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Certain problems concerning polynomials and transcendental entire functions of exponential type

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EN ABRÉGÉ

Soit $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ un polynôme de degré n et $M := \sup_{|z|=1} |P(z)|$. Sans aucne restriction suplémentaire, on sait que $|P'(z)| \leq Mn$ pour $|z| \leq 1$ (inégalité de Bernstein). Si nous supposons maintenant que les zéros du polynôme P sont à l'extérieur du cercle $|z| = k$, quelle amélioration peut-on apporter à l'inégalité de Bernstein? Il est déjà connu [**23**] que dans le cas où $k \geq 1$ on a

$$(*) \qquad |P'(z)| \leq \frac{n}{1+k} M \qquad (|z| \leq 1),$$

qu'en est-il pour le cas où $k < 1$? Quelle est l'inégalité analogue à $(*)$ pour une fonction entière de type exponentiel τ?

D'autre part, si on suppose que P a tous ses zéros dans $|z| \geq k$ $(k \geq 1)$, quelle est l'estimation de $|P'(z)|$ sur le cercle unité, en terme des quatre premiers termes de son développement en série entière autour de l'origine. Cette thèse constitue une contribution à la théorie analytique des polynômes à la lumière de ces questions.

Mots-clés : Inégalité de Bernstein – polynômes et polynômes trigonométriques – fonctions entières de type exponentiel – théorème de Schwarz-Pick – intégrale infinie – théorème des trois cercles d'Hadamard.

ABSTRACT

Let $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ a polynomial of degree n and $M := \sup_{|z|=1} |P(z)|$. Without any additional restriction, we know that $|P'(z)| \leq Mn$ for $|z| \leq 1$ (Bernstein's inequality). Now if we assume that the zeros of the polynomial P are outside the circle $|z| = k$, which improvement could be made to the Bernstein inequality? It is already known [**23**] that in the case where $k \geq 1$, one has

$$(*) \qquad |P'(z)| \leq \frac{n}{1+k} M \qquad (|z| \leq 1),$$

what would it be in the case where $k < 1$? What is the analogous inequality for an entire function of exponential type τ? On the other hand, if we assume that P has all its zeros in $|z| \geq k$ $(k \geq 1)$, which is the estimate of $|P'(z)|$ on the unit circle, in terms of the first four terms of its Maclaurin series expansion. This thesis comprises a contribution to the analytic theory of polynomials in the light of these problems.

Keywords : Bernstein's inequality – polynomial and trigonometric polynomial – entire functions of exponential type – Schwarz-Pick theorem – infinite integral – Hadamard's three circles theorem.

CONTENTS

Dedicated to the memory of
Professor Dr. Qazi Ibadur Rahman
&
to the memory of my parents Naïma and Haroun,
may Allah have Mercy on their souls

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INTRODUCTION

In 1938, Jean Dieudonné published a mémoire entitled “La théorie analytique des polynômes d’une variable”. Eleven years later , appeared “The geometry of the zeros of a polynomial in a complex variable” by Morris Marden. These two monographs are among the publications that announced the birth of the new branch of mathematics called “Analytic theory of polynomials”, a branch about the problems of the zeros and the critical points of a polynomial, the local behaviour of a polynomial in the neighbourhood of a given point, the extremal properties of certain polynomials, etc. The problem of finding an estimate of the modulus of the derivative of a polynomial in terms of its degree and its maximum modulus in a subset of the complex plane is also amongst the problems of the analytic theory of polynomials. Such results are generally called Bernstein type inequalities. In the following example : if P is a polynomial of degree at most n and M is its maximum modulus over the interval $[-1, +1]$, then

$$|P'(x)| \leq Mn^2 \qquad (x \in [-1, +1]),$$

the inequality is called the Markov inequality. Initially, it was a question asked and answered by the imminent chemist D. Mendeleev for $n = 2$ when he represented the graph of the gravity of a solution as a quadratic function of the dissolved substance. The obtained result was mentioned to A. A. Markov, who established the result in its final form above-mentioned. Another example of Bernstein’s type inequality is the Bernstein’s inequality which stipulates that if P is a polynomial of degree n and M its maximum modulus over the unit circle, then we have

$$|P'(z)| < Mn \quad \text{for} \quad |z| \leq 1,$$

and the inequality is sharp except when $P(z) = \lambda z^n$ where λ is of modulus equal to M. Despite what its name suggests, Bernstein’s inequality was not initially proven by Bernstein, it was Fejér, maybe, the first who has done that.

In chapter one, I shall deal with Bernstein’s inequality, I will give two proofs of this inequality within two different approaches and will discuss some refinements of this inequality under different hypothesis for the polynomial P. The following question will be tackled in this opening chapter: What is the sharp upper bound for $\sup_{|z|=1}|P'(z)|$ if P is a polynomial of degree n such that $|P(z)| \leq M$ on the unit circle and $P(z) \neq 0$ for $|z| < k$ for some $k > 0$? Some known results concerning these refinements are presented especially for $k > 1$ but the case $k < 1$ has turned out to be very challenging.

In chapter two, I shall present the extension of Bernstein's inequality to the class of trigonometric polynomials, furthermore I shall tackle the extension of a polynomial's inequality , due to Govil, to a special class of entire functions of exponential type. In spite of Govil's simple proven result, its extension presented in chapter two requires strong results from the classical theory of functions.

In connection with entire functions of exponential type evocated in the precedent chapter, there are situations where an integral, involving in its integrand the maximum modulus $M(r)$ of such function f for $|z| = r$, is required to converge. In chapter three, I shall be interested in such problem by giving the behavior of the $M(r)$ in view of the convergence of a specific integral.

Govil, Rahman and Schmeisser had published a paper in which they established an estimation of the derivative of a polynomial of degree n involving its value at the origin and the values of its two consecutive derivatives at the origin. In chapter four, I will construct a similar estimation of the derivative of the polynomial involving its value at the origin and the values of its first three derivatives at the origin. Actually, the most important task in this chapter is a lemma preceding the main result, which contains an elegant highlight of the Schwarz-Pick theorem.

In the present dissertation, and to make a clear distinction between new results and known ones, theorems, propositions and lemmas are numerated as follows : if the result is known and not my own discovery, then it is numerated using a capital letter preceded by the chapter's number; otherwise, the result is numerated using the arabic numbers preceded by the chapter's number. To make this dissertation self readable, I added four appendices at its end, each one contains some classical theorems from the analytic theory of polynomials and the theory of functions.

Chapter 1

BERNSTEIN'S INEQUALITY FOR POLYNOMIALS ON THE UNIT DISK

1.1. Introduction

In this chapter, I start with an inequality about the growth of a polynomial of degree n. Then, we use it in conjunction with the Gauss–Lucas theorem to prove Bernstein's inequality for the derivative of a polynomial on the unit disk. The idea of the proof via the Gauss–Lucas theorem is almost certainly due to Bernstein himself [**3**]. Next, we present another approach to Bernstein's inequality, which we find very interesting. It readily lends itself to other applications as we shall see. Bernstein's inequality says that if P is a polynomial of degree n such that $|P(z)| \leq M$ on the unit circle, then $|P'(z)| \leq Mn$ for all z on the unit circle. The bound Mn in this inequality is attained only when P has all its n zeros at the origin. This led to the following question: What is the sharp upper bound for $\sup_{|z|=1} |P'(z)|$ if P is a polynomial of degree n such that $|P(z)| \leq M$ on the unit circle and $P(z) \neq 0$ for $|z| < k$ for some $k > 0$? The answer to this question is known in the case where $k \geq 1$. The case $k < 1$ has turned out to be very challenging. I shall briefly but clearly describe the known results and then mention some of my own observations.

1.1.1. An inequality about the growth of polynomials

The following inequality, which is a simple consequence of the maximum modulus principle, is a result of fundamental importance.

Theorem 1-A. *Let P be a polynomial of degree at most n such that $|P(z)| \leq M$ for $|z| = 1$. Then*

$$|P(z)| < M\,|z|^n \qquad (|z| > 1)\,, \tag{1.1}$$

unless $P(z) \equiv M\,\mathrm{e}^{\mathrm{i}\gamma}\,z^n$ for some real γ.

In fact, $Q(z) := z^n\overline{P(1/\overline{z})}$ is a polynomial of degree at most n such that $|Q(z)| = |P(z)| \le M$ for $|z| = 1$ and so by the maximum modulus principle $|Q(z)| < M$ for $|z| < 1$ unless Q is a constant. Therefore Theorem 1-A holds.

1.1.2. Bernstein's inequality for the derivative of a polynomial

Let P be a polynomial of degree at most n such that $|P(z)| \le M$ for $|z| = 1$ and suppose that $P(z) \not\equiv M\,\mathrm{e}^{\mathrm{i}\gamma}\,z^n$ for any real γ. Then by (1.1), $|P(z)| < M\,|z|^n$ for $|z| > 1$. Hence, $P(z) - M\,\lambda\mathrm{e}^{\mathrm{i}\gamma}\,z^n$ cannot vanish for $\lambda \ge 1$ and $\gamma \in \mathbb{R}$. This means that for any $\lambda \ge 1$ and any real γ the polynomial $P(z) - M\,\lambda\mathrm{e}^{\mathrm{i}\gamma}\,z^n$ has all its zeros in the closed unit disk. By the Gauss–Lucas theorem [**32**, p. 71] the derivative

$$P'(z) - Mn\,\lambda\mathrm{e}^{\mathrm{i}\gamma}\,z^{n-1}$$

must also have all its zeros in the closed unit disk. In other words, this polynomial, which depends on the parameters λ and γ, cannot vanish for any z such that $|z| > 1$. It is easily seen that this is possible only if $|P'(z)| < Mn\,|z|^{n-1}$ for $|z| > 1$. Thus, we have proved the following result.

Theorem 1-B. *Let P be a polynomial of degree at most n such that $|P(z)| \le M$ for $|z| = 1$. Then*

$$|P'(z)| < Mn\,|z|^{n-1} \qquad (|z| > 1)\,, \tag{1.2}$$

unless $P(z) \equiv M\,\mathrm{e}^{\mathrm{i}\gamma}\,z^n$ for some real γ.

Letting $|z| \to 1$ in (1.2), we obtain what is known as Bernstein's inequality for the derivative of a polynomial.

Corollary 1-A. [**35**] *Let P be a polynomial of degree at most n such that $|P(z)| \le M$ for $|z| = 1$. Then*

$$|P'(z)| \le Mn \qquad (|z| = 1)\,. \tag{1.3}$$

The example $P(z) := M\,z^n$ shows that the bound given in (1.3) *is sharp.*

1.2. Stronger conclusions

In this section we present De Bruijn's approach to Bernstein's inequality and show how it allows us to say more than what Bernstein's inequality does.

1.2.1. Bernstein's inequality via Laguerre's theorem

The following result which is a simple consequence of Szegő's formulation of Laguerre's theorem [Appendix B, Theorem A-B] has proved to be a very useful tool in connection with Bernstein's inequality.

Lemma 1-A. *Let P be a polynomial of degree not exceeding n. Furthermore, let $S := \{P(z)\ :\ |z| < 1\}$ and $\overline{S} := \{P(z)\ :\ |z| \le 1\}$. Then*

$$P(z) - \frac{1}{n}zP'(z) + \zeta\,\frac{1}{n}P'(z) \in S \qquad (|z| < 1\,,\ |\zeta| < 1)\,, \tag{1.4a}$$

and

$$P(z) - \frac{1}{n}zP'(z) + \zeta\,\frac{1}{n}P'(z) \in \overline{S} \qquad (|z| \le 1\,,\ |\zeta| \le 1)\,, \tag{1.4b}$$

Proof. Let w be any point not belonging to S. Then $f(z) := (P(z) - w)\,/n$ is a polynomial of degree not exceeding n that has no zeros in the open unit disk D_1. Hence, by the Szegő's formulation of Laguerre's theorem [Appendix B , Theorem A-B]

$$n\,\frac{P(z) - w}{n} + (\zeta - z)\frac{P'(z)}{n}$$

has no zeros in D_1 for any ζ such that $|\zeta| < 1$. In other words,

$$P(z) + \frac{\zeta - z}{n}\,P'(z) \ne w \qquad (|z| < 1\,,\ |\zeta| < 1)\,,$$

which proves (1.4a).

In order to prove (1.4b), let w be any point $\notin \overline{S}$. Then $f(z) := (P(z) - w)\,/n$ is a polynomial of degree n that has no zeros in the closed unit disk $\overline{D}_1$. Now apply Szegö's formulation of Laguerre's theorem. □

We wish to indicate how Bernstein's inequality (1.3) follows from Lemma 1-A.

The other proof [11] of Corollary 1-A

Let P be a polynomial of degree at most n satisfying the conditions of Corollary 1-A. Also, let

$$S := \{w = P(z)\ :\ |z| < 1\} \ \text{ and } \ \overline{D}_M := \{w\ :\ |w| \le M\}\,.$$

Then

$$\overline{S} := \{w = P(z)\ :\ |z| \le 1\} \subseteq \overline{D}_M\,.$$

By (1.4b),

$$P(z) - \frac{1}{n}zP'(z) + \zeta\,\frac{1}{n}P'(z)$$

lies in $\overline{S}$ and so in $\overline{D}_M$ for any z such that $|z| \le 1$ and any ζ such that $|\zeta| \le 1$. This means that for any z such that $|z| \le 1$, the disk of radius $|P'(z)|/n$ centered at the point $P(z) - (1/n)zP'(z)$ lies in $\overline{D}_M$. Clearly, any disk that can be placed in $\overline{D}_M$ cannot be of radius larger than M and so $|P'(z)| \le Mn$ for all $z \in \overline{D}_1$. □

Remark. This method of proof allows us to identify (see [**31**, pp. 28–29]) the polynomials for which (1.3) becomes an equality. For this note that in (1.3) equality can hold only if $\overline{S}$ covers every point of $\overline{D}_M$ and is not a proper subset of it. If $P(z)$ is extremal, then, obviously, it cannot be a constant and so $|P(z)|$ cannot be equal to M except for $|z| = 1$. Hence, $|P(\mathrm{e}^{\mathrm{i}\theta})|^2 = M^2$ for infinitely many values of $\theta \in (0\,,\,2\pi]$. Since $|P(\mathrm{e}^{\mathrm{i}\theta})|^2$ is a trigonometric polynomial of degree at most n, we conclude that $|P(z)| = M$ at every point z of the unit circle. Once we know this, there is a classical argument that can be used to conclude that $P(z)$ must be of the form $M\,\mathrm{e}^{\mathrm{i}\gamma}z^n$, $\gamma \in \mathbb{R}$ if it is extremal.

1.2.2. A strengthening of Bernstein's inequality

From Lemma 1-A we can deduce another result which has been found to be very useful.

Lemma 1-B.[**32**, p. 524] *Let $P(z)$ be a polynomial of degree at most n such that $|P(z)| \le M$ for $|z| = 1$ and let $Q(z) := z^n\overline{P(1/\overline{z})}$. Then*

$$|Q'(z)| + |P'(z)| \le Mn \qquad (|z| = 1)\,. \tag{1.5}$$

Proof. From (1.4) it readily follows that

$$|nP(z) - zP'(z)| + |P'(z)| \le Mn \qquad (|z| = 1)\,. \tag{1.6}$$

However, if $z = \mathrm{e}^{\mathrm{i}\theta}, \theta \in \mathbb{R}$, then $Q(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{in\theta}\,\overline{P\left(\mathrm{e}^{\mathrm{i}\theta}\right)}$ and so

$$\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\,Q'(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{i}\,n\mathrm{e}^{\mathrm{i}n\theta}\,\overline{P(\mathrm{e}^{\mathrm{i}\theta})} + \mathrm{e}^{\mathrm{i}n\theta}\,\overline{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\,P'(\mathrm{e}^{\mathrm{i}\theta})}\,,$$

from which it follows that

$$|Q'(z)| = |Q'(\mathrm{e}^{\mathrm{i}\theta})| = \left|n\,\overline{P(\mathrm{e}^{\mathrm{i}\theta})} - \overline{\mathrm{e}^{\mathrm{i}\theta}\,P'(\mathrm{e}^{\mathrm{i}\theta})}\right| = \left|n\,P(\mathrm{e}^{\mathrm{i}\theta}) - \mathrm{e}^{\mathrm{i}\theta}\,P'(\mathrm{e}^{\mathrm{i}\theta})\right|\,.$$

Thus, we see that

$$|nP(z) - zP'(z)| = |Q'(z)| \qquad (|z| = 1)\,. \tag{1.7}$$

Using (1.7) in (1.6) we obtain (1.5). □

This is as good a place as any other to state another auxiliary result which often comes in very handy.

