lines in R_1. Further we have

$$\partial_\alpha f^*(\zeta) = f_2'(f(f_1(\zeta))) f_1'(\zeta) \partial_{\alpha+\beta} f(f_1(\zeta))$$

for almost all $\zeta \in R_1$. Since the derivatives f_2' and f_1' are independent of the direction α, and f satisfies condition 2, this condition is also satisfied by f^*.

The inequality $M' \leq KM$ can now be proved as follows. Since f^* satisfies condition 2, we have $| f_\xi^* |^2 \leq KJ^*$ almost everywhere in R_1, where J^* is the Jacobian of f^*. It follows that

$$\iint_{R_1} | f_\xi^* |^2 \, d\sigma \leq K \iint_{R_1} J^* d\sigma \leq K m(R_2) = KM'. \tag{4.0.4}$$

By Fubini's theorem

$$\iint_{R_1} | f_\xi^* |^2 \, d\sigma = \int_0^1 d\eta \int_0^M | f_\xi^* |^2 \, d\xi,$$

and by Schwarz's inequality

$$\int_0^M | f_\xi^* |^2 \, d\xi \geq \frac{1}{M} \left(\int_0^M | f_\xi^* | \, d\xi \right)^2 .$$

Since f^* is continuous on the boundary of R_1, we have

$$\int_0^M | f_\xi^*(\xi + i\eta) | \, d\xi \geq | f^*(M + i\eta) - f^*(i\eta) | \geq M' \tag{4.0.5}$$

for every η for which f^* is absolutely continuous on every closed subsegment of $I = \{\xi + i\eta : 0 < \xi < M\}$. Since f^* satisfies condition 1 in R_1, 4.0.5 holds for almost all η, $0 < \eta < 1$. Our inequality $M' \leq KM$ now follows from 4.0.4-4.0.5.

□

Let $f: A \to A'$ be a K-quasiconformal mapping and $z \in A$ a point at which f is differentiable. Since

$$\max | \partial_\alpha f | = | \partial f | + | \bar{\partial} f |, \quad \min | \partial_\alpha f | = | \partial f | - | \bar{\partial} f |,$$

the dilatation condition is equivalent to the inequality

$$| \bar{\partial} f(z) | \leq \frac{K-1}{K+1} | \partial f(z) | . \tag{4.0.6}$$

Suppose, in addition that $J_f > 0$. Then $\partial f(z) \neq 0$ and we can form the quotient

$$\mu(z) = \frac{\bar{\partial} f(z)}{\partial f(z)}.$$

The function μ, so defined a.e. in A, is called the complex dilatation of f.
By 4.0.6

$$| \mu(z) | \leq k = \frac{K-1}{K+1} < 1$$

almost everywhere in A.

We shall determine the complex dilatation of a composed mapping $g \circ f$. There is the usual trouble with the notation which is most easily resolved by introducing an intermediate variable $\zeta = f(z)$.

The usual rules are applicable and we find

$$(g \circ f)_z = (g_\zeta \circ f) f_z + (g_{\overline{\zeta}} \circ f) \overline{\overline{f}_z},$$

$$(g \circ f)_{\overline{z}} = (g_\zeta \circ f) f_{\overline{z}} + (g_{\overline{\zeta}} \circ f) \overline{\overline{f}_{\overline{z}}}.$$

They give

$$g_\zeta \circ f = \frac{1}{J}[(g \circ f)_z \overline{f}_{\overline{z}} - (g \circ f)_{\overline{z}} \overline{f}_z],$$

$$g_{\overline{\zeta}} \circ f = \frac{1}{J}[(g \circ f)_{\overline{z}} f_z - (g \circ f)_z f_{\overline{z}}],$$

where $J = \mid f_z \mid^2 - \mid f_{\overline{z}} \mid^2$. We obtain

$$\mu_g \circ f = \frac{f_z}{\overline{f}_{\overline{z}}} \frac{\mu_{gof} - \mu_f}{1 - \overline{\mu}_f \mu_{gof}}.$$

If g is conformal, then $\mu_g = 0$ and we find

$$\mu_{gof} = \mu_f.$$

If f is conformal, $\mu_f = 0$ and

$$\mu_g \circ f = \left(\frac{f'}{\mid f' \mid}\right)^2 \mu_{gof}.$$

In any case, the dilatation is invariant with respect to all conformal transformation. If we set $g \circ f = h$, we find

$$\mu_{hof^{-1}} \circ f = \frac{f_z}{\overline{f}_{\overline{z}}} \frac{\mu_h - \mu_f}{1 - \overline{\mu}_f \mu_h} \tag{4.0.7}$$

Theorem 4.0.28. *(Uniqueness theorem) Let f and g be quasiconformal mappings of a domain A whose complex dilatations agree a.e. in A. Then $f \circ g^{-1}$ is a conformal mapping.*

PROOF. By 4.0.7, the complex dilatation of $f \circ g^{-1}$ vanishes a.e.. So $f \circ g^{-1}$ is conformal. □

Conversely, if $f \circ g^{-1}$ is conformal, we conclude from 4.0.7 that f and g have the same complex dilatation.

A measurable function μ which satisfies

$$\underset{z}{ess \sup} \mid \mu(z) \mid < 1$$

is called a Beltrami differential in its domain. Let μ be a Beltrami differential in its domain. The differential equation

$$\overline{\partial} f = \mu \partial f$$

is called a Beltrami equation.

If f is conformal, μ vanishes identically, and the Beltrami equation becomes the Cauchy-Riemann equation

$$\overline{\partial} f = 0.$$

Theorem 4.0.29. *A homeomorphism f is K-quasiconformal if and only if f is a solution of the Beltrami equation, where $\|\mu\|_\infty < 1$ for almost all z.*

4.0.5. Existence theorem

A quasiconformal mapping f of a domain D induces a bounded function μ_f on D which satisfies $ess\ \sup_z \mid \mu(z) \mid < 1$.

