Assigning Refugees to Landlords in Sweden: Stable Maximum Matchings*

Tommy Andersson† and Lars Ehlers‡

First version: July 18, 2016. This version: December 7, 2016.

Revised version : August 15, 2017

http://www.cireqmontreal.com/wp-content/uploads/2017/12/13-2016-REVISED.pdf

Abstract

The member states of the European Union received 1.2 million first time asylum applications in 2015 (a doubling compared to 2014). Even if asylum will be granted for many of the refugees that made the journey to Europe, several obstacles for successful integration remain. This paper focuses on one of these obstacles, namely the problem of finding housing for refugees once they have been granted asylum. In particular, the focus is restricted to the situation in Sweden during 2015–2016 and it is demonstrated that market design can play an important role in a partial solution to the problem. More specifically, because almost all accommodation options are exhausted in Sweden, the paper investigates a matching system, closely related to the system adopted by the European NGO "Refugees Welcome", and proposes an easy-to-implement algorithm that finds a stable maximum matching. Such matching guarantees that housing is provided to a maximum number of refugees and that no refugee prefers some landlord to their current match when, at the same time, that specific landlord prefers that refugee to his current match.

JEL Classification: C71, C78, D71, D78, F22.

Keywords: refugees, private landlords, forced migration, market design, stable maximum matchings.

*We are grateful to Christian Basteck, Péter Biró, Francis Bloch, Estelle Cantillion, David Delacretaz, Jens Gudmundsson, Guillaume Haeringer, Bettina Klaus, Fuhito Kojima, Scott Duke Kominers, Vikram Manjunath, Michael Ostrovsky, Parag Pathak, Marek Pycia, Alvin Roth, Alexander Teytelboym, and Utku Ünver for many useful and constructive comments. We would also like to thank participants at the NBER Market Design Working Group Meeting (Stanford, 2016) and the CIREQ Microeconomic Theory Conference (Montreal, 2016), as well as the seminar audience at European Center for Advanced Research in Economics and Statistics (Brussels, 2016), for helpful comments. Both authors gratefully acknowledge financial support from the Jan Wallander and Tom Hedelius Foundation. The first author is also grateful to Ragnar Söderbergs Stiftelse for financial support, and the second author is also grateful to the SSHRC (Canada) and the FRQSC (Québec) for financial support.

†Department of Economics, Lund University, Box 7082, 220 07 Lund, Sweden; e-mail: `tommy.andersson@nek.lu.se`.

‡Département de Sciences Économiques and CIREQ, Université de Montréal, C.P. 6128, Succursale Centre-Ville, Montréal, Québec H3C 3J7, Canada; e-mail: `lars.ehlers@umontreal.ca`.

1 Introduction

The European refugee crisis began in 2015 when a rising number of refugees made the journey to Europe to seek asylum. The member states of the European Union received 1.2 million first time asylum applications (more than a doubling compared to 2014).[1] Apart from the Dublin Regulation, which dictates that the member state in which an asylum seeker enters first is obliged to render asylum, there has been no systematic way to divide refugees between the member states. Obviously, this puts great pressure on member states located at the external border of the European Union, and more specifically, on Greece, Hungary and Italy.

In an attempt to reduce pressure on the three member states mentioned above, the European Commission decided in September 2015 on a temporary European relocation scheme for 120,000 refugees who are in need of international protection.[2] The relocation scheme was based on a distribution key, adopted by the European Commission in May 2015, where a specific quota was stated for each member state based on:

> "... objective, quantifiable and verifiable criteria that reflect the capacity of the member states to absorb and integrate refugees, with appropriate weighting factors reflecting the relative importance of such criteria".[3]

The distribution key, however, did not specify which refugees should be relocated to which member states. This specific problem has attracted interest among researchers and more systematic ways to relocate refugees between member states have been proposed. For example, Jones and Teytelboym (2016a) propose a system where member states and refugees submit their preferences about which refugees they most wish to host and which state they most wish to be hosted by, respectively, to a centralized clearing house which matches member states and refugees according to these preferences.

Even if membership quotas are settled and a centralized matching relocation system is in place, several obstacles for successful integration remain. This paper focuses on one of these obstacles, namely the problem of finding housing for refugees once they have been relocated to a European Union membership state, and, in particular, how market design can play an important role in the solution to the problem. The background to the housing problem will be described from the perspective of the situation in Sweden during 2015–2016.

In 2015, the population of Sweden was 9.9 million which accounts for about 1.4 percent of the population in Europe. Yet, 12.4 percent of the asylum seekers in the European Union in 2015 were registered in Sweden which made Sweden the state in the European Union with most asylum seekers per capita.[4] A refugee who enters Sweden is temporarily placed at a Migration

[1]Eurostat News, Release 44/2016, March 4, 2016.

[2]Eurpoean Commission, Statement 15/5697, September 22, 2015.

[3]European Commission, "Communication from the Commission to the European Parliament, the Council, the European Economic and Social Committee and the Committee of the Regions", Annex, May 13, 2015.

[4]Eurostat News, Release 44/2016, March 4, 2016.

Board accommodation facility in anticipation of either a deportation order or a permanent residence permit. The average waiting time for this decision was 15 months in May 2016.[5] Refugees who are granted permanent residence permits are, under Swedish law, entitled to a number of establishment measures, and their legal status is upgraded from "asylum seekers" to "refugees with a permanent residence permit".[6] These establishments measures include, e.g., accommodation, Swedish language courses, and a monthly allowance. An establishment plan is formally established and coordinated between the refugee and the Swedish Employment Service. The local municipality where the refugee is registered has the responsibility to find appropriate accommodation. In this process, the refugee must leave the Migration Board accommodation facility since the legal responsibility for the refugee is transferred from the state to the local municipality.

One problem in Sweden is that almost all accommodation options are exhausted. In March 2015, it was estimated that 9,300 persons with a permanent residence permit still lived in an Migration Board accommodation facility and that, at least, 14,100 residential units were needed before the end of 2016 just to accommodate those who are granted a residence permit.[7] This estimation was updated in February 2016 to at least 20,000 new residential units only in the spring of 2016 provided that there is no drastic increase in the number of asylum seekers.[8] These facts, together with a new legislation, effective from March 1, 2016, stating that all municipalities have to accept refugees puts even more pressure on municipalities to find additional residential units. This has forced some municipalities to consider extraordinary actions. One example is the passenger ship Ocean Gala leased for use as an asylum accommodation with room for nearly 800 people in Utansjö port outside the city Härnösand in the north east of Sweden.[9] Another example is a temporary tent camp with a capacity to accommodate 1,520 asylum seekers that was scheduled to open in December 2015 on Revingehed armor training ground 20 kilometers east of the city of Lund in the south of Sweden.[10] Hence, it is urgent to find residential units for refugees, not only because they are entitled to it under Swedish law, but also because they are blocking asylum seekers from accommodation at Migration Board accommodation facilities.

A key observation, and a possible solution to the above described problem, can be found in a report from "The Swedish National Board of Housing, Building and Planning" in 2013, where it is estimated that 90 percent of the general housing shortage in Sweden can be explained by inefficient use of the existing housing stock.[11] More precisely, due to rent control, tenants tend

[5] Swedish Migration Board, www.migrationsverket.se/Kontakta-oss/Tid-till-beslut.html, May 13, 2016.

[6] The Swedish terminology for "refugees with a permanent residence permit" is "nyanländ" but we will, for convenience, in the remaining part of this paper, slightly abuse the translation of the Swedish word and use "refugees" instead of the correct terminology "refugees with a permanent residence permit".

[7] "Nyanländas boendesituation – delrapport", The Swedish National Board of Housing, Building and Planning, Rapport 2015:10.

[8] "More than 20 000 new places needed in accommodation in the spring", Swedish Migration Board, February 19, 2016.

[9] "Migrationsverket visste inte att miljonbåten var på väg", June 15, 2016, SVT.

[10] "Första asylsökande har flyttat in i tältlägret i Revinge", December 10, 2015, Aftonbladet.

[11] "Bostadsbristen och hyressättningssystemet – ett kunskapsunderlag", Marknadsrapport, The Swedish National

to live in apartments which are too big for their circumstances.

In a recent survey, 31 percent of the Swedish households stated that they are willing to accommodate refugees in their homes.[12] Of course, a stated willingness to accommodate a refugee and actually accommodating a refugee are two different things, and it should also be noted that the general view on refugees in Sweden was not as positive in the spring of 2016 as it was in the fall of 2015.[13] However, there were 4,766,000 households in Sweden in January 2015[14], and if only 1 percent of the households (instead of 30 percent) are willing to accommodate a refugee, there are still 47,660 households that are willing to host refugees. Hence, to release the pressure on municipalities to find housing for refugees, voluntarily supplied private housing can be utilized.[15] In this way, beds that are occupied by refugees with a permanent residence permit at the Migration Board accommodation facilities can be used for asylum seekers.

In several meetings at various levels in the Swedish administration, e.g., with the State Secretary to the Minister of Housing and the Swedish Migration Board, the authors of this paper presented a version of the theoretical matching model described in this paper. The model contains a set of "landlords" (i.e., private persons) with capacity and willingness to accommodate refugees in their private homes and a set of refugee families with permanent residence permits. In the model, a refugee family and a landlord find each other mutually acceptable if they have a spoken language in common and if the number of family members does not exceed the capacity of the landlord. The communication requirement is key and its importance has been stressed by politicians in, e.g., the above mentioned meetings, and it is a requirement in, e.g., the non-centralized system adopted by the European NGO "Refugees Welcome" to match refugees with private persons.

Even if it is natural to assume that landlords have preferences over refugee families, it is not unreasonable to believe that it is difficult for landlords to provide a strict ranking of all refugee families. There are several reasons for this.[16] For example, there are thousands of refugee families in the system and it is probably difficult to gather complete information about all these families and even if such information is available it is not clear how to process it. For this reason, it is, throughout the paper, assumed that landlords only report their spoken languages and their capacity (i.e., the number of available beds).

Board of Housing, Building and Planning, 2013.

[12]"Svenska folkets attityder till flyktingar", September 24, 2015, DN/Ipsos.

[13]"Allmänhetens uppfattning om invandringen", March 25, 2016, Demoskop.

[14]"Antal hushåll i Sverige", 2016, Statistics Sweden.

[15]In fact, many Swedish municipalities are today actively searching for private persons that are willing to accommodate refugees in their private homes and private persons and refugee families are matched in a non-centralized way. Examples of such municipalities include Stockholms stad, Lunds kommun, Ängleholms kommun, Nynäshamns kommun, Kristianstads kommun, Nacka kommun, Botkyrka kommun, Håbo kommun, Härryda kommun, and Lerums kommun.

[16]See Jones et al. (2016) or Jones and Teytelboym (2016a,b) for a discussion about ranking and preferences for local authorities and refugees in a refugee matching context.

Given these reports and three very natural assumptions, it is possible to induce preferences for landlords over refugee families. More specifically, it will be assumed that matched landlords and refugee families must find each other acceptable (in terms of spoken languages and capacities), that private landlords can accommodate at most one refugee family, and that landlords strictly prefer a larger refugee family to a smaller as long as they both are acceptable. Given these assumptions, preferences for landlords can be induced and landlords classify refugee families to belong to different indifference classes. Landlords are indifferent between any two families in the same indifference class, but have strict (induced) preferences over the indifference classes.

Naturally, refugees are also allowed to have preferences over landlords but for the same reasons as above, refugees only report their spoken languages and family size. Based on these reports, it is, again, possible to induce preferences where landlords are classified to be either acceptable or unacceptable.

