

# Single-plateaued choice\*

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**Abstract.** Single-plateaued preferences generalize single-peaked preferences by allowing for multiple best elements. These preferences have played an important role in areas such as voting, strategy-proofness and matching problems. We examine the notion of single-plateauedness in a choice-theoretic setting. Single-plateaued choice is characterized by means of a collinear interval continuity property in the presence of independence of irrelevant alternatives. Further results establish that our notion of single-plateauedness conforms to the motivation underlying the term and we analyze the consequences of alternative continuity properties. The importance of basic assumptions such as closedness and convexity is discussed. *Journal of Economic Literature* Classification Nos.: D11, D71.

**Keywords.** Single-plateauedness, choice correspondences, independence of irrelevant alternatives, continuity.

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# 1 Introduction

The notion of single-peakedness has been of fundamental importance in many fields within economic theory. An early application of single-peakedness is Black's (1948) well-known result that a preference-domain restriction of this nature ensures that the majority rule generates transitive social preferences. The contributions by Inada (1969) and Sen (1970) are other examples for the use of value restrictions that focus on single-peaked preference profiles. Single-peakedness also appears in the study of strategy-proof social choice functions; see, for instance, Moulin (1980) and Sprumont (1991). While much of the work on single-peakedness focuses on one dimension, definitions suitable for higher dimensions are available. These generalizations are used by Barberà, Gul and Stacchetti (1993), Barberà and Jackson (1994), Dutta, Peters and Sen (2002), Ehlers and Storcken (2008) and Le Breton and Weymark (2011), among others. Ballester and Haeringer (2011) characterize one-dimensional single-peaked preference profiles by providing necessary and sufficient conditions for the existence of a single ranking such that all preferences in the profile are single-peaked with respect to this ranking.

Single-peakedness is analyzed in a choice-theoretic setting by Bossert and Peters (2009). The present paper generalizes the analysis carried out there to single-plateaued choice, which within the same framework arises naturally if multi-valued instead of single-valued choice is considered. See Moulin (1984) for an early contribution that employs single-plateauedness in the context of voting theory. Berga (1998), Ehlers (2002), Barberà (2007) and Berga and Moreno (2009), among others, examine strategy-proofness with single-plateaued preferences.

We do not impose any restriction on the dimension of the domain; our results are valid for any fixed-dimensional Euclidean space. The basic object to be studied is a choice correspondence that selects, for any fixed dimension  $n \in \mathbb{N}$ , a non-empty, closed and convex subset of chosen elements from each non-empty, compact and convex subset of  $\mathbb{R}^n$ , the Euclidean  $n$ -dimensional space. As is the case for Bossert and Peters (2009), we concentrate on choice correspondences that satisfy independence of irrelevant alternatives, a contraction consistency condition that is closely related (but, in general, not equivalent) to the rationalizability of a choice correspondence; see, for instance, Richter (1966, 1971). Given the nature of the standard applications of single-peakedness and single-plateauedness, this focus appears to be suitable for the issue to be addressed.

In the presence of independence of irrelevant alternatives (or the weak axiom of revealed preference which is equivalent to independence on our domain), we characterize

single-plateaued choice by means of collinear interval continuity, a continuity property that is restricted to points on the same line. While this result and its proof display some analogy to the single-valued case examined in Bossert and Peters (2009), there are some significant differences in the methods used to establish a link between our formal definition of single-plateauedness and its interpretation. Furthermore, the multi-valued case allows us to work with a considerably richer setup when analyzing various alternative continuity assumptions. In our generalized framework, moving from collinear interval continuity to continuity (or even lower semicontinuity) imposes additional restrictions on possible choice correspondences that do not play any role in the single-valued case.

In the following section, we state the formal definition of the choice correspondences to be considered throughout the paper. The domain of our correspondence consists of all non-empty, compact and convex subsets of  $\mathbb{R}^n$ . The choice correspondence is assumed to be closed-valued and convex-valued. We introduce the axiom of independence of irrelevant alternatives, familiar from the literature on rational choice. Because our domain is closed under intersection, independence of irrelevant alternatives is equivalent to the weak axiom of revealed preference. It is often more convenient to use one of the two equivalent axioms in some of our later proofs and, in order to make the paper self-contained, we provide a short proof of this well-known equivalence result as a preliminary observation.

