Université de Montréal

Induction and plausibility:
A formal approach from the standpoint of artificial intelligence

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Thèse présentée à la Faculté des études supérieures en vue de l'obtention du grade de PhD en Philosophie

Janvier, 2005

Université de Montréal
Faculté des études supérieures

Cette thèse intitulée :

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RÉSUMÉ

Le but de ce travail est analyser la notion d'induction, comprise comme la classe des inférences rationnelles qui ne préservent pas la vérité au sens de la logique non monotone telle que pratiquée en théorie de l'Intelligence Artificielle (IA) dans les vingt-cinq dernières années. En concentrant nos efforts sur le problème de la description de l'induction, nous voulons clarifier la notion d'induction (au sens de Carnap) à l'aide d'une approche purement descriptive et par conséquent exempte des problèmes liés à la justification de l'induction. La notion capitale de plausibilité est comprise ici à la lumière de la notion carnapienne de probabilité pragmatique. En fournissant une analyse formelle et purement descriptive de la notion d'induction, nous avons l'intention aussi de clarifier la notion ordinaire de plausibilité. Une des principales caractéristiques de ce rapport entre induction et plausibilité concerne les deux approches fondamentales qu'on peut adopter pour traiter les contradictions qui surgissent invariablement dans le traitement des inférences inductives. Ces approches sceptiques et crédules de l'induction, comme nous les avons appelées, impliquent deux concepts différents de plausibilité qui touchent directement aux ressources formelles de la logique de l'IA, la logique paraconsistante et la logique paracomplète: alors que la plausibilité sceptique est une notion paracomplète, la plausibilité crédule est une notion paraconsistante. À l'encontre des approches formelles développées en logique philosophique, l'aspect purement descriptif est à l'avant-scène dans la plupart des logiques non monotones. Nous choisissons deux de ces formalismes – la logique du défaut de Reiter et le formalisme paraconsistante de Pequeno – et nous les élaborons de façon à obtenir un système capable d'exécuter la tâche qu'une logique inductive purement descriptive est supposée remplir. Pour compléter notre travail formel, nous développons aussi une logique modale paranormale (c.-à-d., simultanément paracomplète et paraconsistante) pour représenter nos deux notions de plausibilité en logique non monotone. De cette façon, notre travail est aussi une contribution à la logique modale. Pour montrer l'applicabilité de notre système, nous présentons une formalisation du raisonnement abductif et hypothético-déductif appliqué au problème de la confirmation des hypothèses en philosophie des sciences.

MOTS CLÉS: logique philosophique, logique paraconsistente, logique paracomplete, logique modale paranormale, logique nonmonotone, logique *default*, logique inductive, probabilité pragmatique, modèle hypothético-déductif, ambiguïté inductive.

ABSTRACT

The purpose of this work is to analyze the notion of induction, conceived as the class of rational non-truth preserving inferences, from the point of view of the nonmonotonic logical tradition raised inside the field of Artificial Intelligence (AI) in the last twenty-five years. By centering our efforts around what has been called the problem of description of induction, we intend to explicate (in Carnap's sense) the notion of induction through a purely descriptive and consequently justificatory-free approach to induction. Of fundament importance for this enterprise is the notion of plausibility, here understood as the same as Carnap's notion of pragmatical probability. By providing a representational formal analysis of the notion of induction, we also aim to explicate something akin to the ordinary notion of plausibility. One of the main features of this relationship between induction and plausibility concerns the two most basic approaches one can take when dealing with the contradictions that are sure to arise from the use of inductive inferences. These skeptical and credulous approaches to induction, as we have named them, give rise to two different plausibility notions which bear important relations to a domain of formal logic closely connected with AI, the field of paraconsistent and paracomplete logic: while the skeptical plausibility is a paracomplete notion, the credulous plausibility is a paraconsistent one. At the basis of our formal work is the result that in opposition to the formal approaches developed in philosophy, the justificatory-free aspect we are looking for is already present in most nonmonotonic logics. We pick two of these formalisms – Reiter's default logic and Pequeno's paraconsistent default logics – and extend them in such a way as to obtain a system able to perform the task a descriptive logic of induction is supposed to perform. To complete our formal work, we also develop a so-called paranormal (i.e., simultaneously paracomplete and paraconsistent) modal logic to represent the two notions of plausibility and act in conjunction with the mentioned nonmonotonic logic. In this way, our work is also a contribution to the field of modal logic. In order to show the applicability of our system, we present a formalization of the so-called abductive reasoning and hypothetico-deductive reasoning applied to the problem of confirmation of hypotheses in philosophy of science.

KEY WORDS: philosophical logic, paraconsistent logic, paracomplete logic, paranormal modal logic, nonmonotonic logic, default logic, inductive logic, pragmatical probability, hypothetico-deductivism, inductive ambiguity.

Contents

LIST OF ABBREVIATIONS

Symbol	*Meaning*
$=$	Equality
$\neq$	Non-equality
$\equiv$	Syntactical identity
$\in$	Belongs to
$\notin$	Does not belong to
$\subset$	In contained in
$\subseteq$	$\subset$ or $=$
$\cup$	Set union
$\cap$	Set intersection
$\varnothing$	Empty set
$>$	Greater than
$<$	Smaller than
$\geq$	$>$ or $=$
$\leq$	$<$ or $=$
■	End of proof

To my wife Eugênia (Gaura Priya devi dasi) and my daughters Ekarani and Jahnavi, both of whom were born in the middle of my struggle to complete this thesis

ACKNOWLEDGEMENTS

During the research for this thesis and its completion I have been greatly indebted to many people. In special, I would like to thank my supervisors Dr. Tarcisio Pequeno from the Department of Computer Science of the Federal University of Ceará, Brazil and Dr. Yvon Gauthier from the Philosophy Department of the University of Montreal, Dr. Marcelino Pequeno from the Department of Computer Science of the Federal University of Ceará, Brazil, Dr. Arthur Buchsbaum from the Department of Computer Science of the Federal University of Santa Catarina, Brazil, Mr. Manoel Orley, secretary of the Department of Computer Science of the Federal University of Ceará, Brazil, my parents Mr. Armando and Ms. Vanda Silvestre, for the unconditional support that they have always given to me, and all the members of ISKCON Montreal, in special Visvambhara P and Nitai P. and his wife Tulasi M.. Finally, I would like to thank my teacher H. H. Hridayananda das Goswami (Dr. Howard J. Resnick.) for the priceless inspiration to find out the real meaning of the word "rational" and apply it in all aspects of practical life. I am also greatly indebted to the Brazilian National Council of Scientific and Technological Advancement (CNPq) for supporting me with a four-year doctoral fellowship.

Montréal, January, 2005.

CHAPTER 1

INTRODUCTION

This work is primordially concerned with what in the recent philosophical literature has been named constructivist conceptual analysis or concept explication. It consists basically in trying to explicate or clarify a vague, imprecise notion (called the *explicandum*), which may belong to everyday language or to a previous stage in the development of scientific or philosophical language, by replacing it with a clearer, more precise notion (called the *explicatum*) formulated in a systematic context. In our case there will be two *explicanda*: the notion of *induction* and the notion of *plausibility*.

This practice of conceptual clarification is, of course, not new. There have been, in the history of Western philosophy, considerable efforts to analyze concepts in the formulation of problems, and historical research shows many cases where these efforts have been of great importance to eventual achievements. What is rather new is the awareness that the clarification or analysis of concepts is an indispensable step in the process of tackling philosophical problems. As a consequence of that, many philosophers have spent much of their efforts trying to clarify the concepts which they were dealing with and, which is certainly far more remarkable, laying down what we may call theories of concept explication, that is to say, attempts to explicate what the task of concept explication is.

Concerning the different ways this task of explicating a concept may be undertaken, at least three different theoretical positions can be distinguished[1]: the *essentialistic position* (Husserl), which defends that many important concepts may seem at first sight rather vague but yet they have a clear meaning, although it is very difficult to grasp that meaning; the *adaptivistic position* (Wittgenstein, Ryle and Austin), which says that concepts usually have precise rules of usage, and therefore a clear "meaning in use," although it is not easy to make the rules explicit; and the one we are concerned with here, the *constructivistic position*, which defends that most concepts do not have a clear meaning, but they can get one (or more) if our attempts at creating such clear meaning are successful. So therefore the necessity, in the constructivistic case, of having a notion defined in a reasonably precise way (the *explicatum*) whose meaning can be easily grasped and which will itself be the solution for the problem of clarifying the *explicandum*.

[1] Kuipers (1978).

Examples of such constructivistic conceptual analysis are not hard to find in the philosophical literature. Most philosophers agree that Church's proposal to replace the vague term "effectively calculable function" by the mathematically exact term "general recursive function" provides an especially clear example of this sort of concept explication[2]. Other examples usually given are the biologists' replacement of the ordinary-language term "fish" by the scientific term "piscis"[3], Tarski's definition of "true"[4] and the Wiener-Kuratowski reconstruction of the term "ordered pair"[5].

Concerning the several theories of (constructivistic) concept explication proposed by philosophers, surely the most influential of all has been the one proposed by Rudolf Carnap in the first chapter of his "Logical Foundations of Probability."[6] According to Carnap, the very first step we must take before explicating a concept is to make at least practically clear what is meant as the *explicandum*. If the *explicandum* belongs, for example, to a previous stage in the development of scientific or philosophical language, then we should make clear that what we are trying to explicate is such and such term used in such and such way by such and such theorists. Besides this, in order to be an adequate *explicatum*, a particular notion must satisfy four basic criteria: it has to be as precise and as simple as possible, it has to be useful in the sense that it gives rise to the formulation of theories and the solution of problems, and it has to be similar to *explicandum*.

Besides the vagueness and lack of generality, criteria of *explicatum* adequacy such as Carnap's are far from being consensual. For instance, a consequence of Carnap's fourth condition is that the final *explicatum* may have properties which we do not recognize in the *explicandum*. In fact, Carnap goes as far as claiming that "close similarity is not required, and considerable differences are permitted."[7] This seems to be also the position of W. V. Quine and Hans Reichenbach in their theories of concept explication[8]. On the other hand, Nelson Goodman and Alfred Tarski, for example, place considerable emphasis on the similarity in meaning of the reconstructed and unreconstructed concepts[9]. For instance, when speaking about his formal analysis of the notion of truth, Tarski says that "the desired definition does not aim to specify the meaning of a familiar word used to denote a novel notion; on the contrary, it aims to catch hold of the actual meaning of an old notion".[10]

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[2] Hanna (1968).
[3] Carnap (1962).
[4] Ibid and Tarski (1944).
[5] Quine (1960).
[6] Carnap (1962).
[7] Ibid, p. 7.
[8] Quine (1960) and Reichenbach (1938). See Hanna (1968).
[9] Goodman (1965) and Tarski (1944). See Hanna (1968).
[10] Tarski (1944), p. 53.

As far as we are concerned, due to space reasons (and also to our skepticism about the tenability of such sort of "explication of explication" theory), we will not take the trouble of developing a theory of concept explication. Rather, we will take for granted the "concept explication clarification" given by the several theories of concept explication in general and Carnap's in particular as well as by the many attempts to constructively analyze concepts made by theoreticians and lay down our solution for the concept explication problems we are concerned about here[11]. If however knowing the class of concept explication attempts which our *explicata* belong to helps, the solutions for our concept explication problems fall under what is commonly known as *philosophical logic*. More specifically, by reducing our *explicanda* to certain types of sentences, we will develop logical systems which internally incorporate these sentences and consequently explicate through some of their internal features (such as the logical form of the sentences in question and the axioms and semantic rules related to them) our *explicanda*.

As an example of such sort of concept explication, consider traditional modal logic, which is commonly presented in textbooks as the logic of necessity and possibility. By providing a logical analysis of sentences of the type "α is necessary" ($\Box\alpha$) and "α is possible" ($\Diamond\alpha$) modal logic in general and systems like T, S4 and S5 in particular can be quite fairly taken as formalizations of the notions of necessity and possibility. In fact, as exemplified by the work of many theoreticians, the basic framework of modal logic has been shown to be an extremely fruitful way of analyzing concepts which, akin to the notions of necessity and possibility, can be embedded in modal sentences. By interpreting $\Box\alpha$ as "α morally ought to be the case," for example, modal logic D was originally proposed as a formalization of the notion of obligatoriness or moral necessity (D stands for "deontic")[12]. By giving a temporal interpretation of $\Box$ and $\Diamond$ and adding one more pair of such temporal modalities, philosophers have analyzed past and future sentences and created the so-called temporal modal logic[13]. By including symbols to represent agents, some theoreticians have proposed modal logical systems intended to deal with the notions of knowledge and belief ($a\Box\alpha$ will mean something like "agent *a* knowns α")[14] as well as with the notion of agentive action ($a\Box\alpha$ will mean something like "agent *a* acts in such a way as to make α true")[15]. Finally, by making use of the semantic structure used by these logics of action (the so-called structure of ramified time[16]) and combining temporal and alethic modalities, philosophers were able represent the notion of

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[11] Only in the last chapter we will examine to what extent our *explicata* satisfy Carnap's requirements
[12] See Gammut (1991).
[13] See Burgess (1984) and Gabbay et al (1995).
[14] See Gabbay et al (1995).
[15] Chellas (1992) and Belnap & Perloff (1988). See also Silvestre (1998).
[16] Thomason (1970).

historical necessity or inevitability[17]. And even if some of these theoreticians have never made use of the term "explication" in their writings, it is clear that their systems in some very important sense were intended to clarify the concepts they were concerned with[18].

The mention of such modal logic *explicata* is particularly important because the *explicatum* we will give to one of our *explicanda*, the notion of plausibility, belongs to the class of logics known as modal logic. Akin to the examples we have just described, by interpreting modal sentences of the form $\alpha?$ as "α is plausible"[19] and making use of what we shall call the *plurality approach to plausibility*, we will introduce modal logical systems that can be said to explicate the notion of plausibility. It should be said however that there is an important difference between this modal logic of plausibility and the known modal logical systems. Due to the key concept we will analyze here, i.e. induction, the plausibility notion we will be primarily concerned with will be what we can call *inductive plausibility*. When taken in connection with the two most basic ways one can look at the inconsistencies that are sure to arise from the use of inductive inferences[20] – the skeptical and credulous approaches to induction –, this inductive plausibility can be shown to be in fact not one, but two notions: what we call *skeptical plausibility* and *credulous plausibility*. These two plausibility notions will correspond in our system to a pair of dual modalities. Now, a conclusion that arises after a little analysis of these notions is that while the skeptical plausibility is a *paracomplete* notion, the credulous plausibility is a *paraconsistent* one[21]. This makes our system to be closely connected with the recent and growing domain of formal logic concerned with logics that are both paraconsistent and paracomplete: the so-called *paranormal logic*[22]. Thus we have named the logical system through which we intend to explicate the notion(s) of plausibility *paranormal modal logic*[23].

This paranormal modal logic is what we shall call a *general explicatum* of our plausibility *explicandum*. Due to its being not a single but a class of logical systems (akin to the class of normal modal logics), it sets the most basic features any *explicatum* of the notion of plausibility following

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[17] See Thomason (1984) and Vanderveken (1998).

[18] Of course the possible-world style semantics of these logics plays a very crucial role in this clarification endeavour.

[19] As generally done in non-alethic modal logics, we shall not use the traditional square and diamond symbols to represent the notion of plausibility. Rather, we shall use the symbols ! and ? according to a post-fixed notation

[20] This is the so-called problem of inductive inconsistencies, which has been cached the attention of both philosophers and Artificial Intelligence theorists. See for instance Hempel (1960), Coffa (1974), Israel (1980), Perlis (1987) and Pequeno (1990).

[21] The terms "paraconsistent" and "paracomplete" have been so far used almost exclusively in connection with logical systems or components of logical systems. See Arruda (1980) and Loparic & da Costa (1984). In Chapter 3 we shall try to clarify this "conceptual" use of the terms "paraconsistent" and "paracomplete".

[22] Paranormal logics have been studied for example in connection with problems of knowledge representation in Artificial Intelligence. See Rescher & Brandom (1980), Buchsbaum & Pequeno (1993) and Béziau (2001).

the plurality approach to plausibility should possess. What we shall call the *specific* plausibility *explicatum* will be a particular paranormal modal logic containing in addition to a pair of modalities representing the notions of skeptical and credulous plausibility, a pair of modalities intended to represent the notions of *certainty* and *epistemological possibility*. The necessity of considering these two notions is due to the fact that one of the distinguishing features of plausibility, namely its uncertain, refutable character, can be satisfactorily represented inside a monotonic framework only when it is associated with a notion which embodies the opposite feature: certainty or irrefutability.

The other *explicanda*, the notion of induction, will be analyzed through a logical system which falls into the class of nonmonotonic formalism known in Artificial Intelligence (AI) circles as default logic[24]. This is another particularity of our explication endeavor. Even though the similarities between some problems faced in AI practice and some classical ones dealt with within philosophical investigation have been pointed by some theoreticians[25], the effective contribution of ideas, methods and techniques from AI to philosophy and vice-versa is still something hard to be seen. By attacking the classical philosophical problem of concept explication through a formal model based on existing formalisms motivated by reasoning needs in Artificial Intelligence, we present what we believe to be a bridge between these two areas of knowledge that, in addition to its own interest, can also serve as an example and an illustration of a whole lot of connections we hope to come over[26].

Our strategy will be basically to extend two default systems, Reiter's default logic and Pequeno's paraconsistent default logic, in such a way as to allow the full representation of what we call *inductive implication* sentences, i.e., sentences of the form "α inductively implies β unless φ." From the most basic level, these sentences are nothing more than a new interpretation of Reiter's default rules. Instead of reading default

$$\frac{\alpha : \neg\varphi}{\beta}$$

as "if α is the case and it is consistent to assume $\neg\varphi$, then conclude β," we take it as meaning "α inductively implies β unless φ." This is represented in our formalism by $\alpha \succ \beta \precnsim \varphi$. The relation between inductive implications and plausible statements is that the latter generally appears as consequent of the former. This is in accordance with the commonly held view that one of the

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[23] It should be noted that, as far as we known, this paranormal modal logic is a unique contribution of ours to the field of formal logic.
[24] See Reiter (1980) and Poole (1994).
[25] See for example Pollock (1988), Kyburg (1994), Tan (1991), Ford et al.(1995) and Tan (1997).

distinguishing features of induction is that rather than rendering its conclusion certain, it just make them plausible, probable, acceptable, etc. Formally this is achieved by allowing plausibility modal formulae to appear in the consequent of inductive implications (in symbols: $\alpha \succ \beta? \not\precsim \varphi$.) Thus we see the import of having a plausibility *explicandum* for the proper explication of the notion of induction. We also allow the representation of what we can call *calculus of inductive implication*, i.e., a set of logical principles supposedly governing the use of inductive implications. As we shall see, this is a necessary component of the logic of induction that cannot be built inside the basic framework provided by nonmonotonic logics[27].

Our system's belonging to the class of default logics becomes especially relevant when we consider the sort of analysis of the notion of induction we want to develop here. Taken as a class of inferences, induction has two main features: a negative and a positive. While on the one hand the elements of this class are non truth-preserving inferences whose conclusions are uncertain and refutable, on the other hand they must embody some sort of rationality, allowing us to distinguish them from fallacies, which are also non truth-preserving but completely nonsense. Therefore, in the task of analyzing induction philosophers have distinguished two different approaches: one that tries to account for or justify the positive, rational side of inductive inferences, and one which ignores the necessity of such justification and tries simply to describe non truth-preserving patterns of inference traditionally taken as embodying some sort of rationality. While the former sort of analysis has been shown to be extremely problematic and, according to some, even untenable[28], the latter, which is the one we will follow here, has not yet been satisfactorily made (much of which because of the inability of philosophers to keep distance from justificatory issues[29].) Now, since their very genesis nonmonotonic formalisms in general and default logics in particular have centered exclusively on the task of representing inductive inferences without any concern at all about the problem of justifying them. Therefore, by basing our explication endeavors on such sort of formalism, we believe our approach is much more likely to succeed in the task of developing a purely descriptive account of induction than the available philosophical theories of induction[30].

The structure of the thesis is as follows. In Chapter 2 and 3 we try to make practically clear the notions which we intend to explain here. At the same time that by surveying the pertinent literature we try to find the relations that may exist between our *explicanda* and the different accounts philosophers have given of the notions of induction and plausibility, by laying down something

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[26] Some of our efforts in this direction have already appeared in print. See Silvestre & Pequeno (2003) and Silvestre & Pequeno (2004)

[27] Reiter's default logic allows just the representation of what we can call basic inductive implications, that is, formulae of the form $\alpha \succ \beta \not\precsim \varphi$ such that α, β and φ are all ordinary formulae.

[28] See Salmon (1966), chapter II, for instance.

[29] See Glennan (1994) and Fitelsen (2004), for example.

akin to a philosophy of induction and plausibility we set in a systematic way the main features of the concepts we shall try to explicate. This is done in Chapter 2. In Chapter 3 we finish this pre-explication elucidation, focusing primordially on the notion of plausibility. By investigating the so-called inductive ambiguities, we introduce what we shall take as the philosophical basis of our notion of plausibility, the plurality approach to plausibility, as well as the two plausibility concepts implied by it: the skeptical and credulous notions of plausibility. In Chapter 4 we switch from the conceptual problem of induction and plausibility to the way we may try to formally solve it, analyzing some approaches both in the philosophy of science and Artificial Intelligence. It will be in that chapter that we shall investigate to what extent nonmonotonic logics can be useful in accomplishing the task of developing a purely descriptive account of induction. In Chapter 5 we effectively begin our explicative endeavor, laying down what we have called the general *explicatum* of the notion of plausibility: paranormal modal logic. In Chapter 6 we introduce our general *explicatum* of the notion of induction. After presenting our specific plausibility *explicatum*, we use it to introduce several logics of induction and plausibility built inside our general induction *explicatum* and which can be taken as specific *explicata* of the notion of induction. By using one of these logical systems, we show how some quite famous models of confirmation found in the literature of philosophy of science (including the so-called hypothetico-deductive model) can be represented inside our formalism. Finally in Chapter 7 we lay down some conclusive remarks about what we have achieved in the preceding chapters.

[30] See Chapter 4.

CHAPTER 2

ON INDUCTION AND PLAUSIBILITY

In this Chapter, we will try to clarify the basic problem we want to solve here: the explication of the notions of induction and plausibility. We start by carrying out a brief historical analysis of the several meanings which philosophers have attributed to the term "induction." From that, we come to the notion of induction we intend to formally explicate – induction as ampliative or non-truth preserving inferences. All that will be done in Section 2.1. In Section 2.2 we talk about the most influential school of induction of the twentieth century – Carnap's logical school – and the role the notion of probability has played in it. Finally, in Section 2.3 we identify the second notion we aim to explicate – the notion of plausibility – with a specific probability concept dealt with by logical probabilists: the so-called pragmatical notion of probability. It will also be in this section that we will start laying down the philosophy of induction and plausibility that we will try to formally systematize in subsequent chapters.

2.1 From Inductive Generalization to Ampliative Inferences

The most traditional use of the term "induction" is that which equates induction with what today is known as *inductive generalization*, or inference from the particular to the general. Taking a widely used example, if we observe, let us say, 100 ravens and notice that all of them are black, we may generalize that and conclude that all ravens are black. This act of inferring a general statement from particular instances is the first important feature of this traditional meaning of induction. The other is the *purpose* associated with this kind of reasoning. Induction in this sense is conceived as a way of discovering or generating hypotheses, laws or principles; or, broadly speaking, as a sort of logic of discovery.

This use of "induction" has been taken first by Aristotle (at least was him who first used a specific technical term – *epagôgê* – to refer to this inferential process[1]), to whom scientific knowledge is obtained by demonstration from indemonstrable first principles, and knowledge of these first principles is in turn obtained by induction. It is important to remark that, to Aristotle and

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[1] The term "induction" comes from Cicero, who introduced the word *inductio* as an exact equivalent for *epagôgê*.

many others after him, the generalization resultant from an induction is not necessarily of an empirical character. In the words of J. R. Milton[2]:

> Among the truths which Aristotle describes as being reached by induction [...] What we do *not* find are what we are accustomed to think of as empirical generalisations. Aristotle uses the word *epagôgê* and its derivatives over fifty times in his various writings, and the *only* example of a proposition derived by *epagôgê* which could reasonably be described as an empirical generalization is the discussion example of all bileless animals being long-lived which appears in *Prior Analytics*, II.23.

Another important conception of induction is the so-called *singular predictive induction,* or the non-demonstrative inference from the particular to the particular. Taking again our raven example, rather than concluding that all ravens are black, in a singular predictive induction we would conclude that the next raven to be observed will also be black. Historically, it seems that this sense of induction has coexisted with the inductive generalization conception. Diogenes Laertius, for example, distinguished two types of inductive inferences: from particulars to particulars and from particulars to universals.[3] Despite the obvious differences between this meaning and the first one, singular predictive induction can be very fairly taken as a particular case of inductive generalization. We will call this conception of induction understood as inductive generalization and/or singular predictive induction the *classical conception of induction.*

From the time of Aristotle's successors down to the beginning of the seventeenth century, this picture of induction remained as the dominant one. The shift to what we call the *modern conception of induction* took place in 1620 with Francis Bacon's *Novum Organum.* While in this new sense induction remained chiefly conceived as generalization from particulars and as a method of discovery, it started to be taken (as explicitly suggested by Bacon) as the chief method (of discovery) of the newly born natural sciences. Accordingly, all aspects of inductive reasoning, in special its conclusions, were taken as being *empirical* in essence. In this way, we arrive at the modern idea (still in vogue today) according to which all science starts from observation and then slowly and cautiously proceeds to theories, which consist basically of generalizations of such observations.

Another very important part of Bacon's philosophy of science is that he considered pure inductive generalization as a "puerile thing," incapable *per se* of generating any kind of knowledge. In order to generate authentic scientific knowledge, it has to be supplemented with some additional method, in Bacon's case a method of exclusion intended to obtain the right conclusion. As he puts it[4]:

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[2] Milton (1987), p. 53.
[3] There is still a third type that corresponds to a kind of *reductio ad absurdum*. Milton (1987) p. 54.
[4] Bacon (1620), p. 249.

> But the greatest change I introduce is in the form itself of induction and the judgment made thereby. For the induction of which the logicians speak, which proceeds by simple enumeration, is a puerile thing; concludes at hazard [...] Now what the sciences stand in need of is a form of induction which shall analyze experience and take it to pieces, and by the process of exclusion and rejection lead to an inevitable conclusion.

John Stuart Mill, with his methods of agreement, difference, etc, also made use of the same sort of heuristic principle[5].

This heuristic aspect of the modern conception of induction, along with its emphasis on the empirical character of premises and conclusions, is what mostly distinguishes it from the classical conception. However, as mentioned in a previous paragraph, they still share some very fundamental features. First of all, induction in both senses is primarily conceived as a method of discovery (be it of particulars or of general principles). In other words, the role of induction in the scientific enterprise is to produce *new* pieces of scientific knowledge. Another similarity is that both the classical and the modern conceptions can be classified as *structuralist* conceptions of induction, that is to say, the classification of a given reasoning as inductive is based primarily on the analysis of its syntactical structure (whether they go from particulars to general, whether it makes use of such and such heuristic principle, etc.)

There is still a third common trait between the classical and modern conceptions that, unlike the first two, seems to be a much more essential feature of induction. We are talking about the trivial fact that a conclusion got from an inductive generalization or from a singular predictive induction may be false even though their premises are true. In other words, induction either in the classical sense or in the modern sense is a *non truth-preserving* type of reasoning. The main point of course is that the conclusions of inductive generalizations (with or without some heuristic method of conclusion choice) and singular predictive induction contain information that is not contained in the premises. That I have observed 10.000 black ravens says nothing about the features of the next raven I am going to observe or about all ravens. In these cases, the conclusions go beyond what is stated in the premises; they increase our knowledge. And it is exactly this *ampliative* character of induction what makes it non truth-preserving and also so interesting[6].

Now, if the distinguishing "logical" feature of induction is that it is ampliative and consequently non truth-preserving, apart from structural or functional differences, we may say that induction is "logically" indistinguishable from other types of reasoning, such as abduction, which are ampliative too. This viewpoint led some philosophers to extend the meaning of "induction" as to make other ampliative types of reasoning fall under its label. Charles Pierce, for instance, identifies three types of induction: crude induction, quantitative induction and qualitative induction,

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[5] Mill (1896).

[6] It's noteworthy to point out that this feature is the only one which involves only *logical notions*.

where only the first one corresponds to what we have called inductive generalization.[7] If we go on with this meaning extension, we will get to the point of taking induction in a very broad sense and identifying it with the *class of all ampliative* or *non truth-preserving inferences*. That is what we call the *contemporary conception of induction*. Right away we see that this new conception places induction in sharp contrast to deduction: considering that deduction is truth-preserving and consequently non-ampliative, inductive will then mean non-deductive, and deductive non-inductive.

This conception of induction is the one we find in most standard textbooks on logic and induction[8]. It has been taken by most contemporary philosophers of science and, since the beginning of the twentieth century, profoundly influenced the philosophical analysis of induction. For instance, during the 1940's and 1970's a flourishing field of research was completely dedicated to the task of developing a so-called *logic of induction*, conceived in this sense. People like John Keynes, Rudolf Carnap, Carl Hempel, W. Johnson, Richard Jeffrey and Wesley Salmon, just to mention some of the most preeminent, although taking different approaches in their efforts to provide a system of induction, all agreed on the basic point of what we are calling the contemporary conception of induction.

Now, this contemporary notion of induction embodies very significant changes in relation to the earlier conceptions. Maybe one of the most important is that for the first time induction was explicitly seen as a kind of inference or argument, in contrast to a type of reasoning. To make the difference clear, reasoning is a complex structure that, among other things, may contain arguments, definitions, conclusion choice procedures, etc. In its turn, inference is the very cornerstone of reasoning. In the traditional sense, an inference or argument[9] is a logical relation between a set of propositions and a proposition – the first called premises and the second conclusion – according to which, by its very logical nature, the first entails the second. Now, if there is such thing as inductive inference, it should be, due to its very nature, somehow susceptible to a logical analysis. More specifically, by abandoning a simply structural definition and adopting a "logical" one, this contemporary conception of induction placed induction on the same level as deduction and opened the possibility that a logic of induction akin to deductive logic can be developed. As one might expect, these changes brought into scene both those who believe in the existence of such class of inferences and want to develop a logic of induction, as well as those who deny its existence and consequently the possibility of such sort of logic[10].

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[7] Peirce (1931), pp. 756-59.

[8] Such as Copi & Cohen (2001), Hurley (2003), Layman (2002) and Hacking (2000), for instance.

[9] Even though the term "argument" may be taken as something similar to "reasoning," we will use it here in the customary way, as a synonymous of inference.

[10] See Fritz (1960), Sellars (1969) and Machina (1984).

Another significant change entailed by the contemporary conception of induction is concerned with the alleged purpose of induction. According to the classical and modern conceptions, induction was chiefly conceived as a method of discovery. This was not just a policy on the use of inductive inferences; rather, it was part of the very notion of induction. In its turn, induction as conceived by contemporary philosophers rejected this and any other sort of practical purpose. Despite the historical reasons involved, this was a direct consequence of taking induction as a sort of argument. If there is some purpose to be fulfilled in the performance of inductive inferences[11] there must be necessarily reference to procedures foreign to the inferential relation itself. Therefore, despite being possibly connected with each other, the purpose in question cannot be taken (with the risk of nullifying the logical conception) as part of the notion of induction.

So, this contemporary conception of induction is forced, by its own definition, to give up not only the discovery purpose aspect present in the two others conceptions, but actually any other kind of purpose. Induction *per se* is considered to be a purely logical notion. However it may be, this dissociation of induction from the logic of discovery aspect is, as a matter of fact, strongly connected with the great unpopularity that such line of research passed to have among philosophers of science in the twentieth century. It became a consensus to take the way scientific theories are generated simply as outside the scope of the logical analysis of science. As one of the most important critics of this sort of philosophical analysis, Karl Popper, wrote: "The initial stage, the act of conceiving or inventing a theory, seems to me neither to call for logical analysis not to be susceptible of it [...] there is no such thing as a logical method of having new ideas, or a logical reconstruction of this process."[12] [13]

But if induction is not any more conceived as a logic of scientific discovery, what then is its role in the scientific enterprise? In order to answer this question, we have first to notice that an inference (be it deductive or not) from α to β, let us say, may be read in two different ways: "β is obtained from, (somehow) contained in or implied by α" and "the truth of α warrants, supports, justify or confirms the truth of β." While the first way involves a picture where we take a particular α and from it obtain β, the second one involves taking a particular pair of sentences α and β and showing that they are inferentially connected with each other. This latter reading is particularly useful to distinguish (according to our contemporary sense) induction from deduction: while in a deductive inference the premises give full support to the conclusions, in an inductive one the support given is just partial.

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[11] Such as the determination of which hypotheses can be inductively inferred from a given set of evidences.
[12] Popper (1959), pp. 31 and 32.
[13] Even though we may say that in nowadays philosophy of science this meaning of induction as a logic of discovery is practically out of use, in some domains of Artificial Intelligence it remains as fresh as in Bacon's time. See Michalski et al (1983).

Now, if we think of the premises of our inductive inferences as being the observed data and the conclusion as the theory or hypothesis, then the first reading will be obviously related to the process of scientific discovery. However, if we decide to pick the second one and take a particular pair of observed data and hypothesis, then a different process will come up. Now our concern will not be the generation of hypotheses (for they are already there) but their *assessment*. Then, to say that a hypothesis and a body of data are inductively related to each other is to say that the second gives some kind of support (let us call it *evidential support*) to the first. Since the support is not complete but just partial, we cannot say that the observational data, now called *evidences*, demonstrate or prove the hypothesis: they just render them plausible or acceptable. So the place of induction in science is shifted from the *context of discovery* to the *context of justification*: induction is not about how we conceive or create scientific theories, but how they are confirmed by empirical evidences. In fact, to most contemporary philosophers of science, the philosophical investigation of induction is centered around the problem of confirmation of theories[14].

One should not however misunderstand this and think our conclusion that induction in this contemporary sense is purpose-free to be wrong. Of course that if we apply induction to the problem of confirming theories we will have to act according to a specific purpose. But the inductive relation that supposedly exist between hypothesis *h* and evidence *e* is already there, independently of someone even dreaming about making sure that *e* really confirms *h*. In contrast to the ancient and modern conceptions of induction, the contemporary conception of induction is completely apart from the role it plays in the scientific enterprise.

One very important thing we have not mentioned yet concerns the following question: if induction is the class of all ampliative inferences, then how about fallacies? Are they also to be included in the class of inductive arguments or treated separately? Trivially the first alternative is unacceptable: accepting fallacies as a type of inductive argument is simply to give up the soundness we expect to be present in any inductive argument. Then we are left with two options: to distinguish between good and bad induction or to redefine the notion of induction; in both cases as to take fallacies into consideration. Independently of the path we choose, the basic problem is the same: how to distinguish induction (or good induction) from fallacies. Surely the most immediate answer would be to invoke the notion of rationality and say that what distinguishes induction (or good induction) from fallacies is that while the first one is in some sense a reasonable, rational

[14] Since almost all theories of induction mentioned in this text are concerned with the problem of confirmation of hypotheses and theories, references will be given through the whole text. For an interesting anthology of papers on confirmation of theories see Curd & Cover (1998), chaps. 4 and 5.

inference, the steps from premises to conclusion in a fallacy are unwarranted. Wesley Salmon, for instance, says the following[15]:

> If, however, there is any kind of inference whose premises, although not necessitating the conclusions, do lend it weight, support it, or make it probable, then such inferences possess a certain kind of *logical rectitude*. It is not deductively valid, but it is important anyway. Inferences possessing it are correct inductive inferences.

So, this alleged "logical rectitude" is what distinguishes (good) inductive inferences from fallacies. But of course if we just say this we are not saying too much. What precisely is this logical rectitude? What warrants us to classify the inferences that possess it as rational? As one might suspect, this is the famous *problem of justification of induction*, also known as Hume's problem of induction: "How to justify the rationality of inductive arguments?"

The basic difficulty with the problem of justification of induction seems to be that justifying or showing the rationality of an argument is, we fell, tantamount to showing its logical character. Since from a strict point of view there is no logical connection between the premises and conclusion of an inductive inference, we have then a problem that threatens the very foundations of the contemporary conception of induction. In fact, since Aristotle's times, the problem of justifying the reasonableness of non-deductive arguments has been one of the main sources of suspiciousness against induction. Later on, after Hume's famous critics[16] and the recognition of its importance for the scientific method, the justification of induction has occupied a very crucial place in the philosophy of science. Bertrand Russell, for example, has provided a desperate description of the consequences of Hume's conclusion for the theory of science[17]:

> It is important to discover whether there is any answer to Hume [...] If not, there is no intellectual difference between sanity and insanity. The lunatic who believes that he is a poached egg is to be condemned solely on the ground that he is in minority [...] every attempt to arrive at general scientific laws from particular observation is fallacious [...]

Incidentally, since the publication of Hume's *A Treatise of Human Nature* in 1739 up to now, no satisfactory solution to this problem was proposed. Many philosophers go as far as claiming that it will never be: they declare it to be an insoluble problem.[18]

As we have mentioned, even though this problem affects all three conceptions of induction, the effect it has upon each one is not the same. While it may be epistemologically important for the first two conceptions to find a rational justification for the kind of reasoning they are concerned with, the result of such quest does not affect in a decisive way the way they are using to the word

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[15] Salmon (1966), p. 8. The italics are mine.
[16] Hume (1739).
[17] Russell (1946), p. 646. Clearly Russell is using here the modern conception of induction.
[18] For an exposition of the kinds of attempts made to solve the problem of justification of induction see Salmon (1966), chapter II.

"induction." In the case of the contemporary notion the situation is a little different. As we have seen, in order to properly characterize the class of inductive inferences we have to, besides giving a negative criterion (which is of course the argument's being non truth-preserving), also give a positive criterion capable to distinguish inductive arguments from other arguments which also satisfy the negative criterion (read fallacies.) And independently of our appealing or not to the notions of logical rectitude or rationality[19], to give such positive criterion amounts to solving Hume's problem.

This simple but at the same time subtle connection between induction and the problem of justification is at the root of the controversy regarding the existence of inductive arguments and the tenability of the project of building a logic of induction. From a philosophical point of view, the whole thing has to do with the very way we are trying to define the class of inductive inferences, that is to say, *intensionally*. Since we want to give a general criterion to say whether or not a given inference is inductive, we will have inevitably to deal with the problem of justifying why such and such inference is in fact inductive. And since one of the distinguishing features of induction will inevitably be the property of reasonableness or rationality (even if under another label), our criterion will have to give an answer to the question of why such and such inference is rational. Because of that, we say that this contemporary conception of induction is or embodies a sort of *intensional* or *justificatory approach to induction.*

Given all this, it is reasonable to wonder if there is not some other way of defining induction which does not require such sort of justification endeavor. To start with, independently of finding a necessary and sufficient criterion of inductiveness, there is always a class of ampliative inferences that, in a particular period of time, is used in a certain category of practical situations and accepted as sound by a certain community of people. So, one possible alternative is to define inductive inferences as *this set* of accepted ampliative arguments. That is of course an *extensional* definition. And if we still want to refer to the notion of rationality, we can take it as a sort of pragmatical notion: rational is what people of a certain community in a particular period of time consider as so. We will call this approach to the contemporary meaning of induction the *extensional* or *pragmatical approach to induction.*

One may object that this account is unacceptably subjective and leaves room for arbitrariness to come into the characterization of what we call rational. That is not quite so. Since we are making reference to a consensus shared by a community, this approach must rather be classified as inter-subjective, and it is as arbitrary as, let us say, the acceptance of the principle of causality or the existence of the external world. Anyway, if we decide to follow this path, our basic problem

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[19] Which would be quite difficult to do without going astray from the path the philosophical investigation of

regarding the task of defining the class of inductive inferences will be reduced to the problem of describing the accepted patterns of inductive inference, or, in other words, the so-called *problem of description of induction*[20].

It is interesting to observe that according to some philosophers who do not believe in the existence of a (logical) distinction between deductive and inductive arguments, those who believe in it have established such distinction not on logical grounds, but on pragmatic and epistemic ones. Kenton Machina, for instance, says: "As remarkable as it may seem, common attempts to make the primary distinction between inductive and deductive arguments have turned out to generate a pragmatic or epistemic distinction, not a logical one."[21] Later on he adds: "Perhaps, then, the following suggestion will meet with some acceptance: Recognize that the general purpose, all-embracing distinction between deductive and inductive argument belongs to epistemology and rhetoric, not logic."[22]

This somehow corroborates what we have stated above: to intensionally define inductive inferences is tantamount to solving Hume's problem. Since so far this problem has recognizably not been solved, it is expected that the attempts to define induction according to this approach have not been successful either. Therefore, if we think Hume's problem is insoluble or just do not want to go through it, the only approach we are left with is one which avoids these justificatory staffs by adopting an extensional or pragmatical approach to induction.

2.2 Logical Probability and Pragmatical Probability

Now, if we identify induction with the class of all (rational) ampliative arguments, what can we say about the conclusion of such inferences in the case where the premises are true? This question is pertinent because if we take induction as a kind of inference we will expect to *infer* something from the truthfulness of the premises. We know that in a deductive argument, if the premises are true the conclusion will also be so. However, by definition, even when its premises are true it is possible for an inductive inference to have a false conclusion. Therefore, truth does not follow from truth in this sort of inference. But then what can we conclude about the hypothesis of an inductive inference when its evidences or premises are true? Before answering this question, we will have to talk about a very important aspect of contemporary philosophy of induction without which any presentation of the subject would be incomplete: the concept of *probability*.

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induction has historically followed

[20] For a short discussion on the inductive problems of justification and description see Lipton (1991), Chapter 1.

[21] Machina (1984), p. 577.

[22] Ibid. p. 578.

If there is something consensual about induction in the philosophical literature is its connection with probability. To most contemporary authors, the philosophy of induction is essentially the same as the philosophy of probability. Even though this association of induction with probability is not new, it was only in the twentieth century that philosophers explicitly took the philosophical analysis of induction as being for all intents and purposes the same as the investigation of the concept of probability. In the preface to the first edition of his *Logical Foundations of Probability*, Rudolf Carnap expresses this view in a very explicit way: "The theory here presented is characterized by the following *basic* principles: (1) all inductive reasoning, in a wide sense of nondeductive or nondemonstrative reasoning, is reasoning in terms of probability [...]"[23]

It will be useful to name this probability concept applied to (or identified with) induction *inductive probability*. This is necessary because while inductive has almost invariably been taken as probable, the inverse does not hold. The twentieth century saw a remarkable proliferation of different ways of saying what probability is[24]. And many of these so-called interpretations of probability are not concerned at all with induction, in any sense of the word. This is manifestly true, for example, for Karl Popper's propensity interpretation of probability. For other interpretations, the situation is not that straightforward. For example, while many adherents of the relative frequency interpretation claim their theories to be dealing primarily with induction, others defend the thesis that what the frequency theories explicate is a concept intrinsically distinguished from inductive probability[25].

There is nonetheless one school of probability that has explicitly and beyond any controversy taken probability as the key concept in the philosophical investigation of induction. It is the so-called *logical school of probability*. This school has basically taken induction according to what we have named the contemporary conception of induction. In fact, it was in great part due to the efforts of some of its advocates, notably John Keynes and Rudolf Carnap, that this contemporary conception of induction got a definitive, precise shape and was established as the leading one among philosophers. Even though Keynes set down most of the basic principles of the logical school and, according to Carnap, was the first to see probability as inherently relative to a given evidence and consequently meaning something like our inductive probability[26], it was Carnap who further elaborated such principles, cleared up many misconceptions, and gave the main steps towards the chief enterprise of the logical school: the construction of a formal logic of induction.

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[23] Carnap (1962), p. v. Italics in the original.

[24] For a description of the several interpretations of probability see Weatherford (1982), Cohen (1989a), chap. II, Salmon (1966), Chap. V and Hájek (2002).

[25] See Carnap (1962), Salmon (1966), chap. VII and Nagel (1955), pp. 62-66.

[26] Carnap (1962), p. 31.

further elaborated such principles, cleared up many misconceptions, and gave the main steps towards the chief enterprise of the logical school: the construction of a formal logic of induction.

From the point of view of the systematization of principles, Carnap's masterpiece, *Logical Foundations of Probability*, of 1950, represented the great turning point in the contemporary conception of induction. There for the first time, a concise and comprehensive attempt to build a formal system of inductive logic along with a philosophical analysis of both concepts of probability and induction was presented. Carnap's project started in the 1940s[27] and was further developed by Carnap himself and others such as John Kemeny, Richard Jeffrey and Jaakko Hintikka[28] between the 1950s and 1970s. Others such as Carl Hempel[29], even though not directly working on Carnap's systems, have followed the same approach in their treatment of induction. Before Carnap, others such as John Keynes (as we have already mentioned), Harold Jeffreys and B. Koopman have given the same inductive treatment to probability. And before, in the eighteenth century, names like James Bernoulli, Thomas Bayes and Pierre Simon de Laplace were all, according to H. Jeffreys and others, in some sense or another concerned with "the foundations of common sense or inductive inference."[30]

But how precisely does this concept of probability fit into induction? To any person with a little inclination to philosophical thinking the answer will be straightforward. If we reason in terms of certainty and necessity, we may say that a deductive inference is that in which the truth-relation between premises and conclusion is a certain or necessary one: the truthfulness of the conclusion necessarily follows from the truthfulness of the premises. On the other hand, since in an inductive inference the conclusion may be false even when the premises are true, this truth-relation is not certain, but just *probable*: in the case the premises are true, it is just probable, rather than necessary, that the conclusion is also true. If we follow this approach, we will say that inductive inference is the same as *probable inference*; and the sort of conclusion produced by it in the case where the premises are certain is of a probabilistic nature.

However simple this reasoning may appear to be, we should be careful not to overlook the fact that it involves two different and independent positions concerning probability and induction. While the first one makes reference to a relation between two propositions, the other talks about the status of one of these propositions when the other is known to be true. To make sure that there is really a difference, consider a language where we have certain and probable statements. It is quite reasonable to suppose that if h is certain, h is probable. By making use of this inferential schema we will have conclusions marked with a probability modal operator obtained through an inference

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[27] Carnap (1945a) (1945b) (1946) (1947).
[28] Carnap (1952) (1971) (1980), Kenemy (1955), Jeffrey (1957), Hintikka (1966).
[29] Hempel (1943) (1945).

gives inductive support to *h*, but nevertheless *h*'s truthfulness has nothing to do with *e*'s probability. In this case, what is of interest here is an inductive or probable relation concerning the truthfulness of two propositions, which may have nothing to do with other qualities propositions may have. We will call these two positions, respectively, the *status approach* to inductive probability and the *relation approach* to inductive probability.

This first, relational way of understanding inductive probability was the one taken by Carnap's logical school. In addition to conceiving induction in relation to probability, Carnap explicitly identified it as a logical relation of *evidential support* between two propositions, one named hypothesis and the other evidence. While the relation of logical deduction establishes a necessary connection between premises and conclusion, the relation of inductive support establishes just a confirmatory or probable connection between evidences and hypothesis. To Carnap, the confirmation conferred by a piece of evidence to a hypothesis is a purely semantical relation independent of any kind of empirical consideration, be it one's opinion or the relative frequency with which hypotheses of the same kind have occurred in connection with similar evidences. In other words, it is a completely *objective* or *logical* notion. The following quotation illustrates well these points: "Deductive logic may be regarded as the theory of the relation of logical consequence, and inductive logic as the theory of another concept which is likewise objective and logical, viz. $probability_1$ or degree of confirmation."[31] As one might suspect, this conception is essentially the same as the one described in the last section and labeled the contemporary notion of induction.

The use of the index in the above quotation is meant to distinguish the logical, objective concept of probability (called $probability_1$) from the empirical concept of probability dealt with by the relative frequency school of probability (called $probability_2$)[32]. Much more significant for us, however, is the second name given to this logical concept of probability: *degree of confirmation*. As the word "degree" indicates, such conception of probability is intent to be an essentially numerical one. This has to do with Carnap's threefold division of probability concepts. According to him, there are three sorts of logical concepts of confirmation: the *qualitative* (positive or classificatory), the *comparative* and the *quantitative* (or metrical) concepts of confirmation. In his words[33]:

> We may distinguish three concepts of confirmation, concepts which have to do with the logical side only of the problem of confirmation [...] (i) *The positive concept of confirmation* is that relation between two sentences *h* and *e* which is usually expressed by sentences of the following forms: [a] "*h* is confirmed by *e*." [b] "*h* is supported by *e*." [c] "*e* gives some (positive) evidence for *h*." (ii) *The comparative* [...] *concept of confirmation* [which] is usually

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[31] Carnap (1962), p. 43.
[32] See Carnap (1945b) and (1962), chapter II. Recently, the same distinction has been made under the names of "statistical" and "epistemic" probabilities. See, for instance, Pollock (1983).
[33] Carnap (1945b), p. 516. Italics in the original. *h* and *e* mean, respectively, the hypothesis and the evidence.

> We may distinguish three concepts of confirmation, concepts which have to do with the logical side only of the problem of confirmation [...] (i) *The positive concept of confirmation* is that relation between two sentences *h* and *e* which is usually expressed by sentences of the following forms: [a] "*h* is confirmed by *e*." [b] "*h* is supported by *e*." [c] "*e* gives some (positive) evidence for *h*." (ii) *The comparative* [...] *concept of confirmation* [which] is usually expressed in sentences [like] "*h* is more strongly confirmed (or supported, substantiated, corroborated etc.) by *e* than *h*' by *e*'." [and finally] (iii) *The metrical* (or quantitative) *concept of confirmation*, the concept of *degree of confirmation*

This quantitative probability or degree of confirmation is what Carnap, for all intents and purposes, proposes to explicate. However, even though he explicitly takes the quantitative approach as superior to the other two, he did not deny the importance of the qualitative and comparative notions of probability. In fact, Chapter VII of *Logical Foundations of Probability* is dedicated to the elaboration of a comparative inductive logic[34]. Despite this, many authors have overemphasized the numerical aspect of Carnap's approach to the point of taking quantitativeness as an essential feature of inductive logic. Branden Fitelsen, for instance, wrote[35]:

> More precisely, the following three fundamental tenets have been accepted by the vast majority of proponents of modern inductive logic [...] 1. Inductive logic should provide a quantitative generalization of (classical) deductive logic. That is, the relations of deductive entailment and deductive refutation should be captured as limiting (extremes) cases with cases of 'partial entailment' and 'partial refutation' lying somehow on a continuum (or range) between these extremes.

This quotation gives us an important hint about how this quantitative approach has been undertaken by logical probabilists. The point is not only that the proper or best way to refer to the relation of confirmation that exists between *e* and *h* is through a number, but that this number should lie somehow on a continuum whose extremes represent the relations of deductive entailment and refutation. As Fitelsen clearly puts, inductive support is understood as a quantitative generalization of the relation of logical validity. Because of that, almost invariably logical probabilists have taken the calculus of probability as the standard of how degrees of confirmation should logically behave.

Turning back to the discussion about the relation and the status approaches to inductive probability, this distinction is important because the place one puts the notion of probability in his analysis of induction will determine several foundational aspects of his philosophy of induction. In particular, it will allow one to give or not an answer to the question we have posed at the beginning of this section. Clearly, if we adopt an exclusively relational approach, it will be somehow difficult to give any kind of answer to our question. In fact, most philosophers who have taken this position defended that, in an inductive inference, from true premises we cannot infer anything whatsoever.

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[34] See also Carnap (1953).

[35] Fitelsen (2003), p. #. The other two features are the use of the probability and the notion of inductive support as being logical and objective.

probable *per se*, but only in relation with some body of evidence. This of course has implications to the very definition of induction as a kind of inference. Carnap is very clear about that[36]:

> If we wish to use the word 'inference' [...] we may say that the hypothesis *h* is inductively inferred from the evidence *e*. [...] But in this case we must be careful not to overlook the fact that the probability value characterizes not the hypotheses [...] but rather the inference from the evidence to the hypothesis or, more correctly speaking, the logical relation holding between the evidence and the hypothesis [...] Thus we see that from the evidence *e* together with the statement 'the probability of h with respect to e is 1/5' we can infer [...] neither *h* itself, which may be false, nor a statement of the probability of *h*, which would be meaningless. In fact, nothing can be inferred from those two premises.

That position is, for obvious reasons, unsatisfactory. In practical situations, we want to be able not only to assert that *e* gives such and such inductive support to *h* but also, in appropriate cases where *e* is true, to detach *h* from *e* and conclude something about it. For instance, it may happen that a theory or hypothesis has to be highly confirmed before it can be cited as the explanation of anything, or juries have to bring in an unconditional verdict "Guilty" (or highly probably guilty) before the accused can be sentenced. However, according to traditional logical probabilist's view, none of that could be done.

This problem became known among philosophers of science as the problem of *inductive detachment*, i.e., how to detach the (probability qualified) conclusion of an inductive inference from its premises. Trivially, solving this problem means to go from a relation approach to a status one. In the next chapter we will talk in some detail about how philosophers have tried to deal with this problem and which sort of difficulties they have encountered.

Another aspect that will be determined by the position we chose is related to the problem of justification. If we do like the logical probabilists and decide to take induction (or probability) as being an *objective* or *logical* relation between propositions, we will have to show that there is effectively such kind of relation between the propositions we believe are inductively connected to each other. From an analytical point of view, this implies having to disclose the internal structure of the relation and showing it to depend solely on *a priori* principles. In other words, we will have to show that the structure by itself, without any external help, can tell us whether or not (and to what extent) one proposition supports other proposition. Obviously this action of showing is itself a justificatory act. Of course one may object that these *a priori* principles may be self-evident. Even though that is not the case with the existing inductive logics, supposing someone finds such a formulation, the very act of showing the relation to be dependent on these principles is also a justificatory staff. And even if we do not take this analytical point of view, it should be recalled that

[36] Carnap (1962), p. 33. In this and other statements by Carnap to be quoted in this section reference will be made to a numerical value characterizing the inductive relation between hypothesis and evidences. That is due to already mentioned quantitative aspect of Carnap's approach.

logical is an important instance of *rational*. Therefore, to show something to be logical is tantamount to showing it is rational. We see then that, independently of the way we proceed, adopting a relational position brings inevitably the necessity of dealing with the problem of justification. Because of that, we can claim the logical school's position to be essentially in accordance with what we have named the justificatory approach to induction.[37]

These two aspects, the inability to infer anything from inductive inferences and the necessity of dealing with the problem of justification, are the two main (bad) consequences of adopting the first position. But how about the second one? Is the status approach somehow incompatible with the first position? It will be free from the two mentioned problems? To start with, clearly it is not, in any sense, incompatible with the decision of taking probability as a relation between propositions. In fact, it seems to us that the most natural way of dealing with the problem is to consider both the inference itself and its conclusion as probable.

Carnap has already pointed out something very similar to that. While most of the time being very strict about the possibility of inductively inferring something from the truth of an inductive premise, Carnap has given some few hints about how sometimes that movement may after all be possible. For instance, talking about what he called the *methodology of induction*, he says that "If *e* expresses the total knowledge of [an agent] X at the time *t*, that is to say, his total knowledge of the results of his observations, them X is justified at this time to believe *h* to the degree *r* [...]"[38] Elsewhere he says: "If *e and nothing else* is known by X at *t*, then *h* is confirmed by X at *t* to the degree 2/3."[39] In other words, if the mentioned conditions are satisfied, *h* can be taken as a confirmed or probable hypothesis. Then, should we conclude that Carnap is contradicting himself when he says that nothing can be inferred from an inductive inference? Not quite so. Right after the above statement he adds[40]:

> Here, the term 'confirmed' does not mean the logical (semantical) concept of degree of confirmation [...] but a corresponding *pragmatical concept*; the latter is, however, not identical with the concept of degree of (actual) belief but means rather the degree of belief justified by the observational knowledge of X at *t*.

So, we have here a clear distinction between a logical, on the one hand, and a pragmatical concept of probability on the other. This pragmatical concept is an instance of what we have called status approach to inductive probability. Of course, Carnap is here talking about a quantitative concept akin to his degree of confirmation. However, given his previously explained distinction between the qualitative, comparative and quantitative notions of (logical) confirmation, we may

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[37] In chapter 4 we will give more details about this justificatory aspect of the logical school.
[38] Carnap (1962), p. 211.
[39] Carnap (1946), p. 594. Italics in the original.
[40] Ibid. The italics are mine.

fairly suppose that, in addition to what he calls degree of justified belief, there is also a comparative and qualitative pragmatical notions of probability. In what follows, we will make use of the term "pragmatical probability" in a broader, not necessarily quantitative sense.

According to Carnap, the point where the logical and the pragmatical concepts of probability interact is at the application of inductive logic, conceived exclusively as the logic of the relation of inductive support. That is to say, as soon we have such a logic, we can, provided the evidences are known and certain restrictions are satisfied, conclude that the hypothesis at hand is (pragmatically) probable. These restrictions have to do with the expression "and nothing else is known" in the quotation above and have been taken into account in Carnap's philosophy by what he called the *requirement of total evidence*[41]. Briefly put, the requirement of total evidence states that in order to apply inductive logic for, among other things, getting the mentioned pragmatical probability, one must make sure that the evidences represent all the available knowledge. This is of course needed because *e* may be an evidence for *h* when taken in isolation, but against or neutral to it when taken in conjunction with *e'*. It is important to observe nevertheless that the requirement, as Carnap puts it, is not the only way to achieve that sort of (pragmatical) completeness. In chapter 4 we will see how Artificial Intelligence theorists have achieved the same goal by using a somehow different mechanism. In the rest of this chapter we will refer to such sort of restrictions as *total evidence conditions*.

Another important point contained in the quotation above is the reference to belief. According to Carnap, even though this pragmatical concept is not "identical with the concept of degree of (actual) belief," it is still a sort of belief, namely that which is "justified by the observational knowledge of X at *t*." Others like Keynes have made similar points about the connection between logical probability, belief and justified belief (or pragmatical probability): "The theory of probability is logical, therefore, because it is concerned with the degree of belief which is *rational* to entertain in given conditions, and not merely with the actual beliefs of particular individuals, which may or may not be rational."[42]

From this we can lay down two important features of this pragmatical concept of probability. First, it is a sort of belief and, therefore, not a logical, but an *epistemological* notion. For that reason, we will also refer to this new concept as the *epistemic concept of probability*. Second, it is not, properly speaking, the same as beliefs people ordinarily have. Rather, it is that kind of belief which is obtained in a justified or rational way. More specifically, the belief in *h* is rational if and

[41] Carnap (1962), pp. 211-13.

[42] Keynes (1921), p. 4. Because of passages like that, some authors interpret Keynes conception of probability as being essentially epistemic, and not logical. See for instance Fitelsen (2003). As far as we are concerned, we take the traditional interpretation according to which even though Keynes' use of some terms may not be as clear and uniform as Carnap's, his main concern is with a logical concept of probability.

only if there is a proposition *e* such that *e* is certain, *e* gives (logical) evidential support to *h*, and *e* expresses the total available knowledge. Therefore, the pragmatical probability as conceived by logical probabilists is essentially dependent on the logical one. On the other hand, it is in the formation of these rational degrees of belief that the logic of induction finds its more important application.

It is important to resist the temptation to identify this epistemic probability with the probability concept dealt with by the so-called *subjectivistic theories of probability* (traditionally represented by Frank P. Ramsey, Bruno de Finetti and J. Savage and nowadays extensively used by Bayesian confirmation theories[43].) The point is that even though these theories follow a sort of status approach and use terms like "rational degree of belief" and "consistent degree of belief" (which are basically those quantified beliefs that conform to the probability calculus), it is widely acknowledged that they are concerned in fact with what we have been referring to as actual degrees of belief. In general, subjectivists hold that it does not matter how one come to assign a certain degree of belief to a proposition, the important is how he relates this degree with the measures of his other beliefs.

Henry Kyburg has appropriately laid down a threefold division of theories of degree of belief in which the mentioned aspect is one of the criteria used[44]:

> Sometimes [the theory] is explicitly subjective, in the sense that it is intended to capture the psychological state of an individual subject [...] That is, it is a term being employed in a descriptive psychological theory. Sometimes the theory is construed as a theory of 'idealized' subjective belief, that is, there is no argument about whether your degrees of belief are *rational* or not, but we can argue about whether or not they 'fit together.' And finally, the theory is sometimes construed as a general theory of *rational belief* in which we can assess the rationality of particular degrees of belief (given the evidence the agent has at his disposal) as well as the rationality of their relations to each other.

What we have called subjectivistic theories of probability belong to the second kind of theory of Kyburg's classification. Carnap's systems would belong to the third kind.

Inside Carnap's tradition (but not precisely inside Carnap's works), much has been talked about this pragmatical notion of probability. Despite "accidental" references like the ones we have quoted, this notion has been extensively discussed in connection with the problem of inductive detachment. As we have mentioned, due to the necessity of getting something inferred from inductive inferences, many philosophers felt compelled to deal with a status approach. Although many times the terms "rational", "justified" and "warranted belief" were used, as a rule the problem of detachment was discussed with regard to so-called *rules of acceptance*. The idea was that the problem of detachment is to be solved by specifying certain conditions according to which

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[43] See Kyburg & Smokler (1964) and Curd & Cover (1998), chap. 5.

the conclusion of an inductive inference could be detached from the premises and taken as *accepted*. In the 1960's and 1970's philosophers such as Kyburg, Hintikka, Risto Hilpinen and Keith Lehrer have all followed a status approach along with some sort of "acceptance" notion[45]. An important remark is that much of their endeavors were directed towards the solution of a puzzling problem apparently inevitable when one deals inside a numerical probabilist framework: the so-called *lottery paradox*. Nowadays, the relevance of the lottery paradox has not lost its strength, having transcended the boundaries of official philosophical literature and been extensively debated inside artificial intelligence[46]. In the next chapter we will talk about the lottery paradox and the use of the notion of acceptance in the philosophy of induction.

Regarding the second question, whether the status approach will be free from the two mentioned problems, we believe the answer is 'yes.' The first problem, not to allow anything to be concluded from an inductive inference when its premises are true, is trivially solved. After all, the notion of pragmatical probability is defined as that status the conclusion of an inductive inference gets when its premises are known to be true and some total evidence conditions are satisfied. An objection one may raise against this conclusion concerns the apparent fact that this alleged solution is not a solution at all: while our problem concerns inferring something when the premises are true, the pragmatical probability as defined by Carnap can be applied just in those cases where the premises are *known to be true*. This objection can be replied in several different ways. For example, we could say that the levels to which each one of these descriptions refers are not the same: while the definition of pragmatical probability refers to the object-language level, the definition of inductive inference refers to the meta-language level. Letting $\Box$ be a modal operator meant to represent knowledge about truth, this means that while the first definition refers to statements of the form $\Box\alpha$ itself, the second refers to the truthfulness of this sort of statement. Another possible reply is to say that, despite unquestionable differences, the notions of a sentence being true and being known to be true are connected in such a way that if α is true, $\Box\alpha$ is also so. Therefore, since "α is pragmatically probable" is obtained from $\Box\alpha$, it may also be obtained from (and defined through) "α is true". However, independently of the path we chose, this objection raises a very important question concerning the nature of induction: should it be taken in connection with truth or in connection with knowledge about truth? In other words, should induction be considered intrinsically as an ontological or as an epistemological notion? We will

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[44] Kyburg (1994), p. 401. Italics in the original.
[45] Kyburg (1964), Hintikka & Hilpinen (1966), Hilpinen (1968), Lehrer (1970).
[46] See for instance Stalnaker (1984), Pollock (1987), Poole (1991) and Kyburg (1997)..

postpone the analysis of this question to the next section, where we should lay down in a systematic way our view on induction.

Another possible objection to the conclusion that the pragmatical notion of probability solves this first problem is to say that, since the inductive support given by *e* to *h* may change as we add new premises, even if we take an epistemic notion to refer to *h* when *e* is known to be true and some total evidence conditions are satisfied, this epistemic status of *h* will be strongly dependent on the knowledge situation. Keynes is very clear about that[47]:

> The terms *certain* and *probable* describe the various degrees of rational belief about a proposition which different amounts of knowledge authorize us to entertain. All propositions are true or false, but the knowledge we have of them depends on our circumstances; and while it is often convenient to speak of propositions as certain or probable, this express strictly a relationship in which they stand to a *corpus* of knowledge, actual or hypothetical, and not a characteristic of the propositions in themselves. A proposition is capable at the same time of varying degrees of this relationship, depending upon the knowledge to which it is related, so that it is without significance to call a proposition probable unless we specify the knowledge to which we are relating it.

Therefore, even if we take an epistemic notion of probability, we will still not be able to apply it to propositions without reference to a body of evidence. In other words, probability statements like "α is (pragmatically) probable" are incomplete and consequently meaningless.

One important point contained in the above passage is the reference to certainty. After introducing probability along with the notion of certainty, Keynes says that knowledge-dependence applies equally to both certain and probable propositions. In a more general way we can say that in one way or another every epistemic concept is knowledge-dependent. The question then is whether or not this fact will prevent us from speaking of propositions as being certain or probable without mentioning the knowledge situation at hand. To start with, ordinarily people do speak about propositions as being certain and probable without giving a description of all the statements they believe. Second, many sorts of truths are context-dependent. *A posteriori* and synthetic statements, for example, are obviously dependent on time and circumstance. From a formal point of view, this means that we need some interpretation to classify a sentence as true or false. But of course we do not say that statements belonging to a formal language are elliptical or meaningless when no reference to some interpretation is given. They are just uninterpreted statements. Similarly, in ordinary usage we do not require at every utterance of a statement the time and circumstance to be mentioned. We just take some specific context or interpretation as the default one: unless some other time-circumstance pair or interpretation is explicitly mentioned, we take this default one as the one we are referring to. But then why not to allow the same thing to be done with epistemic concepts? We sincerely cannot see how one could reasonably answer this question.

About the second problem, the necessity of dealing with the problem of justification, there are two points to be considered. First, since we do not intend to take into account the inductive relation that allows one to classify a hypothesis as probable but just the status itself, we will not be forced to say why the step from evidences to hypothesis is rational. However, and this is our second point, as we have seen, to Carnap the notion of pragmatical probability is dependent on the relation of inductive confirmation: if *e* gives evidential support to *h*, *e* is known to be true and expresses the total of our knowledge, then *h* is pragmatically probable. It is just because of this connection that we can classify these beliefs (or degrees of beliefs) as rational. Therefore, if we equate epistemic probability with rational (degree of) belief in the way Carnap does we will fall again into justificatory matters. On the other hand, if we take inductive and rational in the way we have suggested at the end of Section 2.1, that is, as depending on the consensus of a community, then we will not have to bother about that sort of justification. And, as it can be clearly seen, in contrast to the relation approach, the position that takes only the epistemological notion of probability is susceptible to this sort of pragmatical approach to induction. We may however object that saying that rational is what is taken as so by a community is tantamount to giving an answer to the problem of justification, by the way a very similar one to Hume's solution. However, that is surely not the sort of justification which philosophers have been looking for. In fact, we wonder even if someone would take it as an answer to Hume's challenge. The point is that, even though we make use of the term "rational," the meaning we are giving to it does not correspond to what we expect rationality to be: to be a consensus seems simply too weak.

2.3 Induction and Plausibility

Now we have arrived at the point where we can start stating in an explicit way the philosophy of induction and plausibility we will develop here. It should be advised that even though many of the ideas contained in this section are not new, for the sake of completeness we will take the trouble of discussing even those established and uncontroversial points concerning induction and probability.

To start with, we are dealing of course with the contemporary conception of induction according to which induction is the class of (rational) ampliative or non truth-preserving inferences. There is however one important feature of this conception that we shall deliberately avoid. As we have seen, the contemporary view places induction in sharp contrast with deduction. Despite this, in a very important aspect it still makes induction undistinguishable from deduction. As we have shown, a very usual way of contrasting deduction with induction is to say that while in a deductive inference the premises give full evidential support to the conclusion, in an inductive one the

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[47] Keynes (1921), pp. 3-4. Italics in the original.

support is just partial. This means that even though the two sorts of supports are different in "quantity," they are still identical in "*quality*." In other words, the nature of the support given by a deductive inference is the same as the one given by an inductive one. This is reflected in the almost universally held opinion that this evidential support varies from 0 to 1, where the bounds 0 and 1 represent, respectively, the relations of logical inconsistency and logical deduction, and in between them lay the several degrees that characterize inductive evidential support. We have already remarked that because of that the probability calculus has invariably been taken as the canon of how probability statements are to behave.

We do not think this is a very trivial matter. To us it seems much more reasonable to hold that the nature of the supports occurring in deductive and inductive inferences are essentially different from each other. If the blatant fail of more than half-century of attempts to provide a logical characterization of inductive inferences does not convince the skeptic, maybe the two centuries and half of unsuccessful attempts to solve Hume's problem will do it, for, as we hope have showed, in order to show that the inductive evidential support has the same (logical) nature of deductive support, one has to justify the logical rectitude of inductive inferences. Because of this fundamental disagreement we will keep the expression "evidential support" to refer exclusively to inductive inferences (and maybe call the support present in deductive arguments demonstration, proof or validity relation) and define induction as that sort of inference whose premises in certain circumstances *serve as evidences* for the conclusion.

Besides historical-foundational arguments of this sort, this position of ours can be supported by observations concerning the use of the probability calculus to derive degrees of inductive support. As many authors such as H. Heidelberger, Francis Dauer, John Pollock and Johathan Cohen have pointed out, approaches like the ones based on the calculus of probability which take induction as a continuum starting from the relation of logical inconsistency and ending at the relation of logical validity seem to be in sharp contrast with some very basic intuitions concerning induction[48]. The most exhaustedly mentioned problem concerns the way confirmed hypotheses are conjoined in the probability calculus. If, for example, we take the testimony of a certain art expert as a strong evidence to believe that such and such pictures are genuine Vermeers, for instance, then his judgments that picture A is a Vermeer and picture B is a Vermeer would make us take not only the hypotheses "A is a Vermeer" and "B is a Vermeer" as credible, but also the hypothesis "A and B are genuine Vermeers" as equally so. This however is in sharp contradiction to the way probability values are conjoined in the probability calculus. In the words of Cohen[49]:

[48] Heidelberger (1963), Dauer (1980), Pollock (1983), Cohen (1970) (1989a).

[49] Cohen (1989a), pp. 19-20.

> [...] this conflicts with Pascalian principles because the above mentioned multiplicative law for the probability of a conjunction ensures that, except in limiting cases, the conjunction is less probable than either conjunct. If $p(A)>0$ and $p(B|A)<1$, $p(A\&B)$ must always be less than $p(A)$, since in accordance with the multiplication principle for conjunction $p(A\&B)$, as we have seen, is equal to a proper fraction of $p(A)$, namely, $p(A) \times p(B|A)$. Of course, the chance of both pictures' being genuine may well be a lot less than the chance of just one's being genuine. But is the credibility of their genuineness to be judged in terms of such chances [...] or in terms of the reputation of the author of the warranties that have been given to you?

Another famous counterintuitive result which shows up when we use the calculus of probability to represent inductive probability values is connected with the additive character of negation in the calculus of chance, which is materialized by the following principle:

$$p(A) + p(\neg A) = 1.$$

The point is that this principle makes sense only if the negation we have in mind is the *incredibility of A*, but not if it is the *credibility of ¬A*. Given a certain body of evidence, the credibility of A and the credibility of ¬A (that is to say, the justification for believing A and the justification for believing ¬A) may be both low. For example, the statement that some accused has never been in the jail before may provide the same amount of evidence for believing that he is innocent as for believing that he is guilty. In this sense of probability, the probability of being innocent will not vary inversely with the probability of being guilty. The negation of the epistemological concept of certainty is similarly non-additive. As the certainty of A goes up, the uncertainty of A goes down. But there may well be very little certainty that A and also very little certainty that ¬A[50].

Another important point of our theory of induction is that we want deliberately to avoid the problem of justification. That position could be defended by referring to the arguments some philosophers have laid down against the solubility of this problem. Even though we do agree on this point, it is not our intention to present it as the rationale of our choice. Rather, we would like to do that by referring to the already mentioned distinction between the problem of justification and the problem of description of induction. Our main goal here is to adopt a *purely descriptive* or *pragmatical approach to induction*. That means that our concern will be exclusively the representation of the established schemas of inductive inference, without any consideration whatsoever for their rationality or logical rectitude. By saying this, one may object that our goal of explicating the notion of induction could not be properly achieved. Since we have defined induction as the class of rational non truth-preserving inferences, in order to explicate it we will have not only to give an account for non truth-preservingness, but also for what means an inference to be rational. The answer to that question will come up some few paragraphs below when we lay

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[50] For a full list of similar paradoxes see Cohen (1970) and (1977). For a critical discussion of some of these problems see Schoeman (1987).

down in more a precise way the role that our second *explicandum* will play in our account of inductive inferences.

At first glance this descriptive approach may appear to be a quite straightforward task. Not quite so. First of all, the attempts to represent some of the simplest inductive patterns of inference in a sound fashion have been sources of insurmountable problems for philosophers[51]. Second, the methodological commitment of avoiding the problem of justification is not so easily fulfilled. It involves decisions concerning the very way one characterizes the class of inductive inferences. For instance, if from the very beginning one decides to define inductive inferences in an intensional way he is sure to deal with the problem of justification. In chapter 4 we will briefly show how one of the most traditional systems of inductive logic, Carnap's logic, which allegedly was concerned only with the problem of description had, in its essence, a justificatory root.

This decision of ours of adopting a descriptive approach and keeping distance from the problem of justification has two immediate and mutually dependent consequences (which were already mentioned at the end of the last section.) First of all, since the use of an intensional approach would inevitably force us to deal with the problem of justification, we are left with no choice but to adopt an extensional or pragmatical approach to induction. Second, since the use of a probability notion as conceived by the logical school would certainly commit us to dealing with the problem of justification, we shall adopt something akin to Carnap's pragmatical concept of probability. Moreover, we are concerned here with what we may call *inferential conception of induction*, that is to say, a view according to which induction is a sort of inference with some kind of inferential power, in the sense of being able to conclude something about its conclusion when its premises are true or know to be true[52]. Therefore, we have to have some epistemic notion able to distinguish the inductively obtained facts from the deductively obtained ones.

About the use we will make of this pragmatical or epistemic notion of probability, we have to acknowledge that using an old term like "probability" (which has not only one but over half dozen different interpretations) with a somehow new meaning may be quite troublesome. For instance, the existence of a established mathematical calculus of probability may somehow play a role in one's decision of representing our notion, which by whatever reason is also referred to by the term "probability," through such calculus. However, the historical roots of the calculus of probability lays on problems not concerned at all with what we are calling here induction. Therefore, in order to apply the calculus of probability to his analysis of induction, our friend will have to reject the (not so precise but still) philosophical motivations behind the development of such calculus and

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[51] For a brief description of some of the difficulties found in the solution of the problem of description of induction see Lipton (1991), pp. 6-22.

conceive another one. But what if there is a real incompatibility between the notion of probability dealt with by the calculus of probability and our intuitions concerning inductive probability?

Because of that, rather than using "probability" to denote our pragmatical notion of probability we will adopt another term which we believe is less susceptible to this sort of trouble: "*plausibility*." Without abandoning completely the terms "pragmatical probability" and "epistemic probability," we will from now on use the term "plausibility" to refer to the epistemological notion that designate that status the conclusion of an inductive inference gets when its premises are known to be true. This choice however is not just a terminological decision or a mere term change. We do believe that the term "plausibility" and the notion it means are much more likely to succeed in the conceptualization of our pragmatical probability than the term "probability."

To start with, we note that like the concept of probability, plausibility has both the negative, uncertain or *defeasible* aspect and the positive, *reasonableness* aspect required by inductive inferences. This means that we can fairly take the conclusions of inductive inferences as plausible or, equivalently, take inductive inferences as sources of plausible facts. This connection is, we may say, the bridge between the two concepts. On the one hand, plausible facts can be refuted: they are not certain, unquestionable facts, but are subject to revision and therefore can be defeated. Also, as we have seen, the truth-relation between premises and conclusion of an inductive inference are not certain: even though in the case where the premises are true and we may accept provisionally the truth of the conclusion, this acceptance may have to be re-evaluated in the presence of additional information. On the other hand, because a plausible fact is plausible, we expect it to have some sort of reasonableness: we are not ready to accept every hypothesis as plausible; some very good reasons are required. And these good reasons will be given, in the case of inductively obtained plausible facts, by the supposed logical rectitude which inductive inferences posses.

A natural question that may arise concerns the other sources of plausible facts. Are we taking plausibility exclusively in connection with induction or in a broader sense as to encompass also other sources of plausible facts? Even though we are above all concerned with what we may call inductive plausibility, there is another way of getting plausible facts that is of particular interest to us. It is commonplace among philosophers to translate the Greek words "*eikos*" and "*endoxa*" as probable or plausible. S. Sambursky, for instance, leaves no doubt that what the ancients meant by these terms was something very close to our epistemic probability: "The Greek equivalent for 'plausible' or 'probable' – eikos – was in use from pre-Socratic literature onwards and well into the Hellenistic period, in the same sense as these terms are applied today, i.e. 'to be expected with

[52] This point may surely be redundant, but given the exposed in the last section we think it is indispensable to stress it.

some degree of certainty.'"[53] Even though there is some controversy over whether by *eikos* the ancient writers really meant something like our inductive probability[54], most philosophers agree that the term "*endoxa*", which literally means standing in *doxa* in respect of all or most of the wise, represents a quite unambiguous sense of probable or plausible. Speaking about Aristotle's use of the term, George Grote writes: "The *Endoxon* may indeed be rightly called probable, because, whenever a proposition is fortified by a certain body of opinion, Aristotle admits a certain presumption (greater or less) that it is true."[55]

The fundamental point here is that plausibility now is taken as the result of a sort of consensus achieved among the most reputable experts on some subject matter (the wises). Nicholas Rescher has called this specific source of plausible facts the *authority-oriented approach to plausibility*. In his words[56]:

> This authority-oriented approach to plausibilistic inference was in fact envisaged at the very origin of the subject in Aristotle's *Topica*. He took the basis of plausibility to lie in the probity of sources: the opinions held 'by all or the majority or the experts or the best and the most reputable among them.' But his approach was broad, and encompassed not only the opinions of authorities as such, but also the cognitive principles on which they are generally founded (such as induction – *epagôgê*)

It is interesting to note that in this case, the rationality of plausible facts comes not from the logical rectitude of some inference, but from the probity of these authorities. In contrast to our inductive plausibility or epistemic probability, the rationality of plausible facts here is not explained by a further reference to the notion of rationality, but by reference to a different and we believe less problematic notion. That is important to us because we will use a sort of authority-oriented approach to semantically explicate the notion of plausibility[57]. That is what will make possible for us to deal with the rationality of plausible facts and inductive inferences without having to handle the problem of justification. Consequently, given the mentioned connection between the rationality of inductive inferences and the rationality of inductive plausible facts, it will also make possible for us to account for this positive side of inductive inferences through an approach which deliberately avoids any consideration of why inductive inferences are rational and restricts itself to just describing inductive patterns of inference.

At the beginning of this section we have mentioned that to us induction is that sort of inference whose premises in certain circumstances serve as evidence for the conclusion. We also have defined plausibility as that status the conclusion of an inductive inference gets when its premises

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[53] Sambursky (1956), p. 36.
[54] See for instance Madden (1957).
[55] Grote (1872), p. 389. Italics in the original.
[56] Rescher (1976), p. 6.
[57] That will be done in Chapters 5 and 6.

are known to be true and some total evidence conditions are satisfied. Bringing the two definitions together we arrive at a preliminary explication, we may say, of the notion of plausibility: plausible facts are those which there are evidences for. In other words, the sentence "α is plausible" is to be taken as the same as "there are evidences for α." This meaning is what we will try to make precise through the formal systems we will introduce in Chapters 5 and 6.

This definition and its use of the notion of evidence bring us to an important difference between our approach and the logical probabilists'. To the logical probabilists, evidence is a way of calling the premises of an inductive inference which reflects the sort of logical relation that exists between them (the premises) and the conclusion. In other words, being an evidence for some hypothesis depends exclusively on the logical form of the evidences and hypothesis and has nothing to do with the knowledge situation at hand. What will depend on that knowledge situation is the satisfaction of some total evidence condition which will determine whether or not the plausibility of *h* can be concluded from the truthfulness of *e*. The point of divergence between this conception and ours is that we take *e* as evidence for *h* only when such total evidence condition is satisfied. In other words, only when we are in position to conclude "h is plausible" is that we take the *e* as being an evidence for *h*. Adopting a very simple sort of total evidence condition, we could say that *e* is inductively related with *h* if and only if, when taken in isolation, *e* is an evidence for *h*.

Now we are in a position to analyze the objection raised against the use of the pragmatical notion of probability at end of the last section which, we have claimed, has important implications for the nature of induction. It was pointed out there that we could not in fact take plausibility or pragmatical probability as an answer to the question as to what we can conclude from an inductive inference when its premises are true because plausibility, as we have defined it, is that status the conclusion of an inductive inference gets, not when its premises are true, but when they are *known to be true* (and some total evidence conditions are satisfied.) The whole point is of course that the notion of plausibility is primarily concerned with knowledge about the world and not with the word itself. Therefore, unless induction is itself redefined in epistemological terms, our use of the notion of plausibility in connection with induction is simply out of place.

Given our previous discussion about the consequences of taking induction as logical and the guidelines presented at the beginning of this section, the choice for an epistemic position follows naturally. If we take plausibility, which we may think of as an epistemic mark we attach to propositions in order not to leave any doubt about their refutable character, as the inherent product of inductive inferences, induction will have an indubitable and quite strong epistemological aspect. More specifically, it will be taken as sorts of epistemological rules of inference intent basically to fill the gap that exists between data and theory, understood in a broad sense. As a consequence of that, the so-called logical rectitude of inductive inferences will be nothing more than a subjective feeling of soundness and reasonableness that cannot be *a prioristically* justified. The only necessary connection that shall exist between the one hundred observed black ravens and the hypothesis that the next raven to be observed will be black is the psychological necessity of inferring things even in the presence of incomplete and imprecise evidences[58].

There are some important consequences of taking inductive inferences as primarily concerned with knowledge of truth rather than with truth itself. First, it places the notion of truth in its right place and leaves the path to *certainty* open. In other words, regarding the application and characterization of inductive inferences, it is not only the truth of the premises what we should look for, but also, and maybe primordially, their certainty: from a pure epistemological point of view, inductive inferences are not those inferences which lead from truth premises to plausible conclusions, but those which lead from *certain* statements to plausible and therefore uncertain ones. Second, as we mentioned at the end of Section 2.1, the distinction between induction and deduction is not a matter of logic, but of epistemology. While deduction applied to matters of knowledge is certainty preserving, induction is not. The perhaps unnoticed but fundamental asymmetry here is that while deduction is concerned with truth and *can be applied* to epistemic concepts, induction is essentially epistemic and consequently cannot be dissociated from knowledge issues. Finally, according to this new conception, what inductive inferences weaken by going from certain statements to plausible ones is not the degree of truth of statements, but their degree of revocability or, in other words, our readiness to give them up. This is in sharp contrast with the many times held view that induction diminishes the degree of truth of the conclusions, and plausible (or probable) statements are a sort of partial truth. This point is particularly important to us because we will take this susceptibility of being given up of a statement as what effectively distinguishes a certain statement from a plausible one. While the first is irrefutable and very hard to be given up, the second is intrinsically refutable, defeasible and therefore susceptible to be reconsidered[59].

All this talk about induction and plausibility (or pragmatical probability) as being intrinsically epistemic is of course not new. After all, the so-called classical interpretation of probability takes probability essentially as a measure of our ignorance. According to Roy Weatherford, for example, "For the most part, classical theorists held that probability is not genuine part of metaphysical reality, but a human invention cleverly designed to assist us in making rational choices when we

[58] This conclusion does not necessarily imply that inductive inferences are irrational or indistinguishable from nonsense. We could, for example, proceed like many philosophers and try to justify inductive arguments by non-logical means without going against our epistemic position regarding induction.
[59] It is important to note that we do not deny that certainty and plausibility have other equally important aspects. For instance, this position of ours neither denies nor agrees that certainty is something like a sort of justified true belief. To our theory, this simply does not matter.

have less than conclusive information"[60]. We have also seen how Keynes has taken probability as intrinsically connected with the notion of certainty and belief. However, it is equally true that in part due to its connection with games of chance and its use in modern and classical physics, probability has many times been understood as an empirical, non-epistemic concept[61]. Probability as conceived by Carnap has also been understood as an intrinsically non-epistemic concept. This seems then only to reinforce the advantages of taking "plausibility" rather the "probability" to refer our pragmatical notion of probability.

One objection one might raise is that such an epistemic view of induction and plausibility would certainly render the philosophy of induction into a psychological enterprise. In his *Logical Foundations of Probability*, Carnap addresses this very point when discussing psychologism in deductive logic[62]:

> Many logicians prefer formulations which may be regarded as a kind of *qualified psychologism*. They admit that logic is not concerned with the actual processes of believing, thinking, inferring, because then it would become part of psychology. But still clinging to the belief that there must somehow be a close relation between logic and thinking, they say that logic is concerned with correct or rational thinking.

The point is that in the same way certainty is a psychological state people may have regarding a specific proposition but nonetheless has a logical aspect which we may say deductive logic formalizes, plausibility and induction also have a logical aspect susceptible to being formalized. And in the same way this logical aspect of certainty is not concerned with how one appraises his certain beliefs, the logic of plausibility and induction is not concerned with any subjective feeling of plausibility people may use in practical judgments. Rather, the goal of the logic of plausibility, which, as we will see in Chapter 4 is distinct from the logic of induction, is just to set the logical constraints plausible beliefs are subject to[63]. In its turn, even though the logic of induction aims to represent inductive patterns of inference, this is done in an unambiguous and objective way completely independent of any subjective matter[64].

There is still another very important point that distinguishes our approach from the majority of theories of probability and induction. As we have already mentioned, almost all theories of inductive probability take the calculus of probability as the basic canon of how to get derived

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[60] Weatherford (1982), p. 44.

