Université de Montréal

Brisure de symétrie par la réduction des groupes de Lie simples à leurs sous-groupes de Lie réductifs maximaux

par

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Thèse présentée à la Faculté des études supérieures
en vue de l'obtention du grade de
Philosophiæ Doctor (Ph.D.)
en Mathématiques

décembre 2012

Université de Montréal
Faculté des études supérieures

Cette thèse intitulée

Brisure de symétrie par la réduction des groupes de Lie simples à leurs sous-groupes de Lie réductifs maximaux

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Thèse acceptée le:
19 novembre 2012

RÉSUMÉ

Dans ce travail, nous exploitons des propriétés déjà connues pour les systèmes de poids des représentations afin de les définir pour les orbites des groupes de Weyl des algèbres de Lie simples, traitées individuellement, et nous étendons certaines de ces propriétés aux orbites des groupes de Coxeter non cristallographiques. D'abord, nous considérons les points d'une orbite d'un groupe de Coxeter fini G comme les sommets d'un polytope (G-polytope) centré à l'origine d'un espace euclidien réel à n dimensions. Nous introduisons les produits et les puissances symétrisées de G-polytopes et nous en décrivons la décomposition en des sommes de G-polytopes. Plusieurs invariants des G-polytopes sont présentés. Ensuite, les orbites des groupes de Weyl des algèbres de Lie simples de tous types sont réduites en l'union d'orbites des groupes de Weyl des sous-algèbres réductives maximales de l'algèbre. Nous listons les matrices qui transforment les points des orbites de l'algèbre en des points des orbites des sous-algèbres pour tous les cas $n \leq 8$ ainsi que pour plusieurs séries infinies des paires d'algèbre-sous-algèbre. De nombreux exemples de règles de branchement sont présentés. Finalement, nous fournissons une nouvelle description, uniforme et complète, des centralisateurs des sous-groupes réguliers maximaux des groupes de Lie simples de tous types et de tous rangs. Nous présentons des formules explicites pour l'action de tels centralisateurs sur les représentations irréductibles des algèbres de Lie simples et montrons qu'elles peuvent être utilisées dans le calcul des règles de branchement impliquant ces sous-algèbres.

Mots clés: groupes de Weyl, algèbres de Lie simples, sous-algèbres réductives maximales, réduction, matrices de projection, centralisateurs.

ABSTRACT

In this work, we exploit properties well known for weight systems of representations to define them for individual orbits of the Weyl groups of simple Lie algebras, and we extend some of these properties to orbits of non-crystallographic Coxeter groups. Points of an orbit of a finite Coxeter group G are considered as vertices of a polytope (G-polytope) centered at the origin of a real n-dimensional Euclidean space. Products and symmetrized powers of G-polytopes are introduced and their decomposition into the sums of G-polytopes is described. Several invariants of G-polytopes are found. The orbits of Weyl groups of simple Lie algebras of all types are reduced to the union of orbits of the Weyl groups of maximal reductive subalgebras of the algebra. Matrices transforming points of the orbits of the algebra into points of subalgebra orbits are listed for all cases $n \leq 8$ and for many infinite series of algebra-subalgebra pairs. Numerous examples of branching rules are shown. Finally, we present a new, uniform and comprehensive description of centralizers of the maximal regular subgroups in compact simple Lie groups of all types and ranks. Explicit formulas for the action of such centralizers on irreducible representations of the simple Lie algebras are given and shown to have application to computation of the branching rules with respect to these subalgebras.

Keywords: Weyl groups, simple Lie algebras, maximal reductive subalgebras, reduction, projection matrices, centralizers.

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REMERCIEMENTS

Je tiens tout d'abord à remercier mon directeur de recherche, le professeur Jiri Patera, sans qui je n'aurais pas entamé ces études doctorales. Je vous remercie, M. Patera, pour votre temps, votre support, votre compréhension, votre confiance, votre écoute. Merci pour toutes les connaissances que vous m'avez transmises, pour toutes ces expériences que vous m'avez permis de vivre.

Je veux également remercier les professeurs Edita Pelantová (Université Technique Tchèque, Prague) et Francis W. Lemire (Université de Windsor, Ontario), ainsi que Maryna Nesterenko (Académie Nationale de Sciences d'Ukraine), avec qui j'ai eu le privilège de collaborer. J'aimerais aussi exprimer ma reconnaissance aux membres du jury, qui ont accepté d'évaluer cette thèse.

Je remercie le département de mathématiques et de statistique de l'Université de Montréal, ainsi que le Laboratoire de physique mathématique du Centre de Recherches Mathématiques, pour leur support et leur encadrement. Je tiens également à exprimer ma profonde gratitude envers les organismes subventionnaires, le CRSNG et le FQRNT, pour leur soutien financier inestimable qui m'a permis de me concentrer sur ma recherche.

Merci à mes parents, qui m'ont tout donné. Qui m'ont transmis l'amour de l'école, qui m'ont soutenue à travers toutes ces années. Merci à mes soeurs et à mes amies, qui m'ont accompagnée dans cette aventure, à coups de soupers et de discussions, me rappelant ce qui est important.

Et finalement, merci à mon mari et à mon fils, qui sont ma vie, mon équilibre.

INTRODUCTION

Les groupes de Lie compacts simples, leurs algèbres de Lie et leurs représentations de dimension finie sont l'une des parties des mathématiques qui a trouvé le plus d'applications au cours du dernier siècle. En effet, on en rencontre de nombreuses en physique des particules élémentaires, en physique atomique et en chimie quantique [**24, 25**], mais également en génie et en mathématiques à proprement dit [**7, 10, 22, 23**]. Toutefois, encore aujourd'hui, on ne comprend pas les racines mêmes des diverses applications des groupes de Lie simples en mathématiques. À titre d'exemple, on a d'abord utilisé les diagrammes de Coxeter-Dynkin dans la classification des groupes de Lie et de leurs groupes de Weyl [**10, 23**]. Ces mêmes diagrammes ont également été employés dans la classification des carquois (*quivers*) de type fini et de leurs représentations indécomposables [**12, 17**]. Puis, on a découvert que les singularités des fronts d'onde ainsi que celles des structures rayonnées pouvaient être classifiées en termes des groupes de réflexion et de leurs diagrammes [**1**]. Pour le moment, on ne saisit pas comment les diagrammes de Coxeter-Dynkin peuvent servir dans des champs d'applications aussi différents. Par conséquent, tout porte à croire qu'on ne devine pas encore toute la diversité et la richesse des applications des groupes de Lie et de leurs représentations.

Tout de même, pendant la deuxième moitié du 20^{e} siècle, les applications des représentations de dimension finie des groupes de Lie semi-simples ont progressé remarquablement en mathématiques, en physique et dans les sciences naturelles en général. De telles représentations sont décrites efficacement par leurs systèmes de poids. Un système de poids consiste en une union de plusieurs orbites du groupe de Coxeter associé au groupe de Lie correspondant.

Déterminer de quelles orbites une représentation particulière est comprise est un problème laborieux pour lequel on a trouvé une solution algorithmique dans les dernières décennies [**16, 44**]. En pratique, il est nécessaire que les calculs impliquant des représentations de grande dimension soient divisés en opérations sur des orbites individuelles. Sans une telle stratégie, l'obtention de certains résultats déjà publiés n'aurait pas été possible [**19, 38**]. Notre motivation générale pour cette thèse s'inscrit dans des contextes de brisure de symétrie dans différents problèmes où une symétrie décrite en termes d'un groupe est réduite à une symétrie décrite en termes d'un sous-groupe. La brisure de symétrie est une approche fréquemment utilisée pour unifier plusieurs phénomènes physiques qui, autrement, devraient être considérés séparément. On en trouve de nombreuses applications en physique nucléaire, en physique atomique et en physique des particules élémentaires [**24, 65**]. La brisure de symétrie est presque toujours décrite en termes de représentations des groupes de Lie. Une description analogue en termes d'orbites individuelles permet plus de liberté et ouvre de nouvelles possibilités.

Les groupes de réflexions finis non commutatifs sont parmi les groupes les plus étudiés et les mieux connus en mathématiques. Notre outil principal dans ce présent travail sera les groupes de réflexions finis dans un espace euclidien réel de dimension finie n ($n \geq 1$), connus sous le nom de groupes de Coxeter finis [**5, 22, 23**]. De tels groupes sont générés par des réflexions par rapport à n miroirs ayant l'origine comme point commun, un miroir étant un sous-espace de dimension $n-1$ de l'espace euclidien. Le type de groupe est déterminé par les angles relatifs entre les miroirs. Si les angles relatifs entre les miroirs sont des multiples rationnels de 2π, l'application de toutes les réflexions possibles à un point quelconque de l'espace produit un ensemble fini de points, appelé l'orbite du groupe de réflexions.

Les groupes de Coxeter finis se divisent en deux classes : les groupes cristallographiques et les groupes non cristallographiques. Les groupes cristallographiques sont les groupes de Weyl des groupes ou algèbres de Lie semi-simples, et sont notés W. Ils sont les groupes de symétrie des réseaux de $\mathbb{R}^n$.

Ils se divisent en quatre familles dites classiques – A_n, B_n, C_n, et D_n – avec cinq exceptions, E_6, E_7, E_8, F_4, et G_2. L'indice n, 6, 7, 8, 4 ou 2, est le rang de l'algèbre de Lie correspondante et la dimension de l'espace euclidien fini dans lequel les réflexions ont lieu. La principale différence entre les deux types de groupes qui est d'intérêt pour nous est que les réseaux munis des W-symétries sont cristallographiques, tandis que ceux de type non cristallographique sont denses partout.

Nous considérons les orbites des groupes de Coxeter finis G, ou G-orbites. Une G-orbite est un ensemble fini de points de l'espace euclidien réel généré par les réflexions du groupe de Coxeter fini et ce, à partir d'un seul point. Le nombre de points de l'orbite divise toujours l'ordre du groupe correspondant. Comme les réflexions ont lieu dans des miroirs passant tous par l'origine, les points d'une même orbite sont à égale distance de l'origine. Géométriquement, les points d'une orbite peuvent être vus comme les sommets d'un polytope G-invariant de n dimensions centré à l'origine, n représentant le nombre de réflexions élémentaires qui génèrent G.

Les orbites des groupes de Weyl sont intimement liées aux systèmes de poids des représentations irréductibles de dimension finie des algèbres de Lie semi-simples. En effet, un système de poids consiste en une union de plusieurs orbites du groupe de Weyl correspondant, une orbite spécifique apparaissant souvent plus d'une fois dans le même système. De quelles orbites une représentation particulière est comprise est maintenant connu, et des tables extensives de multiplicités des poids dominants sont présentées dans [**6**].

Cette thèse est constituée de cinq chapitres, dont quatre sont des articles publiés dans le *Journal of Physics A : Mathematical and Theoretical* [**21, 32, 33, 34**]. La plupart des résultats du chapitre 4 sont présentés dans [**40**], et n'ont donc pas fait l'objet d'un article. Ma contribution dans les quatre articles a été sensiblement la même, c'est-à-dire que j'ai effectué tous les calculs et participé activement à la rédaction de chacun des quatre articles.

Dans les quatre premiers chapitres, nous exploitons des propriétés déjà connues pour les systèmes de poids des représentations [**6, 39**] afin de les définir pour les orbites des groupes de Weyl des algèbres de Lie simples, traitées individuellement, et nous étendons certaines de ces propriétés aux orbites des groupes de Coxeter non cristallographiques.

Le fait de considérer les opérations sur des orbites individuelles plutôt que sur des systèmes de poids entiers présente différents avantages. D'abord, comme nous l'avons déjà mentionné, cela permet d'étendre les résultats aux orbites des groupes non cristallographiques. Ensuite, alors que le nombre de poids d'un système de poids augmente sans cesse avec la dimension de la représentation, le nombre de points d'une orbite individuelle ne dépasse jamais l'ordre du groupe de Weyl correspondant. Lorsque l'on travaille avec des représentations de très grande dimension, on doit souvent séparer le problème en petits problèmes pour des orbites. De plus, considérer les orbites comme des polytopes permet de les utiliser dans différents modèles, tels que ceux de molécules de carbone ou de virus [**69**]. Finalement, alors que les poids des systèmes de poids doivent nécessairement se trouver sur le réseau de poids de l'algèbre de Lie, les points d'une orbite individuelle peuvent être n'importe où dans l'espace euclidien. Cela nous permet d'obtenir des orbites aussi rapprochées que nous le désirons, tel que discuté dans [**21**]. Il serait intéressant d'étudier la brisure de symétrie qui survient si les points d'une orbite sont minimalement déplacés. De telles orbites pourraient déterminer des fonctions spéciales pour des intégrales de Fourier plutôt que pour des séries de Fourier.

La motivation pour notre étude provient de l'exploitation récente des orbites des groupes de Coxeter dans l'analyse de Fourier [**29, 30, 52, 60**], de la théorie des quasicristaux [**9, 50**], ainsi que de l'étude des polynômes orthogonaux [**13**].

Nous présentons d'abord dans le premier chapitre [**21**] des opérations sur les orbites des groupes de Coxeter finis, connues pour les systèmes de poids des représentations des groupes de Lie semi-simples : (i) le produit d'orbites du même groupe de Coxeter G et sa décomposition en une somme (union) de G-orbites ; (ii) la décomposition de k-ièmes puissances d'orbites, symétrisées par le groupe de permutation de k éléments. Nous introduisons ensuite certaines caractéristiques numériques, telles que les classes de congruence et les indices de différents degrés pour les G-orbites, qui reflètent des propriétés similaires des systèmes de poids des représentations.

Nous abordons par la suite, dans les trois chapitres suivants, la réduction d'orbites d'un groupe de Coxeter cristallographique G à une somme ou une union d'orbites d'un sous-groupe de Coxeter cristallographique G' de G. En physique, la réduction d'orbites est souvent appelée *calcul des règles de branchement*.

La liste des réductions possibles pour les orbites de groupes de Weyl est le résultat d'une grande classification effectuée il y a plus de 60 ans, alors que les sous-algèbres réductives maximales des algèbres de Lie simples ont été déterminées [**4, 14**]. Nous utiliserons cette classification. Nous considérons la réduction d'orbites du groupe de Weyl d'une algèbre de Lie L à des orbites des groupes de Weyl des sous-algèbres réductives maximales L', semi-simples et non semi-simples. La réduction $W(L) \to W(L')$, où L' est une sous-algèbre réductive maximale de L, est une transformation linéaire de $\mathbb{R}^n$ vers $\mathbb{R}^m$, où n et m sont les rangs de L et L' respectivement. Les règles de branchement sont uniques, alors que la matrice de la transformation, appelée *matrice de projection*, dépend du choix des bases.

La méthode que nous utilisons pour calculer les règles de branchement est une extension de la méthode utilisée dans [**39, 40, 41, 61**] pour calculer les règles de branchement des représentations irréductibles de dimension finie des algèbres de Lie simples. La réduction d'orbites a déjà été abordée dans la littérature [**18, 67, 68**], où des méthodes spécifiques sont développées pour différentes paires d'algèbres et sous-algèbres. Le principal avantage de la méthode de la matrice de projection est son uniformité, car elle peut être utilisée pour n'importe quelle paire d'algèbre et sous-algèbre. De plus, bien que nous ne l'étudions pas ici, elle pourrait être utilisée pour traiter les réductions impliquant les groupes de Coxeter non cristallographiques.

Les orbites des groupes de Weyl des algèbres de Lie simples de tous types et de tous rangs sont réduites en l'union d'orbites des groupes de Weyl des sous-algèbres réductives maximales de l'algèbre. Nous listons les matrices de projection qui transforment les points des orbites de l'algèbre en des points des orbites des sous-algèbres pour tous les cas $n \leq 8$ ainsi que pour plusieurs séries infinies des paires d'algèbre-sous-algèbre. De nombreux exemples de règles de branchement sont présentés. Nous ne pouvons malheureusement pas présenter tous les cas possibles de règles de branchement, par manque d'espace. Les exemples choisis le sont généralement pour leur concision. Le chapitre 2 [**34**] présente les résultats où l'algèbre de Lie est de type A_n, $n \geq 1$. Le chapitre 3 [**32**] fait de même, mais dans les cas où l'algèbre de Lie est de type B_n, $n \geq 2$, C_n, $n \geq 2$, ou D_n, $n \geq 4$. Les cas exceptionnels sont finalement traités dans le chapitre 4.

Les éléments discrets qui apparaissent lors de la réduction des représentations de dimension finie d'un groupe de Lie simple compact à une somme de représentations d'un sous-groupe maximal régulier semi-simple, et qui commutent avec le sous-groupe, n'ont encore jamais été explorés dans les applications. Les centralisateurs des sous-groupes réguliers maximaux, semi-simples et non semi-simples, des groupes de Lie simples compacts ne sont pas généralement connus, bien que traités dans la littérature il y a longtemps [**4, 15**]. En utilisant de nouvelles méthodes, nous avons revisité le problème et l'avons rendu plus facile à exploiter dans les applications.

Ainsi, dans le chapitre 5 [**33**], nous fournissons la structure du centralisateur de tous les sous-groupes réguliers maximaux, semi-simples et non semi-simples, des groupes de Lie simples compacts de tous types et de tous rangs. Le centralisateur est soit un produit direct de groupes cycliques finis (dans le cas maximal régulier semi-simple), un groupe continu de rang 1, ou un produit, pas nécessairement direct, d'un groupe continu de rang 1 avec un groupe cyclique fini (dans le cas maximal régulier réductif).

Les valeurs propres de l'action des éléments du centralisateur sur les représentations irréductibles des algèbres de Lie simples peuvent être utilisées dans le calcul des règles de branchement impliquant ces sous-algèbres. Les matrices de projection discutées dans les chapitres 2, 3 et 4 transforment les poids d'une représentation irréductible d'une algèbre de Lie en des poids de représentations de la sous-algèbre. Nous pouvons inclure une étiquette additionnelle, la valeur propre de l'action d'un élément du centralisateur, servant à décomposer la représentation de l'algèbre.

L'action du centralisateur permet de séparer les représentations de l'algèbre en classes d'équivalence, que nous appelons *classes de congruence relative*. Dans le chapitre 5, en plus de décrire la structure du centralisateur de tous les sous-groupes réguliers maximaux, semi-simples et non semi-simples, des groupes de Lie simples compacts de tous types et de tous rangs, nous fournissons pour chacun des cas une équation permettant de calculer la classe de congruence relative d'une représentation.

Chapitre 1

THE RINGS OF N-DIMENSIONAL POLYTOPES

Référence complète : L. Háková, M. Larouche et J. Patera, The rings of n-dimensional polytopes, *Journal of Physics A : Mathematical and Theoretical*, 41(49) : 495202, 21, 2008.

Résumé

Les points d'une orbite d'un groupe de Coxeter fini G, générés par n réflexions à partir d'un seul point, sont vus comme les sommets d'un polytope (G-polytope) centré à l'origine d'un espace euclidien réel à n dimensions. Nous rappelons une méthode efficace pour décrire géométriquement les G-polytopes, leurs faces de toutes dimensions et les éléments qui leur sont adjacents. Nous introduisons les produits et les puissances symétrisées de G-polytopes et nous en décrivons la décomposition en des sommes de G-polytopes. Plusieurs invariants des G-polytopes sont présentés, à savoir les analogues des indices de Dynkin de degrés 2 et 4, et les classes de congruence des polytopes. Les définitions s'appliquent aux groupes de Coxeter cristallographiques et non cristallographiques. Des exemples et des applications sont proposés.

Abstract

Points of an orbit of a finite Coxeter group G, generated by n reflections starting from a single seed point, are considered as vertices of a polytope (G-polytope) centered at the origin of a real n-dimensional Euclidean space. A general efficient method is recalled for the geometric description of G-polytopes,

their faces of all dimensions and their adjacencies. Products and symmetrized powers of G-polytopes are introduced and their decomposition into the sums of G-polytopes is described. Several invariants of G-polytopes are found, namely the analogs of Dynkin indices of degrees 2 and 4, anomaly numbers, and congruence classes of the polytopes. The definitions apply to crystallographic and non-crystallographic Coxeter groups. Examples and applications are shown.

1.1. Introduction

Finite groups generated by reflections in a real Euclidean space $\mathbb{R}^n$ of n dimensions, also called finite Coxeter groups, are split into two classes : crystallographic and non-crystallographic groups [**23, 26**]. The crystallographic groups are the Weyl groups of compact semisimple Lie groups. They are an efficient tool for uniform description of the semisimple Lie groups/algebras [**5, 22, 59**], and they have proven to be an indispensable tool in extensive computations with the representations of such Lie groups or Lie algebras (see for example [**19**] and references therein).

Underlying such applications are two facts : (i) most of the computation can be performed in integers by working with the weight systems of the representations involved in a problem, and (ii) the weight system of a representation of a compact semisimple Lie group/Lie algebra consists of several Weyl group orbits of the weights, many of them occurring more than once. Practical importance of the orbits apparently emerged only in [**45, 47**], where truly large scale computations were anticipated.

The crystallographic Coxeter groups are called Weyl groups and denoted by W. Any finite Coxeter group, crystallographic or not, is denoted by G. A difference between the two cases which is of practical importance to us is that lattices with W-symmetries are common crystallographic lattices, while lattices of non-crystallographic types are dense everywhere in $\mathbb{R}^n$.

Non-crystallographic finite Coxeter groups are of extensive use in modeling aperiodic point sets with long-range order ('quasicrystals') [**9, 42, 50**]. Outside traditional mathematics and mathematical physics, a new line of application of Coxeter group orbits can be found in [**28, 69**]; see also the references therein.

Additional applications of Weyl group orbits are found in [**2, 3, 18, 67, 68**]. Both crystallographic and non-crystallographic Coxeter groups can be used for building families of orthogonal polynomials of many variables [**13**].

In recent years, another field of applications of W-orbits is emerging in harmonic analysis. Multidimensional Fourier-like transforms were introduced and are currently being explored in [**29, 30, 52, 60**], where W-orbits are used to define families of special functions, called orbit functions [**60**], which serve as the kernels of the transforms. They differ from the traditional special functions [**31**]. The number of variables, on which the new functions depend, is equal to the rank of a compact semisimple Lie group that provides the Weyl group. Two properties of the transforms stand out : such special functions are orthogonal when integrated over a finite region F, and they are also orthogonal when summed up over lattice points $F_M \subset F$. The lattices can be of any density, their symmetries are prescribed by the Lie groups. Application of the non-crystallographic groups in Fourier analysis is at its very beginning [**43**].

In this paper we have no compelling reason to distinguish crystallographic and non-crystallographic reflection groups of finite order. Hence, we consider all finite Coxeter groups although from the infinitely many finite Coxeter groups in 2 dimensions (symmetry groups of regular polygons), we usually consider only the lowest few.

An orbit $G(\lambda)$ of a Coxeter group G is the set of points in $\mathbb{R}^n$ generated by G from a single seed point $\lambda \in \mathbb{R}^n$. G-orbits are not common objects in the literature, nor is their multiplication, which can be viewed in parallel to the multiplication of G-invariant polynomials $P(\lambda; x)$ introduced in subsection

1.6.1 (for more about the polynomials see [**13**] and the references therein). [1] Indeed, the set of exponents of all the monomials in $P(\lambda; x)$ is the set of points of the orbit $G(\lambda)$.

In this paper, we have adopted a point of view according to which the orbits $G(\lambda)$, being simpler than the polynomials $P(\lambda; x)$ or the weight systems of representations, are the primary objects of study.

The relation between the orbits of W and the weight systems of finite dimensional irreducible representations of semisimple Lie groups/algebras over $\mathbb{C}$, can be understood as follows. The character of a particular representation involves summation over the weight system of the representation, i.e. over several W-orbits. As for which orbits appear in a particular representation, this is a well known question about multiplicities of dominant weights. There is a laborious but rather fast computer algorithm for calculating the multiplicities. Extensive tables of multiplicities can be found in [**6**] ; see also the references therein. Thus one is justified in assuming that the relation between a representation and a particular W-orbit is known in all cases of interest.

Numerical characteristics, such as congruence classes, indices of various degrees, and anomaly numbers, introduced here for W-orbits, mirror similar properties of weight systems from representation theory, which are often used in applications (for example [**18, 39, 66, 67, 68**]).

In this paper, we introduce operations on W-orbits that are well known for weight systems of representations : (i) the product of W-orbits (of the same group) and its decomposition into the sum of W-orbits ; (ii) the decomposition of the k-th power of a W-orbit symmetrized by the group of permutations of k elements. New is the introduction of such operations for the orbits of non-crystallographic Coxeter groups. We intend to describe reductions of G-orbits to orbits of a subgroup $G' \subset G$ in a separate paper [**34**]. Again, the involvement of non-crystallographic groups makes the reduction problem rather unusual. Corresponding applications deserve to be explored.

1. Polynomials in 1.6.1 are the simplest W-invariant ones. We are not concerned about any other of their properties.

The decomposition of products of orbits of Coxeter groups, as introduced here, is the core of other decomposition problems in mathematics, such as the decomposition of direct products of representations of semisimple Lie groups, the decomposition of products of certain special functions [**60**] and the decomposition of products of G-invariant polynomials of several variables [**13**]. The last two problems are completely solved in terms of orbit decompositions. The first problem requires that the multiplicities of dominant weights in weight systems of representations [**6**] be known.

We view the G-orbits from a perspective uncommon in the literature. Namely, the points of a G-orbit are taken to be vertices of an n-dimensional G-invariant polytope centered at origin, n being the number of elementary reflections generating G (at the same time it is the rank of the corresponding semisimple Lie group). The multiplication of two such polytopes/orbits, say P_1 and P_2, is the set of points/vertices obtained by adding to every point of P_1 every point of P_2. The resulting set of points is again G-invariant and thus it is a union (we say 'sum') of several G-orbits (we say 'G-polytopes'). Thus we have a ring of G-polytopes with positive integer coefficients. We recall and illustrate a general method of description of n-dimensional reflection-generated polytopes [**8, 48**].

The core of our geometric interpretation of orbits as polytopes is in the paragraph following equation (1.6.6). A product of orbits is a union of concentric orbits. Geometrically this can be seen as an 'onion'-like layered structure of orbits of different radii. Unlike in representation theory, where orbit points are always points of the corresponding weight lattices, in our case the seed point of an orbit can be anywhere in $\mathbb{R}^n$. In particular, a suitable choice of the seed points of the orbits, which are being multiplied, can bring some of the layers of the 'onion' structure as close or as far apart as desired. Two examples are given in the last section (see (1.9.4) and (1.9.5)).

1.2. Reflections generating finite Coxeter groups

Let α and x be vectors in $\mathbb{R}^n$. We denote by r_α the reflection in the $(n-1)$-dimensional 'mirror' orthogonal to α and passing through the origin. For any $x \in \mathbb{R}^n$, we have

$$r_\alpha x = x - \frac{2\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha\,. \tag{1.2.1}$$

Here $\langle a, b\rangle$ denotes scalar product in $\mathbb{R}^n$. In particular, we have $r_\alpha 0 = 0$ and $r_\alpha \alpha = -\alpha$ so that $r_\alpha^2 = 1$.

A Coxeter group G is by definition generated by several reflections in mirrors that have the origin as their common point. Various Coxeter groups are thus specified by the set $\Pi(\alpha)$ of vectors α, orthogonal to the mirrors and called the simple roots of G. Consequently, G is given once the relative angles between elements of $\Pi(\alpha)$ are given.

A standard presentation of G, generated by n reflections, amounts to the following relations

$$r_k^2 = 1\,, \qquad (r_i r_j)^{m_{ij}} = 1\,, \qquad k, i, j \in \{1, \dots, n\}\,,$$

where we have simplified the notation by setting $r_{\alpha_k} = r_k$, and where m_{ij} are the lowest possible positive integers. The matrix (m_{ij}) specifies the group. The angles between the mirrors of reflections r_i and r_j are determined from the values of the exponents m_{ij}. Indeed, for $m_{ij} = p$, the angle is π/p, while the angle between α_i and α_j is $\pi - \pi/p$.

The classification of finite reflection (Coxeter) groups was accomplished in the first half of the 20th century.

1.2.1. $n = 1$

There is just one group of order 2. Its two elements are 1 and r. We denote this group by A_1. Acting on a point a of the real line, the group A_1 generates its orbit of two points, a and $ra = -a$, except if $a = 0$. Then the orbit consists of just one point, namely the origin.

1.2.2. $n = 2$

There are infinitely many Coxeter groups in $\mathbb{R}^2$, one for each $m_{12} = 2, 3, 4, \ldots$. Their orders are $2m_{12}$. In physics literature, these are the dihedral groups.

Note that for $m_{12} = 2$, the group is a product of two groups from $n = 1$. The reflection mirrors are orthogonal.

Our notation for the lowest five groups, generated by two reflections, and their orders, is as follows :

$$\begin{array}{lcll} m_{12} = 2 & : & A_1 \times A_1\,, & 4 \\ m_{12} = 3 & : & A_2\,, & 6 \\ m_{12} = 4 & : & C_2\,, & 8 \\ m_{12} = 5 & : & H_2\,, & 10 \\ m_{12} = 6 & : & G_2\,, & 12\,. \end{array}$$

1.2.3. General case : Coxeter and Dynkin diagrams

A convenient general way to provide a specific set $\Pi(\alpha)$ is to draw a graph where vertices are traditionally shown as small circles, one for each $\alpha \in \Pi$, and where edges indicate absence of orthogonality between two vertices linked by an edge.

A diagram consisting of several disconnected components means that the group is a product of several pairwise commuting subgroups. Thus it is often sufficient to consider only the groups with connected diagrams.

In this paper, a Coxeter diagram is a graph providing only relative angles between simple roots while ignoring their lengths. This is done by writing m_{ij} over the edges of the diagram. By convention, the most frequently occurring value, $m_{ij} = 3$, is not shown in the diagrams. When $m_{ij} = 2$, the edge is not drawn, i.e. the nodes numbered i and j are not directly connected.

Consider the examples of Coxeter diagrams of all finite non-crystallographic Coxeter groups with connected diagrams (Figure 1.1). Note that we simply write H_2 when $m = 5$.

FIGURE 1.1. Coxeter diagrams of the finite non-crystallographic Coxeter groups.

A Dynkin diagram is a graph providing, in addition to the relative angles, the relative lengths of the vectors from $\Pi(\alpha)$. Dynkin diagrams are used for the crystallographic Coxeter groups, frequently called the Weyl groups. There are four infinite series of classical groups and five isolated cases of exceptional simple Lie groups. Figure 1.2 presents a complete list of Dynkin diagrams of such groups (with connected diagrams) :

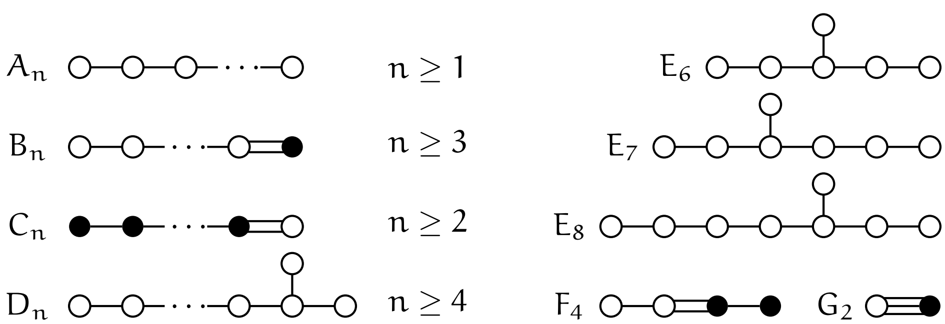


FIGURE 1.2. Dynkin diagrams of the finite crystallographic Coxeter groups.

The names of the groups, as is traditional in Lie theory, are shown on the left of each diagram. Open (black) circles indicate longer (shorter) roots. The ratio of their square lengths is $\langle\alpha_l, \alpha_l\rangle : \langle\alpha_s, \alpha_s\rangle = 2 : 1$ in all cases except for G_2 where the ratio is $3 : 1$. Moreover, we adopt the usual convention that $\langle\alpha_l, \alpha_l\rangle = 2$. A single, double, and triple line indicates respectively the angle $2\pi/3$, $3\pi/4$, and $5\pi/6$ between the roots, or equivalently, the angles $\pi/3$, $\pi/4$, and $\pi/6$ between the reflection mirrors. The absence of a direct link between two nodes implies that the corresponding simple roots, as well as the mirrors, are orthogonal. Note that the relative angles of the mirrors of B_n and C_n coincide. Hence their W-groups are isomorphic. Their simple roots differ by length.

A_n $(n \geq 1)$	B_n $(n \geq 3)$	C_n $(n \geq 2)$	D_n $(n \geq 4)$	E_6	E_7
$(n+1)!$	$2^n n!$	$2^n n!$	$2^{n-1} n!$	$2^7 3^4 5$	$2^{10} 3^4 5 7$
E_8	F_4	G_2	$H_2(m)$	H_3	H_4
$2^{14} 3^5 5^2 7$	$2^7 3^2$	12	$2m$	120	120^2

TABLE 1.1. Orders of the finite Coxeter groups.

We adopt the Dynkin numbering of nodes. The numbering proceeds from left to right $1, 2, \ldots$ In case of D_n and E_6, E_7, E_8, the node above the main line has the highest number, respectively n, 6, 7, 8.

Orders of the finite Coxeter groups are provided in table 1.1 for groups with connected diagrams. When a diagram has several disconnected components, the order is the product of orders corresponding to each subdiagram.

1.3. ROOT AND WEIGHT LATTICES

Information essentially equivalent to that provided by the Coxeter and Dynkin diagrams is also given in terms of $n \times n$ matrices C, called the Cartan matrices. Relative angles and lengths of simple roots can be used to form the Cartan matrix for each group. Its matrix elements are calculated as

$$C = (C_{jk}) = \left(\frac{2\langle \alpha_j, \alpha_k \rangle}{\langle \alpha_k, \alpha_k \rangle} \right) , \qquad j, k \in \{1, 2, \ldots, n\} . \tag{1.3.1}$$

Cartan matrices and their inverses are given in many places, e.g. **[6, 23]**.

The Cartan matrices can be defined for any finite Coxeter group by using formula (1.3.1). For non-crystallographic groups the matrices are

$$C(H_2) = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix} , \quad C(H_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix} , \quad C(H_4) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix} ,$$

where τ is the larger of the solutions of the algebraic equation $x^2 = x + 1$, i.e. $\tau = \frac{1}{2}(1 + \sqrt{5})$.

In addition to the basis of simple roots (α-basis), it is useful to introduce the basis of fundamental weights (ω-basis). Subsequently, most of our computations will be performed in the ω-basis.

$$\alpha = C\omega , \qquad \omega = C^{-1}\alpha .$$

Note the important relation :

$$\langle \alpha_k, \omega_j \rangle = \delta_{jk} \frac{\langle \alpha_k, \alpha_k \rangle}{2}, \qquad j, k \in \{1, 2, \dots, n\}. \tag{1.3.2}$$

Illustrations showing the α- and ω-bases of A_2, C_2, and G_2 are given in Figure 1 of [**48**].

The root lattice Q and the weight lattice P of G are formed by all integer linear combinations of simple roots, respectively fundamental weights, of G,

$$Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n, \qquad P = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n. \tag{1.3.3}$$

Here $\mathbb{Z}$ stands for any integer. For the groups that have simple roots of two different lengths, one may define the root lattice of n linearly independent short roots, which cannot all be simple. In general, $Q \subseteq P$, with $Q = P$ only for E_8, F_4, and G_2.

If G is one of the non-crystallographic Coxeter groups, the lattices Q and P are dense everywhere.

Since α- and ω-bases are not orthogonal and not normalized, it is sometimes useful to work with orthonormal bases. For crystallographic groups, they are found in many places, for example [**5, 6**]. For non-crystallographic groups, H_2, H_3 and H_4 ; see [**9, 50**].

1.4. The orbits of Coxeter groups

1.4.1. Computing points of an orbit

Given the reflections r_α, $\alpha \in \Pi(\alpha)$, of a Coxeter group G, and a seed point $\lambda \in \mathbb{R}^n$, the points of the orbit $G(\lambda)$ are given by the set of distinct points generated by repeated application of the reflections r_α to λ. All points of an orbit are equidistant from the origin. The radius of an orbit is the distance of (any) point of the orbit from the origin.

There are practically important considerations which make it almost imperative that the computation of the points of any orbit of G be carried out in the ω-basis, as follows :

– Every orbit contains precisely one point with nonnegative coordinates in the ω-basis. We specify the orbit by that point, calling it the dominant point of the orbit.

– Given a dominant point λ of the group G in the ω-basis, one readily finds the size of the orbit $G(\lambda)$, i.e. the number of points in the orbit, using the order $|G|$ of the Coxeter group and the order of the stabilizer of λ in G :

$$|G(\lambda)| = \frac{|G|}{|\operatorname{Stab}_G(\lambda)|} \tag{1.4.1}$$

Here $\operatorname{Stab}_G(\lambda)$ is a Coxeter subgroup of G. To find it, one needs to attach the ω-coordinates of λ to the corresponding nodes of the diagram of G. The subdiagram carrying the coordinates 0 is the diagram of $\operatorname{Stab}_G(\lambda)$.

– Due to (1.3.2), the reflections (1.2.1) are particularly simple when applied to ω's :

$$r_k\omega_j = \omega_j - \frac{2\langle\alpha_k, \omega_j\rangle}{\langle\alpha_k, \alpha_k\rangle}\alpha_k = \omega_j - \delta_{jk}\alpha_k\,. \tag{1.4.2}$$

– Starting from the dominant point of an orbit, it suffices to apply, during the computation of the orbit points, only reflections corresponding to positive coordinates of any given weight. All points of the orbit are found in this way.

1.4.2. Orbits of A_2, C_2, G_2, and H_2

We give some examples of orbits. Let $a, b > 0$.

$$\begin{aligned} A_2\ :\quad G((a,0)) =& \{(a,0),\ (-a,a),\ (0,-a)\}\,,\\ G((0,b)) =& \{(0,b),\ (b,-b),\ (-b,0)\}\,,\\ G((a,b)) =& \{(a,b),\ (-a,a+b),\ (b,-a-b),\ (a+b,-b)\,,\\ & (-a-b,a),\ (-b,-a)\}\,. \end{aligned}$$

In particular, the orbit $G((1,1)) = \{(1,1),\ (-1,2),\ (1,-2),\ (2,-1),\ (-2,1),\ (-1,-1)\}$ consists of the vertices of a regular hexagon of radius $\sqrt{2}$. It is the root system of A_2.

$$\begin{aligned} C_2: \quad G((a,0)) &= \{\pm(a,0),\ \pm(-a,a)\}, \\ G((0,b)) &= \{\pm(0,b),\ \pm(2b,-b)\}, \\ G((a,b)) &= \{\pm(a,b),\ \pm(-a,a+b),\ \pm(a+2b,-a-b), \\ &\qquad \pm(-a-2b,b)\}. \end{aligned}$$

In particular, the orbits $G((2,0))$ and $G((0,1))$ of radii $\sqrt{2}$ and 1 are respectively the vertices and midpoints of the sides of a square. Together the two orbits form the root system of C_2.

$$\begin{aligned} G_2: \quad G((a,0)) &= \{\pm(a,0),\ \pm(-a,3a),\ \pm(2a,-3a)\}, \\ G((0,b)) &= \{\pm(0,b),\ \pm(b,-b),\ \pm(-b,2b)\}, \\ G((a,b)) &= \{\pm(a,b),\ \pm(-a,3a+b),\ \pm(2a+b,-3a-b), \\ &\qquad \pm(-2a-b,3a+2b),\ \pm(a+b,-3a-2b), \\ &\qquad \pm(-a-b,b)\}. \end{aligned}$$

In particular, the orbits $G((1,0))$ and $G((0,1))$ are the vertices of regular hexagons of radii $\sqrt{2}$ and $2/\sqrt{3}$, rotated relatively by 30°, i.e. they form a hexagonal star. Together the two orbits form the root system of G_2. The points of $G((a, a\sqrt{3}/\sqrt{2}))$, $a > 0$, are the vertices of a regular dodecahedron of radius $\sqrt{2}a$.

$$\begin{aligned} H_2: \quad G((a,0)) &= \{(a,0),\ (-a,a\tau),\ (a\tau,-a\tau),\ (-a\tau,a),\ (0,-a)\}, \\ G((0,b)) &= \{(0,b),\ (b\tau,-b),\ (-b\tau,b\tau),\ (b,-b\tau),\ (-b,0)\}, \\ G((a,b)) &= \{(a,b),\ (-a,b+a\tau),\ (a\tau+b\tau,-b-a\tau), \\ &\qquad (-a\tau-b\tau,a+b\tau),\ (b,-a-b\tau),\ (a+b\tau,-b), \\ &\qquad (-a-b\tau,a\tau+b\tau),\ (b+a\tau,-a\tau-b\tau), \\ &\qquad (-b-a\tau,a),\ (-b,-a)\}. \end{aligned}$$

In particular, the orbits $G((a,0))$ and $G((0,b))$ are the vertices of regular pentagons of radii $a\sqrt{2}$ and $b\sqrt{2}$, rotated relatively by 36°. The orbit $G((a,a))$ forms a regular decahedron. The orbit $G((\tau,\tau))$ consists of the roots of H_2.

An orbit of A_2 or H_2 contains, with every point (p, q) also the point $(-q, -p)$. Note that in the examples of this subsection the constants a and b do not need to be integers. All one requires is that they are positive. Effects of special choices of these constants are exemplified in (1.9.4) and (1.9.5) below.

