A CHARACTERIZATION OF EXACT NON-ATOMIC MARKET GAMES

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Abstract. Continuous exact non-atomic games are naturally associated to certain operators between Banach spaces. It thus makes sense to study games by means of the corresponding operators. We characterize non-atomic exact market games in terms of the properties of the associated operators. We also prove a separation theorem for weak compact sets of countably additive measures, which is of independent interest.

1. Preliminaries

Given a measurable space (Ω, Σ), a TU (tranferable utility) *game* is a set function $\nu : \Sigma \to \mathbb{R}$ such that $\nu(\varnothing) = 0$. Ω is the set of players, Σ the σ-algebra of admissible coalitions and ν describes the worth of each coalition. In this paper, we deal with games satisfying a certain number of properties such as continuity, non-atomicity, exactness, etc. We recall the main definitions. A coalition $N \in \Sigma$ is *null* if $\nu(A \cup N) = \nu(A)$ for all A in Σ; an *atom* of ν is a non-null coalition A such that for every coalition $B \subset A$ either B or $A \setminus B$ is null. A game ν is *non-atomic* if it has no atoms ([5, p. 14] and [12, p. 55]). Let $\{A_n\}$ be a sequence in Σ, and for each n let A_n^c denotes the complement of A_n. A game ν is *continuous* if $\lim_{n\to\infty} \nu(A_n) = \lim_{n\to\infty} (\nu(\Omega) - \nu(A_n^c)) = 0$ whenever $A_n \searrow \varnothing$. The *core* of a game ν is the set

$$core(\nu) = \{\mu \in ba(\Sigma) : \mu(\Omega) = \nu(\Omega) \text{ and } \mu(A) \geq \nu(A) \text{ for all } A \in \Sigma\}.$$

where $ba(\Sigma)$ denotes the Banach space of charges (= finitely additive measures) on Σ endowed with the variation norm. Clearly, the core is always a

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weak*-compact, convex subset of $ba(\Sigma)$. A game ν is *exact* if $core(\nu) \neq \varnothing$ and

$$\nu(A) = \min_{\mu \in core(\nu)} \mu(A), \quad \forall A \in \Sigma.$$

A game ν is the *lower envelope* of a set K of charges on Σ if it satisfies

$$\nu(A) = \inf_{\mu \in K} \mu(A) \tag{1.1}$$

for all $A \in \Sigma$. It is the *upper envelope* of K if the inf in (1.1) is replaced by a sup. Several classes of games are directly defined as lower/upper envelopes of charges. Examples include the *thin* games of Amarante and Montrucchio [4] and the *symmetric coherent capacities* of Kadane and Wasserman [9]. The first obtain when the set K in (1.1) is a thin set of non-atomic (countably additive) measures [4, Definition 5];[1] the second when all the charges in K satisfy the following symmetry condition [9, Section 1]: there exists a non-atomic probability measure λ such that[2]

$$\lambda(A) = \lambda(B) \Longrightarrow \mu(A) = \mu(B) \qquad for\ all\ \mu \in K$$

Other classes of games, while defined in a different way, are representable as lower/upper envelopes. This is the case for the class of exact games mentioned above: it is easy to see that a game is exact if and only if it is the lower envelope of a (norm-bounded) subset K of $ba_\alpha(\Sigma)$, where $ba_\alpha(\Sigma) = \{\mu \in ba(\Sigma) : \mu(\Omega) = \alpha, \alpha \in \mathbb{R}\}$. Moreover, several important subclasses of exact games (besides the thin games above) are also lower envelope games. In fact, owing to results of Schmeidler [13] and Marinacci-Montrucchio [12, pp. 54-58], *continuous exact games* and *continuous exact non-atomic games* are also lower envelopes games. The former obtain when K is a weak compact subset (in $ba_\alpha(\Sigma)$) of countably additive measures, while the latter requires that, in addition, all the measures be non-atomic. Finally, by virtue of a result of Amarante-Maccheroni-Marinacci-Montrucchio [3, Theorem 3], the class of *exact non-atomic market games* is also a class of lower envelope games, which obtains whenever K is a norm-compact subset of $ba_\alpha(\Sigma)$.