Section 3 focuses on the definition, interpretation and characterization of our notion of single-plateaued choice. We formulate the property by requiring that whenever a point  $x$  is directly revealed preferred to another point  $y$ , then  $x$  must be directly revealed preferred to any point on the half-line that starts at  $x$  and passes through  $y$ . This is in accord with the one-dimensional notion of single-plateauedness for preference relations and we prove that, using our definition, a single-plateaued choice correspondence has an important characteristic that further underlines the appeal of this notion of single-plateauedness. In particular, single-plateauedness implies that there exists a (possibly empty) closed and convex subset  $P$  of  $\mathbb{R}^n$ , to be interpreted as the plateau. Whenever the intersection of this plateau and the feasible set is non-empty, the choice correspondence selects this intersection. If the intersection of  $P$  and the feasible set is empty, the set of chosen points must be contained in the boundary of the feasible set. These properties of  $P$  capture the idea underlying single-plateauedness: there is a set of points that are chosen whenever feasible, and as we move away from this plateau along any half-line, the points become progressively less desirable. It is possible that the plateau is empty; this situation is the analogue of a single-plateaued or single-peaked preference relation that is monotonic in the one-dimensional space.

Collinear interval continuity restricts the familiar continuity requirement to points along a given straight line and our main characterization theorem establishes the equivalence of single-plateauedness and collinear interval continuity in the presence of independence of irrelevant alternatives. Given the observation that independence and the weak axiom are equivalent in our setting, the characterization result can be rephrased using the weak axiom of revealed preference and independence of irrelevant alternatives interchangeably. We conclude Section 3 with the statement of a corollary to this effect.

In Sections 4 and 5, we examine the consequences of using alternative continuity properties and the importance of the closedness and convexity assumptions that we impose on chosen sets.

Our first observation in Section 4 is that, parallel to the single-peaked case discussed in Bossert and Peters (2009), continuity of the choice correspondence is not implied by the conjunction of independence of irrelevant alternatives and collinear interval continuity. Thus, there are additional restrictions on choice correspondences if collinear interval continuity is strengthened to continuity. We prove a theorem showing that, in the continuous case, the set of plateau points  $P$  must be strictly convex rather than merely convex. Moreover, if the intersection of  $P$  and the feasible set is empty, the resulting choice (contained in the boundary of the feasible set) must be single-valued. We also point out that lower semicontinuity, if added to independence and collinear interval continuity, leads to the same additional restrictions: strict convexity of  $P$  and single-valuedness of boundary choices when no plateau points are feasible. In contrast, upper semicontinuity cannot be used in this fashion, as we illustrate with an example. Furthermore, we show that upper semicontinuity or lower semicontinuity alone is not sufficient to imply collinear interval continuity even in the presence of independence of irrelevant alternatives.

Section 5 shows that both closed-valuedness and convex-valuedness of the choice correspondence are essential for our results. To this end, we provide an example of a choice correspondence that satisfies independence, single-plateauedness and convex-valuedness but violates closed-valuedness, and a closed-valued, independent and single-plateaued choice correspondence that is not convex-valued.

Section 6 collects a few concluding remarks.

## 2 Independent choice correspondences

Suppose  $n \in \mathbb{N}$  is fixed and let  $\mathcal{C} = \{C \subseteq \mathbb{R}^n \mid C \text{ is non-empty, compact and convex}\}$ . A choice correspondence is a mapping  $\varphi: \mathcal{C} \rightarrow \mathbb{R}^n$  such that  $\emptyset \neq \varphi(C) \subseteq C$  and  $\varphi(C)$  is closed

and convex for all  $C \in \mathcal{C}$ .

The direct revealed preference relation  $R_\varphi$  of  $\varphi$  is defined as follows. For all  $x, y \in \mathbb{R}^n$ ,

$$xR_\varphi y \Leftrightarrow \text{there exists } C \in \mathcal{C} \text{ such that } x \in \varphi(C) \text{ and } y \in C.$$

The asymmetric part of  $R_\varphi$  is denoted by  $P_\varphi$ .