1.4.3. Orbits of A_3, B_3, C_3, and H_3

We give some examples of orbits. Let $a, b > 0$.

$$
\begin{aligned}
A_3 :\quad G((a,0,0)) =& \{(a,0,0),\ (-a,a,0),\ (0,-a,a),\ (0,0,-a)\},\\
G((0,b,0)) =& \{\pm(0,b,0),\ \pm(b,-b,b),\ \pm(-b,0,b)\},\\
G((1,1,0)) =& \{(1,1,0),\ (-1,2,0),\ (2,-1,1),\ (1,-2,2),\ (-2,1,1),\\
& (2,0,-1),\ (-1,-1,2),\ (1,0,-2),\ (-2,2,-1),\\
& (-1,1,-2),\ (0,-2,1),\ (0,-1,-1)\}.
\end{aligned}
$$

$$
\begin{aligned}
B_3 :\quad G((a,0,0)) =& \{\pm(a,0,0),\ \pm(-a,a,0),\ \pm(0,-a,2a)\},\\
G((0,b,0)) =& \{\pm(0,b,0),\ \pm(b,-b,2b),\ \pm(-b,0,2b),\ \pm(b,b,-2b),\\
& \pm(-b,2b,-2b),\ \pm(2b,-b,0)\},\\
G((0,0,c)) =& \{\pm(0,0,c),\ \pm(0,c,-c),\ \pm(c,-c,c),\ \pm(c,0,-c)\}.
\end{aligned}
$$

$$
\begin{aligned}
C_3 :\quad G((a,0,0)) =& \{\pm(a,0,0),\ \pm(-a,a,0),\ \pm(0,-a,a)\},\\
G((0,b,0)) =& \{\pm(0,b,0),\ \pm(b,-b,b),\ \pm(-b,0,b),\ \pm(b,b,-b),\\
& \pm(-b,2b,-b),\ \pm(2b,-b,0)\},\\
G((0,0,c)) =& \{\pm(0,0,c),\ \pm(0,2c,-c),\ \pm(2c,-2c,c),\ \pm(2c,0,-c)\}.
\end{aligned}
$$

$$H_3: \quad G((a,0,0)) = \{\pm(a,0,0),\ \pm(-a,a,0),\ \pm(0,-a,a\tau),\ \pm(0,a\tau,-a\tau),$$
$$\pm(a\tau,-a\tau,a),\ \pm(-a\tau,0,a)\},$$
$$G((0,0,c)) = \{\pm(0,0,c),\ \pm(0,c\tau,-c), \pm(c\tau,-c\tau,c\tau), \pm(-c\tau,0,c\tau),$$
$$\pm(c\tau,c,-c\tau),\ \pm(-c\tau,c\tau^2,-c\tau),\ \pm(c\tau^2,-c,0),$$
$$\pm(c,-c\tau^2,c\tau^2),\ \pm(-c\tau^2,c\tau,0),\ \pm(-c,-c\tau,c\tau^2)\}.$$

1.5. Orbits as polytopes

In this section, we recall an efficient method **[8]** of description for reflection-generated polytopes in any dimension.

The idea of the method consists in the following. Suppose we have an orbit $G(\lambda)$. Consider its points as vertices (faces of dimension 0) of the polytope also denoted $G(\lambda)$ in $\mathbb{R}^n$. Then for any face f of dimension $0 \le d \le n-1$, we identify its stabilizer $\mathrm{Stab}_{G(\lambda)}(f)$ in G, which is a product of two Coxeter subgroups of G:

$$\mathrm{Stab}_{G(\lambda)}(f) = G_1(f) \times G_2(D)$$

where $G_1(f)$ is the symmetry group of the face, and $G_2(D)$ stabilizes f pointwise, i.e. does not move it at all.

Our method consists in a recursive decoration of the diagram of G, providing at each stage the subdiagrams of $G_1(f) = G(\star)$ and $G_2(D) = G(\circ)$ for faces of one type. The recursive decoration is recursive in the dimension d of the faces. The decoration of the nodes of the diagram indicates to which $G(\star)$ or $G(\circ)$ subgroups of the stabilizer the corresponding reflections belong. For further details, see **[8]**. A much wider application of this method is described in **[48, 49, 51]**, including its exploitation in non Euclidean spaces.

We start with an extreme decoration of the diagram. It is equivalent to stating which coordinates of the dominant weight are positive relative to the ω-basis. The nodes are drawn as either open or black circles, i.e. zero or positive coordinates respectively.

Every possible extreme decoration fixes a polytope. There are only two rules for recursive decoration of the diagrams, starting from one of the extreme ones : (i) a single black circle is replaced by a star ; (ii) open circles, that become adjacent to a star by diagram connectivity, are changed to black ones.

Tables 1.2 and 1.3 show the results of the application of the decoration rules for polytopes in 2D and 3D for all groups with connected diagrams. All polytopes for A_4, B_4, C_4, D_4, and H_4 are described in tables 3 and 4 of [**8**].

		A_2	C_2	G_2	H_2	$H_2(7)$	1	2	3
1	●●	6	8	12	10	14	✓		
2	●○	3	4	6	5	7		✓	
3	○●	3	4	6	5	7			✓
4	⋆●	3	4	6	5	7	✓	✓	
5	●⋆	3	4	6	5	7	✓		✓

TABLE 1.2. The number of faces of 2D polytopes with Coxeter group symmetry. *The first three rows specify representatives of* G-*orbits of* 2D *polytopes. A black (open) dot in the second column stands for a positive (zero) coordinate in the* ω*-basis of the dominant point representing the orbit of vertices. The number of vertices is listed in the subsequent five columns. Rows 4 and 5 refer to the edges of the polytopes. A star in the second column indicates the reflection generating the symmetry group of the edge. The number of edges is shown for each group in subsequent columns. Check marks in one of the last three columns indicate the faces which belong to the polytope described in that column.*

1.5.1. Explanation of the tables

A description of table 1.2 is given in its caption.

Consider table 1.3. The second column contains short-hand notation for several diagrams at once. We call them decorated diagrams. No links between nodes of a diagram are drawn because they would need to be different for each

	Diagram	A_3	B_3	C_3	H_3	1	2	3	4	5	6	7
1	● ● ●	24	48	48	120	✓						
2	● ● ○	12	24	24	60		✓					
3	● ○ ●	12	24	24	60			✓				
4	○ ● ●	12	24	24	60				✓			
5	● ○ ○	4	6	6	12					✓		
6	○ ● ○	6	12	12	30						✓	
7	○ ○ ●	4	8	8	20							✓
8	⋆ ● ●	12	24	24	60	✓		✓				
9	● ⋆ ●	12	24	24	60	✓	✓		✓		✓	
10	● ● ⋆	12	24	24	60	✓		✓				
11	⋆ ● ○	6	12	12	30		✓			✓		
12	○ ● ⋆	6	12	12	30				✓			✓
13	⋆ ⋆ ●	4	8	8	20	✓	✓	✓	✓	✓	✓	
14	⋆ ● ⋆	6	12	12	30	✓		✓				
15	● ⋆ ⋆	4	6	6	12	✓	✓	✓	✓		✓	✓

TABLE 1.3. The number of faces of dimension 0, 1 and 2 of 3D polytopes with Coxeter group symmetry. *Decorated diagrams, rows 1 to 7, specify the polytopes. The dimension of a face equals the number of stars in the diagram, rows 8 to 15.*

group in subsequent columns. The nodes do not reveal the relative lengths of roots, their decoration indicates to which of the pertinent subgroup of the stabilizer of G such a reflection belongs. Thus the diagrams of the second column of the table apply to A_3, B_3, C_3 and H_3 at the same time.

Each line of the table describes one of G-orbits of identical faces. The dimension of the face equals to the number of stars in its decorated diagram. Numerical entries in a row give the number of faces for polytopes of symmetry groups A_3, B_3, C_3 and H_3, shown in the header of the columns. The top seven rows show the starting decorations fixing the polytopes, and also the number of 0-faces (vertices) of the polytopes of each group. The check marks in one of the last seven columns indicate the faces belonging to the same polytope.

Example 1.5.1.

As an example of how to decipher properties of polytope faces, consider rows number 5 and 2. The diagram in row 5 conveys the fact that $\lambda = a\omega_1$ *with* $a > 0$*. The exact value of* a *affects only the size of the polytope, not its shape. The stabilizer of* λ *is given by the subdiagram of open circles, i.e.* r_2 *and* r_3 *generate its stabilizer. For* A_3 *the subdiagram is of type* A_2*, while for* B_3 *and* C_3 *it is of type* C_2*, and for* H_3 *it is of type* H_2*. Hence in row 5 the entries give the number of vertices as* $24/6$*,* $48/8$*,* $48/8$*,* $120/10$ *respectively.*

The check mark in column 5 and row 5 indicates that faces belonging to our polytope are indicated by other check marks in column 5, namely in rows 11 and 13. The diagram of row 11 has just one star, hence the face is 1-dimensional (an edge). Its stabilizer (the subdiagram of stars and open circles) is of type $A_1 \times A_1$ *for all four cases. Hence the number of edges is* $24/4$ *for* A_3*,* $48/4$ *for* B_3 *and* C_3*, and* $120/4$ *for* H_3*. The only type of* 2D *face is given in row 13. The symmetry group of the face is generated by* r_1 *and* r_2*. It is of type* A_2 *for all four cases. Thus there are* $24/6$ *faces in* A_3*,* $48/6$ *in* B_3 *and* C_3*, and* $120/6$ *in* H_3 *polytope.*

Similarly, row 2 indicates that $\lambda = a\omega_1 + b\omega_2$*,* $a, b > 0$*. It is stabilized by the group generated by* r_3*, which is of type* A_1 *for all four cases. Hence the number of vertices equals half of the order of the corresponding Coxeter group. There are two orbits of edges given in rows 9 and 11, while the two orbits of* 2D *faces are given by the check marks in rows 13 and 15.*

Example 1.5.2.

The 2D *faces can actually be constructed knowing their symmetry and the seed point, say* $(a, 0, 0)$*. The diagram of the* 2D *face is* $\star\ \star\ \bullet$*, meaning that the symmetry group of the face is generated by* r_1 *and* r_2*. Moreover, it is of the same type (*A_2*) for all four groups. Then there are just three distinct vertices of the* 2D *face :*

$$(a, 0, 0), \quad r_1(a, 0, 0), \quad r_2 r_1(a, 0, 0)\,.$$

The 2D *face is formed from the seed point* $(a, 0, 0)$ *by application of reflections* r_1 *and* r_2*.*

The vertices of the 2D face are different triangles for each group, because they are given in their respective ω-basis :

$$\begin{aligned}
A_3 &: \quad (a,0,0),\ (-a,a,0),\ (0,-a,a),\\
B_3 &: \quad (a,0,0),\ (-a,a,0),\ (0,-a,2a),\\
C_3 &: \quad (a,0,0),\ (-a,a,0),\ (0,-a,a),\\
H_3 &: \quad (a,0,0),\ (-a,a,0),\ (0,-a,a\tau)\,.
\end{aligned}$$

Example 1.5.3.

Let us consider row 2 in further detail. The starting point is $\lambda = a\omega_1 + b\omega_2$, *where* $a, b > 0$. *There are two orbits of edges given by their endpoints :*

$$((a,b,0), r_1(a,b,0))\,, \qquad ((a,b,0), r_2(a,b,0))\,,$$

and two orbits of 2D faces. Consider just the H_3 *case. The 2D face of row 13 has the symmetry group generated by* r_1, r_2 *(*A_2 *type). It is a hexagon :*

$$\begin{aligned}
&(a,b,0), \quad (-a,a+b,0), \quad (a+b,-b,\tau b), \quad (b,-a-b,\tau(a+b)),\\
&(-a-b,a,\tau b), \quad (-b,-a,\tau(a+b)).
\end{aligned}$$

The 2D face of row 15 has its symmetry group generated by r_2, r_3 *(*H_2 *type). It is a pentagon :*

$$\begin{aligned}
&(a,b,0), \quad (a+b,-b,\tau b), \quad (a+b,\tau b,-\tau b),\\
&(a+\tau^2 b,-\tau b,b), \quad (a+\tau^2 b,0,-b)\,.
\end{aligned}$$

In particular, when $a = b$*, the pentagon and the hexagon are both regular. The polytope is then the familiar fullerene or 'soccer ball'.*

Further questions about the structure of polytopes can be answered within our formalism : How many 2D faces meet in a vertex ? Which 2D faces meet in an edge ? The higher the dimension, the more questions like these can be asked and answered. For more information on such questions and others (e.g. dual pairs of polytopes), we refer to [**8**].

1.6. DECOMPOSITION OF PRODUCTS OF POLYTOPES

1.6.1. Multiplication of G-invariant polynomials

The product of G-polytopes together with its decomposition, as defined in subsection 1.6.2 below, can be simply motivated by its correspondence to the product of more familiar objects than orbits, namely G-invariant polynomials, say $P(\lambda; x)$ and $P(\mu; x)$. Here λ and μ are dominant points of their orbits and x stands for n auxiliary independent variables $x_1, x_2, \ldots, x_n$ whose nature is of no concern to us here. They can be thought of as, for example, complex or real variables. We introduce them in order to make sense of the definitions below.

Denote by $\lambda^{(i)} \in G(\lambda)$ the points of the orbit $G(\lambda)$, and by $\mu^{(k)} \in G(\mu)$, where

$$\lambda^{(i)} = \sum_{p=1}^{n} a_p^{(i)} \omega_p\,, \quad \mu^{(k)} = \sum_{q=1}^{n} b_q^{(k)} \omega_q\,, \qquad 1 \leq i \leq |G(\lambda)|\,, \quad 1 \leq k \leq |G(\mu)|\,. \tag{1.6.1}$$

Here $|G(\lambda)|$ and $|G(\mu)|$ denote the number of points in their orbits. Then we can introduce the polynomials :

$$P(\lambda; x) = \sum_{\lambda^{(i)} \in G(\lambda)} x^{\lambda^{(i)}} := \sum_{i=1}^{|G(\lambda)|} x_1^{a_1^{(i)}} x_2^{a_2^{(i)}} \cdots x_n^{a_n^{(i)}}\,, \tag{1.6.2}$$

$$P(\mu; x) = \sum_{\mu^{(k)} \in G(\mu)} x^{\mu^{(k)}} := \sum_{k=1}^{|G(\mu)|} x_1^{b_1^{(k)}} x_2^{b_2^{(k)}} \cdots x_n^{b_n^{(k)}}\,, \tag{1.6.3}$$

and their product,

$$P(\lambda; x) \otimes P(\mu; x) = \sum_{i=1}^{|G(\lambda)|} \sum_{k=1}^{|G(\mu)|} x_1^{a_1^{(i)}+b_1^{(k)}} x_2^{a_2^{(i)}+b_2^{(k)}} \cdots x_n^{a_n^{(i)}+b_n^{(k)}}\,. \tag{1.6.4}$$

The latter consists of the sum of $|G(\lambda)||G(\mu)|$ monomials which can be decomposed into the sum of polynomials defined by one G-orbit each.

Finally, consider an example : Let G be the group A_2, and $\lambda = (1,0)$ and $\mu = (0,1)$. Therefore $P((1,0); x) = x_1 + x_1^{-1} x_2 + x_2^{-1}$ and $P((0,1); x) = x_2 + x_1 x_2^{-1} + x_1^{-1}$.

Their products decompose as follows :

$$\begin{aligned} P((1,0);x)\otimes P((0,1);x) =& \{x_1x_2+x_1^2x_2^{-1}+x_1^{-1}x_2^2+x_1^{-2}x_2+x_1x_2^{-2}+x_1^{-1}x_2^{-1}\}+3 \\ =& P((1,1);x)+3P((0,0);x)\,, \\ P((1,0);x)\otimes P((1,0);x) =& \{x_1^2+x_1^{-2}x_2^2+x_2^{-2}\}+2\{x_2+x_1x_2^{-1}+x_1^{-1}\} \\ =& P((2,0);x)+2P((0,1);x)\,. \end{aligned}$$

Note that it is equivalent to use formal exponentials instead of polynomials :

$$\sum_{\lambda^{(i)}\in G(\lambda)} x^{\lambda^{(i)}} \leftrightarrow \sum_{\lambda^{(i)}\in G(\lambda)} e^{2\pi i\langle\lambda^{(i)},x\rangle}\,.$$

1.6.2. Products of G-orbits

Suppose we are given two orbits, say $G(\lambda)$ and $G(\mu)$, of the same Coxeter group G. Let $\lambda^{(i)}$ and $\mu^{(k)}$ be the points of $G(\lambda)$ and $G(\mu)$ respectively, numbered in some way. We define the product of two orbits as

$$G(\lambda)\otimes G(\mu) := \bigcup_{\lambda^{(i)}\in G(\lambda),\ \mu^{(k)}\in G(\mu)} (\lambda^{(i)}+\mu^{(k)})\,. \tag{1.6.5}$$

The left side is obviously G-invariant, therefore the right side is also G-invariant. Hence it can be decomposed into a union of several G-orbits. The highest and the lowest components of such a decomposition are easily obtained :

$$G(\lambda)\otimes G(\mu) = G(\lambda+\mu)\cup\cdots\cup G(\lambda+\overline{\mu}). \tag{1.6.6}$$

Here, $\lambda+\mu$ is the sum of the dominant points of the orbits $G(\lambda)$ and $G(\mu)$. The symbol $\overline{\mu}$ stands for the unique lowest point of $G(\mu)$ (all coordinates are non-positive in the ω-basis). Frequently, it happens that $\lambda+\overline{\mu}$ is not a dominant point, i.e. the highest point in its orbit, but it still identifies the orbit uniquely. Note also that $\lambda+\overline{\mu}$ and $\mu+\overline{\lambda}$ always belong to the same G-orbit. The lowest component often appears more than once in the decomposition.

For a geometric interpretation of (1.6.6), recall that all orbits in (1.6.6) are concentric, having the origin as their common center, and that points of one orbit are equidistant from the origin. In physics, the product on the left side of

(1.6.6) can be thought of as a certain 'interaction' between two orbit-layers, resulting on the right side in an 'onion'-like structure of several concentric orbit-layers.

To simplify the notation in the following examples, we write just λ instead of $G(\lambda)$, so that $\lambda \otimes \mu$ means $G(\lambda) \otimes G(\mu)$.

1.6.3. Two-dimensional examples

$$\begin{aligned} A_2 : \quad & (1,0) \otimes (0,1) = (1,1) \cup 3(0,0)\,, \\ & (1,0) \otimes (1,1) = (2,1) \cup 2(1,0) \cup 2(0,2)\,, \\ & (1,1) \otimes (1,1) = (2,2) \cup 2(1,1) \cup 2(3,0) \cup 2(0,3) \cup 6(0,0)\,. \end{aligned}$$

$$\begin{aligned} C_2 : \quad & (1,0) \otimes (0,1) = (1,1) \cup 2(1,0)\,, \\ & (1,0) \otimes (1,1) = (2,1) \cup 2(2,0) \cup 2(0,2) \cup 2(0,1)\,, \\ & (1,1) \otimes (1,1) = (2,2) \cup 2(2,1) \cup 2(4,0) \cup 2(2,0) \cup 2(0,3) \cup \\ & \qquad\qquad\qquad 2(0,1) \cup 8(0,0)\,. \end{aligned}$$

$$\begin{aligned} G_2 : \quad & (1,0) \otimes (0,1) = (1,1) \cup 2(0,2) \cup 2(0,1)\,, \\ & (1,0) \otimes (1,1) = (2,1) \cup (1,2) \cup (1,1) \cup 2(0,4) \cup 2(0,2) \cup 2(0,1)\,, \\ & (1,1) \otimes (1,1) = (2,2) \cup 2(1,1) \cup 2(1,3) \cup 2(3,0) \cup 2(2,0) \cup \\ & \qquad\qquad\qquad 2(1,0) \cup 2(0,5) \cup 2(0,4) \cup 2(0,1) \cup 12(0,0)\,. \end{aligned}$$

$$\begin{aligned} H_2 : \quad & (1,0) \otimes (0,1) = (1,1) \cup (\tau-1,\tau-1) \cup 5(0,0)\,, \\ & (1,0) \otimes (1,1) = (2,1) \cup (\tau,\tau-1) \cup (\tau-1,1) \cup 2(1,0) \cup \\ & \qquad\qquad\qquad 2(0,\tau+1)\,, \\ & (1,1) \otimes (1,1) = (2,2) \cup 2(\tau,\tau) \cup 2(\tau-1,\tau-1) \cup 2(2+\tau,0) \cup \\ & \qquad\qquad\qquad 2(2\tau-1,0) \cup 2(0,2+\tau) \cup 2(0,2\tau-1) \cup 10(0,0)\,. \end{aligned}$$

1.6.4. Three-dimensional examples

$$\begin{aligned} A_3:\quad &(1,0,0)\otimes(0,0,1) = (1,0,1)\cup 4(0,0,0)\,,\\ &(1,0,1)\otimes(0,1,0) = (1,1,1)\cup 3(2,0,0)\cup 4(0,1,0)\cup 3(0,0,2)\,,\\ &(1,1,0)\otimes(0,0,1) = (1,1,1)\cup 3(2,0,0)\cup 2(0,1,0)\,. \end{aligned}$$

$$\begin{aligned} B_3:\quad &(1,0,0)\otimes(0,0,1) = (1,0,1)\cup 3(0,0,1)\,,\\ &(1,0,1)\otimes(0,1,0) = (1,1,1)\cup 2(2,0,1)\cup 3(1,0,1)\cup 2(0,1,1)\cup\\ &\qquad\qquad 3(0,0,3)\cup 6(0,0,1)\,,\\ &(1,1,0)\otimes(0,0,1) = (1,1,1)\cup 2(2,0,1)\cup 2(1,0,1)\cup 2(0,1,1)\,. \end{aligned}$$

$$\begin{aligned} C_3:\quad &(1,0,0)\otimes(0,0,1) = (1,0,1)\cup 2(0,1,0)\,,\\ &(1,0,1)\otimes(0,1,0) = (1,1,1)\cup 2(2,1,0)\cup 2(1,0,1)\cup 4(2,0,0)\cup\\ &\qquad\qquad 4(0,2,0)\cup 4(0,1,0)\cup 3(0,0,2)\,,\\ &(1,1,0)\otimes(0,0,1) = (1,1,1)\cup 2(2,1,0)\cup 2(1,0,1)\cup 4(0,1,0)\,. \end{aligned}$$

$$\begin{aligned} H_3:\quad &(1,0,0)\otimes(0,0,1) = (1,0,1)\cup(0,\tau-1,\tau-1)\cup 5(\tau,0,0)\cup\\ &\qquad\qquad 3(0,0,\tau-1)\,. \end{aligned}$$

1.6.5. Decomposition of products of E_8 orbits

We say that an orbit is fundamental if its dominant weight in the ω-basis has precisely one coordinate equal to 1 and all others are zero. Thus E_8 has 8 fundamental orbits. Their sizes range from 240 to over 17 000.

All 36 different products of fundamental orbits of E_8 were decomposed in **[19]** and are explicitly shown within the tables. They were indispensable in solving the main problem of **[19]**, namely the decomposition of products of fundamental representations of E_8.

1.7. Decomposition of symmetrized powers of orbits

1.7.1. Symmetrized powers of G-polynomials

The product of m identical polynomials, say $P(\lambda; x)$, is the subject of the action of the permutation group S_m of m elements. Thus it can be decomposed into a sum of components with a specific permutation symmetry. It is well known from representation theory that the permutation symmetry commutes with the action of the Weyl group. Consequently, each permutation symmetry component can be decomposed into a sum of polynomials.

Let $\square$ be short-hand notation for a polynomial (1.6.2). The product of two and more copies of $\square$ decomposes into the symmetry components indicated by their Young tableaux :

$$\square\otimes\square=\square\square+\begin{smallmatrix}\square\\ \square\end{smallmatrix}\,,\qquad \square\otimes\square\otimes\square=\square\square\square+2\,\begin{smallmatrix} & \square\\ \square & \square\end{smallmatrix}+\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}\,,\quad \dots \tag{1.7.1}$$

In general, the square stands for a set of G-invariant items, each square containing the same items. Those can be monomials of a polynomial, or weights in the case of the weight system of a representation of a semisimple Lie group/ algebra, or points of a G-orbit. The product of m copies of the same square decomposes into permutational symmetry components according to the representations of the group S_m. The components are identified by their Young tableau. Each of the components is further decomposable into the sum of parts that are labeled by the orbits of the Coxeter group G.

In order to perform such a two-step decomposition, (i) the items of the square need to be numbered consecutively in any convenient way. The items belonging to a particular permutation symmetry component are then determined according to the inequalities, shown in the next subsection, and more generally implied by the corresponding Young tableau. Then (ii) items belonging to a particular Young tableau, which are labeled by points transformed by G, are sorted out into the Coxeter group orbits. Practically it suffices to find the items labeled by dominant points.

1.7.2. Symmetrized powers of G-orbits

For simplicity of notation let us continue to label an orbit $G(\lambda)$ by its dominant point λ. The product of the same two G-orbits decomposes into its symmetric and antisymmetric parts :

$$\lambda \otimes \lambda = (\lambda^2)_{symm} \cup (\lambda^2)_{anti} \tag{1.7.2}$$

Each of the two terms of the right side is further decomposable into the sum of individual orbits. Let $\lambda_1, \lambda_2, \ldots$ be the points of the orbit λ numbered in any order. Then the content of the two parts is determined by the following inequalities, illustrated by their corresponding Young tableau :

$$(\lambda^2)_{symm} \ni \lambda_p + \lambda_q\,, \quad p \geq q\,, \quad \begin{array}{|c|c|}\hline q & p \\ \hline\end{array}\,, \tag{1.7.3}$$

$$(\lambda^2)_{anti} \ni \lambda_p + \lambda_q\,, \quad p > q\,, \quad \begin{array}{|c|}\hline q \\ \hline p \\ \hline\end{array}\,. \tag{1.7.4}$$

The product of 3 copies of λ decomposes likewise

$$\lambda \otimes \lambda \otimes \lambda = (\lambda^3)_{symm} \cup (\lambda^3)_{anti} \cup 2(\lambda^3)_{mixed}\,, \tag{1.7.5}$$

where permutation symmetry components are formed from the N points as follows :

$$(\lambda^3)_{symm} \ni \lambda_p + \lambda_q + \lambda_s\,, \quad p \geq q \geq s\,, \quad \begin{array}{|c|c|c|}\hline s & q & p \\ \hline\end{array}\,, \tag{1.7.6}$$

$$(\lambda^3)_{anti} \ni \lambda_p + \lambda_q + \lambda_s\,, \quad p > q > s\,, \quad \begin{array}{|c|}\hline s \\ \hline q \\ \hline p \\ \hline\end{array}\,, \tag{1.7.7}$$

$$(\lambda^3)_{mixed} \ni \lambda_p + \lambda_q + \lambda_s\,, \quad p \geq q \text{ and } p > s\,, \quad \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{s} \\ \hline \multicolumn{1}{|c|}{q} & \multicolumn{1}{c|}{p} \\ \hline\end{array}\,. \tag{1.7.8}$$

Similarly, any higher power decomposes into permutation symmetry components where each is a sum of individual orbits.

1.7.3. Two-dimensional examples

$$
\begin{aligned}
A_2 \ : \quad & (0,1)^2_{symm} = (1,0) \cup (0,2)\,, \\
& (0,1)^2_{anti} \;\; = (1,0)\,. \\
& (1,1)^2_{symm} = (2,2) \cup (1,1) \cup (3,0) \cup (0,3) \cup 3(0,0)\,, \\
& (1,1)^2_{anti} \;\; = (1,1) \cup (3,0) \cup (0,3) \cup 3(0,0)\,. \\
& (1,0)^3_{symm} = (1,1) \cup (3,0) \cup (0,0)\,, \\
& (1,0)^3_{anti} \;\; = (0,0)\,, \\
& (1,0)^3_{mixed} = (1,1) \cup 2(0,0)\,.
\end{aligned}
$$

$$
\begin{aligned}
C_2 \ : \quad & (0,1)^2_{symm} = (2,0) \cup (0,2) \cup 2(0,0)\,, \\
& (0,1)^2_{anti} \;\; = (2,0) \cup 2(0,0)\,. \\
& (1,0)^3_{symm} = (1,1) \cup (3,0) \cup 2(1,0)\,, \\
& (1,0)^3_{anti} \;\; = (1,0)\,, \\
& (1,0)^3_{mixed} = (1,1) \cup 3(1,0)\,.
\end{aligned}
$$

$$
\begin{aligned}
G_2 \ : \quad & (0,1)^2_{symm} = (1,0) \cup (0,2) \cup (0,1) \cup 3(0,0)\,, \\
& (0,1)^2_{anti} \;\; = (1,0) \cup (0,1) \cup 3(0,0)\,. \\
& (1,0)^3_{symm} = (1,3) \cup (3,0) \cup (2,0) \cup 3(1,0) \cup 2(0,3) \cup 2(0,0)\,, \\
& (1,0)^3_{anti} \;\; = (2,0) \cup 2(1,0) \cup 2(0,0)\,, \\
& (1,0)^3_{mixed} = (1,3) \cup 2(2,0) \cup 5(1,0) \cup 2(0,3) \cup 4(0,0)\,.
\end{aligned}
$$

$$H_2: \quad (0,1)^2_{symm} = (\tau,0) \cup (0,2) \cup (0,\tau-1)\,,$$

$$(0,1)^2_{anti} = (\tau,0) \cup (0,\tau-1)\,.$$

$$(1,0)^3_{symm} = (2-\tau,1) \cup (1,\tau) \cup (3,0) \cup (\tau,0) \cup (0,\tau-1)\,,$$

$$(1,0)^3_{anti} = (\tau,0) \cup (0,\tau-1)\,,$$

$$(1,0)^3_{mixed} = (2-\tau,1) \cup (1,\tau) \cup 2(\tau,0) \cup 2(0,\tau-1)\,.$$

1.7.4. Three-dimensional examples

$$A_3: \quad (1,0,0)^3_{symm} = (1,1,0) \cup (3,0,0) \cup (0,0,1)\,,$$

$$(1,0,0)^3_{anti} = (0,0,1)\,,$$

$$(1,0,0)^3_{mixed} = (1,1,0) \cup 2(0,0,1)\,.$$

$$B_3: \quad (1,0,0)^3_{symm} = (1,1,0) \cup (3,0,0) \cup 3(1,0,0) \cup (0,0,2)\,,$$

$$(1,0,0)^3_{anti} = 2(1,0,0) \cup (0,0,2)\,,$$

$$(1,0,0)^3_{mixed} = (1,1,0) \cup 5(1,0,0) \cup 2(0,0,2)\,.$$

$$C_3: \quad (1,0,0)^2_{symm} = (2,0,0) \cup (0,1,0) \cup 3(0,0,0)\,,$$

$$(1,0,0)^2_{anti} = (0,1,0) \cup 3(0,0,0)\,.$$

$$H_3: \quad (1,0,0)^2_{symm} = (2,0,0) \cup (0,1,0) \cup (0,\tau-1,0) \cup 6(0,0,0)\,,$$

$$(1,0,0)^2_{anti} = (0,1,0) \cup (0,\tau-1,0) \cup 6(0,0,0)\,.$$

1.8. Congruence classes, indices, and anomaly numbers of polytopes

Here we introduce numerical characterizations of W-orbits, analogs of similar quantities known for irreducible representations of semisimple Lie groups, which proved particularly useful in their application.

1.8.1. Congruence classes

Inclusion among the lattices (1.3.3) is an important property of the Weyl group W. The weight lattice P can be understood as a union of several components, each isomorphic to the root lattice Q. The components are shifted relative to each other by some elements of P. An individual component consists of points belonging to one congruence class of P. The index of Q in P, denoted $|Z|$, is the number of distinct congruence classes in P. The value of $|Z|$ reflects other properties of G. For example, it is the order of the center of G, it is a common denominator of coordinates of all points of P when given in the basis of simple roots, etc. One has $|Z| > 1$ for all G but for the exceptional simple Lie groups of types E_8, F_4, and G_2.

The congruence number c is a number attached to points of P. The value of c is common to all points of the same congruence class. It can be defined in a number of equivalent ways. Our definition coincides with that of [**35**]. All points of any W-orbit belong to the same congruence class. Furthermore, orbits obtained from the decomposition of a product belong to the same congruence class, and their congruence number is the sum of the congruence numbers of the orbits of the multiplication. That is also true for the decomposition of symmetrized powers of orbits.

Let $x = (x_1, x_2, \ldots, x_n) \in P$ be a point to consider in the ω-basis. Its congruence number $c(x)$ is given by the following formulas :

$$
\begin{aligned}
A_n \quad &: \quad c(x) = \sum_{k=1}^{n} k x_k \mod (n+1) \\
B_n \quad &: \quad c(x) = x_n \mod 2 \\
C_n \quad &: \quad c(x) = \sum_{k=1}^{[\frac{n+1}{2}]} x_{2k-1} \mod 2 \\
D_n \quad &: \quad c(x) = (c_1(x) \mod 2\,,\ c_2(x) \mod 4), \\
&\quad\ c_1(x) = x_{n-1} + x_n \\
&\quad\ c_2(x) = \begin{cases} 2x_1 + 2x_3 + \cdots + 2x_{n-2} + (n-2)x_{n-1} + nx_n, & n \text{ odd} \\ 2x_1 + 2x_3 + \cdots + 2x_{n-3} + (n-2)x_{n-1} + nx_n, & n \text{ even} \end{cases} \\
E_6 \quad &: \quad c(x) = x_1 - x_2 + x_4 - x_5 \mod 3 \\
E_7 \quad &: \quad c(x) = x_4 + x_6 + x_7 \mod 2
\end{aligned}
\tag{1.8.1}
$$

For E_8, F_4 and G_2 there is only one congruence class, namely $c(x) = 0$ for all $x \in P$. Note also that the roots of any group belong to the congruence class $c(x) = 0$. Hence also the points of the root lattice of any group belong to the congruence class $c(x) = 0$.

The points of any single G-orbit belong to the same congruence class because the difference between any two points of the same orbit is an integer linear combination of simple roots, as can be derived from (1.4.2).

For the non-crystallographic groups, the congruence classes can be similarly defined, involving their appropriate irrationality. It is important to recall that, in these cases, P is a dense lattice. The coordinates of $x \in P$, relative to the ω-basis, are the numbers $a + \tau b$, with $a, b \in \mathbb{Z}$.

$$
H_2 \quad : \quad c(x) = \tau x_1 + 2x_2 \mod 5, \quad \text{where } \tau = 3 \tag{1.8.2}
$$

1.8.2. The second and higher indices

The second and higher indices were defined **[64]** for weight systems of irreducible finite dimensional representations of compact semisimple Lie groups.

Extensive tables of indices of degree 0, 2 and 4 are found in [**39**]. The fact that a weight system is a union of several *W*-orbits suggests that the indices could be introduced for individual orbits. Moreover, we introduce them also for non-crystallographic Coxeter groups with the same formulas.

For any finite Coxeter group G, we define an index $I_\lambda^{(2k)}$ of degree $2k$ of a G-orbit $G(\lambda)$ by

$$I_\lambda^{(2k)} = \sum_{\mu\in G(\lambda)} \langle\mu,\mu\rangle^k = \langle\lambda,\lambda\rangle^k I_\lambda^{(0)}, \qquad k = 0,1,2,\dots, \tag{1.8.3}$$

because points of $G(\lambda)$ are equidistant from the origin. Clearly $I_\lambda^{(0)} = |G(\lambda)|$ is the number of points of the orbit $G(\lambda)$ given by (1.4.1).

Higher indices of products of two orbits, $G(\lambda_1)\otimes G(\lambda_2)$, are also useful in calculating the decompositions. Let r be the rank of G.

$$I^{(2k)}(G(\lambda_1)\otimes G(\lambda_2)) = I^{(2k)}_{\lambda_1\otimes\lambda_2} = I^{(2k)}_{\lambda_1+\lambda_2} + \dots + I^{(2k)}_{\lambda_1+\overline{\lambda_2}} \tag{1.8.4}$$

$$I^{(0)}_{\lambda_1\otimes\lambda_2} = I^{(0)}_{\lambda_1} I^{(0)}_{\lambda_2} \tag{1.8.5}$$

$$I^{(2)}_{\lambda_1\otimes\lambda_2} = I^{(2)}_{\lambda_1} I^{(0)}_{\lambda_2} + I^{(0)}_{\lambda_1} I^{(2)}_{\lambda_2} \tag{1.8.6}$$

$$= I^{(0)}_{\lambda_1} I^{(0)}_{\lambda_2} (\langle\lambda_1,\lambda_1\rangle + \langle\lambda_2,\lambda_2\rangle) \tag{1.8.7}$$

$$I^{(4)}_{\lambda_1\otimes\lambda_2} = I^{(4)}_{\lambda_1} I^{(0)}_{\lambda_2} + \frac{2(r+2)}{r} I^{(2)}_{\lambda_1} I^{(2)}_{\lambda_2} + I^{(0)}_{\lambda_1} I^{(4)}_{\lambda_2} \tag{1.8.8}$$

Table 1.4 presents examples of indices of degree 0, 2, 4, 6 and 8 for individual orbits of A_2, C_2, G_2 and H_2.

1.8.3. Anomaly numbers

Triangle anomaly numbers were introduced in physics [**20, 56, 66**] as quantities assigned to irreducible representations of a few compact semisimple Lie groups and calculated from the weight systems of their representations. Constraints on possible models in particle physics were imposed in terms of admissible values of the anomaly numbers of representations involved in a particular model. Generalization of the concept to all compact semisimple Lie groups and to higher than third degree anomaly number originates in [**62**]. Our goal here is to show that the anomaly numbers can be used also for constituents of the

A_2	$I^{(0)}$	$I^{(2)}$	$3I^{(4)}$	$9I^{(6)}$	$27I^{(8)}$
$(1,0)$	3	2	4	8	16
$(2,0)$	3	8	64	512	4096
$(1,1)$	6	12	72	432	2592
$(2,1)$	6	28	392	5488	76832

C_2	$I^{(0)}$	$I^{(2)}$	$2I^{(4)}$	$4I^{(6)}$	$8I^{(8)}$
$(1,0)$	4	2	2	2	2
$(0,1)$	4	4	8	16	32
$(2,0)$	4	8	32	128	512
$(0,2)$	4	16	128	1024	8192
$(1,1)$	8	20	100	500	2500
$(2,1)$	8	40	400	4000	40000

G_2	$I^{(0)}$	$I^{(2)}$	$3I^{(4)}$	$9I^{(6)}$	$27I^{(8)}$
$(0,1)$	6	4	8	16	32
$(1,0)$	6	12	72	432	2592
$(0,2)$	6	16	128	1024	8192
$(0,3)$	6	36	648	11664	209952
$(2,0)$	6	48	1152	27648	663552
$(1,1)$	12	56	784	10976	153664

H_2	$I^{(0)}$	$(3-\tau)I^{(2)}$	$(3-\tau)^2I^{(4)}$
$(1,0)$	5	10	20
$(2,0)$	5	40	320
$(1,1)$	10	$20(\tau+2)$	$40(\tau+2)^2$
$(2,1)$	10	$10(4\tau+10)$	$10(4\tau+10)^2$

TABLE 1.4. Examples of the indices $I^{(2k)}$, $k = 0, 1, \ldots, 4$.

weight systems of irreducible representations, namely for W-orbits and more generally, for the orbits $G(\lambda)$ of any finite Coxeter group.

The anomaly number $I_\lambda^{(2k-1)}$ of degree $2k-1$ of the orbit $G(\lambda)$ of the Coxeter group G is defined as follows,

$$I_\lambda^{(2k-1)} = \sum_{\mu \in G(\lambda)} \langle \mu, u \rangle^{2k-1}, \qquad k = 1, 2, \dots, \tag{1.8.9}$$

where u is a special vector chosen in a way that its scalar product with any weight gives the lowest possible integer. In particular, $I^{(1)} = 0$ in all cases. The anomaly number of physics literature is $I^{(3)}$, therefore it is the only one we consider.

Frequently used property of $I^{(3)}$ is the decomposition of the product of two orbits, which is the analog of (1.8.6) :

$$I^{(3)}_{\lambda_1 \otimes \lambda_2} = I^{(3)}_{\lambda_1} I^{(0)}_{\lambda_2} + I^{(0)}_{\lambda_1} I^{(3)}_{\lambda_2} = I^{(3)}_{\lambda_1 + \lambda_2} + \dots + I^{(3)}_{\overline{\lambda_1 + \lambda_2}} \tag{1.8.10}$$

In general terms, the direction of u can be characterized as follows. Suppose W in (1.8.9) is the Weyl group of a compact simple Lie group G, and that G has a maximal reductive subgroup of type $U_1 \times G'$. Then the direction of u is given by the direction corresponding to U_1 in the Euclidean space spanned by the roots of G.

The first question to answer is when such a maximal subgroup is present. For a complete list of the cases see below **[4]** :

$$\begin{aligned}
&A_n \supset A_{n-1} \times U_1 && n \geq 2 \\
&A_n \supset A_k \times A_{n-k-1} \times U_1 && n \geq 3\,, \quad 1 \leq k \leq [\tfrac{n-1}{2}] \\
&B_n \supset B_{n-1} \times U_1 && n \geq 3 \\
&C_n \supset A_{n-1} \times U_1 && n \geq 2 \\
&D_n \supset A_{n-1} \times U_1 && n \geq 4 \\
&D_n \supset D_{n-1} \times U_1 && n \geq 5 \\
&E_6 \supset D_5 \times U_1 && \\
&E_7 \supset E_6 \times U_1 &&
\end{aligned} \tag{1.8.11}$$

As long as each orbit of a given group contains with every weight also its negative, the anomaly numbers are equal to zero. Therefore the interesting cases that remain are found in A_n, D_{2k+1}, E_6, and E_7. In physics, however, the only anomaly numbers that we know are used so far are the ones of $A_n \supset A_{n-1} \times U_1$.

Anomaly numbers of H_2, H_3, and H_4 are also defined by (1.8.9). In those cases, however, the direction of u has to be determined differently since there is no U_1 subgroup. Instead, one can require that u be orthogonal to selected simple roots : α_1 for H_2, α_1 and α_2 for H_3, and α_1, α_2, and α_3 for H_4. Anomaly numbers for H_2 are zero for all orbits. They will be considered elsewhere [**21**], along with the anomaly numbers of other non-crystallographic groups.

1.9. Concluding remarks

(1) Useful and interesting objects may turn out to be G-orbits with each point decorated by a sign [**60**] according to the following rule. The dominant point, say λ, and all points obtained from it by an even number of reflections generating G, carry a positive sign, while all points of the orbit obtained from λ by an odd number of reflections carry a negative sign. Let us call an S-orbit a decorated orbit of λ of G, while the orbits without the sign decoration, i.e. all positive signs, are called C-orbits of λ of G. In order to avoid ambiguities, it should be stipulated that λ of an S-orbit must have all coordinates positive in ω-basis.

