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A Generalization of a Theorem of Boyd and Lawton

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A Generalization of a Theorem of Boyd and Lawton

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Résumé

Ce mémoire s'applique à étudier d'abord, dans la première partie, la mesure de Mahler des polynômes à une seule variable. Il commence en donnant des définitions et quelques résultats pertinents pour le calcul de telle hauteur.

Il aborde aussi le sujet de la question de Lehmer, la conjecture la plus célèbre dans le domaine, donne quelques exemples et résultats ayant pour but de résoudre la question.

Ensuite, il y a l'extension de la mesure de Mahler sur les polynômes à plusieurs variables, une démarche semblable au premier cas de la mesure de Mahler, et le sujet des points limites avec quelques exemples.

Dans la seconde partie, on commence par donner des définitions concernant un ordre supérieur de la mesure de Mahler, et des généralisations en passant des polynômes simples aux polynômes à plusieurs variables.

La question de Lehmer existe aussi dans le domaine de la mesure de Mahler supérieure, mais avec des réponses totalement différentes.

À la fin, on arrive à notre objectif, qui sera la démonstration de la généralisation d'un théorème de Boyd-Lawton, ce dernier met en évidence une relation entre la mesure de Mahler des polynômes à plusieurs variables avec la limite de la mesure de Mahler des polynômes à une seule variable.

Ce résultat a des conséquences en termes de la conjecture de Lehmer et sert à clarifier la relation entre les valeurs de la mesure de Mahler des polynômes à une variable et celles des polynômes à plusieurs variables, qui, en effet, sont très différentes en nature.

Mots-clés : la mesure de Maher supérieure, la conjecture de Lehmer, les points limites, les polynômes à plusieurs variables, le théorème de Boyd-Lawton.

Abstract

This thesis applies to study first, in part 1, the Mahler measure of polynomials in one variable. It starts by giving some definitions and results that are important for calculating this height.

It also addresses the topic of Lehmer's question, an interesting conjecture in the field, and it gives some examples and results aimed at resolving the issue.

The extension of the Mahler measure to several variable polynomials is then considered including the subject of limit points with some examples.

In the second part, we first give definitions of a higher order for the Mahler measure, and generalize from single variable polynomials to multivariable polynomials.

Lehmer's question has a counterpart in the area of the higher Mahler measure, but with totally different answers.

At the end, we reach our goal, where we will demonstrate the generalization of a theorem of Boyd-Lawton. This theorem shows a relation between the limit of Mahler measure of multivariable polynomials with Mahler measure of polynomials in one variable. This result has implications in terms of Lehmer's conjecture and serves to clarify the relationship between the Mahler measure of one variable polynomials, and the Mahler measure of multivariable polynomials, which are very different.

Keywords : the higher Mahler measure, Lehmer's question, limit points, multivariable polynomials, Boyd-Lawton theorem.

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À mes parents,

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Introduction

Polynomials have been studied in many areas of mathematics, and they occupy a central place in number theory. Notions like the classical height and length of polynomial can be used to estimate their complexity. Not long ago, in the 1960's, Kurt Mahler gave the idea of a new kind of height, that is related to the roots of the polynomials, and was interested to compare it with other heights. Since then, this new object, named after Mahler, became an interesting topic in number theory with connections to different conjectures in mathematics, such as Lehmer's question, posed by Lehmer 30 years earlier.

In this memoir, we define the Mahler measure of one variable polynomials, extend it to multivariable polynomials, and then to higher Mahler measures, we talk about Lehmer's question for each case. The Boyd-Lawton theorem, mentioned in the title, shows a relation between Mahler measures of one variable polynomials and Mahler measures of multivariable polynomials. Our main goal in this memoir is to prove a generalization of this theorem for higher Mahler measure.

Part 1 The classical Mahler Measure.

Chapter 1

The Mahler measure.

Definition 1.1. *For any non zero polynomial*

$$P(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_0 = a_d \prod_{i=1}^{d} \left(x - \alpha_i \right)$$

in $\mathbb{C}[x]$*, define the Mahler measure of* P *to be* [Ma60]

$$M(P) = |a_d| \prod_{i=1}^{d} \max \left\{ 1, |\alpha_i| \right\}.$$

In this definition, an empty product is assumed to be 1, so the Mahler measure of the non-zero constant polynomial $P(x) = a_0$ is $|a_0|$.

We denote by $m(P) = \log M(P)$ the logarithmic Mahler measure, and we extend the definition to include $m(0) = \infty$.

Mahler called $M(P)$ the measure of the polynomial P, apparently to distinguish it from its classical height. It became known as the Mahler measure because of two papers written by Mahler in the early 1960s.

Mahler's was interested in comparing this construction with other heights such as the classical height and the length of the polynomial. They are defined respectively by

$$H(P) = \max_{0 \le 1 \le d} \left\{ |a_i| \right\},$$

$$\text{and} \quad L(P) = \sum_{i=0}^{d} |a_i|.$$

Mahler found a relation between the three heights [Ma60] [EW99]

$$H(P) \ll M(P) \ll H(P),$$
$$L(P) \ll M(P) \ll L(P),$$

which he later used together with the multiplicative property to prove bounds on the heights of products of polynomials. (Note: for two functions $f(x)$ and $g(x)$, $f \ll g$ means the growth of f is asymptotically bounded by g).

Lemma 1.2. *(Jensen's formula) for any* $\alpha \in \mathbb{C}$

$$\int_0^1 \log\left|\alpha - e^{2\pi i\theta}\right| d\theta = \log \max\{1, |\alpha|\}.$$

Proof. The statement is clear for $\alpha = 0$, so assume that $\alpha \neq 0$.

First assume that $|\alpha| \neq 1$ then

$$\int_0^1 \log\left|\alpha - e^{2\pi i\theta}\right| d\theta = \begin{cases} \log|\alpha| + \int_0^1 \log\left|1 - \alpha^{-1} e^{2\pi i\theta}\right| d\theta & \text{if } |\alpha| > 1, \\ \int_0^1 \log\left|1 - e^{-2\pi i\theta}\alpha\right| d\theta & \text{if } |\alpha| < 1. \end{cases}$$

The integral in the $|\alpha| < 1$ case may also be written (via the substitution $\theta \to -\theta$) as

$$\int_0^1 \log\left|1 - e^{2\pi i\theta}\alpha\right| d\theta.$$

It is therefore enough to prove that, for any β with $|\beta| < 1$,

$$\int_0^1 \log\left|1 - \beta e^{2\pi i\theta}\right| d\theta = 0.$$

Write $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ for the real and imaginary parts of a complex number z.

Notice that $\log|z| = \operatorname{Re}(\log z)$ so

$$\begin{aligned}
&\int_0^1 \log\left|1-\beta e^{2\pi i\theta}\right| d\theta \\
&= \operatorname{Re}\int_0^1 \log\left(1-\beta e^{2\pi i\theta}\right) d\theta \\
&= \operatorname{Re}\int_0^1 \left(-\sum_{n=1}^{\infty} \frac{\beta^n}{n} e^{2\pi i\theta n}\right) d\theta \\
&= \operatorname{Re}\left(-\sum_{n=1}^{\infty} \frac{\beta^n}{n} \int_0^1 e^{2\pi i\theta n} d\theta\right) \\
&= 0,
\end{aligned}$$

where the summation being taken out of the integral is justified because the sum is absolutely convergent.

We are left with the case $\beta = 1$.

Write the integral in the form

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|\alpha - e^{i\theta}\right| d\theta.$$

Assume that $|\alpha| = 1$, indeed after translating by α^{-1} we may as well assume that $\alpha = 1$.

Consider then

$$J = \int_0^{2\pi} \log\left|1-e^{i\theta}\right| d\theta.$$

Since $\left|1-e^{i\theta}\right| = 2\sin\frac{\theta}{2}$ for $\theta \in [0, 2\pi]$, we can replace it in the integral :

$$J = \int_0^{2\pi} \log 2\sin\frac{\theta}{2} d\theta = 2\pi\log 2 + \int_0^{2\pi} \log\sin\frac{\theta}{2} d\theta,,$$

Put $2x = \theta$, then

$$J = 2\pi \log 2 + 2\int_0^{\pi} \log \sin x dx.$$

It is enough to show that

$$J' = \int_0^{\pi} \log \sin x dx = -\pi \log 2.$$

This exists as an improper Riemann integral since $\sin x \sim x$ for small x.

Write $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$, then

$$J' = \pi \log 2 + \int_0^{\pi} \log \sin \frac{x}{2} dx + \int_0^{\pi} \log \cos \frac{x}{2} dx.$$

Substituting $\frac{x}{2} = t$ in the first integral and $\frac{x}{2} = \frac{\pi}{2} - t$ in the second, we get

$$J' = \pi \log 2 + 4\int_0^{\frac{\pi}{2}} \log \sin t dt$$
$$= \pi \log 2 + 2J'.$$

Lemma 1.3 [Ma60][EW99] (Mahler's lemma) *For any non zero polynomial* $P \in \mathbb{C}[x]$

$$m(P) = \int_0^1 \log\left|P(e^{2\pi i\theta})\right| d\theta.$$

Proof. This is a simple consequence of Jensen's formula.

□

By definition, Mahler measure is a positive number bigger than or equal to 1, and it is natural to ask about cases when this number equals 1, or when the logarithmic Mahler measure vanishes. Cases where $m(P) = 0$ can be completely understood using Kronecker's theorem.

Theorem 1.4. *(Kronecker) Suppose that* $\alpha \neq 0$ *is an algebraic integer. If the algebraic conjugates* $\alpha_1 = \alpha, ..., \alpha_d$ *of* α *all have modulus* $|\alpha_j| \leq 1$*, then* α *is a root of unity.*

Proof. Consider the polynomial

$$P_n(x) = \prod_{i=1}^{d} (x - \alpha_i^n)$$

where P_1 is the minimal polynomial for α. The coefficients of P_n are symmetric functions in the algebraic integers α_j so they are rational integers.

Each of the coefficients is uniformly bounded as n varies because $|\alpha_j| \leq 1$ for all j, so the set

$$\{P_n(x)\}_{n \in \mathbb{N}}$$

must be finite. It follows that there is a pair $n_1 \neq n_2$ for which

$$P_{n_1} = P_{n_2},$$

so

$$\left\{\alpha_1^{n_1}, ..., \alpha_d^{n_1}\right\} = \left\{\alpha_1^{n_2}, ..., \alpha_d^{n_2}\right\}.$$

Each permutation $\tau \in S_d$ (the permutation group on d symbols) defines an action on the set of roots by

$$\alpha_i^{n_1} = \alpha_{\tau(i)}^{n_2}.$$

Then if τ has order r in S_d, $\alpha_i^{n_1^r} = \alpha_i^{n_2^r}$, so $\alpha_i^{n_1^r}(\alpha_i^{n_2^r - n_1^r} - 1) = 0$, which shows that α_i must be a root of unity since $\alpha_i \neq 0$.

□

A polynomial in $\mathbb{Z}[x]$ *is called primitive if the coefficients have no non-trivial common factor.*

Theorem 1.5. *Suppose* $P \in \mathbb{Z}[x]$ *is non-zero, primitive and* $P(0) \neq 0$*, then* $m(P) = 0$ *if and only if all the zeros of* $P(x)$ *are roots of unity.*

Proof. Assume that all the zeros of $P(x)$ are roots of unity, then the leading coefficient of $P(x)$ must be ± 1 since $P(x)$ divides $\prod_i \left(x^{N_i} - 1 \right)$ for some $N_i \geq 1$. So, from the definition, $m(P) = 0$.

Conversely, if $m(P) = 0$, then it is clear that P must be a monic polynomial, so all the zeros are algebraic integers, and all must have modulus less than or equal to 1.

Applying Kronecker's theorem, we see they must all be roots of unity.

□

The Mahler's measure has been studied in many branches of mathematics. In the next section we discuss Lehmer's problem, which is the most important open problem in the area.

Chapter 2

Lehmer's Problem.

Lehmer was interested [Le33] in finding large primes, he searched for them amongst the Pierce sequences [Pi17]

$$\Delta_n(P) = \prod_{i=1}^{d} (\alpha_i^n - 1)$$

where the α_i's are the roots of an integer valued monic polynomial $P(x)$ with degree d.

Assume that the α_i's are never roots of unity, since if $\alpha_i^N = 1$ for some N, then $\Delta_n(P) = 0$ wherever n is divisible by N.