To define the direct revealed strict preference relation  $R_\varphi^*$  of  $\varphi$ , let, for all  $x, y \in \mathbb{R}^n$ ,

$$xR_\varphi^* y \Leftrightarrow \text{there exists } C \in \mathcal{C} \text{ such that } x \in \varphi(C) \text{ and } y \in C \setminus \varphi(C).$$

By definition – thus, independently of the assumptions on  $\mathcal{C}$  and  $\varphi$  – the set inclusion  $P_\varphi \subseteq R_\varphi^*$  is valid. To see that this is the case, suppose  $x, y \in \mathbb{R}^n$  are such that  $xP_\varphi y$ . Thus, there exists  $C \in \mathcal{C}$  such that  $x \in \varphi(C)$  and  $y \in C$  and, moreover, there does not exist a  $D \in \mathcal{C}$  such that  $y \in \varphi(D)$  and  $x \in D$ . This implies, in particular, that  $y \notin \varphi(C)$  and, therefore,  $y \in C \setminus \varphi(C)$ . Hence  $xR_\varphi^* y$ , which establishes that  $P_\varphi \subseteq R_\varphi^*$ .

The reverse set inclusion  $R_\varphi^* \subseteq P_\varphi$  does not follow without further assumptions. This inclusion is a generalized version of Samuelson's (1938) weak axiom of revealed preference. See Bossert and Suzumura (2010, p. 17) for a discussion and alternative formulations of the weak axiom of revealed preference. We state the set equality as a formal property.

**Weak axiom of revealed preference.**  $R_\varphi^* = P_\varphi$ .

Below we show that in our framework the weak axiom of revealed preference is equivalent to the condition of independence of irrelevant alternatives, which is a contraction-consistency condition imposed on a choice correspondence. It is often referred to as Arrow's choice axiom (see Arrow, 1959) but, as Shubik (1982, pp. 420–421 and p. 423, footnote 2) remarks, the axiom already appears in 1950 in an informal note authored by Nash. A version for single-valued choice is due to Nash (1950) in the context of axiomatic bargaining theory.

**Independence of irrelevant alternatives.** For all  $C, D \in \mathcal{C}$ , if  $D \subseteq C$  and  $D \cap \varphi(C) \neq \emptyset$ , then  $\varphi(D) = D \cap \varphi(C)$ .

For future reference, we note that, because our domain  $\mathcal{C}$  is closed under intersection (that is, for all  $C, D \in \mathcal{C}$ , the intersection  $C \cap D$  is also in  $\mathcal{C}$  whenever this intersection is non-empty), the weak axiom of revealed preference is equivalent to independence of irrelevant alternatives; see Hansson (1968) for a generalization of this observation. We provide a proof of this known result in order to make our paper self-contained.

**Lemma 1** *A choice correspondence  $\varphi: \mathcal{C} \rightarrow \mathbb{R}^n$  satisfies independence of irrelevant alternatives if and only if  $\varphi$  satisfies the weak axiom of revealed preference.*

**Proof.** Suppose first that  $\varphi$  satisfies independence of irrelevant alternatives. Because  $P_\varphi \subseteq R_\varphi^*$  by definition, we only have to establish the reverse set inclusion. Let  $x, y \in \mathbb{R}^n$  be such that  $xR_\varphi^*y$ , that is, there exists  $C \in \mathcal{C}$  such that  $x \in \varphi(C)$  and  $y \in C \setminus \varphi(C)$ . By definition, this implies  $xR_\varphi y$ . If  $yR_\varphi x$ , there exists  $D \in \mathcal{C}$  such that  $y \in \varphi(D)$  and  $x \in D$ . The intersection  $C \cap D$  is non-empty because it contains  $x$  and  $y$  and, because  $\mathcal{C}$  is closed under intersection,  $C \cap D \in \mathcal{C}$ . Furthermore,

$$x \in C \cap D \cap \varphi(C) \quad \text{and} \quad y \in C \cap D \cap \varphi(D)$$

and, by independence of irrelevant alternatives,

$$\varphi(C \cap D) = C \cap D \cap \varphi(C) \quad \text{and} \quad \varphi(C \cap D) = C \cap D \cap \varphi(D).$$

This implies

$$C \cap D \cap \varphi(C) = C \cap D \cap \varphi(D).$$

Because  $y \in \varphi(D)$  and  $y \notin \varphi(C)$ , this contradicts our hypothesis. Thus,  $xP_\varphi y$ .

Now suppose  $\varphi$  satisfies the weak axiom of revealed preference, that is,  $R_\varphi^* = P_\varphi$ . Let  $C, D \in \mathcal{C}$  be such that  $D \subseteq C$  and  $D \cap \varphi(C) \neq \emptyset$ .