We denote by $L^\infty(D)$ the complex Banach space of all essentially bounded measurable functions on a domain D. Here, the norm is given by

$$\|\mu\|_\infty = ess \sup_{z \in D} \mid \mu(z) \mid, \quad \mu \in L^\infty(D).$$

Let $B(D)$ be the open unit ball $\{\mu \in L^\infty(D) : \|\mu\|_\infty < 1\}$ of $L^\infty(D)$, and call any element of $B(D)$ a Beltrami coefficient on D.

Theorem 4.0.30. *For every Beltrami coefficient $\mu \in B(C)$, there exist a homeomorphism f of $\hat{C}$ onto $\hat{C}$ which is a quasiconforml mapping of C with complex dilatation μ. Moreover, f is uniquely determined by the following normalization:*

$$f(0) = 0, \quad f(1) = 1, \quad f(\infty) = \infty.$$

We call this f, uniquely determined by the normalization conditions, the canonical μ-quasiconformal mapping of $\hat{C}$, or the canonical quasiconformal mapping of $\hat{C}$ with complex dilatation μ, and denote it by f^μ.

Proposition 4.0.31. *Let μ be an arbitrary element of $B(H)$. Then there exists a quasiconformal mapping ω of H onto H with complex dilatation μ. Moreover, such a mapping ω (which can be extended to a homeomorphism of $\overline{H} = H \cup \hat{R}$*

onto itself) is uniquely determined by the following normalization conditions:

$$\omega(0) = 0, \quad \omega(1) = 1, \quad \omega(\infty) = \infty.$$

We call this unique ω satisfying the normalization conditions the canonical μ-quasiconformal mapping of H, and denote it by ω^{μ}.

PROOF. To show the existence, set

$$\hat{\mu}(z) = \begin{cases} \mu(z), & z \in H \\ 0, & z \in R \\ \overline{\mu(\overline{z})}, & z \in H^* \end{cases}.$$

The canonical $\hat{\mu}$-quasiconformal mapping $f^{\hat{\mu}}$ of $\hat{C}$ satisfies

$$f^{\hat{\mu}}(z) = \overline{f^{\hat{\mu}}(\overline{z})}.$$

In particular, we see that $f^{\hat{\mu}}(\hat{R}) = \hat{R}$. Since $f^{\hat{\mu}}$ preserves orientation, $f^{\hat{\mu}}(H) = H$. Hence, the restriction of $f^{\hat{\mu}}$ onto H is the desired one. The uniqueness follows by 4.0.7 and the normalization conditions. □

4.0.6. Quasiconformal mappings of Riemann surfaces

A homeomorphism f between two Riemann surfaces S_1 and S_2 is called K-quasiconformal if for any local parameters h_i of an atlas on S_i, $i = 1, 2$, the mapping $h_2 \circ f \circ h_1^{-1}$ is K-quasiconformal in the set where it is defined. The mapping f is quasiconformal if it is K-quasiconformal for some finite $K \geq 1$.

Suppose that the local parameters h_1, k_1 of S_1 have overlapping domains U_1, V_1 and that $f(U_1 \cap V_1)$ lies in the domains of the local parameters h_2, k_2 of S_2. Using the notation $g = h_1 \circ k_1^{-1}$, $h = k_2 \circ h_2^{-1}$, we then have in $k_1(U_1 \cap V_1)$,

$$k_2 \circ f \circ k_1^{-1} = h \circ (h_2 \circ f \circ h_1^{-1}) \circ g.$$

The mapping h and g are conformal and therefore do not change the complex dilatation.

4.0.7. Quasiconformal deformation of Fuchsian groups

Let G be a group of Möbius transformations. A Beltrami coefficient μ on C is called a Beltrami coefficient on G provided that

$$\mu(g(z))\frac{\overline{g'(z)}}{g'(z)} = \mu(z), \quad g \in G.$$

Then for every $g \in G$, the function $\omega^{\mu}(g(z))$ is a μ-quasiconformal automorphism of $\hat{C}$. Hence, there is a Möbius transformation g_1 with $\omega^{\mu} \circ g = g_1 \circ \omega^{\mu}$. Let $G^{\mu} = \omega^{\mu} G (\omega^{\mu})^{-1}$.

The mapping $G \longrightarrow G^{\mu}$ given by $g \longrightarrow \omega^{\mu} \circ g \circ (\omega^{\mu})^{-1}$ is called a quasiconformal isomorphism defined by μ, or a μ-quasiconformal deformation.

If G is a Fuchsian group and μ is a Beltrami coefficient on G, then G^{μ} is called a quasi-Fuchsian group. If μ also satisfies the condition $\mu(\overline{z}) = \overline{\mu(z)}$, then G^{μ} is again Fuchsian.

4.0.8. Complex dilatation on Riemann surfaces

We can arrive at the complex dilatation of a quasiconformal mapping of a Riemann surface, by lifting the given quasiconformal mapping to a mapping between the universal covering surfaces. Let (D, π_i) be a universal covering surface of $S_i, \quad i = 1, 2$, and G_i the covering group of D over S_i. Consider a lift $\omega \colon D \longrightarrow D$ of the given quasiconformal mapping $f \colon S_1 \longrightarrow S_2$. Since the projection π_1 and π_2 are holomorphic local homeomorphism, ω is quasiconformal. Let μ be the complex dilatation of ω. Because $\omega \circ g \circ \omega^{-1}$ is conformal for every $g \in G_1$, the mapping ω and $\omega \circ g$ have the same complex dilatation. We obtain from 4.0.7

$$\mu = (\mu \circ g)\frac{\overline{g'}}{g'} \tag{4.0.8}$$

for every transformation g. Consequently, a quasiconformal mapping f of a Riemann surface S_1 determines a Beltrami differential for the covering group. This differential is called the complex dilatation of f.

Theorem 4.0.32. *Let μ be a Beltrami differential on a Riemann surface S. Then there is a quasiconformal mapping of S onto another Riemann surface with* *complex dilatation μ. The mapping is uniquely determined up to a conformal mapping.*

PROOF. We consider μ as a Beltrami differential for the covering group G of D over S. By the existence theorem, there is a quasiconformal mapping $f\colon D \to D$ with complex dilatation μ. Since 4.0.8 holds, f and $f \circ g$ have the same complex dilatation for every $g \in G$. Then $f \circ g \circ f^{-1}$ is conformal, and we conclude that f induces an isomorphism of G onto the Fuchsian group $G' = \{f \circ g \circ f^{-1} : g \in G\}$. If π and π' denote the canonical projections of D onto S and $S' = D/G'$, then $\phi \circ \pi = \pi' \circ f$ defines a quasiconformal mapping ϕ of S onto S'. This mapping has the complex dilatation μ.