Lemma 1-C.[**32**, p. 510] *Let $A(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree at most n and $B(z) := \sum_{\nu=0}^{n} b_\nu z^\nu$ a polynomial of exact degree n which means that b_n is different from 0. Suppose that $B(z)$ has all its n zeros in the closed unit disk $\overline{D}_1$ and that $|A(z)| \le |B(z)|$ on the unit circle. Then*

$$|A'(z)| \le |B'(z)| \qquad (|z| \ge 1)\,. \tag{1.8}$$

1.3. Polynomials not vanishing in the disk $|z| < k$

We shall start this section with a result of Turán about polynomials having all their zeros in the closed unit disk.

Let $P(z) := c\prod_{\nu=1}^{n}(z-z_\nu)$ be a polynomial of degree n having all its zeros in $|z| \le k$ for some $k \le 1$. It is easily checked that if $r_\nu := |z_\nu|$, then for any real θ, we have

$$\Re\left(\frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}-z_\nu}\right) \ge \frac{1}{1+r_\nu} \ge \frac{1}{1+k} \qquad (\nu = 1,\dots,n)\,.$$

This implies that

$$\left|\frac{P'(\mathrm{e}^{\mathrm{i}\theta})}{P(\mathrm{e}^{\mathrm{i}\theta})}\right| \ge \Re\left(\frac{\mathrm{e}^{\mathrm{i}\theta}\,P'(\mathrm{e}^{\mathrm{i}\theta})}{P(\mathrm{e}^{\mathrm{i}\theta})}\right) = \sum_{\nu=1}^{n}\Re\left(\frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}-z_\nu}\right) \ge \frac{n}{1+k} \qquad (\theta \in \mathbb{R})\,.$$

Hence, the following result holds.

Theorem 1-C*.[**32**, p. 536] *Let P be a polynomial of degree n having all its zeros in $|z| \le k$ for some $k \le 1$, and let M denote the maximum of $|P(z)|$ on the unit circle. Then*

$$\max_{|z|=1}|P'(z)| \ge M\,\frac{n}{1+k}\,.$$

Turán's inequality

In the case where $k = 1$, which has some special significance, this result was proved by Turán [**38**]. It is convenient to state it separately.

Theorem 1-C. *Let P be a polynomial of degree n having all its zeros in the closed unit disk, and let M denote the maximum of $|P(z)|$ on the unit circle. Then*

$$\max_{|z|=1}|P'(z)| \ge M\,\frac{n}{2}\,. \tag{1.9}$$

A conjecture of Erdős

We really do not know, but we wonder if it was not Theorem 1-C that motivated P. Erdős to wonder if the following statement (to be compared with Corollary 1-A) was true. We call it the conjecture of Erdős because that is the way it has been referred to in the literature.

The conjecture (of Erdős). *If P is a polynomial of degree at most n having no zeros in the open unit disk such that $|P(z)| \le M$ on the unit circle, then*

$$|P'(z)| \le M\,\frac{n}{2} \qquad (|z| = 1)\,. \tag{1.10}$$

In the special case where P has all its zeros on the unit circle, inequality (1.10) was proved by G. Szegö and by G. Pólya independently, using completely different reasonings. Their proofs are published in [**21**], which is a paper of P.D. Lax. Knowing that (1.10) was true in the case where all the zeros of P lay on the unit circle, Lax [**21**] proved that (1.10) must remain true if the zeros of P were allowed to lie anywhere on or outside the unit circle.

A proof of (1.10)

Let P be a polynomial of degree at most n having no zeros in the open unit disk such that $|P(z)| \le M$ on $|z| = 1$ and let $S := \{P(z) : |z| < 1\}$. Then, (1.4a) says that for any z in the open unit disk, the disk of radius $|P'(z)|/n$ centered at $P(z) - (1/n)zP'(z)$ lies in S and so in the punctured disk $\{w : 0 < |w| < M\}$. The radius of any disk, contained in the punctured disk, must be less than $M/2$. Hence, $|P'(z)| < Mn/2$ for any z in the open unit disk and so (1.10) must hold. This is the way De Bruijn [**11**] proved the conjecture of Erdős. □

From Theorem 1-C and (1.10) it follows that *if P is a polynomial of degree n having all its zeros on the unit circle, then*

$$\max_{|z|=1} |P'(z)| = \frac{n}{2} \max_{|z|=1} |P(z)| .$$

A result of M.A. Malik

Let $P(z)$ be a polynomial of degree at most n not vanishing in the open disk $D_k := \{z \; : \; |z| < k\}$ for some $k \ge 1$. By Laguerre's theorem as formulated by Szegö [**36**],

$$nP(z) - zP'(z) + \zeta P'(z) \neq 0$$

for any $z \in D_k$ and any $\zeta \in D_k$. For any given $z \in D_k$ this is possible for all $\zeta \in D_k$ only if

$$|nP(z) - zP'(z)| \ge k|P'(z)| .$$

Hence, if $Q(z) := z^n\overline{P(1/\overline{z})}$, then, in view of (1.7), this is possible for any z on the unit circle only if

$$|Q'(z)| \ge k|P'(z)| \qquad (|z| = 1) . \tag{1.11}$$

Using (1.11) together with (1.5), we obtain the following result.

Theorem 1-D. *Let $P(z)$ be a polynomial of degree at most n such that $|P(z)| \le M$ for $|z| = 1$. Furthermore, let $P(z) \neq 0$ in the disk $D_k := \{z : |z| < k\}$, where $k \ge 1$. Then*

$$|P'(z)| \le M \frac{n}{1+k} \qquad (|z| = 1) . \tag{1.12}$$

□

Theorem 1-D is due to Malik [**23**]. As it has already been pointed out, in the case where $k = 1$ equality holds in (1.12) for any polynomial P which has all its zeros on the unit circle and for which $\max_{|z|=1}|P(z)| = M$. For $k > 1$, the upper bound given in (1.12) is attained for polynomials of the form

$$P(z) := M\,\mathrm{e}^{\mathrm{i}\beta}\left(\frac{z + k\,\mathrm{e}^{\mathrm{i}\alpha}}{1+k}\right)^n\,,\ \alpha \in \mathbb{R}\,,\ \beta \in \mathbb{R}\,.$$

What if $P(z) \neq 0$ in $|z| < k$, where $k < 1$?

This question has turned out to be something very challenging. Not much is known in the case where $k < 1$. Here is what we know.

It is clear that if $P(z) := c\,(z - z_1)$ is a polynomial of degree 1 not vanishing in $|z| < k$, where $k < 1$, then $M := \max_{|z|=1}|P(z)| \geq |c|\,(1+k)$, whereas $\max_{|z|=1}|P'(z)| = |c|$. Thus

$$\max_{|z|=1}|P'(z)| \leq M\,\frac{1}{1+k}\ \left(= M\,\frac{n}{1+k^n}\,,\ n = 1\right).$$

Next, let $P(z)$ be a polynomial of degree 2 having both its zeros on the circle $|z| = k$, where $0 < k < 1$. Without loss of generality, we can suppose that $P(z) := c\,(z - k\,\mathrm{e}^{\mathrm{i}\alpha})(z - k\,\mathrm{e}^{-\mathrm{i}\alpha})$, $0 \leq \alpha \leq \pi/2$. Then

$$\begin{aligned}
\left|P(\mathrm{e}^{\mathrm{i}\theta})\right|^2 &= |c|^2\left|(\mathrm{e}^{\mathrm{i}\theta} - k\,\mathrm{e}^{\mathrm{i}\alpha})(\mathrm{e}^{-\mathrm{i}\theta} - k\,\mathrm{e}^{-\mathrm{i}\alpha})(\mathrm{e}^{\mathrm{i}\theta} - k\,\mathrm{e}^{-\mathrm{i}\alpha})(\mathrm{e}^{-\mathrm{i}\theta} - k\,\mathrm{e}^{\mathrm{i}\alpha})\right|^2\\
&= |c|^2\left\{1 + k^2 - 2k\cos(\theta - \alpha)\right\}\left\{1 + k^2 - 2k\cos(\theta + \alpha)\right\}\\
&= |c|^2\left\{1 + k^4 + 2k^2\cos 2\alpha + 4k^2\cos^2\theta - 4k(1+k^2)(\cos\alpha)\cos\theta\right\}\\
&\leq |c|^2\left\{1 + k^4 + 2k^2\cos 2\alpha + 4k^2 + 4k(1+k^2)(\cos\alpha)\right\} = \left|P(\mathrm{e}^{\mathrm{i}\pi})\right|^2.
\end{aligned}$$

Thus

$$\max_{|z|=1}|P(z)|^2 = |c|^2\left\{(1+k^2)^2 + 4k(1+k^2)\cos\alpha + 4k^2\cos^2\alpha\right\}.$$

Since $P'(z)$ is equal to $c\,(2z - 2k\cos\alpha)$, we readily see that

$$\max_{|z|=1}|P'(z)|^2 = 4|c|^2(1 + 2k\cos\alpha + k^2\cos^2\alpha)\,.$$

Thus

$$\frac{\max_{|z|=1}|P'(z)|^2}{\max_{|z|=1}|P(z)|^2} = \psi(\cos\alpha)\,,$$

where

$$\psi(u) := 4\,\frac{1 + 2ku + k^2u^2}{(1+k^2)^2 + 4k(1+k^2)u + 4k^2u^2}\,.$$

In order to determine how large $\psi(u)$ can be we calculate its derivative and find that

$$\psi'(u) = -32k(1-k^2)\frac{2k^2u^2 + k(3+k^2)u + 1 + k^2}{\{(1+k^2)^2 + 4k(1+k^2)u + 4k^2u^2\}^2}\,,$$

which is clearly negative. This means that $\psi(u)$ decreases as u increases from 0 to 1. So, the largest value of $\psi(u)$ occurs for $u = 0$, that is, for $\alpha = \pi/2$. Thus, we see that

$$\frac{\max_{|z|=1} |P'(z)|^2}{\max_{|z|=1} |P(z)|^2} \leq \frac{4}{(1+k^2)^2}\,.$$

In other words, *if $P(z)$ is a polynomial of degree 2 having both its zeros on the circle $|z| = k$, where $0 < k < 1$, and $|P(z)| \leq M$ for $|z| = 1$, then*

$$|P'(z)| \leq M\,\frac{2}{1+k^2}\ \left(= \frac{n}{1+k^n}\,,\ n = 2\right) \qquad (|z| = 1)\,. \tag{1.13}$$

The bound in (1.13) is attained for polynomials of the form

$$P(z) := M\,\mathrm{e}^{\mathrm{i}\gamma}\frac{z^2+k^2}{1+k^2}\,,\ \gamma \in \mathbb{R}\,.$$

This led to the speculation that (1.13) might be true for any polynomial P of degree 2 which has both its zeros in $|z| \geq k$, where $0 < k < 1$. However, this is not true as the following example found by E. B. Saff shows.

Example. Let $P_0(z) := (z - 1/2)(z + 1/3)$. This is a polynomial of degree 2 whose zeros lie on or outside the circle $|z| = k = 1/3$. By elementary calculation, we find that for any real θ,

$$|P_0(\mathrm{e}^{\mathrm{i}\theta})|^2 = \frac{2}{3}\left(\frac{25}{12} - \frac{5}{12}\cos\theta - \cos^2\theta\right)$$

and so

$$\max_{|z|=1} |P_0(z)| = \max_{\theta\in\mathbb{R}} |P_0(\mathrm{e}^{\mathrm{i}\theta})| = \left[\sqrt{\frac{2}{3}\left(\frac{25}{12} - \frac{5}{12}\cos\theta - \cos^2\theta\right)}\,\right]_{\cos\theta=-5/24} = \frac{35}{12\sqrt{6}}\,.$$

Since $P_0'(z) = 2z - 1/6$, we readily see that $\max_{|z|=1} |P_0'(z)| = 13/6$, so that

$$\frac{\max_{|z|=1} |P_0'(z)|}{\max_{|z|=1} |P_0(z)|} = \frac{26\sqrt{6}}{35} = 1.819620952... > 1.8 = \frac{2}{1+(1/3)^2} = \frac{n}{1+k^n}\,.$$

Let $\mathcal{P}_n$ denote the class of all polynomials of degree at most n. We shall say that $P(z)$ belongs to $\mathcal{P}_{n,k}$ if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in the disk $D_k := \{z : |z| < k\}$. Let

$$\mu_{n,k} := \sup_{P\in\mathcal{P}_{n,k}} \frac{\max_{|z|=1} |P'(z)|}{\max_{|z|=1} |P(z)|}\,.$$

Saff's example shows that, in the case where $n = 2$, $\mu_{n,k}$ may be larger $n/(1 + k^n)$ for some values of k in $(0\,,\,1)$; for $k = 1/3$ it is definitely so. It might seem surprising but we do not know what the value of $\mu_{2\,,\,1/3}$ really is. We see that by restricting ourselves to the case where $n = 2$ we do not get very far because the calculations become quite unmanageable even for a polynomial of the form $(z-k)(z-\ell e^{i\alpha})$, where $\ell > k$ and α is real.

However, by studying the problem for quadratics, we have learnt that determining the exact value of $\mu_{n,k}$ is not a simple matter when $0 < k < 1$.

A result of N. K. Govil

By a theorem of Govil (see [**15**, p. 52]), if $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zeros in $|z| < k$ for some $k < 1$, then

$$\max_{|z|=1} |P'(z)| \le \frac{n}{1+k^n} \max_{|z|=1} |P(z)|\,, \tag{1.14}$$

if the coefficients a_ν are either all non-negative or all non-positive. In fact, he states his theorem as follows.

Theorem 1-E. *Let $P(z)$ be a polynomial of degree n having no zeros in $|z| < k$, $k \le 1$, and let $Q(z) := z^n\overline{P(1/\overline{z})}$. If $\max_{|z|=1} |P'(z)|$ and $\max_{|z|=1} |Q'(z)|$ are attained at the same point of the unit circle $|z| = 1$, then (1.14) holds.*

The bound is attained for the polynomial $P(z) := c\,(z^n + k^n)\,,\ c \in \mathbb{C}$.

In connection with (1.14), another result of Govil [**17**, p. 544] gives an inequality in the opposit direction when all the zeros are inside $|z| < k$ $(k \ge 1)$. It can be stated as follows

Theorem 1-F. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n with $\max\limits_{|z|=1} |p(z)| = 1$ and $p(z)$has all its zeros in the disk $|z| \le k$, $k \ge 1$, then*

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{1+k^n}.$$

The result is best possible with equality for the polynomial $(z) = \dfrac{z^n + k^n}{1+k^n}$.