Let  $x \in D \cap \varphi(C)$ . This implies  $xR_\varphi y$  for all  $y \in C$  and, thus, for all  $y \in D$  because  $D \subseteq C$ . If  $x \notin \varphi(D)$ , let  $z \in \varphi(D)$ . By definition,  $zR_\varphi^*x$  and, therefore,  $zP_\varphi x$  by our hypothesis. This contradicts  $xR_\varphi z$  which follows from  $x \in \varphi(C)$  and  $z \in D \subseteq C$ . Thus,  $x \in \varphi(D)$ .

Now let  $x \in \varphi(D)$  which immediately implies  $x \in D$  and  $xR_\varphi y$  for all  $y \in D$ . If  $x \notin \varphi(C)$ , let  $z \in D \cap \varphi(C)$ . Because  $x \in \varphi(D)$  and  $z \in D$ , we obtain  $xR_\varphi z$ . Furthermore,  $x \in C$  because  $D \subseteq C$  and, because  $x \notin \varphi(C)$ ,  $zR_\varphi^*x$  which is equivalent to  $zP_\varphi x$  by the weak axiom of revealed preference, and we obtain a contradiction to  $xR_\varphi z$ . Thus,  $x \in D \cap \varphi(C)$  which completes the proof. ■

As is well-known (and apparent from the proof), the second implication in the above lemma – the weak axiom of revealed preference implies independence of irrelevant alternatives – is valid even if the domain of a choice correspondence is not closed under intersection.

### 3 Single-plateaued choice correspondences

For distinct  $x, y \in \mathbb{R}^n$ ,  $\ell(x, y)$  denotes the straight line through  $x$  and  $y$  and  $[x, y, \rightarrow)$  is the half-line through  $y$  starting at  $x$ . Furthermore,  $[x, y]$  is the line segment with end points  $x$

and  $y$ . The (relatively) half-open sets  $[x, y)$  and  $(x, y]$ , and the (relatively) open set  $(x, y)$  are defined analogously in the usual way. The boundary of  $C \in \mathcal{C}$  is denoted by  $bd(C)$  and the interior of  $C$  is  $int(C)$ . Convergence of a sequence of sets in  $\mathcal{C}$  is defined in terms of the Hausdorff metric for compact subsets of  $\mathbb{R}^n$ .

Single-plateaued choice correspondences are defined as follows.

**Single-plateauedness.** For all distinct  $x, y \in \mathbb{R}^n$ , if  $xR_\varphi y$ , then  $xR_\varphi z$  for all  $z \in [x, y, \rightarrow)$ .

Thus, single-plateauedness demands that if a point  $x$  is directly revealed preferred to another point  $y$ , then  $x$  is directly revealed preferred to any point on the half-line that starts at  $x$  and passes through  $y$ .

The following theorem provides a motivation of this definition. It states that, given independence of irrelevant alternatives and single-plateauedness, there is a closed and convex set (the plateau) such that, whenever the intersection of this plateau and a feasible set is non-empty, this intersection is equal to the chosen set. If the intersection is empty, only boundary points are chosen.

**Theorem 1** *Let the choice correspondence  $\varphi: \mathcal{C} \rightarrow \mathbb{R}^n$  satisfy independence of irrelevant alternatives and single-plateauedness. Then there exists a closed and convex set  $P \subseteq \mathbb{R}^n$  such that, for all  $C \in \mathcal{C}$ ,*

- (i)  $C \cap P \neq \emptyset \Rightarrow \varphi(C) = C \cap P;$
- (ii)  $C \cap P = \emptyset \Rightarrow \varphi(C) \subseteq bd(C).$

**Proof.** Define

$$P = \{x \in \mathbb{R}^n \mid \text{there exists } C \in \mathcal{C} \text{ such that } x \in int(C) \cap \varphi(C)\}.$$

We first prove that this set  $P$  has property (i) of the theorem statement; (ii) follows immediately from the definition of  $P$ . The proof is concluded by showing that  $P$  is closed and convex.

(i) Suppose  $C \in \mathcal{C}$  is such that  $C \cap P \neq \emptyset$ .