Let ψ be another quasiconformal mapping of S with complex dilatation μ and $w\colon D \to D$ its lift. Then $w \circ f^{-1}\colon D \to D$ is conformal, and so its projection $\psi\colon \phi^{-1}$ is also conformal. □

Theorem 4.0.33. *A Beltrami differential of S defines a conformal structure on S.*

PROOF. Let μ be a Beltrami differential on a Riemann surface S, and h an arbitrary local parameter on S with domain V. From the existence theorem, it follows that there is a complex valued quasiconformal mapping ω of $h(V)$ with complex dilatation $\mu \circ h^{-1}$. Then $f = \omega \circ h$ is a quasiconformal mapping of V into the plane with complex dilatation μ. If f_1 and f_2 are two such mappings with intersecting domains V_1 and V_2, then by the uniqueness theorem, $f_2 \circ f_1^{-1}$ is conformal in $f_1(V_1 \cap V_2)$. □

4.0.9. Universal Teichmüller space

Let us consider the family of all quasiconformal mappings of a fixed domain in the plane. We assume that this domain is the upper half-plane. We will introduce additional structure to this family and begin by regarding two mappings f and g as equivalent if they differ by a conformal mapping, that is, if there exists a conformal mapping h from $f(H)$ to $g(H)$ such that $g = h \circ f$. In view of the

Riemann mapping theorem, we may then restrict ourselves to self-mappings of H as a universal cover of Riemann surfaces and require that they are normalized so as to keep fixed the three boundary points 0, 1, ∞. We denote by F the family of such normalized mappings. Every element of F can be extended to a homeomorphic self-mapping of the closure of H. It is actually the extended mappings to which the normalization requirements apply.

By the existence and uniqueness theorems, there is a one to one correspondence between F and the open unit ball B of the Banach space which consists of all L^∞-functions on H.

Definition 4.0.34. *Two mappings of the family F are equivalent if they agree on the real axis. The complex dilatation of equivalent mappings are also said to be equivalent. The set of equivalence classes is the universal Teichmüller space T.*

We thus have two models for T: Its points are classes of equivalent mappings in the family F or of equivalent functions on the ball B.

4.0.9.1. *Metric on the universal Teichmüller space*

The universal Teichmüller space has a natural metric. We obtain this metric by measuring the distance between quasiconformal mappings in terms of their maximal dilatation.

The distance between the points p and q of T is defined by

$$\tau(p,q) = \frac{1}{2}\min\{\log K_{g\circ f^{-1}} \ : \ f \in p,\ g \in q\},$$

where K denotes the maximal dilatation and $f, g \in F$. This is called the Teichmüller distance between p and q. This metric is non-negative and symmetric, and $\tau(p,p) = 0$. If $\tau(p,q) = 0$, it follows that the mapping $g \circ f^{-1}$ is conformal. Because of the normalization, this mapping is the identity. Hence $f = g$, which implies $p = q$. So τ makes T into a metric space.

If f and g have complex dilatation μ and ν, the norm of the complex dilatation of $g \circ f^{-1}$ is equal to $\|\frac{(\mu - \nu)}{1 - \overline{\mu}\nu}\|_\infty$. Therefore, in terms of complex dilatation, the

Teichmüller distance assumes the form

$$\tau(p,q) = \frac{1}{2}\min\{\log\frac{1+\|\frac{(\mu-\nu)}{1-\overline{\mu}\nu}\|_\infty}{1-\|\frac{(\mu-\nu)}{1-\overline{\mu}\nu}\|_\infty} \;:\; \mu\in p,\ \nu\in q\}.$$

Lemma 4.0.35. *Every Cauchy sequence $\{[f_n]\}$ in (T,τ) contains a subsequence whose points are represented by complex dilatations μ_n with the following properties:*

(1) $\lim \mu_n(z) = \mu(z)$ exists almost everywhere;

(2) $[f_{\mu_n}] \to [f_\mu]$ in the Teichmüller metric.

Proof. In order to simplify the notation, we renumber functions each time that we pass from a sequence to its subsequence. Also, we write $f_n = f_{\mu_n}$.

Let $([f_n])$ be a Cauchy sequence in (t,τ). We shall construct inductively a subsequence with the properties 1 and 2 using suitably chosen mappings f_n.

First, fix a mapping f_i so that

$$\min\log K_{f_{i+l}\circ f_i^{-1}} < \frac{1}{2}, \quad l = 1,2,\dots,$$

Where for each l, the minimum is taken over all mappings of $[f_{i+l}]$. Since $([f_n])$ is a Cauchy sequence, such a mapping f_i exists. We renumber the sequence by setting $f_i = f_1$.

After this, we choose for every $n > 1$ the mapping f_n from its equivalence class so that

$$\log K_{f_n\circ f_1^{-1}} < \frac{1}{2}.$$

>From this new sequence (f_n) we choose a mapping f_k so that

$$\min\log K_{f_{k+l}\circ f_k^{-1}} < \frac{1}{4},$$

where again for each l the infimum is taken over all mappings of the class $[f_{k+l}]$. We set $f_k = f_2$, and for $n > 2$, choose a representative of $[f_n]$ so that

$$\log K_{f_n\circ f_2^{-1}} < \frac{1}{4}.$$

Continuing this procedure we obtain a sequence (f_n), such that $([f_n])$ is a subsequence of the given Cauchy sequence and such that, for any two consecutive indexes, the maximal dilatation satisfy the inequality

$$\log K_{f_{n+1} \circ f_n^{-1}} < 2^{-n}, \quad n = 1, 2, \ldots .$$

It follows that

$$\log K_{f_{n+1} \circ f_n^{-1}} \leq \sum_{j=1}^{l} 2^{-(n+j-1)} < 2^{-n+1}, \tag{4.0.9}$$

for $n,\, l = 1, 2, \ldots$.