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[1] For $M \subset \mathcal{L}^1(\Omega, \Sigma, \lambda)$ and $S \in \Sigma$, the subset $M(S)^\perp \subset \mathcal{L}^\infty(\Omega, \Sigma, \lambda)$ is given by

$$M(S)^\perp = \{\varphi \in \mathcal{L}^\infty(\Omega, \Sigma, \lambda) : \langle f, \varphi \rangle = 0 \text{ for all } f \in M \text{ and } \varphi \chi_{S^c} = 0\}.$$

A set $M \subset \mathcal{L}^1(\Omega, \Sigma, \lambda)$ is *thin*, if and only if $M(S)^\perp \neq \{0\}$ for all S such that $\lambda(S) > 0$ (see [10])

[2] The definition in [9, Section 1] is slightly different, yet obviously equivalent, to the one given in the text.

2. A separation result

For lower/upper envelope games, the link between the game and the set of charges defining it is not always sharp. That is, as the next example of Huber and Strassen [8] shows, different sets of charges might define the same game.

Example 1 (Huber and Strassen [8])**.** *Let $\Omega = \{1, 2, 3\}$ and consider the two measures on Ω defined by $\mu = (\frac{1}{2}, \frac{1}{2}, 0)$ and $\lambda = (\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$. Let ν be the lower envelope game defined by $\mathcal{C}_1 = co\,\{\mu, \lambda\}$ (where co denotes the convex hull). That is,*

$$\nu(A) = \inf_{\xi \in \mathcal{C}_1} \xi(A)$$

for all $A \subset \Omega$. It is easy to check that

(a)

$$core(v) = \mathcal{C}_2 = \left\{ \left(\frac{3+t}{6}, \frac{3-t-s}{6}, \frac{s}{6} \right) : 0 \leq s, t \leq 1 \right\}$$

(b) For all $A \subset \Omega$

$$\min_{\xi \in \mathcal{C}_1} \xi(A) = \nu(A) = \min_{\xi \in \mathcal{C}_2} \xi(A)$$

(c) $\mathcal{C}_1$ is strictly included in $\mathcal{C}_2$.

It is easy to see that the situation described by the example is fairly typical whenever the game has atoms. The next proposition shows, however, that the situation is dramatically different in the non-atomic case.

Proposition 1. *Let $\nu : \Sigma \longrightarrow \mathbb{R}$ be a continuous, exact non-atomic game. Then, ν is the lower envelope of a unique, weak compact (in $ba(\Sigma)$), convex set of non-atomic measures.*

Proof. By the result of Marinacci-Montrucchio mentioned above [12, pp. 54-58], continuous exact non-atomic games are lower envelopes games defined by a weak-compact set $K \subset ba_\alpha(\Sigma)$ consisting of non-atomic measures. By the Bartle-Dunford-Schwartz Theorem (see [6, Cor. 6, p. 14]), the weak compactness of K implies that there exists a non-atomic finite measure λ on Σ such that all the measures in K are absolutely continuous with respect to λ. Thus, by the Radon-Nikodym theorem, K is isometrically isomorphic to a weak compact subset K' of $\mathcal{L}^1(\lambda)$ ([7, Th. IV.9.2]). Since it is continuous, exact and non-atomic, the game ν, however, is also the lower envelope of its core. By Schmeidler's theorem [13] and [7, Th. IV.9.2], this is also isometrically isomorphic to a weak compact subset K'' of $\mathcal{L}^1(\lambda)$. Clearly, the inclusion $K' \subset K''$ always holds. We are going to show that necessarily $K' = K''$ whenever ν is continuous, exact and non-atomic.