First, we establish the set inclusion  $C \cap P \subseteq \varphi(C)$ . Let  $x \in \mathbb{R}^n$  and  $D \in \mathcal{C}$  be such that  $x \in int(D) \cap \varphi(D)$  and  $x \in C$  (and, thus,  $x \in C \cap P$ ). By way of contradiction, suppose  $x \notin \varphi(C)$ . Let  $y \in \varphi(C)$ . Since  $x \in int(D) \cap \varphi(D)$ , there exists  $z \in (x, y] \cap D$  and, by definition, we have  $xR_\varphi z$ . Single-plateauedness implies  $xR_\varphi y$ . Because  $y \in \varphi(C)$  and

$x \in C \setminus \varphi(C)$ , we have  $yR_\varphi^*x$  and, by independence of irrelevant alternatives and Lemma 1, it follows that  $yP_\varphi x$ , a contradiction. Hence  $x \in \varphi(C)$ .

To prove the reverse set inclusion, suppose  $x \in \varphi(C)$ . Let  $y \in C \cap P$ . If  $y = x$ , we are done. If  $y \neq x$ , consider the set  $[x, y] \subseteq C$ . Since  $y \in [x, y] \cap P$ , the set inclusion established in the preceding paragraph implies that  $y \in \varphi([x, y])$ . Furthermore, by independence of irrelevant alternatives,  $x \in \varphi([x, y])$  because  $[x, y] \subseteq C$ . Now let  $D \supseteq [x, y]$  be such that  $x \in \text{int}(D)$ . Then  $y \in \varphi(D)$  by the set inclusion established in the preceding paragraph. Independence of irrelevant alternatives implies  $\varphi([x, y]) = [x, y] \cap \varphi(D)$ . Thus, we must have  $x \in \text{int}(D) \cap \varphi(D)$  and hence  $x \in P$  which, together with  $x \in \varphi(C) \subseteq C$ , implies  $x \in C \cap P$ .

To prove that  $P$  is closed, suppose  $x \in \text{bd}(P) \setminus \text{int}(P)$ . Let  $C \in \mathcal{C}$  be such that  $C \cap P \neq \emptyset$  and  $x \in \text{int}(C)$ . Because  $C \cap P = \varphi(C)$  and  $\varphi(C)$  is closed, we have  $x \in \text{int}(C) \cap \varphi(C)$  and thus  $x \in P$ . Thus,  $P$  is closed.

Finally, we show that  $P$  is convex. If  $P = \emptyset$  or  $P$  is a singleton set, we are done. If  $P$  contains at least two distinct elements  $x$  and  $y$ , let  $C \in \mathcal{C}$  be such that  $[x, y] \subseteq \text{int}(C)$ . Thus,  $C \cap P \neq \emptyset$ . By part (i) established earlier in the proof,  $\varphi(C) = C \cap P$ . Therefore,  $x, y \in \varphi(C)$  and, because  $\varphi$  is convex-valued,  $[x, y] \subseteq \varphi(C)$ . Therefore,  $[x, y] \subseteq \text{int}(C) \cap \varphi(C)$  and, by definition of  $P$ ,  $[x, y] \subseteq P$  which proves that  $P$  is convex. ■

In the presence of independence of irrelevant alternatives, single-plateaued choice correspondences can be characterized by means of a continuity property restricted to half-lines. The axiom is an adaptation of an analogous property formulated for single-valued choice in Bossert and Peters (2009).

**Collinear interval continuity.** For all distinct  $x, y \in \mathbb{R}^n$  and for all sequences  $\langle x^i \rangle_{i \in \mathbb{N}}$  and  $\langle y^i \rangle_{i \in \mathbb{N}}$  such that  $x^i, y^i \in \ell(x, y)$  for all  $i \in \mathbb{N}$ , if  $\lim_{i \rightarrow \infty} x^i = x$  and  $\lim_{i \rightarrow \infty} y^i = y$ , then  $\lim_{i \rightarrow \infty} \varphi([x^i, y^i]) = \varphi([x, y])$ .

The following lemma will be used in the proof of our characterization of single-plateaued choice (Theorem 2).

**Lemma 2** *Let the choice correspondence  $\varphi: \mathcal{C} \rightarrow \mathbb{R}^n$  satisfy independence of irrelevant alternatives and single-plateauedness. Then for all distinct  $x, y \in \mathbb{R}^n$ , if  $xR_\varphi y$ , then  $zR_\varphi z'$  for all  $z' \in [x, y, \rightarrow)$  and for all  $z \in [x, z']$ .*

**Proof.** Let  $x, y, z, z' \in \mathbb{R}^n$  be as in the statement of the lemma. We consider all possible cases to establish the claim.