Given the induced preferences for landlords and refugee families, the idea in this paper is to find a formal procedure for identifying a matching, i.e., a procedure describing which refugees are assigned to which landlords for the given preferences. Because a matching not necessarily is unique, the attention is directed towards matchings with specific properties. Given the acute shortage of residential units in Sweden, a natural requirement on a matching is that it is maximum, i.e., that a maximum number of refugees are matched to (acceptable) landlords or, equivalently, that a maximum number of privately supplied beds are utilized. A second requirement is stability. Given the induced preferences considered in this paper, this axiom means that no refugee family strictly prefers some landlord to being unmatched when, at the same time, that specific landlord strictly prefers that refugee family to his current match. This also means that stability, in this setting, guarantees a lower welfare bound for the participating private landlords as the landlords can be made assure that if they are matched to some refugee family, there is at least no unmatched refugee family that they strictly prefer to their current match. It is also well-known that unstable mechanisms tend to die out while stable mechanisms survive the test of time (Roth, 2008). The main innovation in this paper is the construction of an algorithm, called the Maximality-Improvement-Chains algorithm. This algorithm identifies a stable maximum matching for any given refugee assignment problem.

This paper is related to several strands of matching and market design literature.[17] The basic model is a two-sided matching model with capacities, like, e.g., Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu et al. (2009), Balinski and Sönmez (1999) and Gale and Shapley (1962), where the number of offered beds and the number of needed beds are the capacities of the landlords and the refugee families, respectively. It is, however, not a many-to-many matching model, like, e.g., Echenique and Oviedo (2006) and Konishi and Ünver (2006), because even though landlords offer several beds and refugee families may need multiple beds, agents are matched to at most one agent from the other side of the market, i.e., refugee families are matched to at most

[17] For an overview of the matching and market design literature see, e.g., Roth and Sotomayor (1990) or Sönmez and Ünver (2011).

one landlord and landlords are matched to at most one refugee family.

Furthermore, in contrast to most papers in the two-sided matching literature, this paper is not primarily concerned with stable matchings as the focus instead is directed towards maximum matchings. The first result also states that a maximum matching not necessarily is stable and that a stable matching is not necessarily maximum. However, as we demonstrate, there always exists a maximum matching which in addition is stable. Consequently, even if maximum matchings are central in the analysis, maximality can be achieved without dispensing stability. As already described in the above, stable maximum matchings can be identified by adopting the Maximality-Improvement-Chains algorithm. The main building block in this algorithm is the "execution" of specific "chains of refugees". These executions always strictly increase the cardinality of the matching while maintaining stability. This also means that the model investigated in this paper is located in between school choice with indifferences (Erdil and Ergin, 2008) where a stable-improvement-cycles algorithm is proposed in order to find a stable matching which is not Pareto dominated by another stable matching, and the kidney exchange framework (Roth et al., 2004) where certain trading cycles and chains are executed in order to increase the number of transplanted kidneys.

Even if a variety of problems have been investigated in different market design contexts[18], almost no attention has been directed towards problems related to refugee matching. There are, however, a few exceptions. As already mentioned in the above, Jones and Teytelboym (2016a) propose a partial solution to the European refugee crisis via the construction of a two-sided matching system that assigns refugees to member states of the European Union. More related to this paper is Delacretaz et al. (2016) and Jones and Teytelboym (2016b). Both these papers consider the local refugee matching problem, i.e., the problem of finding out where in a country that refugees should be settled once they have been granted protection. Jones and Teytelboym (2016b) describe in general terms how a two-sided matching system can be constructed when assigning refugees to localities. They also outline in detail how this system can be applied in order to meet the British government's commitment to resettle 20,000 Syrian refugees by 2020. Delacretaz et al. (2016), on the other hand, consider a two-sided matching market for the local refugee match and propose three different refugee resettlement systems that can be used by hosting countries under different circumstances. The common idea in these three systems is to improve match efficiency and to reduce internal movement of refugees across localities. However, none of the matching systems considered in the above three mentioned papers focuses on stable maximum matchings. Neither do they consider matching mechanisms based on the execution of specific chains of refugees as the one proposed in this paper. Instead, they consider versions of the Deferred Acceptance Algorithm (Gale and Shapley, 1962) and the Top Trading Cycles Algorithm (Shapley and Scarf, 1974).

[18]Examples include, e.g., school choice (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003), entry-level job market for doctors (Roth and Peranson, 1999), kidney exchange (Roth et al., 2004), course allocation (Budish and Cantillon, 2012), and cadet-branch matching (Sönmez and Switzer, 2013; Sönmez, 2013).

The remaining part of the paper is outlined as follows. Section 2 introduces the refugee assignment problem and the basic ingredients of the matching model. A number of results related to stable maximum matchings are presented in Section 3. The Maximality-Improvement-Chains algorithm is introduced and analyzed in Section 4. Results related to manipulability and strategy-proofness are stated in Section 5. Some concluding remarks are provided in Section 6. All proofs are relegated to the Appendix.

2 The Model and Basic Definitions

This section starts by presenting the basic ingredients of the refugee assignment problem. Once this problem has been introduced, the preferences of the landlords and refugees can be induced given three very natural assumptions. The section ends by describing the concept of a matching and by stating some desirable properties of matchings for the refugee assignment problem.

2.1 The Refugee Assignment Problem

Each refugee family contains a number of family members that wish to be accommodated by a landlord. Landlords are private persons that voluntarily supply parts of their homes to refugee families. Exactly how many refugees a landlord can accommodate is determined by his capacity. Landlords speak at least one language and have strict preferences over the languages they speak. Refugees speak exactly one language (see Appendix A for an extension to the case with multiple languages). Formally, a refugee assignment problem consists of:

- A set of refugee families $I = \{1, \dots, |I|\}$,
- a set of landlords $C = \{c_1, \dots, c_{|C|}\}$,
- a vector $q_I = (q_1, \dots, q_{|I|})$ specifying the size q_i of each refugee family i,
- a vector $q_C = (q_{c_1}, \dots, q_{c_{|C|}})$ specifying the capacity q_c of each landlord c,
- a set L specifying the languages spoken by the refugee families and the landlords,
- a vector $(l(1), \dots, l(|I|))$ stating the spoken language $l(i)$ for each refugee family i,
- a list of strict preferences $\succeq = (\succeq_{c_1}, \dots, \succeq_{c_{|C|}})$ specifying the strict preference $\succeq_c$ over $L \cup \{c\}$ for each landlord c.

It will sometimes be convenient not to separate refugees from landlords. In this case, we refer to an agent v who belongs to the set $V = C \cup I$. For convenience, the term "refugee" is often used instead of "refugee family" and it is then understood that the refugee is part of a refugee family with a specific number of family members. Moreover, the expression "capacity of refugee i" will often be used to indicate the number of family members in refugee family i. It is also, without loss of generality, assumed that the capacity of the landlord with maximal capacity equals the number of family members of the refugee family with maximal number of family members, i.e., $\max_{c\in C} q_c = \max_{i\in I} q_i$.[19]

For later purposes, the following notation will be important for keeping track of landlords and refugees with specific capacities:

- $C^k = \{c \in C : q_c = k\}$ is the set of landlords with capacity k,
- $I^k = \{i \in I : q_i = k\}$ is the set of refugee families with k family members,
- $I_l = \{i \in I : l(i) = l\}$ is the set of refugees speaking language l,
- $I_l^k = I^k \cap I_l$ is the set of refugee families speaking language l with k family members.

All refugees with at least k family members and all landlords with at least capacity k are gathered in the sets $I^{\geq k} = \cup_{j=k}^n I^j$ and $C^{\geq k} = \cup_{j=k}^n C^j$, respectively.

2.2 Induced Preferences

The notion of acceptable languages and agents are introduced next. Given this notion and three maintained assumptions, the preferences of the refugees and the landlords can be induced from the given refugee assignment problem.

Language l is acceptable for landlord c if $l \succ_c c$. The set of acceptable languages for landlord c is denoted by $A(\succeq_c)$. Both language and capacity constraints play an important role in determining which refugees are acceptable for landlords and vice versa. More precisely, a refugee i is acceptable for landlord c if and only if refugee i speaks a language which is acceptable for landlord c and the capacity (or size) of refugee i does not exceed the capacity of landlord c, i.e., if and only if $l(i) \in A(\succeq_c)$ and $q_i \leq q_c$. By symmetry, landlord c is acceptable for refugee i if and only if refugee i is acceptable for landlord c. An agent that is not acceptable is unacceptable. The following three assumptions will be maintained in the remaining part of this paper.

Assumption 1. Landlords can accommodate at most one refugee family and if a landlord accommodates a refugee family, then the landlord has to accommodate all members of the family.

Assumption 2. Landlords can only accommodate acceptable refugee families and refugee families can only be accommodated by acceptable landlords.

[19] Otherwise, either $q_j = \max_{i\in I} q_i > \max_{c\in C} q_c$ or $\max_{i\in I} q_i < \max_{c\in C} q_c = q_{c'}$. In the first case, refugee j can never be assigned to any landlord, i.e., refugee j may be removed from the problem. For the second case, note that because only one refugee family can be assigned to landlord c' (see Assumption 1), the capacity of landlord c' may, without loss of generality, be set equal to $\max_{i\in I} q_i$.

Assumption 3. Landlords strictly prefer a larger refugee family to a smaller refugee family if both refugee families are acceptable.

The assumptions reflect that landlords should be seen as private persons that voluntarily supply parts of their homes to refugee families. It is then natural to assume that landlords are willing to accommodate at most one refugee family and that local authorities don't allow landlords to accommodate more than one family (there may be legal as well as tax reasons for this). Moreover, it is natural that landlords are able to communicate with the refugee families they accommodate and, from a humanitarian perspective, that refugee families are kept intact. Furthermore, if landlords receive a monetary compensation for accommodating refugee families that, in addition, depends on the size of the refugee family, it is natural that larger refugee families are preferred to smaller (such compensations are paid out by the local authorities to the private landlords in most of the municipalities mentioned in Footnote 15).

Given the notion of acceptability and the above three assumptions, it is possible to derive an induced preference profile R for the agents in V. Let R_c denote the induced preference relation R_c for landlord c over the set $I \cup \{c\}$. Let also P_c and I_c denote the strict and the indifference part of the preference relation R_c, respectively. The induced preference relation R_c for landlord c is described by:

- cP_ci if and only if refugee i is unacceptable,
- iP_cc if and only if refugee i is acceptable,
- iP_cj if refugees $i, j \in I$ are acceptable and $q_i > q_j$,
- iP_cj if refugees $i, j \in I$ are acceptable, $q_i = q_j$ and $l(i) \succ_c l(j)$,
- iI_cj if refugees $i, j \in I$ are acceptable, $q_i = q_j$ and $l(i) = l(j)$.

Similarly as above, let R_i denote the induced preference relation R_i for refugee i over the set $C \cup \{i\}$, and let P_i and I_i denote its strict and indifference relations, respectively. The induced preference relation R_i for refugee i is described by:

- iP_ic if and only if landlord c is unacceptable,
- cP_ii if and only if landlord c is acceptable,
- cI_ic' if landlords $c, c' \in C$ are acceptable.

Let $R = (R_v)_{v \in V}$ denote the induced (preference) profile for the agents in V. The set of all such profiles is denoted by $\mathcal{P}^V$. A profile $R \in \mathcal{P}^V$ may also be written as (R_v, R_{-v}) when the preference relation R_v of agent v is of particular importance. The following example demonstrates how preferences can be induced from the refugee assignment problem.

