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Essais en microéconomie appliquée

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Résumé

La thèse comporte trois essais en microéconomie appliquée. En utilisant des modèles d'*apprentissage (learning)* et d'externalité de réseau, elle étudie le comportement des agents économiques dans différentes situations. Le premier essai de la thèse se penche sur la question de l'utilisation des ressources naturelles en situation d'incertitude et d'*apprentissage (learning)*. Plusieurs auteurs ont abordé le sujet, mais ici, nous étudions un modèle d'*apprentissage* dans lequel les agents qui consomment la ressource ne formulent pas les mêmes croyances a priori. Le deuxième essai aborde le problème générique auquel fait face, par exemple, un fonds de recherche désirant choisir les meilleurs parmi plusieurs chercheurs de différentes générations et de différentes expériences. Le troisième essai étudie un modèle particulier d'organisation d'entreprise dénommé le *marketing multi-niveau* (multi-level marketing).

Le premier chapitre est intitulé *"Renewable Resource Consumption in a Learning Environment with Heterogeneous beliefs"*. Nous y avons utilisé un modèle d'*apprentissage* avec croyances hétérogènes pour étudier l'exploitation d'une ressource naturelle en situation d'incertitude. Il faut distinguer ici deux types d'*apprentissage* : le *adaptive learning* et le *learning* proprement dit. Ces deux termes ont été empruntés à Koulovatianos et al (2009). Nous avons montré que, en comparaison avec le *adaptive learning*, le *learning* a un impact négatif sur la consommation totale par tous les exploitants de la ressource. Mais individuellement certains exploitants peuvent consommer plus la ressource en *learning* qu'en *adaptive learning*. En effet, en *learning*, les consommateurs font face à deux types d'incitations à ne pas consommer la ressource (et donc à investir) : l'*incitation propre* qui a toujours un effet négatif sur la consommation de la ressource et l'*incitation hétérogène* dont l'effet peut être positif ou négatif. L'effet global du *learning* sur la consommation individuelle dépend donc du signe et de l'ampleur de l'*incitation hétérogène*. Par ailleurs, en utilisant les variations absolues et relatives de la consommation suite à un changement des croyances, il ressort que les exploitants ont tendance à converger vers une décision commune.

Le second chapitre est intitulé *"A Perpetual Search for Talent across Overlapping Generations"*. Avec un modèle dynamique à générations imbriquées, nous avons étudié

comment un Fonds de recherche devra procéder pour sélectionner les meilleurs chercheurs à financer. Les chercheurs n'ont pas la même "ancienneté" dans l'activité de recherche. Pour une décision optimale, le Fonds de recherche doit se baser à la fois sur l'ancienneté et les travaux passés des chercheurs ayant soumis une demande de subvention de recherche. Il doit être plus favorable aux jeunes chercheurs quant aux exigences à satisfaire pour être financé. Ce travail est également une contribution à l'analyse des Bandit Problems. Ici, au lieu de tenter de calculer un indice, nous proposons de classer et d'éliminer progressivement les chercheurs en les comparant deux à deux.

Le troisième chapitre est intitulé *"Paradox about the Multi-Level Marketing (MLM)"*. Depuis quelques décennies, on rencontre de plus en plus une forme particulière d'entreprises dans lesquelles le produit est commercialisé par le biais de distributeurs. Chaque distributeur peut vendre le produit et/ou recruter d'autres distributeurs pour l'entreprise. Il réalise des profits sur ses propres ventes et reçoit aussi des commissions sur la vente des distributeurs qu'il aura recrutés. Il s'agit du *marketing multi-niveau* (multi-level marketing, MLM). La structure de ces types d'entreprise est souvent qualifiée par certaines critiques de système pyramidal, d'escroquerie et donc insoutenable. Mais les promoteurs des *marketing multi-niveau* rejettent ces allégations en avançant que le but des MLMs est de vendre et non de recruter. Les gains et les règles de jeu sont tels que les distributeurs ont plus incitation à vendre le produit qu'à recruter. Toutefois, si cette argumentation des promoteurs de MLMs est valide, un paradoxe apparaît. Pourquoi un distributeur qui désire vraiment vendre le produit et réaliser un gain recruterait-il d'autres individus qui viendront opérer sur le même marché que lui? Comment comprendre le fait qu'un agent puisse recruter des personnes qui pourraient devenir ses concurrents, alors qu'il est déjà établi que tout entrepreneur évite et même combat la concurrence. C'est à ce type de question que s'intéresse ce chapitre. Pour expliquer ce paradoxe, nous avons utilisé la structure intrinsèque des organisations MLM. En réalité, pour être capable de bien vendre, le distributeur devra recruter. Les commissions perçues avec le recrutement donnent un pouvoir de vente en ce sens qu'elles permettent au recruteur d'être capable de proposer un prix compétitif pour le produit qu'il désire vendre. Par ailleurs, les MLMs ont une structure semblable à celle des multi-sided markets au sens de Rochet et Tirole (2003, 2006) et Weyl (2010). Le recrutement a un effet externe sur la vente et la vente a un effet externe sur le recrutement, et tout cela est géré par le promoteur de l'organisation. Ainsi, si le promoteur ne tient pas compte de ces externalités dans la fixation des différentes commissions, les agents peuvent se tourner plus ou moins vers le recrutement.

Mots clés Incertitude, learning, ressource naturelle, croyances, bandit problems, chercheur, multi-level marketing, distributeur, two-sided market, effet externe.

Abstract

This thesis includes three essays in applied microeconomics. Using learning and network effects models, we study agents' behavior in various environments through three chapters. The first chapter examines natural resource exploitation under uncertainty with learning and heterogeneous priors. In the second chapter, we examine the problem of research Foundation concerned with finding good quality researchers for today and the future. The third chapter studies the Multi-level marketing organizations.

The first chapter is entitled *"Renewable Resource Consumption in a Learning Environment with Heterogeneous beliefs"*. This work uses a learning model with heterogeneity of beliefs to study natural resources consumption under uncertainty. Following Koulovatianos et al (2009), we distinguish two types of learning process: adaptive learning and learning. I find that learning decreases the total consumption of the resource in comparison with adaptive learning. However, individually and under some conditions, some exploiters could consume more in learning than in adaptive learning. In learning, the exploiter faces two kinds of incentive to invest in the resource: self-incentive which is always positive and heterogeneity incentive which may be negative. The effect of learning on individual consumption depends on the sign and the extent of the heterogeneity incentive. Using absolute change and relative change of consumption due to a change in beliefs, we find that the exploiters tend to converge to a common behaviour.

The second chapter is entitled *"A perpetual Search for Talent across Overlapping Generations"*. We use a dynamic discrete time model with overlapping generations to study how a research Fund should optimally rank and select which researchers to give a grant. The optimal decision rule depends on both the perceived quality of researchers based on past success histories and on age. Between two researchers of equal perceived qualities the Fund should select the youngest. This work contributes to the understanding of the bandit problems sometime known to be untractable and pspace-hard. Here, instead of looking for an index characterization, we propose to rank or eliminate progressively the researchers by comparing them two by two.

The third chapter is entitled *"Paradox about the Multi-Level Marketing"*. Over the past 50 years, the Multi-level Marketing (MLM) has become an important business strategy.

MLM is a marketing method in which independent distributors of a product get profits not only from their own sales, but also from recruiting other distributors. According to the promoters of this organization, the purpose of the business and the will of the distributors are to sell the product and not to set up a scam. The question is the following: if the real intention is to sell, why does a distributor recruit other distributors who will become his competitors? It is well known in classical industrial organization that people avoid competition as well as possible. Why is it not the case in the MLM organization? We explain this paradox by the particular economic structure of the MLM. Distributor could not sell enough if he did not recruit. Recruiting provides a price leading power and allows to sell more and make more profit (in some respect). Moreover, the MLM is similar to the multi-sided market structure a la Rochet and Tirole (2003, 2006) and Weyl (2010). There is network effect between recruitment and selling activity. Thus, the extent of the paradox can also be explained by the different commissions set by the MLM promoters.

Keywords: Learning, beliefs, commons problem, sense of risk, young researcher, senior researcher, equivalence of histories, bandit problem, Gittins index, multi-level marketing, pyramid scheme, direct selling, distributor, upline, downline, salesperson, recruitment, price competition, two-sided market.

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A ma petite soeur Reine Agbo. Peut-être que mon absence a permis les douloureux événements que tu as vécus.

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Introduction Générale

Différents sujets en microéconomie appliquée ont été abordés dans cette thèse à travers trois essais. Le premier essai analyse une situation de prise de décision dans un contexte d'incertitude et d'*apprentissage (learning).* Le deuxième essai traite du choix d'un agent économique qui expérimente. Dans ce modèle d'analyse, le décideur fait son choix de façon à maximiser ses gains, et aussi de façon à disposer d'un stock optimal d'information qui lui permettra de prendre des décisions futures. Quant au troisième et dernier essai, il aborde le *marketing multi-niveau* (multi-level marketing).

Considérons un ensemble de pays qui exploitent une ressource naturelle renouvelable commune. Ces pays ne connaissent pas avec certitude le processus par lequel la ressource se renouvelle. En d'autres termes, si à une période donnée, les pays ont une information parfaite sur le stock disponible de la ressource, ils ne peuvent pas prévoir exactement la quantité qui sera disponible la période suivante. Toutefois, à une période donnée, les pays peuvent observer l'évolution du stock de la ressource au cours des périodes passées. Ils peuvent se baser sur cette information du passé pour améliorer leur niveau d'information sur le stock futur de la ressource. Lorsque les pays agissent ainsi, on dit qu'ils sont dans un processus d'*apprentissage.* L'*apprentissage* décrit donc une situation d'incertitude dans laquelle les agents économiques utilisent les informations qu'ils ont à leur disposition pour réduire l'incertitude. C'est dans ce contexte que se situe le premier chapitre qui étudie l'*apprentissage* dans le cas spécifique de l'exploitation d'une ressource naturelle renouvelable.

Même si certains auteurs ont travaillé dans un contexte déterministe (Dasgupta and Heal (1974), Stiglitz (1974), Levhari and Mirman (1980), Weinstein and Zeckhauser (1975)), l'incertitude a été présente dans de nombreux modèles étudiant l'utilisation des ressources naturelles (voir par exemple Mirman (1971), Brock and Mirman (1972), Mirman and Zilcha (1975), Mendelson and Amihud (1982), Bramoulle and Treich (2009), Mirman (1971) and Brock and Mirman (1972)). Généralement, dans les modèles avec incertitude sans *apprentissage*, les agents connaissent la distribution de probabilité de la variable qui est imparfaitement connue. Avec la distribution de probabilité, la valeur espérée de la variable peut être déterminée afin de procéder à la prise de décision. Par contre, en situation d'*apprentissage*, cette distribution de probabilité n'est pas souvent connue. Les décideurs forment alors des croyances a priori qui seront améliorées avec les informations disponibles (Huffman et al. (2007), Baker (2006), Baker (2009), Grossman et al. (1977), El-Gamal and K. (1993)).

Koulovatianos, Mirman et Santugini (KMS) (2009) ont reconsidéré le modèle de Brock-Mirman en y apportant deux modifications.[1] Dans un premier temps, la fonction décrivant le renouvellement de la ressource dépend d'un paramètre aléatoire η dont la distribution de probabilité n'est pas connue. Cette distribution dépend d'un autre paramètre inconnu θ. Il s'agit d'une incertitude structurelle parce que portant sur la structure même du système de renouvellement de la ressource. Le décideur (un planificateur social qui a une durée de vie infinie) forme des croyances a priori sur la distribution de θ. Dans un second temps, le décideur utilise les observations passées de η pour améliorer les croyances. L'agent décideur décide de la quantité de la ressource qu'il consomme chaque période. KMS introduit ainsi l'*apprentissage* dans le modèle de Brock-Mirman. L'une des limites de KMS est qu'il suppose que la ressource est exploitée par un seul agent. Même si cette hypothèse peut être vérifiée dans certaines situations dans la réalité, il faut reconnaître que dans la plupart des cas, la même ressource est consommée par plusieurs individus.[2] Dans ce chapitre, nous allons donc considérer plusieurs agents qui utilisent la ressource et qui forment différentes croyances. La prise en compte de plusieurs agents présente des intérêts. Cela donne au problème un aspect de jeu dynamique, au lieu d'un simple problème d'optimisation. L'on pourra étudier les interactions gouvernées par l'hétérogénéité des croyances. De même, la présence de plus d'un agent implique la possibilité de l'émergence de la tragédie des biens communs, un problème qui offre un important cadre d'analyse.

Nous avons déterminé l'équilibre de Nash dans chacun des contextes suivants: incertitude sans *apprentissage* et incertitude avec *apprentissage*. Nous avons comparé les niveaux de consommations individuelles en incertitude sans *apprentissage* à ceux de l'*apprentissage*. Nous obtenons que des agents consomment plus en *apprentissage* qu'en non *apprentissage*, et que d'autres consomment moins en *apprentissage*, selon leurs croyances a priori. En effet, en situation d'*apprentissage*, un agent donné fait face à deux types d'incitation: l'*incitation propre* et l'*incitation hétérogène*. L'*incitation propre* a toujours un impact négatif sur le niveau de consommation de l'agent, alors que l'*incitation hétérogène* peut avoir un effet positif ou négatif sur la consommation. Si l'*incitation hétérogène* est positive et domine l'*incitation propre*, alors l'exploitant va consommer plus

[1] Tout au long de ce travail nous utilisons l'acronyme KMS pour référer à Koulovatianos et al. (2009).

[2] Plusieurs pays peuvent exploiter les ressources halieutiques d'un fleuve qui les traverse. Plusieurs multinationales peuvent exploiter un même gisement d'uranium.

en *apprentissage* qu'en non *apprentissage*. Toutefois, globalement les agents consomment moins en *apprentissage* qu'en non *apprentissage*. Par ailleurs, si tous les exploitants de la ressource ont la même croyance, le problème des biens communs est relativement plus prononcé en *apprentissage*. Ceci est dû au fait que le processus d'*apprentissage* est source d'un risque additionnel par rapport à une situation d'incertitude sans *apprentissage*.

Comme le premier chapitre de la thèse, le second chapitre utilise un modèle dynamique en temps discret. En effet, dans ce chapitre nous étudions une classe de problèmes comme celui auquel fait face un Fonds de recherche qui désire sélectionner les meilleurs chercheurs à financer. Les chercheurs n'ont pas la même expérience en terme de nombre d'années dans la recherche, et de succès (publications, découvertes, inventions) dans leur carrière de chercheurs. Considérons un Fonds de recherche qui doit décider de qui financer entre deux chercheurs dont l'un est jeune et l'autre est moins jeune. Chaque chercheur a un projet de recherche qu'il désire exécuter, une fois financé. A première vue, prendre une décision peut paraître simple. En effet, les chercheurs moins jeunes sont souvent plus expérimentés, plus connus et donc sont capables de présenter des projets moins risqués. Leur probabilité de réussir leurs projets pourrait être plus élevée. Par contre, généralement, les jeunes chercheurs manquent d'expérience, font l'objet d'incertitude, et leurs projets peuvent être considérés comme étant à risque. L'on pourrait donc penser que la décision optimale serait d'allouer les fonds de recherche au chercheur moins jeune. Mais en réalité, la question est beaucoup plus complexe. En effet, s'il est d'une assertion évidente que le Fonds de recherche veut financer un chercheur capable de conduire et réaliser avec succès un projet de recherche, il n'en est pas moins que le Fonds se soucie de la perennité de l'activité de recherche, et donc voudrait assurer la relève dans ce domaine. Même si les jeunes chercheurs sont moins expérimentés, ils présentent néanmoins l'avantage d'avoir à passer dans l'activité de recherche plus de temps que les moins jeunes. Par ailleurs, un Fonds de recherche peut aussi être à la quête d'information sur la qualité des chercheurs qui soumettent des demandes de financement. Il pourrait donc décider de financer un chercheur en vue de l'expérimenter et d'avoir des informations sur sa performance. En d'autres termes, le problème d'un Fonds de recherche n'est pas juste une simple sélection de meilleurs projets, mais un problème d'exploitation et d'exploration.

Ce deuxième chapitre, en plus de sa portée théorique, s'inscrit dans un cadre très pratique au regard des défis auxquels fait face la recherche scientifique de nos jours. En effet, comme pour beaucoup d'autres domaines, les ressources disponibles pour la recherche scientifique s'amenuisent alors que la science fait partie des priorités pour beaucoup de gouvernements à travers le monde. Il faudra donc trouver les voies et moyens pour financer de façon efficace cette science. Par ailleurs, il est connu que les jeunes chercheurs éprouvent des difficultés à être financés. Selon un rapport du parlement européen publié en 2009, la valorisation des carrières dans beaucoup de pays est basée plus sur l'ancienneté que sur la compétence. Selon le rapport, cette situation défavorise les jeunes chercheurs. Cependant, les acteurs du monde de la recherche se mettent de plus en plus d'accord sur le fait qu'il faut rendre les conditions de travail plus attrayantes aux jeunes. Dans un rapport du "American Academy of Arts and Sciences" publié en 2008, il est fortement recommandé qu'une attention particulière soit accordée aux chercheurs en début de carrière. Et même la volonté politique semble s'accorder à une telle ligne de conduite dans de nombreux pays. Toutefois, l'on constate un écart important entre cette volonté de financer les jeunes cherheurs et la réalité des conditions de travail de ces derniers. Ceci est dû à l'absence de méthode objective pouvant permettre de faire un arbitrage efficace entre financer un jeune afin d'assurer la relève, et financer un chercheur expérimenté qui présente moins d'incertitude. En effet, le désir d'accorder des fonds de recherche à un jeune ne signifie pas qu'il faut financer n'importe quel jeune chercheur. Cela ne signifie pas aussi qu'il faut croire que les fonds alloués aux chercheurs séniors sont déjà suffisants. La nécessité donc de trouver un bon arbitrage s'avère évidente.

Pour analyser le problème, nous avons utilisé un modèle dynamique à générations imbriquées. Les chercheurs vivent dans l'activité de recherche pendant un nombre fixe et fini de périodes au cours desquelles ils sont actifs. Après ce nombre de périodes, le chercheur prend sa retraite et ne demande plus de fonds de recherche. A chaque période, pendant que certains prennent leur retraite, de nouveaux chercheurs s'engagent dans la recherche et soummettent des demandes de financement. Finalement, à chaque période, le Fonds de recherche doit sélectionner les meilleurs dans un groupe de chercheurs de différents âges. En plus des projets soumis, le Fonds peut observer les performances passées des chercheurs, c'est-à-dire la qualité des résultats qu'ils ont obtenus au cours de leur carrière. Ces performances passées seront utilisées par le Fonds de recherche pour inférer les probabilités de succès futures de chaque chercheur. Les résultats obtenus sont simples et intuitifs. Une décision optimale devra tenir compte à la fois de l'âge du chercheur (nombre d'années dans la recherche) et de ses performances passées. L'âge influence négativement les chances du chercheur à être sélectionné, tandis que les réalisations passées ont une influence positive. En d'autres termes, un jeune chercheur sera préféré à son collègue moins jeune qui a la même performance passée que lui.

Au-delà de l'aspect mis en exergue ci-dessus, ce deuxième chapitre de la thèse constitue également une contribution à l'analyse des *bandit problems*. En effet, comme nous l'avons dit plus haut, le Fonds fait face à un problème d'exploitation et d'exploration. A chaque période, la fondation de recherche fait un choix qui lui permet d'avoir des informations sur le chercheur choisi et aussi d'avoir des gains expérés en terme de qualité de projet réalisé par ce chercheur. A la période suivante, le fonds reprend la même expérience, et ainsi de suite. Nous sommes donc en présence d'un *multi-armed bandit problems.* Dans notre cas ici, le modèle présente une certaine complexité. D'abord, la structure du groupe de chercheur change d'une période à l'autre car de nouveaux jeunes entrent et d'autres prennent leur retraite. Il s'agit d'un *arm acquiring bandit problem.* Ensuite, contrairement au *bandit problems* classique, l'état d'un chercheur change d'une période à l'autre, même si ce dernier n'est pas sélectionné. En effet, l'âge change même si on n'est pas sélectionné. Notre modèle est donc aussi un cas de *restless bandit problems.* Enfin, les chercheurs n'ont pas une durée de vie infinie puisqu'après une période d'activité, ils deviennent inactifs. Notre problème est donc en partie ce que certains auteurs désignent par *mortal multi-armed bandits.* Face à cette complexité, nous avons adopté une démarche assez simple. Au lieu de chercher à définir un indice de prise de décision ou à construire des règles d'arrêt, nous avons proposé de comparer les chercheurs deux à deux afin de les classer.

Contrairement aux deux premiers chapitres, le troisième n'aborde pas un problème dynamique de prise de décision, ou du moins n'analyse pas le sujet en question sous un angle dynamique. Il s'interesse à un type d'organisation d'entreprise connu sous le nom de *marketing multi-niveau (multi-level marketing, MLM).* Le MLM est une stratégie de marketing qui consiste à distribuer le produit par l'intermédiaire de distributeurs qui se font de l'argent non seulement en vendant le produit, mais aussi en recrutant d'autres distributeurs. Chaque distributeur gagne sur ses propres ventes, et sur les ventes ou les achats des individus qu'il auraient recrutés dans le système. Ce type d'organisation est souvent critiqué et qualifié de système pyramidal et d'escroquerie. Selon les détracteurs, avec le MLM, certains agents se font de profits à partir des efforts faits par ceux qui ont été recrutés. Ce système encouragerait un recrutement effréné et insoutenable; et finalement ceux qui sont au bas de la chaîne vont subir de lourdes pertes puisqu'ils n'auront personnes à recruter ou à qui vendre le produit. Cependant, les promoteurs de MLM rejettent ces accusations et argumentent que cette stratégie de marketing encourage plus les distributeurs à vendre qu'à recruter. A en croire les promoteurs, les différents gains sont fixés de sorte que le distributeur gagne plus avec les ventes qu'avec le recrutement. L'ojectif principal est donc de vendre et non de recruter.

Toutefois, une interrogation survient: qu'est-ce qui peut amener un distributeur, qui veut réellement vendre, à recruter d'autres individus qui, en fait, pourraient devenir ses concurrents? Il est connu en économie que tout individu rationnel évite que d'autres s'engagent dans une même opportunité d'affaire que lui, s'il pense réellement que cette opportunité est rentable. Alors nous analysons la question suivante: dans quelle mesure un distributeur va-t-il décider de recruter, quand bien même il a l'intention de vendre? On peut envisager plusieurs réponses à cette question. Certaines analyses estiment que les distributeurs recrutent en vue d'atteindre d'autres marchés. En effet, si un distributeur convoite un marché qui lui est difficile d'accès, il pourra recruter un autre individu pour qui ce marché est d'accès facile. Ainsi, cet individu recruté n'est pas en fait un concurrent mais quelqu'un dont les ventes lui procurent un revenu. C'est l'exemple des distributeurs qui recrutent dans d'autres pays. Cependant, cette justification ne serait tenir pour ce que nous observons habituellement avec les MLMs. L'on rencontre souvent des distributeurs qui recrutent dans leur voisinage immédiat (amis, frères, belle famille, collègues de bureau, etc). Ils recrutent donc des individus qui auront accès au même marché qu'eux. En réalité, ils les recrutent sans se préoccuper du marché sur lequel ces derniers vont vendre. La question demeure donc posée. D'autres analyses pensent les entreprises MLM diversifient suffisamment leurs produits. Ceci permettrait à un distributeur et ceux qu'il a recrutés de ne pas avoir à vendre le même produit que lui. Mais, si on observe le nombre (relativement élevé) de recrutés que certains distributeurs ont dans leur chaîne, il n'y aura pas assez de produits suffisamment différenciés pour limiter la concurrence.

Rappelons qu'un individu qui décide de s'engager dans des activités MLM a deux types possibles d'activité à mener: vendre les produits de la compagnie MLM et recruter d'autres vendeurs pour la compagnie. En réalité, et à examiner de près, les deux types d'activité ne sont pas indépendants, mais s'influencent. Un distributeur qui veut suffisamment vendre, ne doit pas complètement manquer de recruter. Nous avons montré par un modèle simple que l'activité de recrutement permet au distributeur d'augmenter ses opportunités de vente. En effet, si l'objectif des distributeurs est réellement de vendre, alors ces derniers devront faire face à une concurrence de la part des autres distributeurs qui vendent le même produit et de la part des autres entreprises non MLM qui commercialisent le même produit. Etant donné que les distributeurs ne produisent pas eux-mêmes le bien qu'ils vendent, la concurrence sera essentiellement par les prix. Or, pouvoir proposer des prix faibles signifie qu'il faut en avoir les capacités financières. Ainsi, comme le distributeur réalise des gains additionnels (compensation) sur celui qu'il a recruté, cela lui donne ce pouvoir de baisser le prix. En d'autres termes, si un distributeur décide de ne pas recruter alors que les autres le font, il n'aura pas ce pouvoir et donc ne pourra pas vendre puisqu'il ne pourra pas tenir face à la concurrence. Toutefois, pour qu'un MLM puisse réellement vendre, cela dépend de la façon dont la compagnie fixe les différentes compensations pour le recrutement et la vente. En effet, lorsqu'un distributeur recrute, il se confronte à deux effets: un *effet concurrence* et un *effet pécuniaire*. L'effet concurrence est dû au fait que celui qui est recruté peut devenir un concurrent, et l'effet pécuniaire traduit le fait que celui qui a recruté gagne sur celui qui est recruté. Si l'effet pécuniaire est trop forte, les agents auront tendance à seulement recruter et le MLM ne sera qu'un système pyramidal. Par contre si l'effet pécuniaire est relativement faible, les agents vont toujours recruter, mais juste pour avoir un pouvoir de vente.

Par ailleurs, même si les distributeurs vendent plus en recrutant, ils gagnent moins en terme de profit comparativement à ce qu'ils auraient gagné s'ils opéraient dans les mêmes conditions dans une entreprise non MLM. Par contre les promoteurs de MLM vendent plus et réalisent plus de profits.

Chapter 1

Renewable Resource Consumption in a Learning Environment with Heterogeneous Beliefs

1.1 Introduction

Uncertainty plays an important role in the natural resource exploitation literature. Even though a deterministic framework is embedded in many studies (Dasgupta and Heal (1974), Stiglitz (1974), Levhari and Mirman (1980), Weinstein and Zeckhauser (1975)), many papers have introduced uncertainty in their resources consumption models (Mirman (1971), Brock and Mirman (1972), Mirman and Zilcha (1975), Mendelson and Amihud (1982), Bramoulle and Treich (2009)). Mirman (1971) and Brock and Mirman (1972) have modeled uncertainty in outcomes by introducing a random variable in the production function. They assume that agents know the distribution of the random variable and choose the optimal consumption by maximizing the expected sum of the discounted utilities.

Koulovatianos, Mirman and Santugini (KMS) (2009) built a framework within the Brock-Mirman environment by introducing two modifications.[1] First, there is a random variable η in the production function, as in Brock and Mirman, and the distribution of η is not well-known. It depends on the random parameter θ, which constitutes structural uncertainty. The decision-maker (a social planner) formulates prior beliefs about θ. Second, after observing the past realizations of η and using the Bayesian rule, the social planner updates his prior beliefs. In other words, KMS introduced learning with structural uncertainty in the Brock-Mirman environment.

[1] Throughout this paper we use the acronym KMS to refer to Koulovatianos et al. (2009).

One of the main characteristics of KMS is that the decision-maker is a single-agent exploiting the resource. However, in some real-world cases, many exploiters may use a stock of resources (e.g. a river on the border of two countries, a gold field being exploited by many goldsmiths). This paper therefore aims to complement KMS by introducing a multiple-agent setting where the exploiters have different beliefs. In other words, we extend KMS to dynamic game. We chose a multiple-agent framework for several reasons. First, some findings in the single-agent model may not hold when the model embeds more than one agent. Second, studying multiple-agent model with heterogeneity of beliefs allows us to observe mechanisms that are absent in the single-agent problem. Specifically, it allows us to investigate how heterogeneity of beliefs affects the interaction between the exploiters, and how common property of a resource induces commons problem in a learning environment.

We consider three information structures: *full information* as a benchmark, *learning*, and *adaptive learning*.[2] In *full information*, the parameter θ is well-known. In the *learning* structure, θ is not known; the exploiters formulate prior beliefs and anticipate learning. *Adaptive learning* is *learning* without anticipation.

After characterizing the unique Nash-equilibrium in each information structure, we find that anticipation has a negative effect on total consumption. However, its effect on individual consumption may be ambiguous. Indeed, when anticipating learning, each exploiter faces two kinds of incentives to invest in the resource: the self-incentive and the heterogeneity incentive. The effect of the self-incentive on the investment is always positive. However, the effect of the heterogeneity incentive may be negative. If a negative effect exits and overcomes the positive effect, the exploiter will increase its consumption when anticipating. However, in aggregate, the positive effect always overcompensates for the negative effect.

Second, we study the consumption sensitivity to beliefs change. Analyzing the *absolute change* and the *relative change* in consumption, we observe that the exploiters seem to favor mitigated coordination of their decisions. Because each consumer trusts in his personal beliefs while fearing to make irrational decision because he is too optimistic, the players seem to converge to a common consumption decision.

Most of the time, common property of a natural resource leads to the commons problem. We also analyze this aspect of the game. We find that when the exploiters have the same belief, the commons problem is more severe in *learning* than in *adaptive learning* because common property is more harmful for the anticipation effect. In the case of heterogeneity of beliefs, the result depends on the incumbent exploiter's heterogeneity incentive to invest. If this heterogeneity incentive is high then allowing common property

[2]See details in section 1.2.2.

will be more harmful for the anticipation effect on investment.

The rest of the paper is organized as follows. In section 2, we present the model and characterize the Nash-equilibrium of the game under each information structure. In section 3, we study the role played by the heterogeneity of beliefs in the effect of anticipation on consumption decision. In section 4, 5, and 6, we investigate the impact of a change in beliefs on resource exploitation. Section 7 addresses the commons problem. Concluding remarks are presented in section 8.

1.2 Model

In this section, we present the dynamic game under different information structures. The term *information structure* refers to how players are informed or obtain information in the game, and how they use this information. We also characterize the Nash equilibrium of the game.

1.2.1 Set up

There are J countries indexed by $j = 1, \cdots, J$. Each country derives utility from the consumption of a renewable resource, like fish in the ocean. Let y be the stock of fish available in the current period. If the resource is not exploited, then the quantity of fish in the subsequent period is

$$\hat{y} = y^{\eta}, \tag{1.1}$$

where η is a realization of the random variable $\tilde{\eta}$. As in KMS, the p.d.f. of $\tilde{\eta}$ is $\phi(\eta|\theta^*)$ for $\eta \in [0,1]$, which depends on a parameter $\theta^* \in \Theta \subset \mathbb{R}^N$ for $N \in \mathbb{N}$. The relationship between the distribution of $\tilde{\eta}$ and the parameter θ^* is assumed to be strictly monotonic. Expression (1.1) refers to the *production function*, which describes how the resource is renewed. The parameter η is the *production shock*.

Each period, country j extracts c_j, yielding utility $u(c_j) = \ln c_j$. Countries' exploitation of the resource affects the evolution of the quantity of fish, i.e.:

$$\hat{y} = \left(y - \sum_{j=1}^{J} c_j \right)^{\eta}, \tag{1.2}$$

where $y - \sum_{j=1}^{J} c_j$ is an investment or savings for the renewal of the resource. If a country decreases its consumption, then it increases its contribution to the investment. In such cases, we say that the country *invests more in the renewal of the resource*.

We adopt the specific utility and production functions used in KMS because this specification provides tractable characterization of the optimal policies in a learning environment.[3] This class of functions also leads to an equilibrium outcome that is linear in the stock.[4] This linearity will be greatly helpful for further analysis in the paper.

When deciding on the consumption level, each country aims to maximize the expected sum of discounted utilities, where the discount factor is $\delta \in (0,1)$. Expectations are formed regarding the sequence of future biological shocks. The maximization problem depends on the information structure in which the countries are playing. However, before presenting the different information structures, we present two remarks about the production function, which will be helpful for subsequent analysis. In particular, they will help me define the concept of optimism/pessimism in section 1.4.

Remark 1.2.1. *The renewal of the resource depends on stock the y and the shock η. Because the support of η is $[0,1]$, if the stock y is lower than one ($0 < y < 1$) then $y^\eta > y$ for any $\eta \in [0,1]$. The stock of the resource therefore increases from the current period to the subsequent period. As a consequence, if $0 < y < 1$ and the countries invest in the resource, they benefit from this investment. In contrast, if y is greater than one, the stock decreases, i.e. over time, the quantity of the resource goes down. In this case, countries incur a loss when they invest in the resource.*

Remark 1.2.2. *When stock y is higher than one, the production function increases in η. In such a case, an increase of η would be beneficial. In contrast, if y is lower than one then an increase in η will not be beneficial.*

If at a given period the stock of resource y is equal to $y^* = 1$ and the resource is not exploited, then the quantity of resource in the subsequent period is $y^* = 1$, i.e. the same as in the current period. Therefore, the value $y^* = 1$ is the steady state of an exploited resource.

1.2.2 Information structure

In this paper and following KMS, we distinguish three different information structures: *full information*, *learning*, and *adaptive learning*. Both *Learning* and *adaptive learning* refer to the *learning environment.*

Full-Information. First, we revisit the full-information dynamic game of Levhari and Mirman (1980), in which there is no structural uncertainty, i.e., the distribution for the

[3] See KMS or Levhari-Mirman (1980) for more details.

[4] See Gaudet and Lohoues (2008) for some class of functions leading to linear equilibrium decision rules.

dynamics of the resource is known. Specifically, θ^* is known. Given θ^*, the strategies of the other countries and the law of motion defined by (1.2), country j's value function is

$$V_j^{FI}(y,\theta^*) = \max_{c_j \in (0, y - c_{-j})} \ln c_j + \delta \int_0^1 V_j^{FI}\left((y - c_j - c_{-j})^{\eta}, \theta^*\right) \phi(\eta|\theta^*) \mathrm{d}\eta, \tag{1.3}$$

where FI stands for *Full Information* and $c_{-j} \equiv \sum_{k=1,k\neq j}^{J} c_k$. Each country anticipates the effect of its present consumption decision on the future stock of fish and takes into account the consumption decision of the other countries. Even in full information, the shock on production η is not perfectly known. However, the distribution of $\tilde{\eta}$ is perfectly known.

Learning. We now consider the case of learning, as described in KMS for the single-agent model. Here, the parameter θ^* is unknown to all countries, i.e., there is structural uncertainty. Structural uncertainty is characterized by a priori heterogeneous beliefs about θ^*, expressed as a prior p.d.f. ξ_j on Θ for any country $j = 1, ..., J$. That is, country j's probability that $\theta^* \in S$ is $\int_S \xi_j(\theta)\mathrm{d}\theta$ for any $S \subset \Theta$ and $j \in \{1, ..., J\}$. Because the countries do not know θ^*, they form beliefs about it. Structural uncertainty leads to learning, and thus evolves over time.[5] Indeed, each country observes η, which yields information and uses Bayesian methods to learn about θ^*. Formally, given a prior ξ_j, the posterior $\hat{\xi}_j(\cdot|\eta)$ is

$$\hat{\xi}_j(\theta|\eta) = \frac{\phi(\eta|\theta)\xi_j(\theta)}{\int_\Theta \phi(\eta|x)\xi_j(x)\mathrm{d}x} \tag{1.4}$$

for $\theta \in \Theta$ and $j \in \{1, ..., J\}$ by Bayes' Theorem. While all countries observe the same data, their posterior beliefs are different due to heterogeneous prior beliefs. There is a reason to consider that the prior beliefs may be different from one country to another, even if they observe the same data and exploit the same resource. The beliefs may reflect the country's optimism or pessimism about the renewal of the resource being exploited. The factors likely to affect this optimism could be sociological or cultural. Cultural differences between countries could engender a difference in the prior beliefs. Many works have already used models with agents of different beliefs (see Buraschi and Jiltsov (2006), Verardo (2009), Giat et al. (2008)). Buraschi and Jiltsov (2006) argued that heterogeneity of beliefs may be due to private information that each agent has at his disposal. However, this paper does not take this private information into account.

Given the present quantity of fish, country j maximizes the expected sum of discounted utilities. As in the full information case, each country anticipates the effect of its present consumption decision on the future stock of fish, and takes into account the consumption

[5] Structural uncertainty leads to learning because, rationally speaking, countries would like to know more or to make prediction about the parameter.

decision of other countries. Unlike the full information environment, each country anticipates learning. The anticipation of acquiring and using data must be embedded directly in the value function, as shown in KMS for the single-agent growth model.

Formally, using (1.2) and (1.4), the value function of the learning country j is

$$V_j^L(y, \quad \xi_1, ..., \xi_J) \tag{1.5}$$
$$= \max_{c_j \in (0, y - c_{-j})} \ln c_j + \delta \int_0^1 V_j^L \left((y - c_j - c_{-j})^\eta, \hat{\xi}_1(\cdot|\eta), \cdots, \hat{\xi}_J(\cdot|\eta) \right) \left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)\mathrm{d}\theta \right] \mathrm{d}\eta$$

where $c_{-j} \equiv \sum_{k=1, k\neq j}^J c_k$ and L stands for *Learning.* From (1.5) and following KMS, learning increases the uncertainty of future payoffs by introducing two sources of risk: structural uncertainty and uncertainty due to the anticipation of learning. Structural uncertainty comes from the beliefs and from the fact that the true distribution $\phi(\eta|\theta^*)$ of the shock is replaced by the expected p.d.f. of $\tilde{\eta}$, $\int_\Theta \phi(\eta|\theta)\xi_j(\theta)\mathrm{d}\theta$. The second source of risk comes from the updating process. Even though it is based on some known realized shock, future anticipation is a source of risk.

Adaptive Learning. As in KMS, we consider the intermediate case of adaptive learning in which the parameter θ is not known. However, unlike the learning agents, the adaptive learning agents do not anticipate learning, as it is shown in the value function in (1.6).

$$V_j^{AL}(y, \xi_1, ..., \xi_J) = \max_{c_j \in (0, y - c_{-j})} \ln c_j$$
$$+\delta \int_0^1 V_j^{AL} \left(\left(y - \sum_{k=1}^J c_k \right)^\eta, \xi_1, \cdots, \xi_J \right) \left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)\mathrm{d}\theta \right] \mathrm{d}\eta. \tag{1.6}$$

In the continuation utility, the beliefs are not updated. The adaptive learner updates his beliefs, but does so only after observing the realization η. In other words, the new information we get today is used only to update the beliefs of today and not to anticipate the beliefs of tomorrow. Because the adaptive learner has not yet observed the next realization, he considers his future beliefs the same as his current ones.

We therefore restrict attention to stationary Markov decisions.

Observing the different value functions, one can see that each country needs to know the beliefs of other countries to make its decision. We therefore make the following assumption.

Assumption 1. *Beliefs are common knowledge.*

This assumption contradicts that of Aumann (1976).[6] Even though private informa-

[6]According to Aumann's agreement theorem, two people acting rationally (in a certain precise sense)

tion might affect beliefs, these beliefs are mostly exploiters' opinions. Further, we are not modeling the process ruling prior beliefs.

This adapted version of the model in KMS is also used in a working paper by Koulovatianos (2010) to study the effect of environmental awareness campaigns. Koulovatianos (2010) investigates how campaigns for the conservation a resource might fail to reach their intended goal. This work is clearly different from ours; our paper complements KMS by introducing multiple-agent and heterogeneous beliefs to observe mechanisms that are absent in KMS. Even though the two papers use the same model, the motivation, approach and findings are not the same.

1.2.3 Nash equilibrium

The equilibrium concept we consider here is the dynamic Nash equilibrium. In the dynamic games defined by (1.3), (1.5), and (1.6), any country j maximizes its value function given the consumption level of the other countries. At equilibrium, each country should maximize its value function. We therefore give the following definition.

Definition 1.2.1. *In a learning structure, the tuple* $\left\{g_j^L(y,\xi_1,\cdots,\xi_J)\right\}_{j=1,\dots,J}$ *is a Nash equilibrium if for any* j*, given the tuple* $\{c_k\}_{k\neq j}$*, we have* $g_j^L(y,\xi_1,\cdots,\xi_J) = \arg\max_{c_j} V_j^L(y,\xi_1,...,\xi_J)$*.*

Analogically, we also define the Nash equilibrium of the game within *Full information* and *Adaptive learning* structures respectively by the consumption profiles $\left\{g_j^{FI}(y,\theta^*)\right\}_{j=1,\dots,J}$ and $\left\{g_j^{AL}(y,\xi_1,\cdots,\xi_J)\right\}_{j=1,\dots,J}$.

The following results give the optimal consumption under each of the three information structures we describe above: *Full information*, *Learning*, and *Adaptive learning*. This optimal consumption is in fact the Nash equilibrium of the game under each information structure.

Proposition 1.2.1. *From* (1.3)*, the optimal consumption of an informed country* j *is*

$$g_j^{FI}(y,\theta^*) = \frac{1-\delta\mu(\theta^*)}{J+(1-J)\delta\mu(\theta^*)}y \tag{1.7}$$

where $\mu(\theta) \equiv \int_0^1 \eta\phi(\eta|\theta)d\eta$ *for* $\theta \in \Theta$*.*

Proof. See Levhari and Mirman (1980). □

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and with common knowledge of each other's beliefs cannot agree to disagree. Specifically, the theorem says that if two people have the same prior, and their posteriors for a given event A are common knowledge, then these posteriors must be equal. This is so even though they may base their posteriors on different information.

Propositions 1.2.1 and 1.2.2 provide the Nash solutions corresponding to (1.3), (1.5), and (1.6) respectively. In each information structure, the Nash equilibrium exists and is unique. The combination of log utility and the Cobb-Douglas rule of evolution for the resource yields a tractable characterization of optimal consumption in a dynamic game. In particular, the uncertainty in outcomes characterized by $\tilde{\eta}$ affects behavior through its conditional mean, $\mu(\theta) \equiv \int_0^1 \eta\phi(\eta|\theta)\mathrm{d}\eta$ for $\theta \in \Theta$.

Proposition 1.2.2. *From* (1.5), *the optimal consumption of the learning country* j *is*

$$g_j^L(y, \xi_1, \ldots, \xi_J) = \left[\int_\Theta \frac{\mu(\theta)\xi_j(\theta)}{1-\delta\mu(\theta)}d\theta\right]^{-1} \left[\delta + \sum_{k=1}^{J}\left[\int_\Theta \frac{\mu(\theta)\xi_k(\theta)}{1-\delta\mu(\theta)}d\theta\right]^{-1}\right]^{-1} y \tag{1.8}$$

From (1.6), *the adaptive learning countries'optimal consumption is*

$$g_j^{AL}(y, \xi_1, \ldots, \xi_J) = \frac{1-\delta\int_\Theta \mu(\theta)\xi_j(\theta)d\theta}{\int_\Theta \mu(\theta)\xi_j(\theta)d\theta}\left[\delta + \sum_{k=1}^{J}\frac{1-\delta\int_\Theta \mu(\theta)\xi_k(\theta)d\theta}{\int_\Theta \mu(\theta)\xi_k(\theta)d\theta}\right]^{-1} y \tag{1.9}$$

Proof. See Appendix 3.8.1. □

Concerning the functional form of the optimal consumption, our findings are consistent with what that has been seen in KMS. That is, the functional form is the same within the *full information* and *adaptive learning* contexts.[7] Similarly, in each of the three information structures, the functional forms of the optimal consumptions in the single-agent setting and the "multiple-agent" setting are similar.[8] Therefore, introducing more than one agent in the model does not change the functional form of the equilibrium.

1.2.4 Comparative analysis

Throughout this paper, we do comparative analysis. Here, we present how these analysis are done and how they should be interpreted.

When all countries have the same belief about θ^*, we say that there is homogeneity of beliefs. When the countries do not have the same beliefs, there is heterogeneity of beliefs. To capture the effect of heterogeneity on the optimal decision, we compare the multiple-agent problem with homogeneity of beliefs to the multiple-agent problem with heterogeneity of beliefs. However, when we compare the single-agent problem (as seen

[7]In the single-agent model, the optimal consumptions in *full information*, *learning* and *adaptive learning* are respectively $g^{FI}(y, \theta^*) = (1-\delta\mu(\theta^*))\, y$, $g^L(y, \xi) = \left(\int_\Theta \frac{\mu(\theta)\xi(\theta)}{1-\delta\mu(\theta)}d\theta\right)^{-1} y$ and $g^{AL}(y, \xi) = \left(1-\delta\int_\Theta \mu(\theta)\xi(\theta)d\theta\right) y$, where ξ denotes the belief of the exploiter.

[8]To check and see these results clearly, one needs to consider that the countries have the same beliefs.

in KMS) to the multiple-agent problem with homogeneity of beliefs, we find the effect of introducing more than one agent in the model.

Below, we also compare full information and adaptive learning. We then capture the effect of beliefs due to uncertainty. Comparing adaptive learning to learning gives the effect of the anticipation.

1.3 Adaptive Learning vs. Learning: the role of heterogeneity

The difference between *Adaptive Learning* and *Learning* is the anticipation of learning. In this section, we study the effect of this anticipation on resource consumption. In other words, we will compare the consumption decisions in *adaptive learning* and *learning* structures. In KMS, anticipation has a negative effect on consumption. The single agent consumes more in *adaptive learning* than in *learning*.[9] In KMS, anticipation changes the consumption path because it changes the expected future beliefs of the single agent, and subsequently the expected future consumption. Anticipation is a source of risk or uncertainty. This behavior by the single agent can therefore be seen as *precautionary savings* (see Keynes (1930), Leland (1968), Kimball (1990), Dardanoni (1991), Carroll and Kimball (2001)).

We now analyze how anticipation of learning impacts the exploitation of the resource within a context of multiple-exploiters with different beliefs. We do this for both total and individual consumption. Heterogeneity is shown to play an important role in the impact of anticipation on consumption.

Proposition 1.3.1 states that anticipation has a negative effect on total consumption.

Proposition 1.3.1. *Total consumption is higher in adaptive learning than in learning environment, that is:* $\sum_{i=1}^{J} g_i^{AL}(\cdot) > \sum_{i=1}^{J} g_i^{L}(\cdot)$.

Proof. Let $\mathbb{E}_i$ denote the expectation under the distribution ξ_i and $R(x) = x(1-\delta x)^{-1}$. Let $A_i = 1/R\left(\mathbb{E}_i(\mu(\theta))\right)$ and $B_i = 1/\mathbb{E}_i\left(R(\mu(\theta))\right)$. We have $\sum_{i=1}^{J} g_i^{AL}(\cdot) = \sum_{i=1}^{J}\left(A_i/\left(\delta + \sum_{i=1}^{J} A_i\right)\right) y$ and $\sum_{i=1}^{J} g_i^{L}(\cdot) = \sum_{i=1}^{J}\left(B_i/\left(\delta + \sum_{i=1}^{J} B_i\right)\right) y$. Because $A_i > B_i \ \forall i$ (due the convexity of R and Jensen inequality) and $x \mapsto x(\delta + x)^{-1}$ is an increasing function then $\sum_{i=1}^{J} g_i^{AL}(\cdot) > \sum_{i=1}^{J} g_i^{L}(\cdot)$. □

Proposition 1.3.1 is consistent with KMS. Indeed, if total consumption in *adaptive learning* is higher than in the *learning* setting, it means that at least one country consumes

[9] See proposition 4.1. in KMS.

more in *adaptive learning* than in *learning*. Therefore, it would be interesting to ask if all of the countries consume more in *adaptive learning*. Consider the following example involving two countries, 1 and 2.