(i) If  $z = z'$ , the conclusion of the lemma follows immediately because all singleton sets are in the domain  $\mathcal{C}$  of  $\varphi$  and, thus,  $R_\varphi$  is reflexive.

(ii) If  $z = x$ ,  $zR_\varphi z'$  follows from single-plateauedness.

(iii) If  $z \in (x, z')$  and  $z \in \varphi([z, z'])$ , it follows that  $zR_\varphi z'$  by definition of  $R_\varphi$ .

(iv) If  $z \in (x, z')$  and  $z \notin \varphi([z, z'])$ , let  $w \in \varphi([z, z'])$ . Clearly,  $w \in (z, z']$  and we obtain  $wR_\varphi^* z$  and hence  $wP_\varphi z$  by independence of irrelevant alternatives and Lemma 1. By single-plateauedness,  $xR_\varphi w$  and  $wR_\varphi x$  and, thus,  $x, w \in \varphi([x, w])$  by independence of irrelevant alternatives. Because  $\varphi$  is convex-valued, this implies  $z \in \varphi([x, w])$  because  $z \in [x, w]$ . This implies  $zR_\varphi w$ , contradicting  $wP_\varphi z$ . Thus, case (iv) cannot occur and the proof is complete. ■

Our main result is the following characterization of single-plateaued choice.

**Theorem 2** *Let the choice correspondence  $\varphi: \mathcal{C} \rightarrow \mathbb{R}^n$  satisfy independence of irrelevant alternatives. Then  $\varphi$  satisfies single-plateauedness if and only if  $\varphi$  satisfies collinear interval continuity.*

**Proof.** Suppose  $\varphi$  satisfies single-plateauedness. Let  $x, y \in \mathbb{R}^n$  be distinct and consider sequences  $\langle x^i \rangle_{i \in \mathbb{N}}$ ,  $\langle y^i \rangle_{i \in \mathbb{N}}$ ,  $\langle z^i \rangle_{i \in \mathbb{N}}$  and  $\langle \bar{z}^i \rangle_{i \in \mathbb{N}}$  such that  $x^i, y^i \in \ell(x, y)$  for all  $i \in \mathbb{N}$ ,  $\varphi([x^i, y^i]) = [z^i, \bar{z}^i]$  for all  $i \in \mathbb{N}$ ,  $\lim_{i \rightarrow \infty} x^i = x$  and  $\lim_{i \rightarrow \infty} y^i = y$ .

Without loss of generality, assume that the sequences  $\langle z^i \rangle_{i \in \mathbb{N}}$  and  $\langle \bar{z}^i \rangle_{i \in \mathbb{N}}$  converge to  $\underline{z}$  and  $\bar{z}$ , respectively (otherwise consider convergent subsequences). That is,  $\lim_{i \rightarrow \infty} z^i = \underline{z}$  and  $\lim_{i \rightarrow \infty} \bar{z}^i = \bar{z}$ . Clearly,  $\underline{z}, \bar{z} \in [x, y]$ . Let  $[\underline{x}, \bar{y}] = \varphi([x, y])$ . We prove that  $\underline{z} = \underline{x}$  and  $\bar{z} = \bar{y}$ .

Suppose  $\underline{z} \notin [\underline{x}, \bar{y}]$ . Thus,  $\underline{z} \notin \varphi([x, y])$  and, therefore,  $wR_\varphi^* \underline{z}$  for all  $w \in [\underline{x}, \bar{y}]$ . Lemma 1 implies  $wP_\varphi \underline{z}$  for all  $w \in [\underline{x}, \bar{y}]$ . Without loss of generality, suppose  $\underline{x} \in (\underline{z}, \bar{y})$ .

