**Université de Montréal**

# Géométrie nodale et valeurs propres de l'opérateur de Laplace et du $p$-laplacien

par

**Guillaume Poliquin**

Département de mathématiques et de statistique
Faculté des arts et des sciences

Thèse présentée à la Faculté des études supérieures
en vue de l'obtention du grade de
Philosophiæ Doctor (Ph.D.)
en mathématiques

Septembre 2015

# Université de Montréal

Faculté des études supérieures


Cette thèse intitulée

# Géométrie nodale et valeurs propres de l'opérateur de Laplace et du $p$-laplacien

présentée par


## Guillaume Poliquin


a été évaluée par un jury composé des personnes suivantes :

*Marlène Frigon*
(président-rapporteur)

*Iosif Polterovich*
(directeur de recherche)

*Vestislav Apostolov*
(membre du jury)

*Dorin Bucur*
(examinateur externe)

*William J. Mccausland*
(représentant du doyen de la FAS)


Thèse acceptée le
*6 novembre 2015*

# RÉSUMÉ

La présente thèse porte sur différentes questions émanant de la géométrie spectrale. Ce domaine des mathématiques fondamentales a pour objet d'établir des liens entre la géométrie et le spectre d'une variété riemannienne. Le spectre d'une variété compacte fermée $M$ munie d'une métrique riemannienne $g$ associée à l'opérateur de Laplace-Beltrami $\Delta_g$ est une suite de nombres non négatifs,

$$0 = \lambda_0 < \lambda_1(M, g) \leq \lambda_2(M, g) \leq \ldots \nearrow \infty,$$

où la racine carrée de ces derniers représente une fréquence de vibration de la variété. À chaque valeur propre $\lambda = \lambda(M, g)$ du spectre, une fonction propre $u_\lambda$ telle que $\Delta_g u_\lambda = \lambda u_\lambda$ est associée.

Cette thèse présente quatre articles touchant divers aspects de la géométrie spectrale. Le premier article, présenté au Chapitre 1 et intitulé *Superlevel sets and nodal extrema of Laplace eigenfunctions* [**80**], porte sur la géométrie nodale d'opérateurs elliptiques. L'objectif de mes travaux a été de généraliser un résultat de L. Polterovich et de M. Sodin dans [**83**] qui établit une borne sur la distribution des extrema nodaux sur une surface riemannienne pour une assez vaste classe de fonctions, incluant, entre autres, les fonctions propres associées à l'opérateur de Laplace-Beltrami. La preuve fournie dans [**83**] n'étant valable que pour les surfaces riemanniennes, je prouve dans [**80**] une approche indépendante pour les fonctions propres de $\Delta_g$ dans le cas des variétés riemanniennes de dimension arbitraire.

Les deuxième et troisième articles traitent d'un autre opérateur elliptique, le $p$-laplacien. Sa particularité réside dans le fait qu'il est non linéaire. Au Chapitre 2, l'article *Principal frequency of the p-laplacian and the inradius of Euclidean*

*domains* [**79**] se penche sur l'étude de bornes inférieures sur la première valeur propre du problème de Dirichlet du $p$-laplacien en termes du rayon inscrit d'un domaine euclidien. Plus particulièrement, je prouve que, si $p$ est supérieur à la dimension du domaine, il est possible d'établir une borne inférieure sans aucune hypothèse sur la topologie de ce dernier. L'étude de telles bornes a fait l'objet de nombreux articles par des chercheurs connus, tels que W. K. Haymann, E. Lieb, R. Banuelos et T. Carroll, principalement pour le cas de l'opérateur de Laplace. L'adaptation de ce type de bornes au cas du $p$-laplacien est abordée dans mon troisième article, *Bounds on the Principal Frequency of the p-Laplacian* [**78**], présenté au Chapitre 3 de cet ouvrage.

Mon quatrième article, *Wolf-Keller theorem for Neumann Eigenvalues* [**81**], est le fruit d'une collaboration avec Guillaume Roy-Fortin. Le thème central de ce travail gravite autour de l'optimisation de formes dans le contexte du problème aux valeurs limites de Neumann. Le résultat principal de [**81**] est que les valeurs propres de Neumann ne sont pas toujours maximisées par l'union disjointe de disques arbitraires pour les domaines planaires d'aire fixée. Le tout est présenté au Chapitre 4 de cette thèse.

Mots clés : Opérateur de Laplace-Beltrami, $p$-laplacien, géométrie nodale, conditions aux limites de Dirichlet et de Neumann, fonctions propres, valeurs propres, rayon inscrit, optimisation de formes.

# ABSTRACT

The main topic of the present thesis is spectral geometry. This area of mathematics is concerned with establishing links between the geometry of a Riemannian manifold and its spectrum. The spectrum of a closed Riemannian manifold $M$ equipped with a Riemannian metric $g$ associated with the Laplace-Beltrami operator $\Delta_g$ is a sequence of non-negative numbers,

$$0 = \lambda_0 < \lambda_1(M, g) \leq \lambda_2(M, g) \leq \ldots \nearrow \infty.$$

The square root of any number of this sequence represents a frequency of vibration of the manifold. Every eigenvalue $\lambda = \lambda(M, g)$ of the spectrum is associated to an eigenfunction $u_\lambda$ such that $\Delta_g u_\lambda = \lambda u_\lambda$.

This thesis consists of four articles all related to various aspects of spectral geometry. The first paper, *Superlevel sets and nodal extrema of Laplace eigenfunction* [**80**], is presented in Chapter 1. Nodal geometry of various elliptic operators, such as the Laplace-Beltrami operator, is studied. The goal of this paper is to generalize a result due to L. Polterovich and M. Sodin in [**83**] that gives a bound on the distribution of nodal extrema on a Riemann surface for a large class of functions, including eigenfunctions of the Laplace-Beltrami operator. The proof given in [**83**] is only valid for Riemann surfaces. Therefore, in [**80**], I present a different approach to the problem that works for eigenfunctions of the Laplace-Beltrami operator on Riemannian manifolds of arbitrary dimension.

The second and the third papers of this thesis are focused on a different elliptic operator, namely the $p$-Laplacian. This operator has the particularity of being non-linear. The article *Principal frequency of the p-Laplacian and the inradius of Euclidean domains* [**79**] is presented in Chapter 2. It discusses lower bounds on

the first eigenvalue of the Dirichlet eigenvalue problem for the $p$-Laplace operator in terms of the inner radius of the domain. In particular, I show that if $p$ is greater than the dimension, then it is possible to prove such lower bound without any hypothesis on the topology of the domain. Such bounds have previously been studied by well-known mathematicians, such as W. K. Haymann, E. Lieb, R. Banuelos, and T. Carroll. Their papers are mostly oriented toward the case of the usual Laplace operator. The generalization of such lower bounds for the $p$-Laplacian is done in my third paper, *Bounds on the Principal Frequency of the p-Laplacian* [**78**]. It is presented in Chapter 3.

My fourth paper, *Wolf-Keller theorem of Neumann Eigenvalues* [**81**], is a joint work with Guillaume Roy-Fortin. This paper is concerned with the shape optimization problem in the case of the Laplace operator with Neumann boundary conditions. The main result of [**81**] is that eigenvalues of the Neumann boundary problem are not always maximized by disks among planar domains of given area. This joint work is presented in Chapter 4.

Keywords : Laplace-Beltrami operator, $p$-Laplacian, nodal geometry, Dirichlet and Neumann boundary conditions, eigenfunctions, eigenvalues, inradius, shape optimization.

# TABLE DES MATIÈRES

# LISTE DES FIGURES

# LISTE DES TABLEAUX

# REMERCIEMENTS

Je veux d'abord remercier Iosif Polterovich. Il m'a présenté un sujet de recherche avec tant de passion que je n'ai pu faire autrement que d'en devenir moi-même passionné. Il a aussi cru en moi en me proposant de passer directement du baccalauréat au doctorat et m'encourage avec ferveur depuis que j'ai fait ce choix. Il voue tant d'intensité et de dévouement à ses recherches que de le voir travailler me donne le plein d'énergie pour attaquer les miennes avec rigueur. À travers les années, j'ai appris à connaître le chercheur, mais également l'homme et c'est ce dernier qui m'a sans doute le plus impressionné et appris. Un très gros merci, Iosif.

Je tiens à souligner l'apport considérable que Marlène Frigon m'a apporté tout au long de mon parcours à l'Université de Montréal. Ses conseils toujours justes et son dévouement légendaire m'ont permis de faire les bons choix tout au long de mon cheminement. Il m'est impossible de passer sous silence son enseignement rigoureux, clair et concis qui m'a permis d'aborder avec aplomb les questions à saveur analytique de mes recherches. Je tiens également à la remercier de m'avoir fait confiance à mes débuts en me confiant la démonstration du cours d'analyse II. Ma reconnaissance à cet égard lui est infinie. Merci, Marlène.

Ceux qui me connaissent savent à quel point l'enseignement des mathématiques est une passion pour moi. Je me souviens très clairement du premier groupe que l'on m'a confié, un groupe de 150 étudiants en algèbre linéaire. Cette chance, je l'ai eue grâce à Yvan Saint-Aubin. Il a été un mentor pour moi en ce qui concerne l'enseignement. Méticuleux, perfectionniste, créatif, passionné, engagé sont tous des qualificatifs du style d'enseignant qu'est Yvan. Tes conseils, je ne les oublierai pas. Merci, Yvan.

Je dois aussi beaucoup à Dorin Bucur. Un des articles au coeur de cette thèse est basé sur une observation qu'il m'a transmise à la suite d'une présentation à Neuchâtel. Merci, Dorin.

Ma thèse, elle représente beaucoup de choses à mes yeux. C'est normal : sa conception s'étire sur cinq belles années de ma vie. Ces cinq années, en plus de passer beaucoup trop rapidement, m'ont donné l'occasion de rencontrer plusieurs personnes qui sont devenues des amis très chers à mes yeux. Je tiens donc à remercier tout spécialement Janine Bachrachas, Laurence Boulanger, Yaiza Canzani, Layan El Hajj, Patrick Munroe, Helène Péloquin-Tessier, Guillaume Roy-Fortin, Anik Soulière et Yasmine Tawfik. Je remercie aussi Charles Baillargeon d'avoir été là pour me changer les idées lorsque j'en avais besoin. À vous tous, je vous dis merci !

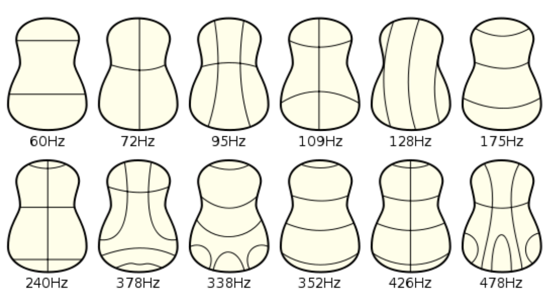
Je termine en temoignant ma gratitude à mes proches, Alexandre Ethier, Suzanne Gélinas, Arya, Gabrielle et Gilbert Poliquin. Je me considère très chanceux de les avoir à mes côtés, notamment parce qu'ils m'ont offert leur support indéfectible, leurs encouragements et leur écoute. Je tiens à leur dire que ce n'est pas passé inaperçu et que je leur en suis très reconnaissant.

# INTRODUCTION

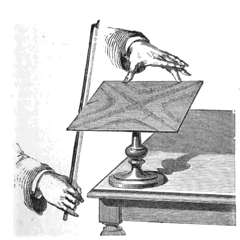
## Le contexte historique entourant la géométrie spectrale

Une des premières apparitions de travaux rigoureux sur des phénomènes liés aux vibrations de plaques ou de membranes remonte à une expérience introduite par Robert Hooke en 1680. Ce dernier a remarqué qu'en faisant vibrer de son archet une plaque couverte de farine, il était possible d'apercevoir des formes étranges dans la farine à certaines fréquences bien précises. Ce n'est qu'environ 100 ans plus tard que cette découverte a été reprise par Georg Christoph Lichtenberg. Elle est ensuite remontée aux oreilles d'Ernst Chladni quelques années plus tard. Ce dernier s'est alors amusé à faire des démonstrations publiques de ce qu'on appelle aujourd'hui, en son honneur, les figures acoustiques de Chladni.

C'était lors d'une soirée de 1809, à Paris, à la résidence officielle de Napoléon Bonaparte, que l'étude plus rigoureuse de ce phénomène est entamée. Napoléon, fort impressionné par la démonstration d'Ersnt Chladni, a décidé d'octroyer une




(a) Figures acoustiques



(b) Expérience de Chladni

Figure 0.1. Images liées à E. Chladni provenant de [**34**].

bourse de 6000 francs à quiconque capable d'expliquer rigoureusement ce phénomène. C'était l'Académie de Paris qui a obtenu le mandat de chapeauter le concours. C'était en quelque sorte une des premières formes de subvention à la recherche octroyée à un chercheur. Par ailleurs, le jury chargé d'évaluer les différentes réponses était, entre autres, composé de Denis Poisson et de Joseph-Louis Lagrange.

Cela n'a pris non pas un, ni deux, mais bien trois essais à une dénommée Sophie Germain pour se mériter le prix. Soulignons au passage qu'il est exceptionnel qu'une femme ait pu obtenir un tel prix, car, à cette époque, les femmes n'étaient point admises à l'Académie. Par conséquent, c'était dans son troisième rapport, soumis le 8 janvier 1816 et intitulé *Recherches sur la théorie des surfaces élastiques*, qu'elle est parvenue à expliquer certaines facettes du phénomène. Il a fallu encore quelques ajustements avant de parvenir à une modélisation cohérente et complète. Pour en savoir davantage sur Sophie Germain, il est possible de consulter [**87**].

Par la suite, en 1877, le baron de Rayleigh, John William Strutt troisième du nom, dans son livre *The Theory of Sound* [**84**], s'est penché sur l'étude des vibrations d'une membrane fixée à ses extrémités (conditions aux limites dites de Dirichlet) et, plus particulièrement, sur les liens entre la forme géométrique d'un tambour et les vibrations qu'il produit. L'ensemble de ces fréquences constituent en fait le spectre d'un opérateur, ici l'ensemble des valeurs propres de l'opérateur de Laplace sous les conditions au bord de Dirichlet. De tels problèmes, appelés «problèmes aux valeurs et aux fonctions propres», sont fondamentaux dans plusieurs champs mathématiques, comme lors de la résolution d'équations aux dérivées partielles ou encore au niveau du problème de Sturm-Liouville. Le baron a émis l'hypothèse que, pour ce qui est de la première fréquence fondamentale d'une membrane attachée à ses extrémités, le disque était la forme permettant d'émettre une fréquence minimale, hypothèse dont la véracité a été démontrée par G. Faber et par E. Krahn de manière indépendante dans les années 1920 (consulter [**35, 56, 57**] pour les articles originaux).

Le développement de toutes formes de bornes inférieures pour la fréquence fondamentale a toujours constitué un défi pour la communauté mathématique, ce qui rendait les résultats de Faber et de Krahn très inspirants. S'en est alors suivie une série de découvertes du même acabit, notamment celles visant à établir une borne inférieure en terme du rayon inscrit de la membrane vibrante, c'est-à-dire du rayon du plus grand disque qu'il est possible d'insérer à l'intérieur de la membrane sans dépasser ses limites. Autrement dit, il s'agissait de savoir si, pour qu'un tambour émette une note arbitrairement basse, il était nécessaire qu'il soit possible de tracer un disque arbitrairement grand sur sa peau. Cela a été prouvé par divers mathématiciens, dont W. K. Haymann et E. Makai dans [**43, 65**]. Cette question a ensuite été transposée à plusieurs contextes, notamment à celui des tambours munis de membranes non élastiques. Diverses approches adaptées à ce cas particulier sont présentées dans le Chapitre 2 et le Chapitre 3.

Lorsque nous faisons vibrer un tambour à une fréquence fondamentale, les formes acoustiques de Chladni nous montrent qu'en fait, la peau de tambour est divisée en un certain nombre de sous-ensembles qui vibrent et qui ont comme frontière une ligne le long de laquelle la membrane ne vibre pas. La Figure 0.2 illustre un exemple de ce qui se passe sur une plaque carrée.

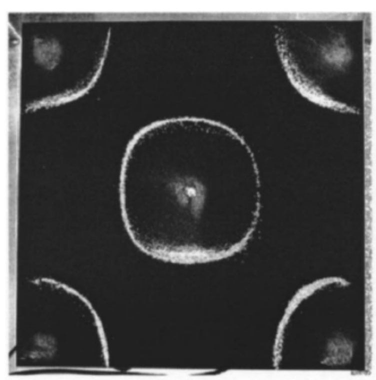


Figure 0.2. Décomposition nodale, image provenant de [**21**].

Sur cette plaque, il est possible de voir six régions fermées, appelées domaines nodaux, où la plaque vibre. Les lignes délimitant ces régions sont appelées lignes nodales et représentent les endroits où la plaque ne bouge pas. Le sable placée sur la plaque vibrante cherche donc à se disperser le long de ces lignes, ce qui nous permet de visualiser les formes acoustiques de Chladni. L'étude mathématique des propriétés des domaines nodaux et des lignes nodales est la branche de la géométrie spectrale appelée la géométrie nodale.

Remarquons également qu'au centre de certaines de ces régions, nous pouvons voir une concentration de poussière. Cette dernière étant très fine et volatile, elle est davantage sensible à la pression de l'air qu'aux vibrations de la plaque. En fait, elle s'accumule là où la pression de l'air est minimale, ce qui se produit aux emplacements où l'amplitude des vibrations est maximale. Dans l'article [**21**], il est expliqué comment il est possible d'observer simultanément les domaines nodaux et les endroits où l'amplitude est maximale sur une plaque vibrante. L'étude de certaines des propriétés de ces zones est traitée dans le Chapitre 1 de cette thèse.

Une tout autre avenue d'étude en théorie spectrale est provenue du fameux article de M. Kac *Can one hear the shape of a drum ?* [**51**] publié en 1966. M. Kac n'arrivait pas à savoir si connaître l'ensemble des fréquences fondamentales d'une membrane vibrante était suffisant pour en déterminer sa forme. Autrement dit, etait-il possible que deux formes de tambours différents aient les mêmes fréquences fondamentales ? Tenter d'expliciter la nature des liens entre la forme de l'objet qui vibre et certains sons émis par le tambour est précisément un des aspects centraux de la géométrie spectrale.

Les résultats issus de l'étude de toutes les questions susmentionnées proviennent surtout du cas des membranes attachées, alors que ce n'est que relativement récemment que le phénomène des membranes libres (conditions à la frontière dites de Neumann) a mené à quelques percées. Dans [**93**], G. Szegö, pendant les années 1950, démontre par son inégalité que, de tous les domaines planaires simplement connexes d'aire fixée, le disque constitue la forme permettant de maximiser la première valeur propre non nulle de Neumann. En 2009, A. Girouard, N. Nadirashvili et I. Polterovich ont démontré que, parmi les domaines simplement connexes d'aire fixée, l'union disjointe de deux disques identiques maximise la seconde valeur propre non nulle du problème de Neumann (voir [**37, 38**]). Il est donc naturel de se demander si les domaines maximiseurs pour les valeurs propres supérieures à la deuxième sont toujours des disques ou des unions de disques. Sinon, quels sont les domaines maximiseurs pour ces mêmes valeurs ? C'est à cette question que tente de répondre le Chapitre 4 de cette thèse.

## L'opérateur de Laplace-Beltrami, ses fonctions et ses valeurs propres

Considérons $(M^N, g)$ une variété riemannienne $N-$ dimensionnelle, lisse, connexe et compacte, que nous supposons orientée si $M$ possède une frontière. Considérons également $\Delta_g : C^\infty(M) \to C^\infty(M)$, l'opérateur de Laplace-Beltrami sur $M$ défini par $\Delta_g u = -\operatorname{div}_g(\nabla_g u)$. En coordonnées locales $\{x_i\}_{i=1}^N$, il s'écrit

$$\Delta_g = \frac{-1}{\sqrt{det(g)}} \sum_{i,j=1}^N \frac{\partial}{\partial x_i}(\sqrt{det(g)} g^{ij} \frac{\partial}{\partial x_j}),$$

où la matrice $(g^{ij})$ est la matrice inverse de $g = (g_{ij})$.

Le problème aux valeurs propres fermé est défini comme étant

$$\Delta_g u_\lambda = \lambda u_\lambda,$$

et, lorsque $M$ possède une frontière, nous imposons les conditions aux limites de Dirichlet,

$$\begin{cases} \Delta_g u = \lambda u \text{ dans } M, \\ u = 0 \text{ sur } \partial M. \end{cases}$$

Ici, le nombre $\lambda \in \mathbb{R}$ est une valeur propre de l'opérateur $\Delta_g$ si la fonction propre associée appartient à l'ensemble des fonctions $u$ non identiquement nulles satisfaisant l'un ou l'autre des problèmes ci-haut. La multiplicité d'une valeur propre $\lambda$ correspond alors à la dimension de l'espace propre correspondant et est toujours finie dans ces contextes.

Ainsi, $\Delta_g$ admet un spectre discret, l'ensemble des modes de vibration de la variété, que nous noterons de la manière suivante :

$$0 = \lambda_0 < \lambda_1(M, g) \leq \lambda_2(M, g) \leq ... \nearrow +\infty,$$

où $\lambda_0$ n'appartient au spectre de $M$ que si $\partial M = \emptyset$. Il est également à noter que l'espace de Hilbert des fonctions dont le carré est intégrable, $L^2(M, g)$, se décompose en termes des espaces propres. Autrement dit, il est possible d'obtenir une base orthogonale de cet espace en terme des fonctions propres de $\Delta_g$.

De manière générale, notons $||.||_p$ pour désigner la norme usuelle $||.||_{L^p(M)}$ et $\sigma$ pour désigner la forme volume riemannienne sur $M$. Nous normalisons $u$ de telle façon que $||u||_2^2 = 1$ et, si $M$ n'admet pas de frontière, nous imposons la condition supplémentaire que $\int_M u d\sigma = 0$. Pour une telle fonction $u$, nous définissons $A$ un domaine nodal comme étant une des composantes connexes de l'ensemble $\{u \neq 0\}$. Nous désignons par $\mathcal{A}(u)$ la famille de tous les domaines nodaux de $u$.

Pour une variété sans bord, le quotient de Rayleigh d'une fonction $u$ non triviale est donnée par

$$\frac{\int_M |\nabla_g u|_g^2 d\sigma}{\int_M |u|_g^2 d\sigma},$$

et sert à établir la caractérisation variationnelle suivante :

$$\lambda_k(M, g) = \inf_u \left( \frac{\int_M |\nabla_g u|_g^2 d\sigma}{||f||_2^2} \right), \tag{0.0.1}$$

où $u \in H^1(M)$ est non nulle et orthogonale aux fonctions propres $u_0, u_1, \dots, u_{k-1}$. Dans le cas d'une variété avec bord, $u$ est plutôt dans $H_0^1(M)$.

Pour une exposition plus détaillée de l'opérateur de Laplace-Beltrami ainsi que pour obtenir un aperçu plus complet des propriétés fondamentales de ses fonctions et valeurs propres, il existe plusieurs références sur le sujet, nommément [**7, 8, 16, 45, 88**].

## La distribution des extrema nodaux

Les extrema nodaux sont définis par $m_{A_i} := \max_{A_i} |u|$, où $A_i \in \mathcal{A}(u)$. Concrètement, la valeur de $m_{A_i}$ correspond à l'amplitude maximale suite à la vibration. Par exemple, dans le cas d'une plaque vibrante, le ou les emplacements où $m_{A_i}$ est atteint correspondent aux endroits où la plaque vibre le plus.

En utilisant la loi de Weyl (voir [**16**, page 9]), il est possible d'obtenir une estimation de la distribution des extrema d'une fonction propre de l'un ou l'autre des problèmes considérés. En effet, la borne de Hormander-Levitan-Avakumovic (voir [**90**], entre autres) nous dit qu'il existe une constante $k_g > 0$ ne dépendant que de la métrique $g$ telle que

$$m_{A_i} \leq ||u_\lambda||_{L^\infty(M)} \leq k_g \lambda^{\frac{N-1}{4}}.$$

En sommant sur tous les domaines nodaux de $u_\lambda$, nous obtenons que

$$\sum_{i=1}^{N_\lambda} ||u_\lambda||_{L^\infty(A_i)} \leq N_\lambda k_g \lambda^{\frac{N-1}{4}} \leq k_g \lambda^{\frac{N}{2}} \lambda^{\frac{N-1}{4}} = k_g \lambda^{\frac{3N-1}{4}}, \tag{0.0.2}$$

où l'on utilise le théorème de Courant. Ce dernier stipule que $N_\lambda \leq \lambda^{\frac{N}{2}}$, où $N_\lambda$ représente le nombre de domaines nodaux de $u_\lambda$ (voir [**16**, page 11]). Or, nous verrons dans les prochaines sections qu'il s'avère que cette borne n'est pas optimale.

## L'approche de L. Polterovich et M. Sodin

Présentons d'abord l'approche de [**83**] pour étudier les extrema nodaux. Considérons $\mathcal{F}_\lambda$ la famille des fonctions lisses sur $M$ telle que $||f||_2^2 = 1$, $||\Delta_g f||_2 \leq \lambda$ et, si $M$ ne possède pas de frontière, $\int_M f d\sigma = 0$. Notons que cette famille est vide si $\lambda < \lambda_1(M, g)$, où $\lambda_1(M, g)$ est la première valeur propre de $M$. En effet, le principe variationnel pour $\lambda_1$ nous donne que

$$\lambda_1(M, g) = \inf_{0 \neq f \in H_0^1(M)} \frac{\int_M |\nabla_g f|_g^2 d\sigma}{||f||_2^2}.$$

Or, en appliquant la formule de Green et ensuite l'inégalité de Cauchy-Schwarz, nous remarquons que

$$\left( \int_M |\nabla f|_g^2 d\sigma \right)^2 = \left( \int_M f \Delta_g f \right)^2 \leq ||f||_2 ||\Delta_g f||_2.$$

En combinant ces observations, nous obtenons alors que

$$\lambda_1(M, g) \leq \frac{\int_M g(\nabla f, \nabla f) d\sigma}{||f||_2^2} \leq \frac{||f||_2 ||\Delta_g f||_2}{||f||_2^2} \leq \lambda,$$

car nous supposons que $||f||_2^2 = 1$ et que $||\Delta_g f||_2 \leq \lambda$, $\forall f \in \mathcal{F}_\lambda$.

Pour les résultats qui suivent, il est naturel de dire que $\lambda \geq \lambda_1(M, g)$. Afin de mieux saisir ce que représente $\mathcal{F}_\lambda$, voici deux exemples de fonctions y appartenant.

**Exemple 0.0.1.** *Toutes les fonctions propres associées au problème aux valeurs propres de* $\Delta_g$ *avec les conditions à la frontière de Dirichlet avec* $||u_\lambda||_2 = 1$ *appartiennent à* $\mathcal{F}_\lambda$. *De plus, toute combinaison linéaire de fonctions propres associées à des valeurs propres inférieures à* $\lambda$ *appartiennent à* $\mathcal{F}_\lambda$.

**Exemple 0.0.2.** *Une autre sous-classe de fonctions appartenant à* $\mathcal{F}_\lambda$ *est donnée par les fonctions propres normalisées du bilaplacien associées au problème de la plaque fixée, à savoir*

$$\Delta_g^2 f = \lambda^2 f, f|_{\partial M} = 0, \nabla_g f|_{\partial M} = 0.$$

Mon objectif est de généraliser le théorème que voici :

**Théorème 0.0.3** (L. Polterovich et M. Sodin, [**83**, Theorem 1.3])**.** *Considérons* $(M^2, g)$ *une surface riemannienne et prenons* $f \in \mathcal{F}_\lambda$. *Alors, il existe une constante* $k_g > 0$ *ne dépendant que de la métrique* $g$ *telle que*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} \leq k_g \lambda,$$

*et*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^2 \leq k_g \lambda.$$

Il est important de noter que l'approche utilisée pour prouver ce résultat est basée sur des techniques géométriques seulement valables dans le cas des surfaces riemanniennes.

### Une première méthode

Soit $\delta \in ]0,1[$. L'ensemble $V_\delta^i$ est défini comme étant $\{x \in A_i : |u_\lambda(x)| \geq \delta m_{A_i}\}$. Ces sous-ensembles des domaines nodaux sont concentrés autour des endroits où $|u_\lambda|$ atteint une valeur maximale et ont comme frontière les courbes de niveau de $|u_\lambda|$. Dans [**80**], je démontre que ces ensembles ne peuvent être arbitrairement petits.

**Théorème 0.0.4** (G. Poliquin, [**80**, Theorem 1.2.4])**.** *Soit* $\delta \in ]0,1[$ *et* $N \geq 2$. *Alors, il existe* $\lambda_0 > 0$ *et* $k_{g,\delta,\lambda_0} > 0$ *tels que, pour tout* $\lambda \geq \lambda_0$,

$$\mathrm{Vol}_g(V_\delta^i) \geq k_{g,\delta} \lambda^{-N/2}, \forall i.$$

Du Théorème 0.0.4, j'obtiens une version du Théorème 0.0.3 applicable aux fonctions propres de l'opérateur de Laplace-Beltrami sur $(M^N, g)$ de dimension $N \geq 2$.

**Théorème 0.0.5** (G. Poliquin, [**80**, Theorem 1.3.1])**.** *Soit* $(M^N, g)$ *une variété riemannienne compacte fermée avec* $N \geq 2$*. Si* $\lambda$ *est suffisamment grand, alors il existe une constante* $k_g > 0$ *telle que*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^p \leq k_g \lambda^{\frac{N}{2}+p\delta(p)}$$

*soit vérifiée quelle que soit la valeur de* $p \geq 2$*. Ici,* $\delta(p)$ *correspond à*

$$\delta(p) = \begin{cases} \dfrac{N-1}{4}\left(\dfrac{1}{2} - \dfrac{1}{p}\right), & 2 \leq p \leq \dfrac{2(N+1)}{N-1}, \\ \dfrac{N}{2}\left(\dfrac{1}{2} - \dfrac{1}{p}\right) - \dfrac{1}{4}, & \dfrac{2(N+1)}{N-1} \leq p \leq +\infty. \end{cases}$$

Une conséquence du Théorème 0.0.5, de la loi de Weyl et du théorème de Courant est consignée dans le corollaire suivant.