Example 1.3.1. *Consider two countries, 1 ($C1$) and 2 ($C2$), with beliefs ξ_1 and ξ_2 respectively. We assume that $\tilde{\eta}$ has a uniform distribution with unknown support $[0, \theta]$, $\theta \in [0, 1]$. ξ_1 and ξ_2 are uniform distributions respectively over $[n_1, 1]$ and $[n_2, 1]$, with $n_1, n_2 \in \mathbb{R}$.*

Figures 1.1 and 1.2 display the best response of each country in *Learning* and *Adaptive Learning* for two different panels of n_1, n_2 and δ.[10]

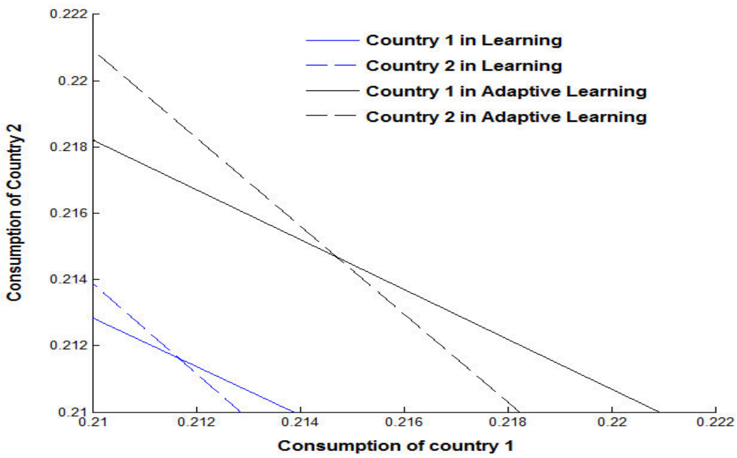

Figure 1.1: Best responses in Learning and Adaptive learning

In the example shown in Figure 1.1, both countries consume more in *adaptive learning*. In contrast, Figure 1.2 shows a consumption path in which country 1 consumes less in *adaptive learning* while country 2 is still consuming more in *adaptive learning*. Therefore, investigate the conditions under which a given country consumes less in *adaptive learning*. In general, for a given country j to have $g_j^{AL} < g_j^L$, the following condition should hold:

$$\delta \left[\mathbb{E}_j\left(R(\mu(\theta))\right) - R\left(\mathbb{E}_j(\mu(\theta))\right)\right] < \sum_{i \neq j} \frac{R\left(\mathbb{E}_j(\mu(\theta))\right)}{R\left(\mathbb{E}_i(\mu(\theta))\right)} - \sum_{i \neq j} \frac{\mathbb{E}_j\left(R(\mu(\theta))\right)}{\mathbb{E}_i\left(R(\mu(\theta))\right)} \tag{1.10}$$

where $\mathbb{E}_i$ denotes the expectation under the distribution ξ_i and $R(x) = x(1 - \delta x)^{-1}$. One can see that this condition does not hold if the countries have the same belief. As a consequence, none of the countries will consume less in *adaptive learning* if they believe

[10]For Figure 1.1, we set $n_1 = n_2 = 0.1$ and $\delta = 0.9$. For Figure 1.2, we set $n_1 = 0.5, n_2 = 0.1$ and $\delta = 0.9$.

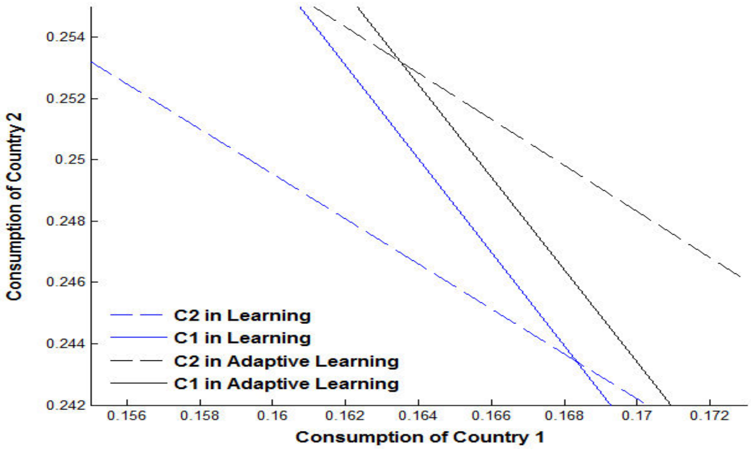

Figure 1.2: Best responses in Learning and Adaptive learning

the same thing about θ. If there is homogeneity of beliefs, each country consumes more in *adaptive learning* than in *learning*.

To understand the condition in inequality (1.10), let us realize that $\delta \mathbb{E}_j \left(R(\mu(\theta)) \right)$ is what country j would expect to gain by investing under a *learning structure*, and $\delta R \left(\mathbb{E}_j(\mu(\theta)) \right)$ is this gain under *adaptive learning* if j were the only exploiter of the resource.[11] Therefore, the expression

$$\delta \left[\mathbb{E}_j \left(R(\mu(\theta)) \right) - R \left(\mathbb{E}_j(\mu(\theta)) \right) \right] \tag{1.11}$$

is the gain of anticipation in terms of investment return. Specifically, it is the gain in terms of investment return if the country moves from the *adaptive learning* to *learning* information structure. As we will examine in detail in section 1.4, in case of heterogeneity, a part of this gain could be extracted by the other countries.[12] In other words, there could be a loss on investment due to heterogeneity, introducing a sort of distortion which is non existent in case of homogeneity of beliefs. The right-side term of condition (1.10) represents the loss due to anticipating. Subsequently, condition (1.10) is easier to understand. It states that the country will consume more in *learning* than in *adaptive learning* if its gain from anticipation in terms of investment return is smaller than the loss. There is therefore a situation in which a country individually consumes more in *learning* than in *adaptive learning*. However, according to proposition 1.3.1, when we consider all the

[11]To express this condition, simply note the single-agent problem under each environment and derive the first-order conditions.

[12]When a country decides to decrease its consumption (and then to increase its investment in the resource), the others increase their consumption. As a consequence, they profit from the investment (or savings) made by this country. In other words, the country that decides to invest more in the resource loses some of the benefit of this investment.

countries, the gain always overcompensates for the loss. Therefore, if a country (for given beliefs) increases its consumption from *adaptive learning* to *learning*, the other countries will not only decrease their consumption but will decrease it to a point that total consumption will be lesser in *learning* than in *adaptive learning*. In other words, if a country decreases its investment from *adaptive learning* to *learning*, the other countries increase their investment more than this decrease.

From this result, the following question may arise: why does the gain always overcompensate for the loss and not compensate it exactly? We answer this question using the case of two countries, 1 and 2. The gain from anticipation of country 1 is $\delta\left[\mathbb{E}_1\left(R(\mu(\theta))\right) - R\left(\mathbb{E}_1(\mu(\theta))\right)\right]$ and its loss due to heterogeneity is $\frac{R(\mathbb{E}_1(\mu(\theta)))}{R(\mathbb{E}_2(\mu(\theta)))} - \frac{\mathbb{E}_1(R(\mu(\theta)))}{\mathbb{E}_2(R(\mu(\theta)))}$. Therefore, its net gain is

$$NG_1 = \underbrace{\delta\left[\mathbb{E}_1\left(R(\mu(\theta))\right) - R\left(\mathbb{E}_1(\mu(\theta))\right)\right]}_{Self\ Incentive} + \underbrace{\frac{\mathbb{E}_1\left(R(\mu(\theta))\right)}{\mathbb{E}_2\left(R(\mu(\theta))\right)} - \frac{R\left(\mathbb{E}_1(\mu(\theta))\right)}{R\left(\mathbb{E}_2(\mu(\theta))\right)}}_{Heterogeneity\ Incentive}. \tag{1.12}$$

For country 2, the net gain is

$$NG_2 = \underbrace{\delta\left[\mathbb{E}_2\left(R(\mu(\theta))\right) - R\left(\mathbb{E}_2(\mu(\theta))\right)\right]}_{Self\ Incentive} + \underbrace{\frac{\mathbb{E}_2\left(R(\mu(\theta))\right)}{\mathbb{E}_1\left(R(\mu(\theta))\right)} - \frac{R\left(\mathbb{E}_2(\mu(\theta))\right)}{R\left(\mathbb{E}_1(\mu(\theta))\right)}}_{Heterogeneity\ Incentive}. \tag{1.13}$$

In condition (1.10) the expression $\delta\left[\mathbb{E}_j\left(R(\mu(\theta))\right) - R\left(\mathbb{E}_j(\mu(\theta))\right)\right]$ can be seen as the relative incentive for country j to invest in *learning*, in comparison with *adaptive learning*. Specifically, it is the incentive coming from country j's side i.e. the incentive to invest if country j is the only exploiter. We call this incentive *self-incentive* because it depends only on the belief of the country in matter. The expression $\sum_{i\neq j} \frac{\mathbb{E}_j(R(\mu(\theta)))}{\mathbb{E}_i(R(\mu(\theta)))} - \sum_{i\neq j} \frac{R(\mathbb{E}_j(\mu(\theta)))}{R(\mathbb{E}_i(\mu(\theta)))}$ is the incentive from other countries due to heterogeneity. We call it the *heterogeneity incentive*. The net gain is therefore the sum of the two types of incentive.

Assuming that the net gain of country 1 is negative, i.e. $NG_1 < 0$, that country's heterogeneity incentive is negative because the self-incentive is always positive. One can also see that if the heterogeneity incentive of country 1 is negative, then the heterogeneity incentive of country 2 is positive. What is lost by country 1 (due to heterogeneity) is recovered by country 2 through heterogeneity as well. As a consequence, if the net gain of country 1 is negative, then both incentives of country 2 are positive. Therefore, in addition to what is recovered from country 1 (through the heterogeneity incentive), country 2 also benefits from its self-incentive. This is why the effect of the gain overcompensates for the loss. Figure 1.3 illustrates the effect of the two kinds of incentive for country 1.

In Figure 1.3, lines $L1$ and $L2$ are the best response functions in *learning* of countries

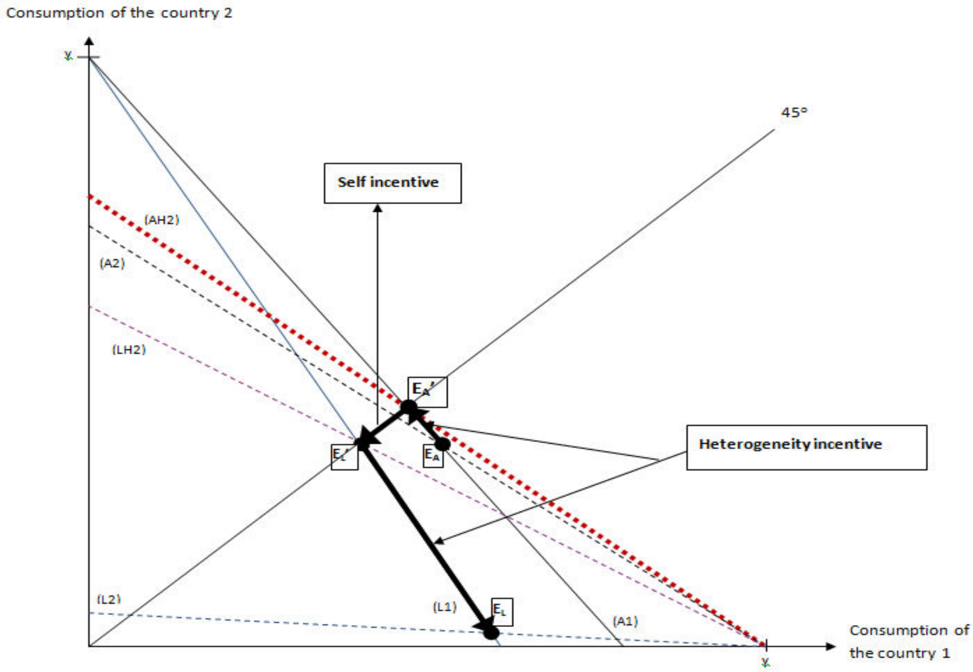

Figure 1.3: Self-incentive and heterogeneity incentive for country 1

1 and 2 respectively. $A1$ and $A2$ are the corresponding best response functions in *adaptive learning*. $LH2$ and $AH2$ would be the best response functions of country 2 in *learning* and *adaptive learning* respectively if it had the same belief as country 1. Therefore, the portion $E'_A - E'_L$ represents the self-incentive of country 1. The heterogeneity incentive (of country 1) is given by the portions $E_A - E'_A$ and $E'_L - E_L$.

Let us now consider Figure 1.4. Except for lines $LF2$ and $AF2$, all other notations in Figure 1.4 remain the same as in Figure 1.3. $LF2$ and $AF2$ would be the best responses of country 2 respectively in *learning* and *adaptive learning* if country 2 had formed beliefs such that the consumption of country 1 is the same in *learning* and in *adaptive learning*, all things being equal. If country 2 forms a belief that puts its best response in *learning* above $LF2$ (and therefore its best response in *adaptive learning* above $AF2$), then country 1 will consume less in *adaptive learning* than in *learning*. The opposite will be true if the best responses of country 2 in *learning* (resp in *adaptive learning*) is below $LF2$ (resp $AF2$). This insight informs Figure 1.5, which displays intuitively the domain of beliefs that supports the effects of the two kinds of incentive. The beliefs comprise the axes. These beliefs are ordered with the first-order stochastic dominance.[13]

Note that Frontiers 1 and 2 are not necessarily straight lines. An application of Figure 1.5 applied to example 1.3.1 leads to Figure 1.6 below.

[13]See section 1.4 for details on the first-order stochastic dominance.

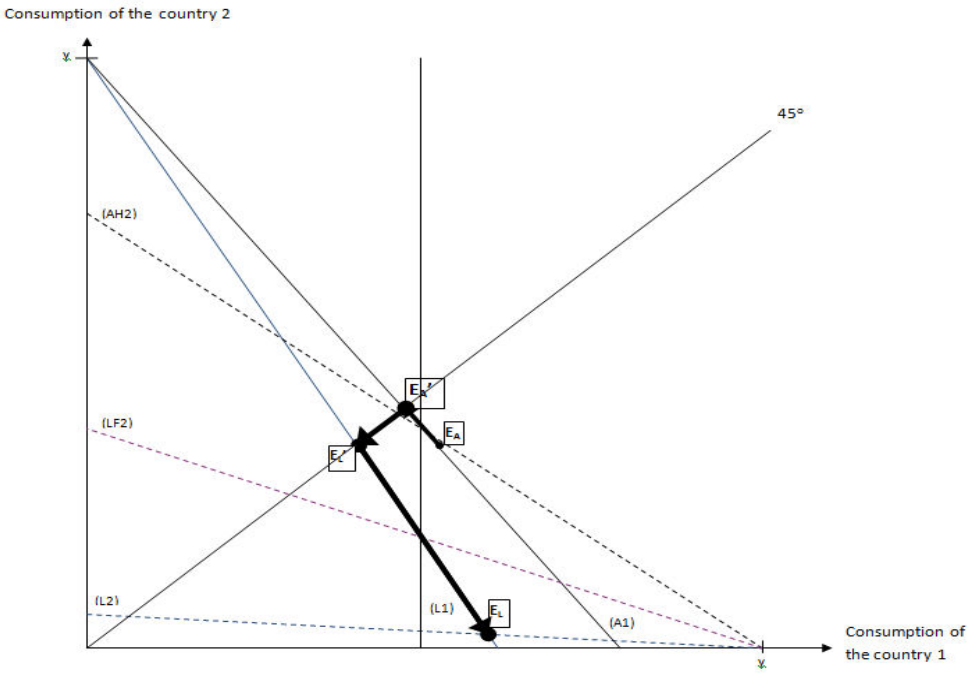

Figure 1.4: Self-incentive and heterogeneity incentive for country 1

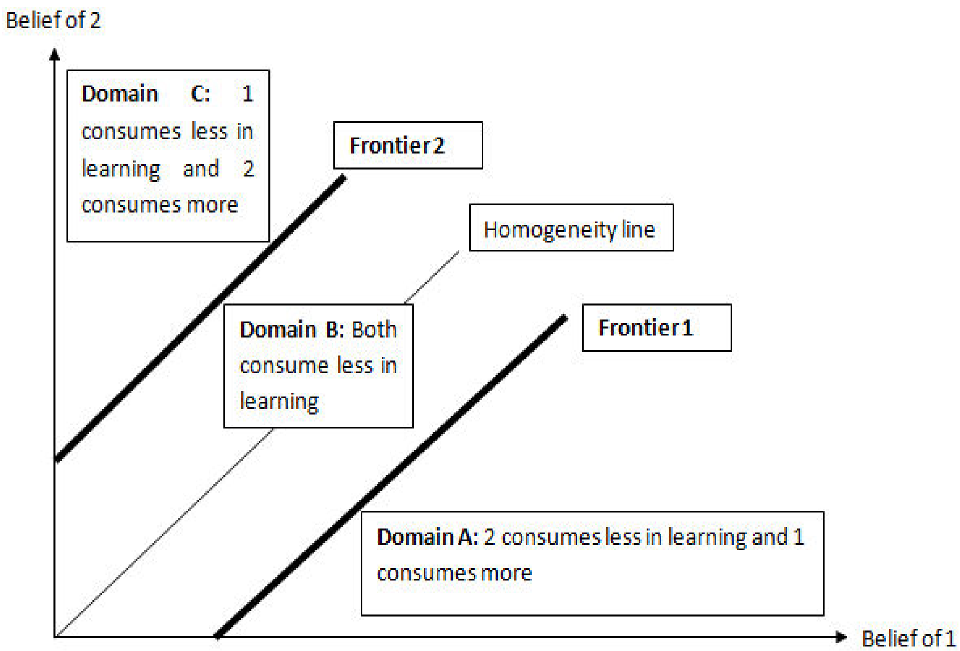

Figure 1.5: Domain of beliefs and incentives for investment

As a conclusion to this part, in case of homogeneity of beliefs, any country consumes more in *adaptive learning* than in *learning*. However, when beliefs are heterogeneous, some

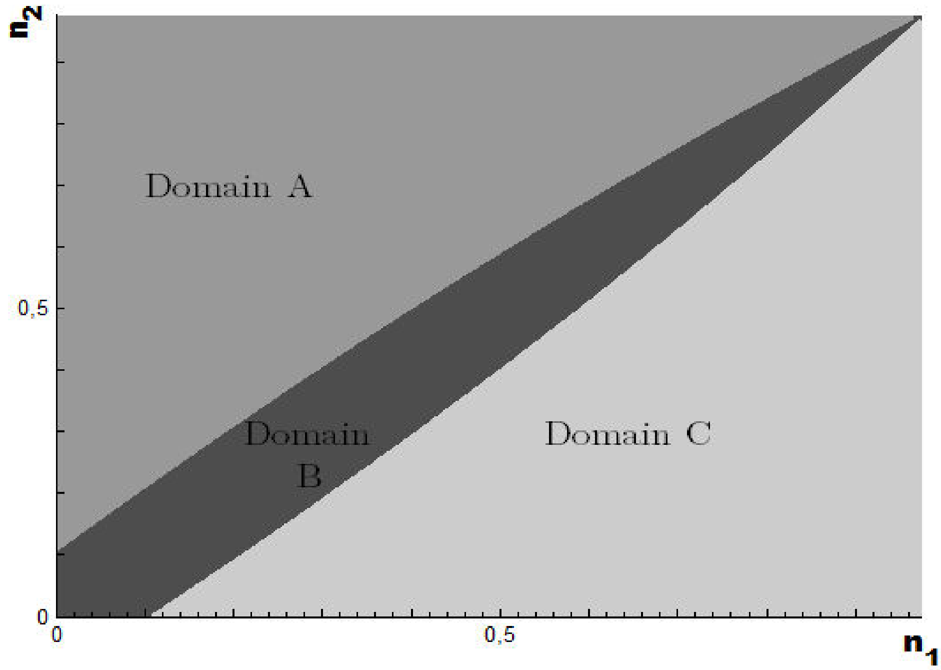

Figure 1.6: Domain of beliefs and incentives for investment (in the context of Example 1.3.1)

countries may consume less in *adaptive learning* than in *learning* because of distortions caused by heterogeneity. However, total consumption is always higher in *adaptive learning* than in *learning*.

1.4 Optimism/Pessimism and resource consumption

In this section, we study the impact of a change in beliefs on the optimal consumption. Specifically, we examine how the countries change their consumption when their beliefs shift in a way that makes them more optimistic or pessimistic about the renewal of the resource. After conducting a comparative analysis, we see how heterogeneity affects this impact of a change in beliefs.

1.4.1 Optimism and Pessimism

Before studying the impact of optimism or pessimism on the consumption decision, we define the meaning this notion.

Insight

Optimism concerns the renewal of the resource being exploited, i.e. the quantity of resources that countries expect to have at their disposal in the future, given the current stock. To illustrate, we consider a state S_1 in which a country expects the future quantity of the resource to be Q_1, given the current stock y. Consider another state S_2 in which the same country expects the future quantity of resource to be Q_2, given the same current stock y. We will say that the country is *more optimistic* (and then *less pessimistic*) in state S_2 than in state S_1 if Q_2 is greater than Q_1. However, this only explains the concept of optimism. We use a slightly different definition based on stochastic dominance.

Characterization of optimism/pessimism: strict first-order stochastic dominance.

Here, we present the methods we used to assess optimism and pessimism. For this purpose, we give the following definition.

Definition 1.4.1. *Consider two distributions defined by the c.d.f.* F^1 *and* F^2 *respectively. The distribution* F^1 *strict first-order stochastically dominates the distribution* F^2*, and we note* $F^1 \succ_1 F^2$*, if for all* x*,* $F^1(x) \leq F^2(x)$ *and there exists some* x_0 *such that* $F^1(x_0) < F^2(x_0)$*.*[14]

According to the definition above, if $F^1 \succ_1 F^2$ then we expect higher values for the variable under the distribution F^1 than under the distribution F^2. Moreover, the countries' expectations of renewal depends on what they expect for η. We also know that the beliefs about θ affect what the countries expect for η through $\mu(\theta)$, the mean of the variable $\tilde{\eta}$. Therefore, we can use the strict first-order stochastic dominance concept on the beliefs to define optimism or pessimism. However, we should also know how these beliefs affect η through μ. This depends on the derivative of μ. If μ is an increasing function, when the countries expect a high (low) value for θ they also expect a high (low) value for η. The opposite is true if μ is a decreasing function. Therefore, for the sake of clarity, we assume hereafter that μ is an increasing function of θ.[15]

Assumption 2. $\mu' > 0$.

Below is the practical definition we use for optimism and pessimism. This definition takes into account our Remarks 1.2.1 and 1.2.2

[14] An equivalent definition is the following: Suppose that the two distributions are of p.d.f. φ^1 and φ^2. The distribution φ^1 strict first-order stochastically dominates the distribution φ^2, and we note $\varphi^1 \succ_1 \varphi^2$, if for every non-decreasing function $\lambda : \mathbb{R} \to \mathbb{R}$, $\int_{\mathbb{R}} \lambda(x)\varphi^1(x)dx \geq \int_{\mathbb{R}} \lambda(x)\varphi^2(x)dx$.

[15] We can also assume that $\mu' < 0$ and any result will be easy to derive.

Definition 1.4.2. *When the current stock of a resource is lower than one, we say that a country gets more optimistic (and then less pessimistic) if its belief changes so that the previous belief strict first-order stochastically dominates the new belief.*[16]

When the current stock of a resource is greater than one, a country is said to get more optimistic (and then less pessimistic) if its belief changes so that the new belief strict first-order stochastically dominates the previous belief.

1.4.2 Optimism/Pessimism and resource consumption

After defining the concept of optimism, we now study the effect of an increase in a country's optimism on resource exploitation. We measure this effect first on the consumption of the country whose optimism increases, and next on the consumption of the other countries, and finally on total consumption (sum of the consumption of each country). The following proposition gives some of the results.

Proposition 1.4.1. *Suppose that the stock of a resource is greater than one (the steady state). Let us consider a J-tuple $\Xi = (\xi_1, \cdots, \xi_i, \cdots, \xi_J)$ of beliefs where ξ_k denotes the beliefs of country k. Consider another J-tuple of beliefs $\tilde{\Xi} = (\xi_1, \cdots, \tilde{\xi}_i, \cdots, \xi_J)$. From Ξ to $\tilde{\Xi}$ belief remains the same for each country except country i. If country i is more optimistic under belief $\tilde{\xi}_i$ than under belief ξ_i, then*

(i) $g_i^L(y, \tilde{\Xi}) < g_i^L(y, \Xi)$.

(ii) $g_i^{AL}(y, \tilde{\Xi}) < g_i^{AL}(y, \Xi)$.

(iii) $g_j^L(y, \tilde{\Xi}) > g_j^L(y, \Xi)$ *for all* $j \in I$ *with* $j \neq i$.

(iv) $g_j^{AL}(y, \tilde{\Xi}) > g_j^{AL}(y, \Xi)$ *for all* $j \in I$ *with* $j \neq i$.

(v) $\sum_{k=1}^{J} g_k^L(y, \tilde{\Xi}) < \sum_{k=1}^{J} g_k^L(y, \Xi)$.

(vi) $\sum_{k=1}^{J} g_k^{AL}(y, \tilde{\Xi}) < \sum_{k=1}^{J} g_k^{AL}(y, \Xi)$.

Proof. See Appendix 3.8.2. □

In proposition 1.4.1 the stock of resource is assumed to be greater than the steady state. Therefore, within the context of this proposition, the optimism is about the loss on the resource's quantity over time. It is the optimism about the loss on investment (or savings) in the renewal of the resource. According to the proposition, for each country, consumption (in *learning* and *adaptive learning*) decreases with its own optimism and

[16] We also say the optimism of the country increases.

increases with the optimism of others. In other words, when a country believes that any reserve will be lost on a higher scale, it decides not to keep a higher quantity in reserve. No country is interested in preserving a rapidly self-depleting resource. The exploiters seem to think that whatever they do, such a resource is likely to become extinct. They also lose part of any investment they make. As a consequence, the less endangered resource will last longer not only because it is less endangered but also because the exploiters preserve it.

In contrast, a country invests less when it knows that the others become more optimistic and will increase their contribution to the investment in the resource. The optimism of a country therefore has two counteracting effects on total consumption: a negative effect from the country in question, and a positive effect from the other countries. All the countries would like to preserve the resource or prevent it from becoming extinct. However, they also want to increase their current utility. As a consequence, each country tries to profit from the investment of other countries. Proposition 1.4.1 states that the negative effect of optimism overcomes the positive effect. Accordingly, when a country decreases its consumption due to optimism, the others do not increase their own to compensate this decrease exactly. This is because even though the exploiters would like to profit from each other's optimism, they also care about the conservation of the resource. They see any optimism in one of them as a good opportunity to protect the resource, rather than as an opportunity to achieve a selfish goal.

Below is an example illustrating proposition 1.4.1.

Example 1.4.1. *Let $\tilde{\eta}$ follow a uniform distribution with unknown support $[0, \theta]$, $\theta \in [0, 1]$, where θ is a random variable. In Table 1.1, four different distributions are presented for θ. The first column gives the possible values taken by θ and the four last*

Table 1.1: Some distributions of θ

Values of θ	**Distribution 1**	**Distribution 2**	**Distribution 3**	**Distribution 4**
0.7	0.3577	0.3473	0.2572	0.0000
0.8	0.3344	0.3178	0.2584	0.3333
0.9	0.1377	0.1571	0.2188	0.3333
1	0.1702	0.1778	0.2656	0.3334

columns display the probability of occurrence of each value under the given distribution. It is easy to see that distribution 4 $\succ_1$ distribution 3 $\succ_1$ distribution 2 $\succ_1$ distribution 1, where "distribution 4 $\succ_1$ distribution 3" means distribution 3 is first order stochastically dominated by distribution 4. Therefore, from distribution 1 to distribution 4, optimism about the renewal is rising. In Table 1.2, we display the consumption path under the

Table 1.2: Beliefs and consumption path

Consumption	Condition 1	Condition 2
For country 1	0.4446	0.4540
For country 2	0.4415	0.4228
Total consumption	0.8861	0.8768

learning environment for two countries (1 and 2). We consider two scenarios, 1 and 2. In scenario 1, the beliefs of countries 1 and 2 are represented respectively by distributions 1 and 2. In scenario 2, beliefs are represented instead by distribution 3 and 4 respectively. Therefore, from scenario 1 to scenario 2, both countries get more optimistic. Total consumption thus decreases as optimism increases. Due to the heterogeneity of beliefs, a country decreases its consumption and another increases it. The country decreasing the consumption is the one for which optimism has increased more. (country 2 in this example). The decrease caused by this agent for total consumption is not offset by the increase provided by the other agent.

We now examine the effect of optimism on resource exploitation when the stock is lower than the steady state. Proposition 1.4.2 gives some results.

Proposition 1.4.2. *Suppose that the stock of resource is lower than the steady state. Let us consider a J-tuple $\Xi = (\xi_1, \cdots, \xi_i, \cdots, \xi_J)$ of beliefs where ξ_k denotes the belief of country k. Consider another J-tuple of beliefs $\tilde{\Xi} = (\xi_1, \cdots, \tilde{\xi}_i, \cdots, \xi_J)$. From Ξ to $\tilde{\Xi}$ belief remains the same for all countries except country i. If country i is more optimistic under belief $\tilde{\xi}_i$ than under belief ξ_i then,*

(i) $g_i^L(y, \tilde{\Xi}) > g_i^L(y, \Xi)$.

(ii) $g_i^{AL}(y, \tilde{\Xi}) > g_i^{AL}(y, \Xi)$.

(iii) $g_j^L(y, \tilde{\Xi}) < g_j^L(y, \Xi)$ *for all* $j \in I$ *with* $j \neq i$.

(iv) $g_j^{AL}(y, \tilde{\Xi}) < g_j^{AL}(y, \Xi)$ *for all* $j \in I$ *with* $j \neq i$.

(v) $\sum_{k=1}^{J} g_k^L(y, \tilde{\Xi}) > \sum_{k=1}^{J} g_k^L(y, \Xi)$.

(vi) $\sum_{k=1}^{J} g_k^{AL}(y, \tilde{\Xi}) > \sum_{k=1}^{J} g_k^{AL}(y, \Xi)$.

Proof. The proof is similar to that of proposition 1.4.1. □

In proposition 1.4.2, we assume that the stock of the resource is lower than the steady state. Therefore, within this context, the optimism is about the regenerative process of the resource, i.e. the positive return on an investment in the resource. Here, the result is the opposite of what we saw when the stock of the resource was lower than the steady state. According to proposition 1.4.2, a country's consumption (both in learning and adaptive learning) increases with its optimism and decreases with the optimism of other countries. In other words, when a country believes that any reserve will be reproductive on a higher scale, it decides to consume more. Countries consequently do not need to invest very much because the return on investment is expected to be higher.

1.5 Optimism and depletion rate: Mitigated trend to a common behavior

We have seen that an increase in optimism of a given agent leads to a decrease or increase in its consumption both in homogenous and heterogenous belief contexts. We have not addressed the extent to which consumption decreases or increases. It would therefore be interesting to investigate the variation of this decrease or increase when beliefs change. Specifically, we are interested in the "speed" of the change in consumption when optimism is increasing. Here, we distinguish two kinds of change in consumption: *absolute change* (decrease or increase) and *relative change*. Absolute change is a measure of the quantity by which an exploiter's consumption changes when its optimism increases. Relative change refers to a measure of change (in proportion to the initial consumption) due to a relative increase in optimism. It is a sort of elasticity. We find that absolute change decreases when optimism is increasing, while relative change is amplified. This result is used to compare the "speeds" of consumption change in heterogenous and homogeneous beliefs. Specifically, we suppose that a given country j has belief ξ_j and the $J-1$ others have the same belief ξ. If $\xi_j \neq \xi$, then there is heterogeneity, and if $\xi_j = \xi$, there is homogeneity. As mentioned above, $R(x) = x(1-\delta x)^{-1}$, $A = \mathbb{E}_j\left(R(\mu(\theta))\right)$, $B = \mathbb{E}\left(R(\mu(\theta))\right)$, where $\mathbb{E}_j$ and $\mathbb{E}$ stand for the expectations under the beliefs ξ_j and ξ respectively. A is the gain of country j from investment in a learning structure. The consumption of country j in *learning* is $g_j^L(.) = \frac{B}{(J-1+\delta B)A+B}y$.[17] What we call *absolute change* (arising from the optimism of j) is the absolute value of the first derivative of $g_j^L(.)$ with respect to A, i.e. $\frac{B(J-1+\delta B)}{[(J-1+\delta B)A+B]^2}$. Absolute change is the change in consumption of country j when it gets more optimistic enough to cause an infinitesimal increase or decrease in A. *Relative change* refers to the elasticity of $g_j^L(.)$ with respect to A, i.e. $\frac{A(J-1+\delta B)}{(J-1+\delta B)A+B}$. In other words, relative change is the variation in percentage of the consumption of country j when it

[17] We can also consider consumption in *adaptive learning*. However, because in this section, any finding is the same in *learning* and *adaptive learning*, we focus on the *learning* outcomes for simplicity.

gets more optimistic such it causes a one-percent increase or decrease in A. We compare the variation of each kind of change in heterogeneity to its variation in homogeneity. Propositions 1.5.1 and 1.5.2 give some results. To simplify the analysis, we consider that the stock of the resource is greater than the steady state. Optimism therefore leads to a decrease in consumption. The result is the same when the stock is lower than the steady state.

Proposition 1.5.1. *We assume that the stock of the resource is greater than the steady state. Let us consider a given country j among the J exploiting the resource. Suppose that j becomes more optimistic, and the other countries' beliefs remain unchanged. Except for j, the $J-1$ others have the same belief.*[18] *Therefore,*

(i) If j were more optimistic than the other countries, then the absolute decrease of its consumption in heterogeneous beliefs would be lower than the absolute decrease in homogeneity.

(ii) If j were less optimistic than the other countries, then the absolute decrease in heterogeneous beliefs would be higher than the absolute decrease in homogeneity.

Proof. We have seen that the absolute decrease is given by $\frac{B(J-1+\delta B)}{[(J-1+\delta B)A+B]^2}$. Because $\frac{B(J-1+\delta B)}{[(J-1+\delta B)A+B]^2}$ is decreasing in A, the proof is completed. □

The *absolute decrease* in optimal consumption in the heterogeneity context depends on the beliefs of a country compared to the beliefs of the others. Based on proposition 1.5.1, we can say that the countries tend to be fairly similar when making their consumption decisions. In other words, even if an exploiter does not have the same belief as others, it tries not to deviate too much from the decision of other exploiters. Therefore, the countries seem to act with caution, leading to a sort of "coordination". This trend toward caution is attributable to the fact that high optimism may lead to irrational decisions (see, for example, Puri and Robinson (2007)). Indeed, every exploiter knows this reality and can think as follows: *If I am so optimistic when the other countries are less optimistic, my belief may wrong. I have to be prudent.* Therefore, fearing they could make an irrational decision, each exploiter avoids "greater loneliness".

However, this trend toward coordination is mitigated according to proposition 1.5.2. *Relative change* moves in the opposite direction from *absolute change*. Even though an exploiter would like to minimize his distance from his peers, he still trusts his own beliefs. As a conclusion, out of fear of making irrational decisions and wanting trusting in their

[18] We assume that the $J-1$ others have the same belief for calculation purposes. It does not affect the intuition of this finding.

personal beliefs, countries seem to resort to mitigated coordination of their consumption decision.

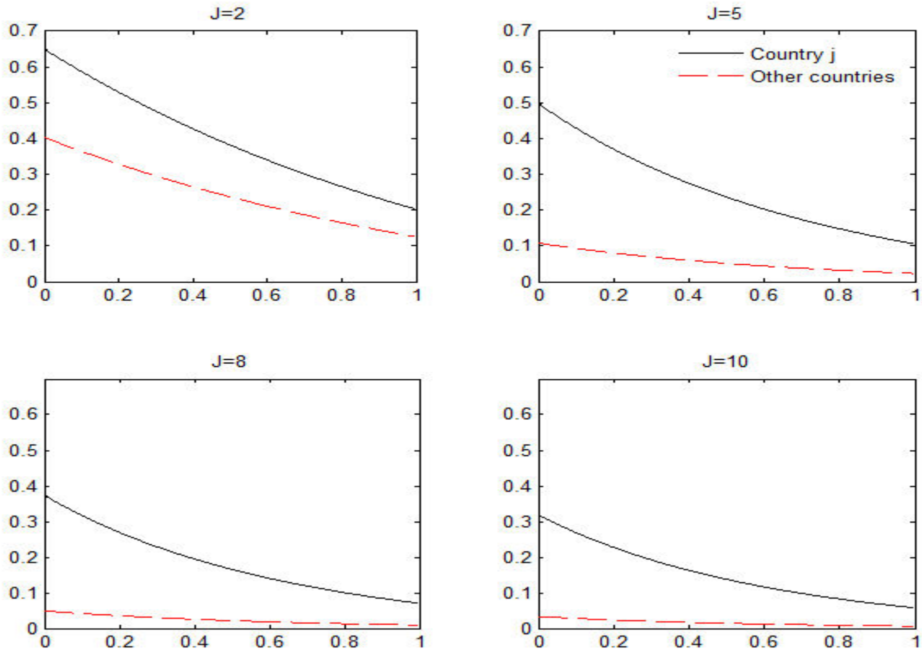

Figure 1.7: Optimism and absolute changes in consumption

Proposition 1.5.2. *We assume that the stock of the resource is greater than the steady state. Consider a given country j among the J exploiting the resource. Suppose that j becomes more optimistic, the other countries' beliefs remaining unchanged. Except for j, the $J-1$ others have the same belief. Therefore,*

(i) If j were more optimistic than the other countries, then the relative decrease in its consumption in heterogenous beliefs would be higher than the relative decrease in homogeneity.

(ii) If j were less optimistic than the other countries then the relative decrease in heterogenous beliefs would be lower than the relative decrease in homogeneity.

Proof. The relative decrease is given by the expression $\frac{A(J-1+\delta B)}{(J-1+\delta B)A+B}$ which is increasing in A. $\square$

The results above are illustrated in Figures 1.7 and 1.8. In Figure 1.7, for different values of J (the number of firms), we draw the absolute change for country j and for the other $J-1$ countries. On the X-axis lies the degree of optimism of country j. We observe that when the country's optimism increases, the two curves seem to approach each other.

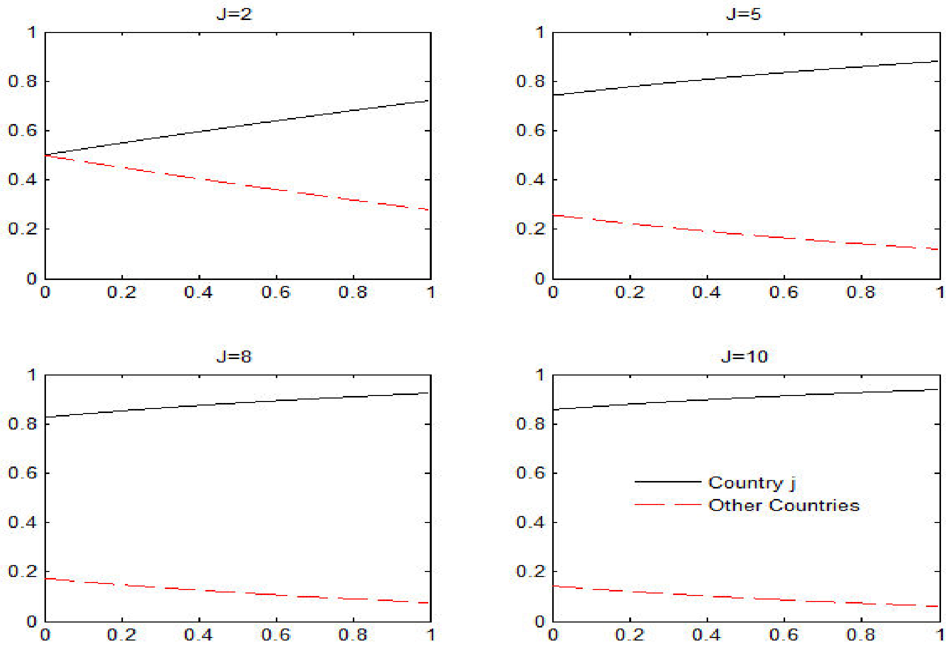

Figure 1.8: Optimism and relative changes in consumption

However, when the optimism decreases, the opposite is observed. In Figure 1.8, we draw the relative changes. Unlike absolute change, optimism pulls the two curves away from each other.

1.6 Beliefs and riskiness: variation in sense of risk

As we did for optimism, we now try to know how consumption changes when the countries believe that there is an increase in risk regarding the renewal of the resource.

1.6.1 Definition of sense of risk

Here, increase in risk means that η becomes more uncertain. In other words, the exploiters sense that $\tilde{\eta}$ is more variable (i.e. more uncertain) than before. When an agent A feels in this way about a given parameter γ, we say that this agent *senses an increase in the risk about* γ. Accordingly, there is an *increase in sense of risk about* γ *for* A.

While *optimism* is defined using strict first-order stochastic dominance, *riskiness* or *sense of risk* is defined by the second order stochastic dominance.[19]

As in the case of optimism, we are interested in the risk related to the renewal of

[19]We define second order stochastic dominance below.

the resource, and then the risk on η. However, the beliefs are about θ. Therefore, we determine the variation in sense of risk about what the countries expect for η through the variation of their risk feeling about θ. Here too, the riskiness about θ impacts that of the renewal through μ, specifically through the concavity of μ. Below, we define two types of second order stochastic dominance, corresponding to the two standard definitions in the literature.

Definition 1.6.1. *Consider two distributions with p.d.f. φ^1 and φ^2. The distribution φ^1 second order stochastically dominates the distribution φ^2 for any risk-averse agent, and we note $\varphi^1 \succ_C \varphi^2$, if for every non-decreasing concave function, $\lambda : \mathbb{R} \to \mathbb{R}$, $\int_{\mathbb{R}} \lambda(x)\varphi^1(x)dx \geq \int_{\mathbb{R}} \lambda(x)\varphi^2(x)dx$.*

Below is the second definition of second-order stochastic dominance.[20]

Definition 1.6.2. *Consider two distributions with p.d.f. φ^1 and φ^2. The distribution φ^1 second order stochastically dominates the distribution φ^2 for any risk seeker, and we note $\varphi^1 \succ_V \varphi^2$, if for every non-decreasing convex function $\lambda : \mathbb{R} \to \mathbb{R}$, $\int_{\mathbb{R}} \lambda(x)\varphi^1(x)dx \geq \int_{\mathbb{R}} \lambda(x)\varphi^2(x)dx$.*

Remark 1.6.1. *(i) $\varphi^1 \succ_C \varphi^2$ means that φ^1 is less risky or "less variable" than φ^2 for the risk-averse agent. Therefore, to avert risk, they should prefer φ^1 to φ^2.*

(ii) $\varphi^1 \succ_V \varphi^2$ means that φ^1 is more risky or "more variable" than φ^2 for the risk seeker. Therefore, the risk seeker would prefer φ^1 to φ^2.

(iii) Consider two variables X and Y with distributions of p.d.f. φ^1 and φ^2 respectively. If $\varphi^1 \succ_C \varphi^2$, we can also say that $X \succ_C Y$, and if $\varphi^1 \succ_V \varphi^2$, we can also say that $X \succ_V Y$.

In the definitions above, the function λ is assumed to be non-decreasing. However, the production function we are targeting here is not always non-decreasing in η. It is non-decreasing in η only if the stock y is greater than the steady state and decreasing if the stock is smaller than the steady state. To circumvent this problem, we resort to the following lemma from Ross (1996, 2nd edition).

Lemma 1.6.1. *See Ross (1996, 2nd edition)*

(i) If X and Y are non-negative random variables such that $\mathbb{E}(X) = \mathbb{E}(Y)$, then $X \succ_V Y$ if and only if $\mathbb{E}(u(X)) \geq \mathbb{E}(u(Y))$ for all convex u.

[20] This type of second order stochastic dominance is also defined in Levy and Wiener (1998), Wong et al. (2006), Al-Zahrani and Stoyanov (2008), Ross (1983), Ross (1996, 2nd edition), Baker (2006).

(ii) If X and Y are non-negative random variables such that $\mathbb{E}(X) = \mathbb{E}(Y)$, then $X \succ_C Y$ if and only if $\mathbb{E}(u(X)) \geq \mathbb{E}(u(Y))$ for all concave u.

$\mathbb{E}(X)$ stands for the expectation of X.

From lemma 1.6.1, we can make the following remark:

Remark 1.6.2. *Let X and Y be two non-negative random variables such that $\mathbb{E}(X) = \mathbb{E}(Y)$. X is more risky than Y for risk seeker if and only if X is more risky than Y for risk averters.*

This remark 1.6.2 allows me to disregard whether the countries are risk averters or risk seekers.

Because $\mu(\theta)$ is always non-negative, we use lemma 1.6.1 to define riskiness about $\mu(\theta)$. We aim to infer the sense of risk about $\mu(\theta)$ from the beliefs (i.e the sense of risk about θ). Hence:

Lemma 1.6.2. *Let X and Y be two variables. Let f be a non decreasing function.*

(i) If $X \succ_C Y$ and $f'' \leq 0$, then $f(X) \succ_C f(Y)$.

(ii) If $X \succ_V Y$ and $f'' \geq 0$, then $f(X) \succ_V f(Y)$.

The proof of lemma 1.6.2 is trivial.

Based on the definitions above, lemmas 1.6.1, and 1.6.2, we give the following definition:

Definition 1.6.3.

- *If $\mu'' \leq 0$, we say that a country senses an increase in risk about the renewal of the resource if its belief changes such that the previous belief ξ second order stochastically dominates the new belief $\tilde{\xi}$ for any risk averse agent, and such that $\mathbb{E}(\mu(\theta)) = \tilde{\mathbb{E}}(\mu(\theta))$.*
- *If $\mu'' \geq 0$, we say that a country senses an increase in risk about the renewal of the resource if its belief changes such that the new belief $\tilde{\xi}$ second-order stochastically dominates the previous belief ξ for any risk seeker, and such that $\mathbb{E}(\mu(\theta)) = \tilde{\mathbb{E}}(\mu(\theta))$,*

where $\mathbb{E}(\mu(\theta))$ and $\tilde{\mathbb{E}}(\mu(\theta))$ stand for the expectation of $\mu(\theta)$ under the distributions ξ and $\tilde{\xi}$ respectively.

The following remark ensues from the definition above:

Remark 1.6.3. *In definition 1.6.3, we allow a change in beliefs that conserves the mean of $\mu(\theta)$. This helps me eliminate the effect of optimism when analyzing riskiness.*[21]

[21] This is known as the Mean Preserving Spread (see Rothschild (1970), Rothschild (1971), Baker (2005), Baker (2009)).

1.6.2 Increase in sense of risk and resource consumption

After defining the concept of *sense of risk*, we now study how optimal consumption varies when the exploiters sense an increase in risk about the renewal of the resource. We examine the effect of this increase on individual and total consumption. The following two propositions give results for *adaptive learning* and *learning*.

Proposition 1.6.1. *Consider a J-tuple $\Xi = (\xi_1, \cdots, \xi_i, \cdots, \xi_J)$ of beliefs where ξ_k denotes the beliefs of country k. Consider another J-tuple of beliefs $\tilde{\Xi} = (\xi_1, \cdots, \tilde{\xi}_i, \cdots, \xi_J)$. From Ξ to $\tilde{\Xi}$, belief remains the same for each country except country i. If country i senses more risk about the renewal under belief $\tilde{\xi}_i$ than it does under the belief ξ_i, then*

(i) $g_i^{AL}(y, \tilde{\Xi}) = g_i^{AL}(y, \Xi)$.

(ii) $g_j^{AL}(y, \tilde{\Xi}) = g_j^{AL}(y, \Xi)$ *for all* $j \in I$ *with* $j \neq i$.

(iii) $\sum_{k=1}^{J} g_k^{AL}(y, \tilde{\Xi}) = \sum_{k=1}^{J} g_k^{AL}(y, \Xi)$.

Proof. The increase in sense of risk preserve the expectation of $\mu(\theta)$. Therefore, the adaptive learning consumption is not affected. □

According to proposition 1.6.1, in *adaptive learning* structure, an increase in sense of risk does not affect the consumption of the countries. However, this may not be the case in *learning* structure, as stated in proposition 1.6.2.

Proposition 1.6.2. *Let us consider a J-tuple $\Xi = (\xi_1, \cdots, \xi_i, \cdots, \xi_J)$ of beliefs where ξ_k denotes the beliefs of country k. Consider another J-tuple of beliefs $\tilde{\Xi} = (\xi_1, \cdots, \tilde{\xi}_i, \cdots, \xi_J)$. From Ξ to $\tilde{\Xi}$, belief remains the same for each country except country i. If country i senses more risk about the renewal under belief $\tilde{\xi}_i$ than it does under the belief ξ_i, then*

(i) $g_i^{L}(y, \tilde{\Xi}) \leq g_i^{L}(y, \Xi)$.

(ii) $g_j^{L}(y, \tilde{\Xi}) \geq g_j^{L}(y, \Xi)$ *for all* $j \in I$ *with* $j \neq i$.