Remark. In Theorem 1-E, requiring $P(z)$ to be different from zero in $|z| < k$ and to have coefficients either all non-negative or all non-positive might seem a bit too restrictive. However, the result is not without interest. We note that any polynomial of the form $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \not\equiv 0$, where

$$a_0 \ge k a_1 \ge \cdots \ge k^\nu a_\nu \ge k^{\nu+1} a_{\nu+1} \ge \cdots \ge k^n a_n \ge 0\,,$$

satisfies the conditions of Theorem 1-E. One should not forget that there are no other large classes of polynomials $P(z)$ which have no zeros in $|z| < k$ for some $k < 1$ and for which the sharp estimates for

$$\frac{\max_{|z|=1} |P'(z)|}{\max_{|z|=1} |P(z)|}$$

are known.

Let $0 < k < 1$. If $P(z) := z^n + k^n$, then

$$\frac{\max_{|z|=1} |P'(z)|}{\max_{|z|=1} |P(z)|} = \frac{n}{1+k^n} = n - nk^n + O(k^{2n})\,,$$

which shows that even when all the zeros of P lie on $|z| = k$, the improvement in the upper bound for $\left(\max_{|z|=1} |P'(z)|\right) / \left(\max_{|z|=1} |P(z)|\right)$ as compared to the one given by (1.3) cannot be greater than $nk^n + O(k^{2n})$, which for any given $k \in (0\,,\,1)$ tends to 0 as $n \to \infty$. We wish we were able to decide if the supremum of $\left(\max_{|z|=1} |P'(z)|\right) / \left(\max_{|z|=1} |P(z)|\right)$ over all polynomials of degree n having no zeros in $|z| < k$ for some $k < 1$ was $n - nk^n + O(k^{2n})$ as $n \to \infty$ but we have not succeeded.

Now, we shall describe what we have been able to prove without imposing any restriction on the degree n.

Theorem 1.1. *Let $P(z)$ be a polynomial of degree at most n having a zero on the circle $|z| = k$ for some $k \in (0\,,\,1)$ and suppose that $|P(z)| \le M$ for $|z| = 1$. Then,*

$$m := \min_{|z|=1} |P(z)| \le \frac{1-k^n}{1+k^n}\, M \tag{1.15}$$

and

$$|P'(z)| \le Mn - \frac{M}{4\pi n}\,(1-a)\,\{1 - a - \sin\,(1-a)\} \qquad (|z| \le 1)\,, \tag{1.16}$$

where

$$a := \frac{1-k^n}{1+k^n}\,.$$

An auxiliary result

The proof of Theorem 1.1 is based on the following result of A. Giroux and Q. I. Rahman (see [**14**, Theorem 1]).

Lemma 1-D. *Let $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree at most n such that $|P(z)| \le M$ for $|z| = 1$. Furthermore, let $\min_{|z|=1} |P(z)| \le a\,M$, where $0 \le a < 1$. Then* (1.16) *holds*

Proof of Theorem 1.1. By a recent result of Qazi and Rahman [**27**] if P is a polynomial of degree at most n such that $m \le |P(z)| \le M$ for $|z| = 1$, then $P(z) \ne 0$ in the open annulus

$$\left(\frac{M-m}{M+m}\right)^{1/n} < |z| < \left(\frac{M+m}{M-m}\right)^{1/n}\,.$$

Hence, if P has a zero on the circle $|z| = k < 1$, then necessarily

$$k \le \left(\frac{M-m}{M+m}\right)^{1/n}$$

and so

$$m \le \frac{1-k^n}{1+k^n}\, M\,,$$

which proves (1.15). Now, Lemma 1 may be applied with $a := (1-k^n)/(1+k^n)$ to complete the proof of Theorem 1.1. □

A remark on (1.15)

The example

$$P(z) := \frac{(M+m)\, z^n - (M-m)}{2}$$

shows that the upper bound for m given in (1.15) cannot be improved even if P is supposed to have all its zeros on $|z| = k$.

If $P(z) := c \prod_{\nu=1}^{n} (z - z_\nu) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros on the circle $|z| = k$, then

$$|a_0| = |c|\, k^n \text{ and } M := \max_{|z|=1} |P(z)| \le |c|\, (1+k)^n$$

so that

$$|a_0| \ge M\, \frac{k^n}{(1+k)^n}\,. \tag{1.17}$$

It is known [**32**, p. 518, see Corollary 14.2.2] that if P is a polynomial of degree n such that $|P(z)| \le M$ on the unit circle, then

$$|P'(z)| \le Mn - \frac{2n}{n+2}\, |P(0)| \qquad (|z| = 1)\,.$$

Using (1.17) in this inequality, we obtain the following result.

Theorem 1.2. *Let P be a polynomial of degree n having all its zeros on the circle $|z| = k$ and let $|P(z)| \le M$ for $|z| = 1$. Then*

$$|P'(z)| \le M \left(n - \frac{2n}{n+2}\, \frac{k^n}{(1+k)^n}\right) \qquad (|z| \le 1)\,.$$

□

Chapter 2

ENTIRE FUNCTIONS OF EXPONENTIAL TYPE NOT VANISHING IN THE HALF-PLANE $\Im Z > K$, WHERE $K > 0$

2.1. Introduction

In this chapter, we present the extension of Bernstein's inequality (1.3) to trigonometric polynomials and also to transcendental entire functions of exponential type. The main result of this chapter is a generalization of Theorem 1-E to entire functions of exponential type.

2.1.1. Bernstein's inequality for trigonometric polynomi- -als

Let $\mathcal{P}_m$ denote the class of all polynomials of degree at most m. From Lemma 1-B it follows that if $P \in \mathcal{P}_n$ and $|P(z)| \le M$ for $|z| = 1$, then

$$\left|\frac{\mathrm{d}}{\mathrm{d}\theta}P(\mathrm{e}^{\mathrm{i}\theta})\right| + \left|-\mathrm{i}nP(\mathrm{e}^{\mathrm{i}\theta}) + \frac{\mathrm{d}}{\mathrm{d}\theta}P(\mathrm{e}^{\mathrm{i}\theta})\right| \le Mn \qquad (\theta \in \mathbb{R})\,.$$

If $t(\theta) := \sum_{\nu=-n}^{n} a_\nu \mathrm{e}^{\mathrm{i}\nu\theta}$ and $|t(\theta)| \le M$ for all real $\theta \in \mathbb{R}$, then $\mathrm{e}^{\mathrm{i}n\theta}t(\theta) = P(\mathrm{e}^{\mathrm{i}\theta})$, where $P \in \mathcal{P}_{2n}$ and $|P(z)| \le M$ for $|z| = 1$. Applying the preceding inequality with $2n$ instead of n, we obtain

$$|\mathrm{i}nt(\theta) + t'(\theta)| + |-\mathrm{i}nt(\theta) + t'(\theta)| \le 2Mn \qquad (\theta \in \mathbb{R})\,. \tag{2.1}$$

In particular, we have

$$|2t'(\theta)| \le |\mathrm{i}nt(\theta) + t'(\theta)| + |-\mathrm{i}nt(\theta) + t'(\theta)| \le 2Mn \qquad (\theta \in \mathbb{R})\,,$$

that is,

$$|t'(\theta)| \le Mn \qquad (\theta \in \mathbb{R})\,. \tag{2.2}$$

This is the famous inequality of S. Bernstein for trigonometric polynomials. It is sharp and in (2.2), equality can hold at any point $\theta \in \mathbb{R}$.

From (2.1) it follows that *if* $t(\theta)$ *is real for all real* θ, *then*

$$n^2 t^2(\theta) + (t'(\theta))^2 \leq M^2 n^2 \qquad (\theta \in \mathbb{R}) \,. \tag{2.3}$$

Remark. Bernstein had proved (2.2) for cosine polynomials and also for sine polynomials. M. Riesz [**34**] seems to have been the first to prove it in its full generality. Inequality (2.3) is a result of J. G. van der Corput and G. Shaake [**10**].

2.1.2. Functions of exponential and Bernstein's inequality

Basic properties of functions of exponential type

A trigonometric polynomial $t(\theta) := \sum_{\nu=-n}^{n} a_\nu \mathrm{e}^{\mathrm{i}\nu\theta}$ is well defined for any θ in the complex plane $\mathbb{C}$ and not only when θ is restricted to the real line. Replacing θ by z, we obtain $t(z) := \sum_{\nu=-n}^{n} a_\nu \mathrm{e}^{\mathrm{i}\nu z}$, which is holomorphic throughout the complex plane. Thus, a trigonometric polynomial $t(\theta)$ can be seen as the restriction of an entire function to the real axis. Unless all the coefficients a_ν except a_0 are zero, $t(z)$ is an entire function of order 1 and of type $T \leq n$. Clearly, there exists a constant C such that $|t(z)| < C\,\mathrm{e}^{n|z|}$ for all $z \in \mathbb{C}$. In other words, $t(z)$ is an entire function of exponential type n. Let us recall that a function $f(z)$ holomorphic in an unbounded domain $\mathcal{D} \subseteq \mathbb{C}$ is said to be of exponential type τ in $\mathcal{D}$ if for any $\varepsilon > 0$, there exists a constant $K(\varepsilon)$ such that $|f(z)| < K(\varepsilon)\,\mathrm{e}^{(\tau+\varepsilon)|z|}$ for all $z \in \mathcal{D}$. In the present context, an interesting example of an unbounded domain is the sector

$$A(\alpha, \beta) := \{z = r\mathrm{e}^{\mathrm{i}\theta} : 0 < r < \infty \,, \; \alpha \leq \theta \leq \beta\} \,,$$

where $\beta \in (\alpha \,, \, \alpha + 2\pi)$, and half-planes have special significance. Some of the important results about functions of exponential type are to be found in what follow.

We know that trigonometric polynomials are 2π-periodic, but an entire function of exponential type may not be periodic at all; $(\sin \tau z)/z$ is such a function. As another example, we wish to mention

$$f(z) := \sum_{\nu=0}^{n} a_\nu \mathrm{e}^{\mathrm{i}\lambda_\nu z} \,, \; \lambda_0 < \cdots < \lambda_n \,, \tag{2.4}$$

which is an entire function of exponential type $\tau := \max\{|\lambda_0| \,, \, |\lambda_n|\}$ but is generally not periodic.

It is known (see [**6**, Theorem 6.10.1]) that if $f(z)$ is an entire function of exponential type τ which is periodic on the real axis with period Δ, then it must be of the form $f(z) = \sum_{\nu=-n}^{n} a_\nu \, \mathrm{e}^{2\pi \mathrm{i}\nu z/\Delta}$ with $n \le \left\lfloor \Delta\tau/(2\pi) \right\rfloor$.

To characterize the dependence of the growth of a function f of exponential type τ in a sector $A(\alpha, \beta)$ on the direction in which z tends to infinity, Phragmén and Lindelöf introduced the function

$$h_f(\theta) := \limsup_{r\to\infty} \frac{\log |f(r\,\mathrm{e}^{\mathrm{i}\theta})|}{r} \qquad (\alpha \le \theta \le \beta)\,,$$

called the *indicator function* of f. It is known that unless $h_f(\theta) \equiv -\infty$, $h_f(\theta)$ is continuous in $\alpha < \theta < \beta$ and that if $\alpha \le \theta < \theta + \pi \le \beta$, then

$$h_f(\theta) + h_f(\theta + \pi) \ge 0\,. \tag{2.5}$$

If f is an entire function of order 1 whose type is τ, then, $h_f(\theta) \le \tau$ for all θ and so, by (2.5), $h_f(\theta) \ge -\tau$. See [**6**, Chapter 5] for these and many other properties of the indicator function.

A basic lemma

The following lemma [**6**, Theorem 6.2.4] serves as a basic tool in the study of functions of exponential type. In [**26**] the reader will find a proof of this result, which contains a thorough discussion of the case of equality.

Lemma 2-A. *Let f be a function of exponential type in the open upper half–plane such that $h_f(\pi/2) \le c$. Furthermore, let f be continuous in the closed upper half–plane and suppose that $|f(x)| \le M$ on the real axis. Then*

$$|f(x + \mathrm{i}y)| < M\,\mathrm{e}^{cy} \qquad (-\infty < x < \infty,\ y > 0) \tag{2.6}$$

unless $f(z) \equiv M\,\mathrm{e}^{\mathrm{i}\gamma}\,\mathrm{e}^{-\mathrm{i}cz}$ for some real γ.

Bernstein's inequality for entire functions of exponential type

Bernstein himself was the first to extend inequality (1.20) to entire function of exponential type. The extended version may be stated as follows.

Theorem 2-A (S. Bernstein [**32**, p. 513]). *Let $f(z)$ be an entire function of exponential type τ such that $|f(x)| \le M$ on the real axis. Then*

$$|f'(x)| \le M\tau \qquad (x \in \mathbb{R})\,. \tag{2.7}$$

In (2.7) equality holds if and only if $f(z)$ is of the form $a\,\mathrm{e}^{-\mathrm{i}\tau z} + b\,\mathrm{e}^{\mathrm{i}\tau z}$, where $|a| + |b| = M$.

If $P(z)$ is a polynomial of degree at most n, then $f(z) := P(\mathrm{e}^{\mathrm{i}z})$ is an entire function of exponential type n. Besides, $|f(x)| \le M$ on the real axis if $|P(z)| \le M$ on the unit circle. Hence, inequality (1.3) is covered by (2.7).

2.1.3. Boas's extension of Lax's inequality (1.10)

If p is a polynomial of degree $n > 0$ such that $|p(z)| \le M$ on the unit circle, then $f(z) := p(\mathrm{e}^{\mathrm{i}z})$ is an entire function of order 1 and type n such that $|f(x)| \le M$ on the real axis. If $p(z) \ne 0$ in the open unit disk, then $f(z) = p(\mathrm{e}^{\mathrm{i}z}) \ne 0$ in the open upper half-plane. In addition,

$$h_f\left(\frac{\pi}{2}\right) = 0\,,$$

since $|p(0)| > 0$. This motivated Boas to ask the following question. How large can $|f'(x)|$ be on the real axis if f is an entire function of exponential type τ having no zeros in the open upper half-plane such that $h_f(\pi/2) = 0$ and $|f(x)| \le M$ for all real x? He answered this question by proving the following result [**7**].

Theorem 2-B. (R.P. Boas). *Let $f(z)$ be an entire function of order 1 and type τ having no zeros in the open upper half-plane. Furthermore, let $|f(x)| \le M$ on the real axis and suppose that $h_f(\pi/2) = 0$. Then*

$$\sup_{-\infty<x<\infty} |f'(x)| \le M\,\frac{\tau}{2}\,. \tag{2.8}$$

Clearly, $f(z) := M\,(\mathrm{e}^{\mathrm{i}\tau z} - 1)/2$ satisfies the conditions of Theorem 2-B and for this function $|f'(x)| = M\,\tau/2$ for all real x. It was proved by Rahman and Tariq [**33**] that *if $f(z)$ is an entire function of exponential type τ with only real zeros such that*

$$\sup_{-\infty<x<\infty} |f(x)| = M, \quad h_f\left(-\frac{\pi}{2}\right) = \tau \quad \textit{and} \quad h_f\left(\frac{\pi}{2}\right) = 0\,,$$

then

$$\sup_{-\infty<x<\infty} |f'(x)| = M\frac{\tau}{2}\,.$$

<u>**Entire functions of exponential type satisfying $f(z) \ne 0$ in $\Im z > k$, $k > 0$**</u>

Now, we shall formulate and prove an extension of Theorem 1-E (a theorem of N.K. Govil) to entire functions of exponential type. Our result may be stated as follows.