**Corollaire 0.0.6** (G. Poliquin, [**80**, Corollary 1.3.4])**.** *Soit* $(M^N, g)$ *une variété riemannienne compacte et fermée. Si* $\lambda$ *est suffisament grand, alors il existe une constante positive* $k_g$ *telle que*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} \leq k_g \lambda^{\frac{N}{2}}. \tag{0.0.3}$$

**Remarque 0.0.7.** *C'est en comparant* (0.0.3) *et* (0.0.2) *que nous constatons que la borne obtenue avec la borne de Hormander-Levitan-Avakumovic n'est pas optimale.*

Je prouve également un résultat analogue pour le cas des variétés avec bord dans le Chapitre 1.

### Une deuxième méthode via les inégalités de Hölder inversées de G. Chiti

Cette autre approche consiste à utiliser une inégalité dite de Hölder inversée issue des travaux de Chiti [**18, 19**]. Cette inégalité est valable pour n'importe quel opérateur elliptique linéaire du second ordre associé au problème aux valeurs propres de Dirichlet. Du coup, cela permet d'obtenir le Théorème 0.0.3 sur un domaine euclidien borné $\Omega$ de $\mathbb{R}^N$, mais pour une famille plus restreinte de fonctions, nommément les fonctions propres d'un opérateur elliptique linéaire de deuxième ordre associé au problème de Dirichlet.

Nous considérons maintenant une grande classe d'opérateurs elliptiques $L$ définis comme suit :

$$L(u) := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + cu.$$

Ici, les coefficients $a_{ij}(x)$ sont des fonctions réelles mesurables satisfaisant

$$a_{ij} = a_{ji}, \forall 1 \leq i, j \leq N.$$

Nous supposons, de plus, que $c$ est une fonction mesurable bornée non négative. Notons que cette dernière hypothèse peut être imposée sans perte de généralité (voir [**45**, Remarque 1.1.3., p. 3]). De plus, nous normalisons les coefficients de telle façon que la constante d'ellipticité soit unitaire. Cette hypothèse se traduit comme suit :

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq |\xi|^2.$$

Soit le problème suivant :

$$\begin{cases} L(u) = \lambda u \text{ dans } \Omega, \\ u = 0 \text{ sur } \partial\Omega. \end{cases} \tag{0.0.4}$$

Dans [**80**], je prouve également le résultat suivant :

**Théorème 0.0.8** (G. Poliquin, [**80**, Theorem 1.4.3])**.** *Soit $C_N$ le volume de la boule de dimension $N$ et considérons $u_\lambda$ une fonction satisfaisant* (0.0.4). *Alors, nous obtenons que*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} \leq K_{N,1}\,\mathrm{Vol}(\Omega)^{\frac{1}{2}}\lambda^{\frac{N}{2}},$$

*et*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^2 \leq K_{N,2}^2\lambda^{\frac{N}{2}},$$

*où $K_{N,p}$ est une constante ne dépendant que de $N$ et de $p$ définie par*

$$K_{N,p} = \frac{2^{1-\frac{N}{2}}(NC_N)^{\frac{-1}{p}}}{\Gamma(\frac{N}{2})\Big(\int_0^{j_{\frac{N}{2}-1}} r^{p-\frac{Np}{2}+N-1} J^p_{\frac{N}{2}-1}(r)dr\Big)^{\frac{1}{p}}}.$$

*Ici, $j_{\frac{N}{2}-1}$ est le premier zéro positif de la fonction de Bessel $J_{\frac{N}{2}-1}$.*

En particulier, si $N = 2$, nous obtenons les inégalités prescrites par le Théorème 0.0.3 avec des constantes explicites.

## Le problème de Dirichlet avec le $p$-laplacien

Soit $1 < p < \infty$ et $\Omega$ un ouvert euclidien borné de $\mathbb{R}^N$. Le $p$-laplacien est un opérateur non linéaire défini comme étant

$$\begin{aligned} \Delta_p : C^\infty(\Omega) &\rightarrow C^\infty(\Omega), \\ u &\rightarrow \Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \end{aligned}$$

où

$$|\nabla u|^{p-2} = \left(\left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n}\right)^2\right)^{\frac{p-2}{2}}.$$

Le problème aux valeurs propres avec cet opérateur consiste à résoudre

$$\Delta_p u + \lambda |u|^{p-2} u = 0 \qquad \text{dans } \Omega, \tag{0.0.5}$$

avec des conditions aux limites à imposer. Ici, nous considérons le problème de Dirichlet où $u = 0$ sur $\partial\Omega$. Notons que, si $p = 2$, le $p$-laplacien correspond précisément à l'opérateur de Laplace usuel.

Le nombre réel $\lambda$ est une valeur propre de $\Delta_p$ si (0.0.5) admet une solution faible non triviale, c'est-à-dire une fonction $u \in W^{1,p}(\Omega)$ telle que

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v - \lambda \int_\Omega |u|^{p-2} u \cdot v = 0,$$

quelle que soit $v \in C_0^\infty(\Omega)$. Nous obtenons ainsi une paire $(\lambda, u)$, une valeur propre et une fonction propre de $\Delta_p$.

La première valeur propre $\lambda_{1,p}$ du problème de Dirichlet pour $\Delta_p$ possède une caractérisation variationnelle bien établie :

$$\lambda_{1,p} = \inf_{0\neq u\in C_0^\infty(\Omega)} \left\{ \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} \right\}.$$

De plus, les propriétés principales de la première valeur propre et fonction propre de l'opérateur de Laplace se transpose au problème du $p$-laplacien : $\lambda_{1,p}$ est simple, isolée et $u_{1,p}$ est l'unique fonction propre ne changeant pas de signe sur $\Omega$. Nous en savons nettement moins sur les autres valeurs propres. Par exemple, il est possible de construire une caractérisation variationnelle pour ces dernières, mais il n'est pas démontré qu'elle soit valide pour le spectre en entier. Pour en savoir davantage sur le $p$-laplacien et ses enjeux, les références suivantes constituent un bon point de départ : [**28, 40, 54, 63, 62, 64**].

Finalement, il vaut la peine de mentionner que cet opérateur sert dans plusieurs modèles physiques entrecoupés d'une relation non linéaire, comme le flot d'un fluide au travers d'une membrane poreuse dans un régime turbulent (voir [**25, 26**] pour les détails menant vers cette modélisation) ou encore la vibration d'une membrane de tambour non élastique (voir [**24, 89**] dans ce cas-ci).

## Bornes inférieures pour la fréquence fondamentale

Trouver une borne inférieure pour la première fréquence fondamentale est un problème classique en géométrie spectrale. La plus connue est sans doute l'inégalité de G. Faber et de E. Krahn stipulant que la première valeur propre du problème de Dirichlet est minimale sur une boule parmi tous les domaines de volume fixé. Cette inégalité est directement généralisable au cas du $p$-laplacien, tel que remarqué dans [**47, 63**].

Pour ma part, je me suis surtout intéressé aux bornes inférieures mettant la première valeur propre en relation avec le rayon inscrit du domaine, noté $\rho_\Omega$. Autrement dit, la question est de déterminer s'il existe une constante positive $\alpha_{N,p}$ telle que

$$\lambda_{1,p}(\Omega) \geq \alpha_{N,p}\rho_\Omega^{-p}. \tag{0.0.6}$$

Ce type de borne a été très étudié dans le cas de l'opérateur de Laplace. Dès les années 1960, J. Hersch a démontré dans [**46**] que, parmi les domaines planaires convexes, cette inégalité est valide avec $\alpha_{2,2} = \frac{\pi^2}{4}$. Le mathématicien hongrois E. Makai a amélioré ensuite ce résultat dans [**65**]. Plusieurs autres mathématiciens ont travaillé sur ce problème, notamment W. K. Haymann et R. Osserman pour ne nommer que ceux-là. Ces derniers, dans [**43, 73**], ont en fait généralisé cette borne au cas des domaines planaires simplement connexes. Le résultalt le plus récent remonte à 1994 et a été obtenu par R. Banuelos et par T. Carroll dans [**6**]. Ils en sont arrivés à $\alpha_{2,2} \approx 0,6197$. Notons que cette question n'a jamais été étudiée dans le cas du $p$-laplacien. En fait, seule G. Bognar, mathématicienne hongroise ayant obtenu son doctorat sous la supervision d'E. Makai, a étudié une question analogue à celle du cas du $p$-laplacien. Elle a essentiellement prouvé une généralisation de la méthode employée par J. Hersch et E. Makai pour le cas du pseudo $p$-laplacien.

Notons que ce n'est pas un hasard si tous les résultats mentionnés plus haut ne sont valables qu'en deuxième dimension. C'est W. K. Hayman, en 1978, qui a fait la remarque suivante à la toute fin de son article [**43**] : si $\Omega$ est une boule de dimension $N \geq 3$ à laquelle nous retirons de très fines « aiguilles », nous ne changeons que très peu la valeur de $\lambda_{1,2}(\Omega)$, alors que $\rho_\Omega \to 0$. Ce qui soutient cette observation, c'est le fait qu'une courbe possède une 2-capacité nulle en dimension $N \geq 3$. Ici, la $p$-capacité d'un sous-ensemble compact $K$ d'une boule $B_r$ de rayon $r$ est définie comme étant

$$\mathrm{Cap}_p(K, B_r) = \inf \left\{ \int_{B_r} |\nabla u|^p dx, u \in C_0^\infty(B_r), u \geq 1 \text{ dans } K \right\}.$$

En étudiant de plus près les propriétés de la $p$-capacité, tel que fait dans [**14, 69**], force est de constater, d'une part, que, si $p > N - 1$, une courbe possède une $p$-capacité positive. Conséquemment, l'observation de W. K. Hayman ne tient plus dans ce cas. D'autre part, si $p > N$, un simple point possède une $p$-capacité non triviale. À la lumière de cette deuxième observation, il devient possible de démontrer des bornes du type (0.0.6) pour le $p$-laplacien dans un contexte plus général que celui des domaines planaires. En effet, je prouve dans [**79**] le théorème suivant.

**Théorème 0.0.9** (G. Poliquin, [**79**, Theorem 1.4.1]). *Si $p > N$ et si $\Omega$ est un domaine euclidien borné, alors il existe une constante positive $\alpha_{N,p}$ telle que*

$$\lambda_{1,p}(\Omega) \geq \alpha_{N,p}\rho_\Omega^{-p}.$$

De plus, il est possible de généraliser les résultats classiques connus pour les domaines planaires simplement connexes au cas où $N - 1 < p < N$.

**Théorème 0.0.10** (G. Poliquin, [**79**, Theorem 1.4.2]). *Si $N - 1 < p < N$ et si $\Omega$ est un domaine euclidien borné tel que $\partial\Omega$ est connexe, alors il existe une constante positive $\alpha_{N,p}$ telle que*

$$\lambda_{1,p}(\Omega) \geq \alpha_{N,p}\rho_\Omega^{-p}.$$

Notons que la plupart des preuves des résultats pour les domaines simplement connexes planaires s'adaptent au cas du $p$-laplacien dans ce même contexte. C'est précisément le fruit du travail que j'ai réalisé dans [**78**]. De surcroît, j'y prouve le théorème suivant sur la densité des lignes nodales de cet opérateur.

**Théorème 0.0.11** (G. Poliquin, [**78**, Theorem 2.5]). *Soit $\Omega$ un domaine planaire borné. Pour une valeur propre $\lambda$ donnée, posons $\mathcal{Z}_\lambda = \{x \in \Omega : u_\lambda = 0\}$. Alors, il existe une constante $M > 0$ telle que, pour toute valeur $\lambda > M$, il est possible de trouver une constante $C_M > 0$ telle que*

$$\mathcal{H}^1(\mathcal{Z}_\lambda) \geq C_M\lambda^{1/p}.$$

*Ici, $\mathcal{H}^1$ représente la mesure de Hausdorff de dimension* 1.

Dans le cas de l'opérateur de Laplace, ce résultat classique a été prouvé par J. Brüning en 1978 dans [**12**].

Il vaut la peine de mentionner une question ouverte pour le $p$-laplacien. Nous ne savons pas si l'intérieur de $\mathcal{Z}_\lambda$ est vide lorsque $p \neq 2$. L'argument utilisé pour démontrer ce fait dans le cas de l'opérateur de Laplace dépend fortement de la linéarité de ce dernier. Évidemment, la borne ci-haut n'aurait aucun intérêt si $\text{int}(\mathcal{Z}_\lambda) \neq \emptyset$. Pour plus de détails sur cette question, les références [**28, 40**] s'avèrent utiles.

En dernier lieu, il est également possible d'étudier la question des extrema nodaux pour le $p$-laplacien. Dans [**80**], je prouve le cas particulier suivant :

**Théorème 0.0.12** (G. Poliquin, [**80**, Theorem 1.7.4])**.** *Soit $\Omega$ un domaine borné de $\mathbb{R}^N$. Soit $u_{p,\lambda}$ une fonction propre du problème de Dirichlet du $p$-laplacien associé à $\lambda$ normalisée de la manière suivante : $||u_{p,\lambda}||_p = 1$. Alors, nous obtenons que*

$$\sum_{i=1}^{|\mathcal{A}(u_{p,\lambda})|} m_{A_i} \leq 4^N \operatorname{Vol}(\Omega)^{1-\frac{1}{p}} \lambda^{\frac{N}{p}}.$$

L'intérêt de ce théorème réside surtout dans le corollaire suivant.

**Corollaire 0.0.13** (G. Poliquin, [**80**, Corollary 1.7.6])**.** *Soit $\Omega$ un domaine borné de $\mathbb{R}^N$. Soit $u_{p,\lambda}$ une fonction propre du problème de Dirichlet du $p$-laplacien associée à $\lambda$ normalisée de la manière suivante : $||u_{p,\lambda}||_p = 1$. Pour n'importe quelle valeur de $a > 0$, il existe une constante $C_a > 0$ telle que le nombre de domaines nodaux où $m_A \geq a$ n'excède pas $C_a a^{-1} \lambda^{\frac{N}{p}}$.*

Dans le cas du $p$-laplacien, le théorème de Courant n'a pas été pleinement démontré. Dans [**31**], il est prouvé que si $u_k$ est une fonction propre associée à la valeur propre $\lambda_k$ obtenue par caractérisation variationnelle, alors son nombre de domaines nodaux est borné par $2k-2$. De plus, il est connu que le nombre de domaines nodaux $N_\lambda$ est borné par $C\lambda^{\frac{N}{p}}$ si $\lambda$ est obtenue par une caractérisation variationnelle. Le Corollaire 0.0.13 établit un résultat analogue pour une valeur propre sans imposer que cette dernière soit obtenue par caractérisation variationnelle.

120

## Le problème de Neumann

Soit $\Omega \subset \mathbb{R}^N$ un domaine borné de volume $|\Omega|$. Insistons sur le fait que $\Omega$ n'est pas nécessairement connexe. Tout au long de cette section, nous normalisons le volume de $\Omega$ de manière à le rendre unitaire. Soit $\Delta := -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$, le laplacien usuel dans $\mathbb{R}^N$. Rappelons que le problème aux valeurs propres de Dirichlet,

$$\Delta u = \lambda u \text{ dans } \Omega \text{ et } u = 0 \text{ sur } \partial\Omega, \qquad \text{(Dirichlet)}$$

admet un spectre discret ordonné de la manière suivante

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq ... \nearrow \infty.$$

Nous omettrons de préciser la dépendance au domaine des valeurs propres si cette dernière est claire. Si nous supposons de plus que $\Omega$ admet une frontière de Lipschitz, alors le problème aux valeurs propres avec les conditions au bord de Neumann,

$$\Delta u = \mu u \text{ dans } \Omega \text{ et } \frac{\partial u}{\partial n} = 0 \text{ sur } \partial\Omega, \qquad \text{(Neumann)}$$

admet également un spectre discret (voir [**45**, p.7]),

$$\mu_0(\Omega) = 0 \leq \mu_1(\Omega) \leq \mu_2(\Omega) \leq ... \nearrow \infty.$$

Notons par $\frac{\partial u}{\partial n}$ la dérivée normale de la fonction $u$ et par $u_i$, la fonction propre associée à la $i^e$ valeur propre du problème considéré. Notons également que la première valeur propre associée au problème de Neumann, $\mu_0$, est toujours nulle, car $u_0$ correspond aux fonctions propres constantes.

**Remarque 0.0.14.** *Le spectre d'un domaine non connexe est constitué de l'union ordonnée du spectre de chacune de ses composantes connexes.*

**Exemple 0.0.15.** *Considérons $\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_n$, un domaine possédant $n$ composantes connexes. Alors, les $n-1$ premières valeurs propres associées au problème de Neumann sont toutes nulles. En effet, la première valeur propre du spectre de chaque composante connexe $\Omega_i$ est $\mu_0(\Omega_i) = 0$ et est associée aux fonctions propres constantes. Ainsi, les $n$ composantes connexes contribuent au spectre du domaine de telle sorte que $\mu_0(\Omega) = \mu_1(\Omega) = ... = \mu_{n-1}(\Omega) = 0$.*

Certains domaines très particuliers permettent de procéder au calcul explicite des valeurs propres grâce à la méthode de séparation de variables (voir notamment [**22**] à cet égard).

**Exemple 0.0.16.** *Considérons un rectangle $\boldsymbol{R}$ de côtés $a$ et $\frac{1}{a}$. Les valeurs propres associées au problème de Dirichlet sont données par*

$$\lambda_{j,k}(\boldsymbol{R}) = \pi^2\left(\frac{j^2}{a^2} + a^2k^2\right), \quad j,k \in \mathbb{N}, \tag{0.0.7}$$

*alors que celles associées au problème de Neumann sont données par*

$$\mu_{j,k}(\boldsymbol{R}) = \pi^2\left(\frac{j^2}{a^2} + a^2k^2\right), \quad j,k \in \mathbb{N} \cup \{0\}. \tag{0.0.8}$$

## L'optimisation de formes

Le problème aux valeurs propres avec conditions à la frontière de Dirichlet avec le volume du domaine fixé en est un de minimisation. Pour voir cela, il suffit de considérer les valeurs propres d'un rectangle données par (0.0.7) en laissant tendre $a \to +\infty$. Cela créera alors une valeur propre arbitrairement grande.

Rappelons que le premier grand résultat à cet égard, obtenu indépendamment par G. Faber et par E. Krahn autour des années 1920, stipule que la première valeur propre $\lambda_1$ est minimisée par une boule (voir [**35, 56**]). La deuxième valeur propre $\lambda_2$, quant à elle, est minimisée par l'union de deux boules disjointes de même volume. Ce résultat, souvent attribué à tort à G. Szegö (voir [**93**] ou même [**99**]), a en fait d'abord été publié par E. Krahn dans [**57**]. Le cas des valeurs propres d'ordre supérieur est plus difficile. Par contre, grâce à l'analyse numérique, plusieurs candidats minimiseurs sont proposés. Par exemple, en ce qui concerne $\lambda_3$, une conjecture, soutenue par une myriade de résultats partiels (voir [**13, 77, 100**]), stipule que le minimiseur est un seul disque.

Dans le cas du problème aux valeurs propres de Neumann, c'est plutôt un problème de maximisation. En effet, il suffit de se baser sur l'Exemple 0.0.15 pour construire un domaine engendrant $\mu_n = 0$. Cela se produit dès que le domaine possède au moins $n+1$ composantes connexes. Il a d'abord été conjecturé par B. Kornhauser et F. Stakgold (voir [**53**]) et ensuite prouvé par G. Szegö et E. Krahn que le disque maximise $\mu_1$ dans [**93**]. La généralisation à une dimension arbitraire est le fruit des travaux de H. F. Weinberger deux années plus tard dans [**98**]. Dans [**37**], un article qui a été publié en 2009, A. Girouard, N. Nadirashvili et I. Polterovich montrent que parmi les domaines planaires simplement connexes d'aire fixée, la deuxième valeur propre non nulle du problème de Neumann est maximisée par une famille de domaines dégénérant vers une union disjointe de deux disques identiques.

À la lumière des résultats obtenus, il est tout naturel de se demander si un seul disque maximise $\mu_3$ comme dans le cas des valeurs propres du problème de Dirichlet. Or, un rectangle ayant pour côté $\sqrt{3}$ et $\frac{1}{\sqrt{3}}$ admet comme troisième valeur propre $\mu_3 \approx 29,610$, excédant du coup celle du disque de $\mu_3 \approx 29,308$. Notons également que trois disques disjoints de même aire donnent une valeur excédant celle d'un seul disque, à savoir $\mu_3 = 3\mu_1 \approx 31,95$. Ces observations donnent le ton à la question suivante : est-ce que le disque ou toute union de disques maximisent $\mu_n$ pour tout $n \in \mathbb{N}$ ?

Dans [**81**], nous en donnons une réponse négative.

**Théorème 0.0.17** (G. Poliquin et G. Roy-Fortin, [**81**, Theorem 1.4])**.** *Dans le cas planaire, la valeur propre de Neumann $\mu_{22}$ n'est maximisée par aucune union disjointe de disques (incluant le cas d'un seul disque).*

La preuve du Théorème 0.0.17 est présentée au Chapitre 4. Le point de départ derrière la réponse à cette question provient d'un résultat obtenu en 1994 par S. A. Wolf et J. B. Keller (voir [**100**, Theorem 8.1]) stipulant que la $n$-ième valeur propre de Dirichlet $\lambda_n(\Omega)$ d'un domaine planaire n'est pas toujours minimisée par une union de disques. Insistons sur le fait que, pour les quatre premières valeurs propres du problème de Dirichlet, il a été démontré (ou conjecturé) que le minimiseur était un disque ou une union de disques (consulter notamment [**45**, p. 83, Figure 5.1]). Plus précisément, S. A. Wolf et J. B. Keller ont montré que la valeur de $\lambda_{13}$ d'un seul carré était inférieure à la valeur obtenue pour une union arbitraire de disques ou pour un seul disque, éliminant du coup les disques comme candidat minimiseur. Quelques années plus tard, E. Oudet dans [**77**] a obtenu numériquement un candidat minimiseur pour $\lambda_5$ qui n'est pas composé de disques.

En dimensions $N \geq 3$, la situation se corse pour le cas de Neumann. En effet, tel que discuté dans [**4**, p. 562], les valeurs propres de la boule n'ont pas encore été étudiées de manière systématique pour $N \geq 4$. En trois dimensions, des formules explicites pour les valeurs propres de la boule peuvent être obtenues en termes des racines $a'_{p,q}$ de la dérivée des fonctions sphériques de Bessel $j_p(x)$ (voir [**102**] pour de l'information sur les fonctions sphériques de Bessel et pour une approximation des zéros $a'_{p,q}$). Il n'a pas été possible de montrer en dimension $N = 3$ que les boules ne sont pas toujours des maximiseurs. Après avoir mené des calculs pour les valeurs $n = 1, 2, ..., 640$, le candidat pour les boules excède toujours celui pour les prismes rectangulaires.

Finalement, il vaut la peine de mentionner qu'avec les méthodes développées dans [**81**], la seconde valeur propre non nulle du problème de Neumann est maximisée par deux boules identiques pour tout $N \geq 3$ parmi les domaines non connexes. Il est donc naturel de se demander s'il est possible d'étendre aux dimensions supérieures le résultat obtenu par A. Girouard, N. Nadirashvili et I. Polterovich dans [**37**].

# Chapitre 1

---

## SUPERLEVEL SETS AND NODAL EXTREMA OF LAPLACE EIGENFUNCTIONS

**Auteur / Author :** Guillaume Poliquin

**Résumé :** Nous estimons le volume des «super- courbes de niveau» des fonctions propres de l'opérateur de Laplace-Beltrami sur une variété riemannienne compacte. La preuve des estimés obtenus repose sur la théorie des fonctions de Green et le principe appelé «Bathtub principle». De ces résultats, il est possible d'établir des bornes supérieures sur la répartition des extrema d'une fonction propre de l'opérateur de Laplace-Beltrami sur ses domaines nodaux. De telles bornes ont d'abord été démontrées par L. Polterovich et M. Sodin dans le cas de surfaces compactes. Notre technique a l'avantage de permettre de généraliser ces bornes à toute dimension. Finalement, nous présentons une autre approche à ce problème basée sur une inéqualité de type Hölder inverse attribuée à G. Chiti.

**Abstract :** We estimate the volume of superlevel sets of Laplace-Beltrami eigenfunctions on a compact Riemannian manifold. The proof uses the Green's function representation and the Bathtub principle. As an application, we obtain upper bounds on the distribution of the extrema of a Laplace-Beltrami eigenfunction over its nodal domains. Such bounds have been previously proved by L. Polterovich and M. Sodin in the case of compact surfaces. Our techniques allow to generalize these results to arbitrary dimensions. We also discuss a different approach to the problem based on reverse Hölder inequalities due to G. Chiti.

## 1.1. Introduction and main results

### 1.1.1. Notation

Let $(M^n, g)$ be a compact, connected $n-$dimensional Riemannian manifold with or without boundary. Let $\Delta_g : C^\infty(M) \to C^\infty(M)$ denote the negative Laplace-Beltrami operator on $M$. In local coordinates $\{x_i\}_{i=1}^n$, we write

$$\Delta_g = \frac{-1}{\sqrt{det(g)}} \sum \frac{\partial}{\partial x_i}(\sqrt{det(g)} g^{ij} \frac{\partial}{\partial x_j}), \tag{1.1.1}$$

where the matrix $(g^{ij})$ is the inverse matrix of $g = (g_{ij})$.

We consider the closed eigenvalue problem,

$$\Delta_g u_\lambda = \lambda u_\lambda, \tag{1.1.2}$$

and when $M$ has a boundary, we impose Dirichlet eigenvalue problem,

$$\begin{cases} \Delta_g u = \lambda u \text{ in } M, \\ u = 0 \text{ on } \partial M. \end{cases} \tag{1.1.3}$$

In both settings, $\Delta_g$ has a discrete spectrum,

$$0 \leq \lambda_1(M, g) \leq \lambda_2(M, g) \leq ... \nearrow +\infty,$$

where $\lambda_1(M, g) > 0$ if $\partial M \neq \emptyset$. Let $||.||_p$ be the usual $||.||_{L^p(M)}$ norm and let $\sigma$ be the Riemannian volume form on $M$ and let $\text{Vol}_g(M)$ denote the Riemannian volume of $M$. We normalize $u$ in such a way that $||u||_2^2 = 1$. If $M$ has no boundary, we require that $\int_M u d\sigma = 0$.

### 1.1.2. Volume of superlevel sets

We define a nodal domain $A$ of an eigenfunction $u_\lambda$ on $M$ as a maximal connected open subset of $\{u_\lambda \neq 0\}$. We denote by $\mathcal{A}(u_\lambda)$ the collection of all its nodal domains.

Let us first consider the Euclidean case. It is known that nodal domains can not be too small. For instance, this can be seen by the Faber-Krahn inequality, stating that given $A_i \in \mathcal{A}(u_\lambda)$,

$$\mathrm{Vol}(A_i) \geq \left(\lambda_1(B)^{n/2}|B|\right)\lambda^{-n/2}, \tag{1.1.4}$$

where $B$ denotes a ball. Denote by $V_\delta^i = \{x \in A_i : |u_\lambda(x)| \geq \delta||u_\lambda||_{L^\infty(A_i)}\}$ the $\delta$-superlevel sets of the restriction of an eigenfunction to one of its nodal domain. The next result can be seen as a refinement of that observation. Indeed, each $\delta$-superlevel set of an eigenfunction can not be too small :

**Lemma 1.1.1.** *Let $n \geq 3$. For all $\delta \in (0,1)$, we have that*

$$\mathrm{Vol}(V_\delta^i) \geq (1-\delta)^{\frac{n}{2}}(2(n-2))^{\frac{n}{2}}\alpha_n\lambda^{-\frac{n}{2}}, \tag{1.1.5}$$

*where $\alpha_n$ stands for the volume of the $n$-dimensional unit ball.*

The preceding lemma and its proof were suggested by F. Nazarov and M. Sodin [**72**].

Letting $\delta \to 0$ in (1.1.5) yields that

$$\mathrm{Vol}(V_0^i) = \mathrm{Vol}(A_i) \geq C_n\lambda^{-\frac{n}{2}},$$

which is an inequality *à la Faber-Krahn* comparable to (1.1.4). However, the constant is not optimal when compared to Faber-Krahn inequality since $C_{n,\delta}$ tends to $C_n = (2(n-2))^{\frac{n}{2}}\alpha_n$ as $\delta \to 0$.