(iii) $\sum_{k=1}^{J} g_k^{L}(y, \tilde{\Xi}) \leq \sum_{k=1}^{J} g_k^{L}(y, \Xi)$.

Proof. See Appendix 3.8.3. □

The results in propositions 1.6.1 and 1.6.2 are well understood. In our model, there are two types of risk: risk due to uncertainty and risk due to anticipation. The *adaptive learning* country faces only the uncertainty risk, while the *learning* country faces both risks. Because in this analysis on *sense of risk*, we are interested in $\mu(\theta)$ (what the countries expect for the shock) and not the shock itself, the uncertainty risk vanishes, but the anticipation risk remains. This is why a variation in *sense of risk* does not affect the consumption decision of the *adaptive learning* country. Concerning the *learning* country, the increase in *risk feeling* leads to an increase in the expected marginal utility of investment, and then to a decrease in consumption.

1.7 Beliefs and tragedy of the commons

In full information and even in uncertainty, it is well known that common property leads to the over-exploitation of the resource, even though this is not in anyone's interest. This result is also true in our model, as shown in proposition 1.7.2. Clearly, if the number of exploiters goes to infinity, the resource is completely depleted only during the first period of exploitation. Bramoullé and Treich (2009), studying the effect of uncertainty on pollution emissions, find that emissions are always lower under uncertainty than under certainty. They therefore assert that uncertainty can alleviate the commons problem. However, when the uncertainty is about a parameter of a given distribution and the agents have some beliefs about this parameter, Bramoullé and Treich may not hold. One can see this in proposition 1.7.1. It depends on how the beliefs affect the countries's optimism, and the functional form of ϕ.

Proposition 1.7.1. *$g_i^{FI}(\cdot)$, $g_i^{AL}(\cdot)$ and $g_i^{L}(\cdot)$ stand for the optimal consumption of country i under full information, adaptive learning and learning assumption, respectively.*

(1) If $\int_\Theta \mu(\theta)\xi_i(\theta)d\theta = \mu(\theta*)$ for all $i \in \{1, ..., J\}$ then $\sum_{i=1}^J g_i^{FI}(\cdot) = \sum_{i=1}^J g_i^{AL}(\cdot) > \sum_{i=1}^J g_i^{L}(\cdot)$
(2) Suppose that countries have the same belief ξ.
If $\int_\Theta \mu(\theta)\xi_i(\theta)d\theta > \mu(\theta*)$ for all $i \in \{1, ..., J\}$ then $\sum_{i=1}^J g_i^{FI}(\cdot) > \sum_{i=1}^J g_i^{AL}(\cdot) > \sum_{i=1}^J g_i^{L}(\cdot)$
If $\int_\Theta \mu(\theta)\xi_i(\theta)d\theta < \mu(\theta*)$ for all $i \in \{1, ..., J\}$ then $\sum_{i=1}^J g_i^{AL}(\cdot) > \sum_{i=1}^J g_i^{FI}(\cdot) > \sum_{i=1}^J g_i^{L}(\cdot)$
(3) Suppose that countries have the same belief ξ and the belief is unbiased about the parameter, $\theta* = \int_\Theta \theta\xi(\theta)d\theta$.
If $\mu" > 0$ then $\sum_{i=1}^J g_i^{FI}(\cdot) > \sum_{i=1}^J g_i^{AL}(\cdot) > \sum_{i=1}^J g_i^{L}(\cdot)$
If $\mu" = 0$ then $\sum_{i=1}^J g_i^{FI}(\cdot) = \sum_{i=1}^J g_i^{AL}(\cdot) > \sum_{i=1}^J g_i^{L}(\cdot)$
If $\mu" < 0$ then $\sum_{i=1}^J g_i^{FI}(\cdot) < \sum_{i=1}^J g_i^{AL}(\cdot)$
If $-2\delta\mu'^2/(1-\delta\mu) < \mu"$ then $\sum_{i=1}^J g_i^{L}(\cdot) < \sum_{i=1}^J g_i^{FI}(\cdot)$
If $-2\delta\mu'^2/(1-\delta\mu) > \mu"$ then $\sum_{i=1}^J g_i^{FI}(\cdot) < \sum_{i=1}^J g_i^{L}(\cdot) < \sum_{i=1}^J g_i^{AL}(\cdot)$
If $-2\delta\mu'^2/(1-\delta\mu) = \mu"$ then $\sum_{i=1}^J g_i^{FI}(\cdot) = \sum_{i=1}^J g_i^{L}(\cdot) < \sum_{i=1}^J g_i^{AL}(\cdot)$.

Proof. See Appendix 3.8.4. □

The goal of this section is the following. Consider that the resource was not of common property and a given country i (the incumbent) was the only country exploiting the resource. Suppose now that the resource becomes a common property one. Therefore, in addition to the incumbent i, $J-1$ other countries start exploiting the resource. One would like to see how harmful this common property characteristic will be for investment within a *learning structure*, compared with *adaptive learning*.

We already know that total consumption is lower in *learning* than in *adaptive learning*, as proposition 1.3.1 states. Therefore, the threat of depletion seems to be lower in *learning* in comparison with *adaptive learning*. However, such an analysis could be incomplete or shortsighted. It would be interesting to know the relative effect of common property in *learning* and *adaptive learning*. Specifically, we should find the increasing rate of total consumption in *learning* and compare this rate to that of *adaptive learning*. The last part of proposition 1.7.2 tries to deal with this aspect of the problem.

Proposition 1.7.2. *Q^1_{FI}, Q^1_{AL} and Q^1_L denote the optimal consumption of the incumbent when the resource was not of common property, for full information, adaptation learning, and learning respectively. ξ_i is the belief of country i about θ. Therefore:*

(1) $\sum_{j=1}^{J} g_j^{FI}(\cdot) > Q^1_{FI}$

(2) $\sum_{j=1}^{J} g_j^{AL}(\cdot) > Q^1_{AL}$

(3) $\sum_{j=1}^{J} g_j^{L}(\cdot) > Q^1_{L}$

(4) $\frac{\sum_{j=1}^{J} g_j^{AL}(\cdot)}{Q^1_{AL}} < \frac{\sum_{j=1}^{J} g_j^{L}(\cdot)}{Q^1_{L}}$ *if there is homogeneity of beliefs.*

(5) In case of heterogeneity of beliefs,

$$\frac{\sum_{j=1}^{J} g_j^{AL}(.)}{Q^1_{AL}} < \frac{\sum_{j=1}^{J} g_j^{L}(.)}{Q^1_{L}} \quad if\ and\ only\ if$$

$$\underbrace{\frac{\sum_{j\neq i} \frac{\mathbb{E}_i(R(\mu(\theta)))}{\mathbb{E}_j(R(\mu(\theta)))}}{\sum_{j\neq i} \frac{R(\mathbb{E}_i(\mu(\theta)))}{R(\mathbb{E}_j(\mu(\theta)))}}}_{Heterogeneity\ incentive\ to\ invest} > \frac{\delta + \sum_{j=1}^{J} \frac{1}{\mathbb{E}_j(R(\mu(\theta)))}}{\delta + \sum_{j=1}^{J} \frac{1}{R(\mathbb{E}_j(\mu(\theta)))}}.$$

Proof. See Appendix 3.8.5. □

When the resource becomes of common property and the exploiters have the same belief, the increasing rate of total consumption is higher in *learning* than in *adaptive learning*. Such a finding is intuitive. Based on what we have found in proposition 1.3.1, anticipation makes (overall effect) people more willing to invest in comparison with *adaptive learning*. In contrast, common property is harmful to investment because each country fears that its investment might benefit the others. In other words, the effect of common property is more pronounced in *learning*.

However, in case of heterogeneity of beliefs, common property might be less pronounced in *learning*. It depends on the incumbent country's heterogeneity incentive to invest. If the incentive the incumbent gets from other countries is high, then the commons problem will be greater in *learning* than in *adaptive learning*. This is because if the heterogeneity incentive of the incumbent is high, the heterogeneity incentive it gives to the entrant exploiters is low. As a consequence, the resource has a high chance of being overexploited. In other words, the position of the incumbent is very important in tackling the commons problem.

Before ending this analysis on the commons problem, let us make a short comment on property rights, one of the solutions to the commons problem. We then try to see how property rights work in our setting. Instead of letting the countries compete for the resource, suppose that each country i is given a proportion α_i, $\alpha \in [0,1]$ of the resource. The total consumption in *learning* will be $g^{LPR} = \sum_{i=1}^{J} \frac{\alpha_i}{\delta \mathbb{E}_i(R(\mu(\theta)))+1} y$. g^{LPR} is smaller than $\frac{\sum_{i=1}^{J} \frac{1}{\mathbb{E}_i(R(\mu(\theta)))}}{\delta + \sum_{i=1}^{J} \frac{1}{\mathbb{E}_i(R(\mu(\theta)))}} y = \sum_{i=1}^{J} g_i^L$. Therefore, when the resource is initially shared such that each country has exploitation rights on a portion, the resource is consumed less. However, in general, the effect of common property is not completely eliminated with these property rights. The interesting result is that, if the resource is adequately shared, total consumption could be even smaller in multiple-agent than in single-agent setting. This will happen if most of the resource is allocated to the more optimistic exploiters, compared to the country exploiting the resource in the single-agent problem. Nonetheless, this situation can never occur if, in addition to the incumbent country, just one firm starts exploiting when there is common property.

1.8 Conclusion

Building on Koulovatianos, Mirman and Santugini (2009), we study a learning model of renewable resource exploitation. Because the resource renewal is subject to uncertainty, the countries formulate prior heterogeneous beliefs and learn about renewal. They can also anticipate learning. Even though the countries are willing to consume the resource in large quantities, they would also like to conserve it to prevent extinction.

We find that anticipation due to learning leads to a decrease in the total quantity consumed by all the countries, and then to an increase in the total investment. However, concerning individual consumption, some countries could increase their consumption when anticipating learning. Indeed, when anticipating learning, each exploiter faces two kinds of incentives to invest in the resource: self-incentive and heterogeneity incentive. The effect of self-incentive on investment is always positive, but the effect of the heterogeneity incentive may be negative. If the negative effect outweighs the positive effect, then the exploiter will increase its consumption when anticipating. However, in aggregate the positive effect, always overcompensates for the negative effect.

We also study the effect of a change in beliefs on resource exploitation. Specifically, we analyze how optimism or pessimism about renewal affects the consumption decision. We find that the result depends on the stock of the resource. If the stock is greater than the steady state, then an increase in the optimism of a given country leads to a decrease in this country's consumption and to an increase in the consumption of others. However, if the stock is lower than the steady state, the opposite result is true. We are also interested in the variation of this change in consumption due to the optimism of some countries. By analyzing *absolute change* and *relative change* in consumption, we observe that the exploiters seem to favor mitigated coordination of their decisions. Because each consumer trusts his personal belief while fearing to make irrational decision by being too optimistic, the players seem to converge to a common consumption decision.

While investigating the change in beliefs, we derive other interesting results about the impact of the exploiters' sense of risk, notably about the production shock. We observe that an increase in risk feeling has no effect on the consumption in *adaptive learning*. However, in *learning*, it decreases the consumption of the agent who feels the increase in risk, and increases the others' consumption.

One of the problems that arise in free resources exploitation is the commons problem. In our model, even if the resource is consumed less in *learning* than in *adaptive learning*, the commons problem may be more or less pronounced in *learning* than in *adaptive learning*. It depends on the incumbent exploiter' heterogeneity incentive to invest. If this heterogeneity incentive is high, then allowing common property will be more harmful for the anticipation effect on investment. In addition, when countries have the same belief, the heterogeneity incentive vanishes and the commons problem is greater in *learning* than in *adaptive learning*.

In this paper, we consider exploiters with infinite life spans and without exploitation cost. It might be interesting to introduce exploitation cost with some finite-time living exploiters. Such a framework could be targeted in future works.

Chapter 2

A Perpetual Search for Talent across Overlapping Generations

2.1 Introduction

The inspiring event of this paper is the famous and controversial speech made by French president against French research system in 2009. This speech was in line with some reforms French government was poised to launch. According to the president, French research system is not productive and some reforms are needed. Any reform should, for him, define strategies to evaluate accurately researchers and incite them to be more productive. The president would like French resources to be at the disposal of those who have the ability to put France among the top nations in research activity. However, here, what we are interested in is not only the speech itself or the underlying reforms. We are also and particularly interested in the debate and protests sparked by them. Indeed, in reply to the speech, many young researchers associations showed their indignation and denounced French government intention to make more precarious and critical the working conditions of young researchers.

In fact, it is generally admitted that young or early-career researchers face specific impediments compared to more senior researchers.[1] These impediments involve atypical forms of salaries, limitations in career opportunities abroad[2], and some difficulties to find secure financing or to enjoy social security programs. In view of this awkward conditions, young researchers would like any reform to be more favorable to them.

The debate in France, after the president's speech, became more interesting when

[1]See report by the American Academy of Arts and Sciences published in 2008 and entitled "Investing in Early-Career Scientists and High-Risk, High-Reward Research".

[2]According to a report by the European Parliament in 2009, career structure in many countries is based on seniority rather than competence. Such a situation decreases opportunities for young researchers to make a career abroad.

French media commentators started commenting the ongoing reforms. For some journalists, the productivity of a researcher would be decreasing in his age, and then it would be more useful to finance young researchers. Seniors should be more and more teaching oriented.[3] They have even argued by resorting to examples of researchers in the past (Albert Einstein, John Nash, Marie Curie[4]) who were productive when they were young.[5] This statement from French journalists sparked indignation from some senior academic researchers who reply by giving examples of researchers who did their most famous work when being of old-age (Louis Pasteur, Charles Darwin, Roentgen,...).[6] This "stormy" debate between journalists and senior researchers is conflicting and even seems to be subjective. Therefore, it would be interesting to conciliate them by proposing an objective criterion that might be broadly accepted.[7]

In addition to French case, some other facts in the world justify this paper and motivate it. Throughout academic world, researchers are given a grant not only by universities or governments but also by industries. Many young researchers are supported by firms that are, most often, interested in qualified young or early-career researchers. However, the economic downturn has put many company research budgets at risk, and today industries give grant to fewer young academics. As a consequence, in addition to the senior researchers, universities and governments should now be in charge of supporting more young researchers than before. Hence, since resources did not increase, we need an objective criterion to allocate research grants to young and senior researchers.

These days, there are many pleas for a funding system that would be more favorable to young researchers. An example is the report by the American Academy of Arts and Sciences published in 2008. This report, addressing federal agencies, universities and private foundations, strongly recommends a special attention to early-career faculty when allocating research grant.[8] However, the will to increase support of young scientists does not mean that support for more senior researchers is sufficient. It does not either mean

[3]On France Info broadcast channel a commentator (Sylvie Pierre-Brossolette, February 9th 2009) said:"It is true that the researchers do their best before 45 years old. Beyond this age they are less productive. It is biological and genetic".

[4]Albert Einstein was remarkably productive before 26 and published the general theory of relativity before 41. John Nash developed Nash equilibrium concept before 22. Marie-Curie won her second Nobel prize before 45.

[5]Such a thought is not carrying only in France. Indeed, in the report by the American Academy of Arts and Sciences referenced above, it is written:"Major creative breakthroughs in science and engineering can occur at all career stages, but many flow from the contributions of talented early-career researchers".

[6]Louis Pasteur developed rabies vaccine at 63, Roentgen discovered X-rays at 50. Darwin published his Theory of Evolution at 50,...

[7]Even if it is our wish, we cannot dare to think that what we propose in this paper will be accepted by everybody. It is a matter of humility that we need in science.

[8]In the report we could read:"We strongly believe that, regardless of overall federal research funding levels, America must invest in young scientists and transformative research in order to sustain its ability to compete in the new global environment".

that any kind of young researcher should be supported. Hence, the question is straightforward: how to make the decision of whom to give the grant, when facing researchers of different ages or seniorities?

This paper intends to fill a gap in economic literature. Browsing this literature, one could easily realize that economists take much interest in decision making system in industries or other institutions. Indeed, there are so many examples of studies that confirm our statement. Concerning firm financing, project selection, production and distribution strategies, we have at disposal different sorts of theoretical and empirical findings. We can observe the same thing concerning the decision about the types of workers to hire or to lay off in a firm. As a conclusion for these different problems, the overall common result is that the decision depends on the resources at disposal and the goal the managers of the firm plan to achieve.

However, economic literature is not so prolix about research activity itself (Che and Gale (2003)). The research on Research-Development has poor content. In other words, one might say: economic research takes large interest in all kind of economic or social activity but the research itself as an economic activity. With this paper, we would like to contribute to fill this gap. Precisely we will analyze a decision mechanism in a context where old (experienced) researchers and young researchers would like their research proposals to be funded by a financing institution.

What shall we say if we have to answer to the following question: between an old (experienced and more known) researcher and a young researcher, who should be supported? and under which condition? At first sight and considering a short term path, the question seems to be trivial because the old researcher should be well known and his probability to succeed in his project performing would be the highest because he is more experienced. Therefore, among two proposals with the same social utility, the old's one would be preferred. However, this kind of analysis might be wrong when the research fund (the financing institution) cares about future. Indeed, financing a research proposal allows the fund to test the ability of the researcher and to find more information about him. There is information learning process with experience. If at each period, the old researcher is selected, the fund is running a risk of facing (in the future) only the researchers whose type is badly known. This fact will deter its capacity to assess accurately some characteristics of the proposers. In addition, the more senior does not have much time left to live in research activity. As a consequence, there is no more chance to get benefit from him for long. Within such a context, the fund may prefer the young to the old since the former might be in charge of the future.

Some may think that our contribution remains less significant by arguing that the previous studies on mechanism design on research projects are so general to cover our scope. To dissuade this thought, we will present briefly some of more recent papers that are close to our framework and discuss, when need be, about what makes difference from ours.

First, it is worthy to tell about the procedure by which research proposals are selected and funded. In most of the cases, research activities in universities and research centers are supported by governments, firms and universities' own resources. The financing process is performed through research Foundations (simply called funds). Each fund is designed for a specific research field. Nevertheless, there are some funds being interested in various topics or fields. In the beginning, the fund calls for research proposals. In response to the call, each researcher or group of researchers works out one or more proposals and submits it to the fund. Finally, through an experts committee, the fund values each proposal and selects the best one according to some given criterions.

This financing procedure sets up the environment we will try to analyze. Specifically, let be aware that we will focus on the last step of the procedure. Therefore, here, the strategy of the submitter to make his project accepted is exogenous, unlike Caillaud and Tirole (2007), Trumbo (1989). Che and Gale (2003) have investigated a similar setup by constructing an optimal design of research contests. In their model, Che and Gale consider an agent (buyer) who wishes to procure an innovation. For this purpose, he invites risk-neutral firms to proposes an innovation through a contest by which he will choose the best innovation. One difference between Che and Gale (2003) and us is the certainty about the utility provided by the innovation. In their paper, once the best innovation is chosen and purchased, the buyer is certain to find the intended utility. In our case, even the best proposal does not guarantee the intended results. It depends on the probability of success of the proposer. Another difference, the most important, is that in Che and Gale what that matters is the characteristics of the proposals and not that of proposers, as it is the case in our work.

Branstetter and Sakakibara (2002) was interested in another aspect of the problem. Instead of dealing with the way to select and finance the best research proposal, they based on Katz (1986) to confirm (empirically) that research consortia are optimal to be supported under some conditions. In other words, governments should rather support research consortia than sole researchers.

Freeman (2005) has proceeded empirically to propose a mechanism to allocate fellowship for graduate research in case of budget constraint. Despite the merit of Freeman's work, the method is so empirical and the proposed mechanism would be valid under a particular kind of data.

As one could see, the general characteristic of the above referred papers is that they suffer from a lack of dynamic structure.[9] In their model, the decision maker does not care about the future, and the seniority of the scientist's characteristics do not matter.

In this work, we will present a dynamic simple model of research funding that will lead us to a decision making tool. We find that, to be optimal, the decision criterion should depend on both the gap between the past performance of the young and the senior, and the gap between their seniorities.

This paper is organized as follows. First, we develop a setting model that helps us to have an overview of the problem and get some intuitions. Secondly we present the model from which we find the decision rule. We end with some concluding remarks.

2.2 A simple setting model

In order to have an overview of the problem we intend to analyze, we start with a simple basic model. The result of this model will help us hereafter. Before introducing this model, it is useful to clarify the meaning of some concepts we use in this work. The concepts of "young" and "senior" are used in relative sense. Thus, a researcher is considered as being young when he is younger than his colleague with whom we compare him. The ages of the researchers are used for this comparison. Here, the word "age" does not necessary mean "length of the whole life" but refers to the seniority in research career. However, because in real life age (in sense of whole life duration) and seniority are highly correlated, we use interchangeably "young and early-career" and "old and senior".

Now, let us consider the following simple framework. At each period t two researchers apply for a grant allowance contest by submitting one proposal. A committee is charged of valuing the proposals and deciding whom to give the grant. One of the two researchers is "old" and the other is young. Each researcher can live just two periods of research career. After living two periods, the researcher quits research activity. The old had submitted a proposal the previous period when he was young. The young will submit next period when he will be old. At each period, the old could be of type H (high quality) with probability α or type L (low quality) with probability $1-\alpha$. The same thing is true for the young, with probabilities α_y and $1-\alpha_y$ respectively. These probabilities come from the assessment of the committee.[10] Conditional on the type $i \in \{H, L\}$, the probability of success is p_i. There is success when the selected researcher performs successfully his research and find the expected innovation. We assume α_y, $p_i \in (0, 1)$ and $p_L < p_H$.

At the current period t, one of the following three states of the nature occurs: N, S, F. The state N occurs at t when at $t-1$ the young researcher (or the current old) is

[9] In line with this literature, we can also reference Bar and Gordon(2009), Knight (2005), Levy (2007).

[10] We will discuss later the way to assess these probabilities. Interestingly, the decision made does not depend on them.

not selected. S occurs at t when at $t-1$ the young researcher (or the current old) is selected and succeeded. When he failed the state is F. Let us denote by α_e the value of α when the state is $e \in \{N, S, F\}$. Let θ_y be the probability of success of the young and θ_e the probability of success of the old in state e. We then have: $\theta_y = \alpha_y p_H + (1-\alpha_y)p_L$. Similarly, $\theta_e = \alpha_e p_H + (1-\alpha_e)p_L$.

It is well admitted that we should take risks if we want to "boost" research activity and get more profit. However, it is also well known that we need to reduce as much as possible these risks. Here, the fund should reduce the risk of failure, and the only way to do so is to base on the past performance of the researchers. Hence, the scientists who had failed most often in the past are considered to be of high-risk, and those who had experienced success are of low-risk. Therefore, at each period, the committee, leaning on what happened the previous period, updates the probability of success of the old researcher. A simple way is to use the bayesian updating rule. The committee updates θ_e by updating α_e, and we have:

$$\alpha_S = Prob(H/S) = \frac{\alpha_y p_H}{\alpha_y p_H + (1-\alpha_y)p_L}$$

and

$$\alpha_F = Prob(H/F) = \frac{\alpha_y(1-p_H)}{\alpha_y(1-p_H) + (1-\alpha_y)(1-p_L)p_L}$$

In case of success the utility is U^C and in case of failure the utility (or payoff) is U^F ($U^C > U^F$). Let us denote by U_t the random variable taking the value U^C or U^F at t. $\mathbb{G}$ represents the probability distribution of success. $\mathbb{G}$ refers to θ_y, θ_S and θ_F.

In this problem, we want to find the probability that the committee will give the grant to the young and the probability to select the old researcher, depending on the state of nature. At t and in state e we denote respectively by P_t^e and P_{t-1}^e the probabilities to choose the young and the old. Let $\mathbb{P}$ be the probability distribution that refers to P_t^e and P_{t-1}^e.

To be optimal, the decision rule should take into account not only what we gain today but also all that we gain in the future. The Bellman equation of our problem is:

$$V(e_t) = \max_{\mathbb{P}} E\{U_t + \delta V(e_{t+1})\} = \max_{\mathbb{P}} E_{\mathbb{P}}\{E_{\mathbb{G}}\{U_t + \delta V(e_{t+1})\}\} \tag{2.1}$$

e_t is the state of nature at time t.

Rewriting, we find:

$$V(N) = \max_{P_t^N, P_{t-1}^N} \{P_t^N[\theta_y[U^C + \delta V(S)] + (1-\theta_y)[U^F + \delta V(F)]] + P_{t-1}^N[\theta_y[U^C + \delta V(N)] + (1-\theta_y)[U^F + \delta V(N)]]\}.$$

$$V(S) = \max_{P_t^S, P_{t-1}^S} \{P_t^S[\theta_y[U^C + \delta V(S)] + (1-\theta_y)[U^F + \delta V(F)]] + P_{t-1}^S[\theta_S[U^C + \delta V(N)] + (1-\theta_S)[U^F + \delta V(N)]]\}.$$

$$V(F) = \max_{P_t^F, P_{t-1}^F} \{P_t^F[\theta_y[U^C + \delta V(S)] + (1-\theta_y)[U^F + \delta V(F)]] + P_{t-1}^F[\theta_F[U^C + \delta V(N)] + (1-\theta_F)[U^F + \delta V(N)]]\}.$$

As one could see, our problem is similar to bandit problem, precisely, acquiring two-armed bandit problem. Indeed, in our setting, the committee faces each period two proposers who are not perfectly known (the payoff from choosing a given proposer is random). After making the decision, he observes the consequences and learns about some characteristics of the chosen researcher. These information learned are used to make future decision. At the current period, the fund is willing to maximize the present value of its choice. In other words, the committee has to make the choice that maximizes the present value.

This kind of problem was introduced by Thompson (1933) within a context of clinical trials of two treatments. The goal of his experimentation is to find at the same time which treatment is efficient and to minimize the expected prevalence. Robbins (1952) reintroduced the problem from a non-Bayes point of view and suggested a search of minimax decision rule. Bradt et al. (1956) developed the bandit problem using Bayes decision rules.

The bandit problem becomes one of useful application when the celebrated theorem of Gittins and Jones (1974) (generalized in Varaiya et al. (1985)) established an important result: the k-armed bandit problem with independent arms and geometric discount can be solved by solving k one-armed bandit problems. Rothschild (1974) analyzed a problem of a single firm facing a market with unknown demand. The problem (clearly a one-armed bandit problem) is to find an optimal sequence of price to learn more about the true demand while maximizing its expected discounted profits. Keller and Rady (1999), and Rustichini and Wolinsky (1995) have developed an extension of Rothschild by considering the problem of a monopolistic facing an unknown demand subject to random changes over time. Weitzman (1979), and Roberts and Weitzman (1981) have analyzed the choice between various research projects as a bandit problem. Multi-armed bandit problem has also been applied in search and matching models (Jovanovic (1979)) and in Finance

(Bergemann and Hege (1998) and Bergemann and Hege (2005)).

Let be aware that the different settings above are slightly different from ours in that in our setting, at each period, we have necessary a bandit (proposer) who is never experienced. Even though our setting could be seen as an acquiring bandit problem, it is different from Whittle (1981). Anyway in the following, when need be, we will discuss about this literature on bandit problem.[11]

We can now come back to our initial problem. Without loss of generality, we can take $U^C = 1$ and $U^F = 0$.[12] Rewriting in the line $P_t^e + P_{t-1}^e = 1$[13], the problem becomes:

$$V(N) = \max_{P_t^N} \left\{ P_t^N \delta \left[\theta_y V(S) + (1-\theta_y) V(F) - V(N) \right] + \theta_y + \delta V(N) \right\}$$

$$V(S) = \max_{P_t^S} \left\{ P_t^S \left[\theta_y - \theta_S + \delta \left(\theta_y V(S) + (1-\theta_y) V(F) - V(N) \right) \right] + \theta_S + \delta V(N) \right\}$$

$$V(F) = \max_{P_t^F} \left\{ P_t^F \left[\theta_y - \theta_F + \delta \left(\theta_y V(S) + (1-\theta_y) V(F) - V(N) \right) \right] + \theta_F + \delta V(N) \right\}$$

Result: There is only one solution: $P_t^N = 1$, $P_t^S = 0$ and $P_t^F = 1$.[14]

This result is intuitive. It states that the young is chosen when both of them are never given a grant or when the old failed. However, if the more experienced researcher used successfully his previous grant, he should be selected once more. In other words, the young is selected in two states (N, F) over three states. The decision in the states S and F can be easily predicted. Indeed, in state S, the senior researcher has good reputation and has gained the confidence of the research foundation. His past proved that he is able to lead and perform successfully his research project. In contrast, in state F, the senior researcher has "bad" reputation and should not be supported by the fund.

Concerning the state N, both scientists have same probability to succeed. Therefore, if the fund leans just on their ability to succeed, it should be indifferent.[15] However, an optimal decision should be to support the younger because the senior has less time left to spend in research career and profit cannot be derived from him for long. The future of science and technology is in the hand of the younger who will be responsible of the research system in the future. Shortly, the fund's decision should be to bequeath the research leadership to the young.

[11]For the literature on bandit problem and the Gittins Index, we refer to Bergemann and Valimaki (2006), Tsitsiklis (1994), Whittle (1980).

[12]We just need to have $U^C > U^F$.

[13]The problem has nothing interesting if it is possible to choose all of the two researchers or none of them.

[14]The different steps of the solving are not presented here. The calculations are heavy and space consuming.

[15]It is easy to see that P_t^N takes any value in $[0, 1]$ when $\delta = 0$.

Let us mention that, according to the solution above, the decision is always expressed in a pure strategy form regardless the state of the nature.[16] It means that, in any case, the committee will always be clear about whom to give the grant. This is an advantage of this simple model. However, some problem could arise. Consider an environment in which usually the researchers succeed in their project. In such a situation, the decision shall consist in using a rotation scheme. That is, from one period to the next, the fund will switch from the young research to the more senior or vice versa. This situation could affect the researchers in their decision to submit, and then could lead to self-selection.[17] In conclusion, we cannot just base on success or failure of the researchers to make a decision. Nevertheless, the results above will be helpful hereafter.

2.3 The model

Let us consider a more generalized case of our setting model. Each researcher quits research activity after K years of research. For simplicity, individuals who are in their k^{th} year of activity are said to be k years old. We assume that, at each period, there is at least one researcher for each age. Furthermore, because one of the most important aspect of our framework is the researcher's age, we assume without loss of generality that their is just one researcher of each age.[18] In other words, for any $k \in [1, K]$, their is just one submitter who is k years old. Finally, there are K researchers at each period. Any researcher who submits at the current period will submit any next period unless he is post aged (K years old).[19] At each period, one new researcher (who is 1 year old) enters and submits.

Contrary to what we have seen in the setting model, instead of a Bernoulli setting, we assume that each project, after being performed, will have a certain measurable quality $Q \in]-\infty, +\infty[$. The quality of a project is the value of what we obtain after performing this project. In real world, this value could be a score according to the journal where it is published, or the contribution of the work to science advancements or to life condition improvement.

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[16]The probabilities are either 0 or 1.

[17]Likewise, in an environment where usually the researchers fail in their project, the fund should always choose the young.

[18]This assumption is just for simplifying the solving of the model. Even if there is more than one researcher of some age, the findings do not change. Assuming that there is one researcher for each age is equivalent to consider a representative researcher of each age.

[19]One could think that it is a strong assumption to suppose that the researcher who submits at the current period will submit the next period. However, if we consider a nation wide fund or a fund that centralizes all research submissions, this assumption is likely to hold. Moreover, the purpose is just for mathematical derivation. Relaxing this assumption does not affect the results. Indeed, even if a researcher has not submitted, we can consider that he has submitted and his project is of zero quality. When he is qualified for the grant, we replace him by the next researcher in the ranking.

As one could see, what that matters is not a success or failure but the quality of the performed project. We are still considering that there are two types of researchers, H and L.[20] Introducing the type of the submitter in the model has the advantage to take into account a possible heterogeneity within the researchers' population. For the types H and L, Q is normally distributed respectively $\mathcal{N}(1,1)$ and $\mathcal{N}(0,1)$.[21] We can also consider a more general distribution setting. For the types H and L, we then consider Q as following respectively the conditional distributions F_H and F_L (F_H and F_L stand for the *c.d.f* of the relevant distributions) with corresponding *p.d.f* f_H and f_L respectively. Indeed, the findings do not change provided that the following reasonable and realistic conditions are fulfilled: f_H and f_L meet the single crossing property as shown in Figure 2.1 below, and F_H dominates stochastically F_L. τ denotes the real number for which $f_H(\tau) = f_L(\tau)$. The assumption of single crossing property considers that the H-type researchers are more likely to procure the higher values of Q and the L-type researchers are more likely to procure the lower values of Q. This assumption is quite reasonable.

With the normal distribution the results are simple and easy to be interpreted. Therefore, in this paper we present both the normal case and the more general case.

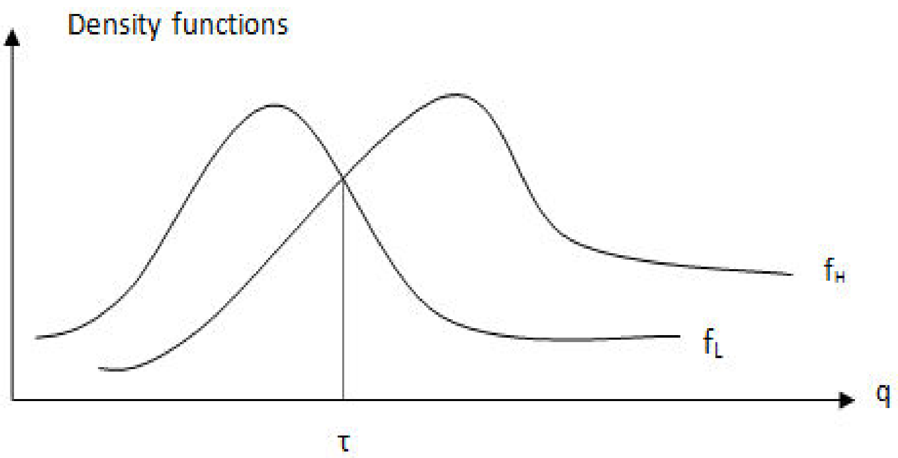


Figure 2.1: Single crossing property of density functions

Let us give the following definition.

Definition 2.3.1. ***History of a researcher, the state of the nature.***
For each researcher aged k, the history is a sequence $a_k = (a_{k1}, a_{k2}, ..., a_{kk-1})$ *that summarizes the quality of his previous performed projects.*

[20]One could consider more than two types.
[21]The findings are not affected if we assume that $Q \rightsquigarrow \mathfrak{N}(m_H, \sigma^2)$ for the type H and $Q \rightsquigarrow \mathfrak{N}(m_L, \sigma^2)$ for the type L, with $m_H > m_L$. The same remains true for a more general setting.

In definition 2.3.1 above, a_{ki} denotes the quality of the project performed by the current aged k researcher when he was i years old and was selected. When at a given period the researcher is not selected, we consider that the quality of his project at this period is not observed by the fund. In such a situation we set $a_{ki} \equiv N$. *The state of the nature* is the history of the submitting group (the group of the researchers who submit a project) and is represented by $A = (a_1, a_2, ..., a_K)$. It is obvious that $a_1 = N$. Sometimes, a_{Ri} may denote the quality of the project performed by the researcher named R when R was i years old and was selected for grant. As one could see, the state of the nature gives information on the age and the anterior achievements of all researchers who apply for financing. Hereafter, for a visibility purpose, we represent the $A = (a_1, a_2, ..., a_K)$ as follows:

$$A = \begin{cases} a_1 = N \\ a_2 = a_{21} \\ a_3 = a_{31}\, a_{32} \\ \vdots \\ a_k = a_{k1}\, a_{k2} \cdots a_{kk-1} \\ \vdots \\ a_K = a_{K1}\, a_{K2} \cdots a_{Kk-1} \cdots a_{KK-1} \end{cases}$$

For any k-aged researcher, we denote by $\theta^k_A(q)$ the probability that the quality of his project will be q, conditionally on the state of the nature A. $P_k(H|A)$ is the probability that the researcher is of type H given the state of the nature. Specifically, we have $P_k(H|A) = P_k(H|a_{k1}, a_{k2}, ..., a_{kk-1})$. Furthermore, let be aware that sometime we will use $\theta^i_A(q)$ to denote this probability for a researcher named i, regardless of his age. In such a case, the reader will be notified. We then have:

$$\theta^k_A(q) = (P_k(H|a_k) f_H(q) + (1 - P_k(H|a_k)) f_L(q)$$

For the new researcher (aged 1), because he is not known, we set his H-type probability at $\alpha \in (0, 1)$. In normal distributions, we find:

$$P_k(H|a_k) = \frac{\alpha \exp \sum_{i=1}^{k-1} (a_{ki} - \frac{1}{2})}{\alpha \exp \sum_{i=1}^{k-1} (a_{ki} - \frac{1}{2}) + 1 - \alpha}$$

and

$$\theta^k_A(q) = \frac{1}{\sqrt{2\pi}} (\exp - \frac{1}{2} q^2) \frac{\alpha \exp\{q - \frac{1}{2} + \sum_{i=1}^{k-1} (a_{ki} - \frac{1}{2})\} + 1 - \alpha}{\alpha \exp \sum_{i=1}^{k-1} (a_{ki} - \frac{1}{2}) + 1 - \alpha}$$

For the general case we easily find:

$$P_k(H|a_k) = \frac{\alpha\Pi_{i=1}^{k-1} f_H(a_{ki})}{\alpha\Pi_{i=1}^{k-1} f_H(a_{ki}) + (1-\alpha)\Pi_{i=1}^{k-1} f_L(a_{ki})}$$

and

$$\theta_A^k(q) = \frac{\alpha f_H(q)\Pi_{i=1}^{k-1} f_H(a_{ki}) + (1-\alpha) f_L(q)\Pi_{i=1}^{k-1} f_L(a_{ki})}{\alpha\Pi_{i=1}^{k-1} f_H(a_{ki}) + (1-\alpha)\Pi_{i=1}^{k-1} f_L(a_{ki})}$$

To make the problem tractable, when $a_{kt} = N$, we set $a_{kt} = \tau$. Let us remember that τ is the real number for which $P_k(H|a_{k1}, ..., a_{kt-1}) = P_k(H|a_{k1}, ..., a_{kt-1}, \tau)$.[22] As one could see, this artifice matches well our benchmark (the setting model). It assumes that when a researcher is not selected at a given period, his H-type probability the following period remains the same as at that period. When there is no new information on the researcher, it is reasonable to consider that his H-type probability remains the same as in previous period.

It is useful to highlight that $\theta_A^k(q)$, in addition to q, only depends on a_k. Therefore, sometimes, if the notation is simple, we will use $\theta_A^k(q)$ and $\theta_{a_k}(q)$ interchangeably. As a consequence, the quality probability of a researcher does not depend on the histories of the other researchers. This setting has nothing unrealistic for two reasons. First, we have no reason to think a priory that the ability of a researcher depends on the ability of the others. Second, even though there is such an interdependence, it should affect the histories of the submitter. Therefore, taking the histories into account might be sufficient to take into account this interdependence.

2.3.1 The Bellman equation of the problem

Let δ be the discount factor and A, the state of the nature. To choose the researcher to give the grant, the fund solves the following problem.

$$V(A) = \max_{i\in\{1,...,K\}} \{V_i(A)\} \tag{2.2}$$

$$where \qquad V_k(A) = \int_{-\infty}^{+\infty} (q + \delta V(A^{(k)}))\theta_A^k(q)dq \qquad \text{for all}\, k \in \{1, ..., K\}$$

$A^{(k)}$ is the state of the next period when the k-aged researcher is chosen in state A. Clearly, if

[22] In normal distributions $\tau = \frac{1}{2}$.

$$A = \begin{cases} a_1 = N \\ a_2 = a_{21} \\ a_3 = a_{31}\, a_{32} \\ \vdots \\ a_k = a_{k1}\, a_{k2} \cdots a_{kk-1} \\ \vdots \\ a_K = a_{K1}\, a_{K2} \cdots a_{Kk-1} \cdots a_{KK-1} \end{cases}$$

then

$$A^{(k)} = \begin{cases} a_1 = N \\ a_2 = N \\ a_3 = a_{21}\, N \\ a_4 = a_{31}\, a_{32}\, N \\ \vdots \\ a_k = a_{k-11}\, a_{k-12} \cdots a_{k-1k-2}\, N \\ a_{k+1} = a_{k1}\, a_{k2} \cdots a_{kk-2}\, a_{kk-1}\, q \\ a_{k+2} = a_{k+11}\, a_{k+12} \cdots a_{k+1k-2}\, a_{k+1k-1}\, a_{k+1k} N \\ \vdots \\ a_K = a_{K-11}\, a_{K-12} \cdots a_{K-1k-1} \cdots a_{K-1K-2}\, N \end{cases}$$

In general $A^{(i_1 i_2 \ldots i_t)}$ denotes the state arising at the t^{th} period subsequent to the period where the state is A. Moreover the notation shows that during the period between the states A and $A^{(i_1 i_2 \ldots i_t)}$ the researchers who are chosen are successively i_1, i_2,..., and i_t.

As one might see, $V(A)$ is the maximal gain the fund could get by making the best choice. It is the value of the submitting group. $V_k(A)$ is the gain from choosing the submitter who is k years old. We call it the value of the researcher. Sometimes $V_R(A)$ will denote the value of the researcher (named R) under the state A. For now, we consider that the fund wants to choose one researcher among the submitters. We study the case in which the fund is willing to choose more than one researcher in the next section.

Our problem may be seen as an acquiring arm bandit problem studied by Whittle (1981) in that at each period new arms appear and are available. Moreover, there is some reason to think that we are in case of restless bandit problem Whittle (1988) because the state of the researcher changes (he gets older) even if he is not selected.[23] However, there is a crucial difference between each of these two problems (arm acquiring bandit

[23]In the classical bandit problem, it is assumed that the state of the arms that are not selected at a given period remains the same the subsequent period. Only the selected arm could move from it previous state to another different state. In contrast, in restless bandit setting, the state of all arms could change. Restless bandit is known to be untractable and pspace hard.

and restless bandit) and ours. They consider that each arm is available forever, that is, lifetime is unlimited. This assumption cannot hold in our setting as it is well known that a researcher cannot live forever.

Recently, this assumption is relaxed by Chakrabarti et al. (2008) within the online advertising context. However, in their paper, it is assumed that the lifetime length of an ads (arm) is related to the number of times this arm is selected. Specifically, the ad dies immediately after it has been selected a given number of times. As one could see, we have no reason to think that researcher's life length is affected when he is chosen to get a grant. In other words, our problem is one that has not yet been solved early. Moreover, Chakrabarti et al did not really analyze the problem. They just provided an near algorithm to pull the ads, assuming that the reward from an arm is a binomial random variable.

2.3.2 The case of K=2

To get the insight of the model, it would be useful to deal with the case where $K = 2$. The problem is

$$V(A) = \max\{V_1(A), V_2(A)\} \qquad where \qquad A = \begin{cases} a_1 = N \\ a_2 = a_{21} \end{cases} \tag{2.3}$$

Here, a_{21} can take different values: $a_{21} = N$ when both researchers have never been selected for grant, or $a_{21} = r \in \mathbb{R}$ when the most experienced was chosen at the previous period and the quality of his performed project was r. In order to simplify the notation, if $a_{21} = N$ (resp r) we set $A = N$ (resp r). The problem becomes:

$$\begin{cases} V(r) = \max\left\{\int_{-\infty}^{+\infty}(q+\delta V(q))\theta_N(q)dq\ ;\ \int_{-\infty}^{+\infty}(q+\delta V(N))\theta_r(q)dq\right\} \\ \\ V(N) = \max\left\{\int_{-\infty}^{+\infty}(q+\delta V(q))\theta_N(q)dq\ ;\ \int_{-\infty}^{+\infty}(q+\delta V(N))\theta_N(q)dq\right\} \end{cases}$$

According to our setting model, the senior researcher is not selected unless he has attained a certain performance (success) in the past. Therefore, leaning on this result, intuitively, we can conjecture that $V(N) = V_1(N)$ and for state $A = r$, there is a threshold of history below which the experienced researcher cannot be chosen. This conjecture is confirmed in the following proposition.

Proposition 2.3.1. *In case of two researchers who live just two periods, the decision of the fund is as follows:*

1. *If both researchers have never been selected then the young researcher is chosen.*

2. If the old was chosen the previous period with history r then it exists r^* such that:

- *if $r > r^*$ then the experienced researcher is selected*
- *if $r < r^*$ then the young researcher is selected*
- *if $r = r^*$ then the fund is indifferent*
- *$r^* > \tau$.*

Proof. See appendix 3.9.1 □

It is easy to check that r^* does not depend on α, even in general case. Hence, in normal distribution we can find that r^* is such that:

$$\frac{F(r^*-1) - F(r^*)exp(r^* - \frac{1}{2})}{1 - (r^* - \frac{1}{2})} = \frac{1+\delta}{\delta}. \tag{2.4}$$

where F is the cumulative density function of the standard normal distribution. In general distributions case, we have:

$$\frac{F_H(r^*)f_L(r^*) - F_L(r^*)f_H(r^*)}{f_L(r^*) - f_H(r^*)} = \frac{1+\delta}{\delta}. \tag{2.5}$$

It would be useful to notify that this finding in proposition 2.3.1 is similar to what that is known generally from classical bandit problem.

2.3.3 The case of $K \geq 2$

Definition 2.3.2. ***Equivalence of histories***

Let us consider two researchers R_1 and R_2 respectively aged k and m. Their histories are respectively $a_k = (a_{k1}, a_{k2}, ..., a_{kk-1})$ and $a_m = (a_{m1}, a_{m2}, ..., a_{mm-1})$.

The histories of the two researchers are said to be equivalent if $\frac{\sum_{i=1}^{k-1} a_{ki} - \sum_{i=1}^{m-1} a_{mi}}{k-m} = \frac{1}{2}$.[24]

History of R_1 is said to be better than History of R_2 if $\frac{\sum_{i=1}^{k-1} a_{ki} - \sum_{i=1}^{m-1} a_{mi}}{k-m} > \frac{1}{2}$.

[24] This definition is for normal distribution setting. In general case (i.e any probability distribution), the histories are equivalent if

$$\frac{\prod_{i=1}^{k-1} f_H(a_{ki})}{\prod_{i=1}^{k-1} f_L(a_{ki})} = \frac{\prod_{i=1}^{m-1} f_H(a_{mi})}{\prod_{i=1}^{m-1} f_L(a_{mi})}$$

and the history of R_1 is said to be better than the history of R_2 if

$$\frac{\prod_{i=1}^{k-1} f_H(a_{ki})}{\prod_{i=1}^{k-1} f_L(a_{ki})} > \frac{\prod_{i=1}^{m-1} f_H(a_{mi})}{\prod_{i=1}^{m-1} f_L(a_{mi})}$$

State A is said to be better than state B (which is written $A > B$)if for all researchers the history in A is equivalent or better in A than in B, and there is at least one researcher whose history is better in A than in B.

Using graph in Figure 2.2, we can say that the history of R_1 is equivalent (respectively better) to the history of R_2 if the slope of the straight line Δ is equal to $\frac{1}{2}$ (respectively greater than $\frac{1}{2}$).

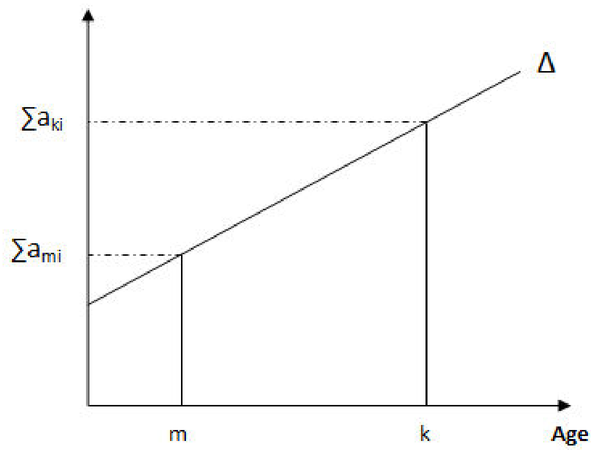


Figure 2.2: Histories comparison

Remark 2.3.1.
- *If two researchers A and B have equivalent histories under state H then $\theta_H^A(q) = \theta_H^B(q)$ for all $q \in \mathbb{R}$.*
- *If the history of A is better than B's history then the distribution corresponding to $\theta_H^B(q)$ is stochastically dominated by the distribution corresponding to $\theta_H^A(q)$.*

In other words, two researchers have equivalent histories if their past makes them to have same quality distribution. The condition to come to such an equivalence is the following: In proportion to their ages, the values of what each of them has done in the past should be equal.