Considering the connection between the maximal dilatation and the norm of the complex dilatation, we deduce from 4.0.9 that the complex dilatations μ_n of f_n satisfy the inequality

$$\|\mu_{n+l} - \mu_n\|_\infty \leq 2\|\frac{\mu_{n+l} - \mu_n}{1 - \overline{\mu_n}\mu_{n+l}}\|_\infty < 2 \tanh 2^{-n}.$$

Thus (μ_n) is a Cauchy sequence in L^∞. Since L^∞ is complete, the limit $\mu = \lim \mu_n$ exists in L^∞. Thus the validity of condition 1 follows. From 4.0.9 we conclude that the mappings f_n are K-quasiconformal for a fixed K. It follows that $\|\mu\|_\infty < 1$. Therefore $[\mu] = \lim[\mu_n]$. This means that the statement 2 is true. □

Theorem 4.0.36. *The universal Teichmüller space is complete.*

PROOF. By the statement 2 in lemma 4.0.35, if a Cauchy sequence contains a convergent subsequence, then the sequence itself is convergent. □

4.0.10. Teichmüller space

We shall now generalize the notion of the universal Teichmüller space and define the Teichmüller space for an arbitrary Riemann surface.

Let us consider all quasiconformal mappings f of a Riemann surface S onto other Riemann surfaces. If two such mappings f_1 and f_2 are declared to be equivalent in the first sense, whenever the Riemann surfaces $f_1(S)$ and $f_2(S)$ are conformally equivalent. The collection of equivalence classes form the Riemann space R_s.

We introduce another equivalence. Let f_1 and f_2 be quasiconformal mappings of a Riemann surface S. Then f_1 and f_2 are said to be equivalent in the second sense, if $f_2 \circ f_1^{-1}$ is homotopic to a conformal mapping of $f_1(S)$ onto $f_2(S)$.

Definition 4.0.37. *The Teichmüller space T_s of the Riemann surface S is the set of the equivalence classes, in the second sense, of quasiconformal mappings of S.*

Theorem 4.0.38. *If $S = H$ then T_s agrees with the universal Teichmüller space.*

PROOF. In applying the above definition of T_s to $S = H$, we first note that all quasiconformal images of H are conformally equivalent. It follows that we may consider only the normalized quasiconformal self mappings f of H with complex dilatation μ. The condition that $f^{\mu_2} \circ (f^{\mu_1})^{-1}$ be homotopic to a conformal mapping is fulfilled if and only if $f^{\mu_2} \circ (f^{\mu_1})^{-1}$ agrees with the identity mapping on the real axis R.

Consequently, f^{μ_1} is equivalent to f^{μ_2} by the above definition if and only if $f^{\mu_1} |_R = f^{\mu_2} |_R$. By the definition of universal Teichmüller space, this is the condition for f^{μ_1} and f^{μ_2} to determine the same point in the universal Teichmüller space. □

The definition of The Teichmüller space T_s can also be formulated in terms of the Beltrami differentials on S. Every quasiconformal mapping of S determines a Beltrami differential on S, namely, its complex dilatation. Conversely, if μ is a Beltrami differential of S, then by theorem 4.0.32 there is a quasiconformal mapping of S whose complex dilatation is μ, and by the uniqueness part of that theorem, all such mappings determine the same point of T_s. Two Beltrami differentials are said to be equivalent if the corresponding quasiconformal mappings are equivalent. Hence, a point of T_s can be thought of as a set of equivalent Beltrami differentials.

4.0.11. Teichmüller space as a subset of the universal space

For a Riemann surface S, we defined the Teichmüller space T_s by means of quasiconformal mappings of S onto Riemann sufaces. Lifting the mappings to mappings between the universal covering surfaces leads to new characterizations of T_s and makes it possible to see the connection between the general space T_s and the universal Teichmüller space. We impose on the Riemann surface S the restriction that it has a half-plane as its universal covering surface. We denote by f^μ the uniquely determined quasiconformal self mapping of H which has the complex dilatation μ and which keeps fixed the points 0, 1, ∞ on the real axis.

Theorem 4.0.39. *The Beltrami differentials μ and ν of S are equivalent if and only if $f^\mu |_R = f^\nu |_R$.*

PROOF. Let us first assume that μ and ν are equivalent. Let ϕ and ψ be quasiconformal mappings of S which lift to f^μ and f^ν, respectively. Then there is a conformal map η: $\phi(S) \longrightarrow \psi(S)$ such that $\eta \cap \phi$ is homotopic to ψ. So we have $f^\nu = h \circ f^\mu$ on R where h as a lift of η, is a Möbius transformation. Since f^μ and f^ν both fix 0, 1, ∞, it follows that h is the identity.

Suppose conversely that $f^\mu = f^\nu$ on the boundary R. Then f^μ and f^ν induce the same isomorphism of the covering group of H over S onto a Fuchsian group G'. The projection of f^μ and f^ν maps S onto the same Riemann surface H'/G'. These projections are homotopic. So μ and ν are equivalent. □

This theorem says that

$$[\mu] \longrightarrow f^\mu |_R$$

is a well-defined injective mapping of the Teichmüller space. In particular, T_s can be characterized as the set of equivalence classes $[f^\mu]$, two mappings being equivalent if they agree on R. We have arrived at the situation which was the starting point of the definition of universal Teichmüller space.

In the general case the complex dilatations of the mappings f^μ are Beltrami differentials for the covering group G. If G is trivial, then T_s is the universal Teichmüller space.

Let S_1 and S_2 be Riemann surfaces and G_1 and G_2 the covering groups of H over S_1 and S_2. If G_1 is a subgroup of G_2, then $T_{s_2} \subset T_{s_1}$. In particular, every

Teichmüller space T_s can be regarded as a subset of the universal Teichmüller space.

4.0.12. Teichmüller metric

Exactly as in the case of the universal Teichmüller space, we define the distance

$$\tau(p,q) = \frac{1}{2}\min\{\log K_{gof^{-1}} \ : \ f \in p,\ g \in q\}$$

between the points p and q of the Teichmüller space. (T_s, τ) is a metric space.