The proof of Lemma 1.1.1 is based on the maximum principle, applied to a precise linear combination of the eigenfunction $u_\lambda$ and of a certain function $w$. The function $w$ is the solution of the following Poisson problem :

$$\Delta w = -\lambda\chi_{V_\delta^i}u_{\lambda,i} \text{ in } \mathbb{R}^n,$$

where $\chi_{V_\delta^i}$ denotes the characteristic function associated to $V_\delta^i$ and $u_{\lambda,i}$ denotes the restriction of $u_\lambda$ to $A_i$. An upper bound on the function $w$ is required while applying the maximum principle. The bound is proved using decreasing rearrangement of functions, as done in [**95**, p. 185]. The next result is a generalization of Lemma 1.1.1, adapted to manifolds of arbitrary dimension :

**Theorem 1.1.2.** *Let $\delta \in (0,1)$ and $n \geq 2$. There exist $\lambda_0 > 0$ and $k_{g,\delta,\lambda_0} > 0$ such that for all $\lambda \geq \lambda_0$, we have that*

$$\mathrm{Vol}_g(V_\delta^i) \geq k_{g,\delta,\lambda_0}\lambda^{-\frac{n}{2}}, \quad \forall i. \tag{1.1.6}$$

The proof of Theorem 1.1.2 for $n \geq 3$ is similar to the proof of its $\mathbb{R}^n$ counterpart. The key idea is to choose a specific linear combination involving $u_{\lambda,i}$ and the solution of the following Poisson problem,

$$\Delta w = -\lambda \chi_{V_\delta^i} u_{\lambda,i} \text{ in } M.$$

In order to apply the maximum principle, it is required to bound the function $w$ in terms of $\lambda$ and of the volume of $V_\delta^i$. The method used to do so differs from the one used in $\mathbb{R}^n$ since decreasing rearrangement of functions no longer works on arbitrary manifolds. Instead, we use an upper bound for Green functions on $M$ in conjunction with a certain form of the Bathtub principle (see [**61**, Theorem 1.14]), that is an upper bound for the integral of a non-negative decreasing radial function :

**Lemma 1.1.3.** *Let $x_0 \in M$. Let $r(x) = d_g(x_0, x)$ the Riemannian distance between $x$ and $x_0$. Let $f(r)$ denote a non-negative strictly decreasing function. Given fixed positive constant $C > 0$, then*

$$\sup_{\Omega \subset M,\ \mathrm{Vol}_g(\Omega)=C} \int_\Omega f(r)d\sigma = \int_{\Omega^*} f(r)d\sigma,$$

*where $\Omega^*$ is the geodesic ball centered at $x_0$ of radius $R$, where $R$ is such that $|\Omega| = |\Omega^*|$.*

Lemma 1.1.3 can also be seen as a weaker form of decreasing rearrangement that has the advantage of being applicable in a more general setting.

For compact surfaces, using a slight adaptation of the result proved in [**67**, Section 3], it is possible to obtain a lower bound on the density of the $\delta$-superlevel set $V_\delta^i$ of an eigenfunction $u_\lambda$ :

**Proposition 1.1.4.** *Let $(M,g)$ be a Riemannian surface and let $\delta \in (0,1)$. For any $p$ such that $u_\lambda(p) = m_{A_i}$, there exists a positive constant $k_{g,\delta}$ such that the* *ball $B_p(k_{g,\delta}\lambda^{-1/2})$ is included in $V_\delta^i$. In particular, this implies that*

$$\rho_\lambda(V_\delta^i) \geq k_{g,\delta}\lambda^{-1/2}, \forall i,$$

*where $\rho_\lambda(V_\delta^i)$ denotes the inner radius of the $\delta$-superlevel set $V_\delta^i$ of the eigenfunction $u_\lambda$.*

Proposition 1.1.4 implies Theorem 1.1.2 in the two dimensional case.

### 1.1.3. Nodal extrema on closed manifolds

The second objective of the paper is to study the distribution of so called nodal extrema, defined as follows :

$$m_{A_i} := \max_{x \in A_i} |u_\lambda(x)|,$$

where $A_i \in \mathcal{A}(u_\lambda)$. Nodal extrema on compact surfaces were previously studied in [**83**]. We consider the more general case of compact Riemannian manifolds of arbitrary dimension. Since the proofs given in [**83**] rely on the classification of surfaces and the existence of conformal coordinates, no direct generalization of their results is possible.

Our first main result in that direction is the following :

**Theorem 1.1.5.** *Let $(M^n, g)$ be a compact closed manifold with $n \geq 2$. If $\lambda$ is large enough, then there exists $k_g > 0$ such that*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^p \leq k_g \lambda^{\frac{n}{2}+p\delta(p)}, \tag{1.1.7}$$

*holds for any $p \geq 2$. Here, $\delta(p)$ corresponds to*

$$\delta(p) = \begin{cases} \dfrac{n-1}{4}\left(\dfrac{1}{2}-\dfrac{1}{p}\right), & 2 \leq p \leq \dfrac{2(n+1)}{n-1}, \\ \dfrac{n}{2}\left(\dfrac{1}{2}-\dfrac{1}{p}\right)-\dfrac{1}{4}, & \dfrac{2(n+1)}{n-1} \leq p \leq +\infty. \end{cases} \tag{1.1.8}$$

Note that $\delta(p)$ is C. Sogge's classical $L^p$ bounds, $||u||_p \leq C\lambda^{\delta(p)}||u||_2$ ([**90**, Ch. 5]). The proof of Theorem 1.1.5 is an application of Theorem 1.1.2.

As an immediate corollary of Theorem 1.1.5, we have the following :

**Corollary 1.1.6.** *Let $(M^n, g)$ be a compact closed manifold. If $\lambda$ is large enough, then there exists $k_g > 0$ such that*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} \leq k_g \lambda^{\frac{n}{2}}. \tag{1.1.9}$$

Indeed, a consequence of Weyl's law and Courant's theorem is that the number of nodal domains $|\mathcal{A}(u_\lambda)|$ is bounded by $k_g \lambda^{n/2}$ (see for instance [**22, 16**]). Using the latter fact and then applying Cauchy-Schwarz inequality yield that

$$\begin{aligned}\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} &\leq \left( \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^2 \cdot \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} 1 \right)^{\frac{1}{2}} \\ &\leq k_g \lambda^{n/4} |\mathcal{A}(u_\lambda)|^{\frac{1}{2}} \leq k_g \lambda^{n/2},\end{aligned}$$

which is the desired result.

**Remark 1.1.7.** *For $p = 1, 2$, it is easy to see that the inequalities are sharp on $\mathbb{T}^n$ ($\prod \sin(nx_i), \lambda = n^2$). For $p > \dfrac{2(n+1)}{n-1}$, extremals are zonal spherical harmonics. Otherwise, the extremals are highest weight spherical harmonics.*

One can visualise inequalities expressed in Theorem 1.1.5 and in Corollary 1.1.6 by considering "fine" dust particles on a vibrating membrane. Indeed, where the membrane's velocity is high, Bernoulli's equation tells us that the air pressure is low. Since the dust particles are most influenced by air pressure, they are swept by the pressure gradient near nodal extrema (see [**21**] for some figures illustrating nodal extrema and for more information on such experiments).

**Remark 1.1.8.** *One can easily obtain bounds on $m_{A_i}$ using the classical Hormander-Levitan-Avakumovic $L^\infty$ bound (see for instance [**90**]). Indeed, it implies that there exists a constant $k_g > 0$ such that $||u_\lambda||_{L^\infty(A_i)} \leq k_g \lambda^{\frac{n-1}{4}}$. Therefore, we have that*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} ||u_\lambda||_{L^\infty(A_i)} \leq k_g |\mathcal{A}(u_\lambda)| \lambda^{\frac{n-1}{4}} \leq k_g \lambda^{\frac{3n-1}{4}},$$

*which is not optimal when compared to the sharp inequality given in Corollary 1.1.6.*

We also obtain a generalization of [**83**, Corollary 1.7]. The result is the following :

**Corollary 1.1.9.** *Given $a > 0$, consider nodal domains such that $m_{A_i} \geq a\lambda^{\frac{n-1}{4}}$. If $\lambda$ is large enough, then there exists $k_g > 0$ such that the number of such nodal domains does not exceed $k_g a^{-\frac{2(n+1)}{n-1}}$. In particular, for fixed $a$, it remains bounded as $\lambda \to \infty$.*

Indeed, letting $N_\lambda$ denote the number of such nodal domains, using (1.1.7) with $p = \frac{2(n+1)}{n-1}$, we have that

$$N_\lambda (a\lambda^{\frac{n-1}{4}})^{\frac{2(n+1)}{n-1}} \leq \sum_{i=1}^{N_\lambda} m_{A_i}^{\frac{2(n+1)}{n-1}} \leq k_g \lambda^{\frac{n+1}{2}},$$

yielding the conclusion.

### 1.1.4. Elliptic operators on Euclidean domains

We obtain analogous results to Theorem 1.1.5. More precisely, we obtain bounds on the distribution of nodal extrema of eigenfunctions associated to the Dirichlet problem of general second order elliptic operators in the divergence form on an Euclidean bounded domain $\Omega$.

Consider the following Dirichlet eigenvalue problem :

$$\begin{cases} L(u) = \lambda u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1.10}$$

where we consider a general elliptic operator $L$ defined as

$$L(u) := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + cu.$$

Here, the coefficients $a_{ij}(x)$ are real measurable functions such that $a_{ij} = a_{ji}, \forall 1 \leq i, j \leq n$. We assume that $c(x)$ is a bounded measurable function such that $c(x) \geq 0$. Note that the non negativity of $c$ can be assumed without loss of generality (see [**45**, Remark 1.1.3, p. 3]). For convenience, we normalize the coefficients in such a way that 1 is the lower ellipticity constant. Thus, the assumption reads

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq |\xi|^2, \forall\, \xi \in \mathbb{R}^n. \tag{1.1.11}$$

We are ready to state the result :

**Theorem 1.1.10.** *Consider* $u_\lambda$ *an eigenvalue of* (1.1.10) *associated to the eigenvalue* $\lambda$*, then*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} \leq K_{n,1}\,\mathrm{Vol}(\Omega)^{\frac{1}{2}}\lambda^{\frac{n}{2}}, \tag{1.1.12}$$

*and*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^2 \leq K_{n,2}^2\lambda^{\frac{n}{2}}. \tag{1.1.13}$$

*The constant* $K_{n,p}$ *depends on* $n$ *and on* $p$ *and is given by*

$$K_{n,p} = \frac{2^{1-\frac{n}{2}}(n\alpha_n)^{\frac{-1}{p}}}{\Gamma(\frac{n}{2})\Big(\int_0^{j_{\frac{n}{2}-1}} r^{p-\frac{np}{2}+n-1}J_{\frac{n}{2}-1}^p(r)dr\Big)^{\frac{1}{p}}}. \tag{1.1.14}$$

The main tool to prove Theorem 1.1.10 is Chiti's reverse Hölder inequality satisfied by any elliptic operator in divergence form with Dirichlet boundary conditions.

**Remark 1.1.11.** *Since Theorem 1.1.10 can be applied to general elliptic operators such as the Laplace-Beltrami operator in local coordinates as defined in* (1.1.1)*, it can also be used with a Laplacian eigenfunction on compact Riemannian manifolds provided that all its nodal domains can always be included in a single chart of* $M$*.*

**Remark 1.1.12.** *A notable feature of* [**83**, Theorem 1.3] *is that the bounds on the distribution of the nodal extrema hold for a larger class of functions defined on compact surfaces, including eigenfunctions associated to the bi-laplacian clamped plate problem. Both approaches can not be extended to the bi-laplacian case since they rely on the maximum principle, which is known not to hold for such operators.*

### 1.1.5. Neumann boundary conditions in the planar case

Let $\Omega$ be a bounded planar domain with piecewise analytic boundary. We consider the Neumann eigenvalue problem on $\Omega$, namely

$$\begin{cases} \Delta u = \mu u \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1.15}$$

Using an argument of [**82**] based on a result of [**96**], it is possible to bound the number of nodal domains touching the boundary of $\Omega$ by $C_\Omega\sqrt{\mu}$. By doing so, it is an easy matter to obtain the following :

**Theorem 1.1.13.** *Let $\Omega$ be a bounded planar domain with piecewise analytic boundary, then there exists $C_\Omega > 0$ and $K_\Omega > 0$ such that*

$$\sum_{i=1}^{|\mathcal{A}(u_\mu)|} m_{A_i} \leq C_\Omega \mu, \tag{1.1.16}$$

*and*

$$\sum_{i=1}^{|\mathcal{A}(u_\mu)|} m_{A_i}^2 \leq K_\Omega \mu. \tag{1.1.17}$$

### 1.1.6. Manifolds with Dirichlet boundary conditions

In order to obtain similar results for manifolds with boundary conditions, one has to use Sogge-Smith's adapted bounds for such setting (see [**91**]). For the sake of clarity, we recall these results here.

Let $(M^n, g)$ be a compact Riemannian manifold with boundary. Let $u_\lambda$ denote a Dirichlet eigenfunction associated to $\lambda$, then there exists $k_g > 0$ such that

$$||u_\lambda||_p \leq k_g \lambda^{\frac{n}{2}(\frac{1}{2}-\frac{1}{p})-\frac{1}{4}} ||u_\lambda||_2, \tag{1.1.18}$$

for $p \geq 4$ if $n \geq 4$, and $p \geq 5$ if $n = 3$. One can easily adapt the proof of Theorem 1.1.5 using Sogge-Smith results to get :

**Theorem 1.1.14.** *Let $(M^n, g)$ be a compact Riemannian manifold with boundary. If $\lambda$ is large enough, there exists $k_g > 0$ such that*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} \leq k_g \lambda^{\frac{n}{2}}, \tag{1.1.19}$$

*and*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^2 \leq k_g \lambda^{\frac{n}{2}}. \tag{1.1.20}$$

*Moreover, we have the following*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^p \leq k_g \lambda^{\frac{n}{2}+\frac{np}{2}(\frac{1}{2}-\frac{1}{p})-\frac{p}{4}}, \tag{1.1.21}$$

*for any $p \geq 4$ if $n \geq 4$, and $p \geq 5$ if $n = 3$.*

In [**91**], it is conjectured that the following bound holds :

$$||u||_p \leq C \lambda^{\alpha(p)} ||u||_2,$$

where

$$\alpha(p) = \begin{cases} \left(\frac{2}{3} + \frac{n-2}{2}\right)\left(\frac{1}{4} - \frac{1}{2p}\right), & 2 \le p \le \frac{6n+4}{3n-4}, \\ \frac{n}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{4}, & \frac{6n+4}{3n-4} \le p \le +\infty. \end{cases} \tag{1.1.22}$$

Hence, a version of Theorem 1.1.14 without the restrictions could be obtained if one showed these latter bounds :

**Conjecture 1.1.15.** *Let $(M^n, g)$ be a manifold with boundary. If $\lambda$ is large enough, then there exists $k_g > 0$ such that*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^p \le k_g \lambda^{\frac{n}{2}+p\alpha(p)}, \tag{1.1.23}$$

*for any $p \ge 2$.*

We also obtain a generalization of [**83**, Corollary 1.7] in the case of manifolds with boundary. Using (1.1.21) with $p = \frac{6n+4}{3n-4}$, we get the following :

**Corollary 1.1.16.** *Given $a > 0$, consider nodal domains such that $m_{A_i} \ge a\lambda^{\frac{n-1}{4}}$. If $\lambda$ is large enough, then there exists $k_g > 0$ such that the number of such nodal domains does not exceed $k_g a^{-\frac{6n+4}{3n-4}}$. In particular, for fixed $a$, it remains bounded as $\lambda \to \infty$.*

### 1.1.7. Bounds for the $p$-Laplacian

For $1 < p < \infty$, the $p-$Laplacian of a function $f$ on an open bounded Euclidean domain $\Omega$ is defined by $\Delta_p f = \mathrm{div}(|\nabla f|^{p-2}\nabla f)$. We consider the following eigenvalue problem :

$$\Delta_p u + \lambda |u|^{p-2} u = 0 \text{ in } \Omega, \tag{1.1.24}$$

where we impose the Dirichlet boundary conditions. We say that $\lambda$ is an eigenvalue of $-\Delta_p$ if (1.1.24) has a nontrivial weak solution $u_{\lambda,p} \in W_0^{1,p}(\Omega)$. That is, for any $v \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda \cdot \nabla v - \lambda \int_\Omega |u_\lambda|^{p-2} u_\lambda v = 0. \tag{1.1.25}$$

The function $u_\lambda$ is then called an eigenfunction of $-\Delta_p$ associated to the eigenvalue $\lambda$. The function $u_\lambda$ is then called an eigenfunction of $-\Delta_p$ associated to $\lambda$. Note that if $p = 2$, the $p$-Laplacian corresponds to the usual Laplacian and is linear. Otherwise, we say that the $p$-Laplacian is "half-linear" in the sense that it is $(p-1)$ homogeneous but not additive.

It is known that the first eigenvalue of the Dirichlet eigenvalue problem of the $p$-Laplace operator, denoted by $\lambda_{1,p}$, is characterized as,

$$\lambda_{1,p} = \min_{0\neq u\in C_0^\infty(\Omega)} \left\{ \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} \right\}. \tag{1.1.26}$$

The infimum is attained for a function $u_{1,p} \in W_0^{1,p}(\Omega)$. In addition, $\lambda_{1,p}$ is simple and isolated. Moreover, the eigenfunction $u_1$ associated to $\lambda_{1,p}$ does not change sign, and it is the only such eigenfunction.

Via, for instance, the Lyusternick-Schnirelmann maximum principle, it is possible to construct $\lambda_{k,p}$ for $k \geq 2$ and hence obtain an increasing sequence of so-called variational eigenvalues of (1.1.24) tending to $+\infty$. There exist other variational characterizations of these eigenvalues. However, no matter which variational characterization one chooses, it always remains to show that all the eigenvalues obtained that way exhaust the whole spectrum of $\Delta_p$.

Less is known about nodal geometry of eigenfunctions for the $p$-Laplace operator. For instance, it is not clear if the the interior of the set $\{x \in \Omega : u_\lambda(x) = 0\}$ is empty or not for $p$-Laplacian eigenfunctions. For more details on nodal geometry of the $p$-Laplace operator, see for instance [**60, 78, 79**].

Nevertheless, using a $L^\infty$ bound obtained in [**63**, Lemma 4.1], one can still obtain an extension of (1.1.12) for the $p$-Laplace operator.

**Theorem 1.1.17.** *Let $\Omega$ be a smooth bounded open set in $\mathbb{R}^n$. Consider $u_{p,\lambda}$ an eigenfunction of the Dirichlet p-Laplacian eigenvalue problem associated to the eigenvalue $\lambda$. Let $||u_{p,\lambda}||_{p,\Omega} = 1$, then we have the following :*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} \leq 4^n \operatorname{Vol}(\Omega)^{1-\frac{1}{p}} \lambda^{\frac{n}{p}}. \tag{1.1.27}$$

Notice that if $p = 2$, this result corresponds to what we expect in the case of the usual Laplace operator.

The Courant nodal theorem combined with the Weyl Law yield that the number of nodal domains of a Dirichlet eigenfunction associated to an elliptic operator $L$ does not exceed $C\lambda^{\frac{n}{2}}$. For the $p$-Laplacian case, the number of nodal domains

$N_\lambda$ associated to an arbitrary eigenfunction is known to be bounded, see [**31**]. It is also shown in [**31**] that the number of nodal domains of an eigenfunction $u_k$ associated to a variational eigenvalue is bounded by $2k-2$. Moreover, it is known that there exists two positive constants depending on $\Omega$ such that $ck^{p/n} \leq \lambda_{k,p} \leq Ck^{p/n}$ (see [**5**]). Combining both results yields that $N_\lambda \leq C\lambda^{n/p}$ if $\lambda$ is a variational eigenvalue. We show that a similar result holds even for non-variational eigenvalue :

**Corollary 1.1.18.** *For any eigenfunction of* (1.1.24) *and any $a > 0$, there exists a positive constant $C > 0$ such that the number of nodal domains $A \in \mathcal{A}(f)$ with $m_A \geq a$ does not exceed $Ca^{-1}\lambda^{\frac{n}{p}}$.*

Indeed, letting $N_\lambda$ denote the number of such nodal domains, using (1.1.27), we have that

$$N_\lambda a \leq \sum_{i=1}^{N_\lambda} m_{A_i} \leq C\lambda^{\frac{n}{p}},$$

yielding the conclusion.

### 1.1.8. Structure of the paper

In Section 1.2, we prove the main results, namely we start with Lemma 1.1.1 in $\mathbb{R}^n$ and then we prove Theorem 1.1.2 for arbitrary compact Riemannian manifolds. This leads to the proof of Theorem 1.1.5 which is an application of Theorem 1.1.2. In Section 1.3, we prove Theorems 1.1.10, **??** and 1.1.17.

## 1.2. Proofs of main results

### 1.2.1. Proof of Lemma 1.1.1

Before proving Theorem 1.1.2 that holds for compact Riemannian manifolds, we give a proof of such result in the Euclidean case to give the intuition behind the proof more clearly.

In order to prove Lemma 1.1.1, we need a technical result concerning Poisson equation. Let $\Omega \subset \mathbb{R}^n, n \geq 3$, denote a bounded domain of $\mathbb{R}^n$ and consider the following problem :

$$\Delta w = f\chi_\Omega \qquad \text{in } \mathbb{R}^n, \tag{1.2.1}$$

where $\chi_\Omega$ is the characteristic function of $\Omega$ and $||f(x)||_{L^\infty(\Omega)} = 1$. It is well known that the solution of such problem is given by $w(x) = (f\chi_\Omega * \Phi)(x)$, where $\Phi(x-y) = \frac{1}{n(n-2)\alpha_n}|x-y|^{2-n}$ is the fundamental solution of the Laplace operator.

**Proposition 1.2.1.** *Let $\Omega \subset \mathbb{R}^n, n \geq 3$ and $||f(x)||_{L^\infty(\Omega)} = 1$. Then, we have that*

$$||w||_{L^\infty(\Omega)} \leq \frac{1}{2(n-2)\alpha_n^{\frac{2}{n}}} \operatorname{Vol}(\Omega)^{2/n}.$$

*Moreover, equality holds if $f \equiv 1$ and if $\Omega$ is a ball.*

Before we give a proof, we give a quick overview of classical rearrangements of functions. Let $u$ be a measurable function defined on an open set $\Omega$. We can form the distribution function of $u$, denoted by $\mu(t)$, the decreasing rearrangement of $u$, $u^*(s)$ into $[0, +\infty]$ and the spherically symmetric rearrangement of $u$, $u^\star$. The distribution function of u

$$\mu(t) = \operatorname{meas}\{x \in \Omega : |u(x)| > t\},$$

is a right-continuous function of $t$, decreasing from $\mu(0) = |\operatorname{supp}(u)|$ to $\mu(+\infty) = 0$ as $t$ increases. The decreasing rearrangement of $u$, a positive, left continuous function into $[0, +\infty]$, is defined as

$$u^*(s) = \inf\{t \geq 0 : \mu(t) < s\}.$$

The spherically symmetric rearrangement of $u$ is a function $u^\star$ from $\mathbb{R}^n$ into $[0, +\infty]$ whose level sets $\{x \in \mathbb{R}^n : u^\star(x) > t\}$ are concentric balls with the same measure as the level sets $\{x \in \Omega : |u(x)| > t\}$. More precisely, $u^\star$ is defined as

$$u^\star(x) = u^*(\alpha_n|x|^n) = \inf\{t \geq 0 : \mu(t) < \alpha_n|x|^n\}.$$

Note that $||u||_\infty = u^*(0) = u^\star(0)$. We refer to [**94**] for more details on rearrangements of functions.

Proof of Proposition 1.2.1. Let us consider first the case where $f \equiv 1$ and if $\Omega$ is a ball centered at $x$ of radius $R$. Straightforward computation shows that

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \chi_\Omega(y)\Phi(x-y)dy\right| &= \frac{1}{n(n-2)\alpha_n}\int_\Omega |x-y|^{2-n}dy \\
&= \frac{1}{(n-2)}\int_0^R r^{2-n}r^{n-1}dr \\
&= \frac{1}{2(n-2)}R^2 = \frac{1}{2(n-2)\alpha_n^{\frac{2}{n}}}\operatorname{Vol}(B_R)^{2/n}.
\end{aligned}$$

Now, for the general case, notice that

$$\begin{aligned}
|w(x)| &= \left|\int_{\mathbb{R}^n} f(y)\chi_\Omega(y)\Phi(x-y)dy\right| \\
&\leq \frac{1}{n(n-2)\alpha_n}\int_{\mathbb{R}^n} |f(y)|\chi_\Omega(y)|x-y|^{2-n}dy \\
&\leq \frac{1}{n(n-2)\alpha_n}\int_{\mathbb{R}^n} \chi_\Omega(y)|x-y|^{2-n}dy.
\end{aligned}$$

The following is a classical result of Hardy and Littlewood that can be found in [**42**] :

$$\int_{\mathbb{R}^n} u(x)v(x)dx \leq \int_{\mathbb{R}^n} u^\star(x)v^\star(x)dx.$$

Therefore, since $\Phi = \Phi^\star$, we get that

$$\begin{aligned}
|w(x)| &\leq \int_{\mathbb{R}^n} \chi_\Omega(y)\Phi(x-y)dy \\
&\leq \int_{\mathbb{R}^n} \chi_{\Omega^\star}(y)\Phi^\star(x-y)dy \\
&= \frac{1}{n(n-2)\alpha_n}\int_{\Omega^\star} |x-y|^{2-n}dy,
\end{aligned}$$

where $\Omega^\star$ denotes a ball centered at $x$ of same volume of $\Omega$. By the previous case, one gets the desired result. □

**Remark 1.2.2.** *The last step of the proof of Proposition 1.2.1 is to show that*

$$\int_\Omega \Phi(x-y)dy \leq \int_{\Omega^\star} \Phi(x-y)dy. \tag{1.2.2}$$

*A generalization of* (1.2.2) *is given by Lemma 1.1.3.*

That being done, we can start the main proof of this section.

Proof of Lemma 1.1.1. Renormalize $u_\lambda$ such that $||u_\lambda||_\infty = 1$. Consider $\delta \in (0,1)$. We want to show that there exists a constant $C_{n,\delta} > 0$ such that

$$\mathrm{Vol}(V_\delta^i) \geq C_{n,\delta}\lambda^{-\frac{n}{2}}.$$

Let $g = u - \delta$. We have that $\Delta g = \Delta u_{\lambda,i} = \lambda u_{\lambda,i}$ in $V_\delta^i$. By Proposition 1.2.1, there exists $w(x)$ satisfying (1.2.1) with $f = -\lambda u_{\lambda,i}$ and $\Omega = V_\delta^i$ such that $||w||_\infty \leq \frac{1}{2(n-2)\alpha_n^{\frac{2}{n}}}\lambda \,\mathrm{Vol}(V_\delta^i)^{\frac{2}{n}}$. Consider the function $g + w$ on $V_\delta^i$. On the boundary, we have that $g + w \leq \frac{1}{2(n-2)\alpha_n^{\frac{2}{n}}}\lambda \,\mathrm{Vol}(V_\delta^i)^{\frac{2}{n}}$. Consider $x_0$ in $V_\delta^i$ such that $u_{\lambda,i}(x_0) = 1 = ||u_\lambda||_\infty$. Thus, we have that $(g+w)(x_0) \geq (1-\delta) - \frac{1}{2(n-2)\alpha_n^{\frac{2}{n}}}\lambda \,\mathrm{Vol}(V_\delta^i)^{\frac{2}{n}}$.

Moreover, since $\Delta(g + w) = \lambda u_{\lambda,i} - \lambda u_{\lambda,i} = 0$, we can use the maximum principle on $g + w$. This implies that

$$\begin{aligned}(1-\delta) - \frac{1}{2(n-2)\alpha_n^{\frac{2}{n}}}\lambda \,\mathrm{Vol}(V_\delta^i)^{\frac{2}{n}} &\leq \frac{1}{2(n-2)\alpha_n^{\frac{2}{n}}}\lambda \,\mathrm{Vol}(V_\delta^i)^{\frac{2}{n}} \\ \iff \mathrm{Vol}(V_\delta^i)^{\frac{2}{n}} &\geq \frac{1}{2}(1-\delta)\left(\frac{\lambda}{2(n-2)\alpha_n^{\frac{2}{n}}}\right)^{-1},\end{aligned}$$

yielding that $\mathrm{Vol}(V_\delta^i) \geq (1-\delta)^{\frac{n}{2}}(2(n-2))^{\frac{n}{2}}\alpha_n\lambda^{-\frac{n}{2}}$. □

### 1.2.2. Proof of Theorem 1.1.2

The proof of Theorem 1.1.2 for manifolds of dimension $n \geq 3$ is in the same spirit as the proof for $\mathbb{R}^n$. The main difference is that we can not use Proposition 1.2.1 which relies on the fundamental solution of the Laplace operator on $\mathbb{R}^n$. We consider instead the Green's representation of the solution to the Poisson problem on $M$.

Let $\Omega$ be a compact smooth domain of $(M^n, g)$ where $n \geq 3$. It is known that there exists a Green's function (see for instance [**88**]), namely a smooth function $G$ defined on $\Omega \times \Omega \setminus \{(x,x) : x \in \Omega\}$ such that

- $G(x,y) = G(y,x), \forall x \neq y$ ;
- For fixed $y$, $\Delta_x G(x,y) = 0, \forall x \neq y$ ;
- $G(x,y) \geq 0$ and $G$ vanishes on the boundary of $\Omega$ ;
- As $x \to y$ for fixed $y$, $G(x,y) \leq \rho(x,y)^{2-n}(1 + o(1)), n \geq 3$, where $\rho(x,y)$ is the geodesic distance between $x$ and $y$ (see [**88**, p. 81]).

Moreover, if we consider the following problem,

$$\begin{cases} \Delta_g w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.3}$$

its unique solution is given by

$$w(y) = \int_\Omega G(x,y) f(x) d\sigma.$$

**Proposition 1.2.3.** *Let $n \geq 3$, $||u_\lambda||_\infty = 1$ and $\delta \in (0,1)$. Let $A_i$ denote a nodal domain of $u_\lambda$ and $V_\delta^i = \{x \in A_i : |u_\lambda(x)| \geq \delta m_{A_i}\}$. There exist $\lambda_0$ and $k_{g,\lambda_0} > 0$ such that $\forall \lambda > \lambda_0$ and for any $x_0 \in V_\delta^i$, we have that*

$$|w(x_0)| \leq k_{g,\lambda_0} \lambda \operatorname{Vol}_g(V_\delta^i)^{\frac{2}{n}},$$

*where $w$ is the solution of problem (1.2.3) with $\Omega = V_\delta^i$ and $f = -\lambda u_\lambda$.*

We want to prove an analogous result to Proposition 1.2.1. To do so, we treat split the argument into two cases depending on if the volume of $V_\delta^i$ is "large" or "small". We define "small $V_\delta^i$" in such a way that we can apply normal coordinates. This becomes handy since Green functions on $M$ behaves roughly like the fundamental solution of the Laplace operator on $\mathbb{R}^n$. Using the Lemma 1.1.3, it is then possible to bound $w$ like claimed.