Proposition 2.3.2. *Two researchers of same age and equivalent history have same value. The best of two researchers of same age is the one of better history.*

Before proving rigorously the proposition 2.3.2, it would be useful to give here the intuition. Indeed, in this model, the value of a researcher depends on his age and his past performance (histories). Hence, if two researchers have same age and their histories unveil same information on their qualities, then they should have same value.

Proof. Because in our framework, what that characterizes a researcher is just his age and his history, we can consider two researchers of same age and different histories as one researcher (named R) observed in different histories, the history of the others remaining the same.
Let j be the age of the researcher, A and B are two different states in which the histories of j are equivalent.

Case 1: j=K[25]

$$V_R(A) = \int_{-\infty}^{+\infty} \left(q + \delta V(A^{(R)})\right) \theta_A^R(q) dq$$
$$V_R(B) = \int_{-\infty}^{+\infty} \left(q + \delta V(B^{(B)})\right) \theta_B^R(q) dq$$

From 2.3.1, $\theta_A^R(q) = \theta_B^R(q)$ for all q. Moreover, $A^{(R)} = B^{(R)}$ for all q (The reason is that R, the only who causes the difference between A and B, will not be present next period). Therefore,

$$V_R(A) = V_R(B)$$

Case 2: j=K-1

$$V_R(A) = \int_{-\infty}^{+\infty} \left(q + \delta V(A^{(R)})\right) \theta_A^R(q)$$
$$V_R(B) = \int_{-\infty}^{+\infty} \left(q + \delta V(B^{(R)})\right) \theta_B^R(q)$$

that is

$$V_R(A) = \int_{-\infty}^{+\infty} \left[q + \delta \max\left\{V_1(A^{(R)}), V_2(A^{(R)}), ..., V_K(A^{(R)})\right\}\right] \theta_A^R(q) dq$$
$$V_R(B) = \int_{-\infty}^{+\infty} \left[q + \delta \max\left\{V_1(B^{(R)}), V_2(B^{(R)}), ..., V_K(B^{(R)})\right\}\right] \theta_B^R(q) dq$$

[25]The reader can remark that the proof is made using an approach similar to backward induction. We start by the last period of research life and move up to the earlier period.

or

$$V_R(A) = \int_{-\infty}^{+\infty} \left[q + \delta \max \left\{ \int_{-\infty}^{+\infty} \left(q + \delta V(A^{(R.1)}) \right) \theta^1_{A^{(R)}}(q)dq; \int_{-\infty}^{+\infty} \left(q + \delta V(A^{(R.2)}) \right) \theta^2_{A^{(R)}}(q)dq; \right. \right.$$

$$\left. \left. ...; V_K(A^{(R)}) \right\} \right] \theta^R_A(q)dq$$

$$V_R(B) = \int_{-\infty}^{+\infty} \left[q + \delta \max \left\{ \int_{-\infty}^{+\infty} \left(q + \delta V(B^{(R.1)}) \right) \theta^1_{B^{(R)}}(q)dq; \int_{-\infty}^{+\infty} \left(q + \delta V(B^{(R.2)}) \right) \theta^2_{B^{(R)}}(q)dq; \right. \right.$$

$$\left. \left. ...; V_K(B^{(R)}) \right\} \right] \theta^R_B(q)dq$$

Obviously, we have $A^{(R.k)} = B^{(R.k)}$ (due to the reason given for the case of $j = K$) and $V_K(A^{(R)}) = V_K(B^{(R)})$ (from the case of $j = K$) for all $k \in \{1, ..., K\}$. Moreover, $\theta^k_{A^{(R)}} = \theta^k_{B^{(R)}}$ for all $k \in \{1, ..., K-1\}$. The reason is that the researchers aged $k \in \{1, ..., K-1\}$ have same history from A to B.

Case 3: $j \in \{1, ..., K\}$
Leaning on what above, for any j=1,...,K we can write $V_R(A)$ and $V_R(B)$ sequentially until the period when R will be post aged and quite. Therefore, we have

$$V_R(A) = \int_{-\infty}^{+\infty} \left[q + \delta \max \left\{ \int_{-\infty}^{+\infty} \left[q + \delta \max \left\{ ..., \int_{-\infty}^{+\infty} \left(q + \delta V(A^{(...1)}) \right) \theta^1_{A^{(...)}}(q)dq, ..., \right\}, \right. \right. \right.$$

$$\left. \left. \left. ..., V_K(A^{(R..)}) \right\}, V_K(A^{(R)}) \right\} \right] \theta^R_A(q)dq$$

$$V_R(B) = \int_{-\infty}^{+\infty} \left[q + \delta \max \left\{ \int_{-\infty}^{+\infty} \left[q + \delta \max \left\{ ..., \int_{-\infty}^{+\infty} \left(q + \delta V(B^{(...1)}) \right) \theta^1_{B^{(...)}}(q)dq, ..., \right\}, \right. \right. \right.$$

$$\left. \left. \left. ..., V_K(B^{(R..)}) \right\}, V_K(B^{(R)}) \right\} \right] \theta^R_B(q)dq$$

For any q and any $k \in \{1, ..., K-2\}$, we have $A^{(...k)} = B^{(...k)}$ and $\theta^k_{A^{(...)}}(q) = \theta^k_{B^{(...)}}(q)$. In addition to what we have seen in cases 1 and 2, we conclude that $V_R(A) = V_R(B)$ □

Remark 2.3.2. *From the proof of proposition 2.3.2, we have $V(A) = V(B)$. Therefore, the value of a submitting group does not change if a researcher is replaced by another researcher of same age and equivalent history. As a consequence, the value of the submitting group is not affected by a permutation on the history of a researcher.*

Proposition 2.3.3. *Let A and B be two different states. If $B > A$ then $V(B) \geq V(A)$. We say that the value of the submitting group is nondecreasing function of state.*

Proof. Let A and B be two different states. The difference between A and B is that there is one researcher whose history is better in B than in A, the history of each of the $K-1$ others remaining the same from A to B. The proof ends if we prove that $V(A) \leq V(B)$, without loss of generality.

As done for proposition 2.3.2, we conduct the proof sequentially. Let R denote the researcher whose history is better in B than in A. $\overline{R}$ denotes the others.

Case 1: R is K years old

$$V_R(A) = \int_{-\infty}^{+\infty} \left(q + \delta V(A^{(R)})\right) \theta_A^R(q)dq$$
$$V_R(B) = \int_{-\infty}^{+\infty} \left(q + \delta V(B^{(R)})\right) \theta_B^R(q)dq$$

In equations above, $V(A^{(R)})$ and $V(B^{(R)})$ do not depend on q because R will not be present next period. Furthermore, from remark 2.3.1, the distribution $\theta_A^R(q)$ is first-order stochastically dominated by the distribution $\theta_B^R(q)$. And then $V_R(A) < V_R(B)$.
Moreover,

$$V_{\overline{R}}(A) = \int_{-\infty}^{+\infty} \left(q + \delta V(A^{(\overline{R})})\right) \theta_A^{\overline{R}}(q)dq$$
$$V_{\overline{R}}(B) = \int_{-\infty}^{+\infty} \left(q + \delta V(B^{(\overline{R})})\right) \theta_B^{\overline{R}}(q)dq$$

Due to some known reasons, $V_{\overline{R}}(A) = V_{\overline{R}}(B)$.

Case 2: R is K-1 years old

$$V_R(A) = \int_{-\infty}^{+\infty} \left(q + \delta V(A^{(R)})\right) \theta_A^R(q)dq$$
$$V_R(B) = \int_{-\infty}^{+\infty} \left(q + \delta V(B^{(R)})\right) \theta_B^R(q)dq$$

that is

$$V_R(A) = \int_{-\infty}^{+\infty} \left[q + \delta \max\left\{V_1(A^{(R)}), V_2(A^{(R)}), ..., V_K(A^{(R)})\right\}\right] \theta_A^R(q)dq$$
$$V_R(B) = \int_{-\infty}^{+\infty} \left[q + \delta \max\left\{V_1(B^{(R)}), V_2(B^{(R)}), ..., V_K(B^{(R)})\right\}\right] \theta_B^R(q)dq$$

or

$$V_R(A) = \int_{-\infty}^{+\infty} \left[q + \delta \max \left\{ \int_{-\infty}^{+\infty} \left(q + \delta V(A^{(R.1)}) \right) \theta^1_{A^{(R)}}(q)dq; \int_{-\infty}^{+\infty} \left(q + \delta V(A^{(R.2)}) \right) \theta^2_{A^{(R)}}(q)dq; \right. \right.$$

$$\left. \left. ...; \int_{-\infty}^{+\infty} \left(q + \delta V(A^{(R.K)}) \right) \theta^K_{A^{(R)}}(q)dq \right\} \right] \theta^R_A(q)dq$$

$$V_R(B) = \int_{-\infty}^{+\infty} \left[q + \delta \max \left\{ \int_{-\infty}^{+\infty} \left(q + \delta V(B^{(R.1)}) \right) \theta^1_{B^{(R)}}(q)dq; \int_{-\infty}^{+\infty} \left(q + \delta V(B^{(R.2)}) \right) \theta^2_{B^{(R)}}(q)dq; \right. \right.$$

$$\left. \left. ...; \int_{-\infty}^{+\infty} \left(q + \delta V(B^{(R.K)}) \right) \theta^K_{B^{(R)}}(q)dq \right\} \right] \theta^R_B(q)dq$$

If eventually $V_R(A) \neq V_R(B)$ then the difference could be due just to the difference between $\theta^K_{A^{(R)}}(q)$ and $\theta^K_{B^{(R)}}(q)$ and/or between $\theta^R_A(q)$ and $\theta^R_B(q)$. In addition, $V(A^{(R.k)})$ and $V(B^{(R.k)})$ do not depend on q (for same reason as in case 1). Therefore, basing on the fact that $\theta^K_{A^{(R)}}(q)$ is first-order stochastically dominated by $\theta^K_{B^{(R)}}(q)$, we conclude that $V_R(A) \leq V_R(B)$.
Similarly,

$$V_{\overline{R}}(A) = \int_{-\infty}^{+\infty} \left[q + \delta \max \left\{ \int_{-\infty}^{+\infty} \left(q + \delta V(A^{(\overline{R}.1)}) \right) \theta^1_{A^{(\overline{R})}}(q)dq; \int_{-\infty}^{+\infty} \left(q + \delta V(A^{(\overline{R}.2)}) \right) \theta^2_{A^{(\overline{R})}}(q)dq; \right. \right.$$

$$\left. \left. ...; \int_{-\infty}^{+\infty} \left(q + \delta V(A^{(\overline{R}.K)}) \right) \theta^K_{A^{(\overline{R})}}(q)dq \right\} \right] \theta^{\overline{R}}_A(q)dq$$

$$V_{\overline{R}}(B) = \int_{-\infty}^{+\infty} \left[q + \delta \max \left\{ \int_{-\infty}^{+\infty} \left(q + \delta V(B^{(\overline{R}.1)}) \right) \theta^1_{B^{(\overline{R})}}(q)dq; \int_{-\infty}^{+\infty} \left(q + \delta V(B^{(\overline{R}.2)}) \right) \theta^2_{B^{(\overline{R})}}(q)dq; \right. \right.$$

$$\left. \left. ...; \int_{-\infty}^{+\infty} \left(q + \delta V(B^{(\overline{R}.K)}) \right) \theta^K_{B^{(\overline{R})}}(q)dq \right\} \right] \theta^{\overline{R}}_B(q)dq$$

Reasoning as above, we find that $V_{\overline{R}}(A) \leq V_{\overline{R}}(B)$

Case 3: $j \in \{1, ..., K\}$

To come to the result, it is easy to write sequentially the value of each researcher up to the period when R quites. The expression of these values are heavy to write down but easy to conceive in mind.
As conclusion $V(B) \geq V(A)$ □

Remark 2.3.3. *1. From the proof of proposition 2.3.3, we find that* $V_i(B) \geq V_i(A)$ *for* *any researcher i.*

2. *If R is the oldest researcher then $V_i(B) = V_i(A)$ for any researcher $i \neq R$.*

3. *If R is not the oldest researcher then $V_i(B) > V_i(A)$ or $V_i(B) = V_i(A)$ for any researcher i, depending on certain conditions. We will discuss these conditions later.*

According to proposition 2.3.3 and remark 2.3.3, the fund will prefer only the researchers of better histories to submit. Each time the history of a researcher improves, the value of the whole submitting group may increase, even if this researcher is not so good to be chosen. The reason is that when the history improves, the value of the corresponding researcher increases. Therefore, there is quality improvement of at least one of the scientists who are potentially the ones to be chosen in the future. For the same reason, the value of each submitter increases when some of their colleagues improve their histories.

It is interesting to be more precise on the third point of remark 2.3.3. We are going to analyze how the characteristics (history) of a researcher could affect the value of another researcher. To see it, let us consider the case where $K = 3$ and try to study how the researcher aged 2 can affect the value of the researcher 3. We have

$$V_3(A) = \int_{-\infty}^{+\infty} q\theta_A^3(q)dq + \delta \max\left\{V_1(A^{(3)}), V_2(A^{(3)}), V_3(A^{(3)})\right\}$$

$$V_3(A) = \int_{-\infty}^{+\infty} q\theta_A^3(q)dq + \delta \max\left\{ \int_{-\infty}^{+\infty} q\theta_A^1(q)dq + \delta \int_{-\infty}^{+\infty} V(A^{(3.1)})\theta_A^1(q)dq; \right.$$

$$\left. \int_{-\infty}^{+\infty} q\theta_A^1(q)dq + \delta \int_{-\infty}^{+\infty} V(A^{(3.2)})\theta_A^1(q)dq; \int_{-\infty}^{+\infty} q\theta_A^2(q)dq + \delta \int_{-\infty}^{+\infty} V(A^{(3.3)})\theta_A^2(q)dq \right\}$$

We can prove that

$$V_3(A) = \int_{-\infty}^{+\infty} q\theta_A^3(q)dq + \delta \max\left\{ \int_{-\infty}^{+\infty} q\theta_A^1(q)dq + \delta \int_{-\infty}^{+\infty} V(A^{(3.1)})\theta_A^1(q)dq; \right.$$

$$\left. \int_{-\infty}^{+\infty} q\theta_A^2(q)dq + \delta \int_{-\infty}^{+\infty} V(A^{(3.3)})\theta_A^2(q)dq \right\}$$

In state A, the history of the 2-aged researcher will affect $V_3(A)$ if

$$\max\left\{ \int_{-\infty}^{+\infty} q\theta_A^1(q)dq + \delta \int_{-\infty}^{+\infty} V(A^{(3.1)})\theta_A^1(q)dq; \int_{-\infty}^{+\infty} q\theta_A^2(q)dq + \delta \int_{-\infty}^{+\infty} V(A^{(3.3)})\theta_A^2(q)dq \right\}$$

$$= \int_{-\infty}^{+\infty} q\theta_A^2(q)dq + \delta \int_{-\infty}^{+\infty} V(A^{(3.3)})\theta_A^2(q)dq$$

It means that $V_3(A)$ is affected by the 2-aged researcher if the fund, when evaluating the 3-aged researcher in state A, expects that the 2-aged researcher will be chosen in the future. The fund has this hope if the 2-aged researcher has a good history in such a way to be likely to be chosen in the future. In other words, the researchers cannot affect the value of their colleagues, only if they have a certain level of history.

We can speculate on the form of the value function of a researcher. To do so, let us use the following lemma.

Lemma 2.3.1. *Let $P_k \equiv \int_{-\infty}^{+\infty} q\theta_A^k(q)dq$. P_k is what the fund is expecting to gain from the k-aged researcher in an immediate future. For any k-aged researcher with $k < K$, $V_k(A) = \Sigma_{i=1}^{K-1}\lambda_i(.)P_i$ and $V_K(A) = \Sigma_{i=1}^{K}\lambda_i(.)P_i$, where the $\lambda_i(.)$ are function of the histories of the researchers.*

Proof. See appendix 3.9.2 □

From this form of the researchers'value, with a close look at its illustration based on $K = 3$, we can see that the $\alpha_i' s$ are also a function of the $P_k' s$, and then, the value of a researcher is a function of the $P_k' s$. This finding will be very helpful hereafter. We can also learn that a researcher affects more his own value than he does for the other researchers because he affects this value in both immediate and distant future.

Still targeting our main question, we would like to deal with the following intuition. When facing two researchers whose histories are equivalent, one could conjecture that the fund should support the younger because the later has more time to live. This intuition is confirmed by the next proposition that we reach to, using the following lemmas.

Lemma 2.3.2. *Let us consider two different states A and B. The k-aged researcher's history in A is $(a_{k1}, a_{k2}, ..., a_{kl}, N, a_{kl+2}, ..., a_{kk-1})$ and $(a_{k1}, a_{k2}, ..., a_{kl}, b, a_{kl+2}, ..., a_{kk-1})$ in B where $b \in \mathbb{R}$. The other researchers have the same history in A and in B. It exists a unique value $\widetilde{b}$ of b such that $V(A) = V(B)$. Moreover, $\widetilde{b} = \tau$.*[26]

Proof. To prove that with $b = \widetilde{b}$, $V(A) = V(B)$, the approach is close to what we did in the proof of proposition 1 and 2. We just need to use the backward induction procedure. Thus, it is easy to prove that the lemma holds for $k = K$. Secondly, we can prove that if the lemma holds for $k = l$ then it holds for $k = l-1$. We should recall (from the definition of τ above) that if $b = \widetilde{b}$ then $\theta_A^k(q) = \theta_B^k(q)$ for all q.

The uniqueness of $\widetilde{b}$ comes from proposition 2.3.2 that states that $V(.)$ is nondecreasing function of state. □

[26]In normal distributions case $\widetilde{b} = \frac{1}{2}$.

Lemma 2.3.3. [27]

$$\textit{Let} \quad A(q) = \begin{cases} a_1 = N \\ a_2 = a_{21} \\ a_3 = a_{31}\, a_{32} \\ \vdots \\ a_k = a_{k1}\, a_{k2} \cdots a_{kk-2}\, q \\ \vdots \\ a_K = a_{K1}\, a_{K2} \cdots a_{Kk-1} \cdots a_{KK-1} \end{cases}$$

Then

$$\int_{-\infty}^{+\infty} V(A(q))\theta^k(q)dq \geq V(A(\tau)).$$

where

$$\theta^k(q) = \frac{\alpha f_H(q)\Pi_{i=1}^{k-2} f_H(a_{ki}) + (1-\alpha) f_L(q)\Pi_{i=1}^{k-2} f_L(a_{ki})}{\alpha \Pi_{i=1}^{k-2} f_H(a_{ki}) + (1-\alpha)\Pi_{i=1}^{k-2} f_L(a_{ki})}.$$

Proof. See appendix 3.9.3 □

Corollary 2.3.1. *A straightforward corollary of the preceding lemma is that it is better for the fund to finance at least one researcher than not to finance any researcher.*

To come to the result above in corollary 2.3.1, the approach is simple. The trick consists in replacing recent history of each researcher (the $a_{mi}s$ at the end of each line in $A(q)$) by N. We then find a state that would arise if in previous period, the fund did not finance any researcher. Hence, the corollary gets trivial.

Remark 2.3.4. *Result in lemma 2.3.3 remains valid if*

$$A(q) = \begin{cases} a_1 = N \\ a_2 = a_{21} \\ a_3 = a_{31}\, a_{32} \\ \vdots \\ a_k = a_{k1}\, a_{k2} \cdots a_{kk-3}\ q\ N \\ \vdots \\ a_K = a_{K1}\, a_{K2} \cdots a_{Kk-1} \cdots a_{KK-1} \end{cases}$$

[27] In normal distributions, $\theta^k(q) = \frac{1}{\sqrt{2\pi}}(\exp -\frac{1}{2}q^2)\frac{\alpha \exp\{q - \frac{1}{2} + \sum_{i=1}^{k-2}(a_{ki} - \frac{1}{2})\} + 1 - \alpha}{\alpha \exp \sum_{i=1}^{k-2}(a_{ki} - \frac{1}{2}) + 1 - \alpha}$ and $\tau = \frac{1}{2}$.

with

$$\theta^k(q) = \frac{\alpha f_H(q)\Pi_{i=1}^{k-2} f_H(a_{ki}) + (1-\alpha) f_L(q)\Pi_{i=1}^{k-2} f_L(a_{ki})}{\alpha\Pi_{i=1}^{k-2} f_H(a_{ki}) + (1-\alpha)\Pi_{i=1}^{k-2} f_L(a_{ki})}.$$

Proposition 2.3.4. *Let us consider two researchers who are respectively k and m years old with $k > m$. If they have equivalent histories then the younger researcher should be chosen.*

Proof. See appendix 3.9.4 □

Proposition 2.3.4 and 2.3.5 are intuitive. The value of a researcher is decreasing in his age. Therefore, the fund is more exacting concerning the experience of senior researcher. To be chosen, senior researcher needs to have better history than his young colleagues do. Proposition 2.3.5 gives insight to select the best researcher, basing on histories and ages. Researchers are compared two by two and the best is chosen. r^* is the benchmark. It does not depend on α (high quality probability for the youngest researcher) when $K = 2$. This result is quite interesting and we would like it to be valid for any K since, for practical purposes, the choice of α could need some (time and resources consuming) investigations.

Besides, r^* depends on the discount δ. As shown in Figure 2.4 and proven in proposition 2.3.6 r^* is an increasing function of δ. The higher is δ the more restricted is the condition for the senior to be supported. It means that when the fund cares about future, it should be favorable to younger researchers. In contrast, if today is more important for, the fund should make things easier for senior researchers. In other words, if the world or the country is facing a problem that is urgent to be solved (new epidemic, urgent need of army,...), the decision-maker should be favorable to seniors or at least to talented seniors. If the goal is to find solutions for problems that have not yet arisen, or to improve standard of living, the fund should be more favorable to talented younger researchers.

We can also learn from these results that, after being graduated, the chance of a young researcher to have his first grant is shrinking as time goes up. As a consequence, if in a given state, the youngest is chosen then none of the other researchers will be selected in the rest of their life.[28]

Proposition 2.3.5. *The state of the nature is A.*

Consider two researchers k and m $(k > m)$, with histories respectively (a_k) and (a_m). If quality distributions are normal then it exists a real r^* (constant) such that:

- *If* $\frac{\sum_{i=1}^{k-1} a_{ki} - \sum_{i=1}^{m-1} a_{mi}}{k-m} > r^*$ *then* $V_k(A) > V_m(A)$

[28]Such a situation could not convenient to in real world. Therefore, we give later a way to proceed in practise.

- *If* $\frac{\sum_{i=1}^{k-1} a_{ki} - \sum_{i=1}^{m-1} a_{mi}}{k-m} < r^*$ *then* $V_k(A) < V_m(A)$
- *If* $\frac{\sum_{i=1}^{k-1} a_{ki} - \sum_{i=1}^{m-1} a_{mi}}{k-m} = r^*$ *then* $V_k(A) = V_m(A)$
- $r^* > \frac{1}{2}$.

In general case (any probability distribution) proposition 2.3.5 recommends to compare r^* with

$$\frac{\prod_{i=1}^{k-1} f_H(a_{ki})}{\prod_{i=1}^{k-1} f_L(a_{ki})} - \frac{\prod_{i=1}^{m-1} f_H(a_{mi})}{\prod_{i=1}^{m-1} f_L(a_{mi})}$$

In this case we rather have $r^* > 0$

Proof. With proposition 2.3.4 and its corollary we have seen that there is some sufficient condition under which $V_k(A) < V_m(A)$.

1- We need to find some condition under which $V_k(A) > V_m(A)$. The condition we are looking for is on $\Delta = \frac{\sum_{i=1}^{k-1} a_{ki} - \sum_{i=1}^{m-1} a_{mi}}{k-m}$ and not on the values of $\sum_{i=1}^{k-1} a_{ki}$ and $\sum_{i=1}^{m-1} a_{mi}$ per se. The intuition here is that if Δ is sufficiently high then $V_k(A)$ will be greater than $V_m(A)$. It is exactly what we come to when we consider a situation in which $\sum_{i=1}^{k-1} a_{ki} \to +\infty$ and $\sum_{i=1}^{m-1} a_{mi} \to -\infty$.

2- If $V(.)$ is a smooth function of the elements of the state then $V_k(A) - V_m(A)$ is a smooth function of Δ. The smoothness of $V(.)$ can be seen with lemma 2.3.1. Therefore, it exists $r^* > \frac{1}{2}$ such that $V_k(A) - V_m(A) = 0$ for $\Delta = r^*$. It is also straightforward that $V_k(A) - V_m(A) < 0$ for $\Delta < r^*$ and $V_k(A) - V_m(A) > 0$ for $\Delta > r^*$.

3- By proposition 2.3.4, r^* is greater than $\frac{1}{2}$. □

Corollary 2.3.2. *(of proposition 2.3.4) Consider two researchers k and m with $k > m$. If the history of m is better than the history of k then m is chosen.*

Proposition 2.3.4 and its corollary give us a pre-selection criterion. For illustration, let us consider Figure 2.3 that is, in fact, a reproduction of Figure 2.2. We can compare two by two the researchers by examining the slope of the segments connecting the top of the vertical lines (of length $\sum_i$). The slope is compared to 0.5. One could see that some researchers should be eliminated. For instance, the researcher 3 is eliminated by the researcher 2, 4 is eliminated by 3 and 5 is dominated by 2.

Even though we know that r^* exists, we do not know its value or functional form. However, we can learn more if we know how r^* varies with certain parameters of the model. Such an information is given by the following proposition.

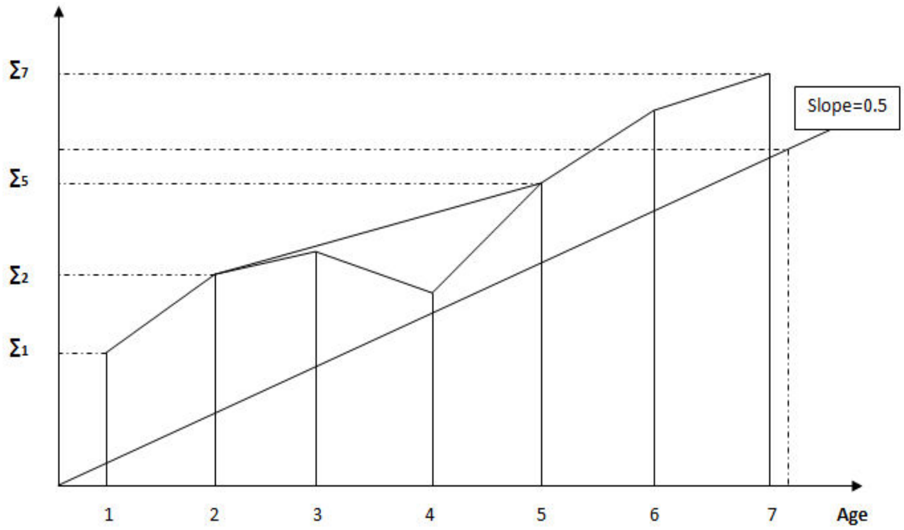


Figure 2.3: First Pre-selection criterion

Proposition 2.3.6. *r^* is decreasing in K (the duration of research life) and increasing in δ (the discount factor).*

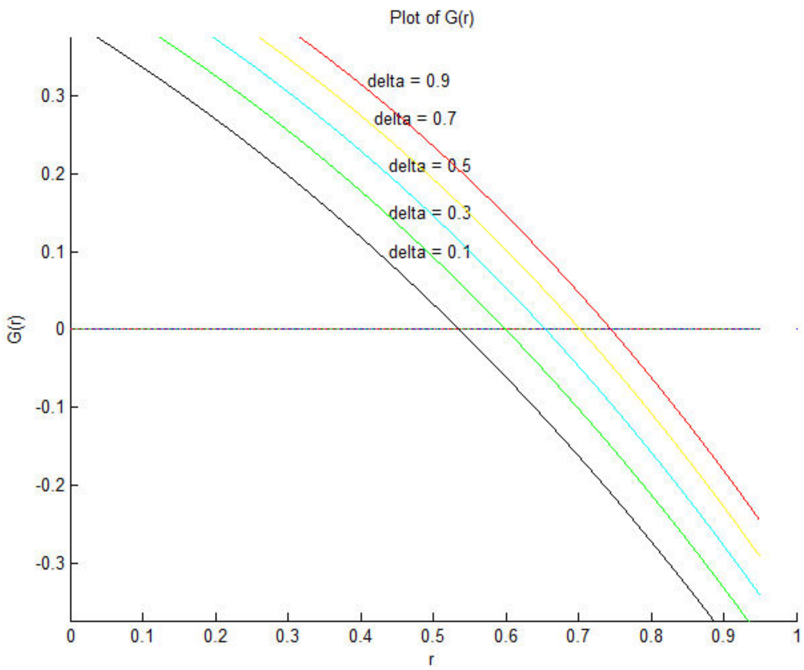


Figure 2.4: r^* in increasing in δ (Case of K=2)

G is a function such that $G(r^*) = 0$ (see proof of proposition 2.3.1 in appendix 3.9.1).

Proposition 2.3.6 is illustrated in Figure 2.4 above.[29] It states that if the retirement age increases then the fund will be more favorable to seniors. The result is not surprising because the reason why the fund is more demanding when deciding to finance an old researcher is that the later has less time to live in the future. In one word, any reform that lengthens research life is more profitable to the post-stage researchers.

Proof. 1- We first prove that r^* is decreasing in K.
What we are interested in is $f(A) = V_k(A) - V_m(A)$. We have seen that $f(A)$ is non decreasing in $\Delta = \frac{\sum_{i=1}^{k-1} a_{ki} - \sum_{i=1}^{m-1} a_{mi}}{k-m}$ since a researcher affects more its own value than he does for the others. To know how r^* varies with K we need to know how $f(A)$ varies with K for a given Δ. If K increases, everything equal elsewhere, it is obvious that $V_k(A)$ and $V_m(A)$ will increase. However, the change of $f(A)$ depends on how large the increase of K affects $V_k(A)$ and $V_m(A)$. If $V_k(A)$ increases more than $V_m(A)$ does, then $f(A)$ will increase. Otherwise $f(A)$ will decrease. The extend to which $V_k(A)$ and $V_m(A)$ increase (due to the increase of K) depends on the histories a_k and a_m. Specifically, if the history of researcher k is better than the history of m then $f(A)$ will increase with K. Because in the neighborhood of r^* the history of k is better than the history of m, $f(A)$ is increasing in K around r^*. Therefore, as illustrated in Figure 2.5 below, r^* in decreasing in K.

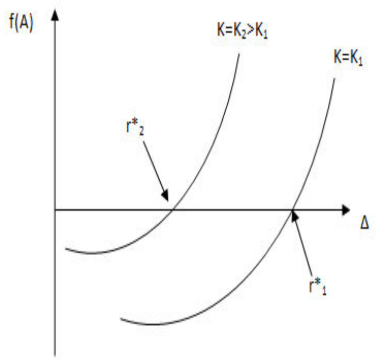


Figure 2.5: r^* is decreasing in K

2- We are going to prove that r^* increases with δ.
It is easy to give the proof for $K = 2$ by using the implicit function theorem applied to equations (3.93) and (3.94) in appendix 3.9.1. For any K the proof consists in showing that $f(A)$ is decreasing in δ. One could easily see that δ affects positively $V_k(A)$ and $V_m(A)$. Therefore, the proof boils down to prove that δ increases more $V_m(A)$ than it does for $V_k(A)$. In other words, we will show that δ affects more the younger than the older. We

[29] In the graph, we draw $G(r)$ for different values of δ. Amongst the values of δ we chose, the black curve is for the smallest δ and the red one is for the highest δ.

use, as usual, the backward induction procedure. Let us prove that $V_K(A) - V_{K-1}(A)$ is decreasing in δ.

$$\begin{aligned} V_K(A) &= \int_{-\infty}^{+\infty} q\theta_A^K(q)dq + \delta V(A^{(K)}). \\ V_{K-1}(A) &= \int_{-\infty}^{+\infty} q\theta_A^{K-1}(q)dq + \delta \int_{-\infty}^{+\infty} V(A^{(K-1)})\theta_A^{K-1}(q)dq. \end{aligned}$$

By lemma 2.3.3 we have $\int_{-\infty}^{+\infty} V(A^{(K-1)})\theta_A^{K-1}(q)dq \geq V(A^{(K)})$, and the result is found.

Assume that $V_{k+1}(A) - V_k(A)$ is decreasing in δ and let us prove the same result for $V_k(A) - V_{k-1}(A)$, with $k \in \{1, ..., K\}$.

$$\begin{aligned} V_{k-1}(A) &= \int_{-\infty}^{+\infty} q\theta_A^{k-1}(q)dq + \delta \int_{-\infty}^{+\infty} V(A^{(k-1)})\theta_A^{k-1}(q)dq. \\ V_k(A) &= \int_{-\infty}^{+\infty} q\theta_A^{k}(q)dq + \delta \int_{-\infty}^{+\infty} V(A^{(k)})\theta_A^{k}(q)dq. \\ V_{k+1}(A) &= \int_{-\infty}^{+\infty} q\theta_A^{k+1}(q)dq + \delta \int_{-\infty}^{+\infty} V(A^{(k+1)})\theta_A^{k+1}(q)dq. \end{aligned}$$

As we did when proving proposition 2.3.4, if we take a close look at the states A^{k-1}, A^k and A^{k+1}, we can realize that they have a similar structure. Therefore, if $\mathbb{E}_{\theta_A^k} V(A^{(k)}) - \mathbb{E}_{\theta_A^{k-1}} V(A^{(k-1)}) < 0$ then $\mathbb{E}_{\theta_A^{k+1}} V(A^{(k+1)}) - \mathbb{E}_{\theta_A^k} V(A^{(k)}) < 0$, where $\mathbb{E}_{\theta_A^i}$ denotes the expectation under the distribution θ_A^i. The proof ends. □

Remark 2.3.5. *The proof about the effect of δ on r^* could also be led by using proposition 2.3.4. From this proposition we can learn the following fact: to be chosen, the old researcher should be at least the one who procures the highest expected utility in the immediate future (that is $\int_{-\infty}^{+\infty} q\theta_A^i(q)dq$). Therefore, the expected utility in the distant future (continuation utility) of the younger is the highest. The intuition is that, because the elder has less time to live, his expected continuation utility is always smaller than the younger's expected continuation utility. Even in case where the elder has good history, this advantage is overcome by the disadvantage induced by its age. In one word, the value of the old stems mainly from his contribution in immediate future and less from what he would give in the future.*

It might be interesting to ask if and how the characteristics of the other researchers could affect r^*. As we have seen in remark 2.3.3, the value of a researcher is affected by the history of the other researchers if the later have good histories. Therefore, when comparing two researchers, the r^* we use might be affected by the histories of the other submitters. Specifically, as we had seen when analyzing the effect of K (retirement age), we can easily prove that r^* is negatively affected by the histories of the rest of the researchers. It means that if the other researchers are recognized to be good, it is more profitable to the elder in comparison with the younger. We can give an intuitive explanation to that finding. First of all, let us recall that r^* is affected by the other researchers who have already good history in such a way to affect the value of the two researchers we are comparing. As we have said above, these (other) researchers might be chosen in the future and such an event is likely if (in the future) they could be better than the very young researchers who will submit in the future but who do not submit at the current period. Therefore, when their histories improve the fund expects that the elder of tomorrow becomes better in comparison with the younger of tomorrow. This perception (about the elder and the younger of tomorrow) has an impact on the perception about the elder and the younger of today. Hence, the fund gets more favorable to the senior by decreasing r^*.

It is as well interesting to understand why the other researchers of worse histories cannot affect r^*. The reason is quite simple: when a researcher has not a sufficient good history, the fund removes him from the submitting group and does not take him into account when comparing the researchers.

Our setting is an example of restless bandit problem. It would then be useful to compare our findings with Whittle (1988). In our work, we are not willing to address the "indexability" of the problem.[30] Even though an index may exist it cannot depend only on the characteristics of the corresponding researcher as usually seen in classical bandit problem and in Whittle (1988) (in case where Whittle derived an index for restless bandit). In our setting, any index might depend on the history of the other researchers. The reason why such a difference arises is that in classical bandit the state of non chosen arms does not change. Therefore, when valuing a researcher, the state of the rest is considered as fixed and then the whole state of the nature evolves according to the state of only the researcher being evaluated. In contrast, in our setting, because of age limitation, the state of all researchers moves from period to period. Moreover, in most of the studies on bandit problem, people consider a markovian evolution of the histories and the state of the nature. Within a markovian setting, the researchers do not affect strongly each other (in terms of value) because researchers affect their colleagues'value through continuation utility.

However, let us notify that our setting is not a case of dependant arms bandit because the quality density function of a researcher does not depend on the state of the other researcher. It is just his continuation utility which is affected by the state of the other researchers who are recognized to be good.

[30]In bandit problem literature, people often study the existence of indices that could be used to make optimal decision. Whittle found a sufficient condition under which the restless bandit is indexable. We refer the reader to the broad literature on bandit problem for details.

2.3.4 Selection criterion

In this section, we summarize the procedure we propose to select the researchers. Let r_2^* be the value of r^* for $K = 2$. In general case ($K \geq 2$), the corresponding notation is r_2^g. According to all we have seen previously, we know that $r^* \in [\frac{1}{2}, r_2^*]$. In general setting, r^* is in $[0, r_2^g]$. The researchers are compared two by two. Suppose we are willing to compare the researcher of age k with the researcher of age m ($k \geq m$). In normal distribution setting the criterion is the following:

- If $\sum_{i=1}^{k-1} a_{ki} - \sum_{i=1}^{m-1} a_{mi} \leq \frac{1}{2}(k-m)$ then the researcher m should be preferred to the older.
- If $\sum_{i=1}^{k-1} a_{ki} - \sum_{i=1}^{m-1} a_{mi} \geq r_2^*(k-m)$ then k should be preferred to m. This point is our second pre-selection criterion.
- If $\sum_{i=1}^{k-1} a_{ki} - \sum_{i=1}^{m-1} a_{mi} \in (\frac{1}{2}(k-m), r_2^*(k-m))$ then we propose to find numerically $V_k(A)$ and $V_m(A)$ and make the comparison.

In non normal distribution setting the criterion is the following:

- If $\frac{\prod_{i=1}^{k-1} f_H(a_{ki})}{\prod_{i=1}^{k-1} f_L(a_{ki})} - \frac{\prod_{i=1}^{m-1} f_H(a_{mi})}{\prod_{i=1}^{m-1} f_L(a_{mi})} \leq 0$ then the researcher m should be preferred to the older.
- If $\frac{\prod_{i=1}^{k-1} f_H(a_{ki})}{\prod_{i=1}^{k-1} f_L(a_{ki})} - \frac{\prod_{i=1}^{m-1} f_H(a_{mi})}{\prod_{i=1}^{m-1} f_L(a_{mi})} \geq r_2^g$ then k should be preferred to m.
- If $\frac{\prod_{i=1}^{k-1} f_H(a_{ki})}{\prod_{i=1}^{k-1} f_L(a_{ki})} - \frac{\prod_{i=1}^{m-1} f_H(a_{mi})}{\prod_{i=1}^{m-1} f_L(a_{mi})} \in (0, r_2^g)$ then we propose to find numerically $V_k(A)$ and $V_m(A)$ and make the comparison.

2.3.5 How to make decision when choosing more than one researcher?

Up to now, we have considered that the fund is willing to choose one researcher among all those who have submitted. However, in most cases, the fund has to choose more than one researcher. The following proposition gives a way to deal with this concern. According to this proposition, if the fund wants to choose s researchers, it just needs to base on the selection criterion above and gives the grant to the s best submitters. This result is interesting for two reasons. First, it is easy to be implemented in real world. Second, it provides interesting results for the multi-armed bandit analysis. Indeed, in the classical bandit problem, the decision maker must choose a single arm, and it is well known that the Gittins' index characterization is rarely possible when the decision maker has to choose more than a single arm at each period. Therefore, proposition 2.3.7 gives an environment where some characterization is possible when more than one arm should be chosen.

Proposition 2.3.7. *If the fund is willing to choose s researchers among K researchers, the best way is to rank them (all Ks) according to the selection criterion and choose the s best researchers.*

Proof. Let S_1 denote a group of s researchers among the K ones. S_2 denotes another group of s researchers in which $s-1$ researchers are the same as in S_1, but the s^{th} one in S_1 is better (according to the criterion above) than the s^{th} one in S_2. Let V_1 and V_2 be the values (gain) from choosing the groups S_1 and S_2 respectively. The proof will end if we prove that $V_1 > V_2$.

As usually A is the current state of the nature and $A^{S_i}(.)$ is the state at the next period if the group S_i is selected.

$$V_1 = \underbrace{\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty}}_{s\ times} \left(\sum_{i=1}^{s} q_{s_i} + \delta V\left(A^{S_1}\left(q_{s_1}, ..., q_{s_s}\right)\right)\right) \prod_{i=1}^{s} \theta_A^{s_i}\left(q_{s_i}\right) dq_{s_1}...dq_{s_s}. \tag{2.6}$$

where, as usually, s_i identifies the researchers.[31] We can write:

$$V_1 = \underbrace{\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty}}_{s-1\ times} \prod_{i=1}^{s-1} \theta_A^{s_i}\left(q_{s_i}\right) dq_{s_1}...dq_{s_{s-1}} \left(\sum_{i=1}^{s} q_{s_i} + \int_{-\infty}^{+\infty} \left(q_{s_s} + \delta V\left(A^{S_1}\left(q_{s_1}, ..., q_{s_s}\right)\right)\right) \theta_A^{s_s}\left(q_{s_s}\right) dq_{s_s} \right).$$

Likewise,

$$V_2 = \underbrace{\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty}}_{s-1\ times} \prod_{i=1}^{s-1} \theta_A^{s_i}\left(q_{s_i}\right) dq_{s_1}...dq_{s_{s-1}} \left(\sum_{i=1}^{s} q_{s_i} + \int_{-\infty}^{+\infty} \left(q_{s_s} + \delta V\left(A^{S_2}\left(q_{s_1}, ..., q_{s_s}\right)\right)\right) \theta_A^{s_s}\left(q_{s_s}\right) dq_{s_s} \right).$$

$$\begin{aligned} Let \quad V^{S_1} &= \int_{-\infty}^{+\infty} \left(q_{s_s} + \delta V\left(A^{S_1}\left(q_{s_1}, ..., q_{s_s}\right)\right)\right) \theta_A^{s_s}\left(q_{s_s}\right) dq_{s_s} \\ and \quad V^{S_2} &= \int_{-\infty}^{+\infty} \left(q_{s_s} + \delta V\left(A^{S_2}\left(q_{s_1}, ..., q_{s_s}\right)\right)\right) \theta_A^{s_s}\left(q_{s_s}\right) dq_{s_s} \end{aligned}$$

V^{S_1} and V^{S_2} might be interpreted as the contribution of the s^{th} researcher to the total

[31] One could also consider a variable $X = \sum_{i=1}^{s} q_{s_i}$ and find the p.d.f f_X of X by convolution product. And then

$$V_1 = \int_{-\infty}^{+\infty} \left(x + \delta V\left(A'(x)\right)\right) f_X(x) dx.$$

gain from the whole groups S_1 and S_2 respectively. It is what he would procure to the fund if he were the only choice, given the performance of the $s-1$ other researchers. Therefore, basing on what we have seen until now, we have $V^{S_1} > V^{S_2}$ and then $V_1 > V_2$. □

This finding may seem to be unusual and surprising since, in general, the characterization found for a single-play does not work for a multiple-plays. The reason why it works here is probably the fact that in bandit problems, commonly, the state of the non chosen arms is considered as not moving. Therefore, the process is not irrevocable. As a consequence, the authors who worked in that field use forward induction approach to derive their solutions. However, in case of multiple-plays this forward induction is not possible since, when valuing a researcher there are some others whose histories are changing because they will also be chosen. In our work, we did not use a forward induction since we know that the state of non chosen researchers are changing. We cannot be surprised that our characterization also works for the multiple-plays.

2.4 Discussion

For a practical purpose, we discuss some parameters and questionable assumptions which may spark some concerns. Indeed, for instance, how to know in practise the probability density functions f_L and f_H? How to choose δ?

One could imagine many ways to find the density functions. Here, we propose to estimate them using a very simple procedure. We can have at disposal some ranking of research centers and universities. This ranking can be used to split universities and research centers into two groups: the top γ as a high quality group and the rest as the low quality group. γ is a percentage and might be equal to 50, 25 percents,..., depending on the ranking we have at hand. The sample of the researchers working in the high quality group is used to estimate f_H. The same is done with the second group to estimate f_L. We propose a non parametrical estimation which is based on the quality of the recent work achieved by the researchers.

The choice of δ might depend on the kind of research field the fund would like to finance or the horizon (long or short run) targeted. δ is supposed to be relatively high if the fund wants the research outcome to solve future problems. Otherwise, δ should be low. However, regardless these considerations above, we can choose δ to be in line with the economic or financial context.

The value of α is directly related to the sampling made to estimate the density functions. For $K = 2$, we have proven that the decision does not depend on α. For $K > 2$, we can give here an insight. α should not affect substantially the decision rule since its value is the same for all researchers. Using the model (or precisely the lemma 2.3.1) one can see that α influences the value of the researchers by affecting $P(H|A)$, the probability that the researcher is of type H given the state. By examining $\frac{\partial P(H|A)}{\partial \alpha}$, we realized that, an increase of α will affect at most not so much the researchers given the properties of the probability density functions.[32]

In this work, we have assumed that if the fund does not select a researcher at a given period, it considers that there is no more information about this researcher at the next period. As we said, one consequence of this assumption is that if at a given state the youngest is chosen then none of the other elders will be selected for a grant in the rest of their life. We think that such a situation might not be convenient in real world. As it is known, a researcher may do some work even if he is not given a grant. Therefore, we propose to put in his history what the researcher would produce when he had no grant. However, the fund would take this work into account only if it does not penalize the researcher. The fund can also take into account the value of the project currently submitted by the researcher.

One may criticize our work because we did not clearly consider some dependance between researchers. Indeed, it might exist some externality through research activity. However, we think that if usually there is sufficient dependance between the researchers, there will also be dependance between their histories. Therefore, taking into account the histories involve any dependance if it is a reciprocal dependance. However, in some cases, such as Ph-D students advising, we recognize that the dependance may not be reciprocal since the younger gets more benefit from the elder's experience than the elder does from the younger.

In our framework, the retirement age K is fixed. Yet one would like K to be a random variable varying across researchers' population. Of course the researchers are free to quit research activity at any age. However, most of the time, people stay in research activity during almost their whole life. One might argue that life length is a random variable, and then K should not be fixed. However, human kind is human kind and we can reasonably expect that the retirement age of the researchers lies in a shrunken set.

We based the history of a researcher on what he had done and we can observe. However, as we can see in other activities, when getting more senior, the researcher may capitalize and acquires abilities, even if he did nothing which is observable. However, this cannot be considered as a shortcoming in our work because research activity is a particular one which involves less routine. Research means innovation and capacity to find new things by new procedures.

[32] We could also ask the young researchers to claim about α and find a way to validate this claim (see for example Al-Najjar and Weinstein (2008)).

People might think that the fund does not need to experiment (choose) a researcher to be able to know his value, in that what the researchers are doing is publicly known. However, we should recognize that the value of a researcher depends also on the condition under which he is working. The productivity of the researcher under a given financing condition may be different from this productivity under another financing condition. Therefore, the fund cannot easily know the true value of a researcher under its own condition if it has never experimented him.

2.5 Conclusion

Nowadays some people think that the future of any nation is in the hand of the youth which is living in it. Such a thought exists in research field as well. One argues that young people are more creative, have more time left to live and the world can get profit from them for long. Therefore, an optimal way to proceed would be to equip young men an women with sufficient resources in order to allow them to solve the problems the world is facing and to increase standard of living. Some others state that human kind can bring innovation at any age. Due to these contradictory ways to think, to allocate resources to researchers, age cannot be the only criterion. We should rather use an objective decision rule.