[61] Popper's and the so-called frequency interpretations of probability are the best examples of this non-epistemic understanding of probability.

[62] Carnap (1962), p. 41. Italics in the original.

[63] The situation is analogous to the epistemological logics developed in philosophy and artificial intelligence. See Gabbay et al (1995).

[64] There is nevertheless an important difference between our conception of the logic of induction and the traditional one concerning what is to be considered sound criteria of confirmation. Due to our choice of taking a purely descriptive approach, our logic of induction will have very little to say about when a statement confirms another one.

There is still another very important point that distinguishes our approach from the majority of theories of probability and induction. As we have already mentioned, almost all theories of inductive probability take the calculus of probability as the basic canon of how to get derived probable facts. This allows us to classify them as quantitative theories of induction. In contrast to that, our approach is essentially a *qualitative* one. Had Carnap employed his threefold taxonomy of confirmation to the pragmatical notion of probability, we could say that our approach is an attempt to *explicate* the *qualitative notion of pragmatical probability*. This should not be seen however as a case against the fruitfulness of a quantitative approach. Even though we are inclined to agree with philosophers such as Ernest Nagel, Keynes, Richard von Mises and Mario Bunge who have claimed numerical inductive probabilities to be applicable only in special cases and therefore a general quantitative account to confirmation to be fundamentally misleading[65], we do not discard the usefulness of a numerical approach to plausibility[66]. It should nevertheless be seen as a case against the way this quantitative approach has been traditionally carried out. As we have already said, we explicitly reject the calculus of probability as suitable to characterize the logic of pragmatical inductive probability.

A last feature of inductive inferences which we cannot avoid mentioning is related to Carnap's requirement of total evidence. As we have said, the raison d'être of any total evidence condition is to make sure that all available evidential knowledge will be taken into account at the time of applying inductive inferences to get pragmatically probable conclusions. One way to understand the necessity of such sort of condition is as follows. In deductive logic, if α is logically deduced from β, it will remain so independently of the amount of additional information we get (in symbols: if $\{\beta\} \vdash \alpha$, then $\{\beta\} \cup A \vdash \alpha$, for any set of formulae A.) That is the well-known monotony feature of classical logic. The relation of inductive support, however, does not have this sort of behavior: the fact that $\{\beta\}$ gives evidential support to α does not guarantee that $\{\beta\} \cup A$ will also do so. And if we allow one to conclude the (pragmatical) probability of α on the basis of this much of evidential support, it may happen that while from β alone we get "it is probable that α", the same thing cannot be concluded from $\{\beta\} \cup A$. In other words, inductive inferences are *nonmonotonic*. This nonmonotonicity is essentially the same as non truth-preservingness applied to a formal inferential system[67]. Therefore, if we want to base our believes, let us say, on what our inductive

[65] Nagel (1955), pp. 68-71, Keynes (1921), chap. III, von Mises (1957), p. ix. The position of M. Bunge was heard by the author at his Fall/2003 seminar at McGill University, entitled "Philosophy of Science."

[66] In fact, in chapter 6 we will present a brief sketch of how our qualitative approach can be very easily turned into a quantitative one.

[67] Even though this feature was well known among classical logical probabilists, the term "nonmonotoniticy" was completely absent from their writings. It was only in the 1970's that AI theorists paid attention to that aspect as the distinguishing feature of commonsense and started developing so-called nonmonotonic logical systems.

One immediate consequence of the nonmonotony of inductive inferences is their *global character*. In order to infer the conclusion of a deductive inference we need to be concerned only with the truthfulness of the premises. We do not need to look at the whole logical theory[68]: a local inspection is enough to warrant the acceptance of the conclusion. On the other hand, an inductive inference does not have this feature. If *e* (taken individually) gives some support to *h*, the same thing may not happen when we take *e* in conjunction with *e'*. Therefore, if we want to inductively conclude *h* from *e* we will have to look not only at the truth value (or, to be more precise, the certainty value) of *e*, but at the whole set of accepted facts, in order to make sure that no one of them defeats the evidential support given to *h* by *e*.

In order to finish this section, some words about the recent history of the term "plausibility" will be opportune. The term "plausibility" is not in any way absent from the philosophical literature. Many probabilists have used it, for example, to designate that status a hypothesis achieves when it has got enough evidential support (or, in other words, our inductive pragmatical probability.) For instance, in one of his 1940's papers Carnap wrote: "[...] the situation is rather this, that we ascribe to the proposition a certain moderate degree of confirmation (or *plausibility*, probability, credibility, acceptability)"[69] From this quotation we can understand the role the term "plausibility" has played in the logical probabilists writings. Invariably, the notion they wanted to explicate (or the *explicanda* of their theories) was the notion of probability or degree of confirmation. The use of terms such as "plausibility," "corroboration" and "credibility" was intent just to clarify the meaning of these *explicanda*, without being themselves subject to the same sort of explicative analysis.

Another probability-dependent use of the term "plausibility" was made by Glen Shafer to designate what A. Dempster originally named "lower probability." The whole idea was to use two numbers to represent our state of uncertainty with regard to a sentence, one meaning the degree of support given by the evidence for the sentence and the other the degree of support given against it. While Dempster named the first lower probability and the second upper probability, Shafer called them plausibility and belief, respectively[70].

A little bit more independent use of the term was provided by those who saw plausibility as a sort of *a priori*, evidence-independent status of reasonableness that hypotheses may have. Peirce, Émile Meyerson, Norwood Hanson, Salmon and Dudley Shapere for example, have taken

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[68] Given a formal language $\mathfrak{I}$, an inferential relation $\vdash$, and a set of formula $A \subseteq \mathfrak{I}$, logicians use the term "theory" or "logical theory" to mean the set of all formulae of $\mathfrak{I}$ that are inferred from A according to $\vdash$. In symbols: $Th(A) = \{\alpha \in \mathfrak{I} \mid A \vdash \alpha\}$.

[69] Carnap (1946), p. 597. The italics are mine.

[70] Dempster (1967) and Shafer (1976).

plausibility in this sense[71]. Salmon is clear about how, according to this view, plausible hypotheses would be distinguished from probable or confirmed ones[72]:

> There are, it seems to me, three logically distinct aspects of the treatment of scientific hypotheses. [...] (1) thinking of the hypothesis, (2) plausibility considerations, and (3) testing or confirmation. Hanson has argued (correctly I think) that there is an important distinction between plausibility arguments and the testing of the hypotheses, but he has (mistakenly I think) conflated plausibility arguments with discovery.

Finally, there are those like Nicholas Rescher, George Polya and René Leclercq who have really engaged themselves in the task of formally explicating something they named "plausibility."[73] While Rescher's theory is somehow similar to the approach we shall develop here, Leclercq's and Polya's seem to be much closer to some modern interpretations of probability. The important point however is that all of them have given priority to the notion of plausibility itself over its connection with induction. In other words, they acknowledge that plausible facts may come from inductive inferences, but the concept is analyzed in a broad way independently of this specific source.

In the field of *Artificial Intelligence* (AI), the term "plausibility" has been used in a very close connection with inductive inferences. A very important task of AI has been the mechanization of the so-called *commonsense reasoning*. One of the most important features of this sort of reasoning is that it is performed in situations of incomplete and imprecise knowledge. An example traditionally given in AI literature is the situation where the only two pieces of information we have is that (1) a certain animal called Twenty is a bird and that (2) birds usually fly, and based on that we wish to conclude whether or not Twenty flies. Even though from the point of view of deductive logic we cannot conclude anything from these premises, in ordinary situations people do conclude things on the basis of typical rules like (2), namely that Twenty does fly. But the inferential step from (1) and (2) to "Twenty flies" is clearly a non truth-preserving one: it may happen that Twenty is a penguin and the proposition that it flies is a false one. Therefore, common sense inferences are inductive in essence[74].

It is important to observe that the term "induction" has rarely been used to refer to this non truth-preserving feature of common sense reasoning. Rather, AI theorists adopted the already mentioned and much more technical term "nonmonotony" or "nonmonotonicity." Accordingly, the

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[71] Peirce (1957), sect. 8.223, Meyerson (1989), Hanson (1961), Salmon (1953), Shapere (1966).
[72] Salmon (1966), p. 114.
[73] Rescher (1976), Polya (1954), Leclercq (1974).
[74] For discussions about the similarities between philosophy and AI concerning inductive inferences see Pollock (1988), Kyburg (1991) (1994) and Tan (1991) (1997).

logical systems developed to formalize this sort of reasoning were named *nonmonotonic logics*[75]. Another important difference is that since the very beginning AI theorists preferred qualitative approaches over quantitative ones. The central point however is that since nonmonotonicity is essentially the same as non truth-preservingness, these common sense inferences are nothing more than a special class of induction conceived according to our contemporary sense[76].

Despite the lack of the term "induction," from the very beginning AI theorists made use of terms like "plausibility" and "plausible" to informally explain the nature of common sense inferences. For instance, at the beginning of his classical paper "A Logic for Default Reasoning", Raymond Reiter says[77]:

> Various forms of default reasoning commonly arise in artificial Intelligence [...] Reasoning patterns of this kind represent a form of *plausible inference* and are typically required whenever conclusions must be drawn despite the absence of total knowledge about the world.

Others such as D. McDermott, Robert Moore, Judea Pearl and Kyburg have equally used expressions like "plausible inference" or "plausible conclusion."[78] However, akin to the work of most logical probabilists, these uses of "plausibility" were mostly intended to help the reader to comprehend the intuitions lying behind the formal systems at hand.

More comprehensive accounts of plausibility has been provided by theorists such as Tarcisio Pequeno, D. Lehmann, Karl Schlechta and David Billington, who someway or another are inclined to classify their systems as logics of plausibility[79]. As would be expected however, all these works emphasize primordially the formal aspects of their systems and their application in AI problems. And even though we may say some important insights about the nature of plausibility have been given in these works, as a rule what we may call "a philosophical explication of a concept" is absent from them.

With exception to the works of Reiter and Pequeno, it is not our purpose here to give details about these plausible logics, both either in philosophy or AI. We mention them just to give the reader an idea of the extent to which the notion of plausibility has been used in these domains.

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[75] For an overview of the filed of nonmonotonic logics see Ginsberg (1987), Lukaszewicz (1990), Brewka (1991) and Gabbay et al (1994).

[76] In Chapter 4 we shall analyse to what extent nonmonotonic logics can be taken as logics of induction in the philosophical sense.

[77] Reiter (1980), p. 81. The italics are mine. See also Reiter & Criscuolo (1981).

[78] McDermott & Doyle (1980), Moore (1985), Pearl & Geffner (1990), Kyburg (1994).

[79] Pequeno & Buchsbaum (1991), Lehmann (1991), Schlechta (1996), Billington & Rock (2001).

CHAPTER 3

INDUCTIVE INCONSISTENCIES AND THE SKEPTICAL AND CREDULOUS APPROACHES TO INDUCTION

In this Chapter, we will investigate an aspect of inductive inferences that will play a very crucial role in our philosophy of induction and plausibility: the appearance of contradictions. After describing the problem, identifying its role in the philosophy of induction, and surveying the pertinent literature (Section 3.1), we turn in Section 3.2 to the two most important positions which one may take when faced with inconsistencies: the so-called skeptical approach to induction and the credulous approach to induction. After explaining these concepts and talking about how philosophers and IA theorists have dealt with them, we finally in Section 3.3 investigate the properties of the two plausibility notions that these approaches give rise: the skeptical notion of plausibility and the credulous notion of plausibility. As it shall become clear later, the study of these two notions is essential for properly explicating the notion of inductive plausibility.

3.1 The Problem(s) of Inductive Inconsistencies

Now we turn to an important facet of induction that will play a very significant role in the task of explicating the notions of induction and plausibility: the problem of *inductive inconsistencies*. In order to give a fairly introduction to that aspect of induction, we will have to speak a little bit about other field of the philosophy of science to which inductive inferences has been of considerable importance: *scientific explanation*.

Traditionally, the expression "scientific inference" has been used to designate those inferences meant to appraise theories and hypotheses. In those cases, the premises are the pieces of evidence and the conclusion is the confirmed (plausible or probable) theory or hypothesis. As we have seen, these inferences are intrinsically inductive. Besides these external (we use this term because this sort of inference is performed, we may say, outside the theory or hypothesis, which is the very thing we want to conclude), confirmatory scientific inferences, there are those performed inside scientific theories and aimed to go from the theory's basic principles to the derived ones. It is through these inferences that predictions and traditionally, but not non-controversially, explanations of singular facts and explanations of derived laws are obtained. These inferences, which we may call *internal scientific inferences*, have been traditionally understood as being strictly deductive.

This deductive view of internal scientific inferences was incorporated in its most precise form by Hempel and Paul Oppenheim's deductive-nonmological (D-N) model of scientific explanation, first published in 1948[1]. According to the D-N model, explanations and predictions of singular facts are deductive arguments whose premises are true, have empirical content, and contain at least one general law essential for the derivation of the conclusion[2]. Later on, in 1962, Hempel proposed what, according to him, would account for the scientific explanations that clearly could not be fitted into the D-N model: the inductive-statistical (I-S) model[3]. The general schema of I-S scientific explanations is the following:

$$\frac{\begin{array}{l} P(G, F) = r \\ Fb \end{array}}{Gb} \; [r]$$

Here the first premise is a statistical law asserting that the relative frequency of Gs among Fs is *r*, *r* being close to 1, the second stands for *b* having the property *F*, and the expression '[r]' next to the double line represents the degree of inductive probability conferred on the conclusion by the premises. We notice that even though the probability value of the statistical law and the argument are identical, they are not the same sort of probability: while the first stands for a statistical nomic connection between two properties of objects, the second stands for a relation of evidential support between premises and conclusions. Because of that the model is called inductive-statistical.

If we ask, for instance, why John Jones (to use Hempel's favorite example) recovered quickly from a streptococcus infection, we would have the following argument as the answer:

$$\frac{\begin{array}{l} P(G, F \wedge H) = r \\ Fb \wedge Hb \end{array}}{Gb} \; [r]$$

where F stands for having a streptococcus infection, H for administration of penicillin, G for quick recovery, b is John Jones, and r is a number close to 1. Given that penicillin was administrated in John Jones case (Hb) and that most (but not all) streptococcus infections clear up quickly when treated with penicillin ($P(G,F \wedge H) = r$), the argument above constitutes the explanation for John Jones's quick recovery.

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[1] Hempel & Oppenheim (1948).

[2] It is important to note that to Hempel, predictions and explanations of singular facts have the same logical structure. The only difference between them is that while in an explanation the conclusion of the inference is already known, in a prediction it is unknown. See Hempel (1965), pp. 366-376.

[3] Hempel (1962).

However, it is known that certain strains of streptococcus bacilli are resistant to penicillin. If it turns out that John Jones is infected with such a strain of bacilli, then the probability of his quick recovery after treatment of penicillin is low. In that case, we could set up the following inductive argument:

$$\frac{\begin{array}{l} P(G, F \wedge H \wedge J) = r' \\ Fb \wedge Hb \wedge Jb \end{array}}{Gb} \quad [r']$$

or, equivalently,

$$\frac{\begin{array}{l} P(\neg G, F \wedge H \wedge J) = 1\text{-}r' \\ Fb \wedge Hb \wedge Jb \end{array}}{\neg Gb} \quad [1\text{-}r']$$

where J stands for the penicillin-resistant character of the streptococcus infection and r' is a number close to zero (consequently, 1 – r' is a number close to 1.) This situation exemplifies what Hempel called explanatory or *inductive ambiguities*. In the case of John Jones's penicillin-resistant infection, we have two inductive arguments where the premises of each argument are logically compatible and the conclusion is the same. Nevertheless, in one argument the conclusion is strongly supported by the premises, whereas in the other the premises strongly undermine the same conclusion.

Since the publication of Hempel's paper, many attempts to solve this problem were proposed. It should be remarked that since philosophers working on the field of scientific explanation are obviously concerned with explanation and not with induction, their solutions to the problem of *explanatory* ambiguities are not necessarily solutions to the problem of *inductive* ambiguities. Among all models of explanation, only those which take inference as the key concept of scientific explanation may be of interest here. And among these inferential models, only those that admit the existence of non-deductive explanations can give us hope to solve the problem of inductive ambiguities[4]. Given this, we will concentrate exclusively on the solution given by Hempel.

Hempel tried to solve the problem of ambiguities by proposing a requirement, called by him the *requirement of maximal specificity* or RMS, which every inductive argument is supposed to satisfy to be classified as an authentic explanation[5]. The RMS was meant basically to prevent the property or class F (to be used in the explanation of Gb) from having a subclass F', let us say, such that the relative frequency of Gs among F's is different from P(G,F). In other words, in order to be used in an explanation, the class F must be *homogeneous* with respect to G. Clearly enough,

[4] For an overview of the problem of inductive ambiguities inside scientific explanation see Salmon (1989), chaps. 2 and 3. See also Coffa (1970).

[5] Hempel (1965), p. 400. Later on, in order to response to a counterexample to RMS construed by Richard Grandy, Hempel proposed a modified version of the RMS called RMS*. See Hempel (1968).

Hempel's solution is able to deal only with those conflicts taking place between a class and one of its subclasses. Moreover, it has the obvious consequence of limiting our inferential power. First, all cases of ambiguities involving non-inclusive classes will be blocked by the RMS. Second, even regarding those cases the RMS was meant for, only in those rare situations where we have a quite complete knowledge about the properties at hand is that we can infer something[6].

Even though it was inside the field of scientific explanation that most of the research on the problem of inductive ambiguities was done, it is important to note that it is not an exclusivity of internal scientific inferences. Elsewhere, Hempel has explained the same problem from the viewpoint of confirmation[7]. That the problem of inductive ambiguities can be explained from the viewpoint of both internal and external inductive inferences is not surprising. Being inductive arguments, the explanatory/predictive inferences may also be seen as sorts of confirmatory inferences. In the example above, we concluded that (1) it is highly probable that John Jones will recover quickly based on the information that (2) he has a streptococcus infection, (3) he took penicillin and that (4) most streptococcus infections clear up quickly when treated with penicillin. As the inference concerning penicillin-resistant bacilli shows, conclusion (1) may be defeated in the presence of additional information. Therefore (2)-(4) do not prove or establish (1) conclusively, but just serve as *evidences* to it. Therefore, forgetting about whether or not we know the truth-value of (1) and taking only the inferential relation between it and (2)-(4), we can fairly name (1) and (2)-(4), respectively, hypothesis and evidences.

This shows that the arising of contradictions is not due to the purpose with which the inductive inference is performed, but in fact to its non truth-preserving nature. To make this point clearer, consider two consistent sets of premises, A and B, each of them deductively related to the conclusions α and β, respectively. Since A and B are consistent, there is at least one interpretation that satisfies both sets of propositions. Let us call this interpretation I. Also, since the relation between A and B and their respective premises is truth-preserving, I satisfies both α and β. Therefore, α and β are necessarily consistent. On the other hand, if the inferential links between A and α and B and β are not deductive but inductive, we cannot be sure that α and β will be satisfied by I. Consequently, since there is no guarantee that there exists at least one interpretation able to satisfy simultaneously α and β, there is no guarantee of consistency between them. In other words, giving up truth-preservingness implies opening the door to ambiguities: even when the premises of two inductive inferences are consistent, they may give evidential support to contradictory conclusions.

[6] See Silvestre & Pequeno (2004).
[7] Hempel (1960) and (1966).

Given the above discussion, it is expected that the problem of inductive ambiguities have appeared in other fields like AI which are also concerned with inductive inferences. Not surprisingly, the very same problem of inductive ambiguities (naturally under a different label) has been one of the main concerns of AI theorists working on the formalization of commonsense reasoning. It has been identified in connection with most of the nonmonotonic formalizations, such as Reiter's default logic, McDermott-Doyle's non-monotonic logic and McCarthy's circumscriptive logic[8]. To many AI experts such as D. Israel, Donald Perlis and Tarcisio Pequeno, the arising of inconsistencies is not simply an unfortunate feature of the available formalisms, but in fact an inevitable and essential characteristic of commonsense reasoning[9]. As Pequeno wrote[10]:

> Inconsistency and nonmonotonic reasoning play complementary roles in the common sense reasoning. Nonmonotonic reasoning can lead to contradictions and to achieve them is just to give the right account for the situation. [...] The achievement of contradictory conclusions is a natural, and possibly unavoidable, companion of nonmonotonic reasoning.

To exemplify this, let us consider one of the most important sorts of commonsense reasoning discussed in AI literature: the so-called *default reasoning*. Consider an expanded version of the example of Twenty given in the last section (which is an instance of default reasoning) where we have two default statements (that is, statements of the form "usually A's are B's."), one universal statement and a singular fact: (1) usually animals do not fly; (2) usually birds fly; (3) birds are animals; (4) Twenty is a bird. If we use (2) along with (4) we will conclude that Twenty flies. But if rather we use (1) along with (3), (4) and *modus ponens*, we will get the contradictory conclusion that Twenty does not fly.

Akin to John Jones' case, we know in the above example that one of the contradictory conclusions is the right one. In the same way that patients infected with penicillin-resistant bacilli exceptionally do not recover quickly; it is true that animals usually do not fly, but birds are an exception to that. Naturally then, the efforts of both AI researchers and philosophers of science were directed towards the elaboration of mechanisms that could prevent the arising of these undesired, anomalous conclusions. It is not surprising then that AI theorists have named the first sort of inconsistency problem to be recognized inside a non-monotonic formalism – Reiter's default logic – the *problem of anomalous extensions*[11].

It is important to note that it is not always the case that we are able to know which one of the statistical laws or default statements has priority over the other. Since inductive reasoning is intrinsically reasoning in the presence of incomplete information, it may happen that we simply do

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[8] Reiter (1980), McDermott & Doyle (1980), McCarthy (1980). See Perlis (1987).
[9] See Israel (1980), Perlis (1987) and Pequeno (1990).
[10] Pequeno (1990), pp. 2 and 4.
[11] Morris (1988).

not know which of the two contradictory conclusions is the right one. In order to illustrate that, we give two very simple examples taken from both AI and philosophy of science literature. Consider first two default statements saying that usually Quakers are pacifists and usually republicans are bellicose. But if we take Richard Nixon, who was both a Quaker and a republican, we will be able to conclude that he is a pacifist and that he is a bellicose, that is, a non-pacifist. Second, consider two statistical laws saying that most Texans are millionaires and most philosophers are not millionaires. But if we take our old friend John Jones, who happens to be a Texan philosopher, we will be able to conclude that John Jones is a millionaire and he is not millionaire[12]. Supposing that in these two examples the mentioned statements are the only information we possess, we will have no choice but to admit our incapability to decide which one of the contradictory conclusions is the right or desirable one.

The similarity between the two above-mentioned examples is not mere coincidence. That is still a relatively unexplored issue, but due to the very fact that default reasoning and inductive-statistical explanation both deal with inductive inferences, they are in a very important sense different manifestations of the same problem[13]. Even though this point may look quite obvious to many (ourselves included), it may be worthy to talk a little bit about it. One of the ways to see the similarities between the two sorts of reasoning is to characterize them in terms of evidential support. We have already done this in connection with explanatory reasoning. Regarding default reasoning, take the first version of our Twenty example. There, we concluded that (1) (it is plausible that) Twenty flies based on the information that (2) it is a bird and (3) usually birds fly. But again, Twenty may be a penguin and not be able to fly. Therefore, (2)-(3) do not conclusively show (1) to be true, but, we may say, just give evidence for it. So, akin to what we have done in John Jones example, we may fairly call (1) the hypothesis and (2)-(3) the evidences. We remark that this translation of any sort of inductive reasoning in terms of hypothesis and evidences is in accordance with our decision to explicate "is plausible that α" through "there are evidences for α."

Another classical philosophical problem concerned with ambiguities is the famous *lottery paradox*, first formulated in 1961 by Henry Kyburg[14]. As mentioned before (Chapter 2), the lottery paradox appeared initially as a consequence of the attempts to detach inductive conclusions, taking them as reasonable or acceptable beliefs. If we do like most inductivists and take the logical confirmation relation as a probability function, one of the most straightforward ways to formulate a so-called rule of acceptance will be that which takes highly probability as our criterion of acceptance:

[12] These examples are due to Reiter & Criscuolo (1981) and Coffa (1974), respectively.
[13] See for instance Tan (1997).
[14] Kyburg (1961).

h is rationally accepted if and only if there is some *e* such that $P(h,e) \geq 0.9$ (or some other high value) and some total evidence criterion is satisfied[15].

Now, consider a lottery with a million tickets that is assumed to be administrated fairly. Since there is a probability of 0.999 on this evidence that ticket no. 1 will not win, we can rationally accept statement "ticket no. 1 will not win." The same of course is valid for all other tickets. But if α and β are both accepted or rationally warranted, it is reasonable to suppose that $\alpha \wedge \beta$ is also so. Therefore, we will have to take the statement "ticket no. 1 will not win & ticket no. 2 will not win & ... & ticket no. 1000 will not win" as rationally warranted. Consequently, we have to accept as true the hypothesis that the lottery will have no winner, which contradicts our initial assumption that the lottery is a fair one.

This same paradox may be formulated in connection with die-tossing. If we take our threshold level of acceptance as being 0.8, for example, then we will have to accept all six statements "this fair and normal die will not come up *i* when tossed", i=1,...,6. But if we accept the conjunction of these six statements as equally reasonable, we will have to admit the absurd conclusion that it is reasonable to belief that no side at all will come up after our fair die is tossed[16].

In contrast to the first problem of inductive inconsistencies, the above contradictions apparently do not rest solely on the non truth-preserving feature of inductive inferences. Beyond the assumption that it is not rational to accept or believe in an inconsistent proposition, the lottery paradox depends on two more postulates: the rule of acceptance itself, which says when a statement is to be rationally accepted, and a sort of acceptance conjunction principle which states that if α and β are both accepted statements then $\alpha \wedge \beta$ is also so. Another important point is that the paradox seems at first glance to belong to that class of "two-all draw" inconsistencies, that is to say, those cases where the knowledge at hand does not give us means to decide for any one of the contradictory conclusions. Like our Texan-philosopher and republican-Quaker examples, we are not ready to give up neither the conclusion that at least one ticket will win nor the conclusion that no ticket will win.

For our purposes here, it suffices to mention four main paths one may follow in order to solve the lottery paradox. The first one is to try to formulate some other acceptance rule (still under a numerical probabilistic framework) that would prevent the paradox from arising. Keith Lehrer has

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[15] P(h,e) represents the conditional probability of h given e. Most formulations of the lottery paradox disregard completely the need of satisfying some sort of total evidence criterion. Exception to this can be found in Cohen (1989a). As we have shown however, any formulation of (detaching) inductive inference that do not do this sort of requirement is prone to be mistaken.

[16] See Rescher (1976), pp. 35-36.

followed this path[17]. A second one is not to put too much emphasis on the rule of acceptance but rather to postulate the thesis that warrant judgments do not respect the conjunction principle: we can infer that each ticket will lose but we cannot go on and infer from this that all tickets will lose. Kyburg has been the main defender of this approach[18]. The third approach is to keep the conjunction principle unchanged but to restrict (even more) our inferential power by saying that in conflicting cases no detachment whatsoever is possible. Hintikka and more recently John Pollock and R. Stalnaker have adopted this way[19]. Still a fourth, not so far well-explored way is to drop the principle of contradiction and admit the acceptance of inconsistent statements without trivializing the logical theory. Kyburg has recently investigated this possibility in connection with the main available paraconsistent logics[20]. In the next section we will examine the implications of each one of these approaches (with the exception of the first) to our analysis of the notion of plausibility.

Many authors such as H. Heidelberger, Francis Dauer and Pollock have taken the lottery paradox as a decisive argument against numerical acceptance rules and for the thesis that the concept of warrant as used in the context of knowledge claims is structurally different from that of probability as defined by the probability calculus[21]. If these philosophers were right in arguing that the lottery paradox evidences the inadequacy of the numerical approach, then a purely qualitative approach would not be susceptible to the same sort of problem. In fact, this is one of the arguments John McCarthy and Pat Hayes used in their classical 1969 paper to defend the use of qualitative models of knowledge representation in AI[22]. However, the same sort of problem has been recently shown to be present also in connection with traditional qualitative approaches to commonsense reasoning[23]. To illustrate this we will cite an example due to Donald Perlis and named by him the *paradox of the Zookeeper*. Consider a zookeeper who takes care of exactly 1000 birds and happens to have strong evidences for the hypothesis that at least one of them is ill. Of course he still knows that usually birds fly. If he makes use of this default statement he will be able to conclude for every one of the 1000 birds that (it is plausible or worth believing that) it flies. Taking then the conjunction of these 1000 conclusions he will have that (it is plausible that) all birds fly, which contradicts the initial hypothesis that at least one of them is incapable of flying[24].

Of course one may object that all the above paradoxes are due to the very specific characteristics of the situations in question, where there is always one assumption (the fairness of

[17] Lehrer (1970).
[18] Kyburg (1964), (1970) and (1997).
[19] Hintikka & Hilpinen (1966), Pollock (1987), Stalnaker (1984).
[20] Kyburg (1997).
[21] Heidelberger (1963), Dauer (1980), Pollock (1983).
[22] McCarthy & Hayes (1969).
[23] See Perlis (1987) and Poole (1991).
[24] Perlis (1987).

the lottery, the necessity of at least on side of the die to come up, and the illness of one of the birds) that contradicts the conjunction of the inductive conclusions. That is to say, it is nothing more than the result of an unhappy combination of more or less artificial conditions, adding nothing to our understanding of the problem of inductive inconsistencies. As a preliminary response to this[25], we can mention the formulation that some authors have made of a lottery-like paradox which shows that, independently of the sort of situation one is dealing with, any *fair* account of (epistemic) inductive reasoning is sure to arrive at contradictions. By fair account of inductive reasoning we mean an account that takes into consideration what Perlis called *introspective inductive reasoning*[26], that is to say, the awareness of the error-prone feature of inductive reasoning.

Since inductive conclusions may be mistaken even when its premises are certain (something the very past use of such sort of inference has shown), any fair account of inductive reasoning should have as premise an axiom saying that, independently of the circumstance we are working on, it is plausible that one of the beliefs we now take as rational is false. Let ? represent the plausibility or acceptability of statements and $\{\alpha_1, \ldots, \alpha_n\}$ a set of statements such that, for $i=1,\ldots,n$, $(\alpha_1)?$ is true. In this way, the mentioned axiom can be represented the formula $(\neg(\alpha_1 \wedge \ldots \wedge \alpha_n))?$. But if all α_i's are plausible, then their conjunction should also be so: $(\alpha_1 \wedge \ldots \wedge \alpha_n)?$. We therefore have a contradiction. Kyburg has used this paradox as one more argument against the conjunction principle, and D. Israel has used it to argue for the inconsistency of commonsense reasoning[27]. Elsewhere, a very similar paradoxical situation has appeared under the name of *paradox of the preface*[28].

3.2 The Skeptical and Credulous Approaches to Induction

What all the discussion of the preceding section shows is that inconsistencies are sure to appear when one deals with inductive conclusions. They are not an accidental phenomenon concerning some sorts of application, but rather something inherent to inductive reasoning. But if this is so, how then are we to proceed when faced with inconsistencies?

Before answering this question, we need to make explicit a distinction we have already mentioned between those inconsistencies which are due to the inability of our formal devices to block the undesired conclusions, and those which we have classified as "two-all draw" inconsistencies, that is to say, inconsistencies which are due to the very nature of our knowledge

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[25] The rest of the response will be given in Section 3.3, where we will show the relevance of the notion of sceptical plausibility or acceptance for the task of explicating the notion of plausibility.
[26] Perlis (1987). As a matter of fact, Perlis uses in this paper the term "introspective default reasoning," rather "introspective inductive reasoning."
[27] Kyburg (1997), p. 113 and Israel (1980).

situation, which does not supply any means for us to decide for one of the contradictory conclusions. For the first sort of case the answer to our question is obvious: find a more powerful representational tool able to draw just the intended conclusion. Concerning the second one, the situation is not so easy. Since in the light of the available knowledge both of the contradictory conclusions seem to be equally well supported by the evidences, the wanted solution of choosing one of the conclusions is simply unrealizable.

Independently of adopting a qualitative or a quantitative account, it seems to us that there are two broad ways one can deal with this question. The first one is to adopt a *skeptical* or cautious position and not to allow any sort of ambiguity. Even in the case where two hypotheses are equally supported by the available evidence, if they contradict each other, both of them are to be rejected. A skeptical posture regarding the lottery paradox, for instance, would not unable us to conclude anything whatsoever about our 1.000 tickets: since they contradict each other, none of them will be take as sound conclusions. The other possible way is, through the adoption of a *credulous* or brave position, to accept any sort of inductive conclusion and try to somehow manage the arising of contradictory conclusions. In our lottery example, this would mean that, independently of the logical result that this may entail, all the 1.000 conclusions about our lottery tickets are to be taken as sound conclusions.

Let K be a consistent set of statements. Supposing the existence of some inductive mechanism of inference (closed under deduction) to be applied to the members of K, we name each maximal consistent set of conclusions obtained from K an *extension*[29]. The cases where there is more than one extension mean that ambiguities are obtained from K. While a skeptical treatment implies to take as sound only those inferences whose conclusions belong to the *intersection* of all extensions, a credulous one recognizes as sound every inference whose conclusion belongs to the *union* of the extensions[30]. We will name these two positions regarding the way one may deal with "two-all draw" contradictions the *skeptical approach to induction* and the *credulous approach to induction*.

An immediate consequence of that is that, if we do like we are doing here and decide to attach the label "plausible" to the conclusions of inductive inferences, then we will have two different ways of evaluating the plausibility of a statement. While the first, skeptical one will take as plausible only those statements which are consistent with all others inductive conclusions (in other words, those which belong to all extensions) the second, credulous one will take any inductive

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[28] Makinson (1965).
[29] The term "extension" is borrowed from nonmonotonic logic terminology.
[30] It should be observed that even though from a strict point of view these two approaches make sense only in connection with "two-all draw" cases of inconsistencies, they can be adopted as general guidelines to be followed in any sort of inconsistency. In fact, having not always been able to realize that their study cases

conclusion as plausible no matter what its relation with the other conclusions is (in other words, everything that belongs to at least one extension.) As a result of these two ways of saying when a statement is plausible, the inevitable conclusion is that the notion of plausibility is in fact what we may term a twofold concept, formed by two different but intimately connected notions of plausibility: the *skeptical plausibility* and the *credulous plausibility*.

According to the way we are taking the plausibility of a statement α, namely as meaning "there are evidences for α," these skeptical and credulous plausibility notions can be distinguished from each other by reference to what will count as evidence. While in the credulous approach the knowledge of the truth of the evidences along with the satisfaction of some total evidence condition is enough to classify the hypothesis as plausible, the skeptical one requires more: the hypothesis must be consistent with all other inductive conclusions. If that is not the case, even when the premises are known to be true and the total evidence condition at hand is satisfied, the premises cannot be taken as evidences for the hypothesis.

This can be more precisely put by bringing together the authority-oriented approach to plausibility mentioned in the last chapter and the notion of extension just explained. More specifically, we can say that each extension is a consistent set of statements representing the views of some leading experts about issues belonging to his domain of research. Since the opinions of such experts may be wrong, their conclusions should not be taken as certain, but just as plausible. The existence of more than one extension will amount to the existence of divergences or contradictions between the views of two or more experts. To an outsider then, there will be two ways of evaluating the plausibility of statements concerning the field at hand: skeptically, requiring all experts to agree on the statement, and credulously, requiring at least one of them to hold it. While for the first approach evidence is something very strong, namely, the consensus among the experts, for the second one it is much weaker: just the conformity of at least one expert. Because of that, we will sometimes refer to the skeptical and credulous plausibility notions, respectively, as the *strong plausibility* and the *weak plausibility*.

From a general point of view, we can say that the credulous and skeptical approaches represent, respectively, minimizing and maximizing strategies of truth assessing. According to the exposed in the last paragraph, if one adopts a credulous position he will not require too much to accept a statement α as plausible. If we follow many logicians and decide to use 1 to represent truth (and 0 to represent falsehood), this can be restated by saying that he will somehow try to *maximize* or bring close to 1 the truth-value of statements like "α is plausible." On the other hand, if one adopts a skeptical positional, he will be more demanding in the matter of accepting α as plausible, which

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could be solved by developing more powerful formalisms, philosophers and AI theorists have many times means that he will try to *minimize* or bring close to 0 the truth-value of statements like "α is plausible." Therefore, while adopting a skeptical position means to be strict in the matter of accepting something as true, in our case the plausibility of sentences, adopting a credulous one means to be tolerant, not so demanding in the matter of taking something as truth. Because of that, we will also refer to the skeptical and credulous points of view as, respectively, the *minimal* and the *maximal* positions.

The reference to the authority-oriented approach to plausibility is significant because it incorporates a very important aspect of the theory of induction and plausibility: the idea of a *plurality* of ways to achieve the same goal. This of course has to do with the picture where several experts are trying to evaluate, according their own criteria (which may of course be different and even incompatible with each other), the truth-value of a specific set of propositions. The same plurality effect could be obtained if we replaced the experts by several competing theories trying to account for the same range of physical phenomena. In this case, while a skeptical view would mean to accept as plausible a statement only if it is implied by all competing theories, a credulous one would mean to consider as plausible every statement that is implied by at least one theory. Another plurality-oriented situation is the one where an annalist, for example, has to make a decision based on the consideration of the different ways a specific situation may evolve. In this case, he will have at his disposal several incompatible scenarios on which he can base his decision. While by adopting a credulous strategy he would take as valuable all facts present in any one of the scenarios, by adopting a skeptical strategy he would consider only those facts present in all scenarios. We will call this way of explicating the notion of plausibility the *plurality approach to plausibility*.

Concerning inductive inferences and the arising of ambiguities, the extensions – consistent sets grouping the conclusions of an inductive mechanism of inference – may be said to represent different ways the conclusions may be extracted. We will have then a similar sort of plurality where each extension can be though of as a plausible scenario. In the case of a Texan philosopher knowledge situation, for example, we will have at least two different plausible scenarios: one in which John Jones is millionaire and other in which he is not. The important question here however is not which scenario is the right one, for, as we have seen, we do not have means to decide between them, but instead *how to proceed given this ignorance of ours and the mentioned plurality of scenarios*.

This brings us to the important relation that exists between induction, plausibility and the skeptical and credulous approaches to induction. As far as we reason inductively, we are sure to arrive at contradictory conclusions and therefore at a plurality of scenarios. But due to the lack of

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stuck to one of these two paths.

criterion of preference to choose one these scenarios, we have, as a matter of acting, to consider at least the skeptical and the credulous positions. Deciding then to call the conclusions of inductive inferences plausible facts will force us to deal with two plausibility concepts: a skeptical, strong or cautious plausibility and a credulous, weak or brave plausibility. The skeptical and credulous positions, which can be made explicit with the help of the two corresponding plausibility concepts, can be said then to be the minimal, necessary approaches that all answers to the mentioned question should include. In other words, we cannot claim to have properly explicated the notions of induction and plausibility without having gone also through these skeptical and credulous approaches to induction. As we will see in Chapter 5, a good part of our theory of plausibility will be devoted to the analysis of the intrinsic characteristics of these skeptical and credulous approaches to induction materialized in the notions of skeptical plausibility and credulous plausibility.

Traces of these two approaches can be found in both philosophical and AI literature of induction. Most of the time however, theorists have adopt exclusively one of the two approaches without neither taking into consideration nor being conscious about the reasonableness of the other approach. Usually, when some sort of distinction was made, it was on a meta-theoretical level. For instance, AI theorists working in the field of nonmonotonic logic have used the distinction between the credulous and the skeptical approaches to classify the different sorts of formalisms to commonsense reasoning[31]. It is widely acknowledged among AI theorists that while Reiter's default logic uses a sort of credulous approach, McCarthy circumscriptive logic takes a skeptical one. However, as pointed out by David Makinson, this classification is more due to an historical accident than to an intrinsic feature of these systems, for it is always possible to use these systems either under a skeptical perspective or under a credulous one[32].

Also very seldom the awareness of these two approaches has given rise to a conceptual distinction between the skeptical and credulous concepts of plausibility. The only exception seems to be the work of Pequeno and his collaborators, who identified both approaches and their respective plausibility concepts as well as incorporated them inside a sole framework[33]. Even though with a somehow different motivation and using another methodology, the distinction made by Pequeno between the skeptical and the credulous notions of plausibility is the same as the one we are making here. Incidentally, our formal analysis of induction and plausibility will be strongly influenced by Pequeno's approach. The description of Pequeno's logical systems as well as the

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[31] D. McDermott (1982) was the first to propose a specific terminology to make such sort of distinction. He used the terms cautious (for skeptical) and brave (for credulous), but others terms such as "conservative" and "liberal" and our "skeptical" (or "sceptical") and "credulous" have also been used. See Ginsberg (1994) and Makinson (1994).

[32] Makinson (1994).

relevant references will be given in the next chapter. In the rest of this section and in the next one we will try to localize in the relevant literature references to this methodological and conceptual dichotomy in hopes of identifying the most relevant features of our two plausibility concepts.

In his classical paper "Studies in the Logic of Confirmation" of 1945, Carl Hempel proposed a set of necessary conditions that any definition of confirmation is supposed to satisfy. By definition of confirmation he meant any way of saying, for any pair of evidence and hypothesis *e-h*, whether or not *e* confirms (or inductively supports) *h*. Among the conditions he lays down, one of them along with its two most important consequences have been particularly controversial[34]:

> (8.3) *Consistency Condition*: Every logically consistent observation report is logically compatible with the class of all the hypotheses which it confirms.
> The two most important implications of this requirement are the followings:
> (8.31) Unless an observation report is self-contradictory, it does not confirm any hypothesis with which it is not logically compatible.
> (8.32) Unless an observation report is self-contradictory, it does not confirm any hypotheses which contradict each other.

The reason of the controversy is described by Hempel himself. Right after the above quotation he adds[35]:

> The first of these corollaries will readily be accepted; the second, however,– and consequently (8.3) itself–will perhaps be felt to embody a too severe restriction. It might be pointed out, for example, that a finite of set of measurements concerning the variation of one physical magnitude, *x*, with another, *y*, may conform to, and this be said to confirm, several different hypotheses as to the particular mathematical functions in terms of which the relationship of *x* and *y* can be expressed; but such hypotheses are incompatible because to at least one value of *x*, they will assign different values of *y*.

However, even after admitting the polemic side of (8.31) as well as the possibility "to liberalize the formal standards" by "dropping (8.3) and (8.32) and retaining only (8.31)."[36], Hempel decides at the end to maintain his original formulation. The important point for us is that Hempel's two options of keeping (8.31) and rejecting it are respectively related to what we have called the skeptical and credulous positions. If we decide to keep (8.31), in the case where *e* gives inductive support to both *h* and *h*' but *h* and *h*' happen to be mutually inconsistent, we are not allowed to say neither that *e* confirms *h* nor that it confirms *h*'. Consequently, even if *e* is true (or certain) and the requirement of total evidence (or some other total evidence condition) is satisfied we cannot say

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[33] See Pequeno & Buchsbaum (1991), Martins (1997) and Buchsbaum et al (2004).
[34] Hempel (1945), p. 105. The italics are in the original. In Section 6.4, we will analyze these and the others conditions laid down by Hempel in the light of a purely descriptive approach to induction.
[35] Ibid, pp. 105-106.
[36] Incidentally, as pointed out by many authors and acknowledged by Hempel himself, the satisfaction of the others conditions (8.1) and (8.2) along with (8.31) logically implies the satisfaction of (8.3). See Coffa (1970).

that h and h' are plausible. Conversely, if we drop (8.31) we are allowed to have in the same set of plausible facts contradictory hypotheses.

Many authors such as Carnap, Popper and Feyerabend have urged against the tenability of (8.32)[37]. The whole point of course is centered around the arising of contradictions. As Hempel himself admitted, we fell that (8.32) is a too strong condition exactly because it prohibits the inevitable cases where the same piece of evidence inductively supports two contradictory hypotheses. According to the ordinary notion of confirmation that laymen and scientists have in mind, the same (consistent) body of evidence do sometimes confirm contradictory hypothesis. Let α, β and φ be three pieces of evidence belonging to the same (consistent) body of evidence in such a way that, according to some criterion of confirmation, α, β and φ confirm, respectively, α', β' and φ'. Suppose still that α' and β' are mutually inconsistent. Accepting (8.32) then entails the disappointing conclusion that even though α and β are supposed to confirm α' and β' by the same criterion that φ confirms φ', we are not entitled at all to say that such confirmation occurs. A credulous approach then seems to be what Carnap, Popper and Feyerabend had in mind when they criticize (8.32) as well as the rationale behind Hempel's alternative of dropping (8.3) and (8.32) and retaining (8.31).

Despite the intuitive appeal, as the lottery paradox shows, the credulous approach has trivially a very problematic side: if we allow every sort of inductive conclusions, we are sure to arrive at inconsistencies. We could of course go further and ask why inconsistencies are problematic, but that seems to be a worthless point. After all, it is because of the inconsistencies that the whole thing is seen as paradoxical. How then AI theorists and philosophers have dealt with that in their attempts to account credulously for the problem of inductive ambiguities? One very common way adopted by AI theorists and well exemplified by Reiter's default logic is to admit contradictory hypothesis but not to bring them together. That is formally obtained by keeping the extensions (which are consistent sets of statements) both from the semantical as well as from the syntactical level separated from each other. Supposing the existence in the formal language of a plausibility modal operator, this would have the consequence of restricting the universe of discourse to one extension at each time in such a way that one would not be able to utter statements of the kind "α is plausible & ¬α is plausible." Each one of these sub-sentences could be uttered, but just inside different extensions.

In a very important sense, this solution adopted by Reiter has the disadvantage of not strictly satisfying Carnap's requirement of total evidence: in the cases where both α and ¬α are plausible, because they are kept separated from each other we will not be able to use both in the same

[37] Carnap (1962), pp. 476-478, Popper (1959), p. 374, Feyerabend (1962). pp. 28-97.

inferential chain. Consequently, important parts of the available knowledge may be left out of the solution of problems which depend on such sort of inferential procedure.[38] Because of that, some authors have acknowledged that this approach does not embody, strictly speaking, a credulous position. Makinson, for example, has created another expression to designate this sort of approach – *choice perspective* – and reserved the term "liberal" to a stricter one where the union of all extensions will effectively be considered as the set of plausible facts[39]. It is interesting to see his opinion about this liberal perspective, which reflects well the traditional view concerning a truly credulous approach: "The liberal perspective is also possible in principle, but is far less interesting: under most [formal systems], whenever there is more than one extension their union is inconsistent."[40]

In the context of the lottery paradox, Kyburg has provided a credulous account that, despite the criticisms, gives some important hints about how to refute Makinson's claim[41]. We have seen that a way of solving the lottery paradox is to reject that the conjunction principle can be applied to acceptance claims. According to this view, the fact that α is acceptable and β is also acceptable does not warrant us to conclude that $\alpha \wedge \beta$ is acceptable. The solution to our problem then comes trivially: even though we may have that α is acceptable and that $\neg\alpha$ is acceptable, we could not conclude that $(\alpha \wedge \neg\alpha)$ is acceptable. That is the way Kyburg claims to have solved the lottery paradox. As a matter of fact, one of his main theses is that the principle of conjunction is to be banished from the logical study of knowledge claims:

> [...] we should be cautions about proceeding from the acceptance of p_1 and the acceptance of p_2 to the acceptance of their conjunction. But that is something we knew all along: acceptability is not adjunctive. An argument that used enough empirical premises (say, a million), could surely use premises that are individually acceptable, but lead to a completely unacceptable conclusion.[42]

In AI, some theorists have also held a similar opinion. After analyzing Pollock's theory, Kevin Korb for example concludes the following: "A philosophy which endorses the Conjunction Principle, therefore, can hardly serve as a framework for developing an artificial intelligence that learns inductively about its world."[43]

Some philosophers such as Lehrer and Jonathan Cohen have taken the exact opposite direction[44]. They maintained that the conjunction of acceptance claims is a very intuitive and

[38] See Pequeno (1990).
[39] Makinson (1994).
[40] Ibid. p. 38.
[41] See Kyburg (1964) and (1970).
[42] Kyburg (1997), p. 119.
[43] Korb (1992), p. 234.
[44] Lehrer (1970), Cohen (1989a).

indeed necessary principle for the logic of induction. Cohen has given the following example[45]. Suppose an art historian declares two pictures to be genuine Vermeers. Intuitively, he seems to be giving us a warranty – the warranty of an expert – to believe that the first picture is a Vermeer, a warranty to believe that the second is, and a warranty (of just the same nature) to believe that they both are. Supposing then that the judgment of an expert is a good reason for us to accept a hypothesis, the claim that we are entitled to accept separately both statements but not their conjunction is simply absurd.

Hempel has also supported this view. Among the other conditions laid down by him in his 1945 paper, one states that "if an observation report confirms every one of a class K of sentences, then it also confirms any sentence which is a logical consequence of K."[46] Therefore, if α and β are both acceptable, all consequences of them, including $\alpha \wedge \beta$ will also be acceptable. Even though in the mentioned paper he does not refer to the concept of acceptance, elsewhere, in his 1962 paper, he states the same condition as well as the controversial consistency condition (8.3) – at this time without any sort of objection – regarding the acceptance of hypotheses:

> (CR1) Any logical consequence of a set of accepted statements is like-wise an accepted statement; or, K [the set of accepted statements] contains all logical consequences of any of its subclasses. [...] (CR2) The set K of accepted statements is logically consistent.[47]

About the other solutions to the problems of inconsistencies mentioned in the previous section, Hempel's RMS, Hintikka's, Pollock's and Stalnaker's all adopt a skeptical approach. By requiring the class F to be *homogeneous* with respect to G, Hempel accepts only those inductive conclusions that do not generate contradictions. Trivially enough, the RMS was conceived exactly to prevent the arising of contradictory explanations. Regarding the mentioned solutions to the lottery paradox, all of them, as we have said, solved the paradox by preventing detachment in case of conflict. Pollock, for instance, proposed what he called the principle of *collective defeat*, which basically states that if we have a set of inductive and therefore defeasible conclusions where for each conclusion there is an argument from the other conclusions and the knowledge situation to its negation, then none of the conclusions is warranted. As he puts it when speaking about the lottery paradox:

> Intuitively, there is no reason to prefer some of the [inductive conclusions] over the others, so we cannot be warranted in believing any of them unless we are warranted in believing all of them. But we cannot be warranted in believing all of them [because of the inconsistency].[48]

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[45] Cohen (1989a), p. 19. We have slightly modified the example in order to make Cohen's point clearer.
[46] Hempel (1945), p. 103.
[47] Hempel (1962). pp. 151-152.
[48] Pollock (1987), p. 494.

As Kevin Korb has pointed out, in the context of the lottery paradox, Pollock's solution can be seen as derivative from the contrapositive of our acceptance conjunction principle: if a conjunction is not acceptable, then not all of its conjuncts are acceptable[49]. But of course the fact that at least one of the conjuncts is not acceptable is quite different from all of conjuncts being unacceptable. Korb takes this as evidence for the absurdity of Pollock's principle of collective defeat and indirectly for the absurdity of our skeptical approach. Such a strong conclusion is clearly due to the fact that, because of the policy of preventing contradictions by blocking inductive conclusions, the skeptical approach restricts in a too severe way our inferential power. According to Kyburg and Harman, for example, this sort of solution to the lottery paradox (the one they analyze is Hintikka's[50]) does not model anything recognizable as scientific induction: the price it pays to avoid contradictions is to restrict acceptance to what are essentially redescriptions of the available evidence[51]. The conclusion of Korb is that Pollock's theory is plagued by the same sort of problem. We have elsewhere pointed out the same think regarding Hempel's RMS[52].

To sum up, we may say that an intrinsic feature of the skeptical approach is the acceptance of the conjunction principle. We have seen how Hempel's theory of confirmation and acceptance – which, despite not have been primordially conceived as a solution to the problem of inconsistencies, addopts a skeptical approach to induction – endorses the conjunction principle. Pollock also explicitly defends it. In fact, in the context of inconsistencies the necessity of dropping the conjunction principle is due to our tolerance regarding inconsistent conclusions. Since the skeptical approach blocks from the very beginning inconsistent hypothesis, there is no need to reject such an intuitive principle like that the conjunction of two accepted facts is likewise accepted.

3.3 Plausibility and Acceptability

We have already observed that, with exception to some AI scientists, in general theorists working on the problem of inductive inconsistencies have not been able to clearly distinguish between the two approaches to induction we are talking about here. The point is that while working on either a skeptical or credulous approach, most philosophers and IA scientists took their respective approaches as *the* solution to the problem of inductive inconsistencies, without recognizing the reasonableness of the other one. An example of this is the dispute over the tenability of the acceptance conjunction principle we have shown in the last section. We are going to argue in this

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49 Korb (1992).
50 Hintikka & Hilpinen (1966).
51 Kyburg (1970) and Harman (1967).
52 Silvestre & Pequeno (2004).

section that such quarrel is due to lack of a somehow more global perspective where both approaches are taken into account. When such position is adopted, the entire dispute is dissolved. In order to understand why, we have to analyze in more detail the twofold conceptual division we mentioned at the beginning of the last section between the skeptical and the credulous concepts of plausibility.

One of the rare occurrences of something akin to the above-mentioned conceptual distinction can be found in section 87 of Carnap's *Logical Foundations of Probability* where Hempel's theory of confirmation and his conditions of adequacy are analyzed[53]. Carnap points out that Hempel's controversy about condition (8.32) is due to a meaning confusion: in contrast with the unambiguous use Hempel thinks to be making of the term "confirmation," there are in fact two quite distinct confirmation concepts involved in his analysis of confirmation. According to Salmon, who has investigated the Hempel-Carnap dispute, this conceptual distinction may be explicated as follows[54]:

> On the one hand, we may intend to say that the special theory [of relativity] has become an accepted part of scientific knowledge and that it is very nearly certain in the light of its supporting evidence. If we admit that scientific hypotheses can have numerical degrees of confirmation, the sentence, on this construal, says that the degree of confirmation of the special theory on the available evidence is high. On the other hand, the same sentence might be used to make a very different statement. It may be taken to mean that some particular evidence – e.g., observations on the lifetimes of mesons – renders the special theory more acceptable or better founded than it was in the absence of this evidence.

He calls these two notions respectively the *absolute* and the *relevance* senses of confirmation. While the relevance sense may be said to fit into condition (8.32), the absolute one is incompatible with it[55]:

> [Hempel's condition 8.32] is suitable for the absolute concept of confirmation, but not for the relevance concept. For, although no two incompatible hypotheses can have degrees of confirmation on the same body of evidence, an observation report can be positively relevant to a number of different and incompatible hypotheses [...] This happens typically when a given observation is compatible with a number of incompatible hypotheses [...]

The important point here is the explicit recognition of two different conceptual approaches we may take regarding the appearance of contradictions: a skeptical and a credulous one. The explanation then for Hempel's apparently paradoxical behavior of defending both the rejection and acceptance of (8.32) is that he was unknowingly dealing with two different concepts of confirmation. Nevertheless, even though Salmon's relevance and absolute notions of confirmation can be said to be in accordance with our skeptical and credulous approach, they are not, strictly

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[53] Carnap (1962), section 87.
[54] Salmon (1975), p. 5.
[55] Ibid. p. 8.

speaking, skeptical and credulous notions of confirmation. If we attach to the absolute confirmation degrees inferior or equal to 0.5, for example, it will not satisfy any more (8.32), and if a hypothesis has a too high degree of confirmation on the background evidence then, no evidence will be able to be relevant both to it and to a hypotheses incompatible with it. Furthermore, these two concepts are concerned with the logical, and not with the epistemic or pragmatical notion of probability. Even though we may say that from the relevance point of view e confirms both h and $\neg h$, we cannot say they are probable or plausible. Also, the two concepts are not equally applied to hypotheses and evidences. While the absolute notion takes a fixed bound to say whether or not e confirms h, the relevance notion compares the degree of confirmation given by e to h on the presence of background evidence i with the degree conferred to h by i alone

More recently, Cohen has urged for a distinction between conditions for rationally *believing p* and conditions for rationally *accepting p*. According to him, the result of these two things' having been treated indistinguishably in much of the philosophical literature is a "widespread tendency to ignore the fact that there are many important differences between belief and acceptance, which are relevant to quite a range of issues in epistemology, the philosophy of science and cognitive science."[56] While talking about the controversy over the reasonableness of the acceptance conjunction principle and the alleged consequence of the introspective inductive reasoning (according to which, since it is plausible that some of our inductive conclusions are mistaken, it is not rational to take their conjunction as likewise plausible), he makes use of the mentioned distinction to show that the whole quarrel rests on a conceptual misunderstanding[57]:

> But here some rather subtle differences between *rationality of belief* and *justifiability of acceptance* are in play. We need to check our intuitions rather carefully. It may be rational to *believe* – in the everyday sense of the expression – that one may sometimes err. But if it is justifiable to *accept* H_1, justifiable to accept H_2, ... and justifiable to accept H_n, it is certainly justifiable to accept both the conjunction $H_1\&H_2\&\ldots\&H_n$ and also any logical consequence of that conjunction.

The suggestion implicit in this passage is that while the introspective principle of inductive reasoning is applicable to beliefs, the conjunction principle applies to accepted hypothesis. Consequently, unless acceptability implies belief (what Cohen explicitly denies), no contradiction will arise. To see how these features of belief and acceptance give rise to a credulous/skeptical conceptual distinction, we need just to require these two principles to be applicable *exclusively* to belief and acceptance, respectively. Letting ? represent rational belief and ! justifiable acceptability, we have then the following picture: (I) $(\neg(H_1 \wedge H_2 \wedge\ldots\wedge H_n))?$ but not $(\neg(H_1 \wedge H_2 \wedge\ldots\wedge H_n))!$; and (II) from $H_1!$, $H_2!$ ... and $H_n!$ we get $(H_1 \wedge H_2 \wedge \ldots\wedge H_n)!$ but from $H_1?$, $H_2?$ ... and $H_n?$ we do not

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[56] Cohen (1989b), p. 367.

[57] Cohen (1989a), p. 207. The italics are mine. See also Cohen (1986).

get $(H_1 \wedge H_2 \wedge \ldots \wedge H_n)$?. Without too much effort, we can see that (I) and (II) are in accordance with our credulous (or weak) and skeptical (or strong) plausibility concepts. First, by admitting that it is rational to believe but not to accept that some of our inductive conclusions are wrong, (I) sets rational belief as something we can arrive at much easier than acceptance. That is to say, our criterion of acceptance is much stronger than our criterion of rational belief. Second, by restricting the conjunction principle only to acceptability, (II) at the same time that leaves the path open so that we may admit contradictory beliefs and therefore have a credulous approach in the style of Kyburg, restricts acceptability to a skeptical treatment in the manner of Pollock. (II) also by itself resolves the lottery paradox and the controversy over the conjunction principle: supposing that the data about the lottery is enough for us to belief but not to accept that each one of the tickets will win, we can go on with our reasoning without bothering about inconsistencies and without going against the intuitiveness of the acceptance conjunction principle.

We have come then to a quite promising solution to the lottery paradox that unites the credulous and skeptical approaches we have shown in the last section. More specifically, we recognize the rationale of both positions concerning the conjunction principle and error-prone feature of inductive reasoning but applicable to only one of the two plausibility concepts. While skeptical plausible facts do conjoint but are not susceptible to the error-prone feature of inductive reasoning, credulous plausible are susceptible to the error-prone feature of inductive reasoning but do not conjoint. In this way the whole controversy is dissolved. While for example it is sound to believe or take as credulously plausible that each one of the tickets will not win, we cannot take as credulously plausible the statement that says that there will be no winner. On the other hand, if we could accept or take as skeptically plausible that each one of the tickets would not win, it would be equally acceptable that that there would be no winner.

Following Cohen's suggestion, we will use the term "acceptability" or "acceptance" to refer to our skeptical notion of plausibility. We will also sometimes use the term "plausibility" without any qualification to mean the credulous notion of plausibility. It should be noted that this terminological choice is not arbitrary. Indeed, all terms we have used so far to refer to our inductive conclusions could be fairly classified according to one of the two approaches. As the laymen and philosopher discourses show, while the terms "plausibility", "probability" and "rational belief" intuitively seem to be weaker and more susceptible to a credulous approach, "confirmation" and "acceptability" seem to be stronger and more subject to a skeptical approach. For instance, to confirm literally means to make firm, rigid, which in a semantical context would mean to be so strongly connected with truth that any change of opinion regarding a confirmed hypothesis would be done only through considerable effort. The same seems to hold for the term "acceptance." Taking for example the expert plurality-oriented model we have mentioned earlier, a statement would be skeptically plausible if and only if there is a consensus among the experts concerning the proposition in question. But it seems to us that being able to form such sort of consensus is the outmost criterion of acceptability of scientific or any other kind of hypothesis. Even though a hypothesis may not be certain (which seldom if ever happens) but all members of the community of experts in that subject find it reasonable, that collective appraisal allows one to take it (the hypothesis) as accepted[58].

A remark we have already made is that these terms have been many times used by philosophers with no connection at all with induction. We have already pointed this out regarding "probability" and "belief." As far as "acceptability" is concerned, there is one very common way of understanding it that should be sharply distinguished from what we can call *inductive acceptability*. We have already seen how in his 1962 paper Hempel supported a skeptical view of acceptability of hypotheses. Besides laying down important features of this inductive acceptability (like conditions CR1 and CR2 which we have quoted before) Hempel analyzed what at that time was the most common way of understanding the notion of acceptability: through some sort of pragmatic-utility approach. At the very beginning of the paper, he explicitly connects his notion of acceptance with inductive inferences and lays down the strong, skeptical aspect of this concept: "Thus, the study of inductive generalization gives rise to the question whether it is possible to formulate criteria for the *rational acceptability* of hypotheses on the basis of information that provides *strong, but not conclusive*, evidence for them." [59] After laying down some necessary conditions for acceptability, Hempel discusses how this notion could be formalized according to this pragmatic-utility approach[60]. After analyzing some possible ways of proceeding in this way, he lays down his conclusion[61]:

> The preceding considerations seem to indicate that it would be pointless to formulate criteria of acceptability by reference to pragmatic utilities; for we are concerned here with purely theoretical (in contrast to applied) explanatory and predictive arguments. We might just add the remark that criteria of rational acceptability based on pragmatic utilities might direct us to accept a certain predictive hypothesis [...] exceedingly improbable on the available evidence, [... for] what is qualified as rational is, properly speaking, not the decision to believe *h* to be true, but the decision to act in the given context as if one believed *h* to be true [...]

Referring to this non-equivalence between the theoretical and applied approaches to acceptability and the exclusively applied treatment given by philosophers at the time to the

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[58] Despite this, we will keep using the term "confirmation" and its derivates in the same way we are using the term "plausibility," that is to say, with both skeptical and credulous meanings. On the other hand, the term "acceptability" or "acceptance," by an imposition of its own nature, we think, will be used exclusively in the skeptical sense.

[59] Hempel (1962), p. 149. The italics are mine.

[60] See Braithwaite (1963). For more bibliographical references see Hempel (192), pp. 149-163.

[61] Hempel (1962), p. 162.

problem, he elsewhere expresses his doubts about the possibility of developing a purely logical account of acceptability[62]:

> Indeed, it is by no means clear whether the conception of basic scientific research as leading to the provisional acceptance or rejection of hypotheses is tenable at all. One of the problems here at issue is whether the notion of accepting a hypothesis independently of any contemplated action can be satisfactorily explicated within the framework of a purely logical and methological analysis of scientific inquiry.

We take this theoretical acceptability of Hempel as being the same as our skeptical or strong plausibility. Consequently, we also accept Hempel's challenge of developing a purely logical analysis of the notion of acceptability independent of any sort of contemplated action. What we believe to be an answer to such challenge will be presented in Chapter 6[63].

To sum up then, we have concluded that while skeptical plausible facts do conjoint but are not susceptible to the error-prone feature of inductive reasoning, credulous plausible are susceptible to the error-prone feature of inductive reasoning but do not conjoint. The difference concerning the error-prone feature can be generalized by saying that while we may admit contradictory plausible facts, we cannot allow α and $\neg\alpha$ to be both accepted or skeptically plausible at the same time. We also have shown that these principles are in accordance with the credulous and skeptical approaches. If by adopting a skeptical approach we prevent from the very beginning the arising of contradictory conclusions, there is no reason to reject such an important logical rule as the conjunction principle. In the case of the weak plausibility and the conjunction principle, its rejection is not only compatible with, but in fact necessary for a credulous approach. If we want to allow things like "α is plausible" and "$\neg\alpha$ is plausible," we have to somehow prevent that from this one conclude "$\alpha \wedge \neg\alpha$ is plausible." That is the essence of Kyburg's solution to the lottery paradox.

Now it is opportune to recall Makinson's claim that the credulous approach is uninteresting and our claim that Kyburg's solution gives some hints of how to refute that. Kyburg's solution of rejecting the conjunction principle prevents contradictions of the sort "$\alpha \wedge \neg\alpha$ is plausible." But how about cases like "α is plausible" and "$\neg\alpha$ is plausible"? Are they unproblematic as Kyburg believes? After all, a knowledge situation containing these two statements will strongly look like an inconsistent theory. Trivially, Makinson's classification of the credulous approach as uninteresting is not because of the inconsistencies *per se*, but because traditionally (that is, according to classical logic) inconsistencies lead to the trivialization of the logical theory. That is the famous *ex-contradictio sequitur quod libet* principle: from inconsistent statements one can infer anything.

[62] Hempel (1966), p. 130.
[63] See Swinburg (1970) for a lucid analysis of when a given theory of confirmation can be said to be also a theory of acceptability.

From a semantic point of view, this trivialization happens because no interpretation or model can satisfy an inconsistent set of statements. Since a statement α is a logical consequence of a set of statements A if and only if every model of A is a model of α, if A has a pair of contradictory sentences, then every α will by vacuity satisfy this criterion.

This seems to make then a credulous approach even in the style of Kyburg's unsusceptible of any sort of worthy logical analysis. This is, to speak the truth, quite disappointing. We have seen how philosophers have defended the rationality of the idea that a consistent body of evidences may inductively support two or more mutually contradictory hypotheses. By doing this however, they did not of course mean that in these circumstances every statement whatsoever would be turned into a "scientific (plausible) truth." Intuitively also we are ready to (rationally, we hope) accept the plausibility of α and the plausibility of $\neg\alpha$ without having to conclude every sort of nonsense. Therefore, both from an intuitive and from a confirmation point of view, the existence of plausible contradictions does not allow the inference of all sorts of statements. In other words, despite the mentioned feature of classical logic, it seems trivial that plausible contradictory facts do not lead to the trivialization of the logical theory to which they belong.

Given the way we have characterized the notions of plausibility and induction, inductive or plausible inconsistencies are a sort of what we can call *epistemic inconsistencies*. This is in sharp contrast with the inconsistencies that classical logic can be said to deal with, which can be named *ontological* inconsistencies. While the first sort of inconsistency concerns our way of seeing the word, the second concerns the world itself[64]. Therefore, it is understandable that classical inconsistencies are not susceptible of having any sort of semantical interpretation, for if they had, it would mean that there could be true contradictions in the world. Even though some philosophers such as Hegel have defended the idea that there are ontological or true inconsistencies, that is a very disputable thesis and one that has never generated any sort of consensus among philosophers and scientists[65]. On the other hand, epistemic inconsistencies are a very reasonable and uncontroversial idea. There is no absurdity in holding two mutually contradictory hypotheses as reasonable or plausible at the same time. It is consensual among scientists and philosophers that two mutually contradictory hypotheses may get the same amount of evidential support from the same body of evidences. The history of science is full of cases where contradictory hypotheses co-existed during considerable periods of time. In the theory of knowledge also, it is consensual that from the standpoint of our perceptions inconsistencies are sure to arise. Therefore, if the scientific and philosophical discourse, which we hope do represent something, contemplate the notion of epistemic inconsistencies, there must be some sort of semantic interpretation able to satisfy

[64] This distinction has firstly appeared in Rescher & Brandom (1980).
[65] That is the so-called *Heraclitus-Hegel's thesis*. See Petrov (1979).

inconsistent sets of plausible statements. And if there is at least one interpretation able to satisfy the set {"α is plausible", "$\neg\alpha$ is plausible"}, then from this set no trivialization should follow. Thus, the just mentioned idea that inductive contradictions are uninteresting is due in fact to a lack of understanding about the nature of inductive inconsistencies.

We see then that in order to deal with plausible contradictions a new sort of logic is required, namely one able to represent inconsistent but nontrivial theories. Since the 1970's, there is a specific term used to refer to those logical systems that do not trivialize in the presence of contradictions: "*paraconsistent*." The origins of such sort of systems go back to the work of J. Lukasiewicz, who published in 1910 a pioneer article proposing to revise some traditional laws of logic, particularly the principle of non-contradiction. In the 1940's, following Lukasiewicz's ideas, S. Jaskowski developed a propositional logical system where contradictions were tolerated without trivialization. Later on in the 1950's, Newton da Costa developed a first-order paraconsistent system with application to set theory that ultimately led to the development of several other systems and the establishment of paraconsistent logic as a independent domain of study[66]. Nowadays, paraconsistent logic is a growing and respectable field of logic with applications in many domains.

We will use the term "paraconsistent" to refer to that inconsistency-tolerance behavior that the notion of plausibility seems to require. Because from a credulous viewpoint plausible inconsistencies are allowed but no trivialization should follow from them, we will say that the very notion of credulous plausibility is essentially *paraconsistent*. This way of speaking of course necessarily involves a somewhat new understanding of the term "paraconsistency," which has traditionally been used in connection with logical systems or components of logical systems (often a paraconsistent logic is said to be one containing a paraconsistent negation) but never in connection with concepts. We may try to precise this by saying that the adjective "paraconsistent" in the expression "the credulous notion of plausibility is paraconsistent," means, in a broad way, something like that: the notion of credulous plausibility, if represented in a formal system, should incorporate some sort of mechanism which do not allow the trivialization of a theory containing plausible inconsistencies[67].

We note in passing that what we are taking as plausible inconsistencies are something like "α is plausible" and "$\neg\alpha$ is plausible" and not "$\alpha \wedge \neg\alpha$ is plausible." Intuitively, this can be justified by making reference to the distinction between epistemic inconsistencies and ontological inconsistencies. If we do not want to commit ourselves to the thesis that there exist in the world real inconsistencies, we have to maintain a minimal rationality regarding what we take as worthy of being called hypothesis.

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[66] See Arruda (1980).

[67] One of the few people to explicitly mention paraconsistent logics in connection with issues related to credulous plausibility was Kyburg. In (1997) he has investigated if paraconsistent logics could be successfully used to solve the lottery paradox. Incidentally, he comes up with a negative answer.

And one of these criteria of rationality is not to set forth contradictory hypotheses. In contradistinction to "α is plausible" and "$\neg\alpha$ is plausible" admitting things like "$\alpha \wedge \neg\alpha$ is plausible" as reasonable is tantamount to admitting the existence of ontological inconsistencies. Incidentally, it is because of this absurdity that the whole controversy around the conjunction principle took place.

Another common way of formally defining paraconsistent systems is to say that they generally do not respect the principle of contradiction[68]. Another Aristotelian principle that has also been rejected by logicians was the *excluded middle principle*. Starting from the 1930's and following the ideas of L. E. J. Brouwer, figures such as A. Heyting, G. Gentzen and S. C. Kleene started developing the so-called *Intuitionistic logic*, which explicitly does not satisfy the principle of excluded middle. As Saul Kripke showed in 1965, from a semantical point of view intuitionistic logic can be defined as that sort of system in which statements and their negations can be both false (but not both true)[69]. More recently, a very similar idea was undertaken without the intuitionistic motivation by da Costa and his collaborators to develop what they called *paracomplete logics*. According to him, while a system is paraconsistent if it can represent nontrivial theories in which certain statements and their negations are true, it is paracomplete "if it can function as the underlying logic of theories in which there are [...] formulas such that these formulas and their negations are simultaneously false."[70] This is of course a semantic definition. In order to give a proof-theoretical definition similar to the one we have given to paraconsistent logics we need to restate in another terms the notion of *complete theory*. Traditionally, a theory T is taken as complete iff T is closed under deduction and for every sentence α, either $\alpha \in T$ or $\neg\alpha \in T$[71]. Inside the framework of classical logic, we can equivalently say that a theory T is *complete* iff T is closed under deduction and, for any sentence α such that $\alpha \notin T$, $T \cup \{\alpha\}$ is a trivial theory. Given this second definition, we can say that a logic is paracomplete iff it can function as the underlying logic of complete theories T such that, for some sentence α, neither $\alpha \in T$ nor $\neg\alpha \in T$.

Trivially, paracomplete logics do not satisfy the excluded middle principle. Besides da Costa, Perlis for example have also worked on something worthy of being called a paracomplete logic[72]. Others such as Rescher, Buchsmaum and da Costa himself have proposed systems that are simultaneously paraconsistent and paracomplete[73], which have been called *paranormal* or *non-*

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[68] Even though this way of defining paraconsistent logic is very often found in the literature, depending on the way we formulate the principle of contradiction, it should be taken only as an informal definition. Outside classical logic, it is not necessarily the case that $\vdash \neg(\alpha \wedge \neg\alpha)$ is equivalent to $\alpha, \neg\alpha \vdash \beta$, for every β.
[69] Kripke (1965). See also Moschovakis (2002).
[70] Loparic & da Costa (1984), p. 119. See also da Costa (1986) and (1989).
[71] Enderton (1972).
[72] Perlis (1989).
[73] Rescher & Brandom (1980), Buchsbaum & Pequeno (1993) and da Costa (1989).

alethic logics[74]. It is interesting to note that in a very important sense paracompleteness and paraconsistency are the dual of each other, and vice-versa. Jean-Yves Béziau has expressed that in the following way[75]:

> Intuitionistic logic appears as a dual of a particular paraconsistent logic. Reverse intuitionistic logic, put his head down and his foots up, his foots will look like a head and his head like some foots, and you will get another logic, a paraconsistent logic. [...] all the reverse of paraconsistent logic are many more than all the possible visions of intuitionistic logic, and form the rich field of *paracomplete logics*. Each paraconsistent logic has a paracomplete dual and each paracomplete logic has a paraconsistent dual. [...] Paraconsistent and paracomplete logic appear therefore like husband and wife.

In Chapter 5 when we show the semantics of our logics, this technical fact will become more evident.

We are mentioning this because it seems that the notion of acceptability or skeptical plausibility has a sort of *paracomplete* behavior. Since plausibility is a knowledge-dependent notion, the truthfulness of plausible statements will be limited by the amount of evidence we are able to gather for them. Therefore, given a statement α, it is completely reasonable to imagine that we are in a position neither to accept α nor to accept $\neg\alpha$, in which case both "α is accepted" and "$\neg\alpha$ is accepted" will be false. In fact, for most part of the statements we can formulate with the help of scientific languages, we do not have evidence neither to speak for nor against them. It is a picture then where apparently the principle of excluded middle is apparently not valid. We will say then that the notion of skeptical plausibility or acceptability is essentially a *paracomplete* one. As we did before, we should also here try to precise the way we are using the term "paracompleteness," which has traditionally been used in connection with logical systems but never in connection with concepts: "paracomplete" in the expression "the skeptical notion of plausibility is paracomplete," means, in a broad way, that a formal system intent to represent the notion of skeptical plausibility should allow what we may call *void acceptance situations*, that is, situations where, for some α, both sentences "α is accepted" and "$\neg\alpha$ is accepted" are false.

These points can be made more accurate with the help of a plurality-oriented semantic model. Let us try to define more precisely the notion of plurality-oriented extension by saying that it is a consistent and deductively closed set of statements build over some specific language which, as we have said, may represent the view of one or more experts, the consequences of one of the competing theories, one of the scenarios speculated by the analyst, a set of consistent conclusions obtained from an inductive reasoning mechanism, etc. We call a specific set of extensions a plurality-oriented interpretation. A statement α will be plausible in a specific interpretation if and

[74] Béziau (1999).
[75] Ibid. p. 12.

only if it holds in at least one extension, and accepted if and only if it holds in all extensions. Trivially, there may be an interpretation I where a specific statement φ is true in one extension but false in another. Therefore we will have that both "φ is plausible" and "¬φ is plausible" are true in I. Consequently, since there is at least one interpretation able to satisfy the theory {"φ is plausible", "¬φ is plausible"}, it does follow that from it we will be able to conclude everything. Also, for the same φ and I, since φ is true in one extension and false in another, neither it not its negation will be taken as accepted in I. Consequently, we may have a complete theory T which does not contain neither "α is accepted" nor "¬α is accepted."[76]

So far then we have that the credulous plausibility does not respect the principle of contradiction and has a paraconsistent behavior. In its turn, the skeptical plausibility respects the principle of contradiction but nevertheless does not respect the principle of excluded middle and has a paracomplete behavior. It remains then to find out whether the credulous plausibility respects or not the principle of excluded middle. At first glance the answer is no. There may be many statements about which no expert at all has any sort of opinion, and consequently neither they nor their negations will get the status of credulously plausible. Only if we conceive our experts as ideal ones, that is, as having opinion about every statement of the language at hand is that we may take the credulous plausibility are respecting the principle of excluded middle. In this case, the situation would be analogous to the traditional epistemic logics where the agent is an ideal one and is aware of all laws of logic as well as of the logical consequences of his beliefs. In Chapter 6 we will investigate the formal properties of both of these two approaches to the credulous plausibility.

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[76] Incidentally, it is by using a structure like that that Kripke arrives at a sound and complete possible-world semantics for intuitionistic logic. This is also the path followed by Buchsbaum and Pequeno (1993) and Rescher & Brandom (1980). For a discussion about the relations between Kripke's semantics and paraconsistent and paracomplete logics see Bèziau (2002).

CHAPTER 4

ON THE LOGIC OF INDUCTION AND PLAUSIBILITY

In this Chapter we will switch from the conceptual problem of induction and plausibility to the way we may try to formally solve this problem. First, in Section 4.1, we will pick the most influential logic of induction developed in philosophy – Carnap's system – and try to figure out the rationale behind today's consensual opinion that the philosophical project of building a logic of induction is, as a whole, a failure. After that we will try to figure out what features an inductive logic should have in order to be in accordance with what we have named in Chapter 2 a purely descriptive or pragmatical approach to induction and perhaps not to face the problems that undermined Carnap's project. In Sections 4.2 and 4.3 we will investigate to what extent AI nonmonotonic logics can be classified as logics of induction and whether they can somehow help us in the task of developing a purely descriptive logic of induction. The nonmonotonic formalisms we will analyze will be Raymond Reiter's default logic and Tarcisio Pequeno's systems of induction and plausibility. It will come up from this analysis that we may indeed have at hand a quite promising framework to be used as the starting point of our formal explication of the notions of induction and plausibility.

4.1 The Logic of Induction: A Dead Horse?

4.1.1 Carnap's System of induction

For the last 15 years or so, it has been commonplace among philosophers of science to consider the whole project of building a logic of induction as conceived by Carnap as fundamentally misleading. In a paper entitled "Why There Can't be a Logic of Induction," Stuart Glennan for example compares such project to a dead horse[1]:

> Carnap's attempt to develop an inductive logic has been criticized on a variety of grounds, and [...] I think it is fair to say that the consensus is that the approach as a whole cannot succeed. In writing a paper on problems with inductive logic [...] I might therefore be accused of beating a dead horse.

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[1] Glennan (1994), p. 78.

A similar statement is found in the entry for "Inductive Logic" in J. Pfeifer's *Philosophy of Science: An Encyclopedia*, written by Branden Fitelsen[2]:

> Moreover, [...] there are further (and some say deeper) problems with Carnapian [...] approaches to logical probability, if they are to be applied to inductive inference generally. The consensus now seems to be that the Carnapian project of characterizing an adequate logical theory of probability is (by his own standards and lights) not very promising.

In order to understand the rationale behind such sort of claims, we have to identify which were the "standards and lights" that Carnap wanted his logic of induction to satisfy. One way this can be done is by paying attention to the features Carnap attributed to the relation of inductive support that is supposed to exist between hypothesis and evidence: "Since we take semantics as the theory of the meanings of expressions in language and specially of sentences [...], the relations [between] *h* and *e* to be studied may be characterized as *semantical*."[3] One very common way Carnap used to use to clarify the nature of this semantical relation was to compare inductive logic with deductive logic: "The principal common characteristic of the statements in both fields [deductive and inductive logic] is their independence of the contingency of facts. This characteristic justifies the application of the common term 'logic' to both fields."[4] Elsewhere he details what this independence of contingent facts is supposed to be[5]:

> It seems to me, however, that an elementary statement in inductive logic [...] expresses a *purely logical relation* between the two sentences involved in the same way that an elementary statement of deductive logic does [...] The relation is in both cases purely logical in the sense that it depends merely upon the meanings of the sentences [...]

In accordance to what we have expounded in Chapter 2 under the label of relation approach to induction, the idea of Carnap's logic of induction therefore was to formalize a purely logical relation of inductive support in the manner as deductive logic formalizes the relation of logical consequence or deductibility. In the same way that by simply giving a semantical structure able to assign meaning to the sentences of a language we automatically set the relation of logical consequence between all these sentences, with a similar endeavor and with no additional non-logical assumption we set a (numerical) relation of confirmation between the sentences. This same idea is found in Hempel's "Studies in the Logic of Confirmation," where he says that the purpose of the logic of confirmation is "to set up purely formal criteria of confirmation in the manner similar to that in which deductive logic provides purely formal criteria for the validity of deductive inferences".[6]

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[2] Fitelsen (2004), p. 9.
[3] Carnap (1962), p. 20. The italics are mine.
[4] Carnap (1962), p. 200.
[5] Carnap (1946), p. 596. The italics are mine.
[6] Hempel (1945), p. 9.

How Carnap tried to achieve this goal can be seen through a quick look at the system of induction he presented in *Logical Foundations of Probability*. Carnap's initial project was to define a sort of function called by him c-function which when applied to hypothesis *h* and evidence *e* would return the degree of confirmation given to *h* by *e* (in symbols: c(h,e).) In order to achieve the goal described in the above quotations, this function would have to be defined in purely semantic grounds depending "merely upon the meanings of the sentences" *h* and *e*. Clearly enough, this requires that no principle other than purely logical ones should be used in the definition of *c*.

The fundamental concept of Carnap's system of inductive logic is the notion of *state-description*. Given some specific language L_N (where N amounts for the number of individual constants of L), a state-description is a sentence which, by affirming or denying each property of each individual, completely describes a state of the world. From this notion of state-description (which can be fairly thought of as a sort of possible world) we get what he calls *range of a sentence*: If *h* is a sentence of L_N, the range of *h* is the class of all state descriptions in which *h* holds. By defining the *weight* of a sentence *h* (in symbols: m(h)) through these two concepts, we can then characterize the degree of confirmation given to *h* by *e* as the ratio between the weight of $h \wedge e$ and the weight of e:

$$c(h,e) = \frac{m(h \wedge e)}{m(e)}$$

The central question now is then how to define the weight m(h) of a sentence. The simplest way to do that is to take m(h) as the proportion of possible worlds in which *h* is true or, in other words, the ratio between the number of state-descriptions in the range of *h* and the total number of state-descriptions. This is of course equivalent to assigning to each state-description the weight of 1/(number of state-descriptions) and define m(h) as the sum of the weights of all state-descriptions which belong to the range of *h*. Carnap calls this weight function and the corresponding c-function obtained from it $m^\dagger$ and $c^\dagger$, respectively. This approach, which Carnap attributes to the early Wittgenstein, is essentially nothing more than the classical definition of probability. The basic difference is that in this case the probability value would be dependent on the language in which the hypothesis and evidences are to be formulated.

The problem that Carnap sees with this $c^\dagger$ c-function is that it would not allow us to learn from experience, that is to say, independently of the evidence *e* we take, $c^\dagger(h,e)$ is always the same. He then proposes a new c-function, c*, that is not plagued by this sort of problem. The distinguishing feature of c* is that it no longer considers all state-descriptions as being equal. Instead, it introduces a definite bias towards uniformity by favoring more homogeneous state-descriptions. To accomplish this, Carnap introduces the notion of *structure-description*: "j is the structure-description corresponding to Z_i (or, Z_i belongs to the structure-description of j) in $L_N =_{df} Z_i$ is a Z in L_N, and j is the disjunction of all Z which are isomorphic to Z_i arranged in lexicographical order."[7]

Two Z's are isomorphic if and only if one can be derived form the other by merely exchanging some individuals for others by means of a one-to-one correlation. The idea of c* then is to treat each of these structures as well as the state-descriptions inside them as equiprobable. That is to say, to each structure-description it will be assigned a weight of 1/(number of structure-descriptions) and to each state-description inside a specific structure-description *s* a weight of (weight of *s*) x (1/(number of state-descriptions inside *s*)). The new weight m*(h) of *h* would then be defined as the sum of the weights of all states descriptions in the range of *h*. As usual, c*(h,e) is defined as the ratio of m*(h ∧ e) to m*(e).

Now we are in a position to analyze the claim that c* satisfies the purpose of the logic of induction. To begin with, we may adopt a sort of orthodox position and state that if some system of induction is to be classified as logical, then it must be not only *a* logic of induction but *the* logic of induction. In the context of Carnap's formalism, this means that the c-function which Carnap takes as the basis of his logical system should be arguably a unique and universal way of assigning degrees of confirmation to pairs of hypothesis/evidence sentences (or at least the core of confirmation reasoning which all the other not-so-universal c-functions should be based on.) It is in this direction for example that Glennan argues for the thesis that there can be no logic of induction "in the sense of no uniquely determined *c* function."[8] The example he gives is a situation where c† would be preferred over c*.