Proof of Proposition 1.2.3. Let $A_i$ a nodal domain of $u_\lambda$ and let $x_0$ be any point such that $u_\lambda(x_0) = m_{A_i}$.

Let $B_{x_0}(r) := \exp_{x_0}(B_0(r))$ denote the geodesic ball of radius $r$ centered at $x_0$. It is known that for $r$ small enough, we have that

$$\operatorname{Vol}_g(B_{x_0}(r)) = r^n \operatorname{Vol}(B_0(1)) \left(1 - \frac{\operatorname{scal}_g(x_0)}{6(n+2)} r^2 + o(r^2)\right),$$

where $\operatorname{scal}_g(x_0)$ denotes the scalar curvature at $x_0$. Therefore, there exists $\epsilon \in (0,1)$ such that for all $0 < r \leq \epsilon \leq \operatorname{injrad}(M,g)$, there exist $A_g > 0$ and $B_g > 0$ such that

$$A_g r^n \leq \operatorname{Vol}_g(B_{x_0}(r)) \leq B_g r^n. \tag{1.2.4}$$

Renormalize $u_\lambda$ such that $||u_\lambda||_\infty = 1$. Fix a nodal domain $A_i$ and $x_0 \in A_i$.

Let $\lambda_0 = B_g^{-2/n}\epsilon^{-2}$. Notice that if $\lambda \geq \lambda_0$ and if $\mathrm{Vol}_g(V_\delta^i) > \mathrm{Vol}_g(B_{x_0}(\epsilon))$, the result holds with $k_g = \frac{A_g}{B_g}$.

On the other hand, if $\lambda \geq \lambda_0$, but $\mathrm{Vol}_g(V_\delta^i) \leq \mathrm{Vol}_g(B_{x_0}(\epsilon))$, it is always possible to pick $R$ such that $\mathrm{Vol}_g(V_\delta^i) = \mathrm{Vol}_g(B_R(x_0))$ and $R \leq \epsilon$ hold.

Let us now work to get an upper bound on $|w(x_0)|$. We have that

$$\begin{aligned} |w(x_0)| &= \left|\lambda \int_{V_\delta^i} G(x,x_0)u_\lambda(x)d\sigma\right| \\ &\leq \lambda \int_{V_\delta^i} G(x,x_0)d\sigma. \end{aligned}$$

Using upper bounds on the Green function (see bounds proved in [**85**]), we have that there exists $C_g > 0$ such that

$$G(x,x_0) \leq C_g \rho(x,x_0)^{2-n}, \ \forall x \neq x_0,$$

implying that

$$|w(x_0)| \leq C_g \lambda \int_{V_\delta^j} \rho(x,x_0)^{2-n} d\sigma.$$

As it was done in $\mathbb{R}^n$, we need to integrate on a ball to obtain a straightforward computable integral. To do so, we use Lemma 1.1.3 whose proof can be found in Section 1.2.4. Applying Lemma 1.1.3, we get the following :

$$C_g\lambda \int_{V_\delta^j} \rho(x,x_0)^{2-n} d\sigma \leq C_g \lambda \int_{(V_\delta^i)^*} \rho^{2-n} d\sigma,$$

where $(V_\delta^i)^* = B_{x_0}(R) = \exp_{x_0}(B_0(R))$.

Using Gauss's Lemma, we now have that

$$\begin{aligned} |w(x_0)| &\leq C_g\lambda \int_{(V_\delta^i)^*} \rho^{2-n}(1 - \frac{1}{6}R_{kl}x^k x^l + O(|x|^3))dx^1 dx^2 \ldots dx^n \\ &\leq C_g \lambda \left( \frac{n\omega_n}{2} R^2 - \frac{n\omega_n Scal_g(x_0)}{6}\frac{R^4}{4} + O(R^5)\right) \\ &\leq C_g\lambda \frac{n\omega_n}{2} R^2 \left(1 - \frac{Scal_g(x_0)}{6}\frac{R^2}{2} + O(R^3)\right) \\ &\leq C_g B_g E_g \lambda \,\mathrm{Vol}_g(B_{x_0}(R))^{\frac{2}{n}} = k_{g,\lambda_0}\lambda\,\mathrm{Vol}_g(V_\delta^i)^{\frac{2}{n}}. \end{aligned}$$

□

The last step to prove Theorem 1.1.2 is very similar to the last step in the proof of Lemma 1.1.1.

Proof of Theorem 1.1.2. Renormalize $u_\lambda$ such that $||u_\lambda||_\infty = 1$. Let $g = u - \delta + w$. On the boundary of $V_\delta^i$, we have that $g = \delta - \delta = 0$. Consider any $x_0$ in $V_\delta^i$ such that $u_{\lambda,i}(x_0) = 1$. By Proposition 1.2.3, we have that $g(x_0) \geq (1-\delta) - C_{g,\lambda_0}\lambda \operatorname{Vol}(V_\delta^i)^{\frac{2}{n}}$.

Moreover, since $\Delta g = \Delta u_{\lambda,i} + \Delta w = \lambda u_{\lambda,i} - \lambda u_{\lambda,i} = 0$ in $V_\delta^i$, we can use the maximum principle on $g$. This implies that

$$(1-\delta) - C_{g,\lambda_0}\lambda \operatorname{Vol}_g(V_\delta^i)^{\frac{2}{n}} \leq 0 \iff \operatorname{Vol}_g(V_\delta^i) \geq k_{g,\lambda_0}(1-\delta)^{\frac{n}{2}}\lambda^{-\frac{n}{2}}.$$

□

We now prove Proposition 1.1.4 which implies Theorem 1.1.2 in the two dimensional.

Proof of Proposition 1.1.4. The proof essentially follows [**67**, Section 3]. It is shown in [**67**] that given a nodal domain $A_i$, there exists a ball $B_p(k_g\lambda^{-1/2}) \subset A_i$ centered at any point $p$ such that $u_\lambda(p) = m_{A_i}$. This implies that

$$\rho_\lambda(A_i) \geq k_g\lambda^{-1/2}.$$

The proof of this fact uses harmonic measure techniques to get a bound on the distance from a point of a set, namely the point $p$ where $u_\lambda(p) = m_{A_i}$, to its boundary. Instead of working on a nodal set $A_i$ of a given eigenfunction $u_\lambda$, one can run the argument on a connected component of the $\delta$-superlevel set containing $p$. Such a modification will only influence the constants in the estimates obtained in [**67**, Section 3]. Thus, arguing in a similar way, one obtains

$$\rho_\lambda(V_i^\delta) \geq k'_{g,\delta}\lambda^{-1/2},$$

which completes the proof of the proposition.

□

### 1.2.3. Proof of Theorem 1.1.5

Let $\delta \in (0,1)$ and $\lambda$ be large enough. Recall that $\mathcal{A}(u) = \{A_i\}_{i=1}^{|\mathcal{A}(u_\lambda)|}$ is the collection of the nodal domains of $u_\lambda$. Consider

$$u_\lambda = \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} u_{\lambda,i} \quad \text{where } u_{\lambda,i} = \begin{cases} u_\lambda & \text{if } x \in \mathcal{A}_i, \\ 0 & \text{elsewhere.} \end{cases} \tag{1.2.5}$$

Observe that $\lambda = \lambda_1(A_i)$ since $u_{\lambda,i}$ does not vanish in $A_i$ (see [**16**] or [**45**]). Apply Theorem 1.1.2 in order to get the following :

$$\begin{aligned} \int_{A_i} |u_{\lambda,i}|^p d\sigma &\geq \int_{V_\delta^i} \delta^p m_{A_i}^p d\sigma \\ &= \delta^p m_{A_i}^p \operatorname{Vol}_g(V_\delta^i) \\ &\geq k_{g,\delta,\lambda_0} m_{A_i}^p \lambda^{-\frac{n}{2}}. \end{aligned}$$

If we sum over all nodal domains, we get that

$$\int_M |u_\lambda|^p d\sigma = \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} \int_{A_i} |u_{\lambda,i}|^p d\sigma \geq k_{g,\delta,\lambda_0} \lambda^{-\frac{n}{2}} \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^p. \tag{1.2.6}$$

To obtain (1.1.7), simply use Sogge's $L^p$ bounds $||u_\lambda||_p \leq \lambda^{\delta(p)} ||u_\lambda||_2$ in (1.2.6).

Notice that one can read off (1.1.9) using the latter argument. Indeed, since

$$\int_M |u_\lambda| d\sigma \leq \operatorname{Vol}_g(M)^{\frac{1}{2}} \left( \int_M |u_\lambda|^2 d\sigma \right)^{\frac{1}{2}} = \operatorname{Vol}_g(M)^{\frac{1}{2}},$$

if we take $p = 1$ in (1.2.6), we get

$$\operatorname{Vol}_g(M)^{\frac{1}{2}} \geq \int_M |u_\lambda| d\sigma \geq k_{g,\delta,\lambda_0} \lambda^{-\frac{n}{2}} \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i},$$

yielding (1.1.9).

### 1.2.4. Proof of Lemma 1.1.3

The proof of Lemma 1.1.3 is an application of the Bathtub principle [**61**, Theorem 1.14] :

**Theorem 1.2.4** (Bathtub principle)**.** *Let $f$ be a real-valued, measurable function on a sigma finite measure space $(X, \Sigma, \mu)$ such that $\mu(\{x : f(x) < t\})$ is finite for all $t \in \mathbb{R}$. Fix $G > 0$ and consider the class of measurable functions on $X$ defined by*

$$\mathcal{C} = \left\{0 \leq g \leq 1 : \int_X g d\mu = G\right\}.$$

*Then, the minimization problem*

$$I = \inf_{g \in \mathcal{C}} \int_X f(x)g(x)d\mu(x)$$

*is solved by $g = \chi_{\{f<s\}}(x) + cs\mu(\{x : f(x) = s\})$, where $s$ is such that*

$$s = \sup_t \{\mu(\{x : f(x) < t\}) \leq G\},$$

*and $c$ is such that*

$$c\mu(\{x : f(x) = s\}) = G - \mu(\{x : f(x) < s\}).$$

*The minimizer is unique if $I$ is finite and if $G = \mu(\{x : f(x) < s\})$ or $G = \mu(\{x : f(x) \leq s\})$.*

Under the assumptions of Lemma 1.1.3, $f$ is a smooth, non negative, strictly decreasing real valued radial function. We prove an equivalent version of Lemma 1.1.3 for strictly increasing functions. In order to obtain the statement for strictly decreasing functions as stated in Lemma 1.1.3, it suffices to replace $f$ by $-f$.

Recall that $r(x) = d_g(x, x_0)$ is the Riemannian distance between $x$ and some fixed point $x_0 \in M$. In that setting, notice that $\mu(\{x : f(r(x)) \leq t\})$ is finite for all $t \in \mathbb{R}$. Moreover, the function $t \to \mathrm{Vol}_g(\{x : f(r(x)) \leq t\})$ is continuous and strictly increasing on $[0, \infty)$. In particular, for all positive constants $G$, there exists $t > 0$ such that $\mathrm{Vol}_g(\{x : f(r(x)) \leq t\}) = G$. Therefore, the solution of the minimization problem stated in the bathtub principle under these assumptions is given by $g = \chi_{\{f \leq R\}}$, where $R$ is such that $\mathrm{Vol}_g(\Omega) = \int \chi_{\{f(r(x)) \leq R\}} d\sigma$. Notice that $\chi_{\{f(r(x)) \leq R\}}$ is the characteristic function of the ball $B_R(x_0)$ of radius $R$ centered at $x_0$ that has the same Riemannian volume as $\Omega$. Thus,

$$I = \inf_{g \in \mathcal{C}} \int_\Omega f(r(x))g(x)d\sigma = \int_{B_R} f(r(x))d\sigma,$$

yielding the desired result.

## 1.3. Proof of Theorems 1.1.10, 1.1.15, and 1.1.17

### 1.3.1. Proof of Theorem 1.1.10

We present the background required to obtain Theorem 1.1.10. For any fixed positive $\lambda$, we consider the $n-$ball,

$$B^n_\lambda = \{x \in \mathbb{R}^n : |x| \leq j_{\frac{n}{2}-1}\lambda^{-\frac{1}{2}}\}, \tag{1.3.1}$$

where $j_{\frac{n}{2}-1}$ is the first positive zero of the Bessel function $J_{\frac{n}{2}-1}$. It is easy to see that the following problem,

$$\begin{cases} \Delta z = \mu z \text{ in } B^n_\lambda, \\ z = 0 \text{ on } \partial B^n_\lambda, \end{cases}$$

has its first eigenvalue equal to $\lambda$, and that the corresponding eigenfunction is given by

$$z(x) = |x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\lambda^{\frac{1}{2}}|x|). \tag{1.3.2}$$

We use the following result, due to G. Chiti (see [**18, 19**]), in the proof :

**Proposition 1.3.1** ([**19**, Theorem 2]). *Let $u$ be a function satisfying* (1.1.10) *and consider $z(x)$, the eigenfunction to the Dirichlet eigenvalue problem on $B^n_\lambda$ defined above. Then, for any $p \geq 1$,*

$$||u||_\infty \left( \int_\Omega |u|^p \right)^{-\frac{1}{p}} \leq ||z||_\infty \left( \int_{B^n_\lambda} z^p \right)^{-\frac{1}{p}}, \tag{1.3.3}$$

*with equality if and only if $\Omega$ is a ball, $c = 0$, $a_{ij} = \delta_{ij}$, $\lambda$ is equal to the first eigenvalue of the equality in* (1.1.10) *and $|\Omega| = |B^n_\lambda|$, where $|E|$ denotes the Lebesgue measure of the set $E$.*

Remark that we can compute the right hand side of (1.3.3) to obtain the following isoperimetric inequality,

$$||u||_\infty \leq K_{n,p}\lambda^{\frac{n}{2p}}||u||_p, \tag{1.3.4}$$

where $K_{n,p}$ is the constant defined in (1.1.14). Indeed, start by computing $||z||_\infty$. The fact that $r^{\frac{n}{2}-1}J_{\frac{n}{2}-1}(r)$ attains its maximum at $r=0$ follows from Poisson's integral (see [**98**, Section 3.3]). Thus, we have that

$$z(0) = \lim_{|x|\to 0} \frac{J_{\frac{n}{2}-1}(\lambda^{\frac{1}{2}}|x|)}{|x|^{\frac{n}{2}-1}} = \frac{\lambda^{\frac{n}{4}-\frac{1}{2}}}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})}. \tag{1.3.5}$$

Since $z(x)$ is a radial function, we get that

$$\left(\int_{B^n_\lambda} z^p\right)^{-\frac{1}{p}} = (nC_n)^{\frac{-1}{p}}\lambda^{\frac{1}{2}-\frac{n}{4}+\frac{n}{2p}}\left(\int_0^{j_{\frac{n}{2}-1}} r^{p-\frac{np}{2}+n-1}J^p_{\frac{n}{2}-1}(r)dr\right)^{\frac{-1}{p}}. \tag{1.3.6}$$

Combine (1.3.5) and (1.3.6), and plug them into (1.3.3) to get (1.3.4).

Proof of Theorem 1.1.10. We start by obtaining (1.1.13). Let us decompose $u_\lambda$ the following way,

$$u_\lambda = \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} u_i \quad \text{where } u_i = \begin{cases} u_\lambda & \text{if } x \in \mathcal{A}_i \\ 0 & \text{elsewhere.} \end{cases} \tag{1.3.7}$$

Since $\operatorname{supp}(u_i)\cap\operatorname{supp}(u_j)=\varnothing$ for $i\neq j$, we note that

$$1 = ||u_\lambda||^2_{L^2(M)} = \int_\Omega \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} u_i^2 = \sum_{i=1}^{|\mathcal{A}(u_\lambda)|}\int_{A_i} u_i^2 = \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} ||u_i||^2_{L^2(A_i)}. \tag{1.3.8}$$

Recall that each $u_i$ corresponds to an eigenfunction of the Dirichlet problem on these nodal domains. Indeed, since $u_i$ does not vanish in $A_i$, it corresponds to the first eigenfunction on $\mathcal{A}_i$ and $\lambda_1(\mathcal{A}_i)=\lambda$ by a corollary of Courant's theorem (see [**45**]).

Thus, we can apply (1.3.4) with $p=2$ to each $u_i$ so that for all $1\le i\le |\mathcal{A}(u_\lambda)|$, we obtain that

$$||u_i||_{L^\infty(\mathcal{A}_i)} \le K_{n,2}\lambda^{\frac{n}{4}}||u_i||_{L^2(\mathcal{A}_i)}.$$

Therefore, we get that

$$m_{A_i} = \sup_{x\in\mathcal{A}_i}|u_i(x)| \le K_{n,2}\lambda^{\frac{n}{4}}||u_i||_{L^2(\mathcal{A}_i)}.$$

Squaring each side and summing over all nodal domains yield that

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{\mathcal{A}_i}^2 \leq K_{n,2}^2 \lambda^{\frac{n}{2}} \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} ||u_i||_{L^2(\mathcal{A}_i)}^2,$$

and we obtain (1.1.13) by applying (1.3.8) to the latter equation. In order to get (1.1.12), we use (1.3.4) with $p = 1$, to get

$$||u_i||_{L^\infty(\mathcal{A}_i)} \leq K_{n,1} \lambda^{\frac{n}{2}} ||u_i||_{L^1(\mathcal{A}_i)}.$$

If we sum over all nodal domains and keep in mind that $\text{supp}(u_i) \cap \text{supp}(u_j) = \varnothing$ for $i \neq j$ , we then get

$$\begin{aligned}\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{\mathcal{A}_i} &\leq K_{n,1} \lambda^{\frac{n}{2}} \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} ||u_i||_{L^1(\mathcal{A}_i)} \\ &= K_{n,1} \lambda^{\frac{n}{2}} ||u_\lambda||_{L^1(\Omega)} \\ &\leq K_{n,1} \lambda^{\frac{n}{2}} ||u_\lambda||_{L^2(\Omega)} \operatorname{Vol}(\Omega)^{\frac{1}{2}}.\end{aligned}$$

The last line follows from Cauchy-Schwartz inequality. Since $||u_\lambda||_{L^2(\Omega)} = 1$, the proof is completed. □

### 1.3.2. Proof of Theorem 1.1.13

Let $I_1$ denote the family of indexes of nodal domains touching the boundary of $\Omega$ and let $I_2 = |\mathcal{A}(u_\mu)| \setminus I_1$. Let us start by obtaining (1.1.17)

Notice that nodal domains whose index is in $I_2$ are such that the eigenfunction $u$ restricted to them corresponds to the first eigenfunction of the Dirichlet eigenvalue problem on such $A_i$, so that $\mu = \lambda_1(A_i)$. Therefore, it is possible to use (1.3.4) with $p = 2$ as done in the proof of Theorem 1.1.10 in order to get that

$$\sum_{i \in I_2} m_{A_i}^2 \leq C\mu.$$

As for nodal domains whose index is in $I_1$, since by the Hormander-Levitan-Avakumovic $L^\infty$ bound, we have that $m_{A_i} \leq C\mu^{1/4}$, we get that

$$\sum_{i \in I_1} m_{A_i}^2 \leq C\sqrt{\mu} \cdot (\mu^{1/4})^2 = C\mu,$$

yielding (1.1.17).

The same reasoning can be applied to obtain (1.1.16), namely

$$\sum_{i\in I_1} m_{A_i} + \sum_{i\in I_2} m_{A_i} \leq C\sqrt{\mu}\cdot\mu^{1/4} + C\mu \leq C'\mu,$$

yielding (1.1.16).

### 1.3.3. Proof of Theorem 1.1.17

The proof is based on the following result :

**Lemma 1.3.2** (Lemma 4.1 in [**63**])**.** *Let $u_{p,1}$ denote the first eigenfunction of the Dirichlet p-Laplacian eigenvalue problem on a bounded Euclidean domain $\Omega \subset \mathbb{R}^n$, then*

$$||u_{p,1}||_{L^\infty(\Omega)} \leq 4^n\lambda^{\frac{n}{p}}||u_{p,1}||_{L^1(\Omega)}.$$

*Note that the constant term $4^n$ is not sharp.*

**Remark 1.3.3.** *One difference between Chiti-type inequalities and the preceding lemma is that Chiti-type inequalities apply to any eigenfunction of the Dirichlet eigenvalue problem rather than only to the first one. However, the generalization of Chiti's results to the p-Laplace operator (see [**2**]) is of the form*

$$||u||_r \leq K(r,q,p,n,\lambda)||u||_q,$$

*where $u$ is any eigenfunction associated to eigenvalue $\lambda$, $0 < q < r \leq +\infty$. It is important to notice that the constant $K(r,q,p,n,\lambda)$ is not explicit (since we can not compute the eigenfunctions of the ball explicitly). Thus, we cannot use it as it was done for the Laplace operator.*

We are ready to prove Theorem 1.1.17.

Proof : Let $||u_{p,\lambda}||_p = 1$. Consider $A_i \subset \Omega$ a nodal domain of $u_{p,\lambda}$. Let us decompose $u_{p,\lambda}$ the following way,

$$u_{p,\lambda} = \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} u_i \quad \text{where } u_i = \begin{cases} u_{p,\lambda} & \text{if } x \in \mathcal{A}_i \\ 0 & \text{elsewhere.} \end{cases} \tag{1.3.9}$$

Since $u_i$ corresponds to the first eigenfunction of the Dirichlet $p$-Laplacian eigenvalue problem on $A_i$, Lemma 1.3.2 yields that

$$||u_i||_{\infty,A_i} \leq 4^n \lambda^{\frac{n}{p}} ||u_i||_{1,A_i}, \qquad \forall \quad 1 \leq i \leq |\mathcal{A}(u_\lambda)|.$$

Therefore, after summing over all nodal domains, we get that

$$\begin{aligned} \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} ||u_i||_{L^\infty(A_i)} = \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} &\leq 4^n \lambda^{\frac{n}{p}} \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} ||u_i||_{L^1(A_i)} \\ &\leq 4^n \lambda^{\frac{n}{p}} ||u_{p,\lambda}||_{L^1(\Omega)} \\ &\leq 4^n \operatorname{Vol}(\Omega)^{1-\frac{1}{p}} \lambda^{\frac{n}{p}} ||u_{p,\lambda}||_{L^p(\Omega)} \\ &= 4^n \operatorname{Vol}(\Omega)^{1-\frac{1}{p}} \lambda^{\frac{n}{p}}. \end{aligned}$$

□

**Acknowledgments**

This research is part of my Ph.D. thesis at the Université de Montreal under the supervision of Iosif Polterovich. The author is grateful to Fedor Nazarov and Mikhail Sodin for communicating the proof of Lemma 1.1.1. The author is indebted to Almut Burchard, for pointing out that Lemma 1.1.3 is in fact an application of the Bathtub principle. The author would also like to thank Laurence Boulanger, Yaiza Canzani, Marlène Frigon, Benoit Kloeckner, Dan Mangoubi, Frédéric Robert, Leonid Polterovich, and Mikhail Sodin for useful discussions.

# Chapitre 2

# PRINCIPAL FREQUENCY OF THE $P$-LAPLACIAN AND THE INRADIUS OF EUCLIDEAN DOMAINS

**Auteur / Author :** Guillaume Poliquin

**Résumé :** Nous étudions les bornes inférieures de la fréquence fondamentale du $p$-laplacien sur des domaines euclidiens $N$-dimensionels. Pour le cas où $p > N$, nous obtenons une borne inférieure de cette valeur propre en terme du rayon inscrit du domaine, sans aucune hypothèse sur la topologie de ce dernier. De plus, nous établissons une borne similaire si $p > N - 1$, cette fois en supposant que la frontière est connexe. Le résultat précédent représente en fait une généralisation des bornes inférieures connues pour la première valeure propre de l'opérateur de Laplace dans le cas de domaines planaires simplement connexes.

**Abstract :** We study the lower bounds for the principal frequency of the $p$-Laplacian on $N$-dimensional Euclidean domains. For $p > N$, we obtain a lower bound for the first eigenvalue of the $p$-Laplacian in terms of its inradius, without any assumptions on the topology of the domain. Moreover, we show that a similar lower bound can be obtained if $p > N - 1$ assuming the boundary is connected. This result can be viewed as a generalization of the classical bounds for the first eigenvalue of the Laplace operator on simply connected planar domains.

## 2.1. Introduction and main results

### 2.1.1. Physical models involving the $p$-Laplacian

Let $\Omega$ be an $N$-dimensional Euclidean bounded domain. The $p$-Laplacian, where $1 < p < \infty,\ p \neq 2$, is a nonlinear operator defined as

$$\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f),$$

for suitable $f$. Notice that the case $p = 2$ corresponds to the well known Laplace operator. The $p$-Laplacian is used to model different physical phenomena, see for instance [**26, 39, 78**].

The Laplacian can be used to describe the vibration of a homogeneous elastic membrane, such as a vibrating drum. The $p$-Laplace operator can also be used to model a vibrating membrane, but composed of a nonelastic membrane under the load $f$,

$$\begin{aligned} -\Delta_p(u) = f \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega. \end{aligned} \tag{2.1.1}$$

The solution $u$ stands for the deformation of the membrane from the equilibrium position (see [**24, 89**]). In that case, its deviation energy is given by $\int_\Omega |\nabla u|^p dx$. Therefore, a minimizer of the Rayleigh quotient,

$$\frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx},$$

on $W_0^{1,p}(\Omega)$ satisfies $-\Delta_p(u) = \lambda_{1,p}|u|^{p-2}u$ in $\Omega$. Here $\lambda_{1,p}$ is usually referred as the principal frequency of the vibrating nonelastic membrane.

### 2.1.2. The eigenvalue problem for the $p$-Laplacian

For $1 < p < \infty$, we study the following eigenvalue problem :

$$\Delta_p u + \lambda |u|^{p-2}u = 0 \text{ in } \Omega, \tag{2.1.2}$$

where we impose the Dirichlet boundary condition and consider $\lambda$ to be the real spectral parameter. We say that $\lambda$ is an eigenvalue of $-\Delta_p$ if (2.1.2) has a nontrivial weak solution $u_\lambda \in W_0^{1,p}(\Omega)$. That is, for any $v \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda \cdot \nabla v - \lambda \int_\Omega |u_\lambda|^{p-2} u_\lambda \cdot v = 0. \tag{2.1.3}$$

The function $u_\lambda$ is then called an eigenfunction of $-\Delta_p$ associated to the eigenvalue $\lambda$. The case $N = 1$ is better understood since explicit solutions in terms of beta functions are known (see [**29, 32**]).

If $N \geq 2$, it is known that the first eigenvalue of the Dirichlet eigenvalue problem of the $p$-Laplace operator, denoted by $\lambda_{1,p}$, is characterized as,

$$\lambda_{1,p} = \inf_{0 \neq u \in C_0^\infty(\Omega)} \left\{ \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} \right\}. \tag{2.1.4}$$

The infimum is attained for a function $u_{1,p} \in W_0^{1,p}(\Omega)$. In addition, $\lambda_{1,p}$ is simple and isolated. Moreover, the eigenfunction $u_1$ associated to $\lambda_{1,p}$ does not change sign, and it is the only such eigenfunction (a proof can be found in [**63**]).

Via, for instance, the Lyusternik-Schnirelmann maximum principle (see [**30**, p. 540]), it is possible to construct $\lambda_{k,p}$ for $k \geq 2$ and hence obtain an increasing sequence of eigenvalues of (2.1.2) tending to $\infty$. There exist other variational characterizations of these eigenvalues. However, no matter which variational characterization one chooses, it always remains to show that all the eigenvalues obtained that way exhaust the whole spectrum of $\Delta_p$.

### 2.1.3. The principal frequency and the inradius

Given a bounded Euclidean domain $\Omega$, the inradius $\rho_\Omega$ is defined as the radius of the largest ball fully contained in $\Omega$, denoted by $B_{\rho_\Omega}$. Obtaining an upper bound for the first eigenvalue of the $p$-Laplacian involving the inradius is immediate. Indeed, noticing that $B_{\rho_\Omega} \subset \Omega$ and using the domain monotonicity property, we have that

$$\lambda_{1,p}(\Omega) \leq \lambda_{1,p}(B_r) = \lambda_{1,p}(B_1)\rho_\Omega^{-p}.$$

On the other hand, lower bounds involving the principal frequency are a greater challenge. A classical lower bound for the case the Laplacian is the Faber-Krahn inequality. It can be adapted to the $p$-Laplacian, $\lambda_{1,p}(\Omega) \geq \lambda_{1,p}(\Omega^*)$, where $\Omega^*$ stands for the $n$-dimensional ball of same volume than $\Omega$ (see [**62**, p. 224]).

It is also possible to obtain a lower bound for the first eigenvalue of the Laplace operator involving the inradius of the domain. That is,

$$\lambda_{1,2}(\Omega) \geq \alpha_{N,2}\ \rho_\Omega^{-2}, \tag{2.1.5}$$

where $\alpha_{N,2} > 0$ is a positive constant. This is equivalent, in the theory of vibrating membranes, to knowing whether for a drum to produce an arbitrarily low note, it is necessary that one can inscribe an arbitrarily large circular drum.

In the case of simply connected planar domains, it is known that (2.1.5) holds with the constant $\alpha_{2,2} = \frac{\pi^2}{4}$ (see [**46, 65, 43, 73**]). More recently, the better constant $\alpha_{2,2} \approx 0.6197$ was found using probabilistic methods in [**6**]. A similar lower bound for domains of connectivity $k \geq 2$ also holds (see [**23**]).

In higher dimensions, it was first noted by W.K. Hayman in [**43**] that if $A$ is a ball with many narrow inward pointing spikes removed from it, then $\lambda_{1,2}(A) = \lambda_{1,2}(\text{Ball})$, but in that case the inradius of $A$ tends to 0. Therefore, bounds of the type,

$$\lambda_{1,2}(\Omega) \geq \alpha_{N,2}\ \rho_\Omega^{-2}, \tag{2.1.6}$$

are generally not possible to obtain even if $\Omega$ is assumed to be simply connected.