This paper intends to participate to this debate. In our model, the research fund lives forever while the submitters live for a finite period. We use a dynamic discrete time model with overlapping generations to propose a simple strategy to rank and select the researchers submitting a project for a financing contest. We have seen that what determines the researchers'selection is both their age and their histories. The history of a researcher is all he has done in the past. To compare the researchers, we propose to use their histories in proportion to their ages. We find that when two researchers have an equivalent history the fund should prefer the younger. It means that the fund should be more exacting towards the elder. The reason is that age deters the value of the researchers because the remaining time to live is determinant. However, when the nation (or a relevant area) is facing a crucial problem which needs to be solved immediately (such as epidemic, natural disaster, economic crises), the fund should be less exacting towards the elder.

We think that this work has also a theoretical contribution. The model we use may be seen as a case of restless bandit problem. Such problems are known to be untractable and pspace-hard due to dimensionality.[33] Here we propose to eliminate or rank progressively

[33]Powell (2007) discussed about some problems of dimensionality in dynamic problems.

the researchers by comparing then two by two, using a simple and intuitive criterion. We also provide a result in case of multiple-plays.

In this paper we consider a single fund facing a certain number of researchers. Rather, we can imagine an environment in which there are several funds competing to keep with them the best researchers. Our model would be similar to a strategic experimentation (see for example Keller et al. (2005), Strulovici (2010). We may also introduce in the model the mobility of the researchers. we expect an ambiguous result since the young people are known to be mobile. This mobility might have two counteracting effects. If a researcher is mobile the fund may tend to choose this researcher in order to encourage him to stay for the next period. However, the fund may not prefer a mobile researcher if it thinks that the probability for this researcher to be present next period is low.

Chapter 3

Paradox about the Multi-Level Marketing

3.1 Introduction

The multi-level marketing (MLM) is a marketing strategy in which independent distributors of the product get profits not only from their own sales, but also from recruiting other distributors. Even though the MLM occupies a prominent place among important in the current issues and debates, it seems neglected by economic theory. A search of the literature reveals a scanty theoretical treatment of the subject. Most of what that has been written about the topic is limited to making judgement about the ethic and the legitimacy of this business, or to proposing means and strategies to make it "fair and legal". It seems that economists consider the MLM as law, marketing or ethical issues and should be analyzed and targeted by the scholars of these fields. Yet the structure of the MLM is more complex, and intrinsically gives rise to problems grounded on merely economic matters. Whatever people legislate and propose without understanding these economic issues, we will still be holding the debate. The purpose of this paper is to use these economic issues to address a specific question: why do people recruit others to be their competitors? What makes such an organization more profitable for the promoters?

Before proceeding with a presentation and a discussion of the results, we need to give the motivation and the relevance of our question. We then describe, briefly, the economic and social context giving rise to the multi-level marketing, and present the current concerns about this business organization.

Marketing strategies refer to the practise consisting in putting a product or service on the market and the techniques and actions by which the sales force succeeds in convincing the consumers to use the product, regardless of the production process itself. Traditionally, the ways to achieve this goal are many and vary from media advertisements, rebates on sales, post sale service, price discrimination, start up special offers, popular events sponsoring, to geographic location and products diversification. The traditional marketing strategies are essentially based on the 4Ps (Product, Price, Promotion and Place) proposed by McCarthy (1960).[1] These strategies may be more or less aggressive. The challenge is to make the product more attractive and to face competition in order to increase profits.

These traditional channels of marketing are still being performed today. However, one should recognize that firms are developing new methods of marketing based either on the traditional one or on some economic considerations. First, knowing the existence of network externality inside the consumers group (see Katz and Shapiro (1985))[2], many firms use relationship or interdependence between consumers to launch a powerful marketing plan (see for example Hill et al. (2006a)). In terms of Katz and Shapiro, there is network externality when the utility a user derives from consumption of a good increases with the number of other agents consuming this good. This network effect may also exist through relationship between consumers and sellers or firm, or between firms. A marketing strategy based on any network effect is referred to as "network marketing".[3] The stated goal is to reach potential consumers that the firm otherwise would not reach with traditional marketing methods. Hill et al. (2006a) used the word "network-based marketing", naming a collection of marketing techniques that take advantage of the ties between consumers to increase sales.

Secondly, these days, many firms are shifting from the traditional marketing towards a more relationship-orientated one (Gummesson (1987), Gronroos (1994)). For the firms' managers, the marketing process consists more and more in establishing, strengthening and developing person to person interactions with the customers. Such a marketing is called thereafter *the relationship marketing* by Berry (1995).[4] The purpose is a long-run issue, creating more customer loyalty (Crosby and Stevens (1987), Berry (1995), Gengler and Popkowski Leszczyc (1997), Macintosh and Lockshin (1997)) and even creating *customer advocates* (Christopher et al. (1991)). The 4Ps principle is becoming obsolete or cannot capture the complexity of today's business world. Some authors have even suggested to "increase the number of Ps". We can read in the literature the 5Ps (Judd (1987)), the 6Ps (Kotler (1986) and even the 15Ps (Baumgartner (1991)). In short, even

[1]Subsequently Borden (1964) named these four variables *the marketing mix*.

[2]See also Katz and Shapiro (1992), Farrell and Saloner (1986), Shapiro and Varian (1998, 1 edition), Liebowitz and Margolis (1994).

[3]Specifically, network marketing is marketing model that relies on a network of different actors operating in the market to grow a business.

[4]For information, even if it is performed with a new focus, Berry (1995) recognized that the relationship marketing idea is an old one.

if our goal in this paper is not to review the evolution of the marketing approaches, we could just mention that the development of new market constraints and challenges has led to the emergence of new marketing strategies based on personal relations, partnerships, cooperation and trust.

Relationship marketing plans are mainly performed through direct marketing and direct selling. Direct marketing consists in reaching the market on a personal basis (home visits, phone calls, private mailings, etc.) or by mass-media campaigns (infomercials, magazine, etc.) that should entice message to get consumers to act. It is a form of advertising that reaches its audience without using traditional formal channels of advertising. The purpose is to communicate straight with the consumer (or potential consumer) with specific advertising techniques.[5] Concerning the direct selling, it is just what it sounds like. The products are directly sold to the consumers away from a fixed retail sales outlet. Direct selling may be accompanied with personal demonstration and explanation of the product's use by the vendor.

These days, with these paradigms (network reality, relationship), things are becoming complex in such a way that it is difficult to consider network marketing and relationship marketing in isolation. Indeed, many firms build huge marketing plans by combining (and even more) network marketing, direct marketing, direct selling and any other kind of relationship marketing. An evidence is the Multi-Level Marketing (MLM). MLM involves network marketing but is more than that. It refers to a marketing method organized by or under the auspices of a company, and in which the distributor of the product makes money not just from his sales, but also by getting commissions from the sales made by the other distributors he had recruited (sponsored) into the MLM. The business is built by creating a tiered network of "independent" distributors. For any distributor participating to the MLM's activities, the task consists in selling the product and/or recruiting new distributors (or participants). MLM is often used interchangeably with *network marketing* (Hill et al. (2006b), Koehn (2001), Kong (2003), Pratt and Rosa (2003), Vander Nat and Keep (2002).[6] Let us mention that, in this paper, we consider the two concepts as referring to two different types of marketing plan while knowing that the MLM uses the network marketing method.

The MLM companies are often accused of being a pyramid scheme which is a form of fraud, an illegal activity and in which there is no independent market value product or service (or the product is just an alibi). Pyramid scheme starts with one person who is on the top of the chain. This person makes money by recruiting other people who pay

[5]Some examples of direct marketing are telemarketing, direct mail marketing, door-to-door solicitations, fax broadcasting.

[6]In Hingley and Lindgreen (2002), the authors seemed to avoid this confusion.

him a fee. The payment of this fee gives these people the right to recruit in turn and make money. A recruiter gets profit from his immediate recruits, and (depending on the pyramid's rules) may also get money from the people recruited by his immediate recruit, and so one. Such a fraudulent business takes many forms: Ponzi scheme, chain letters, etc. An MLM plan will be suspected or considered as being a pyramid scheme if it emphasizes more on recruitment than on the selling of the product or service.[7] Precisely, the MLM, to be fair, should give distributors the incentive to sell the product to the nonparticipants to the MLM, that is the customers outside the organization.[8] These customers should not be distributors. In other words, the compensation commissions paid to distributors should be mostly generated from the sales to nonparticipants. If the compensation plan leads to a system of monetary rewards that favors recruitment over sales then the MLM falls into a pyramid scheme.

However, most of the MLM companies reject these allegations and consider their business to be primarily oriented towards marketable products selling to the outside consumers. Many times Amway has proven that its marketing policies encourage its distributors to sell products to nonparticipants. In other words, members of the MLM have more incentive for selling than recruiting forever. Let us precise that, in this paper, our goal is not to make investigation on the MLMs and sit in judgment. We are not going to tell if an MLM is a pyramid scheme or not. We take the distributors' claim as given, that is, they are really willing to sell. For such a scenario to occur, the distributor should expect to find potential consumers and also to get more profit in selling. If so, a paradox arises. If a business owner is actually willing to sell a product and is expecting to get high profit (as claimed by MLM promoters), why does he allow other "competitors" to take a part of the market? The basic rule in any market policy is to avoid competition as well as possible. If the real intention is to sell products to end consumers, why does salesperson make actions in favor of competition? Yet, it's what that is happening in MLM program. The organizers of the program pretend that their distributors purchase the products for selling them, and at the same time these distributors recruit other salespersons who are in fact their competitors. The goal of this paper is to explain this paradox, basing on the particular characteristics of the Multilevel marketing plan. Why do the distributors

[7] According to the Federal Trade Commission (FTC) in United States, a plan that appears to be heavily based on recruiting others is more likely to be an illegal pyramid scam. In 2007, in United Kingdom, the Department for Business Enterprise and Regulatory Reform (DBEER) presented a petition to wind up Amway, the first ranked MLM company in the world. DBEER alleged that Amway's policies encourage strongly distributors to recruit other participants into the business.

[8] When members of the MLM company buy products for themselves and not for resale, there is internal consumption. According to a correspondence from the FTC to the Direct Selling Association (DSA) in January 2004, pyramid schemes exist when distributors purchase products with the intent of earning income primarily by recruiting participants to do the same, and not for the marketable value of the product or service.

recruit while having more incentive to sell? Precisely, what is the rationale guiding a distributor who recruits even though he is really intending to sell?

One may argue that distributor, by recruiting, expects the recruits to reach other markets that he cannot reach. Such a thought might not fit the MLM's case. Indeed, as we said before, MLM program involves also a relationship marketing. Distributors act with their neighborhood, relatives, friends, people with whom they are in personal relationship. Most of the time, distributors recruit within their neighborhood. This new recruit has a "high chance" to have almost the same neighborhood as his recruiter, and then, is likely to operate in the same market (see for example homophily literature in McPherson et al. (2001) and Currarini et al. (2009)). Even if we consider that distributors are hoping the new recruits to reach other markets, it means that they believe such markets to exist. If so and knowing that the purpose is to make much money (as claimed), why do they not build their own "big business" and reach progressively the markets by themselves? In the same vein, suppose even so that each distributor decides to reach other markets by recruiting people in other areas than the marketplace he is operating in. If we make a global analysis, in this process, any distributor is threatened to see other distributors recruiting in his marketplace. As conclusion, looking for other markets can no longer justify for enough such a situation.

Some others explain that MLM companies diversify their products. For them, when recruiting, distributor and his downline probably agree not to sell the same products. These arguments may not hold in that, with regard to the size of some distributors' downline, the number of products is not sufficiently high to make distributors heterogenous regarding the products they sell. Moreover, distributors are considered to be independent and there is no chance for such agreement to be achieved.

From the MLMs' promoters perspective (the top of the chain), recruiting distributors is normal and profitable if we consider that distributors will, at least, use the products for self-consumption, even if they will not sell. In this situation, the more there are distributors, the higher the sales made by the MLM's promoters. In contrast, for the distributors, the reason is quite different and simple. Since the success of distributors depends mainly on their sales, they will highly desire to sell. As a consequence, a competition should lie between the distributors. Given that distributors do not produce the good themselves, they have less control on its quality, and then any competition will be essentially price competition. Being able to propose a competitive price requires a certain capacity, precisely a financial capacity to survive. The higher is the capacity, the more competitive is the price set by the distributor. Since the sponsor gains from the sales made by his recruit, this gives him a *price leading power* on his recruit. To overcome this leadership of the sponsor, the recruit in turn should recruit. Progressively a trap of recruitment might be built. Distributors will need to recruit more and more to face competition not just with their upline but with the other distributors as well.

The rest of the paper is organized as follows. In section 3.2 we present a description of the pyramid scheme and the multi-level marketing and give some evidences from the real world. Section 3.3 is used to give an insight of our main message through a simple model that is generalized in section 3.4. The two next sections present our results about the price setting decision of the distributors (section 3.5) and about the decisions of the MLM promoters (section 3.6). We end with some concluding remarks (section 3.7).

3.2 The MLM and the pyramid schemes

In this section, we present briefly the structure of an MLM organization and the pyramidal business. As we said before, there is some confusion about these two organizations, even though they may be quite different in some respects.

3.2.1 The pyramid scheme

A Pyramid scheme is a business consisting in making money by recruiting other participants into the business. Each participant, before starting, pays entry fees as a start-up investment. The entry payment (or a part) of a given new recruit goes to his recruiter as the gain from recruiting. In turn, the new recruit will gain from the entry payment made by the participants he would recruit. If he fails to enroll other people he looses money. Pyramid scheme can take many different forms: chain letters, money depository institutions with high deposit return rate, etc. Such a business is considered as being not sustainable since there is no product, and then no value added creation. Even though in some cases a product exists, it is just used by the promoters to disguise the real characteristics of their activity. A close look at the pyramid plan reveals a collection of dues for no service done. For this reason, this kind of activity is illegal in many countries and considered as what it is: a scam and fraud. Many fortunes are ruined by this kind of investment scheme.

3.2.2 The Multilevel marketing plan

Multi-level marketing (MLM) is a type of marketing that combines network marketing, direct marketing and even more, and in which business is built by creating a tiered network of independent distributors to promote and sell a company's products or services. Each distributor, depending on his wish, markets the products and/or recruits other distributors. Participants make money not just from personal sales but also from the sales made by others people they had recruited into the program. For a typical distributor, his *upline* is the group made of his recruiter, the recruiter of his recruiter, and so on up to the top of the organization. The *downline* of this typical distributor is composed of his recruits, the recruits of his recruits, and so on, to the bottom of the chain. The term MLM is often used interchangeably with *network marketing*, but in this work we consider the two concepts as being different from each other. In other words, we are not studying the network marketing.

Many companies use the MLM to sell their products or services. Some evidences are Amway, Herbalife International, Avon Products, Mannatech Inc, Forever Living Products, ACN Inc, DXN, Advocare International. The largest MLM company is *Amway*, founded in 1959 and based in US. It produces and sells beauty, health and home care products. *Avon Products* was founded in 1886 and markets cosmetics, perfume and toy in over 140 countries across the world. *Herbalife* was founded in 1980 and sells nutrition, weight-loss and skin-care products in over 75 countries. *DXN* is a malaysian company founded in 1993 and operating in over 150 countries. It produces and distributes products manufactured from a mushroom (ganoderma). These products are foods and beverages, food supplements, personal care products and water treatment series. *ACN*, founded in 1993, provides telecommunications and connected services, and operates in over 20 countries across 4 continents. Concerning *Forever Living Products*, it was founded in 1978 and markets aloe vera and bee derived drink, cosmetics, nutritional supplements and personal care products in over 145 countries. *Mannatech* is a multinational founded in 1993 and operating in research, development and distribution of glyconutrients (health, weight and fitness, skin-care, etc.).

As some may remark, most of the MLMs sell commodities that are closely tied to the daily life of large number of people. The commodities in matter are used frequently and should be purchased by the consumer repeatedly (Ella (1973) quoted in Vander Nat and Keep (2002)), and not once for all. The unit price of these commodities are up to the purchasing power of large number of consumers. These characteristics of the MLMs products should make them to be demanded continuously by the same people and more. Until today we have not seen an MLM company marketing products like cars, computers, furniture. Moreover, most of the products sold seem to have (in some respect) some particularities that might make them different from those we can get in common markets. Either they are unique (the multi-purpose cleaning product SA8 is sold by only Amway) or the input used to manufacture is not common (Forever Living Product, DXN). This particularity can enhance the valuation of the products by the consumers and provides the company with a "strong" market power.

To describe the complex structure of the MLM scheme, we lean on the Amway's case which is the most important MLM business and which inspires many other MLM companies. In 2007, Amway was ranked by Forbes as one the largest private companies in the US. Like any MLM organization, Amway, after producing, sells its products through business partners (called distributors) who purchase from Amway and distribute products and services as retail sales. Distributors are independent in Amway's terms, in that they can set their own retail price, find their own market, use their own marketing or advertising procedure. In addition to selling and making profit (retail profit in Amway's terminology), the distributor can be sponsor. A sponsor is a distributor who recruits a new distributor to Amway. Therefore, the term *upline of a distributor* can be redefined as referring to all distributors up in the line of the sponsorship of this distributor. The *downline of a distributor* refers to all distributors down in the line of the sponsorship of this distributor. The sponsor earns income in form of bonuses based on the sales volume generated by the distributor in his immediate downline. The sponsor does not necessary earn from the sales made by all his downline, but by his recruits who are in the earlier tiers below him. More precisely, the sponsor get bonus on the sales of his recruits, the recruits of his recruits, and so on until a certain level in the hierarchy, depending on the compensation plan decided by the company. In practice the bonuses are, most of the time, based on the purchases of the downline since it may be delicate to measure accurately the sales volume. To determine the amount of the bonuses, Amway assigns a score (point value in Amway's terminology) to each product or service sold by the distributor and his downline. The distributor's monthly performance bonus depends on the total *point value* in a month. Depending on his sales and the size of his downline, the distributor gets some qualifications and periodically can gain promotion (Silver producer, Gold producer, Platinum, Emerald, etc) if he performs well. Every so often Amway holds seminars, holidays trip, guided tour that benefits distributors. The purpose is to mobilize and motivate the distributors.

Including Amway, a number of MLM companies have been accused of being pyramid scheme. During the last three decades Amway was alleged to be a pyramid scheme in United States, Belgium, United Kingdom, India, etc. However, most of the time these allegations were dismissed by court. In 2004, some former and current Herbalife's distributors were paid 6 millions dollars after filing a class action suit against Herbalife for running a pyramid scheme. However, Herbalife didn't recognize to be a pyramid plan and argued that he made such a payment just to settle the conflict. During the last decade in Canada, Australia and US, ACN was alleged to operate pyramidal system. In this case too, these allegations were considered to be unsubstantiated by court.

From the description we gave above, the characteristics that pyramid scheme and MLM feature in common is on the recruitment side. An MLM would be considered to be a pyramid scam if the compensation plan gives distributors more incentive to recruit than to sell. In practice and over the years, the Federal trade commission (F.T.C) has successively used different tests to characterize an MLM that has strong ties with a pyramid scheme (Koscot test[9], "Amway based" test, etc.). Vander Nat and Keep (2002) proposed a quantitative approach to differentiate MLMs from pyramid schemes.

In this paper, as we said before, we are not intending to find out if an MLM is a pyramid scheme or not. We then consider legal MLM program, and in what following, we summarize the characteristics of the MLMs we are going to analyze.

The MLMs under study

MLM organizations are of a complex nature (Vander Nat and Keep (2002)) and some MLMs may differ from others with respect to some characteristics. It may seem pretentious to try to build a model that encompasses properly all these complexity and diversity. For the purposes of this work, we consider the MLMs having the following features.

1. All the distributors purchase directly from the company. They can use a part of the purchases for their personal consumption.

2. Each distributor is free to set his own retail price and to make any other decision (about quantity to purchase, recruitment, selling, etc.).

3. The recruitment commissions are based on the sales or purchases made by the downline.

4. No considerable cost to entry and to exit the program.

5. No distributor will accept ex-ante zero profit or negative profit.

Even though legitimate MLMs may be of many forms, as we said before, these five points above summarize the common and essential features of most of them. The point five is easily figured out since the purpose of this business is to make money quickly. In practice, no distributor will accept to get zero profit or negative profit, as we can see in common businesses.

[9]According to Koscot test a pyramid scheme is "an arrangement in which participants pay money in return for which they receive (1) the right to sell the product and (2) the right to receive in return for recruiting other participants into the program, rewards that are unrelated to the sale of product to ultimate users"(see FTC vs Koscot case, 1975). Koscot test is inadequate today. According to the "Amway based" test (FTC vs Amway, 1979), an MLM cannot be considered as pyramid scheme if (1) participants do not need to pay large sum of money upfront, (2) the distributors' purchases are largely refundable (90 percent for the buyback rule of Amway), (3) the plan prevents from inventory loading by requiring distributor to sell a large part of their purchases (70 percent and 10 customers rule in Amway's plan), (4) the plan avoids making false and deceptive income claims.

3.3 Simple case

In this paper, we intend to show that distributors gain from recruiting others distributors into the business, even if the commissions for recruitment activity are low. We would like to prove that when people recruit, they can set better (lower) prices, sell more and make more profit (in some respect). We use the following simple model to illustrate this intuition. This simple model does not include all the characteristics featuring the MLM business which is more complex as studied in sections 3.4 and 3.5. However, the current section will help the reader to have an overview of the key information given in this article.

3.3.1 Setup

Let us consider a company that produces a given good and sells it using two retailers 1 and 2. The retailers purchase the product from the company at the wholesale price $c > 0$ and set their retail prices freely. They engage in price competition. Here we distinguish two cases. First, in case 1, we consider a more common setup: the retailers are independent and each of them makes by himself the decision to purchase from the company, resells and makes profit on his own sales. Second, in case 2, we assume that retailer 2 is recruited by 1. Retailer 1 receives not just the profit on his own sales but gets a commission on the sales made by his recruit. The retailers 2 receives the profit made on his own sales.

Case 1: The retailers depend directly on the company

The price for retailer i is p_i. The demand function for 1 and 2 are respectively:

$$D_1(p_1, p_2) = d - ap_1 + bp_2 \quad and \quad D_2(p_1, p_2) = d - ap_2 + bp_1$$

where a, b, $d > 0$ and $a \geq b$. The profits are:

$$\pi_1(p_1, p_2) = (p_1 - c)D_1(p_1, p_2) = (p_1 - c)(d - ap_1 + bp_2)$$
$$\pi_2(p_1, p_2) = (p_2 - c)D_2(p_1, p_2) = (p_2 - c)(d - ap_2 + bp_1)$$

The prices maximizing the profits are characterized by the first-order conditions:

$$\frac{\partial \pi_1}{\partial p_1} = -2ap_1 + bp_2 + ac + d = 0$$
$$\frac{\partial \pi_2}{\partial p_2} = -2ap_2 + bp_1 + ac + d = 0$$

This yields the equilibrium prices:

$$\hat{p}_1 = \hat{p}_2 = \frac{ac+d}{2a-b} \tag{3.1}$$

while the equilibrium demands are

$$\hat{D}_1(p_1,p_2) = \hat{D}_2(p_1,p_2) = d - a\hat{p}_1 + b\hat{p}_2 = d + (b-a)\frac{ac+d}{2a-b} = \frac{ad+abc-a^2c}{2a-b} \tag{3.2}$$

and the total demand is

$$\hat{D}(p_1,p_2) = \hat{D}_1(p_1,p_2) + \hat{D}_2(p_1,p_2) = 2d + 2(b-a)\frac{ac+d}{2a-b} = \frac{2ad+2abc-2a^2c}{2a-b} \tag{3.3}$$

We can then compute the profits of the retailers at equilibrium:

$$\begin{aligned}\hat{\pi}_1 = \hat{\pi}_2 &= (\hat{p}_1 - c)\hat{D}_1(p_1,p_2) \\ &= a\left(\frac{-ac+bc+d}{2a-b}\right)^2\end{aligned} \tag{3.4}$$

Let us now turn to the second case.

Case 2: retailer 2 **is recruited by retailer** 1

In addition to his own profit, retailer 1 received a percentage α on the sales made by 2. Precisely, for each unit of sale made by 2 retailer 1 gets αc as a discount. This discount is paid by the company and not by retailer 2. The recruiter bears a cost A as a percentage of the recruit's sales. This cost A may include some opportunity costs (time, money), training costs (to give training for the recruit) and mainly the cost of having competitor. The profits are then:

$$\begin{aligned}\pi_1^* &= (p_1-c)D_1(p_1,p_2) + \alpha c D_2(p_1,p_2) - AcD_2(p_1,p_2) = (p_1-c)D_1(p_1,p_2) + (\alpha - A)cD_2(p_1,p_2) \\ \pi_2^* &= (p_2-c)D_2(p_1,p_2)\end{aligned}$$

One may ask for the reason why the cost A multiplies the demand $D_2(p_1,p_2)$ of retailer 2. In other words, why does the cost is proportional to the sales of the "recruit"? Many examples are consistent with the existence of this cost. In most cases, the sales are made via internet. The seller then needs a website. In some cases, the downline is affiliated to the website of its sponsor. The volume of sales made by the "recruit" could be a source of internet cost borne by the recruiter (managing cost, congestion cost, etc.). Also, the volume of the downline's activity influences the volume of the upline's activity. Therefore, the more the downline sells, the more the upline might face some fees related to taxation, accounting, etc. Multi-level marketing is a relational business. The actors are willing to build a strong network that could profit them. Therefore, sometimes, the sponsor could accept consciously to leave a part of his market to his downline. The purpose is to help the downline to grow up. However, by doing so, the upline bears an opportunity cost. In the same vein, the upline has the responsibility to give the "recruit" all useful information and to help him managing successfully his business. This responsibility increases with downline's expansion. Another way to see A is to consider that the recruiter bears a cost A to do an action that could generate and amplifie competition against him, because the downline could become his competitor. In fact he makes effort and uses his resources to "buy competition". We then integrate this underlying competition into the cost. It allows us to make difference (concerning the competition size) between the two cases (case 1 and 2). As conclusion, the cost A is the underlying cost implied by the recruitment. Taking the responsibility to bear this cost consists in taking actions necessary to emerge and to succeed in the business. MLM activity could be considered as a race or competition for a sort of prosperity. The first-order conditions characterizing the optimal prices are

$$\frac{\partial \pi_1^*}{\partial p_1} = -2ap_1 + bp_2 + ac + d + (\alpha - A)bc = 0 \quad (3.5)$$

$$\frac{\partial \pi_2^*}{\partial p_2} = -2ap_2 + bp_1 + ac + d = 0 \quad (3.6)$$

Proposition 3.3.1. *Let $\hat{p}_j$ and p_j^* be the prices set by retailer $j = 1, 2$ respectively in case 1 and 2. If $\alpha < A$ then $p_1^* < \hat{p}_1$ and $p_2^* < \hat{p}_2$.*

Proof. From equations (3.5) and (3.6) the equilibrium prices are

$$p_1^* = \frac{ac+d}{2a-b} + \frac{2a(\alpha - A)bc}{(2a-b)(2a+b)} \quad \textit{and} \quad p_2^* = \frac{ac+d}{2a-b} + \frac{(\alpha - A)b^2c}{(2a-b)(2a+b)} \quad (3.7)$$

or

$$p_1^* = \hat{p}_1 + \frac{2a(\alpha - A)bc}{(2a-b)(2a+b)} \quad \textit{and} \quad p_2^* = \hat{p}_2 + \frac{(\alpha - A)b^2c}{(2a-b)(2a+b)} \quad (3.8)$$

□

According to proposition 3.3.1, if the discount for recruiting is low in such a way that $\alpha - A$ is negative then $p_1^* < \hat{p}_1$ and $p_2^* < \hat{p}_2$ (see illustration in Figures 3.1 and 3.2). Thus, in an environment where the recruitment commissions are not important the retailers set lower prices in case 2, compared with the prices set in case 1. Moreover, we can easily check that

$$p_1^* < p_2^*. \tag{3.9}$$

In other words, the retailer who recruits will set the lowest price, will be competitive and then could sell more than in case 1, as stated in proposition 3.3.2 (see illustration in Figure 3.3). We can see this by computing the equilibrium demand $D_1^*(p_1, p_2)$ for the recruiter. We find that

$$D_1^*(p_1, p_2) = \hat{D}_1(p_1, p_2) + \frac{(\alpha - A)bc}{(2a-b)(2a+b)}(b^2 - 2a^2) \geq \hat{D}_1(p_1, p_2). \tag{3.10}$$

The opposite occurs for the recruit who sells less than in case 1 even if his price is lower[10]. The reason is that the recruit does not decrease his price sufficiently to overcome the decrease in price by the recruiter. If retailer 2 is willing to sell he should probably recruit other sellers. In fact, recruitment gives the recruiter a *price leading power*, as shown by Figure 3.1.[11] The *price leading power* is the capacity of a retailer to set a price lower than the other retailers' prices. It is generated both by the recruitment commissions and the competition structure. Here, the *price leading power* of retailer 1 is appreciated by the gap between p_1^* and p_2^*.

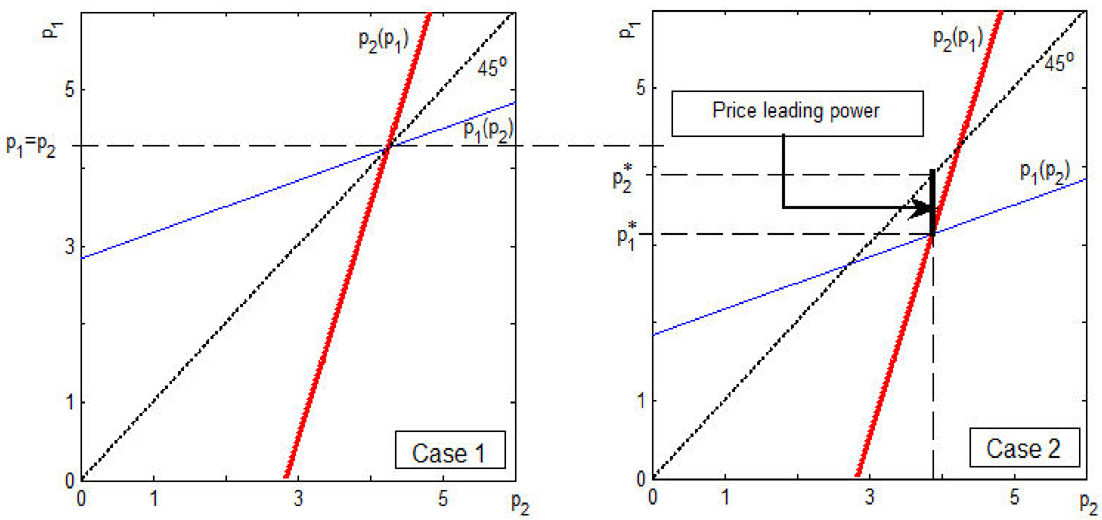


Figure 3.1: Price leading power. In Figure 3.1, we draw the price best response functions of the retailers in each case. $p_1(p_2)$ and $p_2(p_1)$ are respectively the best response functions of retailers 1 and 2.

[10]It is easy to check that $D_2^*(p_1, p_2) = \hat{D}_2(p_1, p_2) + \frac{(\alpha - A)ab^2c}{(2a-b)(2a+b)}$.

[11]We discuss extensively about the *price leading power* concept in section 3.5.

Proposition 3.3.2. *If $\alpha < A$ the retailer 1 sells more in case 2 than in case 1, retailer 2 sells less in case 2 than in case 1 and the whole industry sells more in case 2 than in case 1.*

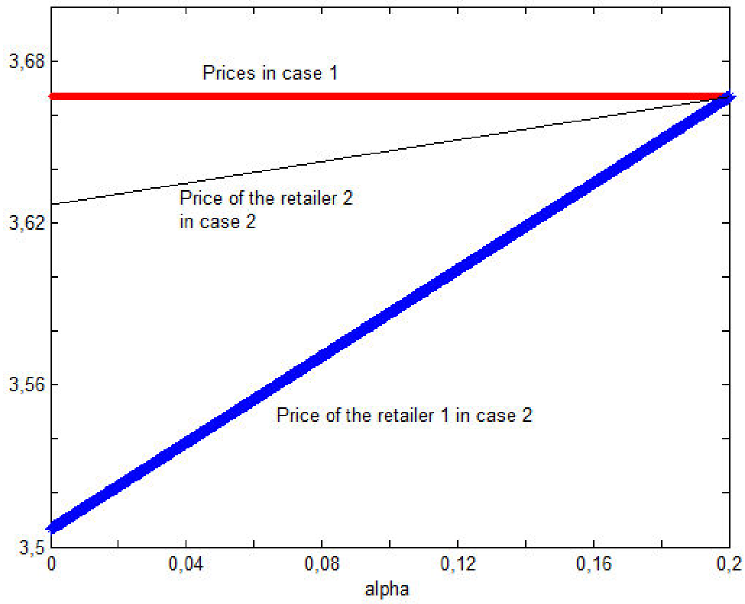


Figure 3.2: Retailers' prices as functions of α. In Figure 3.2, we draw the optimal prices of the retailers as function of the recruitment commission rate α. We choose $a = 2$, $b = 1$, $d = 5$, $A = 0.2$, $c = 3$, $m = 1$.

Another interesting matter is that the company sells more in the business system built in case 2 than it does in the traditionally system (case 1) if recruitment compensation (α) is not too high (see illustration in Figure 3.3). Indeed the total demand in case 2 can be written down as follows:

$$\begin{aligned} D^*(p_1, p_2) &= D_1^*(p_1, p_2) + D_2^*(p_1, p_2) \\ &= \hat{D}(p_1, p_2) + \frac{(\alpha - A)bc}{(2a - b)(2a + b)}(b^2 + ab - 2a^2) \end{aligned} \tag{3.11}$$

Even though the recruit sells less in case 2, the recruiter sells sufficiently to overcompensate this decrease in the retailer 2's sales, and then to increase the sales of the whole industry. In other words, even if the commissions for recruitment is low, the retailers have interest to recruit other retailers into the business. They recruit for the purpose of selling more and, probably, making more profit. In opposite, if the commissions (α) become high, people will still be recruiting, but not for selling more. They will recruit for the gain (commissions) from recruitment itself because if commissions are important the recruiter will decrease his sales in order to give the recruit the opportunity to sell. In

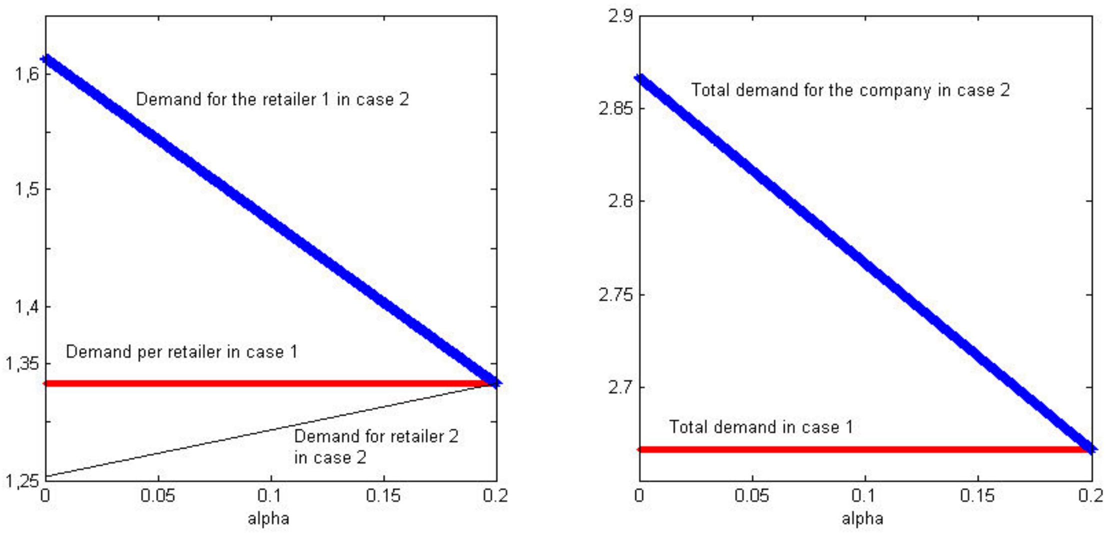


Figure 3.3: Volume of sales as functions of α. We choose $a = 2$, $b = 1$, $d = 5$, $A = 0.2$, $c = 3$, $m = 1$.

fact recruitment has two effects on the "sponsor": *competition effect* in that more retailers implies more competition and *pecuniary effect* based on the commissions paid to the recruiters. If α is high enough then the overall effect is shaped by the *pecuniary effect* and the participants will recruit for the purpose of making money from the recruits and not from selling. This business (case 2) is then likely to fall into pyramid scheme. In contrast, if α is low enough, the *competition effect* will overcome the *pecuniary effect.* In such a situation, people recruit to face competition. Since the "sponsor" gains from the recruits, he might have the capacity to lower his price without loosing money (negative profit). The retailers will then recruit in order to have the capacity to sell more. Actually, the recruit generates some externality for the recruiter. We come later in detail on such externality in section 3.6.

What can we learn from A?: If $A = 0$ then $p_1^* > \hat{p}_1$ and $p_2^* > \hat{p}_2$. As a consequence, if there is no cost borne by the recruiter, the system in case 2 leads to higher prices in comparison with case 1. We can learn much from this fact. At the beginning of an MLM firm's activities, since there is little number of distributors, competition will be at lower level and the effect of A will be close to zero. Thus, prices in case 2 will be higher and demands lower. People recruit then for a recruitment purpose. As time goes up and people are recruiting, competition will intensify and the effect of A will rise until $p_1^* < \hat{p}_1$ and $p_2^* < \hat{p}_2$. At this moment, people will recruit for a selling purpose. Thus, the evolution of an MLM firm may be shaped by two phases as shown in Figure 3.4. During phase 1, people decide to engage in the business because they are attracted by the recruitment opportunity. When the activities of the firm get intensified, it reaches phase 2 in which people realize that they need to sell. It means that firms that are just starting their MLM activities have a chance to fall into pyramid scheme.

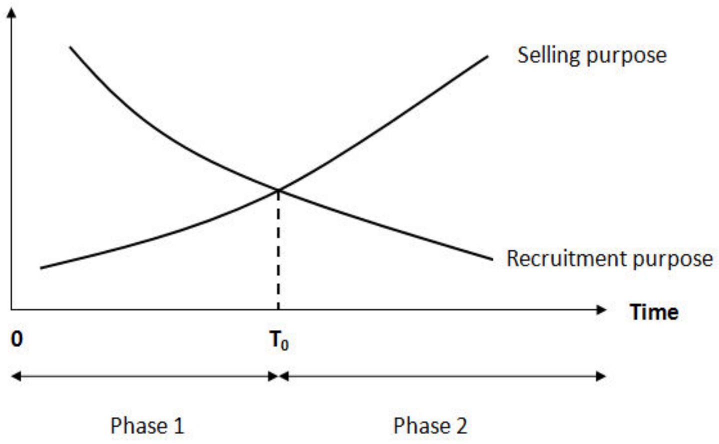


Figure 3.4: Recruitment and selling purpose curves during the firm's life in case 2

We saw that the company sells more in the business plan described in case 2 than it does in case 1. We can also argue that the company might make more profit in case 2 than in case 1, if α is sufficiently low (see illustration in Figure 3.5). We give here an illustration. Let m be the unit cost borne by the company to put the product on the market. The profit made by the firm in case 1 is $\hat{\pi}_C = (c - m)\hat{D}(p_1, p_2)$. In case 2 the profit is:[12]

$$\pi_C^* = (c - m)D^*(p_1, p_2) - \alpha c D_2^*(p_1, p_2). \tag{3.12}$$

For $a = b$, we have:

$$\pi_C^* = \hat{\pi}_C + \frac{c^2\alpha}{3}\left(-\alpha + Aa - \frac{3d}{c}\right). \tag{3.13}$$

If α is chosen such that $\alpha < Aa - \frac{3d}{c}$ then the company makes more profit by using the business plan described in case 2, compared with what happens in case 1.

3.3.2 What about the profits of the retailers?

Even though the company makes more profit and people sell more, the recruitment device of case 2 is not profitable for the retailers, compared with the case 1. As stated in proposition 3.3.3, the retailers make less profits by recruiting or being recruited.

[12]For simplicity we assume that the wholesale price c is the same in cases 1 and 2.

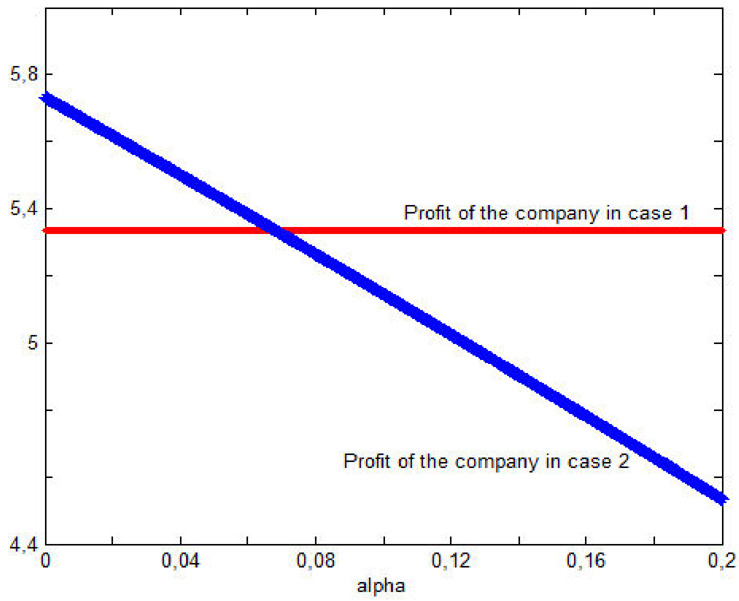


Figure 3.5: Profits of the company as functions of α. We choose $a = 2$, $b = 1$, $d = 5$, $A = 0.2$, $c = 3$, $m = 1$.

Proposition 3.3.3. *If* $\alpha < A$ *then* $\pi_1^* < \hat{\pi}_1$ *and* $\pi_2^* < \hat{\pi}_2$*.*

Proof. For retailer 2, $\pi_2^* < \hat{\pi}_2$ because $p_2^* < \hat{p}_2$ and $D_2^*(p_1, p_2) < \hat{D}_2(p_1, p_2)$. For retailer 1, we have

$$\begin{aligned}
\pi_1^* &= (p_1^* - c)D_1(p_1^*, p_2^*) + (\alpha - A)cD_2(p_1^*, p_2^*) \\
&< (p_1^* - c)D_1(p_1^*, p_2^*) < (p_1^* - c)D_1(p_1^*, \hat{p}_2) \quad \textit{since } p_2^* < \hat{p}_2 \\
&< (\hat{p}_1 - c)D_1(\hat{p}_1, \hat{p}_2) = \hat{\pi}_1.
\end{aligned}$$

□

It is full well known that the purpose of any business is not just to sell, but to earn. Selling might be just a way to achieve this goal. Our question remains then: why do people (the retailers) prefer the business in case 2 to that of case 1? We identify four reasons that justify such a behavior.

First reason: the recruiter does not internalize the competition effect

Retailer 1 recruits and faces pecuniary effect and competition effect. Like the pecuniary effect, competition effect is real and affects the prices. However, suppose that retailer 1, when calculating his profit, does not take into account this competition effect. He may ignore this competition because he does not realize its existence (even if it affect the price he sets), or he falls to assess its size. Thus, he considers his profit to be:

$$\pi_{1E}^{*} = (p_1^* - c)D_1(p_1^*, p_2^*) + \alpha c D_2(p_1^*, p_2^*) - I \tag{3.14}$$

where I is a fixed cost borne by recruiting. Of course I has a link with A. As shown in Figure 3.6, π_{1E}^{*} can be greater than $\hat{p}_1$. In fact, when recruiting, retailer 1 is looking for the reward α but ignores the competition effect generated by this recruitment. The promoters of the company can use the recruitment reward α to attract people and profit from their ignorance of the competition effect. The distributors who realize the existence of this competition will not accept the business or will decide just to recruit, or just to sell. When people recruit and do not sell, they do not face the competition effect. Figure 3.7

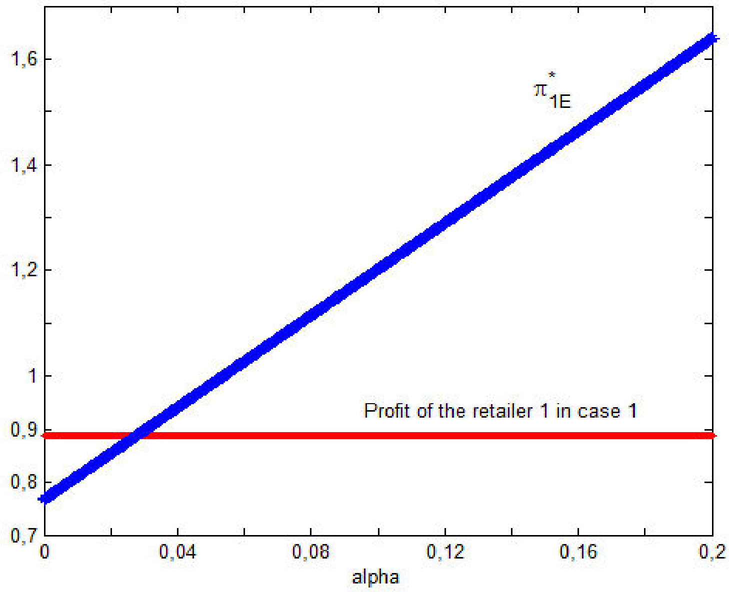


Figure 3.6: π_{1E}^{*} as a function of α. We choose $a = 2$, $b = 1$, $d = 5$, $A = 0.2$, $c = 3$, $m = 1$, $I = 0.05$.

displays the combination of α and A for which the retailor 1 and the company are better off or not with the business in case 2, compared with case 1.

Second reason: compensation on sales

To incite the retailer to sell the company owners can reward them for the volume of their sales. This compensation on sales consists in a discount on the wholesale price c, proportionally to the sales volume of the retailer. We have seen that the retailer who recruits, sells more than the recruit. By recruiting, people expect to gain from the discount on the wholesale price. This kind of compensation plan exists in the multi-level marketing

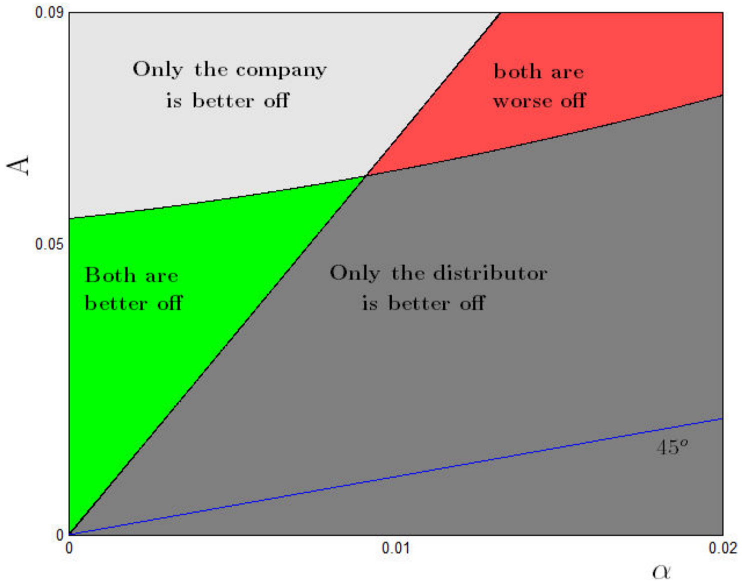


Figure 3.7: No internalization and profits improvement. We choose $a = 2$, $b = 1$, $d = 5$, $c = 3$, $m = 1$, $I = 0.05$.

business and may encourage people to have interest in selling.

Third reason: retailer 1 is at the earlier level of the chain

In the business described in case 2, distributors are organized in hierarchical chain. Some are at the top level and others are at the bottom of this chain. The distributors of the top are those who started their activity in the earlier stage of the company's life. At this stage, people do not face so much competition effect because of the small number of distributors. Thus, for the top level distributors, α can be greater than A and the pecuniary effect will overcome the competition effect. In this case, these distributors gain by recruiting. A prediction from this reality is that the price set by the distributors decrease with time during the life of the company. This fact can give the reason why some distributors (in MLM business), after having built a huge downline, stop or reduce the selling activity in order to avoid the competition effect. They, then, focus on training their recruits or giving conferences.[13] It is also probably the reason why some MLM managers are always trying to launch new product.

[13]We give other reasons for this empirical evidence afterwards.