It should be noted that in the very development of Carnap's inductive system we find some support for this conclusion. While in *Logical Foundations of Probability* Carnap did present c* as the proper c-function of inductive logic, in later works he no longer argued that one c-function is satisfactory in all cases, but tried rather to develop a theoretical description of an infinite continuum of c-functions called λ-continuum (the parameter λ is supposed to indicate how sensitive the corresponding c-function is to "learning from experience.")[9] And as Carnap (1952) himself concedes, no one value of λ is "better *a priori*" than the others. In Carnap's view then, the inexistence of a unique c-function does not seem to be a strong argument against the possibility of a logic of induction. After all, it may happen that even though c* cannot be shown to be the best c-function, it is, as Carnap wished, a purely logical notion.

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[7] Carnap (1962), p. 116.
[8] Glennan (1994), p. 82.
[9] Carnap (1952). In more recent works, Carnap has proposed two more additional adjustable parameters γ and η. See Carnap (1971) and (1980).

In order to appreciate this claim, it is important to note that even though c* may have some advantages over c† in the situations Carnap considers, both of them make use of the same basic principle: the *principle of indifference*. Although in *Logical Foundations of Probability* Carnap denies such dependence and defends that because the mentioned principle "leads sometimes to quite absurd results and in its strongest form even to contradictions, it must be rejected"[10], later he retreated from this and went on to defend that the principle of indifference is in fact to a purely logical assumption[11]:

> [...] the statement of equiprobability to which the principle of indifference leads is, like all the other statements of inductive probability, not a factual but a logical statement. If the knowledge of the observer does not favor any of the possible events, then with respect to this knowledge as evidence they *are* equiprobable. The statement assigning equal probabilities in this case does not assert anything about the facts, but merely the logical relations between the given evidence and each of the hypotheses; namely, that these relations are logically alike.

As would be expected, this point is far from being uncontroversial. In fact, in the same way that the principle of equiprobability has been the most attacked feature of classical systems of probability (as Carnap himself pointed out), it has been one of the most indigestible characteristics of Carnap's inductive logic[12].

Even though we think there are plenty of reasons not to accept Carnap's point that the principle of equiprobability is a logical principle of induction[13], it is not our intent here to engage ourselves in this sort of debate. Rather, we just want to use this controversy as an example of the claim we have made in a good part of chapter 2 that the relation approach to induction inevitably brings us to justificatory issues. Given what we have exposed so far, it quite trivial in which point Carnap gets involved in justificatory issues. Since a semantical notion has to make use of no other principles than purely logical ones, in order to make the point that his concept of degree of confirmation is a logical concept, he has to make sure that all principles his inductive logic is based on are in fact logical. But since one of these principles, the principle of indifference, was not able to form a consensus regarding its logical nature, Carnap had to engage himself in justificatory issues **intent** to show that such principle is in fact a logical one. And exactly because his arguments were not convincing at all, his project as a whole was taken as a fail.

It is interesting to note that this justificatory aspect of Carnap's system has sometimes been overlooked by philosophers. In the early days of Carnap's logical probabilism, for example, John Kemeny wrote the following[14]:

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[10] Carnap (1962), p. 518.
[11] Carnap (1955), p. 22. Italics in the original.
[12] See Weatherford (1982), sections II.11 and III.11, and Salmon (1966), sections V.1 and V.3.
[13] For a couple of arguments against the principle of indifference see Fitelsen (2003).
[14] Kemeny (1963), p. 711.

> The problem of induction [...] has stimulated two different but complementary types of research. First of all there is the problem of how one can justify the inductive inferences that we do as a matter of fact make, a problem whose solution is seems impossible since the days of Hume. The other approach is that of Bacon, Mill, and Laplace, who analyse the way we make inductive inferences. They try to find reasonable methods of inference, without necessarily giving justification that would go counter to Hume's argument. It is this latte problem that was successfully attacked by Carnap.

As we hope to have shown, statements like this are essentially misleading: because of the very goal Carnap proposed to achieve, his logic and all others that have adopted his cannons are justificatory in essence.

4.1.2 Towards a Representational Logic of Induction

At this point one may wonder if what we are have called in Chapter 2 a purely descriptive approach to induction is a possible enterprise. After all, we have seen that the most influential tradition of inductive logic, which was supposed to be essentially descriptive, was not itself able to keep distance from justificatory issues. And this of course was not due, let us say, to the mathematical resources employed by Carnap and his followers, but in fact to the very idea held by these philosophers of what the logic of induction is supposed to be. Therefore, in order to show that a descriptive approach to induction is a tenable project, we will have to somehow rethink the traditional conception of logic of induction in such a way as to make it susceptible to such a purely descriptive account. By so redefining the purpose of the logic of induction, we will try to show that our dead horse is perhaps not so dead after all.

From a general point of view, the task of the logic of induction as conceived in Carnap's tradition could be divided into two:

(i) To set a specific way through which probability values are obtained, that is to say, the conditions according to which one statement gives evidential support to another; and

(ii) To lay down the rules according to which probability values are related to each other or, in other words, the logical relations that are supposed to hold between probable statements.

Let us, for the time being, name the parts of the logic of induction responsible for each one of these tasks, respectively, *model of confirmation* and *calculus of confirmation*. Johnathan Cohen defines these two tasks in the context of a numerical approach as follows[15]:

> Two problems in confirmation theory are not always sufficiently distinguished from one another. [...] On the one hand there is the semantical problem of deciding, in each case, what are the elements of which confirmation-functors are functors and what metric is most appropriate for the assignment of values to these functors. On the other hand there is the syntactical problem of determining any compatibilities or incompatibilities that may hold

[15] Cohen (1966), 463-464.

> universally between such assignments. To construct a calculus of confirmation is to solve the latter, not the former.

Right after the above quotation, Cohen correctly classifies the calculus of probability as a calculus of confirmation. Indeed, the only sort of value-determination the calculus of probability does is to get derived probabilities from prior ones: except in limiting cases such as p(h,h) =1, it says nothing about how to assign such prior probability values. This task is responsibility of what we have called model of confirmation. Using the notation of elementary probability theory, we would say that while the purpose of the model of confirmation is to determine, to any pair of sentences *e* and *h*, the probability value P(h,e) of *h* given *e* or, in inductive logic's terminology, the inductive support given by *e* to *h*, the goal of the calculus of confirmation is to establish the rules according to which different probability statements P(h,e) should be related to each other.

Following Carnap's practice, we will now try to establish a sort of parallel between formal deductive logic and inductive logic, as understood according to above-mentioned division. We first of all note that if we change "probability value" for "true" in the above paragraph, we will get something very similar to the way deductive logic deals with truth-values. What we mean is that in the same way that formal deductive logic gives no sort of effective procedure to decide whether a sentence is true or false (except in limiting cases such as $\alpha \wedge \neg\alpha$) but just sets the logical constraints according to which truth is obtained from truth, the calculus of confirmation also does not say how one sentence confirms another, but just sets the logical cannons which confirmation statements are supposed to satisfy. Not less interesting is the following conclusion: akin to the inferences set by formal classical logic, the inferences set by the calculus of confirmation are, as a quick inspection of the probability calculus will show, deductive rather than inductive. They have the sole purpose of setting the *necessary* and consequently truth-preserving restrictions the reasoning about confirmation is supposed to obey.

The logic of plausibility being the deductive part of the logic of induction, it is needless to say that the model of confirmation will be its inductive part. In fact, as we have said, it is the goal of the model of confirmation to set down the process by which hypotheses are inductively supported by evidences. With this observation in mind and considering the previous paragraph discussion, we note that deductive logic has no component similar in purpose to inductive logic's model of confirmation. The determination of how to assign truth-values to sentences is completely outside the scope of the theorist who is building his logical system: it belongs to the theory of knowledge rather to logic. This is relevant because if we say that inductive logic is a sort of logic in the sense formal deductive logic is, then we are assuming that a component able to determine the truth-value of sentences could be added to formal deductive logic without changing the meaning logicians and philosophers attribute to "logic," however fuzzy it may be. Clearly enough, hardly any one slightly acquainted with logic will take seriously this assumption. If however, for the sake of argument, we accept such postulation, we will have to accept that logic would get merged into the theory of knowledge. As such, it would have to deal with that component of knowledge which, despite being the most controversial of all, has always been present in one way or another in the epistemological theories: the notion of *justification*.

This point is important because, as we have seen, inductive logic does have the above-mentioned component which deductive logic lacks. Therefore, the conclusion we have made regarding the possibility of deductive logic's having added to it a way of getting truth-values applies with the same intensity to inductive logic. In other words, since inductive logic has to somehow determine the degree of confirmation which evidence e gives to hypothesis h, the component responsible for that, the model of confirmation, could be taken in a very important sense as much more concerned with the theory of knowledge than with logic. As such, it will have inevitably to deal in some way or another with the justificatory issues involved in that field. That this is so can also be seen by recalling that inductive inferences, by being ampliative, bring necessarily new pieces of knowledge which, due to their not being contained in the premises, will require some sort of justification.

The important point for us in all that is something we have already observed a couple of times before in another terms: the model of confirmation is, we may say, the window through which the problem of justification of induction comes in the scene. This conclusion is of course anything but surprising: being the only part of inductive logic which deals with inductive inferences, there is no other place the problem of justification of induction could appear except in it. However, from the point of view of our endeavor of conceiving a purely descriptive account of the logic of induction, it is fundamental to know where precisely the problem of justification takes place in order not to take it into account.

It should be observed that the definition of inductive logic's purpose given by our twofold task division does not take into account the task of detaching the hypothesis from the evidences and concluding something like P(h). The reason for that, as we have seen, is that the problem of detachment is according to Carnap not concerned with the logic of induction itself but with its application. This is of course a problem if we want a logic of induction primarily designed to deal with the pragmatical notion of probability rather than with the logical notion of probability. At a first glance, it seems we have two basic alternatives: to include one more component to the above mentioned division in such a way as to take into account the mentioned task or to leave it like that and conceive another logic of induction intent to deal with these "detached" plausible hypothesis. Considering what we have just concluded about the model of confirmation and our willingness of having a purely descriptive account of the logic of induction, it is understandable that we should follow the second alternative and try to discover what such new logic of induction should be.

Given an application of the logic of induction and therefore a set of statements of the form "the degree of inductive support given by *e* to *h* is *x*" or, if we want to stick to a qualitative approach, "*e* inductively supports *h*", our basic problem would be then to formalize the process through which hypothesis *h* is detached from evidence *e*. Since as we have seen this is done when *e* is the case and some total evidence conditions are satisfied, sentence "*e* inductively supports *h*" can be seen as a sort of *inductive implication* where the truth of *e*, we may say, inductively implies the plausibility of *h*. From this perspective, *e* may be seen as the antecedent of the inductive implication, *h* as the consequent and the mentioned process of detachment as a MP-like inferential relation stating that (under the condition that some total evidence condition is satisfied) "*h* is plausible" is can be inductively concluded from "*e* inductively implies *h*" and "*e* is the case." Accordingly, we will call the component of our new inductive logic responsible for such inferential process the *relation of inductive consequence*.

Supposing that we have such inferential mechanism at hand, we will need also to reason about the inductively obtained plausible statements. That is to say, we will need a logical system able to operate on the deductive level for saying which constraints plausible statements are subject to. This, we must concede, is already done by what we have called calculus of confirmation. Taking a quantitative approach based on the probability calculus as example, our detached hypotheses will be probability formulae of the form P(h) = x, whose logic is trivially taken into account by the calculus of probability. However, as the name chosen by Cohen indicates, the calculus of confirmation does a bit more than only reasoning about such plausible formulae: it also reasons about sentences of the form "*e* inductively supports *h*" or, what is the same, inductive implications of the form "*e* inductively implies *h*." In the case of the probability calculus, these two tasks are performed by the same system because P(h,e) and P(h) can always be derived from one another. But of course it does not need to be always like that. Therefore, we will separate these two tasks and call the component of the logic of induction responsible for the first the *calculus* or *logic of plausibility* and the component responsible for the second the *calculus* or *logic of inductive implication*.

In addition to these three parts, the logic of induction should obviously also provide a way to represent the inductive implications and the plausible hypotheses inferred from them. We will name this fourth component the *inductive-plausible language*. Now that we have got a logic of induction with four basic components – the relation of inductive consequence, the logic of plausibility, the logic of inductive implication and the inductive-plausible language – we may wonder if it really has the descriptive purpose our pragmatic approach to induction requires. To start with, we point out that due to its not taking into account the task of saying whether (and to what extent) *e* confirms *h*, our logic of induction will not get involved into the problem of justification of induction. Another consequence of not having nothing akin to the model of confirmation is that the confirmation statements which the logic of inductive implication is supposed to reason about and which the relation of inductive consequence will act upon to "extract" the plausible facts will not be settled by the system, but rather shall come from *outside*. Consequently, rather than being concerned with how facts inductively support others, our logic of induction's main concern will be how to provide a logical framework where inductive implications along with any inferential capability they may posses could be properly *represented*. In other words, our inductive logic's purpose will be shifted from the problem of "generating" confirmation statements to the problem of representing or describing them.

At this point it may be useful to recall our previous discussion about inductive logic and deductive logic to conclude that this new sense of inductive logic perhaps deserves much more the title "logic" than its old justification-laden cousin. As it is widely recognized, one of the main purposes of deductive logic is to serve as a logical framework for representing certain sorts of statements and drawing all logical consequences which may be entailed by them. As we have already observed, nothing is said there about whether or not these statements are correct or true. The responsibility of picking true or reasonable statements belongs to the theorist who will use deductive logic, not to deductive logic itself. Similarly, in our logic of induction, now called *descriptive or representational logic of induction*, nothing is said about how hypotheses are confirmed by evidences or whether such and such evidence confirms such and such hypothesis. Its purpose is rather to serve as a framework for representing inductive implications and drawing the plausible hypotheses entailed by them in a specific knowledge situation. The responsibility concerning the rationality of the represented inductive inferences performed inside inductive logic belongs not to inductive logic itself, but to the knowledge engineer who is making use of it.

Now, if our plausible-inductive language, along with the inferential mechanism provided by the logic of induction, is able to represent the axioms of a calculus of inductive implication[16] which tell us how to obtain inductive implication statements from inductive implication statements, then it sure will also be able to represent specific ways according to which inductive implication statements are obtained from something else (expressible of course in our plausible-inductive language) than inductive implication statements. In other words, it will be able to represent what we have called model of confirmation. In contrast to what one may think, this possibility of representing models of confirmation is in complete accordance with our descriptive approach to the logic of induction. In the same way that, by allowing one to represent what he thinks to be true, deductive logic does not

[16] An instance of such axioms would be what we could call inductive implication transitivity axiom: if α inductively implies β and β inductively implies φ, then α inductively implies φ.

commit itself with the justification of such "true" statements, allowing one to represent the way he thinks inductive statements are "generated" does not commit our inductive logic to the justification of such model of confirmation. The goal of the logic of induction itself is nothing more than to serve as a logical framework where inductive implication axioms of several sorts, including the sort of axioms which could be taken as model of confirmation, can be represented, being the rationality of what these axioms completely outside the scope of the logic. We call the logic of induction so used an *applied logic of induction*.

4.2 Nonmonotonic Logic and the Logic of Induction

As we have mentioned before, from its very beginning Artificial Intelligence was concerned with the formalization of commonsense patterns of reasoning. John McCarthy was perhaps the first to discuss the need for the mechanization of commonsense reasoning, before any theory on the subject existed[17]. It did not take too long for AI theorists to recognize the limitations of classical formal logic to represent commonsense reasoning. One of the first and surely the most famous of such acknowledges was made M. Minsky, one of the founders of AI, in an appendix entitled "Criticism of the Logistic Approach" of an influential paper of 1974 [18]:

> Even if we formulate relevancy restrictions, logistic systems have a problem in using them. In any logistic system, all the axioms are necessarily 'permissive' – they all help to permit new inferences to be drawn. Each added axiom means more theorems, *none can disappear*. There simply is no direct way to add information to tell [...] about the kinds of conclusions that should not be drawn. [...] Because logicians are not concerned with systems that will later be enlarged, they can design axioms that permit only the conclusions they want. In the development of intelligence the situation is different. One has to learn which features of situations are important, and which kinds of deductions are not to be regarded seriously.

Later, when summarizing his criticism of logical (or logistic, as he calls it) systems, by which he meant of course classical formal logic, he writes: "[...] 'Logical' reasoning is not flexible enough to serve as a basis for thinking [...] The *consistency* that Logic absolutely demands is not otherwise usually available – and probably not even desirable [...]"[19]

We see here a direct reference to two topics already mentioned in previous chapters which have a lot to do with the theory of induction we are describing here: classical logic's monotonicity and its intolerance of inconsistencies. According to Minsky, these two things are impediments to the proper treatment of commonsense reasoning. The development of nonmonotonic logics that took place in AI after the publication of Minsky's paper did account for the nonmonotonic aspect required by commonsense without however being able to overcome classical logic's limitation of dealing with

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[17] McCarthy (1958).
[18] Minsky (1974). The italics are mine.

inconsistencies. The mathematical treatment of inconsistent but nontrivial theories begun apart from AI research about two decades had to pass for these two weaknesses of classical logic to be incorporated into a sole theoretical system.

The first languages to have an explicit nonmonotonic component were the *Prolog* programming language developed by A. Colmerauer and his students and the PLANNER language developed by C. E. Hewitt[20]. The formalization of the field of nonmonotonic reasoning, as we know it today, started approximately in 1975/1976, with papers published in the 1977-1979 period[21]. It was however in 1980 that nonmonotonic reasoning obtained its impetus, with the publication of a special issue of the *Artificial Intelligence Journal* devoted exclusively to nonmonotonic reasoning in which the three nowadays main nonmonotonic theories were presented: Reiter's default logic, McDermott-Doyle's non-monotonic logic and McCarthy's circumscriptive logic[22]. Since then up to the middle of the 1990's the amount of research done on nonmonotonic formalisms was such that nowadays nonmonotonic logic is established as an independent and important field inside AI[23].

In earlier chapters we have spoken about the similarities that exist between the philosophical field of inductive logic and nonmonotonic logic. Although not always recognized by philosophers and AI theorists, these two domains deal, we have tried to show, with the same problem. If this is really so, then it is reasonable to expect the tools developed in one field to be somehow useful for the other. More specifically, if nonmonotonic logics in fact deal with the problem of representing patterns of inductive inference, then it may be worthy to find out whether these logics can throw some light upon our project of building a representational logic of induction. Of course this will be so only if nonmonotonic logics are absent from justificatory staffs: if they are plagued with the same problems we have identified in Carnap's inductive logic, then to look the whole thing from AI's perspective would not represent any gain. Keeping these two questions in mind – if nonmonotonic logics are free from the problem of justification of induction and if they deal to some extent with the same problem we are concerned about here – we will investigate one of the most famous nonmonotonic formalisms in AI: Reiter's default logic. All this will be done with the general purpose of finding out to what extent AI nonmonotonic logics in general and Reiter's system in particular can be taken as valuable tools in the philosophical analysis of induction and plausibility.

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[19] Ibid. The italics are mine.
[20] Colmerauer et al (1973), Hewitt (1969).
[21] See for instance McCarthy (1977), Reiter (1978), Clark (1978) and Doyle (1979).
[22] Reiter (1980), McDermott & Doyle (1980), McCarthy (1980).
[23] For a complete overview of the field, see Ginsberg (1987), Lukaszewicz (1990), Brewka (1991) or Gabbay et al (1994).

4.2.1 Default Logic

The main idea behind default logic is to formalize what we have called in Chapter 3 default reasoning. As we have seen there, the core of such reasoning are statements of the form "typically P's are Q's," which we have named default statements. We also have seen that what makes this sort of reasoning non truth-preserving are the exceptions that default statements are susceptible to. Reiter explains this as follows[24]:

> A good deal of what we know about the world is 'almost always' true, with a few exceptions. Such facts usually assume the form 'most P's are Q's' or 'most P's have property Q'. For example most birds fly except for penguins, ostriches, the Maltese falcon etc. Given a particular bird, we will conclude that it flies unless we happen to know that it satisfies one of these exceptions.

After analyzing the possibility of representing this "most" statements by stating all exceptions, he concludes that "what is required is somehow to allow twenty [our old friend bird] to fly by *default*." [25], that is, to conclude that twenty flies even without having all information which may be required to conclude that it is not an exception to the rule that says that all birds fly. It is important to note that despite the above characterization of default statements as sentences of the form "Most P's are Q's", default reasoning is to be, according to Reiter, plainly distinguished from statistical reasoning[26]:

> [In a] purely statistical connotation [such as] 'Most voters prefer Carter' [...] 'most' is being used exclusively in the sense of 'the majority of'. This setting does not lead to default assumptions: given that Maureen is a voter one would not want to assume that Maureen prefers Carter.

The real purpose of default logic is to deal with what Reiter calls prototypical situations[27]:

> [In a] prototypical sense [like] 'Most birds fly' [...] there is a statistical connotation [...] – the majority of birds fly – but there is also the sense that a characteristic of a prototypical or normal bird is being described. Given a bird Polly, one is prepared to assume that it flies unless one has reasons to the contrary. It is towards such prototypical settings that default logic is addressed.

In a footnote to the above quotation, he adds that the best way to distinguish between these two senses of "most" – the statistical one and the prototypical one – is by using the word "typically": while "Typically voters prefer Carter" sounds inappropriate, "Typically birds fly" sounds all right.

Reiter intends to investigate such default statements by interpreting sentence "typically birds fly," for example, as meaning "if x is a bird, then in the absence of any information to the contrary, infer that x can fly." He proposes to formalize this interpretation thought meta-linguistic devices

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[24] Reiter (1980), p. 82.
[25] Ibid.
[26] Reiter & Criscuolo (1981), p. 270.
[27] Ibid.

resembling inference rules called *default rules* (or simply *defaults*.) Let $\alpha(x)$, $\beta(x)$ and $\varphi(x)$ be classical formulae whose free variables are contained in $x = x_1, \ldots, x_m$. A default has the following form:

$$\frac{\alpha(x): \beta(x)}{\varphi(x)}$$

$\alpha(x)$ is called the *prerequisite*, $\beta(x)$ the *justification*, and $\varphi(x)$ the *consequent* of the default[28]. A default is *closed* if and only if none of $\alpha(x)$, $\beta(x)$ and $\varphi(x)$ contain free variables. Commonsense knowledge about the world is represented by what Reiter calls *default theory*: a pair <W, D> where W is a set of closed formulae and D a set of defaults. A default theory <W, D> is called closed if and only if every default of D is closed. Given a default theory <D,W>, Reiter defines through a fixed point operator Γ what he calls the *extension* of <W, D>, that is, the set of conclusions we are allowed to draw from W and D:

> Let S be a set of closed formulae and <W, D> a closed default theory. $\Gamma(S)$ is the smallest set satisfying the following conditions:
>
> (i) $W \subseteq \Gamma(S)$;
>
> (ii) $Th(\Gamma(S)) \subseteq \Gamma(S)$[29];
>
> (iii) If $\frac{\alpha: \beta}{\varphi} \in D$, $\alpha \in \Gamma(S)$ and $\neg\beta \notin S$, then $\varphi \in \Gamma(S)$.
>
> A set of formulae E is an *extension* of <W, D> iff $\Gamma(E) = E$, that is, E is a fixed point of operator Γ.

Condition (i) guarantees that what is known about the world is contained in each extension, (ii) says that the extension has to be closed under deduction, and (iii) has the effect that as many defaults as possible will be applied. The minimality condition does not allow one to draw conclusions for which no argument based on W and D exists.

From the definition of extension we can obtain an inductive inferential relation akin to the inferential relation defined by classical logic, the difference being that it will be inductive (or nonmonotonic) instead of deductive and will be held by default theories and formulae instead of sets of formulae and formulae. Another important difference is that some default theories have more than one extension. As a consequence of that, this default inferential relation may be defined in two different ways: by requiring the formula at hand to belong to at least one extension or requiring it to belong to all extensions. Since the arising of more than one extension is obviously due to existence

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[28] The notation proposed by Reiter in (1980) was slightly different from the standard one we are adopting here.

[29] Th(A) stands for the closure under deduction of the set of formulae A, that is to say, $Th(A) = \{\alpha \mid A \vdash \alpha\}$.

of conflicting inductive conclusions, we have here a formal example of what we have called in Chapter 3 the skeptical and credulous approaches to induction. Formally we have as follows:

> Let <W, D> be a closed default theory and α a closed formula. α is *skeptically inferred from* <W, D> (in symbols: $<W, D> \vdash_s \alpha$) iff, for all extensions E of <W, D>, $\alpha \in E$. α is *credulously inferred* from <W, D> (in symbols: $<W, D> \vdash_c \alpha$) iff, for at least one extension of <W, D>, $\alpha \in E$.

About the meaning and purpose of defaults, default theories and their extensions, Reiter writes the following[30]:

> Imagine a first order formalization of what we know about any reasonably complex world. Since we cannot know everything about that world – there will be gaps in our knowledge – this first order theory will be incomplete. [...] [The] role of a default is to fill in some of the gaps in the knowledge base, i.e. to further complete the underlying incomplete first order theory [...] Defaults therefore function as somewhat like meta-rules; they are instructions about how to create an *extension* of this incomplete theory. Those formulae sanctioned by the defaults and which extend the theory can then be viewed as beliefs about the world.

The set of formulae W of a default theory <W, D> represents therefore the certain, irrefutable knowledge we have about the world. In their turn, the defaults of D instruct us how to extend this certain but nevertheless incomplete knowledge in such a way as to obtain a set of nonmonotonic and therefore defeasible *beliefs* about the world. Since the publication of Reiter's 1980 paper, this interpretation of defaults in terms of beliefs has been the standard one in AI literature. Even though there is no syntactical way in Reiter's logic to distinguish between deductively and inductively obtained conclusions, each extension will contain two sorts of formulae: certain and irrefutable facts obtained exclusively with the help of W, and uncertain and refutable beliefs obtained direct or indirectly with the help of defaults.

If we were to formalize Twenty example, we would have the following schemata of default[31]:

$$\frac{\text{bird}(x)\text{: fly}(x)}{\text{fly}(x)}$$

Taking "Twenty" as the only constant symbol of our language and supposing it to belong to the extension of "bird", we would have as default theory the following pair <W,D>:

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[30] Reiter (1980), p. 86. Italics in the original.

[31] The above definition of extension handles only closed default theories. Defaults containing free variables, which are the more interesting ones, are interpreted as schemata of defaults representing all ground instances of the default.

$$W = \{bird(Twenty)\} \qquad D = \left\{ \frac{bird(Twenty): fly(Twenty)}{fly(Twenty)} \right\}$$

Applying the definition of extension, we will get Th({bird(Twenty), fly(Twenty)}) as the only extension of <W, D>.

If we want to see how nonmonotonicity works in this case, it suffices to add to W the following two formulae: Penguin(Twenty) and $\forall x(Penguin(x) \rightarrow \neg fly(x))$. As one might suspect, the belief fly(Twenty) will not belong any more to the extension of the new default theory. We see therefore that the inferences formalized with the help of defaults are in fact non truth-preserving and consequently in accordance with our contemporary conception of induction.

Another characteristic of default logic determined by the definition of extension concerns the relation we have mentioned in Chapter 2 between inductive inference's nonmonotonicity and global character and what we have called total evidence conditions. As pointed out there, due to the nonmonotonic feature of induction, when performing inductive inferences we need to make a sort of global inspection in order to make sure that all available knowledge is being taken into account. This is done by step (iii) of the definition of extension. Since the test concerning the default's prerequisite and justification is made against the very extension we are trying to create, in order to make a single inductive inference we need to check out the whole set of beliefs warranted by W and D. The main difference between default logic's total evidence condition and Carnap's requirement of total evidence, for example, is that in default logic the whole thing is made not by requiring, as Carnap suggests, all available knowledge to be contained in the evidences, but by inspecting the whole logical theory at the moment of detaching the hypothesis from the evidences.

Defaults like the one above whose justification is equivalent to the consequent are called *normal defaults*. In some cases, normal defaults give rise to unwanted extensions. Consider the following situation to be formalized through default logic[32]:

a) "Typically high school dropouts are not employed"
b) "Typically Adults are employed"
c) "Peter is adult"
d) "Peter is a high school dropout"

This can be represented as follows:

1) $\dfrac{school\_dropout(x): \neg employed(x)}{\neg employed(x)}$

2) $\dfrac{adult(x): employed(x)}{employed(x)}$

3) adult(Peter)
4) school_dropout(Peter)

Given a n-tuple of constant symbols c = <c_1, …, c_n> and an open default labeled *d* with open variables <x_1, …, x_n>, we call d_c the closed default obtained by substituting c_i for x_i, i = 1, …, n. As it can be easily seen, the theory composed by W = {(3), (4)} and D = {$(1)_{<Peter>}$, $(2)_{<Peter>}$} will have two contradictory extensions, one containing employed(Peter) and the other containing ¬employed(Peter).

The point of course is that even though they may be adults, school dropouts are *exceptions* to (b), that is to say, default (1) has priority over (2). Therefore, the extension that says that Peter is employed should be somehow blocked. This can be done with the help of *semi-normal defaults*, that is, defaults where the justification is not equivalent to the consequent but just implies it. Given the conjunctive normal form ($\beta_1 \wedge \ldots \wedge \beta_n$) of the justification β of a default, we will refer to each β_i as a component or part of the justification. If the default is not abnormal[33], we call the β_j that is equivalent to the consequent of the default the *normal part* of the default. The conjunction of the other β_i's we call the *semi-normal part* of the default. The exception condition mentioned above could then be represented with the help of semi-normal defaults as follows:

2') $$\frac{\text{adult}(x)\text{: employed}(x) \wedge \neg\text{school\_dropout}(x)}{\text{employed}(x)}$$

In this way, we will not be able to use $(2')_{<Peter>}$ to conclude employed(Peter), for school-dropout(Peter) will belong to our extension.

4.2.2 Default Logic as a Logic of Induction

At this point we may try to find out to what extent default logic can be taken as a logic of induction as defined by us at the end of Section 4.1. To start with, since the inferences formalized with the help of defaults are in fact non truth-preserving, default logic can be said to incorporate the negative, non truth-preserving feature of inductive inferences. About the positive, rational feature of induction, trivially the whole purpose of default logic (and all other nonmonotonic logics) is to set representational frameworks where cannons of commonsensical reasoning can be represented. Of course we expect these canons to be rational, but default logic itself says nothing about which inferences can be classified as rational. We may use defaults to represent reasonable statements such as "typically birds fly" but also absurd ones such as "typically black cats bring misfortune."

[32] That situation is a slight modification of an example given in Reiter & Criscuolo (1981).
[33] Abnormal defaults are those whose justification does not imply the consequent. As far as we are concerned, the applications we are envisaging here do not require this sort of defaults.

Given such representation, default logic just allows one to calculate the beliefs that may be drawn from them if a certain set of hard facts W is given: if he wants to get only reasonable beliefs from this process, then it is *his* responsibility to come up with reasonable default statements, according to whatever definition of rationality he has.

As it can be easily noted, this is exactly how a system of induction is supposed to behave in order to be said to be free from the problem of justification of induction and consequently satisfy our representative criterion. Therefore, at least at first glance default logic can be considered as a sort of descriptive and justification-free approach to inductive inferences.

But if this is so, that is, if default logic can be said to provide a way to represent inductive inferences, then default rules should be somehow susceptible of being interpreted in terms of confirmation statements or inductive implications. In Chapter 3 we have shown how default reasoning can be understood in terms of hypotheses and evidences. According to that, what the statement "typically birds fly" says is that *x*'s being a bird is an evidence for the hypothesis that *x* flies. Since typically birds fly" may be formalized in default logic through

$$d = \frac{\text{bird}(x)\text{: fly}(x)}{\text{fly}(x)}$$

, the prerequisite and consequent of a default could then be seen, respectively, as the evidence and hypothesis of a confirmation statement or, equivalently, as the antecedent and consequent of an inductive implication. Considering the possibility of an instance of *d*'s having its consequent added to the extension, what d means is something like that: "being a bird inductively implies having the property of flying."

So far so good. But besides a prerequisite and a consequent, a default has also a justification part. So, if defaults can really be read in terms of an inductive implication, where is the place of the justification in the case where it is not equivalent to the consequent? As we have seen at the beginning of this section, the traditional way of reading a default already embodies a sort of implicative procedure:

$$\frac{\alpha\text{: }\beta}{\varphi}$$

means that if α is true and it is consistent to assume β, then conclude φ. This of course is the same as saying that α implies φ only in those cases where it is consistent to assume β. Adapting this to our inductive implication interpretation, we would have that the above default means something like "α inductively implies φ unless $\neg\beta$," being the justification a sort of exception condition to the inductive implication.

In order to appreciate the importance of this conclusion, it will be useful to look at some aspects of the decision of adopting this inductive implication in lieu of the traditional relation of inductive support. First of all, it must be reminded that instead of a capricious choice, the mentioned decision is a necessary step if we are to proceed in a purely descriptive way. As we have seen, the very notion of inductive support as conceived in Carnap's tradition leads to justificatory issues. Therefore, if we want to deal with inductive inferences in a purely descriptive way, we have no choice but to jump the inside aspect of the relation of inductive support and consider just the external process according to which a piece of evidence inductively implies the plausibility of the hypothesis.

An important consequence of this is that the representation of those exception conditions capable of preventing us from concluding the plausibility of the hypothesis will become a necessary part in the task of describing the relation of inductive support between two formulae. Traditionally, only when the detachment of the hypothesis was taken into account was that particular ways of defeating the plausibility of the hypothesis would be considered. Independently of the circumstance being or not favorable for detaching *h*, the relation of inductive support which was supposed to hold between *e* and *h* would remain unchanged. According to this new, purely descriptive position however, a piece of evidence is said to inductively support a hypothesis only if the conditions concerning the justification of the default are satisfied. Consequently, the very notion of evidence will change: given a default

$$\frac{\alpha : \beta}{\varphi}$$

, α would be considered an evidence for φ only if $\neg\beta$ does not hold. In the case of Twenty example, its being a bird gives inductive support to or is an evidence for its flying only if Twenty is not one of the exception to the rule that says that typically birds fly. As one can see, this is in accordance with the philosophy of induction and plausibility we have sketched in Section 2.3. Anyway, the important point is the insight which default logic is giving us here concerning the notion of inductive implication: in order to be properly represented, an inductive implication has to come along with information about the conditions according to which its antecedent may be said to serve as evidence for its consequent.

Of course the raison d'être of all this is the necessity of "detaching" the consequent from the antecedent of these default inductive implications. We then arrive at a point which one may have already realized: besides fulfilling our inductive logic's representative purpose, default logic also has the inductive logic's component that we have named *relation of inductive consequence*, which can be said to be default logic's definition of extension or, from a more general perspective, its corresponding inferential relation $\vdash$ (be it defined skeptically or credulously.) Indeed, interpreting defaults in terms of inductive implications, the whole purpose of the definition of extension is to, given a set of defaults and a knowledge situation, build a set called extension which will contain, besides all logical consequences of the knowledge situation, all hypotheses that could be detached from their evidences in that knowledge situation.

Despite this, we must concede that default logic does not provide what we have called at the end of last section a *calculus of inductive implication*. As a consequence of that, it will not be able either to represent what we have called a *model of confirmation*. The reason for this is that through default logic we can represent only single inductive implication relations. We cannot represent complex relations containing several defaults and other logical connectives to say, for instance, that whenever α inductively implies β and β inductively implies φ, α inductively implies φ; or that α always inductively implies α. In order to perform such task, we would have to have a way of concluding defaults or, we may say, a sort of meta-default logic.

About the "detached" consequents of inductive implications, according to our pragmatical or descriptive approach to induction, they are to be taken as plausible or pragmatically probable hypotheses. As we have said, default theories deal with two sorts of facts: certain and irrefutable formulae, obtained exclusively from W, and uncertain and refutable beliefs, obtained direct or indirectly with the help of defaults. If we interpret defaults as inductive implications, these beliefs will naturally be interpreted as our plausible facts. However, as we have seen, Reiter's logic does not distinguish between these two sorts of facts. Once we have an extension of a certain default theory, we are unable to differentiate between those formulae that were deductively obtained from W from the ones that were obtained through defaults. With few exceptions such as Nute's defeasible logic and its variants[34], almost all nonmonotonic logics share this feature: they make no distinction at all between indefeasible and defeasible facts.

This indistinguishableness between defeasible and indefeasible formulae has some undesirable consequences. The most trivial one is that we will not be able to represent at the same time certain and plausible formulae: at each time that we get a specific set of formulae, we will have to decide if we interpret them as "it is certain that" or as "it is plausible that." This would be harmless if certain facts were logically indistinguishable from plausible facts, that is to say, if at least in certain situations plausible (deductive) reasoning had the same structure as certain reasoning. Unfortunately, this assumption is incompatible with the characterization of plausibility and certainty we have made in Chapter 2: while certain reasoning is something susceptible of being formalized through classical logic, plausible reasoning requires a paraconsistent and paracomplete logic.

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[34] See Nute (1994).

Furthermore, by taking a logic with no sort of qualification on formulae, we would never be able at the language level to distinguish between the skeptical plausibility and the credulous plausibility. Also, with no way to differentiate between certain and plausible facts, the whole conception of induction as those inferences that generate plausible facts could never be properly represented. Given all this, it is clear that default logic lacks our third inductive logic's component: the so-called *logic of plausibility*.

4.2.2 The Logic of Plausibility and the Problem of Anomalous Extensions

Besides these simple philosophically oriented arguments, there are some more technical ones that point out the mentioned indistinguishableness between inductive and deductive facts (or, from a deeper point of view, the lack of a logic of plausibility) as at the root of the problem of anomalous extensions. One of the firsts to call attention to that was Tarcisio Pequeno. In a paper of 1990, he argues that there were fundamentally two problems arising from conflicts among nonmonotonic conclusions that the existing nonmonotonic logics at the time could not give a proper solution[35]:

> 1) In cases where the simultaneous consideration of all the evidence is able to decide in favor of one of the conflicting partial conclusions, how to recognize it in order to select just the intended conclusion and defeat the others? 2) On the other hand, in cases where the conflict cannot be resolved no matter how carefully the available evidence is examined, a situation of real contradiction arises. How should we give a suitable treatment for this situation?

His diagnosis for these two problems rests on the satisfaction of Carnap's requirement of total evidence. According to him, the reason why most nonmonotonic logics are unable to solve these problems is that they do not cope satisfactorily with the mentioned requirement. In order to understand the rationale behind this claim, we need to take a closer look at the problems of anomalous extension as described in AI literature. The first sort of problem mentioned by Pequeno can be illustrated by slightly modifying our Twenty example as follows[36]:

a) Usually animals cannot fly;
b) Winged animals are exceptions to this, they can fly;
c) Birds are animals;
d) Birds normally have wings;
e) Twenty is a bird.

If we were to use default logic, this situation would be formalized as follows:

1) $\dfrac{\text{animal}(x): \neg\text{fly}(x) \wedge \neg\text{winged}(x)}{\neg\text{fly}(x)}$

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[35] Pequeno (1990), pp. 2 and 3.
[36] This example was first given by P. Morris (1988).

2) winged(x) → fly(x)

3) bird(x) → animal(x)

4) $\frac{\text{bird(x): winged(x)}}{\text{winged(x)}}$

5) bird(Twenty)

As it can be easily checked, we will have here two extensions, one containing that Twenty flies and the other that Twenty does not fly. At first glance one may think that this is just one more case of conflict between defaults similar to the school dropout example and can be sorted out by appropriate modifications of the justification part of some default rule. However, the justification part of (1) is already taking into account the priority that (4) is supposed to have over it. The whole problem is that by using (1) firstly, we are able to block the very default that will allow us to conclude the mentioned exception.

As we have mentioned, according to Pequeno the incapacity of default logic to solve problems like this rests on its inability to represent all the relevant knowledge. But what knowledge is lacking in the formulation above? Considering what has been called *defeaters* of defaults, that is, formulae which, being exceptions to an specific default, can defeat its consequent, he points out that we must distinguish between two of them: those that logically contradict the consequent and those present in the semi-normal part of the default that just block its use. These are roughly what John Pollock calls rebutting defeaters and undercutting defeaters, respectively[37]. Adopting Pollock's terminology, while fly(Twenty) is a rebutting defeater of $(1)_{\text{Twenty}}$, winged(Twenty) is an undercutting defeater.

That is the first sort of information that default logic fails to take into account. The second one is related to the treatment that should be given to each sort of defeater. According to Pequeno, we should require much more from rebutting defeaters than from undercutting defeaters. More specifically, rebutting defeaters by his definition should contradict the consequent of the default in a strong, deductive way. As a result, formulae which contradict the consequent and were obtained nonmonotonically would not be taken as rebutting defeaters and consequently will not be able to block the use of the default. On the other hand, formulae that contradict the semi-normal part of the default just in a weak, inductive way are bona-fide undercutting defeaters. Consequently they may by themselves block the use of the default.

Given this, it is trivial what the second relevant information that default logic fails to take into account is: in the case of our Twenty example, that ¬winged(Twenty) was obtained through a default. Since default logic makes no distinction at all between deductively obtained and inductively obtained formulae, it cannot treat rebutting defeaters and undercutting defeaters, or the normal and

[37] Pollock (1987).

semi-normal parts of the default according to the way we have explained above. And it is the non-fulfillment of Carnap's requirement which this inability to distinguish between monotonic and nonmonotonic formulae causes that makes default logic incapable of properly solving the problem illustrated in our last example.

As mentioned by Pequeno, the second class of problems concerns cases of contradictions where "the conflict cannot be resolved no matter how carefully the available evidence is examined" and "a situation of real contradiction arises." Consider the example we gave in Chapter 2 where it is known that (a) Quakers usually are pacifists, (b) republicans usually are not pacifist and (c) Richard Nixon is both a Quaker (d) and a republican. This would be formalized in default logic as follows:

1) $\dfrac{\text{Quaker}(x)\text{: pacifist}(x)}{\text{pacifist}(x)}$

2) $\dfrac{\text{republican}(x)\text{: }\neg\text{pacifist}(x)}{\neg\text{pacifist }(x)}$

3) Quaker(Nixon)

4) republican(Nixon)

Trivially, we will have here two contradictory extensions: one containing pacifist(Nixon) and the other ¬pacifist(Nixon). In contrast to our previous example however, we have here no means to say which default has priority over the other. The knowledge situation simply does not provide such sort of information. The cause of the contradiction is not therefore the representational framework we are using, but the incompleteness of the knowledge we are trying to represent.

Pequeno argues that Reiter's choice approach[38] precludes us from satisfying Carnap's requirement of total evidence. In the case of Nixon's example, keeping the two contradictory conclusions apart from each other will make impossible for us to see from one extension that the other conclusion is equally plausible. That is to say, from the point of view of each extension, we are trivially failing to take into account all relevant knowledge. Of course if we adopt an outside point of view with respect to the extensions we will be able to see that both pacifist(Nixon) and ¬pacifist(Nixon) are plausible. However, since the logical reasoning (both monotonic and nonmonotonic) always takes place inside and not outside each extension (in such a way as to see all extensions at the same time), we can say that the outside point of view leaves the scope of the logical reasoning we are concerned about. Therefore, if we agree that Carnap's requirement is to be satisfied inside those structures in which the reasoning takes place, then the choice account in fact does not satisfy from a strict point of view Carnap's requirement of total evidence.

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[38] We have explained in Section 3.2 why Reiter's logic is better said to formalize a choice approach to induction than a strictly credulous one.

Based on this, Pequeno argues that the proper way to deal with contradictory conclusions obtained through defaults is to keep them in the same extension and at the same time have a paraconsistent mechanism to reason about them. In his words[39]:

> [...] these contradictory conclusions should be assimilated in a single theory and reasoned out just as any other. This would emphasize the need for a better understanding of this kind of situation in order to provide a purely logical analysis for them. In other words, to arrive at these contradictions that emerge in the course of reasoning is just to give the right account of what is going on in the situation. Obviously this could not be done in classical logic. A special logic, a paraconsistent one, is required.

We see then the very same point we have made in Chapter 3 being made in connection with the formalization of commonsense reasoning. Despite the discussion being centered around inconsistency problems arising from the use of a specific formalism, the thesis defended by Pequeno is actually a general one according to which since inconsistency is a natural companion of nonmonotonic reasoning, paraconsistency should play some role in the formalization of such reasoning.

Nowadays, as exemplified by the work of many AI theorists such as Dov Gabbay, Ofer Arieli, Arnon Avron, Johan Akker and Pequeno himself, the importance of paraconsistency for the formalization of monotonic reasoning has been more and more recognized[40]. As far as we are concerned, the important point is that all these theorists took a credulous approach of induction in which the necessity of a paraconsistent treatment of inductive conclusions is explicitly recognized.

4.3 The System IDL & LEI

The problems mentioned in the previous section with default logic motivated some theorists to develop nonmonotonic systems which explicitly distinguish between certain and defeasible facts. As a consequence of that, it became necessary the development of an inferential mechanism meant to replace classical logic as the monotonic basis of the nonmonotonic logic in question and able to reason distinctly about irrefutable, monotonically obtained conclusions on the one hand and refutable, nonmonotonically obtained conclusions on the other.

Incidentally, the special way according to which this new logic is suppose to treat defeasible facts made possible to deal with Minsky's two complaints about classical logic's ability to represent commonsense reasoning which we have shown at the beginning of the previous section. While nonmonotonic logics can be said to cope in a relatively satisfactory way with Minsky's first constraint, regarding the second one no adequate way of dealing with contradictions was seriously

[39] Pequeno & Buchsbaum (1991).
[40] Gabbay & Hunter (1991) (1992), Arieli & Avron (2000), Avron & Lev (2001), Akker & Tan (1993).

considered during all the 1970's and 1980's. Despite the existence since the 1970's of a class of logical systems able to represent inconsistent and nontrivial theories, namely paraconsistent logics, it was not until the beginning of the 1990's that AI theorists came up with a "project" which would effectively allow us to answer Minsky's two criticisms in a cohesive way. Arnon Avron and Ido Lev describe the essence of such project[41]:

> For a long time the research efforts on paraconsistency and on nonmonotonic reasoning were separated. [...] However, in recent years the formal connections between these two areas have begun to be revealed. It is only natural that such a connection would exist, because conclusions that are drawn based on partial information may contradict new and more reliable information, and each new piece of information may contradict previous information and hence force us to revise some of our knowledge.

The idea is to adopt a paraconsistent logic to deal with the contradictions which are sure to arise from reasoning about commonsense. We have already mentioned in the previous section the names of some theorists who have worked with a sort of paraconsistent approach to nonmonotonic reasoning. From the point of view of our division of inductive theories into credulous and skeptical, this step had the important consequence of turning the credulous account of inductive inferences into a logically tenable approach. From a more technical point of view, it showed how we might have a paraconsistent logic as the underlying logic of a specific nonmonotonic inferential system. Since the paraconsistent behavior is primarily directed towards nonmonotonically obtained or plausible facts, these approaches also gave important hints of how a credulous logic of plausibility should proceed in connection with an inductive logic in the style of the existing nonmonotonic logics.

4.3.1 Inconsistent Default Logic

Due to space reasons, we do not intend here to survey all or even the main paraconsistent nonmonotonic logics. Rather, we will try to continue the analysis started in the previous section and show how Reiter's default logic could be expanded in such a way as to allow us to reason paraconsistently about plausible facts.

Incidentally, this is exactly the project proposed by Pequeno in his 1991 paper. There, he presents a new default logic where nonmonotonically obtained facts are syntactically distinguished from certain ones and a paraconsistent logic is used to reason about the nonmonotonic, refutable conclusions. Advancing a little bit our conclusive remarks, we may say that while Reiter's default logic does provide a descriptive account for induction but has nothing which could resemble a logic of plausibility, Pequeno's default logic provides a similar inductive mechanism but at this time in connection with a (exclusively credulous) logic of plausibility. In subsequent works, exploring a

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[41] Avron & Lev (2001), p. 60.

line already present in the 1991 paper, Pequeno extended his system in such a way as to include also a skeptical treatment of plausible conclusions. However, both in the case of Reiter's logic and Pequeno's systems, there is no way one could develop, out of the logical devices available, neither something worthy of being called a calculus of inductive implication nor what we have called a model of confirmation.

In his 1991 article, Pequeno proposes to solve the two problems we have mentioned in the previous section by introducing a logic called by him *Inconsistent Default Logic* (IDL) which modifies Reiter's default in such a way as to explicitly distinguish between monotonic and nonmonotonic formulae on the one hand, and the normal and semi-normal parts of the default on the other. The distinction between rebutting defeater and undercutting defeater comes automatically from that. In order to keep contradictory inductive conclusions in the same extension, he proposes a new, paraconsistent logic in the lieu of classical logic as IDL's underlining monotonic logic. The general IDL default rule is stated below.

$$\frac{\alpha\text{: } \varphi;\ \beta}{\varphi?}$$

Pequeno calls α the *antecedent*, φ the *default condition* and β the *proviso* of the default. The first of the two above-mentioned distinctions is achieved by attaching to the consequent of the default the symbol ?, which is intent to indicate that the formula in question, if inferred, would be done nonmonotonically. The second distinction is reached by splitting the justification part of Reiter's default into two: one representing the normal component (φ), and other representing the semi-normal part (β). He explains these points as follows[42]:

> This rule is a modification of Reiter's rule in accordance with the following considerations: [First] a defeasible conclusion can never have the same epistemic status as an irrefutable one, obtained from deduction. Thus in IDL the former are distinguished from the latter by the use of a question mark (?) suffixing defeasible formulas. [Second] IDL implements the idea of accommodating conflicting views in the same extension. Therefore, in IDL the defeasible negation of a *default condition* (¬B)? [here represented by (¬φ)?] (we call it a *weak contradiction*) does not prevent the application of the default rule. In order to defeat a default application a *strong contradiction* is ¬B required. [Third] The *semi-normal* part of a default is frequently used to express an exception condition. In IDL, C [that here is represented by β] is really taken as a *proviso* for the application of the rule, receiving a different treatment. In order to defeat the application of an IDL default rule by its proviso, a weak contradiction, (¬C)?, suffices.

As usual, a default theory, now called *IDL theory*, is a pair <W, D> where W is a set of closed formulae and D is a set of IDL defaults. Closed defaults and closed theories are also defined as usual. The notion of extension is defined as follows:

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[42] Ibid. Italics in the original.

Let S be a set of closed formulae and <W, D> a closed IDL theory. $\Gamma(S)$ is the smallest set satisfying the following conditions:

(i) $W \subseteq \Gamma(S)$;

(ii) $Th(\Gamma(S)) \subseteq \Gamma(S)$;

(iii) If $\dfrac{\alpha: \varphi; \beta}{\varphi?} \in D$, $\alpha \in \Gamma(S)$, $\neg\varphi \notin Th(W)$ and $(\neg\beta)? \notin S$, then $\varphi? \in \Gamma(S)$

A set of formulae E is an *extension* of <W, D> iff $\Gamma(E) = E$, that is, iff E is a fixed point of the operator Γ.[43]

In order to see how the above definition distinguishes between rebutting defeaters and undercutting defeaters, let us see how it deals with the first sort of problem that default logic was unable to solve. Facts (a)-(e) will be represented in IDL as follows:

1) $\dfrac{\text{animal}(x): \neg\text{fly}(x); \neg\text{winged}(x)}{\neg\text{fly}(x)?}$

2) $\text{winged}(x) \rightarrow \text{fly}(x)$

3) $\text{bird}(x) \rightarrow \text{animal}(x)$

4) $\dfrac{\text{bird}(x): \text{winged}(x)}{\text{winged}(x)?}$

5) bird(Twenty)

We see that (1) and (4) are now written in Pequeno's notation. Assuming Twenty as the only constant symbol of the language, the corresponding IDL theory will be <W, D>, where W = {(2), (3), (5)} and D = $\{(1)_{<Twenty>}, (4)_{<Twenty>}\}$. As before, we can use $(4)_{<Twenty>}$ along with (2) and conclude that Twenty flies. Adopting the other, anomalous path we can conclude animal(Twenty)? from (5) and (2). From it, by using $(1)_{<Twenty>}$, we conclude ¬fly(Twenty)?, and by contrapositive on (2) we get ¬winged(Twenty)?. Notice however that since ¬winged(Twenty)? was obtained through a default, it does not belong to Th(W). As such, by item (iii) of the definition of extension it cannot be a rebutting defeater of $(4)_{<Twenty>}$. Therefore, we can still use $(4)_{<Twenty>}$ to conclude winged(Twenty)?. But since ¬winged(Twenty) is in the semi-normal part of $(1)_{<Twenty>}$, by the same item (iii), we have that the just inferred conclusion winged(Twenty)? is a undercutting defeater of $(1)_{<Twenty>}$. Consequently, it will block the use of $(1)_{<Twenty>}$ and therefore made us realize that its previous utilization was in fact mistaken. Therefore, only one extension containing fly(Twenty)? will be generated.

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[43] Pequeno (1990), p. 20. As confirmed by Pequeno in personal conversation, the original version of the definition of extension contains a misprint in item (iii). What we present above is the corrected definition as laid down by him in the mentioned conversation.

As we have seen, in order to deal with situations such as Nixon example Pequeno proposes to keep the plausible contradictory conclusions in the same extension and reason paraconsistently about them. This is achieved by changing the logic that will define the operator Th( ) in the definition of extension: instead of classical logic, we will have a paraconsistent logic able to reason non-trivially in the presence of plausible contradictions. In his words[44]:

> In the definition of extension above, Th(Γ(S)) stands for the closure under deduction of the set of formulas Γ(S). This deduction is not deduction on classical logic, but a special logic [...] [that] works classically for monotonic conclusions but paraconsistently for nonmonotonic ones (Notice that Γ(S) includes formulas with '?').

Rewritten in IDL's notation, Nixon example becomes almost like the formalization we gave in Reiter's logic:

1) $\dfrac{\text{Quaker(x): pacifist(x);}}{\text{pacifist(x)?}}$
2) $\dfrac{\text{republican(x): } \neg\text{pacifist(x);}}{(\neg\text{pacifist (x))?}}$
3) Quaker(Nixon)
4) republican(Nixon)

As it can be easily seen, in this case we will have only one extension containing both pacifist(Nixon)? and (¬pacifist(Nixon))?. Since the logic responsible for reasoning about the extension's formulae is a paraconsistent one, this contradiction will not trivialize the logical theory.

Concerning the notion of inductive implication we have agreed Reiter's default logic formalizes, Pequeno's IDL does a similar job but with a very important difference: it recognizes the epistemological status of nonmonotonic conclusions, marking them with a special symbol. The other significant difference concerns the interpretation given to the exception part of the inductive implication. Translating Pequeno's default into our language of inductive implication,

$$\frac{\alpha\text{: } \varphi\text{; } \beta}{\varphi?}$$

would mean something like "α inductively implies φ unless ¬φ is the case or ¬β is plausible." Operationally this means that while (¬β)? is enough to prevent α from being taken as an evidence for φ, (¬φ)? is not: we will need to be certain about ¬φ to prevent φ from being inductively implied by α. This, we must concede, is an important refinement of the notion of inductive implication as understood from Reiter's default logic. It has however the disadvantage of restricting the exception of inductive implications only to plausible formulae. It would be much better if we could choose

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[44] Pequeno (1990), p. 21.

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would mean something like "α inductively implies φ unless $\neg\varphi$ is the case or $\neg\beta$ is plausible." Operationally this means that while $(\neg\beta)?$ is enough to prevent α from being taken as an evidence for φ, $(\neg\varphi)?$ is not: we will need to be certain about $\neg\varphi$ to prevent φ from being inductively implied by α. This, we must concede, is an important refinement of the notion of inductive implication as understood from Reiter's default logic. It has however the disadvantage of restricting the exception of inductive implications only to plausible formulae. It would be much better if we could choose

[44] Pequeno (1990), p. 21.

what sort of formulae (plausible or certain) we want to consider in the exception of our inductive implication.

4.3.2 The Logic of Epistemic Inconsistencies (or A Logic of Plausibility)

A very important improvement of IDL was that, by introducing a syntactical mark to distinguish between monotonic and nonmonotonically obtained facts, it left the path to a logic of plausibility open. Indeed, one year after the publication of IDL, in an article written conjointly with Arthur Buchsbaum, Pequeno laid down in an axiomatic and semantic way the paraconsistent logic which is supposed to serve as the monotonic basis of IDL and which could play the role of our logic of plausibility. They named it the *Logic of Epistemic Inconsistency* (LEI)[45]. In addition to the standard formulae of propositional calculus' language, LEI's language $\Im_?$ contains formulae marked with the question mark '?' (in such a way that if α is a formula of $\Im_?$, α? is also so.) Formulae of the form α?, which may be read as "α is plausible"[46], are exactly those formulae obtained with the use of some IDL default and added to the extension with the symbol ? attached to it.

Following the strategy used by Newton da Costa in his paraconsistent calculus C_1[47], LEI's axiomatic arrives at a paraconsistent behavior with regard to ?-marked formulae by modifying the *reductio ad absurdum* axiom in such a way as to allow it to be used only in connection with ?-free formulae. Letting Latin letters denote such ?-free formulae, the *reductio ad absurdum* axiom is rewritten as follows:

$$(\alpha \rightarrow B) \rightarrow ((\alpha \rightarrow \neg B) \rightarrow \neg\alpha)$$

Therefore, ¬ behaves paraconsistently with respect to ?-marked formulae and classically with respect to ?-free formulae. In order to get a negation that behaves classically independently of the form of the formulae, they define a stronger negation as follows:

$$\sim\alpha =_{def} \alpha \rightarrow (p \wedge \neg p)$$

, where p is an arbitrary sentential letter.

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45 Pequeno & Buchsbaum (1991).

46 In the 1991 article, the interpretation given for ?-marked and ?-free formulae is standard one given by nonmonotonic theorists: they are referred with the help of terms like "defeasible" and "indefeasible" or "refutable" and "irrefutable." Only in an article already cited in the 1991 paper but published only two years after is that α? was explicitly taken as meaning "α is plausible" (Buchsbaum & Pequeno (1993).) One year after, in an article written conjointly with Ana Tereza Martins, α? was interpreted simultaneously as "α is plausible" and "there is evidence for α" (Martins & Pequeno (1994).) Despite these and others uses of the term "plausibility," it was only in 1997, in the doctoral dissertation of Martins, co-supervised by Pequeno, that LEI was explicitly presented as something like a logic of plausibility (Martins (1997).)

47 da Costa (1974).

It is interesting to note that in da Costa's calculus C_1 the *reductio ad absurdum* axiom is conditioned to B's explicitly satisfying the law of non-contradiction. Letting B^o be an abbreviation for $\neg(B \wedge \neg B)$, da Costa's *reductio ad absurdum* axiom is stated as $B^o \rightarrow ((A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A))$[48]. In other words, all formulae B are taken as contradictable unless explicitly marked with ^{o}. LEI's presupposition is exactly the opposite: formulae are in general non contradictable, and only those plausible, default generated ones are taken as having a paraconsistent behavior.

Besides the *reductio ad absurdum* axiom, we can mention the following axioms or theorems of LEI:

1. $\alpha \rightarrow \alpha?$	2. $\alpha?? \rightarrow \alpha?$
3. $\neg(\alpha \wedge \neg\alpha)$	4. $(\neg\alpha)? \leftrightarrow \neg(\alpha?)$
5. $\neg\neg\alpha \leftrightarrow \alpha$	6. $(\alpha? \rightarrow \beta?) \rightarrow (\alpha \rightarrow \beta)?$
7. $(\alpha? \vee \beta?) \leftrightarrow (\alpha \vee \beta)?$	8. $(\alpha \wedge \beta)? \rightarrow \alpha? \wedge \beta?$
9. $\neg(\alpha \rightarrow \beta) \leftrightarrow (\alpha \wedge \neg\beta)$	10. $\neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$
11. $\neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$	12. $\alpha \vee \neg\alpha$

The first four schemas require some comments. (1) and (2) lay down two important features of ? according to which irrefutability implies plausibility and additional ?'s are irrelevant to plausible formulae, respectively. From the point of view of a possible world style semantics, they mean that the accessibility relation between what we may call plausible worlds is reflexive and transitive, respectively. As we will see below, the semantics proposed by Pequeno and Buchsbaum, which does follow a Kripke style, really imposes these restrictions on its "accessibility relation." It is quite significant that in 1993 Johan Akker proposed a modified version of the LEI-IDL system (along with practical arguments for it) according to which (2) is not valid and defaults are sensitive to multiple level plausible formulae[49]. Even though he does not mention a semantic component called "accessibility relation," clearly his solution can be characterized as a LEI where such relation is not necessarily transitive. Letting aside questions like which system is the best – with or without (2), Akker's proposal had the merit of showing that logics of plausibility slightly different from LEI in which (2) (or, we may say, (1)) does not hold may be both from a theoretical as well as from a practical point of view worthy of being investigated.

Formula (3), which is a theorem, exemplifies a very interesting feature of LEI. It confronts in a direct way da Costa's statement that a necessary condition that all paraconsistent logic should satisfy is that the principle of non-contradiction must not be a theorem[50]. As it will be clear when we

[48] See da Costa (1974).
[49] Akker & Tan (1993).
[50] See da Costa (1974).

show LEI's semantics, the reason for (3)'s being a theorem is that $\neg(\alpha \wedge \neg\alpha)$ is not an intra-logical representation of the principle of non-contradiction, for $\neg$ is a weak, non-exclusive negation. The non-satisfaction of the mentioned principle is stated by a meta property according to which we may have a non trivial set of formulae A such that, for some formulae α, $A \vdash \alpha?$ and $A \vdash \neg(\alpha?)$.

The right-side of axiom (4), let us call it (4'), has been a very controversial feature of LEI. Intuitively we may accept that the implausibility of α implies the plausibility of $\neg\alpha$ ($\neg(\alpha?) \rightarrow (\neg\alpha)?$). The same however does not seem to hold regarding the plausibility of $\neg\alpha$'s implying the implausibility of α ($(\neg\alpha)? \rightarrow \neg(\alpha?)$). If we interpret $\alpha?$ as meaning "there is evidence for α," then $\neg(\alpha?)$ will mean "there is no evidence for α." Therefore, $(\neg\alpha)? \rightarrow \neg(\alpha?)$ will mean "if there is evidence for $\neg\alpha$, then there is no evidence for α," which is unacceptably counterintuitive.

This objection is important for two reasons. First, from the point of view of default reasoning, it is (4') what guarantees that "actual" $\{\alpha?, \neg(\alpha?)\}$ and not just "apparent" $\{\alpha?, (\neg\alpha)?\}$ plausible contradictions arise from the use of defaults. In Nixon example, from (4') and (¬pacifist(Nixon))? we are able to get ¬(pacifist(Nixon)?), which along with ¬pacifist(Nixon) represents a contradiction that unless treated paraconsistently has the capacity to trivialize the extension. The point however is that if we do not accept (4'), we will not have ¬(pacifist(Nixon)?) and consequently any contradiction of the sort $\{\alpha, \neg\alpha\}$. We at most will be faced with things like $\{\alpha?, (\neg\alpha)?\}$ which may indeed be called plausible contradictions but have no power to trivialize the theory. Consequently, there will be no need of a non-classical, paraconsistent logic to reason about ?-marked formulae.

The problem with all that is that since it is the actual and not apparent plausible contradictions what requires paraconsistency, we may quite fairly doubt if paraconsistency is really, as Pequeno defends, a necessary or even desirable feature of formalized common sense reasoning. If in the very framework that Pequeno built to support his thesis, "actual" inductive contradictions appear only with the help of a quite controversial and artificial axiom without which no paraconsistency would be required, then it seems that rather than supporting his thesis, the system LEI-IDL in fact shows that nonmonotonic reasoning can go on very nicely without paraconsistency.

Of course the conclusion stated above depends on the very particular formalism we are discussing. If, for instance, we follow da Costa's style and take formulae as contradictable unless explicitly stated the opposite, then contradictions coming from the use of defaults would be actual contradictions and consequently require some sort of paraconsistent mechanism to reason about them. It also depends on the use of the phrases "actual contradiction" and "apparent contradiction." What we mean is that our twofold division comprises only "formal contradiction," that is, contradictions depending on the material language one is using. If we adopt a more general, we may say conceptual point of view, then the set {"α is plausible", "$\neg\alpha$ is plausible"} can be taken as an authentic contradiction which, depending on the language used, may or not give rise to a formal actual contradiction. If we use a modal operator in the style of Pequeno's to represent plausibility, only (formal) apparent contradictions will arise from the mentioned set. If instead we follow da Costa's path and decide to mark certain rather than plausible facts, then from the same set we will be faced with (formal) actual contradictions. Therefore, the argument above stated threats Pequeno's thesis only with regard to formal paraconsistency. Even if we drop (4') we are still left with a much more basic form of paraconsistency: what we may call conceptual paraconsistency[51].

The second point is concerned with the relation of plausibility and possibility on the one hand, and LEI and S5 on the other. In Martins (1997), $(\neg\alpha)? \rightarrow \neg(\alpha?)$ was presented as the main difference between the modal operators ? of LEI and $\Diamond$ of S5. In a more general way, it was considered as one of the most basic differences between the notions of plausibility and possibility: if one decides to drop (4'), the two operators ? and $\Diamond$ as well as the notions of plausibility and possibility will become indistinguishable from one another. The problem again is that, since we do not have good reasons to keep (4'), all the use of the term "plausibility" as well as the presentation of LEI as a logic of plausibility loses its sense.

About the framework in which the true-value of ?-marked formulae is analyzed, the semantics presented in the 1991 article suggests a very interesting way of explicating the notions of plausibility. The intuition behind such semantics rests on the consideration of different observations of the same phenomenon. As they write[52]:

> The basic intuition to be captured by the semantics of LEI is truthfulness relative to multiple observations of a same phenomenon, taken under different conditions, when information about these conditions (or even on how observations can be affected by them) is not available. [...] We are facing again a situation of insufficient knowledge leading to disagreement. It parallels our initial motivation about multiple extensions generated by a default theory when lack of knowledge does not enable the control of the selection of plausible alternatives.

Following this idea, a formal semantic model[53] is defined as a non-empty collection C of classical valuations which attribute to each propositional symbol p a truth-value 0 or 1. The function V which will make use of the members of C to evaluated the semantic value of formulae is defined

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[51] In a similar way we can obtain the notion of conceptual paracompleteness. These two notions will play an important role in the discussion we will carry out in the next chapter about the nature of the skeptical and credulous notions of plausibility.

[52] Pequeno & Buchsbaum (1991).

[53] From now on, we will use the term "model" in the same way it is used in standard textbooks of modal logic: rather than denoting an interpretation that satisfies a specific set of formulae, as it does in propositional

through two auxiliary functions V^{max} and V^{min}. Since the truth-value of formula α, let us say, is analyzed with the help of a set of classical valuations representing different interpretations of the same phenomenon, one may adopt two different positions in this endeavor: a skeptical one which by requiring α to be true in all members of C tries to minimize the truth-value of α, and a credulous one which by requiring α to be true in at least one member of C tries to maximize its true-value. It is these two positions what the functions V^{max} and V^{min}, respectively, are meant to represent. Plausible formulae are analyzed through a maximal position, and ?-free formula, which represent irrefutable, certain formulae are analyzed through a minimal perspective. Formally, the functions V, $V^{max}{}_c$ and V^{min} are as follows:

$V(\alpha) = 1$ iff for all $c \in C$, $V^{max}{}_c(\alpha) = 1$;

$V^{max}{}_c(p) = V^{min}{}_c(p) = v_c(p)$;
$V^{max}{}_c(\neg\alpha) = 1$ iff $V^{min}{}_c(\alpha) = 0$;
$V^{min}{}_c(\neg\alpha) = 1$ iff $V^{max}{}_c(\alpha) = 0$;
$V^{max}{}_c(\alpha?) = 1$ iff for at least one $c' \in C$, $V^{max}{}_{c'}(\alpha) = 1$;
$V^{min}{}_c(\alpha?) = 1$ iff for all $c' \in C$, $V^{min}{}_{c'}(\alpha) =1$;
$V^{max}{}_c(\alpha \rightarrow \beta) = 1$ iff $V^{max}{}_c(\alpha) = 0$ or $V^{max}{}_c(\beta) = 1$;
$V^{min}{}_c(\alpha \rightarrow \beta) = 1$ iff $V^{max}{}_c(\alpha) = 0$ or $V^{min}{}_c(\beta) = 1$;
$V^{max}{}_c(\alpha \wedge \beta) = 1$ iff $V^{max}{}_c(\alpha) = 1$ and $V^{max}{}_c(\beta) = 1$;
$V^{min}{}_c(\alpha \wedge \beta) = 1$ iff $V^{min}{}_c(\alpha) = 1$ and $V^{min}{}_c(\beta) = 1$;
$V^{max}{}_c(\alpha \vee \beta) = 1$ iff $V^{max}{}_c(\alpha) = 1$ or $V^{max}{}_c(\beta) = 1$;
$V^{min}{}_c(\alpha \vee \beta) = 1$ iff $V^{min}{}_c(\alpha) = 1$ or $V^{min}{}_c(\beta) = 1$.

The most remarkable feature of this semantic framework to us is the recognition of two postures supposedly necessary for the analysis of plausible facts: a skeptical and a credulous one. Even though ?-marked formulae are analyzed according to a credulous perspective, both functions V^{min} and V^{max} are needed to assign truth-values to formulae. This is due to the way negation formulae are evaluated: the truth-value of negation formulae given by the skeptical function depends on the credulous function, and vice-versa. This strange behavior of negation is at the root of the axiom (4') and consequently of the formal actual paraconsistent behavior of ?. It is also it what differentiates this sort of semantics from traditional modal logic's one: if by lack of philosophical justification, for instance, we drop (4'), we will be left with a semantics that is in all essential aspects indistinguishable from Kripke's possible world semantics. Also noteworthy is the way certain and plausible facts are formally distinguished from each other: while in order for plausible, ?-marked

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and first-order classical logic, a model will be taken here as the interpretation itself, that is, the structure in formulae to be valid the inside sentence has to be true in at least one valuation c, for certain, ?-free formulae the whole formula has to be true in all valuations.

After the publication of the 1990 and 1991 articles, different versions of LEI and IDL were proposed. In Buchsbaum (1995) a first-order axiomatic of LEI, followed by a complete and sound semantics, was presented. In Martins (1997) a first-order sequent calculus presentation was laid down. In Buchsbaum (1995), the exclamation mark operator ! was defined as follows:

$\alpha! =_{def} \sim((\sim\alpha)?)$

Compared to traditional modal logic, ! would play the role of $\Box$: $V^{max}_{c}(\alpha!) = 1$ iff for all $c' \in C$, $V^{max}_{c'}(\alpha) = 1$. However, in contrast to ?, ! has a classical behavior: for formulae marked with !, all classical laws are valid.

The alternative of droping axiom (4') has been formally explored in the most recent works of Pequeno and Buchsbaum. In Pequeno, Buchsbaum & Pequeno (2001) and Buchsbaum, Pequeno & Pequeno (2004), LEI (renamed in these works the *Logic of Appearance* and the *Logic of Plausible Deduction*, respectively) does not contain (4') as an axiom, which makes ? behave from a formal point of view no more paraconsistently. In the 2001 article, ! is introduced as meaning the strong or skeptical plausibility, while ?, which is derived from it ($\alpha! =_{def} \neg((\neg\alpha)?)$) is meant to represent the weak or credulous plausibility. Since now ? is not any more a paraconsistent operator, they came up in these articles with two operators that are both from a syntactical as well as from a semantic point of view identical to S5's $\Diamond$ and $\Box$. Naturally, in these versions the auxiliary functions V^{max} and V^{min} are given up and the semantic follows, in all relevant respects, the traditional style of possible world semantics.

In order to differentiate ? and ! from $\Diamond$ and $\Box$ as well as to justify the use of the term plausibility rather than simply possibility and necessity, they incorporate into the same logical language the symbols $\Diamond$ and $\Box$ (being one primitive and the other derivate – $\Diamond\alpha =_{def} \neg\Box\neg\alpha$) with their traditional meanings. The formal distinction between the two notions of necessity and skeptical plausibility (and consequently between possibility and credulous plausibility) is done syntactically through the axiom $\Box\alpha \rightarrow \alpha!$. Semantically, this is done through the use of two sets of worlds, one called the set of *possible worlds* and the other called the set of *plausible worlds*, where the latter is a proper subset of the former. In this say, every necessary (or possible formula) will also be a skeptically (or credulously) plausible formula and every plausible world will also be a possible world, but not vice-versa. Concerning the philosophical analysis of the notion of plausibility, this

which the truth-value of the formulae of the language will be evaluated.

new formulation has the strange feature of making plausibility dependent on the notion of necessity: if in the language in question the notion of necessity is represented, then we can formalize the notion of plausibility; otherwise ? and ! will simply become the same as ◇ and □. Formally and adopting the standard way of presentation of propositional modal logic semantics, we have as follows:

A model M is a triple $<W, W', v>$ where W is a nonempty set of possible worlds, $W' \subseteq W$ is a set of plausible worlds and v is a function of the form : $P \times W \rightarrow \{0,1\}$ which attributes to each propositional symbol and world a truth-value 0 or 1.

Let $M = <W,W',v>$ be a model. The function $V: \Im \times W \rightarrow \{0,1\}$ which attributes to each formulae and possible world a truth-value 1 or 0 is defined as follows:

$V_w(p) = 1$ iff $v(p) = 1$;

$V_w(\alpha!) = 1$ iff for all $w' \in W'$, $V_{w'}(\alpha) = 1$;

$V_w(\Box\alpha) = 1$ iff for all $w' \in W$, $V_{w'}(\alpha) = 1$[54].

The important point for us in all these so-called logics of plausibility is that for the first time we see an attempt to represent in the same logical system two notions referred to as the skeptical plausibility and credulous plausibility. And not less important, the way the so-called skeptically and credulously plausible formulae inside these systems were obtained is intimately connected with the use of a specific nonmonotonic logic: they consist exactly in the conclusions of the inductive inferences we may represent inside such nonmonotonic logic.

A little bite more fundamental point is the way of representing the two notions of plausibility that these systems suggest. Ignoring secondary formal details, we have at hand a quite promising path where the notions of skeptical plausibility and credulous plausibility are interpreted inside a possible world-like structure, being skeptically plausible facts formalized with the help of a necessity-like operator, and credulously plausible facts with the help of a possibility-like operator. As one can see, this is very close to the explanation of the plurality approach to plausibility we have given in Chapter 3[55]. Considering this suggestion and the two sorts of systems so far exposed, the fundamental choice we will have to make then is if we use a classical structure in the style of traditional modal logics which Pequeno and his collaborators seem to have preferred at the end, or the first one which actually supplies ? with a formal actual paraconsistent behavior (and, as we will see, ! with a formal actual paracomplete behavior). The answer to this question will be postponed

[54] The other connectives are defined as usual.

[55] In Chapter 5, we will elaborate further on this path and present a modal logic where the notions of skeptical plausibility and credulous plausibility are formalized in the above-mentioned way.

until next chapter, where we will try to analyze the properties our two plausibility notions would have concerning the negation operator inside such sort of framework.

4.3.3 IDL: Further Developments

Regarding IDL, the most relevant novelty after the publication of the 1990 and 1991 papers took place between the years 2001 and 2004. In Pequeno, Buchsbaum & Pequeno (2001) and Buchsbaum, Pequeno & Pequeno (2004) a considerably different version of IDL (at this time called the *Logic of Pragmatic Truth* and the *Logic of Plausible Reasoning*, respectively) was proposed. First, the notation was shortened from a three-parameters schema to a two-parameters one intent to represent those generalizations expressed with the help of the conjunction "unless." This is done through the symbol —(, which applied to two modality-free propositions α and β means "α unless β" (in symbols: α —(β.)[56] The whole expression α —(β is called a *generalization* (we will keep calling it default), α the *conjecture* and β the *exception* of the generalization. According to this notation, item (iii) of the definition of extension would be rewritten as follows:

(iii") If α —($\beta \in D$, α is consistent with $\Gamma(S)$ and $\beta \notin S$, then $\alpha? \in \Gamma(S)$

The translation of the old notation into the new one would be as follows:

$(\alpha \rightarrow \varphi)$ —($\neg\beta$

We should observe first that the test of consistency of the consequent of the default is done inside the definition of extension. Besides turning the specification of the normal part of the default into an unnecessary step, this has the consequence of allowing only normal and semi-normal default to be represented: abnormal defaults where the consequent is not implied by the justification are automatically refused. Second, rather than considering the proviso of the default, that is, those formulae which must be consistent with the extension in order for the default to be used (which in Reiter's notation corresponds to the semi-normal part of the justification), it takes the exceptions of the default, that is, those formulae which, by being the negation of the semi-normal part of the justification, are able to block the use of the default.

Despite these unquestionable improvements, this new notation has the disadvantage of restricting the representational power of the logic. As we have seen, the old notation is represented now with the help of the material implication. In this case, the inference of β from α will be done

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[56] The restriction on the form of α and β was adopted only in the 2004 article. In the 2001 article —(is defined for every sort of formula. This has the problem of, in some cases, allowing the nonmonotonic inference of ?-free formulae. For example, according to the 2001 article, $\Box\alpha$ —(β is a well-formed default. Depending then on the IDL theory, $(\Box\alpha)?$ will be included in the extension. However, since $(\Box\alpha)? \rightarrow \Box\alpha$ is a theorem of the multi-modal version of LEI, $\Box\alpha$ will also belong to the extension.

only indirectly: from default $(\alpha \to \varphi)$ —($\neg\beta$ we conclude $(\alpha \to \beta)?$, from which we get $\alpha? \to \beta?$ and finally from that along with $\alpha?$ we get $\beta?$. The point is that this process trivially does not let us to represent a situation where we want $\beta?$ to be nonmonotonically inferred *only* from the certainty of α: if we want to infer β from α we will have to do that through $\alpha?$. Funny enough, this new notation is not powerful enough to represent the old default schema as described in the 1990 article.

The definition of extension presented in the 2001 and 2004 articles is in fact much more complex than the original one. One of its main motivations was to solve the apparent limitation of LEI according to which plausible facts cannot be conjoined. Akin to $\diamond$, formula $\alpha? \wedge \beta? \to (\alpha \wedge \beta)?$ is not a theorem of LEI. That such formula is not desirable can be seen from the situation where we have both $\alpha?$ and $(\neg\alpha)?$. However, in cases where α and β are not contradictories, it must be convenient to get $(\alpha \wedge \beta)?$ from $\alpha?$ and $\beta?$. Roughly put, the new definition of extension solves this problem by taking collections of defaults rather than individual defaults and doing two sorts of tests. Let <W,D> be a default theory with the elements of D being written in the new notation (D is as usual a set of modality free formulae.) In the part of the definition of extensions correspondent to item (iii), for each D' $\subseteq$ D it is first verified whether the conjunction of the conjectures of all elements of D' are consistent with the extension (rather than checking whether the conjecture of each default taken in isolation is consistent with E). After, it is checked if the disjunction of the exceptions of all elements of D' does not belong to E. If these tests are successful, formula $(\alpha_1 \wedge \ldots \wedge \alpha_n)?$, where $\{\alpha_1, \ldots, \alpha_n\} = \{\alpha \mid \text{there exists } \beta \text{ such that } \alpha$ —($\beta \in D'\}$, is added to E. Another purpose achieved by these two new versions was to get from the default theory, besides ?-plausible facts, also skeptically plausible. This step is important because it unites in a decisive way the logic of skeptical plausibility and credulous plausibility with the two positions one may adopt while evaluating the conclusions of defaults: the skeptical and the credulous positions.

To conclude this chapter, we may say that a default logic in the style of Reiter's and Pequeno's may be a quite promising starting-point for our representational logic of induction. First, it sure does not get involved in the problem of justification of induction and has, as we wish, a purely descriptive purpose. Second, it provides a quite satisfactory way to represent our inductive implications along with the inferential power our philosophy of induction requires them to have. Third, following Pequeno's ideas, there is an effective way to embody in our inductive logic a logic of plausibility able to formalize the most important aspect of our skeptical plausibility: its paraconsistency. Besides further refinements, two basic tasks will be left for us: to expand this logic of plausibility in such a way as to consider the skeptical notion of plausibility as well as define its role in connection with inductively obtained formulae and to provide a logical language powerful enough to represent such things as a calculus of inductive implication and models of confirmation.

CHAPTER 5

PARANORMAL MODAL LOGIC AND THE LOGIC OF PLAUSIBILITY

In this chapter we will begin to expose the formal aspects of our study of the notions of induction and plausibility. More specifically, we will lay down the mathematical framework which we intend to use to explicate the notion of plausibility. It should be said that even though this system makes use of some well-known resources of elementary modal logic, due to a specific feature of our twofold division of plausibility statements, it will correspond to an entirely new class of logical systems: what we shall call paranormal modal logic. And even though our logic of plausibility (to be presented in the next chapter) will be a specific instance of such paranormal modal logic, due to this novelty factor we will have to present the whole thing from a general point of view, where the main paranormal modal logics will be introduced without preference to any particular system. Nevertheless, since the very raison d'être of this paranormal modal logic is to formalize what we have called plurality approach to plausibility, the whole class of paranormal modal logics will be taken itself as our first, general solution to concept explication problem of plausibility.

The structure of the chapter is as follows. In the next Section we introduce the philosophical arguments for having paranormal modal logic as a necessary step in the task of explicating the notion of plausibility. In Section 5.2 we introduce some general definitions which will be used by all logics to be presented along the chapter. Akin to traditional modal logic, in paranormal modal logic we have a basic logical system $K_?$ from which several other paranormal modal logics can be obtained. In Section 5.3 this system $K_?$ is presented, both axiomatically and semantically, along with its main theorems. We also present a comparative study between $K_?$, classical logic and modal logic K. Finally in Section 5.4 we present the main paranormal modal systems that can be obtained from $K_?$, first-order paranormal modal logic and the so-called multi-normal modal logic.

5.1 Skeptical Plausibility and Credulous Plausibility as Paranormal Modalities

In Chapter 3 we presented what we have called the skeptical and credulous approaches to induction. As we have seen, these two approaches, which are chiefly related to the problem of inductive inconsistencies, give rise to two different plausibility concepts: the credulous or weak plausibility and the skeptical or strong plausibility (which we have identified with the notion of acceptance or acceptability.) In the same chapter we have tried to explain these two notions (and consequently the two approaches) with the help of what we have named the plurality approach to plausibility. The explanation went like that. Suppose that you have several authorities, experts on some subject matter, holding their points of view about issues belonging to their area. As usually happens in real life, the verdict of these experts may differ from each other. In this case, if we call each consistent set of statements representing the views of one or more of these experts an extension, we will have two or more conflicting extensions. We then argued that to an outsider, there will be basically two ways of evaluating the plausibility of a specific statement α: skeptically, requiring all experts to agree on α, and credulously, requiring at least one expert to hold it. While for the first approach what we may call plausibility criterion is something very strong, namely that α be present in each and every extension, for the second approach it is much weaker: being present in at least one extension will suffice to take α as plausible.

As we have observed in the previous chapter when we exposed the logics of plausibility developed by Pequeno and Buchsbaum, this explanation of the two plausibility concepts strongly resembles the possible world semantic-structure used in modal logic to evaluate the truth-value of modal sentences. Take for example a model M of propositional modal logic composed by a set of worlds W, an accessibility relation R and a function ν which assigns to each propositional symbol p and world $w \in W$ a truth-value 0 or 1. Letting each members of W represent an individual expert, R a trustfulness relation between these experts and ν a function which for each expert w and atomic sentence p returns the opinion of w about the truthfulness of p, we will have that α is skeptically plausible according to expert w (or simply at w) iff $V_{M,w'}(\alpha) = 1$ for every $w' \in W$ such that wRw' and that α is credulously plausible at w iff $V_{M,w'}(\alpha) = 1$ for at least one $w' \in W$ such that wRw', where $V_{M,w}$ is the function that recursively defines the truth-value of the complex sentences from atomic ones at model M and world w. Trivially enough, these are the semantic definitions of the modal operators $\Box$ and $\Diamond$. Let us for the time being adopt the notation introduced in Chapters 3 and 4 and represent skeptical and credulous plausibility, respectively, by the symbols ! and ?, both used according to a post-fixed notation. Let us also, for the sake of simplicity, suppose R to be an equivalence relation. We will then have that $V_{M,w}(\alpha!) = 1$ iff $V_{M,w'}(\alpha) = 1$ for every $w' \in W$ and $V_{M,w}(\alpha?) = 1$ iff $V_{M,w'}(\alpha) = 1$ for at least one $w' \in W$. If $V_{M,w}(\alpha) = 1$ for every world $w \in W$, we say that α is true at M. Otherwise we say it is false at M.

It will not be a surprise for those familiar with modal logic to see that this interpretation of skeptical plausibility and credulous plausibility incorporates the solution we gave in Chapter 3 to the controversy around the conjunction principle. There we have solved this problem by requiring the conjunction principle to be applicable only to skeptical plausibility or acceptance but not to credulous plausibility. If ! and ? play, respectively, the role of the modal operators □ and ◇ of modal logic, then, independently of the features of the accessibility relation, we will have that $\vDash \alpha! \wedge \beta! \rightarrow (\alpha \wedge \beta)!$ but $\nvDash \alpha? \wedge \beta? \rightarrow (\alpha \wedge \beta)?$.

Another interesting feature of this possible world account of the plurality approach to plausibility concerns the paraconsistency and paracompleteness aspects of the notion of plausibility. In Chapter 3 we have argued that paracompleteness and paraconsistency are essential features of the concepts of skeptical plausibility and credulous plausibility, respectively. This was described in the first case as the possibility of for some α having the sentences "α is accepted" and "~α is accepted" as both false[1] and, in the second case, as the possibility of having both "α is plausible" and "~α is plausible" as true without trivializing the theory. Without too much effort one can see that these properties are automatically satisfied by our two modal operators ! and ?. Taking any one of the traditional modal logics (K, T, S4, S5, etc) we will be able to conceive a model M at which both α! and (¬α)! are false: M = <W,R,v> such that, for some w',w"∈W, $V_{M,w'}(\alpha) = 1$ and $V_{M,w''}(\alpha) = 0$. At this same model we will also have α? and (¬α)? as true, which trivially allows us to classify any theory containing the set {α?, (¬α)?} as a non-trivial one.