The aim of this paper is to study the generalization of (2.1.6) to the case of the $p$-Laplacian. That is,

$$\lambda_{1,p}(\Omega) \geq \alpha_{N,p}\ \rho_\Omega^{-p}, \tag{2.1.7}$$

where $\alpha_{N,p} > 0$ is a positive constant. The main results, stated in Section 2.1.4, were announced in [**78**]. Other related lower bounds for the first eigenvalue were also discussed in [**78**].

### 2.1.4. Main results

A striking difference between the usual Laplace operator and the $p$-Laplacian is that it is possible to obtain bounds of the type (2.1.7) in higher dimensions, as long as $p$ is "large enough" compared to the dimension. Indeed, Hayman's observation remains valid in the case $p \leq N - 1$ since for such $p$, every curve has a trivial $p$-capacity. Recall that $p$-capacity can be defined for a compact set

$K \subset B_r$, where $N > 2$, as

$$\mathrm{Cap}_p(K, B_r) := \inf \left\{ \int_{B_r} |\nabla \phi|^p dx, \quad \phi \in C_0^\infty(B_r),\ \phi \geq 1 \ \text{ on } K \right\}.$$

On the other hand, for $p > N$, even a single point has a positive $p$-capacity (see [**69**, Chapter 13, Proposition 5 and its corollary]). Consequently, Hayman's counterexample no longer holds since removing a single point has an impact on the eigenvalues as it can be seen from (1.1.26). This leads to the following theorem :

**Theorem 2.1.1.** *Let $p > N$ and let $\Omega$ be a bounded Euclidean domain. Then, there exists a positive constant $C_{N,p}$ such that*

$$\lambda_{1,p}(\Omega) \geq C_{N,p} \rho_\Omega^{-p}. \tag{2.1.8}$$

It is known that if $p > N-1$, every curve has a positive $p$-capacity. Therefore, Hayman's counterexample does not work in that case as well. Nevertheless, points do not have a positive $p$-capacity if $N - 1 < p \leq N$. Taking into account these observations, we get the following result :

**Theorem 2.1.2.** *Let $\Omega$ be a bounded Euclidean domain such that $\partial\Omega$ is connected. If $p > N - 1$, then there exists a positive constant $C_{N,p}$ such that*

$$\lambda_{1,p}(\Omega) \geq C_{N,p} \rho_\Omega^{-p}. \tag{2.1.9}$$

Theorem 2.1.2 can be viewed as a generalization of classical results for the Laplacian on simply connected planar domains ($p = N = 2$) discussed earlier in Section 1.3. Our techniques allow to generalize these results to arbitrary dimensions without obtaining explicit constants.

Also notice that the result cannot hold if $p = N - 1$, as noted by Hayman for the case $p = 2, N = 3$.

**Remark 2.1.3.** *The proof of Theorem 2.1.2 also holds provided that the connected components of $\Omega^\complement$ are "large enough". It would be interesting to extend Theorem 2.1.2 for domains that are $k$-connected.*

## 2.2. Proofs

### 2.2.1. Proof of Theorem 2.1.1

In order to prove (2.1.8), we need the following lemma :

**Lemma 2.2.1.** *Let $p > N$, then for all $a \in \partial\Omega$, there exists $C_{N,p} > 0$ such that*

$$\int_{B_R(a)} |u(x)|^p dx \leq C_{N,p} R^p \int_{B_{2R}(a)} |\nabla u(x)|^p dx, \quad \forall\ u \in C_0^\infty(\Omega),$$

*where $R > 0$.*

Proof : Let $R$ be any positive number. Since $p > N$, by Morrey's inequality (see [**33**, Theorem 4 and the following remark, pp. 280 - 283]), we get for any $x \in B_R(a)$,

$$|u(x)| = |u(x) - u(a)| \leq C_{N,p} R^{1-N/p} ||\nabla u||_{L^p(B_{2R}(a))}.$$

Raising to the power $p$ and integrating over $B_R(a)$ yield that

$$\begin{aligned} \int_{B_R(a)} |u(x)|^p dx &\leq C_{N,p} Vol(B_R(a)) R^{p-N} \int_{B_{2R}(a)} |\nabla u(x)|^p dx \qquad (2.2.1) \\ &\leq C_{N,p} R^p \int_{B_{2R}(a)} |\nabla u(x)|^p dx. \end{aligned}$$

□

Following Hayman's argument, the next step consists of using [**43**, Lemma 5],

**Lemma 2.2.2.** *$\Omega$ can be covered by balls $B_r(x)$ for $x \in \partial\Omega, r = \rho_\Omega(1+\sqrt{N})$ such that the $B_r(x)$ can be divided into at most $C_2 \leq (2\sqrt{N}+4)^N$ subsets in such way that different balls in the same subset are disjoint.*

Covering $\Omega$ by such balls combined to (2.2.1), one gets that

$$\begin{aligned} \int_\Omega |u|^p dV &\leq C(N,p) r^p \int_\Omega |\nabla u|^p dV \\ &\leq C_1(N,p) \rho_\Omega^p \int_\Omega |\nabla u|^p dV, \end{aligned}$$

for all $u \in C_0^\infty(\Omega)$. This concludes the proof of Theorem 2.1.1.

### 2.2.2. Proof of Theorem 2.1.2

Let $\delta = 4\rho_\Omega(1+\sqrt{N})$. For any $x \in \partial\Omega$, let $B_\delta(x)$ denote a ball of radius $\delta$ centered at $x$. Let $K$ denote the connected component of $\Omega^\complement \cap \overline{B}_\delta(x)$ containing $x$. Notice that $\operatorname{diam}(K) \geq \delta$. Indeed, by contradiction, suppose that $\operatorname{diam}(K) < \delta$. Then, $K \cap \partial B_\delta(x) = \emptyset$, which is equivalent to saying that $\Omega^\complement \cap \partial B_\delta(x) = \emptyset$. This implies that $\Omega^\complement$ is bounded, a contradiction since $\partial\Omega$ is connected and $\Omega$ is bounded.

Let $r = \rho_\Omega(1+\sqrt{N}) = \frac{\delta}{4}$. We need a lower bound on the $p$-capacity of $K$ :

**Proposition 2.2.3** (Lemma 5.2 in [**14**]). *Let $p > N-1$. Let $K \subset \mathbb{R}^N$ be a compact, connected set. Then, for all $x \in K$ and $r < \frac{1}{2}\operatorname{diam}(K)$, we have*

$$\operatorname{Cap}_p(K \cap \overline{B}_r(x), B_{2r}(x)) \geq \frac{\operatorname{Cap}_p([0,1], B_2(0))}{\operatorname{Cap}_p(\overline{B}_1(0), B_2(0))} \operatorname{Cap}_p(\overline{B}_r(x), B_{2r}(x)).$$

For sake of completeness, we present a proof of Proposition 2.2.3 in Subsection 2.2.3, taken from [**14**, Lemma 5.2].

Since $\operatorname{diam}(K) \geq \delta$, we have that $\frac{1}{2}\operatorname{diam}(K) \geq \frac{1}{2}\delta > r$, which allows us to use the Proposition 2.2.3, yielding that

$$\operatorname{Cap}_p(K \cap \overline{B}_r(x), B_{2r}(x)) \geq \underbrace{\frac{\operatorname{Cap}_p([0,1], B_2(0))}{\operatorname{Cap}_p(\overline{B}_1(0), B_2(0))}}_{C_{N,p}} \operatorname{Cap}_p(\overline{B}_r(x), B_{2r}(x)). \tag{2.2.2}$$

Let $F = K \cap \overline{B}_r(x) = \Omega^\complement \cap \overline{B}_r(x)$. Note that $u \equiv 0$ on $F \subset \overline{B}_r(x)$. Therefore, using [**69**, Theorem 14.1.2], we get that

$$\int_{B_r(x)} |u|^p dV \leq \frac{K_{N,p} r^N}{\operatorname{Cap}_p(F, B_{2r}(x))} \int_{B_r(x)} |\nabla u|^p dV, \tag{2.2.3}$$

where $K_{N,p}$ is a non-explicit positive constant. Using (2.2.2), we get that

$$\frac{1}{\operatorname{Cap}_p(F, B_{2r}(x))} \leq \frac{1}{C_{N,p} \operatorname{Cap}_p(\overline{B}_r(x), B_{2r}(x))} = \frac{1}{C_{N,p} r^{N-p} \operatorname{Cap}_p(\overline{B}_1(0), B_2(0))}.$$

Combining the previous equation with (2.2.3), we get that

$$\int_{B_r(x)} |u|^p dV \leq \frac{K_{N,p} r^p}{C_{N,p} \operatorname{Cap}_p(\overline{B}_1(0), B_2(0))} \int_{B_r(x)} |\nabla u|^p dV. \tag{2.2.4}$$

Following Hayman's argument, the next step consists of using Lemma 2.2.2. Covering $\Omega$ by such balls combined to (2.2.4), one gets that

$$\begin{aligned}\int_\Omega |u|^p dV &\leq K_{1,N,p}K_{2,N,p}r^p \int_\Omega |\nabla u|^p dV \\ &\leq K_{3,N,p}\rho_\Omega^p \int_\Omega |\nabla u|^p dV,\end{aligned}$$

for all $u \in C_0^\infty(\Omega)$, yielding the desired result.

### 2.2.3. Proof of Proposition 2.2.3

We follow the argument of [**14**]. Before proving Proposition 2.2.3, we recall two properties of the $p$-capacity, which are proved in [**44**, p. 28] :

• *Monotonicity* : Let $K_1 \subset K_2$ be two compact subsets of $\Omega$, then

$$\mathrm{Cap}_p(K_1, \Omega) \leq \mathrm{Cap}_p(K_2, \Omega).$$

• *Decreasing sequence of compact subsets* : If $K_i$ is a decreasing sequence of compact subsets of $\Omega$ such that $K = \cap_i K_i$, then

$$\mathrm{Cap}_p(K, \Omega) = \lim_{i\to\infty} \mathrm{Cap}_p(K_i, \Omega).$$

In order to prove Proposition 2.2.3, we need to establish the following lemma :

**Lemma 2.2.4.** *Let $N-1 < p \leq N$. Consider a curve from $x$ to $\xi$, denoted by $\gamma_{[x,\xi]}$, such that $\xi \in \partial B_r(x)$. Then, we have that*

$$\mathrm{Cap}_p(\gamma_{[x,\xi]}, B_{2r}(x)) \geq \mathrm{Cap}_p([x,\xi], B_{2r}(x)),$$

*where $[x,\xi]$ denotes the segment with extrema $x$ and $\xi$.*

Proof : For $N-1 < p \leq N$, curves have a positive $p$-capacity. For $\epsilon > 0$, it is possible to pick a function $\phi \in C_0^\infty(B_{2r}(x); \mathbb{R}^+)$ such that

$$\int_{B_{2r}(x)} |\nabla \phi|^p dx \leq \mathrm{Cap}_p(\gamma_{[x,\xi]}, B_{2r}(x)) + \epsilon, \tag{2.2.5}$$

and $\phi \geq 1$ on a neighborhood $U$ of $\gamma_{[x,\xi]}$.

Denote by $\phi^*$ its Steiner symmetrization with respect to the line from $x$ to $\xi$ and by $U^*$, the Steiner symmetrization of $U$ (see for instance [**45**] for details on Steiner symmetrization). Then, $\phi^* \in H_0^{1,p}(B_{2r}(x))$ and $\phi^* \geq 1$ on $U^*$ by construction. Therefore, we have that

$$\mathrm{Cap}_p(U^*, B_{2r}(x)) \leq \int_{B_{2r}(x)} |\nabla \phi^*|^p dx. \tag{2.2.6}$$

Moreover, a well known property of Steiner symmetrization is that

$$\int_{B_{2r}(x)} |\nabla \phi^*|^p dx \leq \int_{B_{2r}(x)} |\nabla \phi|^p dx. \tag{2.2.7}$$

In addition, notice that $[x, \xi] \subset U^*$. By the monotonicity property of the $p$-capacity, we have that

$$\mathrm{Cap}_p([x, \xi], B_{2r}(x)) \leq \mathrm{Cap}_p(U^*, B_{2r}(x)). \tag{2.2.8}$$

Combining (2.2.8), (2.2.6), (2.2.7), (2.2.5) together and letting $\epsilon \to 0$ yield the desired result. □

We are now ready to prove Proposition 2.2.3.

Proof : Let us fix a point $x \in K$ and a ball $B_r(x)$ with $r < \frac{1}{2}\operatorname{diam}(K)$. Given $\delta > 0$, define $K^\delta = \{y \in \mathbb{R}^n | \operatorname{dist}(y, K) < \delta\}$. Notice that $\overline{K}^\delta \subseteq K^{\delta+\epsilon}$ for all $\epsilon > 0$. Thus, using the $p$-capacity property for decreasing sequences of compact subsets, we get that

$$\mathrm{Cap}_p(K \cap \overline{B}_r(x), B_{2r}(x)) = \lim_{\delta \to 0} \mathrm{Cap}_p(K^\delta \cap \overline{B}_r(x), B_{2r}(x)). \tag{2.2.9}$$

The set $K^\delta$ is open and contains $K$. Since $K^\delta$ is not contained in $\overline{B}_r(x)$ by our choice of $r$, there exists a point $\xi \in \partial B_r(x) \cap K^\delta$ and a continuous curve $\gamma_{[x,\xi]}$ which links $x$ to $\xi$ and lies in $\overline{B}_r(x) \cap K^\delta$. Using the monotonicity property and Lemma 2.2.4, we get that

$$\mathrm{Cap}_p(K^\delta \cap \overline{B}_r(x), B_{2r}(x)) \geq \mathrm{Cap}_p(\gamma_{[x,\xi]} \cap \overline{B}_r(x), B_{2r}(x)) \geq \mathrm{Cap}_p([x, \xi] \cap \overline{B}_r(x), B_{2r}(x)).$$

Then, combining (2.2.9) with the previous inequality and letting $\delta \to 0$ yield that

$$\mathrm{Cap}_p(K \cap \overline{B}_r(x), B_{2r}(x)) \geq \mathrm{Cap}_p([x,\xi] \cap \overline{B}_r(x), B_{2r}(x)).$$

The scaling property of the $p$-capacity implies that

$$\frac{\mathrm{Cap}_p([x,\xi] \cap \overline{B}_r(x), B_{2r}(x))}{\mathrm{Cap}_p(\overline{B}_r(x), B_{2r}(x))} = \frac{\mathrm{Cap}_p([0,1], B_2(0))}{\mathrm{Cap}_p(\overline{B}_1(0), B_2(0))},$$

which concludes the proof.

□

ACKNOWLEDGEMENTS. This research is part of the author's Ph.D. thesis under the supervision of Iosif Polterovich. The author is grateful to Dorin Bucur for his insights on the $p$-capacity and for suggesting that Theorems 2.1.1 and 2.1.2 should hold. Moreover, he wishes to thank Dan Mangoubi for useful discussions and the anonymous referee for helpful remarks.

# Chapitre 3

---

## BOUNDS ON THE PRINCIPAL FREQUENCY OF THE $P$-LAPLACIAN

**Auteur / Author :** Guillaume Poliquin

**Résumé :** Certaines bornes inférieures de la première valeur propre du $p$-laplacien sur des domaines euclidiens de dimension $n$ sont étudiées dans le présent article. En particulier, nous généralisons plusieurs bornes classiques en termes du rayon inscrit d'un domaine et de la fréquence fondamentale de l'opérateur de Laplace au cas où $p \neq 2$. Un corollaire de ces résultats permet d'établir une borne sur la taille des domaines nodaux d'une fonction propre du $p$-laplacien sur un domaine planaire.

**Abstract :** This paper is concerned with the lower bounds for the principal frequency of the $p$-Laplacian on $n$-dimensional Euclidean domains. In particular, we extend the classical results involving the inner radius of a domain and the first eigenvalue of the Laplace operator to the case $p \neq 2$. As a by-product, we obtain a lower bound on the size of the nodal set of an eigenfunction of the $p$-Laplacian on planar domains.

## 3.1. Overview of the $p$-Laplacian

### 3.1.1. Physical models involving the $p$-Laplacian

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. For $1 < p < \infty$, the $p-$Laplacian of a function $f$ on $\Omega$ is defined by $\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f)$ for suitable $f$. The $p$-Laplacian can be used to model the flow of a fluid through porous media in turbulent regime (see for instance [**25, 26**]) or the glacier's ice when treated as a non-Newtonian fluid with a nonlinear relationship between the rate deformation tensor and the deviatoric stress tensor (see [**39**]). It is also used in the Hele-Shaw approximation, a moving boundary problem (see [**55**]).

Let us present a model, well known in the case of the Laplace operator, which remains very useful to understand the physical meaning behind some inequalities that we shall prove, in particular those involving the inner radius of $\Omega$. The nonlinearity of the $p$-Laplacian is often used to reflect the impact of non ideal material to the usual vibrating homogeneous elastic membrane, modeled by the Laplace operator. Thus, the following is used to describe a nonlinear elastic membrane under the load $f$,

$$\begin{aligned} -\Delta_p(u) = f \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega. \end{aligned} \tag{3.1.1}$$

The solution $u$ stands for the deformation of the membrane from the rest position (see [**24, 89**]). In that case, its deformation energy is given by $\int_\Omega |\nabla u|^p dx$. Therefore, a minimizer of the Rayleigh quotient,

$$\frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx},$$

on $W_0^{1,p}(\Omega)$ satisfies $-\Delta_p(u) = \lambda_{1,p}|u|^{p-2}u$ in $\Omega$. Here $\lambda_{1,p}$ is usually referred as the principal frequency of the vibrating non elastic membrane.

### 3.1.2. The eigenvalue problem for the $p$-Laplacian

For $1 < p < \infty$, we study the following eigenvalue problem :

$$\Delta_p u + \lambda |u|^{p-2}u = 0 \text{ in } \Omega, \tag{3.1.2}$$

where we impose the Dirichlet boundary condition and consider $\lambda$ to be the real spectral parameter. We say that $\lambda$ is an eigenvalue of $-\Delta_p$ if (3.1.2) has a nontrivial weak solution $u_\lambda \in W_0^{1,p}(\Omega)$. That is, for any $v \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda \cdot \nabla v - \lambda \int_\Omega |u_\lambda|^{p-2}u_\lambda v = 0. \tag{3.1.3}$$

The function $u_\lambda$ is then called an eigenfunction of $-\Delta_p$ associated to the eigenvalue $\lambda$.

If $n \geq 2$ and $p = 2$, it is well known that one can obtain an increasing sequence of eigenvalues tending to $+\infty$ via the Rayleigh-Ritz method. Moreover, those are all the eigenvalues of $\Delta_2$. Linearity is a crucial component in the argument.

For the general case, it is known that the first eigenvalue of the Dirichlet eigenvalue problem of the $p$-Laplace operator, denoted by $\lambda_{1,p}$, is characterized as,

$$\lambda_{1,p} = \min_{0 \neq u \in C_0^\infty(\Omega)} \left\{ \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} \right\}. \tag{3.1.4}$$

The infimum is attained for a function $u_{1,p} \in W_0^{1,p}(\Omega)$. In addition, $\lambda_{1,p}$ is simple and isolated (there is no sequence of eigenvalues such that $\lambda_{k,p}$ tends to $\lambda_{1,p}$ ; see [**62**]). Moreover, the eigenfunction $u_1$ associated to $\lambda_{1,p}$ does not change sign, and it is the only such eigenfunction (a proof can be found in [**62**]). As for $\lambda_{2,p} > \lambda_{1,p}$, it allows a min-max characterization and every eigenfunction associated to $\lambda_{2,p}$ changes sign only once in $\Omega$ (it was first shown in [**3**] ; see also [**28**]). It is not known if $\lambda_{2,p}$ is isolated. Via for instance Lyusternick-Schnirelmann maximum principle, it is possible to construct $\lambda_{k,p}$ for $k \geq 3$ and hence obtain an increasing sequence of eigenvalues of (3.1.2). There exist other variational characterizations of the $\Delta_p$ eigenvalues. However, no matter what variational characterization one chooses, it always remains to show that all the eigenvalues obtained exhaust the whole spectrum of $\Delta_p$.

## 3.2. Introduction and main results

### 3.2.1. The principal frequency and the inradius

Using the domain monotonicity property, it is easy to obtain an upper bound for the principal frequency of the $p$-Laplacian. Indeed, for $B_r \subset \Omega$, we have that $\lambda_{1,p}(\Omega) \leq \lambda_{1,p}(B_r) = \frac{\lambda_{1,p}(B_1)}{r^p}$. Therefore, if we consider the largest ball that can be inscribed in $\Omega$, we get

$$\rho_\Omega \leq \left( \frac{\lambda_{1,p}(B_1)}{\lambda_{1,p}(\Omega)} \right)^{\frac{1}{p}}, \tag{3.2.1}$$

where $\rho_\Omega := \sup\{r : \exists B_r \subset \Omega\}$. Note that unlike the case $p = 2$ corresponding to the Laplace operator, there are no explicit formulas for $\lambda_{1,p}$ of a ball.

Lower bounds involving the principal frequency are a greater challenge. Nevertheless, some results are known. Let us start by recalling that the classical Faber-Krahn inequality can be adapted to the $p$-Laplacian as noted in [**63**, p. 224] and [**47**, p. 3353] (see also the rearrangement results in [**52**]) : among all domains of given $n$-dimensional volume, the ball minimizes $\lambda_{1,p}$. In other words, we have that $\lambda_{1,p}(\Omega) \geq \lambda_{1,p}(\Omega^*)$, where $\Omega^*$ stands for the $n$-dimensional ball of same volume than $\Omega$.

For the Laplacian, lower bounds of the type

$$\lambda_{1,2}(\Omega) \geq \alpha_{n,2}\ \rho_\Omega^{-2}, \tag{3.2.2}$$

where $\alpha_{n,2} > 0$ is a positive constant, have been studied extensively. If $n = 2$, the first result proved in that direction is due to J. Hersch in [**46**], and states that for convex simply connected planar domains, the latter inequality holds with the constant $\alpha_{2,2} = \frac{\pi^2}{4}$. This result was later improved by E. Makai in [**65**]. For all simply connected domains, he obtained the constant $\alpha_{2,2} = \frac{1}{4}$. An adaptation of Makai's method for the pseudo $p$-Laplacian was studied in [**9**] and lead to a similar lower bound for simply-connected convex planar domains.

By a different approach, W. K. Hayman also obtained a bound for simply-connected domains with the constant, $\alpha_{2,2} = 1/900$. R. Osserman (see [**73, 74, 76**]) later improved that result to $\alpha_{2,2} = 1/4$. R. Osserman also relaxed the assumption of simple connectedness of the domain by considering the connectivity $k$ of a planar domain has a parameter. He obtained a similar lower bound for domains of connectivity $k \geq 2$, a result that was improved in [**23**].

In higher dimensions, it was first noted by W. K. Hayman in [**43**] that if $A$ is a ball with many narrow inward pointing spikes removed from it, then $\lambda_{1,2}(A) = \lambda_{1,2}(\text{Ball})$, but in that case the inradius of $A$ tends to 0. Therefore, bounds of the type,

$$\lambda_{1,2}(\Omega) \geq \alpha_{n,2}\ \rho_\Omega^{-2},$$

are generally not possible to obtain even if $\Omega$ is assumed to be simply connected. The higher dimensional case for Euclidean domains is discussed in Section 3.3. Similar bounds on manifolds are presented in Section 3.4.

In the next subsection, we extend some of these results to the case $p \neq 2$. The size nodal set of an eigenfunction of the $p$-Laplacian is also discussed. All proofs are given in Section 3.5.

### 3.2.2. First eigenvalue of the $p$-Laplacian and inradius of planar domains

We present some lower bounds for the principal frequency involving the inradius of a planar domain with an explicit constant,

$$\lambda_{1,p}(\Omega) \geq \alpha_{2,p}\ \rho_\Omega^{-p}.$$

We do so by adapting proofs obtained for the usual Laplacian. We need two main ingredients : a modified Cheeger-type inequality (see the original result for the Laplacian in [**17**] ; a generalized version for the $p$-Laplacian can be found in [**54**]), and a geometric inequality relating the ratio of the length of the boundary of a domain and its area. Regarding the modified Cheeger inequality, it consists of an adapted version of a result in [**75**] :

**Lemma 3.2.1.** *Let $(S, g)$ be a Riemannian surface, and let $D \subset S$ be a domain homeomorphic to a planar domain of finite connectivity $k$. Let $F_k$ be the family of relatively compact subdomains of $D$ with smooth boundary and with connectivity at most $k$. Let*

$$h_k(D) = \inf_{D' \in F_k} \frac{|\partial D'|}{|D'|}, \tag{3.2.3}$$

*where $|D|$ is the area of $D'$ and $|\partial D'|$ is the length of its boundary. Then,*

$$\lambda_{1,p}(D) \geq \left(\frac{h_k(D)}{p}\right)^p. \tag{3.2.4}$$

The crucial point of Lemma 3.2.1 resides in the fact that the Cheeger constant is computed among subdomains that have a connectivity of at most the connectivity of the domain, which allows us to use geometric inequalities accordingly.

We start by proving the following extension of Osserman-Croke's result to the $p$-Laplacian. We actually generalize a stronger result that was implicit in [**65**] and [**73**], but made explicit in [**41**]. Instead of considering the inner radius, we use the reduced inradius, which is defined by :

$$\tilde{\rho}_\Omega := \frac{\rho_\Omega}{1 + \frac{\pi \rho_\Omega^2}{|\Omega|}}.$$

Notice that $\frac{\rho_\Omega}{2} < \tilde{\rho}_\Omega < \rho_\Omega$.

The first main result consists of extending classical planar inradius bounds of the Laplace operator to the case of the $p$-Laplacian :

**Theorem 3.2.2.** *Let $\Omega$ be a domain in $\mathbb{R}^2$. If $\Omega$ is simply connected, then for all $p > 1$, we have that*

$$\lambda_{1,p}(\Omega) \geq \left( \frac{1}{p \ \tilde{\rho}_\Omega} \right)^p. \tag{3.2.5}$$

*If $\Omega$ is of connectivity $k \geq 2$, then for all $p > 1$, we have that*

$$\lambda_{1,p}(\Omega) \geq \frac{2^{p/2}}{k^{p/2} p^p \ \rho_\Omega^p}. \tag{3.2.6}$$

It is hard to say whether these bounds are optimal for the case $p \neq 2$ since we can not compute eigenfunctions and eigenvalues explicitly on simple domains, unlike the case of the usual Laplacian. Also note that for the Laplace operator, instead of the constant $\frac{1}{4}$ given by (3.2.2), the better constant $\approx 0.6197$ was found using probabilistic methods in [**6**].

**Remark 3.2.3.** *For a bounded domain $\Omega$ in $\mathbb{R}^n$, the ground state problem associated to the $\infty$-Laplacian is the following :*

$$\min \{ |\nabla u| - \lambda_{1,\infty} u; -\Delta_\infty u \} = 0, \tag{3.2.7}$$

*where $\Delta_\infty u := \sum_{i,j=1}^n \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}$. It is a notable feature that $\lambda_{1,\infty} = \frac{1}{\rho_\Omega}$, i.e. the value of $\lambda_{1,\infty}$ can immediately be read off the geometry of $\Omega$, without any topological assumptions on $\Omega$ (see [**50**] and reference therein for additional details).*

### 3.2.3. The limiting case p=1

As $p \to 1$, the limit equation formally reads

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \lambda_{1,1}(\Omega) \qquad \text{in } \Omega, \tag{3.2.8}$$
$$u = 0 \qquad \text{on } \partial\Omega,$$

where $\lambda_{1,1}(\Omega) := \lim_{p\to 1^+} \lambda_{1,p}(\Omega) = h(\Omega)$, where $h(\Omega) = \inf_{D\subset\Omega} \frac{|\partial\Omega|}{|\Omega|}$ with $D$ varying over all non-empty sets $D \subset \Omega$ of finite perimeter (see [**54, 53**] for instance). Here, $\Omega$ is assumed to be smooth enough. If we restrict the subdomains considered in the computation of $h(\Omega)$ to smooth simply connected ones, we get

**Proposition 3.2.4.** *If $\Omega$ is a simply connected planar domain, then*

$$\lambda_{1,1}(\Omega) \geq \frac{1}{\tilde{\rho}_\Omega}. \tag{3.2.9}$$

### 3.2.4. A bound on the size of the nodal set in the planar case

This subsection is devoted to the study of the size of the nodal set $\mathcal{Z}_\lambda = \{x \in \Omega : u_\lambda(x) = 0\}$ of an eigenfunction $u_\lambda$ of (3.1.2). Yau's Conjecture (see [**101**]) asserts that the size of the nodal set of an eigenfunction $u_\lambda$ is comparable to $\lambda^{1/2}$.

Donnelly and Fefferman (see [**27**]) proved Yau's Conjecture for real analytic manifolds. However, if one assumes only that $(M, g)$ is smooth, Yau's Conjecture remains partially open. In the planar case, it is known that $\mathcal{H}^1(\mathcal{Z}_\lambda) \geq C_1\lambda^{1/2}$, where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure (see [**12**]). If $n \geq 3$, lower bounds were obtained (see recent works of Sogge-Zelditch in [**92**], Colding-Minicozzi in [**20**], and Mangoubi in [**68**]).

We generalize the lower bound in the planar case for the $p$-Laplacian on a planar bounded domain. However, the situation is slightly different from the one for the Laplace operator mainly since it is still not known whether the interior of $\mathcal{Z}_\lambda$ is empty or not (see for instance [**28, 40**] for a discussion on that matter). The result is the following :

**Theorem 3.2.5.** *Let $\Omega$ be a planar bounded domain. There is a constant $M > 0$ such that for $\lambda > M$, there exists a positive constant $C$ such that*

$$\mathcal{H}^1(\mathcal{Z}_\lambda) \geq C\lambda^{1/p}. \tag{3.2.10}$$

In order to prove the previous theorem, we need to start proving the analog of a classical result of the Dirichlet Laplacian stating that every eigenfunction must vanish in a ball of radius comparable to the wavelength :

**Proposition 3.2.6.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $R = \left(\frac{C}{\lambda}\right)^{\frac{1}{p}}$, where $C > \lambda_{1,p}(B_1)$, the first eigenvalue of a ball of radius 1. Then, any eigenfunction $u_\lambda$ of (3.1.2) vanishes in any ball of radius $R$.*

This result can be deduced from (3.2.1).