Multiplication of such orbits follows simple rules :

$$\text{C-orbit} \times \text{C-orbit} \longrightarrow \text{C-orbits}, \tag{1.9.1}$$

$$\text{C-orbit} \times \text{S-orbit} \longrightarrow \text{S-orbits}, \tag{1.9.2}$$

$$\text{S-orbit} \times \text{S-orbit} \longrightarrow \text{C-orbits}. \tag{1.9.3}$$

In (1.9.1), all coefficients in the decomposition of the product are positive integers, while in (1.9.2) and (1.9.3), all such coefficients are integers, but not all may be positive.

The decomposition of many products of C-orbits with lowest nontrivial S-orbit can be directly inferred from the tables [**6**], using the Weyl character formula.

(2) In the examples, we often required that a G-orbit consist of points of the weight lattice P. Very few properties of the orbits would have been lost, had we instead allowed $\lambda \in \mathbb{R}^n$. The congruence classes would not then be applicable.

Consider the following products of A_2 orbits as examples :

$$(a,0) \otimes (\varepsilon,0) = (a+\varepsilon,0) \cup (a-\varepsilon,\varepsilon)\,, \quad 0<\varepsilon \ll 1,\ a \geq 1, \tag{1.9.4}$$

$$(a,0) \otimes (0,a+\varepsilon) = (a,a+\varepsilon) \cup (0,\varepsilon)\,, \quad 0<\varepsilon \ll 1,\ a \gg 1. \tag{1.9.5}$$

The radii of the two orbits in the decomposition (1.9.4) can be drawn arbitrarily close by a suitable choice of ε, and in (1.9.5) they can be pushed as far apart as desired by the choice of a. The second orbit in (1.9.5) has a radius equal to $\varepsilon\sqrt{\frac{2}{3}}$.

(3) For a geometric interpretation of orbits as polytopes, refer to the paragraph following equation (1.6.6). The 'interaction' (i.e. product) between two concentric orbit-layers results in the layered structure of orbits. They are subject to the equality of indices of various degrees, congruence numbers, relations between anomaly numbers. Speculative interpretation can go further : Consider $I^{(2)}_{\lambda}$ as the 'energy' of the orbit and $I^{(2)}_{\lambda\otimes\lambda'}$ as the 'energy' of the interacting pair, etc.

(4) Although we did not pursue it here, orbit multiplication can be viewed as an 'interaction' between two orbits similarly as used in particle physics to view interacting multiplets of particles. A multiplet is described by the weight system of an irreducible representation of the corresponding Lie group/algebra. Here, the role of the multiplet would be given to the set of points of an orbit. In both cases, such interactions would be governed by the strict equality of indices of various degrees, congruence numbers, relations between anomaly numbers. But there is a price to pay for such a reinterpretation of multiplets : the overall invariance of the theory with respect to the Lie group would be reduced to the invariance with respect to the Coxeter group, or to its (discrete) image 'lifted' into the Lie group [**46**].

(5) It would be useful to ask additional questions about the properties of indices and anomaly numbers of various degrees. Such questions can be answered by adaptation of the methods used for the weight system of representations [**62, 64**].

(6) In place of finite Coxeter groups, we could have chosen to consider other finite groups for similar considerations [**36**]. The immediate motivations for our choice were recent applications in harmonic analysis, where W-orbits are playing a fundamental role. Equally interesting would be to consider orbits of infinite Coxeter groups. (A Coxeter group with connected diagram is of infinite order if its diagram is different from those listed in section 1.2.) The orbits of representations of Kac-Moody algebras would be relatively easily amenable to such a study.

(7) Similarly, we could consider orbits of two or more seed points. A simple example is the root system of the group G_2. Choosing as the two seed points one short root and one long root, say α_2 and $\alpha_1 + 3\alpha_2$, the orbit of the pair is a star-like polygon formed by the root system of G_2.

(8) An interesting problem appears to be to pursue a similar study of orbits of the even subgroups of Coxeter groups, particularly because these subgroups are not Coxeter groups in general.

Acknowledgements

Work supported in part by the Natural Sciences and Engineering Research Council of Canada, the MIND Research Institute. We are grateful for the hospitality extended to us by the Doppler Institute, Czech Technical University (J.P. and M.L.), and by the Centre de Recherches Mathématiques, Université de Montréal (L.H.).

Chapitre 2

BRANCHING RULES FOR THE WEYL GROUP ORBITS OF THE LIE ALGEBRA A_N

Référence complète : M. Larouche, M. Nesterenko et J. Patera, Branching rules for the Weyl group orbits of the Lie algebra A_n, *Journal of Physics A : Mathematical and Theoretical*, 42(48) : 485203, 15, 2009.

Résumé

Les orbites des groupes de Weyl $W(A_n)$ des algèbres de Lie simples de type A_n sont réduites en l'union d'orbites des groupes de Weyl des sous-algèbres réductives maximales de A_n. Les matrices qui transforment les points des orbites de $W(A_n)$ en des points des orbites des sous-algèbres sont listées pour tous les cas $n \leq 8$ ainsi que pour les séries infinies des paires d'algèbre-sous-algèbre suivantes : $A_n \supset A_{n-k-1} \times A_k \times U_1$, $A_{2n} \supset B_n$, $A_{2n-1} \supset C_n$, $A_{2n-1} \supset D_n$. De nombreux cas spéciaux sont inclus et plusieurs exemples sont présentés.

Abstract

The orbits of Weyl groups $W(A_n)$ of simple A_n type Lie algebras are reduced to the union of orbits of the Weyl groups of maximal reductive subalgebras of A_n. Matrices transforming points of the orbits of $W(A_n)$ into points of subalgebra orbits are listed for all cases $n \leq 8$ and for the infinite series of algebra-subalgebra pairs $A_n \supset A_{n-k-1} \times A_k \times U_1$, $A_{2n} \supset B_n$, $A_{2n-1} \supset C_n$, $A_{2n-1} \supset D_n$. Numerous special cases and examples are shown.

2.1. INTRODUCTION

Finite groups generated by reflections in an n-dimensional real Euclidean space $\mathbb{R}^n$ are commonly known as finite Coxeter groups **[22, 23]**. Finite Coxeter groups are split into two classes : crystallographic and non-crystallographic groups. Crystallographic groups are often referred to as Weyl groups of semi-simple Lie groups or Lie algebras. They are symmetry groups of some lattices in $\mathbb{R}^n$. There are four infinite series (as to the admissible values of rank n) of such groups, and five isolated exceptional groups of ranks 2, 4, 6, 7, and 8. Non-crystallographic finite Coxeter groups are the symmetry groups of regular 2D polygons (the dihedral groups), and two exceptional groups, for n=3 (the icosahedral group of order 120) and n=4, which is of order 120^2.

We consider the orbits of the Weyl groups $W(A_n)$ of the simple Lie algebras of type A_n, $n \geq 1$, equivalently the Weyl groups of the simple Lie group $SL(n+1, \mathbb{C})$, or of its compact real form $SU(n+1)$. The order of such a Weyl group is $(n+1)!$. An orbit of $W(A_n)$ is a set of distinct points in $\mathbb{R}^n$, obtained from a chosen single (seed) point, say $\lambda \in \mathbb{R}^n$, by application of $W(A_n)$ to λ. Hence, an orbit W_λ of $W(A_n)$ contains at most $(n+1)!$ points. The points of W_λ are equidistant from the origin. It should be noted that the group $W(A_n)$ is isomorphic to the permutation group of n+1 elements. Although we make no use of this fact here, it reveals a rather different perspective on our problem **[55]**.

Geometrically, points of the same orbit can be seen as vertices of a convex polytope generated from λ. There is a method for counting and describing the faces of all dimensions of such polytopes in the real Euclidean space $\mathbb{R}^n$. It uses an easy recursive decoration of the corresponding Coxeter-Dynkin diagrams **[8]**.

Weyl group orbits are closely related to weight systems of finite-dimensional irreducible representations of corresponding Lie algebras. More precisely, the weight system is a union of several Weyl group orbits. Which orbits are composed into a particular weight system is in principle known. An efficient algorithm for the computation exists [**6**]. The representations are finding innumerable applications in science. Very often, such applications can be carried through just by our knowledge of the corresponding weight system. It is conceivable that some of the applications would find interesting new possibilities when working with individual orbits only.

The list of possible reductions of $W(A_n)$ orbits is a result of a major classification problem solved more than half a century ago, when the maximal reductive subalgebras of simple Lie algebras, in particular of A_n, were determined [**4, 14**]. We exploit that classification without further reference to it.

In this paper, we consider orbits of $W(A_n)$ and their reduction to orbits of the Weyl groups of maximal reductive subalgebras of A_n. In the physics literature, a similar task [**39**] is often called computation of branching rules. We will consider two types of maximal reductive subalgebras, maximal reductive subalgebras that are not semisimple [**4**], and subalgebras that are maximal among reductive subalgebras, but which are in fact semisimple. Thus the second type of subalgebras are obtained from the list of [**14**] by eliminating semisimple subalgebras that are part of the reductive subalgebras classified in [**4**].

The present paper can be understood as a continuation of [**21**], where the orbits are seen as elements of a ring of reflection generated polytopes in $\mathbb{R}^n$. In that paper, the main problem was to reduce products of Weyl group orbits/polytopes into a sum of Weyl group orbits. Here, our problem is to transform/reduce/branch each polytope/orbit into a sum of concentric polytopes with lower symmetry, and often also with lower dimension.

Until recently, W-orbits were used as an efficient computational tool, particularly for large-scale computations (see for example [**6, 19, 45, 47**] and references therein). Their appreciation as point sets defining families of W-invariant special functions of n variables is relatively recent [**29, 30, 60**]. Other possible applications could include an unusual twist of some symmetry breaking problems in physics, where, rather than breaking down weight systems of representations, one would break each orbit independently.

The main advantage of the projection matrices method is the uniformity of its application as to the different algebra-subalgebra pairs, which makes it particularly amenable to computer implementation. Thus in [**61**], branching rules for representations of dimension up to 5000 were calculated for all simple Lie algebras of rank up to 8 and for all their maximal semisimple subalgebras. Corresponding projection matrices were presented as a computational tool only later in [**41**]. Subsequently, the tables [**39**] were also based on their exploitation.

Particular Weyl group orbit reduction has undoubtedly been addressed on many occasions in the literature. As a separate subject of interest, orbit branching rules seem to have been first found in [**40**], where they are used for reduction of many representations as well as orbits of the five exceptional simple Lie algebras. The corresponding projection matrices are shown there too. In [**18**], several generating functions for the reduction problem were derived. It is a very efficient method, in that it solves the problem for all orbits at once. Unfortunately, for each algebra-subalgebra pair, a new generating function needs to be derived. An independent original approach to orbit-orbit branching rules can be found in [**67, 68**], in which essentially combinatorial algorithms are developed for specific series of algebra-subalgebra pairs. For A_n, an algorithm for the equal rank subalgebra series of cases can be found there. It should be compared with subsection 2.4.3 of this paper.

Our problem in this paper is closely related to the computation of branching rules for irreducible finite dimensional representations of simple Lie algebras (equivalently, to branching rules for weight systems of representations). Theoretically, such problems need to be solved while describing symmetry breaking in some physical systems. Practically, the orbit branching rules problem needs to be solved whenever a large-scale computation of branching rules for representations is undertaken. The similarity of the two problems is in the transformation of orbit points (weights) that takes place in both cases. However, there are practically important differences between the two problems. The orbit branching rules are less constrained than those for representations. Some of the differences were already pointed out in [**21**]. Here we underline just two :

(i) While weight systems grow without limits, the larger the representation one has to work with, the orbit size (the number of points in an orbit) is always bounded by the order of the corresponding Weyl group. Without limits, only the distance of the orbit points from the origin can grow, but not their number. A weight system of a representation is a union of several W-orbits. The higher the representation, the more orbits it is comprised of. In general, to determine the orbits that form the weight system of a representation (equivalently, to compute dominant weights multiplicities in a representation) is often a difficult and laborious task (see [**6**] and references therein). Therefore, any large-scale computation with representations practically imposes the need to break a large problem for the weight system, into a series of much smaller ones for individual orbits. Computation of branching rules for the representations is one such problem, decomposition of products of representations into the direct sum of irreducible representations is another problem, which often needs to be carried out for relatively large representations, and which is solved entirely using the weight systems, see for example [**19**].

(ii) A point of a weight system of a representation necessarily belongs to a weight lattice of the Lie algebra. Its coordinates are integers in a suitably chosen basis of $\mathbb{R}^n$, so are the points of orbits after reduction. When we work with an individual orbit, we are free to choose the orbit, that is, the seed point λ, anywhere in $\mathbb{R}^n$, as close or as distant from the origin or from any other lattice point as one desires. After the reduction, some orbits can be very close, while some are far apart. Examples of such effects are shown in the Concluding Remarks of [**21**]. The flexibility thus achieved needs yet to be exploited.

The branching rules for $W(A_n) \to W(L)$, where L is a maximal reductive subalgebra of A_n, is a linear transformation between Euclidean spaces

$\mathbb{R}^n \to \mathbb{R}^m$, where m is the rank of L. The branching rules are unique, unlike transformations of individual orbit points, which depend on the relative choice of bases. In this paper, we provide the linear transformation in the form of an $n \times m$ matrix, the *projection matrix*. A suitable choice of bases allows one to obtain integer matrix elements in all the projection matrices listed here. Note that we use Dynkin notations and numberings for roots, weights and diagrams.

2.2. PRELIMINARIES

The general strategy of our approach can be described as follows.

Consider the pair $L \supset L'$ of Lie algebras of ranks n and m respectively, where L is simple and L' is maximal reductive. In principle, the orbit reduction problem for the pair $W(L) \supset W(L')$ is solved when the $n \times m$ matrix P is found, with the property that points of any orbit of $W(L)$ are transformed/projected by P into points of the corresponding orbits of $W(L')$. Computation of the branching rule for a specific orbit amounts to applying P to the points of the orbit, and to sorting out the projected points according to the orbits of $W(L')$.

This task requires that one be able to calculate the points of any orbit of the Weyl group of any semisimple Lie algebra encountered here. There is a standard method, but we refrain from describing it here once again. Instead we refer to **[21]**, the immediate predecessor of this paper, wherein all orbit points are given relative to the so called ω-basis. Geometric relations between the basis vectors are described by the matrix $(\langle\omega_j, \omega_k\rangle)$ of scalar products of the basis vectors. The matrices are found in **[6]** under the name *quadratic form matrices* for all simple Lie algebras.

The Weyl group of the one-parameter Lie algebra U_1 is trivial, consisting of the identity element only. This algebra is present in reductive non-semisimple Lie algebras. Its irreducible representations are all 1-dimensional, hence its orbits consist of one element. They are labeled by integers. The symbol (k) may stand for either the orbit $\{k, -k\}$ of A_1, or for the U_1-orbit of one point $\{k\}$. Distinction should be made from the context. For example, the orbit $(p)(q)$, where $p \in \mathbb{Z}^{>0},\ q \in \mathbb{Z}$, of $W(A_1 \times U_1)$, has two elements, $\{(p)(q), (-p)(q)\}$.

All orbits of $W(A_n)$ have the following symmetry. For each point $(a_1, a_2, \ldots, a_n)$ that belongs to an orbit, the point $(-a_n, -a_{n-1}, \ldots, -a_2, -a_1)$ also belongs to the same orbit. We say that the orbits of $W(A_n)$ in the following pair are contragredient :

$$(q_1, q_2, \ldots, q_{n-1}, q_n), \qquad (q_n, q_{n-1}, \ldots, q_2, q_1), \qquad q_j \geq 0 \text{ for all } j.$$

Branching rules for contragredient orbits are closely related. They either coincide, or one can be obtained from the other by interchanging $q_k \leftrightarrow q_{n-k}$ components of the dominant points. We list only one such pair of branching rules.

It is known that the fundamental representations, i.e. representations with highest weight equal to ω_j, $j = 1, ..., n$, have weight systems consisting only of the one Weyl group orbit W_{ω_j}. If no other orbits are involved in a branching rule, that rule coincides with the branching rule for representations.

The number of points in a Weyl group orbit, labeled by its unique dominant weight $(a_1, a_2, \ldots, a_n)$, is determined by the a_j's that are strictly positive. In orbits encountered in representation theory, we have $a_j \in \mathbb{Z}^{\geq 0}$. Since we are considering a more general setup, we need require only $a_j \in \mathbb{R}^{\geq 0}$. If all a_j's are strictly positive, the orbit of $W(A_n)$ contains $(n+1)!$ points.

For simplicity of notation we subsequently identify cases by algebra-subalgebra symbols rather than by corresponding Weyl groups. In particular, we speak of an orbit of A_k rather than of an orbit of $W(A_k)$. Subsequently dots in a matrix denote zero matrix elements.

2.3. Construction of projection matrices

The projection matrix P for a given pair $L \supset L'$ of Lie algebras is calculated from one known branching rule. The classification of subalgebras amounts precisely to providing that branching rule. Usually the branching rule is given for the lowest dimensional representation. Then the matrix is obtained using the weight systems of the involved representations.

First, make a suitable (lexicographical) ordering of the weights of L and L'. Then associate the weights on both sides one-by-one according to the chosen order. The matrix is obtained from requiring that each weight of L be transformed to its associate weight of L'.

Example 2.3.1.

Consider the case of $A_3 \supset C_2$ *of subsection 2.5.2. The lowest orbit of* A_3 *contains 4 points. The lowest orbit of* C_2 *also contains 4 points. More precisely, there are two 4-point orbits of* A_3 *and two such orbits of* C_2*. Either of the two* A_3 *orbits can be used for setting up the projection matrix. The two orbits of* C_2 *with dominant weights* $(1,0)$ *and* $(0,1)$ *are different, being related to simple roots of different length. We take the* A_3 *orbit of the dominant point* $(1,0,0)$ *and project it onto the* C_2 *orbit of the point* $(1,0)$*. (See the second option in the last item of Concluding Remarks below.)*

$$(1,0,0) \mapsto (1,0), \qquad (-1,1,0) \mapsto (-1,1),$$

$$(0,-1,1) \mapsto (1,-1), \qquad (0,0,-1) \mapsto (-1,0).$$

Writing the points as column matrices, the projection matrix of subsection 2.5.2 is obtained from the first three. Proceeding one column at a time, we have

$$\begin{pmatrix} 1 & * & * \\ 0 & * & * \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Here, stars denote the entries that are still to be determined. The matrix $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ *then automatically transforms the fourth point* $(0,0,-1)$ *of the* A_3 *orbit as required. This matrix can be used for projecting points of any* A_3 *orbit.*

2.4. Equidimensional orbit branching rules

All orbits $W(A_n)$ are n-dimensional except for the trivial one $\lambda = 0$, which consists of one point, the origin. Points can be seen as vertices of a polytope in $\mathbb{R}^n$ **[21]**. Reduction to orbits of the same dimension happens when reduced orbits have the symmetry of $W(A_r \times A_s \times U_1)$, where $r+s+1 = n$. Clearly, we need to consider only the cases $r \geq s$. Geometrically, the orbit points are not displaced in this case ; rather, they are relabeled by the coordinates given in the standard basis of the subgroup.

In this section, we first consider the lowest special cases in part as transparent illustration and in part because they are most frequently encountered in physics applications. Lastly, we consider the infinite series of cases for all possible values of the rank n $(1 \le n < \infty)$: $W(A_n) \supset W(A_{n-k-1} \times A_k \times U_1)$, $0 \le k \le [\frac{n-1}{2}]$, where $[\frac{n-1}{2}]$ is the integer part of $\frac{n-1}{2}$.

2.4.1. Orbit branching rules for $A_n \supset A_{n-1} \times U_1$

2.4.1.1. $A_1 \supset U_1$

The lowest example is trivial. The Weyl group of A_1 has two elements ; the Weyl group of U_1 is just the identity transformation. An orbit $\{p, -p\}$ of A_1 reduces to two orbits of U_1 :

$$(p) \supset (p)+(-p), \qquad p \in \mathbb{R}^{>0}.$$

The reduction is accomplished by applying the 1×1 projection matrix $P = (\,1\,)$ to each element of the A_1 orbit.

2.4.1.2. $A_2 \supset A_1 \times U_1$

The second lowest example is often used in nuclear and particle physics. In terms of compact Lie groups it is $SU(3) \supset U(2) = SU(2) \times U(1)$. The reduction is accomplished by applying to each element of the A_2 orbit the projection matrix $P = \left(\begin{smallmatrix} 1 & \cdot \\ 1 & 2 \end{smallmatrix}\right)$, and by subsequently regrouping the results into orbits of $A_1 \times U_1$. We find the branching rules for the three types of A_2 orbits :

$$\begin{aligned}
(p, 0) &\supset (p)(p)+(0)(-2p), \\
(0, q) &\supset (q)(-q)+(0)(2q), \\
(p, q) &\supset (p)(p+2q)+(p+q)(p-q)+(q)(-2p-q),
\end{aligned}$$

where $p, q \in \mathbb{R}^{>0}$.

2.4.1.3. $A_3 \supset A_2 \times U_1$

Reduction is achieved by applying to each element of an A_3 orbit the projection matrix

$$\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ 1 & 2 & 3 \end{pmatrix} \tag{2.4.1}$$

and by subsequently regrouping the results into orbits of $A_2 \times U_1$. For all seven types of A_3 orbits, we find the branching rules :

$$\begin{aligned}
(p,0,0) &\supset (p,0)(p)+(0,0)(-3p),\\
(0,q,0) &\supset (0,q)(2q)+(q,0)(-2q),\\
(0,0,r) &\supset (0,0)(3r)+(0,r)(-r),\\
(p,q,0) &\supset (p,q)(p+2q)+(p+q,0)(p-2q)+(q,0)(-3p-2q),\\
(p,0,r) &\supset (p,0)(p+3r)+(p,r)(p-r)+(0,r)(-3p-r),\\
(0,q,r) &\supset (0,q)(2q+3r)+(0,q+r)(2q-r)+(q,r)(-2q-r),\\
(p,q,r) &\supset (p,q)(p+2q+3r)+(p,q+r)(p+2q-r)\\
&\qquad +(p+q,r)(p-2q-r)+(q,r)(-3p-2q-r),
\end{aligned} \tag{2.4.2}$$

where $p, q, r \in \mathbb{R}^{>0}$.

Example 2.4.1.

Let us illustrate the actual computation of branching rules on the example of A_3 *orbit* $(2,0,1)$ *containing 12 points. We write the coordinates of the points as column vectors :*

$$\begin{aligned}
&\begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}-2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\-2\\3\end{pmatrix}, \begin{pmatrix}-2\\3\\-1\end{pmatrix}, \begin{pmatrix}3\\-1\\0\end{pmatrix},\\
&\begin{pmatrix}0\\1\\-3\end{pmatrix}, \begin{pmatrix}1\\-3\\2\end{pmatrix}, \begin{pmatrix}-3\\2\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\-2\end{pmatrix}, \begin{pmatrix}-1\\-2\\2\end{pmatrix}, \begin{pmatrix}-1\\0\\-2\end{pmatrix}.
\end{aligned} \tag{2.4.3}$$

Multiplying each of the points of (2.4.3) *by the matrix* (2.4.1), *one gets the points of the* $A_2 \times U_1$ *orbits written as column vectors. Rewriting them in the horizontal form and remembering that the first two coordinates belong to* A_2 *orbits, the third one belonging to* U_1, *we have the set of projected points. It remains to distribute the points into individual orbits. Practically it suffices to select just the dominant ones (no negative coordinates) because they represent the orbits that are present. Results are given by* (2.4.2), *where* $p = 2$ *and* $r = 1$.

2.4.1.4. $A_n \supset A_{n-1} \times U_1, \quad n \geq 2$

The cases listed in 2.4.1.2 and 2.4.1.3 are special cases of the present one.

$$\left(\begin{array}{ccccccc|c} & & & I_{n-1} & & & & \mathbf{0} \\ \hline 1 & 2 & 3 & \dots & & n-2 & n-1 & n \end{array}\right)$$

Note that, here and everywhere below, I_k denotes the $k \times k$ identity matrix and **0** represents the zero matrix of appropriate dimensions.

We give branching rules for this case for orbits of A_n of order $n+1$, $(n^2+n)/2$ and n^2+n respectively :

$$\begin{aligned} (p,0,0,\dots,0) &\supset (p,0,0,\dots,0)(p)+(0,\dots,0)(-np), \\ (0,q,0,\dots,0) &\supset (0,q,0,\dots,0)(2q)+(q,0,0,\dots,0)(-(n-1)q), \\ (p,0,\dots,0,r) &\supset (p,0,0,\dots,0)(p+nr)+(p,0,0,\dots,0,r)(p-r) \\ &\quad +(0,0,\dots,0,r)(-np-r). \end{aligned}$$

Note that, here and everywhere below, $p, q, r \in \mathbb{R}^{>0}$.

2.4.2. Orbit branching rules for $A_n \supset A_{n-k-1} \times A_k \times U_1$

All the cases so far can be viewed as the special cases of the present one where $k = 0$. Here we are considering the cases with general rank $n \geq 3$ and $1 \leq k \leq [\frac{n-1}{2}]$.

2.4.2.1. $A_3 \supset A_1 \times A_1 \times U_1$

The reduction is accomplished by applying to each element of the A_3 orbit the projection matrix $P = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ 1 & 2 & 1 \end{pmatrix}$, and by subsequently regrouping the results into orbits of $A_1 \times A_1 \times U_1$.

For all types of A_3 orbits, we find :

$$
\begin{aligned}
(p,0,0) &\supset (p)(0)(p)+(0)(p)(-p),\\
(0,q,0) &\supset (0)(0)(2q)+(0)(0)(-2q)+(q)(q)(0),\\
(0,0,r) &\supset (0)(r)(r)+(r)(0)(-r),\\
(p,q,0) &\supset (p)(0)(p+2q)+(p+q)(q)(p)+(q)(p+q)(-p)+(0)(p)(-p-2q),\\
(p,0,r) &\supset (p)(r)(p+r)+(p+r)(0)(p-r)+(0)(p+r)(r-p)+(r)(p)(-p-r),\\
(0,q,r) &\supset (0)(r)(2q+r)+(q)(q+r)(r)+(q+r)(q)(-r)+(r)(0)(-2q-r),\\
(p,q,r) &\supset (p)(r)(p+2q+r)+(p+q)(r+q)(p+r)+(p+q+r)(q)(p-r)\\
&\qquad +(q)(p+q+r)(r-p)+(q+r)(p+q)(-p-r)+(r)(p)(-p-2q-r).
\end{aligned}
$$

2.4.2.2. $A_4 \supset A_2 \times A_1 \times U_1$

In terms of compact Lie groups, this is the case frequently used in particle physics, namely $SU(5) \supset SU(3) \times SU(2) \times U(1)$. The reduction is accomplished by applying to each element of the A_4 orbit the projection matrix

$$
\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 2 & 4 & 6 & 3 \end{pmatrix}
$$

and by subsequently regrouping the results into orbits of $A_2 \times A_1 \times U_1$. For the following types of A_4 orbits, we find :

$$
\begin{aligned}
(p,0,0,0) &\supset (p,0)(0)(2p)+(0,0)(p)(-3p),\\
(0,q,0,0) &\supset (0,q)(0)(4q)+(q,0)(q)(-q)+(0,0)(0)(-6q),\\
(p,0,0,r) &\supset (p,0)(r)(2p+3r)+(p,r)(0)(2p-2r)\\
&\qquad +(0,0)(p+r)(3r-3p)+(0,r)(p)(-3p-2r).
\end{aligned}
$$

2.4.2.3. $A_n \supset A_{n-2} \times A_1 \times U_1$, *for odd* $n \geq 3$

The projection matrix is

$$\left(\begin{array}{c|c} I_{n-2} & \mathbf{0} \\ \hline \mathbf{0} & 0 \quad 1 \\ \hline \underbrace{1 \;\; 2 \;\; 3 \;\; 4 \cdots n{-}3 \;\; n{-}2}_{n-2} & \underbrace{n{-}1 \;\; \tfrac{n-1}{2}}_{2} \end{array}\right) \begin{array}{l} \}\, n-2 \\ \}\, 1 \\ \}\, 1 \end{array}$$

and some of the branching rules are

$$(p,0,0,\ldots,0) \supset (p,0,\ldots,0)(0)(p) + (0,\ldots,0)(p)(-\tfrac{n-1}{2}p),$$

$$(0,q,0,\ldots,0) \supset (0,q,0,\ldots,0)(0)(2q) + (q,0,0,\ldots,0)(q)(-\tfrac{n-3}{2}q)$$
$$+ (0,\ldots,0)(0)((1-n)q),$$

$$(p,0,\ldots,0,r) \supset (p,0,\ldots,0)(r)(p+\tfrac{n-1}{2}r) + (p,0,\ldots,0,r)(0)(p-r)$$
$$+ (0,\ldots,0)(p+r)((r-p)\tfrac{n-1}{2}) + (0,\ldots,0,r)(p)(-r-\tfrac{n-1}{2}p).$$

2.4.2.4. $A_n \supset A_{n-2} \times A_1 \times U_1$, *for even* $n \geq 4$

The projection matrix is

$$\left(\begin{array}{c|c} I_{n-2} & \mathbf{0} \\ \hline \mathbf{0} & 0 \quad 1 \\ \hline \underbrace{2 \;\; 4 \;\; 6 \;\; 8 \cdots 2(n{-}3) \;\; 2(n{-}2)}_{n-2} & \underbrace{2(n{-}1) \;\; n{-}1}_{2} \end{array}\right) \begin{array}{l} \}\, n-2 \\ \}\, 1 \\ \}\, 1 \end{array}$$

and some of the branching rules are

$$(p,0,0,\dots,0) \supset (p,0,\dots,0)(0)(2p)+(0,\dots,0)(p)((1-n)p),$$

$$(0,q,0,\dots,0) \supset (0,q,0,\dots,0)(0)(4q)+(q,0,\dots,0)(q)((3-n)q)$$
$$+(0,\dots,0)(0)(2(1-n)q),$$

$$(p,0,\dots,0,r) \supset (p,0,\dots,0)(r)(2p+(n-1)r)+(p,0,\dots,0,r)(0)(2(p-r))$$
$$+(0,\dots,0)(p+r)((n-1)(r-p))+(0,\dots,0,r)(p)(-2r-(n-1)p).$$

2.4.3. The general case $A_n \supset A_{n-k-1} \times A_k \times U_1$: $1 \leq k \leq [\frac{n-1}{2}]$

The branching rules of subsection 2.4.2 are important special cases of the general case. The projection matrix in the general case can be written as :

$$\left(\begin{array}{c|c|c}
\multicolumn{1}{c|}{I_{n-k-1}} & \multicolumn{2}{c}{\mathbf{0}} \\
\hline
\multicolumn{2}{c|}{\mathbf{0}} & I_k \\
\hline
\underbrace{k{+}1 \;\; 2(k{+}1) \cdots (n{-}k{-}1)(k{+}1)}_{n-k-1} & \underbrace{(n{-}k)(k{+}1)}_{1} & \underbrace{k(n{-}k) \cdots 2(n{-}k) \;\; n{-}k}_{k}
\end{array}\right)
\begin{array}{l}
\}\, n-k-1 \\
\}\, k \\
\}\, 1
\end{array}$$

$$(p,0,0,\dots,0)\supset(p,0,\dots,0)(0,\dots,0)((k+1)p)+(0,\dots,0)(p,0,\dots,0)((k-n)p),$$

$$(0,q,0,\dots,0) \supset (0,q,0,\dots,0)(0,\dots,0)(2(k+1)q) + (q,0,\dots,0)(q,0,\dots,0)((2k+1-n)q)$$
$$+ (0,\dots,0)(0,q,0,\dots,0)(2(k-n)q)\,,$$

$$(p,0,\dots,0,r)\supset(p,0,\dots,0)(0,\dots,0,r)((k+1)p+(n-k)r)$$
$$+(p,0,\dots,0,r)(0,\dots,0)((k+1)(p-r))$$
$$+(0,\dots,0)(p,0,\dots,0,r)((n-k)(r-p))$$
$$+(0,\dots,0,r)(p,0,\dots,0)((-k-1)r+(k-n)p).$$

2.5. BRANCHING RULES FOR MAXIMAL SEMISIMPLE SUBALGEBRAS OF A_n

The simple Lie algebras A_n contain no semisimple subalgebras of the same rank n. Hence all orbit branching rules considered in this section have rank strictly smaller than n. We proceed by increasing rank values until $n = 8$. Then we describe the infinite series involving the Weyl groups of classical Lie algebras, namely $W(A_{2n}) \supset W(B_n),\ n \geq 3$, $W(A_{2n-1}) \supset W(C_n),\ n \geq 2$, and $W(A_{2n-1}) \supset W(D_n),\ n \geq 4$.

We include the low-rank special cases of the three infinite series. We exclude the cases when a subalgebra is maximal among semisimple Lie algebras, but not among reductive algebras. Projection matrices for the latter cases are obtained by striking the last line of the corresponding matrices from the previous section.

2.5.1. Rank 2

There is only one case here, namely $A_2 \supset A_1$, which is often specified in terms of corresponding Lie groups either as $SU(3) \supset O(3)$, if the groups should be compact, or $Sl(3,\mathbb{C}) \supset O(3,\mathbb{C})$, if the groups have complex parameters. Their Weyl group orbits are the same. The projection matrix is $P = (2\ 2)$, so that we obtain the reductions :

$$(p,q) \supset (2p{+}2q){+}(2p){+}(2q), \quad (p,0) \supset (2p){+}(0), \quad (0,q) \supset (2q){+}(0). \tag{2.5.1}$$

Example 2.5.1.

Let us underline the geometrical content of the relations (2.5.1)*. On the left side, there are points in* $\mathbb{R}^2$ *given by their coordinates in* ω*-basis* $\{\omega_1,\ \omega_2\}$ *of* A_2*. The geometric relation of the two basis vectors is given by the* $2{\times}2$ *matrix of scalar products* $\langle\omega_j,\omega_k\rangle$*. In* A_n*, it happens to be the inverse* C^{-1} *of the Cartan matrix of the algebra. In particular, for* A_2*, we have* $C^{-1} = \frac{1}{3}\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$*. It follows that the basis vectors are of equal length* $\sqrt{2/3}$*, and that* $\angle(\omega_1,\omega_2) = 60^\circ$*.*

On the right side of (2.5.1)*, there are the* A_1 *orbit points in* $\mathbb{R}^1 \subset \mathbb{R}^2$*. Applying to* A_1 *the same rules as previously applied to* A_2*, we have* $C = (2)$ *so that* $C^{-1} = (1/2)$*. Thus the basis vector of* A_1*, say* ω*, has the length* $1/\sqrt{2}$*.*

It remains to clarify what are the relative positions of ω_1*,* ω_2*, and* ω*. The theory leaves us several options. A reasonable choice is built-in into the construction of the projection matrix in each case. Justification for this is outside the scope of this paper. For additional information, see* **[41]**. *However, the relative positions of basis vectors in* $\mathbb{R}^2$ *and* $\mathbb{R}^1$ *are established, for example, from*

$$P\omega_1 = (2\ 2)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = P\omega_2 = 2\omega.$$

Since equal-length vectors ω_1 *and* ω_2 *are projected into the same point on the* ω*-axis, the direction of* ω *divides the angle between* ω_1 *and* ω_2 *into equal parts.*

2.5.2. Rank 3

There are just two cases to consider. We write only their projection matrices.

$$A_3 \supset C_2 : \quad \left(\begin{smallmatrix} 1 & \cdot & 1 \\ \cdot & 1 & \cdot \end{smallmatrix}\right), \qquad A_3 \supset A_1 \times A_1 : \quad \left(\begin{smallmatrix} 1 & \cdot & 1 \\ 1 & 2 & 1 \end{smallmatrix}\right).$$

Example 2.5.2.

There are 12 points in (2.4.3)*. Let us transform them by the matrix* $\left(\begin{smallmatrix} 1 & \cdot & 1 \\ \cdot & 1 & \cdot \end{smallmatrix}\right)$*. Two dominant points are found when writing the projected points in horizontal form, namely* $(3, 0)$ *and* $(1, 1)$*. Hence we have the* $A_3 \supset C_2$ *rule* $(2, 0, 1) \supset (3, 0) + (1, 1)$*. The orbit* $(3, 0)$ *contains 4 points and the orbit* $(1, 1)$ *contains 8 points.*

Geometrically, $(2, 0, 1)$ *is a tetrahedron with four cut-off vertices. The planar figure after the projection is the union of the square* $(3, 0)$ *and the octagon* $(1, 1)$*.*

Let us underline the difference between the subalgebra $A_1 \times A_1$ here and the one in subsection 2.4.2.1. Using the corresponding projection matrices, we obtain respectively the reductions

$$(1, 0, 0) \supset (1)(1), \quad \text{and} \quad (1, 0, 0) \supset (1)(0)(1) + (0)(1)(-1).$$

Ignoring the contribution from U_1 in the second branching rule, the four orbit points obtained after the reduction are different in the two cases :

$$(1,0,0) \supset \{(1)(1),\ (-1)(1),\ (1)(-1),\ (-1)(-1)\},$$

$$(1,0,0) \supset \{(1)(0),\ (-1)(0),\ (0)(1),\ (0)(-1)\}.$$

There is an obvious subalgebra A_2 in A_3. Although it is maximal among semisimple subalgebras of A_3, it is not maximal among reductive subalgebras. It coincides with A_2 in subsection 2.4.1.3.

2.5.3. Rank 4

There is only one simple and maximal subalgebra of A_4 among the reductive subalgebras :

$$A_4 \supset C_2 : \quad \left(\begin{smallmatrix} \cdot & 2 & 2 & \cdot \\ 1 & \cdot & \cdot & 1 \end{smallmatrix}\right).$$

The other two semisimple subalgebras of rank 3 of A_4, namely A_3 and $A_1 \times A_2$, can be both extended by U_1 to maximal reductive subalgebras. They are the special cases $n = 4$ found in subsections 2.4.1.4 and 2.4.2.4 respectively.

Some branching rules :

$$(p,0,0,0) \supset (0,p)+(0,0),$$

$$(p,0,0,r) \supset (0,p+r)+(0,p)+(0,r)+(2r,p-r), \quad p > r,$$

$$(p,0,0,p) \supset (0,2p)+2(0,p)+2(2p,0).$$

2.5.4. Rank 5

There are four maximal subalgebras in this case. The first two are special cases of the general inclusions of subsection 2.5.8. The Lie algebras A_3 and D_3 coincide, except that by general convention we agreed not to consider the D_3 form. Therefore $A_5 \supset A_3$ can be read equivalently as $A_5 \supset D_3$, provided that we modify the order of point coordinates as follows : (a, b, c) of A_3 corresponds to (b, a, c) of D_3.

$$A_5 \supset A_3 : \quad \begin{pmatrix} \cdot & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & 2 & 1 & \cdot \end{pmatrix}, \qquad A_5 \supset C_3 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix},$$

$$A_5 \supset A_2 : \quad \begin{pmatrix} \cdot & 1 & 3 & 2 & 2 \\ 2 & 2 & \cdot & 1 & \cdot \end{pmatrix}, \qquad A_5 \supset A_1 \times A_2 : \quad \begin{pmatrix} 1 & \cdot & 1 & \cdot & 1 \\ 1 & 2 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 2 & 1 \end{pmatrix}.$$

In particular, the branching rules for the A_5 orbit of 6 points are :

$$(p, 0, 0, 0, 0) \supset \begin{cases} (0, p, 0) & \text{for} \quad A_3 \\ (p, 0, 0) & \text{for} \quad C_3 \\ (0, 2p) & \text{for} \quad C_2 \\ (p)(p, 0) & \text{for} \quad A_1 \times A_2 \end{cases} \qquad p \in \mathbb{R}^{>0}. \tag{2.5.2}$$

The first two are special cases of (2.5.5) and (2.5.4) respectively.

2.5.5. Rank 6

The only entry here is a special case of $A_{2n} \supset B_n$ of subsection 2.5.8, and its branching rules.

$$A_6 \supset B_3 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 2 & 2 & \cdot & \cdot \end{pmatrix}.$$

$$(p, 0, 0, 0, 0, 0) \supset (p, 0, 0) + (0, 0, 0),$$

$$(p, 0, 0, 0, 0, r) \supset (p+r, 0, 0) + (p, 0, 0) + (r, 0, 0) + (p-r, r, 0), \quad p > r,$$

$$(p, 0, 0, 0, 0, p) \supset (2p, 0, 0) + 2(p, 0, 0) + 2(0, p, 0).$$

2.5.6. Rank 7

The first two of the three cases are restrictions to $n = 7$ of the corresponding general cases of subsection 2.5.8.

$$A_7 \supset C_4 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}, \qquad A_7 \supset D_4 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 2 & 1 & \cdot & \cdot \end{pmatrix},$$

$$A_7 \supset A_1 \times A_3 : \quad \begin{pmatrix} 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 \\ 1 & 2 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 2 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 2 & 1 \end{pmatrix}.$$

In particular, for $A_7 \supset A_1 \times A_3$, we obtain

$$(p,0,0,0,0,0,0) \supset (p)(p,0,0),$$

$$(p,0,0,0,0,0,r) \supset (p{+}r)(p,0,r){+}(p{-}r)(p,0,r){+}(p{+}r)(p{-}r,0,0), \quad p > r,$$

$$(p,0,0,0,0,0,p) \supset (2p)(p,0,p){+}2(0)(p,0,p){+}4(2p)(0,0,0).$$

2.5.7. Rank 8

The first case is a special case of (2.5.3).

$$A_8 \supset B_4 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & 2 & \cdot & \cdot & \cdot \end{pmatrix}, \qquad A_8 \supset A_2 \times A_2 : \begin{pmatrix} 1 & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot \\ \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot & 1 \\ 1 & 2 & 1 & 2 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & 2 & 1 & 2 & 1 \end{pmatrix}.$$

Examples of the branching rules for the second case :

$$(p,0,0,0,0,0,0,0) \supset (p,0)(p,0),$$

$$(p,0,0,0,0,0,0,r) \supset (p,r)(p,r){+}(p{-}r,0)(p,r){+}(p,r)(p{-}r,0), \quad p > r,$$

$$(p,0,0,0,0,0,0,p) \supset (p,p)(p,p){+}3(0,0)(p,p){+}3(p,p)(0,0).$$

2.5.8. Three general rank cases

The cases are presented with examples of branching rules for the orbits $(p,0,\dots,0)$ and $(p,0,\dots,0,r)$, where the parameters p, r are strictly positive and real. We also assume that $p > r$. If $p < r$ the parameters p and r in the branching rule need to be interchanged. The case $p = r$ often needs to be listed separately.