The quantity $\Delta_n(P)$ is always an integer since it is a product of all of the algebraic conjugates of $\alpha^n - 1$.

Lehmer $[$Le33$]$ showed that if P has no roots on the unit circle, then $\Delta_n(P)$ grows like $M(P)^n$. The terms of sequence $\Delta_n(P),\ n \geq 1$ are more likely to be prime if the sequence does not grow too quickly. He measured the rate of growth by considering the ratio of successive terms

$$\left| \frac{\Delta_{n+1}(P)}{\Delta_n(P)} \right|.$$

Lemma 2.1. *Provided no root* α_i *of* P *has* $|\alpha_i| = 1,$ *then*

$$\lim_{n \to \infty} \left| \frac{\Delta_{n+1}(P)}{\Delta_n(P)} \right| = \prod_{i=1}^{d} \max\left\{1, |\alpha_i|\right\} = M(P).$$

Proof. This is clear since we can treat each term in the product $\prod_{i=1}^{d}\left|\alpha_i^n - 1\right|$ separately :

If we take the limit of one of the terms, we get :

$$\lim_{n\to\infty}\frac{\left|\alpha_i^{n+1} - 1\right|}{\left|\alpha_i^n - 1\right|} = \begin{cases}\left|\alpha_i\right| & if \quad \left|\alpha_i\right| > 1\\ 1 & if \quad \left|\alpha_i\right| < 1\end{cases}$$
$$= \max\left\{\left|\alpha_i\right|, 1\right\}.$$

Then taking the product of the limit of the terms we get the desired results.

□

Lehmer produced some large primes as values of $\Delta_n(P)$. For example, if $P(x) = x^3 - x - 1$, Lehmer showed that [Le33]

$$\Delta_{113}(P) = 63088004325217...$$
$$\text{and } \Delta_{127}(P) = 3233514251032733...$$

are primes. The Mersenne sequence arises by choosing the polynomial $P(x) = x - 2$ so that

$$\Delta_n(P) = M_n = 2^n - 1.$$

The numbers in this sequence, for n an integer, are called Mersenne numbers. If M_n is prime then M_n is called Mersenne Prime.

It is necessary that n be prime for the Mersenne number to be prime. The inverse is not always true. It is still unknown if there exists infinitely many Mersenne primes.

Lehmer asked whether, or not, $M(P)$ can be arbitrarily close to 1, if $P(x)$ is a monic integer polynomial with $F = \prod_i P_i\left(x_1^{a_{i1}}, \ldots, x_n^{a_{in}}\right)$.

This has become known as "Lehmer's problem" or "Lehmer's question".

The smallest value of $M(P)>1$ he could find was

$$M(L) =: 1.176280818,$$

which is the Mahler measure of the 10 degree polynomial

$$L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

This polynomial is now called Lehmer's polynomial. Its Mahler measure is still the smallest known until present.

Lehmer's polynomial generates some large primes. Lehmer found that

$$\sqrt{\Delta_{379}(L)} =: 37098890596487,$$

which is prime. The values of Δ_n are squares of symmetric polynomials, so it is natural to look for prime values among their square roots.

Definition 2.2. *Suppose* $P(x) \in \mathbb{C}[x]$ *has degree* d*, write* $P^*(x) = x^d P(x^{-1})$.
Then P *is called reciprocal (or symmetric) if*

$$P(x) = P^*(x)$$

and non reciprocal otherwise.

Lehmer calculated the following measures [Le33]:

$$M(x^2 - x - 1) = 1.618...,$$
$$M(x^3 - x - 1) = 1.324...,$$
$$M(x^4 - x - 1) = 1.380...,$$
$$M(x^5 - x^3 - 1) = 1.362...,$$
$$M(x^6 - x - 1) = 1.370...,$$
$$M(x^7 - x^3 - 1) = 1.379... \ .$$

He also studied reciprocal polynomials, among these he found [Le33]:

$$M(x^6 - x^4 - x^3 - x^2 + 1) = 1.401...$$
$$\text{and} \quad M(x^8 - x^5 - x^4 - x^3 + 1) = 1.280...$$

but found no polynomials with smaller measure than $L(x)$ which is reciprocal.

Lehmer's question was answered by Smyth (see Section 2.3.2 below) for the non reciprocal polynomials and is still open for the reciprocals, for which exist many partial results about lower bounds.

2.1 The conjecture of Schinzel and Zassenhaus

If α is an algebraic integer with conjugates $\alpha_1,...,\alpha_d$, let

$$\lceil \alpha \rceil = \max_k |\alpha_k|.$$

For an algebraic number α, we denote by $M(\alpha)$ the Mahler measure of its minimal polynomial in $\mathbb{Z}[x]$.

Definition 2.1.1 *the minimal polynomial of an algebraic integer* α, *is the unique irreducible monic polynomial* $P(x)$, *of smallest degree with rational coefficients, such that* $P(\alpha) = 0$.

Suppose that the minimal polynomial of α has degree d. We have the obvious inequality [SZ65]:

$$M(\alpha)^{\frac{1}{d}} \leq \lceil \alpha \rceil \leq M(\alpha).$$

If α is a unit then $M(\alpha) = M(\alpha^{-1})$.

So that

$$\max\left(\lceil \alpha \rceil, \left\lceil \frac{1}{\alpha} \right\rceil\right) \leq M(\alpha). \tag{1}$$

In 1965, Schinzel and Zassenhaus [SZ65] proved that if $\alpha \neq 0$ an algebraic integer that is not a root of unity, and if $2s$ of its conjugates are non real, then

$$\lceil\alpha\rceil > 1+4^{-s-2}.$$

This was the first progress towards solving Lehmer's problem since by (1) it implies the same lower bound for $M(\alpha)$. Later they conjectured , that under the same hypothesis,

$$\lceil\alpha\rceil \geq 1+\frac{c}{d} \tag{2}$$

for some absolute constant $c>0$. A positive answer to Lehmer's conjecture would imply (2) because $\lceil\alpha\rceil \geq M(\alpha)^{\frac{1}{d}}$ implies that

$$\lceil\alpha\rceil \geq 1+\frac{\log M(\alpha)}{d}.$$

Thus if $M(\alpha) \geq c_0 \geq 1$, then

$$\lceil\alpha\rceil \geq 1+\frac{\log c_0}{d}.$$

The inequality implies that any progress on Lehmer's problem will lead to a corresponding Schinzel-Zassenhaus inequality.

2.2 Lower bounds.

Lehmer's question has been intensely studied in order to get new bounds. Below we cite some partial lower bounds, and note the relations with the degree d of the polynomial.

Some of the lower bounds are:

1) Blanksby and Montgomery (1971) [BM71]:

$$M(P) > 1+\frac{1}{52d\log(6d)}.$$

2) Stewart (1978) [ST78]:

$$M(P) > 1 + \frac{c}{10^4 d \log d}.$$

3) Dobrowolski (1979) [DO78]:

$$M(P) > 1 + (1-\varepsilon)\left(\frac{\log\log d}{\log d}\right)^3, \quad d > d_0(\varepsilon).$$

4) Cantor and Strauss (1982), [CS82]:

$$M(P) > 1 + (2-\varepsilon)\left(\frac{\log\log d}{\log d}\right)^3, \quad d > d_0(\varepsilon).$$

5) Louboutin (1983) [Lo82]:

$$m(P) > 1 + \left(\frac{9}{4} - \varepsilon\right)\left(\frac{\log\log d}{\log d}\right)^3, \quad d > d_0(\varepsilon).$$

As we can notice, 4 and 5 are improvement on the Dobrowolski method in 3.

2.3 Restricted results of Lehmer's problem

2.3.1 Schinzel theorem

The next lemma is a special case of a more general result due to Schinzel, and it concerns polynomials with strictly real zeros.

Lemma 2.3. *For any $d \ge 1$, let $y_1,...y_d > 1$ be real numbers , then*

$$(y_1 - 1)...(y_d - 1) \le \left((y_1...y_d)^{1/d} - 1\right)^d .$$

proof. Let $y_j = x_i + 1$, and write

$$f\left(x_1,...,x_d\right) = \prod_{i=1}^{d}\left(x_i + 1\right)^{1/d} - 1 - \prod_{i=1}^{d}\left(x_i\right)^{1/d},$$

if we prove this function to be ≥ 0 then the lemma will be true.
We have here a multivariable continuous differentiable function where all x_i's are ≥ 2 since $y_j \geq 1$, and $d \geq 1$, we will find the extremum for this function.
Take the derivative of f with respect of x_1

$$f'_{x_1} = \frac{1}{d}\left(1+x_1\right)^{1/d-1}\left(1+x_2\right)^{1/d}\ldots\left(1+x_d\right)^{1/d} - \frac{1}{d}x_1^{1/d-1}x_2^{1/d}\ldots x_d^{1/d},$$

$$f'_{x_1} = 0 \Leftrightarrow \left(1+x_1\right)^{1-d}\left(1+x_2\right)\ldots\left(1+x_d\right) = x_1^{1-d}x_2\ldots x_d$$

$$\Leftrightarrow x_1^d\left(1+x_1\right)\ldots\left(1+x_d\right) = x_1\ldots x_d\left(1+x_1\right)^d$$

since all the x_i are $\geq 2,$ all the terms above are different from $0,$ and we have equaltiy only when all the x_i's are equals. Now take the second derivative of f with respect to x_1

$$f''_{x_1} = \frac{1}{d}\left(\frac{1}{d}-1\right)\left(1+x_1\right)^{1/d-2}\left(1+x_2\right)^{1/d}\ldots\left(1+x_d\right)^{1/d} - \frac{1}{d}\left(\frac{1}{d}-1\right)x_1^{1/d-2}x_2^{1/d}\ldots x_d^{1/d}.$$

when all the x_i are equals,

$$f''_{x_1} = \frac{1}{d}\left(\frac{1}{d}-1\right)(1+x)^{-1} - \frac{1}{d}\left(\frac{1}{d}-1\right)x^{-1}$$

$$= \frac{1}{d}\left(\frac{1}{d}-1\right)\left(\frac{-1}{x\left(x+1\right)}\right)$$

which is >0 since $d \geq 1$ and $x \geq 2$, then the extremum for this function is a minimum.
Note that this second derivative function can not be =0, so the minimum is unique.
we will get the same results for the derivatives of the function with respect to the other x_i's.

Therefore, $f\left(x_1,\ldots x_d\right)$ has a minimum when all the x_i's are equal

$$f\left(x,\ldots x\right) = \left(\left(1+x\right)^{1/d}\right)^d - 1 - \left(x^{1/d}\right)^d = 0$$

Then f is always positive since it has one minimum 0, thus the proof is complete. □

Theorem 2.4. *(Schinzel) [Sc74] Suppose that* $P \in \mathbb{Z}[x]$ *is monic with degree* d*,* $P(-1)P(1) \neq 0$*, and* $P(0) = \pm 1$*. If the zeros of* P *are all real, then*

$$M(P) \geq \left(\frac{1+\sqrt{5}}{2}\right)^{d/2}.$$

Proof. Consider $E = \prod_{i=1}^{d} \left|\alpha_i^2 - 1\right|$, where the α_i's are roots of $P(x)$, since $P(x)$ is monic, we can write $P(x) = \prod_{i=1}^{d}\left(x - \alpha_i\right)$, $\left|P(1)\right| = \prod_{i=1}^{d}\left|1-\alpha_i\right|$ and $\left|P(-1)\right| = \prod_{i=1}^{d}\left|-1-\alpha_i\right|$, then E can be written $E = \left|P(1)P(-1)\right|$, clearly $E \geq 1$ since P is monic (its leading coefficient is 1), and $P(1)P(-1) \neq 0$ by hypothesis.