Example 1. Let $C = \{c_1, c_2, c_3, c_4, c_5\}$, $I = \{1, 2, 3, 4, 5, 6\}$, $q_{c_1} = q_{c_2} = q_{c_3} = q_{c_4} = 2$, $q_{c_5} = 1$, $q_1 = q_2 = q_3 = 2$, $q_4 = q_5 = q_6 = 1$, and $L = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. Suppose further that $l(i) = s_i$ for all $i \in I$, i.e., all refugees speak different languages. Let the strict preferences over acceptable languages for the landlords be given by:

$$s_1 \succ_{c_1} s_3 \succ_{c_1} c_1,$$
$$s_2 \succ_{c_2} s_4 \succ_{c_2} c_2,$$
$$s_1 \succ_{c_3} c_3,$$
$$s_5 \succ_{c_4} s_2 \succ_{c_4} c_4,$$
$$s_5 \succ_{c_5} s_6 \succ_{c_5} c_5.$$

Then the induced preference profile R over acceptable matches is given by the table below. Note that $2P_{c_4}5$ since $s_2, s_5 \in A(\succeq_{c_4})$ and $q_2 > q_5$.

R_{c_1}	R_{c_2}	R_{c_3}	R_{c_4}	R_{c_5}	R_1	R_2	R_3	R_4	R_5	R_6
1	2	1	2	5	c_1, c_3	c_2, c_4	c_1	c_2	c_4, c_5	c_5
3	4	c_3	5	6						
c_1	c_2		c_4	c_5						

2.3 Matchings and Properties of Matchings

Landlords (refugees) are either unmatched or matched to a refugee (to a landlord) under the restriction that a landlord $c \in C$ is matched to refugee $i \in I$ if and only if refugee i is matched to landlord c. Formally, a matching is a function $\mu : C \cup I \to C \cup I$ such that $\mu(c) \in I \cup \{c\}$ for all $c \in C$, $\mu(i) \in C \cup \{i\}$ for all $i \in I$, and $\mu(c) = i$ if and only if $\mu(i) = c$. Agent v is unmatched at matching μ if $\mu(v) = v$. Given a matching μ, the matched landlords and the matched refugees are collected in the sets $\mu(C) \equiv \{c \in C : \mu(c) \neq c\}$ and $\mu(I) \equiv \{i \in I : \mu(i) \neq i\}$, respectively. A matching μ is feasible at profile R if $\mu(v) R_v v$ for all $v \in V$, i.e., if each agent is matched to an acceptable agent or remains unmatched. Note that only feasible matchings are considered in this paper by Assumption 2. The set of all matchings is denoted by $\mathcal{M}$.

Let $|\mu| = \sum_{i \in \mu(I)} q_i$ denote the cardinality of matching $|\mu|$, i.e., the total number of matched refugee family members at matching μ. A matching μ is maximum at profile $R \in \mathcal{P}^V$ if there exists no other feasible matching μ' such that $|\mu'| > |\mu|$. A matching μ is stable at profile $R \in \mathcal{P}^V$ if it is feasible and there is no blocking pair, i.e., there exist no landlord-refugee pair (c, i) such that $i P_c \mu(c)$ and $c P_i \mu(i)$. Note that given the induced preferences considered in this paper, stability means that no refugee family strictly prefers some landlord to being unmatched when, at the same time, that specific landlord strictly prefers that refugee family to his current match. A stable maximum matching is a matching which is both stable and maximum.

3 Properties of Stable Maximum Matchings

This section presents a number of results related to stable and maximum matchings. These results are crucial in the construction of the Maximality-Improvement-Chains algorithm (called the MIC-algorithm, henceforth). This algorithm will be presented in the next section and it can be adopted in order to identify a stable maximum matching for any refugee assignment problem.

The first result demonstrates (using Example 1) that a stable matching is not necessarily maximum and that a maximum matching is not necessarily stable.

Proposition 1. Let μ be a stable matching and let μ' a maximum matching for some profile $R \in \mathcal{P}^V$. Then μ is not necessarily a maximum matching and μ' is not necessarily a stable matching.

Proof of Proposition 1. Consider profile R from Example 1 and the following three matchings:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ 3 & 2 & 1 & c_4 & 5 \end{pmatrix}, \mu' = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ 3 & 4 & 1 & 2 & 6 \end{pmatrix}, \mu'' = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}.$$

The interpretation of matching μ is that landlord c_1 is matched to refugee 3, landlord c_2 is matched to refugee 2, landlord c_3 is matched to refugee 1, landlord c_5 is matched to refugee 5, and landlord c_4 as well as refugees 4 and 6 are unmatched. From the profile R, it is clear that matching μ is a stable matching. It is, however, not maximum because matching μ' is feasible and has greater cardinality than μ, i.e., $|\mu'| = 8 > 7 = |\mu|$. Matching μ', on the other hand, is maximum but not stable. The latter conclusion follows because landlord c_5 and refugee 5 form a blocking pair. In the example, the unique stable maximum matching is given by μ''. $\square$

The next result establishes that a stable maximum matching exists for any profile R. The underlying reason is that preferences over languages are correlated in the following sense; if a landlord finds a certain language acceptable, then the landlord is indifferent between any two refugee families of the same size that speak this language, and any two refugee families speaking the same language are indifferent between any two acceptable landlords speaking their language.

Theorem 1. For any profile $R \in \mathcal{P}^V$, there exists a stable maximum matching.

Theorem 1 has the important consequence that one only needs to search in the set of stable matchings to find a maximum matching. The general procedure for identifying a stable maximum matching described in this paper (i.e., the MIC-algorithm) will therefore start with an arbitrary stable matching. It is straightforward to find a stable matching for any profile $R \in \mathcal{P}^V$, e.g., by arbitrarily breaking ties in R in case it contains any indifference relations, and by applying the Deferred Acceptance Algorithm (Gale and Shapley, 1962) to the resulting strict profile. The key in the MIC-algorithm is to identify specific subsets of refugees and landlords that can be rematched in such a way that the number of matched refugees increases while maintaining the feasibility and stability properties. The next example illustrates this idea.

Example 2. Consider a modification of Example 1 that contains only the agents in the sets $C^2 = \{c_1, c_2, c_3, c_4\}$ and $I^2 = \{1, 2, 3\}$ with corresponding preferences. For this problem, it is easy to check that the following matching is stable:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ 1 & 2 & c_3 & c_4 \end{pmatrix}.$$

Note that refugee 3 is unmatched at μ and speaks language s_3, language s_3 is spoken by landlord c_1, landlord c_1 is matched to refugee 1 at matching μ, refugee 1 speaks language s_1, and landlord c_3 speaks language s_1 and is unmatched at μ. This sequence of agents and languages can be illustrated as follows:

$$3 \to s_3 \to c_1 \to 1 \to s_1 \to c_3. \tag{1}$$

Note that the sequence (1) contains only two refugees and by picking out these two refugees, the chain $\langle 3, 1\rangle$ is formed. The main idea in the MIC-algorithm is to identify such chains (for a given stable but non-maximal matching) and "execute" them. The execution simply means that each refugee in the chain (except the last) is rematched to the landlord that is matched to the refugee that succeeds the refugee in the chain, and the last refugee in the chain is rematched to some unmatched landlord. In this example, the execution implies that a new matching is obtained by matching the unmatched refugee 3 to landlord c_1 and by matching refugee 1 to the unmatched landlord c_3. Such an execution is possible because, as illustrated in the above sequence (1), all rematched refugees have a language in common with the landlords that they are rematched to and no capacity constraints are violated. Note, finally, that in this process, the number of matched refugees increases while the feasibility and stability properties continue to hold. □

The insights from the above example can be generalized as revealed in the next theorem. More precisely, if a stable matching μ is not maximal, then there must exist a shortest chain of refugees $\langle i_0, i_1, \ldots, i_m\rangle$ that can be executed in such a way that the first refugee in the chain, who is unmatched at μ, becomes matched to a landlord after the execution of the chain and the last refugee in the chain (i) either becomes unmatched or (ii) rematched to a landlord that was unmatched at μ. In both cases, the cardinality of the matching increases because, in the first case, the capacity of the first refugee in the chain is strictly greater than the capacity of the last refugee in the chain and, in the second case, all agents, who were matched before the execution of the chain, remain matched after the execution.

Theorem 2. Let μ a stable matching for a given profile $R \in \mathcal{P}^V$. If μ is not a stable maximum matching, then there exists a chain of refugees $\langle i_0, i_1, \ldots, i_m\rangle$ such that $\mu(i_0) = i_0$, $\mu(i_n) = c_n$ and $i_{n-1} P_{c_n} c_n$ for all $n = 1, \ldots, m$, and either:

(i) $q_{i_m} < q_{i_0} \leq q_{i_n}$ for all $n = 1, \ldots, m-1$, and $c_{n'} P_{c_{n'}} i_n$ for all $n = 0, \ldots, m-2$ and all $n' = n+2, \ldots, m$; or

(ii) $q_{i_0} \leq q_{i_n}$ for all $n = 1, \ldots, m$, $i_m P_{c_{m+1}} c_{m+1}$ for some $c_{m+1} \in C \backslash \mu(C)$, and $c_{n'} P_{c_{n'}} i_n$ for all $n = 0, \ldots, m-2$ and all $n' = n+2, \ldots, m$.

4 The Maximality-Improvement-Chains Algorithm

This section describes the MIC-algorithm and demonstrates that it always identifies a stable maximum matching for any refugee assignment problem. In this algorithm, landlords and refugees with maximal capacity are considered first, and then the landlords and refugees with the second highest capacity, and so on. To separate landlords and refugees with different capacity from each other, the refugee assignment problem will be divided into three generic subproblems denoted by R^n, R^{n-k-1} and $R^{\geq n-k-1}$ where $k = 0, \ldots, n-2$. Because similar but different notation and techniques are required for the different subproblems, the MIC-algorithm is next stated in its generic form (even if the above mentioned subproblems not yet have been defined). The subproblems are separately described in the following three subsections. This separation of subproblems also enables a stepwise introduction of an example (based on Example 1) that illustrates all features of the MIC-algorithm.

> **The Maximality-Improvement-Chains Algorithm.** For any given profile $R \in \mathcal{P}^V$, consider problem R^n and identify a stable maximum matching μ^n. For each iteration $k = 0, \ldots, n-2$:
>
> Step (i). Find a stable maximum matching μ^{n-k-1} for problem R^{n-k-1}.
>
> Step (ii). Find a stable maximum matching $\mu^{\geq n-k-1}$ for problem $R^{\geq n-k-1} = R^{\geq n-k} \cup R^{n-k-1}$.
>
> If $k = n-2$, terminate the process. Otherwise, set $k := k+1$ and go to Step (i). $\square$

The following result is the main result of the paper and it states that the MIC-algorithm always identifies a stable maximum matching for any given refugee assignment problem. The proof of the result follows directly from the results presented in the following three subsections and the intuition is straightforward. More precisely, in the last step of the last iteration of the MIC-algorithm, problem $R^{\geq 1}$ is considered, and because all refugees and all landlords are included in this problem, by definition, the identified stable maximum matching must be a stable maximum matching for the given refugee assignment problem.

Theorem 3. For any given profile $R \in \mathcal{P}^V$, the Maximality-Improvement-Chains algorithm identifies a stable maximum matching.

4.1 Problem R^n: The Initialization of the MIC-algorithm

Problem R^n contains the refugees and the landlords with maximal capacity (i.e., the agents in the set $I^n \cup C^n$) and their preferences. As is clear from the description of the MIC-algorithm, a stable maximum matching μ^n for problem R^n is required to initialize the algorithm. This subsection describes how this matching can be identified given that a stable matching for problem R^n is known.[20] The general idea is, as explained in Example 2, to identify "chains of refugees" that satisfy the requirements in the second part of Theorem 2 and "execute" these chains. As will be demonstrated, this execution always generates a new matching with strictly increased cardinality while maintaining the feasibility and stability properties and it will not leave any previously matched agent unmatched. The process is then repeated for the new matching and after a finite number of repetitions, a stable maximum matching μ^n for problem R^n is obtained.