$A_{2n} \supset B_n, \quad n \geq 3$

$$P = \left(\begin{array}{c|c|c} I_{n-1} & \mathbf{0} & E_{n-1} \\ \hline 0\cdots0 & 2\;2 & 0\cdots0 \end{array}\right) \tag{2.5.3}$$

Note that, here and everywhere below, E_k denotes the $k \times k$ matrix with units on the codiagonal.

$$(p,0,0,\dots,0) {\supset} (p,0,\dots,0){+}(0,\dots,0),$$

$$(p,0,\dots,0,r) {\supset} (p{+}r,0,\dots,0){+}(p,0,\dots,0){+}(r,0,\dots,0){+}(p{-}r,r,0,\dots,0),$$

$$(p,0,\dots,0,p) {\supset} (2p,0,\dots,0){+}2(p,0,\dots,0){+}2(0,p,0,\dots,0).$$

$A_{2n-1} \supset C_n, \quad n \geq 2$

$$P = \left(\begin{array}{c|c|c} I_{n-1} & \mathbf{0} & E_{n-1} \\ \hline 0\cdots0 & 1 & 0\cdots0 \end{array}\right) \tag{2.5.4}$$

$$(p,0,0,\ldots,0) \supset (p,0,\ldots,0),$$
$$(p,0,\ldots,0,r) \supset (p+r,0,\ldots,0)+(p-r,r,0,\ldots,0),$$
$$(p,0,\ldots,0,p) \supset (2p,0,\ldots,0)+2(0,p,0,\ldots,0).$$

$A_{2n-1} \supset D_n, \quad n \geq 4$

$$P = \left(\begin{array}{c|c|c} I_{n-1} & \mathbf{0} & E_{n-1} \\ \hline 0\cdots0\,1 & 2 & 1\,0\cdots0 \end{array}\right) \tag{2.5.5}$$

$$(p,0,0,\ldots,0) \supset (p,0,\ldots,0),$$
$$(p,0,\ldots,0,r) \supset (p+r,0,\ldots,0)+(p-r,r,0,\ldots,0),$$
$$(p,0,\ldots,0,p) \supset (2p,0,\ldots,0)+2(0,p,0,\ldots,0).$$

2.6. CONCLUDING REMARKS

- The pairs $W(L) \supset W(L')$ in this paper involve a maximal subalgebra L' in L. A chain of maximal subalgebras linking L and any of its reductive non-maximal subalgebras L'' can be found. Corresponding projection matrices combine, by the common matrix multiplication, into the projection matrix for $W(L) \supset W(L'')$.
- Projection matrices of section 2.4 are square matrices with determinant different from zero. Hence they can be inverted and used in the opposite direction, as discussed in **[11]**. The inverse matrix transforms an orbit of $W(L')$ into the linear combination of orbits of $W(L)$, where $L' \subset L$. The linear combination has integer coefficients of both signs in general. We know of no interpretation of such 'branching rules' in applied literature,

although they have their place in the Grothendieck rings of representations.

– Curious and completely unexplored relations between pairs of maximal subalgebras, say L' and L'', of the same Lie algebra L can be found by combining the projection matrices $P(L \to L')$ and $P(L \to L'')$ as

$$P(L' \to L'') = P(L \to L'')P^{-1}(L \to L').$$

– The index of a semisimple subalgebra in a simple Lie algebra is an invariant of all branching rules for a fixed algebra-subalgebra pair. It was introduced in [**14**], see Equation (2.26). It is an invariant also for any pair $W(L) \supset W(L')$.

– Congruence classes of representations are naturally extended to congruence classes of W-orbits [**21**]. Comparing the congruence classes of orbits for $W(L) \supset W(L')$ reveals that not all combinations of congruence classes are present. A relative congruence class is a valid and useful concept which deserves investigation. Incidentally, relative congruence classes are studied in chapter 5 [**33**] of this thesis.

– Here, the relations between orbits were defined by the classification of maximal reductive subalgebras in simple type A_n Lie algebras. There exists another relation between such algebras that is not an homomorphism. It is called subjoining [**53, 63**]. Consider an example. The 4-dimensional representation $(1,0,0)$ of A_3 does *not* contain the 5-dimensional representation $(0,1)$ of C_2. In spite of that, the projection matrix that maps the highest weight orbit of A_3 to the orbit $(0,1)$ of C_2 can be obtained. Indeed, that projection matrix is $\left(\begin{smallmatrix}0&2&0\\1&0&1\end{smallmatrix}\right)$. Classification of maximal subjoinings in simple Lie algebras is found in [**53**].

Acknowledgements

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada and by the MIND Research Institute. We are grateful for the hospitality extended to M. N. at the Centre de recherches mathématiques, Université de Montréal, where the work was done.

Chapitre 3

BRANCHING RULES FOR WEYL GROUP ORBITS OF SIMPLE LIE ALGEBRAS B_N, C_N AND D_N

Référence complète : M. Larouche et J. Patera, Branching rules for Weyl group orbits of simple Lie algebras B_n, C_n and D_n, *Journal of Physics A : Mathematical and Theoretical*, 44(11) : 115203, 37, 2011. Sélectionné pour être inclus dans *IOP Select*, Institute of Physics, United Kingdom.

Résumé

Les orbites des groupes de Weyl $W(B_n)$, $W(C_n)$ et $W(D_n)$ des algèbres de Lie simples B_n, C_n et D_n sont réduites en l'union d'orbites des groupes de Weyl des sous-algèbres réductives maximales de B_n, C_n et D_n. Les matrices qui transforment les points des orbites de $W(B_n)$, de $W(C_n)$ et de $W(D_n)$ en des points des orbites des sous-algèbres sont listées pour tous les cas $n \leq 8$ ainsi que pour les séries infinies des paires d'algèbre-sous-algèbre suivantes : $B_n \supset B_{n-1} \times U_1$, $B_n \supset D_n$, $B_n \supset B_{n-k} \times D_k$, $B_n \supset A_1$, $C_n \supset C_{n-k} \times C_k$, $C_n \supset A_{n-1} \times U_1$, $C_n \supset A_1$, $D_n \supset A_{n-1} \times U_1$, $D_n \supset D_{n-1} \times U_1$, $D_n \supset B_{n-1}$, $D_n \supset B_{n-k-1} \times B_k$, $D_n \supset D_{n-k} \times D_k$. De nombreux cas spéciaux sont inclus et plusieurs exemples sont présentés.

Abstract

The orbits of Weyl groups $W(B_n)$, $W(C_n)$ and $W(D_n)$ of the simple Lie algebras B_n, C_n and D_n are reduced to the union of the orbits of Weyl groups of the maximal reductive subalgebras of B_n, C_n and D_n. Matrices transforming points of $W(B_n)$, $W(C_n)$ and $W(D_n)$ orbits into points of subalgebra orbits are

listed for all cases $n \le 8$ and for the infinite series of algebra-subalgebra pairs : $B_n \supset B_{n-1} \times U_1$, $B_n \supset D_n$, $B_n \supset B_{n-k} \times D_k$, $B_n \supset A_1$, $C_n \supset C_{n-k} \times C_k$, $C_n \supset A_{n-1} \times U_1$, $C_n \supset A_1$, $D_n \supset A_{n-1} \times U_1$, $D_n \supset D_{n-1} \times U_1$, $D_n \supset B_{n-1}$, $D_n \supset B_{n-k-1} \times B_k$, $D_n \supset D_{n-k} \times D_k$. Numerous special cases and examples are shown.

3.1. INTRODUCTION

This paper is a continuation of [**34**], in which the analogous problem for Lie algebras A_n of the special linear group $SL(n+1, \mathbb{C})$ was considered. Here the problem is considered for simple Lie algebras B_n and D_n of orthogonal groups $O(2n+1)$ and $O(2n)$ respectively, and for the simple Lie algebra C_n of the symplectic group $Sp(2n)$.

The motivation for the present paper is the same as in [**34**]. There are four important points to note : firstly, orbit branching rules are implicitly required for the computation of branching rules of representations of the same Lie algebra-subalgebra pairs. Hence, projection matrices, an essential part of the method in [**34**], are used as the main tool in the paper. Secondly, it turns out that, for any extensive computation with finite-dimensional representations of simple Lie algebras such as branching rules, the decomposition of tensor products of representations, or discrete Fourier analysis, it is impracticable to avoid decomposing the problem into several subproblems for orbits involved. This is because the dimensions of representations increase without bound, while Weyl group orbits are of finite size in all cases, their size always being a divisor of the order of the corresponding Weyl group. Thirdly, an important property as yet unexploited in applications is the fact that Weyl group orbit points do not need to belong to a lattice. Weyl group orbits that are not on the corresponding weight lattice retain most of the valuable properties of orbits that are on the lattice. In particular, branching rules remain valid even if the coordinates of the orbit points are irrational numbers. Recent interest in special functions defined by Weyl group orbits [**29, 30**] is based on knowledge of orbit properties. Branching rules for orbit functions can be extended to branching rules for polynomials [**54**]. Finally, it should also be noted that Lie algebras of type B_n, C_n and D_n are amenable for a different choice of basis than that used in this paper, namely the orthonormal basis. For some problems, this choice may offer a simplifying advantage in terms of computation. We refrain from using it here in favour of the non-orthogonal root and weight bases, because these offer a remarkable uniformity of computation methods for semisimple Lie algebras of all types.

The paper contains projection matrices for all cases of maximal inclusion for Lie algebras of types B_n, C_n, and D_n for ranks $n \leq 8$, with examples of branching rules for specific orbits. In addition, projection matrices and examples of branching rules for infinite series of selected cases are given. Included are all cases where a maximal reductive subalgebra is of the same rank as B_n, C_n, and D_n.

Branching rules for Weyl group orbits of exceptional simple Lie algebras E_6, E_7, E_8, F_4, and G_2 are found in [**40**] among many other results.

The branching rules for $W(L) \supset W(L')$, where L' is a maximal reductive subalgebra of L, is a linear transformation between Euclidean spaces $\mathbb{R}^n \to \mathbb{R}^{n'}$, where n and n' are the ranks of L and L' respectively. The branching rules are unique, unlike transformations of individual orbit points, which depend on the relative choice of bases. We provide the linear transformation in the form of an $n' \times n$ matrix, the *projection matrix*. A suitable choice of bases allows one to obtain integer matrix elements in all the projection matrices listed here. Note that we use Dynkin notations and numberings for roots, weights and diagrams.

The method we use here is an extension of the method used in [**39, 40, 41, 61**] for the computation of reductions of representations of simple Lie algebras to representations of their maximal semisimple subalgebras. Orbit-orbit branching rules have been discussed for one of the first times in the literature in [**40**]. They were then addressed in [**18, 67, 68**], where specific methods were developed for different algebra-subalgebra pairs. The main advantage of the projection matrix method is its uniformity, as it can be used for any algebra-subalgebra pair. We include here, as we did in [**34**], all the cases when the maximal reductive subalgebra is non-semisimple, i.e when it contains the 1-parametric ideal denoted here U_1.

It should be underlined that each of the numerous examples of orbit branching rules shown here is valid for an infinity of cases. For example, an orbit labeled by $(a, 0, \ldots, 0)$, refers to an uncountable number of orbits with $0 < a \in \mathbb{R}$. Orbits that do not belong to a weight lattice should be of importance in Fourier analysis when considering Fourier integrals rather than Fourier series.

The number attached to each representation of a simple Lie algebra and called the second degree index is an invariant of the representation which has been occasionally used in applications [**66**]. Its useful properties remain valid also for Weyl group orbits. The index of a semisimple subalgebra in a simple Lie algebra is an invariant of all branching rules for a fixed algebra-subalgebra pair. It was introduced in [**14**], see Equation (2.26). It is defined using the second degree indices of representations. We give its value for all our cases, but its properties would merit further investigation, particularly when the orbit points are off the weight lattices.

3.2. Preliminaries

Finite groups generated by reflections in an n-dimensional real Euclidean space $\mathbb{R}^n$ are commonly known as finite Coxeter groups [**23**]. Finite Coxeter groups are split into two classes : crystallographic and non crystallographic groups. Crystallographic groups are often referred to as Weyl groups of semisimple Lie groups or Lie algebras. In $\mathbb{R}^n$ they are the symmetry groups of root lattices of the simple Lie groups. There are four infinite series (as to the admissible values of rank n) of such groups, namely A_n, B_n, C_n, D_n, and five isolated exceptional groups of ranks 2, 4, 6, 7, and 8. The non crystallographic finite Coxeter groups are the symmetry groups of regular 2D polygons (the dihedral groups), with two exceptional groups, one of rank 3 – the icosahedral group of order 120 – and one of rank 4, which is of order 120^2.

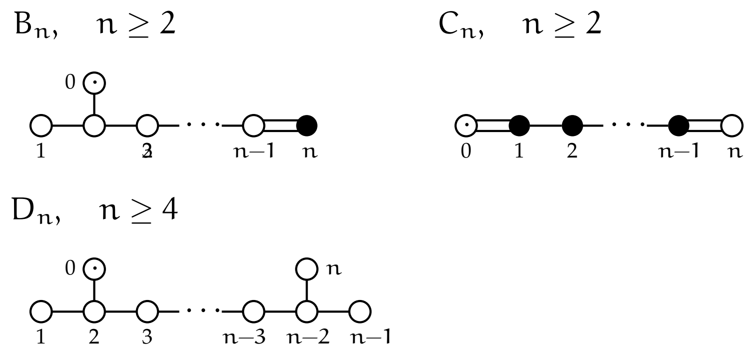


FIGURE 3.1. The Coxeter-Dynkin diagrams of types B_n, C_n and D_n. *The circular nodes stand for the simple roots, with the convention that open (resp. filled) circles indicate long (resp. short) roots. The dotted node is the negative highest root denoted α_0. A link between a pair of roots indicates that the roots are not orthogonal. The Dynkin numbering of the nodes is shown.*

We consider orbits of the Weyl groups $W(B_n)$, $W(C_n)$ and $W(D_n)$ of the simple Lie algebras of type B_n, $n \geq 2$, C_n, $n \geq 2$ and D_n, $n \geq 4$, respectively (Fig. 3.1). The order of such Weyl groups is $2^n n!$ for $W(B_n)$ and $W(C_n)$, while it is $2^{n-1}n!$ for $W(D_n)$. An orbit W_λ of the Weyl group $W(L)$, where L is of rank n, is a finite set of distinct points in $\mathbb{R}^n$, all equidistant from the origin, obtained from a single point $\lambda \in \mathbb{R}^n$ by application of W to λ. Hence, an orbit of $W(B_n)$ or $W(C_n)$ contains at most $2^n n!$ points, and an orbit W_λ of $W(D_n)$ contains at most $2^{n-1}n!$ points.

Consider the pair $W(L) \supset W(L')$, where L' is a maximal reductive subalgebra of a simple Lie algebra L. The orbit reduction is a linear transformation $\mathbb{R}^n \to \mathbb{R}^{n'}$, where n' is the rank of L'. Hence the orbit reduction problem is solved when the $n' \times n$ matrix P is found with the property that points of any orbit of $W(L)$ are projected by P into points of the corresponding orbits of $W(L')$. Computation of the branching rule for a specific orbit of $W(L)$ amounts to applying P to the points of the orbit, and to sorting out the projected points into a sum (union) of orbits of $W(L')$.

Typically the result of the reduction of an orbit W_λ of $W(L)$ is a union of several orbits of $W(L')$. Geometrically the points of W_λ can be understood as vertices of a polytope in $\mathbb{R}^n$. A union of several obits is then an onion-like formation of concentric polytopes **[21]**.

The projection matrix P is calculated from one known branching rule. The classification of maximal reductive subalgebras of simple Lie algebras **[14, 4]** provides the information to find that branching rule. The projection matrix is then obtained using the weight systems of the representations, by requiring that weights of L be transformed by P to weights of L'. Since any ordering of the weights is admissible, the projection matrix is not unique. We choose the natural lexicographical ordering of the weights. The projection matrix obtained can then be used to project points of any orbit of $W(L)$ into points of orbits of $W(L')$. At the end of this section, we consider an example of the construction of a projection matrix for the case $W(B_3) \supset W(G_2)$.

To compute the branching rule for a specific orbit of $W(L)$, all the points of that orbit are listed and then multiplied by the projection matrix. A standard method to calculate points of an orbit of any finite Coxeter group is given in **[21]**, where the points are given in the corresponding basis of fundamental weights, called the ω-basis. All of the orbits appearing here are given in the ω-basis of the corresponding group, linked to the basis of simple roots by the Cartan matrix of the group. Since every orbit contains precisely one point with nonnegative coordinates in the ω-basis, the orbit can be identified by that point, called the dominant point of the orbit. Hence when referring to an orbit, one does not have to list all of the points it contains. The example at the end of this section illustrates the actual computation of branching rules for the case $W(B_3) \supset W(G_2)$.

The Weyl group of the one-parameter Lie algebra U_1 is trivial, consisting of the identity element only. Its irreducible representations are all 1-dimensional, hence its orbits consist of one element. They are labeled by integers, which can also take negative values. The symbol (k) may stand for either the orbit $\{k, -k\}$ of $W(A_1)$, or for the $W(U_1)$ orbit of one point $\{k\}$. Distinction should be made from the context. Since we are working with orbits of the Weyl group of U_1 and the compactness of the Lie group is of no interest to us here, we can allow the orbits of $W(U_1)$ to take real values.

The second degree index for weight systems of irreducible finite dimensional representations of compact semisimple Lie groups was defined in [**64**]. It was then introduced for individual orbits in [**21**]. The second degree index $I^{(2)}_\lambda$ of the orbit W_λ is

$$I^{(2)}_\lambda = \sum_{\mu \in W_\lambda} (\mu|\mu) = (\lambda|\lambda)|W_\lambda| \,,$$

where $|W_\lambda|$ is the size of the orbit and $(\cdot|\cdot)$ is the standard inner product of $\mathbb{R}^n$. The second equality comes from the fact that all points of W_λ are equidistant from the origin. If W_{λ_1} and W_{λ_2} are two orbits of W, then the index of their sum (or union) and the index of their product are given by

$$\begin{aligned} I^{(2)}_{\lambda_1+\lambda_2} &= I^{(2)}_{\lambda_1} + I^{(2)}_{\lambda_2} \\ I^{(2)}_{\lambda_1\times\lambda_2} &= I^{(2)}_{\lambda_1}|W_{\lambda_2}| + I^{(2)}_{\lambda_2}|W_{\lambda_1}| \qquad (3.2.1) \\ &= |W_{\lambda_1}||W_{\lambda_2}|\left((\lambda_1|\lambda_1) + (\lambda_2|\lambda_2)\right) \,. \qquad (3.2.2) \end{aligned}$$

Simple calculations show that if $W^1_{\lambda_1}$ and $W^2_{\lambda_2}$ are two orbits of two different Weyl groups W^1 and W^2, the second degree index of the orbit $\lambda_1\times\lambda_2$ of $W^1\times W^2$ is also given by (1) and (2).

For a fixed pair $W(L) \supset W(L')$ of Weyl groups of an algebra L and its semisimple subalgebra L', the ratio of second degree indices is invariant and is called the index of L' in L. For any orbit $W(L)_\lambda$ reduced to the sum of orbits $\sum_\mu W(L')_\mu$, there exists a positive number $\gamma = \gamma_{L,L'}$ such that

$$I^{(2)}_\lambda = \gamma_{L,L'} \sum_\mu I^{(2)}_\mu \,.$$

We give that number $\gamma_{L,L'}$ for all such pairs of Weyl groups $W(L) \supset W(L')$.

To alleviate notation, we will simply write L instead of $W(L)$ to refer to the Weyl group of the Lie algebra L, and λ instead of W_λ to refer to the orbit of the dominant point λ of the Weyl group W. Subsequently dots in a matrix denote zero matrix elements.

Let us finally consider an example to illustrate how to construct a projection matrix and how to calculate a particular branching rule.

Example 3.2.1.

Consider the case of $B_3 \supset G_2$ *of subsection 3.3.2. From the classification of maximal reductive subalgebras, we know that the lowest orbit of* B_3*, the orbit of the dominant point* $(1, 0, 0)$*, contains 6 points and is projected onto the* G_2*-orbit of the point* $(0, 1)$*, that also contains 6 points. We order the points of the two orbits, and require that points of the first one be transformed into points of the second one in the following manner :*

$$
\begin{aligned}
&(1,0,0) \mapsto (0,1), \qquad &&(-1,1,0) \mapsto (1,-1), \qquad &&(0,-1,2) \mapsto (-1,2),\\
&(0,1,-2) \mapsto (1,-2), \qquad &&(1,-1,0) \mapsto (-1,1), \qquad &&(-1,0,0) \mapsto (0,-1).
\end{aligned}
$$

Writing the points as column matrices, the projection matrix of subsection 3.3.2 is obtained from the first three. Proceeding one column at a time, we have

$$
\begin{pmatrix} 0 & * & * \\ 1 & * & * \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & * \\ 1 & 0 & * \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix},
$$

where stars denote the entries that are still to be determined. The matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ *then automatically transforms the three last points of the* B_3*-orbit as required. This matrix can then be used for projecting points of any* B_3*-orbit. For example, to calculate the reduction of the* B_3*-orbit of* $(0, 2, 0)$*, one has to write the coordinates of the 12 points of the orbit as column vectors :*

$$
\begin{gathered}
\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix},\\
\begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}.
\end{gathered}
\tag{3.2.3}
$$

Multiplying each of the points of (3.2.3) *by the matrix* P*, one gets the points of the* G_2*-orbits written as column vectors. Rewriting them in the horizontal form, we have the set of projected points. To distribute the points into individual orbits, one only has to select the dominant points (no negative coordinates) because they represent the orbits that are present. Hence one gets the following branching rule for that case :*

$$
(0, 2, 0) \supset (2, 0) + (0, 2)\,.
$$

3.3. Reduction of orbits of the Weyl group of B_n

In this section we first consider all cases of dimension (rank of the Lie algebra) up to 8. In the last subsection, 3.3.8, we present infinite series of cases which occur for all values of rank starting from a lowest one. For each case, the projection matrix is given, together with examples of the corresponding reductions/branching rules. For cases involving Weyl groups of a simple algebra L and a maximal reductive semisimple algebra L', we provide the index $\gamma = \gamma_{L,L'}$ of L' in L.

3.3.1. Rank 2

The Lie algebras B_2 and C_2 and their Weyl groups are isomorphic. A practical difference between the two cases is in our numbering convention of simple roots (Fig. 1). In this subsection we work with B_2.

The branching rules for the case $B_2 \supset A_1 \times U_1$ are determined by the projection matrix $\left(\begin{smallmatrix} 2 & 1 \\ \cdot & 1 \end{smallmatrix}\right)$. In particular, for the two lowest orbits each containing 4 points, we have $(1,0) \supset (2)(0) + (0)(2) + (0)(-2)$ and $(0,1) \supset (1)(1) + (1)(-1)$. More generally :

$$
\begin{aligned}
&(a,0) \supset (2a)(0) + (0)(2a) + (0)(-2a)\,, \\
&(0,b) \supset (b)(b) + (b)(-b)\,, \qquad\qquad a, b \in \mathbb{R}^{>0} \\
&(a,b) \supset (2a{+}b)(b) + (2a{+}b)(-b) + (b)(2a{+}b) + (b)(-2a{-}b)\,.
\end{aligned}
$$

Note that the corresponding branching rules for irreducible representations are different in all cases but $(0,1)$.

The maximal subalgebra $A_1 \subset B_2$ is different than the subalgebra A_1 in $A_1 \times U_1 \subset B_2$. Indeed, the projection matrix for the case $B_2 \supset A_1$ is $\left(\begin{smallmatrix} 4 & 3 \end{smallmatrix}\right)$ and yields the following branching rules for the orbits :

$$
\left.
\begin{aligned}
&(a,0) \supset (4a) + (2a)\,, \\
&(0,b) \supset (3b) + (b)\,, \\
&(a,b) \supset (4a{+}3b) + (2a{+}3b) + (4a{+}b) + (|2a{-}b|)\,,
\end{aligned}
\right\} \quad a, b \in \mathbb{R}^{>0}
$$

$$
(a,2a) \supset (10a) + (8a) + (6a) + 2(0)\,.
$$

The index of A_1 in B_2 is $\gamma = \gamma_{B_2,A_1} = 1/5$.

For the $B_2 \supset 2A_1$ case, the projection matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & \cdot \end{smallmatrix}\right)$ applied to the three non zero orbits gives the following branching rules :

$$\begin{aligned} &(a,0) \supset (a)(a)\,, \\ &(0,b) \supset (b)(0) + (0)(b)\,, \qquad\qquad a,b \in \mathbb{R}^{>0} \\ &(a,b) \supset (a{+}b)(a) + (a)(a{+}b)\,. \end{aligned}$$

The index of $2A_1$ in B_2 is $\gamma = \gamma_{B_2,2A_1} = 1$.

Note that in all cases the branching rules hold even if a and b are not integers.

3.3.2. Rank 3

There are four cases to consider. The first one is a special case of the general case of subsection 3.3.8.1, except that it implies a renumbering of simple roots $C_2 \to B_2$ and a corresponding rearrangement of the projection matrix.

$$B_3 \supset C_2 \times U_1 : \begin{pmatrix} \cdot & 2 & 1 \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}\,, \qquad B_3 \supset A_3 : \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & 1 \end{pmatrix}\,,$$
$$B_3 \supset G_2 : \left(\begin{smallmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & 1 \end{smallmatrix}\right)\,, \qquad B_3 \supset 3A_1 : \begin{pmatrix} 1 & 1 & \cdot \\ 1 & 1 & 1 \\ \cdot & 2 & 1 \end{pmatrix}\,.$$

As an example, we give the branching rules for the orbits of B_3 of size 6, 12, 8 and 48 respectively. We also give the index $\gamma = \gamma_{L,L'}$ whenever L' is semisimple.

$B_3 \supset C_2 \times U_1$:

$$\begin{aligned} &(a,0,0) \supset (0,a)(0) + (0,0)(2a) + (0,0)(-2a)\,, \\ &(0,b,0) \supset (2b,0)(0) + (0,b)(2b) + (0,b)(-2b)\,, \\ &(0,0,c) \supset (c,0)(c) + (c,0)(-c)\,, \\ &(a,b,c) \supset (2b{+}c,a)(c) + (2b{+}c,a)(-c) + (c,a{+}b)(2b{+}c) + (c,a{+}b)(-2b{-}c) \\ &\qquad\qquad + (c,b)(2a{+}2b{+}c) + (c,b)(-2a{-}2b{-}c)\,, \end{aligned}$$

$B_3 \supset A_3$:

$$(a,0,0) \supset (0,a,0)\,,$$

$$(0,b,0) \supset (b,0,b)\,,$$

$$(0,0,c) \supset (0,0,c) + (c,0,0)\,,$$

$$(a,b,c) \supset (b,a,b{+}c) + (b{+}c,a,b)\,,$$

$$\gamma = 1\,,$$

$B_3 \supset G_2$:

$$(a,0,0) \supset (0,a)\,,$$

$$(0,b,0) \supset (b,0) + (0,b)\,,$$

$$(0,0,c) \supset (0,c) + 2(0,0)\,,$$

$$(a,a,a) \supset (a,2a) + 2(2a,0) + (a,a) + 2(a,0)\,,$$

$$(a,b,a) \supset (b,2a) + 2(a{+}b,0) + (a,b) + \begin{cases} (a,b{-}a) & \text{if } a < b \\ (b,a{-}b) & \text{if } a > b \end{cases},$$

$$(a,a,c) \supset (a,a{+}c) + (a,c) + 2(a,0) + \begin{cases} (2a,c{-}a) & \text{if } a < c \\ (a{+}c,a{-}c) & \text{if } a > c \end{cases},$$

$$(a,b,c) \supset (b,a{+}c) + \begin{cases} (a{+}b,c{-}a) & \text{if } a < c \\ (b{+}c,a{-}c) & \text{if } a > c \end{cases} + \begin{cases} (a,b{-}a) & \text{if } a < b \\ (b,a{-}b) & \text{if } a > b \end{cases}$$
$$+ \begin{cases} (a,b{+}c{-}a) & \text{if } a < b{+}c \\ (b{+}c,a{-}b{-}c) & \text{if } a > b{+}c \end{cases},$$

$$\gamma = 3/2\,,$$

$B_3 \supset 3A_1$:

$$(a,0,0) \supset (a)(a)(0) + (0)(0)(2a)\,,$$

$$(0,b,0) \supset (b)(b)(2b) + (2b)(0)(0) + (0)(2b)(0)\,,$$

$$(0,0,c) \supset (0)(c)(c) + (c)(0)(c)\,,$$

$$\begin{aligned}(a,b,c) \supset\ & (a{+}b)(a{+}b{+}c)(2b{+}c) + (b)(b{+}c)(2a{+}2b{+}c) + (a)(a{+}2b{+}c)(c) \\ & + (a{+}b{+}c)(a{+}b)(2b{+}c) + (b{+}c)(b)(2a{+}2b{+}c) + (a{+}2b{+}c)(a)(c)\,,\end{aligned}$$

$$\gamma = 3/4\,,$$

where $a, b, c \in \mathbb{R}^{>0}$.

B_3 does not contain the principal 3-dimensional subalgebra A_1 as a maximal subalgebra. The corresponding A_1 occurs in the exceptional chain $B_3 \supset G_2 \supset A_1$. Hence the reduction from $B_3 \supset A_1$ has to be done by multiplying the projection matrices for $B_3 \supset G_2$ and $G_2 \supset A_1$, namely :

$$\begin{pmatrix} 10 & 6 \end{pmatrix} \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & 1 \end{pmatrix} = \begin{pmatrix} 6 & 10 & 6 \end{pmatrix}\,.$$

The projection matrix obtained is the same as the one we would get from the matrix (3.3.1) of the subsection 3.3.8.8 with $n = 3$.

3.3.3. Rank 4

There are six cases to consider. The first two are special cases of the general rank of B_n in subsections 3.3.8.1 and 3.3.8.2 respectively. The next two, $B_4 \supset A_3 \times A_1$ and $B_4 \supset C_2 \times 2A_1$, are also special cases of subsections 3.3.8.3 and 3.3.8.4 respectively, except that they imply a renumbering of simple roots, $A_3 \to D_3$ and $C_2 \to B_2$, and a corresponding rearrangement of the projection matrices. The projection matrix and one example of branching rule in the case of the principal 3-dimensional subalgebra are given for the general rank, $B_n \supset A_1$, in subsection 3.3.8.8.

$$B_4 \supset B_3 \times U_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad B_4 \supset D_4 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & 1 \end{pmatrix}, \quad B_4 \supset A_3 \times A_1 : \begin{pmatrix} \cdot & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & 1 \\ \cdot & \cdot & 2 & 1 \end{pmatrix},$$

$$B_4 \supset C_2 \times 2A_1 : \begin{pmatrix} \cdot & \cdot & 2 & 1 \\ 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & 1 \\ \cdot & 1 & 1 & \cdot \end{pmatrix}, \quad B_4 \supset A_1 : \begin{pmatrix} 8 & 14 & 18 & 10 \end{pmatrix}, \quad B_4 \supset 2A_1 : \begin{pmatrix} 2 & 2 & 4 & 1 \\ 2 & 4 & 4 & 3 \end{pmatrix}.$$

We bring here some examples of branching rules for the $B_4 \supset A_1$ and $B_4 \supset 2A_1$ cases, for orbits of size 8, 24, 32 and 16 respectively, together with their corresponding indices γ.

$$
\begin{aligned}
& B_4 \supset A_1 : \\
& \quad (a,0,0,0) \supset (8a) + (6a) + (4a) + (2a)\,, \\
& \quad (0,b,0,0) \supset (14b) + (12b) + 2(10b) + (8b) + 2(6b) + 2(4b) + 3(2b)\,, \\
& \quad (0,0,c,0) \supset (18c) + (16c) + (14c) + 2(12c) + 2(10c) + (8c) + 2(6c) \\
& \qquad\qquad\qquad + 2(4c) + 2(2c) + 4(0)\,, \\
& \quad (0,0,0,d) \supset (10d) + (8d) + (6d) + 2(4d) + 2(2d) + 2(0)\,, \\
& \quad \gamma = 1/15\,, \\
& B_4 \supset 2A_1 : \\
& \quad (a,0,0,0) \supset (2a)(2a) + (0)(2a) + (2a)(0)\,, \\
& \quad (0,b,0,0) \supset (2b)(4b) + (4b)(2b) + (2b)(2b) + (0)(4b) + (4b)(0) \\
& \qquad\qquad\qquad + 2(0)(2b) + 2(2b)(0)\,, \\
& \quad (0,0,c,0) \supset (4c)(4c) + (0)(6c) + (6c)(0) + (2c)(4c) + (4c)(2c) \\
& \qquad\qquad\qquad + 2(0)(4c) + 2(4c)(0) + (0)(2c) + (2c)(0) + 4(0)(0)\,, \\
& \quad (0,0,0,d) \supset (d)(3d) + (3d)(d) + 2(d)(d)\,, \\
& \quad \gamma = 1/3\,,
\end{aligned}
$$

where $a,b,c,d \in \mathbb{R}^{>0}$.

For cases of rank 5 to 8, we give the projection matrices which are all, except for the $B_7 \supset A_3$ and $B_7 \supset C_2 \times A_1$ ones, special cases of the general rank section. We refrain to give the branching rules here, except for the $B_7 \supset A_3$ and $B_7 \supset C_2 \times A_1$ cases, since they can easily be found in the general rank section, with maximally a minor renumbering of simple roots ($A_3 \to D_3$ and $C_2 \to B_2$).

3.3.4. Rank 5

We give the projection matrices for the six cases to consider. Examples of branching rules can be found in the corresponding subsections of the general rank section 3.3.8.

$$B_5 \supset B_4 \times U_1 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \qquad B_5 \supset D_5 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 \end{pmatrix},$$

$$B_5 \supset B_3 \times 2A_1 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & 1 & 1 \end{pmatrix}, \qquad B_5 \supset D_4 \times A_1 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & 2 & 1 \end{pmatrix},$$

$$B_5 \supset A_3 \times C_2 : \quad \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & 1 & 1 & \cdot & \cdot \end{pmatrix}, \qquad B_5 \supset A_1 : \quad \begin{pmatrix} 10 & 18 & 24 & 28 & 15 \end{pmatrix}.$$

3.3.5. Rank 6

We give the projection matrices for the seven cases to consider. Examples of branching rules can be found in the corresponding subsections of the general rank section 3.3.8.

$$B_6 \supset B_5 \times U_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \qquad B_6 \supset D_6 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \end{pmatrix},$$

$$B_6 \supset B_4 \times 2A_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & 1 \end{pmatrix}, \qquad B_6 \supset D_5 \times A_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 2 & 1 \end{pmatrix},$$

$$B_6 \supset B_3 \times A_3 : \begin{pmatrix} 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & 1 \end{pmatrix}, \qquad B_6 \supset D_4 \times C_2 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot \end{pmatrix},$$

$$B_6 \supset A_1 : \begin{pmatrix} 12 & 22 & 30 & 36 & 40 & 21 \end{pmatrix}.$$

3.3.6. Rank 7

We give the projection matrices of the ten cases to consider. Examples of branching rules for the first eight cases can be found in the corresponding subsections of the general rank section 3.3.8.

$$B_7 \supset B_6 \times U_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \qquad B_7 \supset D_7 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \end{pmatrix},$$

$$B_7 \supset D_6 \times A_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 \end{pmatrix}, \qquad B_7 \supset B_5 \times 2A_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \end{pmatrix},$$

$$B_7 \supset D_5 \times C_2 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \end{pmatrix}, \qquad B_7 \supset B_4 \times A_3 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \end{pmatrix},$$

$$B_7 \supset D_4 \times B_3 : \begin{pmatrix} 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 \end{pmatrix}, \qquad B_7 \supset A_1 : \begin{pmatrix} 14 & 26 & 36 & 44 & 50 & 54 & 28 \end{pmatrix},$$

$$B_7 \supset A_3 : \begin{pmatrix} 1 & \cdot & 1 & 1 & \cdot & 2 & 1 \\ \cdot & 1 & 2 & 1 & 3 & 2 & 1 \\ 1 & 2 & 1 & 3 & 2 & 2 & 1 \end{pmatrix}, \qquad B_7 \supset C_2 \times A_1 : \begin{pmatrix} \cdot & \cdot & 2 & 2 & 4 & 4 & 3 \\ 1 & 2 & 1 & 2 & 1 & 1 & \cdot \\ 2 & 2 & 4 & 2 & 2 & 4 & 1 \end{pmatrix}.$$

We give here some examples of branching rules for the $B_7 \supset A_3$ and $B_7 \supset C_2 \times A_1$ cases, for orbits of size 14, 84 and 128 respectively, together with their corresponding indices γ.

$B_7 \supset A_3$:

$$(a,0,0,0,0,0,0) \supset (a,0,a) + 2(0,0,0)\,,$$

$$(0,b,0,0,0,0,0) \supset (0,b,2b) + (2b,b,0) + 2(0,2b,0) + 4(b,0,b)\,,$$

$$(0,0,0,0,0,0,c) \supset 2(c,c,c) + 4(0,0,2c) + 4(2c,0,0) + 8(0,c,0)\,,$$

$$\gamma = 7/12\,,$$

$B_7 \supset C_2 \times A_1:$

$$(a,0,0,0,0,0,0) \supset (0,a)(2a) + (0,a)(0) + (0,0)(2a)\,,$$

$$\begin{aligned}(0,b,0,0,0,0,0) \supset\ & (2b,0)(4b) + 2(2b,0)(2b) + 3(2b,0)(0) + (0,2b)(2b) \\ & + (0,2b)(0) + (0,b)(4b) + (0,b)(2b) + 2(0,b)(0) + 2(0,0)(4b) \\ & + 4(0,0)(2b)\,,\end{aligned}$$

$$\begin{aligned}(0,0,0,0,0,0,c) \supset\ & (3c,0)(c) + (c,c)(3c) + 2(c,c)(c) + (c,0)(5c) + 3(c,0)(3c) \\ & + 5(c,0)(c)\,,\end{aligned}$$

$$\gamma = 7/16\,,$$

where $a, b, c \in \mathbb{R}^{>0}$.

3.3.7. Rank 8

We give the projection matrices for the nine cases to consider. Examples of branching rules can be found in the corresponding subsections of the general rank section 3.3.8.

$$B_8 \supset B_7 \times U_1: \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \qquad B_8 \supset D_8: \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \end{pmatrix},$$

$$B_8 \supset D_7 \times A_1: \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 \end{pmatrix}, \qquad B_8 \supset B_6 \times 2A_1: \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \end{pmatrix},$$

$$B_8 \supset D_6 \times C_2: \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \end{pmatrix}, \qquad B_8 \supset B_5 \times A_3: \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \end{pmatrix},$$

$$B_8 \supset D_5 \times B_3: \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 \end{pmatrix}, \qquad B_8 \supset B_4 \times D_4: \begin{pmatrix} 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \end{pmatrix},$$

$$B_8 \supset A_1: \begin{pmatrix} 16 & 30 & 42 & 52 & 60 & 66 & 70 & 36 \end{pmatrix}.$$

3.3.8. The general rank cases

In this section we consider infinite series of cases where the ranks of the Lie algebras take all the consecutive values starting from a lowest one. For each case, we give the corresponding projection matrix and some examples of branching rules. When the maximal reductive subalgebra of B_n is semisimple, we provide also its index γ in the Lie algebra B_n.

3.3.8.1. $B_n \supset B_{n-1} \times U_1, \ (n \geq 3)$

$$\left(\begin{array}{c|cc} I_{n-2} & \multicolumn{2}{c}{\mathbf{0}} \\ \hline & 2 & 1 \\ \mathbf{0} & \cdot & 1 \end{array}\right)$$

Note that, here and everywhere below, I_k denotes the $k \times k$ identity matrix, $\mathbf{0}$ represents the zero matrix, and $a, b, c \in \mathbb{R}^{>0}$.

$$(a, 0, 0, \ldots, 0) \supset (a, 0, \ldots, 0)(0) + (0, \ldots, 0)(2a) + (0, \ldots, 0)(-2a)$$

$$(0, b, 0, \ldots, 0) \supset (0, b, 0, \ldots, 0)(0) + (b, 0, \ldots, 0)(2b) + (b, 0, \ldots, 0)(-2b)$$

$$(0, 0, \ldots, 0, c) \supset (0, \ldots, 0, c)(c) + (0, \ldots, 0, c)(-c)$$

Note that, here and everywhere below, in the case of B_2, $(0, b, 0, \ldots, 0)$ becomes $(0, 2b)$.