Theorem 2.1. *Let $f(z)$ be an entire function of order 1 and type τ having no zeros in the half-plane $\Im z > k$ for some $k > 0$. In addition, let $h_f(\pi/2) = 0$ and*

$|f(x)| \le M$ on the real axis. Define $\omega_f(z) := \mathrm{e}^{\mathrm{i}\tau z}\,\overline{f(\overline{z})}$ and suppose that

$$\sup_{-\infty<x<\infty} |f'(x)| \quad \textit{and} \quad \sup_{-\infty<x<\infty} |\omega_f'(x)|$$

are both attained at the same point of the real axis. Then

$$|f'(x)| \le \frac{M\tau}{1+\mathrm{e}^{-\tau k}} \qquad (-\infty < x < \infty)\,. \tag{2.9}$$

The following special case of Theorem 2.1 deserves to be mentioned explicitly. For basic facts about uniformly almost periodic functions, we refer the reader to Appendix D.

Corollary 2.1. *Let $f(z)$ be a uniformly almost periodic entire function of exponential type τ having no zeros in the half-plane $\Im z > k$ for some $k > 0$ and let $h_f(\pi/2) = 0$. In addition, let $|f(x)| \le M$ on the real axis and suppose that the Fourier coefficients of f are all non-negative. Then* (2.9) *holds.*

2.2. Some more auxiliary results

The proof of Theorem 2.1 requires some preparation which consists in recalling certain notions and additional results about entire functions of exponential type.

Definition 2.1. An entire function f of exponential type is said to belong to the *class* **P** if it has no zeros in the open lower half-plane and $h_f(-\pi/2) \ge h_f(\pi/2)$.

Note. From (2.5) it follows that if $f \not\equiv 0$ is an entire function of exponential type 0, then $h_f(\theta) = 0$ for all θ. Hence, any entire function of exponential type 0 having all its zeros in the closed upper half-plane belongs to the class **P**.

It is known (see [**22**] or [**6**, Theorem 7.8.3]) that the Hadamard factorization of a function f belonging to the class **P** has the form

$$f(z) = A\, z^m\, \mathrm{e}^{cz} \prod_{k=1}^{\infty} \left\{ \left(1 - \frac{z}{z_k}\right) \mathrm{e}^{z\,\Re(1/z_k)} \right\}\,, \tag{2.10}$$

where $z_k \ne 0$, $\Im z_k \ge 0$ and $2\Im c = h_f(-\pi/2) - h_f(\pi/2) \ge 0$.

It is also known [**6**, p. 129, Theorem 7.8.1] that if f belongs to **P**, then

$$|f(z)| \ge |f(\overline{z})| \qquad (\Im z < 0)\,. \tag{2.11}$$

From (2.11) it follows that if f belongs to **P**, then $h_f(-\alpha) \ge h_f(\alpha)$ for all $\alpha \in (0\,,\,\pi)$.

The following result (see [**22**, p. 59, Lemma 3] or [**6**, p. 130, Theorem 7.8.6]) is of fundamental importance. Its significance in the present context cannot be over-emphasized.

Lemma 2-B. *Let f be an entire function of order 1 and type τ belonging to the class $\mathbf{P}$. Furthermore, let g be an entire function of exponential type $\sigma \le \tau$ such that*

$$|g(x)| \le |f(x)| \quad \textit{for all} \quad x \in \mathbb{R}\,. \tag{2.12}$$

Then $\phi_\lambda(z) := g(z) - \lambda f(z)$ belongs to $\mathbf{P}$ for any $\lambda \in \mathbb{C}\,,\ |\lambda| > 1$.

Definition 2.2. An additive homogeneous operator $B[f(z)]$ which carries entire functions of exponential type into entire functions of exponential type and leaves the class $\mathbf{P}$ invariant is called (see [**22**, p. 60] or [**6**, p. 225, Definition 11.7.1]) a B-operator.

It may be added that an operator B is *additive* if $B[f+g] = B[f] + B[g]$ and *homogeneous* if $B[cf] = cB[f]$.

Using the representation (2.10), it can be easily shown that *differentiation* is also a B-operator (see [**6**, p. 226]).

Let $f(z)$ be an entire function of order 1 and type τ. Suppose that $|f(x)| \le M$ on the real axis and that $h_f(\pi/2) \le 0$. Then by Lemma 2-A, $|f(z)| < M$ in the open upper half-plane. Hence, $\phi(z) := f(z) - M\mathrm{e}^{-\mathrm{i}\alpha}, \alpha \in \mathbb{R}$, is an entire function of order 1 and type τ which has no zeros in the open upper half-plane. Consequently, the function

$$\omega_\phi(z) := \mathrm{e}^{\mathrm{i}\tau z}\,\overline{\phi(\overline{z})} = \omega_f(z) - M\mathrm{e}^{\mathrm{i}\alpha}\,\mathrm{e}^{\mathrm{i}\tau z}$$

belongs to the class $\mathbf{P}$ and $|\phi(x)| = |\omega_\phi(x)|$ for all real x. By Lemma 2-B, the function $\phi(z) - \lambda\,\omega_\phi(z)$ belongs to the class $\mathbf{P}$ for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Since differentiation is a B-operator, the function $\phi'(z) - \lambda\,\omega'_\phi(z)$ also belongs to the class $\mathbf{P}$ for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. In particular, $\phi'(z) - \lambda\,\omega'_\phi(z) \ne 0$ in the lower half-plane for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. In other words,

$$f'(z) - \lambda\,\left(\omega'_f(z) - M\,\mathrm{i}\tau\,\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\tau z}\right) \ne 0 \tag{2.13}$$

for any z with $\Im z < 0$, for any $\alpha \in \mathbb{R}$ and for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Now, note that f is not a constant and so $\omega_f(z)$ cannot be of the form $M\,\mathrm{e}^{\mathrm{i}\gamma}\,\mathrm{e}^{\mathrm{i}\tau z}$, $\gamma \in \mathbb{R}$. Hence, by Theorem 2-A and Lemma 2-A, $\omega'_f(z) - M\,\mathrm{i}\tau\,\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\tau z}$ is different from zero at every point of the open lower half-plane. Hence (2.13) can hold for any z with $\Im z < 0$, any $\alpha \in \mathbb{R}$ and any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ only if

$$|f'(z)| \le M\tau\mathrm{e}^{-\tau\Im z} - |\omega'_f(z)|\,.$$

Hence, the following result holds. Thus, we have proved that *if f is an entire function of order 1 and type τ such that $|f(x)| \le M$ on the real axis and*

$h_f(\pi/2) \le 0$, *then*

$$|f'(z)| + |\omega_f'(z)| \le M\,\tau\,\mathrm{e}^{-\tau\Im z} \qquad (\Im z < 0)\,.$$

By continuity, the same must be true for z belonging to the real axis. In other words, the following result holds.

Lemma 2-C. *Let f be an entire function of order 1 and type τ. Suppose, in addition, that $|f(x)| \le M$ on the real axis and that $h_f(\pi/2) \le 0$. Then*

$$|f'(z)| + |\omega_f'(z)| \le M\,\tau\,\mathrm{e}^{-\tau\Im z} \qquad (\Im z \le 0)\,. \tag{2.14}$$

2.3. Proof of Theorem 2.1

As the first step towards the proof of Theorem 2.1, we prove the following proposition.

Proposition 2.1. *Let F be an entire function of order 1 and type τ having all its zeros in the half-plane $\{z \in \mathbb{C} : \Im z \ge -k\}$ for some $k > 0$. Suppose that $|F(x)|$ is bounded on the real axis and that $h_F(\pi/2) \le 0$. In addition, let $\omega_F(z) := \mathrm{e}^{\mathrm{i}\tau z}\overline{F(\overline{z})}$. Then*

$$\sup_{-\infty<x<\infty} |\omega_F'(x)| \le \mathrm{e}^{\tau k} \sup_{-\infty<x<\infty} |F'(x)|\,. \tag{2.15}$$

Proof. Suppose that $|F(x)| \le M$ on the real axis. The function $g(z) := F(z-\mathrm{i}k)$ is of order 1 and type τ. Besides, by Lemma 2-A, $|g(x)| \le M\,\mathrm{e}^{\tau k}$ for all real x. We claim that g belongs to the class $\mathbf{P}$ introduced in Definition 2.1. Clearly, g has no zeros in the open lower half-plane. Hence, it is sufficient to check that $h_g(-\pi/2) \ge h_g(\pi/2)$.

Since $|g(x)|$ is bounded on the real axis and $h_g(\pi/2) = h_F(\pi/2) \le 0$, we must necessarily have

$$h_g\left(-\frac{\pi}{2}\right) = h_F\left(-\frac{\pi}{2}\right) = \tau\,,$$

otherwise, by Lemma 2-A, g and so F would not be of order 1 and type τ. Note that τ must be positive because a function of order 1 that is bounded on the real axis or on any line cannot be of type 0. Thus, $h_g(-\pi/2) > 0$ whereas $h_g(\pi/2) \le 0$. Hence in fact, $h_g(-\pi/2) > h_g(\pi/2)$ and so g belongs to $\mathbf{P}$.

Let $\omega_g(z) := \mathrm{e}^{\mathrm{i}\tau z}\overline{g(\overline{z})}$. Then, $|\omega_g(x)| = |g(x)| \le M\,\mathrm{e}^{\tau k}$ for all real x. Besides, $h_{\omega_g}(\pi/2) = -\tau + h_g(-\pi/2) = 0$. Hence, by Lemma 2-A, $|\omega_g(z)| \le M\,\mathrm{e}^{\tau k}$ in the upper half-plane. Since

$$\begin{aligned}\omega_g(z) &= \mathrm{e}^{\mathrm{i}\tau z}\overline{F(\overline{z}-\mathrm{i}k)} \\ &= \mathrm{e}^{\tau k}\,\mathrm{e}^{\mathrm{i}\tau(z+\mathrm{i}k)}\,\overline{F(\overline{z+\mathrm{i}k})} = \mathrm{e}^{\tau k}\omega_F(z+\mathrm{i}k)\end{aligned}$$

we see that

$$h_{\omega_g}\left(-\frac{\pi}{2}\right) = \tau + h_g\left(\frac{\pi}{2}\right) \le \tau$$

and so, by Lemma 2-A, $|\omega_g(z)| \le M\,\mathrm{e}^{\tau(k+|\Im z|)}$ in the lower half-plane. In particular, $\omega_g(z)$ is an entire function of exponential type at most τ.

We have a function g of order 1 and type τ which belongs to the class $\mathbf{P}$. Besides, we have a function $\omega_g(z)$ of exponential type τ such that $|\omega_g(x)| = |g(x)|$ for all real x. So, Lemma 2-B may be applied with g in place of f and ω_g in place of g to conclude that for any λ such that $|\lambda| > 1$, the function $\omega_g(z) - \lambda\, g(z)$ belongs to the class $\mathbf{P}$. Since differentiation is a B-operator, the function $\omega_g'(z) - \lambda\, g'(z)$ also belongs to the class $\mathbf{P}$ for any $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$. In particular, $\omega_g'(z) - \lambda\, g'(z) \neq 0$ if $\Im z < 0$ for any $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$. This is possible only if $|\omega_g'(z)| \le |g'(z)|$ for any z in the open lower half-plane. By continuity, the same must be true for any real z also. Thus, $|\omega_g'(z)| \le |g'(z)|$ for $\Im z \le 0$, which means that

$$\mathrm{e}^{\tau k}\,|\omega_F'(z+\mathrm{i}k)| \le |F'(z-\mathrm{i}k)| \qquad (\Im z \le 0)\,.$$

Taking $z = x - \mathrm{i}k$, in this inequality, we obtain

$$\mathrm{e}^{\tau k}\,|\omega_F'(x)| \le |F'(x-2\mathrm{i}k)| \qquad (-\infty < x < \infty)\,. \tag{2.16}$$

Since F is an entire function of order 1 and type τ, the same can be said about the function F'. Hence, by Lemma 2-A, applied to the function $\overline{F'(\overline{z})}$, we obtain

$$|F'(x-2\mathrm{i}k)| \le \mathrm{e}^{2\tau k} \sup_{-\infty<x<\infty} |F'(x)|$$

for any real x. Combining this with (2.16), we find that

$$|\omega_F'(x)| \le \mathrm{e}^{\tau k} \sup_{-\infty<x<\infty} |F'(x)|$$

for any real x, which is equivalent to (2.15). □

Proposition 2.2. *Let f be an entire function of order 1 and type τ having no zeros in the half-plane $\Im z > k$ where $k > 0$. Besides, let $h_f(\pi/2) = 0$ and suppose that $|f(x)|$ is bounded on the real axis. In addition, let $\omega_f(z) := \mathrm{e}^{\mathrm{i}\tau z}\overline{f(\overline{z})}$. Then*

$$\mathrm{e}^{-\tau k} \sup_{-\infty<x<\infty} |f'(x)| \le \sup_{-\infty<x<\infty} |\omega_f'(x)|\,. \tag{2.17}$$

Proof. Lemma 2-A can be used to see that $h_f(-\pi/2) = \tau$. Hence, $\omega_f(z) := \mathrm{e}^{\mathrm{i}\tau z}\overline{f(\overline{z})}$ is an entire function of order 1 and type τ having all its zeros in the half-plane $\Im z > -k$. Besides, $h_{\omega_f}(\pi/2) = 0$ and $|\omega_f(x)|$ is bounded on the real axis. Hence, ω_f satisfies all the conditions of Proposition 2.1. So, let us apply Proposition 2.1 taking $F = \omega_f$. Clearly, then $\omega_F = \omega_{\omega_f} = f$ and so by (2.15), we have

$$\sup_{-\infty<x<\infty} |f'(x)| \leq e^{\tau k} \sup_{-\infty<x<\infty} |\omega_f'(x)| \,,$$

which proves (2.17). □

Proof of Theorem 2.1. Suppose that

$$\sup_{-\infty<x<\infty} |f'(x)| \quad \text{and} \quad \sup_{-\infty<x<\infty} |\omega_f'(x)|$$

are both attained at the same point x_0 of the real axis.
Combining (2.14) and (2.17), we obtain that

$$\begin{aligned}(1+e^{-\tau k}) \sup_{-\infty<x<\infty} |f'(x)| &\leq \sup_{-\infty<x<\infty} |f'(x)| + \sup_{-\infty<x<\infty} |\omega_f'(x)| \\ &\leq |f'(x_0)| + |\omega_f'(x_0)| \\ &\leq M\tau.\end{aligned}$$

Then

$$\sup_{-\infty<x<\infty} |f'(x)| \leq \frac{\tau}{1+e^{-\tau k}} M,$$

which proves the theorem. □

Chapter 3

ON THE CONVERGENCE OF CERTAIN INTEGRALS

3.1. Introduction

Let $M(r) := \max_{|z|=r} |f(z)|$, where $f(z)$ is an entire function. Also let $\alpha > 0$ and $\beta > 1$. We discuss the behaviour of the integrand $M(r)\mathrm{e}^{-\alpha(\log r)^\beta}$ as $r \to \infty$ if $\displaystyle\int_1^\infty M(r)\mathrm{e}^{-\alpha(\log r)^\beta}\,\mathrm{d}r$ is convergent.