The Teichmüller distance can be expressed in terms of Beltrami differentials:

$$\tau(p,q) = \frac{1}{2}\min\{log\frac{1+\|\frac{\mu-\nu}{1-\overline{\mu}\nu}\|_\infty}{1-\|\frac{\mu-\nu}{1-\overline{\mu}\nu}\|_\infty} \ : \ \mu \in p,\ \nu \in q\}.$$

Let τ and τ_s denote the Teichmüller metrics in the universal Teichmüller space T and the Teichmüller space T_s, respectively. Then the restriction $\tau\ |_{T_s}$ is also a metric in T_s. From the definition of τ and τ_s it follows immediately that

$$\tau\ |_{T_s} \leq \tau_s.$$

T_s does not inherit its metric from the universal Teichmüller space: The metrics τ_s and $\tau\ |_{T_s}$ need not be the same.

Lemma 4.0.35 is true in every Teichmüller space T_s: A Cauchy sequence in (T_s, τ_s) contains always a subsequence whose points have representatives μ_n such that $\lim \mu_n(z) = \mu(z)$ exists almost everywhere, $[f_{\mu_n}] \to [f_\mu]$ in the τ_s-metric.

The proof is the same as in lemma 4.0.35. In this case every μ_n is a Beltrami differential for G, i.e. $(\mu_n \circ g)\frac{\overline{g'}}{g'} = \mu_n$. From $\mu_n(z) \to \mu(z)$ almost everywhere it follows that the $\lim \mu$ also is a Beltrami differential for G, i.e. $[\mu]$ is a point of T_s. From this we obtain a generalization of theorem 4.0.36:

Theorem 4.0.40. *The Teichmüller space (T_s, τ_s) is complete.*

4.0.13. Modular group

Theorem 4.0.41. *The Teichmüller space of two quasiconformally equivalent Riemann surfaces are isometrically bijective.*

PROOF. Let S and S' be Riemann surfaces and h a quasiconformal mapping of S onto S'. The mapping

$$f \longrightarrow f \circ h^{-1}$$

is a bijection of the family of all quasiconformal mappings f of S onto the family of all quasiconformal mappings of S'. If $\omega_i = f_i \circ h^{-1}$, we have $\omega_2 \circ \omega_1^{-1} = f_2 \circ f_1^{-1}$. We first conclude that f_1 and f_2 determine the same point of T_s if and only if ω_1 and ω_2 determine the same point in $T_{s'}$, i.e.

$$[f] \longrightarrow [f \circ h^{-1}] \tag{4.0.10}$$

is a bijective mapping of T_s onto $T_{s'}$. □

Definition 4.0.42. *Let h be a quasiconformal self-mapping of S. Then 4.0.10 defines a bijective isometry of T_s onto itself. The group $Mod(S)$ of all such isomorphisms $[f] \to [f \circ h^{-1}]$ of T_s is called the modular group of T_s.*

Theorem 4.0.43. *The Riemann space is the quotient of the Teichmüller space by the modular group.*

PROOF. Assume first that the points $[f]$ and $[g]$ of T_s are equivalent under $Mod(S)$. We then have a quasiconformal mappings $h\colon S \to S$ such that $f \circ h^{-1}$ is equivalent to g. But this means that there is a conformal mapping of $f(S)$ onto $g(S)$, i.e. f and g determine the same point of R_s.

Conversely, let f and g represent the same point of R_s. Then a conformal mapping $\phi\colon f(S) \to g(S)$ exists, and $h = g^{-1} \circ \phi \circ f$ is a quasiconformal self-mapping of S. From $g = \phi \circ (f \circ h^{-1})$ we see that g and $f \circ h^{-1}$ determine the same point of T_s. □

Chapter 5

COMPLEX AND ALMOST COMPLEX STRUCTURE ON THE SPHERES

The absence of an almost complex structure on S^{4k} for $k \geq 1$ and S^{2n} for $n \geq 4$ was proved by Wu [**28**] and jointly by Borel and Serre [**7**] respectively. Kirchhoff [**18**] has shown that if S^n admits an almost complex structure, then S^{n+1} admits an absolute parallelism, and Adams [**1**] that S^{n+1} admits an absolute parallelism only for $n+1 = 1$, 3 and 7. The result of Adams combined with that of Kirchhoff implies the result of Wu, Borel and Serre. It is well known that the six-dimensional sphere S^6 admits the structure of an almost complex manifold [**10**]. On the other hand, for a given almost complex structure on the 6-sphere, necessary conditions were given in order that it define a complex structure (Ehresmann and Libermann [**9**], Eckmann and Fröhlicher [**8**]).

5.1. Quaternions and octonions

Quaternions are a non-commutative extension of complex numbers. Every quaternion is uniquely expressible in the form $a + bi + cj + dk$ where a, b, c and d are real numbers and i, j and k satisfy:

	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

We denote the algebra of quaternions by H.

Quaternions produce the usual three-dimensional vector product. In general, if we represent a vector (a_1, a_2, a_3) as the quaternion $a_1 i + a_2 j + a_3 k$, we obtain the vector product of two vectors by taking their product as quaternions and deleting the real part of the result (the real part will be the negative of the dot product of the two vectors).

The algebra of octonians, O, has a basis $(I, e_0, e_1, \ldots, e_6)$, where I is called the unit element of O. The products $e_i \cdot e_j$ are given by the equations

$$e_i^2 = -I, \quad e_i \cdot e_j = -e_j \, (i \neq j;\, i, j = 0, 1, \ldots, 6).$$

So we have the following multiplication table:

	e_0	e_1	e_2	e_3	e_4	e_5	e_6
e_0	$-I$	e_2	$-e_1$	e_4	$-e_3$	e_6	$-e_5$
e_1	$-e_2$	$-I$	e_0	$-e_5$	e_6	e_3	$-e_4$
e_2	e_1	$-e_0$	$-I$	e_6	e_5	$-e_4$	$-e_3$
e_3	$-e_4$	e_5	$-e_6$	$-I$	e_0	$-e_1$	e_2
e_4	e_3	$-e_6$	$-e_5$	$-e_0$	$-I$	e_2	e_1
e_5	$-e_6$	$-e_3$	e_4	e_1	$-e_2$	$-I$	e_0
e_6	e_5	e_4	e_3	$-e_2$	$-e_1$	$-e_0$	$-I$

The algebra O is nonassociative; for instance, $(e_1 \cdot e_2) \cdot e_3 \neq e_1 \cdot (e_2 \cdot e_3)$ since the left-hand side is e_4, and the right-hand side is $-e_4$. A vector product for 7-dimensional vectors can be obtained in the same way by using the octonions instead of the quaternions.