Then

$$E = \prod_{\left|\alpha_i\right|}\left|\alpha_i^2 - 1\right| = \prod_{\left|\alpha_i\right|<1}\left|\alpha_i^2 - 1\right| \prod_{\left|\alpha_i\right|>1}\left|\alpha_i^2 - 1\right|$$

$$= \prod_{\left|\alpha_i\right|>1}\left|\alpha_i\right|^2 \frac{1}{M(P)^2} \prod_{\left|\alpha_i\right|<1}\left|\alpha_i^2 - 1\right| \prod_{\left|\alpha_i\right|>1}\left|\alpha_i^2 - 1\right| \frac{P(0)^2}{P(0)^2}$$

$P(0) \neq 0$ and by the factorization of P we can write $P(0) = \prod_{\alpha_i}\left(-\alpha_i\right)$, so

$$E = \prod_{\left|\alpha_i\right|>1}\left|\alpha_i\right|^2 \frac{1}{M(P)^2} \prod_{\left|\alpha_i\right|<1}\left|\alpha_i^2 - 1\right| \prod_{\left|\alpha_i\right|>1}\left|\alpha_i^2 - 1\right| P(0)^2 \frac{1}{\prod_{\left|\alpha_i\right|>1}\left|\alpha_i\right|^2} \frac{1}{\prod_{\left|\alpha_i\right|<1}\left|\alpha_i\right|^2}$$

$= \frac{1}{M(P)^2} \prod_{\left|\alpha_i\right|<1}\left|\alpha_i^{-2} - 1\right| \times \prod_{\left|\alpha_i\right|>1}\left|\alpha_i^2 - 1\right|$. Since $P(0) = \pm 1$ by hypothesis.

Suppose that $\{\alpha_i\}_{i=1,..,j}$ are the roots with absolute value <1, and $\{\alpha_i\}_{i=j+1,..,d}$ are the remaining roots, then

$$E = \frac{1}{M(P)^2}\left(\alpha_1^{-2}-1\right)...\left(\alpha_j^{-2}-1\right)\left(\alpha_{j+1}^{2}-1\right)...\left(\alpha_d^{2}-1\right),$$

we have here a multiplication of d terms where the α_i's are real, so we can apply Lemma 2.3

$$E \le \frac{1}{M(P)^2}\left[\left(\left(\alpha_1^{-2}\right)...\left(\alpha_j^{-2}\right)\left(\alpha_{j+1}^{2}\right)...\left(\alpha_d^{2}\right)\right)^{1/d}-1\right]^d$$

$$\le \frac{1}{M(P)^2}\left[\left(\left(\alpha_1^{-2}\right)...\left(\alpha_j^{-2}\right)\left(\alpha_{j+1}^{2}\right)...\left(\alpha_d^{2}\right)\frac{\left(\alpha_{j+1}^{2}\right)...\left(\alpha_d^{2}\right)}{\left(\alpha_{j+1}^{2}\right)...\left(\alpha_d^{2}\right)}\right)^{1/d}-1\right]^d$$

$$\le \frac{1}{M(P)^2}\left[\left(\frac{1}{\pm P(0)^2}\left(\alpha_{j+1}^{4}\right)...\left(\alpha_d^{4}\right)\right)^{1/d}-1\right]^d$$

$$\le \frac{1}{M(P)^2}\left[\left(\prod_{|\alpha_i|>1}\alpha_i^4\right)^{1/d}-1\right]^d,$$

.

we use the fact that $M(P)=\prod_{|\alpha_i|>1}|\alpha_i|$ to get

$$E \le \frac{1}{M(P)^2}\left(M(P)^{4/d}-1\right)^d$$

$$=\left(M(P)^{2/d}-M(P)^{-2/d}\right)^d.$$

Since $1 \le E$, it follows that

$$M(P)^{2/d} - M(P)^{-2/d} \geq 1.$$

To solve this inequation, put $x = M(P)^{2/d}$, then we have

$$x - x^{-1} \geq 1 \Leftrightarrow x^2 - x - 1 \geq 0,$$

Which is a second degree inequation, true for $x \geq \frac{1+\sqrt{5}}{2}$ and for $x \leq \frac{1-\sqrt{5}}{2}$ (second inequaltity is rejected since it is negative and the Mahler measure is not), so we replace x in $x \geq \frac{1+\sqrt{5}}{2}$ by $M(P)^{2/d}$ to get the desired result

$$M(P) \geq \left(\frac{1+\sqrt{5}}{2}\right)^{d/2}.$$

□

Corollary 2.5. *If* $P \in \mathbb{Z}[x]$ *has all real zeros, with same conditions as in Theorem 2.4, then*

$$M(P) \geq \frac{1+\sqrt{5}}{2}.$$

proof. For $d \geq 2$, it is clear that we have $M(P) \geq \frac{1+\sqrt{5}}{2}$.

When $d = 1$, take $P(x) = x + b$, with $b \in \mathbb{Z}[x]$, since P is monic, and has integer coefficients , we must have $P(0) = \pm 1$, so $P(x) = x \pm 1$, but the condition $P(1)P(-1) \neq 0$ can not be verified with the polynomial $P(x) = x \pm 1$, so no polynomial with degree $d = 1$ verify the corollary then, we will apply this corollary for polynomials with degree $d \geq 2$.

□

2.3.2 Results for non reciprocal polynomials

Polynomials that are not reciprocal have a uniform lower bound for their Mahler measure.

In 1971, C.J. Smyth published the following remarkable theorem.

Theorem 2.6. [Sm71] *(Smyth) If $p(x) \in \mathbb{Z}[x]$ is a non reciprocal polynomial and $p(0)p(1) \neq 0$, then*

$$M(P) \geq M(x^3 - x - 1) = \theta_0 =: 1.3247...$$

where θ_0 is the real zero of $x^3 - x - 1$.

The condition that $p(1) \neq 0$ simply means P is not divisible by $x - 1$. This condition is required for if we multiply any reciprocal polynomial by $x - 1$ the measure does not change, but the polynomial may become non reciprocal. Take for example $P(x)$ of degree d to be a reciprocal polynomial so $P(x) = x^d P\left(x^{-1}\right)$. Put $Q(x) = (x-1)P(x)$, then $x^{d+1}Q\left(x^{-1}\right) = xx^d\left(\frac{1}{x} - 1\right)P\left(x^{-1}\right) = (1-x)P(x) \neq Q(x)$ for every x, so $Q(x)$ is not reciprocal.

Since the Mahler measure is multiplicative, and $M(x-1) = 1$, we have that $M(P) = M(Q)$, which makes the condition $P(1) \neq 0$ necessary regarding the hypothesis in the theorem about the polynomials being non reciprocal.

An algebraic integer α is said to be reciprocal if it is conjugate to α^{-1}.

Smyth's theorem shows that for α a non reciprocal algebraic integer

$$M(\alpha) \geq M(x^3 - x - 1) = \theta_0 =: 1.3247... \quad .$$

Equality $M(\alpha) = \theta_0$ occurs only for α conjugate to $\left(\pm\theta_0\right)^{\pm 1/k}$ for some positive integer k.

Smyth also proved a stronger result [Sm71], also for α non reciprocal algebraic integer

$$M(\alpha) > M(x^3 - x - 1) + 10^{-4}.$$

So that θ_0 is an isolated point in the spectrum of Mahler measures of non reciprocal algebraic integers. Here are some of the known small points in this spectrum [Sm71] :

$$M(x^3 - x - 1) =: 1.324717959...$$
$$M(x^5 - x^4 + x^2 - x + 1) =: 1.349716105...$$
$$M(x^6 - x^5 + x^3 - x^2 + 1) =: 1.359914149...$$
$$M(x^5 - x^2 + 1) =: 1.364199545...$$
$$M(x^9 - x^8 + x^6 - x^5 + x^3 - x + 1) =: 1.367854034...$$

It would be interesting to know more about this spectrum, and maybe it will be possible to find new small points coming from non reciprocal polynomials.

Chapter 3

Mahler's measure in several variables

3.1 Defintion and existence

Definition 3.1. *The Mahler measure of a non zero polynomial*

$$F \in \mathbb{C}\left[x_1^{\pm 1}, \ldots x_n^{\pm 1}\right]$$

is

$$m(F) = \frac{1}{(2\pi i)^n} \int_{\mathrm{T}^n} \log|F| \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n},$$

where $\mathrm{T}^n = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1| = \ldots = |z_n| = 1 \right\}.$

This definition extends to Laurent polynomials $F \in \mathbb{C}\left[x_1^{\pm 1}, \ldots x_n^{\pm 1}\right]$, either by evaluating the integral of the Laurent polynomial, or by noting that the measure satisfies $m(FG) = m(F) + m(G)$.

Lemma 3.2. *The expression $m(F)$ in definition 3.1 always exists as an improper Riemann integral*

Proof. [EW99] Consider $l(F)$ the logarithm of the sum of the absolute values of the coefficients of F. Then, from the triangle inequality, $m(F)$ is bounded above by $l(F)$.

Assume that the lemma is true for all polynomial in $n-1$ variables.

Write F as a polynomial in x_1 with coefficients in $\mathbb{C}[x_2, \ldots, x_n]$

$$F(x_1, \ldots, x_n) = a_d(x_2, \ldots, x_n) x_1^d + \ldots + a_0(x_2, \ldots, x_n).$$

Then factorize F as follows

$$F(x_1,\dots,x_n)=a_d(x_2,\dots,x_n)\prod_{j=1}^{d}(x_1-g_j(x_2,\dots,x_n)).$$

For certain algebraic functions $g_1,\dots,g_d$. Then

$$m(F)=m(a_d(x_2,\dots,x_n))+\sum_{j=1}^{d}\int_0^1\cdots\int_0^1\left(\int_0^1\log\left|e^{2\pi i\theta_1}-g_j(e^{2\pi i\theta_2},\dots,e^{2\pi i\theta_n})\right|d\theta_1\dots d\theta_n\right).$$

Put $J=\int_0^1\log\left|e^{2\pi i\theta_1}-g_j(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_n})\right|d\theta_1.$

By the inductive hypothesis, the first term exists.

Apply Jensen's formula to the integral J, we find that

$$J=\log^{+}\left|g_j\left(e^{2\pi i\theta_2},\dots,e^{2\pi i\theta_n}\right)\right|.$$

(this notation is defined by $\log^{+}\alpha=\log\max\{1,\alpha\}$ for $\alpha\in\mathbb{C}$)

For each $N\in\mathbb{N}$, define

$$\alpha_N=m(a_d(x_2,\dots,x_n))+\sum_{j=1}^{d}\int\cdots\int_{|g_j|\le N}\log^{+}\left|g_j(e^{2\pi i\theta_2},\dots,e^{2\pi i\theta_n})\right|d\theta_2\dots d\theta_n.$$

Then for each N, α_N exists since the integrand is continuous.

Also α_N increases with N, and is bounded above since $m(F)$ is bounded above by $l(F)$, so

$$\alpha_N\to m(F)$$

as $N\to\infty$.

□

In the first chapter, we proved that for a single variable polynomial $P(x)$, $m(P)=0$ in the case $P(x)$ is product of cyclotomic polynomials and monomials, one might ask the same question for multivariable polynomials. Next, we present the main statement in this section, for which the proof is due to Smyth.

Theorem 3.3. [Sm81b] [EW99]. *For any primitive polynomial* $F \in \mathbb{Z}\left[x_1^{\pm 1},\ldots,x_n^{\pm 1}\right]$, $m(F)$ *is zero if and only if* F *is a monomial times a product of cyclotomic polynomials evaluated at monomials i.e.* $F = \prod_i P_i\left(x_1^{a_{i1}},\ldots,x_n^{a_{in}}\right)$.

Definition 3.4. *A non zero polynomial* $P \in \mathbb{C}[x]$ *is said to be unit-monic if* $P(x) = a_d x^d + \ldots + a_0$ *has* $\left|a_d\right| = \left|a_0\right| = 1$.

For unit-monic polynomials, there is a complex analogue of Kronecker's lemma.

Lemma 3.5. *If* $P \in \mathbb{C}[x]$ *is unit monic, then* $m(P) = 0$ *if and only if the zeros of* P *lie on the unit circle.*

(Here the proof is same as in Theorem 1.5, a conclusion from Kronecker's lemma.)

Definition 3.6. *Let* F *be a non zero polynomial in* $\mathbb{C}\left[x_1^{\pm 1},x_2^{\pm 1}\right]$ *written as*

$$F\left(x_1,x_2\right) = \sum_{j \in J} a(j)\, x_1^{j_1} x_2^{j_2}.$$

With J *finite and all* $a(j) \neq 0$, $J = J(F)$ *is called the support of* F.

Define $C(F)$ *to be the Newton Polygon of* F.

Definition 3.7. *In complex algebra, the convex hull for a set of points* X *is the set of all convex combinations of points in* X.

Let F be a non zero polynomial in $\mathbb{C}\left[x_1^{\pm 1},x_2^{\pm 1}\right]$ written as

$$F\left(x_1,x_2\right) = \sum_{j \in J} a(j)\, x_1^{j_1} x_2^{j_2}.$$

Then the Newton polygon $C(F)$ of the polynomial F is defined to be the convex hull of the set $J \subset \mathbb{Z}^2$.