The key in constructing the "chains of refugees" is to make sure that agents that are rematched when a chain is "executed" are matched to agents that speak acceptable languages (because all landlords and refugees have capacity n in this problem, the capacity constraints need not be explicitly considered). To keep track of unmatched agents and their acceptable languages, a specific graph, called the n-MIC-graph, will next be constructed using an iterative process. To formalize the process, suppose that μ is a stable matching for problem R^n and let the set $\underline{L} = \{l(i) : i \in I^n \setminus \mu(I^n)\}$ contain the languages spoken by all unmatched refugees in the set I^n at matching μ, and let the set $\bar{L} = \cup_{c \in C^n \setminus \mu(C^n)} A(\succeq_c)$ contain the acceptable languages of all unmatched landlords in the set C^n at matching μ. In the n-MIC-graph, the notation $l' \to l$ is used to describe a directed edge from l' to l.

> **Construction of the n-MIC-graph for matching μ.** Let the set of nodes be given by the set of languages L. An edge is constructed as follows. Initialize $\mathcal{L}_0 = \underline{L}$. For each $u := 0, 1, \ldots$:
>
> *Iteration* $u+1$. Given $l' \in \cup_{r=0}^{u} \mathcal{L}_r$ and $l \in L \setminus (\cup_{r=0}^{u} \mathcal{L}_r)$, there is a directed edge from l' to l if $l' \in \cup_{c \in C^n : l(\mu(c)) = l} A(\succeq_c)$. Define:
>
> $$\mathcal{L}_{u+1} = \{l \in L \setminus (\cup_{r=0}^{u} \mathcal{L}_r) : l' \to l \text{ for some } l' \in \cup_{r=0}^{u} \mathcal{L}_r\}.$$
>
> If $\mathcal{L}_{u+1} \cap \bar{L} \neq \emptyset$ or $\mathcal{L}_{u+1} = \emptyset$, terminate the process. Otherwise, set $u := u + 1$ and continue. $\square$

Note that the set $\cup_{c \in C^n : l(\mu(c)) = l} A(\succeq_c)$ contains all languages that are acceptable for the landlords in C^n who are matched to a refugee speaking language l at matching μ. The following example illustrates how an edge in the graph can be constructed.

[20] A stable matching for problem R^n can be identified using the Deferred Acceptance Algorithm (Gale and Shapley, 1962) as already explained in Section 3.

Example 3. This example, as well as all remaining examples of the paper, is based on Example 1 with the only modification that refugee 6 and, consequently, language s_6 are deleted from the problem. Because the maximal capacity of any agent equals two, problem $R^n = R^2$ contains the agents in the sets $C^2 = \{c_1, c_2, c_3, c_4\}$ and $I^2 = \{1, 2, 3\}$. For this problem, matching μ from Example 2 is stable. Note next that $\underline{L} = \{s_3\}$ and $\bar{L} = \{s_1, s_2, s_5\}$. Consequently, $\mathcal{L}_0 = \underline{L} = \{s_3\}$ and $L \setminus \mathcal{L}_0 = \{s_1, s_2, s_4, s_5\}$. The construction of an edge in the 2-MIC-graph for matching μ now proceeds as follows:

> *Iteration 1.* Consider the given language $l' = s_3 \in \mathcal{L}_0$, and note that there is a directed edge from $l' = s_3 \in \mathcal{L}_0$ to $l = s_1 \in L \setminus \mathcal{L}_0$, i.e., $s_3 \rightarrow s_1$. This follows because language s_3 is acceptable for landlord c_1 and landlord c_1 is matched to a refugee that speaks language s_1 at matching μ. Consequently, $\mathcal{L}_1 = \{s_1\}$.

Because $\bar{L} = \{s_1, s_2, s_5\}$, it follows that $\mathcal{L}_1 \cap \bar{L} = \{s_1\} \neq \emptyset$ and the process terminates after iteration 1. Hence, for the given stable matching μ, the edge $s_3 \rightarrow s_1$ is identified in the 2-MIC-graph. □

What may not be so obvious is that the construction of the n-MIC-graph is the backbone in the process of identifying a stable maximum matching for problem R^n. To see this, note that a more general description of an edge in the n-MIC-graph is a chain of length m from $\underline{L}$ to $\bar{L}$ such that $l_0 \in \underline{L}$, $l_{m-1} \in \bar{L}$ and $l_u \in \mathcal{L}_u$ for all $u = 1, \ldots, m-1$, i.e.:

$$l_0 \rightarrow l_1 \rightarrow \cdots \rightarrow l_{m-1}. \tag{2}$$

As is demonstrated in the following proposition, the existence of this type of chain, in any stable matching, means that the matching cannot be a stable maximum matching.

Proposition 2. Let μ be a stable matching for problem R^n and $\mathcal{L}_\infty = \cup_{u=0}^{\infty} \mathcal{L}_u$. Then the following are equivalent for n-MIC-graph:

(a) μ is not a maximum matching for problem R^n.

(b) $\mathcal{L}_\infty \cap \bar{L} \neq \emptyset$.

(c) There exists a chain $l_0 \rightarrow l_1 \rightarrow \cdots \rightarrow l_{m-1}$ from $\underline{L}$ to $\bar{L}$ such that $l_0 \in \underline{L}$, $l_{m-1} \in \bar{L}$ and $l_u \in \mathcal{L}_u$ for all $u = 1, \ldots, m-1$.

Proposition 2 reveals that matching μ from Example 2 cannot be a stable maximum matching since $\mathcal{L}_\infty \cap \bar{L} = \{s_1\} \neq \emptyset$. However, this has already been pointed out in Example 2 but the construction of the n-MIC-graph and Proposition 2 combined give a general procedure for determining whether or not a stable matching in addition is maximum. Recall also from Example 2 that languages s_1 and s_3, identified in the sequence (1), enabled the "execution" of the chain of refugees $\langle 3, 1 \rangle$ in the example. Because the construction of the n-MIC-graph is a general method for identifying the relevant languages, the next step is to find a general process for constructing and executing the relevant chains of refugees.

Execution of a chain of length m from $\underline{L}$ to $\bar{L}$ for problem R^n. Consider a given stable matching μ, a given problem R^n, and a given chain of type (2) of length m from $\underline{L}$ to $\bar{L}$ in the n-MIC-graph for matching μ. Choose $i_0 \in I^n \setminus \mu(I^n)$ such that $l(i_0) = l_0$. For each iteration $u := 1, \ldots, m-2$:

Iteration u. Choose $i_u \in \mu(I^n)$ such that $l(i_u) = l_u$ and $l_{u-1} \in A(\succeq_{\mu(i_u)})$.

Iteration $m-1$. Choose $\hat{c} \in C^n \setminus \mu(C^n)$ such that $l_{m-1} \in A(\succeq_{\hat{c}})$.

Matching μ' is then obtained from matching μ as follows: $\mu'(i_u) = \mu(i_{u+1})$ for all $u = 0, \ldots, m-2$, $\mu'(i_{m-1}) = \hat{c}$, and $\mu'(i) = \mu(i)$ for all $i \in I^n \setminus \{i_0, \ldots, i_{m-1}\}$. □

The above process produces a chain of refugees of type $\langle i_0, \ldots, i_{m-1} \rangle$ and a new matching μ' is obtained from the initial matching μ by executing the chain, i.e., each refugee in the chain (except the last) is matched to the landlord which, at matching μ, is matched to the refugee that succeeds the refugee in the chain, and the last refugee in the chain is matched to a landlord which is unmatched at μ. Note also that as long as there is an edge in the n-MIC-graph, a chain of type $\langle i_0, \ldots, i_{m-1} \rangle$ exists.[21] Furthermore, the cardinality of the new matching μ' is strictly greater than the cardinality of the initial matching μ but the feasibility and stability properties are maintained, and all refugees that are matched at μ are also matched at μ'. The latter conclusion follows because $\mu'(I^n) = \mu(I^n) \cup \{i_0\}$ by construction.

The above insights can be applied in the following way. Construct the n-MIC-graph for the given stable matching μ and check whether $\mathcal{L}_\infty \cap \bar{L} \neq \emptyset$. If $\mathcal{L}_\infty \cap \bar{L} = \emptyset$, then by Proposition 2, μ is a stable maximum matching for the problem R^n, and the MIC-algorithm can continue to iteration 0. If not, there exists a chain of type (2) and this chain can be executed to obtain a new stable matching μ' from the initial matching μ. Given matching μ', the procedure is repeated and because the cardinality of the matching strictly increases for each execution, a stable maximum matching μ^n for problem R^n is obtained after a finite number of repetitions.

Example 4. This example is a continuation of Example 3 and it demonstrates how the edge $s_3 \to s_1$ in the 2-MIC-graph can be executed. Because $s_3 \to s_1$, it is clear that $l_0 = s_3 \in \underline{L}$ and $l_1 = s_1 \in \bar{L}$. Since refugee 3 is unmatched at μ and speaks language $l_0 = s_3$, it follows that $i_0 = 3$. The execution of the chain $s_3 \to s_1$ now proceeds as follows:

Iteration 1. Choose $i_1 = 1$ because $l(1) = l_1 = s_1$, $\mu(1) = c_1$ and $l_0 = s_3 \in A(\succeq_{c_1})$.

Iteration 2. Choose $\hat{c} = c_3$ because $l_1 = s_1 \in A(\succeq_{c_3})$.

[21] This follows from the construction of the n-MIC-graph. That is, refugee i_0 exists because $l_0 \in \underline{L} \neq \emptyset$, the agents specified in iterations $1, \ldots, m-2$ exist because $l_{u-1} \to l_u$ for all $u \in \{1, \ldots, m-2\}$, and landlord $\hat{c}$ exists because $l_{m-1} \in \bar{L} \neq \emptyset$.

Hence, $\mu'(3) = \mu'(i_0) = \mu(i_1) = \mu(1) = c_1$ and $\mu'(1) = \mu'(i_1) = \hat{c} = c_3$. Consequently, matching μ' is given by:

$$\mu' = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ 3 & 2 & 1 & c_4 \end{pmatrix}.$$

Note that $\mu'(I^2) = \mu(I^2) \cup \{3\}$ and that the matching μ' is a stable maximum matching for problem R^2. The latter conclusion follows from Proposition 2 because $\mathcal{L}_\infty = \emptyset$ and $\bar{L} = \{s_2, s_5\}$ at matching μ' and, consequently, $\mathcal{L}_\infty \cap \bar{L} = \emptyset$. Hence, $\mu^2 = \mu'$. □

The stable maximum matching μ^n for problem R^n will also be denoted by $\mu^{\geq n}$. Once this matching has been identified, the MIC-algorithm continues with iteration 0 where a stable maximum matching $\mu^{\geq n-1}$ for problem $R^{\geq n-1}$ is identified. As is clear from the description of the algorithm, this iteration contains two separate steps, called Steps (i) and (ii). Because these separate steps can be repeated for any iteration $k = 0, \ldots, n-2$, the following two subsections describe these steps in their generic form, i.e., for an arbitrary iteration $k = 0, \ldots, n-2$.

4.2 Problem R^{n-k-1}: Generic Step (i) of the MIC-algorithm

Problem R^{n-k-1} is generically almost identical to problem R^n. The essential difference is that the former problem does not only contain landlords with capacity $n-k-1$ but also all landlords who remained unmatched after the preceding iterations of the MIC-algorithm (if such landlords exist). More precisely, let the matching identified in the proceeding iteration of the MIC-algorithm be denoted by $\mu^{\geq n-k}$. Then problem R^{n-k-1} contains refugee families with $n-k-1$ family members, landlords with capacity $n-k-1$, all unmatched landlords at matching $\mu^{\geq n-k}$, and the preferences of these agents.