This "non-classical" aspect of modal logic has already bee noticed by a couple of theorists. For instance, in an article entitled "S5 is a Paraconsistent Logic and so is Classical First-Order Logic" Jean-Yves Béziau shows how one may define in S5 an unary operator with the help of ◇ and ~ which has all the relevant properties of a paraconsistent negation[2]. It will be interesting for us to see how he proceeds. Trying first to define what a paraconsistent negation is, he arrives at the following criterion: a paraconsistent negation is any unary operator ¬ that such that (NN) $\alpha, \neg\alpha \nvdash \beta$, for any schema β which is not tautological and (P) ¬ has enough properties to be called a negation. He then defines an operator ¬ in the language of propositional modal logic as follows: $\neg\alpha =_{def} \Diamond{\sim}\alpha$. As in S5 we may have α such that $\alpha, \Diamond{\sim}\alpha \nvdash \beta$, where β is not a tautology, ~ satisfies NN. Regarding P, he shows that ¬ satisfies many classical properties attributed to negation such as $\alpha \vee \neg\alpha$, $\neg(\alpha \wedge \neg\alpha)$, $(\neg\alpha \rightarrow \alpha) \rightarrow \alpha$, $(\alpha \rightarrow \beta) \rightarrow \neg\alpha \vee \beta$ and $\neg\neg\alpha \rightarrow \alpha$. Therefore, ¬ may be fairly considered a paraconsistent negation. Consequently, the system in which it was defined, S5, may be likewise classified as a paraconsistent logic. It is interesting to note that, like almost paraconsistent logics, S5 has two negations, a classical one and a paraconsistent one. The difference is that while in most paraconsistent logics the paraconsistent negation is a primitive

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[1] From this point on, we will slightly change the notation used so far in the other chapters and represent the negation operator through the symbol ~, rather than ¬. The reasons for that will become clear later.

symbol and the classical one is derived from it, in S5 it is the opposite: we have a classical negation as primitive symbol and from it a non-classical, paraconsistent negation is defined.

In the case of Béziau's approach, S5 may really be taken as a paraconsistent logic; not because of ~, but because of the derived operator ¬. In our case, we are not desirous to have such derived negation. It will suffice to have ~ in connection with ! and ? and the other connectives. If we decide to follow this path, then trivially our formalization for the notion of credulous plausibility could not be called from a strictly formal viewpoint a paraconsistent one: along with ?-marked formulae, ~ clearly does not satisfy (NN). It would satisfy (NN) if we had something like $\alpha?, \sim(\alpha?) \nVdash \beta$, and not simply $\alpha?, (\sim\alpha)? \nVdash \beta$. But as we know, $\{\alpha?, \sim(\alpha?)\}$ is a trivial theory in modal logic. The same applies to !. We would have (formal) actual paracompleteness if we could have a model M such that $\alpha!$ and $\sim(\alpha!)$ are both false at M, and not simply $\alpha!$ and $(\sim\alpha)!$. Therefore, from a formal point of view ! and ? when taken along with ~ do not embody actual paracompleteness and actual paraconsistency, but just what we may call apparent paracompleteness and apparent paraconsistency.

One may perhaps take this as a refutation or at least as a weakening of the arguments we have given in Chapter 3 for the thesis that paraconsistency and paracompleteness are intrinsic features of the notions of skeptical plausibility and credulous plausibility, respectively. As reply, it must first be reminded that we have not used the terms "paraconsistent" and "paracomplete" with their usual meaning. Rather, by saying that the credulous plausibility is paraconsistent and that the skeptical plausibility is paracomplete, we meant, respectively, that the notion of credulous plausibility, if represented in a formal system, should incorporate some sort of mechanism for not allowing the trivialization of theories containing plausible inconsistencies and that a formal system intent to represent the notion of skeptical plausibility should allow void acceptance situations[3]. This, one must concede, is trivially satisfied by our modal operators ! and ?. Second, having a modal formal system which fulfills the two above-mentioned requirements but whose operators meant to represent the notions of plausibility are from a formal viewpoint just apparently paraconsistent and paracomplete is a consequence of the particular system we are analyzing. It does not affect the conceptual claim we made about the two plausibility notions. For instance, if we had used a non-modal logic where every sentence α would mean "α is credulously plausible," then in order to fulfill the first requirement our logic would have to be truly paraconsistent (maybe in the style of da Costa's C_1), that is, it would have

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[2] Béziau (2002).

[3] In Chapter 3 void acceptance situations were defined as those situations where it may happen that, for some α, the sentences "α is accepted" and "¬α is accepted" are both false.

to allow things like $\alpha \wedge \sim\alpha$ without trivializing the theory. Even though not embodying formal paracompleteness and formal paraconsistency, the suggested formalization of the notions of skeptical plausibility and credulous plausibility does embody what we have called in the previous chapter conceptual paracompleteness and conceptual paraconsistency.

Given all this, it seems that traditional modal logic is a quite promising way to formally explain the skeptical and credulous notions of plausibility according to our plurality approach to plausibility. After all, besides incorporating the plurality of views required by the mentioned approach, it also provides syntactical devices which, when interpreted as skeptical plausibility and credulous plausibility, satisfy the basic properties we have attributed to these notions. If we decide to follow this path, then what we have called logic of plausibility may be simply taken as traditional modal logic with a new reading, namely $\Box\alpha$ meaning "α is skeptically plausible" and $\Diamond\alpha$ meaning "α is skeptically plausible," which could be made explicit by replacing the traditional modal symbols $\Box$ and $\Diamond$ by ! and ?, on the syntactic side, and the term "possible world" by the term "plausible world" (which could be interpreted as representing an authority on some subject matter, one of several competing theories, a scenario created by an analyst, a specific way of using a set of default statements, etc), on the semantic one. In this way, our task of building a logic of plausibility would be reduced to the task of deciding which modal system (whether K, T, D, S4, S5, etc.) is more adequate to our needs. It in fact would be so if it were not a little detail concerning the intended behavior the negation operator is supposed to have in connection with ! and ? which traditional modal logic, we will try to show, is unable to take into account.

Since plausibility is taken either according to a credulous view or according to a skeptical view, it is fair to suppose that the same should of be done with all concepts which can be derived with the help of these plausibility concepts. More specifically, the notion corresponding to the negating of plausibility, which can be called *implausibility*, should also be taken according to both skeptical and credulous views. In the same way that our logic of plausibility is supposed to explain the meaning of statements like "α is skeptically (credulously) plausible" or "α is plausible according to a skeptical (credulous) position," it should also be automatically able to explain the meaning of "α is skeptically (credulously) implausible" or "α is implausible according to a skeptical (credulous) position."

At first glance this seems to be a quite easy task. After all, as we have said, implausibility is just the negation of plausibility. Therefore, in the same way that we define impossibility through possibility ("α is impossible" is the same as "it is not possible that α" or, formally, $\sim\Diamond\alpha$) we can also define implausibility with the help of the negation operator: "α is implausible" will be the same as "it is not the case that α is plausible" (in symbols, $\sim(\alpha!)$ or $\sim(\alpha?)$). Here however, we should pay attention to the fact that plausibility is being taken under two different views. Therefore, when defining skeptical (or credulous) implausibility of sentence α as the negation of skeptical (or credulous) plausibility of α, we have to decide what we are negating: the plausibility of α according to that view; or the plausibility of α, according to that view. This can be better understood by looking at the two possible ways we can read the sentence "α is not plausible according to a skeptical (or credulous) position":

(i) It is not the case that α is plausible according to a skeptical (credulous) position.

(ii) It is not the case that α is plausible, according to a skeptical (credulous) position. Or: according to a skeptical (credulous) position, α is not plausible.

Using brackets rather than commas we will have:

(i) It is not the case that [α is plausible according to a skeptical (credulous) position.]

(ii) [It is not the case that α is plausible] according to a skeptical (credulous) position.

In order to see that there is really a difference between (i) and (ii), recall the description of the skeptical and credulous approaches we have made in terms of minimizing and maximizing strategies of truth assessing. While in the skeptical case (i) means that we were not able to take "α is plausible" as truth according to a rigid, strict posture, in the credulous one it means that, adopting a tolerant posture concerning truth-assignment, we were not able to classify "α is plausible" as true. On the other hand, (ii) means in the skeptical case that, according to a posture which tries to minimize the truth-value of sentences, α is not plausible or, in other words, we *did* succeed in the task of attributing "true" to the sentence "α is not plausible" according to a posture that requires quite a lot to attach "true" to any sentence, even the sentence "α is not plausible." Similarly for the credulous case: all that (ii) wants to say is that "α is not plausible" is true according to a posture whose goal is to maximize the truth of sentences or, we may say, quite easily classifies statements a true.

Let us try to make this clearer with the help of our expert plurality-oriented possible world semantics. Given a model M, we know that "α is plausible according to a skeptical position" is true at w iff $V_{M,w'}(\alpha) = 1$ for every $w' \in W$ and that "α is plausible according to a credulous position" is true at w iff $V_{M,w'}(\alpha) = 1$ for at least one $w' \in W$. Now what does it mean to say that "α is plausible according to a skeptical position" is not true? It trivially means that it is not the case that $V_{M,w'}(\alpha) = 1$ for every $w' \in W$, which is equivalent to $V_{M,w'}(\alpha) = 0$ for some $w' \in W$. Similarly "α is plausible according to a credulous position" is false iff it is not the case that $V_{M,w'}(\alpha) = 1$ for at least one $w' \in W$, or equivalently, $V_{M,w'}(\alpha) = 0$ for all $w' \in W$. Now, how about (ii)? What means, for instance, trying to assign true to sentence "α is not plausible" according to a posture that tries to maximize the truth-value of sentences? Taking the extremist approach we have adopted, it means that "α is not plausible" will be true in the easiest way, or, making use of our authority-oriented picture, that just one of the experts' disagreeing on the truth of α will be enough for us to take α as not plausible. Therefore, "according to a credulous position, α is not plausible" is true iff $V_{M,w'}(\alpha) = 0$ for at least one w'∈W. Similarly, if we evaluate "α is not plausible" according to a skeptical position, we will be very strict in the matter of accepting "α is not plausible" as true. This means that it will be needed that all the experts agree on the fact that α is false for us to take it as not plausible. Therefore, "according to a skeptical position, α is not plausible" is true iff $V_{M,w'}(\alpha) = 0$ for every w'∈W.

As the second forms of (i) and (ii) show, the whole difference between these two sentences concerns the scope of the negation in each one. While in (i) what is negated is the whole fact that α is plausible according to such and such position, in (ii) only the plausibility of α is negated, and that whole negative statement is interpreted according to a specific viewpoint. Because of that, we can say that while in the first case what is in question is a sort of *external* negation, in the first one we are dealing with an *internal* negation. Despite the specific examples we are using here, as we will see later, these two sorts of negations are not due to the notion of plausibility in its own, but to the use of two different positions to evaluate the truth-value of sentences. If for instance we were considering only one position and not two, there would be no room for such a distinction between an internal and an external negation: since in this case the semantic description could dispense the mention of the position according to which the sentence was evaluated, statements (i) and (ii) would be equivalent. But since in our case we are evaluating sentences through different positions, it is not the case that [α is true according to such and such position] will not necessarily be the same as [is not the case the α is true] according to such and such position.

Another important point is that since the skeptical and credulous positions are used only along with plausibility concepts and in the language we are considering plausibility aspects are represented exclusively through ! and ?, in the evaluation of the truth-value of sentences wherein no plausibility modal operators occur, the two postures will produce identical results. More specifically, since $V_{M,w}(p) = 1$ iff $v_w(p) = 1$, the truth value of *p* according to a skeptical position is the same as the truth-value of *p* according to a credulous position that is the same as the truth-value of *p* according to, we may say, a *neutral* position. Therefore, when applied to modality-free formulae, the internal and external negations will have identical behaviors, which is the same as saying that the skeptical and credulous approaches will have no effect at all.

Now, in the same way that the skeptical and credulous approaches to induction gave rise to two different plausibility concepts, these two negations will give rise to two different pairs of implausibility concepts: the skeptical external and credulous external notions of implausibility and the credulous internal and skeptical internal notions of implausibility. Even though we do thing that these two sorts of implausibility concepts are legitimate, if we accept that all sorts of plausible sentences are to be necessarily analyzed according either to a skeptical or to a credulous position, then it seems that what we mean by skeptical or credulous implausibility are not the external, but the internal notions of implausibility. The external, as the very name suggests, looks much more like an external, meta-conceptual notion indicating a failure in the task of assigning "plausibly true" to a sentence according to a specific position. Therefore, from now on, when we refer to implausibility without qualification we are meaning the internal notion of implausibility.

At this point we can consider a question which may have already come to one's mind: which of the above mentioned negations modal logic's negation ~ corresponds to? Following the path suggested at the beginning of this discussion and obtaining the implausibility of α by negating the plausibility of α, the skeptical and credulous implausibility concepts will be defined, respectively, as $\sim(\alpha!)$ and $\sim(\alpha?)$. It is easy to see that in this case ~ and the corresponding implausibility notions are the external ones: $V_{M,w}(\sim(\alpha!)) = 1$ iff $V_{M,w}(\alpha!) = 0$ iff it is not the case that $V_{M,w'}(\alpha) = 1$ for every $w' \in W$ iff $V_{M,w}(\alpha) = 0$ for at least one $w' \in W$. Similarly, $V_{M,w}(\sim(\alpha?)) = 1$ iff $V_{M,w}(\alpha?) = 0$ iff it is not the case that $V_{M,w'}(\alpha) = 1$ for at least one $w' \in W$ iff $V_{M,w}(\alpha) = 0$ for all $w' \in W$.

Now, how about the internal implausibility notions? Given the above derivations, it seems obvious to us that if we insist on this path and really want to define implausibility of α by negating the plausibility of α, there is no way to obtain the internal notions of implausibility through ~. In other words, unless we decide to define implausibility as something else than the negation of plausibility, what would be quite strange, we will have to have another sort of negation, one that in fact captures the internal behavior necessary to define the second sort of implausibility concepts.

Let us represent this negation by the symbol $\neg$. Then, the skeptical implausibility of α and the credulous implausibility of α will be represented, respectively, by $\neg(\alpha!)$ and $\neg(\alpha?)$. Without making any assumption besides the one we have exposed so far, let us try to find out what will be the behavior of $\neg$ in connection with the operators ! and ?. Since we have agreed to assign true to "α is not plausible according to a skeptical position" iff all experts agree on the fact that α is false, it follows that $V_{M,w}(\neg(\alpha!)) = 1$ iff $V_{M,w'}(\alpha) = 0$ for every $w' \in W$. Consequently, $V_{M,w}(\neg(\alpha!)) = 0$ iff $V_{M,w'}(\alpha) = 1$ for at least one $w' \in W$. Since we know that $V_{M,w}(\alpha!) = 1$ iff $V_{M,w'}(\alpha) = 1$ for every $w' \in W$, we have that $V_{M,w}(\alpha!) = 0$ iff $V_{M,w'}(\alpha) = 0$ for at least one $w' \in W$. Putting these two things together, we have that there may be a model M, namely one having two worlds w' and w'' such that

$V_{M,w'}(\alpha) = 1$ and $V_{M,w''}(\alpha) = 0$, at which both $\alpha!$ and $\neg(\alpha!)$ are false. Consequently, in connection with !, $\neg$ behaves like a paracomplete negation or, equivalently, in connection with $\neg$, ! behaves like a paracomplete modality. Doing the same exercise with ?, we have that "α is not plausible according to a credulous position" is true iff at least one expert holds that α is false. It follows then that $V_{M,w}(\neg(\alpha?)) = 1$ iff $V_{M,w'}(\alpha) = 0$ for at least one $w' \in W$. But since we know that $V_{M,w}(\alpha?) = 1$ iff $V_{M,w'}(\alpha) = 1$ for at least one $w' \in W$, it follows that there may be a model M, namely one having two worlds w' and w" such that $V_{M,w'}(\alpha) = 1$ and $V_{M,w''}(\alpha) = 0$, at which both $\alpha?$ and $\neg(\alpha?)$ are true. Consequently, in connection with ?, $\neg$ behaves like a paraconsistent negation or, equivalently, in connection with $\neg$, ? behaves like a paraconsistent modality. As a consequence of this, the two following schemas of formula are not valid: $\neg(\alpha!) \vee \alpha!$ and $\neg(\alpha? \wedge \neg(\alpha?))$.

What this shows is quite obvious. We have just discovered that if we want to represent skeptical implausibility and credulous implausibility as the negation of skeptical plausibility and credulous plausibility, respectively, our logic will have to have exactly those features which we have agreed would not be necessary to formalize the skeptical and credulous notions of plausibility: (formal) *actual paracompleteness* and (formal) *actual paraconsistency*. In other words, following this path brings the necessity of considering a negation $\neg$ that in connection with ! behaves paracompletly and in connection with ? behaves paraconsistently. Since $\neg$ has both a paraconsistent and a paracomplete behavior, adopting the terminology we have mentioned in Chapter 3 we may say that $\neg$ is a *paranormal* negation. Similarly, since in connection with $\neg$! is a paracomplete modality and ? a paraconsistent one, and since one modality is the dual of the other, we will say that the pair <!,?> is a pair of dual paranormal modalities or simply *paranormal modalities*. Still concerning $\neg$, we note that it is not, strictly speaking, what logicians call paranormal negation, which in general behave paraconsistently *and* paracompletly in connection with the same class of formulae. As we said, the paraconsistent and paracomplete behavior of $\neg$ will depend on the sort of modal formula which is being negated. Besides that, since the skeptical and credulous approaches have no effect when applied to modality-free sentences, $\neg$ has the same behavior as ~ when used along with modality-free formulae. Because of that, besides being an internal negation, $\neg$ may also be classified as a *modality-dependent paranormal negation*. We will call any logic containing a modality-dependent paranormal negation and, consequently, a pair of dual paranormal modalities a *paranormal modal logic*.

Since we do not know any reasonable way to define an implausibility concept other than by deriving it from the negation of its respective plausibility concept and since we do think that the consideration of such implausibility notions is an unavoidable step in the construction of a logic of plausibility, we will have to somehow give an account to this so-called modality-dependent paranormal negation and the paranormal modal logic associated to it. What will follow in the next sections of this chapter is an attempt to formalize such paranormal modal logic.

In a general way, paranormal modal logic shares with traditional modal logic the features of having two or more dual modal operators, being built upon a propositional or first-order language and having the meaning of its operators defined through a possible world semantics. The main difference is that in paranormal modal logic there is a modality-dependent paranormal negation, which implicates the existence of one or more pairs of dual paranormal modalities. An important consequence of that is that many classical logical principles such as the excluded middle principle, the law of non-contradiction, the *reductio ad absurdum* principle and contraposition law do not hold in paranormal modal logic.

The traditional modal logics which we referred to so far are known as *normal modal logics*. They are traditionally defined as those modal systems which have K $(\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta))$ as a valid principle and *modus ponens*, generalization (from α conclude $\Box\alpha$) and the rule of uniform substitution as valid inference rules[4]. Even though not explicitly stated in this definition, normal modal logics are also intent to be conservative extensions of classical logic, as are so all systems traditionally taken as such, such as K, T, S4, S5, etc. As we will see later, all systems to be introduced in this chapter are, according to the mentioned definition, normal modal logics. However, due to the existence of a modality-dependent paranormal negation, paranormal modal logics are not conservative extensions of classical logic. In this way, it seems that the term "paranormal" in the expression "paranormal modal logic" is also appropriate from the point of view of modal logic terminology, for it indicates a normal modal logic that, in contrast to traditional normal logics, has a paraconsistent and paracomplete negation. From now on we will refer to those normal modal logics which are not paranormal simply as *normal modal logics*.

In the construction of our paranormal modal logics, we followed the standard presentation style of modal logics found in classical modal logic textbooks. We will have a basic paranormal modal system, called $K_?$, from which by adding new axioms, on the syntactic side, or by restricting the model structure, on the semantic one, we may obtain several other systems. In the same way that from K we obtain T, D, KT, K45, S4, S5, etc., from $K_?$ we will be able to obtain corresponding paranormal modal systems named $T_?$, $D_?$, $KT_?$, $K45_?$, $S4_?$, $S5_?$, and so on. Besides the stylistic advantages, this will make our systems easily comparable with normal modal logic in such a way that what is really new in our work will be easily identifiable.

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[4] Hughes & Cresswell (1996).

Even though we will proceed in this Chapter as we have done so far and present and justify paranormal modal logic in the light of the philosophy of induction and plausibility we have exposed so far, we do believe that such sort of logic has some import in its own. First of all, it unites two fields of logic that so far have been more or less apart from each other: non-alethic or paranormal logic and modal logic[5]. Second, from the point of view of practical application, paranormal modal logic may be useful not only in the context we have been dealing with so far, but in any situation which requires both a credulous and a skeptical viewpoints to evaluate the sentences. For instance it may be used in connection with general epistemic appraisal (one may believe something from a credulous or skeptical point of view) and in deontic contexts (one may fell that he must do something either from a credulous viewpoint or from a skeptical one). In these cases, it may be useful to use, rather than only one pair of modal operators ! and ?, *n* pairs of such operators representing the skeptical and credulous opinions or obligations of *n* agents.

Besides seminal works such as da Costa's which set the basic principles of paraconsistent and paracomplete logics, the roots of paranormal modal logic as we will present it here go back to the work of Tarcisio Pequeno and Arthur Buschsbaum[6]. Despite having considered only the ? operator[7], these works laid down good part of the technical apparatus needed to formally define a modal logic with the mentioned characteristics. However, due maybe to their initial motivation, their style of presentation (mainly from a semantic point of view) obscured the real interesting novelty of their systems: a modality-dependent paranormal negation. They were also in need of practical applications able to philosophically justify such novelty[8]. If compared to these works, the contribution of this chapter can be said to be threefold. First, for the first time it is presented a sound and complete logic with the paracomplete and paraconsistent operators ! and ?. Second, due to our style of presentation, it will be transparent the fact that the systems in question represent a so far unexplored sort of modal logic. Finally, we have justified in the light of our philosophy of induction and plausibility the import of having such sort of logic.

[5] The reason why we preferred the term "paranormal" over "non-alethic" is that this last has already an established meaning in modal logic: non-alethic modal logics are those modal logics which give to the modal operators $\Box$ and $\Diamond$ an interpretation different from necessity and possibility.

[6] Pequeno & Buchsbaum (1991), Buchsbaum & Pequeno (1993) and Buchsbaum (1995).

[7] As we have seen in the last chapter, Pequeno and Buchsbaum introduced the ! operator but inside a classical framework, which made it from all relevant aspects indistinguishable from normal modal logic operator $\Box$. In an unpublished article however, they presented a system containing both ? and ! along with a modality-dependent paranormal negation. However, besides not having the stylistic features able to classify it as a new sort of modal logic, the axiomatic proposed for the system was apparently not complete (even though it was sound.)

[8] As it is suggested by their last work (Buchsbaum, Pequeno & Pequeno (2004)) they seem to have preferred a classical system with a external negation over the paraconsistent one originally proposed. As we have seen in Chapter 4, the main reason for that seems to have been the lack of justification for the axiom $(\neg\alpha)? \rightarrow \neg(\alpha?)$, which is at the root of the internal negation.

5.2 General Definitions

In this Section we introduce the basic syntactic, semantic and axiomatic notions to be used in the course of the chapter. They are intent to provide a basic framework in which several sorts of modal logic can be formulate. Such general approach will make the movement from one system to another one as well as the comparison between them much more natural. Regarding the semantic and axiomatic definitions, we are in general following the standard style of semantic and syntactic definitions of modal logics found in standard textbooks[9].

5.2.1 Syntactic Definitions

Definition 5.2.1. Let P be an arbitrary (and countable) set of entities called *propositional symbols*. The *propositional language* L is defined as follows:

(i) If $\alpha \in P$, then $\alpha \in L$;

(ii) If $\alpha, \beta \in L$, then $(\neg\alpha)$, $(\alpha \rightarrow \beta)$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta) \in L$;

(iii) Nothing else belongs to L.

The logical symbols $\rightarrow$, $\wedge$ and $\vee$ are interpreted according to their usual meaning. Depending on whether the modal logic that is making use of L is a normal or paranormal one, $\neg$ will behave either as a classical negation or as a modality-dependent paranormal negation, respectively. From now on in this chapter and in subsequent ones, we will use the symbol L to exclusively refer to the propositional language. We will refer to the arbitrary set of propositional symbols upon which L is based, without further mention, simply by the symbol P. Also, when we mention a language $\Im$ without further specification, we are referring to what we can call *normal languages*, that is, languages $\Im$ such that, given two formulae $\alpha, \beta \in \Im$, $\neg\alpha, \alpha \rightarrow \beta, \alpha \wedge \beta, \alpha \vee \beta \in \Im$. $\neg$, $\rightarrow$, $\wedge$ and $\vee$ are called the logical symbols of $\Im$, being $\neg$ the monadic logical symbol and $\rightarrow$, $\wedge$ and $\vee$ the dyadic logical symbols of $\Im$. Finally, when writing down formulae, we will use the standard rules for omitting parentheses[10].

Definition 5.2.2. Let $\Im$ be a language. If $L \subseteq \Im$ we say that $\Im$ is a propositional language.

Definition 5.2.3. Let $\Im$ be a propositional language and $\alpha \in \Im$ any formulae of $\Im$. We define the contradiction and tautology symbols as follows:

$\top =_{def} p \vee \neg p$;

$\bot =_{def} p \wedge \neg p$, where $p \in P$ is an arbitrary propositional symbol.

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[9] Such as, for instance, Chellas (1980), Fitting (1993) and Hughes & Cresswell (1996).

[10] Such as described in Enderton (1972), for example.

Definition 5.2.4. A *vocabulary* K is a quadruple $\langle K_C, K_V, K_F, K_R \rangle$ where K_C is a countable set of constant symbols, K_V a countable set of variable symbols, K_F a countable set of function symbols and K_R a countable set of predicate or relation symbols. Each element *u* of $K_F \cup K_R$ has associated with it a number *n* which we call the arity of *u*, in the case we say *u* is an n-ary symbol.

Definition 5.2.5. Let $K = \langle K_C, K_V, K_F, K_R \rangle$ be a vocabulary. A *term* in K is defined as follows:

(i) If $c \in K_C$ then *c* is a term in K;

(ii) If $x \in K_V$ then *x* is a term in K;

(iii) If $t_1, \ldots, t_n$ are terms in K and $f \in K_F$ is a function of arity *n*, then $f(t_1, \ldots, t_n)$ is a term in K.

Definition 5.2.6. Let $K = \langle K_C, K_V, K_F, K_R \rangle$ be an arbitrary vocabulary. The first-order language $\mathcal{L}$ is defined as follows:

(i) If $t_1, \ldots, t_n$ are terms in K and $r \in K_R$ is a relation of arity *n*, then $r(t_1, \ldots, t_n) \in \mathcal{L}$;

(ii) If $\alpha, \beta \in \mathcal{L}$, then $(\neg\alpha)$, $(\alpha \rightarrow \beta)$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta) \in \mathcal{L}$;

(iii) If $\alpha \in \mathcal{L}$ and $x \in K_V$, then $(\forall x \alpha) \in \mathcal{L}$.

The elements of $\mathcal{L}$ that satisfy item (i) of definition above and the elements of L that satisfy item (i) of definition 5.2.1 are called *atomic formulae*. If $\alpha \in \mathfrak{I}$ is such that either $\alpha \equiv \neg p$, where p is an atomic formula, or α itself is an atomic formula, we say that α is a *basic formula*.

From now on we will use symbol $\mathcal{L}$ to exclusively refer to the first-order language. We will refer to the components of the arbitrary vocabulary K upon which $\mathcal{L}$ is based, without further mention, simply by the symbols K_C, K_V, K_F and K_R. Besides $\neg$, $\rightarrow$, $\wedge$ and $\vee$, $\forall$ is also a logical symbol of $\mathcal{L}$ (a monadic one.) We say that the variable which necessarily comes after $\forall$ is a non-logical complement of $\forall$. $\neg$ has no non-logical complement.

We define a variable *x* as being free in α in the usual way. $\alpha(x)$ means that formula α contains (possibly zero) free occurrences of variable *x*. If, subsequently, we write $\alpha(t)$, we mean the formula that is like $\alpha(x)$ except that occurrences of the term *t* have been substituted for all free occurrences of *x*. We say that such a substitution is admissible if either *t* belongs to K_C or is a variable symbol $z \in K_V$ such that no free occurrence of *x* in $\alpha(x)$ is within the scope of a quantifier $\forall z$.

Definition 5.2.7. Let $\mathfrak{I}$ be a language. If $\mathcal{L} \subseteq \mathfrak{I}$ we say that $\mathfrak{I}$ is a first-order language.

Definition 5.2.8. Let $\mathfrak{I}$ be a first-order language and $\alpha \in \mathfrak{I}$ any formulae of $\mathfrak{I}$. We define the contradiction symbol as follows:

$\top =_{def} r(t_1, \ldots, t_n) \vee \neg r(t_1, \ldots, t_n)$;

$\bot =_{def} r(t_1,\ldots,t_n) \wedge \neg r(t_1,\ldots,t_n)$, where $r \in K_r$ is an arbitrary n-ary relation symbol and $t_1,\ldots,t_n$ are *n* terms in K

Definition 5.2.9. Let $\Im$ be a language and $\alpha,\beta \in \Im$ any two formulae. We define the derivate symbols $\leftrightarrow$ and ~ as follows:

$\alpha \leftrightarrow \beta =_{def} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$;

$\sim\alpha =_{def} \alpha \rightarrow \bot$.

In paranormal modal logic, ~ will be used to simulate classical negation. While ¬ is a modality-dependent paranormal negation which behaves sometimes paraconsistently, sometimes paracompletly and sometimes classically, ~ behaves for any sort of formula, both syntactically and semantically, exactly like classical negation. As it will be made precise later when we show the semantics of paranormal modal logics, while ¬ represents what we have termed the internal negation, ~ will play the role of the external one. Therefore, even with our primitive negation symbol being a modality-dependent paranormal negation, we will still be able, by using ~ along with ! and ?, to formalize the external notions of implausibility. From now on, when introducing the several logics to the described in this chapter, we will refer to ¬ as the modality-dependent paranormal negation or just paranormal negation and ~ as the classical negation of the logic in question.

Definition 5.2.10. A *modal logic basis* ϑ is a pair $<\Theta', \Theta''>$ where Θ' and Θ'' are two possibly empty sets of modal operators, being $\Theta'' \subseteq \Theta'$ and the arity of Θ'' half the arity of Θ'. Θ'' is called the set of *distinguished modal operators*. Letting *n* be the arity of Θ'', we say that ϑ is an n-modal logic basis (or a modal logic basis of arity *n*.)

Definition 5.2.11. Let $\Im$ be a language and $\vartheta = <\Theta', \Theta''>$ a modal logic basis. The *modal language* $\Im_\vartheta$ based on $\Im$ and ϑ is defined as follows:

(i) If $\alpha \in \Im$ is such that it contains no one of $\Im$'s logical symbols, then $\alpha \in \Im_\vartheta$;

(ii) If ⊕ is a monadic logical symbol of $\Im$ along with one of its non-logical complements, if there is any, and $\alpha \in \Im_\vartheta$, then $(\oplus\alpha) \in \Im_\vartheta$;

(iii) If ⊕ is a dyadic logical symbol of $\Im$ and $\alpha,\beta \in \Im_\vartheta$, then $(\alpha \oplus \beta) \in \Im_\vartheta$;

(iv) If $<\oplus',\oplus''>$ is a pair of triadic logical symbols of $\Im$ and $\alpha,\beta,\varphi \in \Im_\vartheta$, then $(\alpha \oplus' \beta \oplus'' \varphi) \in \Im_\vartheta$[11];

(v) If $\theta \in \Theta'$ and $\alpha \in \Im_\vartheta$, then $(\theta\alpha) \in \Im_\vartheta$ (or $(\alpha\theta) \in \Im_\vartheta$, if a post-fixed notation is used);

(vi) Nothing else belongs to $\Im_\vartheta$.

[11] Languages with triadic logical symbols will be introduced only in the next chapter.

The purpose of a modal logic basis is both syntactic and meta-logical. On the syntactic side, as it can be seen in the above definition, Θ' determines how the language $\Im$ will be extended in order to accommodate modal sentences. On the meta-logical side, Θ" will determine whether the logic in question is a mono-modal logic or a multi-modal one. From an axiomatic point of view, this is done by using the elements of Θ" in the formulation of the *n* necessitation rules, that is to say, for each $\theta \in \Theta$" there will be a rule saying that from α one may conclude $\theta\alpha$ (or $\alpha\theta$). From a semantic point of view, this is related to the definition *n* accessibility relations to be used by each one of the *n* elements of Θ". Because of that the members Θ"of are called distinguished modal operators.

An important observation concerns the reason why Θ' may be different from Θ". The whole idea of Θ' is to contain several pairs of dual modal operators in the style of □ and ◇ and ! and ? where one of the members of each pair will also belong to Θ". Now, if for each pair of dual operators we could define one of its members through the distinguished one, then Θ' would effectively be identical to Θ". The point however is that maybe we cannot (or do not want to) define any one of the members of a specific pair of dual operators through the other. For instance, in paranormal modal logic, we cannot define neither ! through ? nor ? through !. Because of that, both have to be introduced as primitive symbols, even though only one of them, namely !, will be taken as a distinguished modal operator.

In this and in the remaining chapters we will define several notions that, like the notion of modal logic basis, depend on some language $\Im$. From now on, we will without further mention refer to notion X applied to propositional language L simply as *propositional X* and to notion X applied to first-order language $\mathcal{L}$ simply as *first-order X*. For instance, in the definition above, if $\Im$ = L we say that $\Im_\vartheta$ is the propositional modal language based on ϑ and if $\Im = \mathcal{L}$ we say that $\Im_\vartheta$ is the first-order modal language based on ϑ.

5.2.2 Semantic Definitions

Definition 5.2.12. A *frame* F of arity *n* (or simply an n-frame F), $n \geq 1$, is a n+1-tuple $<W,R_1, \ldots,R_n>$ where W is a non-empty countable set of entities called *worlds* and $R_1, \ldots, R_n$ are binary relations on W called *accessibility relations*.

Let $F = <W,R_1, \ldots,R_n>$ be an n-frame and $P_1, \ldots, P_2$ the names of the classes of relations which $R_1, \ldots,R_n$, respectively, belongs to. We say that F is a *$<P_1 - \ldots - P_2>$ frame*. If $P_1 = \ldots = P_2$, we call F simply a P_1 frame. For instance, if $F = <W,R,R'>$ is a frame such that R is a reflexive relation and

R' is a reflexive, transitive and symmetric one, F is said to be a <reflexive - reflexive, transitive and symmetric> frame.

Definition 5.2.13. Let $F = \langle W, R_1, \ldots, R_n \rangle$ be an n-frame. F is called an *idealized frame* iff for every $w \in W$ there is at least one $w' \in W$ such that wR_iw', for every $i=1,\ldots,n$.

Definition 5.2.14. Let $\Im$ be a language and $F = \langle W, R_1, \ldots, R_n \rangle$ a frame. A *modal interpretation* in $\Im$ and F is a structure which, along with possibility other parameters, evaluate the truth-value of the basic elements of $\Im$ in each world $w \in W$. If $\Im$ is the propositional language L, a modal interpretation in L and F, called simply a *propositional modal interpretation* in F, is a function v mapping elements of P and W to truth-values 0 and 1.

Definition 5.2.15. Let $\Im$ be a language. A *model* M of arity n (or simply an n-model M) in $\Im$ is a n+2-tuple $\langle W, R_1, \ldots, R_n, v \rangle$ where $F = \langle W, R_1, \ldots, R_n \rangle$ is a n-frame and v is a modal interpretation in $\Im$ and F. We say that the model M is *based on* F and that $w \in W$ is a world of M.

The consideration of more than one accessibility relation is needed to contemplate the so-called multi-modal logics, that is, logics that have more than one distinguished modal operator. In general, to each distinguished modal operator θ_i it will correspond an accessibility relation R_i. From now on, we may refer to some model or frame without any qualification concerning their arity. When this happens, we are either referring to a model or frame of arbitrary arity or to a model or frame of arity unambiguously determined by the context where the reference is made.

The actual meaning of the elements of a model M will depend on the other components of the logical system as well as the aimed application. As explained in the previous section, in the case of a propositional mono-modal system meant to formalize the notions of plausibility, the elements of W may be taken as representing authorities on some field, R a trustfulness relation saying which experts each expert considers trustful and v a function telling the opinion each expert holds about each atomic proposition.

Another possible interpretation for M already mentioned is to take the elements of W as representing competing theories trying to account for the same range of physical phenomena, $R_w = \{w' \mid wRw'\}$ the set of competing theories relevant for a specific situation where w is being taken into consideration, and v a function telling for each theory w and atomic proposition p the truth-value of p according to w. In this case, a sentence α will be credulously plausible iff at least one of the competing theories assigns 1 to it, and skeptically plausible iff according to all theories α is true. Still another possible interpretation is to take each $w \in W$ as representing an extension coming from the use of certain inductive rules (maybe represented in a default rule style) in a specific knowledge situation, $R_w = \{w' \mid wRw'\}$ the set of extensions relevant for a specific situation where w is being taken into consideration, and $\nu_w = \{p \mid \nu_w(p) = 1\}$ the atomic basis of extension w[12].

This variety of interpretations is of course due to the variety of criteria we may conceive to say when a statement is plausible, which is in its turn due to the variety of applications in which the notions of skeptical plausibility and credulous plausibility may be useful. Given all this, we will from now on in the interpretation of plausible models ignore the effective way according to which plausible statements are evaluated and refer to the members of W simply as *plausible worlds*. This helps the general treatment we want to give to our presentation and also reflects the motivation behind the use of a logic of plausibility in the mentioned practical situations (for instance, both the view of each expert, the result of each theory and every and each extension may be seen as sorts of plausible worlds.)

Definition 5.2.16. Let $\Im$ be a language and ϑ a modal logic basis of arity n. A *modal valuation* of arity n (or simply an n-modal valuation) in $\Im$ and ϑ is a function Ψ which, given an n-model $M = \langle W,R,\ldots,v\rangle$ in $\Im$, maps worlds of W, formulae of $\Im_\vartheta$ and possibly another parameters to truth values 0 and 1. In the case of a propositional modal valuation in ϑ, there will be no other parameters besides M, W and α. We represent Ψ applied to M, w and α by $\Psi_{M,w}(\alpha)$.

Definition 5.2.17. Let $\Im$ be a language. A *semantic modal system* $\Lambda°$ of arity n (or simply a semantic n-modal system $\Lambda°$) based on $\Im$ is a triple $\langle\vartheta, \Psi, \Gamma\rangle$ where $\vartheta = \langle\Theta',\Theta''\rangle$ is an n-modal logic basis, Ψ an n-modal valuation in $\Im$ and ϑ and Γ a class of n-frames.

Definition 5.2.18. Let $\Im$ be a language, ϑ an n-modal logic basis, Ψ an n-modal valuation in $\Im$ and ϑ, $M = \langle W,R,\ldots,v\rangle$ an n-model in $\Im$, $w \in W$ a world of M and $\alpha \in \Im_\vartheta$ a formula.

(i) α *is Ψ-satisfied by* M *at* w (in symbols: $M,w \Vdash_\Psi \alpha$) iff $\Psi_{M,w}(\alpha) = 1$;

(ii) α *is Ψ-satisfied by* M (in symbols: $M \Vdash_\Psi \alpha$) iff, for all $w' \in W$, $M,w' \Vdash_\Psi \alpha$.

Definition 5.2.19. Let $\Im$ be a language, $\Lambda° = \langle\vartheta, \Psi, \Gamma\rangle$ a semantic n-modal system based on $\Im$, $\mathcal{M}'$ the set of n-models containing, for each $F \in \Gamma$, all n-models based on $\Im$ and F, and $A \subseteq \Im_\vartheta$ a set of formulae. The function $\mathcal{M}_{\Lambda°}$ is defined as follows:

$$\mathcal{M}_{\Lambda°}(A) = \{M \mid M \in \mathcal{M}' \text{ and } M \Vdash_\Psi \alpha, \text{ for all } \alpha \in A\}.$$

Definition 5.2.20. Let $\Im$ be a language, $\Lambda° = \langle\vartheta, \Psi, \Gamma\rangle$ a semantic n-modal system based on $\Im$, $A,B \subseteq \Im_\vartheta$ two sets of formulae and $\varphi \in \Im_\vartheta$ a formula. φ is a $\Lambda°$-*logical consequence* of A and B, A

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[12] Perhaps the applications which require interpretation like the last two would go best with R as an equivalence relation.

being the set of global premises and B the set of local premises (in symbols: $A \div B \vDash_{\Lambda^\circ} \varphi$), iff, for every n-model $M \in \mathcal{M}_{\Lambda^\circ}(A)$ and every world w of M such that for every $\beta \in B$ $M,w \Vdash_\Psi \beta$, $M,w \Vdash_\Psi \varphi$.

Here we are making use of the important distinction one can make in modal logic between global and local premises[13]. It will play a very important role in our logic of plausibility, to be presented in the next chapter. As we will see below, on the axiomatic side the same distinction is made by restricting the use of the necessitation rule only to global premises.

Definition 5.2.21. Let $\Im$ be a language, $\Lambda^\circ = <\vartheta, \Psi, \Gamma>$ a semantic n-modal system based on $\Im$, $A \subseteq \Im_\vartheta$ a set of formulae and $\alpha \in \Im_\vartheta$ a formulae. α is a Λ°-*logical consequence* of A (in symbols: $A \vDash_{\Lambda^\circ} \alpha$) iff $A \div \varnothing \vDash_{\Lambda^\circ} \alpha$. α is Λ°-*valid* or valid in Λ° (in symbols: $\vDash_{\Lambda^\circ} \alpha$) iff $\varnothing \div \varnothing \vDash_{\Lambda^\circ} \alpha$.

5.2.3 Axiomatic Definitions

Definition 5.2.22. Let $\Im$ be a language and ϑ an n-modal logic basis. The *positive classical axioms* Σ_P in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying one of the following schemas of formula:

P1: $\alpha \to (\beta \to \alpha)$

P2: $(\alpha \to (\beta \to \varphi)) \to ((\alpha \to \beta) \to (\alpha \to \varphi))$

P3: $\alpha \wedge \beta \to \alpha$

P4: $\alpha \wedge \beta \to \beta$

P5: $\alpha \to (\beta \to \alpha \wedge \beta)$

P6: $\alpha \to \alpha \vee \beta$

P7: $\beta \to \alpha \vee \beta$

P8: $(\alpha \to \beta) \to ((\varphi \to \beta) \to (\alpha \vee \varphi \to \beta))$

The positive classical axioms will be used by all modal logics we will present here, be it normal or paranormal.

Definition 5.2.23. Let $\Im$ be a language. An *axiomatic modal system* Λ^* of arity *n* (or simply an axiomatic n-modal system or, still, a modal calculus) based on $\Im$ is a pair $<\vartheta, \Sigma>$ where $\vartheta = <\Theta', \Theta''>$ is a modal logic basis of arity *n* and $\Sigma \subseteq \Im_\vartheta$ a set axioms. We also say that is Λ^* based on ϑ.

[13] See Fitting(1993).

Definition 5.2.24. Let $\Im$ be a language, $\Lambda^* = <\vartheta, \Sigma>$ an axiomatic modal system based on $\Im$ with $\vartheta = <\Theta', \Theta''>$, $A,B \subseteq \Im_\vartheta$ two sets of formulae, $\varphi \in \Im_\vartheta$ a formula and $S = <\lambda_1, \ldots, \lambda_k>$ a sequence of formulae of $\Im_\vartheta$. Let S be divided into two parts: the global part $S_G = <\lambda_1, \ldots, \lambda_n>$ and the local part $S_L = <\lambda_{n+1}, \ldots, \lambda_k>$, $k \geq n$. S is a Λ^*-*derivation* of φ from A and B, A being the set of global premises and B the set of local premises, iff the three conditions below are satisfied:

(i) $\lambda_n \equiv \varphi$;

(ii) For each $\lambda_i \in S$, $i = 1, \ldots, n$, one of the following conditions is met:

a) $\lambda_i \in A$;

b) $\lambda_i \in \Sigma$ is an axiom of Λ^*;

c) There are $\lambda_r, \lambda_s \in S$, $r,s<i$, such that $\lambda_r \equiv \lambda_s \rightarrow \lambda_i$;

d) (Just in case $\Im$ is a first-order language) There is λ_r, $r<i$, such that $\lambda_r \equiv \alpha \rightarrow \beta$, $\lambda_i \equiv \alpha \rightarrow \forall x\beta$ and x has no free occurrences in α;

e) There is λ_r, $r<i$, and $\theta \in \Theta''$ such that $\lambda_i \equiv \theta\lambda_r$ (or $\lambda_i \equiv \theta\lambda_r$, if a post-fixed notation is used.)

(iii) For each $\lambda_i \in S$, $i = n+1, \ldots, k$, one of the following conditions is met:

a) $\lambda_i \in B$;

b) $\lambda_i \in \Sigma$ is an axiom of Λ^*;

c) There are $\lambda_r, \lambda_s \in S$, $r,s<i$, such that $\lambda_r \equiv \lambda_s \rightarrow \lambda_i$;

d) (Just in case $\Im$ is a first-order language) There is λ_r, $r<i$, such that $\lambda_r \equiv \alpha \rightarrow \beta$, $\lambda_i \equiv \alpha \rightarrow \forall x\beta$ and x has no free occurrences in α.

If $B = \emptyset$, then we say simply that S is a Λ^*-derivation of φ from A. If $A = B = \emptyset$, then we say imply that S is a Λ^*-derivation of φ.

As one might suspect, items (ii).c) and (iii).c) correspond to *modus ponens* rule – from α and $\alpha \rightarrow \beta$ conclude β, items (ii).d) and (iii).d) correspond to *generalization* rule (which is applicable of course only in the case $\Im$ is a first-order language) – from $\alpha \rightarrow \beta$ conclude $\alpha \rightarrow \forall x\beta$, and item (ii).d) corresponds to the so-called *necessitation* rule – from α conclude $\theta\alpha$. Concerning the latter, the particularity of our formulation is that, since we allow more than one distinguished modal operator, we also allow more than one necessitation rule, one for each distinguished operator.

Definition 5.2.25. Let $\Im$ be a language, $\Lambda^* = <\vartheta, \Sigma>$ an axiomatic modal system based on $\Im$, $A,B \subseteq \Im_\vartheta$ two sets of formulae and $\varphi \in \Im_\vartheta$ a formula. α is Λ^*-*deducted* from A and B, A being the set of global premises and B the set of local premises (in symbols: $A \div B \vdash_{\Lambda^*} \varphi$), iff there is a Λ^*-derivation of φ from A and B, A being the set of global premises and B the set of local premises.

Definition 5.2.26. Let $\Im$ be a language, $\Lambda^* = <\vartheta, \Sigma>$ an axiomatic modal system based on $\Im$, $A \subseteq \Im_\vartheta$ a set of formulae and $\alpha \in \Im_\vartheta$ a formulae. α is Λ^*-*deducted* from A (in symbols: $A \vdash_{\Lambda^*} \alpha$) iff $A \div \varnothing \vdash_{\Lambda^*} \alpha$. α is a Λ^*-*theorem* or a theorem of Λ^* (in symbols: $\vdash_{\Lambda^*} \alpha$) iff $\varnothing \div \varnothing \vdash_{\Lambda^*} \alpha$.

5.2.4 Modal System

Definition 5.2.27. Let $\Im$ be a language. A *modal system* Λ of arity *n* (or simply an n-modal system Λ) based on $\Im$ is a quadruple $<\vartheta, \Psi, \Gamma, \Sigma>$ where $\Lambda^\circ = <\vartheta, \Psi, \Gamma>$ is a semantic n-modal system based on $\Im$ and $\Lambda^* = <\vartheta, \Sigma>$ is an axiomatic n-modal system based on $\Im$. We also say that say that Λ is the modal system based on Λ° and Λ^*.

Definition 5.2.28. Let $\Im$ be a language, Λ° a semantic modal system based on $\Im$, Λ^* an axiomatic modal system based on $\Im$, Λ the modal system based on Λ° and Λ^*, $A,B \subseteq \Im_\vartheta$ two sets of formulae and $\varphi \in \Im_\vartheta$ a formula.

(i) α is a Λ-*logical consequence* of A and B (in symbols: $A \div B \vDash_\Lambda \varphi$) iff $A \div B \vDash_{\Lambda^\circ} \varphi$;

(ii) α is a Λ-*logical consequence* of A (in symbols: $A \vDash_\Lambda \alpha$) iff $A \div \varnothing \vDash_\Lambda \alpha$;

(iii) α is Λ-*valid* or valid in Λ (in symbols: $\vDash_\Lambda \alpha$) iff $\varnothing \div \varnothing \vDash_\Lambda \alpha$;

(iv) α is Λ-*deducted* from A and B (in symbols: $A \div B \vdash_\Lambda \varphi$) iff $A \div B \vdash_{\Lambda^*} \varphi$;

(v) α is a Λ-*deducted* from A (in symbols: $A \vdash_\Lambda \alpha$) iff $A \div \varnothing \vdash_\Lambda \alpha$;

(vi) α is a Λ-*theorem* or a theorem of Λ (in symbols: $\vdash_\Lambda \alpha$) iff $\varnothing \div \varnothing \vdash_\Lambda \alpha$.

Definition 5.2.29. Let $\Im$ be a language and $\Lambda = <\vartheta, \Psi, \Gamma, \Sigma>$ a modal system based on $\Im$. Λ is *sound* iff for any $A,B \subseteq \Im_\vartheta$ and $\alpha \in \Im_\vartheta$, if $A \div B \vdash_\Lambda \alpha$ then $A \div B \vDash_\Lambda \alpha$. Λ is *complete* iff for any $A,B \subseteq \Im_\vartheta$ and $\alpha \in \Im_\vartheta$, if $A \div B \vDash_\Lambda \alpha$ then $A \div B \vdash_\Lambda \alpha$.

A modal system Λ based on $\Im$ is meant to contain all the elements of a specific modal logic, both syntactic and semantic. It is defined in such a way that, given two n-modal systems Λ and Λ', it will be clear from their very components what makes them different from each other. For instance, normal propositional modal logics K, T, S4, etc. are based on the same language $\Im$, have the same modal logic basis and propositional modal valuation, but differ in their set of frames and axioms. As we will see later, what makes propositional modal system K different from propositional paranormal modal system $K_?$, for instance, is only their modal logic basis, modal valuation and axioms. Since the syntactic shape of a modal symbol is logically irrelevant for the logic which uses it, what in fact makes K and $K_?$ (and T and $T_?$, D and $D_?$, S4 and $S4_?$, and so on) different from each other is their corresponding modal valuations and axioms.

5.3 Paranormal Modal Logic: The System $K_?$

In this section we present the basic paranormal modal system $K_?$. Since all other paranormal modal logics are extensions of $K_?$, this section contains the most fundamental notions of paranormal modal logic. We present $K_?$ both syntactically and semantically. In the next subsection we introduce the language of propositional paranormal modal logic. In Subsections 5.3.2 and 5.3.3 we present, respectively, the basic valuation function and the basic set of axioms that will be used to define the notions of logical consequence and deductibility of paranormal modal logic. In Subsection 5.3.4 we put all these things together to define the system $K_?$. Finally in Subsection 5.3.5 we define classical logic and normal modal logic K with the help of the framework introduced in the last section and lay down some theorems concerning these two logics and $K_?$. For the sake of presentation, the proofs of all theorems of Chapters 5 and 6 in a separated chapter entitled "Appendix: Proof of Theorems."

5.3.1 The Language of Paranormal Modal Logic

Definition 5.3.1. A *?-modal logic basis* ϑ is a pair $<\Theta',\Theta''>$ where $\{!,?\}\subseteq\Theta'$ and $\{!\}\subseteq\Theta''$. The notation adopted for the operators ! and ? is a post-fixed one. We call the *?-modal logic basis* $\vartheta_? = <\{!,?\}, \{!\}>$ the *paranormal modal logic basis*. Letting $\Im$ be a language, we call the modal language based on $\Im$ and $\vartheta_?$, which we will refer to by the symbol $\Im_?$, a *paranormal modal language*.

Given formula α, α? means "α is credulously plausible" or just "α is plausible" and α! "α is skeptically plausible" or "α is accepted." As we noticed above, even though in paranormal modal logic ! and ? form a pair of dual operators (in the same way that $\Box$ and $\Diamond$ are the dual operators of each other), they have both to be introduced as primitive symbols, for no one can be defined through the other, neither with the help of paranormal negation $\neg$ nor with the help of classical negation ~.

Definition 5.3.2. Let $\Im$ be a language, ϑ a ?-modal logic basis and $\alpha\in\Im_\vartheta$ a formula. We say that α is *?-free* iff ? does not occur in α and that α is *!-free* iff ! does not occur in α. We say α is ?!-free iff α is ?-free and !-free.

5.3.2 Paranormal Modal Semantics

Definition 5.3.3. Let $\Im$ be a language, ϑ a ?-modal logic basis of arity *n* and *k* a natural number, 1≥k≥n. A Ω_k*-modal valuation* Ω in $\Im$ and ϑ and a $\mho_k$*-modal valuation* $\mho$ in $\Im$ and ϑ, which will also be referred to as the *max-min* k*-modal valuations* in $\Im$ and ϑ, are n-modal valuations in $\Im$ and ϑ which, given an n-model M = <W,R_1,…,R_k,…,v>, a world $w \in W$, any two formulae $\alpha,\beta \in \Im_\vartheta$ and possibly other parameters, satisfy the following conditions:

(i) $\Omega_{M,w,\ldots}(\neg\alpha) = 1$ iff $\mho_{M,w,\ldots}(\alpha) = 0$;

(ii) $\mho_{M,w,\ldots}(\neg\alpha) = 1$ iff $\Omega_{M,w,\ldots}(\alpha) = 0$;

(iii) $\Omega_{M,w,\ldots}(\alpha \rightarrow \beta) = 1$ iff $\Omega_{M,w,\ldots}(\alpha) = 0$ or $\Omega_{M,w,\ldots}(\beta) = 1$;

(iv) $\mho_{M,w,\ldots}(\alpha \rightarrow \beta) = 1$ iff $\Omega_{M,w,\ldots}(\alpha) = 0$ or $\mho_{M,w,\ldots}(\beta) = 1$;

(v) $\Omega_{M,w,\ldots}(\alpha \wedge \beta) = 1$ iff $\Omega_{M,w,\ldots}(\alpha) = 1$ and $\Omega_{M,w,\ldots}(\beta) = 1$;

(vi) $\mho_{M,w,\ldots}(\alpha \wedge \beta) = 1$ iff $\mho_{M,w,\ldots}(\alpha) = 1$ and $\mho_{M,w,\ldots}(\beta) = 1$;

(vii) $\Omega_{M,w,\ldots}(\alpha \vee \beta) = 1$ iff $\Omega_{M,w,\ldots}(\alpha) = 1$ or $\Omega_{M,w,\ldots}(\beta) = 1$;

(viii) $\mho_{M,w,\ldots}(\alpha \vee \beta) = 1$ iff $\mho_{M,w,\ldots}(\alpha) = 1$ or $\mho_{M,w,\ldots}(\beta) = 1$;

(ix) $\Omega_{M,w,\ldots}(\alpha?) = 1$ iff, for some $w' \in W$ such that wR_kw', $\Omega_{M,w',\ldots}(\alpha) = 1$;

(x) $\mho_{M,w,\ldots}(\alpha?) = 1$ iff, for all $w' \in W$ such that wR_kw', $\mho_{M,w',\ldots}(\alpha) = 1$;

(xi) $\Omega_{M,w,\ldots}(\alpha!) = 1$ iff, for all $w' \in W$ such that wR_kw', $\Omega_{M,w',\ldots}(\alpha) = 1$;

(xii) $\mho_{M,w,\ldots}(\alpha!) = 1$ iff, for some $w' \in W$ such that wR_kw', $\mho_{M,w',\ldots}(\alpha) = 1$.

The purpose of the number *k* is to allow the same structure to be used in multimodal logics where ? and ! are interpreted with the help of the *k*-th accessibility relation R_k.

Definition 5.3.4. Let ϑ be a ?-modal logic basis of arity *n* and *k* a natural number, 1≥k≥n. A *propositional* Ω_k*-modal valuation* Ω in ϑ and a *propositional* $\mho_k$*-modal valuation* $\mho$ in ϑ, which will also be referred to as the *propositional max-min* k*-modal valuations* in ϑ, are the max-min k-modal valuations in L and ϑ which, given a propositional n-model M = <W,R_1,…,R_k,…,v>, a world $w \in W$, and any propositional symbol $p \in P$, satisfy the following condition:

$$\Omega_{M,w}(p) = \mho_{M,w}(p) = 1 \text{ iff } v_w(p) = 1.$$

As it can be seen, a *propositional* Ω_k*-valuation* Ω in ϑ and a *propositional* $\mho_k$*-valuation* $\mho$ in ϑ have as parameters a propositional n-model M, a world *w* of M and a formula α of L_ϑ.

As we have mentioned before, one of the two fundamental differences between two corresponding normal and paranormal modal systems (like K and $K_?$) is their modal valuations. Ω and $\mho$ are called max-min valuations because, depending on the plausible formula they have as parameter, they act either as a maximizing or as a minimizing valuation function. Concerning ?!-free formulae however, the skeptical and credulous positions are, we can say, null: their truth-value are evaluated solely on the base of the modal interpretation v. Therefore, as it can be seen by taking definition 5.3.4 along with items (i)-(viii) of definition 5.3.3, we have that for ?!-free formula paranormal modal logic behaves just like classical logic[14].

Regarding modal formulae, it is easy to see from definition 5.3.3 that Ω evaluates the truth-value of ?-marked formulae according to a maximal or credulous position and of !-marked ones according to a skeptical or minimal posture. Because of this conformity with the meaning we want to give to ! and ?, Ω is will be taken as paranormal modal logic's *principal max-min valuation*, that is to say, the valuation that will be used in the definition of the notion of logical consequence of paranormal modal logic. Formally speaking then, paranormal logic can be characterized as those modal systems whose modal valuation is a Ω_k-modal valuation[15].

Concerning $\mho$ however, it may be surprising to note that it behaves exactly in the opposite way: while it evaluates ?-marked formulae according to a minimal posture, !-marked ones are evaluated according to a maximal one. In other words, when we take $\mho$ into consideration, ? behaves like the skeptical notion of plausibility and ! like the credulous one. In order to understand the necessity of having such an apparently strange function which evaluates plausible formulae in the exact opposite way they are supposed to be evaluated, we have to look a bit closer at how formulae of the form $\neg\alpha$ are to be appraised by a function which is supposed to recursively define the truth-value of sentences sometimes according to a maximizing position and sometimes according to a minimizing one.

If we want to recursively define the truth-value of formulae, we have to define the truth-value of a complex formula in function of its less complex components. More specifically, in order to have a recursive definition of the truth of $\neg\alpha$, we have to take into account the truth of α. Traditionally this is done by assigning true to $\neg\alpha$ only in those cases where α is false. Letting V be such a sort of truth-function, we would have that $V(\neg\alpha) = 1$ iff $V(\alpha) = 0$. Now, for instance, if V

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[14] This observation will made precise in subsection 5.3.5, when we will compare $K_?$ with classical logic.

[15] As we will see later still in this subsection, another reason for taking Ω as the principal max-min valuation is concerned with the definition of the truth-value of implication formulae.

were a truth-function that does its job according to a skeptical position, then $V(\neg\alpha) = 1$ would mean that $\neg\alpha$ is true according to a posture that tries to minimize the truthfulness of $\neg\alpha$. Since $\neg\alpha$'s being true is the same as α's being false, minimizing the truth of $\neg\alpha$ is the same as minimizing the falsehood of α. Therefore, $V(\neg\alpha) = 1$ is the same as saying that α is *false* according to a posture that tries to minimize the *falsehood* of α. Therefore, in order to recursively define the truthfulness of $\neg\alpha$ according to a minimizing evaluation function, we have just to take the falsity of α according to the same minimizing function. Since it seems that this is exactly what $V(\alpha) = 0$ means, that α is false according to the minimizing function V, apparently we can without problem use the traditional definition and say that that $V(\neg\alpha) = 1$ iff $V(\alpha) = 0$. Let us, just for the sake of soundness, check if this path will really lead us to the desired result.

Let us make explicit the position through which V evaluates sentences by writing V_{max} in the case it is a maximizing valuation function, and V_{min} in the case it is a minimizing one. All we want then is $V_{max}(\neg\alpha)$ to give us the truth-value of $\neg\alpha$ according to a credulous, tolerant position which tries to maximize the truthfulness of $\neg\alpha$ and $V_{min}(\neg\alpha)$ the truth-value of $\neg\alpha$ according to a skeptical, rigid posture which tries to minimize the truthfulness of $\neg\alpha$. According to the above suggestion, this will be done by stipulating that $V_{max}(\neg\alpha) = 1$ iff $V_{max}(\alpha) = 0$ and $V_{min}(\neg\alpha) = 1$ iff $V_{min}(\alpha) = 0$. Let us once more make use of our authority-oriented model, but now forgetting about plausible modal operators and evaluating the truth of sentences according exclusively to a maximal or minimal position. In this way, we will have for instance that α is true according to a maximal position, in symbols $V_{max}(\alpha) = 1$, iff at least one expert holds α. Consequently, $V_{max}(\alpha) = 0$ iff *none* of the experts hold α, or in other words, if all experts believe α to be false. This is however clearly not what we mean by $\neg\alpha$ being true according to a tolerant position. Evaluating the truth-value of $\neg\alpha$ according to a tolerant, credulous position means to try to maximize its truthfulness or to require just a little to accept it as true. And this definitely is not what $V_{max}(\alpha) = 0$ does. In fact, by equating $V_{max}(\neg\alpha) = 1$ with $V_{max}(\alpha) = 0$ we do exactly the opposite: we evaluate $\neg\alpha$ according to a skeptical position. Therefore, $V_{max}(\alpha) = 0$ cannot be taken as the definition of $V_{max}(\neg\alpha) = 1$. Similarly, $V_{min}(\alpha) = 1$ iff all experts hold α and $V_{min}(\alpha) = 0$ iff *at least one* of them believe α to be false. But this is not what we mean by $\neg\alpha$ being true according to a rigid position. Evaluating the truth-value of $\neg\alpha$ according to a rigid, skeptical position means to try to minimize its truthfulness or to require a lot to accept $\neg\alpha$ as true, which is not what $V_{min}(\alpha) = 0$ does. By equating $V_{min}(\neg\alpha) = 1$ with $V_{min}(\alpha) = 0$ we rather evaluate $\neg\alpha$ according to a credulous position. Consequently, we cannot take $V_{max}(\alpha) = 0$ as the definition of $V_{max}(\neg\alpha) = 1$.

One may of course wonder why this so. In other words, not considering specific interpretations for V_{max} and V_{min} such as our authority-oriented model, why $V_{max}(\alpha) = 0$ and $V_{min}(\alpha) = 0$ do not mean, respectively, "α is false according to a maximal posture" and "α is false according to a minimal posture"? The key to answer this question rests on the fact that the result of a function applied to a specific value as parameter may be through of as a sort of qualification over this specific value. In the case of a valuation function V applied to α, $V(\alpha) = 0$ may be seen as a qualification over α, namely one which qualifies it as false. Now, V_{max} and V_{min} can be seen as the two functions one may obtain from a truth-function V which receives as parameter in addition to formula α also the position according to which α will be evaluated. Therefore, $V_{max}(\alpha)$ can be seen as an abbreviation for $V(max,\alpha)$ and $V_{min}(\alpha)$ as an abbreviation for $V(min,\alpha)$. Therefore, when we write $V(max,\alpha) = 0$ we are qualifying not only α as false, but α in conjunction with the maximal position or, we may say, the hypothesis that α is true according to a maximal position. Putting in terms of negation of statements, $V_{max}(\alpha) = 0$ then means something like it is not the case that "α according to a maximal position". Note that this is quite different from "it is not the case that α" according to a maximal position, which is our intended meaning for $V_{max}(\neg\alpha) = 1$. Using brackets and replacing α by "α is true" we have in the first case (which is the actual meaning of $V_{max}(\alpha) = 0$) "it is not the case that [α is true according to a maximal position]" and in the second case (which is the meaning we want for $V_{max}(\neg\alpha) = 1$) "[it is not the case that α is true] according to a maximal position." Similarly, $V(min,\alpha) = 0$ means that we are qualifying α in conjunction with the minimal position or the hypothesis that α is true according to such position as false. Putting in terms of negation of statements, $V_{min}(\alpha) = 0$ then means that it is not the case that [α is true according to a minimal position], what is quite different from [it is not the case that α is true] according to a minimal position, which is our intended meaning for $V_{min}(\neg\alpha) = 1$.

Then the inevitable question comes: how then will we recursively define $V_{min}(\neg\alpha) = 1$ and $V_{max}(\neg\alpha) = 1$?. In order to answer this, we have to take a closer look at the relation that exists between the skeptical position and the credulous position when applied to negative formulae. As we have seen, we can soundly read $V_{min}(\neg\alpha) = 1$ as "α is false according to a posture that tries to minimize its falsity" or, equivalently, "we did succeed in the task of evaluating α as false according to a criterion that will be very strict in the matter of qualifying α as false." Now, given that $V_{min}(\neg\alpha)$ is 1, what will happen if we try to evaluate the truth-value of α according to a credulous position? In other words, will we be successful in the task of evaluating α as true according to a tolerant position towards α's truthfulness (in symbols: $V_{max}(\alpha) = 1$)? Clearly we will not, for if we were, our skeptical position, which is supposed to be very strict in the matter of assigning false to α, would classify α as false in a situation in which it could somehow be taken as true. Therefore, it would not be that strict. Thus, $V_{max}(\alpha) = 0$. Conversely, given that $V_{max}(\alpha) = 0$, what will be the result of $\neg\alpha$'s evaluation according to a skeptical posture? If $V_{min}(\neg\alpha)$ equals 0, we would have that the task of evaluating α as false according to a very strict criterion the matter of qualifying α as false was not successful. But then our credulous evaluation of α, which is supposed to be tolerant in the matter of assigning true to it, would not be tolerant at all, because it would fail in the task of classifying α as true in a situation where the only thing that could prevent the falsity of α from a skeptical point of view, namely the truth of α, was present. Thus, $V_{min}(\neg\alpha) = 1$. Therefore, $V_{min}(\neg\alpha) = 1$ iff $V_{max}(\alpha) = 0$.

Similarly, $V_{max}(\neg\alpha) = 1$ means that α is false according to a posture that tries to maximize the falsity of α, or, in other words, that we did succeed in the task of evaluating α as false according to a criterion that will be very tolerant in the matter of qualifying it as false. Now, given that $V_{max}(\neg\alpha)$ is 1, what will happen if we try to classify α as true according to a skeptical position? Will we be successful? Well, if we were, we would have that our skeptical position, which is supposed to be strict in the matter of assigning true to α, would classify α as true in a situation where it could somehow be taken as false. Therefore, it would not be that strict. Consequently, $V_{min}(\alpha) = 0$. Conversely, what will be the result of $\neg\alpha$'s evaluation according to a credulous viewpoint given that we did not succeed in the task of classifying α as true according to a posture that tries to minimize its truthfulness ($V_{min}(\alpha) = 0$)? If $V_{max}(\neg\alpha)$ equals 0, we will have that the task of evaluating α as false according to a criterion that is very tolerant in the matter of qualifying α as false was not successful. But if we suppose that $V_{min}(\alpha) = 0$, we will have that this so-called tolerant criterion is not tolerant at all, because it failed in the task of classifying α as false in a situation where the only thing that could prevent the truth of α from a skeptical point of view, namely the falsity of α, was present. Thus, $V_{max}(\neg\alpha) = 1$. Consequently, $V_{max}(\neg\alpha) = 1$ iff $V_{min}(\alpha) = 0$.

All this can be easily seen if we use our authority-oriented semantic model, again without reference to any plausible modality. $V_{min}(\neg\alpha) = 1$ means that all experts believe $\neg\alpha$ or, equivalently, that all experts disbelieve α. But since $V_{max}(\alpha) = 1$ iff at least one expert holds α, $V_{max}(\alpha) = 0$ iff all experts disbelieve α. Consequently, $V_{min}(\neg\alpha) = 1$ iff $V_{max}(\alpha) = 0$. Similarly, $V_{max}(\neg\alpha) = 1$ means that at least one expert believes $\neg\alpha$ or, equivalently, that at least one expert disbelieve α. But since $V_{min}(\alpha) = 1$ iff all experts hold α, $V_{min}(\alpha) = 0$ iff at least one expert disbelieve α. Consequently, $V_{max}(\neg\alpha) = 1$ iff $V_{min}(\alpha) = 0$

Regarding our functions Ω and $\mho$, as it can be seen from definition 5.3.3, given a modal formulae α, if $\Omega_{M,w}(\alpha)$ returns a truth-value according to maximal posture, $\mho_{M,w}(\alpha)$ will return a truth-value according to a minimal posture; and if $\Omega_{M,w}(\alpha)$ returns a truth-value according to minimal posture, $\mho_{M,w}(\alpha)$ will return a truth-value according to a maximal posture. Trivially then, from the standpoint of each other, Ω and $\mho$ behave exactly like our V_{max} and V_{min} functions. Therefore, the recursive definition of formulae of the form $\neg\alpha$ by Ω and $\mho$ must be done exactly in the same way we did above: $\Omega_{M,w}(\neg\alpha) = 1$ iff $\mho_{M,w}(\alpha) = 0$ and $\mho_{M,w}(\neg\alpha) = 1$ iff $\Omega_{M,w}(\alpha) = 0$. This is why we need function $\mho$ in order for Ω to take account of formulae of the form $\neg\alpha$. If we really want to have a function that recursively defines the truth-value of $\neg\alpha$ from a skeptical (credulous) point of view, we have no choice but to also have a function that defines the truth-value of α from a credulous (skeptical) point of view. Regarding $\neg$ (and, as we will see below, also $\rightarrow$) one view is not complete without the other. Therefore, if we want to analyze $\neg(\alpha!)$ from a skeptical point of view, we have to have a way to analyze $\alpha!$ from a credulous point of view and, similarly, if we want to analyze $\neg(\alpha?)$ from a credulous point of view, we have to have a way to analyze $\alpha?$ from a skeptical point of view.

As one can see, all the exposition we have made in the above paragraphs strongly resembles the distinction we have made in Section 5.1 between the *internal* negation and the *external* negation. As we have explained, the distinction between these two sorts of negations applied to a specific formula φ is that while in the internal case "not φ" means that φ is false according to a skeptical (credulous) position, in the external one it means that it is not the case that "φ is true according to a skeptical (credulous) position." As we can see from definition 5.3.3, $\neg$ is exactly the formalization of the internal negation: $\Omega_{M,w}(\neg(\alpha!)) = 1$ iff $\mho_{M,w}(\alpha!) = 0$ iff for all $w' \in W$ such that wR_kw, $\mho_{M,w'}(\alpha) = 0$ and $\Omega_{M,w}(\neg(\alpha?)) = 1$ iff $\mho_{M,w}(\alpha?) = 0$ iff for at least one $w' \in W$ such that wR_kw, $\mho_{M,w'}(\alpha) = 0$. This is trivially in accordance with the exposition we have made in Section 5.1: α is skeptically implausible iff all experts disbelieve α, and α is credulously implausible iff at least one expert disbelieve α. Therefore, as we have mentioned earlier, $\neg$ as defined by definition 5.3.3 formalizes the internal negation (which, because of the skeptical and credulous positions' being dependent on the modal operator attached to the formulae, is also called a modality-dependent paranormal negation.)

Concerning the external negation, "not $\alpha!$" will mean that it is not the case that "α is plausible according to a skeptical position," and "not $\alpha?$" that it is not the case that "α is plausible according to a credulous position." In Section 5.1 we have made this precise by saying that the first case hold iff at least one expert disbelieves α and the second iff all experts disbelieve α. As we have said earlier, this sort of negation behavior will be formalized by our derived operator ~ (which we have decided to call classical negation.) In this way, $\sim(\alpha!)$ will mean "it is not the case that α is skeptically plausible" and $\sim(\alpha?)$ "it is not the case that α is credulously plausible." Since Ω is the function we will effectively use to evaluate the truth-value of formula (we have called it the principal max-min function of paranormal modal logic), in order to see that this is really the case we have just to take a look at how formulae of the form $\sim(\alpha!)$ and $\sim(\alpha?)$ will evaluated by Ω. Since $\sim(\alpha!)$ is an abbreviation for $\alpha! \rightarrow \bot$, we have that $\Omega_{M,w}(\sim(\alpha!)) = 1$ iff $\Omega_{M,w}(\alpha! \rightarrow \bot) = 1$ iff $\Omega_{M,w}(\alpha!) = 0$ or $\Omega_{M,w}(\bot) = 1$ iff $\Omega_{M,w}(\alpha!) = 0$ iff, for at least one $w' \in W$ such that wR_kw, $\Omega_{M,w'}(\alpha) = 0$. Similarly, since $\sim(\alpha?)$ is an abbreviation for $\alpha? \rightarrow \bot$, we have that $\Omega_{M,w}(\sim(\alpha?)) = 1$ iff $\Omega_{M,w}(\alpha? \rightarrow \bot) = 1$ iff $\Omega_{M,w}(\alpha?) = 0$ or $\Omega_{M,w}(\bot) = 1$ iff $\Omega_{M,w}(\alpha?) = 0$ iff, for all $w' \in W$ such that wR_kw, $\Omega_{M,w'}(\alpha) = 0$.

Now, a last and crucial point concerning the semantics of paranormal modal logic. In classical logic, an implication formula is equivalent to the disjunction of the negation of the antecedent and the consequent. In symbols: $\alpha \rightarrow \beta \equiv \neg\alpha \vee \beta$. From a semantic point of view, this interpretation of implication through disjunction and negation (now stated in the meta-language) has been traditionally used to define the truth-value of implication formulae: $V(\alpha \rightarrow \beta) = 1$ iff $V(\alpha) = 0$ or $V(\beta) = 1$. But since in paranormal modal logic we have two negations, before going on and defining semantically $\rightarrow$-formulae, we have first to decide which of the two negations will be used.

Let us again make use of our functions V_{min} and V_{max} applied to nonmodal formulae. Since V_{min} evaluates the truth-value of formulae according to a minimizing posture, the value of $V_{min}(\alpha \rightarrow \beta)$ should likewise be calculated according to such position. This means that we should not be so tolerant in the matter of assigning 1 to $V_{min}(\alpha \rightarrow \beta)$. If we interpret $\rightarrow$ trough the external negation ~, we will have that $V_{min}(\alpha \rightarrow \beta) = 1$ iff $V_{min}(\alpha) = 0$ or $V_{min}(\beta) = 1$. This means that $\alpha \rightarrow \beta$ is true according to a skeptical position iff at least one expert disbelieves α or all experts believe β. While the part of the consequent poses no problem for the skeptical point of view, clearly the interpretation of the antecedent is not in accordance with a skeptical posture: rather, it is in accordance with a credulous viewpoint. If we take the above definition, we will have that having just one expert against α is enough to take $\alpha \rightarrow \beta$ as true according to a position that is supposed to require quite a lot to classify sentences as true. On the other hand, by using our internal negation we will have that $V_{min}(\alpha \rightarrow \beta) = 1$ iff $V_{max}(\alpha) = 0$ or $V_{min}(\beta) = 1$. In other words $\alpha \rightarrow \beta$ is true according to a skeptical position iff all experts disbelieve α or all experts believe β, what is clearly what we mean by evaluating $\alpha \rightarrow \beta$ skeptically. Similarly, if we define $V_{max}(\alpha \rightarrow \beta) = 1$ as $V_{max}(\alpha) = 0$ or $V_{max}(\beta) = 1$, $\alpha \rightarrow \beta$ will be true according to a credulous position iff all experts disbelieve α or at least one expert believe β. This interpretation is not in accordance with a credulous posture: concerning the falsity of the antecedent, we need all experts to be against α to take $\alpha \rightarrow \beta$ as credulously true. On the other hand, by using the internal negation we will have that $V_{max}(\alpha \rightarrow \beta) = 1$ iff $V_{min}(\alpha) = 0$ or $V_{min}(\beta) = 1$. In other words $\alpha \rightarrow \beta$ is true according to a credulous position iff at least one expert disbelieve α or all at least one expert believe β, what is clearly what we mean by evaluating $\alpha \rightarrow \beta$ credulously.

We can now apply these conclusions to Ω and $\mho$. The correct way for Ω and $\mho$ to evaluate implication formulae is to take $\Omega_{M,w}(\alpha \rightarrow \beta) = 1$ iff $\mho_{M,w}(\alpha) = 0$ or $\Omega_{M,w}(\beta) = 1$ and $\mho_{M,w}(\alpha \rightarrow \beta) = 1$ iff $\Omega_{M,w}(\alpha) = 0$ or $\mho_{M,w}(\beta) = 1$. For instance, since formula $\alpha? \rightarrow \beta?$ should be maximally evaluated, $\Omega_{M,w}(\alpha? \rightarrow \beta?) = 1$ iff $\mho_{M,w}(\alpha?) = 0$ or $\Omega_{M,w}(\beta?) = 1$, what in its turn is the case iff, for at least one $w' \in W$ such that wR_kw', $\mho_{M,w'}(\alpha) = 0$ or, for at least one $w' \in W$ such that wR_kw', $\mho_{M,w'}(\beta) = 1$. Our reasoning will be still valid in the case of formulas like $\alpha? \rightarrow \beta!$ and $\alpha! \rightarrow \beta?$, in which the whole implication formula is evaluated in a mixed way, for both the negation of the antecedent and the consequent will be evaluated according to the posture indicated by the modal symbol attached to them (for instance, we will have that $\Omega_{M,w}(p? \rightarrow q!) = 1$ iff $\mho_{M,w}(p?) = 0$ or $\Omega_{M,w}(q!) = 1$ iff, for at least one $w' \in W$ such that wR_kw', $v_{w'}(p) = 0$ or, for all $w' \in W$ such that wR_kw', $v_{w'}(q) = 1$.)

Perhaps one may be wondering why we did not incorporate these conclusions, which for sure look sound, into definition 5.3.3. As one can see, there is an asymmetry between items (iii) and (iv) of definition 5.3.3: while in item (iv) we have $\mho_{M,w}(\alpha \rightarrow \beta) = 1$ iff $\Omega_{M,w,}(\alpha) = 0$ or $\mho_{M,w}(\beta) = 1$, therefore in accordance with what we have concluded, in item (iii) we have $\Omega_{M,w}(\alpha \rightarrow \beta) = 1$ iff $\Omega_{M,w}(\alpha) = 0$ or $\Omega_{M,w}(\beta) = 1$, what clearly goes against the exposition of the previous paragraphs. The reason for that lies on two words: *modus ponens*. If we equate $\Omega_{M,w}(\alpha \rightarrow \beta) = 1$ with $\mho_{M,w}(\alpha) = 0$ or $\Omega_{M,w}(\beta) = 1$, given a model M and a world w of M, we would have possibly that $M,w \Vdash_{\Omega} \alpha \rightarrow \beta$ and $M,w \Vdash_{\Omega} \beta$ but $M,w \nVdash_{\Omega} \beta$. As a consequence of this, on the semantic side, *modus ponens* would not be valid: given an n-modal system Λ whose n-modal valuation is a Ω_k-modal valuation modified in the suggested way, it may happen that $\{\alpha \to \beta, \alpha\} \nvDash_{\Lambda} \beta$[16]. In order to have *modus ponens* as a valid principle of paranormal modal logic, we have no choice but to define $\Omega_{M,w}(\alpha \to \beta)$ exclusively in terms of Ω: $\Omega_{M,w}(\alpha \to \beta) = 1$ iff $\Omega_{M,w}(\alpha) = 0$ or $\Omega_{M,w}(\beta) = 1$. This of course is tantamount to interpreting $\to$ in terms of ~. Proceeding in this way, we will have that if $M,w \Vdash_{\Omega} \alpha \to \beta$ and $M,w \Vdash_{\Omega} \beta$ then $M,w \Vdash_{\Omega} \beta$ and, consequently, that $\{\alpha \to \beta, \alpha\} \vDash_{\Lambda} \beta$ for any n-modal system Λ whose n-modal valuation is a Ω_k-modal valuation.

An immediate consequence of this unavoidable decision is that many logical laws such as $(\alpha \to \beta) \leftrightarrow \neg\alpha \vee \beta$ and $(\alpha \to \beta) \leftrightarrow (\neg\beta \to \neg\alpha)$ will not be valid in paranormal modal logic. On the other hand, since $\Omega_{M,w}(\sim\alpha) = 1$ iff $\Omega_{M,w}(\alpha) = 0$, all these laws will still be valid if we consider them along with external negation ~. Given any model M, we have that $M \Vdash_{\Omega} (\alpha \to \beta) \leftrightarrow \sim\alpha \vee \beta$ and $M \Vdash_{\Omega} (\alpha \to \beta) \leftrightarrow (\sim\beta \to \sim\alpha)$. In fact, not only these two laws but all other classical laws are valid in paranormal modal logic when $\neg$ is replaced by ~; and if we consider ! and ? exclusively in connection with ~, we will have two modalities undistinguishable from normal modalities $\Box$ and $\Diamond$[17]. Another important observation is that since $\mho$ is defined in the way we agreed it should be, that is, in connection with $\neg$ and not with ~, we have that laws which consist of the negation of an implication formula (such as $\neg(\alpha \to \beta) \leftrightarrow (\alpha \wedge \neg\beta)$, for example) will still be valid in paranormal modal logic (this is the case because only when we negate a specific formula is that we invoke function $\mho$.)

5.3.3 Paranormal Modal Calculus

The calculus of paranormal modal logic is a modal extension of positive classical logic in such a way as to consider a modality-dependent paranormal negation. It will therefore be built by adding extra axioms to the set of classical positive axioms in $L_?$ (as stated in definition 5.2.22).

Definition 5.3.5. Let $\Im$ be a language and ϑ a ?-modal logic basis. The *paranormal classical axioms* Σ_A in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying one of the following schemas of formula:

A1: $(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$ wherein β is ?-free and α is !-free

A2: $\neg\alpha \to (\alpha \to \beta)$ wherein α is ?-free

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[16] See definition 5.2.20.

A3: $\alpha \vee \neg\alpha$ wherein α is !-free

The schemas of formula A1-A3 correspond to the negative axioms of classical logic. Along with schemas P1-P8 of positive classical axioms (definition 5.2.22), they strongly resemble one of the most used axiomatizations for classical logic. The difference is that, as defined above, schemas A1-A3 are not universally applied to any formula: there are restrictions concerning modal formulae. The reason for such restrictions lies on the paraconsistency of ? and paracompleteness of !. Actually, they are the very key of the paracomplete behavior of ! and the paraconsistent behavior of ?. Since the set $\{\alpha?, \neg(\alpha?)\}$ is intent to be an inconsistent but non-trivial theory, schemas A1 and A2 should restrict their use only to ?-free formulae: from $\alpha \to \beta?$ and $\alpha \to \neg(\beta?)$ one should not be able to use A1 to conclude $\neg\alpha$, and from $\alpha?$ and $\neg(\alpha?)$ one should not be able to use A2 to conclude β. Similarly, since we may have both $\alpha!$ and $\neg(\alpha!)$ as false, A1 and A3 have to have the same sort of restriction concerning !-marked formulae: if $\alpha! \to \beta$ and $\alpha! \to \neg\beta$, one should not be able to use A1 to conclude $\neg(\alpha!)$ or A3 to conclude $\alpha! \vee \neg(\alpha!)$. As a consequence of these restrictions, many logical principles whose derivations depend on one of these schemas will not be theorems of paranormal modal calculus. However, since concerning ?!-free formulae schemas A1-A3 can be freely used, we will have that for non-modal formulae, all principles of classical logic will be theorems of paranormal modal calculus.

Definition 5.3.6. Let $\Im$ be a language and ϑ a modal logic basis. The *non-positive additional classical axioms* Σ_N in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying one of the following schemas of formula:

N1: $\neg(\alpha \to \beta) \leftrightarrow (\alpha \wedge \neg\beta)$

N2: $\neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$

N3: $\neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$

N4: $\neg\neg\alpha \leftrightarrow \alpha$

N5: $((\alpha \to \beta) \to \alpha) \to \alpha$

Axiom schemas N1-N5 are meant to restore the deductive power of paranormal modal logic weakened by the restrictions imposed to axioms A1-A3. Taking the {P1-P8, A1-A3} corresponding axiomatization of classical logic, the derivation of each one of the schemas N1-N5 depends on A1, A2 or A3. Therefore, since A1-A3 cannot be unrestrictedly used, such derivations cannot be undergone. As a consequence of that, schemas N1-N5 have to be explicitly stated as axioms of paranormal modal calculus.

[17] These points will be made precise when we compare $K_?$ with classical logic and normal modal logic K in subsection 5.3.5.

Definition 5.3.7. Let $\Im$ be a language and ϑ a ?-modal logic basis. The *paranormal modal axioms* Σ_M in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying one of the following schemas of formula:

K1: $\alpha? \leftrightarrow \sim((\sim\alpha)!)$

K2: $(\neg\alpha)! \leftrightarrow \neg(\alpha!)$

K3: $(\neg\alpha)? \leftrightarrow \neg(\alpha?)$

The paranormal modal axioms K1-K5 set the basic properties of the modal operators ! and ?. K1 states that in connection with external or classical negation ~, ? and ! are the dual operators of each other in the same way that □ is the dual operator of ◇ and ◇ the dual operator of □. The difference is that traditionally ◇ is a taken as a derived operator ($\Diamond\alpha =_{def} \neg\Box\neg\alpha$), although nothing prevents us from taking ◇ as a primitive symbol and adding the extra axiom $\Diamond\alpha \leftrightarrow \neg\Box\neg\alpha$. Independently however of the way we build normal modal logic, K1 and axiom schema $K_?$ (to be defined below) set ! and ? in connection with ~ as indistinguishable from □ and ◇ of normal modal logic.