To better understand Newton polygon, we examine some examples:

1) Let $F(x_1, x_2) = 1 + x_1 + x_2$, the Newton polygon of this polynomial is a triangle:

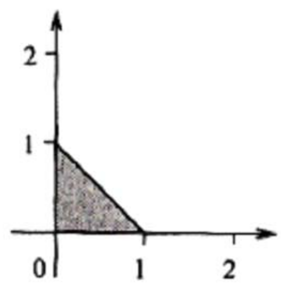

fig.1 [EW99]

2) Let $F_1(x_1, x_2) = 1 + x_1^2 + x_1x_2$ and $F_1(x_1, x_2) = 1 + 2x_1 + x_1^2 + 3x_1x_2$, both polynomials have the same Newton polygon, which is a triangle.

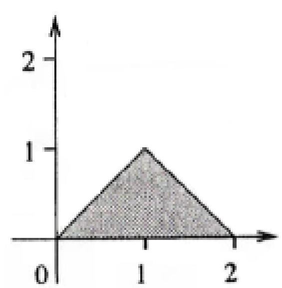

fig.2 [EW99]

An ***extreme point*** *of a convex set S in a real vector space is a point in S which does not lie in any open line segment joining two points of S.*

Definition 3.8. *A non zero polynomial $F \in \mathbb{C}\left[x_1^{\pm 1}, x_2^{\pm 1}\right]$ is extreme monic if $|a(j)| = 1$ for extreme points $j \in C(F)$.*

Lemma 3.9. [EW99] *If F and G are polynomials in $\mathbb{C}\left[x_1, x_2\right]$ then:*

1- $C(FG) = C(F) + C(G)$.

2- Every extreme point of $C(FG)$ is a sum of extreme points of $C(F)$ and $C(G)$ in a unique way.

3- If any two of F, G and FG are extreme monic so is the third.

Lemma 3.10. [EW99] *Suppose $F \in \mathbb{C}\left[x_1^{\pm 1}, x_2^{\pm 1}\right]$ is a non zero Laurent polynomial, then $C(F)$ is a straight line if and only if F is a monomial times a one variable polynomial evaluated at a monomial*

$$F(x_1, x_2) = x_1^a x_2^b G(x_1^c x_2^d)$$

$$\text{for } G \in \mathbb{C}\left[x^{\pm 1}\right].$$

Theorem 3.11. [EW99] *A polynomial $F \in \mathbb{C}\left[x_1^{\pm 1}, x_2^{\pm 1}\right]$ is extreme monic with $m(F) = 0$ if and only if F is a monomial times a product of unit-monic measure zero polynomials evaluated at monomials.*

See [EW99] for more information about the vanishing for the multivariable Mahler measure.

3.2 Explicit values.

Below are some examples of explicit evaluations of Mahler measures of multivariable polynomials. The results are due to Smyth [Sm81a].

Example 3.12. [Sm81a][EW99] Let $F(x_1, x_2) = 2 + x_1 + x_2$, using Jensen's formula twice, we see that

$$\begin{aligned}
m(F) &= \int_0^1 \left(\int_0^1 \log\left| e^{2\pi i\theta_1} + e^{2\pi i\theta_2} + 2 \right| d\theta_1 \right) d\theta_2 \\
&= \int_0^1 \log\max\left\{ 1, \left| e^{2\pi i\theta_2} + 2 \right| \right\} d\theta_2 \\
&= \int_0^1 \log\left| e^{2\pi i\theta_2} + 2 \right| d\theta_2 \\
&= \log 2.
\end{aligned}$$

Example 3.13. [Sm81a][EW99] Let $F(x_1,x_2)=1+x_1+x_2$, by Jensen's formula and a substitution

$$m(1+x_1+x_2)=J=\int_0^1\left(\int_0^1 \log\left|e^{2\pi i\theta_1}+e^{2\pi i\theta_2}+1\right|d\theta_2\right)d\theta_1$$

$$=\int_0^1 \log^+\left|e^{2\pi i\theta_1}+1\right|d\theta_1$$

$$=\int_{-1/2}^{1/2} \log^+\left|1+e^{2\pi i\theta}\right|d\theta.$$

For $\theta\in[-1/2,1/2]$, we have that $\left|1+e^{2\pi i\theta}\right|\geq 1$ only when $-1/3\leq\theta\leq 1/3$.

Hence

$$J=\int_{-1/3}^{1/3} \log\left|1+e^{2\pi i\theta}\right|d\theta$$

$$=2\int_0^{1/3} \log\left|1+e^{2\pi i\theta}\right|d\theta$$

$$=2\,\mathrm{Re}\int_0^{1/3} \log\left(1+e^{2\pi i\theta}\right)d\theta.$$

We can expand the integrand using the Fourrier series

$$\log\left(1+e^{2\pi i\theta}\right)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{e^{2\pi in\theta}}{n}.$$

This series converges uniformly for $\theta\in[0,1/3]$. It follows that

$$J = 2\,\mathrm{Re}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{1/3}\left(\cos 2\pi n\theta + i\sin 2\pi n\theta\right)d\theta$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{1/3}\cos 2\pi n\theta d\theta$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left[\frac{1}{2\pi n}\sin\frac{2\pi n}{3}\right]_0^{1/3}.$$

Notice that

$$\sin\frac{2\pi n}{3} = \left(\frac{n}{3}\right)\frac{\sqrt{3}}{2}.$$

Where $\left(\frac{n}{3}\right) = \chi(n)$ is the Legendre symbol

$$\chi(n) = \begin{cases} 1 & n \equiv 1 \mod 3, \\ -1 & n \equiv 2 \mod 3, \\ 0 & n \equiv 0 \mod 3. \end{cases}$$

Therefore

$$J = \frac{\sqrt{3}}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\chi(n)$$

$$= \frac{\sqrt{3}}{2\pi}\left\{\sum_{n=1}^{\infty}\frac{\chi(2n-1)}{(2n-1)^2} - \sum_{n=1}^{\infty}\frac{\chi(2n)}{(2n)^2}\right\}$$

$$= \frac{\sqrt{3}}{2\pi}\left\{\sum_{n=1}^{\infty}\frac{\chi(n)}{n^2} - 2\sum_{n=1}^{\infty}\frac{\chi(2n)}{(2n)^2}\right\}$$

$$= \frac{\sqrt{3}}{2\pi}\left\{\sum_{n=1}^{\infty}\frac{\chi(n)}{n^2} - \frac{1}{2}\chi(2)\sum_{n=1}^{\infty}\frac{\chi(n)}{n^2}\right\}$$

$$= \frac{\sqrt{3}}{2\pi}\frac{3}{2}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^2} = \frac{3\sqrt{3}}{4\pi}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^2}.$$

Example 3.14. [Bo81a][Sm81a] Let $F(x_1,x_2,x_3)=1+x_1+x_2+x_3$. Starting again with Jensen's formula gives

$$M(ax_3+b)=|a|\max\left(\left|\frac{b}{a}\right|,1\right)=\max\left(|a|,|b|\right).$$

Write $x_2=xx_3$, then

$$\begin{aligned}m(1+x_1+x_2+x_3)&=m(1+x_1+x_3(1+x))\\&=\int_0^1\int_0^1\log\max\left\{\left|1+e^{2\pi i\theta_1}\right|,\left|1+e^{2\pi i\theta_2}\right|\right\}d\theta_1 d\theta_2\\&=I.\end{aligned}$$

Put $t=2\pi\theta_1$, $u=2\pi\theta_2$ then

$$\begin{aligned}I&=\frac{1}{\pi^2}\int_0^{\pi}dt\int_0^{\pi}\max\left(\log\left|1+e^{it}\right|,\log\left|1+e^{iu}\right|\right)du\\&=\frac{2}{\pi^2}\int_0^{\pi}\log\left|1+e^{it}\right|dt\int_t^{\pi}du\\&=\frac{2}{\pi^2}\int_0^{\pi}(\pi-t)\log\left|1+e^{it}\right|dt\\&=\frac{-2}{\pi^2}\int_0^{\pi}t\log\left|1+e^{it}\right|dt.\end{aligned}$$

Since $M(1+z)=1$ so that $m(1+z)=0$.

We use again the expansion

$$\begin{aligned}\log\left|1+e^{it}\right|&=\operatorname{Re}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{itn}\\&=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cos(tn)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{1}{2}\left(e^{(itn)}+e^{(-itn)}\right).\end{aligned}$$

Then

$$I = -\frac{2}{\pi^2}\int_0^{\pi} t\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{1}{2}\left(e^{(itn)}+e^{(-itn)}\right)\right)$$

$$= -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\pi} t\left(e^{(itn)}+e^{(-itn)}\right)dt$$

by integrating by parts, we get

$$\int_0^{\pi} te^{(itn)}dt = \frac{t}{in}e^{(itn)}\Big|_0^{\pi} + \frac{1}{n^2}e^{(itn)}\Big|_0^{\pi} = \frac{\pi(-1)^n}{in}+\frac{(-1)^n}{n^2}-\frac{1}{n^2},$$

and

$$\int_0^{\pi} te^{(-itn)}dt = -\frac{t}{in}e^{(-itn)}\Big|_0^{\pi} + \frac{1}{n^2}e^{(-itn)}\Big|_0^{\pi} = -\frac{\pi(-1)^n}{in}+\frac{(-1)^n}{n^2}-\frac{1}{n^2},$$

So

$$I = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{2(-1)^n}{n^2}-\frac{2}{n^2}\right)$$

for $n = 2k+1$,

$$I = \frac{1}{\pi^2}\sum_{k=0}^{\infty}\frac{4}{(2k+1)^3},$$

We use a special formula of the Riemann zeta function : $\zeta(3) = \frac{8}{7}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}$ to get the desired result

$$I = \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3} = \frac{7}{2\pi^2}\zeta(3).$$

Here $\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$, a number which Apéry has proved to be irrational [Va78].

Chapter 4

Limit points.

Take L to be the set $L = \{M(P) : P \text{ has integer coefficients}\}$. Lehmer's question then, is equivalent to ask whether 1 is a limit point of L.

L is a countable set (according to its construction) and is a semigroup under multiplication (according to the property of Mahler's measure: $M(PQ) = M(P)M(Q)$). Thus, to show that 1 is a limit point of L, we can show that L is dense in $[1, \infty)$.

So in order to find an answer to Lehmer's question, we can study the properties of some of the subsets of L.

Consider now a certain subset S of L called the set of Pisot numbers. A Pisot number is a positive algebraic integer θ greater than 1 all of whose conjugate elements have absolute value less than 1. Clearly $M(P) = \theta,$ in this case since the other conjugates of θ have modulus less than 1, so that $S \subset L$.

Salem [Sa44] showed that S is a closed set (nowhere dense) and that $\min S = \theta_0 > 1$, Siegel [Si44] found that this θ_0 is the real zero of $z^3 - z - 1$, this zero is the same number Smyth showed to be the minimum value of the Mahler's measure of the non-reciprocal polynomials (see Theorem 2.5), answering Lehmer's question for the nonreciprocal polynomials, so here comes another subset of L, which is L_0, the set of $M(P)$ with P non reciprocal.

The set T of Salem numbers is another subset of L, that contains real algebraic integers $\theta > 1$ all of its conjugate roots have absolute value no greater than 1, and at least one has absolute value exactly 1, the last condition forces the polynomial to be reciprocal [Sa45].

We can observe that $\sigma = 1.17628...$ which is the real zero of the Lehmer's 10 degree polynomial $L(x)$ defined earlier in Chapter 2, is a Salem number. Boyd [Bo81a] suggests that σ could be a limit point of L.

Lehmer's question asked if there is a gap between 1 and $1+\varepsilon$ for some positive ε in the set of values of the Mahler measure of single variable integer polynomials. The analogous question may be asked for polynomials in several variables. It turns out that the questions are equivalent because of a surprising approximation result due to Lawton [La77].

Theorem 4.2. [La77] *For a non-zero polynomial* $F \in \mathbb{C}[z,w]$

$$\lim_{N\to\infty} M(F(z,z^N)) = M(F(z,w))$$

Theorem 4.3. [La77] *For a multivariable polynomial* $F \in \mathbb{C}[z_1,...,z_n]$

$$\lim_{r_2\to\infty} \dots \lim_{r_n\to\infty} M(F(z_1^{r_1},...,z_1^{r_n})) = M(F(z_1,...,z_n)),$$

where the limit is taken with all the exponents going to ∞ independently.