3.3.8.2. $B_n \supset D_n, \quad n \geq 4$

$$\left(\begin{array}{c|cc} I_{n-2} & \multicolumn{2}{c}{\mathbf{0}} \\ \hline & 1 & \cdot \\ \mathbf{0} & 1 & 1 \end{array}\right)$$

$$(a,0,0,\ldots,0) \supset (a,0,\ldots,0)$$
$$(0,b,0,\ldots,0) \supset (0,b,0,\ldots,0)$$
$$(0,0,\ldots,0,c) \supset (0,\ldots,0,c) + (0,\ldots,0,c,0)$$
$$\gamma = 1$$

3.3.8.3. $B_n \supset D_{n-1} \times A_1, \quad n \geq 5$

$$\left(\begin{array}{c|ccc} I_{n-3} & & \mathbf{0} & \\ \hline & 1 & 1 & \cdot \\ \mathbf{0} & 1 & 1 & 1 \\ & \cdot & 2 & 1 \end{array}\right)$$

$$(a,0,0,\ldots,0) \supset (a,0,\ldots,0)(0) + (0,\ldots,0)(2a)$$
$$(0,b,0,\ldots,0) \supset (0,b,0,\ldots,0)(0) + (b,0,\ldots,0)(2b)$$
$$(0,0,\ldots,0,c) \supset (0,\ldots,0,c)(c) + (0,\ldots,0,c,0)(c)$$
$$\gamma = n/(n+1)$$

3.3.8.4. $B_n \supset B_{n-2} \times A_1 \times A_1, \quad n \geq 4$

$$\left(\begin{array}{c|cccc} I_{n-4} & & \mathbf{0} & & \\ \hline & 1 & 1 & \cdot & \cdot \\ & \cdot & \cdot & 2 & 1 \\ \mathbf{0} & \cdot & 1 & 1 & \cdot \\ & \cdot & 1 & 1 & 1 \end{array}\right)$$

$$(a,0,0,\dots,0) \supset (a,0,\dots,0)(0)(0) + (0,\dots,0)(a)(a)$$

$$(0,b,0,\dots,0) \supset (0,b,0,\dots,0)(0)(0) + (b,0,\dots,0)(b)(b) + (0,\dots,0)(2b)(0)$$
$$+ (0,\dots,0)(0)(2b)$$

$$(0,0,\dots,0,c) \supset (0,\dots,0,c)(c)(0) + (0,\dots,0,c)(0)(c)$$

$$\gamma = 1$$

3.3.8.5. $B_n \supset B_{n-3} \times A_3, \quad n \geq 6$

$$\left(\begin{array}{c|cccccc} I_{n-6} & & & \mathbf{0} & & & \\ \hline & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot & 2 & 1 \\ \mathbf{0} & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & 1 & 1 & 1 \end{array}\right)$$

$$(a,0,0,\dots,0) \supset (a,0,\dots,0)(0,0,0) + (0,\dots,0)(0,a,0)$$

$$(0,b,0,\dots,0) \supset (0,b,0,\dots,0)(0,0,0) + (b,0,\dots,0)(0,b,0) + (0,\dots,0)(b,0,b)$$

$$(0,0,\dots,0,c) \supset (0,\dots,0,c)(c,0,0) + (0,\dots,0,c)(0,0,c)$$

$$\gamma = 1$$

3.3.8.6. $B_n \supset B_{n-k} \times D_k, \quad n-k \geq k \geq 4$

$$
\left(\begin{array}{c|cccccccccccc}
I_{n-2k} & \multicolumn{12}{c}{\mathbf{0}} \\
\hline
 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & \vdots & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & 1 & 1 & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & 2 & 1 \\
\mathbf{0} & \cdot & 1 & 1 & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & \vdots & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 1 & 1 & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & 1 & 1 & 1
\end{array}\right)
$$

$$
\begin{aligned}
(a,0,0,\ldots,0) &\supset (a,0,\ldots,0)(0,\ldots,0) + (0,\ldots,0)(a,0,\ldots,0) \\
(0,b,0,\ldots,0) &\supset (0,b,0,\ldots,0)(0,\ldots,0) + (b,0,\ldots,0)(b,0,\ldots,0) \\
&\qquad + (0,\ldots,0)(0,b,0,\ldots,0) \\
(0,0,\ldots,0,c) &\supset (0,\ldots,0,c)(0,\ldots,0,c) + (0,\ldots,0,c)(0,\ldots,0,c,0)
\end{aligned}
$$

$\gamma = 1$

3.3.8.7. $B_n \supset D_{n-k} \times B_k, \quad n-k > k \geq 2, \quad n-k \geq 4$

$$
\left(\begin{array}{c|cccccccccccc}
I_{n-2k-1} & \multicolumn{12}{c}{\mathbf{0}} \\
\hline
 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & \vdots & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & 1 & 1 & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 1 & 1 & 1 \\
\mathbf{0} & \cdot & 1 & 1 & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & \vdots & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & 1 & 1 & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & 2 & 1
\end{array}\right)
$$

$$(a,0,0,\dots,0) \supset (a,0,\dots,0)(0,\dots,0) + (0,\dots,0)(a,0,\dots,0)$$

$$(0,b,0,\dots,0) \supset (0,b,0,\dots,0)(0,\dots,0) + (b,0,\dots,0)(b,0,\dots,0)$$
$$+ (0,\dots,0)(0,b,0,\dots,0)$$

$$(0,0,\dots,0,c) \supset (0,\dots,0,c)(0,\dots,0,c) + (0,\dots,0,c,0)(0,\dots,0,c)$$

$$\gamma = 1$$

3.3.8.8. $B_n \supset A_1, \quad n \geq 4$

The projection matrix for that case is given by

$$\begin{pmatrix} p_1 & p_2 & p_3 & \dots & p_{n-1} & p_n \end{pmatrix}$$
$$p_k = k(2n-k+1), \quad 1 \geq k \geq n-1; \qquad p_n = (n+2)(n-1)/2+1\,. \tag{3.3.1}$$

We bring one example of branching rule for that case, together with the index $\gamma = \gamma_{B_n,A_1}$:

$$(a,0,\dots,0) \supset (2na) + ((2n{-}2)a) + ((2n{-}4)a) + \dots + (6a) + (4a) + (2a)\,,$$

$$\gamma = n/(2\sum_{i=1}^{n} i^2)\,.$$

3.4. Reduction of orbits of the Weyl group of C_n

In this section, as in the previous section, we first consider all cases of dimension up to 8. In the last subsection, 3.4.8, we present infinite series of selected cases. For each case of the section the projection matrix is given together with examples of the corresponding reductions/branching rules. For cases involving Weyl groups of a simple algebra L and a maximal reductive semisimple algebra L', we provide the index $\gamma = \gamma_{L,L'}$ of L' in L.

3.4.1. Rank 2

Since the Lie algebras B_2 and C_2 and their Weyl groups are isomorphic, the projection matrices and the branching rules for the C_2 case can be found in subsection 3.3.1. A practical difference between the two cases is in our numbering convention of simple roots (Fig. 3.1). Hence one only needs to interchange the two columns of the projection matrices of B_2, and to switch the two coordinates of the orbits in the branching rules of B_2 to obtain the results for C_2.

3.4.2. Rank 3

There are four cases to consider. The first three are special cases of the general cases presented in the subsections 3.4.8.2, 3.4.8.3 and 3.4.8.5 respectively.

$$C_3 \supset A_2 \times U_1 : \quad \begin{pmatrix} 1 & 1 & \cdot \\ \cdot & 1 & 2 \\ 1 & \cdot & 1 \end{pmatrix}, \qquad C_3 \supset C_2 \times A_1 : \quad \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix},$$

$$C_3 \supset A_1 : \quad \begin{pmatrix} 5 & 8 & 9 \end{pmatrix}, \qquad C_3 \supset 2A_1 : \quad \begin{pmatrix} 1 & \cdot & 1 \\ 2 & 4 & 4 \end{pmatrix}.$$

For all four cases, we give the branching rules for the orbits of C_3 of size 6, 12, 8 and 48 respectively. We also give the index $\gamma = \gamma_{L,L'}$ whenever L' is semisimple.

$C_3 \supset A_2 \times U_1$:

$$(a,0,0) \supset (a,0)(a) + (0,a)(-a)\,,$$

$$(0,b,0) \supset (b,b)(0) + (0,b)(2b) + (b,0)(-2b)\,,$$

$$(0,0,c) \supset (0,2c)(c) + (2c,0)(-c) + (0,0)(3c) + (0,0)(-3c)\,,$$

$$\begin{aligned}(a,b,c) \supset\ & (a{+}b, b{+}2c)(a{+}c) + (b{+}2c, a{+}b)(-a{-}c) + (b, a{+}b{+}2c)(c{-}a) \\ & + (a{+}b{+}2c, b)(a{-}c) + (a,b)(a{+}2b{+}3c) + (b,a)(-a{-}2b{-}3c) \\ & + (a, b{+}2c)(a{+}2b{+}c) + (b{+}2c, a)(-a{-}2b{-}c)\,,\end{aligned}$$

$C_3 \supset C_2 \times A_1$:

$$(a,0,0) \supset (a,0)(0) + (0,0)(a)\,,$$

$$(0,b,0) \supset (0,b)(0) + (b,0)(b)\,,$$

$$(0,0,c) \supset (0,c)(c)\,,$$

$$(a,b,c) \supset (a, b{+}c)(c) + (a{+}b, c)(b{+}c) + (b,c)(a{+}b{+}c)\,,$$

$$\gamma = 1\,,$$

$C_3 \supset A_1$:

$$(a,0,0) \supset (5a) + (3a) + (a)\,,$$

$$(0,b,0) \supset (8b) + (6b) + 2(4b) + 2(2b)\,,$$

$$(0,0,c) \supset (9c) + (7c) + (3c) + (c)\,,$$

$$\gamma = 3/35\,,$$

$C_3 \supset 2A_1$:

$$(a,0,0) \supset (a)(2a) + (a)(0)\,,$$

$$(0,b,0) \supset (0)(4b) + (2b)(2b) + (2b)(0) + 2(0)(2b)\,,$$

$$(0,0,c) \supset (c)(4c) + (3c)(0) + (c)(0)\,,$$

$$\gamma = 3/11\,.$$

For cases of rank 4 to 8, we give the projection matrices for all cases. Whenever a reduction is a special case of the general rank section, we refrain to give the branching rules and the corresponding index γ here since they can easily be found in section 3.4.8.

3.4.3. Rank 4

We give the projection matrices of the five cases to consider. Examples of branching rules for the first four cases can be found in the corresponding subsections of the general rank section 3.4.8.

$$C_4 \supset A_3 \times U_1 : \begin{pmatrix} 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 2 \\ \cdot & 1 & 1 & \cdot \\ 1 & \cdot & 1 & \cdot \end{pmatrix}, \quad C_4 \supset C_3 \times A_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad C_4 \supset 2C_2 : \begin{pmatrix} 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix},$$

$$C_4 \supset A_1 : \begin{pmatrix} 7 & 12 & 15 & 16 \end{pmatrix}, \qquad C_4 \supset 3A_1 : \begin{pmatrix} 1 & \cdot & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 3 & 2 \end{pmatrix}.$$

We give here some examples of branching rules for the $C_4 \supset 3A_1$ case, for orbits of size 8, 24 and 16 respectively, together with the index $\gamma = \gamma_{C_4,3A_1}$.

$C_4 \supset 3A_1$:

$$(a,0,0,0) \supset (a)(a)(a)\,,$$

$$(0,b,0,0) \supset (0)(2b)(2b) + (2b)(0)(2b) + (2b)(2b)(0) + 2(2b)(0)(0) + 2(0)(2b)(0) + 2(0)(0)(2b)\,,$$

$$(0,0,0,c) \supset (2c)(2c)(2c) + (0)(0)(4c) + (0)(4c)(0) + (4c)(0)(0) + 2(0)(0)(0)\,,$$

$$\gamma = 1/3\,.$$

3.4.4. Rank 5

We give the projection matrices of the five cases to consider. Examples of branching rules for the first four cases can be found in the corresponding subsections of the general rank section 3.4.8.

$$C_5 \supset A_4 \times U_1 : \quad \begin{pmatrix} 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & 2 \\ \cdot & 1 & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & 1 \end{pmatrix}, \qquad C_5 \supset C_4 \times A_1 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix},$$

$$C_5 \supset C_3 \times C_2 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \qquad C_5 \supset A_1 : \quad \begin{pmatrix} 9 & 16 & 21 & 24 & 25 \end{pmatrix},$$

$$C_5 \supset C_2 \times A_1 : \quad \begin{pmatrix} \cdot & \cdot & 2 & 4 & 4 \\ 1 & 2 & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & 1 \end{pmatrix}.$$

We give here some examples of branching rules for the $C_5 \supset C_2 \times A_1$ case, for orbits of size 10, 40 and 32 respectively, together with the index $\gamma = \gamma_{C_5, C_2 \times A_1}$.

$C_5 \supset C_2 \times A_1$:

$$(a,0,0,0,0) \supset (0,a)(a) + (0,0)(a)\,,$$

$$(0,b,0,0,0) \supset (0,2b)(0) + (2b,0)(2b) + (0,b)(2b) + 2(2b,0)(0) + 2(0,b)(0) + 2(0,0)(2b)\,,$$

$$(0,0,0,0,c) \supset (4c,0)(c) + (0,2c)(3c) + (0,2c)(c) + (0,0)(5c) + (0,0)(3c) + 2(0,0)(c)\,,$$

$$\gamma = 5/13\,.$$

3.4.5. Rank 6

We give the projection matrices of the seven cases to consider. Examples of branching rules for the first five cases can be found in the corresponding subsections of the general rank section 3.4.8.

$$C_6 \supset A_5 \times U_1 : \begin{pmatrix} 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 2 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & 1 & \cdot \end{pmatrix}, \qquad C_6 \supset C_5 \times A_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix},$$

$$C_6 \supset C_4 \times C_2 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \qquad C_6 \supset 2C_3 : \begin{pmatrix} 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix},$$

$$C_6 \supset A_1 : \begin{pmatrix} 11 & 20 & 27 & 32 & 35 & 36 \end{pmatrix}, \qquad C_6 \supset A_3 \times A_1 : \begin{pmatrix} \cdot & \cdot & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 2 & 3 & 2 \\ 1 & \cdot & 1 & \cdot & 1 & 2 \end{pmatrix},$$

$$C_6 \supset C_2 \times A_1 : \begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ \cdot & \cdot & 1 & 1 & 2 & 1 \\ 2 & 2 & 4 & 2 & 2 & 4 \end{pmatrix}.$$

We give here some examples of branching rules for the $C_6 \supset A_3 \times A_1$ and $C_6 \supset C_2 \times A_1$ cases, for orbits of size 12, 60 and 64 respectively, together with their corresponding indices γ.

$C_6 \supset A_3 \times A_1$:

$$(a,0,0,0,0,0) \supset (0,a,0)(a)\,,$$

$$(0,b,0,0,0,0) \supset (0,2b,0)(0) + (b,0,b)(2b) + 2(b,0,b)(0) + 3(0,0,0)(2b)\,,$$

$$(0,0,0,0,0,c) \supset (2c,0,2c)(2c) + (0,0,4c)(0) + (4c,0,0)(0) + (0,2c,0)(4c) + 2(0,2c,0)(0) + (0,0,0)(6c) + 3(0,0,0)(2c)\,,$$

$$\gamma = 1/3\,,$$

$C_6 \supset C_2 \times A_1$:

$$(a,0,0,0,0,0) \supset (a,0)(2a) + (a,0)(0)\,,$$

$$(0,b,0,0,0,0) \supset (2b,0)(2b) + (0,b)(4b) + 2(0,b)(2b) + (2b,0)(0) + 3(0,b)(0) + 2(0,0)(4b) + 4(0,0)(2b)\,,$$

$$(0,0,0,0,0,c) \supset (2c,c)(4c) + (0,3c)(0) + (2c,c)(0) + (0,c)(8c) + 2(0,c)(4c) + 3(0,c)(0)\,,$$

$$\gamma = 3/11\,.$$

3.4.6. Rank 7

We give the projection matrices of the six cases to consider. Examples of branching rules for the first five cases can be found in the corresponding subsections of the general rank section 3.4.8.

$$C_7 \supset A_6 \times U_1 : \begin{pmatrix} 1&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&2 \\ \cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&1&\cdot&1 \end{pmatrix}, \qquad C_7 \supset C_6 \times A_1 : \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix},$$

$$C_7 \supset C_5 \times C_2 : \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1 \\ \cdot&\cdot&\cdot&\cdot&1&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}, \qquad C_7 \supset C_4 \times C_3 : \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1 \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix},$$

$$C_7 \supset A_1 : \begin{pmatrix} 13&24&33&40&45&48&49 \end{pmatrix}, \qquad C_7 \supset B_3 \times A_1 : \begin{pmatrix} 1&2&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&2&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&2&4&4 \\ 1&\cdot&1&\cdot&1&\cdot&1 \end{pmatrix}.$$

We give here some examples of branching rules for the $C_7 \supset B_3 \times A_1$ case, for orbits of size 14, 84 and 128 respectively, together with the index $\gamma = \gamma_{C_7, B_3 \times A_1}$.

$C_7 \supset B_3 \times A_1$:

$$(a,0,0,0,0,0,0) \supset (a,0,0)(a) + (0,0,0)(a)\,,$$

$$(0,b,0,0,0,0,0) \supset (2b,0,0)(0) + (0,b,0)(2b) + 2(0,b,0)(0) + (b,0,0)(2b) + 2(b,0,0)(0) + 3(0,0,0)(2b)\,,$$

$$(0,0,0,0,0,0,c) \supset (0,0,4c)(c) + (0,2c,0)(3c) + (0,2c,0)(c) + (2c,0,0)(5c) + (2c,0,0)(3c) + 2(2c,0,0)(c) + (0,0,0)(7c) + (0,0,0)(5c) + 3(0,0,0)(3c) + 3(0,0,0)(c)\,,$$

$$\gamma = 7/19\,.$$

3.4.7. Rank 8

We give the projection matrices of the eight cases to consider. Examples of branching rules for the first six cases can be found in the corresponding subsections of the general rank section 3.4.8.

$$C_8 \supset A_7 \times U_1 : \begin{pmatrix} 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&2 \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot \\ \cdot&\cdot&\cdot&1&1&\cdot&\cdot&\cdot \\ \cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&1&\cdot&1&\cdot \end{pmatrix}, \qquad C_8 \supset C_7 \times A_1 : \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix},$$

$$C_8 \supset C_6 \times C_2 : \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}, \qquad C_8 \supset C_5 \times C_3 : \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1 \\ \cdot&\cdot&\cdot&1&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix},$$

$$C_8 \supset 2C_4 : \begin{pmatrix} 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1 \\ \cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}, \qquad C_8 \supset A_1 : \begin{pmatrix} 15&28&39&48&55&60&63&64 \end{pmatrix},$$

$$C_8 \supset D_4 \times A_1 : \begin{pmatrix} 1&2&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&2&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&2&1&2 \\ \cdot&\cdot&\cdot&\cdot&1&2&3&2 \\ 1&\cdot&1&\cdot&1&\cdot&1&2 \end{pmatrix}, \qquad C_8 \supset C_2 : \begin{pmatrix} 1&4&3&4&5&8&7&6 \\ 1&\cdot&2&2&2&\cdot&1&2 \end{pmatrix}.$$

We give here some examples of branching rules for the $C_8 \supset D_4 \times A_1$ and $C_8 \supset C_2$ cases, for orbits of size 16, 112 and 256 respectively, together with their corresponding indices γ.

$C_8 \supset D_4 \times A_1$:

$$(a,0,0,0,0,0,0,0) \supset (a,0,0,0)(a)\,,$$

$$(0,b,0,0,0,0,0,0) \supset (2b,0,0,0)(0) + (0,b,0,0)(2b) + 2(0,b,0,0)(0) + 4(0,0,0,0)(2b)\,,$$

$$(0,0,0,0,0,0,0,c) \supset (0,0,2c,2c)(2c) + (0,0,0,4c)(0) + (0,0,4c,0)(0) + (0,2c,0,0)(4c) + 2(0,2c,0,0)(0) + (2c,0,0,0)(6c) + 3(2c,0,0,0)(2c) + (0,0,0,0)(8c) + 4(0,0,0,0)(4c) + 6(0,0,0,0)(0)\,,$$

$\gamma = 1/3$,

$C_8 \supset C_2$:

$$(a,0,0,0,0,0,0,0) \supset (a,a) + 2(a,0)\,,$$

$$(0,b,0,0,0,0,0,0) \supset (4b,0) + (0,3b) + 3(2b,b) + 6(2b,0) + 4(0,2b) + 9(0,b) + 4(0,0)\,,$$

$$(0,0,0,0,0,0,0,c) \supset (6c,2c) + 2(8c,0) + 3(4c,2c) + 2(2c,4c) + 4(6c,0) + (0,6c) + 6(2c,2c) + 6(4c,0) + 5(0,4c) + 10(2c,0) + 9(0,2c) + 12(0,0)\,,$$

$\gamma = 1/3$.

3.4.8. The general rank cases

In this section, we consider infinite series of cases where the ranks of the Lie algebras take all the consecutive values starting from a lowest one. For each case, we give the corresponding projection matrix and some examples of branching rules. When the maximal reductive subalgebra of C_n is semisimple, we also provide its index γ in the Lie algebra C_n.

3.4.8.1. $C_{2n} \supset A_{2n-1} \times U_1, \quad n \geq 1$

$$\begin{pmatrix}
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & \vdots & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & 1 & 2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\
 & & & & & & \vdots & & & & & & \\
\cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & 1 & \cdot & & 1 & \cdot & 1 & \cdot & 1 & \cdot
\end{pmatrix}$$

$$(a, 0, 0, \ldots, 0) \supset (a, 0, \ldots, 0)(a) + (0, \ldots, 0, a)(-a)$$

$$(0, b, 0, \ldots, 0) \supset (b, 0, \ldots, 0, b)(0) + (0, b, 0, \ldots, 0)(2b) + (0, \ldots, 0, b, 0)(-2b)$$

$$\begin{aligned}
(0, 0, \ldots, 0, c) \supset\ & (\underbrace{0, \ldots, 0}_{n-1}, 2c, \underbrace{0, \ldots, 0}_{n-1})(0) + (\underbrace{0, \ldots, 0}_{n}, 2c, \underbrace{0, \ldots, 0}_{n-2})(2c) \\
& + (\underbrace{0, \ldots, 0}_{n-2}, 2c, \underbrace{0, \ldots, 0}_{n})(-2c) + (\underbrace{0, \ldots, 0}_{n+1}, 2c, \underbrace{0, \ldots, 0}_{n-3})(4c) \\
& + (\underbrace{0, \ldots, 0}_{n-3}, 2c, \underbrace{0, \ldots, 0}_{n+1})(-4c) + \ldots + (0, \ldots, 0, 2c)((2n-2)c) \\
& + (2c, 0, \ldots, 0)(-(2n-2)c) + (0, \ldots, 0)(2nc) + (0, \ldots, 0)(-2nc)
\end{aligned}$$

3.4.8.2. $C_{2n+1} \supset A_{2n} \times U_1, \quad n \geq 1$

$$\begin{pmatrix}
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & \vdots & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & \vdots & & & & & & & \\
\cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & 1 & \cdot & & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1
\end{pmatrix}$$

$$(a,0,0,\dots,0) \supset (a,0,\dots,0)(a) + (0,\dots,0,a)(-a)$$

$$(0,b,0,\dots,0) \supset (b,0,\dots,0,b)(0) + (0,b,0,\dots,0)(2b) + (0,\dots,0,b,0)(-2b)$$

$$\begin{aligned}(0,0,\dots,0,c) \supset\ & (\underbrace{0,\dots,0}_{n},2c,\underbrace{0,\dots,0}_{n-1})(c) + (\underbrace{0,\dots,0}_{n-1},2c,\underbrace{0,\dots,0}_{n})(-c)\\ & + (\underbrace{0,\dots,0}_{n+1},2c,\underbrace{0,\dots,0}_{n-2})(3c) + (\underbrace{0,\dots,0}_{n-2},2c,\underbrace{0,\dots,0}_{n+1})(-3c)\\ & + \dots + (0,\dots,0,2c)((2n-1)c) + (2c,0,\dots,0)(-(2n-1)c)\\ & + (0,\dots,0)((2n+1)c) + (0,\dots,0)(-(2n+1)c)\end{aligned}$$

3.4.8.3. $C_n \supset C_{n-1} \times A_1$, $(n \geq 2)$

$$\left(\begin{array}{c|c} I_{n-2} & \mathbf{0} \\ \hline \mathbf{0} & \begin{smallmatrix} 1 & 1 \\ \cdot & 1 \end{smallmatrix} \end{array}\right)$$

$$(a,0,0,\dots,0) \supset (a,0,\dots,0)(0) + (0,\dots,0)(a)$$

$$(0,b,0,\dots,0) \supset (0,b,0,\dots,0)(0) + (b,0,\dots,0)(b)$$

$$(0,0,\dots,0,c) \supset (0,\dots,0,c)(c)$$

$$\gamma = 1$$

3.4.8.4. $C_n \supset C_{n-k} \times C_k, \quad n-k \geq k \geq 2$

$$
\left(\begin{array}{c|ccccccccccccc}
I_{n-2k} & & & & & & \mathbf{0} & & & & & & \\
\hline
 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & \vdots & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & 1 & 1 & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot & 1 & 1 \\
\mathbf{0} & \cdot & 1 & 1 & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & \vdots & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & 1 & 1 & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)
$$

$$(a,0,0,\dots,0) \supset (a,0,\dots,0)(0,\dots,0) + (0,\dots,0)(a,0,\dots,0)$$

$$(0,b,0,\dots,0) \supset (0,b,0,\dots,0)(0,\dots,0) + (b,0,\dots,0)(b,0,\dots,0)$$
$$+ (0,\dots,0)(0,b,0,\dots,0)$$

$$(0,0,\dots,0,c) \supset (0,\dots,0,c)(0,\dots,0,c)$$

$$\gamma = 1$$

3.4.8.5. $C_n \supset A_1, \quad n \geq 2$

The projection matrix for that case is given by

$$\begin{pmatrix} p_1 & p_2 & p_3 & \dots & p_{n-1} & p_n \end{pmatrix} \qquad p_k = k(2n-k), \qquad 1 \geq k \geq n.$$

We bring one example of branching rule for that case, together with the index $\gamma = \gamma_{C_n,A_1}$:

$$(a,0,\dots,0) \supset ((2n-1)a) + ((2n-3)a) + ((2n-5)a) + \dots + (5a) + (3a) + (a)\,,$$

$$\gamma = n/\sum_{i=1}^{n}(2i-1)^2\,.$$

3.5. Reduction of orbits of the Weyl group of D_n

As in the two previous sections, we first consider all cases of dimension up to 8, and we present infinite series of selected cases in 3.5.7. For each case, the projection matrix is given together with examples of the corresponding branching rules. For cases involving Weyl groups of a simple algebra L and a maximal reductive semisimple algebra L', we provide the index $\gamma = \gamma_{L,L'}$ of L' in L.

3.5.1. Rank 3

Since the Lie algebras D_3 and A_3 and their Weyl groups are isomorphic, the projection matrices and some examples of branching rules for the D_3 case can be found in [**34**]. A practical difference between the two cases is in our numbering convention of simple roots (Fig. 3.1).

For cases of rank 4 to 8, we give the projection matrices for all cases. Whenever a reduction is a special case of the general rank section, we refrain to give the branching rules and the corresponding index γ here since they can easily be found in section 3.5.7, with maximally a minor renumbering of simple roots ($A_3 \to D_3$ and $C_2 \to B_2$).

3.5.2. Rank 4

We give the projection matrices of the five cases to consider. Examples of branching rules for the first three cases can be found in the corresponding subsections of the general rank section 3.5.7.

$$D_4 \supset A_3 \times U_1 : \begin{pmatrix} 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & 1 & \cdot \\ 1 & \cdot & 1 & \cdot \end{pmatrix}, \qquad D_4 \supset B_3 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \end{pmatrix}, \qquad D_4 \supset C_2 \times A_1 : \begin{pmatrix} \cdot & 2 & 1 & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \end{pmatrix},$$

$$D_4 \supset 4A_1 : \begin{pmatrix} \cdot & 1 & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot \end{pmatrix}, \qquad D_4 \supset A_2 : \begin{pmatrix} 1 & \cdot & 1 & 1 \\ 1 & 3 & 1 & 1 \end{pmatrix}.$$

We give here some examples of branching rules for the $D_4 \supset 4A_1$ and $D_4 \supset A_2$ cases, for orbits of size 8, 24 and 8 respectively, together with their corresponding indices γ.

$D_4 \supset 4A_1$:

$$(a,0,0,0) \supset (a)(a)(0)(0) + (0)(0)(a)(a)\,,$$

$$(0,b,0,0) \supset (b)(b)(b)(b) + (2b)(0)(0)(0) + (0)(2b)(0)(0) + (0)(0)(2b)(0) + (0)(0)(0)(2b)\,,$$

$$(0,0,0,c) \supset (0)(c)(c)(0) + (c)(0)(0)(c)\,,$$

$$\gamma = 1\,,$$

$D_4 \supset A_2$:

$$(a,0,0,0) \supset (a,a) + 2(0,0)\,,$$

$$(0,b,0,0) \supset (0,3b) + (3b,0) + 3(b,b)\,,$$

$$(0,0,0,c) \supset (c,c) + 2(0,0)\,,$$

$$\gamma = 2/3\,.$$

3.5.3. Rank 5

We give the projection matrices of the seven cases to consider. Examples of branching rules for the first five cases can be found in the corresponding subsections of the general rank section 3.5.7.

$$D_5 \supset A_4 \times U_1 : \begin{pmatrix} 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & 1 & \cdot & \cdot \\ 2 & \cdot & 2 & -1 & 1 \end{pmatrix}, \qquad D_5 \supset D_4 \times U_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & -1 \end{pmatrix},$$

$$D_5 \supset B_4 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 \end{pmatrix}, \qquad D_5 \supset B_3 \times A_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & 1 \end{pmatrix},$$

$$D_5 \supset 2C_2 : \begin{pmatrix} \cdot & \cdot & 2 & 1 & 1 \\ 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & 1 & 1 & \cdot & \cdot \end{pmatrix}, \qquad D_5 \supset A_3 \times 2A_1 : \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & 1 & 1 & 1 & 1 \\ \cdot & 1 & 1 & \cdot & \cdot \end{pmatrix},$$

$$D_5 \supset C_2 : \begin{pmatrix} 2 & 2 & 4 & 1 & 1 \\ \cdot & 1 & \cdot & 1 & 1 \end{pmatrix}.$$

We give here some examples of branching rules for the $D_5 \supset A_3 \times 2A_1$ and $D_5 \supset C_2$ cases, for orbits of size 10, 40 and 16 respectively, together with their corresponding indices γ.

$D_5 \supset A_3 \times 2A_1$:

$$(a,0,0,0,0) \supset (0,a,0)(0)(0) + (0,0,0)(a)(a)\,,$$

$$(0,b,0,0,0) \supset (0,b,0)(b)(b) + (b,0,b)(0)(0) + (0,0,0)(2b)(0) + (0,0,0)(0)(2b)\,,$$

$$(0,0,0,0,c) \supset (0,0,c)(c)(0) + (c,0,0)(0)(c)\,,$$

$$\gamma = 1\,,$$

$D_5 \supset C_2$:

$$(a,0,0,0,0) \supset (2a,0) + (0,a) + 2(0,0)\,,$$

$$(0,b,0,0,0) \supset (2b,b) + (0,2b) + 3(2b,0) + 4(0,b)\,,$$

$$(0,0,0,0,c) \supset (c,c) + 2(c,0)\,,$$

$$\gamma = 5/6\,.$$

3.5.4. Rank 6

We give the projection matrices of the nine cases to consider. Examples of branching rules for the first six cases can be found in the corresponding subsections of the general rank section 3.5.7.

$$D_6 \supset A_5 \times U_1 : \begin{pmatrix} 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & 1 & \cdot \end{pmatrix}, \qquad D_6 \supset D_5 \times U_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & -1 \end{pmatrix},$$

$$D_6 \supset B_5 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \end{pmatrix}, \qquad D_6 \supset B_4 \times A_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \end{pmatrix},$$

$$D_6 \supset B_3 \times C_2 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot \end{pmatrix}, \qquad D_6 \supset D_4 \times 2A_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & 1 & 1 & 1 \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot \end{pmatrix},$$

$$D_6 \supset 2A_3 : \begin{pmatrix} \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad D_6 \supset 3A_1 : \begin{pmatrix} 2 & 4 & 6 & 6 & 4 & 4 \\ 1 & 2 & 1 & 2 & \cdot & 1 \\ 1 & \cdot & 1 & 2 & 1 & \cdot \end{pmatrix},$$

$$D_6 \supset C_3 \times A_1 : \begin{pmatrix} 1 & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 2 & 1 & 2 & \cdot & 1 \end{pmatrix}.$$

We give here some examples of branching rules for the last three cases, for orbits of size 12, 60 and 32 respectively, together with their corresponding indices γ.

$D_6 \supset 2A_3$:

$$(a,0,0,0,0,0) \supset (0,0,0)(0,a,0) + (0,a,0)(0,0,0)\,,$$

$$(0,b,0,0,0,0) \supset (0,b,0)(0,b,0) + (0,0,0)(b,0,b) + (b,0,b)(0,0,0)\,,$$

$$(0,0,0,0,0,c) \supset (0,0,c)(0,0,c) + (c,0,0)(c,0,0)\,,$$

$$\gamma = 1\,,$$

$D_6 \supset 3A_1$:

$$(a,0,0,0,0,0) \supset (2a)(a)(a) + (0)(a)(a)\,,$$

$$\begin{aligned}(0,b,0,0,0,0) \supset\ & (4b)(2b)(0) + (4b)(0)(2b) + 2(4b)(0)(0) + (2b)(2b)(2b) \\ & + 2(2b)(2b)(0) + 2(2b)(0)(2b) + (0)(2b)(2b) + 4(2b)(0)(0) \\ & + 3(0)(2b)(0) + 3(0)(0)(2b)\,,\end{aligned}$$

$$\begin{aligned}(0,0,0,0,0,c) \supset\ & (4c)(c)(0) + (2c)(c)(2c) + 2(2c)(c)(0) + (0)(3c)(0) \\ & + (0)(c)(2c) + 3(0)(c)(0)\,,\end{aligned}$$

$$\gamma = 3/7\,,$$

$D_6 \supset C_3 \times A_1$:

$$(a,0,0,0,0,0) \supset (a,0,0)(a)\,,$$

$$(0,b,0,0,0,0) \supset (0,b,0)(2b) + (2b,0,0)(0) + 2(0,b,0)(0) + 3(0,0,0)(2b)\,,$$

$$(0,0,0,0,0,c) \supset (0,c,0)(c) + (0,0,0)(3c) + 3(0,0,0)(c)\,,$$

$$\gamma = 1\,.$$

3.5.5. Rank 7

We give the projection matrices of the eleven cases to consider. Examples of branching rules for the first eight cases can be found in the corresponding subsections of the general rank section 3.5.7.

$$D_7 \supset A_6 \times U_1 : \begin{pmatrix} 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & 2 & \cdot & 2 & -1 & 1 \end{pmatrix}, \qquad D_7 \supset D_6 \times U_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 \end{pmatrix},$$

$$D_7 \supset B_6 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \end{pmatrix}, \qquad D_7 \supset B_5 \times A_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \end{pmatrix},$$

$$D_7 \supset B_4 \times C_2 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \end{pmatrix}, \qquad D_7 \supset 2B_3 : \begin{pmatrix} 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & 1 & 1 \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \end{pmatrix},$$

$$D_7 \supset D_5 \times 2A_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \end{pmatrix}, \qquad D_7 \supset D_4 \times A_3 : \begin{pmatrix} 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$D_7 \supset C_2 : \begin{pmatrix} \cdot & 2 & 2 & 6 & 4 & 3 & 3 \\ 2 & 2 & 3 & 1 & 3 & 1 & 1 \end{pmatrix}, \qquad D_7 \supset C_3 : \begin{pmatrix} \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 \\ 1 & \cdot & \cdot & 1 & 3 & 1 & 1 \\ \cdot & 1 & 2 & 1 & \cdot & \cdot & \cdot \end{pmatrix},$$

$$D_7 \supset G_2 : \begin{pmatrix} 1 & \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & 3 & 4 & 3 & 5 & 1 & 1 \end{pmatrix}.$$

We give here some examples of branching rules for the last three cases, for orbits of size 14, 84 and 64 respectively, together with their corresponding indices γ.

$D_7 \supset C_2$:

$$(a,0,0,0,0,0,0) \supset (0,2a) + (2a,0) + (0,a) + 2(0,0),$$

$$(0,b,0,0,0,0,0) \supset (2b,2b) + (0,3b) + 2(2b,b) + (4b,0) + 3(0,2b) + 5(2b,0) + 5(0,b),$$

$$(0,0,0,0,0,0,c) \supset (3c,c) + (c,2c) + 2(3c,0) + 3(c,c) + 4(c,0),$$

$$\gamma = 1/2,$$

$D_7 \supset C_3$:

$$(a,0,0,0,0,0,0) \supset (0,a,0) + 2(0,0,0),$$

$$(0,b,0,0,0,0,0) \supset (b,0,b) + 2(2b,0,0) + 4(0,b,0),$$

$$(0,0,0,0,0,0,c) \supset (c,c,0) + 2(0,0,c) + 4(c,0,0),$$

$$\gamma = 7/6,$$

$D_7 \supset G_2$:

$$(a,0,0,0,0,0,0) \supset (a,0) + (0,a) + 2(0,0)\,,$$

$$(0,b,0,0,0,0,0) \supset (b,b) + (0,3b) + 2(0,2b) + 4(b,0) + 5(0,b)\,,$$

$$(0,0,0,0,0,0,c) \supset (c,c) + 2(0,2c) + 2(c,0) + 4(0,c) + 4(0,0)\,,$$

$$\gamma = 7/8\,.$$

3.5.6. Rank 8

We give the projection matrices of the twelve cases to consider. Examples of branching rules for the first nine cases can be found in the corresponding subsections of the general rank section 3.5.7.

$$D_8 \supset A_7 \times U_1 : \begin{pmatrix} 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot \\ \cdot&\cdot&\cdot&1&1&\cdot&\cdot&\cdot \\ \cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&1&\cdot&1&\cdot \end{pmatrix}, \qquad D_8 \supset D_7 \times U_1 : \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&-1 \end{pmatrix},$$

$$D_8 \supset B_7 : \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1 \end{pmatrix}, \qquad D_8 \supset B_6 \times A_1 : \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&2&1&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1 \end{pmatrix},$$

$$D_8 \supset B_5 \times C_2 : \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&2&1&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1 \\ \cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \end{pmatrix}, \qquad D_8 \supset B_4 \times B_3 : \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&2&1&1 \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1 \end{pmatrix},$$

$$D_8 \supset D_6 \times 2A_1 : \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1 \\ \cdot&\cdot&\cdot&\cdot&1&1&1&1 \\ \cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \end{pmatrix}, \qquad D_8 \supset D_5 \times A_3 : \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1 \\ \cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&1&1&1 \end{pmatrix},$$

$$D_8 \supset 2D_4 : \begin{pmatrix} 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&1&1&1 \\ \cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1 \end{pmatrix}, \qquad D_8 \supset B_4 : \begin{pmatrix} \cdot&\cdot&\cdot&1&1&\cdot&1&\cdot \\ \cdot&\cdot&1&\cdot&1&1&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1 \\ 1&\cdot&1&2&1&2&1&\cdot \end{pmatrix},$$

$$D_8 \supset 2C_2 : \begin{pmatrix} 1&2&1&2&1&2&2&1 \\ \cdot&\cdot&1&1&2&1&\cdot&1 \\ 1&\cdot&1&2&1&2&\cdot&1 \\ \cdot&1&1&\cdot&1&1&1&\cdot \end{pmatrix}, \qquad D_8 \supset C_4 \times A_1 : \begin{pmatrix} 1&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&1&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&1&\cdot&\cdot&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot \\ 1&2&1&2&1&2&\cdot&1 \end{pmatrix}.$$

We give here some examples of branching rules for the last three cases, for orbits of size 16, 112 and 128 respectively, together with their corresponding indices γ.

$D_8 \supset B_4$:

$$(a,0,0,0,0,0,0,0) \supset (0,0,0,a)\,,$$

$$(0,b,0,0,0,0,0,0) \supset (0,0,b,0) + 2(0,b,0,0) + 4(b,0,0,0)\,,$$

$$(0,0,0,0,0,0,0,c) \supset (0,0,c,0) + 2(0,c,0,0) + (2c,0,0,0) + 4(c,0,0,0) + 8(0,0,0,0)\,,$$

$$\gamma = 1\,,$$

$D_8 \supset 2C_2$:

$$(a,0,0,0,0,0,0,0) \supset (a,0)(a,0)\,,$$

$$(0,b,0,0,0,0,0,0) \supset (2b,0)(0,b) + (0,b)(2b,0) + 2(0,b)(0,b) + 2(2b,0)(0,0) + 2(0,0)(2b,0) + 4(0,b)(0,0) + 4(0,0)(0,b)\,,$$

$$(0,0,0,0,0,0,0,c) \supset (c,c)(c,0) + (c,0)(c,c) + 4(c,0)(c,0)\,,$$

$$\gamma = 1\,,$$

$D_8 \supset C_4 \times A_1$:

$$(a,0,0,0,0,0,0,0) \supset (a,0,0,0)(a)\,,$$

$$(0,b,0,0,0,0,0,0) \supset (0,b,0,0)(2b) + (2b,0,0,0)(0) + 2(0,b,0,0)(0) + 4(0,0,0,0)(2b)\,,$$

$$(0,0,0,0,0,0,0,c) \supset (0,0,c,0)(c) + (c,0,0,0)(3c) + 3(c,0,0,0)(c)\,,$$

$$\gamma = 1\,.$$

3.5.7. The general rank cases

In this section we consider infinite series of cases where the ranks of the Lie algebras take all the consecutive values starting from a lowest one. For each case, we give the corresponding projection matrix and some examples of branching rules. When the maximal reductive subalgebra of D_n is semisimple, we provide also its index γ in the Lie algebra D_n.