Fourth reason: competition with a non MLM company

In section 3.3.2, we say that the business in case 2 is not profitable for the retailers, because the retailers' profits in case 2 are lower than their profits in case 1. In other words, case 1 is the benchmark. However, it is possible that for the retailers and even the owners, the benchmark is not case 1. In effect, let us imagine that it exists a non MLM company which is in price competition with the retailers. Suppose that this non MLM company sets a price p and is such that any other firm setting a price higher than p, looses all the market and earn zero-profit. Let us assume now that $p_1^* < p_2^* < p < \hat{p}_1 = \hat{p}_2$. Within this context, by engaging in the business of case 2, the retailers earn positive profit instead of zero-profit.

This simple introductory model shows how a seller may allow other agents to enter the market even though he intends to sell and not to make money from others.[14] Our goal in this section is to give an insight of our explanation of the paradox we mentioned above. Even if the environment described by this introductory model does not fully characterize the MLM plan it shapes it enough. However, we study a more complex model in section 3.4.

3.4 General framework

Analyzing MLM organizations, we can distinguish four groups of agents: the MLM company's managers, the current distributors, the customers and the non MLM companies competing with the MLM company. As we said before, the customers are those outside the MLM program. They are potential consumers and potential distributors. Among them the current distributors will decide whom to recruit and whom to sell the products to.

When we consider a typical distributor, in addition to the MLM company manager, the customers, the non MLM companies, he is playing with his upline, his downline and the other distributors who are not in his network. The interaction between agents can be shown using the following diagram in Figure 3.8. A customer's status (consumer or potential distributor) depends on the reason why the distributor is soliciting him. He will act as a consumer if the distributor is willing to sell him the product. In the same vein, the customer will act as a potential distributor if the distributor's solicitation is of a recruitment purpose.

[14]We can also consider a similar model where the wholesale price c is endogenous and chosen by the company. Thus, at a first step and given c, the retailers (competitors) chose they prices like in equations (3.1), (3.7) and (3.8). At a second step, the company takes into account, in his profit maximization, these reactions of the retailers to c. The results are similar to what we find here.

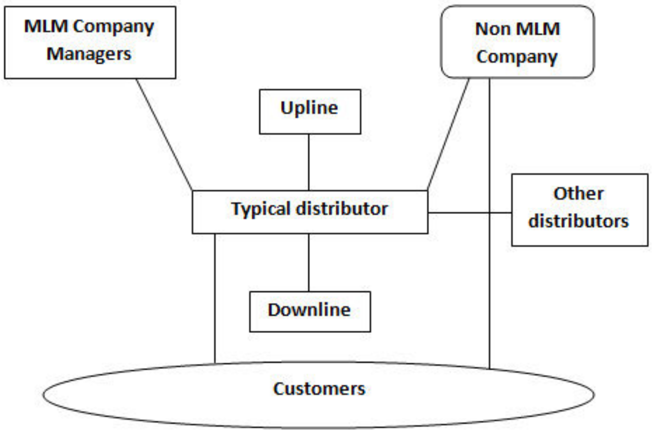


Figure 3.8: Interaction between actors in MLMs

3.4.1 The game

In this part, we briefly describe the game in order to allow a casual reader (or in a hurry) to skip the section 3.4.2, where we give details about the different actors of the game.

We consider four players: a representative consumer, a manager of the MLM, a representative distributor i, a representative distributor j who is representative of the upline of i. The consumer decides to consume the product either from the MLM or from the non MLM company. He can also decide not to consume the product. The consumer derives a utility v per consumption unit. If the price of the good is p then the payoff of the consumer is $v - p$, which is maximized. Concerning the manager, he chooses the wholesale price c at which distributors purchase the product to sell, the commission on sales (denoted by A), the commission on recruitment (denoted by B). In addition to c, A and B, the manager chooses a signal v for the product. The role of the signal is to affect the consumer's valuation of the product. The manager's decisions are made such that his profit is maximized. Distributor i decides the price p_i at which he will sell the product, the quantity q_i to purchase from the MLM, the times τ_i and t_i to allocate respectively to selling and recruitment activities. The distributor can also decide to consume or not the product and derives a utility v^i_{MLM}. The difference between distributor i and his upline j is that the upline receives a commission B on the sales (or purchase) made by distributor i.

3.4.2 Description of the agents

The consumer

The typical consumer c makes the decision to consume either the product from the MLM or from the non MLM company. The consumer's valuation regarding the products in the market is v^c_{MLM} for the MLM and $\bar{v}^c$ for the non MLM. We assume that the valuations of the consumers are continuum and distributed over $\mathbb{R}_+$. Let us consider typical distributor i who sets a price p_i. The price of the other distributors k is denoted by p_k. The price of the non MLM company is $\bar{p}$. The consumer decides to consume or not by solving:

$$\max\{\bar{v}^c - \bar{p}, (v^c_{MLM} - p_i)\mathbf{1}_I, \max_{k\neq i}\{(v^c_{MLM} - p_k)\mathbf{1}_I\}, 0\} \tag{3.15}$$

For simplicity, in the rest of the paper we set $\max_{k\neq i}\{(v^c_{MLM} - p_k)\mathbf{1}_I\} \equiv (v^c_{MLM} - p_{-i})\mathbf{1}_I$. The problem of the consumer is then:

$$\max\{\bar{v}^c - \bar{p}, (v^c_{MLM} - p_i)\mathbf{1}_I, (v^c_{MLM} - p_{-i})\mathbf{1}_I, 0\} \tag{3.16}$$

where $\mathbf{1}_I$ stands for the indicator function for the availability of the product in the market. We introduce this availability indicator function to account for the fact that some MLM's distributors are engaging in MLM program as a part time activity, particularly when they are at the beginning of the business.[15] Thus, it is possible for a given distributor to charge a price and not to be willing to allow sufficient time for selling to all consumers who are ready to buy at this price. The consumer will buy the good when the price is competitive and the product is available. However, we consider that the product is always available in the non MLM company. Most of the time, the actors of non MLM companies are in full time activity. Even if, physically, the product gets unavailable in the non MLMs, we suppose that it is available and is with no valuation for the consumer. An immediate consequence of this setting is that a consumer buys from the MLM if the MLM provides him the best option in comparison with the non MLM. In contrast, the consumer may purchase from the non MLMs even if this alternative is not the best for him. Moreover, when the consumer decides to consume the product, he should also make decision about the quantity q_c of product to consume. As known in the literature, we assume that the consumer's demand from the MLM is a differentiable function of the price p as follows:

$$q_c = f_c(p), \quad \textit{where} \ \frac{\partial f_c}{\partial p} < 0 \tag{3.17}$$

[15]We know full well that some people join the MLMs as full time job. These people are generally unemployed or undereducated. In the same vein, some other enroll to the scheme as a part time activity and go full time afterwards.

Let $\mathbb{C}$ be the following set.

$$\mathbb{C} = \{c \backslash \max\{\bar{v}^c - \bar{p}, (v^c_{MLM} - p_i)\,\mathbf{1}_I, (v^c_{MLM} - p_{-i})\,\mathbf{1}_I\} > \max\{\bar{v}^c - \bar{p}, 0\}\} \tag{3.18}$$

$\mathbb{C}$ is the set of customers who are willing to buy from the MLMs distributors. It is worthy to mention that $f_c(p) = 0$ if c is not in the set $\mathbb{C}$. In other words, if for the consumer the non MLMs product is the best then his demand to the MLMs is equal to zero. In fact $f_c(p)$ is the consumer's demand function towards the seller whose product he decides to buy.

As some might realize in our setup, the total demand is not just determined by the proportion of consumers willing to purchase the good (as for instance in Rochet and Tirole (2003)) but by the quantity purchased by each consumer as well. This is in line with what we said in section 3.2.2. Most of the MLMs' products are repeatedly demanded by the same persons, and an MLM's distributor can make money on people. A consequence of this fact is the following: when distributor gets one more person as costumer (consumer), his sale volume is impacted to (relatively) large extent. In the same vein, loosing a customer is harmful for the demand on a large scale. Distributors would strongly care about loyalty.

The customer as a potential recruiter

As a potential recruiter, customer r decides the time t_r to allocate to recruitment. Any recruiter bears some costs by recruiting (time, equipment, opportunity cost). The costs are fixed cost I and *outside income* $Y_r(t_r)$. The outside income is what the recruiter would earn if he used his time for another activity rather than being engaged in MLM activities. For an individual r, let $\theta_r(t_r)$ be the gain (or expected gain) from recruiting. Thus, any individual r decides to recruit if $\theta_r(t_r) \geq Y_r(t_r) + I$. We assume that the function Y_r is smooth and non decreasing in the time. We have:[16]

$$\theta(t) = B \int_{G(t)} q_r dr \tag{3.19}$$

where B is the commission received from a unit of the downline purchase. B depends on the compensation plan set by the company managers. q_r is the quantity purchased (and sold) by the downline of the recruiter r. $G(t)$ is a set such that:

$$\mid G(t) \mid = \alpha(t) \mid \{r / \exists \underline{\tau}, \theta_r(\underline{\tau}) \geq Y_r(\underline{\tau}) + I\} \mid \tag{3.20}$$

[16]In this work all integral is defined with respect to the Lebesgue measure.

where $0 \leq \alpha(t) \leq 1$ and $\alpha'(t) > 0$. $\{r/\exists \underline{\tau}, \theta_r(\underline{\tau}) \geq Y_r(\underline{\tau}) + I\}$ is the set of all customers to be recruited. $\alpha(t)$ represents the share of this population of customers that the distributor will succeed to recruit if he spends a time t in recruiting. We can also use a subscript i for $\alpha(t)$ (and then for $G(t)$) to denote the customers recruited by a specific distributor i. The $\mid . \mid$ symbol denotes the size of the set with respect to Lebesgue measure. If we had assumed there is a mass of customer normalized to 1 then we could take $\mid G(t) \mid = \alpha(t) Prob\left(\theta_r(\underline{\tau}) \geq Y_r(\underline{\tau}) + I\right)$.

The customer as a salesperson

A salesperson s makes decision about the time τ_s to allocate to selling, the price p_s to set and the quantity q_s of the product to purchase from the MLM's company. He purchases at a unit cost c set by the MLM's managers. We assume that if the seller allows τ_s to selling activity, conditionally to the market characteristics, he will sell $d_s(\tau_s)$. Function $d_s(.)$ is a smooth and non decreasing function of the time. $d_s(.)$ characterizes the *market force* (or the sale force) of the salesperson s. The salesperson could decide to consume himself a certain quantity, $q_s - d_s(\tau_s)$, of the good and derives a utility $v^s_{MLM} - c$, but by forgoing the utility $\bar{v}^s - \bar{p}$ he would derive if he consumes the corresponding product from the non MLM producer. In the same vein, a priori, the salesperson might buy the portion consumed from other salespersons k and would derive the utility $v^s_{MLM} - p_k$, where p_k represents the price set by k. When purchasing from the MLM's company, the salesperson gets commissions A on each unit of purchase. In total, the profit (or gain) from selling is:

$$\begin{aligned} \lambda_s\left(\tau_s, p_s, q_s\right) = p_s d_s(\tau_s) - c q_s + v^s_{MLM}\left(q_s - d_s(\tau_s)\right) \\ - \left(q_s - d_s(\tau_s)\right) \max\left\{\bar{v}^s - \bar{p}, (v^s_{MLM} - p_{-s})\mathbf{1}_I\right\} + A q_s - Y_s(\tau_s) \end{aligned} \tag{3.21}$$

As we will see in proposition 3.4.1 the salesperson will not buy from the other distributors. The payoff from selling becomes:

$$\begin{aligned} \lambda_s\left(\tau_s, p_s, q_s\right) = p_s d_s(\tau_s) - c q_s + v^s_{MLM}\left(q_s - d_s(\tau_s)\right) \\ - \left(\bar{v}^s - \bar{p}\right)\left(q_s - d_s(\tau_s)\right) + A q_s - Y_s(\tau_s) \end{aligned} \tag{3.22}$$

To set the price p_s, the salesperson takes into account the wholesale price c, any payment he receives from the MLMs, and particularly the valuations of the consumers. While these valuations are not really known, the salespersons have to choose the price in a way to be the best for the consumer, in comparison with the non MLMs company. In our model, the problem is not the uncertainty about the valuations, but how to propose the best option to the consumers. If the MLMs company is the only firm producing the good then the managers will act as a monopole. They will fix the wholesale price c, taking into account the production cost m, the commissions and different payment to be borne. The wholesale price should be set in such a way to allow distributors to propose an attractive retail price $\bar{p}$. In turn, the distributors fix the retail price properly. Moreover, if both the MLMs and the non MLMs are producing the good then a price competition may lie between them. Thus, the MLMs wholesales price c would be higher than the non MLMs wholesale price, everything equal elsewhere. However, the retail price set by the distributors may be competitive regarding the retail price of the non MLMs product. Since in our framework the price fixed by the rivals is known, the distributors can set a price just below or around the retail price of the non MLMs product, if such a decision allows them to make positive profit. This could insure them to fill the consumers condition in (3.16). The MLMs promoters should act (make decision) accordingly to allow distributors to set a competitive price while making positive profit. We will come later to the price setting in detail.

In the expression (3.22), $\lambda_s(.)$ might depend on the price of other distributors and the price of the non MLMs as well. Obviously, ex-post, there is a link between $d_s(.)$ and the consumer program in (3.16). In addition to what we have said in section 3.4.2, taking the demand not in term of the number of consumers willing to purchase but in terms of the quantity purchased is crucial to take into account the personal consumption (self-consumption) of distributors.

The company managers

The company managers (also called *parent company* in Vander Nat and Keep (2002)) choose the wholesale price c at which the distributors purchase the product from the MLM. Since the managers should reward the distributors, it is possible they get unable to do price competition with the non MLM's company. Let us remember that what matters for the consumer is not the price per se, but the difference between the price and the valuation. The MLM's managers will also engage in a competition on the valuation of the product (quality competition somewhat). They will try to increase the consumers'valuation of the MLM product by giving a signal v. They do so by using various ways. Either they launch new product which is not existent in the non MLM's company, improve the quality of the product or try to convince that the MLM's product is a particular one of high quality. However, they bear a cost $K(v)$. Let us make the precision that the managers, like any other agent in our framework, do not observe the valuations of the consumers. They can just impact valuations through v. In addition to c and v, the managers should decide the commissions A and the compensation plan that affects B. The payoff of the managers is:

$$\pi_{MLM} = (c - m - A - B)\, Q - K(v) \tag{3.23}$$

where m is the unit production cost and

$$Q = \int_{\mathbf{S}} q_s ds \quad and \quad \mathbf{S} = \{s, \theta(t_s) + \lambda_s\,(\tau_s, p_s, q_s) - Y_s(t_s + \tau_s) - I \geq 0\} \tag{3.24}$$

The managers will choose a value for c that allows to cover A and B. Therefore, increasing A or B may imply a need to increase c. Likewise, increasing c implies an increase of the total amount of commissions since these commissions are calculated as a percentage of the sale volume evaluated in the wholesale price.

Here we consider $K(v)$ as not depending on the production level. It may seem abstract and unrealistic. Of course, improving quality requires investments and costs depending on the quantity produces and sold. Such costs can be included in the production cost m. In our framework, $K(v)$ is the cost borne by the MLMs promoters to improve the quality of the product for the purpose of influencing «ostentatiously» the valuations of the consumers. Precisely, it is a matter of a strategy which is particular to the MLMs organizations. For instance, every so often, the MLMs promoters give training to their representatives on the «in-person presentation» of the products and how to use them. The goal is to improve the quality of the service given by the distributors. Another example is any kind of motivation (holidays trip, guided tour, study trip, etc...) the promoters give to distributors to incite them to be interested in the product consumption. These are some examples of valuation improvement that are specific to the MLMs organizations, and the cost borne does not depend on the production volume.

As one could see, the improvement of valuations through $K(v)$ is based on distributors. An implication is that the distributors will give more value to the MLMs products than the others customers will do. This is the cornerstone of the network marketing and is one of the goals being targeting by the MLMs promoters. In fact, the MLMs promoters are willing to market their products through a network of distributors. Thus, any success requires the promoters to convince the distributors about the valuation of the products.

The current distributor i

The current distributor is a combination of the recruiter and the salesperson. He is willing to sell and to recruit. If an individual i is a distributor then his total payoff from MLM activity is positive.

$$\theta(t_i) + \lambda_i(\tau_i, p_i, q_i) - Y_i(t_i + \tau_i) - I \geq 0 \tag{3.25}$$

As we will see hereafter, no customer will not just be a salesperson. In other word, when people decide to engage in MLMs, they are necessary interested in recruitment. Moreover, most of the MLMs organizations require their distributors to purchase a minimum amount of the product, normally for a selling purpose. Thus, ex-post, the condition for a customer to enter in MLMs plan (to make money) is given by (3.25). The set $G(t)$ in (3.20) can be then rewritten accordingly. Let us mention that, when making his decision, the distributor is not just looking for a positif payoff. He is willing to maximize this payoff. The decision about which activity (selling or recruitment) to concentrate on would depend upon the capacity of the individual to sell, the customers and the market conditions.

The distributor's payoff includes the utility got from his self-consumption of the MLMs product he is selling. A question is to know whether the seller will consume the non MLMs product or not if he is selling the same product. It depends mainly upon the retail price chosen by the non MLMs and the wholesale price c. Using the following proposition 3.4.1 we will argue that the distributor will not consume the non MLMs product. In other words, the MLMs product he is selling will be necessary his best option for consumption.

Proposition 3.4.1. *If the gain from recruitment is minor then the distributor will not consume the non MLMs product if he is selling the same product.*

Proof. If the MLMs firm is the only one producing the good then the result is trivial. Consider now that both the MLMs and the non MLMs are producing. If c is smaller than $\bar{p}$ then the result is figured out since the salesperson purchases from the MLMs at the cost c. Let us now suppose that c is greater than $\bar{p}$. Let us remind that the consumer will not buy from the MLMs if the non MLMs is the best option for him. If so, a high wholesale price is a consequence of two facts: either the MLMs promoters are convinced that the consumers will prefer their product by valuing it accordingly, or the transfers allowed by the compensation plan are considerable (commissions and other payments are relatively high). If the consumers value much and prefer the product then the distributors will value it more (from our comments on $K(v)$ in section 3.4.2), and then, will not consume the non MLMs product. Let us now consider that the reason is the compensation plan. Knowing that the reward for recruitment is minor, the compensation plan is favorable to the distributors who purchase much. As a consequence, if he derives utility from the consumption of the MLMs product the distributor will consume it in order to get commission on the purchases. □

Let us mention that proposition 3.4.1 does not claim that the distributor will necessary consume the MLMs product he is selling, but he will not purchase and consume the same good from the non MLMs firm. He will consume the MLMs product if it provides him with the utility he desires. Moreover, even though the gain from recruitment was considerable we can also argue that distributor would not consume the non MLMs good. As a matter of fact, when the commission from recruitment is higher the distributor would probably prefer allocating his time to recruitment and less to selling. Since the managers require from the distributor a minimum purchase, he will consume the product.

It is worthy to notice that under the assumptions of proposition 3.4.1 the distributor purchases more than he sells to the customers outside the MLMs program, that is, in the expression (3.22), $q_s - d_s(\tau_s)$ is positive.

The non MLM company

The non MLMs sets its price $\bar{p}$ strategically. However, in our framework we are not interested in the non MLM company's actions. The other agents take the non MLM's actions as exogenous. However, we consider any kind of competition between MLM and non MLM companies.

3.4.3 Timing of the game

The timing we consider for the game in the MLMs is as follows:

(i) The MLM's managers observe the non MLM's price and choose the signal for its product valuation. They set the wholesale price c.

(ii) Each current distributor takes the wholesale price as given and sets his price, the time to allocate to recruitment and selling and the quantity of product to purchase from the MLM's company. The decision of a typical distributor may depend on the decision of the other distributors.

(iii) The customers choose the time to allocate to recruitment and selling. They also choose the consumption pattern if they decide to be consumers.

The interaction between players is as described in Figure 3.8.

For this work, we make the following key assumption.

Assumption 3. *The different prices set are publicly known by the distributors, the consumers and all actors in the business.*

Some may think that full information about the prices is a strong assumption. In this article we are intending to study a phenomenon that is specific to the MLMs and which is not observed in the non MLMs business. Even if the prices may not be full well known, this asymmetry may also exist in the non MLMs business. As a consequence, when studying the MLM in comparison with the non MLM, the information asymmetry about prices could not matter as much.

3.5 Distributors' decisions in the MLMs

In this section we discuss how different distributors in MLMs make decisions by giving some simple characterizations of policies. These policies concern essentially prices setting, the allocation of time to each kind of activity (recruitment and selling). We also discuss about some implications of the distributors' strategies.

3.5.1 Distributors and pricing

Inside the distributors' group, the competition is a price competition as well.[17] When a certain number of distributors are operating in the same market, and they are really intending to make sales, they should engage in price competition. Since they are selling a product that they do not produce, they are limited in quality competition.[18] According to the basic Bertrand setting, at equilibrium, the distributors will set the same price since they are marketing the same homogeneous commodity and facing the same marginal cost. With MLM plan, things are more complex than in the traditional Bertrand competition. First, distributors are not necessarily engaging in MLMs as their main activity. Therefore, they may not be intending to satisfy all demand they are facing. They have in mind to make money and will not accept zero profit pattern. Moreover, some of the business transactions are grounded on relationship base. Second, even in MLMs, people have other choice than selling: recruitment. Recruitment and selling are not independent activities.

[17] Some can also imagine other types of competition like advertising, R $\propto$ D, Cournot competition, quality competition; but within the MLM's context the number of distributors may be so high to not allow for Cournot. At the same time it is obvious that distributors can no longer engage in R $\propto$ D since they are not the producers of the good, and then cannot do a quality competition as much. Likewise, traditional advertising is of less importance in direct selling as we said before.

[18] The distributors may also engage in non-price competition by improving their sales force $d_i(.)$ using some specific ways. Among these ways we can list the after sales service, help for the product's use, close relationship with the customers and specific training to improve their service. As an anecdote, I have attended to a presentation by a Forever Living Product distributor. The distributor is a friend of mine and was engaged in beauty products selling. At the end of the presentation, she engaged a private short discussion with each of us in the audience. During this discussion she told us that she is university trained in matters of beauty products. She is among (in her own words) the highly qualified for selling such products. This claim can be seen as a non-price competition strategy.

Third, distributors may use the product for personal consumption, and this personal consumption is not directly affected by the price they set. As a consequence, the findings may be quite different from an usual setting. The following theorem (theorem 3.5.1) gives a first result on the price decision by the distributors.

Theorem 3.5.1. *Two distributors i and j operating in the same market will not set the same price. Moreover, it exists $\varepsilon > 0$ such that $0 < |p_i - p_j| \leq \varepsilon$.*

Proof. See appendix 3.10.1. □

According to some public statements by MLMs officials and distributors, the commissions got for recruitment are by far smaller than what people can be paid by selling. In other words, if a distributor is willing to maximize his profit and make money, he should choose to sell, under some conditions. Naturally, any decision would depend on the ability of the distributor to sell and to recruit. However, if we consider that the goal is to make money mainly from selling then distributors operating in the same market will not set the same price. While this result may not seem expectable, it is fairly intuitive. Indeed, the parameters affecting the price making decision of the distributor are not only the wholesale price c and the commission on sales A, but also the *outside income*(opportunity cost) of the distributor. Since selling implies using one's time, those who have relatively higher outside income (and then higher opportunity cost) would set higher price. If distributors do not have the same outside income they will not set the same price. Moreover, according to the theorem 3.5.1, even in case where distributors have equal outside income, they do not set the same price. Indeed, we would expect, due to competition, the distributors to fix price up to the limits allowed by the wholesale price, the commissions on sales and the outside income.[19] If so, the profit on sale would be equal to zero and the distributors will not accept to run selling activity. This rules out the standard findings in Bertrand competition. The second part of the theorem states that, even though distributors do not set the same price, the gap between the prices will not be high enough for a simple reason: if the gap is sufficiently deep, the distributor who fixes the lower price can increase his price sufficiently enough to offset his outside income and still attracting more consumers. Moreover, since we assume that distributors know the price of each other, it may seem too risky for a given distributor to set a price that is by far higher than that of the other distributors.[20] This finding is pictured in Figure 3.9 where the colored areas represent the couple (p_1, p_2). Actually, as shown in Figure 3.9B, contrary to what we can see in 3.9A,

[19]We are drawing an analogy with Bertrand competition.

[20]We recognize that things might depend on the ability of the distributor to convince the consumer, to show the advantages of the products' use and on the relationship that the distributor had built with the customer. However, since the true valuations of the consumers are not known, the distributor should be prudent, knowing full well that the prices are publicly known.

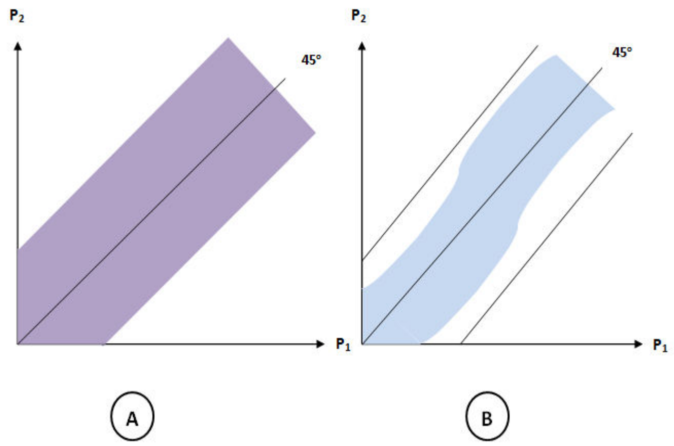


Figure 3.9: Prices setting in MLM

the frontier of the domain where lies (p_1, p_2) is not held by straight lines but may be shaped in any manner by the prices. As we said, the gap between the prices depends as well on the gap between the outside income of the distributors and their gains from the MLM's business. We will see later that it also depends on the capacity of the distributors to face competition in the market. These things are not necessarily linear function of prices.

Knowing full well that two distributors with equal outside income fix different prices, it would be interesting to know, among the two distributors, who is prone to set the higher price. The following theorem shows that if one distributor is the upline (sponsor in Amway's terminology) of the other then the latter will set the higher price.

Theorem 3.5.2. *Let us consider two distributors i and j operating in the same market with j the upline of i. Assume that i has no downline. If i and j have the same outside income and market force, then $p_i > p_j$.*

Proof. Let us consider two distributors 1 and 2 with 2 the upline of 1. Here we will prove that we can have $p_1 < p_2$ at equilibrium. Precisely, we show that if $p_1 < p_2$ then some of the distributors will deviate. In other words, we would like to show that $p_1 < p_2$ is more stable. Leaning on the proof of the second part of the theorem 3.5.1, we can see that the distributor who will set the lowest price is the one who needs more increase of price to accept to sell more. In other words, if the outside option (what the agent looses by taking an action) of 2 is relatively high then distributor 1 can set his price above p_2, and if the outside option is relatively low then $p_1 < p_2$. Moreover, since distributor 2 gains from the time allocated to recruitment, every equal elsewhere, his outside option is higher than that of distributor 1. As a consequence, $p_1 > p_2$. Let us give an illustration. Suppose that the downline of distributor 2 is just distributor 1. If the payoff of distributor 1 is $p_1d_1(\tau_1) - cq_1 + v^1_{MLM}(q_1 - d_1(\tau_1) - (\bar{v}^1 - \bar{p})(q_1 - d_1(\tau_1) + Aq_1 - Y_1(\tau_1) - I$ then the payoff of distributor 2 is $p_2d_2(\tau_2) - cq_2 + v^2_{MLM}(q_2 - d_2(\tau_2) - (\bar{v}^2 - \bar{p})(q_2 - d_2(\tau_2) + Aq_2 + Bq_1 - Y_2(t_2 + \tau_2) - I$. If $p_1 < p_2$, for distributor 1 to accept to sell k more units of the product, he should increase the price by ε such that $(p_1 - c + A)k - Y_1(\tau_1 + \tau_1^k) + Y_1(\tau_1) + \varepsilon\,(d_1(\tau_1) + k) > 0$. Moreover, if distributor 2 increases his sales in such a way to take away a part $\bar{k}$ of the demand for distributor 1 he will loose on what he gains from the downline. As a consequence, if $p_1 > p_2$ then, for distributor 2 to accept to sell k more units of the product, he should increase the price by ε such that $(p_2 - c + A)k - Y_2(t_2 + \tau_2 + \tau_2^k) + Y_2(t_2 + \tau_2) - B\bar{k} + \varepsilon\,(d_2(\tau_2) + k) > 0$. One can see that ε must be higher for distributor 2. □

From the theorem 3.5.2, the upline will propose the most competitive price in comparison with his downline if the latter has not recruited. In fact, in situation of price competition the upline has a *price leading power* on his downline since he gains from the sales made by the latter. He can lower his price and compensates any loss (due to low prices) by what he gets from the sales made by the downline. In this business the upline seems to be 'financially' more 'comfortable' than the downline. Actually, what we are saying is that, in an environment where all the distributors are willing to sell, the upline could be leader in price since the game allows him to be ahead of the downline. This is endogenous and intrinsic to the game. What we are stating has some possible link with the theory about the impact of the financial structure of the firms on competition (Cournot, Bertrand and other) in oligopoly market (see for instance Brander and Lewis (1986), Maksimovic (1988), Showalter (1995), Poitevin (1989)). The financial structure of a firm influences its financial condition (Brander and Lewis (1986)). Under some given conditions (uncertainty about demand or costs) in Cournot competition, the fortune of the firm can be improved if its rivals are in financial distress. Chen et al. (2007), studying the effect of the firms's debt on price competition in retailing industry, argued that in 'distressed industry' the highly levered firm needs to set relatively high price in order to face its financial challenge (debt repayment). In order words, the firms' financial climate affects the competition issue in the market. The difference between this theory is that the former takes the financial structure and the competition as two things apart. The financial structure is not intrinsic to the competition but just affects it. Most often, the pioneers of this theory consider two-stage game where at the first step the firms make decision about their financial structure that will determine their financial condition. At the second step, given the financial structure, competition strategies are played. In our setting, the *price leading power* is tied to the market forces and, then, its strength depends on the actions taken by each actor. This 'financial leg up' got by the upline comes from the market. This advantage given by the market make the upline a sort of leader managing the downline.

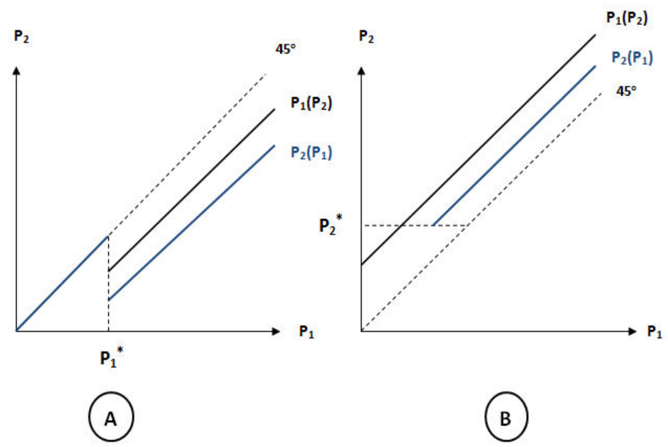


Figure 3.10: Price leading power in MLM

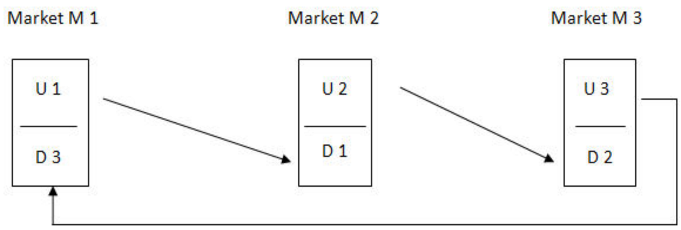
However, the upline cannot be too aggressive in this competition and should allow the downline to sell. The reason is that the upline can no longer hold this *price leading power* if ex-post the downline does not sell, given that the commission are supposed to depend on the downline'sales. The upline will act accordingly and strategically. He will use his power to compete in such a way to be still holding this power. More precisely he can reduce his price down to the minimum price the downline can support. Below this minimal price, the upline looses the *price leading power*. Figure 3.10 gives an illustration. In Figure 3.10, distributor 2 is the sponsor of the distributor 1 and the curves represent the variation of the price of one distributor with the price of the other distributor (best response functions of prices). A given price curve can be seen as the capacity of the corresponding distributor to face the aggressiveness of his rival. Let us assume for simplicity that the market has reached a point where all the distributors would like to fill all the demand and have time to do that. For this illustration we consider the competition to lie between distributors 1 and 2. Figure 3.10A describes a situation in which the dowline has no recruit. Distributor 2 holds price leading power and can set the best price. Strategically, he cannot set his price below p_1^* the minimal price distributor 1 can accept. Otherwise, below p_1^* the upline looses his power and we observe a jump of his price curve.

As we said, this situation seems to put the downline in subordination situation. The recruit could feel to be manipulated by the sponsor who is acting to maintain his power. Thus, he would prefer to overwhelm this price leading power. To do so, the only way is to recruit because more the downline sells, more he provides the upline with leading power. Moreover, he will be able to sell the quantity he desires only if the upline is willing accordingly. If he has no recruit, the downline is not free in his selling decision if they is competition in the market. For this reason, he needs to allocate a part of his time to recruitment in order to overcome and even get the power. If the downline recruits and obtains the price leading power, Figure 3.10B gives an illustration of what will happen. There is no kink here because the price leading power of the downline is not based on the upline's sales since he gains nothing from the upline retail sales.

A lesson from our analysis above is that, contrary to some claims by MLM promoters, the business does not guarantee equal opportunity for all the distributors. Endogenously, the upline has more chance to succeed in making back his investment. Therefore, willing to get the same success opportunity, distributors resort to recruitment.

Some, with good reason, may argue that distributors should just recruit in other areas than their own marketplace if they are really looking for the price leading power. The recruits in other areas provide this power and, in addition, are not really competitors. However, when the distributors choose to recruit just in other areas, each of them is running the risk to see his potential customers (in his marketplace) being recruited by other distributors. These potential customers become his competitors on whom he has no leading power. This might put the business in an embarrassing situation where any distributor may not hold a leading power, as shown in the following example.

Example 3.5.3. *Let us consider three marketplaces M_1, M_2 and M_3, each of them being operated by distributors U_1, U_2 and U_3 respectively. As pictured in the figure below, assume that distributor U_1 recruits the customer D_1 in the marketplace M_2, U_2 recruits D_2 in M_3 and U_3 recruits D_3 in M_1.*



We can prove that none of the distributors U_1, U_2 and U_3 are sure to get and hold a price leading power. Assume that U_1 holds the power to compete successfully with D_3. As a consequence, D_3 cannot sell enough and U_3 will not have a leading power. D_2 can sell, providing a sufficient power to U_2. Thus D_1 will not sell enough and U_1 does not hold the power. This is in contradiction with our starting assumption. As conclusion, even if the distributor recruits in other areas, he will also recruit in his neighborhood.

In example 3.5.3 above D_1, D_2 and D_3 may have problem to find customers to sponsors in their areas. If they get unable to build downline they will probably leave the program.

This can explain the relatively high turnover rate experienced by some MLMs (see for instance Taylor (2011)) even if Vander Nat and Keep (2002) expected that the MLMs' turnover rate will be considerably reduced in comparison with the simple direct selling.

We can also use another way to explain the reason why distributors need to recruit even if they are willing to sell. The compensation plan affects the wholesale price which is the same for all distributors. If the MLM's managers cut or remove the commissions, the wholesale price would be lower and the distributors' income would be just in form of retail profit. In other words, the compensation plan can be seen as a procedure by which the MLM's managers keep with them a part of the distributors' profit and redistribute it afterwards in form of commissions. In this situation, distributors have to recruit in order to take part in the redistribution. In other words, the compensation plan is a sort of "taxes" paid by the distributors and which are redistributed to those who meet some given conditions. This reality strengthens the role of the leading power, in that the distributor had paid fees that finance the commissions. Deciding not to recruit is equivalent to contributing to the power of the rivals.

3.5.2 Discussion

The implications of the *price leading power* have some evidences in real world. Every so often, we observe a sort of arrangements (agreement) between the upline and its downline. As we said, for a distributor, the *price leading power* cannot exist and be sustainable if his downline is not able to sell. Thus, when the downline has difficulty in selling and pursuing the MLM business, his sponsor helps him in various ways. The sponsor either makes monetary transfer profiting the downline, recruits and builds a downline for him, or helps him to sell the product. This strategy is used by the upline to preserve his *price leading power* by preserving his downline. In the same vein, a distributor of high size downline holds high *price leading power.* This power may be harmful (due to competition) for the downline if the distributor uses it to sell. This gives the reason why, most of the time, the distributors who have reached a higher level in the business decide not to sell enough but use their time to train their downline and hold conferences about the MLM's opportunities. This mutual dependence and "solidarity" between the MLM actors can explain the sustainability of such organizations.

3.6 The MLM managers policy: a two-sided market

The managers set the wholesale price and make decision about the compensation plan. In line of the goal of this article, we are interested in the decisions made by the MLM holders and how these decisions impact the distributors' behavior. To do so, we need to understand the environment in which the managers are acting. Knowing this environment could also be helpful to solve simply the problem using previous theories. The MLM managers are considered to be private and guided by profit-maximization ethic. Actually they are operating in a two-sided market (or two-sided platform) built by themselves. It may seem surprising that the multi-level marketing gives rise to a multi-sided market. However, when taking a good look at the program we can see this reality. Of course, the concept of multi-sided market remains ambiguous (Rochet and Tirole (2006)).[21] Most of the time the authors define multi-sided market using typical and canonical examples. In general, a multi-sided market is one in which an operator (platform) acts as an intermediary between two or more groups (sides) of agents (users) and where there is externality through which the members of one group get benefit or damage from the members of other groups. The common examples for these markets are the market of credit cards (MasterCard acts as intermediary between consumers and merchants), newspapers (New York Times is an intermediary between readers and advertisers). In this paper we adopt the concept as viewed by Rochet and Tirole (2003), Rochet and Tirole (2006) or Weyl (2010). Precisely, most of the multi-sided markets shared the following characteristics.

1. The platform acts as an intermediary by proposing to each side a specific service, product or opportunity. It may propose different services to the sides. As such, multi-sided platform can be seen as a multi-product firm. The services can be given from the platform to the sides or from the sides to the platform.

2. There is network effect through which the benefit or damage that the members of one side are provided with is affected by the numbers of users on the other sides. There is cross network effect in the words of Weyl (2010). It may also exist network effect inside the sides.

3. The platform sets different prices (or treatments, rewards) for all sides in compensation of the service provided. The network effect raised should not directly depend on the prices.

The network effect concept raised in the second point is the network effect as pioneered by Katz and Shapiro (1985). If the effect is positive then the users will express less interest in being members of one side if there is no agent willing to be member of the other sides. Similarly, if the network effect is negative then nobody is interested to be member of one side if there is high number of agents on the other sides. Moreover, according to the last point characterizing the multi-sided market, and in line of the first point, the

[21]Rysman (2009) even states that "virtually all markets might be two-sided to some extent".

compensation of the intermediation is not necessary a cost borne by the members on the different sides, but can be a reward benefiting them.

We can argue that, to some extent, the multi-level marketing conforms with the three features above. The company's managers, through the products (or service) they are marketing, propose two different kind of activity to the customers: Recruitment and selling. Those who decide to recruit are said to be on recruitment side and those who sell are on selling side. There is no doubt that more there are sellers more it is interesting to be recruiter. Moreover, from our previous analysis, the more there are recruiters, the more the recruitment side is harmful for the sellers because this prevents the sellers from having the price leading power, and then affects their market force $d_s(.)$. Thus the two-sided market induced by the MLMs is similar to the newspapers' two-sided market. However, it is useful to deal with some possible concerns about the two-sidedness of the MLMs.

First, some may think that the positive network effect that the recruiters enjoy from the sellers is questionable, in that, for a given recruiter not all the sellers provide him network effect but those who are in his downline. This reality is not specific to the MLMs. A given merchant will not get network benefit from all the consumers who are holding visa card, but from those who are inclined to make transaction with him. This aspect of the problem is not necessary of less importance. However, since the literature (on multi-sided platforms) ignores it we then simplify our framework by ignoring it. Second, contrary to Visa, Newspapers or dating platforms, apparently there is no direct interaction between recruiters and sellers. However, such an interaction exists, even though indirect, and is done through the MLM's product and the customers that both recruiters and sellers are targeting. Since recruiters and sellers are competing to reach the same group of customers, they influence each other. In the spirit of Rysman (2009), the customers group could be considered as another side of the market, making the MLM a three-sided platform.[22] However, we can do so because, here, the managers (platform) set no price for the customers. The third concern about the MLM is that the same user can be on the two sides at the same time. The seller can be recruiter as well, and vice versa. This reality in MLM does not conform with the three features that characterize the multi-sided markets. To overcome this problem, we will consider as members of a given side the time allocated to the activity relevant to this side. The members of the recruitment (selling) side are represented by the total amount of time allocated to recruitment (selling) activity. This will not affect our results since in the classical literature (on multi-sided market) the users are assumed to be a continuum on each side (see Rochet and Tirole (2003), or Weyl

[22]In fact, we should recognize that the Multi-level marketing is more than a two-sided market, but for now we are just interesting in what that will be useful to answer our main question in this paper.

(2010)).

After identifying the multi-level marketing as a multi-sided market, a remark is yet to be made. Even if the present setup is similar to the newspapers market, it really has some particularity. The extent of the negative network effect from the recruits to the sellers depends on the network effect that exists inside the selling side through the price leading power the sellers provide to each other. If no seller gets benefit from others, recruitment could not be harmful for sellers.[23] Thus, the MLM's two-sided market is one in which the *cross network effects* stem from the *within network effects*.[24]

In this section, we intend to examine the outcomes if the MLM promoters take into account the network effects we mentioned above. Precisely, we compare the situation where the promoters are guided by profit maximization ethic with a context of social utility maximization.

3.6.1 Profit-maximizing decision

The MLM managers are concerned with their profit that they maximize by choosing optimally the wholesale price c, the signal v and the commissions A (on sales) and B (on recruits' sales). Let us remind that the profit to maximize is

$$\pi_{MLM} = (c - m - A - B)\, Q - K(v) \tag{3.26}$$

where m is the unit production cost and

$$Q = \int_{\mathbf{S}} q_s ds \quad and \quad \mathbf{S} = \{s, \theta(t_s) + \lambda_s\,(\tau_s, p_s, q_s) - Y_s(t_s + \tau_s) - I \geq 0\} \tag{3.27}$$

Q is the total quantity sold by the company. q_s depends on τ_s and t_s (respectively the time allocated to selling and recruitment activities by the distributor s). In turn, τ_s and t_s depend on the variables A, B, c and v which are chosen by the promoters. Thus, q_s depends on the managers' decision variables. In this section, we are not interested in the time allocated by a specific distributor, but the total time allocated by all the distributors to recruitment (size of the recruitment side denoted by t) and to the selling (size of the selling side denoted by τ).

Before moving to the first order conditions characterizing the maximal profit, let us

[23]We do not take into account the pecuniary externality that may exist in the market. See Reisinger et al. (2009) for some analysis on pecuniary externality in two-sided market.

[24]Here the term *within network effects* refers to the externality between the members of the same side. The terminology *cross network effects* is borrowed from Weyl (2010). Evans and Schmalensee (2010) used the terms *direct network effects* and *indirect network effects* respectively for *within network effects* and *cross network effects*.

make the following reasonable assumptions:

$$\frac{\partial t}{\partial A} \leq 0; \frac{\partial \tau}{\partial B} \leq 0; \frac{\partial \tau}{\partial c} \leq 0; \frac{\partial \tau}{\partial A} \geq 0; \frac{\partial t}{\partial B} \geq 0; \frac{\partial t}{\partial v} \geq 0; \frac{\partial \tau}{\partial v} \geq 0. \tag{3.28}$$

These assumptions above are realistic. If the company increases the commissions on sales, distributors will be incited to devote more time to selling and probably less time to recruitment. The analogous reality is true for an increase of commissions on recruitment. Moreover, the wholesale price has negative effect on the capacity to purchase and sell the product, and then on τ. For the signal v, its purpose is to encourage people to get engaged in the MLM activity. Thus v is likely to impact positively both recruitment and selling. Concerning the effect of the wholesale price on the recruitment time, actually, the sign of $\frac{\partial t}{\partial c}$ is ambiguous. An increase of c leads people to decrease τ, and they may turn to recruitment. However, as we said before, recruitment is not profitable if there is no sufficient purchase and sales. An increase of c might discourage people from recruiting. For this reason, sometimes, to make things simple we assume that $\frac{\partial t}{\partial c} = 0$. The first-order conditions maximizing the profit in (3.28) are:

$$-Q + (c - m - A - B)\frac{\partial Q}{\partial A} = 0 \tag{3.29}$$

$$-Q + (c - m - A - B)\frac{\partial Q}{\partial B} = 0 \tag{3.30}$$

$$Q + (c - m - A - B)\frac{\partial Q}{\partial c} = 0 \tag{3.31}$$

$$(c - m - A - B)\frac{\partial Q}{\partial v} - \frac{\partial K}{\partial v} = 0 \tag{3.32}$$

In our introductory model in section 3.2, we saw that the recruitment plan is beneficial for the company in the sense that it allows the company to sell more. It is worthy to check this result in a more complex setup. Precisely, we intend to ask if the total quantity Q increases with t (the total time allocated to recruitment), and under which condition. For τ, the time allocated to selling, we can argue that $\frac{\partial Q}{\partial \tau} > 0$. If distributors devote more time to selling, everything equal elsewhere, they will sell more and then the purchase from the company will increase. In contrast, concerning the sign of $\frac{\partial Q}{\partial t}$, things are not so clear for a simple reason. If t increases, due to expression in (3.20), the size of $G(t)$ will increase and then $\int_{G(t)} q_r dr$ (the purchase from the company) will increase, everything equal elsewhere. Let us remind that $G(t)$ is the set of people who will be engaged in MLM program if the total time devoted to recruitment is t. At the same time, more recruitment may deter selling incentive and then the purchasing incentive. However, the following proposition states that the promoters make their decisions in such a way that recruitment allows them to sell their product.

Proposition 3.6.1. *At the point where the managers maximize their profit, recruitment is favorable for the company's sales, that is* $\frac{\partial Q}{\partial t} > 0$.

Proof. Equations (3.29) and (3.30) can be rewritten as follows:

$$-Q + (c - m - A - B)\left(\frac{\partial Q}{\partial t}\frac{\partial t}{\partial A} + \frac{\partial Q}{\partial \tau}\frac{\partial \tau}{\partial A}\right) = 0 \tag{3.33}$$

$$-Q + (c - m - A - B)\left(\frac{\partial Q}{\partial t}\frac{\partial t}{\partial B} + \frac{\partial Q}{\partial \tau}\frac{\partial \tau}{\partial B}\right) = 0 \tag{3.34}$$

$$We have then \quad \frac{\partial Q}{\partial t}.\frac{\partial t}{\partial A} + \frac{\partial Q}{\partial \tau}\frac{\partial \tau}{\partial A} = \frac{\partial Q}{\partial t}\frac{\partial t}{\partial B} + \frac{\partial Q}{\partial \tau}\frac{\partial \tau}{\partial B} \tag{3.35}$$

Rearranging (3.35), we find:

$$-\frac{\frac{\partial Q}{\partial \tau}}{\frac{\partial Q}{\partial t}} = \frac{\frac{\partial t}{\partial A} - \frac{\partial t}{\partial B}}{\frac{\partial \tau}{\partial A} - \frac{\partial \tau}{\partial B}} \tag{3.36}$$

Under assumptions above and with (3.36), it turns out that $\frac{\partial Q}{\partial \tau}/\frac{\partial Q}{\partial t} > 0$, and then $\frac{\partial Q}{\partial t} > 0$ since $\frac{\partial Q}{\partial \tau} > 0$. □

Proposition 3.6.1 gives an answer to the question of why some firms owners adopt the MLM. If decisions (about commissions and other decision variables) are made accordingly, recruitment provides the company with an important selling tool. When the recruitment size increases, the number of people being engaged in the MLM activities rises and the opportunity to sell is enhanced for the company. However, it is worthy to precise that recruitment is not always good for the company but at the point where the profit is maximized. Thus, if the managers do not make rationally their marketing policies, recruitment might be harmful.