Given problem R^{n-k-1} and matching $\mu^{\geq n-k}$, the $(n-k-1)$-MIC-graph can now be constructed in the same fashion as the n-MIC-graph but for the above sets of agents. Consequently, the relevant chains of refugees can be identified and executed by using identical techniques as for problem R^n. Recall also that Proposition 2 demonstrated that these techniques in fact always generate a stable maximum matching for problem R^n. This result holds also for problem R^{n-k-1} even if the result not is formally stated in this paper (the proof follows directly by changing some notation in the proof of Proposition 2).

Example 5. Consider Examples 3 and 4, and note that $R^{\geq 2} = R^2$, $\mu' = \mu^2 = \mu^{\geq 2}$ and $R^{n-k-1} = R^1$. The latter follows since $n = 2$ and $k = 0$. Problem R^1 is defined by $I^1 = \{4, 5\}$ and $C^{\geq 1} \setminus \{\mu^{\geq 2}(C^{\geq 2})\} = \{c_4, c_5\}$. In this example, the 1-MIC-graph need not be constructed as the only refugee in I^1 which is acceptable for the landlords in $C^{\geq 1}$ is refugee 5, i.e., refugee 5 is acceptable for both landlords c_4 and c_5. Consequently, the following matching is a stable maximum matching for problem R^1:

$$\mu^1 = \begin{pmatrix} c_4 & c_5 \\ c_4 & 5 \end{pmatrix}.$$

□

After a stable maximum matching μ^{n-k-1} for problem R^{n-k-1} has been identified, the MIC-algorithm continues to its generic Step (ii) for the given iteration k. This step is necessary because the already identified matchings $\mu^{\geq n-k}$ and μ^{n-k-1} are obtained almost separate from each other (even if there may be some overlap, i.e., a landlord may be included problems $R^{\geq n-k}$ and R^{n-k-1}). Hence, even if $\mu^{\geq n-k}$ and μ^{n-k-1} are stable maximum matchings for their corresponding problems, the union of these matchings need not be a stable maximum matching for the union of their corresponding problems. This can also seen in the following example.

Example 6. Consider Examples 3–5 and matchings $\mu^{\geq 2}$ and μ^1, and note that:

$$\mu'' = \mu^{\geq 2} \cup \mu^1 = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ 3 & 2 & 1 & c_4 & 5 \end{pmatrix}.$$

But μ'' is not a stable maximum matching. This is clear because the following matching is stable and its cardinality is strictly greater than the cardinality of matching μ'':

$$\mu^* = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}. \qquad \square$$

4.3 Problem $R^{\geq n-k-1}$: Generic Step (ii) of the MIC-algorithm

The objective of Step (ii) in the MIC-algorithm is to find a stable maximum matching $\mu^{\geq n-k-1}$ for problem $R^{\geq n-k-1}$. Because this problem is defined as the union of the problems $R^{\geq n-k}$ and R^{n-k-1}, it contains all agents that have been included in the previous iterations and steps of the MIC-algorithm. Hence, problem $R^{\geq n-k-1}$ contains all refugees with at least $n-k-1$ family members, all landlords with at least capacity $n-k-1$, and the preferences of these agents, i.e., the agents in the set $I^{\geq n-k-1} \cup C^{\geq n-k-1}$ and their preferences.

A stable maximum matching for problem $\mu^{\geq n-k-1}$ is found by, again, constructing a specific graph, called the ($\geq n-k-1$)-MIC-graph, for the known matching $\mu = \mu^{\geq n-k} \cup \mu^{n-k-1}$, and by executing the relevant chains. This graph needs to be constructed differently compared to the corresponding graphs for problems R^n and R^{n-k-1} because the rematching procedure (or, equivalently, the execution of the relevant chains of refugees) differs since a rematched landlord may be assigned a refugee with fewer family members than the initial match. This is not possible in problems R^n and R^{n-k-1} because rematched landlords in these problems are, by construction, always rematched to a refugee family with a weakly larger family size compared to the initial match. For this reason, it is not only important to keep track of the languages spoken by the refugees and landlords, it is also important to keep track of what is "demanded" and what can be "supplied" of these languages in terms of capacity. Consider, for instance, matching μ'' from Example 5. At this matching, refugee family 4 and landlord c_4 are unmatched. Because refugee family 4 have one family member and speaks language s_3, language s_3 "demands" capacity 1. Similarly, since landlord c_4 speaks languages s_2 and s_5 and have the capacity to accommodate a refugee family of size 1 or size 2, languages s_2 and s_5 can "supply" capacity 1 and capacity 2, respectively.

To keep track of what is demanded and what can be supplied of the languages in terms of capacity, the language notation needs to be augmented. Let l^t denote a language $l \in L$ that demands capacity t and/or supplies capacity t. Language l^t is then said to be a capacity-augmented language. The set of all capacity-augmented languages for problem $R^{\geq n-k-1}$ is denoted by:

$$L^a = \{l^t : l \in L \text{ and } t \geq n-k-1\}.$$

To construct the $(\geq n-k-1)$-MIC-graph, it will now be important to identify the set of capacity-augmented languages of all unmatched refugees and the set of acceptable capacity-augmented languages for all unmatched landlords, denoted by $\underline{L}$ and $\bar{L}$, respectively. These sets are formally defined as:

$$\begin{aligned} \underline{L} &= \{l^t : \text{ there exists } \hat{i} \in I^{\geq n-k-1} \backslash \mu(I^{\geq n-k-1}) \text{ such that } l(\hat{i}) = l \text{ and } q_{\hat{i}} = t\}, \\ \bar{L} &= \{l^t : \text{ there exists } \hat{c} \in C^{\geq n-k-1} \backslash \mu(C^{\geq n-k-1}) \text{ such that } l \in A(\succeq_{\hat{c}}) \text{ and } q_{\hat{c}} \geq t\}. \end{aligned}$$

The $(\geq n-k-1)$-MIC-graph can now be constructed as follows for the given matching μ.

Construction of the $(\geq n-k-1)$-MIC-graph for matching $\mu = \mu^{\geq n-k} \cup \mu^{n-k-1}$. Let the set of nodes be given by the set of capacity-augmented languages L^a. The edges are constructed as follows. Initialize $\mathcal{L}_0 = \underline{L}$. For each iteration $u := 0, 1, \ldots$:

Iteration $u+1$. Given $\hat{l}^{\hat{t}} \in \cup_{r=0}^{u} \mathcal{L}_r$ and $l^t \in L \backslash (\cup_{r=0}^{u} \mathcal{L}_r)$ there is a directed edge from $\hat{l}^{\hat{t}}$ to l^t if $\hat{l} \in \cup_{i \in \mu(I_l^t): q_{\mu(i)} \geq \hat{t}} A(\succeq_{\mu(i)})$. Define:

$$\mathcal{L}_{u+1} = \{l^t \in L^a \backslash (\cup_{r=0}^{u} \mathcal{L}_r) : \hat{l}^{\hat{t}} \to l^t \text{ for some } \hat{l}^{\hat{t}} \in \cup_{r=0}^{u} \mathcal{L}_r\}.$$

If $\mathcal{L}_{u+1} \cap \bar{L} \neq \emptyset$ or $\mathcal{L}_{u+1} = \emptyset$, terminate the process. Otherwise, set $u := u+1$ and continue. □

Note that the set $\cup_{i \in \mu(I_l^t): q_{\mu(i)} \geq \hat{t}} A(\succeq_{\mu(i)})$ contains all languages that are acceptable for the landlords in $C^{\geq n-k-1}$ that are matched to a refugee with capacity t and speaking language l at matching μ. A more general description of an edge in the $(\geq n-k-1)$-MIC-graph is a chain of length m from $\underline{L}$ to $\bar{L}$ such that $l_0^{t_0} \in \underline{L}$, $l_{m-1}^{t_j} \in \bar{L}$, and $l_u^{t_u} \in \mathcal{L}_u$ for all $u = 1, \ldots, m-1$, i.e.:

$$l_0^{t_0} \to l_1^{t_1} \to \cdots \to l_{m-1}^{t_{m-1}}. \tag{3}$$

Proposition 3. Let μ be a stable matching for problem $R^{\geq n-k-1}$ and $\mathcal{L}_\infty = \cup_{u=0}^{\infty} \mathcal{L}_u$. Then the following are equivalent for the $(\geq n-k-1)$-MIC-algorithm:

(a) μ is not a maximum matching for problem $R^{\geq n-k-1}$.

(b) $\mathcal{L}_\infty \cap \bar{L} \neq \emptyset$.

(c) There exists a chain $l_0^{t_0} \to l_1^{t_1} \to \cdots \to l_{m-1}^{t_{m-1}}$ from $\underline{L}$ to $\bar{L}$ such that $l_0^{t_0} \in \underline{L}$, $l_{m-1}^{t_{m-1}} \in \bar{L}$, and $l_u^{t_u} \in \mathcal{L}_u$ for all $u = 1, \ldots, m-1$.

Using an almost identical procedure as the one described for problem R^n, a chain of type (3) can be executed as described next.

Execution of a chain of length m from $\underline{L}$ to $\bar{L}$ for problem $R^{\geq n-k-1}$. Consider a given stable matching $\mu = \mu^{\geq n-k} \cup \mu^{n-k-1}$, a given problem $R^{\geq n-k-1}$, and a given chain of type (3) of length m from $\underline{L}$ to $\bar{L}$ in the $(\geq n-k-1)$-MIC-graph for matching μ. Choose $i_0 \in I_{l_0}^{t_0} \backslash \mu(I_{l_0}^{t_0})$ such that $l(i_0) = l_0$. For each $u := 1, \ldots, m-2$:

Iteration u. Choose $i_u \in \mu(I_{l_u}^{t_u})$ such that $l(i_u) = l_u$, $l_{u-1} \in A(\succeq_{\mu(i_u)})$, and $q_{\mu(i_u)} \geq t_{u-1}$.

Iteration $m-1$. Choose $\hat{c} \in C^{\geq n-k-1} \backslash \mu(C^{\geq n-k-1})$ such that $l_{m-1} \in A(\succeq_{\hat{c}})$ and $q_{\hat{c}} \geq t_{m-1}$.

From matching μ, matching μ' is obtained as follows: $\mu'(i_u) = \mu(i_{u+1})$ for all $u = 0, \ldots, m-2$, $\mu'(i_{m-1}) = \hat{c}$, and $\mu'(i) = \mu(i)$ for all $i \in I^{\geq n-k-1} \backslash \{i_0, \ldots, i_{m-1}\}$. $\square$

Note that any execution of a chain from $\underline{L}$ to $\bar{L}$ strictly increases the cardinality of the initial matching while maintaining stability, and it does not leave any previously matched agent unmatched. The latter conclusion is important since it implies that one need not return to problem $R^{\geq n-k}$. Moreover, the findings in this subsection can be used as follows. Construct the $(\geq n-k-1)$-MIC-graph for the given stable matching μ and check whether $\mathcal{L}_\infty \cap \bar{L} \neq \emptyset$. If $\mathcal{L}_\infty \cap \bar{L} = \emptyset$, then matching μ is a stable maximum matching for problem R^{n-k-1} by Proposition 3, and the algorithm can continue to iteration $k+1$. Otherwise, there is a chain of type (3) and this chain can be executed to obtain a new stable matching μ' from the initial matching μ. Given matching μ', the procedure is repeated and after a finite number of iterations, stable maximum matching $\mu^{\geq n-k-1}$ for problem $R^{\geq n-k-1}$ is obtained. This is in fact also the proof of Theorem 3. Too see this, note that in the last step of the last iteration of the MIC-algorithm (i.e., Step (ii) of iteration $n-2$), all refugees and all landlords are included in problem $R^{\geq 1}$ by construction. Then because a stable maximum matching is identified for problem $R^{\geq 1}$, it must also be a stable maximum matching for the given refugee assignment problem.