The dual of $\mathcal{L}^1(\lambda)$ is the Banach space $\mathcal{L}^\infty(\lambda)$ of Σ-measurable, λ-essentially bounded functions on Ω. Let Φ denote the intersection of the positive cone with the unit ball in $\mathcal{L}^\infty(\lambda)$. Since λ is finite and non-atomic, the indicator functions are weak*-dense in Φ (see [10]). Now, suppose that the exists a $\kappa \in K'' \backslash K'$. The sets K' and $\{\kappa\}$ are both weak compact. Thus, by the Separating Hyperplane Theorem (see [7, Th. V.2.10]) there exists a $\varphi_0 \in \mathcal{L}^\infty(\lambda) - \{0\}$ and a $\varepsilon > 0$ such that

$$\int \varphi_0 k d\lambda + 2\varepsilon < \min_{\tilde{k} \in K'} \int \varphi_0 \tilde{k} d\lambda$$

Wlog, we can assume that $\varphi_0 \in \Phi$ (otherwise take $\frac{\varphi_0 - \text{essinf}\varphi_0}{\|\varphi_0 - \text{essinf}\varphi_0\|}$).

Consider the functions

$$F : \Phi \times \{k\} \longrightarrow \mathbb{R} \quad , \quad F(\varphi, k) = \int \varphi k d\lambda$$

$$G : \Phi \times K' \longrightarrow \mathbb{R} \quad , \quad G(\varphi, \tilde{k}) = \int \varphi \tilde{k} d\lambda$$

where Φ is endowed with the weak*-topology (and is compact in this topology) and K' is endowed with the weak-topology (and is compact in this topology). By the continuity of the function F. there exists a weak*-neighborhood of φ_0, $W(\varphi_0)$, such that

$$\varphi \in W(\varphi_0) \qquad \Longrightarrow \qquad \left| \int \varphi_0 k d\lambda - \int \varphi k d\lambda \right| < \varepsilon$$

Since the function G is jointly continuous and K' is compact, the Maximum Theorem (see [1, Theorem 17.31]) implies that there exists a weak*-neighborhood of φ_0, $U(\varphi_0)$, such that

$$\varphi \in U(\varphi_0) \qquad \Longrightarrow \qquad \left| \min_{\tilde{k} \in K'} \int \varphi_0 \tilde{k} d\lambda - \min_{\tilde{k} \in K'} \int \varphi \tilde{k} d\lambda \right| < \varepsilon$$

Thus, for $\varphi \in U(\varphi_0) \cap W(\varphi_0)$ we have

$$\begin{aligned} \min_{\tilde{k} \in K'} \int \varphi \tilde{k} d\lambda &> \min_{\tilde{k} \in K'} \int \varphi_0 \tilde{k} d\lambda - \varepsilon \\ &> \int \varphi_0 k d\lambda + 2\varepsilon - \varepsilon \\ &> \int \varphi k d\lambda + 2\varepsilon - \varepsilon - \varepsilon = \int \varphi k d\lambda \end{aligned}$$

Since the indicator functions are weak*-dense in Φ, one can take $\varphi = \chi_A$, the indicator function of the set $A \in \Sigma$, and the preceding states that there is a measure $\mu_k \in core(\nu)$ such that

$$\mu_k(A) < \min_{\mu \in K} \mu(A)$$

which contradicts the fact that ν is the lower envelope of both K and $core(\nu)$.

□

The result of Proposition 1 subsumes earlier results of Amarante-Maccheroni[2], who obtained it for thin cores, and of Amarante-Maccheroni-Marinacci-Montrucchio [3], who obtained it for norm compact cores. With obvious modifications (i.e., by replacing the core with the anti-core), it also applies to symmetric coherent capacities as the symmetry condition of Kadane and Wasserman [9] implies that those cores are weak compact (see [11]). The same argument as that used in the proof of Proposition 1 leads to the following Corollary, which is of independent interest.