Axiom schemas K2 and K3 set a sort of internalization and externalization of ¬ with respect to ! and ?, respectively. From the point of view of the theory of plausibility we developed so far, K2 and K3 state, respectively, that the skeptical plausibility of ¬α is equivalent to the skeptical implausibility of α, and that the credulous plausibility of ¬α is equivalent to the credulous implausibility of α. This is clearly in accordance with what we have agreed in Section 5.1 that skeptical implausibility and credulous implausibility should be. Also from K2 and K3 the internal aspect of ¬ becomes evident: by taking (¬α)! and ¬(α!), and (¬α)? and ¬(α?) as equivalent, it becomes explicit that what is negated by ¬ is the inside of the modal formula. In fact, it is axioms K2 and K3 what sets the internal character of ¬ and consequently makes possible the formalization of the notions of skeptical implausibility and credulous implausibility as the negation of skeptical plausibility and credulous plausibility, respectively.

It is worthy noticing that $(\neg\alpha)? \rightarrow \neg(\alpha?)$ is the same controversial axiom of LEI that we have discussed in Chapter 4. The argument we presented against it there went like that: since α? means "there is some evidence for α" and ¬(α?) means "there is no evidence for α," trivially the fact that there is some evidence for ¬α does not imply that there is no evidence for α. Given what we have exposed in this chapter, the fallacy in this argument trivially comes from interpreting ¬(α?) independently of the position according to which plausible statements are being evaluated. If this is done, the only negation available for us (to be used in connection with plausible formulae) will be the external one. Consequently, $\neg(\alpha?)$ (what in our symbolism corresponds to $\sim(\alpha?)$) will mean "there is no evidence for α." However, as we have seen, as important as the external negation is a negation that acts internally in the plausible formulae, without which it is not possible to formalize the notion of credulous implausibility, that is to say, α's being taken as implausible according to a tolerant position. And it is this negation that this controversial axiom is about.

Definition 5.3.8. Let $\Im$ be a language and ϑ a ?-modal language basis. The $K_?$ *axioms* $\Sigma_{K?}$ in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying the following schema of formula:

$K_?$: $(\alpha \rightarrow \beta)! \rightarrow (\alpha! \rightarrow \beta!)$

5.3.4 The System $K_?$

With the semantic and syntactic elements we have defined in Section 5.2 and so far in this section we can give a precise definition of paranormal modal logic. We start with the basic system upon which all others will be based: $K_?$. This name is a direct reference both to the way it is axiomatically obtained – adding $\Sigma_{K?}$ to the system obtained by Σ_P, Σ_N, Σ_M, necessitation rule and *modus ponens* – as well as to normal modal system K, which is obtained by adding axiom K ($\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$) and necessitation rule to classical logic.

Definition 5.3.9. The propositional paranormal modal logic $K_?$ is the propositional modal system $<\vartheta_?, \Omega_?, \Gamma, \Sigma>$ where $\vartheta_? = <\{!,?\}, \{!\}>$ is the paranormal modal logic basis, $\Omega_?$ is the *propositional* Ω_1*-modal valuation* Ω in $\vartheta_?$, Γ is the set of all frames and

$\Sigma = \Sigma_P \cup \Sigma_A \cup \Sigma_N \cup \Sigma_M \cup \Sigma_{K?}$

where Σ_P are the positive classical axioms in $L_?$, Σ_A the paranormal classical axioms in $L_?$, Σ_N the non-positive additional classical axioms in $L_?$, Σ_M the paranormal modal axioms in $L_?$ and $\Sigma_{K?}$ the $K_?$ axioms in $L_?$.

As we mentioned in Section 5.1, $K_?$ in particular and paranormal modal logic in general are in fact normal modal logics: they have *modus ponens* and necessitation (definition 5.2.24), a K-like axiom schema (definition 5.3.8) and the rule of uniform substitution (which comes from the very way we are defining the axioms, as can be seen in definitions 5.2.22, 5.3.6, 5.3.7 and 5.3.8.)

Theorem 5.3.1. $K_?$ is sound and complete (that is, for any $A,B \subseteq L_?$ and $\alpha \in L_?$, $A \div B \vdash_{K?} \alpha$ iff $A \div B \vDash_{K?} \alpha$.)

Theorem 5.3.2. Some formulae of $L_?$ that satisfy one of the following schemas of formula are *not* $K_?$-theorems (and consequently not $K_?$-valid.)

$(\alpha \to \beta) \to (\alpha \to \neg\beta) \to \neg\alpha \qquad \neg\alpha \to (\alpha \to \beta)$

$\neg\alpha \vee \alpha \qquad \neg(\alpha \wedge \neg\alpha)$

$(\alpha \to \beta) \to (\neg\beta \to \neg\alpha) \qquad (\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$

$\neg\alpha \vee \beta \to (\alpha \to \beta) \qquad (\alpha \to \beta) \to \neg\alpha \vee \beta$

$(\alpha \to \beta) \to \neg(\alpha \wedge \neg\beta) \qquad \neg(\alpha \wedge \neg\beta) \to (\alpha \to \beta)$

$\neg\alpha \to (\alpha \to \neg\beta) \qquad (\alpha \to \neg\alpha) \to \neg\alpha$

Theorem 5.3.2 indicates from a proof-theoretical point of view in which respect paranormal logic differs from classical logic (and consequently also from normal modal logic.) Taking the {P1-P8, A1-A3} corresponding axiomatization of classical logic, all the above schemas of formula need one of the axioms A1-A3 to be derived. But since A1-A3 can be used only if certain restrictions are satisfied, none of the above schemas can be unrestrictedly derived

Theorem 5.3.3. All formulae of $L_?$ that satisfy one of the following schemas of formula, wherein α and β are ?!-free formulae, are $K_?$-theorems (and consequently $K_?$-valid.)

$(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha) \qquad \neg\alpha \to (\alpha \to \beta)$

$\neg\alpha \vee \alpha \qquad \neg(\alpha \wedge \neg\alpha)$

$(\alpha \to \beta) \to (\neg\beta \to \neg\alpha) \qquad (\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$

$\neg\alpha \vee \beta \to (\alpha \to \beta) \qquad (\alpha \to \beta) \to \neg\alpha \vee \beta$

$(\alpha \to \beta) \to \neg(\alpha \wedge \neg\beta) \qquad \neg(\alpha \wedge \neg\beta) \to (\alpha \to \beta)$

$\neg\alpha \to (\alpha \to \neg\beta) \qquad (\alpha \to \neg\alpha) \to \neg\alpha$

Theorem 5.3.4. All formulae of $L_?$ that satisfy one of the following schemas of formula are $K_?$-theorems (and consequently $K_?$-valid.)

$(\alpha \to \beta) \to (\alpha \to {\sim}\beta) \to {\sim}\alpha \qquad {\sim}\alpha \to (\alpha \to \beta)$

${\sim}\alpha \vee \alpha \qquad {\sim}(\alpha \wedge {\sim}\alpha)$

$(\alpha \to \beta) \to ({\sim}\beta \to {\sim}\alpha) \qquad ({\sim}\beta \to {\sim}\alpha) \to (\alpha \to \beta)$

${\sim}\alpha \vee \beta \to (\alpha \to \beta) \qquad (\alpha \to \beta) \to {\sim}\alpha \vee \beta$

$(\alpha \to \beta) \to {\sim}(\alpha \wedge {\sim}\beta) \qquad {\sim}(\alpha \wedge {\sim}\beta) \to (\alpha \to \beta)$

${\sim}\alpha \to (\alpha \to {\sim}\beta) \qquad (\alpha \to {\sim}\alpha) \to {\sim}\alpha$

While theorem 5.3.3 shows that paranormal modal logic behaves like classical logic when only !?-free formulae are taken into account, theorem 5.3.4 shows that in fact paranormal modal logic behaves like classical logic when we consider just the external negation ~.

Theorem 5.3.5. All formulae of $L_?$ that satisfy one of the following schemas of formula are $K_?$-theorems (and consequently $K_?$-valid.)

$$(\alpha \to \beta)! \to (\alpha! \to \beta!) \qquad (\alpha \to \beta)! \to (\alpha? \to \beta?)$$

$$(\alpha \wedge \beta)! \leftrightarrow \alpha! \wedge \beta! \qquad (\alpha \wedge \beta)? \to (\alpha? \wedge \beta?)$$

$$\alpha! \vee \beta! \to (\alpha \vee \beta)! \qquad (\alpha? \vee \beta?) \leftrightarrow (\alpha \vee \beta)?$$

$$(\alpha \to \beta)? \leftrightarrow (\alpha! \to \beta?) \qquad (\alpha \vee \beta)! \to (\alpha! \vee \beta?)$$

$$\sim(\alpha!) \leftrightarrow (\sim\alpha)? \qquad \sim(\alpha?) \leftrightarrow (\sim\alpha)!$$

$$\sim(\alpha!) \vee (\sim\alpha)? \qquad \sim(\alpha?) \vee (\sim\alpha)!$$

Theorem 5.3.6. The following schemas of relations between sets of formulas and formula are sound.

$$\{\alpha \to \beta\} \vdash_{K?} (\alpha \to \beta)! \qquad \{\alpha \to \beta\} \vDash_{K?} (\alpha \to \beta)!$$

$$\{\alpha \to \beta\} \vdash_{K?} \alpha! \to \beta! \qquad \{\alpha \to \beta\} \vDash_{K?} \alpha! \to \beta!$$

$$\{\alpha \to \beta\} \vdash_{K?} \alpha? \to \beta? \qquad \{\alpha \to \beta\} \vDash_{K?} \alpha? \to \beta?$$

Theorems 5.3.5 and 5.3.6 show the similarity between ! and ? and normal modal operators □ and ◇. First, when only positive schemas of formula are considered, every theorem of normal modal logic is also a theorem in paranormal modal logic. (Here we are considering just $K_?$, but clearly, as it will become evident later, this applies to all extensions of $K_?$.) Second, when we consider external negation ~, all theorems of normal modal logic, without exception, are also theorems of paranormal modal logic. The difference between ! and ? and □ and ◇ will appear only when we consider internal negation ¬[18].

Theorem 5.3.7. Some formulae of $L_?$ that satisfy one of the following schemas of formula are *not* $K_?$-theorems (and consequently not $K_?$-valid.)

$$\alpha? \to \neg((\neg\alpha)!) \qquad \alpha! \to \neg((\neg\alpha)?)$$

$$\neg((\neg\alpha)!) \to \alpha? \qquad \neg((\neg\alpha)?) \to \alpha!$$

$$\neg(\alpha!) \to (\neg\alpha)? \qquad \neg(\alpha?) \to (\neg\alpha)!$$

$$(\neg\alpha)? \to \neg(\alpha!) \qquad (\neg\alpha)! \to \neg(\alpha?)$$

$$\alpha! \vee \neg(\alpha!) \qquad \neg(\alpha! \wedge \neg(\alpha!))$$

$$\neg(\alpha? \wedge \neg(\alpha?)) \qquad \alpha? \vee \neg(\alpha?)$$

Theorem 5.3.7 shows the distinguishing features of ! and ? when taken in connection with ¬. It is interesting to note that it is not only ? that disrespects the principle of non-contradiction in its intra-logical form: ! does not satisfy it either. And it is not only ! that disrespects the middle excluded principle: ? does not satisfy it either. This is because the following sorts of formulae are not $K_?$-theorems: $\neg(\alpha?! \wedge \neg(\alpha?!))$ and $\alpha!? \vee \neg(\alpha!?)$. But then, since $\neg(\alpha?) \vee \alpha?$ and $\neg(\alpha! \wedge \neg(\alpha!))$ are not $K_?$-theorems, besides classifying ? and ! as paraconsistent and paracomplete modalities, respectively, should we also take ? as a paracomplete modality and ! as a paraconsistent one? We will postpone the answer to this question to the next chapter, when we will lay down a finer definition of paraconsistency and paracompleteness applied to modal operators. For the time being, let us keep utilizing these terms in the more or less informal way we have being adopting up to this point[19].

Definition 5.3.10. Let $\Im$ be a language and ϑ a ?-modal logic basis. For any $n \geq 0$ we define the following abbreviation:

(i) $\alpha!^0 =_{def} \alpha$;

(ii) $\alpha!^{n+1} =_{def} (\alpha!^n)!$, $n \geq 1$.

Theorem 5.3.8. Let $A,B \subseteq L_?$ be two sets of formulae and $\alpha,\varphi \in L_?$ two formulae. $A \div B \cup \{\varphi\} \vdash_{K?} \alpha$ iff $A \div B \vdash_{K?} \varphi \rightarrow \alpha$.

Theorem 5.3.9. Let $A,B \subseteq L_?$ be two sets of formulae and $\alpha,\varphi \in L_?$ two formulae. $A \cup \{\varphi\} \div B \vdash_{K?} \alpha$ iff, for some $n \geq 0$, $A \div B \cup \{\varphi!^0, \varphi!^1, \ldots, \varphi!^n\} \vdash_{K?} \alpha$.

Theorem 5.3.10. Let $A,B \subseteq L_?$ be two sets of formulae and $\alpha,\varphi \in L_?$ two formulae. $A \div B \cup \{\varphi\} \vDash_{K?} \alpha$ iff $A \div B \vDash_{K?} \varphi \rightarrow \alpha$.

Theorem 5.3.11. Let $A,B \subseteq L_?$ be two sets of formulae and $\alpha,\varphi \in L_?$ two formulae. $A \cup \{\varphi\} \div B \vDash_{K?} \alpha$ iff, for some $n \geq 0$, $A \div B \cup \{\varphi!^0, \varphi!^1, \ldots, \varphi!^n\} \vDash_{K?} \alpha$.

Theorems 5.3.8 to 5.3.11 lay down the syntactic and semantic forms of both local (theorems 5.3.8 and 5.3.9) and global (theorems 5.3.10 and 5.3.11) deduction theorems of paranormal modal logic. They are equivalent to deduction theorems of normal modal logic as stated, for example, in Fitting(1993).

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[18] The result of all theorems from 5.3.3 to 5.3.6 will be stated in a more general way when we compare $K_?$, classical logic and normal modal logic K in the next section.

[19] It should be noted that the informal way we are talking about concerns only the notion of paraconsistency and paracompleteness in connection with modal operators. Concerning the other uses of paraconsistency and paracompleteness (such as the notions of paraconsistent and paracomplete logical systems, for example), their degrees of formality fit just right the needs of this work.

5.3.5 $K_?$, Classical Logic and K

For the sake of uniformity and comparison, we will define classical logic and normal modal logic K through the framework we have introduced in Section 5.2.

$K_?$ and Classical Logic

Definition 5.3.11. The *trivial modal logic basis* $\vartheta_\varnothing$ is the pair $<\varnothing, \varnothing>$.

Definition 5.3.12. Let $\Im$ be a language. The modal language $\Im_\varnothing$ based on $\vartheta_\varnothing$ is called the *trivial modal language* based on $\Im$.

Clearly, for any language $\Im$, $\Im_\varnothing = \Im$.

Definition 5.3.13. A *trivial frame* is a frame $F = <W,R>$ where W is an unary set of worlds. A model M based on some trivial frame F is called a *trivial model.*

Definition 5.3.14. Let $\Im$ be a language and ϑ a modal logic basis of arity *n*. A *normal modal valuation* Ψ in $\Im$ and ϑ is a n-modal valuation in $\Im$ and ϑ which, given an n-model M = $<W,R,\ldots,v>$, a world $w \in W$, any two formulae $\alpha,\beta \in L_\vartheta$ and possibly other parameters, satisfies the following conditions:

(i) $\Psi_{M,w,\ldots}(\neg\alpha) = 1$ iff $\Psi_{M,w,\ldots}(\alpha) = 0$;

(ii) $\Psi_{M,w,\ldots}(\alpha \to \beta) = 1$ iff $\Psi_{M,w,\ldots}(\alpha) = 0$ or $\Psi_{M,w,\ldots}(\beta) = 1$;

(iii) $\Psi_{M,w,\ldots}(\alpha \wedge \beta) = 1$ iff $\Psi_{M,w,\ldots}(\alpha) = 1$ and $\Psi_{M,w,\ldots}(\beta) = 1$;

(iv) $\Psi_{M,w,\ldots}(\alpha \vee \beta) = 1$ iff $\Psi_{M,w,\ldots}(\alpha) = 1$ or $\Psi_{M,w,\ldots}(\beta) = 1$.

Definition 5.3.15. Let ϑ be a modal logic basis of arity *n*. A *propositional normal valuation* Ψ in ϑ is a normal valuation in L and ϑ which, given an n-model M = $<W,R,\ldots,v>$, a world $w \in W$ and any propositional symbol $p \in P$, satisfies the following condition:

$\Psi_{M,w}(p) = 1$ iff $v_w(p) = 1$.

A propositional n-normal valuation Ψ in ϑ has as parameters a propositional n-model M, a world *w* of M and a formula α of L_ϑ. If $\vartheta = \vartheta_\varnothing$ we call the propositional normal valuation in it, which we refer to by the symbol $\Psi_\varnothing$, the *trivial propositional valuation.*

Definition 5.3.16. Let $\Im$ be a language and ϑ an n-modal logic basis. The *non-positive classical axioms* Σ_C in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying one of the following schemas of formula:

P9: $(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$

P10: $\neg\alpha \to (\alpha \to \beta)$

P11: $\alpha \vee \neg\alpha$

Definition 5.3.17. The *trivial propositional modal logic* or simply *propositional classical logic* C is the propositional modal system $<\vartheta_\varnothing, \Psi_\varnothing, \Gamma, \Sigma>$ where $\vartheta_\varnothing = <\varnothing, \varnothing>$ is the trivial modal logic basis, $\Psi_\varnothing$ is the trivial propositional valuation, Γ is the set of all trivial frames and $\Sigma = \Sigma_P \cup \Sigma_C$, where Σ_P and Σ_C are, respectively, the positive classical axioms and the non-positive classical axioms in $L_\varnothing$.

Trivially, restricting $\vdash_C$ and $\vDash_C$ to their two parameters forms ($A \div \varnothing \vdash_C \alpha$ and $A \div \varnothing \vDash_C \alpha$), C is the same as propositional classical logic.

Theorem 5.3.12. Let $\alpha \in L_\varnothing$. If $\vdash_C \alpha$, then $\vdash_{K?} \alpha$.

Theorem 5.3.13. Let $\alpha \in L_\varnothing$. If $\vDash_C \alpha$, then $\vDash_{K?} \alpha$.

Theorems 5.3.12 and 5.3.13 state in a general way what theorem 5.3.3 only indicated: when we consider only non-modal formulae, paranormal modal logic is undistinguishable from classical logic.

$K_?$ and K

Definition 5.3.18. A *$\diamond$-modal logic basis* ϑ is a pair $<\Theta',\Theta''>$ where $\{\Box,\diamond\}\subseteq\Theta'$ and $\{\Box\}\subseteq\Theta''$. The notation adopted in the operators $\Box$ and $\diamond$ is a pre-fixed one. We call the *$\diamond$-modal logic basis* $\vartheta_\diamond = <\{\Box,\diamond\}, \{\Box\}>$ the *normal modal logic basis*.

We say that the propositional modal language based on L and $\vartheta_\diamond$ is the *propositional normal modal language*. It will be referred to by the symbol $L_\diamond$. Just for the sake of comparison with $K_?$, we will here take normal modal logic with $\diamond$ as a primitive symbol, rather as a derived one obtained from $\Box$.

Definition 5.3.19. Let $\Im$ be a language. A *normal modal valuation* Ψ in $\Im$ is a normal valuation in $\Im$ and $\vartheta_\diamond$ which, given a model M = <W,R,v>, a world $w \in W$ and any two formulae $\alpha,\beta \in \Im_\diamond$, satisfies the following conditions:

(i) $\Psi_{M,w}(\diamond\alpha) = 1$ iff, for some $w' \in W$ such that wRw', $\Psi_{M,w'}(\alpha) = 1$.

(ii) $\Psi_{M,w}(\Box\alpha) = 1$ iff, for all $w' \in W$ such that wRw', $\Psi_{M,w'}(\alpha) = 1$.

Definition 5.3.20. Let $\Im$ be a language and ϑ a $\Diamond$-modal logic basis. The *possibility-necessity axioms* Σ_{NP} in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying the following schema of formula:

NP: $\Diamond\alpha \leftrightarrow \neg\Box\neg\alpha$

Axiom NP is needed because of our decision of taking $\Diamond$ as a primitive symbol.

Definition 5.3.21. Let $\Im$ be a language and ϑ a $\Diamond$-modal logic basis. The K *axioms* Σ_K in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying the following schema of formula:

K: $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$

Definition 5.3.22. The propositional normal modal logic K is the propositional modal system $<\vartheta_\Diamond, \Psi_\Diamond, \Gamma, \Sigma>$ where $\vartheta_\Diamond = <\{\Box,\Diamond\},\{\Box\}>$ is the normal modal logic basis, $\Psi_\Diamond$ is the propositional normal modal valuation, Γ is the set of all frames and

$\Sigma = \Sigma_P \cup \Sigma_C \cup \Sigma_{NP} \cup \Sigma_K$,

where Σ_P are the positive classical axioms in $L_\Diamond$, Σ_C the non-positive classical axioms in $L_\Diamond$, Σ_{NP} the possibility-necessity axioms in $L_\Diamond$ and Σ_K the K axioms in $L_\Diamond$.

From definitions 5.3.22 and 5.3.9 we can see in which respect paranormal modal logic $K_?$ and normal modal logic K differ from each other. Disregarding syntactical differences concerning the shape of the modal operators and their positions in the formulae, what makes $K_?$ and K different from each other is exclusively their modal valuation and axioms: both language and set of frames are identical. Also the structure upon which formulae are semantically evaluated of both systems, that is the model, is identical. We can also see from a proof-theoretical point of view the similarities between $K_?$ from K: both $K_?$'s and K's axiomatic use positive classical axioms, a possibility-necessity axiom (K1 of definition 5.3.7), a K-like axiom, *modus ponens* and necessitation. The difference is that while K has to add to this list just non-positive classical axioms P9-P11, $K_?$ has to take a restricted version of them along with some additional non-positive classical axioms and the paranormal modal axioms which define the behavior of $\neg$ in connection with ! and ?.

Definition 5.3.23. We define the function Φ: $L_\Diamond \rightarrow L_?$ as follows:

(i) $\Phi(p) = p$, where $p \in P$;

(ii) $\Phi(\neg\alpha) = \sim\Phi(\alpha)$;

(iii) $\Phi(\alpha \ominus \beta) = \Phi(\alpha) \oplus \Phi(\beta)$, where $\ominus \in \{\wedge, \vee, \rightarrow\}$.

(iv) $\Phi(\Diamond\alpha) = \Phi(\alpha)?$

(v) $\Phi(\Box\alpha) = \Phi(\alpha)!$

Theorem 5.3.14. Let $\alpha \in L_\Diamond$. If $\vdash_K \alpha$, then $\vdash_{K?} \Phi(\alpha)$.

Theorem 5.3.15. Let $\alpha \in L_\Diamond$. If $\models_K \alpha$, then $\models_{K?} \Phi(\alpha)$.

Theorems 5.3.14 and 5.3.15 state in a precise way what theorems 5.3.4 and 5.3.5 just indicated: when we consider only ~, paranormal modal logic is indistinguishable from normal modal logic. Since normal modal logic is a proper extension of classical logic, we have also that ~ behaves exactly like classical negation.

Definition 5.3.24. We define the functions Π and $\amalg$ of the form : $L_? \rightarrow L_\Diamond$ as follows:

(i) $\Pi(p) = \amalg(p) = p$;

(ii) $\Pi(\alpha?) = \Diamond\Pi(\alpha)$;

(iii) $\amalg(\alpha?) = \Box\amalg(\alpha)$;

(iv) $\Pi(\alpha!) = \Box\Pi(\alpha)$;

(v) $\amalg(\alpha!) = \Diamond\amalg(\alpha)$;

(vi) $\Pi(\neg\alpha) = \neg\amalg(\alpha)$;

(vii) $\amalg(\neg\alpha) = \neg\Pi(\alpha)$;

(viii) $\Pi(\alpha \oplus \beta) = \Pi(\alpha) \oplus \Pi(\beta)$, where $\oplus \in \{\wedge, \vee, \rightarrow\}$;

(ix) $\amalg(\alpha \oplus \beta) = \amalg(\alpha) \oplus \amalg(\beta)$, where $\oplus \in \{\wedge, \vee\}$;

(x) $\amalg(\alpha \rightarrow \beta) = \Pi(\alpha) \rightarrow \amalg(\beta)$.

Definition 5.3.25. We define the functions Δ and ∇ of the form : $L_\Diamond \rightarrow L_?$ as follows:

(i) $\Delta(p) = \nabla(p) = p$;

(ii) $\Delta(\Diamond\alpha) = \Delta(\alpha)?$;

(iii) $\nabla(\Diamond\alpha) = \nabla(\alpha)!$;

(iv) $\Delta(\Box\alpha) = \Delta(\alpha)!$;

(v) $\nabla(\Box\alpha) = \nabla(\alpha)?$;

(vi) $\Delta(\neg\alpha) = \neg\nabla(\alpha)$;

(vii) $\nabla(\neg\alpha) = \neg\Delta(\alpha)$;

(viii) $\Delta(\alpha \oplus \beta) = \Delta(\alpha) \oplus \Delta(\beta)$, where $\oplus \in \{\wedge, \vee, \rightarrow\}$;

(ix) $\nabla(\alpha \oplus \beta) = \nabla(\alpha) \oplus \nabla(\beta)$, where $\oplus \in \{\wedge, \vee\}$;

(x) $\nabla(\alpha \rightarrow \beta) = \Delta(\alpha) \rightarrow \nabla(\beta)$.

Definition 5.3.26. Let $A \subseteq L_?$ and $B \subseteq L_\diamond$.

(i) $\Pi(A) = \{\Pi(\alpha) \mid \alpha \in A\}$;

(ii) $\amalg(A) = \{\amalg(\alpha) \mid \alpha \in A\}$;

(iii) $\Delta(B) = \{\Delta(\alpha) \mid \alpha \in B\}$;

(iv) $\nabla(B) = \{\nabla(\alpha) \mid \alpha \in B\}$.

Definitions 5.3.25 and 5.3.26 formalize a translation that comes naturally when we look at the semantics of normal and paranormal modal logics. With the help of them, we will be able to lay down important results concerning the normal and paranormal relations of deductibility and logical consequence and, in a more general way, the very expressiveness of both logics.

Theorem 5.3.16. Let $A,B \subseteq L_\diamond$ and $\alpha \in L_\diamond$. $A \div B \vdash_K \alpha$ iff $\Delta(A) \div \Delta(B) \vdash_{K?} \Delta(\alpha)$.

Theorem 5.3.17. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \div B \vdash_{K?} \alpha$ iff $\Pi(A) \div \Pi(B) \vdash_K \Pi(\alpha)$.

Theorem 5.3.18. Let $A,B \subseteq L_\diamond$ and $\alpha \in L_\diamond$. $A \div B \vDash_K \alpha$ iff $\Delta(A) \div \Delta(B) \vDash_{K?} \Delta(\alpha)$.

Theorem 5.3.19. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \div B \vDash_{K?} \alpha$ iff $\Pi(A) \div \Pi(B) \vDash_K \Pi(\alpha)$.

Theorems 5.3.16-5.3.17 and 5.3.18-5.3.19 show, respectively, that from a proof theoretical and a semantic point of view, K and $K_?$ are translatable into each other. In other words, while by using the function Δ we can translate any inferential relation in K into an inferential relation in $K_?$, by using Π we can translate any inferential relation in $K_?$ into an inferential relation in K. As a consequence of this, both K and $K_?$ can be defined through each other: with paranormal modal logic at hand we can obtain normal modal logic and vice versa.

5.4 Other Paranormal Modal Logics

In this section we will show how $K_?$ can be extended in such a way as to obtain other paranormal modal logics. In the next subsection we consider how by adding extra axioms on the axiomatic side or by restricting the set of frames on the semantic side we can do this. The procedure is exactly identical to the way we extend K in order to obtain other normal modal logics. In Subsection 5.4.2 we will proceed to consider first order paranormal modal logic. Since the definition of other first-order systems is identical to the propositional case, we will just consider first-order paranormal modal logic $K_?$. Finally, in Subsection 5.4.3, we will consider multi-modal logics which contain both normal and paranormal modal operators. We call them multi-normal modal logics.

5.4.1 Extensions of $K_?$

In this subsection we present just some of the main paranormal modal logics. As it will become clear when we start our exposition, for each normal modal system N there is a corresponding paranormal system $N_?$. And the way $N_?$ is obtained from $K_?$ is identical from the way N is obtained from K. For instance, in the same say that we obtain T from K by taking into account only reflexive frames, on the semantic side, and by adding the axiom schema $\Box\alpha \rightarrow \alpha$ to K's axiomatic, on the axiomatic one, we obtain $T_?$ from $K_?$ by restricting ourselves to reflexive frames and by adding the axiom schema $\alpha! \rightarrow \alpha$ to $K_?$'s axiomatic. Therefore, given the extensive literature on normal modal logic, the exposition of the many paranormal modal systems we can obtain from $K_?$ is practically unnecessary. Being aware of the great diversity of meanings that each one of these logics give to the modal operators ! and ? (that, in principle, is as great as the diversity of meanings the several normal modal logics attribute to $\Box$ and $\Diamond$), we will postpone to the next chapter the consideration of which one of them best formalizes the notions of skeptical plausibility and credulous plausibility according to the applications we have in mind.

The System $D_?$

Definition 5.4.1. Let $\Im$ be a language and ϑ a ?-modal language basis. The $D_?$ *axioms* $\Sigma_{K?}$ in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying the following schema of formula:

$D_?$: $\alpha! \rightarrow \alpha?$

Definition 5.4.2. Let $K_? = <\vartheta_?, \Omega_?, \Gamma_K, \Sigma_{K?}>$. The propositional paranormal modal logic $D_?$ is the propositional modal system $<\vartheta_?, \Omega_?, \Gamma_D, \Sigma_{K?} \cup \Sigma_{D?}>$, where Γ_D is the set of all *idealized* frames and $\Sigma_{D?}$ are the $D_?$ axioms in $L_?$.

Theorem 5.4.1. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \div B \vdash_{D?} \alpha$ iff $A \cup \Sigma_{D?} \div B \vdash_{K?} \alpha$.

Theorem 5.4.2. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \div B \vDash_{D?} \alpha$ iff $A \cup \Sigma_{D?} \div B \vDash_{K?} \alpha$.

Theorem 5.4.3. $D_?$ is sound and complete.

Theorem 5.4.4. All formulae of $L_?$ that satisfy one of the following schemas of formula are $D_?$-theorems (and consequently $D_?$-valid.)

$(\alpha \rightarrow \alpha)?$

$((\alpha \rightarrow \beta)! \rightarrow (\alpha! \rightarrow \beta!))?$

The System $T_?$

Definition 5.4.3. Let $\Im$ be a language and ϑ a ?-modal language basis. The $T_?$ *axioms* $\Sigma_{K?}$ in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying the following schema of formula:

$T_?$: $\alpha! \rightarrow \alpha$

Definition 5.4.4. Let $K_? = <\vartheta_?, \Omega_?, \Gamma_K, \Sigma_{K?}>$. The propositional paranormal modal logic $T_?$ is the propositional modal system $<\vartheta_?, \Omega_?, \Gamma_T, \Sigma_{K?} \cup \Sigma_{T?}>$, where Γ_T is the set of all *reflexive* frames and $\Sigma_{T?}$ are the $T_?$ axioms in $L_?$.

Theorem 5.4.5. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \div B \vdash_{T?} \alpha$ iff $A \cup \Sigma_{T?} \div B \vdash_{K?} \alpha$.

Theorem 5.4.6. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \div B \models_{T?} \alpha$ iff $A \cup \Sigma_{T?} \div B \models_{K?} \alpha$.

Theorem 5.4.7. $T_?$ is sound and complete.

Theorem 5.4.8. All formulae of $L_?$ that satisfy one of the following schemas of formula are $T_?$-theorems (and consequently $T_?$-valid.)

$\alpha \rightarrow \alpha?$

$(\alpha \rightarrow \alpha!)?$

$\alpha! \rightarrow \alpha?$

The System $B_?$

Definition 5.4.5. Let $\Im$ be a language and ϑ a ?-modal language basis. The $B_?$ *axioms* $\Sigma_{B?}$ in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying the following schema of formula:

$B_?$: $\alpha \rightarrow \alpha?!$

Definition 5.4.6. Let $K_? = <\vartheta_?, \Omega_?, \Gamma_{K?}, \Sigma_K>$. The propositional paranormal modal logic $B_?$ is the propositional modal system $<\vartheta_?, \Omega_?, \Gamma_B, \Sigma_{K?} \cup \Sigma_{T?} \cup \Sigma_{B?}>$, where Γ_B is the set of all *reflexive* and *symmetric* frames, $\Sigma_{T?}$ are the $T_?$ axioms in $L_?$ and $\Sigma_{B?}$ the $B_?$ axioms in $L_?$.

Theorem 5.4.9. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \div B \vdash_{B?} \alpha$ iff $A \cup \Sigma_{T?} \cup \Sigma_{B?} \div B \vdash_{K?} \alpha$.

Theorem 5.4.10. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \dot{-} B \vDash_{B?} \alpha$ iff $A \cup \Sigma_{T?} \cup \Sigma_{B?} \dot{-} B \vDash_{K?} \alpha$.

Theorem 5.4.11. $B_?$ is sound and complete.

The System $S4_?$

Definition 5.4.7. Let $\Im$ be a language and ϑ a ?-modal language basis. The $4_?$ *axioms* $\Sigma_{4?}$ in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying the following schema of formula:

$4_?$: $\alpha! \rightarrow \alpha!!$

Definition 5.4.8. Let $K_? = <\vartheta_?, \Omega_?, \Gamma_{K?}, \Sigma_K>$. The propositional paranormal modal logic $S4_?$ is the propositional modal system $<\vartheta_?, \Omega_?, \Gamma_{S4}, \Sigma_{K?} \cup \Sigma_{T?} \cup \Sigma_{4?}>$, where Γ_{S4} is the set of *reflexive* and *transitive* frames, $\Sigma_{T?}$ are the $T_?$ axioms in $L_?$ and $\Sigma_{4?}$ the $4_?$ axioms in $L_?$.

Theorem 5.4.12. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \dot{-} B \vdash_{S4?} \alpha$ iff $A \cup \Sigma_{T?} \cup \Sigma_{4?} \dot{-} B \vdash_{K?} \alpha$.

Theorem 5.4.13. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \dot{-} B \vDash_{S4?} \alpha$ iff $A \cup \Sigma_{T?} \cup \Sigma_{4?} \dot{-} B \vDash_{K?} \alpha$.

Theorem 5.4.14. $S4_?$ is sound and complete.

Theorem 5.4.15. All formulae of $L_?$ that satisfy one of the following schemas of formula are $S4_?$-theorems (and consequently $S4_?$-valid.)

$\alpha?? \rightarrow \alpha?$	$\alpha?!? \rightarrow \alpha?$
$\alpha! \leftrightarrow \alpha!!$	$\alpha? \leftrightarrow \alpha??$
$\alpha?! \leftrightarrow \alpha?!?!$	$\alpha!? \leftrightarrow \alpha!?!?$

The System $S5_?$

Definition 5.4.9. Let $K_? = <\vartheta_?, \Omega_?, \Gamma_K, \Sigma_{K?}>$. The propositional paranormal modal logic $S5_?$ is the propositional modal system $<\vartheta_?, \Omega_?, \Gamma_{S5}, \Sigma_{K?} \cup \Sigma_{T?} \cup \Sigma_{B?} \cup \Sigma_{4?}>$, where Γ_{S5} is the set of *reflexive, transitive* and *symmetric* frames, $\Sigma_{T?}$ are the $T_?$ axioms in $L_?$, $\Sigma_{B?}$ the $B_?$ axioms in $L_?$ and $\Sigma_{4?}$ the $4_?$ axioms in $L_?$.

Theorem 5.4.16. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \dot{-} B \vdash_{S5?} \alpha$ iff $A \cup \Sigma_{T?} \cup \Sigma_{B?} \cup \Sigma_{4?} \dot{-} B \vdash_{K?} \alpha$.

Theorem 5.4.17. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \dot{-} B \vDash_{S5?} \alpha$ iff $A \cup \Sigma_{T?} \cup \Sigma_{B?} \cup \Sigma_{4?} \dot{-} B \vDash_{K?} \alpha$.

Theorem 5.4.18. $S5_?$ is sound and complete.

Theorem 5.4.19. All formulae of $L_?$ that satisfy one of the following schemas of formula are $S5_?$-theorems (and consequently $S5_?$-valid.)

$\alpha? \leftrightarrow \alpha?!$	$\alpha! \leftrightarrow \alpha!?$
$\alpha? \leftrightarrow \alpha??$	$\alpha! \leftrightarrow \alpha!!$
$(\alpha \vee \beta!)! \leftrightarrow (\alpha! \vee \beta!)$	$(\alpha \vee \beta?)! \leftrightarrow (\alpha! \vee \beta?)$
$(\alpha \wedge \beta?)? \leftrightarrow (\alpha? \wedge \beta?)$	$(\alpha \wedge \beta!)? \leftrightarrow (\alpha? \wedge \beta!)$

5.4.2 First-order Paranormal Modal Logic

Definition 5.4.10. Let F = <W,R,...> be an n-frame. A *first-order modal interpretation* ν in F, which is a modal interpretation in $\mathcal{L}$ and F, is a quadruple $<D,V_C,V_F,V_R>$ where D is a function which maps each $w \in W$ to some non-empty set called the domain of *w*, V_C is a function which assigns to each $w \in W$ and $c \in K_C$ an element of D(w), V_F is a function which assigns to each n-ary function symbol $f \in K_F$ and $w \in W$ a function from $D(w)^n$ to D(w) and V_R is a function which assigns to each n-ary relation symbol $r \in K_R$ and world $w \in W$ a subset of $D(w)^n$.

Definition 5.4.11. Let M = <W,R,...,ν> be an n-model. We say M is a *first-order model* of arity *n* (or simply a first-order n-model) if ν is a first-order modal interpretation.

Definition 5.4.12. Let $F = <W,R_1,\ldots,R_n>$ be an n-frame and $\nu = <D,V_C,V_F,V_R>$ a first-order modal interpretation in F. We say ν is *monotonic* iff, for every $w,w' \in W$, if wR_iw' then $D(w) \subseteq D(w')$, for any i=1,...,n. We call the first-order n-model M based on F a *monotonic first-order n-model.*

Definition 5.4.13. Let F = <W,R,...> be an n-frame and $\nu = <D,V_C,V_F,V_R>$ a first order modal interpretation in F. We say ν is a *rigid* first order modal interpretation iff, for each $c \in K_c$ and $w,w' \in W$, $V_C(w,c) = V_C(w',c)$ and for every $f \in K_F$ and $w,w' \in W$, $V_F(f,w) = V_F(f,w')$. We call the first-order n-model M based on F a *rigid first-order n-model.*

From now on in this section, we will consider only monotonic and rigid first-order n-models in such a way that, when we speak of a first-order n-model, we are meaning a monotonic and rigid first-order n-model.

Definition 5.4.14. Let M = <W,R,..., ν> be a first-order n-model. An *assignment* in M is a function *s* that assigns to each $x \in K_V$ an element s(x). (It is not required that s(x) be in the domain of every world.) We write s[x|a] for the mapping that is like s on all variables except x and which maps x to a[20].

Definition 5.4.15. Let M = <W,R,…,ν> be a first order n-model with $\nu = \langle D, V_C, V_F, V_R \rangle$, $w \in W$ a world of M and s an assignment in M. The *denotation function* $\Phi_{M,w,s}$ is defined as follows:

(i) If $c \in K_C$ then $\Phi_{M,w,s}(c) = V_C(w,c)$;

(ii) If $x \in K_V$ then $\Phi_{M,w,s}(x) = s(x)$;

(iii) If $f \in K_F$ and $t_1, \ldots, t_n$ are terms in K, then $\Phi_{M,w,s}(f(t_1, \ldots, t_n)) = V_F(w,f)(\Phi_{M,w,s}(t_1), \ldots, \Phi_{M,w,s}(t_n))$.

Definition 5.4.16. Let ϑ be a ?-modal logic basis of arity n. A *first-order* Ω_k*-modal valuation* Ω in ϑ and a *first-order* $\mho_k$*-modal valuation* $\mho$ in ϑ, which will also be referred to as the *first-order max-min* k*-modal valuations* in ϑ, are max-min k-modal valuations in $\mathcal{L}$ and ϑ, which, given a first-order n-model M = <W,R,…,ν> with $\nu = \langle D, V_C, V_F, V_R \rangle$, an assignment s in M, a world $w \in W$, any formulae $\alpha \in \mathcal{L}_\vartheta$, any m-ary relation symbol $r \in K_R$ and any m-tuple of terms in K $t_1, \ldots, t_n$, satisfy the following conditions:

(i) $\Omega_{M,w,s}(r(t_1, \ldots, t_n)) = \mho_{M,w,s}(r(t_1, \ldots, t_n)) = 1$ iff $\langle \Phi_{M,w,s}(t_1), \ldots, \Phi_{M,w,s}(t_n) \rangle \in V_R(w,r)$;

(ii) $\Omega_{M,w,s}(\forall x \alpha) = 1$ iff, for all $d \in D(w)$, $\Omega_{M,w,s[x,d]}(\alpha) = 1$;

(iii) $\mho_{M,w,s}(\forall x \alpha) = 1$ iff, for all $d \in D(w)$, $\mho_{M,w,s[x,d]}(\alpha) = 1$.

A *first-order* Ω_k*-valuation* Ω in ϑ and a *first-order* $\mho_k$*-valuation* $\mho$ in ϑ have as parameters a propositional n-model M, a world w of M, assignment s in M and a formula α of $\mathcal{L}_\vartheta$.

Definition 5.4.17. Let ϑ be an n-modal logic basis. The *quantifier axioms* Σ_Q in $\mathcal{L}_\vartheta$ is the set composed by all formulae of $\mathcal{L}_\vartheta$ satisfying the following schema of formula:

Q: $\forall x \alpha(x) \rightarrow \alpha(t)$, where the substitution of t for x is admissible

Definition 5.4.18. The first-order paranormal modal logic $K_?^1$ is the first-order modal system $\langle \vartheta_?, \Omega^1, \Gamma, \Sigma \rangle$ where $\vartheta_? = \langle \{!,?\}, \{!\} \rangle$ is the paranormal modal logic basis, Ω^1 is the first-order Ω_1-modal valuation in $\vartheta_?$, Γ is the set of all frames and

$$\Sigma = \Sigma_P \cup \Sigma_A \cup \Sigma_N \cup \Sigma_M \cup \Sigma_{K?} \cup \Sigma_Q,$$

[20] Fitting (1993), p. 422.

where Σ_P are the positive classical axioms in $\mathcal{L}_?$, Σ_A the paranormal classical axioms in $\mathcal{L}_?$, Σ_N the non-positive additional classical axioms in $\mathcal{L}_?$, Σ_M the paranormal modal axioms in $\mathcal{L}_?$, $\Sigma_{K?}$ the $K_?$ axioms in $\mathcal{L}_?$ and Σ_Q the quantifier axioms in $\mathcal{L}_?$.

Theorem 5.4.20. $K_?^1$ is sound and complete (that is, for any $A,B \subseteq \mathcal{L}_?$ and $\alpha \in \mathcal{L}_?$, $A \div B \vdash_{K1?} \alpha$ iff $A \div B \vDash_{K1?} \alpha$.)

About $K_?^1$-theorems and $K_?^1$-valid formulae, things work exactly like in propositional paranormal modal logic $K_?$: all theorems from 5.3.2 to 5.3.8 restated in terms of $K_?^1$ and $\mathcal{L}$ are valid. More generally, and that is a quite trivial point, the differences between $K_?$ and $K_?^1$ start only when we consider quantified formulae. A not so straightforward observation is that, both from a proof-theoretical and from a semantic point of view, the differences between $K_?$ and $K_?^1$ are equivalent to the differences between the propositional and first-order cases of normal modal logic K. More generally, we can say that, given a specific propositional paranormal modal logic $P_?$, the way it is extended into first-order paranormal modal logic $P_?^1$ is exactly the same as the way propositional normal modal logic P is extended into first-order normal modal logic P^1. Therefore, given the amount of literature about the connections between (normal) propositional and first-order modal logic, we will proceed without elaborating further on the first-order features of $K_?^1$.

An important aspect of first-order modal logic concerns how one will deal with the designation of terms in the several worlds. As far as we are concerned, we are considering only *monotonic* and *rigid* first-order models. This means first that every constant symbol *c* and function symbol *f* name the same *things* no matter what plausible world we are considering and second that everything that exists in a given world also exists in any world accessible from it. From a proof-theoretical point of view, this is the sort of model we obtain when we extend propositional modal logic into first-order modal logic through the simplest way: by adding axiom Q and generalization rule. In this formulation, even though the converse Barcan formula holds ($\Box\forall x\alpha \rightarrow \forall x\Box\alpha$), the so-called Barcan formula ($\forall x\Box\alpha \rightarrow \Box\forall x\alpha$) does not. However, when we consider logics with symmetric frames such as S5 and B, both Barcan formulae are valid. The justification for choosing this specific way of extending $K_?$ in $K_?^1$ rests on the same point of the preceding paragraph. Since it is not our purpose to get into first-order details such as technical and philosophical discussions about the way quantifiers should be treated, for the sake of completeness we just picked up the simplest way of extending propositional paranormal modal logic into first-order paranormal modal logic.

5.4.2 Multi-normal Modal Logic

What we call multi-normal modal logic is any normal, in the traditional sense, modal logic that contains both normal and paranormal modalities. It therefore includes modal systems of arity greater than or equal to 2. For the sake of simplicity, we will consider here only the simplest case where there exist two pairs of dual modal operators, one normal and the other paranormal.

Definition 5.4.19. The *multi-normal modal logic basis* $\vartheta_{?\diamond}$ is the pair $<\{\Box,\diamond,!,?\},\{\Box,!\}>$. $\Box$ and $\diamond$ are used in a pre-fixed notation and ! and ? are used in a post-fixed notation. Given a language $\Im$, we refer to the modal logic language based on $\Im$ and $\vartheta_{?\diamond}$ by $\Im_{?\diamond}$.

In multi-normal modal logic, ! and ? are the paranormal modal operators and $\Box$ and $\diamond$ the normal ones. Therefore, the same negation operator $\neg$ will behave sometimes as a modality-dependent paranormal negation and sometimes as a (modal) classical one.

Definition 5.4.20. Let $\Im$ be a language. A *multi-normal modal* Ω*-valuation* $\Omega_{?\diamond}$ in $\Im$ and a *multi-normal modal* $\mho$*-valuation* $\mho_{?\diamond}$ in $\Im$, which will also be referred to as the *multi-normal max-min modal valuations* in $\Im$, are, respectively, a Ω_2*-modal valuation* in $\Im$ and $\vartheta_{?\diamond}$ and a $\mho_2$*-modal valuation* in $\Im$ and $\vartheta_{?\diamond}$ which, given a 2-model $M = <W,R_\diamond,R_?,v>$, a world $w\in W$ and any two formulae $\alpha,\beta\in\Im_{?\diamond}$ and possibly other parameters, satisfy the following conditions:

(i) $\Omega_{?\diamond M,w,...}(\diamond\alpha) = 1$ iff, for some $w'\in W$ such that $wR_\diamond w'$, $\Omega_{?\diamond M,w',...}(\alpha) = 1$;

(ii) $\mho_{?\diamond M,w,...}(\diamond\alpha) = 1$ iff, for some $w'\in W$ such that $wR_\diamond w'$, $\mho_{?\diamond M,w',...}(\alpha) = 1$;

(iii) $\Omega_{?\diamond M,w,...}(\Box\alpha) = 1$ iff, for all $w'\in W$ such that $wR_\diamond w'$, $\Omega_{?\diamond M,w',...}(\alpha) = 1$;

(iv) $\mho_{?\diamond M,w,...}(\Box\alpha) = 1$ iff, for all $w'\in W$ such that $wR_\diamond w'$, $\mho_{?\diamond M,w',...}(\alpha) = 1$.

A model of multi-normal modal logic is then a 2-model M with two accessibility relations where one is used to evaluate ? and !-marked formulae and the other to evaluate $\Box$ and $\diamond$-marked ones.

Definition 5.4.21. The *first-order multi-normal modal* Ω*-valuation* is the modal valuation $\Omega_{?\diamond}^1$ which is both a first-order Ω_2-modal valuation in $\vartheta_{?\diamond}$ and a multi-normal modal Ω-valuation in $\mathcal{L}$.

Definition 5.4.22. Let $\Im$ be a language and ϑ a $\diamond$-modal logic basis. The *negation necessity axioms* Σ_{NN} in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying the following schema of formula:

NN: $\sim\Box\sim\alpha \leftrightarrow \neg\Box\neg\alpha$

Axiom NN is needed in multi-normal modal logic in order to set the normal behavior of $\Box$ (and, consequently, of $\diamond$.)

Definition 5.4.23. The first-order multi-normal modal logic $K_?K$ is the first-order 2-modal system $<\vartheta_{?\diamond}, \Omega_{?\diamond}{}^1, \Gamma, \Sigma>$ where $\vartheta_{?\diamond} = <\{\Box,\diamond,!,?\},\{\Box,!\}>$ is the multi-normal modal logic basis, $\Omega_{?\diamond}{}^1$ is the fist-order multi-normal modal Ω-valuation, Γ is the set of all 2-frames and

$$\Sigma = \Sigma_P \cup \Sigma_A \cup \Sigma_N \cup \Sigma_M \cup \Sigma_{K?} \cup \Sigma_{NP} \cup \Sigma_K \cup \Sigma_Q \cup \Sigma_{NN}$$

where Σ_P are the positive classical axioms in $\mathcal{L}_{?\diamond}$, Σ_A the paranormal classical axioms in $\mathcal{L}_{?\diamond}$, Σ_N the non-positive additional classical axioms in $\mathcal{L}_{?\diamond}$, Σ_M the paranormal modal axioms in $\mathcal{L}_{?\diamond}$, $\Sigma_{K?}$ the $K_?$ axioms in $\mathcal{L}_{?\diamond}$, Σ_{NP} the possibility-necessity axioms in $\mathcal{L}_{?\diamond}$, Σ_K the K axioms in $\mathcal{L}_{?\diamond}$, Σ_Q the quantifier axioms in $\mathcal{L}_{?\diamond}$ and Σ_{NN} the negation necessity axioms in $\mathcal{L}_{?\diamond}$.

Theorem 5.4.21. $K_?K$ is sound and complete.

$K_?K$ is the most basic first-order multi-normal modal system and, strictly speaking, does not assign any "complete" meaning to its modal symbols. If we want to take $\Box$ and $\diamond$ along with their traditional meanings of necessity and possibility, and ! and ? as meaning our skeptical plausibility and credulous plausibility, it seems that at least the axiom schema above should be added to $K_?K$.

Definition 5.4.24. Let $\Im$ be a language and $\vartheta_{?\diamond}$ the multi-normal modal logic basis. The *possibility-plausibility axioms* Σ_{PP} in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying the following schema of formula:

PP: $\Box\alpha \rightarrow \alpha!$

Intuitively, PP means that if α is necessary, then it is also skeptically plausible. From PP we can derive $\alpha? \rightarrow \diamond\alpha$, which means that if α is possible, then it is (credulously) plausible. From a semantic point of view, this implies taking only those frames $<W,R_\diamond,R_?>$ which, for each $w,w' \in W$, if $wR_\diamond w'$, then $wR_?w$. Calling $R_\diamond(w) = \{w' \mid wR_\diamond w'\}$ the set of possible worlds accessible from w

and $R_?(w) = \{w' \mid wR_?w'\}$ the set of plausible worlds accessible from *w*, we have that for each world $w \in W$, every plausible world (of *w*) is also one of a plausible world (of *w*), but not the converse.

Definition 5.4.25. Let $<\vartheta_{?\diamond}, \Omega_{?\diamond}{}^1, \Gamma_{KK?}, \Sigma_{KK?}>$ be first-order multi-modal logic $K_?K$. The first-order multi-normal modal logic $PPK_?K$ is the first-order 2-modal system $<\vartheta_{?\diamond}, \Omega_{?\diamond}{}^1, \Gamma_{PP}, \Sigma_{KK?} \cup \Sigma_{PP}>$, where Γ_{PP} is the set of all 2-frames $<W,R_\diamond,R_?>$ which, for each $w,w' \in W$, if $wR_\diamond w'$, then $wR_?w'$ (we call the members of Γ_{PP} PP-frames), and Σ_{PP} are the possibility-plausibility axioms in $\mathcal{L}_{?\diamond}$.

Theorem 5.4.22. $PPK_?K$ is sound and complete.

Theorem 5.4.23. The following schemas of relations between sets of formulas and formula are correct.

$\{\Box\alpha\} \vdash_{PPK?K} \alpha!$	$\{\Box\alpha\} \models_{PPK?K} \alpha!$
$\{\alpha?\} \vdash_{PPK?K} \diamond\alpha$	$\{\alpha?\} \models_{PPK?K} \diamond\alpha$

According to this interpretation of our modal symbols, $PPK_?K$ can be said to be the most basic logic of plausibility and possibility. From it, many logics such as $PPB_?S4$ and $PPS5_?S5$ can be defined. In the next chapter, we will use a specific extension of $PPK_?K$ to build our logic of plausibility. While we will still of course interpret ! and ? as our skeptical plausibility and credulous plausibility, □ will be taken as meaning epistemological certainty rather than necessity.

CHAPTER 6

A NEW APPROACH TO THE LOGIC OF INDUCTION AND PLAUSIBILITY

In this chapter we will continue the formal exposition started in the previous chapter and lay down our solution for the concept explication problem of induction and plausibility. We will follow the kind of explication delineated in the previous chapter and, rather than presenting a sole inductive system as our induction *explicatum*, introduce several logics of induction, which, taken together, will be our explication of the notion of induction. By doing this we will also be providing what we have called a purely descriptive logic of induction. All that will be done in Sections 6.1 and 6.3. In Section 6.2 we adopt a less general approach and present one of the paranormal modal logics introduced in the previous chapter as our specific explication of the notion of plausibility. It will be this logic what will play the role of the logic of plausibility of the descriptive logics of induction we will introduce in Section 6.3. Finally, in Section 6.4 we pick one of these inductive logics and try to formalize some models of confirmation found in the literature of philosophy of science, including the so-called hypothetico-deductive model.

6.1 The Logic of Induction

In this section we will effectively make use of the basic framework introduced in Chapter 5 and lay down in a formal way what we have called a *descriptive logic of induction* (or simply a logic of induction.) We will try to follow the same sort of general approach outlined in the previous chapter and present our inductive logic in such a way that most of its elements will be taken as parameters (rather than fixed for good.) As a consequence of that, what will come up from this exposition will not be, strictly speaking, a logic of induction as we have defined in Chapter 4, but, we may say, a restatement of this definition inside a particular logicomathematical framework. In other words, through such logicomathematical notation we shall set in a precise way the components which form a logic of induction as well as how they are supposed to interact with one another. In the same way that we have presented the whole class of paranormal modal logics as an explication of the notion of plausibility, this inductive logic schema will be taken as our general induction *explicandum*. The exposition of structures that satisfy this schema and consequently can be taken as specific solutions for the concept explication problem of induction will be done in Section 6.3.

6.1.1 The Inductive Language and Inductive-Modal Language

Definition 6.1.1. Let $\Im$ be a language. The *inductive language* $\Im_{\succ}$ built over $\Im$ is defined as follows:

(i) If $\alpha \in \Im$ is such that it contains no one of the logical symbols of $\Im$, then $\alpha \in \Im_{\succ}$;

(ii) If $\oplus$ is a monadic logical symbol of $\Im$ along with one of its non-logical complements, if there is any, and $\alpha \in \Im_{\succ}$, then $(\oplus\alpha) \in \Im_{\succ}$;

(iii) If $\oplus$ is a dyadic logical symbol of $\Im$ and $\alpha, \beta \in \Im_{\succ}$, then $(\alpha \oplus \beta) \in \Im_{\succ}$;

(iv) If $\alpha, \beta, \varphi \in \Im_{\succ}$, then $(\alpha \succ \beta \not\lesssim \varphi) \in \Im_{\succ}$;

(v) Nothing else belongs to $\Im_{\succ}$.

We say that $<\succ, \not\lesssim>$ is a pair of triadic logical symbols of $\Im_{\succ}$. Formulae built with the help of these symbols, such as $\alpha \succ \beta \not\lesssim \varphi$, for example, are read as "α inductively implies β unless φ." They are trivially meant to capture the notion of inductive implication introduced in Chapter 4. The inspiration for its form comes from the analysis of default logics we have made in that chapter. α, which is called the *antecedent* of the inductive implication, represents what Reiter calls the prerequisite of the default; β, which will be called the *consequent* of the implication, plays the role of Reiter's consequent; and φ, which is named the *exception* of the inductive implication, corresponds to the negation of the semi-normal part of Reiter's default or the exception part of Pequeno's generalization. From now on we will refer to these formulae as *inductive implications*. For the sake of clarity, we will refer to all other formulae which are not inductive implications as *ordinary formulae*

As we have started explaining in Chapter 4, the conception of inductive implication in terms of antecedent, consequent and exception has important consequences for the notion of evidence. As we have mentioned in Chapter 2, traditionally the notion of evidence has been conceived in such a way that *e*'s inductively supporting *h* depends exclusively on the logical form of *e* and *h* and has nothing to do with the knowledge situation at hand. What will depend on the knowledge situation is whether or not we will be able to infer the plausibility of *h* from the truthfulness of *e*, which will not, it must be said, affect the relation of evidential support that is supposed to exist between *e* and *h*. In our case, since we have decided to take the description of the exception as an essential part of the task of specifying that *e* gives evidential support to *h*, the knowledge situation will be transformed into a third parameter of the relation of inductive support. This is so because in order for α to be an evidence for β (or to inductively imply β) as specified in formula $\alpha \succ \beta \not\lesssim \varphi$, the knowledge situation must be such that φ is not true in it. Otherwise, despite the truthfulness of α and $\alpha \succ \beta \precnsim \varphi$, we are not entitled to attribute any sort of inductive connection between α and β. A trivial consequence of this is that there will be no such thing as a sentence *e* being, due to its logical form or whatever, an evidence for *h*: *e* might be taken as evidence for *h* but only when a specific knowledge situation is considered.

Another aspect of considering the exception condition as part of the logical form of inductive implications concerns the necessity of having an external or purely mechanical account of inductive support. We have said in Chapter 4 that in order to deal with inductive inferences in a purely descriptive way we have to somehow "jump" the inside aspect of the relation of evidential support and consider only the external process according to which a piece of evidence inductively implies the plausibility of the hypothesis. In a nutshell, we have to deal exclusively with the problem of inductive detachment. As we have observed in Chapter 2, in order to properly detach a hypothesis from its evidences, it is necessary that some sort of total evidence condition be satisfied. After all, the purpose of such conditions is exactly to guarantee that the relation of inductive support will produce its "inferential fruits," that is to say, the plausibility of the hypothesis. Therefore, if we want to embody in such relation (or perhaps to reduce it to) the mentioned fruit, we have to consider the conditions which must be satisfied (or, equivalently, the state of affairs which cannot not be satisfied) for the hypothesis to be properly detached from the evidences. All this shall be formally stated in Subsection 6.1.3, when we lay down our definition of the notion of extension.

A very important difference between our approach and the nonmonotonic formalisms so far exposed is that in our language the inductive connectives play the same role as all other logical symbols. One should remember that in the nonmonotonic logics we have examined, defaults (or generalizations) can be built only from ordinary formulae and never from other defaults. Also, there is no possibility in those logics to interact defaults and formulae with the help of standard logical connectives. In our turn, due to the very way $\Im_{\succ}$ is constructed. it is possible to have inductive implications appearing as the consequent, antecedent or exception of another inductive implication ($\alpha \succ (\beta \succ \varphi \precnsim \lambda) \precnsim \phi$, $(\beta \succ \varphi \precnsim \lambda) \succ \alpha \precnsim \phi$ and $\phi \succ \alpha \precnsim (\beta \succ \varphi \precnsim \lambda)$, respectively.) It is also possible, with the help of the other logical symbols available, to have inductive implications and ordinary formulae interacting with each other. For example, the following strings are well-formed formulae: $\alpha \wedge (\varphi \succ \lambda \precnsim \beta)$, $(\alpha \succ \beta \precnsim \varphi) \rightarrow (\beta \rightarrow (\beta \succ \alpha \precnsim \lambda))$, $\alpha \succ (\alpha \wedge (\beta \succ \varphi \precnsim \lambda)) \succ (\phi \vee (\phi \succ \beta \precnsim \neg\alpha))$. As we have agreed in Chapter 4, this is the sort of representational mechanism we need to build calculi of inductive implication and models of confirmation.

Definition 6.1.2. Let $\Im$ be a language and $\vartheta = <\Theta', \Theta''>$ a modal logic basis. The *inductive modal language* $\Im_{\succ\vartheta}$ is the modal language based on $\Im_{\succ}$ and ϑ.

The above definition sets the sort of language our logic of induction will be based on: a modal language built over an inductive language. In the case where ϑ is a $\diamond$-modal logic basis for example, we will be able to construct sentences like $\Box\alpha \succ \beta \precsim \diamond\varphi$ or $\alpha \vee (\beta \succ \Box\varphi \succ \phi)$. As we will see later, we will also account for the important aspect of allowing epistemological constraints on the components of inductive implications. For instance, if ϑ is a ?-modal logic basis too and ? is taken as a formalization of the notion of inductive plausibility, then it is natural to require the consequents of all inductive implications to be of the form α?. Similarly, if besides interpreting α? as "α is plausible" we take $\Box\alpha$ as meaning "α is certain," then it will be possible to formalize the definition of induction presented in Chapter 2 according to which induction is the class of ampliative inferences which lead from certainty to plausibility: we have just to require all inductive implications to be of the form $\Box\alpha \succ \beta? \precsim \varphi$.

Definition 6.1.3. Let $\Im$ be a language, $\vartheta = <\Theta', \Theta''>$ a modal logic basis, and $\alpha,\beta \in \Im_{\succ\vartheta}$ two formulae. We define the following abbreviations in $\Im_{\succ\vartheta}$:

(i) $\alpha \succ \beta =_{def} \alpha \succ \beta \precsim \bot$;

(ii) $\alpha \precsim \beta =_{def} \top \succ \alpha \precsim \beta$;

(iii) $\alpha^{\circ} =_{def} \top \succ \alpha \precsim \bot$.

Definition 6.1.4. Let $\Im$ be a language, $\vartheta = <\Theta', \Theta''>$ a modal logic basis, $\theta \in \Theta'$ a modal operator and ϕ an inductive implication ($\phi \in \Im_{\succ\vartheta}$ and $\phi \notin \Im_{\vartheta}$). ϕ is θ-free[1].

6.1.2 The Logic of Plausibility and the Inductive-Plausible Language

Definition 6.1.5. Let $\Im$ be a language and $\Lambda^* = <\vartheta, \Sigma_1 \cup \ldots \cup \Sigma_m>$ an axiomatic n-modal system based on $\Im$ such that Σ_1 are the A_1 axioms in $\Im_{\vartheta}$, Σ_2 the A_2 axioms in $\Im_{\vartheta}$, ..., and Σ_m the A_m axioms in $\Im_{\vartheta}$. The *pseudo-inductive modal calculus* Λ^* based on $\Im$ and Λ^* is the axiomatic n-modal system $<\vartheta, \Sigma_{1\succ} \cup \ldots \cup \Sigma_{m\succ}>$ based on $\Im_{\succ}$, where Σ_1 are the A_1 axioms in $\Im_{\succ\vartheta}$, Σ_2 the A_2 axioms in $\Im_{\succ\vartheta}$, ... and Σ_m the A_m axioms in $\Im_{\succ\vartheta}$.

[1] In the case where ϑ is a ?-modal logic basis, this definition then complements definition 5.3.2.

Given a specific modal logic, a pseudo-inductive modal calculus is simply an extension of such logic (we call it a $\succ$-*extension*) in which inductive implications are added to the language and treated as atomic formulae. Given $\Im_{\succ\vartheta}$'s ability to represent constraints about inductive implications[2], a pseudo-inductive modal calculus provides us the deductive apparatus needed to represent what we have called calculus of inductive implication and model of confirmation. The term "pseudo-inductive" indicates that the calculus in question is deductive rather than inductive but nevertheless contains and reasons (deductively) about inductive implications.

Definition 6.1.6. Let A be a set. We define $\aleph(A)$ as the power set of the set of all finite sequences made out of the elements of A.

Definition 6.1.7. Let $\Im$ be a language. A *pseudo-inductive logic of plausibility* ψ based on $\Im$ is a quadruple $<\vartheta, \Sigma, \Theta_p, \Theta_t>$ where $\Lambda^* = <\vartheta, \Sigma>$ is a pseudo-inductive modal calculus based on $\Im$, with $\vartheta = <\Theta', \Theta''>$, and $\Theta_p, \Theta_t \in \aleph(\Theta')$ are two sets of modal operators called, respectively, the *inductive plausibility modal operators* and the *basic truth modal operators*. We also say that ψ is based on Λ^*.

The reason for classifying ψ as a pseudo-inductive logic is the same as why we have taken Λ^* a pseudo-inductive modal calculus. In contrast to Λ^* however, ψ has what we have called inductive plausibility modal operators and basic truth modal operators. They are meant to allow us to set constraints on the logical form of inductive implications as we have mentioned above. As one might expect, the inductive plausibility modal operators are intent to represent the plausible, refutable formulae inferred through inductive implications. In order to be distinguished from deductively obtained formulae, the consequent part of inductive implications will be necessarily marked with such modalities. The basic truth modal operators in their turn are meant to represent these irrefutable, deductively obtained formulae which will represent, we may say, the most basic "type of truth" we will deal with. For instance, if we really want to go deep into the idea that induction is in essence an epistemological notion and has no direct connection with ontological concepts such as the notion of truth, then it will be paradoxical to allow statements such as "α is true" to occur in our inductive theories in general and in the antecedent part of inductive implications in particular. One way to sort this out is to require *every formula* to be marked with a modal operator meant to represent a sort of basic epistemological truth. In the situation we have just

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[2] Such as, for example, the inductive implication transitivity axiom: $\alpha \succ \beta \rightarrow ((\beta \succ \varphi) \rightarrow (\alpha \succ \varphi))$.

considered above, this would be done by taking $\Box$ as meaning "is certain that" as the sole member of Θ_t.

As its name indicates, the pseudo-inductive logic of plausibility ψ contains the component of the logic of induction which we have named *logic of plausibility*. It is however not identical to it. Given a pseudo-inductive logic of plausibility $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ based on $\Im$, the modal system Λ^* which $\Lambda^* = <\vartheta, \Sigma>$ is based on is what could be more fairly taken as the logic of plausibility, for it is, we may say, the core of the logical machinery responsible for reasoning about plausible facts. Λ^* and ψ would be better taken as sorts of extended logics of plausibility able to reason (monotonically) about plausible facts obtained through inductive implications as well as about inductive implications themselves. It is interesting to observe that we can classify Λ^* and Λ^* as logics of plausibility only due to their being part of ψ. That is to say, only because of the interpretation we have given to the members of Θ_p and their intent use is that we can claim Λ^* to be at least in principle able to reason about inductively obtained formulae and consequently entitled to be classified as a formalization of the notion of plausibility.

The reason for defining Θ_p and Θ_t as sets of sequences of modal operators and not simply as a sets of modal operators is that besides primitive operators, we want also to allow derived operators (which may contain more than one primitive modal operator) to be taken as inductive plausibility modal operators and basic truth modal operators. For instance, in the next section we will introduce a sort of credulous plausibility composed by ! and ? ($\alpha\mathbf{?} =_{def} \alpha!?$) which has some advantages over the primitive symbol ? and consequently may be worthy of being used as an inductive plausibility modal operator.

Another important consequence of having Θ_p and Θ_t is that since formulae marked with Θ_p's operators will correspond to nonmonotonically inferred formulae, by the use of inductive implications we will be able to make explicit the refutable aspect of plausible formulae we have mentioned in Chapter 2. There we have said that what effectively distinguishes certain statements from plausible ones is that while the first is irrefutable and very hard to be given up, the second is intrinsically refutable, defeasible and therefore susceptible to be reconsidered. If for example we take $\Theta_p = \{?\}$ and $\Theta_t = \{\Box\}$, $\Box$ meaning "it is certain that," then the mentioned distinction will become evident: while in order for "certain formulae" to be reconsidered we have to manually alter the theory, formulae of the form $\alpha?$ are automatically reconsidered as soon as some incompatibility with the inductive implications that introduced them arises. This will be formally described below in definition 6.1.11.

The exact way which inductive plausibility modal operators and basic truth modal operators are used is shown below.

Definition 6.1.8. Let $\Im$ be a language, $\vartheta = <\Theta',\Theta''>$ a modal logic basis and $\Theta \in \aleph(\Theta')$ a set of modal operators. The notion of Θ-*formula* is defined as follows:

(i) If $\alpha \in \Im_\vartheta$ and $\Theta = \varnothing$, then α is a Θ-formula;

(ii) If $\alpha \in \Im_\vartheta$ and $\theta \in \Theta$, then $\theta\alpha$ is a Θ-formula[3];

(iii) Nothing else is a Θ-formula.

Definition 6.1.9. Let $\Im$ be a language, $\vartheta = <\Theta',\Theta''>$ a modal logic basis and $\Theta_p,\Theta_t \in \aleph(\Theta')$ two sets of modal operators. The notions of $\Theta_p\Theta_t$-*inductive formula* and $\Theta_p\Theta_t$-*formula* are defined as follows:

(i) If $\alpha \in \Im_\vartheta$ is a Θ_p-formula, then α is a $\Theta_p\Theta_t$-inductive formula;

(ii) If $\alpha \in \Im_{\succ\vartheta}$ is a $\Theta_p\Theta_t$-inductive formulae, then $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$ and $\forall x\alpha$ are also $\Theta_p\Theta_t$-inductive formulae;

(iii) If $\alpha \in \Im_\vartheta$ is a Θ_t-formula, then α is a $\Theta_p\Theta_t$-formula;

(iv) $\top$ is a $\Theta_p\Theta_t$-formula and $\bot$ is a $\Theta_p\Theta_t$-inductive formula;

(v) If $\alpha,\beta \in \Im_{\succ\vartheta}$ are $\Theta_p\Theta_t$-formulae, then $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$, $\neg\alpha$ and $\forall x\alpha$ are also $\Theta_p\Theta_t$-formulae;

(vi) If $\alpha \in \Im_{\succ\vartheta}$ is a $\Theta_p\Theta_t$-formula, $\beta \in \Im_{\succ\vartheta}$ is a $\Theta_p\Theta_t$-inductive formula and $\varphi \in \Im_{\succ\vartheta}$ is such that it is either a $\Theta_p\Theta_t$-formula or a $\Theta_p\Theta_t$-inductive formula, then $\alpha \succ \beta \precsim \varphi$ is both a $\Theta_p\Theta_t$-formula and a $\Theta_p\Theta_t$-inductive formula;

(vii) Nothing else is a $\Theta_p\Theta_t$-formula or a $\Theta_p\Theta_t$-inductive formula.

As we have said above, the purpose of inductive plausibility modal operators and basic truth modal operators is to restrict the form of inductive implications and ordinary formulae in such a way that our epistemological constraints will be satisfied. This is effectively materialized by the notions of ψ-inductive-plausible language and ψ-theory, to be presented below. As we will see right after that in Subsection 6.1.3, only ψ-theories will be allowed to use ψ's inductive inferential mechanism[4].

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[3] Here $\theta\alpha$ is the result of applying θ to α in such a way that the notations of *all* operators which compose θ are taken into account. For instance, if $\theta = \Box?$, then $\theta\alpha \equiv (\Box\alpha)?$; if $\theta = \Box!?\Diamond$, then $\theta\alpha \equiv \Diamond((\Box\alpha)!?)$.

[4] Regarding Θ_p, our policy does not of course apply to inductive implications themselves. If we consider nested inductive implications, that is, inductive implications with other inductive implications as their consequent, we will be interested in distinguishing only the consequent of the innermost inductive

Definition 6.1.10. Let $\Im$ be a language and $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\Im$. The ψ-*inductive-plausible language* $\Im_\psi$ is defined as follows: if $\alpha \in \Im_{\succ\vartheta}$ is a closed $\Theta_p\Theta_t$-formula[5], then $\alpha \in \Im_\psi$. We call any set $A \subseteq \Im_\psi$ a ψ-*theory*.

As the name indicates, given a pseudo-inductive logic of plausibility ψ, the corresponding ψ-inductive-plausible language $\Im_\psi$ will play the role of the inductive-plausible language of the logic of induction ψ will be related to.

6.1.3 The Relation of Inductive Consequence

Definition 6.1.11. Let $\Im$ be a language, $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\Im$ and $\Lambda^* = <\vartheta, \Sigma>$, with $\vartheta = <\Theta', \Theta''>$, and $A \subseteq \Im_{\succ\vartheta}$ a ψ-theory. The function $\Upsilon_{\psi,A}$, which assigns to each set of formulae $S \subseteq \Im_{\succ\vartheta}$ a set of formulae $\Upsilon_{\psi,A}(S) \subseteq \Im_{\succ\vartheta}$, is defined as follows. Let $S \subseteq \Im_{\succ\vartheta}$ be a set of formulae of $\Im_{\succ\vartheta}$. $\Upsilon_{\psi,A}(S)$ is the smallest set satisfying the following conditions:

(i) $A \subseteq \Upsilon_{\psi,A}(S)$;

(ii) If $\Upsilon_{\psi,A}(S) \vdash_{\Lambda^*} \alpha$ then $\alpha \in \Upsilon_{\psi,A}(S)$;

(iii) If $\alpha \succ \beta \precsim \varphi \in \Upsilon_{\psi,A}(S)$, $\alpha \in \Upsilon_{\psi,A}(S)$, $\varphi \notin S$ and $\sim\beta \notin S$[6], then $\beta \in \Upsilon_{\psi,A}(S)$.

Definition 6.1.12. Let $\Im$ be a language, $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\Im$, $A \subseteq \Im_\vartheta$ a ψ-theory and $E \subseteq \Im_{\succ\vartheta}$ a set of formulae. E is a ψ-*inductive extension* of A iff $E = \Upsilon_{\psi,A}(E)$.

Definitions 6.1.11 and 6.1.12 follow the same style of extension definition of default logics exposed by us in Chapter 4. As such, they contain the core of inductive logic's component which we have named *relation of inductive consequence*. Indeed, given a pseudo-inductive logic of induction ψ and a ψ-theory A possibly containing formulae of the form $\alpha \succ \beta \precsim \varphi$, the raison d'être of an ψ-inductive extension of A is to be a maximal consistent set of all inductive

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implications and not the inductive implications themselves. From now on when explaining aspects of our system, we will most of the time forget about these so-called nested inductive implications and consider only basic inductive implications (that is, inductive implications $\alpha \succ \beta \precsim \varphi$ such that β is an ordinary formula.).

[5] Here *closed formula* is taken, as usually, as formulae (be it ordinary formulae or inductive implications) with no free variable.

consequences that may be obtained from A. From this notion of ψ-inductive extension we define what shall be our inductive logic's relation of inductive consequence.

Definition 6.1.13. Let $\Im$ be a language, $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\Im$, $A \subseteq \Im_{\succ\vartheta}$ a ψ-theory and $\alpha \in \Im_{\succ\vartheta}$ a formulae. α is a *ψ-s-inductive consequence* of A (in symbols: A $\vdash_s^\psi \alpha$) iff, for all ψ-inductive extensions E of A, $\alpha \in E$. We call $\vdash_s^\psi$ a *ψ-relation of inductive consequence*, in this case a skeptical one.

Definition 6.1.14. Let $\Im$ be a language, $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\Im$, $A \subseteq \Im_{\succ\vartheta}$ a ψ-theory and $\alpha \in \Im_{\succ\vartheta}$ a formulae. α is a *ψ-c-inductive consequence* of A (in symbols: A $\vdash_c^\psi \alpha$) iff, for at least one ψ-inductive extension E of A, $\alpha \in E$. We call $\vdash_c^\psi$ a *ψ-relation of inductive consequence*, in this case a credulous one.

We are here defining the notion of inductive consequence according to both skeptical and credulous positions. As we have seen in Chapter 4, the necessity of defining such relation according either to a skeptical approach or to a credulous one is due to the lack of guarantee that every default theory will have only one extension. Since we are following a general path where several sorts of logics of induction could be represented[7], we leave room for the possibility of having inductive logics with either a skeptical or a credulous relation of inductive consequence. From now on in this section we will use the symbol $\vdash^\psi$ to speak generally about $\vdash_s^\psi$ and $\vdash_c^\psi$.

About the technicalities of our formulation, we first observe that in order to represent default statements such as "typically birds fly" we do not need to consider schemas of defaults (like Reiter does.) Rather, we just act as if we were representing an universal statement in classical logic and use the universal quantifier $\forall$ along with an open formula, in this case an inductive implication. For example, if we were to represent the just mentioned default statement we would take $\forall$ in conjunction with bird(x) $\succ$ flies(x) and get the universal inductive implication $\forall$x(bird(x) $\succ$ flies(x)). For the conclusion of flies(b) for some *b* which has the property of being a bird, supposing that the pseudo-inductive modal calculus Λ^* which ψ is based on has axiom Q ($\forall x\alpha(x) \rightarrow \alpha(t)$), then automatically all instances of "typically birds fly" would be taken into account at the time of calculating the extension. This is of course due to ψ's being able to reason deductively not only

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[6] ~ is like described in definition 5.2.9.

[7] Default logic for example is obtained by taking $\psi = <\vartheta_\Im, \Sigma_\Im, \varnothing, \varnothing>$, where $\Lambda_\Im^* = <\vartheta_\Im, \Sigma_\Im>$ is the pseudo inductive modal calculus based on the calculus of trivial modal logic (that is, classical logic.)

about plausible formulae but also about inductive implications. As we have already observed, this is one of the main innovations of our approach: we can reason deductively about inductive implications in exactly the same way as we reason about ordinary formulae.

Second, following Pequeno's approach we make the test of consistency of the consequent inside the very definition of extension. Because of that we do not have to consider it among the exceptions of the inductive implication. This, we must concede, is a quite natural and desirable thing. About the exceptions to the claim that α inductively implies β, one that will appear in all cases, independently of the form of α and β, is $\sim\beta$. Therefore, nothing more natural than not requiring $\sim\beta$ to be informed at every time we write an inductive implication; and making the test of the consistency of the consequent compulsory for all inductive implications. As we have mentioned in Chapter 4, another consequence of proceeding in this way is that we will not allow the representation of so-called abnormal defaults, which are quite counterintuitive and one of the main sources of anomalous extensions.

Third, given a particular pseudo-inductive logic of plausibility ψ, the relation of inference $\vdash^{\psi}$ will be in fact non truth-preserving or nonmonotonic: even though we may have $A \vdash^{\psi} \alpha$, it is not necessarily the case that $A \cup B \vdash^{\psi} \alpha$ for any set B. Take, for instance, $A = \{\lambda \succ \beta \precsim \varphi, \lambda\}$, $\alpha \equiv \beta$ and $B = \{\varphi\}$. Trivially in this case $A \vdash^{\psi} \alpha$ but $A \cup B \nvdash^{\psi} \alpha$. Of course this feature concerns only inductive implications. If the set of premises A does not contain any formula of the form $\alpha \succ \beta \precsim \varphi$, then $A \vdash^{\psi} \alpha$ entails $A \cup B \vdash^{\psi} \alpha$ for any set of formulae B.

Finally, considering what can be expressed with the help of ψ's language, our relation of inductive conclusion automatically satisfies what we have called total evidence condition. According to the general interpretation we have given to Carnap's requirement of total evidence, since *e* may be an evidence for *h* when taken in isolation but against or neutral to it when taken in conjunction with e', all available information should be taken into account at the time of saying whether or not *e* inductively confirms *h*. Following the way an extension is defined in default logic, in order for the inductive consequent β to be taken as a ψ-inductive consequence of A, the checks concerning β itself, the antecedent and the exception of the inductive implication which β belongs to are made from a global perspective, in connection with all inductive and deductive consequences of A (item (ii) of definition 6.1.11).

For the same of simplicity, we have considered in the definition of $\vdash^{\psi}$ only the so-called global premises. We show below the version of definitions 6.1.11, 6.1.12, 6.1.13 and 6.1.14 which also takes into account the local premises.

Definition 6.1.15. Let $\Im$ be a language, $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\Im$ and $\Lambda^* = <\vartheta, \Sigma>$, with $\vartheta = <\Theta', \Theta''>$, and $A,B \subseteq \Im_{\succ\vartheta}$ two ψ-theories. We define the function $\Upsilon_{\psi,A,B}$: $\aleph(\Im_{\succ\vartheta}) \rightarrow \aleph(\Im_{\succ\vartheta})$ as follows. Let $S \subseteq \Im_{\succ\vartheta}$ be a set of formulae of $\Im_{\succ\vartheta}$. $\Upsilon_{\psi,A,B}(S) = \Upsilon^G_{\psi,A,B}(S) \cup \Upsilon^L_{\psi,A,B}(S)$ is the smallest set satisfying the following conditions:

(i) $A \subseteq \Upsilon^G_{\psi,A,B}(S)$ and $B \subseteq \Upsilon^L_{\psi,A,B}(S)$;

(ii) If $\Upsilon^G_{\psi,A,B}(S) \div \varnothing \vdash_{\Lambda^*} \alpha$ then $\alpha \in \Upsilon^G_{\psi,A,B}(S)$;

(iii) If $\Upsilon^G_{\psi,A,B}(S) \div \Upsilon^L_{\psi,A,B}(S) \vdash_{\Lambda^*} \alpha$ and $\alpha \notin \Upsilon^G_{\psi,A,B}(S)$ then $\alpha \in \Upsilon^L_{\psi,A,B}(S)$;

(iv) If $\alpha \succ \beta \not\precsim \varphi \in \Upsilon^L_{\psi,A,B}(S)$, $\alpha \in \Upsilon_{\psi,A,B}(S)$, $\varphi \notin S$ and $\sim\beta \notin S$, then $\beta \in \Upsilon^L_{\psi,A,B}(S)$;

(v) If $\alpha \succ \beta \not\precsim \varphi \in \Upsilon^G_{\psi,A,B}(S)$, $\alpha \in \Upsilon_{\psi,A,B}(S)$, $\varphi \notin S$ and $\sim\beta \notin S$, then $\beta \in \Upsilon^G_{\psi,A,B}(S)$.

Definition 6.1.16. Let $\Im$ be a language, $\psi = <\vartheta, \Sigma, \Theta>$ a pseudo-inductive logic of plausibility based on $\Im$, $A,B \subseteq \Im_{\succ\vartheta}$ two ψ-theories and $E \subseteq \Im_{\succ\vartheta}$ a set of formulae. E is a ψ-*extension* of A and B iff $E = \Upsilon_{\psi,A,B}(E)$.

Definition 6.1.17. Let $\Im$ be a language, $\psi = <\vartheta, \Sigma, \Theta>$ a pseudo-inductive logic of plausibility based on $\Im$, $A,B \subseteq \Im_{\succ\vartheta}$ two ψ-theories and $\alpha \in \Im_{\succ\vartheta}$ a formulae. α is a ψ-*inductive consequence* of A and B (in symbols: $A \div B \vdash^{\psi} \alpha$) iff, for every (at least one) ψ-inductive extension E of A, $\alpha \in E$.

6.1.4 The Logic of Induction

Definition 6.1.18. Let $\Im$ be a language. A *logic of induction* (or inductive logic) ζ based on $\Im$ is a triple $<\psi, \Im_\psi, \vdash>$ where ψ is a pseudo-inductive logic of plausibility based on $\Im$, $\Im_\psi$ the ψ-inductive-plausible language and $\vdash$ a ψ-relation of inductive consequence.

We may sometimes refer to ψ, $\Im_\psi$ and $\vdash$, respectively, as ζ's logic of induction (in symbols: ψ_ζ), ζ's inductive-plausible language (in symbols: $\Im_\zeta$) and ζ's relation of inductive consequence (in symbols: $\vdash_\zeta$.)

Definition 6.1.19. Let $\Im$ be a language, $\zeta = <\psi, \Im_\psi, \vdash>$ a logic of induction based on $\Im$, $T \subseteq \Im_\psi$ a set of formulae called inductive axioms, $A \subseteq \Im_\psi$ a set formulae and $\alpha \in \Im_\psi$ formula. α is a T-ζ-*inductive consequence* of A (in symbols: $A \vdash_{T\text{-}\zeta} \alpha$) iff $T \cup A \vdash \alpha$. We call the pair $\zeta' = <\zeta, T>$ an *applied logic of induction* and $\vdash_{T\text{-}\zeta}$ the ζ'*-relation of inductive consequence* (in symbols: $\vdash_{\zeta'}$.)

It should be noted that we are not following strictly the inductive logic's structure presented in subsection 4.1.2. Rather than being a quadruple composed by a logic of plausibility, an inductive-plausible language, a relation of inductive consequence and a calculus of inductive implication, a logic of induction ζ is a triple composed only by the first three components. The reason for that is a technical one. Since we have based ζ's inferential relation on ψ, the aspects concerned with the calculus of inductive implication will be built over this structure. In other words, what we have called the axioms of the calculus of inductive implication will be represented through a set of inductive axioms T belonging to $\Im_\psi$ and the calculus itself will be the resultant applied inductive logic as defined above. This makes even more sense if we consider that through set T we can represent many things besides a calculus of inductive implication, such as a model of confirmation or some axioms of ψ which cannot be represented deductively. Moreover, this approach makes explicit the independent character that a calculus of inductive implication is supposed to have, being always possible that the same T be used long with several different pseudo-inductive logics of plausibility sharing the same $\Im$, modal logic basis and modal operators. If however one wishes to stick to the old definition, he can take the quadruple $<\psi, \Im_\psi, \vdash_{T\text{-}\zeta}, T>$ and define it as our actual inductive logic.