With the result in proposition 3.6.1 we should ask if distributors sell more or allocate more time to selling activity at equilibrium when recruitment time rises. Let τ^* and t^* denote respectively the selling time and recruitment time which maximize the managers's profit. Thus, it exists Q^* such that $Q(\tau^*, t^*) = Q^*$ at equilibrium. As a consequence, $-\frac{\partial Q}{\partial t}/\frac{\partial Q}{\partial \tau}$ is an expression of the sensitivity of the selling time to the recruitment time. We know from (3.36) that $-\frac{\partial Q}{\partial t}/\frac{\partial Q}{\partial \tau} < 0$. As a consequence, at the point where the company maximizes its profit, recruitment allows the whole company to get rid of the product while it deters the distributors' incentive to devote more time to selling activity. We can learn two things from this finding. First, recruitment increases the market share of the company because it gives recruiters sufficient *price leading power* in order to sell without devoting much time to selling activity. It means that people who recruit are those who sell, and they recruit for the purpose of getting the *price leading power*. In this case, the managers' policies incite distributors to recruit just for selling purpose. Second, maybe recruitment increases the company's market share because it increases the number of people involved in the MLM plan and purchasing from the company. People purchase from the company for personal consumption and there's a chance that unsold inventory exists. In such a situation, people recruit just for the recruitment purpose.

If managers are willing to incite distributors to devote time to selling activity, they can use the commission on sales (A) or the wholesale price (c). The strategy would be to increase A while maintaining constant B or to decrease c. The question is to know between increasing A and decreasing c, which policy incites more distributors to allocate their time to selling. Proposition 3.6.2 gives an answer.

Proposition 3.6.2. *Assume* $\frac{\partial t}{\partial c} = 0$ *and the profit-maximizing managers are willing to increase selling time; then increasing* A *is better decision than decreasing* c*.*

Proof. From equations (3.29) and (3.31)

$$-\frac{\frac{\partial Q}{\partial \tau}}{\frac{\partial Q}{\partial t}} = \frac{\frac{\partial t}{\partial A} + \frac{\partial t}{\partial c}}{\frac{\partial \tau}{\partial A} + \frac{\partial \tau}{\partial c}} < 0 \tag{3.37}$$

If $\frac{\partial t}{\partial c} = 0$ then $\mid \frac{\partial \tau}{\partial A} \mid > \mid \frac{\partial \tau}{\partial c} \mid$. □

Proposition 3.6.2 states that increasing the commissions rate on sales incites the distributors to turn to selling activity, more than decreasing the wholesale price. The reader may not expect this result in that decreasing c would allow distributors to set competitive price, to expect to attract more consumers and then to get interested in being engaged in selling. The commission rate A is a discount on sales and then a deferred reduction on wholesale price. Distributors would prefer an immediate reduction of cost to a deferred reduction. However, reality in MLM business is quite different and not in line with such an analysis. People are involved in MLM to make money, so much money more than what they can earn in other business. In comparison with c, the commission rate A is one of the characteristics that make MLM different from other businesses. When people are engaged in MLM they are targeting A more than c. Moreover, decreasing c, per se, is in fact profitable not just for those who want to sell but for everybody, including those who purchase the product for personal consumption. In conclusion, decreasing the wholesale price and increasing the commission on sales are not equivalent policies.

It would be also interesting to discuss about how the signal v impact distributors' selling behavior. We can easily find

$$-\frac{\frac{\partial Q}{\partial t}}{\frac{\partial Q}{\partial \tau}} = \frac{\frac{\partial \tau}{\partial A}\frac{\partial K}{\partial v} - \frac{\partial \tau}{\partial v}Q}{\frac{\partial t}{\partial A}\frac{\partial K}{\partial v} - \frac{\partial t}{\partial v}Q} < 0.$$

Under our assumptions,

$$\frac{\partial \tau}{\partial A}\frac{\partial K}{\partial v} > \frac{\partial \tau}{\partial v}Q.$$

or

$$\frac{\frac{\partial \tau}{\partial A}}{Q} > \frac{\frac{\partial \tau}{\partial v}}{\frac{\partial K}{\partial v}} \tag{3.38}$$

The left hand side of expression (3.38) can be interpreted as the benefit-cost ratio of an increase of the commission rate on sales. The right hand side is the benefit-cost ratio of an increase the signal. Let us remind that the signal is the actions initiated by the managers in favor of recruitment and selling and for which the cost does depend on the quantity produced or sold. Therefore, according to (3.38), the signal contributes less to selling time than the commission rate on sales. This result can explain why the company initiates such actions not for all the distributors but for a minority which has been involved for long time.

Concerning the recruitment commission B, we try to compare its effect on recruitment and selling time with the effect of other variables. The result is ambiguous because of the double and uncertain effect of recruitment on selling activity. Recruitment may give strong price leading power to those who are willing to sell, or deter selling activity because of competition environment it gives rise to. Even at the point where the promoters maximize their profit, the commission on recruitment may have an undesirable effect on selling. In other words, the profit-maximizing managers may fail to set an appropriate B which is not harmful for selling.

We now see what the social utility maximizer would adopt.

3.6.2 Social utility maximizing decision

In this part the goal is to consider the total utility got by all the actors of the program while taking into account the network effect interaction between them. Let us remind that, within the contest of the present section, people acting in the business are the promoters, the recruiters and the sellers.[25] The utility got by the recruitment side is $B \int_{G(t)} q_r dr$ with $r \in G(t)$. t is total time in recruitment for all distributors. For the selling side, the utility is $\lambda(\tau, .)$. The costs are $Y(\tau + t) + I$. Thus the total utility is

$$V = (c - m - A - B)\, Q - K(v) + B \int_{G(t)} q_r dr + \lambda(\tau, .) - Y(\tau + t) - I \tag{3.39}$$

At equilibrium we should have

$$Q = \int_{G(t)} q_r dr.$$

Moreover $\lambda(\tau, .)$ includes AQ and cQ. Thus, the social utility becomes

$$V = -mQ - K(v) + \xi(.) - Y(\tau + t) - I \tag{3.40}$$

where $\xi(.) = \lambda(\tau, .) - AQ + cQ$.
Observing equation (3.40) reveals that the only term that could make V positive is $\xi(.)$. In other words, if there is no sufficient profitable sales, MLM cannot be socially profitable. This shows the importance of the selling side. The first-order conditions maximizing V are

$$-m\frac{\partial Q}{\partial A} + \frac{\partial \xi}{\partial A} - \frac{\partial Y}{\partial A} = 0 \tag{3.41}$$
$$-m\frac{\partial Q}{\partial B} + \frac{\partial \xi}{\partial B} - \frac{\partial Y}{\partial B} = 0 \tag{3.42}$$
$$-m\frac{\partial Q}{\partial c} + \frac{\partial \xi}{\partial c} - \frac{\partial Y}{\partial c} = 0 \tag{3.43}$$
$$-m\frac{\partial Q}{\partial v} - \frac{\partial K}{\partial v} + \frac{\partial \xi}{\partial v} - \frac{\partial Y}{\partial v} = 0 \tag{3.44}$$

In the following proposition 3.6.3 we give a characterization describing the optimal decision.

[25]We do not the consumers group because they do not matter for the present analysis. They are not included in the two-sided market we study and the managers (platform) do not act directly with them.

Proposition 3.6.3. *At the point where the social welfare is maximized we have:*

$$\frac{\frac{\partial \tau}{\partial A}}{\frac{\partial t}{\partial A}} = \frac{\frac{\partial \tau}{\partial B}}{\frac{\partial t}{\partial B}} = \frac{\frac{\partial \tau}{\partial c}}{\frac{\partial t}{\partial c}} \tag{3.45}$$

Proof. From equations (3.41) - (3.43) we can write:

$$\left(-m\frac{\partial Q}{\partial t} + \frac{\partial \xi}{\partial t} - \frac{\partial Y}{\partial t}\right)\frac{\partial t}{\partial A} + \left(-m\frac{\partial Q}{\partial \tau} + \frac{\partial \xi}{\partial \tau} - \frac{\partial Y}{\partial \tau}\right)\frac{\partial \tau}{\partial A} = 0 \tag{3.46}$$

$$\left(-m\frac{\partial Q}{\partial t} + \frac{\partial \xi}{\partial t} - \frac{\partial Y}{\partial t}\right)\frac{\partial t}{\partial B} + \left(-m\frac{\partial Q}{\partial \tau} + \frac{\partial \xi}{\partial \tau} - \frac{\partial Y}{\partial \tau}\right)\frac{\partial \tau}{\partial B} = 0 \tag{3.47}$$

$$\left(-m\frac{\partial Q}{\partial t} + \frac{\partial \xi}{\partial t} - \frac{\partial Y}{\partial t}\right)\frac{\partial t}{\partial c} + \left(-m\frac{\partial Q}{\partial \tau} + \frac{\partial \xi}{\partial \tau} - \frac{\partial Y}{\partial \tau}\right)\frac{\partial \tau}{\partial c} = 0. \tag{3.48}$$

Rearranging we find:

$$-\frac{-m\frac{\partial Q}{\partial t} + \frac{\partial \xi}{\partial t} - \frac{\partial Y}{\partial t}}{-m\frac{\partial Q}{\partial \tau} + \frac{\partial \xi}{\partial \tau} - \frac{\partial Y}{\partial \tau}} = \frac{\frac{\partial \tau}{\partial A}}{\frac{\partial t}{\partial A}} = \frac{\frac{\partial \tau}{\partial B}}{\frac{\partial t}{\partial B}} = \frac{\frac{\partial \tau}{\partial c}}{\frac{\partial t}{\partial c}}. \tag{3.49}$$

□

From proposition 3.6.3 we can write $\left|\frac{\frac{\partial \tau}{\partial A}}{\frac{\partial \tau}{\partial B}}\right| = \left|\frac{\frac{\partial t}{\partial A}}{\frac{\partial t}{\partial B}}\right|$ if social utility is maximized. $\left|\frac{\partial \tau}{\partial A}\right|$ is an effect (benefit) of the sales commission rate A on the selling side and $\left|\frac{\partial \tau}{\partial B}\right|$ is a cost (damage) of the recruitment commission B on the selling side. Similarly $\left|\frac{\partial t}{\partial B}\right|$ and $\left|\frac{\partial t}{\partial A}\right|$ are respectively a gain from B and a cost from A on the recruitment side. Thus, $\left|\frac{\frac{\partial \tau}{\partial A}}{\frac{\partial \tau}{\partial B}}\right|$ and $\left|\frac{\frac{\partial t}{\partial A}}{\frac{\partial t}{\partial B}}\right|$ are respectively the relative gain on selling side and relative cost on recruitment side induced by the sales commission rate A. We can then say that, according to proposition 3.6.3, the relative gain on selling side should be equal to the relative cost on recruitment side. This is a standard result when studying internalization of network effects or of any kind of externality. Let us tell that the profit maximizer does not necessary come to this condition. As a consequence, in case of deviation from the social welfare maximizer's decision, the profit-maximizer can incite distributors to turn more or less to non optimal recruitment.

Another way to understand proposition 3.6.3 could be as follows. Since $\left|\frac{\frac{\partial \tau}{\partial A}}{\frac{\partial t}{\partial A}}\right| = \left|\frac{\frac{\partial \tau}{\partial B}}{\frac{\partial t}{\partial B}}\right|$ we can say that at equilibrium the net gain from A is equal to the net cost from B.

Due to the network effect we identified before, we can assume that $\left|\frac{\partial \tau}{\partial A}\right| > \left|\frac{\partial t}{\partial A}\right|$ (the sale commission rate is more profitable on the selling side than it is harmful on recruitment side). As consequence $\left|\frac{\partial \tau}{\partial B}\right| > \left|\frac{\partial t}{\partial B}\right|$. Thus, to meet the condition in proposition 3.6.3 the commissions rate should be chosen such that $A > B$. If for instance the profit-maximizer chooses $B > A$, it can lead to over recruitment. We prove that

$$\underbrace{-\frac{-m\frac{\partial Q}{\partial t} + \frac{\partial \xi}{\partial t} - \frac{\partial Y}{\partial t}}{-m\frac{\partial Q}{\partial \tau} + \frac{\partial \xi}{\partial \tau} - \frac{\partial Y}{\partial \tau}}}_{Value\ of\ recruitment\ side} = \frac{\frac{\partial \tau}{\partial A}}{\frac{\partial t}{\partial A}} = \frac{\frac{\partial \tau}{\partial B}}{\frac{\partial t}{\partial B}} = \frac{\frac{\partial \tau}{\partial c}}{\frac{\partial t}{\partial c}}. \tag{3.50}$$

where the first term of the equations can be interpreted as the value of recruitment side in comparison with the value of selling side. The social utility maximizer can choose the commissions such that[26]

$$\frac{A}{B} = \left|\frac{\frac{\partial \tau}{\partial A}}{\frac{\partial t}{\partial A}}\right| = \left|\frac{\frac{\partial \tau}{\partial B}}{\frac{\partial t}{\partial B}}\right|$$

Contrary to the case of the profit-maximizing decision, here if $\frac{\partial t}{\partial c} = 0$ then $\frac{\partial \tau}{\partial c} = 0$ for a simple reason. In traditional business[27], the wholesale price c has no impact on social welfare because c is lost by the distributors but is recovered by the firm managers. Here in MLM, the main difference from the traditional business is recruitment. Thus, if c has no effect on recruitment time then its effect on selling time should be insignificant. As in section 3.6.1, to impact selling time, the better tool remains A in comparison with the wholesale price c.

3.7 Conclusion

To our knowledge, this paper is the first that makes an economic analysis of the multi-level marketing organizations. Most of the authors having studied the topic, have just discussed the fairness of the business and proposing mathematical device to make it legitimate and

[26] This expression of $\frac{A}{B}$ is not derived from the social welfare maximizer program. It is just an intuitive insight.

[27] As the reader could remark the term *"traditional business"* refers to any business organization other than the MLM.

legal. They did so without exploring the real economic incentive guiding behavior in multi-level marketing. We make here two contributions. First, we build a simple, yet sufficient, model to understand why rational people solicit others to be their competitors in a profit-making activity such as the multi-level marketing where distributors recruit other distributors to do the same activity as them. This model also explains the motive guiding some business holders towards the MLM instead of the other classical marketing organizations. Second, we identify the MLM as a two-sided market in the spirit of Rochet and Tirole (2003), Rochet and Tirole (2006) or Weyl (2010), and derive implications for the decisions made by the actors of the business.

We find that recruitment gives the recruiters a *price leading power* allowing them to propose a competitive price to consumers. By profiting from the sales made by other distributors, the recruiter gets the capacity to lower his price in order to attract more consumers. Thus, recruitment helps distributors to face competition with the upline and with other distributors. For the distributor's perspective, recruiting and selling represent a multi-product firm he is managing and in which he uses the gain from one product to finance the others. A consequence is that, to solve recruitment problem, the solution cannot no longer be to limit the maximum size of the downline of each distributor because the *price leading power* depends not on the number of recruits but on the capacity of the recruits to sell and purchase the product. Since any new recruit needs to get the leading power, the system could fall into a recruitment trap if no policy is undertaken to incite people to sell.

When we stay at the promoters level, the multi-level marketing is an example of two-sided market. The platform is represented by the promoters and the sides are the recruitment side and the selling side. We see that the decision of the managers may vary depending on whether they internalize the network effect inside and between the two sides (social welfare maximizer) or not (profit-maximizing decision). The level of the commissions rate on sales and recruitment are crucial instrument to meet the challenge network effect internalization. If the managers do not choose adequately the commissions rate, distributors can be turned towards more or less recruitment.

In empirical point of view, many studies have used data to understand the network marketing in general. However, within the context of the present analysis, much needs to be done. It would be interesting to build an empirical device to estimate the *price leading power*. Meeting this empirical challenge is delicate in that we need to observe the same MLM firm in two states: its natural state as MLM and an hypothetical state where we suppose the firm is operating classical business other than MLM. Much remains also to be done to confront some predictions in this paper with reality in order to understand well the MLM organization and to measure the extent of the network effect mentioned in this work.

At the theoretical level, we still need to study in detail the behavior of consumers, their valuation of the products and to obtain an explicit demand function which may be particular to the MLM. The time allocated to selling and recruitment affects the consumers' behavior as well and a complete analysis of MLM as multi-sided market could integrate consumers and interaction between them. Furthermore, much remains to be learned about the probability for an individual to accept to be recruited into the MLM program. Introducing this probability can help to understand the arrangements between upline and downline we mentioned in section 3.5.2.

Finally, we consider the multi-level marketing as being a promising research field for economists. We can introduce, as done for other classical business, information asymmetry (moral hazard, adverse selection,...), theory of contracts, experimentation, auctions and public procurements.

Conclusion Générale

A travers trois chapitres, cette thèse apporte trois grandes contributions. La première concerne l'exploitation d'une ressource naturelle en situation d'incertitude et de *learning*. Nous avons expliqué en partie pourquoi certains pays (ou agents économiques) peuvent consommer plus ou moins la ressource, comparativement à d'autres. Cette disparité dans les niveaux de consommation s'explique d'une part par les croyances a priori quant au processus de renouvellement de la ressource, et d'autre part par le fait que les pays soient engagés ou non dans un processus de *learning*. Les croyances reflètent la perception ou le degré d'optimisme des agents à propos de la disponibilté future de la ressource. Supposons que la ressource en est une dont le stock décroît avec le temps (renouvellement négatif).[28] Si les agents exploitants sont optimistes et croît que la perte sur le stock de la ressource n'est pas très importante, ils vont préférer réduire leur consommation et ainsi conserver la ressource. Etant donné qu'il s'agit d'une ressource épuisable, l'on souhaiterait la conserver si on sait que la non consommation va préserver la ressource. Considérons maintenant une ressource dont le stock augmente avec le temps (renouvellement positif). Dans ce cas, l'optimisme augmente la consommation de la ressource. En effet, si la ressource se regénère facilement, c'est-à-dire sa vitesse de création est relativement élevée, les agents vont se permettre de la consommer beaucoup plus. Toutefois, la consommation d'un agent ne dépend pas seulement de son propre degré d'optimisme, mais aussi de celui des autres exploitants. En effet, un agent qui estime que les autres vont réduire (augmenter) leur consommation, va accroître (réduire) la sienne. L'on y observe ainsi un comportement de passager clandestin. Cependant, l'existence d'un tel comportement doit être relativisée lorsque l'on prend en compte le *learning*. Lorsqu'un individu est seul à consommer une ressource, il consomme moins en *learning* qu'en non *learning*. Mais si cet individu exploite la ressource en commun avec d'autres, le *learning* peut avoir un effet positif ou négatif sur sa consommation. De par leurs croyances, les exploitants se donnent mutuellement des incitations à conserver ou non la ressource. Cette incitation dépend de ce qu'un agent gagne quand il décide d'investir dans la conservation de la ressource, c'est-à-dire de réduire

[28] La qualité de l'eau d'une nappe phréatique peut baisser avec le temps à cause d'une certaine forme de pollution ou de contamination.

sa consommation. Supposons qu'un agent veut réduire sa consommation avec le *learning*, mais constate que les autres ne désirent pas faire autant. Alors sa volonté de conserver la ressource sera anéantie par le comportement des autres. Ce résultat peut expliquer l'échec de certains accords entre pays sur la protection de l'environnement. En effet, certains pays voudraient bien respecter les termes de l'accord, mais vont être découragés par le comportement non exemplaire des autres signataires.

La deuxième contribution se situe au niveau de l'analyse des *bandits problem* et de leur application à un problème générique de sélection de chercheurs qui n'ont pas même expérience en terme de nombre d'années et de réalisations dans la profession. Le problème que nous avons tenté de résoudre est un exemple relativement complexe combinant le *arm acquiring bandit problem*, le *restless bandit problems*, et le *mortal multi-armed bandits*. Dans ce travail, au lieu de calculer des indices (comme l'indice de Gittins), nous avons proposé de comparer les chercheurs deux-à-deux, de les classer afin de sélectionner les meilleurs. Par ailleurs, ce travail est aussi une contribution au débat sur l'opportunité de financer les jeunes chercheurs. En effet, il est souvent reconnu que les jeunes chercheurs ont des difficultés à avoir accès aux sources de financement. Certaines gens estiment qu'il est risqué d'accorder assez de financement à un jeune chercheur mal connu et qui n'a pas encore eu le temps de faire ses preuves. Par contre, d'autres suggèrent que l'on fasse confiance aux jeunes qui peuvent être capables de réaliser avec succès des projets. Notre travail consiste donc à participer à ce débat en essayant de concilier les deux points de vue. Nous proposons de tenir compte du fait que les chercheurs expérimentés ont certes l'avantage d'avoir de l'expérience, mais présentent aussi l'inconvénient de ne plus avoir beaucoup de temps à passer dans la recherche. Il faudra donc faire un arbitrage entre le gain expéré futur et le gain espéré dans le présent (ou futur proche).

La troisième contribution est relative au *multi-level marketing (MLM)*. Le MLM est une stratégie de marketing qui consiste à distribuer le produit par l'intermédiaire de distributeurs qui se font de l'argent non seulement en vendant le produit, mais aussi en recrutant d'autres distributeurs. Chaque distributeur gagne sur ses propres ventes, et sur les ventes ou les achats des individus qu'il auraient recrutés dans le système. C'est une pratique très répandue de nos jours à travers le monde et qui fait énormément parler d'elle dans les médias et chez certains spécialistes comme les sociologues, les hommes de loi, et les gestionnaires d'entreprises. Les détracteurs pensent qu'il s'agit d'un système pyramidal qui n'a pour but que de ruiner les économies des familles au profit d'un groupe d'individus profiteurs. Ils estiment que dans le cadre des activités de MLM, l'objectif des acteurs n'est pas de vendre un produit à valeur marchande, mais plutôt de se faire de l'argent sur le dos des autres. Par contre, les promoteurs de ce type de marketing tentent de convaincre sur le fait que le but est de vendre un produit et que les distributeurs ont plus d'incitation à vendre qu'à recruter. D'autres analystes, plus modérés, suggèrent qu'il y ait une réglementation pouvant contraindre les MLMs à ne pas balancer dans la catégorie des structures pyramidales. Malgré l'existence d'un tel débat, les économistes semblent ne pas être concernés par le phénomène. A notre connaissance, ce travail constitue la première analyse réellement économique du *multi-level marketing*. Une telle analyse économique est pourtant nécessaire puisqu'elle permet de comprendre les bases économiques qui gouvernent ce type d'organisation. Quelle que soit la réglementation mise en place, si on ne comprend pas la structure économique des MLMs, on continuera à faire face aux mêmes problèmes.

L'une des aspects qui distinguent le MLM est le recrutement. Nous avons montré que le recrutement, dans une certaine mesure, peut aider à la vente du produit. Si les managers de l'entreprise choisissent convenablement les différents niveaux de compensation, le *multi-level marketing (MLM)* ne court pas le risque de devenir un système pyramidal. Par ailleurs, par rapport au système traditionnel (non MLM) et dans les mêmes conditions, les promoteurs de MLMs gagnent plus et les distributeurs gagnent moins.

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Appendices

3.8 Appendix for chapter 1

3.8.1 Proof of proposition 1.2.2

We first give proof for the *learning* country.
The program of the learning country j is:

$$V_j^L(y,\xi_1,...\xi_J) = \max_{q_j}\left\{\ln q_j + \delta\int_0^1 V_j^L\left[\left(y-\sum_{i=1}^J q_i\right)^{\eta}, \hat{\xi}_j(.|\eta)\right]\left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)d\theta\right]d\eta\right\} \quad (3.51)$$

Let us make the conjecture that $V_j^L(y,\xi_1,...\xi_J) = K_1(\xi_j)\ln y + K_2(\xi_1,\ldots,\xi_J)$. We then have:

$$V_j^L(y,\xi_1,...\xi_J) = \max_{q_j}\left\{\begin{array}{c}\ln q_j + \delta\ln\left(y-\sum_{i=1}^J q_i\right)\int_0^1 K_1\left(\hat{\xi}_j(.|\eta)\right)\eta\left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)d\theta\right]d\eta \\ +\delta\int_0^1 K_2\left(\hat{\xi}_1(.|\eta),\cdots,\hat{\xi}_J(.|\eta)\right)\left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)d\theta\right]d\eta\end{array}\right\} \quad (3.52)$$

The first-order condition (FOC) is:

$$\frac{1}{q_j} - \frac{\delta}{y-\sum_{i=1}^J q_i}\int_0^1 K_1\left(\hat{\xi}_j(.|\eta)\right)\eta\left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)d\theta\right]d\eta = 0 \quad (3.53)$$

$$y - q_j - \sum_{\substack{i=1\\ i\neq j}}^J q_i = \delta q_j\int_0^1 K_1\left(\hat{\xi}_j(.|\eta)\right)\eta\left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)d\theta\right]d\eta \quad (3.54)$$

Taking into account the FOC of each of the J countries, we find that:

$$\begin{pmatrix} 1+\delta A_1 & 1 & 1 & \cdots & 1 \\ 1 & 1+\delta A_2 & 1 & \cdots & 1 \\ 1 & 1 & 1+\delta A_3 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1+\delta A_J \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_J\end{pmatrix} = y\begin{pmatrix}1\\1\\1\\ \vdots \\ 1\end{pmatrix} \quad (3.55)$$

where $A_j = \int_0^1 K_1\left(\hat{\xi}_j(.|\eta)\right)\eta\,[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)d\theta]\,d\eta$.
Rearranging the system above yields:

$$\begin{array}{r} (1+\delta A_1)q_1 + q_2 + \cdots + q_J = y \\ \delta A_2 q_2 - \delta A_3 q_3 = 0 \\ \delta A_2 q_2 - \delta A_4 q_4 = 0 \\ \vdots \\ \delta A_2 q_2 - \delta A_J q_J = 0 \\ \delta A_2 q_2 - \delta A_1 q_1 = 0 \end{array} \quad or \quad \begin{array}{r} (1+\delta A_1)q_1 + q_2 + \cdots + q_J = y \\ A_1 q_1 = A_2 q_2 = \cdots = A_J q_J \end{array} \tag{3.56}$$

$$q_j = \left[1 + \delta A_j + A_j\left(\sum_{\substack{i=1\\i\neq j}}^{J}\frac{1}{A_i}\right)\right]^{-1} y = \frac{1}{A_j}\left[\delta + \sum_{i=1}^{J}\frac{1}{A_i}\right]^{-1} y \tag{3.57}$$

$$y - \sum_{i=1}^{J} q_i = \delta A_j \left[1 + \delta A_j + A_j\left(\sum_{\substack{i=1\\i\neq j}}^{J}\frac{1}{A_i}\right)\right]^{-1} y \tag{3.58}$$

and

$$y - \sum_{\substack{i=1\\i\neq j}}^{J} q_i = (1+\delta A_j)\left[1 + \delta A_j + A_j\left(\sum_{\substack{i=1\\i\neq j}}^{J}\frac{1}{A_i}\right)\right]^{-1} y \tag{3.59}$$

Plugging in the value function yields:

$$\begin{aligned} V_j^L(y,\xi_j) \quad &= \quad -\ln(1+\delta A_j) + \ln(1+\delta A_j) - \ln\left(1 + \delta A_j + A_j\left(\sum_{\substack{i=1\\i\neq j}}^{J}\frac{1}{A_i}\right)\right) + \ln y + \\ &\left[\ln \delta A_j - \ln\left(1 + \delta A_j + A_j\left(\sum_{\substack{i=1\\i\neq j}}^{J}\frac{1}{A_i}\right)\right) + \ln y\right]\delta A_j + \delta B \end{aligned} \tag{3.60}$$

where

$$B = \int_0^1 K_2\left(\hat{\xi}_1(.|\eta),\cdots,\hat{\xi}_J(.|\eta)\right)\left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)d\theta\right]d\eta \tag{3.61}$$

Rearranging, we come to:

$$V_j^L(y,\xi_j) = (1+\delta A_j)\ln y - (1+\delta A_j)\ln\left[1+\delta A_j + A_j\left(\sum_{\substack{i=1\\ i\neq j}}^{J}\frac{1}{A_i}\right)\right] + \delta A_j \ln(\delta A_j) + \delta B \tag{3.62}$$

It is easy to see that: $K_1(\xi_j) = 1+\delta\int_0^1 K_1\left(\hat{\xi}_j(.|\eta)\right)\eta\left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)d\theta\right]d\eta$. From KMS (2009), $K_1(\xi_j) = \int_\Theta \frac{\xi_j(\theta)}{1-\delta\mu(\theta)}d\theta$ with $\mu(\theta) = \int_0^1 \eta\phi(\eta|\theta)d\eta$. Rewriting $A_j = \frac{1}{\delta}\left(K_1(\xi_j)-1\right)$, we come to:

$$q_j = \left[K_1(\xi_j) + \frac{1}{\delta}\left(K_1(\xi_j)-1\right)\sum_{i\neq j}\frac{1}{\frac{1}{\delta}\left(K_1(\xi_j)-1\right)}\right]^{-1} y$$
$$= \left[K_1(\xi_j) + \left(K_1(\xi_j)-1\right)\sum_{i\neq j}\frac{1}{\left(K_1(\xi_j)-1\right)}\right]^{-1} y \tag{3.63}$$

$$q_j^L(y,\xi_1,\cdots,\xi_J) = \left[\int_\Theta \frac{\xi_j(\theta)}{1-\delta\mu(\theta)}d\theta + \sum_{i\neq j}\frac{\int_\Theta \frac{\xi_j(\theta)\mu(\theta)}{1-\delta\mu(\theta)}d\theta}{\int_\Theta \frac{\xi_i(\theta)\mu(\theta)}{1-\delta\mu(\theta)}d\theta}\right]^{-1} y. \tag{3.64}$$

Another expression of q_j^L is

$$q_j^L(y,\xi_1,\cdots,\xi_J) = \left[\int_\Theta \frac{\xi_j(\theta)\mu(\theta)}{1-\delta\mu(\theta)}d\theta\right]^{-1}\left[\delta + \sum_{i=1}^{J}\left(\int_\Theta \frac{\xi_i(\theta)\mu(\theta)}{1-\delta\mu(\theta)}d\theta\right)^{-1}\right]. \tag{3.65}$$

Adaptive learning country

We now give the proof for the adaptive learning country.

$$V_j^{AL}(y,\xi_1,...\xi_J) = \max_{q_j}\left\{\ln q_j + \delta\int_0^1 V_j^{AL}\left[\left(y-\sum_{i=1}^{J}q_i\right)^\eta, \xi_1,...\xi_J\right]\left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)d\theta\right]d\eta\right\}$$

As before, we assume that $V_j^{AL}(y,\xi_1,...\xi_J) = K_1(\xi_j)\ln y + K_2(\xi_1,\ldots,\xi_J)$.

We have then

$$V_j^{AL}(y,\xi_1,...\xi_J) = \max_{q_j}\left\{\begin{array}{c}\ln q_j + \delta\ln\left(y-\sum_{i=1}^{J}q_i\right)\int_0^1 K_1(\xi_j)\eta\left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)d\theta\right]d\eta \\ +\delta\int_0^1 K_2(\xi_1,\ldots,\xi_J)\left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)d\theta\right]d\eta\end{array}\right\}$$

FOC:

$$\frac{1}{q_j} - \frac{\delta}{y-\sum_{i=1}^{J}q_i}\int_0^1 K_1(\xi_j)\eta\left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)d\theta\right]d\eta = 0$$

or

$$y - \sum_{i=1}^{J} q_i = \delta C_j q_j$$

where $C_j = \int_0^1 K_1(\xi_j)\eta \left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)d\theta\right] d\eta$

Considering the FOC of all J countries, we find that

$$\begin{pmatrix} 1+\delta C_1 & 1 & 1 & \cdots & 1 \\ 1 & 1+\delta C_2 & 1 & \cdots & 1 \\ 1 & 1 & 1+\delta C_3 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1+\delta C_J \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_J \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Rearranging, we have

$$\begin{aligned} (1+\delta C_1)q_1 + q_2 + \cdots + q_J &= y \\ \delta C_2 q_2 - \delta C_3 q_3 &= 0 \\ \delta C_2 q_2 - \delta C_4 q_4 &= 0 \\ &\vdots \\ \delta C_2 q_2 - \delta C_J q_J &= 0 \\ \delta C_2 q_2 - \delta C_1 q_1 &= 0 \end{aligned} \quad \begin{aligned} (1+\delta C_1)q_1 + q_2 + \cdots + q_J &= y \\ C_1 q_1 = C_2 q_2 = \cdots &= C_J q_J \end{aligned}$$

and

$$q_j^{AL} (= \left[1 + \delta C_j + C_j \left(\sum_{i \neq j} \frac{1}{C_i}\right)\right]^{-1} y = \frac{1}{C_j}\left[\delta + \sum_{i=1}^{J} \frac{1}{C_i}\right]^{-1} y$$

$$y - \sum_{i=1}^{J} q_i = \delta C_j \left[1 + \delta C_j + C_j \left(\sum_{i \neq j} \frac{1}{C_i}\right)\right]^{-1} y = \delta \left[\delta + \sum_{i=1}^{J} \frac{1}{C_i}\right]^{-1} y.$$

Plugging q_j^{AL} into the value function yields

$$V_j^{AL}(y, \xi_1, \cdots, \xi_J) = -\ln\left[1 + \delta C_j + C_j \left(\sum_{i \neq j} \frac{1}{C_i}\right)\right] + \ln y + \delta C_j \left[-\ln\left(1 + \frac{1}{\delta}\sum_{i=1}^{J} \frac{1}{C_i}\right) + \ln y\right] + \delta D.$$

One can see that $K_1(\xi_j) = 1 + \delta \int_0^1 K_1(\xi_j)\eta \left[\int_\Theta \phi(\eta|\theta)\xi_j(\theta)d\theta\right] d\eta \Longrightarrow K_1(\xi_j) = \frac{1}{1-\delta\int_\Theta \mu(\theta)\xi_j(\theta)d\theta}$. Similarly,

$$C_j = \frac{1}{\delta}\left(K_1(\xi_j) - 1\right) = \frac{\int_\Theta \mu(\theta)\xi_j(\theta)d\theta}{1 - \delta\int_\Theta \mu(\theta)\xi_j(\theta)d\theta}$$

or

$$q_j^{AL}(.) = \left[\frac{1}{1-\delta\int_\Theta \mu(\theta)\xi_j(\theta)d\theta} + \frac{1}{1-\delta\int_\Theta \mu(\theta)\xi_j(\theta)d\theta}\left(\int_\Theta \mu(\theta)\xi_j(\theta)d\theta\right)\sum_{i\neq j}\frac{1-\delta\int_\Theta \mu(\theta)\xi_j(\theta)d\theta}{\int_\Theta \mu(\theta)\xi_j(\theta)d\theta}\right]^{-1} y$$

$$q_j^{AL}(y,\xi_1,\cdots,\xi_J) = \left(1-\delta\int_\Theta \mu(\theta)\xi_j(\theta)d\theta\right)\left[1+\left(\int_\Theta \mu(\theta)\xi_j(\theta)d\theta\right)\sum_{i\neq j}^{J}\frac{1-\delta\int_\Theta \mu(\theta)\xi_j(\theta)d\theta}{\int_\Theta \mu(\theta)\xi_j(\theta)d\theta}\right]^{-1} y$$

3.8.2 Proof of proposition 1.4.1

.

Let $R(x) = x(1-\delta x)^{-1}$. For all $k \in I$, we use the following notations: $A_k = \frac{1}{\mathbb{E}_k(R(\mu(\theta)))}$, $B_k = \frac{1}{R(\mathbb{E}_k(\mu(\theta)))}$, $\tilde{A}_k = \frac{1}{\tilde{\mathbb{E}}_k(R(\mu(\theta)))}$, $\tilde{B}_k = \frac{1}{R\left(\tilde{\mathbb{E}}_k(\mu(\theta))\right)}$, where $\tilde{\mathbb{E}}_k$ and $\mathbb{E}_k$ stand for the expectations under the distributions $\tilde{\xi}_k$ and ξ_k respectively. Because $\tilde{\xi}_i \succ_1 \xi_i$, we have:

$$\tilde{A}_k < A_k \; and \tilde{B}_k < B_k \quad for\ all\ k \in I. \tag{3.66}$$

(i) We have

$$g_i^L(y,\tilde{\Xi}) = \frac{\tilde{A}_i}{\delta + \tilde{A}_i + \sum_{k\neq i} A_k} \quad and \quad g_i^L(y,\Xi) = \frac{A_i}{\delta + A_i + \sum_{k\neq i} A_k}. \tag{3.67}$$

Because the function $x \mapsto \frac{x}{\delta + x + \sum_{k\neq i} A_k}$ is increasing then, from (3.66), we have $g_i^L(y,\tilde{\Xi}) < g_i^L(y,\Xi)$.

(ii) We have

$$g_i^{AL}(y,\tilde{\Xi}) = \frac{\tilde{B}_i}{\delta + \tilde{B}_i + \sum_{k\neq i} B_k} \quad and \quad g_i^{AL}(y,\Xi) = \frac{B_i}{\delta + B_i + \sum_{k\neq i} B_k}. \tag{3.68}$$

The proof is similar to what we did in (i).

(iii)

$$g_j^L(y,\tilde{\Xi}) = \frac{A_j}{\delta + \tilde{A}_i + \sum_{k\neq i} A_k} \quad and \quad g_j^L(y,\Xi) = \frac{A_j}{\delta + A_i + \sum_{k\neq i} A_k}. \tag{3.69}$$

From (3.66), we have $g_j^L(y,\tilde{\Xi}) > g_j^L(y,\Xi)$.

(iv)

$$g_j^{AL}(y,\tilde{\Xi}) = \frac{B_j}{\delta + \tilde{B}_i + \sum_{k\neq i} B_k} \quad and \quad g_j^{AL}(y,\Xi) = \frac{B_j}{\delta + B_i + \sum_{k\neq i} B_k}. \tag{3.70}$$

The proof is similar to what we did in (iii).

(v)

$$\sum_{k=1}^{J} g_k^L(y,\tilde{\Xi}) = \frac{\tilde{A}_i + \sum_{k\neq i} A_k}{\delta + \tilde{A}_i + \sum_{k\neq i} A_k} \quad and \quad \sum_{k=1}^{J} g_k^L(y,\Xi) = \frac{A_i + \sum_{k\neq i} A_k}{\delta + A_i + \sum_{k\neq i} A_k}. \tag{3.71}$$

Since the function $x \mapsto \frac{x+\sum_{k\neq i} A_k}{\delta+x+\sum_{k\neq i} A_k}$ is increasing then, from (3.66), we have $\sum_{k=1}^{J} g_k^L(y,\tilde{\Xi}) < \sum_{k=1}^{J} g_k^L(y,\Xi)$.

(vi)

$$\sum_{k=1}^{J} g_k^{AL}(y,\tilde{\Xi}) = \frac{\tilde{B}_i + \sum_{k\neq i} B_k}{\delta + \tilde{B}_i + \sum_{k\neq i} B_k} \quad and \quad \sum_{k=1}^{J} g_k^{AL}(y,\Xi) = \frac{B_i + \sum_{k\neq i} B_k}{\delta + B_i + \sum_{k\neq i} B_k}. \tag{3.72}$$

The proof is similar to what we did in (v).

3.8.3 Proof of proposition 1.6.2

As usual, $R(x) = x(1-\delta x)^{-1}$. $\mathbb{E}_j$ and $\tilde{\mathbb{E}}_j$ denote respectively the expectations under the beliefs ξ_j and $\tilde{\xi}_j$ for any $j = 1, ..., J$. We know that:

$$g_i^L(y,\Xi) = \frac{\frac{1}{\mathbb{E}_i(R(\mu(\theta)))}}{\delta + \sum_{k=1}^{J} \frac{1}{\mathbb{E}_k(R(\mu(\theta)))}}, \quad and$$
$$g_i^L(y,\tilde{\Xi}) = \frac{\frac{1}{\tilde{\mathbb{E}}_i(R(\mu(\theta)))}}{\delta + \frac{1}{\tilde{\mathbb{E}}_i(R(\mu(\theta)))} + \sum_{k=1,k\neq i}^{J} \frac{1}{\mathbb{E}_k(R(\mu(\theta)))}} \tag{3.73}$$

We also have:

$$g_j^L(y,\Xi) = \frac{\frac{1}{\mathbb{E}_j(R(\mu(\theta)))}}{\delta + \sum_{k=1}^{J} \frac{1}{\mathbb{E}_k(R(\mu(\theta)))}}, \quad and$$
$$g_j^L(y,\tilde{\Xi}) = \frac{\frac{1}{\mathbb{E}_j(R(\mu(\theta)))}}{\delta + \frac{1}{\tilde{\mathbb{E}}_i(R(\mu(\theta)))} + \sum_{k=1,k\neq i}^{J} \frac{1}{\mathbb{E}_k(R(\mu(\theta)))}} \; for \; any \; j \neq i \tag{3.74}$$

Total consumption is:

$$\sum_{k=1}^{J} g_k^L(y,\Xi) = \frac{\sum_{k=1}^{J} \frac{1}{\mathbb{E}_k(R(\mu(\theta)))}}{\delta + \sum_{k=1}^{J} \frac{1}{\mathbb{E}_k(R(\mu(\theta)))}}, \quad and$$
$$\sum_{k=1}^{J} g_k^L(y,\tilde{\Xi}) = \frac{\frac{1}{\tilde{\mathbb{E}}_i(R(\mu(\theta)))} + \sum_{k=1,k\neq i}^{J} \frac{1}{\mathbb{E}_k(R(\mu(\theta)))}}{\delta + \frac{1}{\tilde{\mathbb{E}}_i(R(\mu(\theta)))} + \sum_{k=1,k\neq i}^{J} \frac{1}{\mathbb{E}_k(R(\mu(\theta)))}} \tag{3.75}$$

For this proof, we denote by $\tilde{\theta}$ the variable following the distribution ξ_i.

Case 1: $\mu" \leq 0$. If country i feels that $\tilde{\xi}_i$ is more risky than ξ_i then $\xi_i \succ_C \tilde{\xi}_i$ i.e $\theta \succ_C \tilde{\theta}$ according to i. As a consequence, $\mu(\theta) \succ_C \mu(\tilde{\theta})$ because μ is concave. Because R is convex we then have:

$$\begin{aligned}
\mathbb{E}(-R(\mu(\theta))) &\geq \mathbb{E}(-R(\mu(\tilde{\theta}))) \\
\mathbb{E}(R(\mu(\tilde{\theta}))) &\geq \mathbb{E}(R(\mu(\theta))) \\
\tilde{\mathbb{E}}_i(R(\mu(\theta))) &\geq \mathbb{E}_i(R(\mu(\theta))) \\
\frac{1}{\tilde{\mathbb{E}}_i(R(\mu(\theta)))} &\leq \frac{1}{\mathbb{E}_i(R(\mu(\theta)))}
\end{aligned} \tag{3.76}$$

From (3.73), (3.74), (3.75) and (3.76) the proof of the statements (i), (ii), (iii) in proposition 1.6.2 is completed for $\mu" \leq 0$.

Case 2: $\mu" \geq 0$
If country i feels that $\tilde{\xi}_i$ is more risky than ξ_i then $\tilde{\xi}_i \succ_V \xi_i$. Because μ and R are both convex and R is increasing then $Ro\mu$ is convex. This leads to $\tilde{\mathbb{E}}_i(Ro\mu(\theta)) \geq \mathbb{E}_i(Ro\mu(\theta))$ or $\frac{1}{\tilde{\mathbb{E}}_i(R(\mu(\theta)))} \leq \frac{1}{\mathbb{E}_i(R(\mu(\theta)))}$. The proof is then completed.

3.8.4 Proof of proposition 1.7.1

Before giving the proofs, it is useful to know that under the homogenous beliefs assumption, proving an inequality on individual consumption is equivalent to proving this inequality on total consumption.

(1) From proposition 1.2.1, $g_i^{FI}(\cdot) = \frac{1-\delta\mu(\theta^*)}{J+(1-J)\delta\mu(\theta^*)}y$. or $\sum_{i=1}^J g_i^{FI}(\cdot) = Jg_i^{FI}(\cdot) = \frac{1-\delta\mu(\theta^*)}{1+\frac{1-J}{J}\delta\mu(\theta^*)}y$. Similarly, if $\int_\Theta \mu(\theta)\xi_i(\theta)d\theta = \mu(\theta^*)$ for all i, then $g_i^{AL}(\cdot) = \frac{1-\delta\mu(\theta^*)}{\mu(\theta^*)}\left[\delta + J\frac{1-\delta\mu(\theta^*)}{\mu(\theta^*)}\right]^{-1}y$. Therefore, $\sum_{i=1}^J g_i^{AL}(\cdot) = Jg_i^{AL}(\cdot) = \frac{1-\delta\mu(\theta^*)}{J+(1-J)\delta\mu(\theta^*)}y = \sum_{i=1}^J g_i^{FI}(\cdot)$

(2) If countries have same belief ξ then $g_i^{AL}(\cdot) = \frac{1-\delta\int_\Theta \mu(\theta)\xi_i(\theta)d\theta}{\int_\Theta \mu(\theta)\xi_i(\theta)d\theta}\left[\delta + J\frac{1-\delta\int_\Theta \mu(\theta)\xi_i(\theta)d\theta}{\int_\Theta \mu(\theta)\xi_i(\theta)d\theta}\right]^{-1}y$ or $\sum_{i=1}^J g_i^{AL}(\cdot) = Jg_i^{AL}(\cdot) = \frac{1-\delta\int_\Theta \mu(\theta)\xi_i(\theta)d\theta}{1+\frac{1-J}{J}\delta\int_\Theta \mu(\theta)\xi_i(\theta)d\theta}y$. Using the fact that $x \mapsto \frac{1-\delta x}{1+\frac{1-J}{J}\delta x} \equiv S(x)$ is decreasing the proof ends.

(3) We have $\mu(\mathbb{E}(\theta)) = \mu(\theta^*)$ for each country. Furthermore, $\sum_{i=1}^J g_i^{FI}(\cdot) = S(\mathbb{E}(\theta))y$. We also know that $\sum_{i=1}^J g_i^{AL}(\cdot) = S(\mathbb{E}(\mu(\theta)))y$. If $\mu" > 0$ then $\mathbb{E}(\mu(\theta)) > \mu(\mathbb{E}(\theta)) = \mu(\theta^*)$. Because S is decreasing then $\sum_{i=1}^J g_i^{FI}(\cdot) > \sum_{i=1}^J g_i^{AL}(\cdot)$. The result is also straightforward for $\mu" < 0$ and $\mu" = 0$.
Now let $W(\theta) \equiv \frac{J+(1-J)\delta\mu(\theta)}{J(1-\delta\mu(\theta))}$. We have $\sum_{i=1}^J g_i^{FI}(\cdot) = y/W(\theta^*)$ and $\sum_{i=1}^J g_i^{L}(\cdot) = y/\mathbb{E}(W(\theta))$.

If $-2\delta\mu'^2/(1-\delta\mu) < \mu$", then W is convex. Therefore, $\mathbb{E}(W(\theta)) > W(\mathbb{E}(\theta)) = W(\theta^*)$ and $\sum_{i=1}^J g_i^{FI}(\cdot) > \sum_{i=1}^J g_i^L(\cdot)$. I then also prove the remaining inequalities.