Example 7. This example builds on Examples 3–6. Note first that $I^{\geq 1} = I^1 \cup I^2 = \{1,2,3\} \cup \{4,5\} = I$ and $C^{\geq 1} = C^1 \cup C^2 = \{1,2,3,4\} \cup \{5\} = C$. Moreover, because 2 is the maximal capacity of any agent in problem $R^{\geq 1}$, it follows that $L^a = L^1 \cup L^2$ where $L^t = \{s_1^t, \ldots, s_5^t\}$ for $t = 1, 2$. Recall now from Example 6 that only landlord c_4 and refugee 4 are unmatched at $\mu'' = \mu^{\geq 2} \cup \mu^1$. Because refugee 4 only speaks language s_4 and only has one family member, it follows $\underline{L} = \{s_4^1\}$. Furthermore, landlord c_4 speaks both languages s_2 and s_5 and has the capacity to accommodate refugee families of size 1 and 2. Hence, $\bar{L} = \{s_2^1, s_2^2, s_5^1, s_5^2\}$.

Start by constructing an edge in the (≥ 1)-MIC-graph for matching μ''. Set $\mathcal{L}_0 = \underline{L} = \{s_4^1\}$ and, consequently, $s_2^2 \in L^a \backslash \mathcal{L}_0 = L^a \backslash \{s_4^1\}$. The construction of an edge in the (≥ 1)-MIC-graph for matching μ'' now proceeds as follows:

> *Iteration 1.* Consider the given language $l' = s_4^1 \in \mathcal{L}_0$, and note that there is a directed edge from $l' = s_4^1 \in \mathcal{L}_0$ to $l = s_2^2 \in L^a \setminus \mathcal{L}_0$, i.e., $s_4^1 \to s_2^2$. This follows because language s_4 is acceptable to landlord c_2, landlord c_2 is matched to refugee 2 at matching μ'', and refugee 2 has capacity 2 and speaks language s_2. Consequently, $\mathcal{L}_1 = \{s_2^2\}$.

Because $s_2^2 \in \bar{L}$, it follows that $\mathcal{L}_1 \cap \bar{L} = \{s_2^2\} \neq \emptyset$ and the process terminates. Execute next the identified chain $s_4^1 \to s_2^2$. Note first that $l_0 = s_4^1 \in \underline{L}$ and $l_1 = s_2^2 \in \bar{L}$. Since refugee 4 is unmatched at μ'' and speaks language $l_0 = s_4$, it follows that $i_0 = 4$. Furthermore:

> *Iteration 1.* Choose $i_1 = 2$ because $l(2) = s_2$ and $q_2 = 2$, $\mu''(2) = c_2$, and $l_0 = s_4 \in A(\succeq_{c_2})$.
>
> *Iteration 2.* Choose $\hat{c} = c_4$ because $l_1 = s_2 \in A(\succeq_{c_4})$ and $q_2 \leq q_{c_4}$.

Now, matching μ''' is given by:

$$\begin{aligned} \mu'''(4) &= \mu'''(i_0) = \mu''(i_1) = \mu''(2) = c_2, \\ \mu'''(2) &= \mu'''(i_1) = \hat{c} = c_4, \\ \mu'''(j) &= \mu''(j) \text{ for } j \in \{1, 3, 5\}. \end{aligned}$$

Hence:

$$\mu''' = \mu^* = \mu^{\geq 1} = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}.$$

That matching $\mu^{\geq 1}$ is a stable maximum matching for the considered refugee assignment problem follows from Proposition 3 since $\bar{L} = \emptyset$ at matching $\mu^{\geq 1}$. □

Remark 1. A stable maximum matching can be identified in polynomial time for any refugee assignment problem. To see this, note that in each iteration of the MIC-algorithm, (a) a stable matching needs to be identified, (b) edges in the MIC-graph needs to be constructed, and (c) chains need to be executed. Part (a) can be achieved in polynomial time using the Deferred Acceptance Algorithm (Gale and Shapley, 1962). Part (b) can also be achieved in polynomial time because at most $2n + |C|$ edges must be identified in total and each edge can be identified in polynomial time using methods introduced by Ford and Fulkerson (1956). Finally, we note that the relevant chains in part (c) are implicitly identified in (b).

To see that at most $2n + |C|$ edges must be identified in total, note that an edge in the MIC-graph is constructed either to (b.1) verify that a matching is a stable maximum matching or to (b.2) execute a chain from $\underline{L}$ to $\bar{L}$ to increase the cardinality of the matching. A verification of type (b.1) is needed for Steps (i) and (ii) in each iteration k, i.e., in total $2n$ times since the MIC-algorithm contains exactly n iterations (including the initialization step). An execution of type (b.2) can take place at most $|C|$ times. This follows because only chains of type (ii) from Theorem 2 are executed, since exactly one unmatched landlord becomes matched after each execution, and because a stable maximum matching is obtained if all landlords are matched to some refugee family (i.e., if $\bar{L} = \emptyset$). □

Remark 2. Finally, we remark that if the matching of the MIC-algorithm is not (Pareto) efficient, then the matching $\mu^{\geq 1}$ can be used as endowments for the landlords and the Top Trading Cycles Algorithm (Shapley and Scarf, 1974) can be applied in a setting where landlords exchange refugee families of equal size. Then, the cardinality of the matching can never decrease and the resulting matching is stable because any landlord only points to refugees speaking one of its acceptable languages. □

5 Strategy-Proofness

When designing matching mechanisms, it is always a concern that agents may be able manipulate the outcome of the mechanism in their advantage (see, e.g., Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2009; Balinski and Sönmez, 1999; Dubins and Freedman, 1981; Roth, 1982; Roth et al., 2004). Unfortunately, it is known that there exists no matching mechanism on two-sided matching markets that produces an outcome which is stable and at the same time give the agents on both sides of the market incentives to truthfully report their preferences (Roth, 1982). As can be expected from this result, most of the results related manipulability presented in this section are negative. Note also that refugee families only report their family size and the language they speak and it is not very unrealistic to believe that both these variables are difficult to misrepresent as they easily can be verified. However, landlords have some degree of freedom as they can rank languages they speak (refugees can also be allowed to report multiple languages, see Appendix A) and it is therefore important to investigate strategic properties related to the refugee assignment problem.

A matching mechanism is, in this section, described as a function $f : \mathcal{P}^V \to \mathcal{M}$ choosing a feasible matching for any profile $R \in \mathcal{P}^V$. Let also $f_v(R)$ denote the match for agent v at matching $f(R)$. The mechanism f is a maximum matching mechanism if $f(R)$ is a maximum matching for any profile $R \in \mathcal{P}^V$, and the mechanism f is a stable matching mechanism if $f(R)$ is a stable matching for any $R \in \mathcal{P}^V$. A matching mechanism is strategy-proof if no agent can gain by misrepresenting their true preferences (say by reporting $R_{v'}$ instead of the true preferences R_v) at any profile given the reports of the other agents. Formally, matching mechanism f is strategy-proof if for all $R \in \mathcal{P}^V$ and all $v \in V$, there exists no $(R'_v, R_{-v}) \in \mathcal{P}^V$ such that $f_v(R'_v, R_{-v}) P_v f_v(R)$. Here, the profile resulting from unilateral deviation by agent v must still belong to the set of our problems (in the sense that c may misreport its ranking over languages and i may misreport its spoken language).[22]

Proposition 4. There is no maximum matching mechanism which is strategy-proof for the landlords.

Because any stable maximum matching mechanism is a maximum matching mechanism, Proposition 4 implies that there does not exist any stable maximum matching mechanism which is strategy-proof for the landlords in the refugee assignment problem. While this may not be entirely surprising, it is next demonstrated that there exist a stable mechanism which is strategy-proof for the landlords but no stable mechanism which is strategy-proof for both landlords and refugees. The mechanism which is strategy-proof for the landlords always selects a C-optimal matching. Here, a matching μ is C-optimal if it is stable and there exists no other stable matching μ' such that $\mu'(c) R_c \mu(c)$ for all $c \in C$ and $\mu'(c) P_c \mu(c)$ for some $c \in C$. To demonstrate these results let, without loss of generality, $q_{c_1} \geq q_{c_2} \geq \cdots \geq q_{c_{|C|}}$, and let t stand for breaking ties according to the natural order $>_t$, i.e., $c_1 >_t c_2 >_t \cdots >_t c_{|C|}$ for $C = \{c_1, \ldots, c_{|C|}\}$ and $i_1 >_t i_2 >_t \cdots >_t i_{|I|}$ for $I = \{i_1, \ldots, i_{|I|}\}$. Let $DA^t(R)$ denote the landlord-proposing Deferred Acceptance Algorithm where, for any profile R, ties are broken according to t.

Theorem 4. Let $R \in \mathcal{P}^V$. Then:

(a) $DA^t(R)$ always produces a C-optimal matching.

(b) $DA^t(R)$ is strategy-proof for C.

(c) No stable mechanism is strategy-proof for both landlords and refugees.

Note also that the last part of the theorem is related to the corresponding strategy-proofness result for the marriage market (Gale and Shapley, 1962) where it is known that a mechanism that selects the male-optimal (female-optimal) stable matching is generally strategy-proof for the males (females) but not for the females (males). See, e.g., Roth (1982) or Dubins and Freedman (1981).

[22] This paper does not consider the possibility for groups of agents to manipulate the mechanism. See Barberà et al. (2016) for the a recent paper on the relation between individual manipulability and group manipulability.

6 Concluding Remarks

This paper is one of the first to investigate a matching model related to refugee resettlement and refugee assignment. In fact, we are only aware of a few other matching contributions investigating this specific problem which have been cited in the Introduction.

The point of departure has been the European refugee crisis during 2015–2016 and, more specifically, the acute problem to find housing for refugees in Sweden. Even if the presented matching model is stylized, in the sense that landlords and refugees only are allowed to submit limited information related to ability to communicate (i.e., language) and capacity (i.e., number of beds available/needed), the model is still relevant from a policy perspective because the MIC-algorithm is easy-to-implement and it can be adopted as is without any modifications. The MIC-algorithm can therefore be seen as a first emergency measure to release pressure on the municipalities in their attempts to find additional residential units. Even if it has not been discussed in the paper, one can also imagine that preferences are induced based on other criteria than communication (e.g., geographical preferences) or a combination of several variables (see, e.g., Delacretaz et al., 2016). Hence, this paper should be seen as a first step to solve an acute problem but future research is needed to find alternative proposals. It is, however, clear that the investigated problem is on the highest political agenda in all member states of the European Union and it is therefore crucial that the market design community continues to investigate problems related to refugee matching and refugee resettlement.

The paper has also contributed to the matching and market design literature in a broader sense since it has provided a new type of matching mechanism for two-sided matching markets. This mechanism is not limited to the refugee assignment problem and the techniques developed in the paper can be applied to any problem where agents rank indifference classes of agents rather than individual agents and where stable maximum matchings are of importance.

Finally, the key in the method to identify a stable maximum matching, considered in this paper, is the construction of the edges in the MIC-graph. A possible alternative method is to consider a bipartite graph whose vertices are given by the disjoint sets C and I, and where an edge connects landlord $c \in C$ and refugee $i \in I$ if and only if landlord c is acceptable to refugee i and refugee i is acceptable to landlord c. By assigning specific edge weights, based on preferences over languages and family size, a stable maximum matching may then be identified in polynomial time by adopting the Hungarian method of Kuhn (1955) and Munkres (1957). The appropriate construction of such problem and an investigation of the properties of the solution to the problem is left for future research.

Appendix A: Multiple Languages for Refugees

Suppose now that refugees are allowed to speak multiple languages. This case can be covered by the framework investigated in this paper as explained in this Appendix.