We give now an example (also is an application of Theorem 4.3) taken from Boyd's paper [Bo81a], about an interesting limit point which is the Mahler measure of the two variables polynomial $F(z_1,z_2) = 1+z_1+z_2$.

$$\beta = M(1+z_1+z_2) =: 1.38135...$$

Smyth has shown that

$$\log\beta = \frac{3\sqrt{3}}{4\pi}\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{1}{n^2}.$$

(The details of the calculations were given in Example 3.13)

Boyd showed that

$$m(1+z+z^n) = m(1+z_1+z_2) + \frac{c(n)}{n^2} + O(\frac{1}{n^3}),$$

where $c(n)=\begin{cases} -\sqrt{3}\pi/6 & n\equiv 2 \mod 3, \\ \sqrt{3}\pi/18 & n\equiv 0 \quad or \quad 1 \mod 3. \end{cases}$

So that β is a limit point in the set L, and it seems also that β is the smallest limit point of the set L_0. A result by Salem shows that each element in L_0 is a limit point in L. In the case of non-reciprocal polynomials, it is easier to show that $M(F(z_1,...,z_n))$ is a limit point in the set L.

If we show the following statement suggested and proved by Boyd [Bo81a]

$$m(1+z+z^n)=m(1+z_1+z_2)+\frac{c(n)}{n^2}+O(\frac{1}{n^3}).$$

We can then deduce easily that β is a limit point. A detailed proof of this can be found in [Bo81a].

Part 2 The higher Mahler Measure

Chapter 5

Definitions and examples

Definition 5.1[KLO08] *Given a non zero polynomial* $P(x) \in \mathbb{C}[x]$ *and a positive integer* k, the k-higher Mahler measure of P is defined by:

$$m_k(P) = \frac{1}{2\pi i} \int_{|x|=1} \log^k |P(x)| \frac{dx}{x}.$$

Or equivalently, by

$$m_k(P) = \int_0^1 \log^k \left| P\left(e^{2\pi i\theta}\right) \right| d\theta.$$

We observe that for $k = 1$, $m_1(P)$ is the classical logarithmic Mahler measure (Definition 1.1).

Definition 5.2 *For a finite collection of non-zero polynomials* $P_1, ..., P_\ell \in \mathbb{C}[x]$, their *multiple Mahler measure is defined by*

$$m(P_1, ..., P_\ell) = \frac{1}{2\pi i} \int_{|x|=1} \log|P_1(x)| ... \log|P_\ell(x)| \frac{dx}{x}.$$

The generalization in Definition 3.1 can be extended to the multiple higher Mahler measure:

Definition 5.3. *Let* $P_1, ..., P_\ell \in \mathbb{C}\left[x_1, ..., x_n\right]$ *be nonzero polynomials. Then, we define* $m\left(P_1, ..., P_\ell\right)$ *as*

$$\frac{1}{(2\pi i)^n}\int_{|x_1|=1}\cdots\int_{|x_n|=1}\log\left|P_1\left(x_1,\dots,x_n\right)\right|\dots\log\left|P_\ell\left(x_1,\dots,x_n\right)\right|\frac{dx_1}{x_1}\dots\frac{dx_n}{x_n}.$$

5.1 Higher Mahler measure for one variable polynomial

In general, Higher Mahler measures are very hard to compute, even for simple linear polynomials in one variable. In this section, we consider the simplest example : $P(x)=1-x.$

Theorem 5.4 shows a direct connection between the higher Mahler measure of $P(x)=1-x$ and the Riemann Zeta function.

Theorem 5.4. [KLO08]

$$m_k(1-x)=\sum_{b_1+\dots+b_h=k,b_t\geq 2}\frac{(-1)^k\,k!}{2^{2h}}\zeta(b_1,\dots,b_h),$$

where $\zeta(b_1,\dots,b_h)$ *denotes a multi zeta value, i.e.*

$$\zeta(b_1,\dots,b_h)=\sum_{0<\ell_1<\dots<\ell_h}\frac{1}{\left(\ell_1\right)^{b_1}\dots\left(\ell_h\right)^{b_h}}.$$

Example 5.5. [KLO08] Applying theorem 5.5, we are able to compute $m_k(1-x)$, here are the first few examples :

$$m_2(1-x)=\frac{\zeta(2)}{2},$$

$$m_3(1-x)=-6\left(\frac{\zeta(3)}{4}\right)=-\frac{3}{2}\zeta(3),$$

$$m_4(1-x)=24\left(\frac{\zeta(4)}{4}+\frac{\zeta(2,2)}{16}\right)=6\zeta(4)+3\frac{\left(\zeta(2)^2-\zeta(4)\right)}{4}=\frac{3\zeta(2)^2+21\zeta(4)}{4}=\frac{19\pi^2}{240},$$

$$m_5(1-x)=-120\left(\frac{\zeta(5)}{4}+\frac{\zeta(2,3)+\zeta(3,2)}{16}\right)=-\frac{15\zeta(2)\zeta(3)+45\zeta(5)}{2},$$

$$m_6(1-x)=720\left(\frac{\zeta(6)}{4}+\frac{\zeta(3,3)}{16}+\frac{\zeta(2,4)+\zeta(4,2)}{16}+\frac{\zeta(2,2,2)}{64}\right)$$

$$=180\zeta(6)+\frac{45\left(\zeta(3)^2-\zeta(6)\right)}{2}+45\left(\zeta(2)\zeta(4)-\zeta(6)\right)+\frac{45\left(2\zeta(6)-3\zeta(2)\zeta(4)+\xi(2)^3\right)}{4.6}$$

$$=\frac{930\zeta(6)+180\zeta(3)^2+315\zeta(2)\zeta(4)+15\zeta(2)^3}{8}=\frac{45}{2}\zeta(3)^2+\frac{275}{1344}\pi^6.$$

5.2 Multiple Mahler measure for several linear polynomials

As before, it is easier to consider the one-variable polynomial case. The next theorem allows us to give some examples of higher measure for several polynomials.

Theorem 5.6[KLO08] *For* $0\le\alpha\le1$

$$m\left(1-x,1-e^{2\pi i\alpha}x\right)=\frac{\pi^2}{2}\left(\alpha^2-\alpha+\frac{1}{6}\right).$$

Proof. By definition

$$m\left(1-x,1-e^{2\pi i\alpha}x\right)=\int_0^1\operatorname{Re}\log\left(1-e^{2\pi i\theta}\right)\operatorname{Re}\log\left(1-e^{2\pi i(\theta+\alpha)}\right)d\theta$$

$$=\int_0^1\left(-\sum_{k=1}^{\infty}\frac{1}{k}\cos2\pi k\theta\right)\left(-\sum_{\ell=1}^{\infty}\frac{1}{\ell}\cos2\pi\ell(\theta+\alpha)\right)d\theta$$

$$=\sum_{k,\ell\ge1}\frac{1}{k\ell}\int_0^1\cos(2\pi k\theta)\cos\left(2\pi\ell(\theta+\alpha)\right)d\theta.$$

On the other hand

$$\int_0^1\cos(2k\pi\theta)\cos\left(2\pi\ell(\theta+\alpha)\right)d\theta$$

$$=\begin{cases}\frac{1}{2}\cos(2k\pi\alpha) & \ell=k;\\ 0 & otherwise.\end{cases}$$

Putting everything together, we get

$$m\left(1-x,1-e^{2\pi i\alpha}x\right)=\frac{1}{2}\sum_{k=1}^{\infty}\frac{\cos(2\pi k\alpha)}{k^2}$$
$$=\frac{\pi^2}{2}\left(\alpha^2-\alpha+\frac{1}{6}\right)$$

In order to prove this we will find the Fourier series of the polynomial $\pi^2\alpha^2-\pi^2\alpha+\frac{\pi^2}{6}$.

Cosider first the case for $0<\alpha<1$, then $0<2\pi\alpha<2\pi$,

put $y=2\pi\alpha$, and replace it in the polynomial, so we will have the function:

$f(y)=\frac{y^2}{4}-\frac{\pi y}{2}+\frac{\pi^2}{6}$ for $0<y<2\pi$.

again put $x=y-\pi$ so that we have the function

$$g(x)=\frac{x^2}{4}-\frac{\pi^2}{12}$$

$g(x)$ is a periodic function of period 2π with $-\pi<x<\pi$.

The Fourier series of such function is given by:

$$S(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left[a_k\cos kx+b_k\sin kx\right]$$

Where

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}g(x)dx,$$

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}g(x)\cos kxdx,$$

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}g(x)\sin kxdx.$$

Therefore

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{x^2}{4}-\frac{\pi^2}{12}\right)dx=\frac{1}{\pi}\left(\frac{x^3}{12}-\frac{\pi^2 x}{12}\right)_{-\pi}^{\pi}=0.$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{x^2}{4}-\frac{\pi^2}{12}\right)\cos kx dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\frac{x^2 \cos kx}{4}dx - \frac{2}{\pi}\int_0^{\pi}\frac{\pi^2 \cos kx}{12}dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2 \sin kx}{k} - 2x\frac{-\cos kx}{k^2} + 2\frac{-\sin kx}{k^3}\right]_0^{\pi} - \frac{2}{\pi}\left[\frac{\pi^2}{12}\frac{\sin kx}{k}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{2\pi \cos k\pi}{k^2}\right] = \frac{(-1)k}{k^2}.$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{x^2}{4}-\frac{\pi^2}{12}\right)\sin kx dx = 0 \text{ since the integrand is an odd function.}$$

(Note: $f(x)$ is said to be an even function if $f(x) = f(-x)$, in this case $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$,

for $a \in \mathbb{R}$, and is said to be an odd function if $f(x) = -f(-x)$ and in this case $\int_{-a}^{a} f(x)dx = 0$)

So we obtain the following Fourier series

$$g(x) = \frac{x^2}{4} - \frac{\pi^2}{12} = \sum_{k=1}^{\infty}\frac{(-1)^n}{k^2}\cos kx.$$

Now we replace x by $y - \pi$,

note that $\cos k(y-\pi) = \cos ky \cos k\pi - \sin ky \sin k\pi = (-1)^k \cos k\pi$ by the trigonometric identity $\cos(A-B) = \cos A \cos B - \sin A \sin B$,

so

$$f(y) = \frac{y^2}{4} - \frac{\pi y}{2} + \frac{\pi^2}{6} = \sum_{k=0}^{\infty}\frac{\cos ky}{k^2},$$

finally for $y = 2\pi\alpha$, and $0 < \alpha < 1$,

$$\pi^2\left(\alpha^2 - \alpha + \frac{1}{6}\right) = \sum_{k=0}^{\infty}\frac{\cos 2\pi k\alpha}{k^2}.$$

For the case when $\alpha = 0$ or α=1, the value of the polynomial is $\frac{\pi^2}{6}$, on the other side, we have the same value in the cosine series, for $\alpha = 0$ or α=1, which is

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2) = \frac{\pi^2}{6} (\text{Riemann Zeta function for } n = 2),$$

thus the proof is complete.

□

Example 5.7. Applying the Theorem 5.6 above, we obtain the following examples

$$m(1-x, 1-x) = \frac{\pi^2}{12},$$

$$m(1-x, 1+x) = -\frac{\pi^2}{24},$$

$$m(1-x, 1 \pm ix) = -\frac{\pi^2}{96},$$

$$m\left(1-x, 1-e^{2\pi i\alpha}x\right) = 0 \Leftrightarrow \alpha = \frac{3 \pm \sqrt{3}}{6}.$$

Using the method of Theorem 5.6, we obtain a higher analogue of Jensen's formula.

Lemma 5.8. [KLO08] *for* $\alpha, \beta \in \mathbb{C}$

$$m(1-\alpha x, 1-\beta x) = \begin{cases} \frac{1}{2} \operatorname{Re} Li_2\left(\alpha \overline{\beta}\right) & if \quad |\alpha|, |\beta| \le 1, \\ \frac{1}{2} \operatorname{Re} Li_2\left(\frac{\alpha \beta}{|\alpha|^2}\right) & if \quad |\alpha| \ge 1, |\beta| \le 1, \\ \frac{1}{2} \operatorname{Re} Li_2\left(\frac{\alpha \overline{\beta}}{|\alpha\beta|^2}\right) + \log|\alpha| \log|\beta| & if \quad |\alpha|, |\beta| \ge 1. \end{cases}$$

Where Li_2 *is the dilogarithm function defined by the sum*

$$Li_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \; for \quad |z| \le 1,$$

or the integral

$$Li_2(z) = -\int_0^z \frac{\log(1-t)\,dt}{t}.$$

The previous computation may be extended to multiple higher Mahler measure involving more than two linear polynomials:

Lemma 5.9. [KLO08]

$$m\left(1-x, 1-e^{2\pi i\alpha}x, 1-e^{2\pi i\beta}x\right)$$
$$= -\frac{1}{4} \sum_{r,s \in \mathbb{Z} \setminus \{0\}, r+s>0} \frac{\cos 2\pi(r\alpha + s\beta)}{|rs(r+s)|}.$$

It would be interesting to compute higher Mahler measure for polynomials whose roots have absolute value $\neq 1$, and degree >1.