5.2. An almost complex structure on the 2 and 6-sphere

In this section we consider the three dimensional case R^3, where the vector cross product derives from the multiplicative properties of imaginary quaternions. Indeed, as a linear space imaginary quaternions coincide with R^3. Thus the unit sphere $S^2 \subset R^3$ is isomorphic with the space of unit imaginary quaternions, and it acquires a natural almost complex structure from the action of the quaternionic vector cross product in R^3. We have exactly the same approach for the 6-sphere; S^6, when considered as the set of unit norm imaginary octonions, inherits an almost complex structure from the octonion multiplication.

Explicitly this quaternionic or octonionic almost complex structure on S^2 or S^6 is constructed as follows:

The dot or internal product $\langle w, v \rangle$ can be defined for any two vectors w and v in R^n. However, we shall show that the cross product $w \times v$, can only be defined for vectors in R^3 or R^7. The cross product has the following properties:

(1) $w \times v = -v \times w$.

(2) $w \times v = 0$ if and only if v is a multiple of w.

(3) $w \times v$ is orthogonal to both w and v.

(4) $w \times (w \times v) = \langle w, v \rangle w - \langle w, w \rangle v$.

These properties of the cross product allows us to see the sphere S^2 and S^6 as almost complex manifolds. Recall that S^2 (resp. S^6) is the set of all vectors w in R^3 (R^7) such that $\langle w, w \rangle = 1$. Let v be any vector in R^3 (resp. R^7) which is tangential to S^2 (resp. S^6) at the point w, so that $\langle v, w \rangle = 0$. Then $w \times v$ is tangential to S^2 (resp. S^6) at w and $w \times (w \times v) = -v$. Hence, the transformation which sends v to $w \times v$ is linear and its square $w \times (w \times v) = -v$ is equal to $-I$, so it defines an almost complex structure on S^2 (resp. S^6).

Here we explain the case S^6 in more detail **[12]** **[22]**. In order to introduce a vector product in R^7, we shall consider R^7 as the hyperplane of R^8 consisting of the imaginary octonions. A general element of O may be written as

$$xI + X, \quad x \in R,$$

where R is the set of all real numbers, and

$$X = \sum_{i=0}^{6} x^i e_i, \quad x^i \in R, \quad i = 0, 1, \ldots, 6.$$

If $x = 0$, the element is called a purely imaginary octonian number. All octonian numbers form an 8-dimensional vector space, which we denote by O, over the real numbers, and all purely imaginary octonian numbers form a 7-dimensional subspace R^7 of O. Let

$$Y = \sum_{i=0}^{6} y^i e_i, \quad y^i \in R, \quad i = 0, 1, \ldots, 6.$$

Then we define

$$X \cdot Y = -\langle X, Y\rangle I + X \times Y,$$

where

$$\langle X, Y\rangle = \sum_{i=0}^{6} x^i y^i,$$
$$X \times Y = \sum_{i \neq j} x^i y^j \, e_i \cdot e_j$$

are respectively the scalar product and the vector product of X and Y in R^7.

The operation of the vector product is bilinear. Now let us write

$$e_i \cdot e_j = \sum_{k=0}^{6} \varepsilon_{ij}^k e_k, \quad i, j = 0, 1, \ldots, 6.$$

Then from the octonions multiplication table, it follows that $\varepsilon_{ij}^k = -\varepsilon_{ji}^k$ are $0, 1$ or -1, and that

$$X \times Y = \sum_k \sum_j x_j^k y^j e_k,$$

where

$$x_j^k = \sum_{i=0, i\neq j}^{6} x^i \varepsilon_{ij}^k.$$

From the multiplication table, we obtain the whole matrix (x_i^j):

$$\begin{pmatrix} x_0^0 & x_1^0 & x_2^0 & x_3^0 & x_4^0 & x_5^0 & x_6^0 \\ x_0^1 & x_1^1 & x_2^1 & x_3^1 & x_4^1 & x_5^1 & x_6^1 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 & x_6^3 \\ x_0^4 & x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 & x_6^4 \\ x_0^5 & x_1^5 & x_2^5 & x_3^5 & x_4^5 & x_5^5 & x_6^5 \\ x_0^6 & x_1^6 & x_2^6 & x_3^6 & x_4^6 & x_5^6 & x_6^6 \end{pmatrix} = \begin{pmatrix} 0 & -x^2 & x^1 & -x^4 & x^3 & -x^6 & x^5 \\ x^2 & 0 & -x^0 & x^5 & -x^6 & -x^3 & x^4 \\ -x^1 & x^0 & 0 & -x^6 & -x^5 & x^4 & x^3 \\ x^4 & -x^5 & x^6 & 0 & -x^0 & x^1 & -x^2 \\ -x^3 & x^6 & x^5 & x^0 & 0 & -x^2 & -x^1 \\ x^6 & x^3 & x^4 & -x^1 & x^2 & 0 & -x^0 \\ -x^5 & -x^4 & -x^3 & x^2 & x^1 & x^0 & 0 \end{pmatrix}.$$

Thus we can show that

$$X \times Y = -Y \times X,$$

and that $X \times Y$ is orthogonal to both X and Y, that is,

$$\langle X, X \times Y\rangle = 0, \quad \langle Y, X \times Y\rangle = 0.$$

Moreover,

$$X \times (X \times Y) = \langle X, Y\rangle X - \langle X, X\rangle Y.$$

Now that we have a vector product in R^7, we can go back and repeat the construction of an almost complex structure on the 6-sphere. Consider the unit 6-sphere S^6 in R^7:

$$S^6 = \{x \in R^7 : \langle X, X\rangle = 1\}.$$

The tangent space $T_X(S^6)$ of S^6 at $X \in S^6$ can naturally be identified with the subspace of R^7 orthogonal to X. Define the endomorphism J_X on $T_X(S^6)$ by

$$J_X Y = X \times Y \quad for \quad Y \in T_X(S^6).$$

Then

$$J_X^2 Y = J_X(X \times Y) = X \times (X \times Y) = -Y,$$

which implies that $J_X^2 = -I$; here I is the identity operator. Thus the correspondence $X \longrightarrow J_X$ defines a J such that $J^2 = -I$, and hence defines an almost complex structure on S^6.