3.1.1. Convergence of integrals vis-à-vis convergence of series

There is one fundamental property of a convergent infinite series in regard to which the analogy between infinite series and infinite integrals breaks down. If $\sum_{n=1}^\infty \theta(n)$ is convergent, then $\theta(n) \to 0$ as $n \to \infty$; but it is not always true, even when $\theta(r)$ is always positive, that if $\int_a^\infty \theta(r)\,\mathrm{d}r$ is convergent, then $\theta(r) \to 0$ as $r \to \infty$. It is however true that if $\int_a^\infty \theta(r)\,\mathrm{d}r$ converges and $\theta(r)$ is non-negative, then

$$\liminf_{r\to\infty} r\,(\log r)(\log\log r)\cdots(\ell_k r)\,\theta(r) = 0\,,$$

where $\ell_k r$ is the k-th iterate of $\log r$. If this was not true, then there would exist positive numbers c and R_0 such that for all $R > R_0$, we would have

$$\int_R^{\mathrm{e}^R} \theta(r)\,\mathrm{d}r > \int_R^{\mathrm{e}^R} \frac{c}{r\,(\log r)(\log\log r)\cdots(\ell_k r)}\,\mathrm{d}r = c\,\left(\ell_k R - \ell_{k+1} R\right)$$

and then $\int_R^{\mathrm{e}^R} \theta(r)\,\mathrm{d}r$ could not be made arbitrarily small by taking R sufficiently large, contradicting the convergence of the integral $\int_a^\infty \theta(r)\,\mathrm{d}r$. On the other hand, it is well known that if $\theta(r)$ is positive and non-increasing, then $\int_a^\infty \theta(r)\,\mathrm{d}r$ can converge only if $r\,\theta(r) \to 0$ as $r \to \infty$. The same conclusion can be drawn if $\theta(r)$ is the product of a monotonic function $\varphi(r)$ and a non-negative function $L(r)$ which is continuous and $L(cr) \sim L(r)$ as $r \to \infty$ (i.e. $\displaystyle\lim_{r\to+\infty}\frac{L(cr)}{L(r)} = 1$). This can be explained as follows. Let ε be any given positive number. Then for all sufficiently large values of u, we have

$$\varepsilon > \left| \int_u^{2u} \varphi(r)\, L(r)\, \mathrm{d}r \right| \geq \min\{|\varphi(u)|\,,\, |\varphi(2u)|\} \int_u^{2u} L(r)\, \mathrm{d}r = |\varphi(2u)| \int_u^{2u} L(r)\, \mathrm{d}r$$

(say). That $u\theta(u) \to 0$ as $u \to \infty$, now follows from the fact (see Lemma 3-A bellow)

$$\int_a^u L(r)\, \mathrm{d}r \sim uL(u)\,.$$

Lemma 3-A. ([**20**], Lemma 4) *The condition*

$$\varphi_1(t) = \int_1^t \varphi(u)\mathrm{d}u \sim t\varphi(t)$$

is equivalent to

$$\varphi(kt) \sim_{t\to\infty} \varphi(t) \qquad \textit{for every fixed positive } k.$$

3.1.2. A special kind of integrands

Let $M(r) := \max_{|z|=r} |f(z)|$, where $f(z)$ is an entire function. In his work on Carlson's theorem [**6**, Chapter 9] for entire functions of exponential type Rahman [**30**, Theorem 7] had a situation where the integral $\int^\infty r^{2Q} M(r)\, \mathrm{e}^{-\pi r}\, \mathrm{d}r$ was convergent and he wanted to know the behavior of $M(r)$ for large values of r. He noted [**30**, Lemma 6] that $r^{2Q} M(r)\, \mathrm{e}^{-\pi r} \to 0$ as $r \to \infty$. In order to prove it we do not require anything more than the fact that $M(r)$ is a non-decreasing function of r. However, $M(r)$ is not just a non-decreasing function of r but also $\log M(r)$ is a downward convex function of $\log r$. Thus $r^{2Q} M(r) = o\left(\mathrm{e}^{\pi r}\right)$ was not expected to be all that the convergence of $\int^\infty r^{2Q} M(r)\, \mathrm{e}^{-\pi r}\, \mathrm{d}r$ would imply. Recently, Qazi [**25**] has proved the following stronger result, which is "essentially" best possible.

Theorem 3-A. *Let* $M(r) := \max_{|z|=r} |f(z)|$, *where* f *is an entire function and suppose that* $\int_0^\infty r^\alpha M(r)\, \mathrm{e}^{-\beta r}\, \mathrm{d}r < \infty$ *for some* $\alpha > 0$ *and some* $\beta > 0$. *Then* $\sqrt{r} \cdot r^\alpha M(r)\, \mathrm{e}^{-\beta r} = O(1)$ *as* $r \to \infty$.

3.2. The main result

An entire function f is a polynomial if and only if there exists a positive number k such that $M(r) := \max_{|z|=r} |f(z)| = O(r^k)$ as $r \to \infty$. The degree of f is the infimum of all such numbers k. If f is a transcendental entire function, then

$$\frac{\log M(r)}{\log r} \to \infty \quad \text{as} \quad r \to \infty\,;$$

however, $M(r)\,\mathrm{e}^{-\alpha(\log r)^\beta}$ may tend to zero as $r \to \infty$ for some $\alpha > 0$ and some $\beta > 1$. This can happen if f is an entire function of order 0, that is, if

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = 0\,.$$

In connection with Theorem 3-A, one may then ask the following question: *What can we say about the behavior of $M(r)$ as $r \to \infty$ if f is an entire function such that $\int_1^\infty M(r)\,\mathrm{e}^{-\alpha(\log r)^\beta}\,\mathrm{d}r$ converges for some $\alpha > 0$ and some $\beta > 1$?*

We give an answer to this question. The proof of Theorem 3-A as given by Qazi [**25**] is based on the use of the well known Stirling's formula for Euler's Gamma function. This was somehow natural because of the integrand in $\int_0^\infty r^\alpha\, M(r)\,\mathrm{e}^{-\beta r}\,\mathrm{d}r$ having $\mathrm{e}^{-\beta r}$ as a factor. Since the integrand does not anymore have such a factor, the use of Stirling's formula is more or less out of the question. So, we have to use some other ideas. In addition to Stirling's formula, Qazi's proof of Theorem 3-A uses Hadamard's three-circles theorem. That remains available to us and we have tried to use it as efficiently as we could.

Theorem 3.1. *Let $M(r) := \max_{|z|=r} |f(z)|$, where f is an entire function and suppose that $\int_1^\infty M(r)\,\mathrm{e}^{-\alpha\,(\log r)^\beta}\,\mathrm{d}r < \infty$ for some $\alpha > 0$ and some $\beta > 1$. Then, for any $\varepsilon > 0$,*

$$\lim_{r\to\infty} r\,(\log r)^{-\gamma-\varepsilon}\cdot M(r)\,\mathrm{e}^{-\alpha\,(\log r)^\beta} = 0\,,$$

where $\gamma := \max\{0\,,\,(\beta-2)/2\}$.

3.3. Proof of Theorem 3.1

We present the proof in several steps.

Step I. First we prove that

$$\frac{R}{(\log R)^{\beta-1}}\,M(R)\,\mathrm{e}^{-\alpha(\log R)^\beta} \to 0 \ \text{ as } \ R\to\infty\,. \tag{3.1}$$

Take any $\varepsilon > 0$ and note that

$$\frac{M(r)}{(\log r)^{\beta-1}(1/r)}$$

is an increasing function of r for all large r. Hence, if R is large enough, then

$$\begin{aligned}\frac{R\,M(R)}{\alpha\beta(\log R)^{\beta-1}}\int_R^{R^2}\alpha\,\beta\,(\log r)^{\beta-1}\left(\frac{1}{r}\right)\mathrm{e}^{-\alpha\,(\log r)^\beta}\,\mathrm{d}r \ &\le\ \int_R^{R^2} M(r)\,\mathrm{e}^{-\alpha\,(\log r)^\beta}\,\mathrm{d}r\\ &<\ \varepsilon\,,\end{aligned}$$

that is,

$$\frac{R}{(\log R)^{\beta-1}}M(R)\,\left(\mathrm{e}^{-\alpha\,(\log R)^\beta} - \mathrm{e}^{-\alpha\,(2\log R)^\beta}\right) < \alpha\beta\varepsilon\,,$$

which implies (3.1).

Step II. Next, we prove that for all large r,

$$M(S)\,\mathrm{e}^{-\alpha\,(\log S)^\beta} < \frac{(\log S)^\gamma}{S} \quad \text{for some} \;\; S = S(r) \in \left(r\,,\, r + \frac{r}{(\log r)^\gamma}\right). \tag{3.2}$$

If this was not true, then for all $t \in (r\,,\, r + r/(\log r)^\gamma)$, which in the case where $1 < \beta \le 2$ means "for all $t \in (r\,,\, 2r)$", we would have

$$M(t)\,\mathrm{e}^{-\alpha\,(\log t)^\beta} \ge \frac{(\log t)^\gamma}{t}\,.$$

This would imply that

$$\begin{aligned}\int_r^{r+r/(\log r)^\gamma} M(t)\,\mathrm{e}^{-\alpha\,(\log t)^\beta}\,\mathrm{d}t \;&\ge \int_r^{r+r/(\log r)^\gamma} \frac{(\log t)^\gamma}{t}\,\mathrm{d}t \\ &= \frac{1}{\gamma+1}\left\{\left(\log\left(r + \frac{r}{(\log r)^\gamma}\right)\right)^{\gamma+1} - (\log r)^{\gamma+1}\right\}.\end{aligned}$$

It is easily checked that the last expression is equal to $\log 2$ if γ is zero and is $1 + o\,(1)$ if γ is positive. Thus the integral $\int_1^\infty M(t)\,\mathrm{e}^{-\alpha\,(\log t)^\beta}\,\mathrm{d}t$ would not be convergent, contradicting our hypothesis. Hence (3.2) holds. This means that for all large r,

$$M(\lambda r) < \frac{(\log \lambda r)^\gamma}{\lambda r}\,\mathrm{e}^{\alpha\,(\log \lambda r)^\beta} \quad \text{for some} \;\; \lambda \in \left(1\,,\, 1 + \frac{1}{(\log r)^\gamma}\right). \tag{3.3}$$

Step III. Since $\log M(r)$ is a convex function of $\log r$, we have

$$(M(r))^2 \le M\left(\frac{r}{\lambda}\right)\, M(\lambda r) \qquad (\lambda > 0)\,. \tag{3.4}$$

This is our main tool. We use (3.1) and (3.3) in (3.4) to conclude that

$$\lim_{r\to\infty} r\,(\log r)^{-(\gamma+\beta-1)/2}\, M(r)\,\mathrm{e}^{-\alpha\,(\log r)^\beta} = 0\,. \tag{3.5}$$

If r is sufficiently large and

$$\lambda \in \left(1\,,\, 1 + \frac{1}{(\log r)^\gamma}\right) \tag{3.6}$$

is chosen such that (possible by (3.3))

$$M(\lambda r) < \frac{(\log \lambda r)^\gamma}{\lambda r}\,\mathrm{e}^{\alpha\,(\log \lambda r)^\beta}\,,$$

then using this and (3.1) in (3.4), we obtain

$$(M(r))^2 \le c_1(r)\,\frac{(\log(r/\lambda))^{\beta-1}}{r/\lambda}\,\mathrm{e}^{\alpha\,(\log(r/\lambda))^\beta}\cdot\frac{(\log(\lambda r))^\gamma}{\lambda r}\,\mathrm{e}^{\alpha\,(\log(\lambda r))^\beta}\,,$$

where $c_1(r) = o(1)$ as $r \to \infty$. Now, note that

$$\begin{aligned}
(\log(r/\lambda))^{\beta-1}(\log(\lambda r))^{\gamma} &= (\log r)^{\gamma+\beta-1}\left\{\left(1-\frac{\log\lambda}{\log r}\right)^{\beta-1}\left(1+\frac{\log\lambda}{\log r}\right)^{\gamma}\right\} \\
&\leq (\log r)^{\gamma+\beta-1}\left\{\left(1-\frac{\log\lambda}{\log r}\right)^{\beta-1}\left(1+\frac{\log\lambda}{\log r}\right)^{\beta-1}\right\} \\
&< (\log r)^{\gamma+\beta-1}
\end{aligned}$$

because $\gamma < \beta - 1$. Hence

$$\begin{aligned}
&(M(r))^2 \\
&\leq c_1(r)\frac{(\log r)^{\gamma+\beta-1}}{r^2}\exp\left\{\alpha(\log r)^{\beta}\left(\left(1-\frac{\log\lambda}{\log r}\right)^{\beta}+\left(1+\frac{\log\lambda}{\log r}\right)^{\beta}\right)\right\} \\
&= c_1(r)\frac{(\log r)^{\gamma+\beta-1}}{r^2}\exp\left\{\alpha(\log r)^{\beta}\left(2+(\beta(\beta-1)+c_2(r))\left(\frac{\log\lambda}{\log r}\right)^{2}\right)\right\} \\
&= c_1(r)\frac{(\log r)^{\gamma+\beta-1}}{r^2}\exp\left\{\alpha(\log r)^{\beta}\left(2+(\beta(\beta-1)+c_2(r))(\log r)^{-2\gamma-2}\right)\right\},
\end{aligned}$$

where $c_2(r) = o(1)$ as $r \to \infty$, and where we have used (3.6) in the last line. Note that $\beta - 2\gamma - 2$ is negative if $1 < \beta < 2$ and zero if $\beta \geq 2$. Hence $(\log r)^{\beta-2\gamma-2} = O(1)$ as $r \to \infty$. This allows us to conclude that

$$M(r) \leq c_3(r)\,\frac{(\log r)^{(\gamma+\beta-1)/2}}{r}\,\mathrm{e}^{\alpha(\log r)^{\beta}} = c_3(r)\,\frac{(\log r)^{\gamma+(\beta-1-\gamma)/2}}{r}\,\mathrm{e}^{\alpha(\log r)^{\beta}},$$

where $c_3(r) = o(1)$ as $r \to \infty$; which is equivalent to (3.5).

Inequality (3.5) is considerably stronger that (3.1) and provides a better estimate for $M(r/\lambda)$ in (3.4). Using (3.5) and (3.3) in (3.4) the way (3.1) and (3.3) were used above in (3.4), we obtain

$$\lim_{r\to\infty} r\,(\log r)^{-\gamma-(\beta-1-\gamma)/2^2}\,M(r)\,\mathrm{e}^{-\alpha(\log r)^{\beta}} = 0\,, \tag{3.7}$$

which may in turn be used to conclude that

$$\lim_{r\to\infty} r\,(\log r)^{-\gamma-(\beta-1-\gamma)/2^3}\,M(r)\,\mathrm{e}^{-\alpha(\log r)^{\beta}} = 0\,. \tag{3.8}$$

Clearly, (3.8) is stronger than (3.7). Since this process can go on indefinitely, we see that for any positive integer k, we have

$$\lim_{r\to\infty} r\,(\log r)^{-\gamma-(\beta-1-\gamma)/2^k}\,M(r)\,\mathrm{e}^{-\alpha(\log r)^{\beta}} = 0\,,$$

from which the desired result follows. □

Remark. The proof of Theorem 3.1 is of a somewhat wider scope than it might appear. In fact, the property of the function $M(r)$ by which $\log M(r)$ is a convex function of $\log r$ is shared by some other functions associated with an entire function f. For example, if

$$\mathfrak{M}_p(r) := \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(r\,\mathrm{e}^{\mathrm{i}\theta})\right|^p \,\mathrm{d}\theta\right)^{1/p}, \; p > 0\,,$$

then, $\log \mathfrak{M}_p(r)$ is a convex function of $\log r$ for any $p > 0$. This is a well-known result of G. H. Hardy [**19**]. The reader might find [**28**] to be of some interest in this connection.

Also, if $f(z) := \sum_{n=0}^{\infty} a_n z^n$, then for any $r > 0$, the maximum of $|a_n| r^n$ for $n \in \{0, 1, 2, \ldots\}$ is called the maximum term. It is usually denoted by $\mu(r)$ and $\log \mu(r)$ is known [**39**, pp. 30–31] to be a convex function of $\log r$.

Chapter 4

FUNCTIONS HOLOMORPHIC AND BOUNDED IN THE UNIT DISK

4.1. Introduction

In this chapter, I present a generalization of an inequality due to Govil, Rahman and Schmeisser [**18**] giving an estimation of the value of a polynomial of degree n on the unit circle in terms of its value at the origin, and the values of its two consecutive derivatives at the origin. My result is involving the value of its third derivative at the origin in addition to the known result. An elegant highlight of the Schwarz-Pick theorem is used in the proof of my result.