## 3.3. Lower bounds involving the inradius in higher dimensions

### 3.3.1. Lieb's and Maz'ya-Shubin's approaches to the inradius problem

Recall that if $p \leq n-1$, W. K. Hayman's counterexample holds. Therefore, it is not possible to get such lower bounds in that case even if $\Omega$ is simply connected or if $\partial\Omega$ connected. Nevertheless, E. Lieb obtained a similar lower bound by relaxing the condition that the ball has to be completely inscribed in $\Omega$. By doing so, one can also relax some hypotheses on $\Omega$. Instead, throughout this section, we shall only assume that $\Omega$ is an open subset of $\mathbb{R}^n$. No assumptions on the boundedness or on the smoothness of the boundary of $\Omega$ are required. We denote the bottom of the spectrum of $-\Delta_p$ by $\lambda_{1,p}(\Omega)$. In the case that $\Omega$ is a bounded domain, $\lambda_{1,p}(\Omega)$ corresponds to the lowest eigenvalue of $-\Delta_p$ with Dirichlet boundary condition as defined in (3.1.4). In the general case, we write

$$\lambda_{1,p}(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \left\{ \frac{\int_\Omega |\nabla|^p dx}{\int_\Omega |u|^p dx} \right\}. \tag{3.3.1}$$

Lieb's method to get such a bound is to allow a fixed fraction $\alpha \in (0,1)$ of the Lebesgue measure of the ball to stick out of $\Omega$. The result can be found in [**60**,

p. 446]. It states that for a fixed $\alpha \in (0,1)$, if

$$\sigma_{n,p}(\alpha) = \lambda_{1,p}(B_r)\delta_n^{p/n}(\alpha^{-1/n} - 1)^p,$$

where $\delta_n = \frac{r^n}{|B_r|}$, then

$$\lambda_{1,p}(\Omega) \geq \frac{\sigma_{n,p}(\alpha)}{(r^L_{\Omega,\alpha})^p}. \tag{3.3.2}$$

Here $r^L_{\Omega,\alpha} = \sup\{r : \exists B_r \text{ such that } |B_r \setminus \Omega| \leq \alpha |B_r|\}$. Notice that the constant in the lower bound is not totally explicit since it depends on $\lambda_{1,p}(B_1)$, which is not known explicitly for $p \neq 2$.

Maz'ya and Shubin obtained a similar bound, but instead of using Lebesgue measure, they considered the Wiener capacity. The goal of this section is to generalize results of [**70**] to the $p$-Laplacian to cover the case where $p < n$. To do so, we mainly use results stated in [**69**] and simplify the approach used in [**70**], while losing the explicit constants in the bounds. Recall that $p$-capacity is defined for a compact set $F \subset \Omega$, where $n > 2$, as

$$\operatorname{cap}_p(F) = \inf_u \left\{ \int_{\mathbb{R}^n} |\nabla u|^p : u \in C_0^\infty(\Omega), u|_F \geq 1 \right\}. \tag{3.3.3}$$

Fix $\gamma \in (0,1)$. A compact set $F \subset B_r$ is said to be $(p,\gamma)$-negligible if

$$\operatorname{cap}_p(F) \leq \gamma \operatorname{cap}_p(\overline{B}_r). \tag{3.3.4}$$

We are ready to state the main result of this section and its corollaries :

**Theorem 3.3.1.** *If $1 < p \leq n$, there exist two positive constants $K_1(\gamma,n,p)$ and $K_2(\gamma,n,p)$ that depend only on $\gamma, n, p$ such that*

$$K_1(\gamma,n,p) r_{\Omega,\gamma}^{-p} \leq \lambda_{1,p}(\Omega) \leq K_2(\gamma,n,p) r_{\Omega,\gamma}^{-p}, \tag{3.3.5}$$

*where $r_{\Omega,\gamma} = \sup\left\{r : \exists B_r, \overline{B}_r \setminus \Omega \text{ is } (p,\gamma) - \text{negligible}\right\}$ is the interior $p$-capacity radius.*

A direct application of Theorem 3.3.1 is the following :

**Corollary 3.3.2.** *If $1 < p \leq n$, then $\lambda_{1,p}(\Omega) > 0 \iff r_{\Omega,\gamma} < +\infty$.*

Corollary 3.3.2 gives a necessary and sufficient condition of strict positivity of the operator $-\Delta_p$ with Dirichlet boundary conditions. For instance, let $\Omega$ be the complement of any Cartesian grid in $\mathbb{R}^{n\geq 3}$ (this example is adapted from [**69**, p. 789]. Clearly, for any $\gamma \in (0,1)$, $r_{\Omega,\gamma} = +\infty$; thus, $\lambda_{1,p}(\Omega) = 0$. However, if $\Omega$ is a narrow strip, $r_{\Omega,\gamma} < +\infty$, implying that $\lambda_{1,p}(\Omega) > 0$.

Note that since $\lambda_{1,p}(\Omega) > 0$ does not depend on $\gamma$, we immediately get the following :

**Corollary 3.3.3.** *If $1 < p \leq n$, the conditions $r_{\Omega,\gamma} < +\infty$ for different $\gamma$'s are all equivalent.*

Also, one can show that for $1 < p \leq n$, the lower bound given by Theorem 3.3.1 implies the lower bound obtained earlier by Lieb, (3.3.2). To do so, one needs to use an isocapacity inequality that can be found in [**69**, Section 2.2.3], stating that if $F$ be a compact subset of $\mathbb{R}^n$, then

$$\operatorname{cap}_p(F) \geq \omega^{p/n} n^{(n-p)/n} \left(\frac{n-p}{p-1}\right) |F|^{(n-p)/n}, \tag{3.3.6}$$

where equality occurs if and only if $F$ is a ball.

**Proposition 3.3.4.** *If $\alpha = \gamma^{n/(n-p)}$ and $1 < p \leq n$, then $r^L_{\Omega,\alpha} \geq r_{\Omega,\gamma}$. In particular, Theorem 3.3.1 implies* (3.3.2).

PROOF : Let $C = \omega^{p/n} n^{(n-p)/n} \left(\frac{n-p}{p-1}\right)$ and fix $\gamma \in ]0,1[$. Suppose that

$$\operatorname{cap}_p(\overline{B}_r \setminus \Omega) \leq \gamma \operatorname{cap}_p(\overline{B}_r);$$

therefore, using (3.3.6), we get that

$$\begin{aligned} |\overline{B}_r \setminus \Omega| &\leq C^{-\frac{n}{n-p}} \operatorname{cap}_p(\overline{B}_r \setminus \Omega) \\ &\leq C^{-\frac{n}{n-p}} \gamma^{\frac{n}{n-p}} \operatorname{cap}_p(\overline{B}_r)^{\frac{n}{n-p}} \\ &= \alpha |\overline{B}_r|, \end{aligned}$$

yielding the desired result. □

### 3.3.2. Convex domains in $\mathbb{R}^n$

Another way to avoid such difficulty is to consider convex domains in $\mathbb{R}^n$. Doing so, we can prove the following :

**Proposition 3.3.5.** *If $\Omega$ be a convex body in $\mathbb{R}^n$, then the following inequality*

$$\lambda_{1,p}(\Omega) \geq \left(\frac{1}{p\rho_\Omega}\right)^p, \tag{3.3.7}$$

*holds for all $p > 1$ :*

The proof is based on two key facts (see [**76**, p. 26]), namely that the inequality,

$$|\partial\Omega| \geq h(\Omega)|\Omega|,$$

is required to be true only for subdomains bounded by level surfaces of the first eigenfunction of $\Omega$ (see the proof of Lemma 3.2.1), and that if $\Omega$ is convex, then those subdomains $\Omega'$ are also convex (see [**10**, Theorem 1.13]). Recalling that $\frac{1}{\rho_{\Omega'}} \geq \frac{1}{\rho_\Omega}$, the proposition then follows easily.

### 3.3.3. Further discussion

A striking difference between the usual Laplace operator and the $p$-Laplacian is that it is possible to obtain bounds of the type (3.2.2) in higher dimensions, as long as $p$ is "large enough" compared to the dimension. Indeed, W.K. Hayman's observation remains valid in the case $p \leq n-1$ since for such $p$, every curve has a trivial $p$-capacity. On the other hand, for $p > n$, even a single point has a positive $p$-capacity (see [**69**, Chapter 13, Proposition 5 and its corollary]). Consequently, W. K. Hayman's counterexample no longer holds since removing a single point has an impact on the eigenvalues as it can be seen from (3.1.4). Taking the latter observation into account, it is possible to prove the following theorem :

**Theorem 3.3.6.** *Let $p > n$ and let $\Omega$ be a bounded domain. Then there exists a positive constant $C_{n,p}$ such that*

$$\lambda_{1,p}(\Omega) \geq \frac{C_{n,p}}{\rho_\Omega^p}. \tag{3.3.8}$$

It is known that if $p > n-1$, every curve has a positive $p$-capacity. Therefore, W. K. Hayman's counter example does not work in that case, leading to the following result :

**Theorem 3.3.7.** *For $p > n-1$, suppose that $\Omega$ is a bounded domain such that $\partial\Omega$ is connected, then there exists $C_{n,p} > 0$ such that*

$$\lambda_{1,p}(\Omega) \geq \frac{C_{n,p}}{\rho_\Omega^p}.$$

This result can be viewed as a generalization of classical results for the Laplacian on simply connected planar domains ($p = n = 2$) discussed earlier however, without the explicit constant. Also notice that the result can not hold if $p = n-1$ as noted by W. K. Hayman for the case $p = 2, n = 3$.

Theorems 3.3.6 and 3.3.7 were suggested by D. Bucur and their proofs can be found in [**79**].

## 3.4. Lower bounds involving eigenvalues of the $p$-Laplace operator on manifolds

### 3.4.1. A two-sided inradius bound for nodal domains on closed Riemannian surfaces

Some results concerning the inner radius of nodal domains on manifolds were obtained in [**68, 66**]. As suggested by D. Mangoubi, it is possible to extend a two sided estimate valid for closed surfaces, [**68**, Theorem 1.2] to the case of the $p$-Laplacian. Let $U_\lambda$ denote the $\lambda$-nodal domain associated to the eigenfunction $u_{p,\lambda}$. The result is the following :

**Theorem 3.4.1.** *Let $(S, g)$ be a closed surface. Then, there exists two positive constants $c, C$ such that*

$$c\lambda^{-1/p} \leq \rho_{U_\lambda} \leq C\lambda^{-1/p}.$$

The proof of the case $p = 2$ relies on two main tools, namely Faber-Krahn, which still holds for the $p$-Laplace operator, and a Poincaré inequality, [**68**, Theorem 2.4]. The latter can also be generalized to our setting. We have the following result :

**Lemma 3.4.2.** *Let $Q \subseteq \mathbb{R}^2$ be a cube whose edge is of length $a$. Fix $\alpha \in (0,1)$ and let $u$ be in $C_0^\infty(Q)$ such that $u$ vanishes on a curve whose projection on one* *of the edges of $Q$ has size $\geq \alpha a$. Then, there exists a constant $C(\alpha)$ such that*

$$\int_Q |u|^p dx \leq C(\alpha) a^p \int_Q |\nabla u|^p dx.$$

Since the proof of Theorem 3.4.1 and Lemma 3.4.2 are direct extensions of what is done in [**67**], we omit details.

### 3.4.2. Lower bounds involving the inradius on surfaces

We can adapt some inradius results of [**73**] valid for simply connected domains of surfaces $(S, g)$ with controlled Gaussian curvature to the case of the $p$-Laplacian :

**Proposition 3.4.3.** *Let $S$ be a surface. Let $D \subset S$ be a simply connected domain. Denote by $K$ its Gaussian curvature and by $\beta$ the quantity $\int_D K^+$, where $K^+ = \max\{K, 0\}$. If $\beta \leq 2\pi$ the inequality,*

$$\lambda_{1,p}(D) \geq \left(\frac{1}{p\ \rho_D}\right)^p, \tag{3.4.1}$$

*holds.*

If the surface has a negative Gaussian curvature, then we have the following :

**Proposition 3.4.4.** *Let $S$ be a simply connected surface. Let $D \subset S$ be a simply connected domain such that $K \leq -\alpha^2$ on $D$, $\alpha > 0$, where $K$ stands for its Gaussian curvature. The stronger inequality*

$$\lambda_{1,p}(D) \geq \left(\frac{\alpha \coth(\alpha \rho_D)}{p}\right)^p \tag{3.4.2}$$

*holds.*

*Furthermore, if $S$ is a complete surface, then one has the following,*

$$\lambda_{1,p}(D) \geq \left(\frac{\alpha \coth(\alpha R_D)}{p}\right)^p, \tag{3.4.3}$$

*where $R_D$ stands for the circumradius of $D$.*

### 3.4.3. Negatively curved manifolds

The first result, called McKean's theorem (see [**71**]) in the case of the Laplace operator, concerns manifolds of negative sectional curvature :

**Proposition 3.4.5.** *Let $(M, g)$ be a complete and simply connected Riemannian manifold. Let $D \subset M$ be a domain such that its sectional curvature is bounded by $\leq -\alpha^2, \alpha > 0$ , then*

$$\lambda_{1,p}(D) \geq \frac{(n-1)^p \alpha^p}{p^p}. \tag{3.4.4}$$

The next result is valid on an arbitrary surface (without additional assumptions, such as being simply connected), but it is only valid for doubly-connected domains.

**Proposition 3.4.6.** *Let $(S, g)$ be a surface. Let $D \subset S$ be a doubly-connected domain such that $K \leq -1$ where $K$ stands for its Gaussian curvature, then*

$$\lambda_{1,p}(D) \geq \frac{1}{p^p}. \tag{3.4.5}$$

### 3.4.4. Minimal submanifolds in $\mathbb{R}^n$

It is also possible to prove the following using similar arguments :

**Proposition 3.4.7.** *Let $D$ be a domain on a $m$-dimensional minimal submanifold in $\mathbb{R}^n$. If $D$ lies in a ball of radius $R$, then*

$$\lambda_{1,p}(D) \geq \left(\frac{m}{p}\right)^p. \tag{3.4.6}$$

## 3.5. Proofs

### 3.5.1. Proof of Lemma 3.2.1

By (3.2.3), it follows that if one proves (3.2.4) for all domains in a regular exhaustion of $D$, then (3.2.4) will also hold for $D$. Knowing that every finitely-connected domain has a regular exhaustion by domains of the same connectivity, we may assume that $D$ has a smooth boundary. Hence, the $p$-Laplacian Dirichlet eigenvalue problem admits a solution $u_1$ corresponding to $\lambda_{1,p}$ and may be chosen without loss of generality such that $u_1 \geq 0$.

Let $g = u_1^p$. Then, Hölder's inequality implies that

$$\begin{aligned} \int_D |\nabla g(x)| dx &= p \int_D |u_1(x)|^{p-1} |\nabla u_1(x)| dx \\ &\leq p ||u_1||_p^{p-1} ||\nabla u_1||_p. \end{aligned}$$

Dividing by $||u_1||_p^p$, one gets

$$\frac{1}{p}\frac{\int_D |\nabla g(x)|dx}{\int_D |g(x)|dx} \leq \frac{||\nabla u_1||_p}{||u_1||_p} = \lambda_{1,p}(D)^{1/p}. \tag{3.5.1}$$

For regular values $t$ of the function $g$, we define the set

$$D_t = \{y \in D : g(y) > t\}.$$

We want to show that the connectivity of $D_t$ is at most $k$, since it will imply by (3.2.3) that

$$L(t) \geq h_k(D)A(t), \tag{3.5.2}$$

for all regular values of $t$. Here, $L(t)$ is the length of the boundary of $D_t$ and $A(t)$ the area of $D_t$.

Since $t$ is regular, the boundary $\partial D_t$ consists of a finite number, say $m$, of smooth curves $C_1, C_2, ..., C_m$ along which $\nabla g \neq 0$. Let $D_t'$ be any connected component of $D_t$. If the connectivity of $D_t'$ were greater than $k$, then the complement of $D_t$ would contain a component lying completely in $D$ of boundary say, $C_l$. Since $u_1 \in C(D) \cap W^{1,p}(D)$ and

$$\int_D |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla v \; dx = \lambda_{1,p} \int_D |u_1|^{p-2} u_1 v \; dx \geq 0,$$

for all non negative $v$ in $C_0^\infty(D)$, $u_1$ is $p$-superharmonic (see [**64**, Theorem 5.2]). By definition of $p$-superharmonicity (again [**64**, Definition 5.1]), the comparison principle holds. Therefore, since $u_1 = \sqrt[p]{t}$ on $C_l$, it follows that $u_1^p \geq t$ in the internal region of $C_l$, which contradicts the fact that the internal region lies in the complement of $D_t'$, so that the function $g$ has to be such that $g < t$.

Now, since the set of singular values of $g$ is a closed set of measure zero by Sard's theorem, its complement is a countable union of open intervals $I_n$. Thus, we can define the following

$$E_n = \{p \in D : g(p) \in I_n\}.$$

Then, by the coarea formula,

$$\int_D |\nabla g(x)| \; dx \;\; \geq \;\; \sum_{n=1}^{\infty} \int_{E_n} |\nabla g(x)| \; dx = \sum_{n=1}^{\infty} \int_{I_n} L(t) \; dt$$

$$\geq \quad h_k(D) \sum_{n=1}^{\infty} \int_{I_n} A(t)\ dt = h_k(D) \int_D g(x)\ dx. \qquad (3.5.3)$$

Combining (3.5.1) with (3.5.3) completes the proof.

### 3.5.2. Proof of Theorem 3.2.2, of Propositions 3.4.3, and 3.4.4

We use the following lemma which was proved in [**41**] :

**Lemma 3.5.1.** *If* $\Omega' \subset \Omega$, *then*

$$\tilde{\rho}_{\Omega'} \leq \tilde{\rho}_{\Omega}.$$

Proof of Theorem 3.2.2. Combining Lemma 3.2.1 with Lemma 3.5.1 and Bonnesen's inequality (see [**11**]) then yields the desired result.

In order to adapt the argument to planar domains of connectivity $k$, one must use a generalized geometric inequality that can be found in [**23**, Theorem 1], stating that

$$\frac{|\partial\Omega|}{|\Omega|} \geq \frac{\sqrt{2}}{\sqrt{k\rho}}.$$

Combining Lemma 3.2.1, [**23**, Theorem 1], and the fact that $\rho_{\Omega'} \leq \rho_{\Omega}$ provided that $\Omega' \subset \Omega$ yields the desired result. □

Proof of Proposition 3.4.3. We start by using Burago and Zalgaller's inequality that can be found in [**15**], and that states that

$$\rho_D|\partial D| \geq |D| + \left(\pi - \frac{1}{2}\beta\right)\rho_D^2 \quad \Longleftrightarrow \quad \frac{|\partial D|}{|D|} \geq \frac{1 + \frac{(\pi-\frac{1}{2}\beta)\rho_D^2}{|D|}}{\rho_D} \geq \frac{1}{\rho_D}. \qquad (3.5.4)$$

By definition of $h_1(D)$, together with (3.5.4), one gets the desired result.

□

Proof of Proposition 3.4.4. Since $S$ is simply connected, one can use [**48**, Theorem 1] or [**76**, Theorem 8(c)] to get that

$$|\partial D| \geq \frac{\alpha|D|}{\tanh \alpha\rho_D} + \frac{2\pi}{\alpha}\tanh\frac{\alpha\rho_D}{2} \geq \alpha|D|\coth(\alpha\rho_D).$$

Combining the previous inequality with Lemma 3.2.1 and the fact that $\rho_{\Omega'} \leq \rho_\Omega$ provided that $\Omega' \subset \Omega$ proves the first part of Theorem 3.4.4. To conclude the proof, one must use [**76**, Theorem 8], which states that

$$|\partial D| \geq \alpha|D|\coth(\alpha R_D).$$

Combining the latter inequality with Lemma 3.2.1 and the fact that for any subdomain $D'$, its circumradius satisfies $R' \leq R, \coth(\alpha R') \geq \coth(\alpha R)$ yields the desired result. □

### 3.5.3. Proof of Proposition 3.2.6

Consider a ball $B_R$ and suppose that $u_\lambda \neq 0$ inside $B_R$. Since $u_\lambda \neq 0$ in $B_R$, there exists a nodal domain $A$ of $u_\lambda$ such that $B_R \subset A$. By (3.2.1), we have that

$$R \leq \rho_A \leq \left(\frac{\lambda_{1,p}(B_1)}{\lambda_{1,p}(A)}\right)^{\frac{1}{p}}.$$

Since the restriction of $u_\lambda$ corresponds to the first eigenfunction on $A$, $\lambda_{1,p}(A) = \lambda$. Thus, we get that

$$R \leq \left(\frac{\lambda_{1,p}(B_1)}{\lambda}\right)^{\frac{1}{p}},$$

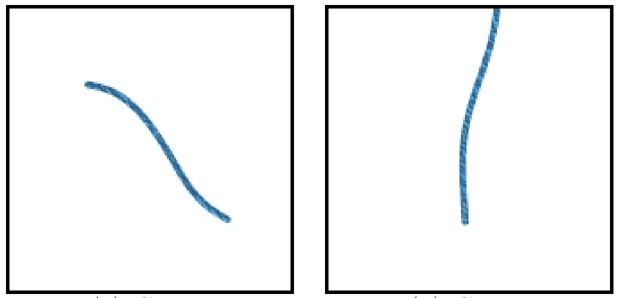
leading to a contradiction.

### 3.5.4. Proof of Theorem 3.2.5

Notice that for the $p$-Laplacian, it is still unclear whether $\mathrm{int}(\mathcal{Z}_\lambda)$ is empty or not. Nevertheless, if $\mathrm{int}(\mathcal{Z}_\lambda) \neq \emptyset$, then $\mathcal{H}^1(\mathcal{Z}_\lambda) = +\infty$.

Suppose that $\mathrm{int}(\mathcal{Z}_\lambda) = \emptyset$. We need to show that $\exists C > 0$ such that $\mathcal{H}^1(\mathcal{Z}_\lambda) \leq C\lambda^{\frac{1}{p}}$ for $\lambda$ large enough.

By Proposition 3.2.6, $\Omega$ can be split into squares $S_c$ of side $c = \mathrm{Area}(\Omega)\lambda^{-1/p}$ such that each square contains a zero of $u_\lambda$. Take $\lambda$ large enough to allow that the center of each square corresponds to a zero of $u_\lambda$. We represent the various



(a) Case 3 (b) Case 4

Figure 3.1. Nodal lines in a square.

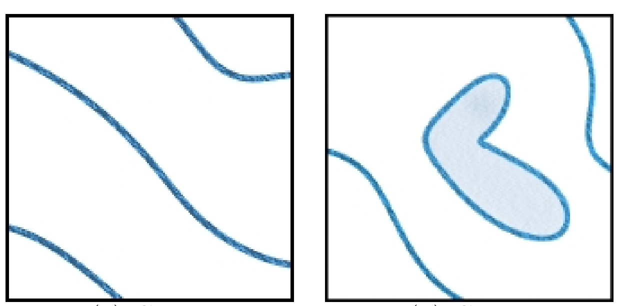
cases of nodal lines in a square in Figure 3.1 and Figure 3.2. Recall the following Harnack inequality :

**Theorem 3.5.2** (Theorem 1.1 of [**97**])**.** *Let $K = K(3\rho) \subset \Omega$ be a cube of length $3\rho$. Let $u_\lambda$ be a solution of* (3.1.2) *associated to the eigenvalue $\lambda$ such that $u_\lambda(x) \geq 0$ for all $x \in K$, then*

$$\max_{K(\rho)} u_\lambda(x) \leq C \min_{K(\rho)} u_\lambda(x),$$

*where $C$ is a positive constant that depends on $n, p$ and on $\lambda$.*

Notice that in order for Theorem 3.5.2 to fail, we know that every neighborhood of the boundary of the nodal domain must contain points such that $u_\lambda$ changes sign. To do so, the nodal line must be closed. Therefore, nodal lines depicted in Figure 3.1 can not occur.



(a) Case 1 (b) Case 2

Figure 3.2. Closed nodal lines in a square.

Suppose that there is a closed nodal line inside a square. If it were the case, it would mean that there exists a nodal domain $A$ included the square. Since the eigenfunction restricted to $A$ would then correspond to the first one, domain monotonicity would yield a contradiction. Indeed, we would have the following

$$\lambda_{1,p}(S_1)\lambda\,\mathrm{Area}(\Omega)^{-p} = \lambda_{1,p}(S_c) \le \lambda_{1,p}(A) = \lambda,$$

yielding that

$$\lambda_{1,p}(S_1) \le \mathrm{Area}(\Omega)^p,$$

a contradiction (simply rescale $\Omega$ if necessary).

Therefore, any nodal line inside any such square must be at least of length greater than or equal to $C_1\lambda^{-1/p}$ . Since $\mathrm{Area}(\Omega) = \mathrm{Area}(S_c)\lambda^{2/p}$, there are roughly $\lambda^{2/p}$ squares covering $\Omega$, implying that

$$\mathcal{H}^1(\mathcal{Z}_\lambda) \ge C_2\lambda^{2/p}\lambda^{-1/p} = C_2\lambda^{1/p}.$$

### 3.5.5. Proof of Theorem 3.3.1

Using [**69**, Theorem 14.1.2], we get the following lemma :

**Lemma 3.5.3.** *Let $F$ be a compact subset of $\overline{B}_r$.*
*1. If $1 < p \le n$, for all $u \in C^\infty(\overline{B}_r)$ such that $u \equiv 0$ on $F$, there exists a positive constant $C_1(n,p)$ depending only on $n$ and $p$ such that*

$$\mathrm{cap}_p(F) \le \frac{C_1(n,p)}{r^{-n}} \frac{\int_{\overline{B}_r} |\nabla u|^p}{\int_{\overline{B}_r} |u|^p}. \tag{3.5.5}$$

*2. If $1 < p \le n$, for all $u \in C^\infty(\overline{B}_r)$ such that $u \equiv 0$ on $F$, where $F$ is a negligible subset of $\overline{B}_r$, and*

$$||u||_{L^p(\overline{B}_{r/2})} \le C||\nabla u||_{L^p(\overline{B}_r)}, \tag{3.5.6}$$

*then there exists a positive constant $C_2(n,p)$ depending only on $n$ and $p$ such that*

$$\mathrm{cap}_p(F) \ge \frac{C_2(n,p)}{r^{-n}} \frac{\int_{\overline{B}_r} |\nabla u|^p}{\int_{\overline{B}_r} |u|^p}. \tag{3.5.7}$$

Let us also recall the following two properties of the $p$-capacity (which are proved in [**69**]) :

– The $p$-capacity is monotone : $F_1 \subset F_2 \implies \mathrm{cap}_p(F_1) \le \mathrm{cap}_p(F_2)$.

– The $p$-capacity of a closed ball of radius $r$ can be computed explicitly :

$$\operatorname{cap}_p(\overline{B}_r) = r^{n-p}\operatorname{cap}_p(\overline{B}_1) = r^{n-p}\omega_n\left(\frac{|n-p|}{p-1}\right)^{p-1}, \tag{3.5.8}$$

where $\omega_n$ is the area of the unit sphere $S^{n-1}\subset\mathbb{R}^n$ and $p\neq n$.

We are ready to begin the proof of the lower bound. The ideas used in the following proof come from [**70**].

Lower bound of Theorem 3.3.1. Fix $\gamma\in(0,1)$ and choose any $r>r_{\Omega,\gamma}$. Then, any ball $\overline{B}_r$ is of non-negligible intersection, i.e.

$$\operatorname{cap}_p(\overline{B}_r\setminus\Omega)\geq\gamma\operatorname{cap}_p(\overline{B}_r).$$

Since any $u\in C_0^\infty(\Omega)$ vanishes on $\overline{B}_r\setminus\Omega$, one can use Lemma 3.5.3, part 1. Using (3.5.9) and the explicit value of the $p$-capacity of a closed ball, one gets the following :

$$\begin{aligned}\int_{\overline{B}_r}|u|^p dx &\leq \frac{C_1(n,p)}{r^{-n}\operatorname{cap}_p(\overline{B}_r\setminus\Omega)}\int_{\overline{B}_r}|\nabla u|^p dx \qquad (3.5.9)\\ &\leq \frac{C_1(n,p)}{r^{-n}\gamma\operatorname{cap}_p(\overline{B}_r)}\int_{\overline{B}_r}|\nabla u|^p dx\\ &\leq \frac{C_1(n,p)}{r^{-p}\gamma\operatorname{cap}_p(\overline{B}_1)}\int_{\overline{B}_r}|\nabla u|^p dx.\end{aligned}$$

Choose a covering of $\mathbb{R}^n$ by balls $\overline{B}_r=\overline{B}_r^{(k)}$, $k=1,2,\ldots$, so that the multiplicity of this covering is at most $N=N(n)$, which is bounded since for example, for $n\geq 2$, the following estimate is valid (see [**86**, Theorem 3.2]) :

$$N(n)\leq n\log(n)+n\log(\log(n))+5n.$$

Sum up (3.5.9) to get the following :

$$\begin{aligned}\int_{\mathbb{R}^n}|u|^p dx &\leq \sum_k\int_{\overline{B}_r^{(k)}}|u|^p dx\\ &\leq \frac{C_1(n,p)}{r^{-p}\gamma\operatorname{cap}_p(\overline{B}_1)}\sum_k\int_{\overline{B}_r^{(k)}}|\nabla u|^p dx\end{aligned}$$

$$\leq \frac{C_1(n,p)N(n)}{r^{-p}\gamma \operatorname{cap}_p(\overline{B}_1)} \int_{\mathbb{R}^n} |\nabla u|^p dx.$$

Since for all $u \in C_0^\infty(\Omega)$, we have that

$$\frac{\gamma \operatorname{cap}_p(\overline{B}_1) r^{-p}}{C_n N(n)} \leq \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx},$$

we get that

$$\lambda_{1,p}(\Omega) \geq K_1(\gamma,n,p) r^{-p} = \frac{\gamma \operatorname{cap}_p(\overline{B_1})}{C_1(n,p)N(n)} r^{-p}. \tag{3.5.10}$$

Taking the limit of (3.5.10) as $r \searrow r_{\Omega,\gamma}$ yields the desired result. □

The proof of the upper bound is very similar to the last one, but uses the second part of Lemma 3.5.3. However, this proof is different from the one given in [**70**], but has the disadvantage of not yielding an explicit constant. Nevertheless, no such constant are known in the case of the $p$-Laplacian (recall that Lieb's constant for the lower bound and that the upper bound given in (3.2.1) are not totally explicit since they both depend on $\lambda_{1,p}(B)$).