Suppose now that each refugee $i \in I$ is allowed to speak a subset of languages $L(i) \subseteq L$. This translates to the "best" language for any landlord c in the following way. Let $l_c(i) = \max_{\succeq_c} L(i) \cup \{\emptyset\}$ be the most preferred spoken language of refugee i from the perspective of landlord c. Then refugee i is acceptable for landlord c if and only if $l_c(i) \in A(\succeq_c)$ and $q_i \leq q_c$. Now, when defining the preference R_c of landlord c, language $l(i)$ is simply replaced by language $l_c(i)$.

Similarly, the preferences R_i of refugee i is derived as follows: (a) $cP_i i \Leftrightarrow l_c(i) \in A(\succeq_c)$ and $q_i \leq q_c$ and (b) $cI_i c' \Leftrightarrow l_c(i) \in A(\succeq_c)$ and $q_i \leq q_c$, and $l_{c'}(i) \in A(\succeq_{c'})$ and $q_i \leq q_{c'}$. Note here that (b) means that a refugee is indifferent between any acceptable landlords that have the "best" language in common with the refugee.

Let $R = (R_v)_{v \in V}$ denote the derived profile. Then it is easy to check that all results presented in this paper holds also for this profile.

Appendix B: Proofs

For some of the coming results, it will be necessary to introduce the notion of tie-breaking. For this purpose, let $\mathcal{R}^V$ denote the set of all profiles including those which do not necessarily respect language constraints. A profile $R \in \mathcal{R}^V$ is strict if R_v is strict for all $v \in V$. Consider now the profiles $R^t, R \in \mathcal{R}^V$ and the matching μ. A profile R^t breaks ties in R if and only if (i) for all landlords $c \in C$ and all distinct agents $v, v' \in I \cup \{c\}$, $vP_c v'$ implies $vP_c^t v'$, and (ii) for all refugees $i \in I$ and all distinct agents $v, v' \in C \cup \{i\}$, $vP_i v'$ implies $vP_i^t v'$. A profile R^t breaks ties in R in favor of μ if and only if (i) R^t breaks ties in R, (ii) for any landlord $c \in C$ and agent $v \in I \cup \{c\}$ with $v \neq \mu(c)$, $\mu(c) R_c v$ implies $\mu(c) P_c^t v$, and (iii) for any landlord $i \in C$ and agent $v \in C \cup \{i\}$ with $v \neq \mu(i)$, $\mu(i) R_i v$ implies $\mu(i) P_i^t v$. The first result in this Appendix relates the notions of stability and tie-breaking.

Lemma 1. Let μ be a matching and $R \in \mathcal{R}^V$. Then matching μ is stable at profile R if and only if there exists a strict profile $R^t \in \mathcal{R}^V$ that breaks ties in R such that μ is stable under R^t.

Proof. Suppose that the matching μ is stable at profile $R \in \mathcal{R}^V$, and let R^t be a strict profile that breaks ties in R in favor of μ. Then, the matching μ is stable at profile R^t by construction.

In showing the converse, let R^t be a strict profile that breaks ties in R. Suppose that μ is stable under R^t. Then μ is feasible under R. Furthermore, if for some (c, i) we have both $iP_c\mu(c)$ and $cP_i\mu(i)$, then we have both $iP_c^t\mu(c)$ and $cP_i^t\mu(i)$ since R^t breaks ties in R, contradicting stability of matching μ at profile R^t. □

Proof of Theorem 1. Let μ be a maximum matching and let R^t be a strict profile breaking ties in R in favor of μ. If μ is stable under R^t, then by Lemma 1, μ is a stable maximum matching and the proof is completed. Otherwise, there exists a blocking pair (c, i) for matching μ and profile R^t. Note now that it cannot be the case that $\mu(i) = i$ and $\mu(c) = c$, because then μ cannot be a maximum matching. Moreover, it cannot be the case that $\mu(i) \neq i$ and $\mu(c) = c$ since refugee i then cannot strictly gain from blocking by construction of the preferences. Hence, it must be the case that $\mu(c) \neq c$. Since (c, i) is a blocking pair, it now follows that either:

- $q_{\mu(c)} < q_i \leq q_c$ and $l(i) \in A(\succeq_c)$, or;
- $q_{\mu(c)} = q_i \leq q_c$ and $l(i) \succ_c l(\mu(c))$.

Consider now matching μ' where $\mu'(c) = i$, $\mu'(\mu(c)) = \mu(c)$, and $\mu'(v) = \mu(v)$ for all $v \in V \setminus \{c, i, \mu(c)\}$. Then $|\mu'| \geq |\mu|$ and μ' is a maximum matching. Let R'^t be a strict profile breaking ties in R in favor of μ'. If μ' is stable under R'^t, then the proof is again completed by Lemma 1. Otherwise we continue as above and improve the landlords' rankings and, at some point, a stable maximum matching must be found. □

Proof of Theorem 2. Let μ be a stable matching for profile R, and let R^k be a strict profile that breaks ties in R in favor of μ. Consequently, μ is a stable matching also for the strict profile R^k. Suppose now that μ is not a maximum matching. Then there exists a feasible matching μ' such that $|\mu'| > |\mu|$. Without loss of generality, the matching μ' may be chosen such that the number $|\{v \in V : \mu(v) \neq \mu'(v)\}|$ is minimal among all feasible matchings having strictly greater cardinality than μ.

Note first that there exists a refugee $i \in I$ such that $\mu(i) = i$ and $\mu'(i) \neq i$. This follows because if $\mu(i) = i$ implies $\mu'(i) = i$ for all unmatched refugees at matching μ, then $|\mu'| \leq |\mu|$ which contradicts the assumption that $|\mu'| > |\mu|$. Suppose now that $\mu(i) = i$ and $\mu'(i) = c$ for some refugee $i \in I$ and some landlord $c \in C$. Then, by feasibility of matching μ', it follows that $l(i) \succ_c c$ and $q_i \leq q_c$. Because matching μ is stable at profile R^k, it must be the case that $\mu(c) P_c^k i$. It will next be demonstrated that also $\mu(c) P_c i$.

To see that also $\mu(c) P_c i$, note first that, by stability of matching μ, the case $i P_c \mu(c)$ can be excluded. Suppose instead that $\mu(c) I_c i$ and let $\mu(c) = j$. This also means that $q_j = q_i$ and $l(i) = l(j)$. But then instead of considering matching μ', we may ignore refugee j and consider the matching μ'' where $\mu''(i) = \mu'(j)$ and $\mu''(i') = \mu'(i')$ for all $i' \in I \setminus \{j\}$. Then $|\mu''| = |\mu'| > |\mu|$ but $|\{v \in V : \mu(v) \neq \mu''(v)\}| < |\{v \in V : \mu(v) \neq \mu'(v)\}|$, which contradicts the choice of matching μ'. Thus, it must be the case that $j P_c i$ or, equivalently, that $\mu(c) P_c i$. This also means that either $q_j > q_i$, or $q_j = q_i$ and $l(j) \succ_c l(i)$.

Because $\mu(i) = i$, $\mu'(i) = c$, $\mu(c) = j$, there exists a sequence of refugees $\langle i_0, \ldots, i_m \rangle$ such that $\mu(i_0) = i_0$ and $\mu(i_n) = c_n$ for all $n = 1, \ldots, m$, where $\mu'(i_n) = \mu(i_{n+1}) = c_{n+1}$ for all $n = 0, 1, \ldots, m-1$ and either (a) $\mu'(i_m) = i_m$ or (b) $\mu'(i_m) = c_{m+1}$ and $\mu(c_{m+1}) = c_{m+1}$. Note also that $i_{n-1} P_{c_n} c_n$ for all $n = 0, 1, \ldots, m$ since matching μ' is feasible. By the choice of matching μ', we may, without loss of generality, assume that the sequence $\langle i_0, \ldots, i_m \rangle$ is both unique and shortest. But then $c_{n'} P_{c_{n'}} i_n$ for all $n = 0, \ldots, m-2$ and all $n' = n+2, \ldots, m$, because otherwise refugee i_n may be assigned to landlord $c_{n'}$ to obtain a shorter sequence $\langle i_0, \ldots, i_n, i_{n'}, \ldots, i_m \rangle$. Cases (a) and (b) are next considered separately.

Case (a) will prove Part (i) of the theorem. That $q_{i_m} < q_{i_0}$ follows directly since $\mu'(i_m) = i_m$ and $|\mu'| > |\mu|$. To see that also $q_{i_0} \leq q_{i_n}$ for all $n = 1, \ldots, m-1$, suppose that $q_{i_n} < q_{i_0}$ for some n. But then it is possible to choose a shorter sequence of refugees $\langle i_0, \ldots, i_n \rangle$ leaving refugee i_n unmatched and at the same time increase the cardinality of matching μ. It then follows that $l(i_0) \notin A(\succeq_{c_n})$ for all $n = 2, \ldots, m$, otherwise it is again possible to consider a shorter sequence of refugees $\langle i_0, i_n, i_{n+1}, \ldots, i_m \rangle$ and obtain a matching with strictly greater cardinality, which is a contradiction. Hence, $c_{n'} P_{c_{n'}} i_0$ for all $n' = 2, \ldots, m$. Similarly, refugee i_1 cannot be matched to any of the landlords $c_3, c_4, \ldots, c_m$ because if this is the case, it is again possible to consider a shorter sequence. Thus, $l(i_1) \notin A(\succeq_{c_n})$ or $q_{i_1} > q_{c_n}$ for all $n = 3, \ldots, m$. By repeating the arguments for the remaining refugees in the chain, it follows that $c_{n'} P_{c_{n'}} i_n$ for all $n = 0, \ldots, m-2$ and all $n' = n+2, \ldots, m$.

Case (b) will prove Part (ii) of the theorem. Note first that because μ' is a feasible matching, $\mu'(i_m) = c_{m+1}$ and $\mu(c_{m+1}) = c_{m+1}$, it follows that $i_m P_{c_{m+1}} c_{m+1}$ and $c_{m+1} \in C \backslash \mu(C)$. Similarly as above, because there is no shorter sequence of refugees than $\langle i_0, \ldots i_m \rangle$, it must hold that $c_{n'} P_{c_{n'}} i_n$ for all $n = 0, 1, \ldots, m-2$ and all $n' = n+2, \ldots, m$. □

Proof of Proposition 2. (a)⇔(c). It is only shown that (a) implies (c) since the proof that (c) implies (a) is analogous. If μ is not a stable maximum matching, then by Theorem 2, there exists a shortest chain of refugees $\langle i_0, i_1, \ldots, i_j \rangle$ such that $\mu(i_0) = i_0$, $\mu(i_u) = c_u$ and $i_{u-1} P_{c_u} c_u$ for all $u = 1, \ldots, j$, and either part (i) or part (ii) of the theorem holds. Because all refugees have capacity n, it cannot be the case that $q_{i_0} < q_{i_j}$ and, consequently, part (i) of the theorem cannot hold. Suppose instead that part (ii) holds and consider the n-MIC-graph for matching μ and a shortest chain $\langle i_0, i_1, \ldots, i_j \rangle$. Then, by construction of the (shortest) chain, $l(i_0) \in \underline{L} = \mathcal{L}_0$, $l(i_0) \to l(i_1)$, $l(i_1) \in \mathcal{L}_1$, $l(i_u) \to l(i_{u+1})$ for all $u = 1, \ldots, j-1$, and $l(i_j) \in \bar{L}$. Because $\langle i_0, i_1, \ldots, i_j \rangle$ is a shortest chain, it is clear that $l(i_1) \in L \setminus \mathcal{L}_0$ and $l(i_{u+1}) \in L \setminus \cup_{r=0}^{u} \mathcal{L}_r$ for all $u = 1, \ldots, j-1$. But then $l_u \equiv l(i_u) \in \mathcal{L}_u$ for all $u = 1, \ldots, j-1$ by definition of the set $\mathcal{L}_u$. Hence, (a) implies (c).