Corollary 1. *Let C_1 and C_2 be two convex, weak-compact subsets of $ba(\Sigma)$, which consists of countably additive non-atomic measures. If $C_1 \cap C_2 = \varnothing$, then there exists $A \in \Sigma$ such that*

$$\min_{\mu \in C_1} \mu(A) > \max_{\mu \in C_2} \mu(A)$$

3. From games to operators

Another property common to both thin games and exact non-atomic market games is that they are both naturally associated with certain compact operators. To see this, let us begin by observing that any continuous exact game is naturally associated with a linear operator $\mathcal{L}^\infty(\lambda) \to l^\infty(\mathcal{I})$. For if ν is continuous and exact, then $core(\nu)$ is isometrically isomorphic (by Schmeidler's theorem) to a subset $\mathcal{F} = \{f_i\}_{i \in \mathcal{I}} \subset \mathcal{L}^1(\lambda)$, and we can define the operator $T_\nu : \mathcal{L}^\infty(\lambda) \to l^\infty(\mathcal{I})$ by

$$T_\nu \varphi = (\int \varphi f_i d\lambda)_{i \in \mathcal{I}}.$$

T_ν is the integral of the vector measure $\boldsymbol{\mu} : \Sigma \longrightarrow l^\infty(\mathcal{I})$ defined by

$$A \longmapsto (\int \chi_A f_i d\lambda)_{i \in \mathcal{I}}.$$

We observe that (a) $\mathcal{F}$ weakly compact $\Longrightarrow \boldsymbol{\mu}$ bounded and countably additive; and (b) the Bartle-Dunford-Schwartz theorem implies that T_ν is always weak* to weak continuous (see [6, Corollary 7, p. 14]).

Theorem 1. *T_ν is a compact operator when ν is either an exact non-atomic market game or ν is a thin game.*

Proof. (1) If ν is an exact non-atomic market game, then $core(\nu) \sim \mathcal{F} = \{f_i\}_{i\in\mathcal{I}} \subset \mathcal{L}^1(\lambda)$ and is compact in the $\mathcal{L}^1$-norm by [3, Theorem 3]. Let $R \equiv T_\nu \mid_{B_{\mathcal{L}^\infty(\lambda)}}$, that is R is the restriction of T_ν to the unit ball $B_{\mathcal{L}^\infty(\lambda)}$ in $\mathcal{L}^\infty(\lambda)$. We are going to show that R is weak* to norm continuous, which immediately implies compactness of T_ν. To this end, consider the family of linear functionals $\mathcal{F} = \left\{\int \cdot f_i d\lambda \mid i \in \mathcal{I}\right\}$ on $\mathcal{L}^\infty(\lambda)$. By considering the restrictions of the functionals to $B_{\mathcal{L}^\infty(\lambda)}$, we can view $\mathcal{F}$ as a subset of $\mathcal{C}(B_{\mathcal{L}^\infty(\lambda)})$. Since $\mathcal{F}$ is norm-compact, it is equicontinuous by Arzelà-Ascoli's theorem. Hence, for any $\varphi^* \in B_{\mathcal{L}^\infty(\lambda)}$ and $\forall\varepsilon > 0$, there exists a weak* neighborhood $U(\varphi^*)$ such that

$$\varphi \in U(\varphi^*) \Longrightarrow \|R\varphi - R\varphi^*\|_\infty = \sup_{i\in\mathcal{I}} \left| \int \varphi f_i d\lambda - \int \varphi^* f_i d\lambda \right| < \varepsilon,$$

which proves that T_ν is bounded-weak* to norm continuous.

(2) Let $core(\nu) \sim \mathcal{F} = \{f_i\}_{i\in\mathcal{I}} \subset \mathcal{L}^1(\lambda)$. If ν is thin, then the norm-closure of the range of the vector measure $\boldsymbol{\mu}$ is convex [10]. By letting K_0 denote the set of indicator functions in $\mathcal{L}^\infty(\lambda)$, and by $K = [\mathcal{L}^\infty(\lambda)]_1$, we have

$$T_\nu(K) = \overline{T_\nu(K_0)}^w = \overline{\boldsymbol{\mu}(K_0)}^w = \overline{\boldsymbol{\mu}(K_0)}.$$