Now that we have finally arrived at a complete characterization of what a descriptive logic of induction is supposed to be, a short summary may be appropriate. First we have an inductive language $\Im_{\succ}$ which, by considering a modal logic basis $\vartheta = <\Theta', \Theta''>$, will be transformed into an inductive modal language $\Im_{\succ\vartheta}$. We then take a modal calculus Λ^* based on $\Im$ and ϑ and expand it in such a way as to obtain what we call a pseudo-inductive modal calculus $\Lambda^{\dot{*}}$, that is, an extension of Λ^* able to reason deductively about inductive implications. By considering the epistemological restrictions one wish to impose on inductive implications with the help of some modal operators of ϑ, we then obtain a pseudo-inductive logic of plausibility ψ. When considered inside ψ, Λ^* is taken as the *calculus of plausibility* of our logic of induction. The language $\Im_\psi$ resultant from the appliance of the restrictions specified by ψ plays the part of the *inductive-plausible language*. By taking only sets of formulae belonging to $\Im_\psi$, we then define with the help of the notion of ψ-inductive extension the *relation of inductive consequence* of our inductive logic (which can be done either skeptically or credulously.) Finally, by considering a specific set of inductive implications belonging to $\Im_\psi$, we can apply our logic of induction to obtain, among other things, a *calculus of inductive implication*.

Before finishing this section we have to note that a logic of induction ζ does not in fact give any indication of how to confirm a hypothesis from a piece of evidence. The sole purpose of ζ is to allow one to represent through inductive inferences the way he thinks things are to be concluded inductively from others. The responsibility of coming up with rational inductive patters of inference belongs to the knowledge engineer who will make use of ζ. We have tried to put in our definition of ζ only what we think to be the essential aspects a descriptive logic of induction should deal with, namely the logical form of inductive implications, the need (or, we may say, the possibility) of distinguishing inductively obtained ordinary formulae from deductively obtained ones and the way inductive conclusions are calculated. Given these, let us say, universal aspects of a descriptive logic of induction, by providing a pseudo-inductive logic of plausibility along with some inductive implication axioms one may get a specific logic of induction useful in a particular range of applications.

6.2 The Logic of Plausibility and Certainty

6.2.1 The Logic of Skeptical Plausibility and Credulous Plausibility

In this section we will introduce our specific solution to the concept explication problem of plausibility, which will also play the part of the logic of plausibility of the inductive logics we shall present in the next section. We will refer to such system as the *logic of skeptical plausibility and credulous plausibility* or simply *the logic of two plausibility notions* (LP^2, for short.)

LP^2 is a multi-normal modal logic in the style of those introduced in Section 5.4 with two pairs of dual operators: the paranormal modalities ! and ? and the normal modalities $\Box$ and $\Diamond$. As we have mentioned at the end of Section 5.4, while $\alpha!$ and $\alpha?$ will as usual be interpreted as "α is skeptically plausible" (or "α is accepted") and "α is credulously plausible," respectively, $\Box\alpha$ and $\Diamond\alpha$ will be taken as meaning "α is certain" and "α is epistemologically possible," respectively. Taking all formulae as instances of some sort of hypothesis, we would say that while $\alpha!$ and $\alpha?$

mean, respectively, "α is an accepted hypothesis" and "α is a plausible hypothesis," □α and ◇α mean "α is a certain hypothesis" and "α is an epistemologically possible hypothesis," respectively. The reason why certainty and epistemological possibility are formalized through normal and not through paranormal modalities is quite obvious: in contrast to plausibility, we analyze the certainty of a statement from a sole point of view. If we take, for example, each possible world as being a world compatible with all facts the agent in question knows, then to know α means simply that α is true in all such compatible worlds. Being true in at least one of these worlds does not represent a new sort of knowledge. It means simply that the statement in question is not known to be false, but simply, from the agent's epistemological point of view, a possible truth.

An important point related to the meaning of all sorts of formulae in general and non-modal formulae in particular concerns the place they will appear in the relation of deductibility or logical consequence. We recall that the relations of deductibility and logical consequence have two sets of premises: the so-called global premises and the local premises. From a proof-theoretical point of view, the difference between them is that only those formulae obtained exclusively with the help of the global premises will be able to make use of the necessitation rules N (α / □α) and $N_?$ (α / α!.)[8] From the viewpoint of epistemic modal logic, this distinction is important because whether a formula α belongs to the set of global premises or to the set of local premises will determine its ultimate meaning. More specifically, while an arbitrary formula α belonging to the set of global premises A can be said to mean "α is true" or "α is a true hypothesis", a formulae β belonging to the set B of local premises means simply "β is a hypothesis."[9] The rationale behind this reading can be seen both from a proof-theoretical as well as from a semantic point of view. Concerning the global premise α, since α is a true hypothesis, we sure can claim it to be skeptically plausible (α!) as well as to be certain about its truth (□α). This stands in contrast to the local hypothesis β, which, being absent of any sort of modal qualification, does not entitle us to lay down any epistemological claim about it. Also, the fact that we can semantically conclude □α and α! from α, which is due to all models taken into account being exactly those which satisfy α (that is, α is true in all their worlds), reflects the idea that α is being taken as a true hypothesis and not just as a certain or accepted one. In its turn, β helps to select, out of the multitude of worlds belonging to some of these models, the individual worlds that will be used to evaluate φ[10]. It therefore functions like a local,

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[8] See definitions 5.2.20, 5.2.24 and 5.2.28.

[9] Incidentally, this interpretation of non-modal formulae in terms of truthfulness is the standard one in epistemic modal logic. See Gabbay et al (1995).

[10] See definitions 5.2.19 and 5.2.20.

hypothetical premise whose truth is guaranteed not in all, but only in a few possible worlds of the model in question.

The need of considering the notion of certainty has already been emphasized in Chapter 2. In the same way that induction leads from certainty to plausible, uncertain knowledge, deduction leads from certainty to certainty. They are like the opposite of each other: while certain knowledge is something safe, irrefutable, plausible knowledge is intrinsically defeasible and refutable. In fact, we have somewhere pointed out that the susceptibility of being refuted or given up is one of the distinguishing features of the notion of plausibility. Formally, this will be made precise by requiring the consequent of inductive implications to be marked with some plausibility operator (through the Θ_p component of the pseudo-inductive logic of plausibility that will make use of LP^2.) In this way, at the same time that we impart refutability unto plausible facts, we also provide the positive, rational side the notion of plausibility is supposed to have. Of course this will be done only when we have at hand a logic of induction ζ which makes use of a pseudo-inductive logic of plausibility ψ made out of LP^2. As far as the deductive, logic of plausibility component is concerned, such sort of characterization of plausibility in terms of its refutable aspect is of course not possible. However, if we have at hand a characterization of certain, irrefutable facts, by establishing the relations that are supposed to exist between certain hypotheses and plausible ones we can in principle obtain a satisfactory account of what uncertain, refutable facts are. Moreover, in the same Chapter 2 we have proposed a definition of induction according to which inductive arguments are those inferences which lead not from truth premises to plausible conclusions, but from certain statements to plausible and therefore uncertain ones. This has to do with our decision of taking induction as primarily concerned with knowledge of truth rather than with truth itself. Therefore, if we want to fully represent this epistemological feature of induction we have to have some way to represent not only plausible facts, but also certain, irrefutable hypotheses as well[11]. Furthermore, by allowing LP^2 to represent certain facts we provide a framework where inductive inferences are always analyzed from an epistemological perspective, for, from a proof-theoretical point of view, even non-modal true formulae and non-modal hypothetical ones are so only due to their connection with the certainty operator.

Definition 6.2.1. Let $\Im$ be a language and ϑ a $\Diamond$-modal logic basis. The D *axioms* Σ_D in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying the following schema of formula:

D: $\Box\alpha \rightarrow \Diamond\alpha$

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[11] As we will see in the next section, such conception of induction shall be formalized by setting $\Theta_p = \{?\}$ and $\Theta_t = \{\Box\}$.

Definition 6.2.2. Let $\Im$ be a language and ϑ a $\diamond$-modal logic basis. The B *axioms* Σ_B in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying the following schema of formula:

B: $\alpha \rightarrow \Box\diamond\alpha$

Definition 6.2.3. Let $\Im$ be a language and ϑ a $\diamond$-modal logic basis. The 4 *axioms* Σ_4 in $\Im_\vartheta$ is the set composed by all formulae of $\Im_\vartheta$ satisfying the following schema of formula:

4: $\Box\Box\alpha \rightarrow \Box\alpha$

Definition 6.2.4. Let $<\vartheta_{?\diamond}, \Omega_{?\diamond}{}^1, \Gamma_{PP}, \Sigma_{PP}>$ be the first-order multi-normal modal logic $PPK_?K$. The (first-order) logic of skeptical plausibility and credulous plausibility (or simply LP^2) is the first-order 2-modal system $<\vartheta_{?\diamond}, \Omega_{?\diamond}{}^1, \Gamma_{LP2}, \Sigma_{LP2}>$ where $\Gamma_{LP2} \sqsubseteq \Gamma_{PP}$ is the set of all idealized 2-frames $<W,R_\diamond,R_?>$ belonging to Γ_{PP} such that $R_\diamond$ is symmetric and transitive and $R_?$ is symmetric (we call the members of Γ_{LP2} LP^2-*frames*), and $\Sigma_{LP2} = \Sigma_{PP} \cup \Sigma_{D?} \cup \Sigma_{B?} \cup \Sigma_D \cup \Sigma_B \cup \Sigma_4$, where $\Sigma_{D?}$ are the $D_?$ axioms in $\mathcal{L}_{?\diamond}$, $\Sigma_{B?}$ the $B_?$ axioms in $\mathcal{L}_{?\diamond}$, Σ_D the D axioms in $\mathcal{L}_{?\diamond}$, Σ_B the B axioms in $\mathcal{L}_{?\diamond}$ and Σ_4 the 4 axioms in $\mathcal{L}_{?\diamond}$.

6.2.2 On the Axiomatic of LP^2

As one can see, LP^2 is the first-order 2-modal system $PPK_?KD_?DB_?B4$. The axiomatic 2-modal system $<\vartheta_{?\diamond}, \Sigma_{LP2}>$ based on $\mathcal{L}$, which we represent by the symbol CP^2, is called the LP^2 *calculus of plausibility*. For the sake of comprehensibility, we write below all axioms and inference rules of CP^2.

Positive Classical Axioms

P1: $\alpha \rightarrow (\beta \rightarrow \alpha)$

P2: $(\alpha \rightarrow (\beta \rightarrow \varphi)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \varphi))$

P3: $\alpha \wedge \beta \rightarrow \alpha$

P4: $\alpha \wedge \beta \rightarrow \beta$

P5: $\alpha \rightarrow (\beta \rightarrow \alpha \wedge \beta)$

P6: $\alpha \rightarrow \alpha \vee \beta$

P7: $\beta \rightarrow \alpha \vee \beta$

P8: $(\alpha \rightarrow \beta) \rightarrow ((\varphi \rightarrow \beta) \rightarrow (\alpha \vee \varphi \rightarrow \beta))$

Paranormal Classical Axioms

A1: $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha)$ wherein β is ?-free and α is !-free

A2: $\neg\alpha \rightarrow (\alpha \rightarrow \beta)$ — wherein α is ?-free

A3: $\alpha \vee \neg\alpha$ — wherein α is !-free

Non-positive Additional Classical Axioms

N1: $\neg(\alpha \rightarrow \beta) \leftrightarrow (\alpha \wedge \neg\beta)$

N2: $\neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$

N3: $\neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$

N4: $\neg\neg\alpha \leftrightarrow \alpha$

N5: $((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$

Paranormal Modal Axioms

K1: $\alpha? \leftrightarrow \sim((\sim\alpha)!)$

K2: $(\neg\alpha)! \leftrightarrow \neg(\alpha!)$

K3: $(\neg\alpha)? \leftrightarrow \neg(\alpha?)$

$K_?$: $(\alpha \rightarrow \beta)! \rightarrow (\alpha! \rightarrow \beta!)$

$D_?$: $\alpha! \rightarrow \alpha?$

$B_?$: $\alpha \rightarrow \alpha?!$

Normal Modal Axioms

NP: $\Diamond\alpha \leftrightarrow \neg\Box\neg\alpha$

K: $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$

NN: $\sim\Box\sim\alpha \leftrightarrow \neg\Box\neg\alpha$

D: $\Box\alpha \rightarrow \Diamond\alpha$

B: $\alpha \rightarrow \Box\Diamond\alpha$

4: $\Box\alpha \rightarrow \Box\Box\alpha$

Multi-normal Modal Axioms

PP: $\Box\alpha \rightarrow \alpha!$

First-order Axioms

Q: $\forall x\alpha(x) \rightarrow \alpha(t)$ — wherein the substitution of t for x is admissible

Rules of Inference

MP: $\alpha \rightarrow \beta, \alpha / \beta$

G: $\alpha \rightarrow \beta / \alpha \rightarrow \forall x\beta$ — wherein x has no free occurrences in α

N: $\alpha / \Box\alpha$

$N_!$: $\alpha / \alpha!$

We have already commented the axioms of paranormal modal logic $K_?$ (P1-P8, A1-A3, N1-N5, K1-K3, $K_?$), the modal axioms of normal modal logic K (NP, K) and the multi-normal axiom NN. D and $D_?$ guarantee, respectively, that what is certain is also epistemologically possible and what is skeptically plausible is also credulous plausible. B and $B_?$ say, respectively, that if α is a true or an unqualified hypothesis, then it is certain that α is epistemologically possible and it is skeptically plausible that α is credulously plausible. The reasonableness of these principles is self-evident in the case where α is a true hypothesis. Concerning the local, unqualified hypothesis case, B and $B_?$ state a sort of minimal rationality principle about the hypotheses we can consider. Even though the hypotheses we lay down may be neither plausible nor epistemologically possible, they must be so from a second-order point of view. 4 is a sort of principle of positive introspection: if we know that α, then we know that we know that α. From B and 4 we deduce 5, $\neg\Box\alpha \rightarrow \Box\neg\Box\alpha$, which is a principle of negative introspection: if we are not certain about α, then we are certain that we are not certain about α. PP, which could be fairly called here PC or the plausibility-certainty axiom, states that if α is certain then it is also an accepted hypothesis. From it, along with MP and K1, we conclude $\alpha? \rightarrow \Diamond\alpha$, that is, if α is (credulously) plausible, then it is epistemologically possible.

The reason why we have excluded axioms T ($\Box\alpha \rightarrow \alpha$) and $T_?$ ($\alpha! \rightarrow \alpha$) is that they represent a kind of principle of epistemological arrogance undesirable in the case of both certainty and skeptical plausibility. Taking α as meaning "α is true," while T means that if we are certain that α is true then it is true, $T_?$ means that accepting α as true entails that it is true. On similar grounds, $T_?$ and T cannot be accepted if we take α as representing an unqualified hypothesis. While from $T_?$ along with K1 we conclude $\alpha \rightarrow \alpha?$, which means that every conceivable hypothesis is automatically a plausible one, from T we derive $\alpha \rightarrow \neg\Box\neg\alpha$, which means that every conceivable hypothesis is an irrevocable one or, in other words, that from a hypothesis α, which we may have conceived quite arbitrarily, it follows that we will never be able to be certain about its negation $\neg\alpha$. $4_?$ ($\alpha! \rightarrow \alpha!!$) was not included on account of the desirableness of allowing gradations of credulous plausibility ($T_?$ along with K1 entails $\alpha?? \rightarrow \alpha?$), from which it is possible to develop a quantitative theory of plausibility.

LP^2 is not only an epistemic logic but also an *autoemistemic logic*. Due to axiom 4 ($\Box\alpha \rightarrow \Box\Box\alpha$) and theorem 5 ($\neg\Box\alpha \rightarrow \Box\neg\Box\alpha$), we are aware of those facts which we known, as well as of those which we do not know. N does the same job but only in connection with true formulae: from the fact that α is true (and α can be of any form) it follows that we know that it is true. Like 4 and 5, N is a sort of positive autoepistemic principle: we are always positively aware of those statements we take as true. But how about those statements whose truth we have no hint about? Suppose that Th(A) is all we can conclude from knowledge situation A. By our positive autoepistemic principle then, for each $\alpha \in Th(A)$, we will have that we know that α ($\Box\alpha$.) But how about those statements β which do not belong to Th(A)? Restricting ourselves to knowledge situation A, it is reasonable that for all β such that $\beta \notin Th(A)$ we conclude $\neg\Box\beta$. This is what we could call a *negative autoepistemic principle*. It is trivially a non-monotonic rule: if from A we conclude $\neg\Box\beta$, from $A \cup \{\beta\}$ the same inference could not be done. Consequently, it cannot be formalized inside a purely deductive logic such as LP^2. Only when we start presenting our logics of induction in Section 6.3 is that we will be able to have a fully autoepistemic logic of plausibility[12].

Theorem 6.2.1. LP^2 is sound and complete.

Theorem 6.2.2. The following schemas of relations between sets of formulas and formula are correct.

$\{\Box\alpha\} \vdash_{LP2} \alpha!$

$\{\alpha!\} \vdash_{LP2} \alpha?$

$\{\alpha?\} \vdash_{LP2} \Diamond\alpha$

According to theorem 6.2.2 then, we have the following hierarchy: $\Box\alpha \vdash_{LP2} \alpha! \vdash_{LP2} \alpha? \vdash_{LP2} \Diamond\alpha$, which means that α's being certain entails that α is an accepted hypothesis, which entails that α is credulously plausible, which in its turn entails that α is epistemologically possible. In other words, certainty implies acceptability, which implies plausibility, which implies possibility.

6.2.3 On the Semantics of LP^2

Regarding the semantics of LP^2, we first note that, given a LP^2-frame $<W,R_\Diamond,R_?>$ and a world $w \in W$, the sets $R_\Diamond(w) = \{w' \mid wR_\Diamond w'\}$ and $R_?(w) = \{w' \mid wR_? w'\}$ represent, respectively, what we may call the *epistemologically possible worlds* of *w* and the *plausible worlds* of *w*. This was already outlined in Section 5.4. The relations $R_\Diamond$ and $R_?$ correspond, respectively, to what we may call the *certainty accessibility relation* and the *plausibility accessibility relation*. Second, these LP-frames belong to the class of P^2-frames, that is, 2-frames $<W,R_\Diamond,R_?>$ such that, for each $w,w' \in W$, if

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[12] For more about autoepistemic logics see Konolige (1994).

$wR_{\diamond}w'$ then $wR_{?}w'$. This means that, given a world of reference *w*, every plausible world of *w* is also an epistemologically possible world of *w* (in symbols: $R_{\diamond}(w) \subseteq R_{\diamond}(w)$). As we have seen in Chapter 5, from a proof-theoretical point view this property corresponds to axiom P^2 ($\Box\alpha \rightarrow \alpha!$.) Besides this, every LP-frame is also a idealized 2-frame, that is to say, for every $w \in W$, there is at least one $w' \in W$ and at least one $w'' \in W$ such that $wR_{\diamond}w'$ and $wR_{?}w''$. This is proof-theoretically obtained by axioms D and $D_?$. Finally, while $R_{\diamond}$ is a symmetric and transitive relation, $R_?$ is only a symmetric one, which corresponds, respectively, to axioms B and 4 and axiom $B_?$.

From this explanation of the semantic elements of LP^2 one can better understand our claim that the modal operators ! and ? formalize or explicate the notions of skeptical plausibility and credulous plausibility, respectively. As we have said in Chapter 2, as a result of accepting inductive conclusions as plausible or pragmatically probable hypothesis and induction itself as that sort of inference whose premises, in certain circumstances, serve as evidences for its conclusion, "α is plausible" was taken as the same as "there are evidences for α." Whether we are talking about the skeptical or about the credulous plausibility will be made precise by the strength of what we are calling evidence: while "α is skeptically plausible" means "there are strong evidences for α", "α is credulously plausible" means "there are weak evidences for α." Now the crucial point is how our semantic formulation relates to this notion of evidence. According to the traditional sense of the term "evidence," we would say that each plausible world where α is true *indicates*, we may say, the existence of plain evidences for the truthfulness of α. It would be more or less as if through some class of mechanisms for evaluating sentences we get different world pictures of what is true and false. Considering that there must be some rationale behind the way according to which each one of these mechanisms arrive at their worldviews, the fact that α is true in one of these worldviews entitles us to believe that there is some sort of plain evidence for α (based on which the mechanism in question decided to include it in its worldview.) Now, the indication that there exists this sort of evidence may itself be taken as an evidence, of a second-order we may say, for the truthfulness of α. In other words, the very fact that there is a specific world view where α is true, which was rationally produced with the help of plain evidences, can itself be considered as an evidence for α, for it sure leads us to believe in its plausibility in the same way that the plain evidences when rationally analyzed do. This is what our plausible worlds are intent to be. And if we decide to really adopt this second-order meaning of evidence, α's being true in at least one of such worldviews will signify that there are evidences for α, but only of a weak type, for α might be true in all worldviews at hand, which would mean that we had a sort of strong evidence for the truthfulness of α.

One important feature of this way of explicating the notion(s) of plausibility is that we do not give details about what the mechanisms which generate the plausible worlds are. They can be several experts trying to evaluate according to their own criteria the truthfulness of a specific set of propositions, but also several competing theories trying to account for the same range of physical phenomena or the different ways an annalist can look at a specific situation. The important aspect captured by our formal model concerns what we have called plurality approach to plausibility, or in other words, the idea that the distinguishing feature of the notion of plausibility is the necessity to refer to a plurality of ways to achieve some specific goal. What these ways and their corresponding goals are do not matter to the level of investigation we are doing here: only at the time of applying the logic to solve some specific problem is that we will consider a particular mechanism for generating plausible worlds.

Instead of an arbitrary choice, the decision of letting the mentioned mechanisms undefined is a crucial aspect of our project of developing a pragmatical account of induction and plausibility. As said in Chapter 2, in order not to go through justificatory issues we have decided to adopt what we have called an extensional or pragmatical approach to induction. According to this approach, inductive inference is simply that class of ampliative inferences that in a particular period of time is used in some practical situations and accepted as sound by a certain community of people. Accordingly, the notion of rationality itself, which must be present as the positive side of inductive inferences, will also be taken pragmatically: instead of being defined through some intensional criterion, rational will be what people of a particular community in a particular period of time take as so. That is precisely what our formal semantic model captures. From the strongest point of view, the plausible and consequently the rational is the pragmatical consensus among a particular class of entities. And again, what these entities are is less important for our general approach: the significant point is that from a plurality of views about a specific subject matter we arrive at something we can quite fairly call plausible, pragmatically probable or rationally acceptable[13]. Only because of that is that we can claim to have given some sort of account for the rationality of inductive conclusions (without also having tried to give an answer to Hume's problem of justification) as well as answered Hempel's challenge of developing a purely logical analysis of the notion of acceptability independent of any sort of contemplated goal.

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[13] It is worthy to note that this (skeptical) way of defining the notion of pragmatical probability is very similar to Newton da Costa's formalizaton of what he calls *pragmatic truth*. Despite technical details, one of the main differences between his approach and ours is that while in his case the same idea of plurality of views is there, there is no stress on the positive side of what we are calling plausible worlds: besides being possible, compatible worlds, plausible worlds should also be the result of some pragmatical method of truth-assessing (in our case, a specific set of inductive implications.) See da Costa et al (1998).

6.2.4 On LP^2 Plausibility

An important result achieved by our formalization concerns the controversy around the lottery paradox and the conjunction principle. We have already sketched informally in Chapter 3 (and formally but in not very much detail in Chapter 4) how the consideration of two plausibility notions could bring this controversy to an end. The whole point centers around the strength of the support given by the body of evidence to the two hypotheses to be conjoint. In the case of credulously plausible facts, the support given by the evidences is weak, fragile we may say. It is as if, by adopting a credulous position, we were committing ourselves to the hypothesis in question in such a way that we are allowed to take mutually contradictory hypotheses as plausible without compromising the self-consistency of our body of evidence. However, at the same time that this is so, some special care is needed at the moment of dealing with such mutually contradictory plausible hypotheses. Since the support given to α and β, for example, is weak, the fact that α is plausible and β is plausible does not allow us to take $\alpha \wedge \beta$ as equally plausible, for since such plausible judgments may have used the mentioned flexibility of the credulous position to support mutually contradictory hypotheses, to take $\alpha \wedge \beta$ as weakly plausible may imply an actual collapse of our logical theory.

On the other hand, if α and β are skeptically plausible, the support given to them by the body of evidences is strong, safe. It is as if we were very strongly committed to the hypotheses expressed by α and β. This of course has the consequence that we will be committed in the same strong way to $\alpha \wedge \beta$, but also that we will not be able to support mutually contradictory hypotheses without compromising our own internal consistency. Then we will have a picture where we can allow mutually contradictory hypotheses (such as the ones of the lottery paradox) to be each one of them separately plausible without having any self-contradictory hypothesis as plausible as well as to maintain the conjunction principle with respect to a stronger sort of plausibility. This, as we have seen, is formally obtained by the very way we semantically defined the notions of skeptical plausibility and credulous plausibility: while $\models \alpha! \wedge \beta! \rightarrow (\alpha \wedge \beta)!$ is the case, $\alpha? \wedge \beta? \rightarrow (\alpha \wedge \beta)?$ is not a valid schema of LP^2.

While this way of doing things manages quite well a controversy apparently without solution, it leaves a very important problem without answer. As we explained above, in the case of $\alpha?$ and $(\neg\alpha)?$ it is clear that we should not allow the conjunction principle to work. But how about two formulae α and β which are not inconsistent with each other? Clearly, if α and β are not incompatible, the fact that there are weak evidences for α and there are weak evidences for β should imply that there are weak evidences for $\alpha \wedge \beta$. The danger of conjoining weakly plausible hypotheses has to do only with the mutually contradictory hypotheses case. Therefore, if α and β are not inconsistent with each other, the following principle should be valid: $\alpha? \wedge \beta? \rightarrow (\alpha \wedge \beta)?$. The obvious problem is that this principle cannot be obtained monotonically. Let A be a set of formulae and α, β two formulae such that α and β are mutually consistent with respect to A, that is to say, $\{\alpha,\beta\} \cup A \nvdash \bot$. Applying the mentioned principle, we will have that $\{\alpha,\beta\} \cup A \vdash \alpha? \wedge \beta? \rightarrow (\alpha \wedge \beta)?$. But now consider A augmented by formula $\alpha \rightarrow \neg\beta$. Since α and β are not mutually consistent with respect to $A \cup \{\alpha \rightarrow \neg\beta\}$, the relation $\{\alpha,\beta\} \cup A \cup \{\alpha \rightarrow \neg\beta\} \vdash \alpha? \wedge \beta? \rightarrow (\alpha \wedge \beta)?$ is not valid, despite the fact that $\{\alpha,\beta\} \cup A \vdash \alpha? \wedge \beta? \rightarrow (\alpha \wedge \beta)?$ is still valid. Our solution, to be detailed in the next section, consists in introducing this principle as a nonmonotonic inference. More specifically, we will take the following formula as an inductive axiom of one of the applied logics of induction to be introduced in the next section: $\alpha? \wedge \beta? \succ (\alpha \wedge \beta)?$[14].

Another important feature of our framework concerns the criticisms we have made at the beginning of Section 2.3 against the probability calculus. There we have talked about the idea defended by those who use the probability calculus to formalize inductive probability that deduction and induction are distinguished from one another only by the degree of evidential support given to the conclusion by the premises: concerning the nature of such support, they are for all intents and purposes indistinguishable from each other. Our position is that this is manifestly mistaken. We have given two practical examples to support this: the way that confirmed hypotheses are conjoined in the probability calculus and the additive character of negation in such system.

In order to see how our logic deals with this point, compare the way we are representing certain, irrefutable facts with the way we are representing plausible, refutable ones. Despite the connection that exists between these two sorts of hypotheses ($\Box\alpha \rightarrow \alpha!$ or $\alpha? \rightarrow \Diamond\alpha$), both notions are defined with the help of quite different ontological resources. In contrast to the probability calculus, here certainty is not a particular case of plausibility: the highest degree of plausibility does not correspond to certainty, and the lowest degree of plausibility does not correspond to epistemological impossibility. It is also to easy that LP^2 is not plagued by the problem of conjunction of probable hypotheses. In contrast to the probability calculus, there is no decrease of plausibility when we put two plausible hypotheses together: as we have mentioned above, $\alpha! \wedge \beta! \rightarrow (\alpha \wedge \beta)!$ is a theorem of LP^2. Concerning ?, even though $\alpha? \wedge \beta? \rightarrow (\alpha \wedge \beta)?$ is not a theorem of LP^2, as we have mentioned, through $\succ$ we can obtain a similar axiom that satisfies both our intuitions as well as the formal constraints of such plausibility conjunctive principle.

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[14] Another point that will have to wait for the next section concerns the problem we have mentioned in

Regarding the problem of additive character of negation, the whole point is that besides considering the improbability or incredibility of A, we need also to deal with the probability or credibility of $\neg$A, which, as shown by the law $p(A) + p(\neg A) = 1$, is incompatible with the way the probability calculus deals with negation. In a qualitative approach like ours, the law $p(A) + p(\neg A) = 1$ would correspond to a version of the excluded middle principle according to which, independently of the body of evidence at hand, we would always be sure about either the plausibility of α or the plausibility of not α. This does not mean however that a plausibility system in which sentence "α is plausible or not (α is plausible)" is not valid will solve the problem. In order for "α is plausible or not (α is plausible)" not to be valid, we need just to have an instance of α able to make "α is plausible or not (α is plausible)" false. But here what we want is a little bit different: we want "α is plausible or not (α is plausible)" to be aprioristically false for all non-tautological and non-contradictory formulae α. Only in this way is that we will make sure that it will never be possible to make such sort of aprioristic claims about the plausibility of sentences. We will call this principle the *contrary plausibility excluded middle principle*[15].

Trivially, this principle is satisfied by our skeptical operator !: for any non-tautological and non-contradictory formula α, $\alpha! \vee (\neg\alpha)!$ (as well as $\alpha! \vee \neg(\alpha!)$) is not valid in LP^2. It is worthy to note that the principle represented by law $p(A) + p(\neg A) = 1$, which is quite acceptable if we take $p(\neg A)$ as meaning the improbability of A, holds in LP^2 when we take instead of $\neg$ the external negation ~: $\alpha! \vee \sim(\alpha!)$ is a theorem of LP^2. Concerning ?, however, things are not so straightforward. As it can be easily seen, ? does not satisfy the contrary plausibility excluded middle principle: even though $\alpha? \vee (\neg\alpha)?$ and $\alpha? \vee \neg(\alpha?)$ are not theorems of LP^2, for most formulae $\varphi \in \Im_{?\circ}$(namely those to which axiom A3 can be applied), schema $\varphi? \vee (\neg\varphi)?$ is LP^2-valid. It is as if it does not matter the quality or quantity of the body of evidence or even if φ is related to it, we will always have that either φ or $\neg\varphi$ is plausible. The same applies to ~: $\alpha? \vee (\sim\alpha)?$ is a theorem of LP^2.

An important conclusion one may draw from this is that paracompleteness as we have informally defined is not enough for a plausibility modality to satisfy our intuitions concerning the excluded middle principle. As exemplified by the behavior of ?, there may be a paracomplete operator that, as such, does not satisfy the excluded middle principle but nevertheless disrespects the inverse plausibility excluded middle principle. But we may wonder, is this way of defining paracomplete modalities reasonable? We know that $\alpha? \vee \neg(\alpha?)$ is not theorem only because α may contain some instance of !: for all other sorts of α's, $\alpha? \vee \neg(\alpha?)$ is LP^2 valid. This means that ? can

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Chapter 3 about the error-prone feature of inductive reasoning.

[15] The reason for the term "contrary" will become clear later.

be classified as a paracomplete operator only because of the possibility of using it along with a truly paracomplete modality in the same formula. Similarly for !. Only due to ?'s properties is that $\neg(\alpha! \wedge \neg(\alpha!))$ is not a LP^2-theorem. Consequently, ! may be entitled to be classified as a paraconsistent operator not due to ! itself, but to the possibility of using it along with ?. Given all this, we can fairly say that some other definition of paracompleteness and paraconsistency applied to modalities is in need here. Since the whole motivation of this discussion is the satisfaction or not of the contrary plausibility excluded middle principle, we will restrict ourselves to intra-logical accounts of paracompleteness and paraconsistency, which are based on the excluded middle and non-contradiction principles, respectively[16].

We start by suggesting that perhaps the notions of paracomplete modality and paraconsistent modality are primordially concerned with the satisfaction of what we may call the contraries of the non-contradiction and middle excluded principles, respectively, rather than with the non-satisfaction of the principles themselves. Let θ be a modal operator and $\sim$ a negation symbol. What we call the $\sim\theta$-middle excluded principle and the $\sim\theta$-non-contradiction principle may be represented, respectively, by the statements "for all formulae α, $\alpha\theta \vee \sim(\alpha\theta)$ is the case" and "for all formulae α, $\sim(\alpha\theta \wedge \sim(\alpha\theta))$ is the case." In this way, the contrary of the $\sim\theta$-excluded middle principle and the contrary of the $\sim\theta$-non-contradiction principle is, respectively, "for all non-tautological and non-contradictory formulae α, $\alpha\theta \vee \sim(\alpha\theta)$ is not (aprioristically) true" and "for all non-tautological and non-contradictory formulae α, $\sim(\alpha\theta \wedge \sim(\alpha\theta))$ is not (aprioristically) true." We then say that θ is paracomplete in relation to $\sim$ if and only if the contrary of the $\sim\theta$-excluded middle principle is satisfied, and paraconsistent in relation to $\sim$ if and only if the contrary of the $\sim\theta$-non-contradiction principle is satisfied. In this way, it is clear that only ? is paraconsistent in relation to $\neg$ and only ! is paracomplete in relation to $\neg$. The theorem below states such result.

Theorem 6.2.3. Let $\alpha \in \mathcal{L}_{?\circ}$ be such that $\nvdash_{LP2} \alpha$ and $\nvdash_{LP2} \neg\alpha$. The following formulae are *not* LP^2 theorems:

$\alpha! \vee \neg(\alpha!)$ $\qquad\qquad$ $\neg(\alpha? \wedge \neg(\alpha?))$

It should be noted however that unlike the above schemas of formulae, the contrary plausibility excluded middle principle is not about the plausibility of α and the implausibility of α, but about the plausibility of α and the plausibility of not α. It looks more like what we have called conceptual paracompleteness (Chapters 4 and 5) than the formal paracompleteness we have defined above. Of course, if we express the contrary plausibility excluded middle principle in terms of $\neg$, these two

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[16] An example of an extra-logical account of paracompleteness and paraconsistency was given in Section 3.3, when we have defined the notions of paracomplete and paraconsistent logics.

notions of paracompleteness will be equivalent: $\alpha! \vee \neg(\alpha!) \leftrightarrow \alpha! \vee (\neg\alpha)!$. Regarding ~ things are a little bit different. Since $\alpha! \vee \sim(\alpha!)$ is logically distinct from $\alpha! \vee (\sim\alpha)!$, we need to lay down another pair of notions similar to the pair $\tilde{\ }\theta$-excluded middle principle - $\tilde{\ }\theta$-non-contradiction principle which reflects this conceptual character of the contrary plausibility excluded middle principle. Let θ be a modal operator and $\tilde{\ }$ a negation symbol. We call the statements "for all formulae α, $\alpha\theta \vee (\tilde{\ }\alpha)\theta$ is the case" and "for all formulae, $\tilde{\ }(\alpha\theta \wedge (\tilde{\ }\alpha)\theta)$," respectively, the conceptual $\tilde{\ }\theta$-middle excluded principle and the conceptual $\tilde{\ }\theta$-non-contradiction principle. The contraries of these notions are defined in the same way as their non-conceptual counterparts. We then say that θ is conceptually paracomplete in relation to $\tilde{\ }$ if and only if the contrary of the conceptual $\tilde{\ }\theta$-excluded middle principle is satisfied, and conceptually paraconsistent in relation to $\tilde{\ }$ if and only if the contrary of the conceptual $\tilde{\ }\theta$-non-contradiction principle is satisfied. The theorems below state the conceptual paracompleteness of ! in relation to ¬ and ~, and the conceptual paraconsistency of ? in relation to ¬ and ~.

Theorem 6.2.4. Let $\alpha \in \mathcal{L}_{?\diamond}$ be such that $\nvdash_{LP2} \alpha$ and $\nvdash_{LP2} \neg\alpha$. The following formulae are *not* LP^2 theorems:

$\alpha! \vee (\neg\alpha)!$ $\qquad\qquad$ $\neg(\alpha? \wedge (\neg\alpha)?)$

Theorem 6.2.5. Let $\alpha \in \mathcal{L}_{?\diamond}$ be such that $\nvdash_{LP2} \alpha$ and $\nvdash_{LP2} \sim\alpha$. The following formulae are *not* LP^2 theorems[17]:

$\alpha! \vee (\sim\alpha)!$ $\qquad\qquad$ $\sim(\alpha? \wedge (\sim\alpha)?)$

The left-side formulae of the above theorems show that the contrary plausibility excluded middle principle is indeed satisfied by ! in connection with both ¬ and ~.

Turning back to ?'s non-satisfaction of the contrary plausibility excluded middle principle, or, according to our new definition, its non-conceptual paracompleteness, this apparent inadequacy can be justified by making reference to the strength of the plausibility notion that the credulous approach is supposed to entail. More specifically, the sort of evidential support required to classify a formula as credulously plausible may be so weak (in fact the weakest one) that, independently of the strength of the body of evidence at hand, it always transforms, for almost all formulae α, either α or $\tilde{\ }\alpha$ into a plausible formula (here $\tilde{\ }$ may be both ¬ or ~.) This of course makes real sense only when we take into account the epistemologically idealized character of our logic[18]. In the same way

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[17] It is worthy of mention that we could have defined the notions of conceptual paracompleteness and conceptual paraconsistency in relation to ~ in the standard way, without making use of the contraries of the excluded middle and non-contradiction principles.

[18] This was already mentioned at the end of Section 3.3.

that the principle α / $\Box\alpha$ makes sense only when we consider idealized agents who know all logical truths, formula $\alpha? \vee (\tilde{}\,\alpha)?$ makes sense only when we consider idealized bodies of evidence. This means that, independently of the agent's being or not able to conclude $\alpha?$ or $(\tilde{}\,\alpha)?$, the body of evidence at hand will always gives some sort of support to either $\alpha?$ or $(\tilde{}\,\alpha)?$. It is as if the sorts of bodies of evidence which will create our epistemical reality have to keep a minimum of similarity with the ontological reality: in the same way that in reality either α is true or $\tilde{}\,\alpha$ is true, concerning the weakest way we may evaluate the plausibility of sentences, independently of our even being able to inductively inferring α or $\tilde{}\,\alpha$, it is sure that at least one of them is plausible. This sort of evidential completeness is reflected in our semantic by the fact that each plausible world has attached to itself a complete propositional evaluation: for every propositional symbol p, either $v_w(p) = 1$ or $v_w(p) = 0$.

One may object that even though this explanation may be acceptable in the case of ontological concepts such as the notion of truth, it is untenable if we are talking about epistemic concepts which clearly involve some sort of constructive process. In contrast to the process of evaluating the truth of statements, which we may suppose is there independently of our knowing it, to classify a statement as plausible we have to somehow construct the way that goes from the evidences to the plausibility of the statement at hand. Therefore, it is simply incorrect to assume $\alpha? \vee (\tilde{}\,\alpha)?$ as aprioristically true, even if only for some non-tautological and non-contradictory subset of the logical language. In order to reply to this objection, it will be useful to recall the general purpose of idealization in the problem of concept explication. When idealized objects or idealized situations are used, the purpose is not to lay down an exact description of the "conceptual reality" we are analyzing, but rather to obtain a sort of approximation of it. Even though these idealized objects or situations do not correspond exactly to the way we think things are, through them we can better understand the essence of such things and, which is perhaps more interesting, by some sort of refinement to obtain a more accurate description of the conceptual reality in question. In our case, for example, through this "unrealistic" notion of plausibility, which, because of being idealized, is also the most basic and weakest sort of plausibility we can conceive, we can arrive at a stronger and "more realistic" notion of credulous plausibility. For instance, we can through the basic operators ? and ! define a stronger (in the sense of more realistic) notion of credulous plausibility which, due to its being conceptually paraconsistent and paracomplete in relation to $\neg$ and $\sim$ and paraconsistent and paracomplete in relation to $\neg$, completely satisfies (we hope) our intuitions concerning the principles of non-contradiction and excluded middle. What follows below is an attempt to formalize such derived notion of plausibility.

Definition 6.2.5. Let $\Im$ be a language, ϑ a ?-modal logic basis and $\alpha \in \Im_\vartheta$ a formula. We define below the derived operator **?** which is intent to formalize the notion of credulous-skeptical plausibility:

$\alpha\mathbf{?} =_{def} \alpha!?$

In a very trivial way then, we obtain an operator that incorporates at the same time the properties of paraconsistency and paracompleteness. If we use our expert plurality-oriented model for example, we will have that, in order for α to be credulously-skeptically plausible to a certain expert *w*, it will have necessarily to be an accepted hypothesis to at least one of the experts who *w* has trust in (which, naming such expert *w*', means that α has to be true to all experts who *w*' has trust in.) In this case, accepting the consensus between several experts as more relevant or stronger that the opinion of only one expert, then **?** will be a stronger sort of plausibility than ?. However, since $R_?$ is not transitive, the set of reliable experts of *w*' is not contained in the set of reliable experts of *w*, where $wR_?w'$. Therefore, if one takes the opinion of one reliable expert as more relevant than the consensus among several experts who are, in the opinion of this very reliable expert, equally reliable, then **?** will be weaker than ?. To this we can add that while $\alpha?$ can be read as "it is plausible that α is a true hypothesis", $\alpha\mathbf{?}$ means "it is plausible that α is an accepted hypothesis," which evidences that **?** is indeed a weaker sort of plausibility. The theorems below state both sorts of paraconsistency and paracompleteness of **?** in relation to $\neg$, and its conceptual paraconsistency and paracompleteness in relation to $\sim$.

Theorem 6.2.6. Let $\alpha \in \mathcal{L}_{?\diamond}$ be such that $\nvdash_{LP2} \alpha$ and $\nvdash_{LP2} \neg\alpha$. The following formulae are *not* LP^2 theorems:

$\alpha\mathbf{?} \vee \neg(\alpha\mathbf{?})$ $\qquad\qquad$ $\neg(\alpha\mathbf{?} \wedge \neg(\alpha\mathbf{?}))$

$\alpha\mathbf{?} \vee (\neg\alpha)\mathbf{?}$ $\qquad\qquad$ $\neg(\alpha\mathbf{?} \wedge (\neg\alpha)\mathbf{?})$

Theorem 6.2.7. Let $\alpha \in \mathcal{L}_{?\diamond}$ be such that $\nvdash_{LP2} \alpha$ and $\nvdash_{LP2} \sim\alpha$. The following formulae are *not* LP^2 theorems:

$\alpha\mathbf{?} \vee (\sim\alpha)\mathbf{?}$ $\qquad\qquad$ $\sim(\alpha\mathbf{?} \wedge (\sim\alpha)\mathbf{?})$

In order to better lay down the properties of **?**, we define an operator intent to formalize the notion of skeptical-skeptical plausibility.

Definition 6.2.6. Let $\Im$ be a language, ϑ a ?-modal logic basis and $\alpha \in \Im_{\vartheta}$ a formula. We define below the derived operator **!** which is intent to formalize the notion of *skeptical-skeptical plausibility*:

$\alpha\mathbf{!} =_{def} \alpha!!$

Theorem 6.2.8. All formulae of $\mathcal{L}_{?\diamond}$ that satisfy one of the following schemas of formula are LP^2-theorems (and consequently LP^2-valid.)

$(\neg\alpha)\mathbf{?} \leftrightarrow \neg(\alpha\mathbf{?})$	$(\alpha \rightarrow \beta)\mathbf{!} \rightarrow (\alpha\mathbf{?} \rightarrow \beta\mathbf{?})$
$(\alpha\mathbf{?} \vee \beta\mathbf{?}) \rightarrow (\alpha \vee \beta)\mathbf{?}$	$(\alpha \wedge \beta)\mathbf{?} \rightarrow (\alpha\mathbf{?} \wedge \beta\mathbf{?})$
$(\alpha \rightarrow \beta)\mathbf{?} \rightarrow (\alpha\mathbf{!} \rightarrow \beta\mathbf{?})$	$\alpha\mathbf{!} \rightarrow \alpha\mathbf{?}$
$\alpha\mathbf{!} \rightarrow \alpha\mathbf{?!}$	$\alpha\mathbf{?} \rightarrow \alpha\mathbf{??}$[19]

6.3 The Logic of Induction and its Applications

In this section we will deal with some instances of the inductive logic schema we have introduced in Section 6.1. Even though we will always utilize the same pseudo-inductive modal calculus, namely the $\succ$-extension of LP^2, we will introduce here what we think to be the most interesting logics of induction that can be obtained out of such pseudo-inductive modal calculus. Each one of them will correspond to a different philosophy of induction and plausibility. Consequently, they can also be taken as different solutions to the concept explication problem of induction. Through these logics of induction, we will also try to show how some of the problems raised in the course of the previous chapters and sections can be solved inside the general induction *explicatum* we have presented in Section 6.1. All this will of course point also to the fruitfulness and representative power of our approach.

Definition 6.3.1. Let $CP^2 = \langle\vartheta_{?\diamond}, \Sigma_{LP2}\rangle$ (based on $\mathcal{L}$) be LP^2 calculus of plausibility. We call the pseudo-inductive modal calculus based on $\mathcal{L}$ and CP^2 the *pseudo-inductive* LP^2 *calculus of plausibility*, which we represent by the symbol $CP^2_{\succ}$.

$CP^2_{\succ}$ is the $\succ$-extension of LP^2 we have mentioned above. As we have said, it will serve as the basis for all systems of induction we will present in this section.

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[19] It is worthy to note that all this analysis we made about the paracompleteness and paraconsistency of ?, ! and **?** applies not only to LP^2, but to all paranormal modal logics.

6.3.1 A Basic Logic of Induction

Definition 6.3.2. Let $CP^2_{\succ} = <\vartheta_{?\diamond}, \Sigma_{\succ}>$ be the pseudo-inductive LP^2 calculus of plausibility. The *basic plausible logic* $\psi_?$ is the pseudo-inductive logic of plausibility $<\vartheta_{?\diamond},\Sigma_{\succ},\{?\},\varnothing>$.

Definition 6.3.3. The *basic logic of induction* $\zeta_?$ is the logic of induction $<\psi_?, \mathcal{L}_{\psi?}, \vdash_c{}^{\psi?}>$, where $\psi_?$ is the basic plausible logic, $\mathcal{L}_{\psi?}$ is $\psi_?$'s inductive-plausible language and $\vdash_c{}^{\psi?}$ is the credulous $\psi_?$-relation of inductive consequence.

$\psi_?$ and $\zeta_?$ formalize the basic philosophy of induction we have sketched in chapters 2, 3 and 4 according to which the main epistemological feature of inductive inferences is that they "produce" credulously plausible hypotheses. This is done by requiring the consequent of inductive implications to be marked with the plausibility operator ?. A very important point implied by this restriction concerns the meaning of formulae of the form $\alpha \succ \beta \precsim \varphi$. In Section 6.1 we have interpreted such formulae as meaning "α inductively implies β unless φ." Now that β is necessarily a ?-marked formula or an inductive implication that ultimately leads to a ?-marked formula, there will be, according to our philosophy of induction and plausibility, an effective relation of confirmation between α and λ or, we may say, between α and β, for β will ultimately lead to the formulae with which α is so related. Therefore, in the case of $\Im_{\psi?}$, we can fairly say that formulae of the form $\alpha \succ \beta \precsim \varphi$ mean "α (weakly) confirms β unless φ."[20]

Within $\zeta_?$ we can lay down some inductive axioms that will help us to solve some of the problems we have mentioned previously. These axioms will be used to form what we have called in Section 6.1 an applied logic of induction.

Definition 6.3.4. Let $\Im$ be a language and $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\Im$ wherein $? \in \Theta_p$. The *plausibility conjunction axioms* $\Sigma_{?\wedge}$ in $\Im_\psi$ is the set composed by all formulae of $\Im_\psi$ satisfying the following schema of formula:

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[20]A point we cannot afford to mention concerns our decision of taking $\psi_?$'s credulous relation of inductive consequence $\vdash_c{}^{\psi?}$ as $\zeta_?$'s relation of inductive consequence. The reason for that has to do with the credulous purpose already embodied in $\zeta_?$ by $\psi_?$'s inductive plausibility modal operator set. Since some $\psi_?$-theories may have more than one extension, if we had used $\psi_?$'s skeptical relation of inductive consequence we would be forced to reject some inductive conclusions and consequently go against $\psi_?$'s policy of being tolerant in the matter of judging the plausibility of statements.

$C_{?\wedge}$: $\alpha? \wedge \beta? \succ (\alpha \wedge \beta)?$

As we have mentioned in the last section, axiom $C_{?\wedge}$ will be responsible for conjoining credulously plausible hypotheses in the case where they are not mutually contradictory. This, as we have seen, is a necessary step if we are to have a satisfactory explication of the notion of plausibility. We notice that $\alpha? \wedge \beta? \succ (\alpha \wedge \beta)?$ is an abbreviation for $\alpha? \wedge \beta? \succ (\alpha \wedge \beta)? \not\lesssim \bot$. Therefore, the only situations able to block the use of $C_{?\wedge}$ are those in which $\sim(\alpha \wedge \beta)$ is the case, or, in other words, in which the conjunction of α and β implies the trivialization of the theory. By managing things in this way, $C_{?\wedge}$ puts the last touches on the controversy around the conjunction principle and the lottery paradox we have been talking about since Chapter 3.

Definition 6.3.5. Let $\Im$ be a language and ψ = $<\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\Im$ wherein $?\in\Theta_p$. The *negative auto-epistemic axioms* Σ_{NA} in $\Im_\psi$ is the set composed by all formulae of $\Im_\psi$ satisfying the following schema of formula:

NA: $((\neg\Box\alpha)?)^\circ$

NA is the axiom which will transform $\zeta_?$ into a truly autoepistemic logic of plausibility. Note that $((\neg\Box\alpha)?)^\circ$ is an abbreviation for $\top \succ (\neg\Box\alpha)? \not\lesssim \bot$. Therefore, independently of the knowledge situation at hand, if it does not contain $\sim((\neg\Box\alpha)?)$ we will be able to infer nonmonotonically that $\neg\Box\alpha$ is plausible. The purpose of this is of course to make explicit that our agent does not know about the truthfulness of those formulae whose certainty cannot be inferred from his knowledge base. In the cases where $\Box\alpha$ does not belong to the logical theory, that is to say, α is not known, $(\neg\Box\alpha)?$ will be the case. We have already explained why such autoemistemic inference has to be represented nonmonotonically through an inductive implication. One may be thinking that because what we conclude through NA is $(\neg\Box\alpha)?$ and not $\neg\Box\alpha$, NA does not in fact perform the task we are claiming it performs. Not quite so. Since $\alpha? \rightarrow \Diamond\alpha$ (which is obtained from PP, K1 and $\Diamond\alpha \leftrightarrow \sim\Box\sim\alpha$), from $(\neg\Box\alpha)?$ we get $\Diamond\neg\Box\alpha$. From that, along with NP, we get $\neg\Box\neg\neg\Box\alpha$, which is equivalent to $\neg\Box\Box\alpha$. Since $\neg\Box\Box\alpha \rightarrow \neg\Box\alpha$, we have then that $\neg\Box\alpha$[21].

[21] Some of these theorems will be proved in a separated appendix.

Definition 6.3.6. Let $\Im$ be a language and $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\Im$ wherein $? \in \Theta_p$. The *introspective inductive axioms* $\Sigma_{I?}$ in $\Im_\psi$ is the set composed by all formulae of $\Im_\psi$ satisfying the following schema of formula:

$I_?$: $\alpha_1? \wedge \ldots \wedge \alpha_n? \succ (\neg(\alpha_1 \wedge \ldots \wedge \alpha_n))? \precsim \beta?$, wherein $\alpha_1, \ldots, \alpha_n$ and β are *different* basic formulae

Axiom $I_?$ tries to formalize what we have called in Chapter 3 introspective inductive reasoning, or the awareness of the error-prone feature of inductive reasoning. As have explained there, since inductive conclusions may be mistaken even when its premises are true (something the very past use of such sort of inference has shown), any fair account of inductive reasoning should have an axiom saying that, independently of the circumstances we are working on, it is plausible that one of the beliefs we now take as rational is false. Following the suggestion we have given in the mentioned chapter, $I_?$ formalizes that by saying that if *n* basic formulae[22] are plausible, then it is also plausible that some of them are false (or, as we wrote, that the negation of their conjunction is plausible.) The exception part of $I_?$ is intent to guarantee that no plausible atomic formula will be out of the conjunction $\alpha_1? \wedge \ldots \wedge \alpha_n?$: if this is the case, then the induction implication at hand cannot be used.

Definition 6.3.7. The *auto-epistemic introspective logic of induction* ζ_{AI} is the applied logic of induction $<\zeta_?, T_{CAI}>$, where $\zeta_?$ is the basic logic of induction and $T_{CAI} = \Sigma_{?\wedge} \cup \Sigma_{NA} \cup \Sigma_{I?}$, where $\Sigma_{?\wedge}$ are the plausibility conjunction axioms in $\mathcal{L}_{\psi?}$, Σ_{NA} the negative auto-epistemic axioms in $\mathcal{L}_{\psi?}$ and $\Sigma_{I?}$ the introspective inductive axioms in $\mathcal{L}_{\psi?}$.

6.3.2 Calculating Skeptically Plausible Hypotheses

A very important point we have not mentioned so far concerns the way according to which skeptically plausible facts will be introduced. Independently of the philosophical guidelines we adopt, one must agree that such skeptically plausible hypotheses are to be inferred in the same nonmonotonic way as weakly plausible ones. This means that we will have to extend $\psi_?$ in such a way as to allow !-marked formulae to be present in the consequent of inductive implications as well as to find out how !-marked formulae will be obtained inside the corresponding logic of induction

[22] In Chapter 5 we have defined a basic formula as a atomic formula or a negation of an atomic formula.

Definition 6.3.8. Let $CP^2_{\succ} = <\vartheta_{?\circ}, \Sigma_{\succ}>$ be the pseudo-inductive LP^2 calculus of plausibility. The *skeptical-credulous plausible logic* (or simply s-c plausible logic) $\psi_{?!}$ is the pseudo-inductive logic of plausibility $<\vartheta_{?\circ},\Sigma_{\succ},\{?,!\},\varnothing>$.

Definition 6.3.9. *The skeptical-credulous logic of induction* $\zeta_{?!}$ is the logic of induction $<\psi_{?!}, \mathcal{L}_{\psi?!}, \vdash_c^{\psi?!}>$, where $\psi_{?!}$ is the skeptical-credulous plausible logic, $\mathcal{L}_{\psi?!}$ is $\psi_{?!}$'s inductive-plausible language and $\vdash_c^{\psi?!}$ is the credulous $\psi_{?!}$-relation of inductive consequence.

As one would expect, $\zeta_{?!}$ allows both sorts of plausible statements to be inductively inferred through inductive implications. Since $\zeta_{?!}$ is supposed to embody both credulous and skeptical approaches, there was no philosophically determinant reason for having taken the credulous $\psi_{?!}$-relation of inductive consequence instead of the skeptical one. Concerning the point of how to obtain !-marked formulae, the most straightforward solution would be to let our system user to specify in the form of !-consequent inductive implications ($\alpha \succ \beta! \precsim \varphi$) his own rules for inferring strongly plausible facts. In the same way that a descriptive logic of induction is not supposed to say when a piece of evidence weakly confirms a hypothesis, it is also not supposed to say when a hypothesis is strongly confirmed by a piece of evidence: all it has to do is to set the basic representational tools one needs to represent such strong plausibility rules of inference. From a formal point of view, this means that the formulae which will effectively introduce the skeptically plausible hypothesis will be contained in the set A which will make use of $\zeta_{?!}$'s relation of inductive consequence.

While this looks all right, we know that, from a semantic point of view, the notions of skeptical plausibility and credulous plausibility are connected in such a way that skeptically plausible facts are obtained from a specific set of credulously plausible ones. In all examples we have considered so far, the notion of acceptability has been explained as that consensus one obtains when a hypothesis is taken as plausible from all different viewpoints. Therefore, at the same time that we should allow the user to set skeptically plausible inductive implications, it seems reasonable also to expect that the logic of induction itself sets in the form of a special set of inductive axioms a logical way through which skeptically plausible hypotheses are obtained from credulously plausible ones.

The form of such inductive axioms will of course depend on the philosophy of induction we want of formalize within $\psi_{?!}$. For instance, if we adopt a very simple sort of confirmation by enumeration philosophy, α will be taken as accepted only after it has got enough weak confirmation. It is as if, by observing one black raven we turn the hypothesis "all ravens are black" into a very weakly plausible one; by observing another one we increase a little bit its degree of plausibility; and so and so forth, until that, after have observed a certain number of black ravens, say *n*, we raise the hypothesis in question to the status of an accepted or strongly plausible statement. In order to formalize that, we need of course to somehow quantify how much a hypothesis was weakly confirmed or, in the context of taking weak confirmation and credulous plausibility as the same, how weakly plausible a hypothesis is.

The most straightforward way to do that inside LP^2 is to count in how many plausible worlds a hypothesis is true. If α is true in at least one plausible world we write $\alpha?_1$; if it is true in at least two plausible worlds we write $\alpha?_2$ … until it is true in at least *n* plausible worlds, in the case we write $\alpha?_n$ or $\alpha!$. What follows is an attempt to set within LP^2 such quantified notion of plausibility along with the notion of acceptability entailed by it.

Definition 6.3.10. Let $\Im$ be a language, $\vartheta = <\Theta', \Theta''>$ a ?-modal logic basis, and $\alpha,\beta \in \Im_\vartheta$ two formulae. We define the following abbreviations in $\Im_\vartheta$:

(i) $\alpha?_1 =_{def} \alpha?$;

(ii) $\alpha?_2 =_{def} (\alpha \wedge q)? \wedge (\alpha \wedge \neg q)?$, where q is an arbitrary atomic formula of $\Im_\vartheta$;

(iii) $\alpha?_n =_{def} (\alpha \wedge p_1 \wedge q)? \wedge \ldots \wedge (\alpha \wedge p_m \wedge q)? \wedge (\alpha \wedge p_1 \wedge \neg q)? \wedge \ldots \wedge (\alpha \wedge p_m \wedge \neg q)?$, where $n = 2^{k+1}$, $m = 2^k$, $k>0$, $\alpha?_m \equiv (\alpha \wedge p_1)? \wedge \ldots \wedge (\alpha \wedge p_m)?$ and q is an arbitrary atomic formula of $\Im_\vartheta$ which do not occur in p_i;

(iv) $\alpha?_n =_{def} (\alpha \wedge p_1)? \wedge \ldots \wedge (\alpha \wedge p_n)?$, where $2^{k+1}>n>2^k$ and $\alpha?_{n+1} \equiv (\alpha \wedge p_1)? \wedge \ldots \wedge (\alpha \wedge p_n)? \wedge (\alpha \wedge p_{n+1})?$.

$\alpha?_n$ may be understood as meaning "the degree of plausibility of α is *n*." As we have mentioned above, such meaning is achieved by counting in how many plausible worlds α is true, which is performed with the help of the classical feature of plausible worlds. Given an atomic formula q, we know that q and $\neg$q cannot be true at the same time in world *w*. Therefore, if $(\alpha \wedge q)?$ and $(\alpha \wedge \neg q)?$ are true, then the plausible worlds which make these two formulae true cannot be the same. Consequently, α is necessarily true in at least two worlds. Similarly, given an atomic formula p distinct from q, $(\alpha \wedge q \wedge p)? \wedge (\alpha \wedge \neg q \wedge p)? \wedge (\alpha \wedge q \wedge \neg p)?$ means that α is true in at least three worlds, $(\alpha \wedge q \wedge p)? \wedge (\alpha \wedge \neg q \wedge p)? \wedge (\alpha \wedge q \wedge \neg p)? \wedge (\alpha \wedge \neg q)?$ that α is true in at least four worlds, and so on and so forth. The way this numerical (credulous) notion of plausibility is used to obtain the skeptical plausibility is shown below[23].

Definition 6.3.11. Let $\Im$ be a language, $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\Im$ wherein $\{?,!\} \subseteq \Theta_p$ and $n \geq 1$ is natural number. The n-*numerical acceptability axioms* $\Sigma_{!n}$ in $\Im_\psi$ is the set composed by all formulae of $\Im_\psi$ satisfying the following schema of formula:

$!_n$: $\alpha?_n \succsim \alpha! \not\precsim (\neg\alpha)?$

The reason why we have decided to represent $!_n$ as an inductive implication and not deductively through $\rightarrow$ has to do with the nature of acceptance in the context of a confirmation by enumeration philosophy. As one can see, the above way of obtaining skeptically plausible hypotheses is very similar to the way many probabilists define the notion of acceptability (as we have seen in Chapter 3): all one has to do is to pick a number and use it as the threshold level of acceptance according to which weakly plausible hypotheses will be transformed into strongly plausible ones. The obvious problem with this approach is that even after having taken the hypothesis "all ravens are black" (let us call it H) as accepted (due to the observation of *n* black ravens, let us say), we will have still to consider the defeasible nature of accepted hypotheses. What we mean is that "H is accepted" can be falsified even after getting the status of an accepted hypothesis. In the case where H! is falsified through the observation of one non-black raven, there will be no problem at all, for the very weak instances of "H is plausible" will be reconsidered. But how about if we have a weaker falsification where $\neg$H is discovered to be (weakly of strongly) plausible? In this case, if we write $\alpha?_n \rightarrow \alpha!$ instead of $!_?$, H! will not be defeated and probably two incompatible extensions will arise. The important point however concerns the soundness of taking H as an accepted hypothesis in the presence of the plausibility of $\neg$H. More specifically, is it reasonable to keep H! even after having discovered that $\neg$H is weakly plausible, for example?

All this has of course to do with the above mentioned error-phone feature of inductive reasoning. In the same way that we should be aware that our weakly plausible hypotheses may be false, we should also consider that our accepted hypotheses may be falsified both from a strong, truthfulness way, as well as from a weak, plausible way. Because of that, we have written the principle in question as an inductive implication and put $(\neg\alpha)?$ in its exception part. Moreover, by writing $\alpha?_n \rightarrow \alpha!$ we would be imposing a quite heavy limitation on the semantic models which would be taken into account. More specifically, we would restrict ourselves only to models M such

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[23] This numerical skeptical plausibility could also be defined in a somehow opposite way: rather than looking inside the set of plausible worlds of world w ($R_?(w)$), we could look at the sets of plausible worlds of each

that, for every world w of M, $R_?(w)$ has exactly n elements. Because of that, it would be impossible, for instance, to have in the same extension $\alpha?_n$ and $(\neg\alpha)?$. What follows below is the definition of what we can call the n-numerical logic of induction.

Definition 6.3.12. Let $n \geq 1$ be a natural number. The *n-numerical logic of induction* ζ_n is the applied logic of induction $<\zeta_{?!}, \Sigma_{!n}>$, where $\zeta_{?!}$ is the skeptical-credulous logic of induction and $\Sigma_{!n}$ are the n-numerical acceptability axioms in $\mathcal{L}_{\psi?!}$.

Another way to "automatically" get skeptically plausible facts from credulously plausible ones is to use the consistency approach adopted my most AI theorists to define the skeptical and credulous approaches to induction. As we have seen in Chapters 3 and 4, the way these approaches are defined in AI depends on the notion of extension, understood as a maximal consistent set of conclusions obtained deductive-inductively from a specific knowledge situation. As a consequence of that, what we may call the criterion of identity of extensions is based on the existence or not of contradictory formulae in the extension: extension E is different from extension E' iff, for some formulae φ, $\varphi \in E$ and $\sim\varphi \in E'$. Now, if we define skeptically and credulously plausible facts in terms of extensions, this criterion trivially entails that, given a specific set of extensions, if α is credulously plausible and it is not the case that $\sim\alpha$ is credulously plausible then α is skeptically plausible, and vice-versa.

If we wish to fully represent this equivalence between the skeptical and the credulous plausibility, we will have to deal with a very trick difficulty: the necessity of representing that $(\sim\alpha)?$ is not the case. The solution is to represent half the equivalence (the one that interests us of course) as an inductive implication with $(\sim\alpha)?$ as its exception part and the credulous plausibility of α as its antecedent. In this way, we will have that if $\alpha?$ is the case, then $\alpha!$ can be inductively inferred unless $(\sim\alpha)?$ is the case.

Definition 6.3.13. Let $\Im$ be a language and $\psi = <\vartheta,\Sigma,\Theta_p,\Theta_t>$ a pseudo-inductive logic of plausibility based on $\Im$ wherein $\{?,!\} \subseteq \Theta_p$. The *consistency acceptability axioms* $\Sigma_{!\perp}$ in $\Im_\psi$ is the set composed by all formulae of $\Im_\psi$ satisfying the following schema of formula:

$!_\perp$: $\alpha? \succ \alpha! \precnsim (\sim\alpha)?$

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member of $R_?(w)$: (i) $\alpha?_1 = \alpha?$ and (ii) $\alpha?_{n-1} = \alpha?_n?$.

Definition 6.3.14. The *consistency acceptability logic of induction* ζ_{CA} is the applied logic of induction $<\zeta_{?!}, \Sigma_{!\perp}>$, where $\zeta_{?!}$ is the skeptical-credulous logic of induction and $\Sigma_{!\perp}$ are the consistency acceptability axioms in $\mathcal{L}_{\psi?!}$.

It should be noted that despite the consistency acceptability axiom's motivation's being, as we have explained, the way the skeptical and credulous views are understood in AI, the way the term "extension" was used to explain such axiom is not identical to the way it is used in definition 6.1.12. From a semantic point of view, the use of the term "extension" in the mentioned explanation comes in the context of a specific application of our semantic model where each plausible world, here called extension, is identified solely on the basis of the truth-values of atomic formulae. It is what we may call an intra-logical use of the term. Trivially then, the satisfaction of the definition of extension as an inferentially maximal consistent set of deductively-inductively obtained formulae will be achieved only indirectly: while the maximal aspect is obtained in conjunction with a modal valuation which allows us to know the true value of all non-modal formulae, the inferential aspect will be obtained by supposing that the plausible world in question is the result of some inductive inferential mechanism. In definition 6.1.12 on the other hand, the term "extension" is used with an extra-logical meaning, where a so-called extension is the end product of the whole inferential mechanism of ζ applied to a set of formulae A. To see that there is really a difference, note that if A $\vdash_{\zeta?!}$ α? and A $\vdash_{\zeta?!}$ $(\neg\alpha)$?, from the intra-logical perspective there will be at least two extensions (plausible worlds) where α is true in one and $\neg\alpha$ is true in the other. On the other hand, from the extra-logical perspective, α? and $(\neg\alpha)$? will belong to the same and possibly the only one extension. Similarly, while from the internal perspective there is no way of simultaneously dealing with α! and $(\sim\alpha)$? (which in formal terms means that no model can satisfy at the same time the two formulae), from the external perspective such situation is easily treatable by putting the two formulae in separated extensions.

In order to finish this section, we should mention the role of all the logics of induction presented so far in the task of explicating the notion of plausibility. As we have mentioned in Section 6.2, unless we are able to represent the susceptibility of plausible statements of being refuted or given up we will not have a satisfactory solution for the concept explication problem of plausibility. Since this task is an eminently nonmonotonic one, it cannot be done inside LP^2. Therefore, by allowing us to infer nonmonotonically plausible facts through inductive implications, logics $\zeta_?$, ζ_{AI}, $\zeta_{?!}$, ζ_n and ζ_{CA} complement LP^2 in the task of explicating the notion of plausibility.

6.3.3 Towards an Epistemic Logic of Induction

In the logics of induction we have been discussing so far, we have used exclusively ? as the credulous plausibility marker. However, as we have seen in Section 6.2, due to its idealized character, in special its behavior in connection with the middle excluded principle, a more "realistic" application may require a stronger (or, if you wish, weaker) plausibility symbol. What follows below is an attempt to provide such a more "realistic" and consequently epistemologically sounder logic of induction where $\mathbf{?}$ is used instead of ? as the inductive plausibility modal operator.

Definition 6.3.15. Let $CP^2_{\succ} = <\vartheta_{?\circ}, \Sigma_{\succ}>$ be the pseudo-inductive LP^2 calculus of plausibility. The *s-basic plausible logic* $\psi_{\mathbf{?}}$ is the pseudo-inductive logic of plausibility $<\vartheta_{?\circ}, \Sigma_{\succ}, \{!?\}, \varnothing>$.

Definition 6.3.16. The *s-basic logic of induction* $\zeta_{\mathbf{?}}$ is the logic of induction $<\psi_{\mathbf{?}}, \mathcal{L}_{\psi\mathbf{?}}, \vdash_c^{\psi\mathbf{?}}>$, where $\psi_{\mathbf{?}}$ is the s-basic plausible logic, $\mathcal{L}_{\psi?}$ is $\psi_{\mathbf{?}}$'s inductive-plausible language and $\vdash_c^{\psi\mathbf{?}}$ is the credulous $\psi_{\mathbf{?}}$-relation of inductive consequence.

About the axioms we have presented in Subsection 6.3.1 to be used along with $\zeta_?$ and build what we have called the auto-epistemic introspective logic of induction, all of them, from definitions 6.3.4 to 6.3.6, can be easily transformed into $\psi_{\mathbf{?}}$ axioms by replacing ? by $\mathbf{?}$: the plausibility conjunction axiom will become $\alpha\mathbf{?} \wedge \beta\mathbf{?} \succ (\alpha \wedge \beta)\mathbf{?}$, the negative auto-epistemic axiom $((\neg\Box\alpha)\mathbf{?})^\circ$ and the introspective inductive axiom $\alpha_1\mathbf{?} \wedge \ldots \wedge \alpha_n\mathbf{?} \succ (\neg(\alpha_1 \wedge \ldots \wedge \alpha_n))\mathbf{?} \not\precsim \beta\mathbf{?}$. It is worthy to note that axiom $C_{?\wedge}$ could be kept in its ?-form if ? were one of the inductive plausibility modal operators and $C_{?\wedge}$'s sole purpose were to allow the conjunction of $\mathbf{?}$-marked formulae: from $\alpha!? \wedge \beta!?$ we could inductively get $(\alpha! \wedge \beta!)?$, from which, along with $\alpha! \wedge \beta! \leftrightarrow (\alpha \wedge \beta)!$, we would obtain $(\alpha \wedge \beta)!?$. Another important point is that, akin to $((\neg\Box\alpha)?)^\circ$, $((\neg\Box\alpha)\mathbf{?})^\circ$ will also produce the intended result of concluding $\neg\Box\alpha$: from $\alpha? \rightarrow \Diamond\alpha$ we get $\alpha?! \rightarrow (\Diamond\alpha)!$, from which, along with $(\Diamond\alpha)! \rightarrow (\Diamond\alpha)?$ we get $\alpha?! \rightarrow (\Diamond\alpha)?$. From that, along with $(\Diamond\alpha)? \rightarrow \Diamond\Diamond\alpha$ we get $\alpha?! \rightarrow \Diamond\Diamond\alpha$. From that, along with $(\neg\Box\alpha)!?$, we get $\Diamond\Diamond\neg\Box\alpha$ that implies $\neg\Box\Box\Box\alpha$ and, consequently, $\neg\Box\alpha$. Similar adjustments can me made in order to obtain stronger versions of what we called in Subsection 6.3.2 the n-numerical logic of induction and the consistency acceptability logic of induction (in which cases both operators ! or $\mathbf{!}$ may be used.)

Another feature of the logics of induction we have presented ($\zeta_?$ included) is that we have not imposed any sort of restriction on the form of ordinary formulae. More specifically, all pseudo-inductive logics of plausibility we have considered had $\varnothing$ as their sets of basic truth modal operators. This means that the most basic claim unit of the languages $\Im_\psi$ we have analyzed are the truth of statements. While from a technical viewpoint this does not seem to be any problem – after all, we will have even more freedom at the time of writing our formulae –, from the epistemological position we have adopted in Chapter 2 it has some troublesome implications.

We recall that in that chapter we have taken induction as primarily concerned with our limitations of knowing. This leaded us to the new understanding of accepting induction not as those inferences which lead from truth premises to plausible conclusions, but as those which lead from certain statements to plausible and therefore uncertain ones. Put differently, induction, as we understand it, is something essentially epistemic and cannot, in all its relevant aspects, be dissociated from knowledge issues. Now, if our pseudo-inductive logic of plausibility $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ is such that $? \in \Theta_p$ and $\Theta_t = \varnothing$, for example, then the plausible-inductive language of our logic of induction will have formulae of the form $\alpha \succ \beta? \precsim \varphi$ (α and β being non-modal formulae), which can be taken as meaning something like "from the *truth* of α, conclude inductively the plausibility of β, unless φ is true."

As can be easily seen, this goes against the epistemological philosophy of induction we have just sketched. Since according to this philosophy all components of an inductive inference should be about our knowledge of reality and not about reality itself, from the point of view of our formal framework we will have that all components of an inductive implication should have some epistemic modal operator attached to it. Going a little bit deeper, we can say that since our purpose is to provide a theory able to perform inductive inferences and since this will be achieved by making inductive implications (formed, as we have agreed, exclusively with the help of epistemic formulae) and ordinary formulae to interact, it does not make sense to allow ordinary formulae to be of any other sort than likewise epistemic. That is to say, since the whole purpose of our endeavor is to account for a sort of inference whose raison d'être is exactly our failure to dispose of precise and complete information, it is at least incoherent to suppose that it (the failure) affects solely inductive implications. Put differently, having a really comprehensive account of the plausible or imperfect but at the same time epistemologically sound reasoning implies having to take all formulae as being primarily concerned with the same sort of imprecision that makes inductive inferences into a plausible subject matter.

In our mathematical framework, what all that means is that the set of basic truth modal operators Θ_t should consist of a set composed exclusively of epistemological operators. What follows below is an attempt to formally materialize such philosophical guidelines.

Definition 6.3.17. Let $CP^2_{\succ} = <\vartheta_{?\diamond}, \Sigma_{\succ}>$ be the pseudo-inductive LP^2 calculus of plausibility. The epistemic plausible logic $\psi_{\Box?}$ is the pseudo-inductive logic of plausibility $<\vartheta_{?\diamond},\Sigma_{\succ},\{\Box?,\Box!\},\{\Box\}>$.

Definition 6.3.18. The *epistemic logic of induction* $\zeta_{\Box?}$ is the logic of induction $<\psi_{\Box?}, \mathcal{L}_{\psi\Box?}, \vdash_s{}^{\psi\Box?}>$, where $\psi_{?!}$ is the epistemic plausible logic, $\mathcal{L}_{\psi\Box?}$ is $\psi_{\Box?}$'s inductive-plausible language and $\vdash_s{}^{\psi\Box?}$ is the skeptical $\psi_{\Box?}$-relation of inductive consequence.

$\zeta_{\Box?}$ formalizes the philosophy of induction we have sketched in Chapter 2 and recapitulated some lines above. By equating Θ_t with $\{\Box\}$, we make sure that all our claims will be about our knowledge of the truth, and not about truth itself. Rather than claiming that the earth is flat, for example, we will be able just to claim that *we know* that the earth is flat. It is as if, due to the limitations of our knowledge acquiring abilities, in order for a claim to be acceptable as meaningful it should never refers to reality itself, but always to the way which reality appears to be to us. $\zeta_{\Box?}$ can therefore be taken as a sort of a logic of appearance wherein not only epistemological claims are allowed, but also no sort of non-epistemological claim is tolerated.

An important consequence of this radical epistemological commitment is that all claims we are able to represent inside such a strictly epistemological logic will have a sort of paracomplete behavior. This is a point we have not mentioned before, but if we adopt a "realistic" epistemological point of view, given a formal language $\Im$ there will be trivially many statements of $\Im$ which we do not have any sort of knowledge about. For many statements of such language, we simply will not be able to claim neither that they are true nor that they are false. This of course means that what we may call the certainty middle excluded principle – "α is certain or $\neg\alpha$ is certain" – should not be valid in a truly epistemic logic of certainty. As a consequence of that, the logical machinery that will reason about these claims necessarily has to be a paracomplete one. This is achieved by $\zeta_{\Box?}$ by the already mentioned conceptual paracompleteness of $\Box$ in LP^2. This is formally described in the theorem below:

Theorem 6.3.1. If $\alpha \in \mathcal{L}_{?\diamond}$ is such that $\nvdash_{LP2} \alpha$ and $\nvdash_{LP2} \neg\alpha$, then $\Box\alpha \vee \Box(\neg\alpha)$ is *not* LP^2 theorem. If $\alpha \in \mathcal{L}_{?\diamond}$ is such that $\nvdash_{L\backslash P2} \alpha$ and $\nvdash_{LP2} {\sim}\alpha$, then $\Box\alpha \vee \Box({\sim}\alpha)$ is *not* LP^2 theorem.

The reason why we have taken $\Theta_p = \{\Box?, \Box!\}$ is twofold. First, by taking $\Box?$ and $\Box!$ instead of ? and ! we act in accordance with the strictly epistemological philosophy of induction we have been talking about: rather than speaking about the plausibility of α's being true, we speak about the plausibility of it's being certain. Second, by using the derived plausibility symbol $\Box?$ instead of ?, we automatically satisfy what we have called in Section 6.2 the contrary plausibility excluded middle principle: representing $(\Box\alpha)?$ as $\alpha\mathbf{?}_\Box$ and $(\Box\alpha)!$ as $\alpha\mathbf{!}_\Box$ we have the following theorems.

Theorem 6.3.2. Let $\alpha\in\mathcal{L}_{?\diamond}$ be such that $\nvdash_{LP2}\alpha$ and $\nvdash_{LP2}\neg\alpha$. The following formulae are *not* LP^2 theorems:

$\alpha\mathbf{!}_\Box \vee (\neg\alpha)\mathbf{!}_\Box$ $\qquad\qquad$ $\neg(\alpha\mathbf{?}_\Box \wedge (\neg\alpha)\mathbf{?}_\Box)$

Theorem 6.3.3. Let $\alpha\in\mathcal{L}_{?\diamond}$ be such that $\nvdash_{LP2}\alpha$ and $\nvdash_{LP2}{\sim}\alpha$. The following formulae are *not* LP^2 theorems:

$\alpha\mathbf{!}_\Box \vee ({\sim}\alpha)\mathbf{!}_\Box$ $\qquad\qquad$ ${\sim}(\alpha\mathbf{?}_\Box \wedge ({\sim}\alpha)\mathbf{?}_\Box)$

It worthy to note that despite our having committed ourselves to a certain number of logical principles concerning the notions of plausibility and certainty, namely the ones incorporated by LP^2, we have still at our disposal a considerable variety of representational systems of induction that can be made out of it. Therefore, the logics of induction as well and the applied logics of induction we have introduced here should be seen more as an illustration of the fruitfulness and representative power of our framework than a definitive position of ours about what a proper system of induction is supposed to be. In fact, this has been one of the main guidelines of this work. Rather than proposing a definitive approach to induction and plausibility, we intend to show how some AI techniques may be extremely fruitful in clarifying the notions of induction and plausibility.

6.4 An Application in the Philosophy of Science

In this section we will consider how the inductive logics introduced in the last section can be of some help in the problem of confirmation of hypotheses and theories in the philosophy of science. We will focus on Hempel's conditions of adequacy and the hypothetico-deductive method. Even though we will make use exclusively of $\zeta_?$ in the formalization of these models, the applied logics of induction to be presented here could have been made out of any one of the logics of induction we have introduced in the last section. We picked $\zeta_?$ just because it is the simplest of all.

6.4.1 Hempel's Calculus of Inductive Implication

The logics of induction we have considered so far are not exactly what we have called a calculus of inductive implication. Rather, the T components of the applied logics of induction introduced in Subsections 6.3.1 and 6.3.2 are simple representations of some properties we wish ? and ! to posses which cannot be represented through a deductive calculus. On the other hand, a calculus of inductive implication is meant to set the logical cannons which inductive implications are supposed to obey. As an example of such cannons, we have spoken about inductive implication's supposed transitivity property. What will follow in this subsection is an attempt to extend $\zeta_?$ in such a way as to obtain something worthy of being called a calculus of inductive implication.

In Chapter 3 we have talked about the conditions laid down by Carl Hempel which any definition of confirmation is supposed to satisfy[24]. By definition of confirmation Hempel meant any way of saying whether or not *e* confirms or inductively supports *h*, for any pair of sentences *e-h*. Since the end product of a model of confirmation is, among other things, a set of statements of the form "*e* inductively supports *h*," Hempel's conditions can be said to define the logical restrictions which these statements are supposed to obey. Consequently, forgetting about the way these statements are generated, they can also be taken as the basis of a calculus of inductive confirmation. We list below the conditions which Hempel accepts as sound[25]:

0. *General Applicability condition*: a definition of confirmation should be applicable to the confirmation of statements of any logical form, producing then confirmation statements formed by statements of any form;

I. *Entailment condition*: if statement α entails (that is, logically implies) statement β, then β should be confirmed by α;

II. *Consequence condition*: if statement α confirms statement β and β logically implies statement φ, then α should also confirm φ;

III. *Equivalence condition*: if statement α confirms statement β and β is logically equivalent to φ, then α should also confirm φ;

IV. *Weak Consistency condition*: if statement α confirms statement β and α is not self-contradictory, then α and β should be logically compatible;

IV'. *Strong Consistency condition*: if statement α confirms statements β and φ and α is not self-contradictory, then β and φ should be logically compatible.

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[24] Hempel (1945). For a discussion on Hempel's conditions see Scheffler (1963) part II, Skirms (1966), Hesse (1970) and Hanen (1971).

[25] As shall see below, Hempel lays down some other conditions which he rejects as unacceptably problematic. We have here slightly modified the way Hempel originally presented his conditions.

We first observe that the part of condition 0 which interests us is automatically satisfied by $\zeta_?$: taking $\alpha \succ \beta?$ as meaning "α confirms β," it is clear that α and β can be of any logical form. Second, conditions IV and IV' are, respectively, conditions 8.31 and 8.32 which we have discussed in Chapter 3. Considering then what we have concluded in that chapter, IV' is not to be incorporated into a calculus of inductive implication formalized within $\zeta_?$ (although it could be incorporated into a logic of induction whose pseudo-inductive logic of plausibility $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ were such that $\Theta_p = \{!\}$, for example). Of course this is true only when confirmation statements of the form "α confirms β" are represented by $\alpha \succ \beta?$. If we take $\alpha \succ \beta$ as meaning "α confirms β" then condition IV' will be automatically satisfied: since every β in this case will be a formula of the form $\varphi?$, if $\alpha \succ \beta$ and $\alpha \succ \lambda$, then β and λ will always be compatible with each other.

We shall now show how these conditions, with the exception of 0 and IV', can be represented inside $\mathcal{L}_{\psi?}$. We should however remind that by doing that, we are not taking the side of Hempel and defending the reasonableness or even the tenability of his conditions. Rather, our purpose is just to show how our formalism can very easily incorporate something worthy of being called a calculus of inductive implication.

Definition 6.4.1. Let $\mathfrak{I}$ be a language and $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\mathfrak{I}$ wherein $? \in \Theta_p$. The *Hempel confirmation axioms* Σ_H in $\mathfrak{I}_\psi$ is the set composed by all formulae of $\mathfrak{I}_\psi$ satisfying the following schemas of formula:

$I_\succ$:	$(\alpha \rightarrow \beta) \rightarrow (\alpha \succ \beta?)$	*Entailment*
$II_\succ$:	$(\alpha \succ \beta?) \rightarrow ((\beta \rightarrow \varphi) \rightarrow (\alpha \succ \varphi?))$	*Consequence*
$III_\succ$:	$(\alpha \succ \beta?) \rightarrow ((\beta \leftrightarrow \varphi) \rightarrow (\alpha \succ \varphi?))$	*Equivalence*
$IV_\succ$:	$(\alpha \succ \beta?) \rightarrow ((\alpha \wedge \beta \rightarrow \bot) \rightarrow \bot)$	*Weak consistency*

About this formulation of Hempel's conditions, we just observe that we have represented the weak consistency condition through an axiom that basically turns any theory containing a formulae of the form $\alpha \succ \beta?$ where α and β are mutually contradictory into an inconsistent one.

Now, some philosophers have tried to extend this list by proposing additional conditions which seem to be in accordance with Hempel's idea. We skip the informal presentation and lay down below some of these conditions represented in our formal inductive language.

Definition 6.4.2. Let $\Im$ be a language and $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\Im$ wherein $?\in\Theta_p$. The *Hempel additional confirmation axioms* $\Sigma_{H'}$ in $\Im_\psi$ is the set composed by all formulae of $\Im_\psi$ satisfying the following schemas of formula:

$V_{\succ}$: $(\alpha \succ \beta?) \rightarrow ((\alpha \leftrightarrow \varphi) \rightarrow ((\varphi \succ \beta?))$ *Inverse Equivalence*

$VI_{\succ}$: $(\alpha \succ \beta?) \rightarrow ((\beta \succ \varphi?) \rightarrow (\alpha \succ \varphi?))$ *Transitivity*

$V_{\succ}$ is a sort of inverse of $III_{\succ}$ which applies the equivalence principle present in $III_{\succ}$ to the evidences rather than to the hypothesis. $VI_{\succ}$ is the transitivity principle we have already mentioned[26]. Together with $I_{\succ}$, $VI_{\succ}$ implies what we can call the *material-inductive transitivity* or simply *inverse consequence* condition: $(\alpha \rightarrow \beta) \rightarrow ((\beta \succ \varphi?) \rightarrow (\alpha\succ \varphi?))$. Now can we put these axioms together to define what we call the Hempel logic of induction.

Definition 6.4.3. The *Hempel logic of induction* ζ_H is the applied logic of induction $<\zeta_?, T_H>$, where $\zeta_?$ is the basic logic of induction and $T_H = \Sigma_H \cup \Sigma_{H'}$, where Σ_H are the Hempel confirmation axioms in $\mathcal{L}_{\psi?}$ and $\Sigma_{H'}$ the Hempel additional confirmation axioms in $\mathcal{L}_{\psi?}$[27].