Upper bound of Theorem 3.3.1. Fix $\gamma \in (0,1)$ . Consider $r_{\Omega,\gamma}$. By definition, we know that

$$\operatorname{cap}_p(\overline{B}_{r_{\Omega,\gamma}} \setminus \Omega) \leq \gamma \operatorname{cap}_p(\overline{B}_{r_{\Omega,\gamma}}).$$

We know want to use Lemma 3.5.3, part 2. Let $F = \overline{B}_{r_{\Omega,\gamma}} \setminus \Omega$. Clearly, $F$ is a negligible subset of $\overline{B}_{r_{\Omega,\gamma}}$. It is also clear that any test function $u \in C_0^\infty(\Omega)$ will vanish identically on $F$. Therefore, for any such function, using Poincaré inequality, we get

$$||u||_{L^p(\overline{B}_{r_{\Omega,\gamma}/2})} \leq ||u||_{L^p(\overline{B}_{r_{\Omega,\gamma}})} \leq C||\nabla u||_{L^p(\overline{B}_{r_{\Omega,\gamma}})}.$$

Therefore, one can use Lemma 3.5.3 part 2 and get :

$$\begin{aligned} \int_{\overline{B}_{r_{\Omega,\gamma}}} |u|^p dx &\geq \frac{C_2(n,p)}{r_{\Omega,\gamma}^{-n} \operatorname{cap}_p(\overline{B}_{r_{\Omega,\gamma}} \setminus \Omega)} \int_{\overline{B}_{r_{\Omega,\gamma}}} |\nabla u|^p dx \\ &\geq \frac{C_2(n,p)}{r_{\Omega,\gamma}^{-n} \gamma \operatorname{cap}_p(\overline{B}_{r_{\Omega,\gamma}})} \int_{\overline{B}_{r_{\Omega,\gamma}}} |\nabla u|^p dx \end{aligned}$$

$$\geq \quad \frac{C_2(n,p)}{r_{\Omega,\gamma}^{-p}\gamma\,\mathrm{cap}_p(\overline{B}_1)}\int_{\overline{B}_{r_{\Omega,\gamma}}}|\nabla u|^p dx.$$

Choose a covering of $\mathbb{R}^n$ by balls $\overline{B}_{r_{\Omega,\gamma}} = \overline{B}^{(k)}_{r_{\Omega,\gamma}}$, $k = 1, 2, ...$, so that the multiplicity of this covering is at most $N = N(n)$, and get

$$\begin{aligned}\int_{\mathbb{R}^n}|\nabla u|^p dx &\leq \sum_k \int_{\overline{B}^{(k)}_{r_{\Omega,\gamma}}}|\nabla u|^p dx \\ &\leq \frac{r_{\Omega,\gamma}^{-p}\gamma\,\mathrm{cap}_p(\overline{B}_1)}{C_2(n,p)}\sum_k \int_{\overline{B}^{(k)}_{r_{\Omega,\gamma}}}|u|^p dx \\ &\leq \frac{r_{\Omega,\gamma}^{-p}\gamma\,\mathrm{cap}_p(\overline{B}_1)N(n)}{C_2(n,p)}\int_{\mathbb{R}^n}|u|^p dx.\end{aligned}$$

For such $u \in C_0^\infty(\Omega)$, we have that

$$\lambda_{1,p}(\Omega) \leq \frac{\int_{\mathbb{R}^n}|\nabla u|^p dx}{\int_{\mathbb{R}^n}|u|^p dx} \leq \frac{r_{\Omega,\gamma}^{-p}\gamma\,\mathrm{cap}_p(\overline{B}_1)N(n)}{C_2(n,p)};$$

thus, yielding the desired result with $K_2(\gamma,n,p) = \frac{\gamma\,\mathrm{cap}_p(\overline{B}_1)N(n)}{C_2(n,p)}$. □

### 3.5.6. Proofs of Propositions 3.4.5, 3.4.6, and 3.4.7

The proofs are very straightforward and consist of combining Lemma 3.2.1 with use [**76**, p.26, eq. (121)-(122)], [**74**, p. 1208, Eq. (4.31)], or [**75**, Eq. (14)] respectively.

Acknowledgements. This paper is part of the author's Ph.D. thesis under the supervision of Iosif Polterovich. The author is thankful to Iosif Polterovich and to Dima Jakobson for suggesting the problem. Moreover, he wishes to thank Dorin Bucur, Yaiza Canzani, Daniel Grieser, Dan Mangoubi, Pavel Drabek, and Robert G. Owens for useful discussions and references. Finally, the author thanks Gabrielle Poliquin for her help with the figures.

# Chapitre 4

## WOLF-KELLER THEOREM FOR NEUMANN EIGENVALUES

**Auteurs / Authors :** Guillaume Poliquin and Guillaume Roy-Fortin

**Résumé :** L'inégalité classique de Szegő–Weinberger affirme que, parmi les domaines planaires bornes d'aire fixe, la premère valeur propre non-nulle de Neumann est maximisée par un disque. Récemment, il a été montré dans [**37**] que, pour les domaines simplement connexes d'aire donnée, la deuxième valeur propre non-nulle de Neumann est maximisée à la limite par une suite de domaines dégénerant vers l'union disjointe de disques identiques. Nous montrons que les valeurs propres de Neumann ne sont pas toujours maximisées par l'union disjointe de disques arbitraires pour les domaines planaires d'aire fixe. Il s'agit de l'analogue d'un résultat obtenu plus tôt par Wolf and Keller pour les valeurs propres de Dirichlet.

**Abstract :** The classical Szegő–Weinberger inequality states that among bounded planar domains of given area, the first nonzero Neumann eigenvalue is maximized by a disk. Recently, it was shown in [**37**] that, for simply connected planar domains of given area, the second nonzero Neumann eigenvalue is maximized in the limit by a sequence of domains degenerating to a disjoint union of two identical disks. We prove that Neumann eigenvalues of planar domains of fixed area are not always maximized by a disjoint union of arbitrary disks. This is an analogue of a result by Wolf and Keller proved earlier for Dirichlet eigenvalues.

## 4.1. Introduction and main results

### 4.1.1. Dirichlet and Neumann eigenvalue problems

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain (not necessarily connected) of volume $|\Omega|$. Throughout the paper, we assume that $|\Omega| = 1$. Let $\Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ be the Laplace operator. The Dirichlet eigenvalue problem,

$$-\Delta u = \lambda u \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega, \tag{Dirichlet}$$

has discrete spectrum (see [**45**, p.7])

$$0 < \lambda_1 \leq \lambda_2 \leq ... \nearrow \infty.$$

According to a classical result of Faber and Krahn, the first eigenvalue $\lambda_1$ is minimized by a ball. Furthermore, we have the Krahn inequality, obtained from the Faber-Krahn result, stating that $\lambda_2$ is minimized by the disjoint union of two identical balls (see, for instance, [**45**], [**56**] or [**16**]).

If $\Omega$ has Lipschitz boundary, then it is well known that the Neumann eigenvalue problem,

$$-\Delta u = \mu u \text{ in } \Omega \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{Neumann}$$

has discrete spectrum (see [**45**, p.113]),

$$\mu_0 = 0 \leq \mu_1 \leq \mu_2 \leq ... \nearrow \infty.$$

Here $\frac{\partial}{\partial n}$ denotes the outward normal derivative. The first eigenvalue of the Neumann problem $\mu_0 = 0$ corresponds to constant eigenfunctions.

**Remark 4.1.1.** *For a disconnected domain $\Omega = \Omega_1 \cup \Omega_2$, the spectrum $\sigma(\Omega)$ is the ordered union of $\sigma(\Omega_1)$ and $\sigma(\Omega_2)$. If $\Omega$ has $n$ connected components, then the Neumann eigenvalues $\mu_0 = \mu_1 = ... = \mu_{n-1} = 0$.*

We know, from a classical result of Szegő-Weinberger (see [**99**] or [**93**]), that the ball maximizes $\mu_1$ in all dimensions.

**Remark 4.1.2.** *In some rare cases Neumann eigenvalues can be calculated explicitly (see* [**22**]*). For instance, the rectangle with sides a and b has eigenvalues*

$$\mu_{j,k}(\Omega) = \pi^2\Big(\frac{j^2}{a^2} + \frac{k^2}{b^2}\Big); \quad j,k \in \mathbb{N}\cup\{0\}, \tag{4.1.1}$$

*and the disk of unit area has eigenvalues*

$$\mu_{m,n}(\Omega) = \pi j'^2_{m,n}; \quad m \in \mathbb{N}\cup\{0\}, n \in \mathbb{N}. \tag{4.1.2}$$

*Here, $J_n$ is the n-th Bessel function of the first kind and $j'_{m,n}$ is the m-th zero of its derivative $J'_n$.*

### 4.1.2. Main results

The starting point of our research is a theorem by Wolf and Keller (see [**100**, Theorem 8.1]), stating that the Dirichlet eigenvalues $\lambda_n$ of planar domains are not always minimized by disjoint unions of disks (note that for $n = 1, 2, 3, 4$ it is either known or conjectured that the minimizers are disks or disjoint unions of disks).

More precisely, Wolf and Keller showed that $\lambda_{13}$ of any disjoint union of disks is bigger than $\lambda_{13}$ of a single square (see [**100**, p. 408]). Later, Oudet obtained numerical candidates that were no longer disjoint unions of disks starting with $\lambda_5$ (see [**77**]).

In the present paper, we ask

**Question 4.1.3.** *Are disjoint unions of disks maximizing $\mu_n$ for all n ?*

Let us emphasize that, as in [**100**], we allow disjoint unions of *arbitrary* disks.

In a recent paper of Girouard, Nadirashvili and Polterovich (see [**37**]) it is shown that for simply-connected planar domains of fixed area, the second positive Neumann eigenvalue $\mu_2$ is maximized by a family of domains degenerating to the disjoint union of two disks of equal area. The same authors also made a remark (see [**37**, Remark 1.2.8]) that a disjoint union of $n$ identical disks can not maximize $\mu_n$ for sufficiently large values of $n$.

In the present paper, we give a negative answer to Question 4.1.3 :

**Theorem 4.1.4.** *$\mu_{22}$ is not maximized by any disjoint union of disks.*

To prove this theorem, we present an adaptation of Wolf-Keller's result (see [**45**, p. 74]) to the Neumann case. We shall use the same kind of notation as in Wolf-Keller's paper. Let $\mu_n^* = \sup \mu_n(\Omega)$, which is finite (see [**58**]). Assuming that the preceding supremum is attained for a certain domain, let $\Omega_n^*$ be a maximizer of $\mu_n$ among all domains of unit volume. Also, we denote by $\alpha\Omega$ the image of $\Omega$ by a homothety with $\alpha$.

**Theorem 4.1.5.** *Let $n \geq 2$. Suppose that $\Omega_n^*$ is the disjoint union of less than $n$ domains in $\mathbb{R}^N$, each of positive volume, such that their total volume equals 1. Then*

$$(\mu_n^*)^{N/2} = (\mu_i^*)^{N/2} + (\mu_{n-i}^*)^{N/2} = \max_{1 \leq j \leq \frac{n}{2}} \left\{ (\mu_j^*)^{N/2} + (\mu_{n-j}^*)^{N/2} \right\}, \tag{4.1.3}$$

*where $i$ is an integer maximizing $(\mu_j^*)^{N/2} + (\mu_{n-j}^*)^{N/2}$ for $j \leq \dfrac{n}{2}$. Moreover, we have*

$$\Omega_n^* = \left( \left( \frac{\mu_i^*}{\mu_n^*} \right)^{\frac{1}{2}} \Omega_i^* \right) \bigcup \left( \left( \frac{\mu_{n-i}^*}{\mu_n^*} \right)^{\frac{1}{2}} \Omega_{n-i}^* \right), \tag{4.1.4}$$

*where the union above is disjoint.*

Thus, if $\mu_i^*$ is known for $1 \leq i \leq n$, using formula (4.1.3) we can find whether there exists a disconnected domain $\Omega_n^*$ that would achieve $\mu_n^*$. If $(\mu_i)^{N/2} + (\mu_{n-i})^{N/2} < (\mu_n^*)^{N/2}$ for all $i$, $1 \leq i \leq \dfrac{n}{2}$, then it is clear that $\Omega_n^*$ must be connected. Furthermore, if we know $\mu_i^*$ for $i \leq i \leq n-1$, but don't know $\mu_n^*$, we can sometimes show that $\Omega_n^*$ is connected :

**Corollary 4.1.6.** *Let $\Omega$ be a domain such that $|\Omega| = 1$ and*

$$(\mu_n(\Omega))^{N/2} > (\mu_i^*)^{N/2} + (\mu_{n-i}^*)^{N/2} \tag{4.1.5}$$

*for all $i$, $1 \leq i \leq \frac{n}{2}$, then $\Omega_n^*$ must be connected.*

In order to prove Theorem 4.1.4, we use Theorem 4.1.5 iteratively. In other words, to find $\mu_n^*$ in a specific class of domains (i.e., disjoint unions of either disks or squares), we use the results obtained already for $\mu_k^*$, $k = 1, .., n-1$ in this class and choose the eigenvalue of either the connected domain or of the "best" disjoint union given by (4.1.3), whichever is bigger. In this way we obtain sharp upper bounds for arbitrary disjoint unions of either disks or squares, which we in turn compare between themselves in order to find a case where biggest eigenvalue yielded by the disks is lower than that of the squares (see section 4.2.2 for details).

### 4.1.3. Discussion

In dimensions $N \geq 3$, the situation is more complicated than in the planar case. Indeed, as discussed in [**4**, p. 562], the eigenvalues of the ball have not yet been studied systematically for $N \geq 4$. In three dimensions, explicit formulas for eigenvalues of a ball can be obtained in terms of the roots $a'_{p,q}$ of the derivative of the spherical Bessel function $j_p(x)$ (see [**1**] for details regarding the spherical Bessel functions and refer to [**102**] for a table of their zeros $a'_{p,q}$). In an attempt to answer the analogue of Question 4.1.3 in the three–dimensional case, we conducted numerical experiments for $n = 1, ..., 640$. However, for all these $n$, there exists a disjoint union of balls whose corresponding eigenvalue is bigger than that of the cube.

Finally, we remark that among disconnected domains, the second nonzero Neumann eigenvalue is maximized by a disjoint union of two identical balls for all $N \geq 3$ by Theorem 4.1.5. Also, $\mu_1 = \mu_2 = ... = \mu_N$ for an $N$-dimensional ball, and therefore a single ball always yields a lower second nonzero eigenvalue than the disjoint union mentioned above. Taking this into account together with the results of [**37**], we may pose the following

**Question 4.1.7.** *Is the disjoint union of two identical balls maximizing $\mu_2$ in all dimensions ?*

Going back to the planar case, we conclude the discussion by the following result, whose analogue for the Dirichlet eigenvalues was proved in [**100**, p. 399] :

**Proposition 4.1.8.** *Consider the first and the second nonzero eigenvalues $\mu_1, \mu_2$ of the Neumann problem and their respective maxima $\mu_1^* = \pi\, j_{1,1}^{'2}, \mu_2^* = 2\pi\, j_{1,1}^{'2}$ in the class of disjoint unions of simply connected domains of total unit area. Then, for $i = 1, 2$, there exists a domain $\Omega_t$ in that class such that*

$$\mu_i(\Omega_t) = t,$$

*for all values of $t$ in the interval $[0, \mu_i^*]$.*

## 4.2. Proofs

### 4.2.1. Maximization of Neumann eigenvalues for disconnected domains

In this section, we present the proofs of Theorem 4.1.5 and Corollary 4.1.6.

Proof of Theorem 4.1.5. Our proof is similar to the proof of [**100**, Theorem 8.1]. Let $\Omega_n^*$ be the disjoint union of $\Omega_1$ and $\Omega_2$ with $|\Omega_1| > 0$, $|\Omega_2| > 0$ and $|\Omega_1| + |\Omega_2| = 1$. Let $u_n$ be an eigenfunction corresponding to the eigenvalue $\mu_n^*$ on $\Omega_n^*$. Then, $u_n$ is not identically zero on one of the components of $\Omega_n^*$, say, on $\Omega_1$ and we have $\mu_n^*{=}\mu_i(\Omega_1)$ for some $0 \leq i \leq n$.

At the same time, since $\sigma(\Omega_n^*)$ is the ordered union of $\sigma(\Omega_1)$ and $\sigma(\Omega_2)$, we have $\mu_n^* \leq \mu_{n-i}(\Omega_2)$. Assume $\mu_n^* < \mu_{n-i}(\Omega_2)$. Then, since $\mu_n^* = \min\{\mu_i, \mu_{n-i}\}$, we can increase the value of $\mu_n^*$ by increasing the volume of $\Omega_1$ and by decreasing the volume of $\Omega_2$ while keeping the total volume equal to 1. This would contradict the definition of a maximizer. Therefore, $\mu_n^* = \mu_i(\Omega_1) = \mu_{n-i}(\Omega_2)$. Note that $i$ can not be either 0 or $n$ since it would imply that $\mu_n^* = 0$.

We now optimize our choice of domains. Replacing $\Omega_1$ by $|\Omega_1|^{\frac{1}{N}}\Omega_i^*$, we have a domain that has the same volume as $\Omega_1$ (note that $|\alpha\Omega| = \alpha^N|\Omega|$). Moreover, since $\Omega_i^*$ maximizes $\mu_i$, we get a bigger associated eigenvalue. We do the same for $\Omega_2$. Thus, we get

$$\mu_n^* = \mu_i(\Omega_1) = \mu_i(|\Omega_1|^{\frac{1}{N}}\Omega_i^*) = \frac{1}{|\Omega_1|^{2/N}}\mu_i(\Omega_i^*) = \frac{1}{|\Omega_1|^{2/N}}\mu_i^*, \tag{4.2.1}$$

and

$$\mu_n^* = \mu_{n-i}(\Omega_2) = \mu_{n-i}(|\Omega_2|^{\frac{1}{N}}\Omega_{n-i}^*) = \frac{1}{|\Omega_2|^{2/N}}\mu_{n-i}(\Omega_{n-i}^*) = \frac{1}{|\Omega_2|^{2/N}}\mu_{n-i}^*. \tag{4.2.2}$$

Using (4.2.1), we find that $|\Omega_1|^{2/N} = \dfrac{\mu_i^*}{\mu_n^*}$, and similarly, we get $|\Omega_2|^{2/N} = \dfrac{\mu_{n-i}^*}{\mu_n^*}$ from (4.2.2). Since $|\Omega_1|{+}|\Omega_2| = 1$, we have $(\mu_n^*)^{N/2} = (\mu_i^*)^{N/2} + (\mu_{n-i}^*)^{N/2}$. This implies the first equality of (4.1.3) and (4.1.4).

Let us now consider $\widetilde{\Omega}_j$ for $j = 1, 2, ..., n-1$ defined by

$$\widetilde{\Omega}_j = \left[\left(\frac{(\mu_j^*)^{N/2}}{(\mu_j^*)^{N/2} + (\mu_{n-j}^*)^{N/2}}\right)^{\frac{1}{N}} \Omega_j^*\right] \bigcup \left[\left(\frac{(\mu_{n-j}^*)^{N/2}}{(\mu_j^*)^{N/2} + (\mu_{n-j}^*)^{N/2}}\right)^{\frac{1}{N}} \Omega_{n-j}^*\right].$$

Each $\widetilde{\Omega}_j$ has a unit volume. Moreover, we must remark that

$$\begin{aligned}
\mu_j\left(\left(\frac{(\mu_j^*)^{N/2}}{(\mu_j^*)^{N/2} + (\mu_{n-j}^*)^{N/2}}\right)^{\frac{1}{N}} \Omega_j^*\right) &= \mu_{n-j}\left(\left(\frac{(\mu_{n-j}^*)^{N/2}}{(\mu_j^*)^{N/2} + (\mu_{n-j}^*)^{N/2}}\right)^{\frac{1}{N}} \Omega_{n-j}^*\right) \\
&= \left[\left(\frac{(\mu_j^*)^{N/2} + (\mu_{n-j}^*)^{N/2}}{(\mu_{n-j}^*)^{N/2}}\right)^{\frac{1}{N}}\right]^2 \mu_{n-j}(\Omega_{n-j}^*) \\
&= ((\mu_j^*)^{N/2} + (\mu_{n-j}^*)^{N/2})^{\frac{2}{N}}.
\end{aligned}$$

Since $\mu_n^* \geq \mu_n(\widetilde{\Omega}_j)$ for all $j$, $1 \leq j \leq n-1$ and $\mu_n^* = \mu_n(\widetilde{\Omega}_i)$ for some $i$, $1 \leq i \leq n-1$, we have that $\mu_n^*$ is the maximum value of $\mu_n(\widetilde{\Omega}_j)$. Hence $\Omega_n^* = \widetilde{\Omega_j}$ for any index $j$ realizing the maximum, which proves the last equality in formula (4.1.3). Clearly, if $j \leq \dfrac{n}{2}$, then $n - j \geq \dfrac{n}{2}$, and therefore we only need to consider values of $j$ less than or equal to $\dfrac{n}{2}$.

□

Proof of Corollary 4.1.6. By hypothesis, we have that $(\mu_n(\Omega))^{N/2} > (\mu_i^*)^{N/2} + (\mu_{n-i}^*)^{N/2}$ and by the definition of a maximizer, we know that $\mu_n^* \geq \mu_n(\Omega)$. Then, Theorem 4.1.5 can not hold for a disconnected domain. □

### 4.2.2. Disks do not always maximize $\mu_n$

In this section, we give details of computations that led to Theorem 4.1.4.

We calculate the first twenty-two nonzero eigenvalues of the Neumann problem for disjoint unions of squares (right hand side of the Table 4.1) and for disjoint unions of disks (left hand side of the Table 4.1). Let $\mu_n(\mathbf{D})$ be the eigenvalues obtained from a single disk, $\mu_n^*(\mathbf{UD})$ be the eigenvalues obtained from disjoint unions of disks using Theorem 4.1.5 and $\mu_n^*$ be the maximizer among all disjoint unions of disks.

Here is the legend for Table 4.1 :

- First column represents the index $n$ of the eigenvalue $\mu_n$ ;
- Second column is the eigenvalue $\mu_n$ for a single disk computed from formula (4.1.2) ;
- Third column gives the numerical value of $\mu_n$ ;
- In the fourth column, we use Theorem 4.1.5 to provide the numerical value of $\mu_n^*$ under the assumption that the maximizing domain is disconnected.
- Fifth column represents the maximum of third and fourth columns in terms of $\mu_j$ of a disk. This yields the geometry of the maximizing domain ;
- Sixth column gives the numerical value of $\mu_n^*$ for disjoint unions of disks ;

TABLE 4.1. Maximal eigenvalues for disjoint unions of disks and disjoint unions of squares computed using Theorem 4.1.5.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| n | $\mu_n(\mathbf{D})$ | $\mu_n(\mathbf{D})$ | $\mu_n^*(\mathbf{UD})$ | $\mu_n^*$ | $\mu_n^*$ | $(j^2+k^2)$ | $\mu_n^*(\mathbf{US})/\pi$ | $\mu_n^*$ | $\mu_n^*$ |
| 1 | $\pi j_{1,1}^{'2}$ | 10.650 | - | $\mu_1$ | 10.65 | 1+0 | - | $\mu_1$ | 9.87 |
| 2 | $\pi j_{1,1}^{'2}$ | 10.650 | 21.300 | $2\mu_1$ | 21.30 | 0+1 | 2 | $2\mu_1$ | 19.74 |
| 3 | $\pi j_{2,1}^{'2}$ | 29.306 | 31.950 | $3\mu_1$ | 31.95 | 1+1 | 3 | $3\mu_1$ | 29.61 |
| 4 | $\pi j_{2,1}^{'2}$ | 29.306 | 42.599 | $4\mu_1$ | 42.60 | 4+0 | 4 | $4\mu_1 = \mu_4$ | 39.48 |
| 5 | $\pi j_{0,2}^{'2}$ | 46.125 | 53.249 | $5\mu_1$ | 53.25 | 0+4 | 5 | $5\mu_1$ | 49.35 |
| 6 | $\pi j_{3,1}^{'2}$ | 55.449 | 63.899 | $6\mu_1$ | 63.90 | 4+1 | 6 | $6\mu_1$ | 59.22 |
| 7 | $\pi j_{3,1}^{'2}$ | 55.449 | 74.549 | $7\mu_1$ | 74.55 | 1+4 | 7 | $7\mu_1$ | 69.09 |
| 8 | $\pi j_{4,1}^{'2}$ | 88.833 | 85.199 | $\mu_8$ | 88.83 | 4+4 | 8 | $8\mu_1 = \mu_8$ | 78.96 |
| 9 | $\pi j_{4,1}^{'2}$ | 88.833 | 99.483 | $\mu_8 + \mu_1$ | 99.48 | 9+0 | 9 | $9\mu_1 = \mu_9$ | 88.83 |
| 10 | $\pi j_{1,2}^{'2}$ | 89.298 | 110.133 | $\mu_8 + 2\mu_1$ | 110.13 | 0+9 | 10 | $10\mu_1$ | 98.70 |
| 11 | $\pi j_{1,2}^{'2}$ | 89.298 | 120.783 | $\mu_8 + 3\mu_1$ | 120.78 | 9+1 | 11 | $11\mu_1$ | 108.57 |
| 12 | $\pi j_{5,1}^{'2}$ | 129.308 | 131.432 | $\mu_8 + 4\mu_1$ | 131.43 | 1+9 | 12 | $12\mu_1$ | 118.44 |
| 13 | $\pi j_{5,1}^{'2}$ | 129.308 | 142.081 | $\mu_8 + 5\mu_1$ | 142.08 | 9+4 | 13 | $13\mu_1 = \mu_{13}$ | 128.30 |
| 14 | $\pi j_{2,2}^{'2}$ | 141.284 | 152.732 | $\mu_8 + 6\mu_1$ | 152.73 | 4+9 | 14 | $14\mu_1$ | 138.17 |
| 15 | $\pi j_{2,2}^{'2}$ | 141.284 | 163.382 | $\mu_8 + 7\mu_1$ | 163.38 | 16+0 | 15 | $\mu_{15}$ | 157.91 |
| 16 | $\pi j_{0,3}^{'2}$ | 154.624 | 177.666 | $2\mu_8$ | 177.67 | 0+16 | 16+1=17 | $\mu_{15} + \mu_1$ | 167.78 |
| 17 | $\pi j_{6,1}^{'2}$ | 176.774 | 188.316 | $2\mu_8 + \mu_1$ | 188.32 | 16+1 | 18 | $\mu_{15} + 2\mu_1$ | 177.65 |
| 18 | $\pi j_{6,1}^{'2}$ | 176.774 | 198.965 | $2\mu_8 + 2\mu_1$ | 198.97 | 1+16 | 19 | $\mu_{15} + 3\mu_1$ | 187.52 |
| 19 | $\pi j_{3,2}^{'2}$ | 201.829 | 209.615 | $2\mu_8 + 3\mu_1$ | 209.62 | 9+9 | 20 | $\mu_{15} + 4\mu_1$ | 197.39 |
| 20 | $\pi j_{3,2}^{'2}$ | 201.829 | 220.265 | $2\mu_8 + 4\mu_1$ | 220.27 | 16+4 | 21 | $\mu_{15} + 5\mu_1$ | 207.26 |
| 21 | $\pi j_{1,3}^{'2}$ | 228.924 | 230.915 | $2\mu_8 + 5\mu_1$ | 230.92 | 4+16 | 22 | $\mu_{15} + 6\mu_1$ | 217.13 |
| 22 | $\pi j_{1,3}^{'2}$ | 228.924 | 241.565 | $2\mu_8 + 6\mu_1$ | 241.56 | 16+9 | 23 | $\mu_{22}$ | 246.74 |
| 23 | $\pi j_{7,1}^{'2}$ | 231.156 | 252.215 | $2\mu_8 + 7\mu_1$ | 252.21 | 9+16 | 25+1=26 | $\mu_{22} + \mu_1$ | 256.61 |
| 24 | $\pi j_{7,1}^{'2}$ | 231.156 | 266.499 | $3\mu_8$ | 266.50 | 25+0 | 27 | $\mu_{22} + 2\mu_1$ | 266.48 |
| 25 | $\pi j_{4,2}^{'2}$ | 270.689 | 277.148 | $3\mu_8 + \mu_1$ | 277.15 | 0+25 | 28 | $\mu_{22} + 3\mu_1$ | 276.35 |

– Seventh column gives the values of $(j+k)$ computed using formula (4.1.1) for eigenvalues of a square ;
– Eighth column gives $\frac{\mu_n^*}{\pi^2}$ computed using Theorem 4.1.5 ;
– Ninth column gives the maximum of seventh and eighth columns in terms of $\mu_j$ of a square. This yields the geometry of the maximizing domain ;
– Last column gives the numerical value for the maximum of $\mu_n$ for disjoint unions of squares, i.e.
$\max\{\text{column } 7, \text{column } 8\}\pi$.