Chapter 6

Lehmer's question for higher Mahler measure

Since Lehmer's question is interesting for the classical Mahler measure, one might ask about the analogous question for the higher Mahler measures.

We said earlier that Lehmer's question can be rephrased as whether 0 is a limit point for logarithmic Mahler measures. The question is the same for m_k, but here we distinguish between two cases : the odd one and the even one.

A theorem about lower bounds for the even higher Mahler measure m_{2h} reveals that Lehmer's question is answered negatively for this case (see Theorem 6.1 below).
A second theorem shows that the limit for the odd higher Mahler measure m_{2h+1} is zero, so the answer to Lehmer's question for this case is positive (see Theorem 6.2 below).

Theorem 6.1. [LS11] *If* $P(x) \in \mathbb{Z}[x]$ *is not a monomial then for any* $h \geq 1$

$$m_{2h}(P) \geq \begin{cases} \left(\dfrac{\pi^2}{12}\right)^h & \text{if} \quad P(x) \quad \text{is} \quad \text{reciprocal}, \\[2ex] \left(\dfrac{\pi^2}{48}\right)^h & \text{if} \quad P(x) \quad \text{is} \quad \text{non-reciprocal}. \end{cases}$$

Theorem 6.1 provides a general lower bound for both kinds of polynomials (reciprocal or non reciprocal). There are no conditions on the polynomials, unlike in the case of the classical Mahler measure where lower bounds come with restrictions on the polynomials (see Chapter 2). This theorem answers Lehmer's question negatively for the even higher Mahler measure since as we can see from the lower bound, the limit is not zero.

Theorem 6.2 [LS11] *Let* $P_n(x) = \dfrac{x^n - 1}{x - 1}$. *For* $h \geq 1$ *fixed,*

$$\lim_{n \to \infty} m_{2h+1}(P_n) = 0.$$

We obtain in this way a positive answer for Lehmer's question for m_{2h+1}.

Note that the sequence $m_{2h+1}\left(P_n\right)$ in Theorem 2.6 is a nonconstant sequence, which means that it can not not be identically zero, we can find in [LS11] in section 5.3 a discussion about this, and a proof that $m_{2h+1}\left(P_n\right)$ behaves like a nonzero constant times $\frac{\log^{2h-1} n}{n}$ when n goes to infinity.

See [KLO08] and [LS11] for more details about the higher Mahler measure and its connection with the Lehmer problem.

Chapter 7

Limit values for higher Mahler measures.

We arrive now to the main goal of the thesis, which is a generalization of an important theorem conjectured by Boyd and completely proved by Lawton [La83]. The reader will find an article by Zahraa Issa and Matilde Lalin where they prove a generalization of this theorem to higher Mahler measure and generalized Mahler measure.

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A GENERALIZATION OF A THEOREM OF BOYD AND LAWTON

Abstract. The Mahler measure of a nonzero n-variable polynomial P is the integral of $\log|P|$ on the unit n-torus. A result of Boyd and Lawton says that the Mahler measure of a multivariate polynomial is the limit of Mahler measures of univariate polynomials. We prove the analogous result for different extensions of Mahler measure such as generalized Mahler measure (integrating the maximum of $\log|P|$ for possibly different P's), multiple Mahler measure (involving products of $\log|P|$ for possibly different P's), and higher Mahler measure (involving $\log^k|P|$).

7.1 Introduction

The Mahler measure of a nonzero polynomial $P\left(x_1,\dots,x_n\right)\in\mathbb{C}\left[x_1,\dots,x_n\right]$ is defined by

$$m(P):=\frac{1}{(2\pi i)^n}\int_{\mathrm{T}^n}\log\left|P\left(x_1,\dots,x_n\right)\right|\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n},$$

where $\mathrm{T}^n = \left\{ \left(z_1, \ldots, z_n \right) \in \mathbb{C}^n : \left| z_1 \right| = \ldots \left| z_n \right| = 1 \right\}$ is the unit torus in dimension n.

This formula has a particularly simple expression for univariate polynomials. If $P(x) = a \prod_i \left(x - \alpha_i \right)$, Jensen's formula implies that $m(P) = \log|a| + \sum_i \max\left\{ 0, \log\left| \alpha_i \right| \right\}$. In fact, Lehmer [Le33] considered first the measure for univariate polynomials which was later extended to multivariate polynomials by Mahler [Ma62]. Lehmer's motivation for considering this object was a method to construct large prime numbers that generalizes Mersenne's sequence. Mahler, on the other hand, was interested in relating heights of products of polynomials with the heights of the factors. The Mahler measure is a height which is multiplicative, and therefore it was a natural object for Mahler to consider.

Boyd and Lawton proved the following useful and interesting result.

Theorem 7.1.1 [Bo81a, Bo81b, La83] *Let* $P\left(x_1, \ldots, x_n \right) \in \mathbb{C}\left[x_1, \ldots, x_n \right]$ *and* $r = \left(r_1, \ldots, r_n \right), r_i \in \mathbb{Z}_{>0}$

Define $P_r(x)$ *as*

$$P_r(x) = P\left(x^{r_1}, \ldots, x^{r_n} \right),$$

And let

$$q(r) = \min\left\{ H(s) : s = \left(s_1, \ldots, s_n \right) \in \mathbb{Z}^n, s \neq (0, \ldots, 0), \sum_{j=1}^{n} s_j r_j = 0 \right\},$$

where $H(s) = \max\left\{ \left| s_j \right| : 1 \leq j \leq n \right\}$. *Then*

$$\lim_{q(r) \to \infty} m\left(P_r \right) = m(P).$$

This result implies that the multivariate Mahler measure is a limit of univariate Mahler measures. In particular, it gives evidence that the extension to multivariate polynomials is the right generalization.

The Mahler measure of multivariate polynomials often yields special values of the Riemann zeta function and L-functions, thus one can construct sequences of numbers that approach these special values in this way.

In addition, this theorem has consequences in terms of limit points of Mahler measure. The most famous open question in this area is the so called Lehmer's question. *Is there a constant* $c > 0$ *such that for every polynomial* $P \in \mathbb{Z}[x]$ *with* $m(P) > 0$*, then* $m(P) \geq c$*?* Thus, Theorem 7.1.1 tells us that given a multivariate polynomial whose measure is smaller than a certain constant c, we can generate infinitely many univariate polynomials with the same property.

In this work, we are going to consider two extensions of Mahler measure.

Given $P_1,\dots,P_s \in \mathbb{C}\left[x_1,\dots,x_n\right]$, (not necessarily distinct) nonzero polynomials, the *generalized Mahler measure* is defined in [GO04] by

$$m_{\max}\left(P_1,\dots,P_s\right) := \frac{1}{(2\pi i)^n}\int_{\mathrm{T}^n} \max\left\{\log\left|P_1\left(x_1,\dots,x_n\right)\right|,\dots,\log\left|P_s\left(x_1,\dots,x_n\right)\right|\right\}\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

On the other hand, the *multiple Mahler measure* is defined in [KLO08] by

$$m\left(P_1,\dots,P_s\right) := \frac{1}{(2\pi i)^n}\int_{\mathrm{T}^n} \log\left|P_1\left(x_1,\dots,x_n\right)\right|\dots\log\left|P_s\left(x_1,\dots,x_n\right)\right|\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

For the particular case in which $P_1 = \dots = P_s = P$, the multiple Mahler measure is called *higher Mahler measure*

$$m_s(P) := \frac{1}{(2\pi i)^n}\int_{\mathrm{T}^n} \log^s\left|P\left(x_1,\dots,x_n\right)\right|\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

These objects have been related to special values of the Riemann zeta function and L-functions ([GO04,La08] for generalized Mahler measure, [KLO08,Sa10,Sa,BS,BBSW] for multiple Mahler measure), but the nature of this relationship is less well understood than in the classical case.

Our goal in this note is to prove the equivalent for Theorem 7.1.1 for these generalizations.

Theorem 7.1.2. *Let $P_1,\dots,P_s \in \mathbb{C}\left[x_1,\dots,x_n\right]$ and r as before. Then*

(1)

$$\lim_{q(r)\to\infty} m_{\max}\left(P_{1r},\dots,P_{sr}\right) = m_{\max}\left(P_1,\dots,P_s\right).$$

(2)

$$\lim_{q(r)\to\infty} m\left(P_{1r},\dots,P_{sr}\right) = m\left(P_1,\dots,P_s\right).$$

(3)If $P_1 = \dots = P_s = P$,

$$\lim_{q(r)\to\infty} m_s\left(P_r\right) = m_s(P).$$

7.2. Some Preliminary Results

The difficulty in obtaining Theorem 7.1.2 lies in the case where (some of) the polynomials vanish in the domain of integration and the logarithm is not bounded. This problem already appears in the proof of Theorem 7.1.1. The key result of solving this is a theorem by Lawton [La83].

Let μ_n denote the Lebesgue measure in the torus T^n.

Theorem 7.2.1. ([La83], Theorem 1). *Let $P(x) \in \mathbb{C}[x]$ be a monic polynomial and let k =number of nonzero coefficients of P. Then if $k \geq 2$, there is a positive constant C_k that depends only on k such that*

$$\mu_1\left(\left\{z \in \mathrm{T} : \left|P(z)\right| < y\right\}\right) \leq C_k y^{\frac{1}{k-1}},$$

for any real number $y > 0$.

The strength of this result lies in the fact that the constant is absolute and depends on the number of nonzero coefficients of P but it does not depend on P.

Notice that we can always assume that the polynomials involved in multiple Mahler measure have at least two nonzero monomials since $\log\left|ax^k\right|$ is a constant and can be easily extracted from the integral. It should be noted that the above theorem remains true for $k = 1$ if y is sufficiently small (i.e. $y < |a|$) and $C_1 = 0$.

It is not hard to prove a result where the constant depends on P. For example,

Lemma 7.2.2. ([EW99], Lemma 3.8, pg. 58) *Let $P\left(x_1, \ldots, x_n\right) \in \mathbb{C}\left[x_1, \ldots, x_n\right]$. There are constants C_P, δ_P that depend on P such that*

$$\mu_n\left(\left\{\left(z_1, \ldots, z_n\right) \in \mathrm{T}^n : \left|P\left(z_1, \ldots, z_n\right)\right| < y\right\}\right) \leq C_P y^{\delta_P}, \tag{2.1}$$

for small $y > 0$.

In what follows, we will denote by

$$S_n(P, y) = \left\{\left(z_1, \ldots, z_n\right) \in \mathrm{T}^n : \left|P\left(z_1, \ldots, z_n\right)\right| < y\right\},$$

where the n depends on the number of variables involved. Thus $n \geq$ number of variables of P. We will write $S(P, y)$ for $S_1(P, y)$.

This result is weaker than Lemma 7.2.1, because the constant depends on P, but it has the advantage that can be applied to T^n, as opposed to Theorem 7.2.1 which is only valid for T.

The following elementary lemma will be useful to bound integrals.