3.5.7.1. $D_{2n} \supset A_{2n-1} \times U_1, \quad n \geq 2$

$$\begin{pmatrix}
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & \vdots & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\
 & & & & & & \vdots & & & & & & \\
\cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & 1 & \cdot & & 1 & \cdot & 1 & \cdot & 1 & \cdot
\end{pmatrix}$$

$$(a,0,0,\dots,0) \supset (a,0,\dots,0)(a) + (0,\dots,0,a)(-a)$$

$$(0,b,0,\dots,0) \supset (b,0,\dots,0,b)(0) + (0,b,0,\dots,0)(2b) + (0,\dots,0,b,0)(-2b)$$

$$(0,0,\dots,0,c) \supset \begin{cases}
(\underbrace{0,\dots,0}_{n-1},c,\underbrace{0,\dots,0}_{n-1})(0) + (\underbrace{0,\dots,0}_{n+1},c,\underbrace{0,\dots,0}_{n-3})(2c) \\
+(\underbrace{0,\dots,0}_{n-3},c,\underbrace{0,\dots,0}_{n+1})(-2c) + (\underbrace{0,\dots,0}_{n+3},c,\underbrace{0,\dots,0}_{n-5})(4c) \\
+(\underbrace{0,\dots,0}_{n-5},c,\underbrace{0,\dots,0}_{n+3})(-4c) + \dots \\
+(0,\dots,0,c,0)((n-2)c) + (0,c,0,\dots,0)(-(n-2)c) \\
+(0,\dots,0)(nc) + (0,\dots,0)(-nc) & n \text{ even} \\
\\
(\underbrace{0,\dots,0}_{n-1},c,\underbrace{0,\dots,0}_{n-1})(0) + (\underbrace{0,\dots,0}_{n+1},c,\underbrace{0,\dots,0}_{n-3})(2c) \\
+(\underbrace{0,\dots,0}_{n-3},c,\underbrace{0,\dots,0}_{n+1})(-2c) + (\underbrace{0,\dots,0}_{n+3},c,\underbrace{0,\dots,0}_{n-5})(4c) \\
+(\underbrace{0,\dots,0}_{n-5},c,\underbrace{0,\dots,0}_{n+3})(-4c) + \dots + (0,\dots,0,c)((n-1)c) \\
+(c,0,\dots,0)(-(n-1)c) & n \text{ odd}
\end{cases}$$

3.5.7.2. $D_{2n+1} \supset A_{2n} \times U_1, \quad n \geq 2$

$$
\left(\begin{array}{cccccccccccccc}
1 & 1 & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & \vdots & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & \vdots & & & & & & & \\
\cdot & 1 & 1 & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & 2 & \cdot & 2 & \cdot & & 2 & \cdot & 2 & \cdot & 2 & -1 & 1
\end{array}\right)
$$

$$(a,0,0,\dots,0) \supset (a,0,\dots,0)(2a) + (0,\dots,0,a)(-2a)$$

$$(0,b,0,\dots,0) \supset (b,0,\dots,0,b)(0) + (0,b,0,\dots,0)(4b) + (0,\dots,0,b,0)(-4b)$$

$$
(0,0,\dots,0,c) \supset \begin{cases}
(\underbrace{0,\dots,0}_{n},c,\underbrace{0,\dots,0}_{n-1})(c) + (\underbrace{0,\dots,0}_{n-2},c,\underbrace{0,\dots,0}_{n+1})(-3c) \\
+(\underbrace{0,\dots,0}_{n+2},c,\underbrace{0,\dots,0}_{n-3})(5c) + (\underbrace{0,\dots,0}_{n-4},c,\underbrace{0,\dots,0}_{n+3})(-7c) \\
+\dots + (c,0,\dots,0)(-(2n-1)c) + (0,\dots,0)((2n+1)c) \qquad n \text{ even} \\
\\
(\underbrace{0,\dots,0}_{n},c,\underbrace{0,\dots,0}_{n-1})(c) + (\underbrace{0,\dots,0}_{n-2},c,\underbrace{0,\dots,0}_{n+1})(-3c) \\
+(\underbrace{0,\dots,0}_{n+2},c,\underbrace{0,\dots,0}_{n-3})(5c) + (\underbrace{0,\dots,0}_{n-4},c,\underbrace{0,\dots,0}_{n+3})(-7c) \\
+\dots + (0,\dots,0,c)((2n-1)c) + (0,\dots,0)(-(2n+1)c) \qquad n \text{ odd}
\end{cases}
$$

3.5.7.3. $D_n \supset D_{n-1} \times U_1, \quad n \geq 5$

$$
\left(\begin{array}{c|c}
I_{n-3} & \mathbf{0} \\
\hline
\mathbf{0} & \begin{array}{ccc} 1 & \cdot & \cdot \\ 1 & 1 & 1 \\ \cdot & 1 & -1 \end{array}
\end{array}\right)
$$

$$(a,0,0,\dots,0) \supset (a,0,\dots,0)(0) + (0,\dots,0)(2a) + (0,\dots,0)(-2a)$$

$$(0,b,0,\dots,0) \supset (0,b,0,\dots,0)(0) + (b,0,\dots,0)(2b) + (b,0,\dots,0)(-2b)$$

$$(0,0,\dots,0,c) \supset (0,\dots,0,c,0)(c) + (0,\dots,0,c)(-c)$$

3.5.7.4. $D_n \supset B_{n-1}, \quad n \geq 4$

$$\left(\begin{array}{c|cc} I_{n-2} & \multicolumn{2}{c}{\mathbf{0}} \\ \hline \mathbf{0} & 1 & 1 \end{array}\right)$$

$$\begin{aligned} (a,0,0,\dots,0) &\supset (a,0,\dots,0) + 2(0,\dots,0) \\ (0,b,0,\dots,0) &\supset (0,b,0,\dots,0) + 2(b,0,\dots,0) \\ (0,0,\dots,0,c) &\supset (0,\dots,0,c) \\ \gamma &= n/(n-1) \end{aligned}$$

3.5.7.5. $D_n \supset B_{n-2} \times A_1, \quad n \geq 4$

$$\left(\begin{array}{c|ccc} I_{n-3} & \multicolumn{3}{c}{\mathbf{0}} \\ \hline & 2 & 1 & 1 \\ \mathbf{0} & \cdot & 1 & 1 \end{array}\right)$$

$$\begin{aligned} (a,0,0,\dots,0) &\supset (a,0,\dots,0)(0) + (0,\dots,0)(2a) + 2(0,\dots,0)(0) \\ (0,b,0,\dots,0) &\supset (0,b,0,\dots,0)(0) + (b,0,\dots,0)(2b) + 2(b,0,\dots,0)(0) \\ &\quad + 2(0,\dots,0)(2b) \\ (0,0,\dots,0,c) &\supset (0,\dots,0,c)(c) \\ \gamma &= 1 \end{aligned}$$

3.5.7.6. $D_n \supset B_{n-k-1} \times B_k, \quad n-k-1 \geq k \geq 2, \quad n \geq 5$

$$
\left(\begin{array}{c|cccccccccccc}
I_{n-2k-1} & \multicolumn{12}{c}{\mathbf{0}} \\
\hline
 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & \vdots & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & 1 & 1 & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 2 & 1 & 1 \\
\mathbf{0} & \cdot & 1 & 1 & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & \vdots & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & 1 & 1 & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & 1 & 1
\end{array}\right)
$$

$$
\begin{aligned}
(a,0,0,\ldots,0) &\supset (a,0,\ldots,0)(0,\ldots,0) + (0,\ldots,0)(a,0,\ldots,0) + 2(0,\ldots,0)(0,\ldots,0) \\
(0,b,0,\ldots,0) &\supset (b,0,\ldots,0)(b,0,\ldots,0) + (0,b,0,\ldots,0)(0,\ldots,0) \\
&\quad + (0,\ldots,0)(0,b,0,\ldots,0) + 2(b,0,\ldots,0)(0,\ldots,0) \\
&\quad + 2(0,\ldots,0)(b,0,\ldots,0) \\
(0,0,\ldots,0,c) &\supset (0,\ldots,0,c)(0,\ldots,0,c)
\end{aligned}
$$

$\gamma = n/(n-1)$

3.5.7.7. $D_n \supset D_{n-2} \times A_1 \times A_1, \quad n \geq 6$

$$
\left(\begin{array}{c|ccccc}
I_{n-5} & \multicolumn{5}{c}{\mathbf{0}} \\
\hline
 & 1 & 1 & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & 1 & 1 & \cdot \\
\mathbf{0} & \cdot & \cdot & 1 & \cdot & 1 \\
 & \cdot & 1 & 1 & 1 & 1 \\
 & \cdot & 1 & 1 & \cdot & \cdot
\end{array}\right)
$$

$$(a, 0, 0, \ldots, 0) \supset (a, 0, \ldots, 0)(0)(0) + (0, \ldots, 0)(a)(a)$$

$$(0, b, 0, \ldots, 0) \supset (0, b, 0, \ldots, 0)(0)(0) + (b, 0, \ldots, 0)(b)(b) + (0, \ldots, 0)(2b)(0)$$
$$+ (0, \ldots, 0)(0)(2b)$$

$$(0, 0, \ldots, 0, c) \supset (0, \ldots, 0, c)(c)(0) + (0, \ldots, 0, c, 0)(0)(c)$$

$$\gamma = 1$$

3.5.7.8. $D_n \supset D_{n-3} \times A_3, \quad n \geq 7$

$$\left(\begin{array}{c|ccccccc}
I_{n-7} & & & & \mathbf{0} & & & \\
\hline
 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\
\mathbf{0} & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1
\end{array}\right)$$

$$(a, 0, 0, \ldots, 0) \supset (a, 0, \ldots, 0)(0, 0, 0) + (0, \ldots, 0)(0, a, 0)$$

$$(0, b, 0, \ldots, 0) \supset (0, b, 0, \ldots, 0)(0, 0, 0) + (b, 0, \ldots, 0)(0, b, 0) + (0, \ldots, 0)(b, 0, b)$$

$$(0, 0, \ldots, 0, c) \supset (0, \ldots, 0, c)(0, 0, c) + (0, \ldots, 0, c, 0)(c, 0, 0)$$

$$\gamma = 1$$

3.5.7.9. $D_n \supset D_{n-k} \times D_k, \quad n-k \geq k \geq 4$

$$
\left(\begin{array}{c|cccccccccccc}
I_{n-2k} & \multicolumn{12}{c}{\mathbf{0}} \\
\hline
 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & \vdots & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & 1 & 1 & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & 1 & 1 & 1 & 1 \\
\mathbf{0} & \cdot & 1 & 1 & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & \vdots & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 1 & 1 & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & 1 & \cdot & 1
\end{array}\right)
$$

$$(a,0,0,\dots,0) \supset (a,0,\dots,0)(0,\dots,0) + (0,\dots,0)(a,0,\dots,0)$$

$$(0,b,0,\dots,0) \supset (0,b,0,\dots,0)(0,\dots,0) + (b,0,\dots,0)(b,0,\dots,0)$$
$$+ (0,\dots,0)(0,b,0,\dots,0)$$

$$(0,0,\dots,0,c) \supset (0,\dots,0,c)(0,\dots,0,c) + (0,\dots,0,c,0)(0,\dots,0,c,0)$$

$$\gamma = 1$$

3.6. CONCLUDING REMARKS

– The pairs $W(L) \supset W(L')$ in this paper involve a maximal subalgebra L' in L. A chain of maximal subalgebras linking L and any of its reductive non-maximal subalgebras L'' can be found. Corresponding projection matrices combine, by common matrix multiplication, into the projection matrix for $W(L) \supset W(L'')$.

– Projection matrices of $W(L) \supset W(L')$ when the ranks of L and L' are the same, are square matrices with determinant different from zero. Hence they can be inverted and used in the opposite direction, as discussed in **[11]**. The inverse matrix transforms an orbit of $W(L')$ into the linear combination of orbits of $W(L)$, where $L' \subset L$. The linear combination has integer coefficients of both signs in general. We know of no interpretation of such 'branching rules' in applied literature, although they have their place in the Grothendieck rings of representations.

- Weyl group orbits retain most of their useful properties, such as decomposition of their products and branching rules, even when their points are off the weight lattice. Two applications of such orbits can be anticipated. First they could serve as models of molecules that have full Weyl group symmetry without having the rigid regularity of distances between their points/atoms. Another application is undoubtedly Fourier analysis, when Fourier integral expansions are studied rather than discrete ones.

- Curious and completely unexplored relations between pairs of maximal subalgebras, say L' and L'', of the same Lie algebra L can be found by combining the projection matrices $P(L \supset L')$ and $P(L \supset L'')$ as

$$P(L' \rightarrow L'') = P(L \supset L'')P^{-1}(L \supset L').$$

  Here L' must be of the same rank as L for $P(L \supset L')$ to be invertible. We write $L' \rightarrow L''$ instead of $L' \supset L''$ here because L'' is obviously not a subalgebra of L'.

- Congruence classes of representations are naturally extended to congruence classes of W-orbits [**21**]. Comparing the congruence classes of orbits for $W(L) \supset W(L')$ reveals that not all combinations of congruence classes are present. A relative congruence class is a valid and useful concept which deserves investigation. Incidentally, relative congruence classes are studied in chapter 5 [**33**] of this thesis.

- Following the experience gained from applications of finite dimensional representations of semisimple Lie algebras, one could also study, in the case of Weyl group orbits, their anomaly numbers [**58, 62**] and indices of higher than second degree [**37, 57, 64**].

– Subjoining among semisimple Lie resembles inclusion because it allows one to calculate ‘branching rules’. Projection matrices are perfectly adequate for this task [**63**]. But it is not an homomorphism, therefore it is a different relation. All maximal subjoinings have been classified [**53**].
  Consider an example of subjoining. The 4-dimensional representation $(1,0,0)$ of A_3 does *not* contain the 5-dimensional representation $(0,1)$ of C_2. In spite of this, the projection matrix that maps the highest weight orbit of A_3 (and any other orbit of A_3) into the orbit $(0,1)$ of C_2 can be obtained. Indeed, that projection matrix is $\left(\begin{smallmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right)$.

Acknowledgements

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada and by the MIND Research Institute of Santa Ana, California. M. L. is grateful for the support she receives from the Alexander Graham Bell Scholarship.

Chapitre 4

RÉDUCTION D'ORBITES DES GROUPES DE WEYL DES GROUPES DE LIE EXCEPTIONNELS

4.1. Introduction

Les règles de branchement des orbites des groupes de Weyl des algèbres de Lie classiques à l'union d'orbites des groupes de Weyl des sous-algèbres réductives maximales, calculées à l'aide des matrices de projection, ont été traitées dans les deux chapitres précédents. Afin de proposer un document complet sur la réduction des orbites des groupes de Weyl des algèbres de Lie, nous présentons, de manière succinte, les matrices de projection ainsi que des règles de branchement des orbites des groupes de Weyl des algèbres de Lie exceptionnelles. Dans le but de limiter les répétitions, nous nous abstenons d'expliquer à nouveau les concepts liés aux règles de branchement et aux matrices de projection, et référons plutôt le lecteur aux chapitres 2 et 3.

Les matrices de projection pour les cas exceptionnels sont listées dans [**40**], mais les règles de branchement qui y sont présentées sont celles des systèmes de poids des représentations des algèbres de Lie exceptionnelles, plutôt que celles d'orbites individuelles.

Pour chacun des cas exceptionnels où la sous-algèbre est réductive maximale, nous donnons la matrice de projection et listons les règles de branchement pour différentes orbites de $\mathbb{R}^n$. Nous incluons, comme l'un des cas spéciaux, l'orbite qui représente le poids le plus élevé de la représentation adjointe de l'algèbre exceptionnelle.

Il est à noter que dans toutes les matrices du présent chapitre, un point dans une matrice représente l'élément matriciel 0.

4.2. Réduction d'orbites du groupe de Weyl de E_6

Les matrices de projection pour les huit sous-algèbres réductives maximales de E_6 sont :

$$E_6 \supset D_5 \times U_1 : \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & -1 & \cdot & 1 & -1 & \cdot \end{pmatrix}, \qquad E_6 \supset A_5 \times A_1 : \begin{pmatrix} \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & 1 & \cdot & 1 \end{pmatrix},$$

$$E_6 \supset 3A_2 : \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & 1 & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & 1 & \cdot & 1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \end{pmatrix}, \qquad E_6 \supset A_2 \times G_2 : \begin{pmatrix} 1 & 2 & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & 2 & 1 & 1 \\ \cdot & 1 & 1 & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & 1 & 1 \end{pmatrix},$$

$$E_6 \supset C_4 : \begin{pmatrix} \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & 2 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \qquad E_6 \supset F_4 : \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix},$$

$$E_6 \supset A_2 : \begin{pmatrix} 2 & 2 & 5 & 5 & 2 & 1 \\ 2 & 5 & 5 & 2 & 2 & 4 \end{pmatrix}, \qquad E_6 \supset G_2 : \begin{pmatrix} \cdot & 1 & \cdot & 1 & \cdot & 1 \\ 2 & 2 & 5 & 2 & 2 & 1 \end{pmatrix}.$$

Nous donnons des exemples de règles des branchement pour les orbites du groupe de Weyl de E_6 d'ordre 27, 216, 720 et 72 respectivement. Ici, et jusqu'à la fin du présent chapitre, a, b, c et d sont dans $\mathbb{R}^{>0}$.

$E_6 \supset D_5 \times U_1$:

$$(a,0,0,0,0,0) \supset (0,0,0,0,a)(a) + (a,0,0,0,0)(-2a) + (0,0,0,0,0)(4a)\,,$$

$$(0,b,0,0,0,0) \supset (b,0,0,0,b)(-b) + (0,0,b,0,0)(2b) + (0,b,0,0,0)(-4b)$$
$$+ (0,0,0,0,b)(5b)\,,$$

$$(0,0,c,0,0,0) \supset (c,0,c,0,0)(0) + (0,c,0,c,0)(3c) + (0,c,0,0,c)(-3c)$$
$$+ (0,0,c,0,0)(-6c) + (0,0,c,0,0)(6c)\,,$$

$$(0,0,0,0,0,d) \supset (0,d,0,0,0)(0) + (0,0,0,d,0)(3d) + (0,0,0,0,d)(-3d)\,,$$

$E_6 \supset A_5 \times A_1$:

$$(a,0,0,0,0,0) \supset (0,0,0,a,0)(0) + (a,0,0,0,0)(a)\,,$$

$$(0,b,0,0,0,0) \supset (b,0,0,b,0)(b) + (0,0,b,0,b)(0) + (0,b,0,0,0)(2b) + (2b,0,0,0,0)(0)\,,$$

$$(0,0,c,0,0,0) \supset (c,0,c,0,c)(c) + (0,c,0,c,0)(2c) + (0,c,0,0,2c)(0) + (2c,0,0,c,0)(0) + (0,0,2c,0,0)(0) + (0,0,c,0,0)(3c)\,,$$

$$(0,0,0,0,0,d) \supset (0,0,d,0,0)(d) + (d,0,0,0,d)(0) + (0,0,0,0,0)(2d)\,,$$

$E_6 \supset 3A_2$:

$$(a,0,0,0,0,0) \supset (a,0)(0,0)(0,a) + (0,a)(0,a)(0,0) + (0,0)(a,0)(a,0)\,,$$

$$(0,b,0,0,0,0) \supset (b,b)(0,b)(0,b) + (b,0)(b,0)(b,b) + (0,b)(b,b)(b,0) + (2b,0)(0,0)(b,0) + (b,0)(0,2b)(0,0) + (0,2b)(b,0)(0,0) + (0,b)(0,0)(0,2b) + (0,0)(2b,0)(0,b) + (0,0)(0,b)(2b,0)\,,$$

$$(0,0,c,0,0,0) \supset (c,c)(c,c)(c,c) + (2c,c)(0,c)(c,0) + (2c,0)(c,0)(2c,0) + (2c,0)(0,2c)(0,c) + (c,2c)(c,0)(0,c) + (c,0)(c,2c)(c,0) + (0,2c)(2c,0)(c,0) + (c,0)(0,c)(2c,c) + (c,0)(2c,0)(0,2c) + (0,2c)(0,c)(0,2c) + (0,c)(0,2c)(2c,0) + (0,c)(2c,c)(0,c) + (0,c)(c,0)(c,2c) + (3c,0)(0,0)(0,0) + (0,3c)(0,0)(0,0) + (0,0)(3c,0)(0,0) + (0,0)(0,3c)(0,0) + (0,0)(0,0)(3c,0) + (0,0)(0,0)(0,3c)\,,$$

$$(0,0,0,0,0,d) \supset (d,0)(0,d)(d,0) + (0,d)(d,0)(0,d) + (d,d)(0,0)(0,0) + (0,0)(d,d)(0,0) + (0,0)(0,0)(d,d)\,,$$

$E_6 \supset A_2 \times G_2$:

$$(a,0,0,0,0,0) \supset (a,0)(0,a) + (0,2a)(0,0) + 2(a,0)(0,0)\,,$$

$$(0,b,0,0,0,0) \supset (2b,0)(b,0) + (0,b)(0,2b) + 2(0,b)(b,0) + (b,2b)(0,b) + 3(2b,0)(0,b) + 2(0,b)(0,b) + 2(b,2b)(0,0) + 2(0,b)(0,0)\,,$$

$$(0,0,c,0,0,0) \supset (c,c)(c,c) + (2c,2c)(c,0) + (0,3c)(0,2c) + (3c,0)(0,2c)$$
$$+ (0,0)(0,3c) + 2(0,3c)(c,0) + 2(3c,0)(c,0) + 2(c,c)(0,2c)$$
$$+ 3(0,0)(c,c) + 3(2c,2c)(0,c) + 2(c,c)(c,0) + 2(0,3c)(0,c)$$
$$+ 2(3c,0)(0,c) + 6(0,0)(c,0) + 2(c,c)(0,c) + 2(0,3c)(0,0)$$
$$+ 2(3c,0)(0,0) + 3(0,0)(0,c)\,,$$

$$(0,0,0,0,0,d) \supset (d,d)(0,d) + (0,0)(d,0) + 3(0,0)(0,d) + 2(d,d)(0,0)\,,$$

$E_6 \supset C_4$:

$$(a,0,0,0,0,0) \supset (0,a,0,0) + 3(0,0,0,0)\,,$$

$$(0,b,0,0,0,0) \supset (b,0,b,0) + 3(0,0,0,b) + 3(2b,0,0,0) + 2(0,b,0,0)\,,$$

$$(0,0,c,0,0,0) \supset (0,0,2c,0) + (2c,0,0,c) + 2(0,c,0,c) + 2(2c,c,0,0)$$
$$+ 2(c,0,c,0) + 6(0,0,0,c) + 6(2c,0,0,0)\,,$$

$$(0,0,0,0,0,d) \supset (0,0,0,d) + (2d,0,0,0) + 2(0,d,0,0)\,,$$

$E_6 \supset F_4$:

$$(a,0,0,0,0,0) \supset (0,0,0,a) + 3(0,0,0,0)\,,$$

$$(0,b,0,0,0,0) \supset (0,0,b,0) + 3(b,0,0,0) + 2(0,0,0,b)\,,$$

$$(0,0,c,0,0,0) \supset (0,c,0,0) + 2(c,0,0,c) + 2(0,0,c,0) + 6(c,0,0,0)\,,$$

$$(0,0,0,0,0,d) \supset (d,0,0,0) + 2(0,0,0,d)\,,$$

$E_6 \supset A_2$:

$$(a,0,0,0,0,0) \supset (2a,2a) + (0,3a) + (3a,0) + 2(a,a) + 3(0,0)\,,$$

$$(0,b,0,0,0,0) \supset (2b,5b) + (5b,2b) + 3(3c,3c) + (6b,0) + (0,6b) + 5(b,4b)$$
$$+ 5(4b,b) + 4(2b,2b) + 7(0,3b) + 7(3b,0) + 8(b,b) + 6(0,0)\,,$$

$$(0,0,c,0,0,0) \supset (5b,5b) + 3(3b,6b) + 3(6b,3b) + (0,9b) + (9b,0)$$
$$+ 4(4b,4b) + 5(b,7b) + 5(7b,b) + 8(5b,2b) + 8(2b,5b) + 6(0,6b)$$
$$+ 9(3b,3b) + 6(6b,0) + 14(b,4b) + 14(4b,b) + 8(2b,2b) + 15(0,3b)$$
$$+ 15(3b,0) + 14(b,b) + 12(0,0)\,,$$

$$(0,0,0,0,0,d) \supset (b,4b) + (4b,b) + 2(2b,2b) + 3(3b,0) + 3(0,3b) + 5(b,b)\,,$$

$E_6 \supset G_2$:

$$(a,0,0,0,0,0) \supset (0,2a) + (a,0) + 2(0,a) + 3(0,0)\,,$$

$$\begin{aligned}(0,b,0,0,0,0) \supset{}& (b,2b) + (2b,0) + 3(0,3b) + 5(b,b) + 4(0,2b) + 7(b,0)\\ &+ 8(0,b) + 6(0,0)\,,\end{aligned}$$

$$\begin{aligned}(0,0,c,0,0,0) \supset{}& (0,5c) + (3c,0) + 3(c,3c) + 5(2c,c) + 4(0,4c) + 8(c,2c)\\ &+ 6(2c,0) + 9(0,3c) + 14(c,c) + 8(0,2c) + 15(c,0) + 14(0,c)\\ &+ 12(0,0)\,,\end{aligned}$$

$$(0,0,0,0,0,d) \supset (d,d) + 2(0,2d) + 3(d,0) + 5(0,d)\,.$$

4.3. Réduction d'orbites du groupe de Weyl de E_7

Les matrices de projection pour les onze sous-algèbres réductives maximales de E_7 sont listées ci-dessous. La première matrice $E_7 \to A_1$ est celle de la sous-algèbre principale de E_7, alors que la deuxième est celle de la sous-algèbre sous-principale de E_7.

$$E_7 \supset E_6 \times U_1 : \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \end{pmatrix}, \qquad E_7 \supset A_7 : \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$E_7 \supset A_5 \times A_2 : \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdot & 1 \\ \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \qquad E_7 \supset D_6 \times A_1 : \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 \end{pmatrix},$$

$$E_7 \supset C_3 \times G_2 : \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 & \cdot \end{pmatrix}, \qquad E_7 \supset F_4 \times A_1 : \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 2 & 1 & 2 & 1 & 1 \end{pmatrix},$$

$$E_7 \supset G_2 \times A_1 : \begin{pmatrix} \cdot & 1 & 1 & \cdot & 1 & 1 & 1 \\ 2 & 2 & 4 & 4 & 1 & \cdot & 1 \\ 2 & 4 & 4 & 5 & 4 & 1 & 3 \end{pmatrix}, \qquad E_7 \supset A_2 : \begin{pmatrix} 4 & 9 & 11 & 10 & 6 & 6 & 7 \\ 4 & 6 & 11 & 7 & 6 & \cdot & 4 \end{pmatrix},$$

$$E_7 \supset 2A_1 : \begin{pmatrix} 4 & 10 & 18 & 12 & 8 & 6 & 8 \\ 6 & 10 & 12 & 11 & 8 & 3 & 7 \end{pmatrix}, \qquad E_7 \supset A_1 : \begin{pmatrix} 34 & 66 & 96 & 75 & 52 & 27 & 49 \end{pmatrix},$$

$$E_7 \supset A_1 : \begin{pmatrix} 26 & 50 & 72 & 57 & 40 & 21 & 37 \end{pmatrix}.$$

Nous donnons des exemples de règles des branchement pour les orbites du groupe de Weyl de E_7 d'ordre 126 et 56 respectivement :

$E_7 \supset E_6 \times U_1$:

$$(a,0,0,0,0,0,0) \supset (0,0,0,0,0,a)(0) + (0,0,0,0,a,0)(2a) + (a,0,0,0,0,0)(-2a)\,,$$

$$(0,0,0,0,0,b,0) \supset (b,0,0,0,0,0)(b) + (0,0,0,0,b,0)(-b) + (0,0,0,0,0,0)(3b) + (0,0,0,0,0,0)(-3b)\,,$$

$E_7 \supset A_7$:

$$(a,0,0,0,0,0,0) \supset (0,0,0,a,0,0,0) + (a,0,0,0,0,0,a)\,,$$

$$(0,0,0,0,0,b,0) \supset (0,b,0,0,0,0,0) + (0,0,0,0,0,b,0)\,,$$

$E_7 \supset A_5 \times A_2$:

$$(a,0,0,0,0,0,0) \supset (a,0,0,0,a)(0,0) + (0,a,0,0,0)(0,a) + (0,0,0,a,0)(a,0) + (0,0,0,0,0)(a,a)\,,$$

$$(0,0,0,0,0,b,0) \supset (0,0,b,0,0)(0,0) + (b,0,0,0,0)(b,0) + (0,0,0,0,b)(0,b)\,,$$

$E_7 \supset D_6 \times A_1$:

$$(a,0,0,0,0,0,0) \supset (0,a,0,0,0,0)(0) + (0,0,0,0,a,0)(a) + (0,0,0,0,0,0)(2a)\,,$$

$$(0,0,0,0,0,b,0) \supset (0,0,0,0,0,b)(0) + (b,0,0,0,0,0)(b)\,,$$

$E_7 \supset C_3 \times G_2$:

$$(a,0,0,0,0,0,0) \supset (0,a,0)(0,a) + (0,0,0)(a,0) + (2a,0,0)(0,0) + 2(0,a,0)(0,0) + 3(0,0,0)(0,a)\,,$$

$$(0,0,0,0,0,b,0) \supset (b,0,0)(0,b) + (0,0,b)(0,0) + 2(b,0,0)(0,0)\,,$$

$E_7 \supset F_4 \times A_1$:

$$(a,0,0,0,0,0,0) \supset (a,0,0,0)(0) + (0,0,0,a)(2a) + 2(0,0,0,a)(0) + 3(0,0,0,0)(2a)\,,$$

$$(0,0,0,0,0,b,0) \supset (0,0,0,b)(b) + (0,0,0,0)(3b) + 3(0,0,0,0)(b)\,,$$

$E_7 \supset G_2 \times A_1$:

$$(a,0,0,0,0,0,0) \supset (0,2a)(2a) + (a,0)(2a) + (0,2a)(0) + (0,a)(4a) + 2(a,0)(0) + 3(0,a)(2a) + 4(0,a)(0) + (0,0)(4a) + 5(0,0)(2a)\,,$$

$$(0,0,0,0,0,b,0) \supset (b,0)(b) + (0,b)(3b) + 2(0,b)(b) + (0,0)(3b) + 3(0,0)(b)\,,$$

$E_7 \supset A_2$:

$$(a,0,0,0,0,0,0) \supset (4a,4a) + (5a,2a) + (2a,5a) + (6a,0) + 2(3a,3a) + (0,6a) + 2(4a,a) + 2(a,4a) + 3(2a,2a) + 3(3a,0) + 3(0,3a) + 5(a,a)\,,$$

$$(0,0,0,0,0,b,0) \supset (6b,0) + (0,6b) + (4b,b) + (b,4b) + 2(2b,2b) + 2(3b,0) + 2(0,3b) + 2(b,b) + 2(0,0)\,,$$

$E_7 \supset 2A_1$:

$$(a,0,0,0,0,0,0) \supset (4a)(6a) + (6a)(4a) + (8a)(2a) + (2a)(6a) + 2(4a)(4a) + 2(6a)(2a) + (8a)(0) + (0)(6a) + 3(2a)(4a) + 4(4a)(2a) + 2(6a)(0) + 3(0)(4a) + 5(2a)(2a) + 4(4a)(0) + 6(0)(2a) + 6(2a)(0)\,,$$

$$(0,0,0,0,0,b,0) \supset (6b)(3b) + (2b)(5b) + (4b)(3b) + (6b)(b) + (0)(5b) + 2(2b)(3b) + 2(4b)(b) + 2(0)(3b) + 3(2b)(b) + 3(0)(b)\,,$$

$E_7 \supset A_1$ (principale) :

$$(a,0,0,0,0,0,0) \supset (34a) + (32a) + (30a) + (28a) + 2(26a) + 2(24a) + 3(22a) + 3(20a) + 4(18a) + 4(16a) + 5(14a) + 5(12a) + 6(10a) + 6(8a) + 6(6a) + 6(4a) + 7(2a)\,,$$

$$(0,0,0,0,0,b,0) \supset (27b) + (25b) + (23b) + (21b) + (19b) + 2(17b) + 2(15b) + 2(13b) + 2(11b) + 3(9b) + 3(7b) + 3(5b) + 3(3b) + 3(b)\,,$$

$E_7 \supset A_1$ (sous-principale) :

$$
\begin{aligned}
(a,0,0,0,0,0,0) \supset\ & (26a) + (24a) + 2(22a) + 2(20a) + 3(18a) + 4(16a) \\
& + 5(14a) + 5(12a) + 7(10a) + 7(8a) + 8(6a) + 8(4a) + 9(2a) \\
& + 2(0)\,, \\
(0,0,0,0,0,b,0) \supset\ & (21b) + (19b) + (17b) + 2(15b) + 2(13b) + 3(11b) \\
& + 3(9b) + 3(7b) + 4(5b) + 4(3b) + 4(b)\,.
\end{aligned}
$$

4.4. Réduction d'orbites du groupe de Weyl de E_8

Les matrices de projection pour les onze sous-algèbres réductives maximales de E_8 sont listées ci-dessous. La première matrice $E_8 \to A_1$ est celle de la sous-algèbre principale de E_8, alors que la deuxième est celle de la sous-algèbre sous-principale de E_8.

$$
E_8 \supset A_8 : \begin{pmatrix}
\cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot \\
1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot \\
\cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot
\end{pmatrix}, \qquad
E_8 \supset 2A_4 : \begin{pmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 \\
\cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & 1 & 2 & 1 & \cdot & 1 \\
\cdot & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & 1
\end{pmatrix},
$$

$$
E_8 \supset D_8 : \begin{pmatrix}
\cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 \\
\cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}, \qquad
E_8 \supset E_7 \times A_1 : \begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1
\end{pmatrix},
$$

$$
E_8 \supset E_6 \times A_2 : \begin{pmatrix}
\cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 \\
\cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot
\end{pmatrix}, \qquad
E_8 \supset F_4 \times G_2 : \begin{pmatrix}
\cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot \\
1 & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\
1 & 1 & 2 & 1 & 1 & 2 & 1 & \cdot
\end{pmatrix},
$$

$$
E_8 \supset A_2 \times A_1 : \begin{pmatrix}
1 & 4 & 4 & 6 & 8 & 4 & 2 & 3 \\
1 & 1 & 4 & 6 & 5 & 4 & 2 & 3 \\
6 & 10 & 14 & 16 & 22 & 16 & 8 & 12
\end{pmatrix}, \qquad
E_8 \supset C_2 : \begin{pmatrix}
2 & 8 & 8 & 12 & 16 & 8 & 4 & 6 \\
3 & 3 & 7 & 8 & 9 & 8 & 4 & 6
\end{pmatrix},
$$

$$
E_8 \supset A_1 : \begin{pmatrix} 58 & 114 & 168 & 220 & 270 & 182 & 92 & 136 \end{pmatrix},
$$

$$
E_8 \supset A_1 : \begin{pmatrix} 46 & 90 & 132 & 172 & 210 & 142 & 72 & 106 \end{pmatrix},
$$

$$
E_8 \supset A_1 : \begin{pmatrix} 38 & 74 & 108 & 142 & 174 & 118 & 60 & 88 \end{pmatrix}.
$$

Nous donnons des exemples de règles des branchement pour les orbites du groupe de Weyl de E_8 d'ordre 240 :

$E_8 \supset A_8$:

$$(a,0,0,0,0,0,0,0) \supset (0,0,a,0,0,0,0,0) + (0,0,0,0,0,a,0,0) + (a,0,0,0,0,0,0,a)\,,$$

$E_8 \supset 2A_4$:

$$(a,0,0,0,0,0,0,0) \supset (a,0,0,0)(0,0,a,0) + (0,a,0,0)(a,0,0,0) + (0,0,a,0)(0,0,0,a) + (0,0,0,a)(0,a,0,0) + (a,0,0,a)(0,0,0,0) + (0,0,0,0)(a,0,0,a)\,,$$

$E_8 \supset D_8$:

$$(a,0,0,0,0,0,0,0) \supset (0,0,0,0,0,0,0,a) + (0,a,0,0,0,0,0,0)\,,$$

$E_8 \supset E_7 \times A_1$:

$$(a,0,0,0,0,0,0,0) \supset (a,0,0,0,0,0,0)(0) + (0,0,0,0,0,a,0)(a) + (0,0,0,0,0,0,0)(2a)\,,$$

$E_8 \supset E_6 \times A_2$:

$$(a,0,0,0,0,0,0,0) \supset (0,0,0,0,0,a)(0,0) + (a,0,0,0,0,0)(a,0) + (0,0,0,0,a,0)(0,a) + (0,0,0,0,0,0)(a,a)\,,$$

$E_8 \supset F_4 \times G_2$:

$$(a,0,0,0,0,0,0,0) \supset (0,0,0,a)(0,a) + (a,0,0,0)(0,0) + 2(0,0,0,a)(0,0) + (0,0,0,0)(a,0) + 3(0,0,0,0)(0,a)\,,$$

$E_8 \supset A_2 \times A_1$:

$$(a,0,0,0,0,0,0,0) \supset (a,a)(6a) + (3a,0)(4a) + (0,3a)(4a) + (2a,2a)(2a) + 3(a,a)(4a) + 2(3a,0)(2a) + 2(0,3a)(2a) + (2a,2a)(0) + 2(0,0)(6a) + 5(a,a)(2a) + 2(3a,0)(0) + 2(0,3a)(0) + 4(0,0)(4a) + 6(a,a)(0) + 8(0,0)(2a) + 2(0,0)(0)\,,$$

$E_8 \supset C_2$:

$$(a,0,0,0,0,0,0,0) \supset (2a,3a) + (6a,0) + (0,4a) + 2(4a,a) + 3(2a,2a)$$
$$+ 4(0,3a) + 4(4a,0) + 6(2a,a) + 6(0,2a) + 9(2a,0) + 10(0,a)$$
$$+ 4(0,0)\,,$$

$E_8 \supset A_1$ (principale) :

$$(a,0,0,0,0,0,0,0) \supset (58a) + (56a) + (54a) + (52a) + (50a) + (48a)$$
$$+ 2(46a) + 2(44a) + 2(42a) + 2(40a) + 3(38a) + 3(36a) + 4(34a)$$
$$+ 4(32a) + 4(30a) + 4(28a) + 5(26a) + 5(24a) + 6(22a) + 6(20a)$$
$$+ 6(18a) + 6(16a) + 7(14a) + 7(12a) + 7(10a) + 7(8a) + 7(6a)$$
$$+ 7(4a) + 8(2a)\,,$$

$E_8 \supset A_1$ (sous-principale) :

$$(a,0,0,0,0,0,0,0) \supset (46a) + (44a) + (42a) + (40a) + 2(38a) + 2(36a)$$
$$+ 3(34a) + 3(32a) + 3(30a) + 4(28a) + 5(26a) + 5(24a) + 6(22a)$$
$$+ 6(20a) + 7(18a) + 7(16a) + 8(14a) + 8(12a) + 9(10a) + 9(8a)$$
$$+ 9(6a) + 9(4a) + 10(2a) + 2(0)\,,$$

$E_8 \supset A_1$:

$$(a,0,0,0,0,0,0,0) \supset (38a) + (36a) + 2(34a) + 2(32a) + 2(30a) + 3(28a)$$
$$+ 4(26a) + 4(24a) + 6(22a) + 6(20a) + 7(18a) + 8(16a) + 9(14a)$$
$$+ 9(12a) + 10(10a) + 10(8a) + 11(6a) + 11(4a) + 12(2a) + 4(0)\,.$$

4.5. Réduction d'orbites du groupe de Weyl de F_4

Les matrices de projection pour les cinq sous-algèbres réductives maximales de F_4 sont :

$$F_4 \supset 2A_2 : \begin{pmatrix} \cdot & \cdot & 1 & 1 \\ \cdot & 2 & 1 & \cdot \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & \cdot \end{pmatrix} , \qquad F_4 \supset B_4 : \begin{pmatrix} \cdot & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \end{pmatrix} ,$$

$$F_4 \supset C_3 \times A_1 : \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 2 & 3 & 2 & 1 \end{pmatrix} , \quad F_4 \supset G_2 \times A_1 : \begin{pmatrix} \cdot & 1 & 1 & \cdot \\ 1 & 1 & \cdot & 1 \\ 4 & 4 & 4 & 2 \end{pmatrix} ,$$

$$F_4 \supset A_1 : \begin{pmatrix} 22 & 42 & 30 & 16 \end{pmatrix} .$$

Nous donnons des exemples de règles des branchement pour les orbites du groupe de Weyl de F_4 d'ordre 24, 96, 96, et 24 respectivement :

$F_4 \supset 2A_2$:

$$(a,0,0,0) \supset (0,2a)(a,0) + (2a,0)(0,a) + (0,0)(a,a) ,$$

$$\begin{aligned}(0,b,0,0) \supset\ & (2b,2b)(b,b) + (0,2b)(2b,b) + (2b,0)(b,2b) + (4b,0)(b,0) \\ & + (0,4b)(0,b) + (0,0)(3b,0) + (0,0)(0,3b) ,\end{aligned}$$

$$\begin{aligned}(0,0,c,0) \supset\ & (c,c)(c,c) + (2c,c)(c,0) + (c,2c)(0,c) + (0,c)(2c,0) \\ & + (c,0)(0,2c) + (0,3c)(0,0) + (3c,0)(0,0) ,\end{aligned}$$

$$(0,0,0,d) \supset (d,0)(d,0) + (0,d)(0,d) + (d,d)(0,0) ,$$

$F_4 \supset B_4$:

$$(a,0,0,0) \supset (0,a,0,0) ,$$

$$(0,b,0,0) \supset (b,0,b,0) ,$$

$$(0,0,c,0) \supset (c,0,0,c) + (0,0,c,0) ,$$

$$(0,0,0,d) \supset (0,0,0,d) + (d,0,0,0) ,$$

$F_4 \supset C_3 \times A_1$:

$$(a,0,0,0) \supset (0,0,a)(a) + (2a,0,0)(0) + (0,0,0)(2a) ,$$

$$(0,b,0,0) \supset (0,0,b)(3b) + (0,2b,0)(2b) + (2b,0,b)(b) + (0,0,2b)(0) ,$$

$$(0,0,c,0) \supset (0,c,0)(2c) + (c,c,0)(c) + (c,0,c)(0)\,,$$

$$(0,0,0,d) \supset (d,0,0)(d) + (0,d,0)(0)\,,$$

$F_4 \supset G_2 \times A_1$:

$$(a,0,0,0) \supset (0,a)(4a) + (a,0)(0) + (0,a)(0)\,,$$

$$(0,b,0,0) \supset (b,b)(4b) + (b,0)(8b) + (0,3b)(0) + (b,b)(0) + 2(b,0)(4b) + (0,b)(8b) + (0,b)(0)\,,$$

$$(0,0,c,0) \supset (c,0)(4c) + (0,2c)(2c) + (0,c)(6c) + 2(c,0)(2c) + (0,2c)(0) + (0,c)(4c) + (0,c)(2c) + 2(0,0)(6c) + 2(0,0)(0)\,,$$

$$(0,0,0,d) \supset (0,d)(2d) + (0,d)(0) + (0,0)(4d) + 2(0,0)(2d)\,,$$

$F_4 \supset A_1$:

$$(a,0,0,0) \supset (22a) + (20a) + (18a) + (14a) + (12a) + 2(10a) + (8a) + (6a) + (4a) + 2(2a)\,,$$

$$(0,b,0,0) \supset (42b) + (40b) + (38b) + (36b) + 2(34b) + (32b) + 4(30b) + (28b) + 4(26b) + 3(24b) + 2(22b) + 3(18b) + 3(16b) + 4(14b) + (12b) + 2(10b) + 2(8b) + 6(6b) + (4b) + 3(2b) + 4(0)\,,$$

$$(0,0,c,0) \supset (30c) + (28c) + 2(26c) + 3(24c) + 2(22c) + 3(20c) + 5(18c) + 3(16c) + 3(14c) + 4(12c) + 4(10c) + 2(8c) + 4(6c) + 4(4c) + 3(2c) + 8(0)\,,$$

$$(0,0,0,d) \supset (16d) + (14d) + (12d) + (10d) + 2(8d) + 2(6d) + 2(4d) + 2(2d)\,.$$

4.6. Réduction d'orbites du groupe de Weyl de G_2

Les matrices de projection pour les trois sous-algèbres réductives maximales de G_2 sont :

$$G_2 \supset A_2 : \begin{pmatrix} 1 & 1 \\ 1 & \cdot \end{pmatrix}\,, \quad G_2 \supset 2A_1 : \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}\,, \quad G_2 \supset A_1 : \begin{pmatrix} 10 & 6 \end{pmatrix}\,.$$

Nous donnons des exemples de règles des branchement pour les orbites du groupe de Weyl de G_2 d'ordre 6, 6 et 12 respectivement :

$G_2 \supset A_2$:

$(a, 0) \supset (a, a)$,

$(0, b) \supset (b, 0) + (0, b)$,

$(a, b) \supset (a + b, a) + (a, a + b)$,

$G_2 \supset 2A_1$:

$(a, 0) \supset (a)(3a) + (2a)(0), ,$

$(0, b) \supset (b)(b) + (0)(2b)$,

$(a, b) \supset (a + b)(3a + b) + (2a + b)(b) + (a)(3a + 2b)$,

$G_2 \supset A_1$:

$(a, 0) \supset (10a) + (8a) + (2a)$,

$(0, b) \supset (6b) + (4b) + (2b)$,

$(a, a) \supset (16a) + 2(14a) + (10a) + (6a) + 2(0)$,

$$(a, b) \supset (10a + 6b) + (10a + 4b) + (8a + 6b) + (8a + 2b) + (2a + 4b) + (2|a - b|) .$$

Chapitre 5

CENTRALIZERS OF MAXIMAL REGULAR SUBGROUPS IN SIMPLE LIE GROUPS AND RELATIVE CONGRUENCE CLASSES OF REPRESENTATIONS

Référence complète : M. Larouche, F. W. Lemire et J. Patera, Centralizers of maximal regular subgroups in simple Lie groups and relative congruence classes of representations, *Journal of Physics A : Mathematical and Theoretical*, 44(41) : 415204, 23, 2011. Sélectionné pour être inclus dans *IOP Select*, Institute of Physics, United Kingdom.