5.3. Dimension of a composition algebra

The preceding construction of almost complex structure only works on the spheres S^2 and S^6. This is due to the fact, that R^3 and R^7 are the only vector spaces where one can define an antisymmetric bilinear cross product of vectors. The existence of a vector cross product on R^3 and R^7 reflects the fact that besides real and complex numbers, quaternions and octonions are the only normed division algebras.

In this section, we show that the possible dimensions of a composition algebra are 1, 2, 4 or 8 [**26**]. Our starting problem is to understand composition algebras, then, instead of composition algebras we look at the equivalent notion of vector product algebras. These algebras can be obtained by rewriting the axioms of a composition algebra in terms of the pure vectors.

5.4. Composition algebras and vector products

Definition 5.4.1. *A composition algebra consists of a vector space C together with*

(1) a symmetric bilinear form $\langle\, ,\rangle$ on C,

(2) a linear map $C \otimes C \to C$, $x \otimes y \mapsto x \cdot y$,

(3) an element $0 \neq e \in C$,

(4) $e \cdot x = x \cdot e = x$,

(5) with $N(x) = \langle x, x \rangle$, $N(x \cdot y) = N(x)\, N(y)$.

Definition 5.4.2. *A vector product algebra consists of a vector space* V *together with*

(1) a symmetric bilinear form $\langle\, , \rangle$ *on* V,

(2) a linear map $V \otimes V \to V$, $x \otimes y \mapsto x \times y$,

(3) $\langle x \times y, z\rangle$ *is alternating in* x, y, z ,

(4) $(x \times y) \times x = \langle x, x\rangle y - \langle x, y\rangle x$.

Vector product algebras and composition algebras are equivalent notions. Namely, given a composition algebra C, let $V = \langle e\rangle^{\perp}$ and put

$$x \times y = \frac{1}{2}(x \cdot y - y \cdot x).$$

Conversely, given a vector product algebra V, put $C = \langle e\rangle \perp V$ and define the product on C by

$$(ae + x) \cdot (be + y) \;=\; (ab - \langle x, y\rangle)e + ay + bx + x \times y.$$

This equivalence between composition algebras and vector product algebras seems to provide a convenient way to comprise some well known rules in composition algebras.

5.5. The contraction of $\langle\, , \rangle$

Let V be a finite-dimensional vector product algebra and let $(e_i)_i$ be an orthonormal basis of V. Put

$$d \;=\; \sum_i \langle e_i, e_i\rangle.$$

In the following we will apply the third property of definition 5.4.2 in the formulation

$$\langle x \times y, z\rangle \;=\; \langle x, y \times z\rangle, \tag{5.5.1}$$

$$y \times x = -x \times y. \tag{5.5.2}$$

The fourth property of definition 5.4.2 will be used also in the following forms which are obtained by polarizing and from the third property:

$$(x \times y) \times z + x \times (y \times z) = 2\langle x, z\rangle y - \langle x, y\rangle z - \langle z, y\rangle x, \tag{5.5.3}$$

$$\langle x \times y, z \times t\rangle + \langle y \times z, t \times x\rangle = 2\langle x, z\rangle\langle y, t\rangle - \langle x, y\rangle\langle z, t\rangle - \langle y, z\rangle\langle t, x\rangle. \tag{5.5.4}$$

Other relations to be used are

$$\sum_i e_i \times (v \times e_i) = \sum_i \langle e_i, e_i\rangle v - \sum_i \langle e_i, v\rangle e_i = dv - v = (d-1)v \tag{5.5.5}$$

and

$$\sum_{i,j} \langle e_i \times e_j, e_i \times e_j\rangle = \sum_{i,j} \langle e_i, e_j \times (e_i \times e_j)\rangle = (d-1)\sum_i \langle e_i, e_i\rangle = d(d-1). \tag{5.5.6}$$

We first consider vector product algebras which correspond to associative composition algebras.

Proposition 5.5.1. *Suppose that*

$$(x \times y) \times z = \langle x, z\rangle y - \langle y, z\rangle x \tag{5.5.7}$$

holds. Then $d(d-1)(d-3) = 0$.

PROOF. Consider

$$A = \sum_{i,j,k} \langle e_i \times (e_k \times e_i), e_j \times (e_k \times e_j)\rangle.$$

By 5.5.5 we have

$$A = \sum_k (d-1)^2 \langle e_k, e_k\rangle = d(d-1)^2.$$

On the other hand, using 5.5.7 and 5.5.6 one finds

$$\begin{aligned} A &= \sum_{i,j,k} \langle (e_i \times (e_k \times e_i)) \times e_j, e_k \times e_j\rangle \\ &= \sum_{i,j,k} \langle \langle e_i, e_j\rangle e_k \times e_i - \langle e_k \times e_i, e_j\rangle e_i, e_k \times e_j\rangle \\ &= \sum_{i,k} \langle e_k \times e_i, e_k \times e_i\rangle - \sum_{i,j,k} \langle e_k \times e_i, e_j\rangle\langle e_i \times e_k, e_j\rangle \\ &= 2\sum_{i,k} \langle e_k \times e_i, e_k \times e_i\rangle = 2d(d-1). \end{aligned}$$

So

$$0 = A - A = d(d-1)(d-3).$$

□

Theorem 5.5.2. *Let V be a finite dimensional vector product algebra. One has the relation*

$$d(d-1)(d-3)(d-7) = 0.$$

PROOF. Put

$$h(u,v) = \sum_i (u \times e_i) \times (e_i \times v).$$

The following formula has been introduced by T.A. Springer.