Lemma 7.2.3. *Let ℓ be a positive integer and $y, \delta > 0$. Then*

$$J_{\ell,\delta}(y) := (-1)^{\ell} \int_0^y \log^{\ell} z d\left(z^{\delta}\right)$$

$$= y^{\delta}\left((-1)^{\ell}\log^{\ell} y + \frac{\ell}{\delta}(-1)^{\ell-1}\log^{\ell-1} y + \frac{\ell(\ell-1)}{\delta^2}(-1)^{\ell-2}\log^{\ell-2} y + \ldots\right.$$

$$\left. + \frac{\ell(\ell-1)\ldots 2}{\delta^{\ell-1}}(-1)\log y + \frac{\ell!}{\delta^{\ell}}\right)$$

Proof. First we consider the integral for $\ell = 1$, then

$$J_{1,\delta} = -\int_0^y \log z dz^{\delta},$$

If we consider $u = z^{\delta}$, then we can rewrite the integral

$$J_{1,\delta} = -\int_0^{y^{\delta}} \log z^{\frac{1}{\delta}} dz = -\frac{1}{\delta}\int_0^{y^{\delta}} \log z dz$$

$$= -\frac{1}{\delta}\left[z\log z - z\right]_0^{y^{\delta}} = -\frac{1}{\delta}y^{\delta}\log y^{\delta} + \frac{1}{\delta}y^{\delta}$$

$$= -y^{\delta}\log y + \frac{1}{\delta}y^{\delta} = y^{\delta}\left(-\log y + \frac{1}{\delta}\right).$$

So the lemma is true for $\ell = 1$. Suppose the lemma is true for $\ell - 1$
now

$$J_{\ell,\delta} = (-1)^{\ell} \int_0^y \log^{\ell} z dz^{\delta} = (-1)^{\ell} \frac{1}{\delta^{\ell}} \int_0^{y^{\delta}} \log^{\ell} z dz$$

by integration by part:

$$J_{\ell,\delta} = \frac{(-1)^{\ell}}{\delta^{\ell}} \left[z \log^{\ell} z \Big|_0^{y^{\delta}} - \ell \int_0^{y^{\delta}} \log^{\ell-1} z dz \right]$$

$$= (-1)^{\ell} y^{\delta} \log^{\ell} y + \frac{\ell}{\delta} J_{\ell-1,\delta}$$

since $J_{\ell-1,\delta}$ verify the lemma by the recursion hypothesis, we can replace it by its value for ℓ-1

$$= (-1)^{\ell} y^{\delta} \log^{\ell} y + y^{\delta} \frac{\ell}{\delta} \Big((-1)^{\ell-1} \log^{\ell-1} y + \frac{\ell-1}{\delta} (-1)^{\ell-2} \log^{\ell-2} y$$

$$+ \frac{(\ell-1)(\ell-2)}{\delta^2} (-1)^{\ell-3} \log^{\ell-3} y + \ldots + \frac{(\ell-1)(\ell-2)\ldots 2}{\delta^{\ell-2}} (-1) \log y + \frac{(\ell-1)!}{\delta^{\ell-1}} \Big)$$

$$= y^{\delta} \Big((-1)^{\ell} \log^{\ell} y + (-1)^{\ell-1} \frac{\ell}{\delta} \log^{\ell-1} y + \frac{(\ell)(\ell-1)}{\delta^2} (-1)^{\ell-2} \log^{\ell-2} y$$

$$+ \frac{\ell(\ell-1)(\ell-2)}{\delta^3} (-1)^{\ell-3} \log^{\ell-3} y + \ldots + \frac{\ell(\ell-1)(\ell-2)\ldots 2}{\delta^{\ell-1}} (-1) \log y + \frac{\ell!}{\delta^{\ell}} \Big),$$

so the lemma is true for ℓ and the proof is complete.

□

Corollary 7.2.4. *For* $0 < y \le 1$ *we have*

$$0 \le J_{\ell,\delta}(y) \le y^{\delta} (\ell+1)! \max \left\{ \frac{1}{\delta}, (-\log y) \right\}^{\ell}$$

In other words,

$$\lim_{y \to 0} J_{\ell,\delta}(y) = 0 .$$

For the remainder of the chapter, we will denote by

$$I_{\ell,k}(y) = J_{\ell,\frac{1}{k-1}} = (-1)^{\ell} \int_0^y \log^{\ell} z d\left(z^{\frac{1}{k-1}} \right). \tag{2.2}$$

We finish this section by recalling the statement of the following extension of Holder inequality:

Lemma 7.2.5. *Let* S *a measurable set of* $\mathbb{R}^n$ *or* $\mathbb{C}^n$ *and* $f_1,...,f_s$ *measurable complex or real valued functions.*
Then

$$\int_S \left|f_1...f_s\right| dx \leq \left(\int_S \left|f_1\right|^s dx\right)^{\frac{1}{s}} ... \left(\int_S \left|f_s\right|^s\right)^{\frac{1}{s}}.$$

7.3. Integration over combinations of $S(P,y)$

In this section, we consider the integration over sets resulting from combining the different $S(P,y)$'s.

Lemma 7.3.1. *Let* $P(x) \in \mathbb{C}[x]$ *be a polynomial having* $k \geq 2$ *non-zero complex coefficients each having modulus* ≥ 1. *Let* $0 < y \leq 1$. *Then*

$$0 \leq (-1)^\ell \int_{S(P,y)} \log^\ell \left|P(x)\right| \frac{dx}{x} \leq C_k I_{\ell,k}(y).$$

Analogously, if $P\left(x_1,...,x_n\right) \in \mathbb{C}\left[x_1,...,x_n\right]$ *and* $0 < y$ *small enough to satisfy equation (2.1),*

$$0 \leq (-1)^\ell \int_{S_n(P,y)} \log^\ell \left|P\left(x_1,...,x_n\right)\right| \frac{dx_1}{x_1} ... \frac{dx_n}{x_n} \leq C_P J_{\ell,\delta_P}(y).$$

Proof. The case $\ell = 1$ is Lemma 4 in [La83]. The general proof starts in the same way. Define for $0 < z \leq 1$

$$h(z) := \mu_1(S(P,z)),$$

where we recall that μ_1 stands for the Lebesgue measure of the set. Let the leading coefficient of $P(x)$ be a with $|a| \geq 1$. Then $a^{-1}P$ is monic and so Theorem 7.2.1 implies that

$$h(z) \leq C_k \left(\frac{z}{|a|}\right)^{\frac{1}{k-1}} \leq C_k z^{\frac{1}{k-1}}.$$

Now we compute the desired integral.

$$(-1)^{\ell} \int_{S(P,y)} \log^{\ell}|P(x)|\frac{dx}{x}$$

$$=(-1)^{\ell}\int_{z=0}^{z=y}\int_{\substack{|x|=1\\|P(x)|=x}} \log^{\ell} z\frac{dx}{x}dz$$

$$=(-1)^{\ell}\int_{0}^{y}\log^{\ell} zg(z)dz$$

Where $g(z)=\mu(|P(x)|=z)$ so that $h(z)$ is almost everywhere differentiable and $dh(z)=g(z)$

$$=(-1)^{\ell}\log^{\ell} yh(y)-\int_{0}^{y}\frac{d}{dz}\left[(-\log z)^{\ell}\right]h(z)dz$$

$$\le(-1)^{\ell}C_k\int_{0}^{y}\log^{\ell} zd\left(z^{\frac{1}{k-1}}\right)-\int_{0}^{y}\frac{d}{dz}\left[(-\log z)^{\ell}\right]C_k z^{\frac{1}{k-1}}dz$$

where the last inequality is consequence of the fact that $(-\log z)^{\ell}$ is a positive decreasing function and its derivative is negative. By applying integration by parts again we obtain

$$\le(-1)^{\ell}C_k\int_{0}^{y}\log^{\ell} zd\left(z^{\frac{1}{k-1}}\right),$$

which finishes the proof of the first statement by Lemma7.2.3 and equation (2.2).

The proof of the second statement follows along the same lines.

□

Lemma 7.3.2. *Let* $P_1(x),...,P_s(x)\in\mathbb{C}[x]$ *be polynomials having* $k_1,...,k_s$ *nonzero complex coefficients with absolute value greater than 1 and* $0<y_1,...,y_s\le 1$. *Let* $1\le n\le s$. *Then*

$$0 \le (-1)^s \int_{\cap_{i=1}^n S(P_i, y_i) \setminus \cup_{i=n+1}^s S(P_i, y_i)} \log|P_1(x)| \dots \log|P_s(x)| \frac{dx}{x}$$

$$\le \left(C_{k_1} I_{n,k_1}(y_1) \dots C_{k_n} I_{n,k_n}(y_n) \right)^{\frac{1}{n}} (-1)^{s-n} \log y_{n+1} \dots \log y_s .$$

Proof. Notice that $0 \le -\log|P(x)| \le -\log y$ for $x \notin S(P, y)$ for $0 < y \le 1$. Therefore,

$$(-1)^s \int_{\cap_{i=1}^n S(P_i, y_i) \setminus \cup_{i=n+1}^s S(P_i, y_i)} \log|P_1(x)| \dots \log|P_s(x)| \frac{dx}{x}$$

$$\le (-1)^s \log y_{n+1} \dots \log y_s \int_{\cap_{i=1}^n S(P_i, y_i) \setminus \cup_{i=n+1}^s S(P_i, y_i)} \log|P_1(x)| \dots \log|P_n(x)| \frac{dx}{x}$$

$$\le (-1)^s \log y_{n+1} \dots \log y_s \int_{\cap_{i=1}^n S(P_i, y_i)} \log|P_1(x)| \dots \log|P_n(x)| \frac{dx}{x}$$

$$\le (-1)^{s-n} \log y_{n+1} \dots \log y_s \left(C_{k_1} I_{n,k_1}(y_1) \dots C_{k_n} I_{n,k_n}(y_n) \right)^{\frac{1}{n}}$$

By Lemma 7.2.5 and Lemma 7.3.1.

□

Lemma 7.3.3. *Let* $P_1(x), \dots, P_s(x) \in \mathbb{C}[x]$ *be polynomials having* $k_1, \dots, k_s$ *nonzero complex coefficients with absolute value greater than 1 and* $0 < y_1, \dots, y_s \le 0$. *Then*

$$0 \le (-1)^s \int_{S(P_1, y_1) \cup \dots \cup S(P_s, y_s)} \log|P_1(x)| \dots \log|P_s(x)| \frac{dx}{x}$$

$$\le \sum_{A \subset \{1, \dots, s\}} \prod_{i \in A} \left(C_{k_i} I_{|A|, k_i}(y_i) \right)^{\frac{1}{|A|}} \prod_{i \in \{1, \dots, s\} \setminus A} (-\log y_i).$$

*Proof***.** We start with the observation that

$$\bigcup_{i=1}^s S(P_i, y_i) = \bigcup_{A \subset \{1, \dots, s\}} \left(\bigcap_{i \in A} S(P_i, y_i) \setminus \bigcup_{i \in \{1, \dots, s\} \setminus A} S(P_i, y_i) \right).$$

By applying Lemma 7.3.2, we get

$$(-1)^s \int_{S(P_1,y_1)\cup\ldots\cup S(P_s,y_s)} \log|P_1(x)|\ldots\log|P_s(x)|\frac{dx}{x}$$

$$\leq \sum_{A\subset\{1,\ldots,s\}} (-1)^s \int_{\bigcap_{i\in A} S(P_i,y_i)\setminus \bigcup_{i\in\{1,\ldots,s\}\setminus A} S(P_i,y_i)} \log|P_1(x)|\ldots\log|P_s(x)|\frac{dx}{x}$$

$$\leq \sum_{A\subset\{1,\ldots,s\}} \prod_{i\in A}\left(C_{k_i} I_{|A|,k_i}(y_i)\right)^{\frac{1}{|A|}} \prod_{i\in\{1,\ldots,s\}\setminus A}(-\log y_i)$$

□

Setting $y_1 = \ldots = y_s = y$ and letting $y \to 0$, we get the following result by Corollary 7.2.4.

Corollary 7.3.4. *Let $P_1(x),\ldots,P_s(x)\in\mathbb{C}[x]$ be polynomials having $k_1,\ldots,k_s$ nonzero complex coefficients with absolute value greater than 1. Let $0<y<1$. As y approaches 0, we obtain*

$$\lim_{y\to 0}\int_{S(P_1,y)\cup\ldots\cup S(P_s,y)} \log|P_1(x)|\ldots\log|P_s(x)|\frac{dx}{x}=0,$$

where the speed of convergence is independent of the polynomials $P_1(x),\ldots,P_s(x)$.

Lemma 7.3.5. *Let $P_1(x),\ldots,P_s(x)\in\mathbb{C}[x]$ be polynomials having $k_1,\ldots,k_s$ nonzero complex coefficients with absolute value greater than 1 and $0<y_1,\ldots,y_s\leq 1$. Then*

$$0\leq(-1)^s\int_{S(P_1,y_1)\cap\ldots\cap S(P_s,y_s)} \log|P_1(x)|\ldots\log|P_s(x)|\frac{dx}{x}$$

$$\leq\left(C_{k_1} I_{s,k_1}(y_1)\ldots C_{k_s} I_{s,k_s}(y_s)\right)^{\frac{1}{s}}.$$

Proof. This is a simple sequence of lemma 7.3.2 with $n=s$.