4.2. Functions holomorphic and bounded in $|z| < 1$

4.2.1. Some basic facts

Let $f(z) := \sum_{\nu=0}^{\infty} a_\nu z^\nu$ be holomorphic with $|f(z)| \le 1$ for $|z| < 1$. Then

$$\sum_{\nu=0}^{\infty} |a_\nu|^2 r^{2\nu} = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \,d\theta \le 1 \qquad (0 < r < 1)\,,$$

from which it follows that $\sum_{\nu=0}^{\infty} |a_\nu|^2 \le 1$. In particular, we have

$$|a_\nu| \le 1 \qquad (\nu = 0, 1, 2, \ldots)\,. \tag{4.1}$$

It is also known that [**37**, p. 212] that

$$|f(z)| \le \frac{|z| + |f(0)|}{|f(0)||z| + 1} = \frac{|z| + |a_0|}{|a_0||z| + 1} \qquad (|z| < 1)\,. \tag{4.2}$$

In (4.2) equality holds for functions of the form

$$f(z) := \frac{ze^{i\gamma} + a_0}{\overline{a}_0 z e^{i\gamma} + 1}\,,\ |a_0| < 1,\ \gamma \in \mathbb{R}\,. \tag{4.3}$$

Another inequality which we wish to recall says that

$$|f(z)|^2 + (1-|z|^2)|f'(z)| \le 1 \qquad (|z|<1)\,.$$

It is known as the Schwarz–Pick theorem [**1**, p. 3]. In particular, it says that if $f(z) := \sum_{\nu=0}^{\infty} a_\nu z^\nu$ is holomorphic with $|f(z)| \le 1$ for $|z|<1$, then

$$|a_0|^2 + |a_1| \le 1\,. \tag{4.4}$$

In (4.4) equality holds for any function of the form (4.3). In fact, if $f(z)$ is as in (4.3), then $f'(0) = (1-|a_0|^2)\mathrm{e}^{\mathrm{i}\gamma}$.

Let $\mathcal{B}^1$ denote the class of all functions $f(z) := \sum_{\nu=0}^{\infty} a_\nu z^\nu$ holomorphic in the open unit disk D such that $|f(z)| \le 1$ for all $z \in D$. Note that f belongs to $\mathcal{B}^1$ if and only if $\mathrm{e}^{\mathrm{i}\alpha} f$ belongs to $\mathcal{B}^1$ for all $\alpha \in \mathbb{R}$. So, we may suppose that $a_0 = f(0) \ge 0$. Then (4.4) says that if $f(z) := \sum_{\nu=0}^{\infty} a_\nu z^\nu \in \mathcal{B}^1$ and $a_0 \in [0\,,\,1)$, then a_1 lies in the disk of radius $1-|a_0|^2$ centered at the origin.

Next, we note that $f(z) := \sum_{\nu=0}^{\infty} a_\nu z^\nu$ belongs to $\mathcal{B}^1$ if and only if

$$\mathrm{e}^{\mathrm{i}\alpha} f(z\mathrm{e}^{\mathrm{i}\beta}) = a_0 \mathrm{e}^{\mathrm{i}\alpha} + a_1 \mathrm{e}^{\mathrm{i}(\alpha+\beta)} z + \sum_{\nu=2}^{\infty} a_\nu\, \mathrm{e}^{\mathrm{i}(\alpha+\nu\beta)} z^\nu\,,\ \alpha \in \mathbb{R}\,,\ \beta \in \mathbb{R}$$

does. By choosing α and β appropriately, we can arrange that

$$a_0 \mathrm{e}^{\mathrm{i}\alpha} = |a_0| \ge 0 \quad \text{and} \quad a_1 \mathrm{e}^{\mathrm{i}(\alpha+\beta)} = |a_1| \ge 0\,.$$

There is therefore no loss of generality in assuming $f(0) \ge 0$ and $f'(0) \ge 0$ in the following question.

Question. *Let f belong to $\mathcal{B}^1$ and suppose that $f(0)$ and $f'(0)$ are given. What kind of values can $f''(0)$ take?*

Here is the answer to this question.

Theorem 4.1. *Let $f(z) := \sum_{\nu=0}^{\infty} a_\nu z^\nu$ be holomorphic with $|f(z)| \le 1$ for $|z|<1$. Furthermore, let $f(0) = a_0 = a \ge 0\,,\ f'(0) = a_1 = b \ge 0$. Then*

$$\left| a_2 - \left(-\frac{ab^2}{1-a^2} \right) \right| \le \frac{(1-a^2)^2 - b^2}{1-a^2}\,. \tag{4.5}$$

Inequality (4.5) says that a_2 lies in the closed disk $D(\mathbf{c}\,;\,\rho)$ whose center is $\mathbf{c}$ and the radius is ρ, where

$$\mathbf{c} := -\frac{ab^2}{1-a^2} \quad \text{and} \quad \rho := \frac{(1-a^2)^2 - b^2}{1-a^2}\,.$$

Any point in $D(\mathbf{c}\,;\,\rho)$ is a possible value for a_2 as we proceed to explain.

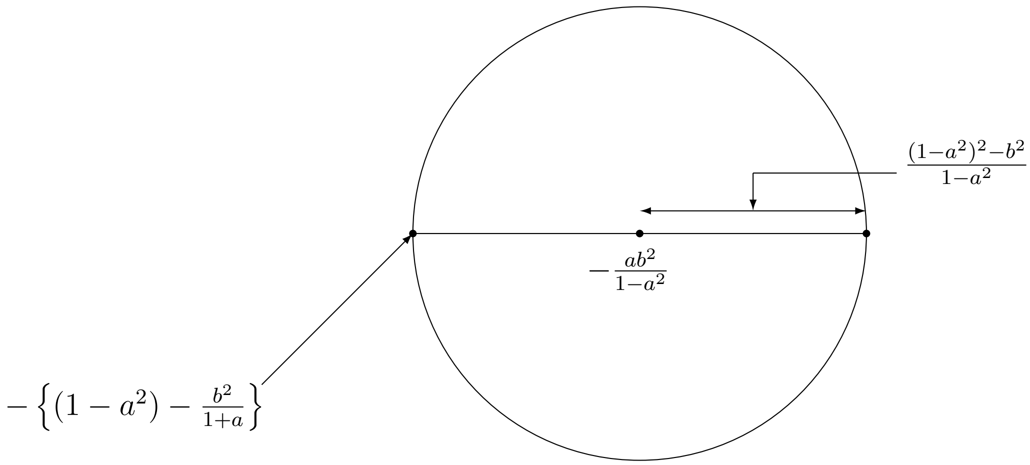

FIG. 4.1. The disk where $\frac{c}{2}$ lies

Let us take the point

$$P_\gamma \; : \quad -\frac{ab^2}{1-a^2} - \frac{(1-a^2)^2-b^2}{1-a^2}\mathrm{e}^{\mathrm{i}\gamma}\,,\; 0 \le \gamma < 2\pi$$

belonging to the boundary of $D(\mathbf{c}\,;\,\rho)$. We are going to construct a function $f(z) := \sum_{\nu=0}^{\infty} a_\nu z^\nu$ belonging to $\mathcal{B}^1$ for which

$$a_0 = a\,,\; a_1 = b \quad \text{and} \quad a_2 = -\frac{ab^2}{1-a^2} - \frac{(1-a^2)^2-b^2}{1-a^2}\mathrm{e}^{\mathrm{i}\gamma}\,. \tag{4.6}$$

Since $f(0) = a_0 = a \ge 0$, the function

$$F(z) := \frac{f(z)-a}{af(z)-1} = -\frac{f'(0)}{1-a^2}\,z + \text{ terms in higher powers of } z \tag{4.7}$$

belongs to $\mathcal{B}^1$ also. Now, note that the function

$$z\,\frac{z\,\mathrm{e}^{\mathrm{i}\gamma} - \frac{b}{1-a^2}}{-\frac{b}{1-a^2}\,z\,\mathrm{e}^{\mathrm{i}\gamma}+1}$$

belongs to $\mathcal{B}^1$. Its Maclaurin series expansion is

$$z\,\frac{z\,\mathrm{e}^{\mathrm{i}\gamma} - \frac{b}{1-a^2}}{-\frac{b}{1-a^2}\,z\,\mathrm{e}^{\mathrm{i}\gamma}+1} = -\frac{b}{1-a^2}\,z + \text{ terms in higher powers of } z \tag{4.8}$$

and

$$\left|z\,\frac{z\,\mathrm{e}^{\mathrm{i}\gamma} - \frac{b}{1-a^2}}{-\frac{b}{1-a^2}\,z\,\mathrm{e}^{\mathrm{i}\gamma}+1}\right| \le r_0\,\frac{r_0 + \frac{b}{1-a^2}}{\frac{b}{1-a^2}r_0+1} \le r_0 \qquad (|z| \le r_0 < 1)\,. \tag{4.9}$$

Comparing (4.7) and (4.8), we see that in order to have a function $f \in \mathcal{B}^1$ with $f(0) = a$ and $f'(0) = b$, we may consider f such that

$$\frac{f(z)-a}{af(z)-1} = z\,\frac{z\,\mathrm{e}^{\mathrm{i}\gamma} - \frac{b}{1-a^2}}{-\frac{b}{1-a^2}\,z\,\mathrm{e}^{\mathrm{i}\gamma}+1} =: w\,. \tag{4.10}$$

Solving for $f(z)$, we find that

$$f(z) = \frac{a-w}{1-aw}$$

and so in view of (4.9), we have

$$|f(z)| \le \frac{r_0 + a}{ar_0+1} < 1 \qquad (|z| \le r_0 < 1)\,.$$

Thus, we know that the function f satisfying (4.10) belongs to $\mathcal{B}^1$. Now, let us calculate the first three terms of its Maclaurin series. We easily see that

$$\begin{aligned}
f(z) &= \frac{a + \frac{b}{1-a^2}(1-a\,\mathrm{e}^{\mathrm{i}\gamma})z - \mathrm{e}^{\mathrm{i}\gamma}z^2}{1-\frac{b}{1-a^2}(\mathrm{e}^{\mathrm{i}\gamma}-a)z - a\,\mathrm{e}^{\mathrm{i}\gamma}z^2} \\
&= \left\{a + \frac{b}{1-a^2}(1-a\,\mathrm{e}^{\mathrm{i}\gamma})z - \mathrm{e}^{\mathrm{i}\gamma}z^2\right\}\left\{1 + \frac{b}{1-a^2}(\mathrm{e}^{\mathrm{i}\gamma}-a)z \right. \\
&\qquad \left. + a\mathrm{e}^{\mathrm{i}\gamma}z^2 + \frac{b^2}{(1-a^2)^2}(\mathrm{e}^{\mathrm{i}\gamma}-a)^2z^2 + \cdots\right\} \\
&= a + bz + \\
&\quad \left\{a^2\mathrm{e}^{\mathrm{i}\gamma} + \frac{ab^2}{(1-a^2)^2}\,(\mathrm{e}^{\mathrm{i}\gamma}-a)^2 + \frac{b^2}{(1-a^2)^2}\,(1-a\,\mathrm{e}^{\mathrm{i}\gamma})(\mathrm{e}^{\mathrm{i}\gamma}-a) - \mathrm{e}^{\mathrm{i}\gamma}\right\} z^2 + \cdots \\
&= a + bz + \left\{-(1-a^2)\mathrm{e}^{\mathrm{i}\gamma} + \frac{b^2}{(1-a^2)^2}(\mathrm{e}^{\mathrm{i}\gamma}-a)(a\mathrm{e}^{\mathrm{i}\gamma}-a^2+1-a\mathrm{e}^{\mathrm{i}\gamma})\right\} z^2 + \cdots \\
&= a + bz + (1-a^2)\left\{-\mathrm{e}^{\mathrm{i}\gamma} + \frac{b^2}{(1-a^2)^2}(\mathrm{e}^{\mathrm{i}\gamma}-a)\right\} z^2 + \cdots \\
&= a + bz + \left\{-\frac{ab^2}{1-a^2} - \frac{(1-a^2)^2-b^2}{1-a^2}\,\mathrm{e}^{\mathrm{i}\gamma}\right\} z^2 + \cdots\,,
\end{aligned}$$

which proves that

$$f_1(z) := \frac{a + \frac{b}{1-a^2}(1-a\,\mathrm{e}^{\mathrm{i}\gamma})z - \mathrm{e}^{\mathrm{i}\gamma}z^2}{1-\frac{b}{1-a^2}(\mathrm{e}^{\mathrm{i}\gamma}-a)z - a\,\mathrm{e}^{\mathrm{i}\gamma}z^2} \tag{4.11}$$

belongs to $\mathcal{B}^1$ and has the properties stipulated in (4.6).

Replacing γ by $\gamma+\pi$, we obtain the function

$$f_0(z) := \frac{a + \frac{b}{1-a^2}(1+a\,\mathrm{e}^{\mathrm{i}\gamma})z + \mathrm{e}^{\mathrm{i}\gamma}z^2}{1+\frac{b}{1-a^2}(\mathrm{e}^{\mathrm{i}\gamma}+a)z + a\,\mathrm{e}^{\mathrm{i}\gamma}z^2} = \sum_{\nu=0}^{\infty} a_\nu z^\nu\,, \tag{4.12}$$

which belongs to $\mathcal{B}^1$ and for which

$$a_0 = f_0(0) = a,\ a_1 = f_0'(0) = b \ \text{ and } \ a_2 = \frac{1}{2}f_0''(0) = -\frac{ab^2}{1-a^2} + \frac{(1-a^2)^2-b^2}{1-a^2}\mathrm{e}^{\mathrm{i}\gamma}\,.$$

Any point of the diameter of $D(\mathbf{c}\,;\,\rho)$ that joins the points

$$P_\gamma : -\frac{ab^2}{1-a^2} - \frac{(1-a^2)^2-b^2}{1-a^2}\mathrm{e}^{\mathrm{i}\gamma} \ \text{ and } \ P_{\gamma+\pi} : -\frac{ab^2}{1-a^2} + \frac{(1-a^2)^2-b^2}{1-a^2}\mathrm{e}^{\mathrm{i}\gamma}$$

can be written as

$$\lambda\left\{-\frac{ab^2}{1-a^2}-\frac{(1-a^2)^2-b^2}{1-a^2}\mathrm{e}^{\mathrm{i}\gamma}\right\}+(1-\lambda)\left\{-\frac{ab^2}{1-a^2}+\frac{(1-a^2)^2-b^2}{1-a^2}\mathrm{e}^{\mathrm{i}\gamma}\right\}$$

for some $\lambda\in[0\,,\,1]$. Clearly, $f_\lambda(z):=\lambda f_1(z)+(1-\lambda)f_0(z)$ belongs to $\mathcal{B}^1$. Furthermore, $f_\lambda(0)=a$, $f'_\lambda(0)=b$ and

$$\frac{1}{2}f''_\lambda(0)=\lambda\left\{-\frac{ab^2}{1-a^2}-\frac{(1-a^2)^2-b^2}{1-a^2}\mathrm{e}^{\mathrm{i}\gamma}\right\}$$
$$+(1-\lambda)\left\{-\frac{ab^2}{1-a^2}+\frac{(1-a^2)^2-b^2}{1-a^2}\mathrm{e}^{\mathrm{i}\gamma}\right\}.$$

Since γ can be any real number, we can find a function $f(z):=\sum_{\nu=0}^{\infty}a_\nu z^\nu$ of the form $\lambda f_1(z)+(1-\lambda)f_0(z)$, $0\le\lambda\le 1$, $\gamma\in\mathbb{R}$ for which

$$a_0=f(0)=a\quad,\quad a_1=f'(0)=b$$

and a_2 is an arbitrarily chosen point of $D(\mathbf{c}\,;\,\rho)$.