Résumé

Dans cet article, nous fournissons une nouvelle description, uniforme et complète, des centralisateurs des sous-groupes réguliers maximaux des groupes de Lie simples de tous types et de tous rangs. Le centralisateur est soit un produit direct de groupes cycliques finis, un groupe continu de rang 1, ou un produit, pas nécessairement direct, d'un groupe continu de rang 1 avec un groupe cyclique fini. Nous présentons des formules explicites pour l'action de tels centralisateurs sur les représentations irréductibles des algèbres de Lie simples.

Abstract

In the paper we present a new, uniform and comprehensive description of centralizers of the maximal regular subgroups in compact simple Lie groups of all types and ranks. The centralizer is either a direct product of finite cyclic

groups, a continuous group of rank 1, or a product, not necessarily direct, of a continuous group of rank 1 with a finite cyclic group. Explicit formulas for the action of such centralizers on irreducible representations of the simple Lie algebras are given.

5.1. INTRODUCTION

Let G be a connected simple Lie group with corresponding Lie algebra denoted by L. Let L' be a maximal regular semisimple Lie subalgebra of L with corresponding subgroup G'. The goal of this paper is to study the centralizer of G' in G and its action on the representations of the Lie algebra L. In general these centralizers are abelian subgroups of G. The first complete description of the continuous centralizers, whenever they exist, was given by Borel and de Siebenthal [**4**], while the cases of discrete centralizers were first described by Dynkin and Oniščik [**15**].

In this paper we reformulate the results of [**4**] and [**15**] in a more accessible manner, using tools which were not available to the original authors. The existence and structure of the centralizer is made immediately visible from a decoration of the extended Dynkin-Coxeter diagram. In addition we provide explicit formulas for the actions of these centralizers on the finite-dimensional irreducible representations of L and apply this information to the branching rules of L with respect to L'. We observe in particular that the centralizer of G' in G is either a direct product of finite cyclic groups (in the maximal regular semisimple case), a continuous group of rank 1 or a product, not necessarily direct, of a continuous group of rank 1 with a finite cyclic group (in the maximal regular reductive case).

The eigenvalues of these operators serve to decompose the irreducible representations of L into representations of L'. Projection matrices provided in [**32, 34, 40**] transform the weights of an irreducible representation of L into weights of the representations of the subalgebra. We can include, as an additional label, the eigenvalue of the action of the centralizer vector that serves to decompose the irreducible representation of L into a direct sum of representations of L'. Note that the representation of L' corresponding to a fixed eigenvalue may not be irreducible.

In physics the importance of the centralizers has been recognized for a long time. One of the best known examples occurs in the case $SU(3) \supset SU(2) \times U_1$. Here the centralizer is a continuous 1-parametric subgroup denoted U_1. The existence, structure and application of the centralizers in specific representations is not as well known. As two of the lowest examples one can point out the cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ in $Sp(2) \supset SU(2) \times SU(2) \times \mathbb{Z}_2$ and $G_2 \supset SU(3) \times \mathbb{Z}_3$ respectively. One of the consequences of the presence of a centralizer $\mathbb{Z}_n$ is that it splits irreducible representations of the subalgebra/subgroup into n equivalence classes. Undoubtedly such classes would find a physical interpretation in some cases. We call them *relative congruence classes* in this paper.

Discrete centralizers of maximal regular semisimple subalgebras are found in all simple Lie algebras except A_n $(1 \leq n < \infty)$. In all cases they are formed as a product of up to three cyclic groups. Continuous centralizers of maximal regular reductive subalgebras appear in all simple Lie algebras, except in G_2, F_4, and E_8.

Note that we use Dynkin notations and numberings for roots, weights and diagrams.

5.2. THE CENTER OF G

We start by reviewing the well-known results concerning the center of the simple Lie groups. We use the standard notation to identify the simple Lie groups G and their corresponding simple Lie algebras, namely there are four infinite classes denoted A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 2)$ and D_n $(n \geq 4)$ as well as five exceptional groups/algebras denoted by E_6, E_7, E_8, F_4 and G_2. The structure and properties of these Lie groups and their corresponding Lie algebras is encoded in their so-called decorated extended Dynkin diagrams (see Figure 5.1). The node in these diagrams labelled by 0 denotes α_0, the negative of the highest root of the algebra. The remaining nodes represent the simple roots $\{\alpha_1, \dots, \alpha_n\}$ of the algebra. The mark m_k on the simple root α_k for $k = 1, \dots, n$ denotes the coefficient of α_k in the expansion of the highest root $-\alpha_0$ in terms of the simple roots α_i (see Figure 5.1). The mark on α_0, by convention, is 1. Note that the algebras B_2 and C_2 are isomorphic, and the extended Dynkin diagram of B_2 is the same as the one of C_2, with the only difference being in the numbering of the nodes : the roots α_1 and α_2 are interchanged.

From **[27]** we know that the conjugacy classes of elements of finite order in G of rank n are specified in a bijective fashion by the set of all $(n+1)$-tuples of relatively prime non-negative integers. To each such $(n+1)$-tuple $[s_0, s_1, \dots, s_n]$ with $s_i \in \mathbb{Z}_{\geq 0}$ we associate the point X in the fundamental region of G given by

$$X = \frac{s_1}{M}\omega_1 + \cdots + \frac{s_n}{M}\omega_n$$

where $M = s_0 + \sum_{i=1}^{n} m_i s_i$ and the $\omega_i's$ denote the fundamental weights of the algebra. The order of the element of G corresponding to such an X is M.

The elements of the center $Z(G)$ of the simple Lie group G are in one-to-one correspondence with the nodes of the corresponding extended diagram that carry marks equal to 1. They are in fact associated with the corners of the fundamental region of G. The extension node, which always has its mark equal to 1, refers to the identity element of G. Explicitly, if $\{\hat{\omega}_i | i = 1, \dots, n\}$ denotes the basis of the Cartan subalgebra $\mathcal{H}$ of L which is dual to the base of simple roots $\{\alpha_i | i = 1, \dots, n\}$ of $\mathcal{H}^*$ in the sense that $\alpha_i(\hat{\omega}_j) = \delta_{i,j}$, then the elements of the center of G consist of all elements $e^{2\pi i \hat{\omega}_k}$ where α_k has mark $m_k = 1$. In table 5.5, for each simple Lie group admitting a non-trivial center, we list for reference the group structure as well as a generator of the center.

For any irreducible representation of the group G the central elements act as multiples of the identity. The collection of all finite-dimensional irreducible representations can then be partitioned according to the action of the central elements. Each equivalence class of irreducible representations with respect to this equivalence is called a *congruence class*. The concept of congruence classes has application in the decomposition of representations such as tensor products of irreducible representations, see for example **[35]**.

Let us consider an irreducible finite-dimensional representation of G having highest weight $\lambda = \sum_{i=1}^{n} m_i \omega_i$. Let $z = e^{2\pi i \hat{\omega}_j}$ be a non trivial element of the center $Z(G)$. Then the eigenvalue of z acting on this representation is given by $e^{2\pi i \lambda(\hat{\omega}_j)}$.

If we write

$$\hat{\omega}_j = \frac{1}{C} \sum_{i=1}^{n} r_i \hat{\alpha}_i$$

where $\{\hat{\alpha}_1, \dots, \hat{\alpha}_n\}$ is the basis of $\mathcal{H}$ dual to the basis of fundamental weights $\{\omega_1, \dots, \omega_n\}$ i.e. $\omega_i(\hat{\alpha}_j) = \delta_{i,j}$, and where C is the determinant of the Cartan matrix of G, we have that

$$\lambda(\hat{\omega}_j) = \frac{1}{C} \sum_{i=1}^{n} r_i m_i \, .$$

Since the eigenvalue of the central element z is $e^{2\pi i \lambda(\hat{\omega}_j)}$, we are really interested in the value of $\lambda(\hat{\omega}_j) \mod \mathbb{Z}$, which is uniquely determined by $\zeta_z := \sum_{i=1}^{n} r_i m_i \mod C$. The values ζ_z are listed in table 5.5 for each non-trivial central element of G. By convention, we list the value ζ_z, where $z = e^{2\pi i \hat{\omega}_j}$, next to the j^{th} node in the extended Dynkin diagram of G. We write 1 next to the extension node since it represents the identity of G.

5.3. BRANCHING RULES AND PROJECTION MATRICES

Reduction of weight systems of irreducible finite-dimensional representations of simple Lie algebras to weight systems of representations of their maximal reductive subalgebras has been addressed several times in the literature **[39, 40, 41, 61]**. In physics that problem is often referred to as the *computation of branching rules*.

The branching rule for $L \supset L'$, where L' is a maximal reductive subalgebra of L, is a linear transformation between Euclidean spaces $\mathbb{R}^n \to \mathbb{R}^m$, where n and m are the ranks of L and L' respectively. This linear transformation can be expressed in the form of an $m \times n$ matrix, the *projection matrix*. A suitable choice of bases allows us to obtain integer matrix elements in all the projection matrices we use here. The main advantage of the projection matrix method is the uniformity of its application as to the different algebra-subalgebra pairs, which makes it particularly amenable to computer implementation.

The projection matrix method, used in [**39, 40, 41, 61**], can also be extended to compute the branching rules of orbits of Weyl groups $W(L)$ of semisimple Lie algebras L. An orbit of $W(L)$ is a finite set of points of $\mathbb{R}_n$ obtained from the action of $W(L)$ on a single point of $\mathbb{R}_n$. Weyl group orbits are closely related to weight systems of finite-dimensional irreducible representations of semisimple Lie algebras. Indeed a weight system consists of many orbits of the corresponding Weyl group, a specific orbit often appearing more than once. Which orbits a particular representation is comprised of is well known, and extensive tables of multiplicities of dominant weights can be found in [**6**]. Considering the reduction of individual orbits rather than of entire weight systems offers some advantages, one of which is computational : while the number weights of a weight system grows without limits with the dimension of the representation, the number of points of an individual orbit is at most the order of the corresponding Weyl group. When dealing with large-scale computation for representations, one often needs to break down the problem into smaller ones for individual orbits. Orbit-orbit branching rules are computed with the projection matrix method for orbits of $W(A_n)$ in [**34**] and for orbits of $W(B_n)$, $W(C_n)$ and $W(D_n)$ in [**32**].

The projection matrix P for a particular pair $L \supset L'$ is calculated from one known branching rule. The classification of maximal reductive subalgebras of simple Lie algebras [**4, 14**] provides the information to find that branching rule. The projection matrix is then obtained using the weight systems of the representations, by requiring that weights of L be transformed by P to weights of L'. Since any ordering of the weights is admissible, the projection matrix is not unique. However, by ordering the weights of L by levels and by doing the same with the weights of L', the projection matrix obtained is convenient for large-scale computation, because dominant weights of L' will always be in the first half of the weights found by multiplying the weights of L. Hence the problem is already reduced by half.

The projection matrices we will use in this paper are the ones provided by **[34]** for reductions involving the Lie algebra A_n, by **[32]** for reductions involving the Lie algebras B_n, C_n and D_n, and by **[40]** for the ones involving the exceptional Lie algebras.

5.4. Discrete centralizers

Let G be a connected simple Lie group with its corresponding Lie algebra of rank n denoted by L. Any maximal regular semisimple subalgebra of L having rank n can be realized in terms of the extended Dynkin diagram of L. In fact any such subalgebra L', with corresponding subgroup G', corresponds to the Dynkin diagram resulting from deleting one node having prime mark from the extended Dynkin diagram of L. Clearly such maximal regular semisimple subalgebras occur for all simple Lie algebras except A_n $(1 \leq n < \infty)$. Since G' is a maximal regular semisimple subgroup of G the centralizer $C_G(G')$ of G' in G consists of all elements $e^{2\pi i h}$ where $h \in \mathcal{H}$ has the property that for all roots β of the subalgebra L' we have $e^{2\pi i \beta(h)} = 1$ or equivalently $\beta(h) \in \mathbb{Z}$. It follows that the centralizer is a discrete abelian subgroup of the group G. In fact, the centralizer contains the center $Z(G)$ of the group G, the center $Z(G')$ of the group G' as well as the group generated by the element $e^{\frac{2\pi i}{m_k}\hat{\omega}_k}$, where α_k denotes the deleted node with prime mark m_k and $\hat{\omega}_k$ is the element of the Cartan subalgebra $\mathcal{H}$ of L such that $\alpha_i(\hat{\omega}_k) = \delta_{i,k}$ for all $i = 1, \ldots, n$. This situation could be complicated by the fact that these three discrete groups have a non trivial intersection, but the following lemma simplifies it.

Lemma 5.4.1. *Let G be a connected simple Lie group, with corresponding Lie algebra L, and G' be a maximal regular semisimple Lie subgroup of G with corresponding subalgebra L'. Let α_k denote the deleted node from the extended Dynkin diagram of G, having prime mark m_k. Then*

(i) $C_G(G')/Z(G) \cong \mathbb{Z}_{m_k}$ *and*

(ii) $C_G(G') = Z(G')$.

PROOF. *(i)* From the construction of G' from the Dynkin diagram of G, it is clear that

$$C_G(G') = \left\langle Z(G), Z(G'), e^{\frac{2\pi i}{m_k}\hat{\omega}_k} \right\rangle$$

and that

$$Z(G') \subseteq \left\langle Z(G), e^{\frac{2\pi i}{m_k}\hat{\omega}_k} \right\rangle .$$

By combining those two observations, we thus find

$$C_G(G') = \left\langle Z(G), e^{\frac{2\pi i}{m_k}\hat{\omega}_k} \right\rangle$$

and we have the first result

$$C_G(G')/Z(G) \cong \mathbb{Z}_{m_k} .$$

(ii) To prove the second result, it suffices to find an element in the center of G' which is not in the center of G. Suppose such an element x exists :

$$\exists x \in Z(G')\backslash Z(G) \Rightarrow \langle x + Z(G)\rangle \cong \mathbb{Z}_{m_k}$$

since $\mathbb{Z}_{m_k}$ is a simple group and has no proper subgroup. But from *(i)* we have that

$$\mathbb{Z}_{m_k} \cong C_G(G')/Z(G)$$

and thus find

$$Z(G') = C_G(G') .$$

Now, it remains to demonstrate that such an element x exists in all cases. It is easy to see that in all cases except for the $D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$ case, whenever a node having prime mark is deleted from the extended diagram of G, a new node in the diagram of G' has its mark equal to 1. Hence a new element is added to the center of G'. In the $D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$ case, all the remaining nodes after the deletion already had their marks equal to 1 in the diagram of $G = D_4$. However, the extended node, the α_0 node, which corresponded to the identity in $G = D_4$, becomes a non trivial element in $G' = A_1 \oplus A_1 \oplus A_1 \oplus A_1$ and so is the element x we were looking for. □

5.4.1. Discrete centralizers and representations

Let $\phi(G)$ be an irreducible finite-dimensional representation of G, acting as a set of linear transformations in V_ϕ. An element $z = e^{ix}$ of the centralizer $C_G(G')$ of G' in G acts on any finite-dimensional irreducible representation $\phi_\lambda(G')$, arising during the restriction of a representation $\phi(G)$ to the subgroup :

$$\phi(G) = \bigoplus_\lambda \phi_\lambda(G') \quad \Longleftrightarrow \quad \phi(L) = \bigoplus_\lambda \phi_\lambda(L')$$

as a multiple of the identity matrix :

$$\phi(z)\phi(G)\phi(z^{-1}) = \phi(z)\left(\bigoplus_\lambda \phi_\lambda(G')\right)\phi(z^{-1}) = \bigoplus_\lambda \kappa_\lambda \phi_\lambda(G')\,.$$

If $z^N = 1$, the eigenvalues κ_λ are N-th roots of 1.

A discrete centralizer $C_G(G')$ is a product of cyclic groups. Hence it consists of elements of G which are of finite order. For our task it suffices to describe just one element which generates each cyclic subgroup in the centralizer. More precisely, we need to determine the eigenvalues κ_λ of such elements on every $\phi_\lambda(G')$.

We are interested, within each G-conjugacy class, by its unique element represented by a diagonal matrix in every V_ϕ. A general method of describing diagonal representatives of conjugacy classes of elements of finite order in G is found in **[45]**. Here we use it just for the elements of the centralizers.

Suppose $\phi(z)$ is the diagonal matrix representing the element $z = e^{ix} \in G$ in V_ϕ. Suppose further that V_ϕ is decomposed into the sum of its weight subspaces :

$$V_\phi = \sum_\mu V_\phi(\mu)\,.$$

Then for any vector $v \in V_\phi(\mu)$ we have

$$\phi(z)v = \kappa_\mu v$$

where

$$\kappa_\mu = e^{i\mu(x)}\,.$$

For example, if we take z to be the element $e^{\frac{2\pi i}{m_k}\hat{\omega}_k} \in C_G(G')$, then the eigenvalue κ_μ can be calculated in the same fashion as in section 5.2. If $\mu = \sum_{i=1}^{n} m_i\omega_i$, by writing $\hat{\omega}_k$ in terms of the $\hat{\alpha}_i$'s, we get :

$$\hat{\omega}_k = \frac{1}{C}\sum_{i=1}^{n} r_i\hat{\alpha}_i$$

where C is the determinant of the Cartan matrix of G, and therefore have that

$$\frac{\mu(\hat{\omega}_k)}{m_k} = \frac{1}{m_k C}\sum_{i=1}^{n} r_i m_i\,.$$

Again, we are really interested in the value of $\frac{\mu(\hat{\omega}_k)}{m_k} \mod \mathbb{Z}$, which can be given by a congruence equation of the form $\sum_{i=1}^{n} r_i m_i \mod m_k C$.

5.4.2. Relative congruence classes and branching rules

The decomposition of an irreducible representation of L into a sum of irreducible representations of L' is known as the branching rule for the pair $L \supset L'$. In general, if we start with a finite-dimensional representation of G then the elements in the centralizer of G' can be used to provide partial invariants for the summands in the branching rule. The sets of weights of L on which the centralizer elements take on constant values are called *relative congruence classes*.

Example 5.4.2 below illustrates the use of relative congruence classes in branching rules.

5.4.3. Explanation of tables 5.6 and 5.7

Tables 5.6 and 5.7 present the structure of the centralizers and the relative congruence classes for all maximal regular semisimple subalgebras in classical and exceptional simple Lie algebras, respectively. For each such algebra-subalgebra pair $L \supset L'$, with associated groups $G \supset G'$, we give the structure of the centralizer of G' in G, $C_G(G')$, which is always a product of cyclic groups.

Since $C_G(G') = Z(G')$, we give the generators of the centralizer by computing the generator of the center of each simple part of the subgroup G'. The embedding we choose for our task is the one provided by the corresponding projection matrix, which can be found in **[32]** for the classical cases and in **[40]** for the exceptional ones. As it was discussed in section 5.3, this particular choice offers computation efficiency. Assume that $\{\alpha_1, \dots, \alpha_n\}$ are the simple roots of the simple Lie algebra L and let $-\alpha_0$ denote the highest root. A maximal regular semisimple subalgebra L' of L can be realized as the subalgebra with simple roots $\{\alpha_0, \alpha_1, \dots, \alpha_n\} \setminus \{\alpha_k\}$ where α_k is a simple root of L with prime mark m_k. We saw in Lemma 5.4.1 that the centralizer of G' in G is generated by the center of the Lie group G together with the element

$$e^{\frac{2\pi i}{m_k}\hat{\omega}_k}.$$

In order to relate this information in the context of projection matrices we note that there exists a Weyl automorphism σ of L which transforms the subalgebra L' to the corresponding subalgebra L'' used to produce the projection matrix for the branching rule for $L \supset L''$. If G'' is the Lie group associated with L'', it follows that the centralizer of G'' in G is generated by the center of G together with

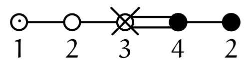
$$e^{\frac{2\pi i}{m_k}\sigma(\hat{\omega}_k)}.$$

As explained in subsection 5.4.1, the eigenvalue of the action of an element of $C_G(G')$ on a $\mu = \sum_{i=1}^{n} m_i\omega_i$ weight subspace is uniquely determined by the value of its exponent, which can be given in the form of a congruence equation. In tables 5.6 and 5.7, for each pair $L \supset L'$, we provide a generator of the center of each simple part of L' as a congruence equation. Furthermore, we give the structure of the quotient $C_G(G')/Z(G)$ and an element that generates it. More precisely, if $C_G(G')/Z(G) = \langle x + Z(G) \rangle$, we give the element x, again as a congruence equation. That particular equation is really the *relative congruence equation*, as it provides the new partial invariants for the summands in the branching rule. Since the element associated to the deleted node α_k is certainly a suitable x, we take x to be $e^{\frac{2\pi i}{m_k}\sigma(\hat{\omega}_k)}$, where σ is the automorphism corresponding to the projection matrix.

Note that for all cases where the index k appears in table 5.6, for example $B_n \supset B_k \oplus D_{n-k}$, the inequality $k \geq n - k$ holds.

Example 5.4.1.

Let us consider the case $F_4 \supset A_2 \oplus A_2$*. In this case the simple root* α_2 *with mark 3 is deleted from the extended Dynkin diagram, where the dotted node represents the extension* (α_0) *:*



First we determine the Weyl automorphism σ*, which transforms the subalgebra* $(A_2 \oplus A_2)'$ *having coroots* $\{\hat{\alpha}_0, \hat{\alpha}_1\}$ *and* $\{\hat{\alpha}_3, \hat{\alpha}_4\}$ *to the corresponding subalgebra* $A_2 \oplus A_2$ *used to produce the projection matrix for the branching rule for* $F_4 \supset A_2 \oplus A_2$*. This can accomplished by noting from the projection matrix* **[40]**

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

that

$$\sigma(\hat{\alpha}_0) = \hat{\alpha}_3 + \hat{\alpha}_4; \sigma(\hat{\alpha}_1) = 2\hat{\alpha}_2 + \hat{\alpha}_3; \sigma(\hat{\alpha}_3) = \hat{\alpha}_1 + 2\hat{\alpha}_2 + \hat{\alpha}_3 + \hat{\alpha}_4; \sigma(\hat{\alpha}_4) = \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3.$$

Therefore the coroots of $A_2 \oplus A_2$ *in our chosen embedding are*

$$\{\hat{\alpha}_3 + \hat{\alpha}_4; 2\hat{\alpha}_2 + \hat{\alpha}_3\}; \{\hat{\alpha}_1 + 2\hat{\alpha}_2 + \hat{\alpha}_3 + \hat{\alpha}_4; \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3\}.$$

Applying the results of table 5.5 for A_n *to the case* $n = 2$*, we know that the eigenvalue of the action of a generator of* $Z(A_2)$ *on a representation of highest weight* $\mu = m_1\omega_1 + m_2\omega_2$ *is given by the congruence equation*

$$m_1 + 2m_2 \mod 3\,.$$

In our situation, this information can be easily translated : the eigenvalue of the action of a generator of the center of the first A_2 *on a* $\mu = \sum_{i=1}^{4} m_i\omega_i$ *weight subspace is given by the congruence equation*

$$(m_3 + m_4) + 2(2m_2 + m_3) \equiv m_2 + m_4 \mod 3\,.$$

Similarly, for the second A_2 *we find*

$$(m_1 + 2m_2 + m_3 + m_4) + 2(m_1 + m_2 + m_3) \equiv m_2 + m_4 \mod 3\,.$$

Therefore, if we define $a := m_2 + m_4 \mod 3$*, we have that*

$$C_{F_4}(A_2 \oplus A_2) \cong \mathbb{Z}_3 = \langle a \rangle\,.$$

Now, since the center of F_4 *is trivial, we already know that*

$$C_{F_4}(A_2 \oplus A_2)/Z(F_4) \cong \mathbb{Z}_3 = \langle a \rangle \, .$$

However, let us present a uniform method to compute a generator of the quotient. We want to find the action of $e^{\frac{2\pi i}{3}\sigma(\hat{\omega}_2)}$. *First, we compute*

$$\hat{\omega}_2 = 3\hat{\alpha}_1 + 6\hat{\alpha}_2 + 4\hat{\alpha}_3 + 2\hat{\alpha}_4 \, .$$

Since $\hat{\alpha}_0 = -2\hat{\alpha}_1 - 3\hat{\alpha}_2 - 2\hat{\alpha}_3 - \hat{\alpha}_4$ *we conclude that*

$$\sigma(\hat{\alpha}_2) = -\hat{\alpha}_1 - 3\hat{\alpha}_2 - 2\hat{\alpha}_3 - \hat{\alpha}_4 \, .$$

Therefore, by substitution, we have

$$\sigma(\hat{\omega}_2) = -2\hat{\alpha}_2 - 3\hat{\alpha}_3 - 2\hat{\alpha}_4.$$

From this we have that the action of $e^{\frac{2\pi i}{3}\sigma(\hat{\omega}_2)}$ *on a weight* $\mu = \sum_{i=1}^{4} m_i\omega_i$ *is given in modular form by*

$$a : m_2 + m_4 \mod 3$$

and so that

$$C_{F_4}(A_2 \oplus A_2)/Z(F_4) \cong \mathbb{Z}_3 = \langle a \rangle \, .$$

In particular we have the branching rule

$$(1,0,0,0) \supset (0,0)(1,1)[0] + (0,2)(1,0)[1] + (2,0)(0,1)[2] + (1,1)(0,0)[0]$$

where the term in square brackets is the relative congruence class and is to be interpreted modulo 3.

Example 5.4.2.

Consider $B_3 \supset A_3$. *We know from* **[32]** *that the projection matrix for that case is*

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \, ,$$

and from table 5.6 (from the line $B_n \supset D_n$*, n odd, with* $n = 3$*) that the relative congruence equation is*

$$a := 2m_1 + 3m_3 \mod 4 \, .$$

Now consider the branching rule for the irreducible B_3 *representation with highest weight* $\omega_1 = (1,0,0)$. *We have :*

Weight	$2m_1 + 3m_3 \mod 4$	*Image under P*
(1,0,0)	2	(0,1,0)
(-1,1,0)	2	(1,-1,1)
(0,-1,2)	2	(-1,0,1)
(0,0,0)	0	(0,0,0)
(0,1,-2)	2	(1,0,-1)
(1,-1,0)	2	(-1,1,-1)
(-1,0,0)	2	(0,-1,0)

TABLE 5.1. The irreducible B_3 representation with highest weight $(1, 0, 0)$.

and we can conclude that the relative congruence classes split entirely the two representations of A_3 *here. The branching rule is*

$$(1, 0, 0) \supset (0, 1, 0)[2] + (0, 0, 0)[0]$$

where the term in square brackets is to be interpreted modulo 4.

The same exercise with the irreducible B_3 *representation with highest weight* $\omega_3 = (0, 0, 1)$ *gives us :*

Weight	$2m_1 + 3m_3 \mod 4$	*Image under P*
(0,0,1)	3	(0,0,1)
(0,1,-1)	1	(1,0,0)
(1,-1,1)	1	(-1,1,0)
(-1,0,1)	1	(0,-1,1)
(1,0,-1)	3	(0,1,-1)
(-1,1,-1)	3	(1,-1,0)
(0,-1,1)	3	(-1,0,0)
(0,0,-1)	1	(0,0,-1)

TABLE 5.2. The irreducible B_3 representation with highest weight $(0, 0, 1)$.

and again we can conclude that the relative congruence classes split entirely the two representations of A_3*. The branching rule is*

$$(0,0,1) \supset (0,0,1)[3] + (1,0,0)[1]$$

where the term in square brackets is to be interpreted modulo 4.

The difference in labels in the two cases – 0 *and* 2 *for the first one,* 1 *and* 3 *for the second – is caused by the fact that these two representations of* B_3 *belong to two different congruence classes : in the first case* $m_3 \equiv 0 \mod 2$ *whereas in the second case* $m_3 \equiv 1 \mod 2$*.*

5.5. Continuous centralizers

The maximal regular reductive subalgebras of a simple Lie algebra L can again be easily described in terms of the Dynkin diagram of L. Explicitly any such subalgebra arises as the semisimple Lie algebra having its Dynkin diagram given by deleting one node of the Dynkin diagram of L having mark equal to 1 direct sum with the 1-dimensional subalgebra $\mathbb{C}h_0$ consisting of the intersection of the kernels of the remaining roots. If α_k denotes the node of mark 1 deleted from the Dynkin diagram of L, we observe that the centralizer $C_G(G')$ of G' in G is generated by the center $Z(G)$ of G, the center $Z(G')$ of G' together with the rank 1 subgroup $U_1 := \left\langle e^{i\theta\hat{\omega}_k} \mid \theta \in \mathbb{R}\right\rangle$. In all cases it is easily verified that the center $Z(G')$ of G' is contained in the subgroup $Z(G) \times U_1$. The centralizers of maximal regular reductive subalgebras separate into two types. Either $e^{2\pi i\hat{\omega}_k}$ generates the center of G in which case the centralizer of G' in G is U_1 or $e^{2\pi i\hat{\omega}_k}$ generates a proper subgroup of $Z(G)$ in which case the centralizer properly contains U_1 – in fact, we have

$$C_G(G')/U_1 \simeq Z(G)/\left\langle e^{2\pi i\hat{\omega}_k}\right\rangle \; .$$

In some cases this second type of centralizer cannot be expressed as a direct product of subgroups.

The definition of *relative congruence classes* introduced in subsection 5.4.2 is also true for maximal regular reductive subalgebras.

5.5.1. Explanation of table 5.8

Table 5.8 presents the structure of the centralizers and the relative congruence relations for all maximal regular reductive subalgebras in classical and exceptional simple Lie algebras, whenever such a subalgebra is present.

For each such algebra-subalgebra pair $L \supset L'$, with associated groups $G \supset G'$, we give the structure of the centralizer of G' in G, $C_G(G')$, which is either a continuous group of rank 1 or a product, not necessarily direct, of a continuous group of rank 1 with a finite cyclic group. Furthermore, if $L' = L'' \oplus H_1$ and $G' = G'' \times U_1$, we give the modular relations associated with the centers $Z(G)$ and $Z(G'')$, the structure of H_1 as well as the relative congruence relation provided by the centralizer $C_G(G')$.

As we did for the discrete centralizers in section 5.4, the embedding of the subalgebra L' we choose for our task is the one provided by the corresponding projection matrix, which can be found in **[32, 34, 40]**.

Assume that $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\omega_1, \ldots, \omega_n\}$ are the simple roots and the fundamental weights of the simple Lie algebra L, respectively. The semisimple part L'' of a maximal regular reductive subalgebra L' of L (i.e. $L' = L'' \oplus H_1$) can be realized as the subalgebra with simple roots $\{\alpha_1, \ldots, \alpha_n\} \backslash \{\alpha_k\}$ where α_k is a simple root of L with mark $m_k = 1$. We have

$$U_1 := \left\langle e^{i\theta\hat{\omega}_k} \mid \theta \in \mathbb{R} \right\rangle$$

or equivalently

$$H_1 := \mathbb{C}(\hat{\omega}_k)\,.$$

In order to present this information in terms of the embedding provided by the projection matrix we first note that there exists a Weyl automorphism σ of L which transforms the subalgebra L'' to the corresponding subalgebra L''' used to produce the projection matrix for the branching rule for $L \supset L'''$. We can then determine the simple roots of L''', with associated Lie group G''', and write them in terms of the fundamental weights ω_i's. To compute the H_1 summand, one only has to find the element of the Cartan subalgebra that is in the intersection of the kernels of these weights. This in turn provides a relative congruence relation. The net effect of all these calculations is that the projection matrix of $L \supset L''' \oplus H_1$ should be written as the projection matrix of $L \supset L'''$ with an additional row at the bottom – when a weight of an irreducible representation of L is multiplied by this projection matrix, the last coordinate will yield the relative congruence value for the weight.

Now, we know the continuous rank 1 group U_1 is contained in the centralizer of $G''' \times U_1$ in G, because

$$C_G(G''' \times U_1) = \langle Z(G), Z(G''' \times U_1), U_1 \rangle \ .$$

And since it is easy to show that

$$Z(G''' \times U_1) \subseteq \langle Z(G), U_1 \rangle \ ,$$

it remains to determine whether or not the center of G is contained in U_1 to be able to finally give the structure of the centralizer.

Example 5.5.1.

Let us consider the case $E_6 \supset D_5 \oplus H_1$. We first note from **[40]** *that the projection matrix for $E_6 \supset D_5$ is given by*

$$P = \begin{pmatrix} 0&1&1&1&0&0 \\ 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&1&0 \\ 1&1&0&0&0&0 \end{pmatrix} \ .$$

(Note that we could use directly the projection matrix for $E_6 \supset D_5 \oplus H_1$, also presented in **[40]***, but we choose to show here the reasoning behind that last line of the matrix.)*

Since all of the roots of E_6 have the same length we can see that the base of simple roots for D_5 is given by

$$\{\alpha_2+\alpha_3+\alpha_4, \alpha_6, \alpha_3, \alpha_4+\alpha_5, \alpha_1+\alpha_2\}\,.$$

Writing the roots in terms of the fundamental weights of E_6, we get

$$\{-\omega_1+\omega_2+\omega_4-\omega_5-\omega_6, -\omega_3+2\omega_6, -\omega_2+2\omega_3-\omega_4-\omega_6, -\omega_3+\omega_4+\omega_5, \omega_1+\omega_2-\omega_3\}\,.$$

We can determine the element of the Cartan subalgebra that is in the intersection of the kernels of these weights by simply solving a homogeneous system of linear equations, and we find :

$$H_1 := \mathbb{C}(\hat{\alpha}_1 - \hat{\alpha}_2 + \hat{\alpha}_4 - \hat{\alpha}_5)\,.$$

Using table 5.5 we find that the center of E_6 *is generated by*

$$e^{2\pi i\hat{\omega}_1} = e^{\frac{2\pi i}{3}(4\hat{\alpha}_1+5\hat{\alpha}_2+6\hat{\alpha}_3+4\hat{\alpha}_4+2\hat{\alpha}_5+3\hat{\alpha}_6)}\,.$$

It follows that the congruence class of an irreducible representation with highest weight $\lambda = \sum_{i=1}^{6} m_i\omega_i$ *is determined by the value*

$$m_1 + 2m_2 + m_4 + 2m_5 \equiv m_1 - m_2 + m_4 - m_5 \mod 3\,.$$

Again from table 5.5 we have that the center of D_5 *is generated by*

$$e^{\frac{2\pi i}{4}(2(\hat{\alpha}_2+\hat{\alpha}_3+\hat{\alpha}_4)+2\hat{\alpha}_3+3(\hat{\alpha}_4+\hat{\alpha}_5)+5(\hat{\alpha}_1+\hat{\alpha}_2))}$$

which reduces to the modular condition

$$m_1 + 3m_2 + m_4 + 3m_5 \equiv m_1 - m_2 + m_4 - m_5 \mod 4\,.$$

Finally we observe that the continuous rank 1 group

$$U_1 := \left\langle e^{i\theta(\hat{\alpha}_1-\hat{\alpha}_2+\hat{\alpha}_4-\hat{\alpha}_5)} \mid \theta \in \mathbb{R} \right\rangle$$

is contained in the centralizer of $D_5 \times U_1$ *in* E_6 *and further that the center of* E_6 *is contained in* U_1 *(take* $\theta = \frac{2\pi}{3}$*) and that the center of* D_5 *is contained in* U_1 *(take* $\theta = \frac{2\pi}{4}$*). So we naturally have that the centralizer of* $D_5 \times U_1$ *is equal to* U_1*. For this embedding the relative congruence condition can be written as*

$$m_1 - m_2 + m_4 - m_5\,.$$

Note that the effect of having the center of E_6 *in* U_1 *is that all weights in an irreducible* E_6 *representation will yield relative congruence values, i.e. the values of* $m_1-m_2+m_4-m_5$*, that will be congruent modulo 3. In other words if the* E_6 *irreducible representation has congruence class 0 then all the relative congruence values on the weights of this representation will be congruent to 0 modulo 3* $(0, \pm 3, \pm 6, \ldots)$*.*

All these calculations imply that the projection matrix of $E_6 \supset D_5 \oplus H_1$ *should be written as the projection matrix of* $E_6 \supset D_5$ *with an additional row at the bottom given by*

$$\begin{pmatrix} 1 & -1 & 0 & 1 & -1 & 0 \end{pmatrix}\,.$$

When a weight of an irreducible representation of E_6 *is multiplied by this projection matrix, the last coordinate will yield the relative congruence value for the weight.*

Now consider the branching rule for the irreducible 27-dimensional E_6 *representation with highest weight* $\omega_1 = (1,0,0,0,0,0)$. *We have :*

Weight	$m_1-m_2+m_4-m_5$	*Image under P*
(1,0,0,0,0,0)	1	(0,0,0,0,1)
(-1,1,0,0,0,0)	-2	(1,0,0,0,0)
(0,-1,1,0,0,0)	1	(0,0,1,0,-1)
(0,0,-1,1,0,1)	1	(0,1,-1,1,0)
(0,0,0,-1,1,1)	-2	(-1,1,0,0,0)
(0,0,0,1,0,-1)	1	(1,-1,0,1,0)
(0,0,0,0,-1,1)	1	(0,1,0,-1,0)
(0,0,1,-1,1,-1)	-2	(0,-1,1,0,0)
(0,0,1,0,-1,-1)	1	(1,-1,1,-1,0)
(0,1,-1,0,1,0)	-2	(0,0,-1,1,1)
(1,-1,0,0,1,0)	1	(-1,0,0,1,0)
(0,1,-1,1,-1,0)	1	(1,0,-1,0,1)
(-1,0,0,0,1,0)	-2	(0,0,0,1,-1)
(1,-1,0,1,-1,0)	4	(0,0,0,0,0)
(0,1,0,-1,0,0)	-2	(0,0,0,-1,1)
(1,-1,1,-1,0,0)	1	(-1,0,1,-1,0)
(-1,0,0,1,-1,0)	1	(1,0,0,0,-1)
(-1,0,1,-1,0,0)	-2	(0,0,1,-1,-1)
(1,0,-1,0,0,1)	1	(-1,1,-1,0,1)
(-1,1,-1,0,0,1)	-2	(0,1,-1,0,0)
(1,0,0,0,0,-1)	1	(0,-1,0,0,1)
(0,-1,0,0,0,1)	1	(-1,1,0,0,-1)
(-1,1,0,0,0,-1)	-2	(1,-1,0,0,0)
(0,-1,1,0,0,-1)	1	(0,-1,1,0,-1)
(0,0,-1,1,0,0)	1	(0,0,-1,1,0)
(0,0,0,-1,1,0)	-2	(-1,0,0,0,0)
(0,0,0,0,-1,0)	1	(0,0,0,-1,0)

TABLE 5.3. The irreducible E_6 representation with highest weight $(1,0,0,0,0,0)$.

and we can conclude that the relative congruence classes split entirely the three representations of D_5 *here. The branching rule is*

$$(1,0,0,0,0,0) \supset (0,0,0,0,1)[1] + (1,0,0,0,0)[-2] + (0,0,0,0,0)[4]$$

where the term in square brackets is the relative congruence class.