Theorem 4.1.5 is used iteratively in order to obtain the maximizer among disks and squares. Column five allows to recover the maximizing domain for $\mu_n$ of disks. For instance, in the case of $\mu_j$ for $2 \leq j \leq 7$, the maximizer is given by a disjoint union of $j$ disks of the same area, since any other combination would yield a lower value of $\mu_j$. For $\mu_8$, the maximizer is a single disk. Therefore, the maximizer for $\mu_j$ for $9 \leq j \leq 15$ is one big disk and $j-8$ smaller ones. For $\mu_{16}$, we consider two disks of same area, and for $\mu_j$ for $17 \leq j \leq 22$, we have two big disks and $j-16$ smaller disks. As for the squares, for $\mu_{15}$, the maximizer is a single square. As a result, for $16 \leq j \leq 21$, we have a big square and $j-15$ smaller squares. For $\mu_{22}$, the eigenvalue of a single square is bigger than the possible value obtained using any combination of squares.

Proof of Theorem 4.1.4. We see that for $n = 22$, the value of the maximizer of any disjoint union of disks is smaller than the corresponding value of a single square, i.e. $241.56 < 246.74$. From column 5, we can recover the geometry of the maximizer among the disjoint unions of disks (see Figure 4.1 on page 90). □

**Remark 4.2.1.** *We see that $\mu_{22}$ and $\mu_{23}$ of disjoint unions of squares are bigger than any disjoint union of disks, but we have to wait until $\mu_{83}$ to see that happen once again.*

Numerical experiments performed in [**36**] show that $\Omega_3^*$ is a connected domain different from a disk. Taking into account this observation and Theorem 4.1.4, as well as the results of [**93, 99, 37**] one may reformulate Question 4.1.3 in the following way :

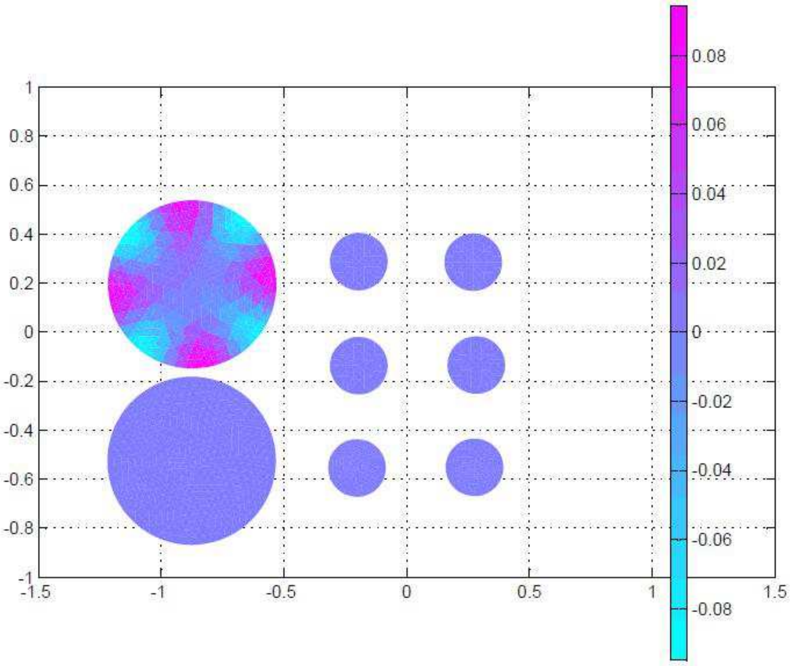



FIGURE 4.1. A disconnected domain $\Omega$ maximizing $\mu_{22}$ among disjoint unions of disks ; $\mu_{22}(\Omega) \approx 241.56$

**Question 4.2.2.** *For which $n > 3$ is the set $\Omega_n^*$ a disjoint union of disks ?*

Numerical and analytic results in the Dirichlet case show that the optimal domain $\Omega_n^*$ is sometimes connected, $n = 1, 3, 5, 6, 8, 9, 10$ and sometimes not, $n = 2, 4, 7$ (see [**45**, Figure 5.1] or [**77**]). Let us conclude this section by yet another question regarding maximizers for Neumann eigenvalues :

**Question 4.2.3.** *For which $n > 3$ are the optimal domains $\Omega_n^*$ connected ?*

### 4.2.3. Proof of Proposition 4.1.8

PROOF : For $t = 0$ and $i = 1$ (resp. $i = 2$), it suffices to build any domain with 2 (resp. 3) or more connected components. We now consider the case $t > 0$.

We begin with the case $i = 1$. We let $\Omega_a$ be the ellipse with axes $a/\sqrt{\pi}$ and $1/\sqrt{\pi}a$, for values of $a$ in $(0, 1]$. Every such ellipse is bounded and convex in $\mathbb{R}^2$ and, by a result from Kröger (see Theorem 1 in [**59**]), we have $\mu_m(\Omega_a)d_\Omega^2 \leq (2j_{0,1} + (m-1)\pi)^2$, where $d_{\Omega_a}$ is the diameter of the convex domain, namely $2/\sqrt{\pi}a$ in this case. When applied to the first eigenvalue, the result yields the inequality $\mu_1(\Omega_a) \leq \pi j_{0,1}' a^2$. This means that, for any $t$ in $(0, \pi j_{1,1}'^2]$, there exists $\alpha$ in $(0,1]$ such that $\mu_1(\Omega_\alpha) \leq t$. We conclude this part of the proof by invoking a result from Chenais (see [**45**, p. 35 ]), which shows that once $t$ and $\alpha$ are picked as above, the first eigenvalue $\mu_1$ of the family of ellipses $\Omega_a$ varies continuously from $\pi j_{1,1}'^2$ to $t$, as $a$ decreases from 1 to $\alpha$.

We now consider the case $i = 2$. We know that $\mu_2^* = 2\pi j_{1,1}'^2$. Let $t \in (0, 2\pi j_{1,1}'^2]$. Depending on the value of $t$, we construct $\Omega_t$ as a disjoint union of rectangles and circles.

We use formulas (4.1.1) and (4.1.2) to compute explicitly the eigenvalues.

We first define $a := \frac{t-\epsilon}{\pi\sqrt{t}}$ and $b := \frac{\pi}{\sqrt{t}}$ and consider the disjoint union of a rectangle with sides $a, b$ and a disk of radius $\sqrt{\frac{\epsilon}{t\pi}}$. The union has total volume one and it's first nonzero eigenvalue comes from the rectangle for a small enough $\epsilon > 0$. Since $b \geq 1$ for $t \in (0, \pi^2]$, the first nonzero eigenvalue of the rectangle is $\pi^2/b^2 = t$, thus allowing us to cover the desired range.

The idea is similar when $t$ is in the interval $(\pi^2, \pi j_{1,1}'^2]$. For $b := \pi/\sqrt{t}$, we consider the rectangle of sides $b$ and $b - \epsilon$. Letting the other component being the disk whose radius brings the disjoint union to a total area of 1, we choose $\epsilon > 0$ small enough so that the first nonzero eigenvalue comes from the rectangle. Doing so, we get a domain for which $\mu_2 = t$, for all t in $(\pi^2, \pi j_{1,1}'^2]$, as desired.

We finally consider the disjoint union of two identical disks of area 1/2 and scale the first one by a ratio of $j_{1,1}'(2\pi/t)^{1/2}$. Scaling the second one so that the union has unit area allows us to cover the range $t$ in $(\pi j_{1,1}'^2, 2\pi j_{1,1}'^2)$ and concludes the proof.

□

**Acknowledgments**

The authors are grateful to Iosif Polterovich for suggesting the problem, as well as for his continuous support throughout this project. We would like to thank Dan Mangoubi for invaluable help during the first stages of this research. We are also thankful to Alexandre Girouard for useful discussions.

# CONCLUSION

Tout au long de cette thèse, diverses questions concernant la géométrie nodale et les valeurs propres du laplacien et du $p$-laplacien ont été étudiées.

D'abord, il a été question des extrema nodaux des fonctions propres de l'opérateur de Laplace-Beltrami sur une variété compacte avec ou sans bord. Une approche alternative à celle présentée par L. Polterovich et M. Sodin a notamment été démontrée au Chapitre 1. Elle permet notamment d'étendre leurs résultats au cas des variétés riemaniennes compactes de dimension arbitraire. Pour poursuivre le travail entamé sur ce sujet, il serait intéressant de voir s'il est possible de modifier la démarche contenue dans [**83**] de telle sorte qu'elle devienne applicable au cas du problème de Neumann sur une surface riemanienne.

Ensuite, les valeurs propres du $p$-laplacien ont été étudiées. Dans le Chapitre 2 et le Chapitre 3, je prouve une série de bornes inférieures pour la première valeur propre du problème de Dirichlet du $p$-laplacien sur un domaine euclidien de $\mathbb{R}^n$. Je m'attarde surtout aux bornes inférieures en terme du rayon inscrit. Plus particulièrement, je prouve que, si $p$ est supérieure à la dimension du domaine, il est possible d'établir une borne inférieure sans aucune hypothèse sur la topologie de ce dernier.

Rappelons que les articles sur la géométrie spectrale du $p$-laplacien pour le problème au bord de Dirichlet sont très récents. Il reste donc toute une série de questions à étudier pour cet opérateur. Par exemple, il serait intéressant d'étudier l'optimisation de formes dans le contexte du problème de Neumann pour $\Delta_p$. Il n'a pas été démontré de manière générale que la première valeur propre non nulle de ce problème est maximisée par le disque, comme dans le cas de l'opérateur de Laplace.

Finalement, le Chapitre 4 était consacré à l'optimisation de formes du problème aux valeurs propres de l'opérateur de Laplace avec comme conditions limites celles de Neumann. Guillaume Roy-Fortin et moi montrons que, sous la contrainte d'une aire fixée, les domaines planaires permettant de maximiser les valeurs du spectre de Neumann ne sont pas toujours des disques. Dans [**49**], M. Iversen et M. van den Berg étudient le problème variationnel pour les valeurs propres du problème de Dirichlet suivant :

$$\inf \{\lambda_k(\Omega), \Omega \subset \mathbb{R}^n, T(\Omega) \leq 1\},$$

où $T$ est une fonction non négative définie sur des ouverts et ayant certaines propriétés. Par exemple, $T$ pourrait être la mesure de Lebesgue. Les auteurs prouvent notamment que le nombre de composantes connexes du domaine minimiseur d'une valeur propre donnée est borné supérieurement si $T$ satisfait une certaine relation préservant l'échelle de mesure. Dans le cas du problème de Neumann, le nombre de composantes connexes d'un domaine maximiseur de la valeur propre $\mu_k$ est trivialement borné par $k$. Il serait intéressant de voir si les méthodes développées dans [**49**] permettent d'obtenir une borne non triviale dans ce cas. Notons que ce projet serait réalisé en collaboration avec Guillaume Roy-Fortin.

# BIBLIOGRAPHIE


[1] Abramowitz, M. et I. A. Stegun. 1964, «Handbook of Mathematical Functions», *National Bureau of Standards Applied Mathematics Series*, vol. 55, U.S. Governement Printing Office, Washington, DC.

[2] Alvino, A., Ferone, V. et G. Trombetti. 1998, «On the properties of some non linear eigenvalues», *SIAM Journal on Mathematical Analysis*, vol. 29, n$^o$2, p. 437-451.

[3] Anane, A., Charkrone, O., Filali. M. et B. Karim. 2007, «Nodal domains for the $p$-Laplacian», *Advances in Dynamical Systems and Applications*, vol. 2, n$^o$ 2, p. 135-140.

[4] Ashbaugh, M. S. et R. D. Benguria. 1993, «Universal bounds for the low eigenvalues of the Neumann Laplacians in N dimensions», *SIAM Journal on Mathematical Analysis*, vol. 24, n$^o$ 3, p. 557-570.

[5] Azorera, J. G. et I. Peral Alonzo. 1987, «Existence and nonuniqueness for the $p$-Laplacian : non linear eigenvalues», *Communications in Partial Differential Equations*, vol. 12, p. 1389-1430.

[6] Banuelos, R. et T. Carroll. 1994, «Brownian motion and the fundamental frequency of a drum», *Duke Mathematical Journal*, vol. 75, n$^o$ 3, p. 575-602.

[7] Bérard, P. H. 1986, «Spectral geometry : direct and inverse problems», *Lecture Notes in Mathematics*, vol. 1207, Springer-Verlag, Berlin.

[8] Berger, M., Gauduchon, P. et E. Mazet. 1971, «Le spectre d'une variété riemannienne», *Lecture Notes in Mathematics*, vol. 194, Springer-Verlag, Berlin.

[9] Bognar, G. 1993, «A lower bound for the smallest eigenvalue of some non-linear eigenvalue problems on convex domains in two dimensions», *Applicable Analysis*, vol. 51, n$^{os}$ 1-4, p. 277-288.

[10] Brascamp, H. J. et E. H. Lieb. 1975, «Some inequalities for Gaussian measures and the long-rage order of one-dimensional plasma», *Functional Integration and its Applications (A.M. Arthurs, ed.)*, Clarendon Press, Oxford.

[11] Bonnesen, T. 1921, «Über eine Verschäfung der isoperimetrischen Ungleichheit des Kreises in der Ebene und auf der Kugeloberfläche nebst einer Anwendung auf eine Minkowskische Ungleichheit für convexe Körper», *Mathematische Annalen*, vol. 84, p. 216-227.

[12] Brüning, J. 1978, «Uber Knoten von Eigenfunktionen des Laplace-Beltrami-Operators», *Mathematische Zeitschrift*, vol. 158, n$^{o}$ 1, p. 15-21.

[13] Bucur, D. et A. Henrot. 2000, «Minimization of the third eigenvalue of the Dirichlet Laplacian», *Proceedings of the Royal Society of London*, vol. 456, p. 985–996.

[14] Bucur, D. et P. Trebeschi. 1998, «Shape Optimization Problems Governed by Nonlinear State Equations», *Proceedings of the Royal Society of Edinburgh : Section A Mathematics*, vol. 128, n$^{o}$ 05, p. 945-963.

[15] Burago, J.D. et V. A. Zalgaller. 1967, «Isoperimetric problems for regions on a surface having restricted width, Two-Dimensional Manifolds of Bounded Curvature», *Proceedings of the Steklov Institute of Mathematics*, n$^{o}$ 76, p. 100-108. (English translation in American Mathematical Society, Providence, 1967.)

[16] Chavel, I. 1984, «Eigenvalues in Riemannian geometry», *Academic Press*, p. 1-362.

[17] Cheeger, J. 1970, «A lower bound for the smallest eigenvalue of the Laplacian», *Problems in Analysis, A Symposium in Honor of Salomon Bochner, R.C. Gunning Ed., Princeton Univ. Press*, p. 195–199.

[18] Chiti, G. 1982, «A reverse Hölder inequality for the eigenfunctions of linear second order elliptic operators», *Journal of Applied Mathematics and Physics (ZAMP)*, vol. 33, p. 143-147.

[19] Chiti, G. 1982, «An isoperimetric inequality for the eigenfunctions of linear second order elliptic operators», *Bollettino della Unione matematica italiana*, vol. 1, n°i :1, p. 145-151.

[20] Colding T. H. et W. P. Minicozzi. 2011, «Lower bounds for nodal sets of eigenfunctions», *Communications in Mathematical Physics*, vol. 306, n° 3, p. 777–784.

[21] Comer, J. R., Henriksen P. N., Shepard M. J. et R. D. Ramsier. 2004, «Chladni plates revisited», *American Journal of Physics*, vol. 72, n° 10.

[22] Courant, H. et D. Hilbert. 1953, «Methods of mathematical physics», vol. 1, deuxième édition.

[23] Croke, C. B. 1981, «The first eigenvalue of the Laplacian for plane domains», *Proceedings of the American Mathematical Society*, vol. 81, p. 304-305.

[24] Cuccu, F., Emamizadeh, B. et G. Porru. 2006, «Nonlinear elastic membranes involving the $p$-Laplacian operator», *Electronic Journal of Differential Equations*, vol. 2006, n° 49, p. 1-10.

[25] Diaz, J. I., Hernandez, J. et L. Tello. 1997, «On the multiplicity of equilibrium solutions to a nonlinear diffusion equation on a manifold arising in climatology», *Journal of Mathematical Analysis and Applications*, vol. 216, p. 593-613.

[26] Diaz, J. I. et F. Thelin. 1994, «On a nonlinear parabolic problem arising in some models related to turbulent flows», *SIAM Journal on Mathematical Analysis*, vol. 25, n° 4, p. 1085-1111.

[27] H. Donnelly et C. Fefferman. 1988, «Nodal sets of eigenfunctions on Riemannann manifolds», *Inventiones mathematicae*, vol. 93, n° 1, p. 161-183.

[28] Drabek, P. et S. B. Robinson. 2002, «On the Generalization of the Courant Nodal Domain Theorem», *Journal of Differential Equations*, vol. 181, p. 58-71.

[29] Drabek, P. 1992, «Solvability and bifurcations of nonlinear equations», *Pitman Research Notes in Mathematics Series*, vol. 264, Longman, Harlow.

[30] Drabek, P. et J. Milota. 2013, «Methods of Nonlinear Analysis, Applications to Differential Equations», *Springer Basel*, deuxième édition, p. 1-658.

[31] Drabek, P. et S. B. Robinson. 2002, «On the Generalization of the Courant Nodal Domain Theorem», *Journal of Differential Equations*, vol. 181, p. 58-71.

[32] Elbert, A. 1979, «A half-linear second order differential equation», *Colloquia Mathematica Societatis*, Janos Bolyai, vol. 30, p. 158-180.

[33] Evans, L. C. 2010, «Partial Differential Equations», *American Mathematical Society*, Second edition, p. 280-283.

[34] *Wikipedia's page on Ernst Chladni. http : //en.wikipedia.org/wiki/Ernst_Chladni* (pageweb consultée le 2 juin 2015).

[35] Faber, G. 1923, «Beweis, dass unter allen homogenen membranen von gleicher flüche und gleicher spannung die kreisfürmige den tiefsten grundton gibt», *Jahrbuch der Bayerischen Akademie der Wissenschaften*, p. 169–172.

[36] Freitas, P. 2009, *Private communication.*

[37] Girouard A, Nadirashvili N. et I. Polterovich. 2009, «Maximization of the second positive Neumann eigenvalue for planar domains», *Journal of Differential Geometry*, vol. 83, n° 3, p. 637-662.

[38] Girouard, A. et I. Polterovich. 2010, «Shape optimization for low Neumann and Steklov eigenvalues», *Special issue on "Complex-Analytic Methods", Mathematical Methods in the Applied Sciences*, vol. 33, n° 4, p. 501-516.

[39] Glowinski, R. et J Rappaz. 2003, «Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid model in glaciology», *Mathematical Modelling and Analysis*, vol. 37, n° 1, p. 175–186.

[40] Granlund, S. et N. Marola. 2012, «Vanishing properties of sign changing solutions to the $p$-laplace type equations in the plane», preprint.

[41] Grieser, D. 2006, «The first eigenvalue of the Laplacian, isoperimetric constants and the Max Flow Min Cut Theorem», *Archiv der Mathematik*, vol. 87, p. 75-85.

[42] Hardy, G.H. L, Littlewood J.E. et G. Poyla. 1929, «Some simple inequalities satisfied by convex functions», *Messenger of Mathematics*, vol. 58.

[43] Hayman, W. K. 1978, «Some bounds for principal frequency», *Applicable Analysis : An International Journal*, vol. 7, p. 247-254.

[44] Heinonen, J., Kilpeläinen, T. et O. Martio. 2012, «Nonlinear Potential Theory of Degenerate Elliptic Equations», *Courier Corportation*, p. 416.

[45] Henrot, A. 2006, «Extremum Problems for Eigenvalues of Elliptic Operators», *Birkhauser Verlag*, Basel.

[46] Hersch, J. 1960, «Sur la fréquence fondamentale d'une membrane vibrante. Évaluation par défaut et principe du maximum», *Zeitschrift für Angewandte Mathematik und Mechanik, Journal of Applied Mathematics and Mechanics*, vol. 11, p. 387-413.

[47] Huang, Y.X. 1997, «On the eigenvalues of the $p$-Laplacian with varying $p$», *Proceedings of the American Mathematical Society*, vol. 125, p. 3347–3354.

[48] Ionin, V.K. 1969, «Isoperimetric and various other inequalities on a manifold of bounded curvature», *Siberian Mathematical Journal*, vol. 10, n° 2, p. 233-243.

[49] Iversen, M. et M. van den berg. 2013, «On the minimization of Dirichlet eigenvalues of the Laplace operator», *Journal Of Geometric Analysis*, vol. 23, p. 660-676.

[50] Juutinen, P. et P. Lindqvist. 2005, «On the higher eigenvalues for the $\infty$-eigenvalue problem», *Calculus of Variation and Partial Differential Equations*, vol. 23, p. 169-192.

[51] Kac, M. Avril 1966, «Can One Hear the Shape of a Drum ?», *American Mathematical Monthly*, vol. 73, n° 4, partie 2.

[52] Kawohl, B. 1985, «Rearrangements and Convexity of Level Sets in PDE», *Springer Lecture Notes in Mathematics*, vol. 1150.

[53] Kawohl, B. et F. Schuricht. 2007, «Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem», *Communications in Contemporary Mathematics*, vol. 9, p. 515–543.

[54] Kawohl, B. et V. Fridman. 2003, «Isoperimetric estimates for the first eigenvalue of the p-Laplace operators and the Cheeger constants», *Commentationes Mathematicae Universitatis Carolinae*, vol. 44, n° 4, p. 659–667.

[55] King, J. R. et S. W. McCue. 2009, «Quadrature domains and $p$-Laplacian growth», *Complex Analysis and Operator Theory*, vol. 3, p. 453-469.

[56] Krahn, E. 1924, «Uber eine von Rayleigh formulierte Minimaleigenschaft des Kreises», *Mathematische Annalen*, vol. 5, p. 97-100.

[57] Krahn, E. 1926, «Über Minimaleigenshaften der Kugel in drei und mehr Dimensionen», *Acta et Commentationes Universitatis Tartuensis (Dorpatensis)*, vol. A9, p. 1–44.

[58] Kroger, P. 1992, «Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space», *Journal of Functional Analysis*, vol. 106, p. 353-357.

[59] Kroger, P. 1999, «On upper bounds for high order Neumann eigenvalues of convex domains in Euclidean space», *Proceedings of the American Mathematical Society*, vol. 127, p. 1665-1669.

[60] Lieb, E. H. 1983, «On the lowest eigenvalue of the Laplacian for the intersection of two domains», *Inventiones mathematicae*, Springer-Verlag, vol. 74, p. 441-448.

[61] Lieb, E. H. et M. Loss. 2001, «Analysis, Graduate Studies in Mathematics», *American Mathematical Society*, Providence, vol. 14, deuxième édition.

[62] Lindqvist P. 1990, «On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$», *Proceedings of the American Mathematical Society*, p. 157-164.

[63] Lindqvist, P. 1992, «A note on the nonlinear Rayleigh quotient», *Analysis, Algebra and Computers in Mathematical Research, edited by M. Gyllenberg and L.E. Persson*, Eds., Marcel Dekker Lecture Notes in Pure and Applied Mathematics, vol. 156, p. 223–231.

[64] Lindqvist P. 2006, «Notes on the $p$-Laplace equation», *University of Jyväskylä - Lectures notes.*

[65] Makai, E. 1995, «A lower estimation of the principal frequencies of simply connected membranes», *Acta Mathematica Hungarica*, vol. 16, p. 319-323.

[66] Mangoubi, D. 2008, «Local asymmetry and the inner radius of nodal domains», *Communications in Partial Differential Equations*, vol. 33, n$^{os}$ 7-9, p. 1611-1621.

[67] Mangoubi, D. 2008, «On the Inner Radius of Nodal Domains», *Canada Mathematical Bulletin*, vol. 51, p. 249-260.

[68] Mangoubi, D. 2011, «A remark on recent lower bounds for nodal sets», *Communications in Partial Differential Equations*, vol. 36, nº 12, p. 2208–2212.

[69] Maz'ya, V.. 2011, «Sobolev Spaces, with applications to elliptic partial differential equations», *Springer-Verlag*.

[70] Maz'ya, V. et M. Shubin. 2005, «Can one see the fundamental frequency of a drum ?», *Letters in Mathematical Physics*, vol. 74, nº 2, p. 135–151.

[71] McKean, H. P. 1970, «An upper bound to the spectrum of $\Delta$ on a manifold of negative curvature», *Journal of Differential Geometry*, vol. 4, p. 359-366.

[72] Nazarov, F. et M. Sodin. 2012, *Private communication*.

[73] Osserman, R. 1977, «A note on Hayman's theorem on the bass note of a drum», *Comment. Math. Helv.*, vol. 52, p. 545-555.

[74] Osserman, R. 1978, «The isoperimetric inequality», *Bulletin of the American Mathematical Society*, vol. 84, p. 1182-1238.

[75] Osserman, R. 1978, «Isoperimetric Inequalities and Eigenvalues of the Laplacian», *Proceedings of the International Congress of Mathematics*, Helsinki.

[76] Osserman, R. 1979, «Bonnesen-style isoperimetric inequalities», *American Mathematical Monthly*, vol. 86, p. 1-29.

[77] Oudet, E. 2004, «Some numerical results about minimization problems involving eigenvalues», *ESAIM : Control, Optimisation and Calculus of Variations*, vol. 10, p. 315-335.

[78] Poliquin, G. 2014, «Bounds on the Principal Frequency of the $p$-Laplacian», *Contemporary Mathematics*, vol. 630, American Mathematical Society, Providence, RI, p. 349-366.

[79] Poliquin, G. 2015, «Principal frequency of the $p$-Laplacian and the inradius of Euclidean domains», *Journal of Topology and Analysis*, vol. 07, nº 505, p. 1-7.

[80] Poliquin, G. 2015, «Superlevel sets and nodal extrema of Laplace eigenfunctions», Accepté comme publication dans le «Journal of Spectral Theory».

[81] Poliquin, G. et G. Roy-Fortin. 2012, «Wolf-Keller theorem for Neumann eigenvalues», *Annales des Sciences Mathématiques du Québec*, vol. 36, p. 169-178.

[82] Polterovich, I. 2009, «Pleijel's nodal domain theorem for free membranes», *Proceedings of the Americal Mathematical Society*, vol. 137, p. 1021-1024.

[83] Polterovich, L. et M. Sodin. 2007, «Nodal inequalities on surfaces», *Mathematical Proceedings of Cambridge Philosophical Society*, vol. 143, n° 2, p. 459-467.

[84] Rayleigh, Baron J. W. S. 1894, «The Theory of Sound», vol. 1 et 2, *Macmillan*.

[85] Robert, F. 2010, «Existence et asymptotiques optimales des fonctions de Green des opérateurs elliptiques d'ordre deux», disponible au http ://www.iecn.u-nancy.fr/~frobert/index_fran.html (pageweb consultée le 26 juin 2013).

[86] Rogers, C. A. 1964, «Packing and covering», *Cambridge University Press*.

[87] Ross, A. 2009, «Dossier grands mathématiciens : Sophie Germain», *Accromath*, vol. 4, été-automne 2009, p. 16-17.

[88] Shoen, R. et S.-T. Yau. 2010, «Lectures on differential geometry», *International Press of Boston*, p. 1-432.

[89] Silva, M. A. J. 2012, «On a viscoelastic plate equation with history setting and perturbation of $p$-Laplacian type», *IMA Journal of Applied Mathematics Advance Access*, p. 1-17.

[90] Sogge, C. D. 1993, «Fourier integrals in classical analysis», *Cambridge Tracts in Mathematics*, vol. 105, Cambridge University Press.

[91] Sogge, C. D. et H. F. Smith. 2007, «On the $L^p$ norm of spectral clusters for compact manifolds with boundary», *Journal of Acta Mathematica*, vol. 198, n° 1, p. 107-153.

[92] Sogge, C. D. et S. Zelditch. 2011, «Lower bounds on the Hausdorff measure of nodal sets», *Mathematical Research Letters*, vol. 18, n° 1, p. 25–37.

[93] Szegő, G. 1954, «Inequalities for certain eigenvalues of a membrane of given area», *Journal of Rational Mechanics and Analysis*, vol. 3.

[94] Talenti, G. 1976, «Elliptic equations and rearrangements», *Annali della Scuola Normale Superiore di Pisa*, tome 3, n° 4, p. 697-718.

[95] Talenti, G. 1994, «Inequalities in rearrangement invariant function spaces», *Nonlinear Analysis, Function Spaces and Applications, Proceedings of the Spring School held in Prague, May 23-28, 1994*, vol. 5, Prometheus Publishing House, Praha, p. 177-230.

[96] Toth, J. et S. Zelditch. 2009, «Counting nodal lines which touch the boundary of an analytic domain», *Journal of Differential Geometry*, vol. 81, n° 3, p. 649–686.

[97] Trudinger, N. S. 1967, «On Harnack Type Inequalities and Their Applications to Quasilinear Elliptic Equations», *Communications on pure and applied mathematics*, vol. 20, p. 721-747.

[98] Watson, G. N. 1944, «A treatise on the Theory of Bessel Functions», *Cambridge, Cambridge University Press*, deuxième édition.

[99] Weinberger, H. F. 1956, «An isoperimetric inequality for the N-dimensionnal free membrane problem», *Journal of Rational Mechanics and Analysis*, vol. 5, p. 633-636.

[100] Wolf, S.A. et J. B. Keller. 1994, «Range of the first two eigenvalues of the Laplacian», *Proceedings of the Royal Society of London A*, vol. 447, p. 397-412.

[101] Yau, S. T. 1982, «Problem section», *Seminar on Differential Geometry, Ann. of Math. Stud.*, vol. 102, Princeton University Press, Princeton, N.J., p. 669-706.

[102] «Royal Society Mathematical Tables, Bessel functions, Part III. Zeros and associated values», *Cambridge University Press*, Cambridge, England, vol. 7, 1960.