3.8.5 Proof of proposition 1.7.2

3.8.6 Proof of (1)

From Levhari and Mirman (1980), we have

$$Q_{FI}^1(y,\theta^*) = (1-\delta\mu(\theta^*))\,y \quad \textit{and} \quad g_j^{FI}(y,\theta^*) = \frac{1-\delta\mu(\theta^*)}{J+(1-J)\delta\mu(\theta^*)}y. \tag{3.77}$$

$$\begin{aligned}\sum_{j=1}^J g_j^{FI}(y,\theta^*) = Jg_j^{FI}(y,\theta^*) &= \frac{J\,(1-\delta\mu(\theta^*))}{J+(1-J)\delta\mu(\theta^*)}y \\ &= \frac{1-\delta\mu(\theta^*)}{1+\frac{1-J}{J}\delta\mu(\theta^*)}y > (1-\delta\mu(\theta^*))\,y = Q_{FI}^1(y,\theta^*)\end{aligned}$$

3.8.7 Proof of (2), (3), (4) and (5)

There are J countries indexed by $j = 1,...,J$ and of beliefs $\xi_1,...,\xi_J$. We suppose that i is the only firm exploiting the resource before common property. i is one of the Js in $\{1,...,J\}$. For any country $j = 1,...,J$, we use the following notations: $A_j = \mathbb{E}_j(R(\mu(\theta)))$ and $B_j = R(\mathbb{E}_j(\mu(\theta)))$ where $\mathbb{E}_j$ stands for the expectation under belief ξ_j. We then have:

$$Q_L^1 = \frac{1}{1+\delta A_i}y \quad \textit{and} \quad Q_{AL}^1 = \frac{1}{1+\delta B_i}y \tag{3.78}$$

We also have

$$\sum_{j=1}^J g_j^L(.) = \frac{\sum_{j=1}^J \frac{1}{A_j}}{\delta + \sum_{j=1}^J \frac{1}{A_j}}y \quad \textit{and} \quad \sum_{j=1}^J g_j^{AL}(.) = \frac{\sum_{j=1}^J \frac{1}{B_j}}{\delta + \sum_{j=1}^J \frac{1}{B_j}}y \tag{3.79}$$

Proof of (2)

From (3.79), we can write:

$$\sum_{j=1}^J g_j^{AL}(.) = \frac{1}{1+\frac{\delta}{\sum_{j=1}^J \frac{1}{B_j}}}y \tag{3.80}$$

From (3.78) and (3.80), we only need to prove that

$$\frac{1}{\sum_{j=1}^{J}\frac{1}{B_j}} < B_i. \tag{3.81}$$

$$\frac{1}{\sum_{j=1}^{J}\frac{1}{B_j}} = \frac{1}{\frac{1}{B_i}+\sum_{j\neq i}\frac{1}{B_j}} < \frac{1}{\frac{1}{B_i}} = B_i \quad because \quad \sum_{j\neq i}\frac{1}{B_j} > 0.$$

Proof of (3)

From (3.79), we can write:

$$\sum_{j=1}^{J} g_j^L(.) = \frac{1}{1+\frac{\delta}{\sum_{j=1}^{J}\frac{1}{A_j}}}y \tag{3.82}$$

From (3.78) and (3.82), we only need to prove that

$$\frac{1}{\sum_{j=1}^{J}\frac{1}{A_j}} < A_i. \tag{3.83}$$

$$\frac{1}{\sum_{j=1}^{J}\frac{1}{A_j}} = \frac{1}{\frac{1}{A_i}+\sum_{j\neq i}\frac{1}{A_j}} < \frac{1}{\frac{1}{A_i}} = A_i \quad because \quad \sum_{j\neq i}\frac{1}{A_j} > 0.$$

Proof of (4) and (5) From (3.78) and (3.79), we write:

$$\frac{\sum_{j=1}^{J} g_j^L(.)}{Q_L^1} = \frac{(1+\delta A_i)\left(\sum_{j=1}^{J}\frac{1}{A_j}\right)}{\delta+\sum_{j=1}^{J}\frac{1}{A_j}} = \frac{\sum_{j=1}^{J}\frac{1}{A_j}+\delta\sum_{j=1}^{J}\frac{A_i}{A_j}}{\delta+\sum_{j=1}^{J}\frac{1}{A_j}} \tag{3.84}$$

and

$$\frac{\sum_{j=1}^{J} g_j^{AL}(.)}{Q_{AL}^1} = \frac{(1+\delta B_i)\left(\sum_{j=1}^{J}\frac{1}{B_j}\right)}{\delta+\sum_{j=1}^{J}\frac{1}{B_j}} = \frac{\sum_{j=1}^{J}\frac{1}{B_j}+\delta\sum_{j=1}^{J}\frac{B_i}{B_j}}{\delta+\sum_{j=1}^{J}\frac{1}{B_j}}. \tag{3.85}$$

Therefore,

$$\begin{aligned}\frac{\sum_{j=1}^{J} g_j^L(.)}{Q_L^1} & -\frac{\sum_{j=1}^{J} g_j^{AL}(.)}{Q_{AL}^1} \\ = & \frac{\left(\delta+\sum_{j=1}^{J}\frac{1}{B_j}\right)\left(\sum_{j=1}^{J}\frac{1}{A_j}+\delta\sum_{j=1}^{J}\frac{A_i}{A_j}\right)-\left(\delta+\sum_{j=1}^{J}\frac{1}{A_j}\right)\left(\sum_{j=1}^{J}\frac{1}{B_j}+\delta\sum_{j=1}^{J}\frac{B_i}{B_j}\right)}{\left(\delta+\sum_{j=1}^{J}\frac{1}{A_j}\right)\left(\delta+\sum_{j=1}^{J}\frac{1}{B_j}\right)}\end{aligned}$$

Comparing $\frac{\sum_{j=1}^{J} g_j^L(.)}{Q_L^1}$ to $\frac{\sum_{j=1}^{J} g_j^{AL}(.)}{Q_{AL}^1}$ is equivalent to determining the sign of

$$A = \left(\delta + \sum_{j=1}^{J} \frac{1}{B_j}\right)\left(\sum_{j=1}^{J} \frac{1}{A_j} + \delta \sum_{j=1}^{J} \frac{A_i}{A_j}\right) - \left(\delta + \sum_{j=1}^{J} \frac{1}{A_j}\right)\left(\sum_{j=1}^{J} \frac{1}{B_j} + \delta \sum_{j=1}^{J} \frac{B_i}{B_j}\right)$$

$$\begin{aligned}
A &= \delta \sum_{j=1}^{J} \frac{1}{A_j} + \delta^2 \sum_{j=1}^{J} \frac{A_i}{A_j} + \left(\sum_{j=1}^{J} \frac{1}{A_j}\right)\left(\sum_{j=1}^{J} \frac{1}{B_j}\right) + \delta \left(\sum_{j=1}^{J} \frac{1}{B_j}\right)\left(\sum_{j=1}^{J} \frac{A_i}{A_j}\right) \\
&- \delta \sum_{j=1}^{J} \frac{1}{B_j} - \delta^2 \sum_{j=1}^{J} \frac{B_i}{B_j} - \left(\sum_{j=1}^{J} \frac{1}{A_j}\right)\left(\sum_{j=1}^{J} \frac{1}{B_j}\right) - \delta \left(\sum_{j=1}^{J} \frac{1}{A_j}\right)\left(\sum_{j=1}^{J} \frac{B_i}{B_j}\right) \\
&= \delta^2 \sum_{j \neq i} \frac{A_i}{A_j} - \delta^2 \sum_{j \neq i} \frac{B_i}{B_j} + \delta \sum_{j=1}^{J} \frac{1}{A_j} - \delta \sum_{j=1}^{J} \frac{1}{B_j} \\
&+ \delta \left(\sum_{j=1}^{J} \frac{1}{B_j}\right)\left(1 + \sum_{j \neq i} \frac{A_i}{A_j}\right) - \delta \left(\sum_{j=1}^{J} \frac{1}{A_j}\right)\left(1 + \sum_{j \neq i} \frac{B_i}{B_j}\right) \\
&= \delta^2 \sum_{j \neq i} \frac{A_i}{A_j} - \delta^2 \sum_{j \neq i} \frac{B_i}{B_j} + \delta \left(\sum_{j=1}^{J} \frac{1}{B_j}\right)\left(\sum_{j \neq i} \frac{A_i}{A_j}\right) - \delta \left(\sum_{j=1}^{J} \frac{1}{A_j}\right)\left(\sum_{j \neq i} \frac{B_i}{B_j}\right)
\end{aligned}$$

$$A = \delta \left(\delta + \sum_{j=1}^{J} \frac{1}{B_j}\right)\left(\sum_{j \neq i} \frac{A_i}{A_j}\right) - \delta \left(\delta + \sum_{j=1}^{J} \frac{1}{A_j}\right)\left(\sum_{j \neq i} \frac{B_i}{B_j}\right) \tag{3.86}$$

From (3.86), $A > 0$ if and only if

$$\frac{\sum_{j \neq i} \frac{A_i}{A_j}}{\sum_{j \neq i} \frac{B_i}{B_j}} > \frac{\delta + \sum_{j=1}^{J} \frac{1}{A_j}}{\delta + \sum_{j=1}^{J} \frac{1}{B_j}}. \tag{3.87}$$

We have

$$\frac{\sum_{j=1}^{J} g_j^L(.)}{Q_L^1} > \frac{\sum_{j=1}^{J} g_j^{AL}(.)}{Q_{AL}^1} \quad if\ and\ only\ if$$

$$\frac{\sum_{j \neq i} \frac{\mathbb{E}_i(R(\mu(\theta)))}{\mathbb{E}_j(R(\mu(\theta)))}}{\sum_{j \neq i} \frac{R(\mathbb{E}_i(\mu(\theta)))}{R(\mathbb{E}_j(\mu(\theta)))}} > \frac{\delta + \sum_{j=1}^{J} \frac{1}{\mathbb{E}_j(R(\mu(\theta)))}}{\delta + \sum_{j=1}^{J} \frac{1}{R(\mathbb{E}_j(\mu(\theta)))}}. \tag{3.88}$$

If the countries have the same belief, the condition in (3.88) always holds because $A_j > B_j$ for all j.

3.8.8 Proposition on comparative analysis

Proposition 3.8.1. *Let* ${}^k g_i^L$ *and* ${}^k g_i^{AL}$ *be the optimal consumption by country* i *respectively under learning and adaptive learning settings, when the conditional p.d.f of* η *is* ϕ^k.
(1) If $\phi^1 \succ_1 \phi^2$ *then* $\sum_{i=1}^J {}^1 g_i^L \leq \sum_{i=1}^J {}^2 g_i^L$ *and* $\sum_{i=1}^J {}^1 g_i^{AL} \leq \sum_{i=1}^J {}^2 g_i^{AL}$
(2) If $\phi^1 \succ_2 \phi^2$ *then* $\sum_{i=1}^J {}^1 g_i^L = \sum_{i=1}^J {}^2 g_i^L$ *and* $\sum_{i=1}^J {}^1 g_i^{AL} = \sum_{i=1}^J {}^2 g_i^{AL}$
(3) Suppose that $\xi^1 \succ_1 \xi^2$ *and the countries have the same belief. Let* $g_i^L(\cdot, \xi)$ *and* $g_i^{AL}(\cdot, \xi)$ *be the optimal consumption by country* i *under the belief* ξ.
If $\mu' > 0$ *then* $\sum_{i=1}^J g_i^L(\cdot, \xi^1) \leq \sum_{i=1}^J g_i^L(\cdot, \xi^2)$
If $\mu' < 0$ *then* $\sum_{i=1}^J g_i^L(\cdot, \xi^1) \geq \sum_{i=1}^J g_i^L(\cdot, \xi^2)$
If $\mu' = 0$ *then* $\sum_{i=1}^J g_i^L(\cdot, \xi^1) = \sum_{i=1}^J g_i^L(\cdot, \xi^2)$.
(4) Suppose that $\xi^1 \succ_2 \xi^2$ *and the countries have the same belief.*
If $\mu" < -2\delta\mu'^2/(1-\delta\mu)$ *then* $\sum_{i=1}^J g_i^L(\cdot, \xi^1) \leq \sum_{i=1}^J g_i^L(\cdot, \xi^2)$
If $\mu" > -2\delta\mu'^2/(1-\delta\mu)$ *then* $\sum_{i=1}^J g_i^L(\cdot, \xi^1) \geq \sum_{i=1}^J g_i^L(\cdot, \xi^2)$
$\mu" = -2\delta\mu'^2/(1-\delta\mu)$ *then* $\sum_{i=1}^J g_i^L(\cdot, \xi^1) = \sum_{i=1}^J g_i^L(\cdot, \xi^2)$.

Where $\succ_1$ refers to the first-order stochastic dominance and $\succ_2$ is the second order stochastic dominance defined with respect to risk aversion.

Proof. Before giving the proofs, it is useful to know that under homogenous beliefs assumption, proving an inequality on individual consumption is equivalent to proving this inequality on total consumption.

(1) Let $\mu^k(\theta)$ be the conditional expectation of η under the distribution ϕ^k. For each country i let ${}^k B_i = 1/\mathbb{E}_i(R(\mu^k(\theta)))$. $\sum_{i=1}^J {}^k g_i^L = V\left(\sum_{i=1}^J {}^k B_i\right) y$ where $V(x) = x/(\delta + x)$. If $\phi^1 \succ_1 \phi^2$ then $\mu^1(\theta) > \mu^2(\theta)$. Using the fact that R and V are increasing functions we prove the inequality.

(2) If $\phi^1 \succ_2 \phi^2$ then $\mu^1(\theta) = \mu^2(\theta)$. Leaning on what we got in (1), the result is trivial.

(3) Let $Z(x) = \frac{J+(1-J)\delta x}{J(1-\delta x)}$. $\sum_{i=1}^J g_i^L(\cdot, \xi) = y/\mathbb{E}(Z(\mu(\theta)))$. Because Z is increasing, the proof of (3) ends.

(4) We have seen that $\sum_{i=1}^J g_i^L(\cdot, \xi) = y/\mathbb{E}(W(\theta))$. If $\mu" < -2\delta\mu'^2/(1-\delta\mu)$, W is concave and $\mathbb{E}_1(W(\theta)) > \mathbb{E}_2(W(\theta))$. The result is then straightforward, even for $\mu" > -2\delta\mu'^2/(1-\delta\mu)$ and $\mu" = -2\delta\mu'^2/(1-\delta\mu)$. □

3.9 Appendix for chapter 2

3.9.1 Proof of proposition 2.3.1

Proof. Let us give a simple proof for the normal distributions case. For the general case, the proof is done analogically. Our approach is to assume that the proposition is true and try to find r^*. Then,

$$V(N) = \int_{-\infty}^{+\infty} (q + \delta V(q))\theta_N(q)dq.$$

$$if \quad r < r^* \quad then \quad V(r) = \int_{-\infty}^{+\infty} (q + \delta V(q))\theta_N(q)dq \equiv V^-(r).$$

$$if \quad r > r^* \quad then \quad V(r) = \int_{-\infty}^{+\infty} (q\theta_r(q))dq + \delta V(N).$$

$$if \quad r = r^* \quad then \quad \int_{-\infty}^{+\infty} (q+\delta V(q))\theta_N(q)dq = \int_{-\infty}^{+\infty} (q\theta_r(q))dq+\delta V(N). \quad So \quad V^-(r) = V(N).$$

We can then write:

$$V^-(r) = \int_{-\infty}^{+\infty} q\theta_N(q)dq+\delta\left[\int_{-\infty}^{r^*} V(N)\theta_N(q)dq + \int_{r^*}^{+\infty}\left(\int_{-\infty}^{+\infty}(x\theta_q(x))dx + \delta V(N)\right)\theta_N(q)dq\right].$$

$$V^-(r) = \alpha + \delta V(N)\int_{-\infty}^{r^*} \theta_N(q)dq + \delta\int_{r^*}^{+\infty}\int_{-\infty}^{+\infty} x\theta_q(x)\theta_N(q)dxdq + \delta^2 V(N)\int_{r^*}^{+\infty}\theta_N(q)dq.$$

$$V^-(r) = \alpha + \delta V(N)\left[\alpha F(r^*-1) + (1-\alpha)F(r^*)\right] + \delta\int_{r^*}^{+\infty} h(q)\theta_N(q)dq$$
$$+ \delta^2 V(N)\left[1 - \alpha F(r^*-1) - (1-\alpha)F(r^*)\right].$$

where

$$h(q) = \frac{\alpha\exp(q-\frac{1}{2})}{\alpha\exp(q-\frac{1}{2}) + (1-\alpha)} \tag{3.89}$$

and F is the $c.d.f$ of the standard normal distribution. We can now write:

$$\int_{r^*}^{+\infty} h(q)\theta_N(q)dq = \alpha(1 - F(r^*-1)). \tag{3.90}$$

Therefore, we find

$$V(N) = \frac{\alpha(1+\delta-\delta F(r^*-1))}{(1-\delta)\left[1+\delta-\delta\left[\alpha F(r^*-1) + (1-\alpha)F(r^*)\right]\right]} \tag{3.91}$$

If $r = r^*$, we had

$$\int_{-\infty}^{+\infty} (q + \delta V(q))\theta_N(q)dq = \int_{-\infty}^{+\infty} q\theta_r(q)dq + \delta V(N). \tag{3.92}$$

As a consequence

$$(1-\delta)V(N) = \frac{\alpha\exp(r^* - \frac{1}{2})}{\alpha\exp(r^* - \frac{1}{2}) + 1 - \alpha}. \tag{3.93}$$

3.89 and 3.90 yield

$$\frac{\alpha(1+\delta-\delta F(r^*-1))}{1+\delta-\delta\left[\alpha F(r^*-1)+(1-\alpha)F(r^*)\right]} = \frac{\alpha\exp(r^* - \frac{1}{2})}{\alpha\exp(r^* - \frac{1}{2}) + 1 - \alpha} \tag{3.94}$$

$$Let \quad G(r) = \frac{\alpha(1+\delta-\delta F(r-1))}{1+\delta-\delta\left[\alpha F(r-1)+(1-\alpha)F(r)\right]} - \frac{\alpha\exp(r - \frac{1}{2})}{\alpha\exp(r - \frac{1}{2}) + (1 - \alpha)} \tag{3.95}$$

.[29] If equation $G(r) = 0$ has solution then r^* exists and is this solution. Let us remark that G is a smooth function. Therefore, a part of the proof ends since

$$\lim_{r\to+\infty} G(r) = \alpha - 1 < 0 \quad and \quad \lim_{r\to-\infty} G(r) = \alpha > 0$$

We should now prove that $r^* > \frac{1}{2}$. The proof is trivial. □

3.9.2 Proof of lemma 2.3.1

Proof. The proof has nothing difficult or ingenious. The calculation is just heavy to manipulate. Therefore, we will consider the case of $K = 3$ and try to find $V_1(A)$.

$$\begin{aligned} V_1(A) &= P_1 + \delta\int_{-\infty}^{+\infty} V(A^{(1)})\theta_A^1(q)dq \\ &= P_1 + \delta\int_{-\infty}^{+\infty} \theta_A^1(q)dq \max\left\{V_1(A^{(1)}), V_2(A^{(1)}), V_3(A^{(1)})\right\} \end{aligned}$$

[29] In the general case we find:

$$G(r) = \frac{1+\delta-\delta F_H(r)}{1+\delta-\delta\alpha F_H(r)-\delta(1-\alpha)F_L(r)} - \frac{f_H(r)}{\alpha f_H(r)+(1-\alpha)f_L(r)}$$

Let $H_i^{(1)}$ be the set of q over which $\max\left\{V_1(A^{(1)}), V_2(A^{(1)}), V_3(A^{(1)})\right\} = V_i(A^{(1)})$ where $i \in \{1, 2, 3\}$. We then have

$$\begin{aligned}
V_1 \quad & (A) = P_1 + \delta \int_{H_1^{(1)}} V_1(A^{(1)})\theta_A^1(q)dq + \delta \int_{H_2^{(1)}} V_2(A^{(1)})\theta_A^1(q)dq + \delta \int_{H_3^{(1)}} V_3(A^{(1)})\theta_A^1(q)dq \\
= \quad & P_1 + \delta \int_{H_1^{(1)}} \theta_A^1(q)dq \left[P_1 + \delta \int_{-\infty}^{+\infty} V(A^{(1.1)})\theta_{A^{(1)}}^1(.)d(.) \right] \\
+ \quad & \delta \int_{H_2^{(1)}} \theta_A^1(q)dq \left[P_2(q) + \delta \int_{-\infty}^{+\infty} V(A^{(1.2)})\theta_{A^{(1)}}^2(.)d(.) \right] \\
+ \quad & \delta \int_{H_3^{(1)}} \theta_A^1(q)dq \left[P_2 + \delta \int_{-\infty}^{+\infty} V(A^{(1.3)})\theta_{A^{(1)}}^3(.)d(.) \right]
\end{aligned}$$

Developing, we find:

$$\begin{aligned}
V_1 \quad & (A) = P_1 + \delta P_1 \int_{H_1^{(1)}} \theta_A^1(q)dq \\
+ \quad & \delta^2 \int_{H_1^{(1)}} \theta_A^1(q)dq \int_{-\infty}^{+\infty} \theta_{A^{(1)}}^1(.)d(.) \max\left\{V_1(A^{(1.1)}), V_2(A^{(1.1)}), V_3(A^{(1.1)})\right\} \\
+ \quad & \delta P_1 \int_{H_2^{(1)}} f_H(q)dq + \delta^2 \int_{H_2^{(1)}} \theta_A^1(q)dq \int_{-\infty}^{+\infty} \theta_{A^{(1)}}^2(.)d(.) \max\left\{V_1(A^{(1.2)}), V_2(A^{(1.2)}), V_3(A^{(1.2)})\right\} \\
+ \quad & \delta P_2 \int_{H_3^{(1)}} \theta_A^1(q)dq + \delta^2 \int_{H_3^{(1)}} \theta_A^1(q)dq \int_{-\infty}^{+\infty} \theta_{A^{(1)}}^3(.)d(.) \max\left\{V_1(A^{(1.3)}), V_2(A^{(1.3)}), V_3(A^{(1.3)})\right\}
\end{aligned}$$

Or

$$\begin{aligned}
V_1 \quad & (A) = P_1 + \delta P_1 \int_{H_1^{(1)}} \theta_A^1(q)dq + \delta^2 \int_{H_1^{(1)}} \theta_A^1(q)dq \left[\int_{H_1^{(1.1)}} V_1(A^{(1.1)})\theta_{A^{(1)}}^1(.)d(.) \right. \\
+ \quad & \left. \int_{H_2^{(1.1)}} V_2(A^{(1.1)})\theta_{A^{(1)}}^1(.)d(.) + \int_{H_3^{(1.1)}} V_3(A^{(1.1)})\theta_{A^{(1)}}^1(.)d(.) \right] \\
+ \quad & \delta P_1 \int_{H_2^{(1)}} f_H(q)dq + \delta^2 \int_{H_2^{(1)}} \theta_A^1(q)dq \left[\int_{H_1^{(1.2)}} V_1(A^{(1.2)})\theta_{A^{(1)}}^2(.)d(.) + \int_{H_2^{(1.2)}} V_2(A^{(1.2)})\theta_{A^{(1)}}^2(.)d(.) \right. \\
+ \quad & \left. \int_{H_3^{(1.2)}} V_3(A^{(1.2)})\theta_{A^{(1)}}^2(.)d(.) \right] \\
+ \quad & \delta P_2 \int_{H_3^{(1)}} \theta_A^1(q)dq + \delta^2 \int_{H_3^{(1)}} \theta_A^1(q)dq \left[\int_{H_1^{(1.3)}} V_1(A^{(1.3)})\theta_{A^{(1)}}^3(.)d(.) + \int_{H_2^{(1.3)}} V_2(A^{(1.3)})\theta_{A^{(1)}}^3(.)d(.) \right. \\
+ \quad & \left. \int_{H_3^{(1.3)}} V_3(A^{(1.3)})\theta_{A^{(1)}}^3(.)d(.) \right]
\end{aligned}$$

Rearranging we find the result. □

3.9.3 Proof of lemma 2.3.3

Proof. Without loss of generality, let us give the proof for the normal distribution case. We use the backward induction procedure.
We first prove the lemma for $k = K$.

$$\int_{-\infty}^{+\infty} V(A(q))\theta^k(q)dq = \int_{-\infty}^{+\infty} \max\{V_1(A(q)), V_2(A(q)), ..., V_K(A(q))\}\,\theta^k(q)dq$$
$$= \int_{-\infty}^{+\infty} \max\left\{\int_{-\infty}^{+\infty}\left(q_1 + \delta V(A_1^{(1)})\right)\theta^1_{A(q)}(q_1)dq_1, ..., \int_{-\infty}^{+\infty}\left(q_K + \delta V(A_K^{(1)})\right)\theta^K_{A(q)}(q_K)dq_K\right\}\theta^k(q)dq$$
$$= \int_{-\infty}^{+\infty} \max\left\{\int_{-\infty}^{+\infty}\left(q_1 + \delta V(A_1^{(1)})\right)\theta^1_{A(q)}(q_1)dq_1, ..., \int_{-\infty}^{+\infty} q_K\theta^K_{A(q)}(q_K)dq_K + \delta V(A_K^{(1)})\right\}\theta^k(q)dq$$
$$\geq \max\left\{\int_{-\infty}^{+\infty}\left(q_1 + \delta V(A_1^{(1)})\right)\theta^1_{A(q)}(q_1)dq_1, ..., \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} q_K\theta^K_{A(q)}(q_K)\theta^k(q)dq_Kdq + \delta V(A_K^{(1)})\right\}$$

In addition,

$$V\left(A\left(\frac{1}{2}\right)\right) = \max\left\{\int_{-\infty}^{+\infty}\left(q_1 + \delta V(A_1^{(1)})\right)\theta^1_{A(\frac{1}{2})}(q_1)dq_1, ..., \int_{-\infty}^{+\infty} q_K\theta^K_{A(\frac{1}{2})}(q_K)dq_K + \delta V(A_K^{(1)})\right\}$$

To compare $\int_{-\infty}^{+\infty} V(A(q))\theta^k(q)dq$ with $V\left(A\left(\frac{1}{2}\right)\right)$ we just need to compare $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} q_K\theta^K_{A(q)}(q_K)\theta^k(q)dq_Kdq$ with $\int_{-\infty}^{+\infty} q_K\theta^K_{A(\frac{1}{2})}(q_K)dq_K$.

It is easy to find that $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} q_K\theta^K_{A(q)}(q_K)\theta^k(q)dq_Kdq = \int_{-\infty}^{+\infty} q_K\theta^K_{A(\frac{1}{2})}(q_K)dq_K$.

Now, assume that the lemma is true for $k = l + 1$ and prove it for $k = l$.

$$\int_{-\infty}^{+\infty} V(A(q))\theta^l(q)dq \geq \max\left\{\int_{-\infty}^{+\infty} V_1(A(q))\theta^l(q)dq, \int_{-\infty}^{+\infty} V_2(A(q))\theta^l(q)dq, ..., \int_{-\infty}^{+\infty} V_K(A(q))\theta^l(q)dq\right\}$$

$$= \max\left\{\begin{array}{l}
\int_{-\infty}^{+\infty} q_1\theta^1_{A(q)}(q_1)dq_1 + \delta\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} V\left(A_1^{(1)}(q,.)\right)\theta^1_{A(q)}(q_1)\theta^l(q)dq_1dq, ...,\\
..., \int_{-\infty}^{+\infty} q_{l-1}\theta^{l-1}_{A(q)}(q_{l-1})dq_{l-1} + \delta\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} V\left(A_{l-1}^{(1)}(q,.)\right)\theta^{l-1}_{A(q)}(q_{l-1})\theta^l(q)dq_{l-1}dq,\\
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} q_l\theta^l_{A(q)}(q_l)\theta^l(q)dq_ldq + \delta\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} V\left(A_l^{(1)}(q,.)\right)\theta^l_{A(q)}(q_l)\theta^l(q)dq_ldq,\\
\int_{-\infty}^{+\infty} q_{l+1}\theta^{l+1}_{A(q)}(q_{l+1})dq_{l+1} + \delta\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} V\left(A_{l+1}^{(1)}(q,.)\right)\theta^{l+1}_{A(q)}(q_{l+1})\theta^l(q)dq_{l+1}dq, ...,\\
..., \int_{-\infty}^{+\infty} q_K\theta^K_{A(q)}(q_K)dq_K + \delta\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} V\left(A_K^{(1)}(q,.)\right)\theta^K_{A(q)}(q_K)\theta^K(q)dq_Kdq.
\end{array}\right\}$$

$$= \max\left\{\begin{array}{l}
\int_{-\infty}^{+\infty} q_1\theta^1_{A(q)}(q_1)dq_1 + \delta\int_{-\infty}^{+\infty}\theta^1_{A(q)}(q_1)dq_1\int_{-\infty}^{+\infty} V\left(A_1^{(1)}(q,.)\right)\theta^l(q)dq, ...,\\
..., \int_{-\infty}^{+\infty} q_{l-1}\theta^{l-1}_{A(q)}(q_{l-1})dq_{l-1} + \delta\int_{-\infty}^{+\infty}\theta^{l-1}_{A(q)}(q_{l-1})dq_{l-1}\int_{-\infty}^{+\infty} V\left(A_{l-1}^{(1)}(q,.)\right)\theta^l(q)dq,\\
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} q_l\theta^l_{A(q)}(q_l)\theta^l(q)dq_ldq + \delta\int_{-\infty}^{+\infty}\theta^l(q)dq\int_{-\infty}^{+\infty} V\left(A_l^{(1)}(q,.)\right)\theta^l_{A(q)}(q_l)dq_l,\\
\int_{-\infty}^{+\infty} q_{l+1}\theta^{l+1}_{A(q)}(q_{l+1})dq_{l+1} + \delta\int_{-\infty}^{+\infty}\theta^{l+1}_{A(q)}(q_{l+1})dq_{l+1}\int_{-\infty}^{+\infty} V\left(A_{l+1}^{(1)}(q,.)\right)\theta^l(q)dq, ...,\\
..., \int_{-\infty}^{+\infty} q_K\theta^K_{A(q)}(q_K)dq_K + \delta\int_{-\infty}^{+\infty}\theta^K_{A(q)}(q_K)dq_K\int_{-\infty}^{+\infty} V\left(A_K^{(1)}(q,.)\right)\theta^K(q)dq.
\end{array}\right\}$$

Moreover,

$$V\left(A\left(\frac{1}{2}\right)\right) = \max\left\{V_1\left(A\left(\frac{1}{2}\right)\right), ..., V_K\left(A\left(\frac{1}{2}\right)\right)\right\}$$

$$V\left(A\left(\tfrac{1}{2}\right)\right) = \max\left\{\begin{array}{l} \int_{-\infty}^{+\infty} q_1\theta^1_{A(1/2)}(q_1)dq_1 + \delta\int_{-\infty}^{+\infty} V\left(A_1^{(1)}\left(1/2,.\right)\right)\theta^1_{A(1/2)}(q_1)dq_1, ..., \\ ..., \int_{-\infty}^{+\infty} q_{l-1}\theta^{l-1}_{A(1/2)}(q_{l-1})dq_{l-1} + \delta\int_{-\infty}^{+\infty} V\left(A_{l-1}^{(1)}\left(1/2,.\right)\right)\theta^{l-1}_{A(1/2)}(q_{l-1})dq_{l-1}, \\ \int_{-\infty}^{+\infty} q_l\theta^l_{A(1/2)}(q_l)dq_l + \delta\int_{-\infty}^{+\infty} V\left(A_l^{(1)}\left(1/2,.\right)\right)\theta^l_{A(1/2)}(q_l)dq_l, \\ \int_{-\infty}^{+\infty} q_{l+1}\theta^{l+1}_{A(1/2)}(q_{l+1})dq_{l+1} + \delta\int_{-\infty}^{+\infty} V\left(A_{l+1}^{(1)}\left(1/2,.\right)\right)\theta^{l+1}_{A(1/2)}(q_{l+1})dq_{l+1}, ..., \\ ..., \int_{-\infty}^{+\infty} q_K\theta^K_{A(1/2)}(q_K)dq_K + \delta\int_{-\infty}^{+\infty} V\left(A_K^{(1)}\left(1/2,.\right)\right)\theta^K_{A(1/2)}(q_K)dq_K \end{array}\right\}$$

For $k \in [1, K]$ and $k \neq l$, we have

$$\int_{-\infty}^{+\infty} V\left(A_k^{(1)}(q)\right)\theta^l(q)dq \geq V\left(A_k^{(1)}(1/2)\right)$$
$$(\textit{due to the backward induction assumption and remark } 4)$$

$$\textit{and} \quad \theta^k_{A(q)}(q_k) = \theta^k_{A(1/2)}(q_k).$$

$$\textit{Therefore,} \quad \int_{-\infty}^{+\infty} V_k\left(A(q)\right)\theta^l(q)dq \geq V_k\left(A(1/2)\right) \quad \textit{for } k \in \{1, ..., K\} \textit{ and } k \neq l.$$

We now need to compare

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} q_l\theta^l_{A(q)}(q_l)\theta^l(q)dq_l dq + \delta\int_{-\infty}^{+\infty}\theta^l(q)dq\int_{-\infty}^{+\infty} V\left(A_l^{(1)}(q,.)\right)\theta^l_{A(q)}(q_l)dq_l$$
with
$$\int_{-\infty}^{+\infty} q_l\theta^l_{A(1/2)}(q_l)dq_l + \delta\int_{-\infty}^{+\infty} V\left(A_l^{(1)}(1/2,.)\right)\theta^l_{A(1/2)}(q_l)dq_l.$$

From the part 1 of this proof, we have proven that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} q_l\theta^l_{A(q)}(q_l)\theta^l(q)dq_l dq = \int_{-\infty}^{+\infty} q_l\theta^l_{A(1/2)}(q_l)dq_l$$

Therefore, we need to compare $\int_{-\infty}^{+\infty}\theta^l(q)dq\int_{-\infty}^{+\infty} V\left(A_l^{(1)}(q,.)\right)\theta^l_{A(q)}(q_l)dq_l$ with $\int_{-\infty}^{+\infty} V\left(A_l^{(1)}(1/2,.)\right)\theta^l_{A(1/2)}(q_l)dq_l$.

Using the backward induction assumption and remark 1, we have:

$$\int_{-\infty}^{+\infty} V\left(A_l^{(1)}(q,.)\right)\theta^l_{A(q)}(q_l)dq_l \geq V\left(A_l^{(1)}(q,1/2,.)\right)$$

In addition

$$\theta^l_{A(1/2)}(q_l)dq_l = \theta^l(q_l)dq_l$$

Therefore

$$\int_{-\infty}^{+\infty}\theta^l(q)dq\int_{-\infty}^{+\infty} V\left(A_l^{(1)}(q,.)\right)\theta^l_{A(q)}(q_l)dq_l \geq \int_{-\infty}^{+\infty} V\left(A_l^{(1)}(1/2,.)\right)\theta^l_{A(1/2)}(q_l)dq_l$$

and the proof ends. □

3.9.4 Proof of proposition 2.3.4

Proof. Proving proposition 2.3.4 boils down to prove that the value of a researcher decreases with his age when his history remains constant.[30] Let us consider the following state.

$$A = \begin{cases} a_1 = N \\ a_2 = a \\ a_3 = a\,N \\ \vdots \\ a_k = a\,N\cdots N\,N \\ a_{k+1} = a\,N\cdots N\,N\,N \\ \vdots \\ a_K = a\,N\cdots N\cdots N\,N\,N\,N\,N \end{cases}$$

The real a is such that all researchers have an equivalent history. In other words $a = \frac{1}{2}$. Without loss of generality, the proof will end if we prove that for any k, $V_{k+1}(A) < V_k(A)$. We will use the backward induction procedure.

1- Assume that $V_{k+2}(A) < V_{k+1}(A)$ and let us prove that $V_{k+1}(A) < V_k(A)$. It is useful to note that $\theta^i_A(q)$ is the same for all researchers. Therefore, we set $\theta^i_A(q) = \theta_A(q)$ for all

[30]An history is said to be constant from period t to period t+1 if the histories at t and t+1 are equivalent.

$i \in \{1, ..., K\}$.

$$\begin{aligned} V_k(A) &= \int_{-\infty}^{+\infty} q\theta_A(q)dq + \delta \int_{-\infty}^{+\infty} V(A_k^{(1)})\theta_A(q)dq. \\ V_{k+1}(A) &= \int_{-\infty}^{+\infty} q\theta_A(q)dq + \delta \int_{-\infty}^{+\infty} V(A_{k+1}^{(1)})\theta_A(q)dq. \\ V_{k+2}(A) &= \int_{-\infty}^{+\infty} q\theta_A(q)dq + \delta \int_{-\infty}^{+\infty} V(A_{k+2}^{(1)})\theta_A(q)dq. \end{aligned}$$

where

$$A_k^{(1)} = \begin{cases} a_1 = N \\ a_2 = N \\ a_3 = a\,N \\ \vdots \\ a_k = a\,N \cdots N\,N \\ a_{k+1} = a\,N \cdots N\,N\,q \\ a_{k+2} = a\,N \cdots N\,N\,N\,N \\ \vdots \\ a_K = a\,N \cdots N \cdots N\,N\,N\,N\,N \end{cases} \qquad A_{k+1}^{(1)} = \begin{cases} a_1 = N \\ a_2 = N \\ a_3 = a\,N \\ \vdots \\ a_k = a\,N \cdots N\,N \\ a_{k+1} = a\,N \cdots N\,N\,N \\ a_{k+2} = a\,N \cdots N\,N\,N\,q \\ a_{k+3} = a\,N \cdots N\,N\,N\,N\,N \\ \vdots \\ a_K = a\,N \cdots N \cdots N\,N\,N\,N\,N \end{cases}$$

$$and \quad A_{k+2}^{(1)} = \begin{cases} a_1 = N \\ a_2 = N \\ a_3 = a\,N \\ \vdots \\ a_k = a\,N \cdots N\,N \\ a_{k+1} = a\,N \cdots N\,N\,N \\ a_{k+2} = a\,N \cdots N\,N\,N\,N \\ a_{k+3} = a\,N \cdots N\,N\,N\,N\,q \\ a_{k+4} = a\,N \cdots N\,N\,N\,N\,N\,N \\ \vdots \\ a_K = a\,N \cdots N \cdots N\,N\,N\,N\,N \end{cases}$$

$$\begin{aligned} V_k(A) - V_{k+1}(A) &= \delta \int_{-\infty}^{+\infty} \left[V(A_k^{(1)}) - V(A_{k+1}^{(1)}) \right] \theta_A(q)dq \\ &= \delta \int_{-\infty}^{+\infty} \max \left\{ V_1(A_k^{(1)}), \cdots, V_k(A_k^{(1)}), \cdots, V_K(A_k^{(1)}) \right\} \theta_A(q)dq \\ &- \max \int_{-\infty}^{+\infty} \left\{ V_1(A_{k+1}^{(1)}), \cdots, V_{k+1}(A_{k+1}^{(1)}), \cdots, V_K(A_{k+1}^{(1)}) \right\} \theta_A(q)dq \end{aligned}$$

Moreover,

$$\begin{aligned} V_{k+1}(A) - V_{k+2}(A) &= \delta \int_{-\infty}^{+\infty} \max\left\{V_1(A^{(1)}_{k+1}), \cdots, V_{k+1}(A^{(1)}_{k+1}), \cdots, V_K(A^{(1)}_{k+1})\right\} \theta_A(q)dq \\ &- \max \int_{-\infty}^{+\infty} \left\{V_1(A^{(1)}_{k+2}), \cdots, V_{k+2}(A^{(1)}_{k+2}), \cdots, V_K(A^{(1)}_{k+2})\right\} \theta_A(q)dq \end{aligned}$$

Observing $A^{(1)}_k$, $A^{(1)}_{k+1}$ and $A^{(1)}_{k+2}$, we can see that $V_k(A) - V_{k+1}(A)$ and $V_{k+1}(A) - V_{k+2}(A)$ have a certain similarity that leads us to say that $V_k(A) - V_{k+1}(A)$ and $V_{k+1}(A) - V_{k+2}(A)$ should have the same sign. As a consequence, $V_k(A) - V_{k+1}(A) > 0$.

2- Now, assume that $k = K-1$ and $k+1 = K$. We want to prove that $V_k(A) > V_{k+1}(A)$.

$$V_k(A) = \int_{-\infty}^{+\infty} q\theta_A(q)dq + \delta \int_{-\infty}^{+\infty} V(A^{(1)}_k)\theta_A(q)dq. \tag{3.96}$$

$$V_{k+1}(A) = \int_{-\infty}^{+\infty} q\theta_A(q)dq + \delta V(A^{(1)}_{k+1}) \quad since\, A^{(1)}_{k+1}\, does\, not\, depend\, on\, q. \tag{3.97}$$

By lemma 2.3.3, we have

$$\int_{-\infty}^{+\infty} V(A^{(1)}_k)\theta_A(q)dq > V(A^{(1)}_{k+1}). \tag{3.98}$$

The proof ends. □

3.10 Appendix for chapter 3

3.10.1 Proof of theorem 3.5.1

We first prove the first part of the theorem. Let us consider two distributors 1 and 2 who set p_1 and p_2 respectively. Assume that $p_1 = p_2$. We should prove that this cannot occur. Precisely we prove that at least one of the distributors will deviate at $p_1 = p_2$.

Case 1: None of the distributors is willing to sell more at this price. We have then

$$\forall k > 0, (p_i - c + A)k - Y_i(t_i + \tau_i + \tau_i^k) + Y_i(t_i + \tau_i) \leq 0, \quad for\ i = 1, 2. \tag{3.99}$$

τ_i^k is the time needed by distributor i to sell k more units of products.

Suppose that a typical distributor j increases the price by ε in such a way that the demand to him decreases by a given k. His payoff will vary by $\Delta = (-p_j + c - A)k - Y_j(t_j + \tau_j - \tau_j^k) + Y_j(t_j + \tau_j) + \varepsilon(d_j(\tau_j) - k)$. Δ is positif if $(p_j - c + A)k - Y_j(t_j + \tau_j) + Y_j(t_j + \tau_j - \tau_j^k) <$

$\varepsilon(d_j(\tau_j) - k)$. ε can be chosen such that

$$\varepsilon > \frac{1}{d_j(\tau_j) - k}\Big((p_j - c + A)k - Y_j(t_j + \tau_j) + Y_j(t_j + \tau_j - \tau_j^k)\Big) \tag{3.100}$$

Let us mention that $(p_j - c + A)k - Y_j(t_j + \tau_j) + Y_j(t_j + \tau_j - \tau_j^k) > 0$ because if not distributor j will not use his time up to τ_j in selling.
However, for (3.100) to hold we need to have $k < d_j(\tau_j)$. In other words, for a price increase to be profitable, the distributor does have a minimal demand. To do so, the distributor should meet the two following challenges. *(1)* fulfill requirements in the program (3.16) for the consumers and *(2)* Not all consumers turn to the other distributors because of the price increase.
The first challenge is easily met by choosing ε sufficiently small and the second challenge is overcome due to (3.99).

However, we should face another challenge. ε and k are not independent since ε implies k through the consumers'demand function. When the distributor chooses the ε such that (3.100) holds, he should be sure that the chosen ε implies the given k or less. In other words we need to insure that it exists a given k for which it exists a corresponding ε that fits (3.100). If the distributor increases the price by ε to fill the requirements of the set $\mathbb{C}$ defined in (3.18), the maximum decrease in demand will be

$$k_\varepsilon = \int_{\mathbb{C}} \left(f_c(p_j) - f_c(p_j + \varepsilon)\right) dc \tag{3.101}$$

In fact k_ε would be the decrease in demand to distributor j if all of the consumers willing to buy from the MLMs had bought from distributor j. We have $k_\varepsilon > k$. From (3.101), since the consumer demand is smooth and monotonic it exists a function S such that $\varepsilon = S(k_\varepsilon)$. S is an increasing function, so $S(k_\varepsilon) > S(k)$. Moreover, it also exists an increasing function h such that $h(k) = \varepsilon$. As a consequence,

$$\varepsilon = h(k) = S(k_\varepsilon) > S(k). \tag{3.102}$$

Let us remind that we are intending to prove the existence of ε and k such that (3.100) holds. In other words we will prove the existence of k such that the following holds:

$$h(k) > \frac{1}{d_j(\tau_j) - k}\Big((p_j - c + A)k - Y_j(t_j + \tau_j) + Y_j(t_j + \tau_j - \tau_j^k)\Big) \tag{3.103}$$

Let $\varphi(k) = \frac{1}{d_j(\tau_j)-k}\Big((p_j - c + A)k - Y_j(t_j + \tau_j) + Y_j(t_j + \tau_j - \tau_j^k)\Big)$. Let us remark that the expression $(p_j - c + A)k - Y_j(t_j + \tau_j) + Y_j(t_j + \tau_j - \tau_j^k)$ is his gain if distributor j

increases his sales by k without varying his price and such that the time allocated to selling shifts from $\tau_j - \tau_j^k$ to τ_j. In other words, $(p_j - c + A)k - Y_j(t_j + \tau_j) + Y_j(t_j + \tau_j - \tau_j^k)$ is increasing in k and then φ is increasing in k.
It is obvious that $\varphi(0) = 0$. We also have $S(0) = 0$, because, if there is no price increase there is no decrease in demand, everything equal elsewhere.
Since the functions $S(.)$ and $\varphi(.)$ are both smooth, increasing and meet at 0, it exists a real k in the neighborhood of 0 such that $S(k)$ and $\varphi(k)$ are so close ($S(k) \approx \varphi(k)$), and then, due to (3.102), $h(k) > \varphi(k)$.

Case 2: Some of the distributors are willing to sell more at this price

Let us denote by i the distributors who are willing to sell more at the current price. It exists $\hat{k} > 0$ such that $(p_i - c + A)\hat{k} - Y_i\left(t_i + \tau_i + \tau_i^{\hat{k}}\right) + Y_i(t_i + \tau_i) > 0$. Suppose that distributor i decreases his price by $\bar{\varepsilon}$ in order to have more sales of $\bar{k}$ units. It would be profitable for him if

$$\bar{\varepsilon} < \frac{1}{d_i(\tau_i) + \bar{k}}\left[(p_i - c + A)\bar{k} - Y_i\left(t_i + \tau_i + \tau_i^{\bar{k}}\right) + Y_i(t_i + \tau_i)\right] \tag{3.104}$$

$\bar{k}$ is the increase of the demand for distributor i if he decreases the price by ε_0. Here we are intending to prove that such $\bar{\varepsilon}$ and $\bar{k}$ exist.
As in case 1, for each ε it exists k_ε such that $\varepsilon = S(k_\varepsilon)$. $S(.)$ is smooth and increasing. This k_ε is what that is predicted by the demand function. Let $\varphi(k) = (p_i - c + A)k - Y_i\left(t_i + \tau_i + \tau_i^k\right) + Y_i(t_i + \tau_i)$ for any $k \geq 0$. $\varphi(\hat{k}) > 0$ and $\varphi(0) = 0$. We can prove that it exists and interval $[0, \lambda)$ over which φ is increasing. λ is the highest real for which φ is greater than zero.
Assume that distributor i chooses $\bar{\varepsilon} = \varepsilon_0$ such that the increase in demand predicted by the demand function is $k_0 < 0$. As a consequence, $\varepsilon_0 = S(k_0)$. Actually the true increase (denoted by k^*) of the demand will be greater than k_0 since the consumers purchasing from the other distributor will now prefer the distributor i. Thus, the additional sales of distributor i can be up to $\tilde{k} = \min(k^*, \lambda) > k_0$. The distributor should make his additional sale choice $\bar{k}$ between k_0 and $\tilde{k}$. Moreover, since $\varphi(\bar{k}) > 0$ it exists $A > 0$ such that $\varphi(\bar{k}) > A$. Distributor i can decide appropriately A and chooses k_0 such that $S(k_0) \leq A$.

We now move to the second part of the theorem claiming that $0 < |p_1 - p_2| \leq \varepsilon$. From the first part of the theorem, $p_1 < p_2$ or $p_1 > p_2$. Here we are going to show that the gap between p_1 and p_2 should not be so hight because if so, some of the distributors will profitably deviate. Let us assume that $p_1 < p_2$.[31] If at this price setting, both of the

[31] The same reasoning is valid for $p_1 > p_2$.

distributors sell then, for any k, $(p_1 - c + A)k - Y_1(t_1 + \tau_1 + \tau_1^k) + Y_1(t_1 + \tau_1) \leq 0$. It is obvious that distributor 1 will not decrease his price. It may be possible for him to increase the price. Assume that he increases his price by γ to be able to accept additional sales of k_γ. He should increase the price while setting the best price, in comparison with distributor 2. His new payoff is then

$$\theta(t_1) + (p_1 + \gamma)\,(d_1(\tau_1) + k_\gamma) - c(q_1 + k_\gamma) + v^1_{MLM}\,(q_1 - d_1(\tau_1)) \\ -(\bar{v}^1 - \bar{p})\,(q_1 - d_1(\tau_1)) + A(q_1 + k_\gamma) - Y_1(t_1 + \tau_1 + \tau_1^{k_\gamma}) - I \tag{3.105}$$

This new payoff is higher than the previous one if

$$(p_1 - c + A)k_\gamma - Y_1(t_1 + \tau_1 + \tau_1^{k_\gamma}) + Y_1(t_1 + \tau_1) + \gamma\,(d_1(\tau_1) + k_\gamma) > 0. \tag{3.106}$$

(3.106) will hold if γ is sufficiently high. This is possible if p_1 is not too close to p_2. Moreover, due to the consumer program, any increase in price should be done while being sure that the non MLMs is not getting the best. Thus, it exists $\varepsilon > 0$ such that $0 < |p_i - p_j| \leq \varepsilon$.