Finally, we discuss an example where the centralizer of the maximal regular reductive subalgebra is of the second type, i.e. where $e^{2\pi i \hat{\omega}_k}$ generates a proper subalgebra of $Z(G)$ in which case the centralizer properly contains U_1.

Example 5.5.2. *Let us consider the case* $D_4 \supset A_3 \oplus H_1$*. We use directly the projection matrix for* $D_4 \supset A_3 \oplus H_1$*, that can be found in* **[32]** *:*

$$P = \begin{pmatrix} 1&1&0&0 \\ 0&0&0&1 \\ 0&1&1&0 \\ 1&0&1&0 \end{pmatrix} .$$

Since all of the roots of D_4 *have the same length we can see that the base of simple roots for* A_3 *is given by*

$$\{\alpha_1+\alpha_2, \alpha_4, \alpha_2+\alpha_3\} .$$

Writing the roots in terms of the fundamental weights of D_4*, we get*

$$\{\omega_1+\omega_2-\omega_3-\omega_4, -\omega_2+2\omega_4, -\omega_1+\omega_2+\omega_3-\omega_4\} .$$

We can determine the element of the Cartan subalgebra that is in the intersection of the kernels of these weights by simply solving a homogeneous system of linear equations, and we find :

$$H_1 := \mathbb{C}(\hat{\alpha}_1 + \hat{\alpha}_3) .$$

Using table 5.5 we find that the center of D_4 *is generated by*

$$e^{2\pi i \hat{\omega}_1} = e^{\frac{2\pi i}{2}(2\hat{\alpha}_1+2\hat{\alpha}_2+\hat{\alpha}_3+\hat{\alpha}_4)}$$

and

$$e^{2\pi i \hat{\omega}_4} = e^{\frac{2\pi i}{2}(\hat{\alpha}_1+2\hat{\alpha}_2+\hat{\alpha}_3+2\hat{\alpha}_4)} .$$

It follows that the congruence class of an irreducible representation with highest weight $\lambda = \sum_{i=1}^{4} m_i \omega_i$ *is determined by the values*

$$m_3 + m_4 \mod 2$$

and

$$m_1 + m_3 \mod 2\,.$$

Again from table 5.5 we have that the center of A_3 *is generated by*

$$e^{\frac{2\pi i}{4}((\hat{\alpha}_1+\hat{\alpha}_2)+2\hat{\alpha}_4+3(\hat{\alpha}_2+\hat{\alpha}_3))} = e^{\frac{2\pi i}{4}(\hat{\alpha}_1+4\hat{\alpha}_2+3\hat{\alpha}_3+2\hat{\alpha}_4)}$$

which reduces to the modular condition

$$m_1 + 3m_3 + 2m_4 \mod 4\,.$$

Finally we observe that the continuous rank 1 group

$$U_1 := \left\langle e^{i\theta(\hat{\alpha}_1+\hat{\alpha}_3)} \mid \theta \in \mathbb{R} \right\rangle$$

is contained in the centralizer of $A_3 \times U_1$ *in* D_4 *and further that the center of* A_3 *is contained in* $Z(D_4) \times U_1$ *(take* $e^{2\pi i \hat{\omega}_1}$ *and* $\theta = \frac{2\pi}{4}$*). However, the center of* D_4 *is not contained in* U_1 *: instead, we have that* $Z(D_4) \cap U_1$ *is a proper subgroup of* $Z(D_4)$ *that yields the modular condition*

$$m_1 + m_3 \mod 2\,.$$

In short, we know that

$$C_{D_4}(A_3 \times U_1) = \langle Z(D_4), U_1 \rangle$$

and that

$$Z(D_4) \cong \mathbb{Z}_2 \times \mathbb{Z}_2\,,$$

and thus using the fact that

$$Z(D_4) \cap U_1 \cong \mathbb{Z}_2\,,$$

we find that

$$C_{D_4}(A_3 \times U_1) \cong U_1 \times \mathbb{Z}_2\,.$$

For this embedding the relative congruence condition can be written as

$$m_1 + m_3\,,$$

which coincides with the last line of P.

Note that the effect of not having the center of D_4 *contained in* U_1 *is that knowing the relative congruence value does not tell us which congruence class we are* *dealing with. For example the modules with highest weights* $(1,0,0,0)$ *and* $(0,0,1,0)$ *will both have odd relative congruence labels but these two modules are in different congruence classes –* $m_3 + m_4 \equiv 0 \mod 2$ *for* $(1,0,0,0)$*, and* $m_3 + m_4 \equiv 1 \mod 2$ *for* $(0,0,1,0)$ *– and hence their weight spaces must be distinguished by the action of the whole centralizer. However when trying to reduce a representation of* D_4*, the only information that can help splitting the* A_3 *representations is the relative congruence condition.*

Let us consider the branching rule for the irreducible 8-dimensional D_4 *representation with highest weight* $\omega_1 = (1,0,0,0)$*. We have :*

Weight	m_1+m_3	*Image under P*
(1,0,0,0)	1	(1,0,0)[1]
(-1,1,0,0)	-1	(0,0,1)[-1]
(0,-1,1,1)	1	(-1,1,0)[1]
(0,0,-1,1)	-1	(0,1,-1)[-1]
(0,0,1,-1)	1	(0,-1,1)[1]
(0,1,-1,-1)	-1	(1,-1,0)[-1]
(1,-1,0,0)	1	(0,0,-1)[1]
(-1,0,0,0)	-1	(-1,0,0)[-1]

TABLE 5.4. The irreducible D_4 representation with highest weight $(1,0,0,0)$.

and we can conclude that the relative congruence classes split entirely the two representations of A_3 *here. The branching rule is*

$$(1,0,0,0) \supset (1,0,0)[1] + (0,0,1)[-1]$$

where the term in square brackets is the relative congruence class (and the value of the H_1 *term).*

Finally, the branching rule for the irreducible 28-dimensional D_4 *representation with highest weight* $\omega_2 = (0,1,0,0)$ *is*

$$(0,1,0,0) \supset (1,0,1)[0] + (0,1,0)[2] + (0,1,0)[-2] + (0,0,0)[0]\,.$$

We have here an example where the representation of A_3 *corresponding to a fixed eigenvalue is not be irreducible : the* A_3 *representations with highest weights* $(1, 0, 1)$ *and* $(0, 0, 0)$ *share the same relative congruence label* $[0]$.

Acknowledgements

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada and by the MIND Research Institute of Santa Ana, California. M. L. is grateful for the support she receives from the Alexander Graham Bell Scholarship. We would like to thank Iryna Kashuba, who was at the beginning of this work several years ago.

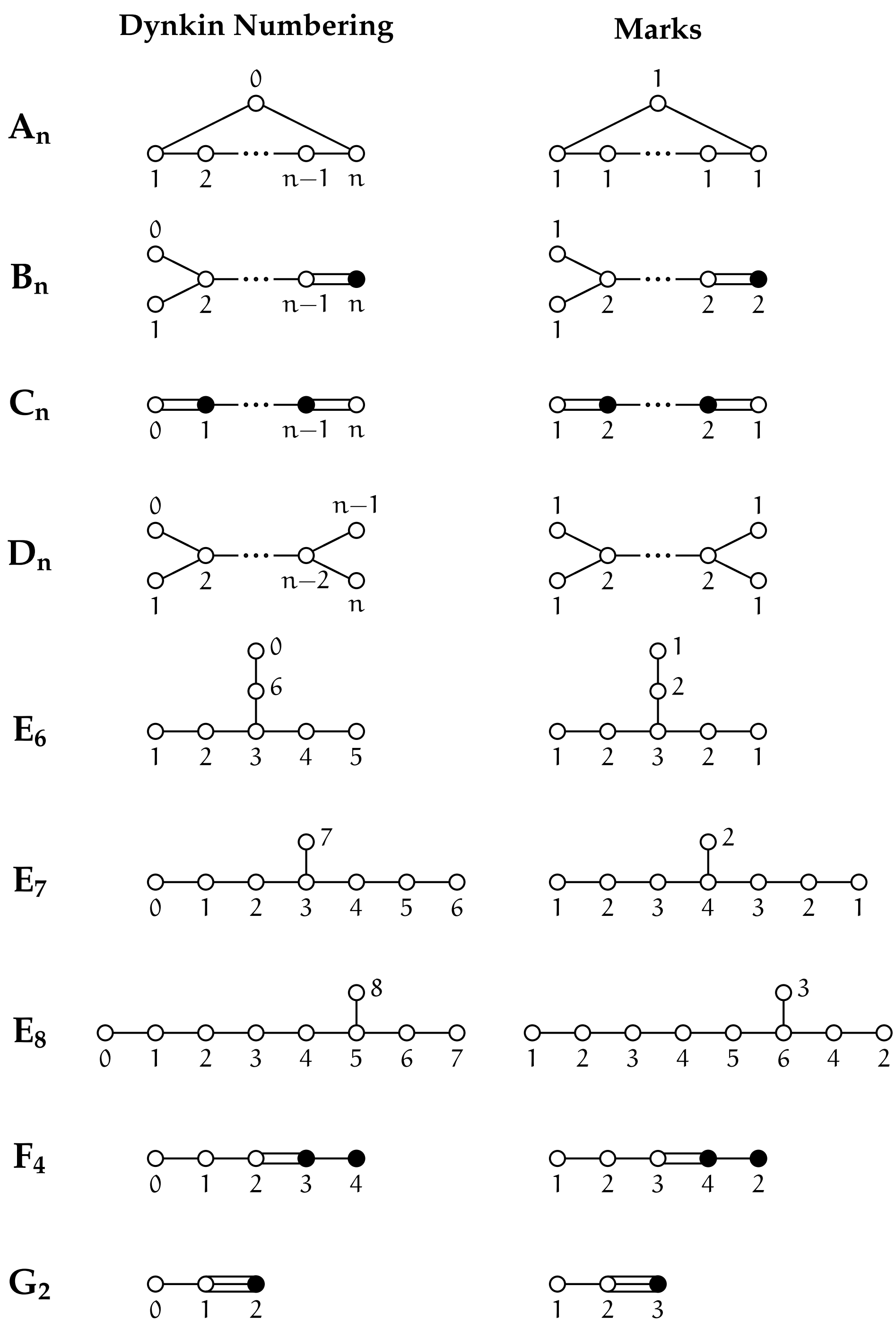


FIGURE 5.1. The Dynkin numbering of the extended diagrams with marks.

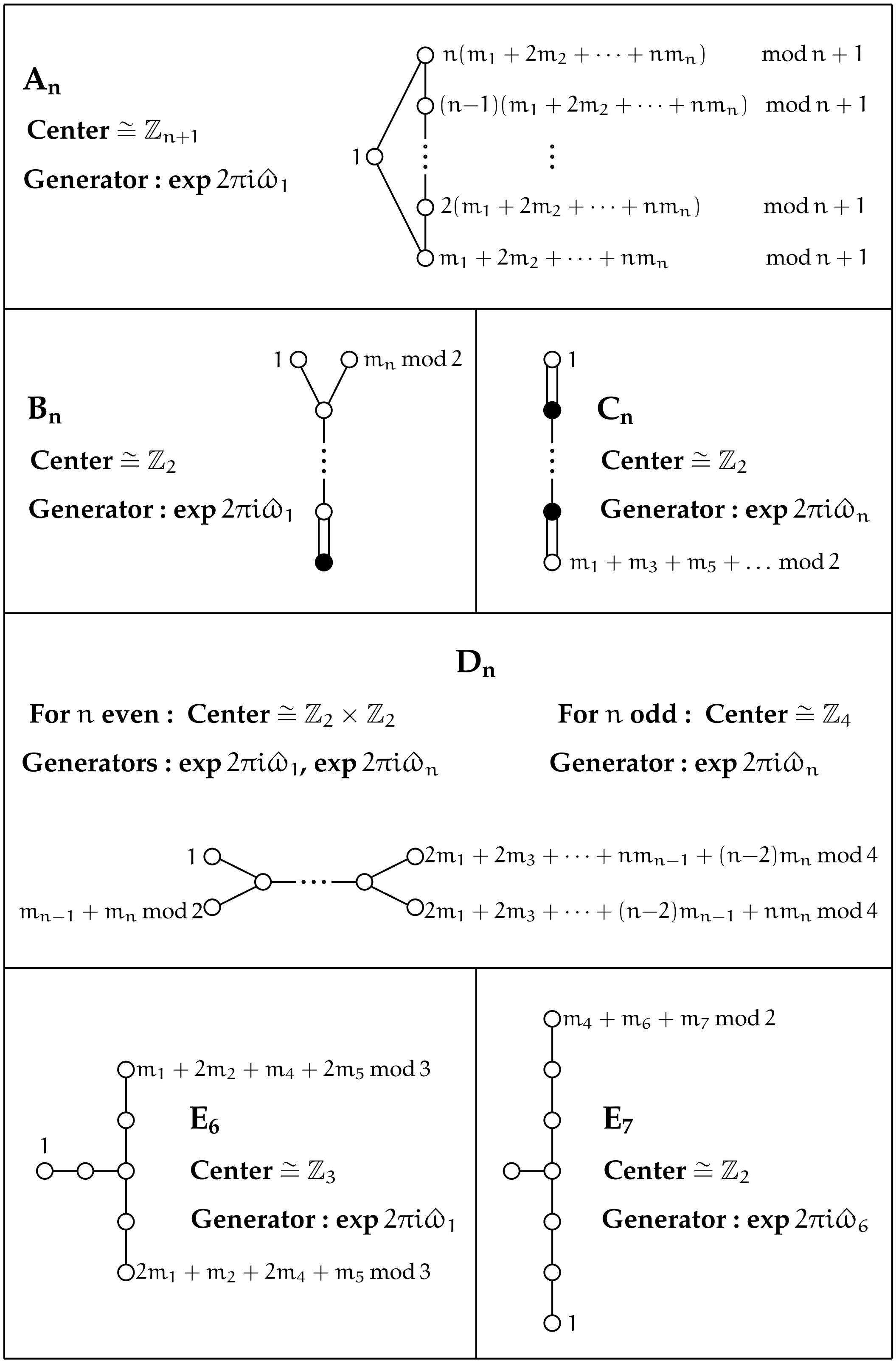


TABLE 5.5. Eigenvalues of central elements $z = \exp 2\pi i\hat{\omega}_j$ on an irreducible representation of highest weight $\lambda = \sum_{i=1}^{n} m_i\omega_i$.

$\mathbf{L \supset L'}$	$\mathbf{C_G(G')}$	$\frac{\mathbf{C_G(G')}}{\mathbf{Z(G)}}$
$B_2 \supset A_1 \oplus A_1$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$	$\mathbb{Z}_2$
A_1	$a : m_1 + m_2 \bmod 2$	a
A_1	$b : m_1 \bmod 2$	
$B_n \supset B_{n-2} \oplus A_1 \oplus A_1$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$	$\mathbb{Z}_2$
B_{n-2}	$a : m_n \bmod 2$	
A_1	$b : m_{n-2} + m_{n-1} + m_n \bmod 2$	b
A_1	$a{+}b : m_{n-2} + m_{n-1} \bmod 2$	
$B_4 \supset A_1 \oplus A_3$	$\mathbb{Z}_4 = \langle a \rangle$	$\mathbb{Z}_2$
A_1	$2a : m_4 \bmod 2$	a
A_3	$a : 2m_1 + 3m_4 \bmod 4$	
$B_n \supset B_{n-3} \oplus A_3$	$\mathbb{Z}_4 = \langle a \rangle$	$\mathbb{Z}_2$
B_{n-3}	$2a : m_n \bmod 2$	a
A_3	$a : 2m_{n-4} + 2m_{n-3} + 3m_n \bmod 4$	
$B_n \supset D_n$	(n odd) $\mathbb{Z}_4 = \langle a \rangle$	$\mathbb{Z}_2$
D_n	$a : 2m_1 + 2m_3 + \cdots + 2m_{n-2} + nm_n \bmod 4$	a
$B_n \supset D_n$	(n even) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$	$\mathbb{Z}_2$
D_n	$a : m_1 + m_3 + \cdots + m_{n-1} + \frac{n}{2} m_n \bmod 2$ and $b : m_n \bmod 2$	a
$B_n \supset D_{n-1} \oplus A_1$	(n odd) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$	$\mathbb{Z}_2$
D_{n-1}	$a : m_1 + m_3 + \cdots + m_{n-2} + m_{n-1}$ $+\frac{n-1}{2} m_n \bmod 2$ and $b : m_n \bmod 2$	a
A_1	$b : m_n \bmod 2$	
$B_n \supset D_{n-1} \oplus A_1$	(n even) $\mathbb{Z}_4 = \langle a \rangle$	$\mathbb{Z}_2$
D_{n-1}	$a : 2m_1 + 2m_3 + \cdots + 2m_{n-3} + (n-1)m_n \bmod 4$	a
A_1	$2a : m_n \bmod 2$	
$B_n \supset B_k \oplus D_{n-k}$	($n{-}k$ even) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$	$\mathbb{Z}_2$
B_k	$a : m_n \bmod 2$	
D_{n-k}	a and $b : \sum_{i=0}^{(n-k-4)/2} (m_{2k-n+4i+2} + m_{2k-n+4i+3})$ $+m_{n-2} + m_{n-1} + \frac{n-k}{2} m_n \bmod 2$	b

TABLE 5.6. Discrete centralizers and relative congruence classes of irreducible representations of the classical simple Lie algebras.

$\mathbf{L} \supset \mathbf{L}'$	$\mathbf{C_G(G')}$	$\frac{\mathbf{C_G(G')}}{\mathbf{Z(G)}}$
$B_n \supset B_k \oplus D_{n-k}$	$(n-k \text{ odd}) \quad \mathbb{Z}_4 = \langle a \rangle$	$\mathbb{Z}_2$
B_k	$2a : m_n \bmod 2$	
D_{n-k}	$a : \sum_{i=0}^{(n-k-3)/2} 2(m_{2k-n+4i+2} + m_{2k-n+4i+3}) + (n-k)m_n \bmod 4$	a
$B_n \supset D_k \oplus B_{n-k}$	$(n, k \text{ odd}) \quad \mathbb{Z}_4 = \langle a \rangle$	$\mathbb{Z}_2$
D_k	$a : \sum_{i=0}^{(2k-n-3)/2} 2m_{2i+1} + \sum_{j=0}^{(n-k-2)/2} 2(m_{2k-n+4j} + m_{2k-n+4j+1}) + km_n \bmod 4$	a
B_{n-k}	$2a : m_n \bmod 2$	
$B_n \supset D_k \oplus B_{n-k}$	$(n \text{ even}, k \text{ odd}) \quad \mathbb{Z}_4 = \langle a \rangle$	$\mathbb{Z}_2$
D_k	$a : \sum_{i=0}^{(2k-n-2)/2} 2m_{2i+1} + \sum_{j=0}^{(n-k-3)/2} 2(m_{2k-n+4j+2} + m_{2k-n+4j+3}) + km_n \bmod 4$	a
B_{n-k}	$2a : m_n \bmod 2$	
$B_n \supset D_k \oplus B_{n-k}$	$(n, k \text{ even}) \quad \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$	$\mathbb{Z}_2$
D_k	$a : m_n \bmod 2$ and $b : \sum_{i=0}^{(2k-n-2)/2} m_{2i+1} + \sum_{j=0}^{(n-k-2)/2} (m_{2k-n+4j+2} + m_{2k-n+4j+3}) + \frac{k}{2} m_n \bmod 2$	b
B_{n-k}	$a : m_n \bmod 2$	
$B_n \supset D_k \oplus B_{n-k}$	$(n \text{ odd}, k \text{ even}) \quad \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$	$\mathbb{Z}_2$
D_k	$a : m_n \bmod 2$ and $b : \sum_{i=0}^{(2k-n-3)/2} m_{2i+1} + \sum_{j=0}^{(n-k-1)/2} (m_{2k-n+4j} + m_{2k-n+4j+1}) + \frac{k}{2} m_n \bmod 2$	b
B_{n-k}	$a : m_n \bmod 2$	

TAB. 5.6. (continued)

$\mathbf{L} \supset \mathbf{L}'$	$\mathbf{C_G}(\mathbf{G}')$	$\frac{\mathbf{C_G}(\mathbf{G}')}{\mathbf{Z}(\mathbf{G})}$
$C_n \supset C_{n-1} \oplus A_1$	(n even) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$	$\mathbb{Z}_2$
C_{n-1}	$a : m_1 + m_3 + \cdots + m_{n-1} + m_n \bmod 2$	a
A_1	$b : m_n \bmod 2$	
$C_n \supset C_{n-1} \oplus A_1$	(n odd) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$	$\mathbb{Z}_2$
C_{n-1}	$a : m_1 + m_3 + \cdots + m_{n-2} \bmod 2$	a
A_1	$b : m_n \bmod 2$	
$C_n \supset C_k \oplus C_{n-k}$	(n, k odd) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$	$\mathbb{Z}_2$
C_k	$a : \sum_{i=1}^{(2k-n+1)/2} m_{2i-1} + \sum_{j=1}^{(n-k)/2} (m_{2k-n+4j-1} + m_{2k-n+4j}) \bmod 2$	a
C_{n-k}	$b : \sum_{j=1}^{(n-k)/2} (m_{2k-n+4j-2} + m_{2k-n+4j-1}) \bmod 2$	
$C_n \supset C_k \oplus C_{n-k}$	(n, k even) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$	$\mathbb{Z}_2$
C_k	$a : \sum_{i=1}^{(2k-n)/2} m_{2i-1} + \sum_{j=1}^{(n-k)/2} (m_{2k-n+4j-3} + m_{2k-n+4j-2}) \bmod 2$	a
C_{n-k}	$b : \sum_{j=1}^{(n-k)/2} (m_{2k-n+4j-2} + m_{2k-n+4j-1}) \bmod 2$	
$C_n \supset C_k \oplus C_{n-k}$	(n even, k odd) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$	$\mathbb{Z}_2$
C_k	$a : \sum_{i=1}^{(2k-n)/2} m_{2i-1} + \sum_{j=1}^{(n-k+1)/2} (m_{2k-n+4j-3} + m_{2k-n+4j-2}) \bmod 2$	a
C_{n-k}	$b : \sum_{j=1}^{(n-k-1)/2} (m_{2k-n+4j-2} + m_{2k-n+4j-1}) + m_n \bmod 2$	

TAB. 5.6. (continued)

$\mathbf{L} \supset \mathbf{L}'$	$\mathbf{C_G}(\mathbf{G}')$	$\frac{\mathbf{C_G}(\mathbf{G}')}{\mathbf{Z}(\mathbf{G})}$
$C_n \supset C_k \oplus C_{n-k}$	(n odd, k even) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b\rangle$	$\mathbb{Z}_2$
C_k	$a: \sum_{i=1}^{(2k-n+1)/2} m_{2i-1} + \sum_{j=1}^{(n-k-1)/2} (m_{2k-n+4j-1} + m_{2k-n+4j}) \bmod 2$	a
C_{n-k}	$b: \sum_{j=1}^{(n-k-1)/2} (m_{2k-n+4j-2} + m_{2k-n+4j-1}) + m_n \bmod 2$	
$D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b, c\rangle$	$\mathbb{Z}_2$
A_1	$a: m_2 + m_3 \bmod 2$	b
A_1	$b: m_2 + m_4 \bmod 2$	
A_1	$c: m_1 + m_2 + m_3 + m_4 \bmod 2$	
A_1	$a{+}b{+}c: m_1 + m_2 \bmod 2$	
$D_n \supset D_{n-2} \oplus A_1 \oplus A_1$	(n even) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b, c\rangle$	$\mathbb{Z}_2$
D_{n-2}	$a: m_1 + m_3 + \cdots + m_{n-5} + m_{n-2} + \frac{n}{2} m_{n-1} + (1 + \frac{n}{2}) m_n \bmod 2$ and $b: m_{n-1} + m_n \bmod 2$	a
A_1	$c: m_{n-3} + m_{n-2} + m_{n-1} + m_n \bmod 2$	
A_1	$b{+}c: m_{n-3} + m_{n-2} \bmod 2$	
$D_n \supset D_{n-2} \oplus A_1 \oplus A_1$	(n odd) $\mathbb{Z}_4 \times \mathbb{Z}_2 = \langle a, b\rangle$	$\mathbb{Z}_2$
D_{n-2}	$a: 2m_1 + 2m_3 + \cdots + 2m_{n-4} + 2m_{n-3} + n m_{n-1} + (n-2) m_n \bmod 4$	a
A_1	$b: m_{n-3} + m_{n-2} + m_{n-1} + m_n \bmod 2$	
A_1	$2a{+}b: m_{n-3} + m_{n-2} \bmod 2$	
$D_6 \supset A_3 \oplus A_3$	$\mathbb{Z}_4 \times \mathbb{Z}_2 = \langle a, a{+}b\rangle$	$\mathbb{Z}_2$
A_3	$a: 2m_2 + 2m_3 + m_5 + 3m_6 \bmod 4$	a
A_3	$b: 2m_1 + 2m_2 + 3m_5 + 3m_6 \bmod 4$	
$D_n \supset D_{n-3} \oplus A_3$	(n even) $\mathbb{Z}_4 \times \mathbb{Z}_2 = \langle a, a{+}b\rangle$	$\mathbb{Z}_2$
D_{n-3}	$a: 2m_1 + 2m_3 + \cdots + 2m_{n-7} + 2m_{n-4} + 2m_{n-3} + (n-1) m_{n-1} + (n+1) m_n \bmod 4$	a
A_3	$b: 2m_{n-5} + 2m_{n-4} + 3m_{n-1} + 3m_n \bmod 4$	

TAB. 5.6. (continued)

$\mathbf{L} \supset \mathbf{L}'$	$\mathbf{C_G}(\mathbf{G}')$	$\frac{\mathbf{C_G}(\mathbf{G}')}{\mathbf{Z}(\mathbf{G})}$
$D_n \supset D_{n-3} \oplus A_3$	(n odd) $\mathbb{Z}_2 \times \mathbb{Z}_4 = \langle a, b \rangle$	$\mathbb{Z}_2$
D_{n-3}	$a : m_1 + m_3 + \cdots + m_{n-6} + m_{n-5} + m_{n-2}$ $+\frac{n-1}{2} m_{n-1} + \frac{n+1}{2} m_n \bmod 2$ and $2b : m_{n-1} + m_n \bmod 2$	a
A_3	$b : 2m_{n-5} + 2m_{n-4} + 3m_{n-1} + 3m_n \bmod 4$	
$D_n \supset D_k \oplus D_{n-k}$	(n, k odd) $\mathbb{Z}_4 \times \mathbb{Z}_2 = \langle a, b \rangle$	$\mathbb{Z}_2$
D_k	$a : \sum_{i=1}^{(2k-n+1)/2} 2m_{2i-1}$ $+ \sum_{j=1}^{(n-k-2)/2} 2(m_{2k-n+4j-1} + m_{2k-n+4j})$ $+k(m_{n-1} + m_n) \bmod 4$	a
D_{n-k}	$2a : m_{n-1} + m_n \bmod 2$ and $b : \sum_{j=1}^{(n-k-2)/2} (m_{2k-n+4j-2} + m_{2k-n+4j-1})$ $+m_{n-2} + \frac{n-k-2}{2} m_{n-1} + \frac{n-k}{2} m_n \bmod 2$	
$D_n \supset D_k \oplus D_{n-k}$	(n, k even) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b, c \rangle$	$\mathbb{Z}_2$
D_k	$a : m_{n-1} + m_n \bmod 2$ and $b : \sum_{i=1}^{(2k-n)/2} m_{2i-1}$ $+ \sum_{j=1}^{(n-k)/2} (m_{2k-n+4j-3} + m_{2k-n+4j-2})$ $+\frac{k}{2}(m_{n-1} + m_n) \bmod 2$	b
D_{n-k}	$a : m_{n-1} + m_n \bmod 2$ and $c : \sum_{j=1}^{(n-k-2)/2} (m_{2k-n+4j-2} + m_{2k-n+4j-1})$ $+m_{n-2} + \frac{n-k-2}{2} m_{n-1} + \frac{n-k}{2} m_n \bmod 2$	
$D_n \supset D_k \oplus D_{n-k}$	(n even, k odd) $\mathbb{Z}_4 \times \mathbb{Z}_2 = \langle a, a+b \rangle$	$\mathbb{Z}_2$
D_k	$a : \sum_{i=1}^{(2k-n)/2} 2m_{2i-1}$ $+ \sum_{j=1}^{(n-k-1)/2} 2(m_{2k-n+4j-3} + m_{2k-n+4j-2})$ $+k(m_{n-1} + m_n) \bmod 4$	a
D_{n-k}	$b : \sum_{j=1}^{(n-k-1)/2} 2(m_{2k-n+4j-2} + m_{2k-n+4j-1})$ $+(n-k-2)m_{n-1} + (n-k)m_n \bmod 4$	

TAB. 5.6. (continued)

$\mathbf{L} \supset \mathbf{L}'$	$\mathbf{C_G(G')}$	$\frac{\mathbf{C_G(G')}}{\mathbf{Z(G)}}$
$D_n \supset D_k \oplus D_{n-k}$	(n odd, k even) $\mathbb{Z}_2 \times \mathbb{Z}_4 = \langle a, b \rangle$	$\mathbb{Z}_2$
D_k	$a : \sum_{i=1}^{(2k-n+1)/2} m_{2i-1} + \sum_{j=1}^{(n-k-1)/2} (m_{2k-n+4j-1} + m_{2k-n+4j}) + \frac{k}{2}(m_{n-1} + m_n) \bmod 2$ and $2b : m_{n-1} + m_n \bmod 2$	a
D_{n-k}	$b : \sum_{j=1}^{(n-k-1)/2} 2(m_{2k-n+4j-2} + m_{2k-n+4j-1}) + (n-k-2)m_{n-1} + (n-k)m_n \bmod 4$	

TAB. 5.6. (continued)

$\mathbf{L} \supset \mathbf{L}'$	$\mathbf{C_G(G')}$	$\frac{\mathbf{C_G(G')}}{\mathbf{Z(G)}}$
$E_6 \supset A_5 \oplus A_1$	$\mathbb{Z}_6 = \langle a \rangle$	$\mathbb{Z}_2$
A_5	$a : 4m_1 + 5m_2 + 3m_3 + m_4 + 2m_5 + 3m_6 \bmod 6$	$3a$
A_1	$3a : m_2 + m_3 + m_4 + m_6 \bmod 2$	
$E_6 \supset A_2 \oplus A_2 \oplus A_2$	$\mathbb{Z}_3 \times \mathbb{Z}_3 = \langle a, b \rangle$	$\mathbb{Z}_3$
A_2	$a : m_1 + m_5 + m_6 \bmod 3$	b
A_2	$b : 2m_2 + m_4 + m_5 + 2m_6 \bmod 3$	
A_2	$2a{+}b : 2m_1 + 2m_2 + m_4 + m_6 \bmod 3$	
$E_7 \supset A_5 \oplus A_2$	$\mathbb{Z}_6 = \langle a \rangle$	$\mathbb{Z}_3$
A_5	$a : 2m_2 + m_4 + 4m_5 + 3m_6 + 5m_7 \bmod 6$	a
A_2	$2a : 2m_2 + m_4 + m_5 + 2m_7 \bmod 3$	
$E_7 \supset A_7$	$\mathbb{Z}_4 = \langle a \rangle$	$\mathbb{Z}_2$
A_7	$a : 2m_1 + 2m_2 + m_4 + m_6 + 3m_7 \bmod 4$	a
$E_7 \supset D_6 \oplus A_1$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$	$\mathbb{Z}_2$
D_6	$a : m_2 + m_3 + m_6 \bmod 2$ and $b : m_4 + m_6 + m_7 \bmod 2$	b
A_1	$a{+}b : m_2 + m_3 + m_4 + m_7 \bmod 2$	
$E_8 \supset D_8$	$\mathbb{Z}_2 = \langle a \rangle$	$\mathbb{Z}_2$
D_8	$a : m_1 + m_2 + m_5 + m_6 \bmod 2$ (and $0 \bmod 2$)	a
$E_8 \supset A_8$	$\mathbb{Z}_3 = \langle a \rangle$	$\mathbb{Z}_3$
A_8	$a : m_1 + m_4 + 2m_6 + 2m_7 + m_8 \bmod 3$	a
$E_8 \supset E_7 \oplus A_1$	$\mathbb{Z}_2 = \langle a \rangle$	$\mathbb{Z}_2$
E_7	$a : m_5 + m_8 \bmod 2$	a
A_1	$a : m_5 + m_8 \bmod 2$	
$E_8 \supset A_4 \oplus A_4$	$\mathbb{Z}_5 = \langle a \rangle$	$\mathbb{Z}_5$
A_4	$a : m_1 + 3m_2 + m_3 + 4m_6 + 4m_7 + 2m_8 \bmod 5$	a
A_4	$3a : 3m_1 + 4m_2 + 3m_3 + 2m_6 + 2m_7 + m_8 \bmod 5$	
$E_8 \supset E_6 \oplus A_2$	$\mathbb{Z}_3 = \langle a \rangle$	$\mathbb{Z}_3$
E_6	$a : m_3 + 2m_7 + m_8 \bmod 3$	a
A_2	$a : m_3 + 2m_7 + m_8 \bmod 3$	

TABLE 5.7. Discrete centralizers and relative congruence classes of irreducible representations of the exceptional simple Lie algebras.

$\mathbf{L} \supset \mathbf{L}'$	$\mathbf{C_G}(\mathbf{G}')$	$\frac{\mathbf{C_G}(\mathbf{G}')}{\mathbf{Z}(\mathbf{G})}$
$G_2 \supset A_1 \oplus A_1$	$\mathbb{Z}_2 = \langle a \rangle$	$\mathbb{Z}_2$
A_1	$a : m_1 + m_2 \bmod 2$	a
A_1	$a : m_1 + m_2 \bmod 2$	
$G_2 \supset A_2$	$\mathbb{Z}_3 = \langle a \rangle$	$\mathbb{Z}_3$
A_2	$a : m_2 \bmod 3$	a
$F_4 \supset A_2 \oplus A_2$	$\mathbb{Z}_3 = \langle a \rangle$	$\mathbb{Z}_3$
A_2	$a : m_2 + m_4 \bmod 3$	a
A_2	$a : m_2 + m_4 \bmod 3$	
$F_4 \supset B_4$	$\mathbb{Z}_2 = \langle a \rangle$	$\mathbb{Z}_2$
B_4	$a : m_3 + m_4 \bmod 2$	a
$F_4 \supset C_3 \oplus A_1$	$\mathbb{Z}_2 = \langle a \rangle$	$\mathbb{Z}_2$
C_3	$a : m_2 + m_4 \bmod 2$	a
A_1	$a : m_2 + m_4 \bmod 2$	

TAB. 5.7. (continued)

$\mathbf{A_n \supset A_k \oplus A_{n-k-1} \oplus H_1}$	$C_G(G') \cong U_1 \times Z(A_n)$ $C_G(G')/U_1 \cong \mathbb{Z}_d\,,\ d = \gcd(k+1, n+1)$
$Z(A_n)$	$m_1 + 2m_2 + \cdots + nm_n \mod (n+1)$
$Z(A_k)$	$m_1 + 2m_2 + \cdots + km_k \mod (k+1)$
$Z(A_{n-k-1})$	$m_{k+2} + 2m_{k+3} + \cdots + (n-k-1)m_n \mod (n-k)$
H_1	$\mathbb{C}\frac{1}{n+1}\left((n-k)\sum_{i=1}^{k+1} i\hat{\alpha}_i + (k+1)\sum_{i=1}^{n-k-1}(n-k-i)\hat{\alpha}_{k+1+i}\right)$
Relative congruence relation	$(n-k)\sum_{i=1}^{k+1} im_i + (k+1)\sum_{i=1}^{n-k-1}(n-k-i)m_{k+1+i}$
$\mathbf{B_n \supset B_{n-1} \oplus H_1}$	$C_G(G') \cong U_1$
$Z(B_n)$	$m_n \bmod 2$
$Z(B_{n-1})$	$m_n \bmod 2$
H_1	$\mathbb{C}\frac{1}{2}\,(\hat{\alpha}_n)$
Relative congruence relation	m_n
$\mathbf{C_n \supset A_{n-1} \oplus H_1}$	$C_G(G') \cong U_1$
$Z(C_n)$	$m_1 + m_3 + \cdots + m_{2[\frac{n+1}{2}]-1} \bmod 2$
$Z(A_{n-1})$	$m_1 + m_3 + \cdots + m_{2[\frac{n+1}{2}]-1} \bmod n$
H_1	$\mathbb{C}\frac{1}{2}\left(\hat{\alpha}_1 + \hat{\alpha}_3 + \cdots + \hat{\alpha}_{2[\frac{n+1}{2}]-1}\right)$
Relative congruence relation	$m_1 + m_3 + \cdots + m_{2[\frac{n+1}{2}]-1}$
$\mathbf{D_n \supset A_{n-1} \oplus H_1}$	(n even) $C_G(G') \cong U_1 \times \mathbb{Z}_2$
$Z(D_n)$	$m_{n-1} + m_n \bmod 2$ and $m_1 + m_3 + \cdots + m_{n-3} + (1+\frac{n}{2})m_{n-1} + \frac{n}{2}m_n \bmod 2$
$Z(A_{n-1})$	$m_1 + m_3 + \cdots + m_{n-3} + (1+\frac{n}{2})m_{n-1} + \frac{n}{2}m_n \bmod n$
H_1	$\mathbb{C}\frac{1}{2}\,(\hat{\alpha}_1 + \hat{\alpha}_3 + \cdots + \hat{\alpha}_{n-1})$
Relative congruence relation	$m_1 + m_3 + \cdots + m_{n-1}$
$\mathbf{D_n \supset A_{n-1} \oplus H_1}$	(n odd) $C_G(G') \cong U_1$
$Z(D_n)$	$2m_1 + 2m_3 + \cdots + 2m_{n-2} + (n-2)m_{n-1} + nm_n \bmod 4$
$Z(A_{n-1})$	$m_1 + m_3 + \cdots + m_{n-2} + \frac{n-1}{2}m_{n-1} + \frac{n+1}{2}m_n \bmod n$
H_1	$\mathbb{C}\frac{1}{4}\,(2\hat{\alpha}_1 + 2\hat{\alpha}_3 + \cdots + 2\hat{\alpha}_{n-2} - \hat{\alpha}_{n-1} + \hat{\alpha}_n)$
Relative congruence relation	$2m_1 + 2m_3 + \cdots + 2m_{n-2} - m_{n-1} + m_n$

TABLE 5.8. Continuous centralizers and relative congruence classes of irreducible representations of the simple Lie algebras.

$\mathbf{D_n \supset D_{n-1} \oplus H_1}$	(n even) $\quad C_G(G') \cong U_1 \times \mathbb{Z}_2$
$Z(D_n)$	$m_{n-1} + m_n \bmod 2$ and $m_1 + m_3 + \cdots + m_{n-3} + (1+\frac{n}{2})m_{n-1} + \frac{n}{2} m_n \bmod 2$
$Z(D_{n-1})$	$2m_1 + 2m_3 + \cdots + 2m_{n-3} + (n-1)(m_{n-1} + m_n) \bmod 4$
H_1	$\mathbb{C}\frac{1}{2}(\hat{\alpha}_{n-1} - \hat{\alpha}_n)$
Relative congruence relation	$m_{n-1} - m_n$

$\mathbf{D_n \supset D_{n-1} \oplus H_1}$	(n odd) $\quad C_G(G') \cong U_1 \times Z(D_n)$ $C_G(G')/U_1 \cong \mathbb{Z}_2$
$Z(D_n)$	$2m_1 + 2m_3 + \cdots + 2m_{n-2} + (n-2)m_{n-1} + nm_n \bmod 4$
$Z(D_{n-1})$	$m_{n-1} + m_n \bmod 2$ and $m_1 + m_3 + \cdots + m_{n-2} + \frac{n-1}{2}(m_{n-1} + m_n) \bmod 2$
H_1	$\mathbb{C}\frac{1}{2}(\hat{\alpha}_{n-1} - \hat{\alpha}_n)$
Relative congruence relation	$m_{n-1} - m_n$

$\mathbf{E_6 \supset D_5 \oplus H_1}$	$C_G(G') \cong U_1$
$Z(E_6)$	$m_1 - m_2 + m_4 - m_5 \bmod 3$
$Z(D_5)$	$m_1 - m_2 + m_4 - m_5 \bmod 4$
H_1	$\mathbb{C}\frac{1}{3}(\hat{\alpha}_1 - \hat{\alpha}_2 + \hat{\alpha}_4 - \hat{\alpha}_5)$
Relative congruence relation	$m_1 - m_2 + m_4 - m_5$

$\mathbf{E_7 \supset E_6 \oplus H_1}$	$C_G(G') \cong U_1$
$Z(E_7)$	$m_4 + m_6 + m_7 \bmod 2$
$Z(E_6)$	$m_4 + m_6 + m_7 \bmod 3$
H_1	$\mathbb{C}\frac{1}{2}(\hat{\alpha}_4 + \hat{\alpha}_6 + \hat{\alpha}_7)$
Relative congruence relation	$m_4 + m_6 + m_7$

TAB. 5.8. (continued)

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