The first equality follows from the fact – noted above – that T_ν is always weak* to weak continuous, the second is a definition and the third follows from $\overline{\boldsymbol{\mu}(K_0)}^w = \overline{\left[\overline{\boldsymbol{\mu}(K_0)}\right]}^w = \overline{\boldsymbol{\mu}(K_0)}$. In fact, $\overline{\boldsymbol{\mu}(K_0)}$ is convex, and we have $\overline{\left[\overline{\boldsymbol{\mu}(K_0)}\right]}^w = \overline{\left[\overline{\boldsymbol{\mu}(K_0)}\right]} = \overline{\boldsymbol{\mu}(K_0)}$. Now, $T_\nu(K)$ norm-compact $\Longrightarrow$ $T_\nu(B_{\mathcal{L}^\infty(\lambda)}) = T_\nu(K - K)$ – the image of the ball – is norm-compact. Hence, T_ν is a compact operator. $\square$

4. A characterization of exact non-atomic market games

Thin games need not be market games as thin sets need not be norm-compact (see [4]). We now give a necessary and sufficient condition for a continuous exact game (hence, a thin game) to be a market game. It is stated in terms of the properties of the operator T_ν encountered above.

Theorem 2. *A continuous exact game ν is a market game if and only if the associated operator T_ν is bounded-weak* to norm continuous.*[3]

Proof. If ν is a market game, then $core(\nu) \sim \mathcal{F} = \{f_i\}_{i\in\mathcal{I}} \subset \mathcal{L}^1(\lambda)$ and is compact in the $\mathcal{L}^1$-norm by [3, Theorem 3]. By the proof of Theorem 1 part (1), T_ν is bounded-weak* to norm continuous.

In the converse direction, assume that T_ν is bounded-weak* to norm continuous. We are going to show that this implies that the support functional of $\mathcal{F}$, $\sigma_{\mathcal{F}} : \mathcal{L}^\infty(\lambda) \to \mathbb{R}$,

$$\sigma_{\mathcal{F}}(\varphi) = \sup_{i\in\mathcal{I}} \int \varphi f_i d\lambda,$$

is continuous for the bounded weak*-topology. In fact, let $\varphi^* \in \mathcal{L}^\infty(\lambda)$. There exists $\alpha \in \mathbb{R}$ such that $\varphi^* \in \alpha B_{\mathcal{L}^\infty(\lambda)}$. Since $T_\nu \mid_{\alpha B_{\mathcal{L}^\infty(\lambda)}}$ is weak*-to-norm continuous, $\forall \varepsilon > 0$ here exists a weak* neighborhood $U(\varphi^*)$ such that

$$\varphi \in U(\varphi^*) \cap \alpha B_{\mathcal{L}^\infty(\lambda)} \Longrightarrow \sup_{i\in I} \left| \int \varphi f_i d\lambda - \int \varphi^* f_i d\lambda \right| < \varepsilon.$$

Combining this with the elementary inequality: If $w, z \in \mathbb{R}^I$ are bounded, then

$$\left| \sup_I w - \sup_I z \right| \leq \sup_I |w - z|$$

we get (by observing that for $\varphi \in B_{\mathcal{L}^\infty(\mu)}$ and $f_i \in \mathcal{L}^1(\lambda)$, the mapping $w = (\int \varphi f_i d\mu)_{i\in I}$ is a bounded element of $\mathbb{R}^I$)

$$\varphi \in U(\varphi^*) \cap \alpha B_{\mathcal{L}^\infty(\lambda)} \Longrightarrow |\sigma_{\mathcal{F}}(\varphi) - \sigma_{\mathcal{F}}(\varphi^*)| \leq \sup_{i\in I} \left| \int \varphi f_i d\lambda - \int \varphi^* f_i d\lambda \right| < \varepsilon.$$

Now, $\sigma_{\mathcal{F}}$ continuous for the bounded weak*-topology implies, by Lemma 3 in [3], that $\mathcal{F}$ is compact in the norm topology. That is, ν is market game. □

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