Now we turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. Without loss of generality, it has been assumed that $a_0=f(0)=a\ge 0$ and $a_1=f'(0)=b\ge 0$. By (4.4), $a^2+b\le 1$. We may suppose that $a<1$ since otherwise f would be identically 1 and a_2 would be 0. Let us consider the function

$$F(z):=\frac{f(z)-a}{af(z)-1}\,,$$

introduced in (4.7). As we have already indicated, it belongs to $\mathcal{B}^1$. It is to be noted that

$$F(0)=0\quad\text{and}\quad F'(0)=-\frac{f'(0)}{1-a^2}=-\frac{b}{1-a^2}\,.$$

Besides, simple calculation shows that

$$F''(z)=-(1-a^2)\frac{f''(z)(1-af(z))+2\,a(f'(z))^2}{(1-af(z))^3}$$

and so

$$F''(0)=-\frac{2\,ab^2+2\,a_2(1-a^2)}{(1-a^2)^2}\,.$$

Thus,

$$\begin{aligned}F(z)&=F(0)+\frac{1}{1!}F'(0)z+\frac{1}{2!}F''(0)z^2+\cdots\\&=-\frac{b}{1-a^2}z-\frac{ab^2+a_2(1-a^2)}{(1-a^2)^2}z^2+\cdots\,.\end{aligned}$$

Since $F(0)=0$, the function $\phi(z):=F(z)/z$ also belongs to $\mathcal{B}^1$. Its Maclaurin series expansion being

$$\phi(z)=-\frac{b}{1-a^2}-\frac{ab^2+a_2(1-a^2)}{(1-a^2)^2}z+\cdots,$$

it follows from (4.4) that

$$\frac{b^2}{(1-a^2)^2}+\frac{|ab^2+a_2(1-a^2)|}{(1-a^2)^2}\leq 1,$$

which can also be written as (4.5) □

Lemma 4-A. (Govil, Rahman, Schmeisser) [**18**] **.** *If $f(z)$ is an analytic function and $|f(z)|\leq 1$ in $|z|<1$, then*

$$|f(z)|\leq\frac{(1-|a|)\,|z|^2+|bz|+|a|\,(1-|a|)}{|a|\,(1-|a|)\,|z|^2+|bz|+(1-|a|)}\qquad(|z|<1),\tag{4.13}$$

where $a=f(0)$, $b=f'(0)$.

Let

$$f(0)=a,\ f'(0)=b,\ f''(0)=c.$$

Then, by Theorem 4.1, we have

$$t:=|2ab^2+c\,(1-a^2)|\leq 2\,(1-a^2-b)(1-a^2+b).\tag{4.14}$$

Let $\phi(z):=F(z)/z:=\frac{1}{z}\frac{f(z)-a}{af(z)-1}$. Set

$$A:=|\phi(0)|=\frac{b}{1-a^2}\ ,\ B:=\frac{1}{2}\frac{|2ab^2+c(1-a^2)|}{(1-a^2)^2}$$

and apply the lemma of Govil, Rahman and Schmeisser (Lemma 4-A) to the function $\phi(z)$. We obtain

$$|F(z)|\leq R(\rho):=\frac{(1-A)\rho^3+B\rho^2+A(1-A)\rho}{A(1-A)\rho^2+B\rho+1-A}\qquad(|z|=\rho<1).$$

Hence, for $|z|=r<\rho$:

$$\begin{aligned}|f(z)| &\leq \frac{R(\rho)+a}{1+aR(\rho)}\\ &= \frac{\frac{(1-A)\rho^3+B\rho^2+A(1-A)\rho}{A(1-A)\rho^2+B\rho+1-A}+a}{1+\frac{a(1-A)\rho^3+aB\rho^2+aA(1-A)\rho}{A(1-A)\rho^2+B\rho+1-A}}\\ &= \frac{(1-A)\rho^3+\{B+aA(1-A)\}\rho^2+\{aB+A(1-A)\}\rho+a(1-A)}{a(1-A)\rho^3+\{aB+A(1-A)\}\rho^2+\{B+aA(1-A)\}\rho+1-A}.\end{aligned}$$

We have

$$1-A=\frac{1-a^2-b}{1-a^2}\,,\;A(1-A)=\frac{b(1-a^2-b)}{(1-a^2)^2}\,,\;B=\frac{1}{2}\,\frac{|2ab^2+c(1-a^2)|}{(1-a^2)^2}\,.$$

Using the parameter t introduced in (4.14), we find that

$$B+aA(1-A)=\frac{1}{(1-a^2)^2}\left\{\frac{1}{2}\,t+ab(1-a^2-b)\right\}\,,$$

$$aB+A(1-A)=\frac{1}{(1-a^2)^2}\left\{\frac{a}{2}\,t+b(1-a^2-b)\right\}$$

and

$$a(1-A)=\frac{a(1-a^2-b)}{1-a^2}\,.$$

Thus, setting

$$\begin{aligned}N(t):=&\,(1-a^2)(1-a^2-b)(\rho^3+a)+\\&(1-a^2-b)b(a\rho^2+\rho)+\left(\frac{1}{2}\rho^2+\frac{a}{2}\rho\right)t\end{aligned}\tag{4.15}$$

and

$$\begin{aligned}D(t):=&\,(1-a^2)(1-a^2-b)(a\rho^3+1)+\\&b(1-a^2-b)(\rho^2+a\rho)+\left(\frac{a}{2}\rho^2+\frac{1}{2}\rho\right)t\end{aligned}\tag{4.16}$$

we see that

$$|f(z)|\le\psi(t):=\frac{N(t)}{D(t)}\qquad(|z|=r\le\rho)\,,\tag{4.17}$$

where t is subject to (4.14).

We note that $\psi(t)$ is an increasing function of t. For this we calculate $\psi'(t)$ and find that

$$\begin{aligned}&D^2(t)\psi'(t)\\&=\left(\frac{1}{2}\rho^2+\frac{a}{2}\rho\right)\{(1-a^2)(1-a^2-b)(a\rho^3+1)+b(1-a^2-b)(\rho^2+a\rho)\}\\&\;-\left(\frac{a}{2}\rho^2+\frac{1}{2}\rho\right)\{(1-a^2)(1-a^2-b)(\rho^3+a)+b(1-a^2-b)(a\rho^2+\rho)\}\\&=\frac{1}{2}(1-a^2)(1-a^2-b)\{(\rho^2+a\rho)(a\rho^3+1)-(a\rho^2+\rho)(\rho^3+a)\}\\&\;+\frac{1}{2}b(1-a^2-b)\{(\rho^2+a\rho)^2-(a\rho^2+\rho)^2\}\\&=\frac{1}{2}(1-a^2)^2(1-a^2-b)(\rho^2-\rho^4)-\frac{1}{2}b(1-a^2-b)(1-a^2)(\rho^2-\rho^4)\\&=\frac{1}{2}(1-a^2-b)^2(1-a^2)(\rho^2-\rho^4)>0\end{aligned}$$

for $0 < \rho < 1$. This proves that $\psi(t)$ is an increasing function of t. By (4.14), $t \le 2\,(1-a^2-b)(1-a^2+b)$ and therefore for $|z| \le \rho$ we have

$$
\begin{aligned}
|f(z)| &\le \psi(t) \le \psi\Big(2\,(1-a^2-b)(1-a^2+b)\Big)\\
&= \frac{(1-a^2)\rho^3 + \{1-a^2+ab+b\}\rho^2 + \{a(1-a^2+b)+b\}\rho + a(1-a^2)}{a(1-a^2)\rho^3 + \{a(1-a^2+b)+b\}\rho^2 + \{1-a^2+b+ab\}\rho + (1-a^2)}\\
&= \frac{(1-a)\rho^3 + (1-a+b)\rho^2 + (a-a^2+b)\rho + a - a^2}{(a-a^2)\rho^3 + (a-a^2+b)\rho^2 + (1-a+b)\rho + 1 - a}\\
&= \frac{(1-a)\rho^2 + b\rho + a(1-a)}{a(1-a)\rho^2 + b\rho + 1 - a}\,.
\end{aligned}
$$

Thus, the bound for $|f(z)|$ given by (4.17) is generally smaller than the one given by (4.13). Of course, (4.17) requires us to know $f''(0)$ in addition to $f(0)$ and $f'(0)$ whereas (4.13) takes into account $f(0)$ and $f'(0)$ only. As an immediate application of the previous lemma, we have the following theorem.

Theorem 4.2. *Let $P(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$ be a polynomial of degree n having all its zeros in $|z| \ge k \ge 1$ and let consider the following quantities*

$$\lambda := \frac{k}{n}\frac{P'(0)}{P(0)} = \frac{k}{n}\frac{c_1}{c_0}, \mu := \frac{k^2}{n}\frac{P''(0)}{P(0)} = 2\frac{k^2}{n}\frac{c_2}{c_0} \text{ and } \tau := \frac{k^3}{n}\frac{P'''(0)}{P(0)} = 6\frac{k^3}{n}\frac{c_3}{c_0}.$$

Then

$$\max_{|z|=1} |P'(z)| \le nR(k) \max_{|z|=1} |P(z)|,$$

where

$$R(k) := \frac{(1-A) + \{B + |\lambda| A(1-A)\}k + \{|\lambda| B + A(1-A)\}k^2 + |\lambda|(1-A)k^3}{\{B + |\lambda| A(1-A)\}k(1+k^2) + \{|\lambda| B + A(1-A)\}k^2 + (1-A)(1+k^4)},$$

$$A := \frac{|\mu - (n-1)\lambda^2|}{1 - |\lambda|^2}$$

$$\text{and } B := \frac{|\tau(1-|\lambda|^2) + 2\mu^2\bar{\lambda} + (3n-4)\mu\lambda + (7n-8)\mu\lambda|\lambda|^2 + 2(n-1)^2\lambda^3|}{2(1-|\lambda|^2)^2}.$$

Proof of Theorem 4.2. Let us recall the polynomial $Q(z) := z^n\overline{P(\frac{1}{\bar{z}})}$ and consider the function

$$f(z) := \frac{kP'(kz)}{nP(kz) - kzP'(kz)}.$$

Since $P(z)$ is not vanishing for $|z| < k$, then by the Laguerre's theorem (Theorem A-B in the Appendix B)

$$nP(z) + (\zeta - z)P'(z) \ne 0 \qquad (|\zeta| < k \text{ and } |z| < k),$$

hence

$$|nP(z) - zP'(z)| \ne |\zeta||P'(z)| \qquad (|\zeta| < k \text{ and } |z| < k)$$

and consequently f is holomorphic on the unit disk and $|f(z)| \leq 1$ for $|z| < 1$. Furthermore, a long calculation gives $f(0) = \lambda$, $f'(0) = \mu - (n-1)\lambda^2$ and $f''(0) = \tau + 2(n-1)^2\lambda^3 + (3n-4)\lambda\mu$. In view of (4.17) we have, for $|w| = \rho < 1$,

$$|f(w)| \;\leq\; \frac{(1-A)\rho^3 + \{B + |\lambda|A(1-A)\}\rho^2 + \{|\lambda|B + A(1-A)\}\rho + |\lambda|(1-A)}{|\lambda|(1-A)\rho^3 + \{|\lambda|B + A(1-A)\}\rho^2 + \{B + |\lambda|A(1-A)\}\rho + 1 - A}.$$

In the last inequality, if we replace w by $\frac{z}{k}$ where $|z| = 1$, then we obtain

$$k|P'(z)| \leq \frac{r(k)}{k^3 r\left(\frac{1}{k}\right)}|Q'(z)| \qquad (|z| = 1)$$

where $r(k) := (1-A) + \{B + |\lambda|A(1-A)\}k + \{|\lambda|B + A(1-A)\}k^2 + |\lambda|(1-A)k^3$. By (1.5) we have

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)| \qquad (|z| = 1)|.$$

Using this two inequalities, we obtain

$$\left\{1 + \frac{k^4}{r(k)} r\left(\frac{1}{k}\right)\right\} |P'(z)| \leq |P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)| \qquad (|z| = 1)|,$$

and consequently

$$\max_{|z|=1} |P'(z)| \leq nR(k) \max_{|z|=1} |P(z)|,$$

where $R(k) := \dfrac{r(k)}{r(k) + k^4 r\left(\frac{1}{k}\right)}$. □

Remark. Our result in Theorem 4.2. contains the result of Malik (Theorem 1.D.) as it is shown in what follows.
Recall that $R(k) = \dfrac{r(k)}{r(k) + k^4 r\left(\frac{1}{k}\right)}$, where

$$r(k) := (1-A) + \{B + |\lambda|A(1-A)\}k + \{|\lambda|B + A(1-A)\}k^2 + |\lambda|(1-A)k^3.$$

We have to show that $R(k) \leq \frac{1}{1+k}, (k \geq 1)$, which is equivalent to show that $r(k) \leq k^3 r\left(\frac{1}{k}\right)$. Let check the sign of $k^3 r\left(\frac{1}{k}\right) - r(k)$. We have

$$\begin{aligned} k^3 r\left(\frac{1}{k}\right) - r(k) &= (1-A)k^3 + \{B + |\lambda|A(1-A)\}k^2 + \{|\lambda|B + A(1-A)\}k \\ &+ |\lambda|(1-A) \\ &= (1-|\lambda|)(1-A)(k^3-1) + (1-|\lambda|)k(k-1)\{B - A(1-A)\} \\ &= (1-|\lambda|)(k-1)\{(1-A)(k^2+k+1) + kB - kA(1-A)\} \\ &= (1-|\lambda|)(k-1)\{(1-A)k^2 + [B + (1-A)^2]k + (1-A)\} \end{aligned}$$

Using the condition $|f(z)| \leq 1$ for $|z| < 1$ and the Schwarz-Pick theorem, we deduce that $|\lambda| \leq 1$ and $A = \dfrac{|f'(0)|}{1-|f(0)|^2} \leq 1$. Furthermore $B \geq 0$, hence $k^3 r\left(\frac{1}{k}\right) - r(k) \geq 0$, and consequently $R(k) \leq \frac{1}{k+1}$ for $k \geq 1$.

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CONCLUSION

In this dissertation, we were able to extend some results given by Govil, Rahman, and Schmeisser and to describe the behaviour of $M(r)\mathrm{e}^{-\alpha(\log r)^\beta}$ as $r \to \infty$ when $\int_1^\infty M(r)\mathrm{e}^{-\alpha(\log r)^\beta}\mathrm{d}r$ is convergent. Moreover, we extended a result of Govil (Theorem 1-E) to a special class of entire functions of exponential type (Theorem 2.1).

However, in view of this thesis, some comments and questions which can be considered in future works arise :

1- It seems that the sharp estimation of $|P'(z)|$ over the unit circle, in the case where $P(z) \neq 0$ for $|z| < k$ when $k < 1$, requires new most sophisticated methods, since the improvement in the case where $P(z)$ having its zeros on the circle $|z| = k$ is very small.

2- More generaly, how large can be

$$\frac{\max_{|z|=1} |P'(z)|}{\max_{|z|=1} |P(z)|}$$

when $P(z)$ lies among all the polynomials of degree n having m zeros inside the circle $|z| = k$?

3- In Theorem 2.1 (and analogously Theorem 1-E) there is a condition which should be verified, i.e. "$\sup_{-\infty<x<\infty} |f'(x)|$ and $\sup_{-\infty<x<\infty} |w'_f(x)|$ are both attained at the same point of the real axis". This hypothesis is very restrictive, can it be replaced by a weakly one? Even in the polynomial case, the problem turned to be challenging.

Appendix A

THE GAUSS–LUCAS THEOREM

Let g be a polynomial. We shall use $\mathcal{K}(g)$ to denote the convex hull of the zeros of g. It may be noted that $\mathcal{K}(g)$ is the smallest convex set containing all the zeros of g.