(c)⇔(b). This equivalence follows immediately from the construction of the n-MIC-graph for matching μ. □

Proof of Proposition 3. The proof is analogous to the proof of Proposition 3. (a)⇔(c). Again, it is only demonstrated that (a) implies (c) since the proof that (c) implies (a) is analogous. If μ is not a stable maximum matching, then by Theorem 2, there exists a shortest chain $\langle i_0, i_1, \ldots, i_j \rangle$ such that $\mu(i_0) = i_0$, $\mu(i_u) = c_u$ and $i_{u-1} P_{c_u} c_u$ for all $u = 1, \ldots, j$, and either part (i) or part (ii) of the theorem holds. Suppose first that part (i) holds, i.e., that $q_{i_j} < q_{i_0} \leq q_{i_u}$ for all $u = 1, \ldots, j-1$. If $q_{i_j} \geq n-k$, then matching $\mu^{\geq n-k}$ cannot be a maximum matching since the chain $\langle i_0, i_1, \ldots, i_j \rangle$ then is present also at problem $R^{\geq n-k}$. Thus, $q_{i_j} = n-k-1$. But then $q_{i_u} \geq q_{i_0} \geq n-k$ for all $u = 1, \ldots, j-1$, and it, again, follows that matching $\mu^{\geq n-k}$ cannot be maximum for problem $R^{\geq n-k}$. Thus, only part (ii) of the theorem can occur. Because $\mu^{\geq n-k}$ is a maximum matching for problem $R^{\geq n-k}$, it must be the case that $q_{i_0} = n-k-1$ and $q_{i_u} \geq n-k$ for some $u = 1, \ldots, j$.

Consider now the ($\geq n-k-1$)-MIC-graph for matching μ and a shortest chain $\langle i_0, i_1, \ldots, i_j \rangle$. Then, by construction of the chain, $l(i_0)^{q_{i_0}} \in \underline{L} = \mathcal{L}_0$, $l(i_0)^{q_{i_0}} \to l(i_1)^{q_{i_1}}$, $l(i_1)^{q_{i_1}} \in \mathcal{L}_1$, $l(i_u)^{q_{i_u}} \to l(i_{u+1})^{q_{i_{u+1}}}$ for all $u = 1, \ldots, j-1$, and $l(i_j)^{q_{i_j}} \in \bar{L}$. Because $\langle i_0, i_1, \ldots, i_j \rangle$ is a shortest chain, it is clear that $l(i_1)^{q_{i_1}} \in L^a \setminus \mathcal{L}_0$ and $l(i_{u+1})^{q_{i_{u+1}}} \in L^a \setminus \cup_{r=0}^{u} \mathcal{L}_r$ for all $u = 1, \ldots, j-1$. But then, by setting $t_u = q_{i_u}$ for all $u = 1, \ldots, j-1$, it follows that $l_u^t \equiv l(i_u)^{t_u} \in \mathcal{L}_u$ for all $u = 1, \ldots, j-1$ by definition of the set $\mathcal{L}_u$. Hence, (a) implies (c).

(c)⇔(b). This equivalence is immediate from the construction of the ($\geq n-k-1$)-MIC-graph for matching μ. □

Proof of Proposition 4. Let $C = \{c_1, c_2, c_3\}$, $I = \{1, 2, 3\}$ and $q_v = 1$ for all $v \in V$, $L = \{s_1, s_2, s_3\}$, and $l(i) = s_i$ for all $i \in I$. The strict preferences over acceptable languages for the landlords are given by $s_1 \succ_{c_1} s_3 \succ_{c_1} c_1$, $s_2 \succ_{c_2} s_3 \succ_{c_2} c_2$, and $s_2 \succ_{c_3} s_1 \succ_{c_3} c_3$. The induced preference profile R is then given by:

R_{c_1}	R_{c_2}	R_{c_3}	R_1	R_2	R_3
1	2	2	c_1, c_3	c_2, c_3	c_1, c_2
3	3	1			

Then matchings μ and μ' are the only maximum matchings at profile R where:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 \\ 1 & 3 & 2 \end{pmatrix}, \mu' = \begin{pmatrix} c_1 & c_2 & c_3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Because $f(R)$ is a maximum matching, it must be the case that $f_{c_1}(R) = 3$ or $f_{c_2}(R) = 3$. These cases are next considered separately.

Suppose first that $f_{c_1}(R) = 3$. In this case, landlord c_1 can manipulate by misrepresenting preferences over acceptable languages $\succeq'_{c_1}$ such that $s_1 \succ'_{c_1} c_1$. In this case, μ is the unique maximum matching for profile (R'_{c_1}, R_{-c_1}) and, consequently, $f(R'_{c_1}, R_{-c_1}) = \mu$. But then $f_{c_1}(R'_{c_1}, R_{-c_1}) P_{c_1} f_{c_1}(R)$, which means that landlord c_1 has a profitable deviation from R.

Suppose next that $f_{c_2}(R) = 3$. In this case, landlord c_2 can manipulate by misrepresenting preferences over acceptable languages $\succeq'_{c_2}$ such that $s_2 \succeq'_{c_2} c_2$. In this case, μ' is the unique maximum matching for (R'_{c_2}, R_{-c_2}) and, consequently, $f(R'_{c_2}, R_{-c_2}) = \mu'$. But then $f_{c_2}(R'_{c_2}, R_{-c_2}) P_{c_2} f_{c_2}(R)$, which means that landlord c_2 has a profitable deviation from R. □

Proof of Theorem 4. To show part (a), suppose that matching μ is selected by $DA^t(R)$ but that matching μ not is a C-optimal matching. Then there exists a stable matching μ' such that $\mu'(c) R_c \mu(c)$ for all $c \in C$ with $\mu'(\hat{c}) P_{\hat{c}} \mu(\hat{c})$ for some landlord $\hat{c}$. By stability of matching μ, there is no refugee $i \in I$ such that $\mu(i) = i$, $\mu'(i) \neq i$ and $i P_{\mu'(i)} \mu(\mu'(i))$. Thus, if $\mu(i) = i$ and $\mu'(i) \neq i$, then $i I_{\mu'(i)} \mu(\mu'(i))$. But then, for $\mu(\mu'(i)) = j$, we may suppose, without loss of generality, that $\mu(i) = \mu'(i)$ and $\mu(j) = j$ as this does not change the welfare of $\mu'(i)$ and does not alter stability. But now matching μ must admit a stable improvement cycle à la Erdil and Ergin (2008), say $\langle c_1, \ldots, c_m \rangle$, i.e, $\mu(c_{k+1}) P_{c_k} \mu(c_k)$ for all k. Because each refugee speaks exactly one language, ties are broken according to $>_t$ and each landlord c_k applies to $\mu(c_{k+1})$ before applying to $\mu(c_k)$, this implies that $c_{k+1} >_t c_k$ for all k, and $c_1 >_t c_m$. But then $c_m >_t c_{m-1} >_t \cdots >_t c_1 >_t c_m$, which is a contradiction to the fact that the same ordering $>_t$ is used to break ties in refugees preferences.

Part (b) of the theorem follows directly because the Deferred Acceptance algorithm is strategy-proof for C on the strict domain and landlords break ties in favor of larger capacities. Hence, it is clear that $DA^t(R)$ is strategy-proof for C for all profiles $R \in \mathcal{P}^V$.

To prove part (c), suppose that $C = \{c_1\}$, $I = \{1, 2\}$, $q_v = 1$ for all v, and $L = \{s_1, s_2, s_3\}$. Let the strict preferences over acceptable languages $\succeq_{c_1}$ for landlord c_1 be given by $s_1 \succ_{c_1} s_2 \succ_{c_1} s_3 \succ_{c_1} c_1$. Suppose further that refugees 1 and 2 speaks languages s_2 and s_3, respectively. Then by stability, refugee 1 is assigned to landlord c_1 and refugee 2 remains unmatched. But if refugee 2 changes his language from s_3 to s_1, then refugee 2 is assigned to landlord c_1. Hence, even if the stable mechanism is strategy-proof for the landlords, it can be manipulated by the refugees. □

References

Abdulkadiroğlu, A., Sönmez, T., 2003. "School choice – A mechanism design approach," Amer. Econ. Rev. 93, 729–747.

Abdulkadiroğlu, A., Pathak, P., Roth, A.E., 2009. "Strategy-proofness versus efficiency in matching with indifferences: Redesigning the NYC high school match," Amer. Econ. Rev. 99, 1954–1978.

Barberà, S., Berga, D., Moreno, B., 2016. "Group strategy-proofness in private good economies," Amer. Econ. Rev. 106, 1073–99.

Balinski, M., Sönmez, T., 1999. "A tale of two mechanisms: Student placement," J. Econ. Theory 84, 73–94.

Budish, E., Cantillon, E., 2012. "The multi-unit assignment problem: Theory and evidence from course allocation at Harvard," Amer. Econ. Rev. 102, 2237–2271.

Delacretaz, D., Kominers, S., Teytelboym, A., 2016. "Refugee resettlement," Working Paper.

Dubins, L., Freedman, D., 1981. "Machievelli and the Gale-Shapley algorithm," Amer. Math. Monthly 88, 485–494.

Echenique, F., Oviedo, J., 2006. "A theory of stability in many-to-many matching markets," Theoretical Econ. 1, 233–273.

Erdil, A., Ergin, H., 2008. "What's the matter with tie-breaking? Improving efficiency in School Choice," Amer. Econ. Rev. 98, 669–689.

Ford, L.R., Fulkerson, D.R. (1956). Maximal flow through a network. Canadian J. Math. 8, 299–404.

Gale, D., Shapley, L., 1962. “College admissions and the stability of marriage,” Amer. Math. Monthly 69, 9–15.

Jones, W., Teytelboym, A., 2016a. “The refugee match: A system that respects refugees’ preferences and the priorities of states,” Working Paper.

Jones, W., Teytelboym, A., 2016b. “The local refugee match: Aligning refugees’ preferences with the capacities and priorities of localities,” Working Paper.

Konishi, H., Ünver, M.U., 2006. “Credible group-stability in many-to-many matching problems,” J. Econ. Theory 129, 57–80.

Kuhn, H.W., 1955. “The Hungarian method for the assignment problem,” Naval Research Logistics Quarterly 2, 83–97.

Munkres, J., 1957. “Algorithms for the assignment and transportation problems,” Journal of the Society for Industrial and Applied Mathematics 5, 32–38.

Roth, A.E., 1982. “The economics of matching: Stability and incentives,” Math. Oper. Research 7, 617–628.

Roth, A.E., 2008.“Deferred acceptance algorithms: History, theory, practice, and open questions,” Int. J. Game Theory 36, 537–569.

Roth, A.E., Peranson, E., 1999. “The redesign of the matching market for American physicians: Some engineering aspects of economic design,” Amer. Econ. Rev. 89, 748–82.

Roth, A.E., Sönmez, T., Ünver, U., 2004. “Kidney exchange,” Quart. J. Econ. 119, 457–488.

Roth, A.E., Sotomayor, M., 1990. “Two-sided matching: A study in game-theoretic modeling and analysis,” Econometric Society Monograph, Cambridge: Cambridge University Press.

Shapley, L., Scarf, H., 1974. “On cores and indivisibility,” J. Math. Econ. 1, 23–38.

Sönmez, T., 2013. “Bidding for army career specialties: Improving the ROTC branching mechanism,” J. Polit. Econ. 121, 186–219.

Sönmez, T., and Switzer, T., 2013. “Matching with (branch-of-choice) contracts at the United States military academy,” Econometrica 81, 451–488.

Sönmez, T., Ünver, U., 2011. “Matching, allocation, and exchange of discrete resources,” in Jess Benhabib, Alberto Bisin, and Matthew O. Jackson (eds.), Handbook of Social Economics, Vol. 1A, pp. 781–852, The Netherlands: North-Holland.