□

Lemma 7.3.6. *Let $P_1(x),\ldots,P_s(x)\in\mathbb{C}[x]$ be polynomials having $k_1,\ldots,k_s$ nonzero polynomials coefficients with absolute value greater than 1 and $0<y_1,\ldots,y_s\leq 1$. Then*

$$0 \le \int_{S(P_1,y_1)\cap\ldots\cap S(P_s,y_s)} \max_{1\le i\le s}\left\{\log\left|P_i(x)\right|\right\}\frac{dx}{x}$$

$$\le (2\pi)^{1-\frac{1}{s}}\left(C_{k_1} I_{s,k_1}(y_1)\ldots C_{k_s} I_{s,k_s}(y_s)\right)^{\frac{1}{s}}.$$

Proof. Notice that $\max_{1\le i\le s}\left\{\log\left|P_i(x)\right|\right\} = -\min_{1\le i\le s}\left\{-\log\left|P_i(x)\right|\right\}$.

In $S(P_1,y_1)\bigcap\ldots\bigcap S(P_s,y_s)$ we have $0 \le \min_{1\le i\le s}\left\{-\log\left|P_i(x)\right|\right\} \le -\log\left|P_i(x)\right|$ for any $i = 1,\ldots,s$. Thus

$$\left(-\max_{1\le i\le s}\left\{\log\left|P_i(x)\right|\right\}\right)^s = \left(\min_{1\le i\le s}\left\{-\log\left|P_i(x)\right|\right\}\right)^s$$

$$\le (-1)^s \log\left|P_1(x)\right|\ldots\log\left|P_s(x)\right|.$$

By applying holder inequality, and taking into account that the measure of the whole space is 2π, we get

$$0 \le \int_{S(P_1,y_1)\cap\ldots\cap S(P_s,y_s)} -\max_{1\le i\le s}\left\{\log\left|P_i(x)\right|\right\}\frac{dx}{x}$$

$$\le (2\pi)^{1-\frac{1}{s}}\left(\int_{S(P_1,y_1)\cap\ldots\cap S(P_s,y_s)}\left(-\max_{1\le i\le s}\left\{\log\left|P_i(x)\right|\right\}\right)^s\frac{dx}{x}\right)^{\frac{1}{s}}$$

$$\le (2\pi)^{1-\frac{1}{s}}\left(C_{k_1} I_{s,k_1}(y_1)\ldots C_{k_s} I_{s,k_s}(y_s)\right)^{\frac{1}{s^2}}.$$

□

Again, we let $y_1 = \ldots = y_s = y$ and $y \to 0$ and we conclude the following result.

Corollary 7.3.7. *Let $P_1(x),\ldots,P_s(x) \in \mathbb{C}[x]$ be polynomials having $k_1,\ldots,k_s$ nonzero complex coefficients with absolute value greater than 1. Let $0 < y \le 1$. As y approaches 0, we obtain*

$$\lim_{y\to 0}\int_{S(P_1,y)\cap\ldots\cap S(P_s,y)} \max_{1\le i\le s}\left\{\log\left|P_i(x)\right|\right\}\frac{dx}{x} = 0,$$

where the speed of convergence is independent of the polynomials $P_1(x),\dots,P_s(x)$.

Observe that when $k_i = 1$, the previous result is trivially true since the set $S(P_i, y)$ becomes empty for y sufficiently small.

Remark 7.3.8. *Results analogous to Corollary 7.3.4 and Corollary 7.3.7 can be proved for the case where* $P_1(x_1,\dots,x_n),\dots,P_s(x_1,\dots,x_n)$ *are fixed polynomials in* $\mathbb{C}[x_1,\dots,x_n]$.

7.4. Proof of Theorem 7.1.2.

We begin by proving that the extended versions of the Mahler measures always exist (i.e. that the integrals always converge). This was used repeatedly in previous works but the details have never been written and we include them here for completeness.

Theorem 7.4.1. *Let* $P_1(x_1,\dots,x_n),\dots,P_s(x_1,\dots,x_n) \in \mathbb{C}[x_1,\dots,x_n]$ *nonzero polynomials. Then the integrals giving the generalized Mahler measure and the multiple Mahler measure converge, i.e.,*

(1) $$\left| m_{\max}(P_1,\dots,P_s) \right| < \infty,$$

(2) $$\left| m(P_1,\dots,P_s) \right| < \infty,$$

(3) *If* $P_1 = \dots = P_s = P$

$$m_s(P) < \infty.$$

Proof. (1) Let $y > 0$. We write

$$\int_{\mathbb{T}^n} \max_{1\le i\le s}\left\{\log\left|P_i(x_1,\dots,x_n)\right|\right\}\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}$$

$$= \int_{S_n(P_1,y)\cap\dots\cap S_n(P_s,y)} \max_{1\le i\le s}\left\{\log\left|P_i(x_1,\dots,x_n)\right|\right\}\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}$$

$$+ \int_{S(P_1,y)^c\cup\dots\cup S(P_n,y)^c} \max_{1\le i\le s}\left\{\log\left|P_i(x_1,\dots,x_n)\right|\right\}\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

The second integral converges because the Mahler measure of a single polynomial converges absolutely and thus is the integral of a smaller function in a smaller set,

while the first integral approaches 0 as $y \to 0$ by Corollary 7.3.7 and Remark 7.3.8. Therefore the integral on the left converges.
(2) For $y > 0$. We consider

$$\int_{T^n} \log\left|P_1\left(x_1,\dots,x_n\right)\right| \dots \log\left|P_s\left(x_1,\dots,x_n\right)\right| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n}$$

$$= \int_{S_n(P_1,y)\cup \dots S_n\left(P_s,y\right)} \log\left|P_1\left(x_1,\dots,x_n\right)\right| \dots \log\left|P_s\left(x_1,\dots,x_n\right)\right| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n}$$

$$+ \int_{S_n(P_1,y)^c \cap \dots \cap S_n\left(P_n,y\right)^c} \log\left|P_1\left(x_1,\dots,x_n\right)\right| \dots \log\left|P_s\left(x_1,\dots,x_n\right)\right| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n}.$$

As before, the second integral converges, while the first integral approaches 0 as $y \to 0$ by the Corollary 7.3.4 and Remark 7.3.8. Thus the first integral converges.
(3) This statement is a particular case of (2) .

□

Proof of Theorem 7.1.2. (1) Following [La83], we define $F : T^n \to \mathbb{R}$ by $F(w) = \max_{1 \le i \le s} \left\{ \log\left|P_i(w)\right| \right\}$ for $w \in T^n$. It suffices to prove that

$$\lim_{q(r)\to\infty} \left| \int_{T} F_r - \int_{T^n} F \right| = 0 .$$

Without loss of generality, we may assume that each coefficient of P_i has modulus ≥ 1, and therefore the same is true for $P_{i,r}$ for $q(r)$ sufficiently large. For any $1 \ge y \ge 0$ we construct a continuous function $g_y : T^n \to \mathbb{R}$ such that $0 \le g_y(w) \le 1$ for all $w \in T^n$, $g_y(w) = 1$ for $\max_{1 \le i \le s} \left\{ \left|P_i(w)\right| \right\} \ge y$, and $g_y(w) = 0$ for $\max_{1 \le i \le s} \left\{ \left|P_i(w)\right| \right\} \le \frac{1}{2} y$. Therefore, $g_y F_r$ is a continuous function on T^n for $1 \ge y \ge 0$. Since $F = g_y F + \left(1 - g_y\right) F$, the triangular inequality implies that

$$\limsup_{q(r)\to\infty}\left|\int_{\mathrm{T}} F_r - \int_{\mathrm{T}^n} F\right| \le \limsup_{q(r)\to\infty}\left|\int_{\mathrm{T}}\left[g_y F\right]_r - \int_{\mathrm{T}^n} g_y F\right|$$
$$+\limsup_{q(r)\to\infty}\left|\int_{\mathrm{T}}\left[\left(1-g_y\right)F\right]_r\right| + \limsup_{q(r)\to\infty}\left|\int_{\mathrm{T}^n}\left(1-g_y\right)F\right| \tag{4.1}$$

Now, by Weierstrass approximation theorem, the first term goes to zero since $g_y F$ is continuous on T^n. The function $\left[\left(1-g_y\right)F\right]_r = \left(1-g_{y,r}\right)F_r$ vanishes in the set $\bigcup S\left(P_{i,r},y\right)^c = \left(\bigcap S\left(P_{i,r},y\right)\right)^c$ and it is bounded below by 0 and above by F_r in $\bigcap S\left(P_{i,r},y\right)$. This implies

$$0 \le \limsup_{q(r)\to\infty}\left|\int_{\mathrm{T}}\left[\left(1-g_y\right)F\right]_r\right| \le \limsup_{q(r)\to\infty}\left|\int_{\bigcap S\left(P_{i,r},y\right)} F_r\right|,$$

which goes to zero as $y \to 0$ by Corollary 7.3.7. Finally, the third term in (4.1) tends to zero as $y \to 0$ since F is integrable over T^n by Theorem 7.4.1 (1).

Thus, $\limsup_{q(r)\to\infty}\left|\int_{\mathrm{T}} F_r - \int_{\mathrm{T}^n} F\right| = 0$ since it is independent of y and tends to zero as $y \to 0$.

(2) We proceed as before. We define $F:\mathrm{T}^n \to \mathbb{R}$ by $F(w) = \prod_{i=1}^{s}\left(-\log\left|P_i(w)\right|\right)$ for $w \in \mathrm{T}^n$.Without loss of generality, we may assume that each coefficient of P_i has modulus ≥ 1, and therefore the same is true for $P_{i,r}$ for $q(r)$ sufficiently large. For any $1 \ge y \ge 0$ we construct a continuous function $g_y:\mathrm{T}^n \to \mathbb{R}$ such that $0 \le g_y(w) \le 1$ for all $w \in \mathrm{T}^n$, $g_y(w) = 1$ if $\left|P_i(w)\right| \ge y$ for all i, and $g_y(w) = 0$ if there is an i such that

$\left|P_i(w)\right| \le \frac{1}{2}y$. Therefore, $g_y F$ is a continuous function on T^n for $1 \ge y \ge 0$. The triangle inequality implies that

$$\limsup_{q(r)\to\infty}\left|\int_{\mathrm{T}} F_r - \int_{\mathrm{T}^n} F\right| \le \limsup_{q(r)\to\infty}\left|\int_{\mathrm{T}}\left[g_y F\right]_r - \int_{\mathrm{T}^n} g_y F\right|$$

$$+ \limsup_{q(r)\to\infty}\left|\int_{\mathrm{T}}\left[\left(1-g_y\right)F\right]_r\right| + \limsup_{q(r)\to\infty}\left|\int_{\mathrm{T}^n}\left(1-g_y\right)F\right| \tag{4.2}$$

The Weierstrass approximation theorem implies that the first term goes to zero since $g_y F$ is continuous on T^n. Now the function $\left[\left(1-g_y\right)F\right]_r = \left(1-g_{y,r}\right)F_r$ vanishes on the set $\bigcap S\left(P_{i,r}, y\right)^c = \left(\bigcup S\left(P_{i,r}, y\right)\right)^c$ and it is bounded below by 0 and above by F_r in $\bigcup S\left(P_{i,r}, y\right)$. Combining all of this,

$$0 \le \limsup_{q(r)\to\infty}\left|\int_{\mathrm{T}}\left[\left(1-g_y\right)F\right]_r\right| \le \limsup_{q(r)\to\infty}\left|\int_{\bigcup S\left(P_{i,r},y\right)} F_r\right|.$$

The term in the right goes to zero as $y \to 0$ by Corollary 7.3.4. The third term in (4.2) tends to 0 as $y \to 0$ since F is integrable over T^n by Theorem 7.4.1 (2).

Finally, $\limsup_{q(r)\to\infty}\left|\int_{\mathrm{T}} F_r - \int_{\mathrm{T}^n} F\right| = 0$ since it is independent of y and tends to zero as $y \to 0$.

(3)This case follow from (2) by setting $P_1 = \ldots = P_s = P$. This concludes the proof of the theorem.

□

Conclusion

The Mahler measure has additional connections to ergodic theory [EW99], to the Weil height [Sm08], and plays a role in approximation theory and diophantine approximation [Wa00]. Mahler measure has also application to special values of zeta functions [Bo98].

It would be interesting to explore these topics as a continuation of this work.