The role of axioms $I_{\succ}$-$VI_{\succ}$ in the above-defined logic of induction evidences a very important difference between Hempel's project and ours. The whole purpose of Hempel's laying down his conditions of adequacy was to set the basic features which a good definition of confirmation, or in our terminology, a good model of confirmation is supposed to possess. And it is quite remarkable how Hempel tried to achieve this. Rather than directly addressing the internal mechanism which determines whether or not hypothesis *h* is confirmed by evidence *e*, he addressed the "output" of the model: if the set of confirmation statements produced by a model of confirmation is such that it satisfies all the conditions, then the model in question can be taken as a good definition of confirmation. Now, in the case of ζ_H there is no such thing as models of confirmation to be evaluated. The purpose of axioms $I_{\succ}$-$VI_{\succ}$ is what we can call an axiomatic one: at the same time that they stipulate some properties we wish our confirmation statements to have, they also set the reasoning canons which any confirmation statement to be represented inside shall ζ_H follow: {$\alpha \succ$

[26] Formulations of $V_{\succ}$ and $VI_{\succ}$ have appeared, for instance, in Scheffler (1963) and Hesse (1970), respectively.

[27] Since $II_{\succ}$ can be obtained from $VI_{\succ}$ and $I_{\succ}$ and $III_{\succ}$ from $II_{\succ}$, they ($II_{\succ}$ and $III_{\succ}$) did not have in fact to be present in ζ_H's axiomatization. It is just for the same of clarity of presentation that we take them into account as well.

$\beta?, \beta \succ \varphi?\} \vdash_{\zeta_H} \alpha \succ \varphi?$, $\{\alpha \succ \beta?, \beta \leftrightarrow \varphi\} \vdash_{\zeta_H} \alpha \succ \varphi?$ and so on and so forth. Therefore we call ζ_H a calculus of confirmation.

As perhaps one may have already noticed, there is a little problem with axioms $I_\succ$ and $II_\succ$. Due to the properties of the material implication, these axioms do not actually achieve the goal they are designed to achieve. Take for example axiom $I_\succ$. Clearly enough, $I_\succ$'s purpose is to represent the intuitive principle according to which if α entails β, then α also confirms β. But suppose that β is a tautology. Trivially, for any formula α, $\alpha \to \beta$. Then, by axiom $II_\succ$ we will have that any formula whatsoever confirms a tautology β. Besides the question whether it is meaningful a tautology's being confirmed by a formula, the problem with this is that by $I_\succ$ two formula which are not in any imaginable way inductively connected with each other will be such that one gives evidential support to the other. Similarly, if α is a contradiction, we have that $\alpha \to \beta$ for any formula β and, by $I_\succ$, that any formula is confirmed by a contradictory formula. $II_\succ$ and the derived principle which we have called inverse consequence condition have got the same sort of problem.

Even though these problems cannot be sorted out inside a purely classical framework, they can be easily solved if we take full advantage of the representational apparatus provided by $\mathcal{L}_{\psi?}$. While still making use of material implication, we just represent the above-mentioned principles as inductive implications and consider the anomalous cases in their exception parts:

$I_\succ$: $(\alpha \to \beta) \succ (\alpha \succ \beta?) \precsim ((\alpha \to \bot) \vee (\top \to \beta))$

$II_\succ$: $(\beta \to \varphi) \succ ((\alpha \succ \beta?) \succ (\alpha \succ \varphi?)) \precsim (\top \to \varphi)$

What follows below is an amended version of Hempel's logic of induction.

Definition 6.4.4. Let $\mathfrak{I}$ be a language and $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\mathfrak{I}$ wherein $? \in \Theta_p$. The *Hempel+ confirmation axioms* Σ_{H+} in $\mathfrak{I}_\psi$ is the set composed by all formulae of $\mathfrak{I}_\psi$ satisfying the following schemas of formula:

$I_\succ$:	$(\alpha \to \beta) \succ (\alpha \succ \beta?) \precsim ((\alpha \to \bot) \vee (\top \to \beta))$	*Entailment*
$II_\succ$:	$(\beta \to \varphi) \succ ((\alpha \succ \beta?) \to (\alpha \succ \varphi?)) \precsim (\top \to \varphi)$	*Consequence*
$III_\succ$:	$(\alpha \succ \beta?) \to ((\beta \leftrightarrow \varphi) \to (\alpha \succ \varphi?))$	*Equivalence*
$IV_\succ$:	$(\alpha \succ \beta?) \to ((\alpha \wedge \beta \to \bot) \to \bot)$	*Weak consistency*

Due to $I_{\succ}$'s being represented as an inductive implication with an exception condition, the inverse consequence principle, which itself must have an exception condition to block the cases where α is a tautology, cannot any more be obtained from $I_{\succ}$ and $VI_{\succ}$. We therefore have to take such principle as an axiom of our amended Hempel logic of induction. Also, since our new version of $I_{\succ}$ excludes any tautological or contradictory formulae, we will have that for those formulae the principle according to which any formula confirms itself, which we may call *inclusion principle*, will not be hold[28]. Therefore, it too will have to be included in our new Hempel logic.

Definition 6.4.5. Let $\Im$ be a language and $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\Im$ wherein $?\in\Theta_p$. The *Hempel+ additional confirmation axioms* $\Sigma_{H+'}$ in $\Im_\psi$ is the set composed by all formulae of $\Im_\psi$ satisfying the following schema of formula:

$VII_{\succ}$: $(\alpha \to \beta) \succ ((\beta \succ \varphi?) \to (\alpha \succ \varphi?)) \precsim (\alpha \to \bot)$ *Inverse Consequence*

$VIII_{\succ}$: $\alpha \succ \alpha?$ *Inclusion*

Definition 6.4.6. The *Hempel+ logic of induction* ζ_{H+} is the applied logic of induction $<\zeta_?, T_{H+}>$, where $\zeta_?$ is the basic logic of induction and $T_{H+} = \Sigma_{H+} \cup \Sigma_{H'} \cup \Sigma_{H+'}$ where Σ_{H+} are the Hempel+ confirmation axioms in $\mathcal{L}_{\psi?}$, $\Sigma_{H'}$ the Hempel additional confirmation axioms in $\mathcal{L}_{\psi?}$ and $\Sigma_{H+'}$ the Hempel+ additional confirmation axioms in $\mathcal{L}_{\psi?}$[29].

6.4.2 Abduction and Hypothetico-Deductivism

As we have said in Chapter 4, the task of the calculus of inductive implication or calculus of confirmation is to set the rules according to which confirmation statements should be related to each other, or the rules according to which derived confirmation statements are to be obtained from prior ones. This, we should acknowledge, is what all axioms of ζ_{H+} do. All with the exception of $I_{\succ}$[30].

Since $I_{\succ}$ is has as its antecedent a material implication statement and as its consequent an inductive implication, it is in an important sense an instruction about how to derive confirmation statements out of something else than inductive implications. It functions then like a criterion to obtain the so-

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[28] The same holds for $II_{\succ}$ and $III_{\succ}$: $II_{\succ}$ cannot any more be derived from $VI_{\succ}$ and $I_{\succ}$ and $III_{\succ}$ cannot any more be derived from $II_{\succ}$. We postpone to the future the task of developing a complete calculus of inductive implication with all logical relations that are supposed to exist between $\succ$, $\precsim$ and $\to$.

[29] In Silvestre & Pequeno (2005) we have presented a version of the applied logics of induction introduced here where the exception part of the inductive implications is considered.

called prior confirmation statements. This means that besides being a calculus of confirmation, ζ_{H+} also embodies in its axiomatic what we have named a model of confirmation.

The model of confirmation contained in ζ_{H+} belongs to a class of definitions of confirmation which try to derive a relation of inductive support from some special instance of the deductive entailment relation. In the case of I, this from-deduction-to-induction confirmation mechanism is obtained by saying that if α entails β (and neither α is contradiction nor β is a tautology), then α confirms β. It should be said nevertheless that this is not exactly what we expect from a definition of confirmation. Since induction inferences are supposed to be ampliative and non truth-preserving, by just setting material implication as a special case of confirmation we do not provide any sort of genuinely inductive relation of evidential support.

In the same section of Hempel's "Studies in the Logic of Confirmation" where conditions I-IV are introduced, there is mention of a condition which may serve as a starting point for a truly model of confirmation:

IX. *Converse Consequence condition*: if statement α confirms statement β and statement φ logically implies β, then α confirms φ[31].

Even though VII still derives confirmation statements from confirmation statements, accepting what we have called the inclusion principle (which is a consequence of Hempel's formulation of the entailment condition) will entail the following additional condition:

X. *Converse Entailment condition*: if statement α logically implies statement β, then β confirms α.

Trivially enough, IX and X are a formulation of the reasoning pattern known as *abduction*: if α logically entails β and β is true, then conclude α. In our formalism, this abduction model of confirmation could be formalized through the following two axioms:

$IX_{\succ}$: $(\alpha \succ \beta?) \rightarrow ((\varphi \rightarrow \beta) \rightarrow (\alpha \succ \varphi?))$ *Converse Consequence*

$X_{\succ}$: $(\alpha \rightarrow \beta) \rightarrow (\beta \succ \alpha?)$ *Converse Entailment*

Now one may think that we can just add these two axioms to the Hempel+ logic of induction and obtain a sort of Hempelian Abductive model of confirmation. Not quite so. As Hempel and

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[30] And perhaps $II_{\succ}$ and $VII_{\succ}$ too.

[31] Instance of such principle can be found in the confirmation of Newton's mechanics through the confirmation of Kepler's laws of planetary motion: observation of the orbit of Mars confirm Kepler's laws of planetary motion; Newton's laws of mechanics entail Kepler's laws of planetary motions; ergo, observations of the orbit of Mars confirm Newton's laws of mechanics.

others have shown, IX and X lead to very counterintuitive results when taken along with Hempel's previous conditions. Consider for example the Stark effect (Se) which is known to confirm quantum mechanics (Qm). Trivially, Qm in conjunction with, let us say, the metaphysical principle known as physicalism according to which matter and energy are the cause of everything that exists (Ph) logically implies Qm. Then, since Se confirms Qm and $Qm \wedge Ph \rightarrow Qm$, by IX we will have the conclusion that the Stark effect confirms the conjunction of quantum mechanics and physicalism. Now, given that Se confirms $Qm \wedge Ph$ and $Qm \wedge Ph \rightarrow Ph$, by II we will have the even more unacceptable conclusion that the Stark effect gives evidential support to such a metaphysical principle as physicalism. In fact, it will confirm not only Ph, but any statement expressible in the language at hand. Things get still worse when we consider X. While IX allows us to make such sort of claim only in connection with statements which serve as evidence for some hypothesis, X leads to the conclusion that any pair of statement *e-h* whatsoever is such that *e* confirms *h*[32]. In our notation we would have that since $\alpha \wedge \beta \rightarrow \alpha$, by $X_{\succ}$ $\alpha \succ (\alpha \wedge \beta)?$. But since $\alpha \wedge \beta \rightarrow \beta$, by $II_{\succ}$ we have $\alpha \succ \beta?$.

The same unwanted conclusion could be derived if we consider disjunctive statements rather than conjunctive ones. Since $\alpha \rightarrow \alpha \vee \beta$, by I we have that α confirms $\alpha \vee \beta$. But since $\beta \rightarrow \alpha \vee \beta$, by IX we have that α confirms β[33]. Similarly, by X we have that $\alpha \vee \beta$ confirms β. Since by I α confirms $\alpha \vee \beta$, by the transitivity condition we have that α confirms β. We will call these two sources of anomaly in the use of IX and X in conjunction with I and II, respectively, the *conjunctive* and *disjunctive problems of abduction*.

Because of these problems, Hempel rejected this converse consequence condition along with the definition of confirmation which brought it into the discussion: the *prediction-criterion of confirmation*. This prediction-criterion of confirmation is nothing less than a formulation of the so-called *Hypothetico-Deductive* (or simply *H-D*) *model* of confirmation, whose importance for the contemporary theory of science is such that some philosophers went so far as claiming it to be the official "scientist's philosophy of science."[34] Due to that, it may be worthy to take a closer look at the H-D model and how it is connected with the problems we have mentioned so far. Below we show Hempel's formulation of the H-D model[35].

> Let *h* be a hypothesis and B a class of observational statements. B is said to confirm *h* if B can be divided into two mutually exclusive subclasses B' and B" such that B" is not empty and

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[32] See Hempel (1945), Hesse (1970), Hanen (1971) and Le Morvan (1999).
[33] See Skyrms (1966) and Brody (1968).
[34] Lipton (1991).
[35] Hempel (1945). See also Hempel (1966), Merril (1979) and Earman & Salmon (1992).

every sentence of B" can be logically deduced from B' in conjunction with *h*, but not from B' alone.

Considering statements instead of sets of statements and not taking the observational nature of the members of B into account we have as follows:

e confirms *h* if

(i) $\vdash e \leftrightarrow e' \wedge e''$
(ii) $\{e' \wedge h\} \vdash e''$
(iii) $\{e'\} \nvdash e''$ [36]

As one might expect, this definition of confirmation satisfies condition IX. Also, if we take $e' \equiv \top$ we will have that if $h \vdash e$, then *e* confirms *h*. Therefore, condition X is also satisfied. But here this submodel of H-D, which we may call the *abduction model of confirmation*, is not different from our formulation of condition X. We therefore see how a principle in the style of Hempel's conditions can be used either as a condition to be satisfied by a model of confirmations or as a model of confirmation itself.

It is not very difficult to see why the H-D model is so appealing. According to it, a hypothesis or theory is supported when it, along with some other statements, deductively entails some observed statement. Take the big-bang theory of the origin of the universe for example. Obviously the statements which compose this theory cannot be directly observed. However, along with other statements the big bang theory entails that we ought to find ourselves today traveling though a uniform background radiation, like the ripples left by a rock falling into a pond. Now it seems almost self-evident that the fact that we do now observe this radiation (or effects of it) should somehow count as a good reason for us to take the big bang theory as a plausible hypothesis. Therefore a theory or hypothesis, even though appealing to unobservable entities or processes, is supported or confirmed by its successful predictions.

There have been a couple of different formulations of the H-D model. R. Braithwaite, for example, takes it as something like our condition X, identifying thus hypothetico-deductivism with abduction[37]. More recently, in his famous book "Theory and Evidence" Clark Glymour has discussed a version of the H-D model[38] which due to the spreading of Glymour's criticisms was

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[36] The conditions that B' and B" be two mutually exclusive classes and B" be nonempty (which is equivalent to requiring e" not to be a tautology) are automatically satisfied by (iii).
[37] Braithwaite (1953). See Grimes (1990).
[38] Glymour (1980a).

responsible for a remarkable proliferation of new formulations of the H-D method[39]. Glymour's formulation goes as follows:

e confirms *h* with respect to a theory T if

(i) $h \wedge T$ is consistent

(ii) $\{h \wedge T\} \vdash e$

(iii) $\{T\} \nvdash e$

Besides the test of consistency between *h* and T, the main difference here is that *e*'s confirming *h* will depend always on some background theory T.

Now, Glymour have pointed out three main problems with this formulation of the H-D model. According to him, these problems lead to the inevitable conclusion that the H-D method is hopelessly untenable[40]. The first difficulty he finds stems from the fact that a consequence of a given theory cannot be confirmed with respect to that theory (condition (iii)); it can only be confirmed with respect to some subtheory. This might not be troublesome, for as long as the subtheory conjoined with the hypothesis is logically equivalent to the original theory, we should be able to use the subtheory for H-D confirmation. Despite the intuitive appeal of this reasoning, it is inadequate. Glymour points out that if we adhere to this rationale, any true evidential consequence of T confirms almost any hypothesis in theory T with respect to some subtheory. For if $\{T\} \vdash e$, $\{T\} \vdash h$, and $\{h \rightarrow T\} \nvdash e$, then according to condition (i)-(iii), *e* confirms *h* with respect to subtheory $h \rightarrow T$. The second difficulty involves the problem that according to conditions (i)-(iii), any true piece of evidence confirms almost any sentence with respect to some true theory. This follows from the fact that if *e* is any true and not valid statement, conditions (i)-(iii) are satisfied by letting $T \equiv e \rightarrow h$ and h equal any statement that does not imply $\sim e$ and that is not a consequence of $\sim e$[41]. Finally, he points out that if T, h and e satisfy conditions (i)-(iii), then T, $(h \wedge A)$ and e also satisfy these conditions for any A that is consistent with $h \wedge T$. Therefore, if *e* confirms *h* with respect to T, then *e* will also confirm $h \wedge A$ with respect to T.

One is perhaps wondering what is the relation between these problems and the ones we have shown in connection with conditions IX and X. In a nutshell, they all come from the irrelevance feature of classical entailment, be it in the form of the inferential relation $\vdash$ or in the form of the material implication connective[42]. When, for instance, we say that if α entails β then β confirms α,

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[39] Horwich (1983), Giere (1984), Waters (1987), Grimes (1990) and Gemes (1993), just to mention a few.

[40] Glymour (1980a). See also Glymour (1980b).

[41] Glymour mentions only the latter condition. But if h implied $\sim e$, then $T \wedge h$ would not be consistent. See Waters (1987).

[42] Some few philosophers have already pointed this out. See for instance Waters (1987).

we expect that all parts of α are necessary for the derivation of β and therefore deductively connected with it. Now, if α and β are such connected and we conjoin φ to α, trivially β is logically entailed by $\varphi \wedge \alpha$; but φ plays no role at all in the derivation of β from $\alpha \wedge \varphi$. Therefore we are not in any way ready to say that $\alpha \wedge \varphi$ confirms β, even though α alone does. The same thing happens when we take the disjunction of β and φ. All the incompatibility between IX and X and Hempel's former conditions as well as the problems we have identified with axioms $I_{\succ}$ and $II_{\succ}$ come from this irrelevance feature of classical entailment.

Concerning Glymour's criticisms, for the third one it is clear that its cause is classical entailment irrelevance. After all, it is just a reformulation of the first sort of conjunction problem of abduction we have shown in terms of the H-D model itself. For the first problem, *e* confirms *h* with respect to $H \rightarrow T$ because $h \rightarrow T$ and H are consistent, $\{(h \rightarrow T) \wedge h\} \vdash e$ and $\{h \rightarrow T\} \nvdash e$. But notice that $(h \rightarrow T) \wedge h$ is equivalent to $(\neg h \wedge h) \vee (T \wedge h)$, which in turn is equivalent to $T \wedge h$. Since therefore $\{T\} \vdash e$, the relation $\{(h \rightarrow T) \wedge h\} \vdash e$ is not, we may say, a relevant one. For the second one, we have that *e* confirms *h* with respect to $h \rightarrow e$ because $h \rightarrow e$ and h are consistent, $\{(h \rightarrow e) \wedge h\} \vdash e$ and $h \rightarrow e \nvdash e$. But again $(h \rightarrow e) \wedge h$ is equivalent to $(\neg h \wedge h) \vee (e \wedge h)$ which in turn is equivalent to $e \wedge h$. Since $e \vdash e$, the relation $\{(h \rightarrow e) \wedge h\} \vdash e$ is not a relevant one. It is worthy to notice that the problems with conditions I and II that forced us to reformulate axioms $I_{\succ}$ and $II_{\succ}$ are also due to classical entailment irrelevance, for $\alpha \rightarrow \bot$ and $\top \rightarrow \beta$ sure do not satisfy our relevance entailment criterion.

Now the crucial question: is our framework able to properly represent these abductive or hypothetico-deductivist models of confirmation in such a way that the mentioned problems will not arise? As shown by what we have discussed in the previous paragraphs, all problems which philosophers have for decades attributed to the H-D model in its several formulations are due not the H-D model itself, but to the framework in which these formulations have been described[43]. The H-D model, by its very nature, requires some sort of relevance entailment relation, for if we establish a confirmation relation between $e' \wedge e''$ and h from the basic fact that $\{e' \wedge h\} \vdash e''$, the minimum we can expect is that all parts of $\{e' \wedge h\}$ will be needed to infer all parts of e''. Therefore, since the classical relation of entailment does not incorporate such sort of relevance behavior, it is understandable why describing the H-D model inside classical logic is so much troublesome.

Even though the deductive logic we are using (LP^2) along with our logics of induction does not have such sort of relevant entailment relation, with the help of our inductive connectives and the intuition behind the H-D model we are able to simulate an inductive implication and therefore have a proper representation of the hypothetico-deductive model of confirmation.

Definition 6.4.7. Let $\Im$ be a language, ϑ a ?-modal logic basis and $\alpha,\beta \in \Im_{\succ\vartheta}$ any two formulae of $\Im_{\succ\vartheta}$. We define the following abbreviations in $\Im_{\succ\vartheta}$:

(i) $\alpha \Leftrightarrow \beta =_{def} (\alpha \to \beta) \vee (\beta \to \alpha)$

(ii) $\alpha \rhd \beta =_{def} \beta \succ \alpha?$

(iii) $\alpha \unrhd \beta =_{def} (\alpha \to \beta) \wedge (\alpha \rhd \beta)$

(iv) $\alpha \not\rhd \beta =_{def} (\alpha \rhd \beta) \succ \bot?$

$\alpha \Leftrightarrow \beta$ is a simple abbreviation meaning that α and β are "implicationally connected" to each other. $\alpha \rhd \beta$ is an alternative way of writing $\alpha \succ \beta?$ which will be of some help in our task of representing the abductive method of confirmation. $\alpha \rhd \beta$ can be read as "α is confirmed by β." $\alpha \unrhd \beta$ is intent to represent a situation where α relevantly implies β. It depends directly on what we have called abduction model of confirmation: if α (relevantly) implies β, then β confirms α. That is to say, supposing that we have such a model, if β confirms α and $\alpha \to \beta$, then α relevantly implies β. Finally, $\alpha \not\rhd \beta$ represents a situation where α and β are such that, due to the lack of a relevant entailment connection between α and β, α cannot be confirmed by β through the abduction model[44]. We show below the basic axioms which will make use of these abbreviations.

Definition 6.4.8. Let $\Im$ be a language and ϑ a ?-modal logic basis. The *abduction axioms* Σ_{Ab} in $\Im_{\succ\vartheta}$ is the set composed by all formulae of $\Im_{\succ\vartheta}$ satisfying one of the following schemas of formula:

$X_{\succ}$: $(\alpha \to \beta) \succ (\alpha \rhd \beta) \precsim (\alpha \not\rhd \beta)$

Ab1: $((\alpha \leftrightarrow \alpha' \wedge \alpha'') \wedge (\alpha' \unrhd \beta)) \succ (\alpha \not\rhd \beta) \precsim ((\alpha'' \unrhd \beta) \vee (\alpha' \Leftrightarrow \alpha''))$

Ab2: $((\beta \leftrightarrow \beta' \vee \beta'') \wedge (\alpha \unrhd \beta')) \succ (\alpha \not\rhd \beta) \precsim ((\alpha \unrhd \beta'') \vee (\beta'' \Leftrightarrow \beta'))$

The purpose of the above axioms is basically to define what we have been calling abductive confirmation. $X_{\succ}$, which is a more sophisticated formulation of condition X, sets

[43] See Waters (1987).

[44] Recall that in LP^2 $\bot? \to \bot$. See the Appendix "Proof of Theorems."

the basic abductive criterion according to which formula α confirms formula β: if $\alpha \rightarrow \beta$ then α is confirmed by β. However, as we have seen, material implication does not embody the relevant aspects required by an abductive model of confirmation: sometimes even though $\alpha \rightarrow \beta$, due to α's not being relevantly connected with β, α is not confirmed by α. It is the goal of the exception part of $X_{\triangleright}$, $\alpha \ntriangleright \beta$, to block these non-relevance cases and therefore prevent $\alpha \triangleright \beta$ from being concluded from $\alpha \rightarrow \beta$. These non-relevance cases are formally defined by axioms Ab1 and Ab2, which basically take into account the conjunction and disjunctive problems of abduction which we have discussed at the beginning of this subsection.

Ab1 says that if α is equivalent to the conjunction of α' and α'', and α' relevantly implies β, then to conjoin α' and α'' and write $\alpha' \wedge \alpha'' \rightarrow \beta$ will be a trivialization with no relevance content. Therefore $\alpha \ntriangleright \beta$. Of course there are exceptions to this. The first one is α'' relevantly implying β, in which case $\alpha' \wedge \alpha''$ should be confirmed by β (which will be obtained by using $\alpha' \wedge \alpha'' \rightarrow \beta$ along with $X_{\triangleright}$.) Also, if $\alpha' \rightarrow \alpha''$ or $\alpha'' \rightarrow \alpha'$ then $\alpha \ntriangleright \beta$ should not be the case, for if $\alpha' \rightarrow \alpha''$ then α' will be equivalent to $\alpha' \wedge \alpha''$, and if $\alpha'' \rightarrow \alpha'$, by transitivity $\alpha'' \rightarrow \beta$ and therefore $\alpha'' \triangleright \beta$. Hence, $\alpha' \wedge \alpha'' \triangleright \beta$. One could think that this second part of $\alpha' \Leftrightarrow \alpha''$ was not needed at all, for, since $\alpha'' \triangleright \beta$ (which is obtained by using $X_{\triangleright}$ along with $\alpha'' \rightarrow \beta$,) the situation was already contemplated by the inductive implication Ab2. However, taking Ab1 without $\alpha'' \rightarrow \alpha'$ in its exception part, and $\alpha'' \rightarrow \alpha'$ and $\alpha' \trianglerighteq \beta$ as valid formulae (which implies $\alpha'' \leftrightarrow \alpha' \wedge \alpha''$) entails a conflict between $X_{\triangleright}$ and Ab1: by using Ab1 first and concluding $\alpha'' \ntriangleright \beta$ (which could be done because we have not used yet $X_{\triangleright}$ to conclude $\alpha'' \triangleright \beta$ and be able to block Ab1) we will not be able to use $X_{\triangleright}$ and conclude $\alpha'' \triangleright \beta$. Therefore two extensions would arise. In order to prevent that, we have to consider $\alpha'' \rightarrow \alpha'$ in the very exception part of Ab1.

For Ab2 the reasoning is almost the same. If β is equivalent to the disjunction of β' and β'', and α relevantly implies β', then to write $\alpha \rightarrow \beta' \vee \beta''$ means to go against our relevance principle, for β'' plays no role at all in the derivation of $\beta' \vee \beta''$ from α. Therefore

$\alpha \not\triangleright \beta$. About the exceptions, we have first that if α relevantly implies β" then α should be confirmed by $\beta' \vee \beta''$. Also, if $\beta'' \rightarrow \beta'$ or $\beta' \rightarrow \beta''$ then $\alpha \not\triangleright \beta$ should not be the case, for if $\beta'' \rightarrow \beta'$ then β' will be equivalent to $\beta' \vee \beta''$, and if $\beta' \rightarrow \beta''$ then by transitivity $\alpha \rightarrow \beta''$ and therefore $\alpha \triangleright \beta''$. Hence $\alpha \triangleright \beta' \vee \beta''$. About the objection that it is not necessary to consider $\beta' \rightarrow \beta''$ as an exception, taking Ab2 without $\beta' \rightarrow \beta''$ in its exception, and $\beta' \rightarrow \beta''$ and $\alpha \trianglerighteq \beta'$ as valid formulae (which implies $\beta'' \leftrightarrow \beta' \vee \beta''$) entails a conflict between $X_{\succ}$ and Ab2: by using Ab2 first and concluding $\alpha \not\triangleright \beta''$ (which could be done because we have not used yet $X_{\succ}$ to conclude $\alpha \triangleright \beta''$ and be able to block Ab2) we will not be able to use $X_{\succ}$ and conclude $\alpha \triangleright \beta''$. Therefore two extensions would arise.

It is easy to see that this formulation of the abduction model of confirmation is not plagued by the relevance problems we have been talking about here. Below we have what we can call the abduction logic of induction.

Definition 6.4.9. The *abduction logic of induction* ζ_{HAb} is the applied logic of induction $<\zeta_?, \Sigma_{Ab}>$, where $\zeta_?$ is the basic logic of induction and Σ_{Ab} the abduction axioms in $\mathcal{L}_{\psi?}$.

With the help of these abduction axioms we can have a complete Hempelian logic of induction which formalizes all conditions I, II, III, IV, V, VI, VII, IX and X. Notice that since we now dispose of a relevance implication, axioms $I_{\succ}$, $II_{\succ}$ and $VI_{\succ}$ can be formalized without making use of the exception part. Due to that, axiom $VIII_{\succ}$ (inclusion) does not have to be included in this new formalization.

Definition 6.4.10. Let $\mathfrak{I}$ be a language and $\psi = <\vartheta, \Sigma, \Theta_p, \Theta_t>$ a pseudo-inductive logic of plausibility based on $\mathfrak{I}$ wherein $? \in \Theta_p$. The *Hempel$\trianglerighteq$ confirmation axioms* $\Sigma_{H\trianglerighteq}$ in $\mathfrak{I}_\psi$ is the set composed by all formulae of $\mathfrak{I}_\psi$ satisfying the following schemas of formula:

$I_{\succ}$:	$(\alpha \trianglerighteq \beta) \succ (\alpha \succ \beta?)$	*Entailment*
$II_{\succ}$:	$(\beta \trianglerighteq \varphi) \succ ((\alpha \succ \beta?) \succ (\alpha \succ \varphi?))$	*Consequence*
$III_{\succ}$:	$(\alpha \succ \beta?) \rightarrow ((\beta \leftrightarrow \varphi) \rightarrow (\alpha \succ \varphi?))$	*Equivalence*
$IV_{\succ}$:	$(\alpha \succ \beta?) \rightarrow ((\alpha \wedge \beta \rightarrow \bot) \rightarrow \bot)$	*Weak consistency*
$VII_{\succ}$:	$(\alpha \trianglerighteq \beta) \succ ((\beta \succ \varphi?) \rightarrow (\alpha \succ \varphi?))$	*Inverse Consequence*

$IX_{\succ}$: $(\alpha \succ \beta?) \rightarrow ((\varphi \trianglerighteq \beta) \rightarrow (\alpha \succ \varphi?))$ *Converse Consequence*

Definition 6.4.11. The *Hempel-abduction logic of induction* ζ_{HAb} is the applied logic of induction $<\zeta_?, T_{HAb}>$, where $\zeta_?$ is the basic logic of induction and $T_{HAb} = \Sigma_{H\trianglerighteq} \cup \Sigma_{H'} \cup \Sigma_{Ab}$, where $\Sigma_{H\trianglerighteq}$ are the Hempel$\trianglerighteq$ confirmation axioms in $\mathcal{L}_{\psi?}$, $\Sigma_{H'}$ the Hempel additional confirmation? axioms in $\mathcal{L}_{\psi?}$ and Σ_{Ab} the abduction axioms in $\mathcal{L}_{\psi?}$.

With these abduction axioms at hand we can also define a relevance-problem-free H-D model.

Definition 6.4.12. Let $\Im$ be a language and ϑ a ?-modal logic basis. The H-D *axioms* $\Sigma_{H\text{-}D}$ in $\Im_{\succ\vartheta}$ is the set composed by all formulae of $\Im_{\succ\vartheta}$ satisfying the following schema of formula:

H-D: $((\beta \leftrightarrow \beta' \wedge \beta'') \wedge (\alpha \wedge \beta' \rightarrow \beta'')) \succ (\beta \succ \alpha?) \not\precsim (\alpha \wedge \beta' \nrightarrow \beta'') \vee (\top \rightarrow \alpha))$

We are using the formulation proposed by Hempel which we have shown at the beginning of this subsection: If β is composed by two statements β' and β'' and β'' can be logically deducted from β' in conjunction with α, then α is confirmed by β. There will be two kinds of exceptions to this rule. The first obviously are situations where $\alpha \wedge \beta'$ does not relevantly imply β''. The second are cases where α is a tautology. The reason for this second sort of exception is that since $\beta' \leftrightarrow \top \wedge \beta'$, we do not want to take $\top \wedge \beta' \rightarrow \beta''$ as an irrelevant implication. Therefore $\top \wedge \beta' \nrightarrow \beta''$ will not be the case. But we are also not ready to say that $\beta' \wedge \beta''$ confirm T. The only alternative then is to consider this case as a separated exception. Concerning Hempel's three conditions, we note that the third one ($\{e'\} \nvdash e''$, which would be represented in our notation by introducing $\beta' \rightarrow \beta''$ in the exception part of H-D) is already contemplated by $\alpha \wedge \beta' \nrightarrow \beta''$.

Definition 6.4.13. The *H-D logic of induction* $\zeta_{H\text{-}D}$ is the applied logic of induction $<\zeta_?, T_{H\text{-}D}>$, where $\zeta_?$ is the basic logic of induction and $T_{H\text{-}D} = \Sigma_{Ab} \cup \Sigma_{H\text{-}D}$, where Σ_{Ab} are the abduction axioms in $\mathcal{L}_{\psi?}$ and $\Sigma_{H\text{-}D}$ the H-D axioms in $\mathcal{L}_{\psi?}$.

One thing we should mention about all the logics of induction we have introduced in this section concerns the status of true and false statements. Take ζ_{Ab} for example. As we have said, $\alpha' \rightarrow \alpha''$ in the exception part of Ab1 takes into account the cases where $\alpha' \leftrightarrow \alpha' \wedge \alpha''$. This of course includes a situation where $\alpha'' \leftrightarrow \top$, for $\alpha' \rightarrow \top$ and $\alpha' \leftrightarrow \alpha' \wedge \top$. But now consider a set of formulae A and a formula φ such that $A \vdash_{\zeta_{Ab}} \varphi$. Due to $X_{\succ}$, we will have that $A \vdash_{\zeta_{Ab}} \lambda \succ (\lambda \wedge$

$\varphi)?$ for any formula λ, or still that $A \vdash_{\zeta Ab} Pa \succ (\forall x(Px) \wedge \varphi)?$ for every propositional symbol P (which seem to be standard cases of non-relevance anomalous confirmation.) But $A \vdash_{\zeta Ab} Pa \succ (\forall x(Px) \wedge \varphi)?$ is the case because Ab1 was not able to infer $\forall x(Px) \wedge \varphi \not\succ Pa$, which happened because $\forall x(Px) \rightarrow \varphi$, which in its turn is the case because $A \vdash_{\zeta Ab} \varphi$. That is to say, from the point of view of A, φ is such that $\forall x(Px)$ is equivalent to $\forall x(Px) \wedge \varphi$ or, in other words, statements φ such that $A \vdash_{\zeta Ab} \varphi$ have the same status as tautologies.

In contrast to what one may be thinking, this feature of ζ_{Ab} is not bad at all. The abduction and H-D models of confirmation operate on statements which are deductively connected to each other: if $\alpha \rightarrow \beta$, then $\beta \succ \alpha?$. If we allow such deductively connected statements to be only the ones provided by the axiomatic of ζ_{Ab}, then the range of application of our logic will decrease enormously. Consider a situation where we want to evaluate the plausibility of a hypothesis *h* on the basis of some experimental result *e* which makes use of principles of a highly accepted theory T whose confirmation is simply not part of our business. Now, the situation is such that *e* is logically entailed by the conjunction of *h* and T but not by *h* alone. Then, unless we take theory T as true, we will not be able to get a confirmatory relation between *e* and *h* (since $T \vdash_{\zeta Ab} h \rightarrow e$, we will have that $T \vdash_{\zeta Ab} e \succ h?$, which is quite similar to Glymour's formulation according to which *e* confirms *h* with respect to some background theory T.) But since T is taken as true, its presence in any formula will be simply innocuous. Moreover, there are no formal logical devices presently available to us to distinguish, from the logical language point of view, between true formulae and tautologies (that is to say, once we have set a set A to make use of our inferential relation $\vdash$, there is no way we can find out whether α such that $A \vdash \alpha$ is true to A or due to the logical laws of $\vdash$.)

CHAPTER 7

Conclusion

In this thesis we have addressed the problem of explicating the notions of induction and plausibility. As a final result of our endeavors, we came up with a class of formal systems of induction and plausibility, the *explicata* of our analysis, which made use of some formal devices traditionally related to AI's research on the formalization of commonsense reasoning. More specifically, we have picked one of the most wide spread nonmonotonic logics – default logic – and after finding out to what extent it could be considered as an inductive logic in the philosophical sense as well as which sort of adjustments should be made to transform it into such logic, we extended it in such a way as to obtain what we have called a representative logic of induction. The use of a nonmonotonic logic is this context was especially relevant because our approach to the concept explication problem of induction involved what we have called a purely descriptive approach to induction: as far as we know, all nonmonotonic formalisms can be said to be free from the justificatory problems that undermined the Carnapian project of building a logic of induction. The explicative power of our default logic of induction can be said to rest basically on its representation of the logical form of inductive implication sentences, which are the core of our account of inductive inferences, along with the formal mechanism used to "detach" the consequent of such inductive implications from its antecedent and exception parts.

Of fundamental importance for our enterprise was the awareness of the importance of the notion of plausibility for any descriptive account of induction in general and of its double aspect as a paraconsistent and paracomplete notion in particular. This brings us to the second component of our representative logic of induction: a paraconsistent and paracomplete modal logic (named by us paranormal modal logic) intended to formalize the concepts of skeptical and credulous plausibility and to act in conjunction with our extended default logic in order to represent the conclusions obtained with the help of inductive implications. The explicative power of this paranormal modal logic is basically derived from the axiomatic and semantic resources traditionally used in philosophical logic to explicate "modal notions" such as the concepts of necessity and possibility. At the same that the axioms and logical theorems make explicit some key formal properties of plausible sentences, the semantics tells us what these sentences actually mean. About this latter, it should be said that by incorporating what we have called plurality approach to plausibility and unclosing the paraconsistent and paracomplete behaviors of ? and !, our semantics provided a quite satisfactory explication of the notions of skeptical and credulous plausibility. This is especially important because due to the connection between inductive inferences and plausibility, the explicative power of the logic of plausibility that composes a specific logic of induction is automatically transferred to the clarification of the notion of induction itself.

Besides having contributed to the clarification of the concepts of induction and plausibility, our work can be said to be have contributed to the fields of philosophy and AI as follows. First, it provided a genuinely descriptive approach to induction. Second, besides showing a relevant connection between the fields of nonmonotonic logic and inductive logic, it demonstrated how these two fields can practically benefit from each other. More specifically, it demonstrated how the philosophical insights concerning inductive inferences can be useful to better understand the nature and limitations of default logic, and how this latter can be used to perform the task of a logic of induction. Third, as a result of our endeavors to explicate the notion of plausibility, we came up with an entirely new sort of modal logic – paranormal modal logic –, which we believe might be of some relevance both from the purely technical and practical point of view, along with a philosophically sound motivation for it. Finally, we have showed how our system can be used in the formalization of a traditional problem in philosophy of science, namely the problem of confirmation of hypotheses through Hempel's conditions of adequacy and the abductive and hypothetico-deductive models.

About the shortcomings of the content of the thesis, we should first mention that the formal treatment given to the monotonic and nonmonotonic parts of our system were not symmetric. Undoubtedly, we did not provide a minimally complete analysis of the formal features of our extension of default logic. Also, as we have pointed out, we did not develop a complete calculus of inductive implication akin to the calculus of material implication contained in classical logic. To this we should add that we did not offer any sort of semantic analysis of inductive implication sentences. All this should be done by us in the near future. Also, it should be said that the philosophical analysis of induction and plausibility provided in chapters 2 and 3 is far from deserving being classified as a philosophy of induction and plausibility. As we have pointed out in chapter 1, the purpose of such chapters was to make clear what would be formally explicated in the next chapters. Of course that in order to do that we had to engage ourselves in some sort of philosophical analysis of what the concepts of induction and plausibility mean.

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APPENDIX

PROOF OF THEOREMS

In this appendix we will show the proof of the theorems mentioned in Chapters 5 and 6. In the following section we lay down some introductory followed by the lemmas that will be used in the proofs of some theorems. Sections A.2 and A.3 present the proofs of theorems of Chapters 5 and 6, respectively.

A.1 Lemmas, Definitions and Preliminary Remarks

In proving the theorems of chapters 5 and 6 and auxiliary lemmas, we will have sometimes to mention another theorem whose proof is shown only later on in the text. It must be noted nevertheless that this is just a matter of order of presentation, being clear that no sort of circularity is involved. Due to space limitations, another policy we will adopt is to show in detail only the key proofs. For those theorems whose proofs are trivial or follow a pattern already explained, we will only give an indication of how the proof can be constructed. Other important remark is that we will, in our proofs, very often make use of well-established results of symbolic logic without showing the respective proofs.

When showing derivations in axiomatic-related theorems (or lemmas), we will adopt some conventions. Let us explain them with the help of an example: the derivation of the relation $\vdash_{K?} \sim\alpha \rightarrow (\alpha \rightarrow \beta)$.

1.	$\neg p \rightarrow (p \rightarrow \beta)$	A2
2.	$p \rightarrow (\neg p \rightarrow \beta)$	P(1) 1
3.	$p \wedge \neg p \rightarrow \beta$	P(3) 2
4.	$(\alpha \rightarrow \bot) \rightarrow ((\bot \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta))$	P(2)
5.	$(\bot \rightarrow \beta) \rightarrow ((\alpha \rightarrow \bot) \rightarrow (\alpha \rightarrow \beta))$	P(1) 4
6.	$(\alpha \rightarrow \bot) \rightarrow (\alpha \rightarrow \beta)$	MP 4,5

A2 on line 1 means that $\neg p \rightarrow (p \rightarrow \beta)$ is an instance of axiom A2. P(2) on the right-side of line 4 means that $(\alpha \rightarrow \bot) \rightarrow ((\bot \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta))$ is an instance of schema of formulae (or the schema of relation of the form $\vdash_{K?} \alpha$) called P(2). P(1) 1 on the right-side of line 2 means that from an instance of relation P(1) along with formula of line 1 we can derive $p \to (\neg p \to \beta)$. If we wished to detail this procedure, we would have something like that:

1. $\neg p \to (p \to \beta)$ A2
2. $\neg p \to (p \to \beta) \vdash_{K?} p \to (\neg p \to \beta)$ instance of P(1)
3. $(\neg p \to \beta)$ since 1 and 2, therefore 3

For the sake of compactness, we skip these detailed steps and show only the final result. MP 4,5 on line 6 justifies the presence of $(\alpha \to \bot) \to (\alpha \to \beta)$ by saying that it is the result of applying *modus ponens* (MP) to the formulae of lines 4 and 5 (that is, it satisfies items (ii).b or (iii).b of definition 5.2.24.) We adopt a similar procedure when other inference rules such as necessitation are involved. For the sake of readability, we omit the definition in which axiom A2 appears as well as the lemmas to which the schemas of relation P(1) and P(2) belong to.

It is also worthy of mention that for axioms or schemas of formula whose form is $\alpha \leftrightarrow \beta$ (which is an abbreviation of $\alpha \to \beta \wedge \beta \to \alpha$), we may sometimes refer to them by writing only one side of the implication, skipping in this way the steps that go from $\alpha \to \beta \wedge \beta \to \alpha$ to $\alpha \to \beta$ or $\beta \to \alpha$.

Lemma A.1. The following schemas of relations between sets of formulae and formula are correct[1]:

P(1): $\alpha \to (\beta \to \varphi) \vdash_{K?} \beta \to (\alpha \to \varphi)$

P(2): $\alpha \to \beta, \beta \to \varphi \vdash_{K?} \alpha \to \varphi$

$\vdash_{K?} (\alpha \to \beta) \to ((\beta \to \varphi) \to (\alpha \to \varphi))$[2]

P(3): $\alpha \to (\beta \to \varphi) \vdash_{K?} \alpha \wedge \beta \to \varphi$

P(4): $\alpha \to (\beta \to \varphi), \varphi \to \lambda \vdash_{K?} \alpha \to (\beta \to \lambda)$

P(5): $\vdash_{K?} \alpha \to \alpha$

P(6): $\alpha \to \beta, \varphi \to \phi \vdash_{K?} \alpha \to (\varphi \to (\beta \wedge \phi))$

P(7): $\alpha \to (\beta \to \varphi), \phi \to (\lambda \to \varphi) \vdash_{K?} (\alpha \wedge \phi) \wedge (\beta \vee \lambda) \to \varphi$

P(8): $\alpha \to \beta, \alpha \vee \varphi \vdash_{K?} \beta \vee \varphi$

P(9): $\alpha \vee \beta \vdash_{K?} (\alpha \to \varphi) \to \beta \vee \varphi$

P(10): $\alpha \to (\beta \to \varphi), \phi \to (\lambda \to \varphi) \vdash_{K?} \alpha \vee \phi \to (\beta \wedge \lambda \to \varphi)$

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[1] For the sake of simplicity, when writing schemas of relation we will omit the set-delimiting brackets. In this way, rather than writing $\{\alpha,\beta\} \vdash \varphi$, we will simply write $\alpha,\beta \vdash \varphi$.

[2] In our derivations, we will refer to the "implicative" and "inferential" forms of theorem P(2) through the same index.

P(11): $\alpha \to (\beta \to \varphi), \phi \to (\lambda \to \varphi) \vdash_{K?} \alpha \wedge \phi \to (\beta \vee \lambda \to \varphi)$

P(12): $\vdash_{K?} (\alpha \to \beta \wedge \varphi) \to (\alpha \to \beta) \wedge (\alpha \to \varphi)$

P(13): $(\alpha \to \beta) \to \varphi, \lambda \to (\phi \to \beta) \vdash_{K?} (\alpha \to \lambda) \to ((\alpha \to \phi) \to \varphi)$

P(14): $\alpha \to \beta \vdash_{K?} \alpha \vee \varphi \to \beta \vee \varphi$

P(15): $\vdash_{K?} (\alpha \to \beta) \to (\varphi \wedge \alpha \to \varphi \wedge \beta)$

P(16): $(\alpha \to \beta) \to ((\alpha \to \varphi) \to \lambda) \vdash_{K?} (\alpha \to \beta \wedge \varphi) \to \lambda$

P(17): $\alpha \to \beta, \phi \to \varphi \vdash_{K?} \alpha \vee \phi \to \beta \vee \varphi$

P(18): $\alpha \vee \beta \vdash_{K?} \beta \vee \alpha$

P(19): $\alpha \vee \beta, \varphi \vee \phi \vdash_{K?} (\alpha \wedge \varphi) \vee (\beta \vee \phi)$

P(20): $\alpha, \beta \vdash_{K?} \alpha \wedge \beta$

P(21): $\alpha \to \beta, \varphi \to \phi \vdash_{K?} \alpha \wedge \varphi \to \beta \wedge \phi$

Proof. All these schemas of relation hold in positive propositional classical logic (P1-P8+MP)[3], that is, classical logic without negation. Since $K_?$ is a conservative extension of positive classical logic, the derivations of these schemas of relations from P1-P8+MP can be taken without any modification as derivations in $K_?$[4]. As a consequence of this, these schemas are valid also in any paranormal modal logic. ■

Definition A.1. Let φ be a schema of formula of some language $\Im$ and p an atomic formula of $\Im$. We define function ρ as follows:

(i) $\rho(p) \equiv p$;

(ii) If $\varphi \equiv \neg\alpha$, then $\rho(\varphi) = \sim\rho(\alpha)$;

(iii) If $\varphi \equiv \alpha \oplus \beta$, then $\rho(\alpha \oplus \beta) = \rho(\alpha) \oplus \rho(\beta)$, where $\oplus \in \{\wedge, \vee, \to\}$.

Definition A.2. Let A be a set of schemas of formula of some language $\Im$. $\rho(A) = \{\rho(\alpha) \mid \alpha \in A\}$.

Lemma A.2. If the schema of relation $A \vdash_c \alpha$ is correct (where $\vdash_c$ is the relation of deduction of propositional classical logic), then the schema of relation $\rho(A) \vdash_{K?} \rho(\alpha)$ is also correct.

Proof: In order to prove this lemma, we need to show that, given a specific axiomatics for classical logic A + MP, where A is a set of schemas of formula, all schemas of formula of $\rho(A)$ are valid in $K_?$. Let us consider the axiomatic for classical logic P1-P11+MP[5]. Since the schemas of formula P1-

[3] Definitions 5.2.22 and 5.2.24.

[4] The derivation of many of these schemas of relation is available in Detlovs & Podnieks (2004).

[5] Definitions 5.2.22, 5.2.24 and 5.3.16.

P8 have no occurrence of the negation symbol $\neg$ and since they belong to axiomatic of $K_?$, the schemas of formula $\rho(P1)$, …, $\rho(P8)$ are automatically valid in $K_?$. It rests then to consider schemas P9-P11, that is to say, to show that the schemas of formulae $\rho(P9)$, $\rho(P10)$ and $\rho(P11)$ are also valid in $K_?$:

$\rho(P9)$: $(\rho(\alpha) \to \rho(\beta)) \to ((\rho(\alpha) \to \sim\rho(\beta)) \to \sim\rho(\alpha))$

$\rho(P10)$: $\sim\rho(\alpha) \to (\rho(\alpha) \to \rho(\beta))$

$\rho(P11)$: $\rho(\alpha) \vee \sim\rho(\alpha)$

In order to do that, we have just to prove that the following schemas of formulae are valid in $K_?$:

P9': $(\alpha \to \beta) \to ((\alpha \to \sim\beta) \to \sim\alpha)$

P10': $\sim\alpha \to (\alpha \to \beta)$

P11': $\alpha \vee \sim\alpha$

Below we have the derivation of each one of these schemas of formulae:

P9': $(\alpha \to \beta) \to ((\alpha \to \sim\beta) \to \sim\alpha)$

1. $(\alpha \to (\beta \to \bot)) \to ((\alpha \to \beta) \to (\alpha \to \bot))$ — P1
2. $(\alpha \to \beta) \to ((\alpha \to (\beta \to \bot)) \to (\alpha \to \bot))$ — P(1) 1

P10': $\sim\alpha \to (\alpha \to \beta) \to \rho(\beta))$

1. $\neg p \to (p \to \beta)$ — A2
2. $p \to (\neg p \to \beta)$ — P(1) 1
3. $\bot \to \beta$ — P(3) 2
4. $(\alpha \to \bot) \to ((\bot \to \beta) \to (\alpha \to \beta))$ — P(2)
5. $(\bot \to \beta) \to ((\alpha \to \bot) \to (\alpha \to \beta))$ — P(1) 4
6. $(\alpha \to \bot) \to (\alpha \to \beta)$ — MP 3,5

In order to present the derivation of $\rho(P11)$, we need the following theorems:

Aux1: $\vdash_{K?} (\alpha \to \beta) \to (\sim\beta \to \sim\alpha)$

Aux2: $\vdash_{K?} \sim\sim\alpha \to \alpha$

In classical logic, $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$ is deduced from P1, P2, P9 and MP. Therefore, in order to prove $\vdash_{K?} (\alpha \to \beta) \to (\sim\beta \to \sim\alpha)$ we must just rewrite the derivation of $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$ in classical logic replacing P9 by its $\sim$ version P9', which we have just proved above. Below we show the derivation of $\sim\sim\alpha \to \alpha$.

1. $\neg p \to (p \to \alpha)$ — A2
2. $p \to (\neg p \to \alpha)$ — P(1) 1

3.	$\bot \to \alpha$	P(3) 2
4.	$((\alpha \to \bot) \to \alpha) \to \alpha$	N5
5.	$((\alpha \to \bot) \to \bot) \to ((\bot \to \alpha) \to ((\alpha \to \bot) \to \alpha))$	P(2)
6.	$(\bot \to \alpha) \to (((\alpha \to \bot) \to \bot) \to ((\alpha \to \bot) \to \alpha))$	P(1) 5
7.	$((\alpha \to \bot) \to \bot) \to ((\alpha \to \bot) \to \alpha)$	MP 3,6
8.	$((\alpha \to \bot) \to \bot) \to \alpha$	P(2) 4,7

P11': $\alpha \vee \sim\alpha$

1.	$\alpha \to \alpha \vee \sim\alpha$	P6
2.	$\sim\alpha \to \alpha \vee \sim\alpha$	P7
3.	$(\alpha \to \alpha \vee \sim\alpha) \to (\sim(\alpha \vee \sim\alpha) \to \sim\alpha)$	Aux1
4.	$\sim(\alpha \vee \sim\alpha) \to \sim\alpha$	MP 1,3
5.	$(\sim\alpha \to \alpha \vee \sim\alpha) \to (\sim(\alpha \vee \sim\alpha) \to \sim\sim\alpha)$	Aux1
6.	$\sim(\alpha \vee \sim\alpha) \to \sim\sim\alpha$	MP 2,5
7.	$(\sim(\alpha \vee \sim\alpha) \to \sim\alpha) \to ((\sim(\alpha \vee \sim\alpha) \to \sim\sim\alpha) \to \sim\sim(\alpha \vee \sim\alpha))$	P9'
8.	$(\sim(\alpha \vee \sim\alpha) \to \sim\sim\alpha) \to \sim\sim(\alpha \vee \sim\alpha)$	MP 4,7
9.	$\sim\sim(\alpha \vee \sim\alpha)$	MP 6,8
10.	$\sim\sim(\alpha \vee \sim\alpha) \to (\alpha \vee \sim\alpha)$	Aux2
11.	$\alpha \vee \sim\alpha$	MP 9,10 ■

Lemma A.3. The following schemas of relation between sets of formulae and formula are correct:

~(1): $\vdash_{K?} (\alpha \to \beta) \to ((\alpha \to \sim\beta) \to \sim\alpha)$

~(2): $\vdash_{K?} \sim\alpha \to (\alpha \to \beta)$

~(3): $\vdash_{K?} \alpha \vee \sim\alpha$

~(4): $\vdash_{K?} (\sim\alpha \to \beta) \leftrightarrow (\sim\beta \to \alpha)$

~(5): $\vdash_{K?} \sim(\alpha \to \beta) \leftrightarrow \alpha \wedge \sim\beta$

~(6): $\vdash_{K?} \sim(\alpha \wedge \beta) \leftrightarrow \sim\alpha \vee \sim\beta$

~(7): $\vdash_{K?} \sim(\alpha \vee \beta) \leftrightarrow \sim\alpha \wedge \sim\beta$

~(8): $\alpha \vee \beta \vdash_{K?} \sim\alpha \to \beta$

$\vdash_{K?} \alpha \vee \beta \to (\sim\alpha \to \beta)$

~(9): $\vdash_{K?} \sim\sim\alpha \leftrightarrow \alpha$

~(10): $\vdash_{K?} (\alpha \rightarrow \beta) \leftrightarrow (\sim\beta \rightarrow \sim\alpha)$

Proof. If we replace ~ by ¬, all these schemas of relation will be valid in classical logic. From lemma A.2 then, we have that they are valid in $K_?$. ■

Lemma A.4. The following schemas of relation between sets of formulae and formula are correct:

K(1): $\alpha \rightarrow \beta \vdash_{K?} \alpha? \rightarrow \beta?$

K(2): $\vdash_{K?} \sim(\alpha!) \leftrightarrow (\sim\alpha)?$

K(3): $\vdash_{K?} (\alpha \rightarrow \beta)! \rightarrow (\alpha? \rightarrow \beta?)$

K(4): $\alpha \rightarrow \beta \vdash_{K?} \alpha! \rightarrow \beta!$

K(5): $\vdash_{K?} \sim(\alpha?) \leftrightarrow (\sim\alpha)!$

K(6): $\alpha \rightarrow (\varphi \rightarrow \bot) \vdash_{K?} \alpha? \rightarrow (\varphi! \rightarrow \bot)$

K(7): $\vdash_{K?} \bot \rightarrow \beta$

K(8): $\vdash_{K?} (\sim\alpha)? \vee \alpha!$

Proof. If we apply the inverse of function Φ^6 to each one of the schemas of formula that appear in the above relations, then we will obtain a set of schemas of relation which are valid in normal modal logic K. It follows then from theorem 5.3.14 that the schemas as presented above are all valid in $K_?$. K(7) is an easy derivation from A2 and P(3). ■

Lemma A.5. Let M be a model and *w* a world of M. If $M,w \Vdash_{\Psi\diamond} \alpha$, then $M,w \Vdash_{\Omega?} \Phi(\alpha)$.

Proof. We first note that, regarding Ω, the function Ʊ is invoked only when negation formulae are considered. Formulae of the form ~α are analyzed without the help of function Ʊ: $\Omega_{M,w}(\alpha \rightarrow \bot) = 1$ iff $\Omega_{M,w}(\alpha) = 0$ or $\Omega_{M,w}(\bot) = 1$ iff $\Omega_{M,w}(\alpha) = 0$. With this remark in mind, it becomes trivial that if $M,w \Vdash_{\Psi\diamond} \alpha$ then $M,w \Vdash_{\Omega?} \Phi(\alpha)$. ■

Lemma A.6. Let $\alpha \in L_?$ and $\beta \in L_\diamond$. $\Delta(\Pi(\alpha)) = \alpha$ and $\Pi(\Delta(\beta)) = \beta$. Let $A \subseteq L_?$ and $B \subseteq L_\diamond$. $\Delta(\Pi(A)) = A$ and $\Pi(\Delta(B)) = B$.

Proof. Trivially, Δ and Π are the inverse functions of each other. ■

Lemma A.7. Let $A,B \subseteq L_\diamond$ and $\alpha \in L_\diamond$. If $A \div B \vdash_K \alpha$ then $\Delta(A) \div \Delta(B) \vdash_{K?} \Delta(\alpha)$.

Proof. We are going to prove this theorem by induction on the size of the K-derivation of α from A and B. Let $\alpha_1, \ldots, \alpha_n$ be the K-derivation from $A \div B$ to α.

Base of induction: derivation of size 1: $\alpha_1 \equiv \alpha$

- Case 1: $\alpha \in A \cup B$.

Trivially, $\Delta(\alpha) \in \Delta(A \cup B)$. Therefore, $\Delta(A) \dotplus \Delta(B) \vdash_{K?} \Delta(\alpha)$.

- Case 2: α is an axiom of K.

In order to deal with this case, we have to analyze the possibility of α's being an instance of each one of K's axiom schemas. For each one of these possibilities, we will show that there is a $K_?$-derivation of $\Delta(\alpha)$, what implies that there exists a $K_?$-derivation of $\Delta(\alpha)$ from $\Delta(A)$ and $\Delta(B)$ and therefore that $\Delta(A) \dotplus \Delta(B) \vdash_{K?} \Delta(\alpha)$. Regarding K's classical axioms, the cases where α is an instance of P1-P8 are trivial, for these schemas of axiom appear in $K_?$'s axiomatic without any modification. Just for the sake of illustration, we show how this analysis would be done regarding axiom schema P1.

P1: $\alpha \equiv \varphi \rightarrow (\beta \rightarrow \varphi)$. $\Delta\alpha \equiv \Delta(\varphi \rightarrow (\beta \rightarrow \varphi)) \equiv \Delta\varphi \rightarrow (\Delta\beta \rightarrow \Delta\varphi)$. Since $\Delta\varphi \rightarrow (\Delta\beta \rightarrow \Delta\varphi)$ is an instance of $K_?$'s P1, $\Delta\alpha$ is a $K_?$-derivation of $\Delta\alpha$.

Now, it lacks to show that if α is an instance of P9, P10, P11, K or NP, then there is a $K_?$-derivation of $\Delta\alpha$.

P9: $\alpha \equiv (\varphi \rightarrow \beta) \rightarrow ((\varphi \rightarrow \neg\beta) \rightarrow \neg\varphi)$. $\Delta((\varphi \rightarrow \beta) \rightarrow ((\varphi \rightarrow \neg\beta) \rightarrow \neg\varphi)) \equiv (\Delta\varphi \rightarrow \Delta\beta) \rightarrow ((\Delta\varphi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\varphi)$. We have to prove then that there exists a $K_?$-derivation of $(\Delta\varphi \rightarrow \Delta\beta) \rightarrow ((\Delta\varphi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\varphi)$. In order to do that, will have to make use of a result which will be proved only when we consider the case where α is an instance of P10 and which we will refer to as P10*: for every $\beta \in \Im_\diamond$ there is a $K_?$-derivation of $\neg\nabla\beta \rightarrow (\Delta\beta \rightarrow \Delta\varphi)$, where $\varphi \in \Im_\diamond$ is an arbitrary formula. Let us do the proof by induction on the size of φ.

Base of induction: φ has size 1. $\varphi \equiv p$. $\Delta\alpha \equiv (\Delta\varphi \rightarrow \Delta\beta) \rightarrow ((\Delta\varphi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\varphi) \equiv \Delta\alpha \equiv (\Delta p \rightarrow \Delta\beta) \rightarrow ((\Delta p \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla p) \equiv (p \rightarrow \Delta\beta) \rightarrow ((p \rightarrow \neg\nabla\beta) \rightarrow \neg p)$.

1.	$(p \rightarrow q) \rightarrow ((p \rightarrow \neg q) \rightarrow \neg p)$	A1
2.	$(p \rightarrow \bot) \rightarrow \neg p$	P(16) 1
3.	$\neg\nabla\beta \rightarrow (\Delta\beta \rightarrow \bot)$	P10*
4.	$\Delta\beta \rightarrow (\neg\nabla\beta \rightarrow \bot)$	P(1) 3
5.	$(p \rightarrow \Delta\beta) \rightarrow ((p \rightarrow \neg\nabla\beta) \rightarrow \neg p)$	P(13) 2,4

[6] Definition 5.3.23.

Hypothesis of Induction: Take an arbitrary formula φ of size n. Suppose that, for formulae ϕ of size m<n, there is a $K_?$-derivation of $(\Delta\phi \to \Delta\lambda) \to ((\Delta\phi \to \neg\nabla\lambda) \to \neg\nabla\phi)$. We will show that, if this is the case, there is necessarily a $K_?$-derivation of $(\Delta\varphi \to \Delta\beta) \to ((\Delta\varphi \to \neg\nabla\beta) \to \neg\nabla\varphi)$. Before considering all forms φ may have, we will prove the following auxiliary result: $(\Delta\phi \to \Delta\beta) \to ((\Delta\phi \to \neg\nabla\beta) \to \neg\nabla\phi) \vdash_? (\Delta\phi \to \bot) \to \neg\nabla\phi$.

Aux: $(\Delta\phi \to \Delta\beta) \to ((\Delta\phi \to \neg\nabla\beta) \to \neg\nabla\phi) \vdash_? (\Delta\phi \to \bot) \to \neg\nabla\phi$

1. $(\Delta\phi \to \Delta\beta) \to ((\Delta\phi \to \neg\nabla\beta) \to \neg\nabla\phi)$ — Hyp.
2. $(\Delta\phi \to \Delta\beta) \wedge (\Delta\phi \to \neg\nabla\beta) \to \neg\nabla\phi$ — P(3) 1
3. $(\Delta\phi \to \Delta\beta \wedge \neg\nabla\beta) \to (\Delta\phi \to \Delta\beta) \wedge (\Delta\phi \to \neg\nabla\beta)$ — P(12)
4. $(\Delta\phi \to \Delta\beta \wedge \neg\nabla\beta) \to \neg\nabla\phi$ — P(2) 2,3
5. $(\Delta\phi \to \bot) \to (\Delta\phi \to \Delta\beta \wedge \neg\nabla\beta)$ — ~(2)
6. $(\Delta\phi \to \bot) \to \neg\nabla\phi$ — P(2) 4,5

<u>$\varphi \equiv \Box\phi$</u>. $\Delta\alpha \equiv (\Delta\varphi \to \Delta\beta) \to ((\Delta\varphi \to \neg\nabla\beta) \to \neg\nabla\varphi) \equiv (\Delta\Box\phi \to \Delta\beta) \to ((\Delta\Box\phi \to \neg\nabla\beta) \to \neg\nabla\Box\phi) \equiv (\Delta\phi! \to \Delta\beta) \to ((\Delta\phi! \to \neg\nabla\beta) \to \neg(\nabla\phi?))$. Since ϕ's size is smaller than n, by the hypothesis of induction we have that there is a $K_?$-derivation of $(\Delta\phi \to \Delta\beta) \to ((\Delta\phi \to \neg\nabla\beta) \to \neg\nabla\phi)$.

1. $(\Delta\phi \to \Delta\beta) \to ((\Delta\phi \to \neg\nabla\beta) \to \neg\nabla\phi)$ — H. Induction[7]
2. $(\Delta\phi \to \bot) \to \neg\nabla\phi$ — Aux 1
3. $(\Delta\phi \to \bot)? \to (\neg\nabla\phi)?$ — K(1) 2
4. $(\Delta\phi! \to \bot) \to (\Delta\phi \to \bot)?$ — K(2)
5. $(\Delta\phi! \to \bot) \to (\neg\nabla\phi)?$ — P(2) 3,4
6. $(\neg\nabla\phi)? \to \neg(\nabla\phi?)$ — K3
7. $(\Delta\phi! \to \bot) \to \neg(\nabla\phi?)$ — P(2) 5,6
8. $\neg\nabla\beta \to (\Delta\beta \to \bot)$ — P10*
9. $\Delta\beta \to (\neg\nabla\beta \to \bot)$ — P(1) 8

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[7] "H. Induction." means that the result stated before is part of the hypothesis of induction of the respective proof.

10. $(\Delta\phi! \rightarrow \Delta\beta) \rightarrow ((\Delta\phi! \rightarrow \neg\nabla\beta) \rightarrow \neg(\nabla\phi?))$ P(13) 7,9

<u>$\varphi \equiv \Diamond\phi$</u>. $\Delta\alpha \equiv (\Delta\varphi \rightarrow \Delta\beta) \rightarrow ((\Delta\varphi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\varphi) \equiv (\Delta\Diamond\phi \rightarrow \Delta\beta) \rightarrow ((\Delta\Diamond\phi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\Diamond\phi) \equiv (\Delta\phi? \rightarrow \Delta\beta) \rightarrow ((\Delta\phi? \rightarrow \neg\nabla\beta) \rightarrow \neg(\nabla\phi!))$. Since ϕ's size is smaller than n, by the hypothesis of induction we have that there is a $K_?$-derivation of $(\Delta\phi \rightarrow \Delta\beta) \rightarrow ((\Delta\phi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\phi)$.

1. $(\Delta\phi \rightarrow \Delta\beta) \rightarrow ((\Delta\phi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\phi)$ H. Induction
2. $(\Delta\phi \rightarrow \bot) \rightarrow \neg\nabla\phi$ Aux 1
3. $(\Delta\phi \rightarrow \bot)! \rightarrow (\neg\nabla\phi)!$ K(4) 2
4. $(\Delta\phi? \rightarrow \bot) \rightarrow (\Delta\phi \rightarrow \bot)!$ K(5)
5. $(\Delta\phi? \rightarrow \bot) \rightarrow (\neg\nabla\phi)!$ P(2) 3,4
6. $(\neg\nabla\phi)! \rightarrow \neg(\nabla\phi!)$ K2
7. $(\Delta\phi? \rightarrow \bot) \rightarrow \neg(\nabla\phi!)$ P(2) 5,6
8. $\neg\nabla\beta \rightarrow (\Delta\beta \rightarrow \bot)$ P10*
9. $\Delta\beta \rightarrow (\neg\nabla\beta \rightarrow \bot)$ P(1) 8
10. $(\Delta\phi? \rightarrow \Delta\beta) \rightarrow ((\Delta\phi? \rightarrow \neg\nabla\beta) \rightarrow \neg(\nabla\phi!))$ P(13) 7,9

<u>$\varphi \equiv \neg\phi$</u>. $\Delta\alpha \equiv (\Delta\varphi \rightarrow \Delta\beta) \rightarrow ((\Delta\varphi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\varphi) \equiv (\Delta\neg\phi \rightarrow \Delta\beta) \rightarrow ((\Delta\neg\phi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\neg\phi) \equiv (\neg\nabla\phi \rightarrow \Delta\beta) \rightarrow ((\neg\nabla\phi \rightarrow \neg\nabla\beta) \rightarrow \neg\neg\Delta\phi)$. Since ϕ's size is smaller than n, by the hypothesis of induction we have that there is a $K_?$-derivation of $(\Delta\phi \rightarrow \Delta\beta) \rightarrow ((\Delta\phi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\phi)$.

1. $(\Delta\phi \rightarrow \Delta\beta) \rightarrow ((\Delta\phi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\phi)$ H. Induction
2. $(\Delta\phi \rightarrow \bot) \rightarrow \neg\nabla\phi$ Aux 1
3. $((\Delta\phi \rightarrow \bot) \rightarrow \neg\nabla\phi) \rightarrow ((\neg\nabla\phi \rightarrow \bot) \rightarrow \Delta\phi)$ ~(4)
4. $(\neg\nabla\phi \rightarrow \bot) \rightarrow \Delta\phi$ MP 2,3
5. $\Delta\phi \rightarrow \neg\neg\Delta\phi$ N4
6. $(\neg\nabla\phi \rightarrow \bot) \rightarrow \neg\neg\Delta\phi$ P(2) 4,5
7. $\neg\nabla\beta \rightarrow (\Delta\beta \rightarrow \bot)$ P10*
8. $\Delta\beta \rightarrow (\neg\nabla\beta \rightarrow \bot)$ P(1) 6

9. $(\neg\nabla\phi \rightarrow \Delta\beta) \rightarrow ((\neg\nabla\phi \rightarrow \neg\nabla\beta) \rightarrow \neg\neg\Delta\phi)$ P(13) 6,8

$\underline{\varphi \equiv \phi \rightarrow \lambda}$. $\Delta\alpha \equiv (\Delta\varphi \rightarrow \Delta\beta) \rightarrow ((\Delta\varphi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\varphi) \equiv (\Delta(\phi \rightarrow \lambda) \rightarrow \Delta\beta) \rightarrow ((\Delta(\phi \rightarrow \lambda) \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla(\phi \rightarrow \lambda)) \equiv ((\Delta\phi \rightarrow \Delta\lambda) \rightarrow \Delta\beta) \rightarrow (((\Delta\phi \rightarrow \Delta\lambda) \rightarrow \neg\nabla\beta) \rightarrow \neg(\Delta\phi \rightarrow \nabla\lambda))$. Since λ's size is smaller than n, by the hypothesis of induction we have that there is a $K_?$-derivation of $(\Delta\lambda \rightarrow \Delta\beta) \rightarrow ((\Delta\lambda \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\lambda)$.

1. $(\sim\Delta\lambda \rightarrow \neg\nabla\lambda) \rightarrow (\Delta\phi \wedge \sim\Delta\lambda \rightarrow \Delta\phi \wedge \neg\nabla\lambda)$ P(15)
2. $(\Delta\lambda \rightarrow \Delta\beta) \rightarrow ((\Delta\lambda \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\lambda)$ H. Induction
3. $\sim\Delta\lambda \rightarrow \neg\nabla\lambda$ Aux 2
4. $\Delta\phi \wedge \sim\Delta\lambda \rightarrow \Delta\phi \wedge \neg\nabla\lambda$ MP 1,3
5. $\Delta\phi \wedge \neg\nabla\lambda \rightarrow \neg(\Delta\phi \rightarrow \nabla\lambda)$ N1
6. $\Delta\phi \wedge \sim\Delta\lambda \rightarrow \neg(\Delta\phi \rightarrow \nabla\lambda)$ P(2) 4,5
7. $\sim(\Delta\phi \rightarrow \Delta\lambda) \rightarrow \Delta\phi \wedge \sim\Delta\lambda$ ~(5)
8. $((\Delta\phi \rightarrow \Delta\lambda) \rightarrow \bot) \rightarrow \neg(\Delta\phi \rightarrow \nabla\lambda)$ P(2) 6,7
9. $\neg\nabla\beta \rightarrow (\Delta\beta \rightarrow \bot)$ P10*
10. $\Delta\beta \rightarrow (\neg\nabla\beta \rightarrow \bot)$ P(1) 9
11. $((\Delta\phi \rightarrow \Delta\lambda) \rightarrow \Delta\beta) \rightarrow (((\Delta\phi \rightarrow \Delta\lambda) \rightarrow \neg\nabla\beta) \rightarrow \neg(\Delta\phi \rightarrow \nabla\lambda))$ P(13) 8,10

$\underline{\varphi \equiv \phi \wedge \lambda}$. $\Delta\alpha \equiv (\Delta\varphi \rightarrow \Delta\beta) \rightarrow ((\Delta\varphi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\varphi) \equiv (\Delta(\phi \wedge \lambda) \rightarrow \Delta\beta) \rightarrow ((\Delta(\phi \wedge \lambda) \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla(\phi \wedge \lambda)) \equiv ((\Delta\phi \wedge \Delta\lambda) \rightarrow \Delta\beta) \rightarrow (((\Delta\phi \wedge \Delta\lambda) \rightarrow \neg\nabla\beta) \rightarrow \neg(\nabla\phi \wedge \nabla\lambda))$. Since ϕ's and λ's sizes are smaller than n, by the hypothesis of induction we have that there is a $K_?$-derivation of $(\Delta\phi \rightarrow \Delta\beta) \rightarrow ((\Delta\phi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\phi)$ and $(\Delta\lambda \rightarrow \Delta\beta) \rightarrow ((\Delta\lambda \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\lambda)$.

1. $(\Delta\phi \rightarrow \Delta\beta) \rightarrow ((\Delta\phi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\phi)$ H. Induction
2. $\sim\Delta\phi \rightarrow \neg\nabla\phi$ Aux 1
3. $(\Delta\lambda \rightarrow \Delta\beta) \rightarrow ((\Delta\lambda \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\lambda)$ H. Induction
4. $\sim\Delta\lambda \rightarrow \neg\nabla\lambda$ Aux 3
5. $\sim\Delta\phi \vee \sim\Delta\lambda \rightarrow \neg\nabla\phi \vee \neg\nabla\lambda$ P(17) 2,4
6. $\sim(\Delta\phi \wedge \Delta\lambda) \rightarrow \sim\Delta\phi \vee \sim\Delta\lambda$ ~(6)

7. $\sim(\Delta\phi \wedge \Delta\lambda) \rightarrow \neg\nabla\phi \vee \neg\nabla\lambda$ P(2) 5,6

8. $\neg\nabla\phi \vee \neg\nabla\lambda \rightarrow \neg(\nabla\phi \wedge \nabla\lambda)$ N2

9. $((\Delta\phi \wedge \Delta\lambda) \rightarrow \bot) \rightarrow \neg(\nabla\phi \wedge \nabla\lambda)$ P(2) 7,8

10. $\neg\nabla\beta \rightarrow (\Delta\beta \rightarrow \bot)$ P10*

11. $\Delta\beta \rightarrow (\neg\nabla\beta \rightarrow \bot)$ P(1) 10

12. $((\Delta\phi \wedge \Delta\lambda) \rightarrow \Delta\beta) \rightarrow (((\Delta\phi \wedge \Delta\lambda) \rightarrow \neg\nabla\beta) \rightarrow \neg(\nabla\phi \wedge \nabla\lambda))$ P(13) 9,11

<u>$\varphi \equiv \phi \vee \lambda$</u>. $\Delta\alpha \equiv (\Delta\varphi \rightarrow \Delta\beta) \rightarrow ((\Delta\varphi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\varphi) \equiv (\Delta(\phi \vee \lambda) \rightarrow \Delta\beta) \rightarrow ((\Delta(\phi \vee \lambda) \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla(\phi \vee \lambda)) \equiv ((\Delta\phi \vee \Delta\lambda) \rightarrow \Delta\beta) \rightarrow (((\Delta\phi \vee \Delta\lambda) \rightarrow \neg\nabla\beta) \rightarrow \neg(\nabla\phi \vee \nabla\lambda))$. Since ϕ's and λ's sizes are smaller than *n*, by the hypothesis of induction we have that there is a $K_?$-derivation of $(\Delta\phi \rightarrow \Delta\beta) \rightarrow ((\Delta\phi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\phi)$ and of $(\Delta\lambda \rightarrow \Delta\beta) \rightarrow ((\Delta\lambda \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\lambda)$.

1. $(\Delta\phi \rightarrow \Delta\beta) \rightarrow ((\Delta\phi \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\phi)$ H. Induction

2. $\sim\Delta\phi \rightarrow \neg\nabla\phi$ Aux 1

3. $(\Delta\lambda \rightarrow \Delta\beta) \rightarrow ((\Delta\lambda \rightarrow \neg\nabla\beta) \rightarrow \neg\nabla\lambda)$ H. Induction

4. $\sim\Delta\lambda \rightarrow \neg\nabla\lambda$ Aux 3

5. $\sim\Delta\phi \wedge \sim\Delta\lambda \rightarrow \neg\nabla\phi \wedge \neg\nabla\lambda$ P(21) 2,4

6. $\sim(\Delta\phi \vee \Delta\lambda) \rightarrow \sim\Delta\phi \wedge \sim\Delta\lambda$ ~(7)

7. $\sim(\Delta\phi \vee \Delta\lambda) \rightarrow \neg\nabla\phi \wedge \neg\nabla\lambda$ P(2) 5,6

8. $\neg\nabla\phi \wedge \neg\nabla\lambda \rightarrow \neg(\nabla\phi \vee \nabla\lambda)$ N3

9. $((\Delta\phi \vee \Delta\lambda) \rightarrow \bot) \rightarrow \neg(\nabla\phi \vee \nabla\lambda)$ P(2) 7,8

10. $\neg\nabla\beta \rightarrow (\Delta\beta \rightarrow \bot)$ P10*

11. $\Delta\beta \rightarrow (\neg\nabla\beta \rightarrow \bot)$ P(1) 10

12. $((\Delta\phi \vee \Delta\lambda) \rightarrow \Delta\beta) \rightarrow (((\Delta\phi \vee \Delta\lambda) \rightarrow \neg\nabla\beta) \rightarrow \neg(\nabla\phi \vee \nabla\lambda))$ P(13) 9,11

P10 and P11: In order to prove that there is a $K_?$-derivation of $\Delta\alpha$ in the cases where α is either an instance of P10 or an instance of P11, we need to consider in the same proof by induction the two mentioned cases. In this circumstance, $\Delta\alpha$ has one of the following forms:

$\alpha \equiv \neg\varphi \rightarrow (\varphi \rightarrow \beta)$. $\Delta\alpha \equiv \Delta(\neg\varphi \rightarrow (\varphi \rightarrow \beta)) \equiv \neg\nabla\varphi \rightarrow (\Delta\varphi \rightarrow \Delta\beta)$.

$\alpha \equiv \varphi \vee \neg\varphi$. $\Delta\alpha \equiv \Delta(\varphi \vee \neg\varphi) \equiv \Delta\varphi \vee \neg\nabla\varphi$.

As before, we will do the proof that by induction on the size of φ.

Base of induction: φ has size 1. $\varphi \equiv p$. In this case, $\Delta\alpha$ has one of the following forms:

$\Delta\alpha \equiv \neg\nabla\varphi \to (\Delta\varphi \to \Delta\beta) \equiv \neg\nabla p \to (\Delta p \to \Delta\beta) \equiv \neg p \to (p \to \Delta\beta)$.

$\Delta\alpha \equiv \Delta\varphi \vee \neg\nabla\varphi \equiv \Delta p \vee \neg\nabla p \equiv p \vee \neg p$

In the first case, $\Delta\alpha$ is an instance of A2 and, therefore, itself a $K_?$-derivation of $\Delta\alpha$. In the second case, $\Delta\alpha$ is an instance of A3 and, therefore, itself a $K_?$-derivation of $\Delta\alpha$.

Hypothesis of Induction: Take an arbitrary formula φ of size *n*. Suppose that, for any arbitrary formula ϕ of size m<n, there is a $K_?$-derivation of $\neg\nabla\phi \to (\Delta\phi \to \Delta\beta)$, where $\beta \in L$ is an arbitrary formula, and of $\Delta\phi \vee \neg\nabla\phi$. We will show that, if this is the case, then there is also a $K_?$-derivation of $\neg\nabla\varphi \to (\Delta\varphi \to \Delta\beta)$ and of $\Delta\varphi \vee \neg\nabla\varphi$. Let us first consider all possible forms φ may have in the case α is an instance of C10, and after all possible forms it may have in the case α is an instance of C11.

P10: $\alpha \equiv \neg\varphi \to (\varphi \to \beta)$. $\Delta\alpha \equiv \Delta(\neg\varphi \to (\varphi \to \beta)) \equiv \neg\nabla\varphi \to (\Delta\varphi \to \Delta\beta)$.

<u>$\varphi \equiv \Box\phi$</u>. $\Delta\alpha \equiv \neg\nabla\varphi \to (\Delta\varphi \to \Delta\beta) \equiv \neg\nabla(\Box\phi) \to (\Delta(\Box\phi) \to \Delta\beta) \equiv \neg(\nabla\phi?) \to (\Delta\phi! \to \Delta\beta)$.

Since ϕ's size is smaller than *n*, by the hypothesis of induction we have that there is a $K_?$-derivation of $\neg\nabla\phi \to (\Delta\phi \to \Delta\lambda)$. Let $\lambda \equiv \bot$. $\Delta\lambda \equiv \Delta(p \wedge \neg p) \equiv \Delta p \wedge \neg\nabla p \equiv p \wedge \neg p \equiv \bot$.

1. $\neg\nabla\phi \to (\Delta\phi \to \bot)$ — H. Induction
2. $(\neg\nabla\phi)? \to (\Delta\phi! \to \bot)$ — K(6) 1
3. $\neg(\nabla\phi?) \to (\neg\nabla\phi)?$ — K3
4. $\neg(\nabla\phi?) \to (\Delta\phi! \to \bot)$ — P(2) 2,3
5. $\bot \to \Delta\beta$ — K(7)
6. $\neg(\nabla\phi?) \to (\Delta\phi! \to \Delta\beta)$ — P(4) 4,5

<u>$\varphi \equiv \Diamond\phi$</u>. $\Delta\alpha \equiv \neg\nabla\varphi \to (\Delta\varphi \to \Delta\beta) \equiv \neg\nabla(\Diamond\phi) \to (\Delta(\Diamond\phi) \to \Delta\beta) \equiv \neg(\nabla\phi!) \to (\Delta\phi? \to \Delta\beta)$.

Since ϕ's size is smaller than *n*, by the hypothesis of induction we have that there is a $K_?$-derivation of $\neg\nabla\phi \to (\Delta\phi \to \Delta\lambda)$. Let $\lambda \equiv \bot$. $\Delta\lambda \equiv \Delta(p \wedge \neg p) \equiv \Delta p \wedge \neg\nabla p \equiv p \wedge \neg p$.

1. $\neg\nabla\phi \to (\Delta\phi \to \bot)$ — H. Induction
2. $\Delta\phi \to (\neg\nabla\phi \to \bot)$ — P(1) 1

3. $\Delta\phi? \rightarrow ((\neg\nabla\phi)! \rightarrow \bot)$ K(6) 2

4. $(\neg\nabla\phi)! \rightarrow (\Delta\phi? \rightarrow \bot)$ P(1) 3

5. $\neg(\nabla\phi!) \rightarrow (\neg\nabla\phi)!$ K2

6. $\neg(\nabla\phi!) \rightarrow (\Delta\phi? \rightarrow \bot)$ P(2) 4,5

7. $\bot \rightarrow \Delta\beta$ K(7)

8. $\neg(\nabla\phi!) \rightarrow (\Delta\phi? \rightarrow \Delta\beta)$ P(4) 6,7

$\underline{\varphi \equiv \neg\phi}$. $\Delta(\alpha) \equiv \neg\nabla\varphi \rightarrow (\Delta\varphi \rightarrow \Delta\beta) \equiv \neg\nabla(\neg\phi) \rightarrow (\Delta(\neg\phi) \rightarrow \Delta\beta) \equiv \neg\neg\Delta\phi \rightarrow (\neg\nabla\phi \rightarrow \Delta\beta)$. Since ϕ's size is smaller than *n*, by the hypothesis of induction we have that there is a $K_?$-derivation of $\neg\nabla\phi \rightarrow (\Delta\phi \rightarrow \Delta\beta)$.

1. $\neg\nabla\phi \rightarrow (\Delta\phi \rightarrow \Delta\beta)$ H. Induction

2. $\Delta\phi \rightarrow (\neg\nabla\phi \rightarrow \Delta\beta)$ P(1) 1

3. $\neg\neg\Delta\phi \rightarrow \Delta\phi$ N4

4. $\neg\neg\Delta\phi \rightarrow (\neg\nabla\phi \rightarrow \Delta\beta)$ P(2) 2,3

$\underline{\varphi \equiv \phi \rightarrow \lambda}$. $\Delta(\alpha) \equiv \neg\nabla\varphi \rightarrow (\Delta\varphi \rightarrow \Delta\beta) \equiv \neg\nabla(\phi \rightarrow \lambda) \rightarrow (\Delta(\phi \rightarrow \lambda) \rightarrow \Delta\beta) \equiv \neg(\Delta\phi \rightarrow \nabla\lambda) \rightarrow ((\Delta\phi \rightarrow \Delta\lambda) \rightarrow \Delta\beta)$. Since ϕ's and λ's sizes are smaller than *n*, by the hypothesis of induction we have that there exists a $K_?$-derivation of $\neg\nabla\phi \rightarrow (\Delta\phi \rightarrow \Delta\mu)$, $\neg\nabla\lambda \rightarrow (\Delta\lambda \rightarrow \Delta\mu)$ and $\Delta\phi \vee \neg\nabla\phi$. Let $\mu \equiv \bot$. $\Delta\mu \equiv \Delta(p \wedge \neg p) \equiv \Delta p \wedge \neg\nabla p \equiv p \wedge \neg p$.