Theorem A-A (K.F. Gauss & F. Lucas) *Let $f(z) := c\prod_{\nu=1}^{n}(z - z_\nu)$ be a polynomial of degree n. Then $\mathcal{K}(f') \subseteq \mathcal{K}(f)$.*

Proof. We need to show that any critical point ζ of f (a zero of f') can be expressed as a convex linear combination of the zeros $z_1, \ldots, z_n$. It is clear that ζ is either a multiple zero of f or else

$$\sum_{\nu=1}^{n} \frac{1}{\zeta - z_\nu} = 0\,. \tag{A.1}$$

If $\zeta = z_\mu$, then we can write $\zeta = \sum_{\nu=1}^{n} \lambda_\nu z_\nu$, where

$$\lambda_\nu := \begin{cases} 1 & \text{if } \nu = \mu\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Now, let us turn to the case where (A.1) holds. Then $\zeta \neq z_\nu$ for all ν. Since a complex number vanishes if and only if its conjugate does, (A.1) holds if and only if

$$\sum_{(\nu=1}^{n} \frac{\zeta - z_\nu}{|\zeta - z_\nu|^2} = 0\,,$$

that is, if and only if

$$\left(\sum_{\mu=1}^{n} \frac{1}{|\zeta - z_\mu|^2}\right)\zeta = \sum_{\nu=1}^{n} \frac{\zeta}{|\zeta - z_\nu|^2} = \sum_{\nu=1}^{n} \frac{z_\nu}{|\zeta - z_\nu|^2}\,,$$

that is, if and only if

$$\zeta = \sum_{\nu=1}^{n} \left(\frac{|\zeta - z_\nu|^{-2}}{\sum_{\mu=1}^{n} |\zeta - z_\mu|^{-2}}\right) z_\nu\,.$$

This says that $\zeta = \sum_{\nu=1}^{n} \lambda_\nu \, z_\nu$, where

$$\lambda_\nu := \frac{|\zeta - z_\nu|^{-2}}{\sum_{\mu=1}^{n} |\zeta - z_\mu|^{-2}} \qquad (\nu = 1, \dots, n)\,.$$

Clearly, the numbers λ_ν are all positive and and their sum is 1. $\square$

Appendix B

A THEOREM OF E. LAGUERRE

By a *circular domain* we mean a domain $\mathcal{D}$ in $\mathbb{C}$ which is of one of the following forms:

$$\text{(i)}\{z : |z-a| < \rho\}\,, \text{(ii)}\{z : |z-a| \le \rho\}\,, \text{(iii)}\{z : |z-a| > \rho\}\,, \text{(iv)}\{z : |z-a| \ge \rho\}\,,$$

$$\text{(v)}\ \{z\ :\ \Im\{(z-z_0)\,\mathrm{e}^{\mathrm{i}\gamma}\} < 0\}\,,\ \text{(vi)}\ \{z\ :\ \Im\{(z-z_0)\,\mathrm{e}^{\mathrm{i}\gamma}\} \le 0\}\,,$$

where $z_0 \in \mathbb{C}$ and $\gamma \in \mathbb{R}$.

The following result is Laguerre's theorem as formulated by Szegö [**36**, p. 33]. In this form it has proved to be a very efficient tool.

Theorem A-B *Let $f(z)$ be a polynomial of degree at most n having no zeros in the circular domain $\mathcal{D}$. Then*

$$n\,f(z) + (\zeta - z)f'(z) \ne 0 \qquad (z \in \mathcal{D}\,,\ \zeta \in \mathcal{D})\ . \qquad (B.1)$$

Proof. Let $f(z) := c\prod_{\nu=1}^{m}(z - z_\mu)$, $m \le n$ and suppose that $\mathcal{D}$ is of the form (i). For any $z \in \mathcal{D}$, we have

$$\frac{f'(z)}{f(z)} = \sum_{\mu=1}^{m} \frac{1}{z - z_\mu}$$

and so

$$\begin{aligned} n + (\zeta - z)\,\frac{f'(z)}{f(z)} &= n + \sum_{\mu=1}^{m} \frac{\zeta - z}{z - z_\mu} = \sum_{\mu=1}^{m} \left(\frac{\zeta - z}{z - z_\mu} + 1\right) + n - m \\ &= n \cdot \frac{1}{n}\left(\sum_{\mu=1}^{m} \frac{\zeta - z_\mu}{z - z_\mu} + \sum_{n-m\ \text{terms}} 1\right) \end{aligned}$$

For $z \in \mathcal{D}$ and $\zeta \in \mathcal{D}$, arbitrarily chosen but fixed, we can find $\rho_0 < \rho$ such that z and ζ still belong to $|z - a| < \rho_0$. Then, according to our hypothesis, the points $z_1, \ldots, z_n$ all lie outside the circle $C := \{z \in \mathbb{C} : |z - a| = \rho_0\}$.

As usual, let $\widehat{\mathbb{C}}$ denote the extended complex plane, that is, $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and consider the Möbius transformation

$$T(w) := \frac{\zeta - w}{z - w}\,.$$

Under this transformation, no point in $\mathbb{C}\backslash\{|w-a| < \rho_0\}$ goes to the point at infinity. Also the point at infinity does not; in fact, the point at infinity goes to 1 under this transformation. Thus we see that the transformation T maps $\widehat{\mathbb{C}}\backslash\{|w-a| < \rho\}$ onto a closed disk Δ having 1 as one of its interior points and the circle C onto a circle Γ containing Δ inside it. It is important to note that 0 lies *outside* the circle C whereas Δ lies *inside* it. Note that

$$\frac{\zeta - z_1}{z - z_1}\,, \ldots\,, \frac{\zeta - z_n}{z - z_m}$$

all lie in Δ and so does 1. With this we see that

$$\frac{1}{n}\left(\sum_{\mu=1}^{m} \frac{\zeta - z_\mu}{z - z_\mu} + \sum_{n-m\,\text{terms}} 1\right)$$

is the arithmetic mean (center of gravity) of n numbers all lying in the closed disk Δ. So, it must lie in Δ too and cannot be 0 since 0 lies outside Δ. Thus

$$n + (\zeta - z)\,\frac{f'(z)}{f(z)} = n \cdot \frac{1}{n}\left(\sum_{\mu=1}^{m} \frac{\zeta - z_\mu}{z - z_\mu} + \sum_{n-m\,\text{terms}} 1\right) \neq 0\,,$$

which implies (B.1).

Now let $\mathcal{D}$ be of the form (ii). In this case, there exists $\rho_1 > \rho$ such that p has no zeros in the disk $\mathcal{D}_1 := \{z| :: \ |z-a| < \rho_1$. By what we have already proved

$$nf(z) + (\zeta - z)f'(z) \neq 0 \qquad (z \in \mathcal{D}_1\,,\ \zeta \in \mathcal{D}_1)\ ,$$

which covers (B.1).

Using appropriate linear transformations we can deduce the desired result in a case where $\mathcal{D}$ is of one of the remaining four forms. □

Appendix C

HADAMARD'S THREE CIRCLES THEOREM

In this section we collect some basic facts about the function $M(r)$ which we have used in our proof of Theorem 3.1.

Convexity of $\log M(r)$ *as a function of* $\log r$

Hadamard's three-circles theorem [**37**, p. 172] says: *Let $f(z)$ be an analytic function, regular for $r_1 \leq |z| \leq r_3$. Furthermore, let $r_1 < r_2 < r_3$, and let M_1, M_2, M_3 be the maxima of $|f(z)|$ on the three circles $|z| = r_1, r_2, r_3$, respectively. Then*

$$M_2^{\log(r_3/r_1)} \leq M_1^{\log(r_3/r_2)}\, M_3^{\log(r_2/r_1)}\,. \tag{C.1}$$

Since we may write $(C.1)$ in the form

$$\log M(r_2) \leq \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} \log M(r_1) + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} \log M(r_3)\,, \tag{C.2}$$

Hadamard's three-circles theorem may be expressed by saying that $\log M(r)$ is a convex function of $\log r$.

In our case, $M(r) := \max_{|z|=r} |f(z)|$, where f is an entire function. Unless f is a constant, $M(r)$ is a strictly increasing function of r. It is easily seen that $f(z)$ is a polynomial of degree n, that is $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$, $a_n \neq 0$ if and only if $\frac{\log M(r)}{\log r} \to n$ as $r \to \infty$. From $(C.2)$ it follows that if $f(z)$ is a transcendental entire function, then there exists a number r_0 such that $\frac{\log M(r)}{\log r}$ is an unbounded strictly increasing function of r for $r \geq r_0$.

We know that $\log M(r)$ is continuous. In addition, it is a convex function of $\log r$. It is known (see [**8**, p. 142]) that *a continuous convex function has finite right-hand and left-hand derivatives at each point, and that these derivatives themselves are nondecreasing functions.*

Appendix D

ALMOST PERIODIC FUNCTIONS

This section contains some basic facts about uniformly almost periodic functions.

A periodic function f with period Δ has the property that $f(x+\mathfrak{t}) = f(x)$ for all real x and any $\mathfrak{t} \in \{0, \pm\Delta, \pm 2\Delta, \ldots\}$. The set $E := \{0, \pm\Delta, \pm 2\Delta, \ldots\}$ has the property that every interval $(\alpha\,,\, \alpha+\ell)$ of length greater than Δ contains at least one element of E.

In order to explain what an almost periodic function means we need to introduce the following definition.

Definition A-D. A set E of real numbers is said to be *relatively dense* if there exists a positive number ℓ such that any interval $(\alpha\,,\, \alpha+\ell)$ contains at least one number of E.

The set $E := \{0, \pm\Delta, \pm 2\Delta, \ldots\}$ associated with a Δ-periodic function f is relatively dense. On the hand, the set $\{\pm n^2 \,:\, n \in \mathbb{N}\}$ is not relatively dense.

For a periodic function f the equation $f(x+\mathfrak{t}) = f(x)$ is satisfied for all real x and infinitely many values of $\mathfrak{t}$ which form a relatively dense set. If f is not periodic but the equation $f(x+\mathfrak{t}) = f(x)$ is satisfied with arbitrary degree of accuracy for an infinitely many values of $\mathfrak{t}$ which form a relatively dense set, then we say that f is almost periodic. In somewhat more precise terms, f is almost periodic if for any positive ε, there exists a relatively dense set $E = E(\varepsilon)$ such that $|f(x+\mathfrak{t}) - f(x)| \le \varepsilon$ for all real x and all $\mathfrak{t} \in E$.

Definition B-D. A real number $\mathfrak{t} = \mathfrak{t}\,(\varepsilon) = \mathfrak{t}_f(\varepsilon)$ is called a *translation number* of f corresponding to ε provided that $|f(x+\mathfrak{t}) - f(x)| \le \varepsilon$ for all real x.

We are now ready to give the precise definition of a uniformly almost periodic function.

Definition C-D. A continuous function $f : \mathbb{R} \to \mathbb{C}$ is called *uniformly almost periodic function, u.a.p.* for short, if there exists a relatively dense set of translation numbers of f corresponding to any given $\varepsilon > 0$. In other words, for any $\varepsilon > 0$ we can find a positive number $\ell = \ell(\varepsilon)$ such that an arbitrary interval of length ℓ contains at least one translation number $\mathfrak{t}(\varepsilon)$.

Clearly, any continuous periodic function is uniformly almost periodic. If f is uniformly almost periodic then so is cf for any constant c; also $|f|$ is u.a.p. since $||f(x+\mathfrak{t})| - |f(x)|| \leq |f(x+\mathfrak{t}) - f(x)|$. Here are some other properties of u.a.p. functions. We refer the reader to [**4.**, pp. 2–6, 12–15] for proofs.

- A u.a.p. function f is bounded on the real axis, i.e. there exists a constant $M = M(f)$ such that $|f(x)| \leq M$ for all real x.
- A u.a.p. function f is uniformly continuous for $-\infty < x < \infty$, i.e. to an arbitrary $\varepsilon > 0$ there corresponds a $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x_1) - f(x_2)| \leq \varepsilon \qquad (|x_1 - x_2| \leq \delta)\,.$$

- The limit function $f(x)$ of a sequence of u.a.p. functions $f_1(x), \ldots, f_n(x), \ldots$, uniformly convergent for $-\infty < x < \infty$, is also u.a.p.
- If f and g are two u.a.p. functions, then $f + g$ and fg are both u.a.p.
- If f is u.a.p., then $(1/T)\int_0^T f(x)\,\mathrm{d}x$ tends to a finite limit as $T \to \infty$. The limit is denoted by $\mathcal{M}\{f\}$ and is called the mean value. Since $|f|$ is u.a.p. because f is, we may add that also $(1/T)\int_0^T |f(x)|\,\mathrm{d}x$ tends to a finite limit as T tends to ∞.

Let f be uniformly almost periodic. Since $g(x) := \mathrm{e}^{-\mathrm{i}\lambda x}$ is periodic and so uniformly almost periodic, the product $f(x)\,\mathrm{e}^{-\mathrm{i}\lambda x}$ is uniformly almost periodic. Hence,

$$a(\lambda) = \mathcal{M}\{f(x)\,\mathrm{e}^{-\mathrm{i}\lambda x}\} := \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)\,\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x \qquad (D.1)$$

exists and this for *any* $\lambda \in \mathbb{R}$. The following result (see [**4.**, pp. 16–18] for a proof) plays a key role in the theory of u.a.p. functions.

Theorem A-D. *Let f be uniformly almost periodic, and let $a(\lambda)$ be as in (D.1). Then $a(\lambda) = 0$ except for a countable number of λ's.*

Let f be uniformly almost periodic. Denote the values of λ for which $a(\lambda)$ differs from zero by $\Lambda_1, \Lambda_2, \ldots$, and write $A_n := a(\Lambda_n)$ for $n \in \mathbb{N}$. We call the numbers $\Lambda_1, \Lambda_2, \ldots$, *Fourier exponents* and the numbers $A_1, A_2, \ldots$ *Fourier coefficients* of the function f. The formal series

$$\sum_{n=1}^{\infty} A_n\,\mathrm{e}^{\mathrm{i}\Lambda_n x} = A_1\,\mathrm{e}^{\mathrm{i}\Lambda_1 x} + A_2\,\mathrm{e}^{\mathrm{i}\Lambda_2 x} + \cdots$$

is called the *Fourier series* of f.

The following result for whose proof we refer the reader to [**4**., pp. 21–28] is called "Bohr's fundamental theorem" or "Parseval equation".

Theorem B-D. *Let $f(x)$ be any u.a.p. function and let $\sum_{n=1}^{\infty} A_n\, e^{i\Lambda_n x}$ be its Fourier series. Then*

$$\mathcal{M}\{|f|^2\} = \sum_{n=1}^{\infty} |A_n|^2 \,.$$

By a result in [**4**., pp. 51–52], the Fourier series $\sum_{n=1}^{\infty} A_n\, e^{i\Lambda_n x}$ of a uniformly almost periodic function is absolutely convergent if the Fourier coefficients $A_1,\, A_2, \ldots$ are positive. This result has been extended; the extended version [**33**, Lemma 1] says that the Fourier series of a uniformly almost periodic function is absolutely convergent as long as the coefficients $A_1,\, A_2, \ldots$ lie in a sector of opening $\gamma \in (0\,,\, \pi)$.

It is useful to know (see for example [**5**]) that the Fourier exponents Λ_n of a uniformly almost periodic entire function of exponential type are bounded. The following lemma clearly shows this.

Lemma A-D. *Let f be an entire function of exponential type τ bounded on the real axis, and let λ be any real number such that $|\lambda| > \tau$. Then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(x)\, e^{i\lambda x}\, dx = 0\,. \qquad (D.2)$$