$$h(u,v) = (d-4) u \times v. \tag{5.5.8}$$

To check it one uses 5.5.3 with $x = u$, $y = e_i$ and $z = e_i \times v$ and finds

$$\begin{aligned} h(u,v) &= -\sum_i u \times (e_i \times (e_i \times v)) + 2\sum_i \langle u, e_i \times v\rangle e_i \\ &\quad -\sum_i \langle u, e_i\rangle e_i \times v - \sum_i \langle e_i \times v, e_i\rangle u \\ &= (d-1)u \times v + 2\sum_i \langle v \times u, e_i\rangle e_i - u \times v - \sum_i \langle v, e_i \times e_i\rangle u \\ &= (d-1)u \times v - 2u \times v - u \times v - 0 = (d-4)u \times v. \end{aligned}$$

Formulas 5.5.8 and 5.5.6 make it easy to compute the sum

$$\begin{aligned} B &= \sum_{i,k} \langle h(e_i, e_k), h(e_k, e_i)\rangle \\ &= (d-4)^2 \sum_{i,k} \langle e_i \times e_k, e_k \times e_i\rangle = -d(d-1)(d-4)^2. \end{aligned}$$

We nest compute B in a different way. One has

$$B = \sum_{i,j,k,l} \langle (e_i \times e_j) \times (e_j \times e_k) \times (e_k \times e_l) \times (e_l \times e_i)\rangle.$$

Applying 5.5.4 shows

$$B + B' = 2C - D - D',$$

where

$$\begin{aligned}
B' &= \sum_{i,j,k,l} \langle (e_j \times e_k) \times (e_k \times e_l) \times (e_l \times e_i) \times (e_i \times e_j) \rangle, \\
C &= \sum_{i,j,k,l} \langle e_i \times e_j, e_k \times e_l \rangle \langle e_j \times e_k, e_l \times e_i \rangle, \\
D &= \sum_{i,j,k,l} \langle e_i \times e_j, e_j \times e_k \rangle \langle e_k \times e_l, e_l \times e_i \rangle, \\
D' &= \sum_{i,j,k,l} \langle e_j \times e_k, e_k \times e_l \rangle \langle e_l \times e_i, e_i \times e_j \rangle.
\end{aligned}$$

By reindexing one finds $B = B'$ and $D = D'$. Therefore

$$B = C - D.$$

We compute C and D:

$$\begin{aligned}
C &= \sum_{i,j,k,l} \langle e_i, e_j \times (e_k \times e_l) \rangle \langle (e_j \times e_k) \times e_l, e_i \rangle \\
&= \sum_{j,k,l} \langle e_j \times (e_k \times e_l), (e_j \times e_k) \times e_l \rangle \\
&= \sum_{j,k,l} \langle (e_j \times (e_k \times e_l)) \times ((e_j \times e_k) \times e_l) \rangle \\
&= -\sum_{k,l} \langle h(e_k \times e_l, e_k), e_l \rangle = -(d-4) \sum_{k,l} \langle (e_k \times e_l) \times e_k, e_l \rangle \\
&= -(d-1)(d-4) \sum_{l} \langle e_l, e_l \rangle = -d(d-1)(d-4), \\
D &= \sum_{i,j,k,l} \langle e_i, e_j \times (e_j \times e_k) \rangle \langle (e_k \times e_l) \times e_l, e_i \rangle \\
&= \sum_{j,k,l} \langle e_j \times (e_j \times e_k), (e_k \times e_l) \times e_l \rangle \\
&= \sum_{k} (d-1)(d-1) \langle e_k, e_k \rangle = d(d-1)^2.
\end{aligned}$$

Hence

$$B = -d(d-1)(d-4) - d(d-1)^2 = -d(d-1)(2d-5).$$

Finally

$$
\begin{aligned}
0 = B - B &= -d(d-1)(2d-5) + d(d-1)(d-4)^2 \\
&= d(d-1)(d^2 - 10d + 21) \\
&= d(d-1)(d-3)(d-7).
\end{aligned}
$$

□

5.6. Complex structure on 2-dimensional manifolds

From the Newlander-Nirenberg theorem that we mentioned in chapter 2, we shall show any two-dimensional almost complex structure is integrable. A familiar example is the two dimensional sphere S^2 which inherits its complex structure from the complex plane C as we saw in chapter 3. But in higher dimensions a given almost complex structure on a manifold is not necessarily integrable, and an almost complex manifold does not need to be a complex manifold.

We show that $N_J(X, Y)$ vanishes, for every almost complex structure J on a 2-dimensional orientable manifold M. For any vector field X on M, we have

$$N_J(X, X) = J[X, X] - J[JX, JX] - [JX, X] - [X, JX] = 0.$$

On the other hand, in a neighborhood of a point where X is not 0, every vector field Y is a linear combination of X and JX. Hence $N_J = 0$, proving our assertion.

CONCLUSION

After introducing an almost complex structure on the 2-sphere and the 6-sphere and relinquishing the study of complex structure on the 6-sphere, our new objective was to study complex structures on surfaces and construct the equivalence classes of these structures. We solved this problem by studying the complex structures on Euclidean spaces. We even obtained the result for infinite dimensional vector spaces by defining the **C**-isomorphisms.

We continued constructing complex structures on manifolds, based on the property that locally look like Euclidean spaces, and proved that two complex structures on a differentiable manifold are equivalent if the corresponding almost complex structures are equivalent. Then we focused on Riemann surface as one dimensional complex manifolds and we represented (most of) them as quotient spaces of the upper half plane by Fuchsian groups. By this way, we arrived at Riemann's moduli space, which consists of the conformal equivalence classes of Riemann surfaces.

Along the way, we modified our definition of equivalent Riemann surface structures. The new equivalence relation was stricter than simply biholomorphic equivalence. Indeed, two complex structures on the topological base surface, both being quasiconformally diffeomorphic to the initial Riemann surface structure, were defined to be Teichmüller equivalent if they were biholomorphically equivalent via a quasiconformal mapping which was also required to be homotopic to the identity map. Finally, The relation between moduli space and Teichmüller space was shown by using modular groups.

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