1. $\neg(\Delta\phi \rightarrow \nabla\lambda) \rightarrow (\Delta\phi \wedge \neg\nabla\lambda)$ N1

2. $\Delta\phi \vee \neg\nabla\phi$ H. Induction

3. $(\Delta\phi \rightarrow \Delta\lambda) \rightarrow (\neg\nabla\phi \vee \Delta\lambda)$ P(9) 2

4. $\neg(\Delta\phi \rightarrow \nabla\lambda) \rightarrow ((\Delta\phi \rightarrow \Delta\lambda) \rightarrow (\Delta\phi \wedge \neg\nabla\lambda) \wedge (\neg\nabla\phi \vee \Delta\lambda))$ P(6) 1,3

5. $\neg\nabla\phi \rightarrow (\Delta\phi \rightarrow \bot)$ H. Induction

6. $\Delta\phi \rightarrow (\neg\nabla\phi \rightarrow \bot)$ P(1) 5

7. $\neg\nabla\lambda \rightarrow (\Delta\lambda \rightarrow \bot)$ H. Induction

8. $(\Delta\phi \wedge \neg\nabla\lambda) \wedge (\neg\nabla\phi \vee \Delta\lambda) \rightarrow \bot$ P(7) 5,7

9. $\neg(\Delta\phi \rightarrow \nabla\lambda) \rightarrow ((\Delta\phi \rightarrow \Delta\lambda) \rightarrow \bot)$ P(4) 4,8

10. $\bot \rightarrow \Delta\beta$ K(7)

11. $\neg(\Delta\phi \rightarrow \nabla\lambda) \rightarrow ((\Delta\phi \rightarrow \Delta\lambda) \rightarrow \Delta\beta)$ P(4) 9, 10

$\underline{\varphi \equiv \phi \wedge \lambda}$. $\Delta(\alpha) \equiv \neg\nabla\varphi \rightarrow (\Delta\varphi \rightarrow \Delta\beta) \equiv \neg\nabla(\phi \wedge \lambda) \rightarrow (\Delta(\phi \wedge \lambda) \rightarrow \Delta\beta) \equiv \neg(\nabla\phi \wedge \nabla\lambda) \rightarrow (\Delta\phi \wedge \Delta\lambda \rightarrow \Delta\beta)$. Since ϕ's and λ's sizes are smaller than n, by the hypothesis of induction we have that there exists a $K_?$-derivation of $\neg\nabla\phi \rightarrow (\Delta\phi \rightarrow \Delta\beta)$ and $\neg\nabla\lambda \rightarrow (\Delta\lambda \rightarrow \Delta\beta)$.

1. $\neg\nabla\phi \rightarrow (\Delta\phi \rightarrow \Delta\beta)$ H. Induction
2. $\neg\nabla\lambda \rightarrow (\Delta\lambda \rightarrow \Delta\beta)$ H. Induction
3. $\neg\nabla\phi \vee \neg\nabla\lambda \rightarrow (\Delta\phi \wedge \Delta\lambda \rightarrow \Delta\beta)$ P(10) 1,2
4. $\neg(\nabla\phi \wedge \nabla\lambda) \rightarrow \neg\nabla\phi \vee \neg\nabla\lambda$ N2
5. $\neg(\nabla\phi \wedge \nabla\lambda) \rightarrow (\Delta\phi \wedge \Delta\lambda \rightarrow \Delta\beta)$ P(2) 3,4

$\underline{\varphi \equiv \phi \vee \lambda}$. $\Delta(\alpha) \equiv \neg\nabla\varphi \rightarrow (\Delta\varphi \rightarrow \Delta\beta) \equiv \neg\nabla(\phi \vee \lambda) \rightarrow (\Delta(\phi \vee \lambda) \rightarrow \Delta\beta) \equiv \neg(\nabla\phi \vee \nabla\lambda) \rightarrow (\Delta\phi \vee \Delta\lambda \rightarrow \Delta\beta)$. Since ϕ's and λ's sizes are smaller than n, by the hypothesis of induction we have that there exists a $K_?$-derivation of both $\neg\nabla\phi \rightarrow (\Delta\phi \rightarrow \Delta\beta)$ and $\neg\nabla\lambda \rightarrow (\Delta\lambda \rightarrow \Delta\beta)$.

1. $\neg\nabla\phi \rightarrow (\Delta\phi \rightarrow \Delta\beta)$ H. Induction
2. $\neg\nabla\lambda \rightarrow (\Delta\lambda \rightarrow \Delta\beta)$ H. Induction
3. $\neg\nabla\phi \wedge \neg\nabla\lambda \rightarrow (\Delta\phi \vee \Delta\lambda \rightarrow \Delta\beta)$ P(11) 1,2
4. $\neg(\nabla\phi \vee \nabla\lambda) \rightarrow \neg\nabla\phi \wedge \neg\nabla\lambda$ N3
5. $\neg(\nabla\phi \vee \nabla\lambda) \rightarrow (\Delta\phi \vee \Delta\lambda \rightarrow \Delta\beta)$ P(2) 3,4

P11: $\alpha \equiv \varphi \vee \neg\varphi$. $\Delta\alpha \equiv \Delta(\varphi \vee \neg\varphi) \equiv \Delta\varphi \vee \neg\nabla\varphi$.

$\underline{\varphi \equiv \Box\phi}$. $\Delta\alpha \equiv \Delta\varphi \vee \neg\nabla\varphi \equiv \Delta\Box\phi \vee \neg\nabla\Box\phi \equiv \Delta\phi! \vee \neg(\nabla\phi?)$. Since ϕ's size is smaller than n, by the hypothesis of induction we have that there is a $K_?$-derivation of $\Delta\phi \vee \neg\nabla\phi$.

1. $\Delta\phi \vee \neg\nabla\phi$ H. Induction
2. $\sim\Delta\phi \rightarrow \neg\nabla\phi$ ~(8) 1
3. $(\sim\Delta\phi)? \rightarrow (\neg\nabla\phi)?$ K(1) 2
4. $(\neg\nabla\phi)? \rightarrow \neg(\nabla\phi?)$ K3
5. $(\sim\Delta\phi)? \rightarrow \neg(\nabla\phi?)$ P(2) 3,4
6. $(\sim\Delta\phi)? \vee \Delta\phi!$ K(8)
7. $\neg(\nabla\phi?) \vee \Delta\phi!$ P(8) 5,6

8. $\Delta\phi! \vee \neg(\nabla\phi?)$ P(18) 7

<u>$\varphi \equiv \Diamond\phi$</u>. $\Delta\alpha \equiv \Delta\varphi \vee \neg\nabla\varphi \equiv \Delta\Diamond\phi \vee \neg\nabla\Diamond\phi \equiv \Delta\phi? \vee \neg(\nabla\phi!)$. Since ϕ's size is smaller than n, by the hypothesis of induction we have that there is a $K_?$-derivation of $\Delta\phi \vee \neg\nabla\phi$.

1. $\Delta\phi \vee \neg\nabla\phi$ H. Induction
2. $\sim\Delta\phi \rightarrow \neg\nabla\phi$ ~(8) 1
3. $(\sim\Delta\phi)! \rightarrow (\neg\nabla\phi)!$ K(4)
4. $(\neg\nabla\phi)! \rightarrow \neg(\nabla\phi!)$ K2
5. $(\sim\Delta\phi)! \rightarrow \neg(\nabla\phi!)$ P(2) 3,4
6. $(\sim\sim\Delta\phi))? \vee (\sim\Delta\phi)!$ K(8)
7. $\sim\sim\Delta\phi \rightarrow \Delta\phi$ ~(9)
8. $(\sim\sim\Delta\phi)? \rightarrow \Delta\phi?$ K(1) 7
9. $\Delta\phi? \vee (\sim\Delta\phi)!$ P(8) 6,8
10. $(\sim\Delta\phi)! \vee \Delta\phi?$ P(18) 9
11. $\neg(\nabla\phi!) \vee \Delta\phi?$ P(8) 5,10
12. $\Delta\phi? \vee \neg(\nabla\phi!)$ P(18) 11

<u>$\varphi \equiv \neg\phi$</u>. $\Delta\alpha \equiv \Delta\varphi \vee \neg\nabla\varphi \equiv \Delta\neg\phi \vee \neg\nabla\neg\phi \equiv \neg\nabla\phi \vee \neg\neg\Delta\phi$. Since ϕ's size is smaller than n, by the hypothesis of induction we have that there is a $K_?$-derivation of $\Delta\phi \vee \neg\nabla\phi$.

1. $\Delta\phi \vee \neg\nabla\phi$ H. Induction
2. $\Delta\phi \rightarrow \neg\neg\Delta\phi$ N4
3. $\neg\neg\Delta\phi \vee \neg\nabla\phi$ P(8) 1,2
4. $\neg\nabla\phi \vee \neg\neg\Delta\phi$ P(18) 3

<u>$\varphi \equiv \phi \rightarrow \lambda$</u>. $\Delta\alpha \equiv \Delta\varphi \vee \neg\nabla\varphi \equiv \Delta(\phi \rightarrow \lambda) \vee \neg\nabla(\phi \rightarrow \lambda) \equiv (\Delta\phi \rightarrow \Delta\lambda) \vee \neg(\Delta\phi \rightarrow \nabla\lambda)$. Since ϕ's and λ's sizes are smaller than n, by the hypothesis of induction we have that there exists a $K_?$-derivation of $\Delta\phi \vee \neg\nabla\phi$, of $\Delta\lambda \vee \neg\nabla\lambda$ and of $\neg\nabla\phi \rightarrow (\Delta\phi \rightarrow \Delta\mu)$. Let $\mu \equiv \perp$. $\Delta\mu \equiv \Delta(p \wedge \neg p)$ $\equiv \Delta p \wedge \neg\nabla p \equiv p \wedge \neg p$.

1. $\Delta\phi \vee \neg\nabla\phi$ H. Induction
2. $\Delta\lambda \vee \neg\nabla\lambda$. H. Induction

3. $\neg\nabla\lambda \vee \Delta\lambda$ P(18) 2

4. $(\Delta\phi \wedge \neg\nabla\lambda) \vee (\neg\nabla\phi \vee \Delta\lambda)$ P(19) 1,3

5. $(\Delta\phi \wedge \neg\nabla\lambda) \rightarrow \neg(\Delta\phi \rightarrow \nabla\lambda)$ N1

6. $\neg(\Delta\phi \rightarrow \nabla\lambda) \vee (\neg\nabla\phi \vee \Delta\lambda)$ P(8) 4,5

7. $\neg\nabla\phi \rightarrow (\Delta\phi \rightarrow \bot)$ H. Induction

8. $\Delta\phi \rightarrow (\neg\nabla\phi \rightarrow \bot)$ P(1) 7

9. $(\Delta\phi \rightarrow \sim\neg\nabla\phi) \rightarrow ((\sim\neg\nabla\phi \rightarrow \Delta\lambda) \rightarrow (\Delta\phi \rightarrow \Delta\lambda))$ P(2)

10. $(\sim\neg\nabla\phi \rightarrow \Delta\lambda) \rightarrow (\Delta\phi \rightarrow \Delta\lambda)$ MP 8,9

11. $\neg\nabla\phi \vee \Delta\lambda \rightarrow (\sim\neg\nabla\phi \rightarrow \Delta\lambda)$ ~(8)

12. $\neg\nabla\phi \vee \Delta\lambda \rightarrow (\Delta\phi \rightarrow \Delta\lambda)$ P(2) 10,11

13. $(\neg\nabla\phi \vee \Delta\lambda) \vee \neg(\Delta\phi \rightarrow \nabla\lambda)$ P(18) 6

14. $(\Delta\phi \rightarrow \Delta\lambda) \vee \neg(\Delta\phi \rightarrow \nabla\lambda)$ P(8) 12,13

$\underline{\varphi \equiv \phi \wedge \lambda}$. $\Delta\alpha \equiv \Delta\varphi \vee \neg\nabla\varphi \equiv \Delta(\phi \wedge \lambda) \vee \neg\nabla(\phi \wedge \lambda) \equiv (\Delta\varphi \wedge \Delta\lambda) \vee \neg(\nabla\phi \wedge \nabla\lambda)$. Since ϕ's and λ's sizes are smaller than n, we have that there exists a $K_?$-derivation of $\Delta\phi \vee \neg\nabla\phi$ and of $\Delta\lambda \vee \neg\nabla\lambda$.

1. $\Delta\phi \vee \neg\nabla\phi$ H. Induction

2. $\Delta\lambda \vee \neg\nabla\lambda$. H. Induction

3. $(\Delta\phi \wedge \Delta\lambda) \vee (\neg\nabla\phi \vee \neg\nabla\lambda)$ P(19) 1,2

4. $(\neg\nabla\phi \vee \neg\nabla\lambda) \vee (\Delta\phi \wedge \Delta\lambda)$ P(18) 3

5. $\neg\nabla\phi \vee \neg\nabla\lambda \rightarrow \neg(\nabla\phi \wedge \nabla\lambda)$ N2

6. $\neg(\nabla\phi \wedge \nabla\lambda) \vee (\Delta\phi \wedge \Delta\lambda)$ P(8) 4,5

7. $(\Delta\phi \wedge \Delta\lambda) \vee \neg(\nabla\phi \wedge \nabla\lambda)$ P(18) 6

$\underline{\varphi \equiv \phi \vee \lambda}$. $\Delta\alpha \equiv \Delta\varphi \vee \neg\nabla\varphi \equiv \Delta(\phi \vee \lambda) \vee \neg\nabla(\phi \vee \lambda) \equiv (\Delta\varphi \vee \Delta\lambda) \vee \neg(\nabla\phi \vee \nabla\lambda)$. Since ϕ's and λ's sizes are smaller than n, we have that there exists a $K_?$-derivation of $\Delta\phi \vee \neg\nabla\phi$ and of $\Delta\lambda \vee \neg\nabla\lambda$.

1. $\Delta\phi \vee \neg\nabla\phi$ H. Induction

2. $\neg\nabla\phi \vee \Delta\phi$ P(18) 1

3. $\Delta\lambda \vee \neg\nabla\lambda.$ H. Induction

4. $\neg\nabla\lambda. \vee \Delta\lambda$ P(18) 3

5. $(\neg\nabla\phi \wedge \neg\nabla\lambda) \vee (\Delta\phi \vee \Delta\lambda)$ P(19) 2,4

6. $(\neg\nabla\phi \wedge \neg\nabla\lambda) \rightarrow \neg(\nabla\phi \vee \nabla\lambda)$ N3

7. $\neg(\nabla\phi \vee \nabla\lambda) \vee (\Delta\phi \vee \Delta\lambda)$ P(8) 5,6

8. $(\Delta\phi \vee \Delta\lambda) \vee \neg(\nabla\phi \vee \nabla\lambda)$ P(18) 7

K: $\alpha \equiv \Box(\varphi \rightarrow \phi) \rightarrow (\Box\varphi \rightarrow \Box\phi)$. $\Delta\alpha \equiv \Delta(\Box(\varphi \rightarrow \phi) \rightarrow (\Box\varphi \rightarrow \Box\phi)) \equiv (\Delta\varphi \rightarrow \Delta\phi)! \rightarrow (\Delta\varphi! \rightarrow \Delta\phi!)$. We have therefore to prove that there exists a $K_?$-derivation of $(\Delta\varphi \rightarrow \Delta\phi)! \rightarrow (\Delta\varphi! \rightarrow \Delta\phi!)$. Since $(\Delta\varphi \rightarrow \Delta\phi)! \rightarrow (\Delta\varphi! \rightarrow \Delta\phi!)$ is an instance of $K_?$, it itself is the derivation we are looking for.

NP: $\alpha \equiv \Diamond\varphi \leftrightarrow \neg\Box\neg\varphi.\Delta\alpha \equiv \Delta(\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi) \equiv \Delta((\Diamond\varphi \rightarrow \neg\Box\neg\varphi) \wedge (\neg\Box\neg\varphi\rightarrow \Diamond\varphi)) \equiv \Delta(\Diamond\varphi \rightarrow \neg\Box\neg\varphi) \wedge \Delta(\neg\Box\neg\varphi \rightarrow \Diamond\varphi) \equiv \Delta(\Diamond\varphi) \rightarrow \Delta(\neg\Box\neg\varphi) \wedge \Delta(\neg\Box\neg\varphi) \rightarrow \Delta(\Diamond\varphi) \equiv \Delta\varphi? \rightarrow \neg\nabla(\Box\neg\varphi) \wedge \neg\nabla(\Box\neg\varphi) \rightarrow \Delta\varphi? \equiv \Delta\varphi? \rightarrow \neg((\nabla\neg\varphi)?) \wedge \neg((\nabla\neg\varphi)?) \rightarrow \Delta\varphi? \equiv \Delta\varphi? \rightarrow \neg((\neg\Delta\varphi)?) \wedge \neg((\neg\Delta\varphi)?) \rightarrow \Delta\varphi?$

1. $\Delta\varphi \rightarrow \neg\neg\Delta\varphi$ N4

2. $\Delta\varphi? \rightarrow (\neg\neg\Delta\varphi)?$ K(1) 1

3. $(\neg\neg\Delta\varphi)? \rightarrow \neg((\neg\Delta\varphi)?)$ K3

4. $\Delta\varphi? \rightarrow \neg((\neg\Delta\varphi)?)$ P(2) 2,3

5. $\neg\neg\Delta\varphi \rightarrow \Delta\varphi$ N4

6. $(\neg\neg\Delta\varphi)? \rightarrow \Delta\varphi?$ K(1) 5

7. $\neg((\neg\Delta\varphi)?) \rightarrow (\neg\neg\Delta\varphi)?$ K3

8. $\neg((\neg\Delta\varphi)?) \rightarrow \Delta\varphi?$ P(2) 6,7

9. $\Delta\varphi? \rightarrow \neg((\neg\Delta\varphi)?) \wedge \neg((\neg\Delta\varphi)?) \rightarrow \Delta\varphi?$ P(20) 4,8

We therefore have proved that in the case where α is an instance of one of the axioms of K, if $A \div B \vdash_K \alpha$ then $\Delta(A) \div \Delta(B) \vdash_{K?} \Delta(\alpha)$. This completes the basis of induction of our proof. Now we will proceed to consider the case where the size of the K-derivation of α from A and B is greater than 1.

Hypothesis of induction: Let n>1 be the size of the K-derivation of α from A and B. Suppose that for K-derivations of sizes smaller than *n* the result holds. That is to say, if A ∸ B $\vdash_K \varphi$ and the size of the derivation of φ from A and B is smaller than *n*, then $\Delta(A)$ ∸ $\Delta(B) \vdash_{K?} \Delta(\varphi)$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be the K-derivation of α from A and B. By definition 5.2.24, $\alpha_n \equiv \alpha$ should satisfy one of the following conditions:

(i) α_n is an axiom of K;

(ii) There are $\alpha_i, \alpha_j \in \{\alpha_1, \ldots, \alpha_n\}$, i,j<n, such that $\alpha_j \equiv \alpha_i \rightarrow \alpha_n$;

(iii) There is $\alpha_i \in \{\alpha_1, \ldots, \alpha_n\}$, i<j, such that $\alpha_n \equiv \Box\alpha_i$, and no element of B appears in the derivation of α_i.

We have just considered the first case when we dealt with derivations of size 1. Let us now consider the two other cases.

Case (ii): $\alpha_j \equiv \alpha_i \rightarrow \alpha_n$. $\Delta(\alpha_j) \equiv \Delta(\alpha_i) \rightarrow \Delta(\alpha_n)$. Since the size of the K-derivations of α_i and α_j from A and B are smaller than *n*, by the hypothesis of induction we have that there is a $K_?$-derivation of $\Delta(\alpha_i)$ from $\Delta(A)$ and $\Delta(B)$ and a $K_?$-derivation of $\Delta(\alpha_j) \equiv \Delta(\alpha_i) \rightarrow \Delta(\alpha_n)$ from $\Delta(A)$ and $\Delta(B)$. Therefore, taking these two $K_?$-derivations together and considering items (ii).c) and (iii).c) of definition 5.2.24 (MP rule), we conclude that there is a $K_?$-derivation of $\Delta(\alpha_n)$ from $\Delta(A)$ and $\Delta(B)$.

Case (iii): $\alpha_n \equiv \Box\alpha_i$. $\Delta(\alpha_n) \equiv \Delta(\alpha_i)!$. Since no element of B appears in the K-derivation of α_i from A and B, we are sure that there is a K-derivation of α_i from A and $\varnothing$. Since the size of such derivation is smaller than *n*, by the hypothesis of induction we have that there is a $K_?$-derivation of $\Delta(\alpha_i)$ from $\Delta(A)$ and $\varnothing$. Given this, and taking this $K_?$-derivation along with item (ii).e) of definition 5.2.24 (rule N), we conclude that that there is a $K_?$-derivation of $\Delta(\alpha_n) \equiv \Delta(\alpha_i)!$ from $\Delta(A)$ and $\Delta(B)$. ■

Lemma A.8. The following schemas of relations between sets of formulas and formula are correct:

$K_\circ(1)$: $\vdash_K \neg\alpha \leftrightarrow \sim\alpha$

$K_\circ(2)$: if $\vdash_K \alpha \leftrightarrow \beta$ and $\vdash_k \varphi$, wherein α occur in φ, then $\vdash_K \varphi[\alpha/\beta]$

$K_\circ(3)$: $\vdash_K (\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \varphi) \rightarrow \lambda) \vdash_K (\alpha \rightarrow \beta \wedge \varphi) \rightarrow \lambda$

$K_\circ(4)$: $\vdash_K \Box\neg\alpha \leftrightarrow \neg\Diamond\alpha$

$K_\circ(5)$: $\vdash_K \Diamond\neg\alpha \leftrightarrow \neg\Box\alpha$

Proof. $K_\diamond(2)$ is the theorem of replacement we know is valid in normal modal logic K, $K_\diamond(3)$ is a classical logic theorem and $K_\diamond(4)$ and $K_\diamond(5)$ are known K's theorems. Concerning $K_\diamond(1)$, we think it is worthy to show its derivation:

$K_\diamond(1)$: $\vdash_K \neg\alpha \leftrightarrow \sim\alpha$

1. $\neg\alpha \rightarrow (\alpha \rightarrow \bot)$ P10
2. $(\alpha \rightarrow p) \rightarrow ((\alpha \rightarrow \neg p) \rightarrow \neg\alpha)$ P9
3. $(\alpha \rightarrow \bot) \rightarrow \neg\alpha$ $K_\diamond(3)$ 2 ■

Lemma A.9. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. If $A \dotplus B \vdash_{K?} \alpha$ then $\Pi(A) \dotplus \Pi(B) \vdash_K \Pi(\alpha)$.

Proof. To prove this theorem, we should follow the same path of the previous proof. If $A \dotplus B \vdash_{K?} \alpha$, then there is a $K_?$-derivation of α from A and B. We then have to prove that if this is the case, there is K-derivation of $\Pi(\alpha)$ from $\Pi(A)$ and $\Pi(B)$.

Base of induction: derivation of size 1: $\alpha_1 \equiv \alpha$.

- Case 1: $\alpha \in A \cup B$.

Trivially, $\Pi(\alpha) \in \Pi(A \cup B)$. Therefore, $\Pi(A) \dotplus \Pi(B) \vdash_K \Pi(\alpha)$.

- Case 2: α is an axiom of $K_?$.

To deal with this case, we have to analyze the possibility of α's being an instance of each one of $K_?$'s axiom schemas. For each one of these possibilities, we shall show then that there is a K-derivation of $\Pi(\alpha)$, what implies that there exists a K-derivation of $\Pi(\alpha)$ from $\Pi(A)$ and $\Pi(B)$ and therefore that $\Pi(A) \dotplus \Pi(B) \vdash_K \Pi(\alpha)$. The cases where α is instance of one of $K_?$'s positive classical axioms, paranormal classical axioms and non-positive additional classical axioms are trivial, for these schemas of axiom are valid in K: P1-P8 and A1-A3 belong to the axiomatic of K and N1-N5 are easily derived from them along with MP (in fact, all of them are theorems of classical logic). Below we consider the cases where α is an instance of K1-K5 or $K_?$ and show that, in these cases, there is a K-derivation of $\Pi(\alpha)$. In order to simplify our exposition, we prove below that $\Pi(\sim\varphi) \equiv \sim\Pi(\varphi)$:

$\Pi(\sim\varphi) \equiv \Pi(\varphi \rightarrow (p \wedge \neg p)) \equiv \Pi\varphi \rightarrow \Pi(p \wedge \neg p) \equiv \Pi\varphi \rightarrow (\Pi p \wedge \Pi\neg p) \equiv \Pi\varphi \rightarrow (\Pi p \wedge \neg\Box p) \equiv \Pi\varphi \rightarrow (p \wedge \neg p) \equiv \sim\Pi\varphi$

K1: $\alpha \equiv \varphi? \leftrightarrow \sim((\sim\varphi)!)$. $\Pi\alpha \equiv \Pi(\varphi? \leftrightarrow \sim((\sim\varphi)!)) \equiv \Pi((\varphi? \rightarrow \sim((\sim\varphi)!)) \wedge (\sim((\sim\varphi)!) \rightarrow \varphi?)) \equiv \Pi((\varphi? \rightarrow \sim((\sim\varphi)!)) \wedge \Pi(\sim((\sim\varphi)!) \rightarrow \varphi?)) \equiv \Pi(\varphi?) \rightarrow \Pi(\sim((\sim\varphi)!)) \wedge \Pi(\sim((\sim\varphi)!)) \rightarrow \Pi(\varphi?) \equiv$

$\Pi(\varphi?) \to \sim\Pi((\sim\varphi)!) \wedge \sim\Pi((\sim\varphi)!) \to \Pi(\varphi?) \equiv \Diamond\Pi\varphi \to \sim\Box\Pi(\sim\varphi) \wedge \sim\Box\Pi(\sim\varphi) \to \Diamond\Pi\varphi \equiv \Diamond\Pi\varphi \to \sim\Box\sim\Pi\varphi \wedge \sim\Box\sim\Pi\varphi \to \Diamond\Pi\varphi \equiv \Diamond\Pi\varphi \leftrightarrow \sim\Box\sim\Pi\varphi$

1. $\neg\Pi\varphi \leftrightarrow \sim\Pi\varphi$ — $K_{\diamond}(1)$
2. $\Diamond\Pi\varphi \leftrightarrow \neg\Box\neg\Pi\varphi$ — NP
3. $\Diamond\Pi\varphi \leftrightarrow \neg\Box\sim\Pi\varphi$ — $K_{\diamond}(2)$ 1,2
4. $\neg\Box\sim\Pi\varphi \leftrightarrow \sim\Box\sim\Pi\varphi$ — $K_{\diamond}(1)$
5. $\Diamond\Pi\varphi \leftrightarrow \sim\Box\sim\Pi\varphi$ — $K_{\diamond}(2)$ 3,4

K2: $\alpha \equiv (\neg\varphi)! \leftrightarrow \neg(\varphi!)$. $\Pi\alpha \equiv \Pi((\neg\varphi)! \to \neg(\varphi!) \wedge \neg(\varphi!) \to (\neg\varphi)!) \equiv \Pi((\neg\varphi)! \to \neg(\varphi!)) \wedge \Pi(\neg(\varphi!) \to (\neg\varphi)!) \equiv \Pi((\neg\varphi)!) \to \Pi(\neg(\varphi!)) \wedge \Pi(\neg(\varphi!)) \to \Pi((\neg\varphi)!) \equiv \Box\Pi(\neg\varphi) \to \neg\amalg(\varphi!) \wedge \neg\amalg(\varphi!) \to \Box\Pi(\neg\varphi) \equiv \Box\neg\amalg\varphi \to \neg\Diamond\amalg\varphi \wedge \neg\Diamond\amalg\varphi \to \Box\neg\amalg\varphi \equiv \Box\neg\amalg\varphi \leftrightarrow \neg\Diamond\amalg\varphi$. Since $\Box\neg\amalg\varphi \leftrightarrow \neg\Diamond\amalg\varphi$ is an instance of $K_{\diamond}(4)$, it itself is the derivation we are looking for.

K3: $\alpha \equiv (\neg\varphi)? \leftrightarrow \neg(\varphi?)$. $\Pi\alpha \equiv \Pi((\neg\varphi)? \to \neg(\varphi?) \wedge \neg(\varphi?) \to (\neg\varphi)?) \equiv \Pi((\neg\varphi)? \to \neg(\varphi?)) \wedge \Pi(\neg(\varphi?) \to (\neg\varphi)?) \equiv \Pi((\neg\varphi)?) \to \Pi(\neg(\varphi?)) \wedge \Pi(\neg(\varphi?)) \to \Pi((\neg\varphi)?) \equiv \Diamond\Pi(\neg\varphi) \to \neg\amalg(\varphi?) \wedge \neg\amalg(\varphi?) \to \Diamond\Pi(\neg\varphi) \equiv \Diamond\neg\amalg\varphi \to \neg\Box\amalg\varphi \wedge \neg\Box\amalg\varphi \to \Diamond\neg\amalg\varphi \equiv \Diamond\neg\amalg\varphi \leftrightarrow \neg\Box\amalg\varphi$. Since $\Diamond\neg\amalg\varphi \leftrightarrow \neg\Box\amalg\varphi$ is an instance of $K_{\diamond}(5)$, it itself is the K-derivation we are looking for.

$K_?$: $\alpha \equiv (\varphi \to \phi)! \to (\varphi! \to \phi!)$. $\Pi\alpha \equiv \Pi((\varphi \to \phi)! \to (\varphi! \to \phi!)) \equiv \Pi(\varphi \to \phi)! \to \Pi(\varphi! \to \phi!) \equiv \Box\Pi(\varphi \to \phi) \to (\Pi(\varphi!) \to \Pi(\phi!)) \equiv \Box(\Pi\varphi \to \Pi\phi) \to (\Box\Pi\varphi \to \Box\Pi\phi)$. Since $\Box(\Pi\varphi \to \Pi\phi) \to (\Box\Pi\varphi \to \Box\Pi\phi)$ is an instance of K, it itself is the K-derivation we are searching for.

We have proved then that, in the case where α is an instance of one of the axioms of $K_?$, if $A \div B \vdash_{K?} \alpha$ then $\Pi(A) \div \Pi(B) \vdash_K \Pi(\alpha)$. This completes the basis of induction of the proof. Let us examine now the case where the size of the K-derivation of α from A and B is greater than 1.

Hypothesis of induction: Let $n>1$ be the size of the $K_?$-derivation of α from A and B. Suppose that, for $K_?$-derivations of sizes smaller than *n* the result holds. That is to say, if $A \div B \vdash_{K?} \varphi$ and the size of the derivation of φ from A and B is smaller than *n*, then $\Pi(A) \div \Pi(B) \vdash_K \Pi(\varphi)$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be the $K_?$-derivation of α from A and B. By definition 5.2.24, $\alpha_n \equiv \alpha$ may have been obtained in one of the following ways:

(i) α_n is an axiom of K;

(ii) There are $\alpha_i, \alpha_j \in \{\alpha_1, \ldots, \alpha_n\}$, $i,j<n$, such that $\alpha_j \equiv \alpha_i \to \alpha_n$;

(iii) There is $\alpha_i \in \{\alpha_1, \ldots, \alpha_n\}$, i<j, such that $\alpha_n \equiv \alpha_i!$ and no element of B appears in the derivation of α_i.

We have just considered the first case when we dealt with derivations of size 1. Let us now consider the two other cases.

Case (ii): $\alpha_j \equiv \alpha_i \rightarrow \alpha_n$. $\Pi(\alpha_j) \equiv \Pi(\alpha_i) \rightarrow \Pi(\alpha_n)$. Since the size of the K-derivations of α_i and α_j from A and B is smaller than I, by the hypothesis of induction we have that there is a K-derivation of $\Pi(\alpha_i)$ from $\Pi(A)$ and $\Pi(B)$ and a K-derivation of $\Pi(\alpha_j) \equiv \Pi(\alpha_i) \rightarrow \Pi(\alpha_n)$ from $\Pi(A)$ and $\Pi(B)$. Therefore, taking these two K-derivations together and considering items (ii).c) and (iii).c) of definition 5.2.24 (MP rule), we conclude that there is a K-derivation of $\Pi(\alpha_n)$ from $\Pi(A)$ and $\Pi(B)$.

Case (iii): $\alpha_n \equiv \alpha_i!$, $\Pi(\alpha_n) \equiv \Box\Pi\alpha_i$. Since no element of B appears in the $K_?$-derivation of α_i from A and B, we are sure that there is a $K_?$-derivation of α_i from A and $\varnothing$. Since the size of such derivation is smaller than *n*, by the hypothesis of induction we have that there is a K-derivation of $\Pi(\alpha_i)$ from $\Pi(A)$ and $\varnothing$. Given this and taking this K-derivation along with (ii).e) of definition 5.2.24 (rule N), we conclude that there is a K-derivation of $\Pi(\alpha_n) \equiv \Box\Pi\alpha_i$ from $\Pi(A)$ and $\Pi(B)$. ■

Lemma A.10. Let $\alpha \in L_\diamond$ be a formula, M = <W,R,v> a model and $w \in W$ a world of M. $M,w \Vdash_{\Psi\diamond} \alpha$ iff $M,w \Vdash_{\Omega?} \Delta(\alpha)$ or, equivalently, $\Psi_{\diamond M,w}(\alpha) = 1$ iff $\Omega_{?M,w}(\Delta(\alpha)) = 1$.

Proof. We will prove this lemma by induction on the size of α.

Base of induction: $\alpha \equiv p$. In this case the result trivially holds, for $\Delta(p) = p$.

Hypothesis of induction: Let α be an arbitrary formula. Suppose the result holds for all formulae φ of size m<n, where *n* is α's size. We have to prove that, if this is the case, the result also holds for α. This will be done by considering all possible forms α may have. The only situation which poses some difficulty is the case where $\alpha \equiv \neg\varphi$. For all others, the proof is trivial. For the sake of illustration, we show below the proof for the case where $\alpha \equiv \Box\varphi$, and after consider the really interesting case where $\alpha \equiv \neg\varphi$.

<u>$\alpha \equiv \Box\varphi$</u>. $\Delta\alpha \equiv \Delta\Box\varphi \equiv (\Delta\varphi)!$. If $\Psi_{\diamond M,w}(\Box\varphi) = 1$, then for all $w' \in W$ such that wRw' $\Psi_{\diamond M,w'}(\varphi) = 1$. But since φ's size is smaller than α's, by the hypothesis of induction we have that, for all for all $w' \in W$ such that wRw', $\Omega_{?M,w'}(\Delta\varphi) = 1$. Trivially then, we will have that $\Omega_{?M,w}((\Delta\varphi)!) = 1$. If $\Omega_{?M,w}((\Delta\varphi)!) = 1$, then, for all $w' \in W$ such that wRw', $\Omega_{?M,w'}(\Delta\varphi) = 1$. Since φ's size is smaller than

n, by the hypothesis of induction we have that, for all $w' \in W$ such that wRw', $\Psi_{\diamond M,w'}(\varphi) = 1$. Therefore, $\Psi_{\diamond M,w}(\Box\varphi) = 1$.

<u>$\alpha \equiv \neg\varphi$</u>. $\Delta\alpha \equiv \Delta(\neg\varphi) \equiv \neg\nabla\varphi$. We have then to prove that $\Psi_{\diamond M,w}(\neg\varphi) = 1$ iff $\Omega_{?M,w}(\neg\nabla\varphi) = 1$. That will be done by induction on the size of φ.

Basis of Induction: $\varphi = p$. This case is trivial, for $\nabla p = \Delta p = p$.

Hypothesis of induction (which, in order to be distinguished from the first hypothesis of induction, will be referred to as the second hypothesis of induction): Suppose that the result holds for formulae of size smaller than φ's size. We will show that, if this supposition holds, independently of the form of φ, the general result that $\Psi_{\diamond M,w}(\neg\varphi) = 1$ iff $\Omega_{?M,w}(\neg\nabla\varphi) = 1$ also holds. As usual, we will consider all forms φ may have.

$\varphi \equiv \Box\phi$. $\neg\nabla\varphi \equiv \neg\nabla\Box\phi \equiv \neg(\nabla\phi?)$. $\Omega_{?M,w}(\neg(\nabla\phi?)) = 1$ iff $\mho_{?M,w}(\nabla\phi?) = 0$ iff, for at least one $w' \in W$ such that wRw', $\mho_{?M,w'}(\nabla\phi) = 0$. $\Psi_{\diamond M,w}(\neg\Box\phi) = 1$ iff $\Psi_{\diamond M,w}(\Box\phi) = 0$ iff, for at least one $w' \in W$ such that wRw', $\Psi_{\diamond M,w'}(\phi) = 0$. If $\Psi_{\diamond M,w}(\neg\Box\phi) = 1$, then, for at least one $w' \in W$ such that wRw', $\Psi_{\diamond M,w'}(\phi) = 0$ or, equivalently, $\Psi_{\diamond M,w'}(\neg\phi) = 1$. Since ϕ's size is smaller than φ's, by our second hypothesis of induction we have that $\Omega_{?M,w'}(\neg\nabla\phi) = 1$. Since $\Omega_{?M,w'}(\neg\nabla\phi) = 1$ iff $\mho_{?M,w'}(\nabla\phi) = 0$, we have that, for at least one $w' \in W$ such that wRw', $\mho_{?M,w'}(\nabla\phi) = 0$, which implies that $\Omega_{?M,w}(\neg(\nabla\phi?)) = 1$. If $\Omega_{?M,w}(\neg(\nabla\phi?)) = 1$, then, for at least one $w' \in W$ such that wRw', $\mho_{?M,w'}(\nabla\phi) = 0$, or, equivalently, $\Omega_{?M,w'}(\neg\nabla\phi) = 1$. Since ϕ's size is smaller than φ's, by our second hypothesis of induction we have that $\Psi_{\diamond M,w'}(\neg\phi) = 1$. Since $\Psi_{\diamond M,w'}(\neg\phi) = 1$ iff $\Psi_{\diamond M,w'}(\phi) = 0$, we have that, for at least one $w' \in W$ such that wRw', $\Psi_{\diamond M,w'}(\phi) = 0$, which implies that $\Psi_{\diamond M,w}(\neg\Box\phi) = 1$.

$\varphi \equiv \diamond\phi$. $\neg\nabla\varphi \equiv \neg\nabla\diamond\phi \equiv \neg(\nabla\phi!)$. The proof of this case is almost identical to the previous one. We have just to change the occurrences of ! by ?, and of □ by □, and where it appears the expression "for at least one" we write "for all."

$\varphi \equiv \neg\phi$. $\neg\nabla\neg\phi \equiv \neg\neg\Delta\phi$. $\Omega_{?M,w}(\neg\neg\Delta\phi) = 1$ iff $\mho_{?M,w}(\neg\Delta\phi) = 0$ iff $\Omega_{?M,w}(\Delta\phi) = 1$. $\Psi_{\diamond M,w}(\neg\neg\phi) = 1$ iff $\Psi_{\diamond M,w}(\neg\phi) = 0$ iff $\Psi_{\diamond M,w}(\phi) = 1$. If $\Psi_{\diamond M,w}(\neg\neg\phi) = 1$, then $\Psi_{\diamond M,w}(\phi) = 1$. Since ϕ's size is smaller than α's, by the (first) hypothesis of induction, we have that $\Omega_{?M,w}(\Delta\phi) = 1$. Therefore, $\Omega_{?M,w}(\neg\neg\Delta\phi) = 1$. If $\Omega_{?M,w}(\neg\neg\Delta\phi) = 1$, then we have $\Omega_{?M,w}(\Delta\phi) = 1$. Since ϕ's size is smaller than α's, by the (first) hypothesis of induction, we have that $\Psi_{\diamond M,w}(\phi) = 1$. Therefore, $\Psi_{\diamond M,w}(\neg\neg\phi) = 1$.

$\varphi \equiv \phi \rightarrow \lambda$. $\neg\nabla(\phi \rightarrow \lambda) \equiv \neg(\Delta\phi \rightarrow \nabla\lambda)$. $\Omega_{?M,w}(\neg(\Delta\phi \rightarrow \nabla\lambda)) = 1$ iff $\mho_{?M,w}(\Delta\phi \rightarrow \nabla\lambda) = 0$ iff $\Omega_{?M,w}(\Delta\phi) = 1$ and $\mho_{?M,w}(\nabla\lambda) = 0$. $\Psi_{\diamond M,w}(\neg(\phi \rightarrow \lambda)) = 1$ iff $\Psi_{\diamond M,w}(\phi \rightarrow \lambda) = 0$ iff $\Psi_{\diamond M,w}(\phi) = 1$ and $\Psi_{\diamond M,w}(\lambda) = 0$. If $\Psi_{\diamond M,w}(\neg(\phi \rightarrow \lambda)) = 1$, then $\Psi_{\diamond M,w}(\phi) = 1$ and $\Psi_{\diamond M,w}(\lambda) = 0$, which is equivalent to $\Psi_{\diamond M,w}(\phi) = 1$ and $\Psi_{\diamond M,w}(\neg\lambda) = 1$. Since ϕ's size is smaller than α's, by the (first) hypothesis of induction we have that $\Omega_{?M,w}(\phi) = 1$, and since λ's size is smaller than φ's, by the second hypothesis of induction we have that $\Omega_{?M,w}(\neg\nabla\lambda) = 1$, which is equivalent to $\mho_{?M,w}(\nabla\lambda) = 0$. We therefore have $\Omega_{?M,w}(\neg(\Delta\phi \rightarrow \nabla\lambda)) = 1$. If $\Omega_{?M,w}(\neg(\Delta\phi \rightarrow \nabla\lambda)) = 1$, then $\Omega_{?M,w}(\Delta\phi) = 1$ and $\mho_{?M,w}(\nabla\lambda) = 0$, which is equivalent to $\Omega_{?M,w}(\Delta\phi) = 1$ and $\Omega_{?M,w}(\neg\nabla\lambda) = 1$. Since ϕ's size is smaller than α's, by the first hypothesis of induction we have that $\Psi_{\diamond M,w}(\phi) = 1$, and since λ's size is smaller than φ, by the second hypothesis of induction we have that $\Psi_{\diamond M,w}(\neg\lambda) = 1$, which is equivalent to $\Psi_{\diamond M,w}(\phi) = 1$ and $\Psi_{\diamond M,w}(\lambda) = 0$. Therefore, $\Psi_{\diamond M,w}(\neg(\phi \rightarrow \lambda)) = 1$.

$\varphi \equiv \phi \wedge \lambda$. $\neg\nabla(\phi \wedge \lambda) \equiv \neg(\nabla\phi \wedge \nabla\lambda)$. $\Omega_{?M,w}(\neg(\nabla\phi \wedge \nabla\lambda)) = 1$ iff $\mho_{?M,w}(\nabla\phi \wedge \nabla\lambda) = 0$ iff $\mho_{?M,w}(\nabla\phi) = 0$ and $\mho_{?M,w}(\nabla\lambda) = 0$. $\Psi_{\diamond M,w}(\neg(\phi \wedge \lambda)) = 1$ iff $\Psi_{\diamond M,w}(\phi \wedge \lambda) = 0$ iff $\Psi_{\diamond M,w}(\phi) = 0$ and $\Psi_{\diamond M,w}(\lambda) = 0$. If $\Psi_{\diamond M,w}(\neg(\phi \wedge \lambda)) = 1$, then $\Psi_{\diamond M,w}(\phi) = 0$ and $\Psi_{\diamond M,w}(\lambda) = 0$, which is equivalent to $\Psi_{\diamond M,w}(\neg\phi) = 1$ and $\Psi_{\diamond M,w}(\neg\lambda) = 1$. Since ϕ's and λ's sizes are smaller than φ's, by the second hypothesis of induction we have that $\Omega_{?M,w}(\neg\nabla\phi) = 1$ and $\Omega_{?M,w}(\neg\nabla\lambda) = 1$, which is equivalent to $\mho_{?M,w}(\nabla\phi) = 0$ and $\mho_{?M,w}(\nabla\lambda) = 0$. Therefore, $\Omega_{?M,w}(\neg(\nabla\phi \wedge \nabla\lambda)) = 1$. If $\Omega_{?M,w}(\neg(\nabla\phi \wedge \nabla\lambda)) = 1$, then $\mho_{?M,w}(\nabla\phi) = 0$ and $\mho_{?M,w}(\nabla\lambda) = 0$, which is equivalent to $\Omega_{?M,w}(\neg\nabla\phi) = 1$ and $\Omega_{?M,w}(\neg\nabla\lambda) = 1$. Since ϕ's and λ's sizes are smaller than φ, by the second hypothesis of induction we have that $\Psi_{\diamond M,w}(\neg\phi) = 1$ and $\Psi_{\diamond M,w}(\neg\lambda) = 1$, which is equivalent to $\Psi_{\diamond M,w}(\phi) = 0$ and $\Psi_{\diamond M,w}(\lambda) = 0$. Therefore, $\Psi_{\diamond M,w}(\neg(\phi \wedge \lambda)) = 1$.

$\varphi \equiv \phi \vee \lambda$. The proof of this case is almost identical to the previous one. We have just to replace all occurrences of $\wedge$ by $\vee$ and the relevant occurrences of "and" by "or." ▨

Lemma A.11. Let $\alpha \in L_?$ a formula, M be a model and w a world of M. M,w $\Vdash_{\Omega?}$ α iff M,w $\Vdash_{\Psi\diamond}$ $\Pi(\alpha)$ or, equivalently, $\Omega_{?M,w}(\alpha) = 1$ iff $\Psi_{\diamond M,w}(\Pi\alpha) = 1$.

Proof. The proof of this lemma is almost identical to lemma A.10's. All we have to do is to properly erase the occurrences of Δ and consider function Π along with Ψ. ■

A.2 Theorems from Chapter 5

Theorem 5.3.1. $K_?$ is sound and complete (that is, for any $A,B \subseteq L_?$ and $\alpha \in L_?$, $A \div B \vdash_{K?} \alpha$ iff $A \div B \vDash_{K?} \alpha$.)

Proof. Let us first prove the left-right direction (soundness): for any $A,B \subseteq L_?$ and $\alpha \in L_?$, if $A \div B \vdash_{K?} \alpha$ then $A \div B \vDash_{K?} \alpha$. Suppose that $A \div B \vdash_{K?} \alpha$ and $A \div B \nvDash_{K?} \alpha$. If $A \div B \nvDash_{K?} \alpha$, by theorem 5.3.19 we have that $\Pi(A) \div \Pi(B) \nvDash_{K} \Pi(\alpha)$. By the soundness theorem of normal modal logic K[8], we have $\Pi(A) \div \Pi(B) \nvdash_{K} \Pi(\alpha)$. From that, along with theorem 5.3.17, we have that $A \div B \nvdash_{K?} \alpha$, which is a contradiction. Therefore, if $A \div B \vdash_{K?} \alpha$ then $A \div B \vDash_{K?} \alpha$. The right-left direction (completeness) is proved through the same reference to normal modal logic K. Suppose that $A \div B \vDash_{K?} \alpha$ and $A \div B \nvdash_{K?} \alpha$. If $A \div B \nvdash_{K?} \alpha$, by theorem 5.3.17 we have that $\Pi(A) \div \Pi(B) \nvdash_{K} \Pi(\alpha)$. By the completeness theorem of normal modal logic K, we have then that $\Pi(A) \div \Pi(B) \nvDash_{K} \Pi(\alpha)$. From that, along with theorem 5.3.19, we have that $A \div B \nvDash_{K?} \alpha$, which is a contradiction. Therefore, if $A \div B \vDash_{K?} \alpha$ then $A \div B \vdash_{K?} \alpha$. ■

Theorem 5.3.2. Some formulae of $L_?$ that satisfy one of the following schemas of formula are *not* $K_?$-theorems (and consequently not $K_?$-valid.)

$(\alpha \to \beta) \to (\alpha \to \neg\beta) \to \neg\alpha$	$\neg\alpha \to (\alpha \to \beta)$
$\neg\alpha \vee \alpha$	$\neg(\alpha \wedge \neg\alpha)$
$(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$	$(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$
$\neg\alpha \vee \beta \to (\alpha \to \beta)$	$(\alpha \to \beta) \to \neg\alpha \vee \beta$
$(\alpha \to \beta) \to \neg(\alpha \wedge \neg\beta)$	$\neg(\alpha \wedge \neg\beta) \to (\alpha \to \beta)$
$\neg\alpha \to (\alpha \to \neg\beta)$	$(\alpha \to \neg\alpha) \to \neg\alpha$

Proof. In order to proof this theorem, it suffices to show an instance of each one of these schemas of formula that is not $K_?$-valid. (From theorem 5.3.2 it follows that it is also not $K_?$-theorem.) Having picked such a formula, we need then only to show a model $M = \langle W,R,v \rangle$ and a world

[8] For the proof of soundness and completeness of normal modal logic K with local and global premises see Fitting (1993). See also Chellas (1980) and Hughes & Cresswell (1996).

$w \in W$ such that the formula is not $\Omega_?$-satisfied by M at w. Bellow we show, for each one of the above schemas, the mentioned instance along with the features of the falsifying model. $p,q \in P$ are propositional symbols.

1. $(q \to p?) \to ((q \to \neg(p?)) \to \neg q)$	$w \in W$ such that $v_w(q) = 1$ and there are $w', w'' \in W$ such that wRw' and wRw'' such that $v_{w'}(p) = 1$ and $v_{w''}(p) = 0$.
2. $\neg(p!) \vee p!$	$w \in W$ is such that there are $w', w'' \in W$ such that wRw' and wRw'' and $v_{w'}(p) = 1$ and $v_{w''}(p) = 0$.
3. $(q \to p?) \to (\neg(p?) \to \neg q)$	The same as 1.
4. $\neg(p?) \vee q \to (p? \to q)$	$w \in W$ such that $v_w(q) = 0$ and there are $w', w'' \in W$ such that wRw' and wRw'' such that $v_{w'}(p) = 1$ and $v_{w''}(p) = 0$.
5. $(q \to p?) \to \neg(q \wedge \neg(p?))$	The same as 1.
6. $\neg(p?) \to (p? \to \neg q)$	The same as 1.
7. $\neg(p?) \to (p? \to q)$	The same as 4.
8. $\neg(p? \wedge \neg(p?))$	The same as 2.
9. $(\neg(p!) \to \neg q) \to (q \to p!)$	The same as 1.
10. $(p! \to q) \to \neg(p!) \vee q$	The same as 4.
11. $\neg(q \wedge \neg(p!)) \to (q \to p!)$	The same as 1.
12. $(p! \to \neg(p!)) \to \neg(p!)$	The same as 2. ■

Theorem 5.3.3. All formulae of $L_?$ that satisfy one of the following schemas of formula, wherein α and β are ?!-free formulae, are $K_?$-theorems (and consequently $K_?$-valid.)

$(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$	$\neg\alpha \to (\alpha \to \beta)$
$\neg\alpha \vee \alpha$	$\neg(\alpha \wedge \neg\alpha)$
$(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$	$(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$
$\neg\alpha \vee \beta \to (\alpha \to \beta)$	$(\alpha \to \beta) \to \neg\alpha \vee \beta$
$(\alpha \to \beta) \to \neg(\alpha \wedge \neg\beta)$	$\neg(\alpha \wedge \neg\beta) \to (\alpha \to \beta)$
$\neg\alpha \to (\alpha \to \neg\beta)$	$(\alpha \to \neg\alpha) \to \neg\alpha$

Proof. Since all the above schemas of formula are theorems of classical logic and since all of their instances are modality-free, it follows from theorem 5.3.12 that they are $K_?$-theorems. ■

Theorem 5.3.4. All formulae of $L_?$ that satisfy one of the following schemas of formula are $K_?$-theorems (and consequently $K_?$-valid.)

$(\alpha \to \beta) \to (\alpha \to \sim\beta) \to \sim\alpha$	$\sim\alpha \to (\alpha \to \beta)$

$\sim\alpha \vee \alpha$	$\sim(\alpha \wedge \sim\alpha)$
$(\alpha \rightarrow \beta) \rightarrow (\sim\beta \rightarrow \sim\alpha)$	$(\sim\beta \rightarrow \sim\alpha) \rightarrow (\alpha \rightarrow \beta)$
$\sim\alpha \vee \beta \rightarrow (\alpha \rightarrow \beta)$	$(\alpha \rightarrow \beta) \rightarrow \sim\alpha \vee \beta$
$(\alpha \rightarrow \beta) \rightarrow \sim(\alpha \wedge \sim\beta)$	$\sim(\alpha \wedge \sim\beta) \rightarrow (\alpha \rightarrow \beta)$
$\sim\alpha \rightarrow (\alpha \rightarrow \sim\beta)$	$(\alpha \rightarrow \sim\alpha) \rightarrow \sim\alpha$

Proof. If we apply the inverse of function ρ[9] to each one of the above schemas, we will obtain a set of schemas which are valid in classical logic. From lemma A.2 then, we have that the schemas as presented above are all valid in $K_?$. ■

Theorem 5.3.5. All formulae of $L_?$ that satisfy one of the following schemas of formula are $K_?$-theorems (and consequently $K_?$-valid.)

$(\alpha \rightarrow \beta)! \rightarrow (\alpha! \rightarrow \beta!)$	$(\alpha \rightarrow \beta)! \rightarrow (\alpha? \rightarrow \beta?)$
$(\alpha \wedge \beta)! \leftrightarrow \alpha! \wedge \beta!$	$(\alpha \wedge \beta)? \rightarrow (\alpha? \wedge \beta?)$
$\alpha! \vee \beta! \rightarrow (\alpha \vee \beta)!$	$(\alpha? \vee \beta?) \leftrightarrow (\alpha \vee \beta)?$
$(\alpha \rightarrow \beta)? \leftrightarrow (\alpha! \rightarrow \beta?)$	$(\alpha \vee \beta)! \rightarrow (\alpha! \vee \beta?)$
$\sim(\alpha!) \leftrightarrow (\sim\alpha)?\backslash$	$\sim(\alpha?) \leftrightarrow (\sim\alpha)!$
$\sim(\alpha!) \vee (\sim\alpha)?$	$\sim(\alpha?) \vee (\sim\alpha)!$

Proof. If we apply the inverse of function Φ to each one of the above schemas, we will obtain a set of schemas which are valid in normal modal logic K. It follows then from theorem 5.3.14 that the schemas as presented above are all valid in $K_?$. ■

Theorem 5.3.6. The following schemas of relations between sets of formulas and formula are correct.

$\{\alpha \rightarrow \beta\} \vdash_{K?} (\alpha \rightarrow \beta)!$	$\{\alpha \rightarrow \beta\} \vDash_{K?} (\alpha \rightarrow \beta)!$
$\{\alpha \rightarrow \beta\} \vdash_{K?} \alpha! \rightarrow \beta!$	$\{\alpha \rightarrow \beta\} \vDash_{K?} \alpha! \rightarrow \beta!$
$\{\alpha \rightarrow \beta\} \vdash_{K?} \alpha? \rightarrow \beta?$	$\{\alpha \rightarrow \beta\} \vDash_{K?} \alpha? \rightarrow \beta?$

Proof. Applying the inverse of function Φ to each one of the schemas of formula that appear in the above relations, then we will obtain a set of schemas of relation which are valid in normal modal logic K. From theorem 5.3.14 then, we have that the schemas as presented above are all valid in $K_?$. ■

Theorem 5.3.7. Some formulae of $L_?$ that satisfy one of the following schemas of formula are *not* $K_?$-theorems (and consequently not $K_?$-valid.)

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[9] Definition A.1.

$\alpha? \to \neg((\neg\alpha)!)$ $\qquad$ $\alpha! \to \neg((\neg\alpha)?)$

$\neg((\neg\alpha)!) \to \alpha?$ $\qquad$ $\neg((\neg\alpha)?) \to \alpha!$

$\neg(\alpha!) \to (\neg\alpha)?$ $\qquad$ $\neg(\alpha?) \to (\neg\alpha)!$

$(\neg\alpha)? \to \neg(\alpha!)$ $\qquad$ $(\neg\alpha)! \to \neg(\alpha?)$

$\alpha! \vee \neg(\alpha!)$ $\qquad$ $\neg(\alpha! \wedge \neg(\alpha!))$

$\neg(\alpha? \wedge \neg(\alpha?))$ $\qquad$ $\alpha? \vee \neg(\alpha?)$

Proof. The procedure to prove this theorem is the same as the one we used to prove theorem 5.3.2. ■

Theorem 5.3.8. Let $A,B \subseteq L_?$ be two sets of formulae and $\alpha,\varphi \in L_?$ two formulae. $A \div B \cup \{\varphi\} \vdash_{K?} \alpha$ iff $A \div B \vdash_{K?} \varphi \to \alpha$.

Proof. Suppose that $A \div B \cup \{\varphi\} \vdash_{K?} \alpha$ but $A \div B \nvdash_{K?} \varphi \to \alpha$. If $A \div B \nvdash_{K?} \varphi \to \alpha$, by theorem 5.3.17, $\Pi(A) \div \Pi(B) \nvdash_K \Pi(\varphi \to \alpha) \equiv \Pi(\varphi) \to \Pi(\alpha)$. Then, by the K's local deduction theorem[10], we have that $\Pi(A) \div \Pi(B) \cup \{\Pi(\varphi)\} \nvdash_K \Pi(\alpha)$, what is the same as $\Pi(A) \div \Pi(B \cup \{\varphi\}) \nvdash_K \Pi(\alpha)$. But then, by theorem 5.3.17 again, we have that $A \div B \cup \{\varphi\} \nvdash_{K?} \alpha$, what is a contradiction. Therefore, if $A \div B \cup \{\varphi\} \vdash_{K?} \alpha$ then $A \div B \vdash_{K?} \varphi \to \alpha$. Suppose now that $A \div B \vdash_{K?} \varphi \to \alpha$ but $A \div B \cup \{\varphi\} \nvdash_{K?} \alpha$. If $A \div B \cup \{\varphi\} \nvdash_{K?} \alpha$, by theorem 5.3.17, $\Pi(A) \div \Pi(B \cup \{\varphi\}) \nvdash_K \Pi(\alpha)$, what is the same as $\Pi(A) \div \Pi(B) \cup \Pi(\varphi) \nvdash_K \Pi(\alpha)$. Then, by K's local deduction theorem, $\Pi(A) \div \Pi(B) \nvdash_K \Pi(\varphi) \to \Pi(\alpha) \equiv \Pi(\varphi \to \alpha)$. But by theorem 5.3.17, we have that $A \div B \nvdash_{K?} \varphi \to \alpha$, what is a contradiction. Therefore, if $A \div B \vdash_{K?} \varphi \to \alpha$ then $A \div B \cup \{\varphi\} \vdash_{K?} \alpha$. ■

Theorem 5.3.9. Let $A,B \subseteq L_?$ be two sets of formulae and $\alpha,\varphi \in L_?$ two formulae. $A \cup \{\varphi\} \div B \vdash_{K?} \alpha$ iff, for some $n \geq 0$, $A \div B \cup \{\varphi!^0, \varphi!^1, \ldots, \varphi!^n\} \vdash_{K?} \alpha$.

Proof. The proof of the axiomatic version of $K_?$'s global deduction theorem goes in the same way as $K_?$'s local deduction theorem. We have just to repeat what we have done in the proof of theorem 5.3.8, but now using K's global deduction theorem instead of K's local theorem. ■

Theorem 5.3.10. Let $A,B \subseteq L_?$ be two sets of formulae and $\alpha,\varphi \in L_?$ two formulae. $A \div B \cup \{\varphi\} \vDash_{K?} \alpha$ iff $A \div B \vDash_{K?} \varphi \to \alpha$.

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[10] See Fitting (1993).

Theorem 5.3.11. Let $A,B \subseteq L_?$ be two sets of formulae and $\alpha,\varphi \in L_?$ two formulae. $A \cup \{\varphi\} \div B \vDash_{K?} \alpha$ iff, for some $n \geq 0$, $A \div B \cup \{\varphi!^0, \varphi!^1, \ldots, \varphi!^n\} \vDash_{K?} \alpha$.

Proof. The proofs of theorems 5.3.10 and 5.3.11 go exactly in the same way as the proofs of theorems 5.3.8 and 5.3.9. We have just to use theorem 5.3.19 instead of theorem 5.3.17 and consider the semantic version of K's local and global deduction theorems. ■

Theorem 5.3.12. Let $\alpha \in L_\varnothing$. If $\vdash_C \alpha$, then $\vdash_{K?} \alpha$.

Proof. The easiest way to prove this theorem is to show that, concerning non-modal formulae, the axiomatic of C is included in the axiomatic of $K_?$. This is a trivial result, for the positive classical axioms Σ_P in $L_\varnothing$ and the non-positive classical axioms Σ_C in $L_\varnothing$ are automatically included in the set of axioms of $K_?$ (definitions 5.3.5 and 5.3.9) ■

Theorem 5.3.13. Let $\alpha \in L_\varnothing$. If $\vDash_C \alpha$, then $\vDash_{K?} \alpha$.

Proof. If $\vDash_C \alpha$, then for all trivial models M, $M \Vdash_{\Psi\varnothing} \alpha$. Since M contains only one world w, this means that α is valid in all possible classical valuations. Trivially then, being α a non-modal formula, it will be $\Psi_?$-satisfied by M at w, for any model M and world w of M, and therefore $K_?$-valid. ■

Theorem 5.3.14. Let $\alpha \in L_\diamond$. If $\vdash_K \alpha$, then $\vdash_{K?} \Phi(\alpha)$.

Proof. In order to prove this theorem, we have to show that, for all axiom-schemas α of K, $\vdash_{K?} \Phi(\alpha)$ is a valid schema of relation. Let us do that by induction on the size of the K-derivation $\alpha_1, \ldots, \alpha_n \equiv \alpha$ we know there exists.

Base of induction: derivation of size 1: $\alpha_1 \equiv \alpha$.

In this case, α is an axiom of K. The cases where α is an instance of P1, ... or P8 are trivial, for $\Phi(\alpha)$ will be automatically an axiom of $K_?$. If α is an instance of P9, P10 or P11, $\Phi(\alpha)$ will be an instance of P9', P10' or P11', which, as we have showed in the proof of lemma A.2, are $K_?$-theorems. If α is an instance of NP ($\Diamond\alpha \leftrightarrow \neg\Box\neg\alpha$), $\Phi(\alpha)$ will be an instance of K1 ($\alpha? \leftrightarrow \sim((\sim\alpha)!)$). And, finally, if α is an instance of K ($(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$), $\Phi(\alpha)$ will be an instance of $K_?$ ($(\alpha \rightarrow \beta)! \rightarrow (\alpha! \rightarrow \beta!)$).

Hypothesis of Induction: Suppose that the K-derivation of α is of size n and that the result holds for derivations of size smaller than n. Let $\alpha_1, \ldots, \alpha_{n-1}, \alpha_n \equiv \alpha$ be the derivation of α. α_n presence in the derivation may be justified in one of the following ways: (i) α_n is an axiom of K; (ii) there are $i,j<n$ such that $\alpha_i \equiv \alpha_j \rightarrow \alpha_n$; and (iii) there is $i<n$ such that $\alpha_n \equiv \Box\alpha_i$. We have just considered the first

case. About the second, since the derivation of α_i and of α_j are of sizes smaller than n, by the hypothesis of induction we have that there is a $K_?$-derivation of both $\Phi(\alpha_i) \equiv \Phi(\alpha_j \rightarrow \alpha_n) \equiv \Phi(\alpha_j) \rightarrow \Phi(\alpha_n)$ and $\Phi(\alpha_j)$. Applying *modus ponens* then, we have that there is a $K_?$-derivation of $\Phi(\alpha_n)$. About the third case, since the derivation of α_i is of size smaller than n, by the hypothesis of induction we have that there is a $K_?$-derivation of $\Phi(\alpha_i)$. Applying $N_!$ then, we have that there is a $K_?$-derivation of $\Phi(\alpha_n) \equiv \Phi(\Box\alpha_i) \equiv \Phi(\alpha_i)!$. ■

Theorem 5.3.15. Let $\alpha \in L_\diamond$. If $\models_K \alpha$, then $\models_{K?} \Phi(\alpha)$.

Proof. If $\models_K \alpha$, then for all models M and all worlds w of M, $M,w \Vdash_{\Psi\diamond} \alpha$. From lemma A.5, we have that $M,w \Vdash_{\Omega?} \Phi(\alpha)$. Consequently, $\models_{K?} \Phi(\alpha)$. ■

Theorem 5.3.16. Let $A,B \subseteq L_\diamond$ and $\alpha \in L_\diamond$. $A \div B \vdash_K \alpha$ iff $\Delta(A) \div \Delta(B) \vdash_{K?} \Delta(\alpha)$.

Proof. By lemma A.7, if $A \div B \vdash_K \alpha$ then $\Delta(A) \div \Delta(B) \vdash_{K?} \Delta(\alpha)$. Suppose that $\Delta(A) \div \Delta(B) \vdash_{K?} \Delta(\alpha)$ but $A \div B \nvdash_K \alpha$. If $A \div B \nvdash_K \alpha$, by lemma A.6, we have that $\Pi(\Delta(A)) \div \Pi(\Delta(B)) \nvdash_K \Pi(\Delta(\alpha))$. By lemma A.9, we have then that $\Delta(A) \div \Delta(B) \nvdash_{K?} \Delta(\alpha)$, which is a contradiction. Therefore, if $\Delta(A) \div \Delta(B) \vdash_{K?} \Delta(\alpha)$ then $A \div B \vdash_K \alpha$. ■

Theorem 5.3.17. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \div B \vdash_{K?} \alpha$ iff $\Pi(A) \div \Pi(B) \vdash_K \Pi(\alpha)$.

Proof. By lemma A.9, if $A \div B \vdash_{K?} \alpha$ then $\Pi(A) \div \Pi(B) \vdash_K \Pi(\alpha)$. Suppose that $\Pi(A) \div \Pi(B) \vdash_K \Pi(\alpha)$ but $A \div B \nvdash_{K?} \alpha$. If $A \div B \nvdash_{K?} \alpha$, by lemma A.6, we have that $\Delta(\Pi(A)) \div \Delta(\Pi(B)) \nvdash_{K?} \Delta(\Pi(\alpha))$. By lemma A.7, we have then that $\Pi(A) \div \Pi(B) \nvdash_K \Pi(\alpha)$, which is a contradiction. Therefore, if $\Pi(A) \div \Pi(B) \vdash_K \Pi(\alpha)$ then $A \div B \vdash_{K?} \alpha$. ■

Theorem 5.3.18. Let $A,B \subseteq L_\diamond$ and $\alpha \in L_\diamond$. $A \div B \models_K \alpha$ iff $\Delta(A) \div \Delta(B) \models_{K?} \Delta(\alpha)$.

Proof. Suppose that $A \div B \models_K \alpha$ but $\Delta(A) \div \Delta(B) \not\models_{K?} \Delta(\alpha)$. If $\Delta(A) \div \Delta(B) \not\models_{K?} \Delta(\alpha)$, then there is a model M and a world w of M such that $M \Vdash_{\Omega?} \Delta(\phi)$, for all $\Delta(\phi) \in \Delta(A)$, $M,w \Vdash_{\Omega?} \Delta(\lambda)$, for all $\Delta(\lambda) \in \Delta(B)$, and $M,w \nVdash_{\Omega?} \Delta(\alpha)$. But if $M \Vdash_{\Omega?} \Delta(\phi)$ for all $\Delta(\phi) \in \Delta(A)$, $M,w \Vdash_{\Omega?} \Delta(\lambda)$ for all $\Delta(\lambda) \in \Delta(B)$ and $M,w \nVdash_{\Omega?} \Delta(\alpha)$, by lemma A.10 we have that $M \Vdash_{\Psi\diamond} \phi$ for all $\phi \in A$, $M,w \Vdash_{\Psi\diamond} \lambda$ for all $\lambda \in B$ and $M,w \nVdash_{\Psi\diamond} \alpha$. Consequently, $A \div B \not\models_K \alpha$, which is a contradiction. Therefore, if $A \div B \models_K \alpha$, $\Delta(A) \div \Delta(B) \models_{K?} \Delta(\alpha)$. Suppose now that $\Delta(A) \div \Delta(B) \models_{K?} \Delta(\alpha)$ but $A \div B \not\models_K \alpha$. If $A \div B \not\models_K \alpha$ then there is a model M and a world w of M such that $M \Vdash_{\Psi\diamond} \phi$ for all $\phi \in A$, $M,w \Vdash_{\Psi\diamond}$

λ for all $\lambda \in B$ and M,w $\nVdash_{\Psi\circ} \alpha$. But if M $\Vdash_{\Psi\circ} \phi$ for all $\phi \in A$, M,w $\Vdash_{\Psi\circ} \lambda$ for all $\lambda \in B$ and M,w $\nVdash_{\Psi\circ}$ α, then by lemma A.10 M $\Vdash_{\Omega?} \Delta(\phi)$ for all $\Delta(\phi) \in \Delta(A)$, M,w $\Vdash_{\Omega?} \Delta(\lambda)$ for all $\Delta(\lambda) \in \Delta(B)$ and M,w $\nVdash_{\Omega?} \Delta(\alpha)$. Consequently, $\Delta(A) \div \Delta(B) \nvDash_{K?} \Delta(\alpha)$, which is a contradiction. Therefore, if $\Delta(A) \div \Delta(B) \vDash_{K?} \Delta(\alpha)$ then $A \div B \vDash_K \alpha$. ■

Theorem 5.3.19. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \div B \vDash_{K?} \alpha$ iff $\Pi(A) \div \Pi(B) \vDash_K \Pi(\alpha)$.

Proof. The proof of this theorem follows the same idea of theorem 5.3.18's. We have just to properly erase the occurrences of Δ and consider function Π along with Ψ as well as to use lemma A.11 instead of lemma A.10. ■

Theorem 5.4.1. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \div B \vdash_{D?} \alpha$ iff $A \cup \Sigma_{D?} \div B \vdash_{K?} \alpha$.

Proof. Suppose that $A \div B \vdash_{D?} \alpha$. We can easily extend the proof of theorem 5.3.17 ($A \div B \vdash_{K?} \alpha$ iff $\Pi(A) \div \Pi(B) \vdash_K \Pi(\alpha)$) in such a way as to prove that $A \div B \vdash_{D?} \alpha$ iff $\Pi(A) \div \Pi(B) \vdash_D \Pi(\alpha)$, where D is normal modal logic D (we have just to consider the additional axiom $\alpha! \rightarrow \alpha?$ in $D_?$ and $\Box\alpha \rightarrow \Diamond\alpha$ in D). With this result, we have that $\Pi(A) \div \Pi(B) \vdash_D \Pi(\alpha)$. Given then the known result that $A \div B \vdash_D \alpha$ iff $A \cup \Sigma_D \div B \vdash_K \alpha$ (where Σ_D is the set of all instances of axiom D)[11] we have then that $\Pi(A) \cup \Sigma_D \div \Pi(B) \vdash_K \Pi(\alpha)$. Since $\Sigma_D = \Pi(\Sigma_{D?})$, $\Pi(A) \cup \Sigma_D \div \Pi(B) \vdash_K \Pi(\alpha)$ is the same as $\Pi(A) \cup \Pi(\Sigma_{D?}) \div \Pi(B) \vdash_K \Pi(\alpha)$ or $\Pi(A \cup \Sigma_{D?}) \div \Pi(B) \vdash_K \Pi(\alpha)$. By theorem 5.3.17 therefore, we have that $A \cup \Sigma_{D?} \div B \vdash_{K?} \alpha$. The right-left side of the proof follows the same reasoning. ■

Theorem 5.4.2. Let $A,B \subseteq L_?$ and $\alpha \in L_?$. $A \div B \vDash_{D?} \alpha$ iff $A \cup \Sigma_{D?} \div B \vDash_{K?} \alpha$.

Proof. The proof of this theorem is almost identical to theorem 5.4.1's. All we have to do is to replace the relation of deductibility by the relation of logical consequence and instead of theorem 5.3.17 consider theorem 5.3.19. Of course, we have also to take the semantic version of the theorem that links D to K. ■

Theorem 5.4.3. $D_?$ is sound and complete.

Proof. Theorem 5.4.3 follows immediately from theorems 5.4.1, 5.4.2 and 5.3.1. ■

Theorem 5.4.4. All formulae of $L_?$ that satisfy one of the following schemas of formula are $D_?$-theorems (and consequently $D_?$-valid.)

[11] See Fitting (1993).

$(\alpha \to \alpha)?$

$((\alpha \to \beta)! \to (\alpha! \to \beta!))?$

Proof. We can easily extend the proof of theorem 5.3.14 (If $\vdash_K \alpha$, then $\vdash_{K?} \Phi(\alpha)$) in such a way as to prove that If $\vdash_D \alpha$, then $\vdash_{K?} \Phi(\alpha)$. Therefore, since the schemas of formula above are valid in D if we replace ! by $\Box$ and ? by $\Diamond$, all of them are valid in $D_?$. ■

The proofs of theorems 5.5.5 to 5.5.19 follow the same reasoning of the proofs of theorems 5.4.1-5.4.3 we showed above. We have just to replace D and $D_?$ by the normal and paranormal modal logics in question.

Theorem 5.4.20. $K_?^1$ is sound and complete (that is, for any $A,B \subseteq \mathcal{L}_?$ and $\alpha \in \mathcal{L}_?$, $A \div B \vdash_{K1?} \alpha$ iff $A \div B \vDash_{K1?} \alpha$.)

Proof. In the same way that in the proof of theorem 5.3.1 we benefited from the theorems of soundness and completeness of propositional normal modal logic K, we can also make use of the theorems of soundness and completeness of first-order normal modal logic K to prove the soundness and completeness of first-order paranormal modal logic $K_?^1$. If we follow this path, the proof of theorem 5.4.20 will have the same structure of theorem 5.3.1's. The difference is that we will have in addition to consider axiom Q and the rule of generalization in the corresponding expansion of theorems 5.3.16 and 5.3.17 and, in the expansion of theorems 5.3.18 and 5.3.19, the first-order way of evaluating atomic propositions and quantified formulae. ■

Theorem 5.4.21. $K_?K$ is sound and complete.

Proof. By following the suggestion above and considering at this time the proof of soundness and completeness of normal 2-modal logic KK (that is, the normal multi-modal logic with two pairs of dual modalities and the axioms $\Box_1(\alpha \to \beta) \to (\Box_1\alpha \to \Box_1\beta)$ and $\Box_2(\alpha \to \beta) \to (\Box_2\alpha \to \Box_2\beta)$), we can construct the proof of soundness and completeness of $K_?K^{12}$. From a general point of view, this method can be used to prove the completeness and soundness of any paranormal or multi-normal modal logic: we have just to take the corresponding mono or multi-modal normal modal logic along with its theorem of completeness and soundness and consider the steps not taken into account by the proofs of theorems 5.3.16 - 5.3.19. We will call this method the *from normal to paranormal method of completeness and soundness proof* (or, for short, the from normal to paranormal method). It is worthy of mention that it was already used in the proof of theorem 5.4.3. ■

Theorem 5.4.22. $PPK_?K$ is sound and complete.

Proof. The proof of this theorem is done by applying the from normal to paranormal method along with the proof of completeness and soundness of normal 2-modal logic PPKK (here, axiom PP will be simply an axiom stating the relation between the two modalities: $\Box_2\alpha \rightarrow \Box_1\alpha$, for example.) ■

Theorem 5.4.23. The following schemas of relations between sets of formulas and formula are correct.

$\{\Box\alpha\} \vdash_{PPK?K} \alpha!$ $\{\Box\alpha\} \vDash_{PPK?K} \alpha!$

$\{\alpha?\} \vdash_{PPK?K} \Diamond\alpha$ $\{\alpha?\} \vDash_{PPK?K} \Diamond\alpha$

Proof. $\{\Box\alpha\} \vdash_{PPK?K} \alpha!$ follows directly from the use of axiom PP. From PP and ~(10) we get $\sim((\sim\alpha)!) \rightarrow \sim\Box\sim\alpha$. From P2 along with NN and NP we get $\Diamond\alpha \leftrightarrow \sim\Box\sim\alpha$. From that, along with $\sim((\sim\alpha)!) \rightarrow \sim\Box\sim\alpha$ and K1 and P2 we get $\alpha? \rightarrow \Diamond\alpha$, from which $\{\alpha?\} \vdash_{PPK?K} \Diamond\alpha$ follows directly. The other two relations follow from theorem 5.4.22 applied to the two just proved results. ■

A.3 Theorems from Chapter 6

Theorem 6.2.1. LP^2 is sound and complete.

Proof. Again here we can use the from normal to paranormal method to prove the soundness and completeness of LP^2. The difference is that at this time we will have to consider the normal 2-modal logic $PPK_1K_2D_1D_2B_1B_14_1$, where axiom A_n is the axiom A written in terms of modality $\Box_n$. ■

Theorem 6.2.2. The following schemas of relations between sets of formulas and formula are correct.

$\{\Box\alpha\} \vdash_{LP2} \alpha!$ $\{\Box\alpha\} \vDash_{LP2} \alpha!$

$\{\alpha!\} \vdash_{LP2} \alpha?$ $\{\alpha!\} \vDash_{LP22} \alpha?$

$\{\alpha?\} \vdash_{LP2} \Diamond\alpha$ $\{\alpha?\} \vDash_{LP2} \Diamond\alpha$

Proof. The proof of the first and third schemas of relation is identical to the proof of theorem 5.4.23. For the second, it follows directly from axiom $D_?$. ■

Theorem 6.2.3. Let $\alpha \in \mathcal{L}_{?\Diamond}$ be such that $\nvdash_{LP2} \alpha$ and $\nvdash_{LP2} \neg\alpha$. The following formulae are *not* LP^2 theorems:

$\alpha! \vee \neg(\alpha!)$ $\neg(\alpha? \wedge \neg(\alpha?))$

[12] For meta-results on multi-modal logics see ... **

Proof. Let us suppose that $\alpha! \vee \neg(\alpha!)$ is a LP^2 theorem. From theorem 6.2.1 then, we have that it is LP^2 valid. Consequently, for every first-order 2-model M and world w of M, $\alpha! \vee \neg(\alpha!)$ is $\Omega_{?\diamond}$-satisfied by M at w, that is to say, $\Omega_{?\diamond M,w}(\alpha! \vee \neg(\alpha!)) = 1$, which is equivalent to $\Omega_{?\diamond M,w}(\alpha!) = 1$ or $\Omega_{?\diamond M,w}(\neg(\alpha!)) = 1$. $\Omega_{?\diamond M,w}(\alpha!) = 1$ iff, for all $w' \in W$ such that $wR_? w'$, $\Omega_{?\diamond M,w'}(\alpha) = 1$. In its turn, $\Omega_{?\diamond M,w}(\neg(\alpha!)) = 1$ iff $\mho_{?\diamond M,w}(\alpha!) = 0$ iff, for all $w' \in W$ such that $wR_? w'$, $\mho_{?\diamond M,w'}(\alpha) = 0$. We see that these two alternatives are not such that one of them has necessarily to be the case. This will happen only if either $\models_{LP2} \alpha$ or $\models_{LP2} \neg\alpha$, which, by theorem 6.2.1, contradicts the condition that $\nvdash_{LP2} \alpha$ and $\nvdash_{LP2} \neg\alpha$. Therefore, $\alpha! \vee \neg(\alpha!)$ is not a LP^2 theorem. The proof of the other part of the theorem follows the same reasoning. ■

Theorem 6.2.4. Let $\alpha \in \mathcal{L}_{?\diamond}$ be such that $\nvdash_{LP2} \alpha$ and $\nvdash_{LP2} \neg\alpha$. The following formulae are *not* LP^2 theorems:

$\alpha! \vee (\neg\alpha)!$ $\qquad\qquad$ $\neg(\alpha? \wedge (\neg\alpha)?)$

Proof. The proof of this theorem follows from theorem 6.2.3 along with the observation that, due to axioms K2 and K3, $\vdash_{LP2} \alpha! \vee (\neg\alpha)! \leftrightarrow \alpha! \vee \neg(\alpha!)$ and $\vdash_{LP2} \neg(\alpha? \wedge (\neg\alpha)?) \leftrightarrow \neg(\alpha? \wedge \neg(\alpha?))$. ■

Theorem 6.2.5. Let $\alpha \in \mathcal{L}_{?\diamond}$ be such that $\nvdash_{LP2} \alpha$ and $\nvdash_{LP2} \sim\alpha$. The following formulae are *not* LP^2 theorems:

$\alpha! \vee (\sim\alpha)!$ $\qquad\qquad$ $\sim(\alpha? \wedge (\sim\alpha)?)$

Theorem 6.2.6. Let $\alpha \in \mathcal{L}_{?\diamond}$ be such that $\nvdash_{LP2} \alpha$ and $\nvdash_{LP2} \neg\alpha$. The following formulae are *not* LP^2 theorems:

$\alpha\mathbf{?} \vee \neg(\alpha\mathbf{?})$ $\qquad\qquad$ $\neg(\alpha\mathbf{?} \wedge \neg(\alpha\mathbf{?}))$

$\alpha\mathbf{?} \vee (\neg\alpha)\mathbf{?}$ $\qquad\qquad$ $\neg(\alpha\mathbf{?} \wedge (\neg\alpha)\mathbf{?})$

Theorem 6.2.7. Let $\alpha \in \mathcal{L}_{?\diamond}$ be such that $\nvdash_{LP2} \alpha$ and $\nvdash_{LP2} \sim\alpha$. The following formulae are *not* LP^2 theorems:

$\alpha\mathbf{?} \vee (\sim\alpha)\mathbf{?}$ $\qquad\qquad$ $\sim(\alpha\mathbf{?} \wedge (\sim\alpha)\mathbf{?})$

The proof of theorems 6.2.5, 6.2.6 and 6.2.7 follows the same reasoning as theorem 6.2.3's.

Theorem 6.2.8. All formulae of $\mathcal{L}_{?\diamond}$ that satisfy one of the following schemas of formula are LP^2-theorems (and consequently LP^2-valid.)

$(\neg\alpha)\mathbf{?} \leftrightarrow \neg(\alpha\mathbf{?})$

$(\alpha\mathbf{?} \vee \beta\mathbf{?}) \rightarrow (\alpha \vee \beta)\mathbf{?}$

$(\alpha \rightarrow \beta)\mathbf{?} \rightarrow (\alpha\mathbf{!} \rightarrow \beta\mathbf{?})$

$\alpha\mathbf{!} \rightarrow \alpha\mathbf{?!}$

$(\alpha \rightarrow \beta)\mathbf{!} \rightarrow (\alpha\mathbf{?} \rightarrow \beta\mathbf{?})$

$(\alpha \wedge \beta)\mathbf{?} \rightarrow (\alpha\mathbf{?} \wedge \beta\mathbf{?})$

$\alpha\mathbf{!} \rightarrow \alpha\mathbf{?}$

$\alpha\mathbf{?} \rightarrow \alpha\mathbf{??}$

Proof. We show below the derivation of each one of the above stated schemas of formula.

$(\neg\alpha)\mathbf{?} \leftrightarrow \neg(\alpha\mathbf{?})$

1. $(\neg(\alpha!))? \rightarrow \neg(\alpha!?)$ — K3
2. $(\neg\alpha)! \rightarrow \neg(\alpha!)$ — K2
3. $(\neg\alpha)!? \rightarrow (\neg(\alpha!))?$ — K(1) 2
4. $(\neg\alpha)!? \rightarrow \neg(\alpha!?)$ — P(2) 1,3
5. $\neg(\alpha!?) \rightarrow (\neg(\alpha!))?$ — K3
6. $\neg(\alpha!) \rightarrow (\neg\alpha)!$ — K2
7. $(\neg(\alpha!))? \rightarrow (\neg\alpha)!?$ — K(1) 6
8. $\neg(\alpha!?) \rightarrow (\neg\alpha)!?$ — P(2) 5,7

With similar steps we get $(\neg\alpha)!? \rightarrow \neg(\alpha!?)$ ...

$(\alpha \rightarrow \beta)\mathbf{!} \rightarrow (\alpha\mathbf{?} \rightarrow \beta\mathbf{?})$

1. $(\alpha! \rightarrow \beta!)! \rightarrow (\alpha!? \rightarrow \beta!?)$ — Th. 5.3.5
2. $(\alpha \rightarrow \beta)! \rightarrow (\alpha! \rightarrow \beta!)$ — $K_?$
3. $(\alpha \rightarrow \beta)!! \rightarrow (\alpha! \rightarrow \beta!)!$ — K(4) 2
4. $(\alpha \rightarrow \beta)!! \rightarrow (\alpha!? \rightarrow \beta!?)$ — P(2) 1,3

$(\alpha\mathbf{?} \vee \beta\mathbf{?}) \rightarrow (\alpha \vee \beta)\mathbf{?}$

1. $(\alpha!? \vee \beta!?) \rightarrow (\alpha! \vee \beta!)?$ — Th. 5.3.5
2. $\alpha! \vee \beta! \rightarrow (\alpha \vee \beta)!$ — Th. 5.3.5
3. $(\alpha! \vee \beta!)? \rightarrow (\alpha \vee \beta)!?$ — K(1) 2
4. $(\alpha!? \vee \beta!?) \rightarrow (\alpha \vee \beta)!?$ — P(2) 1,3

$(\alpha \wedge \beta)\mathbf{?} \rightarrow (\alpha\mathbf{?} \wedge \beta\mathbf{?})$

1. $(\alpha! \wedge \beta!)? \rightarrow (\alpha!? \wedge \beta!?)$ — Th. 5.3.5
2. $(\alpha \wedge \beta)! \rightarrow \alpha! \wedge \beta!$ — Th. 5.3.5
3. $(\alpha \wedge \beta)!? \rightarrow (\alpha! \wedge \beta!)?$ — K(1) 2
4. $(\alpha \wedge \beta)!? \rightarrow (\alpha!? \wedge \beta!?)$ — P(2) 1,3

$(\alpha \rightarrow \beta)\mathbf{?} \rightarrow (\alpha\mathbf{!} \rightarrow \beta\mathbf{?})$

1. $(\alpha! \rightarrow \beta!)? \rightarrow (\alpha!! \rightarrow \beta!?)$ — Th. 5.3.5

2. $(\alpha \to \beta)! \to (\alpha! \to \beta!)$ $K_?$

3. $(\alpha \to \beta)!? \to (\alpha! \to \beta!)?$ K(1) 2

4. $(\alpha \to \beta)!? \to (\alpha!! \to \beta!?)$ P(2) 1,3

$\alpha\mathbf{!} \to \alpha\mathbf{?}$

1. $\alpha!! \to \alpha!?$ $D_?$

$\alpha\mathbf{!} \to \alpha\mathbf{?!}$

1. $\alpha! \to \alpha!?!$ $B_?$

2. $\alpha!! \to \alpha!?!!$ K(4) 1

$\alpha\mathbf{?} \to \alpha\mathbf{??}$

1. $\alpha! \to \alpha!?!$ $B_?$

2. $\alpha!? \to \alpha!?!?$ K(1) 1 ■