If  $\underline{z}^i \in [x, \underline{z}]$  for all sufficiently large  $i \in \mathbb{N}$ , then  $\underline{z}^i R_\varphi w$  for all  $w \in [\underline{x}, \bar{y}]$  and hence  $\underline{z} R_\varphi w$  for all  $w \in [\underline{x}, \bar{y}]$  by Lemma 2, a contradiction. Analogously, if  $\underline{z}^i \in (\underline{z}, \underline{x})$  for all sufficiently large  $i \in \mathbb{N}$ , then  $wP_\varphi \underline{z}^i$  for all  $w \in [\underline{x}, \bar{y}]$  and also  $\underline{z}^i R_\varphi w$  for all  $w \in [\underline{x}, \bar{y}]$ , again a contradiction. Thus,  $\underline{z} \in [\underline{x}, \bar{y}] = \varphi([x, y])$ . Using a parallel argument, it follows that  $\bar{z} \in [\underline{x}, \bar{y}] = \varphi([x, y])$  and, therefore,  $[\underline{z}, \bar{z}] \subseteq [\underline{x}, \bar{y}] = \varphi([x, y])$ .

Suppose, without loss of generality, that  $\underline{z} \in (\underline{x}, \bar{z})$  and  $\bar{z} \in [\underline{z}, \bar{y}]$ . For  $w \in (\underline{x}, \underline{z})$ , we have  $w \in \varphi([x, y])$  and thus  $wR_\varphi \underline{z}$ . Lemma 2 then implies  $wR_\varphi \underline{z}^i$  for all sufficiently large  $i \in \mathbb{N}$  but we also have  $\underline{z}^i P_\varphi w$  (by  $\varphi([x^i, y^i]) = [z^i, \bar{z}^i]$  and Lemma 1), a contradiction. Hence

$$[\underline{z}, \bar{z}] = [\underline{x}, \bar{y}] = \varphi([x, y]).$$

Now suppose  $\varphi$  satisfies collinear interval continuity. Consider distinct  $x, y \in \mathbb{R}^n$  such that  $xR_\varphi y$  and let  $z \in [x, y, \rightarrow)$ . We have to show that  $xR_\varphi z$ . If  $z \in [x, y]$ , the conclusion follows immediately from independence of irrelevant alternatives. If  $z \in [x, y, \rightarrow) \setminus [x, y]$ , suppose that, by way of contradiction,  $\neg xR_\varphi z$ . This implies  $x \notin \varphi([x, z])$ . Let  $v \in \varphi([x, z])$ . By independence of irrelevant alternatives,  $v \in \varphi([x, v])$  and, using independence of irrelevant alternatives again (since  $x \in \varphi([x, y])$ ),  $v \in (y, z]$ . For all  $w \in [y, v]$ , we must have either  $x \in \varphi([x, w])$  or  $\varphi([x, w]) \subseteq (y, w]$ . Let  $w^\beta = \beta v + (1 - \beta)y$  for all  $\beta \in [0, 1]$ . Define  $\beta^* = \inf\{\beta \in [0, 1] \mid x \notin \varphi([x, w^\beta])\}$ . The existence of this infimum follows from the observation that  $w^1 = v \in \varphi([x, v])$ . Thus,  $x \in \varphi([x, w^\beta])$  for all  $\beta \in [0, \beta^*)$ . By collinear interval continuity,  $x \in \varphi([x, w^{\beta^*}])$  and  $\varphi([x, w^{\beta^*}]) \subseteq [y, w^{\beta^*}]$ , a contradiction. ■

Combining Lemma 1 and Theorem 2, we obtain the following corollary, which separates conditions on the choice correspondence from conditions on the associated revealed preference.

**Corollary 1** *A choice correspondence satisfies independence of irrelevant alternatives and collinear interval continuity if and only if it satisfies the weak axiom of revealed preference and single-plateauedness.*

## 4 Continuity

The standard continuity axiom familiar from the relevant literature is a strengthening of collinear interval continuity.

**Continuity.** For all  $C \in \mathcal{C}$  and for all sequences  $\langle C^i \rangle_{i \in \mathbb{N}}$  with  $C^i \in \mathcal{C}$  for all  $i \in \mathbb{N}$ , if  $\lim_{i \rightarrow \infty} C^i = C$ , then  $\lim_{i \rightarrow \infty} \varphi(C^i) = \varphi(C)$ .

Continuity is equivalent to the combination of upper and lower semicontinuity.

**Upper semicontinuity.** For all  $C \in \mathcal{C}$ , for all sequences  $\langle C^i \rangle_{i \in \mathbb{N}}$  with  $C^i \in \mathcal{C}$  for all  $i \in \mathbb{N}$ , for all  $x \in \mathbb{R}^n$  and for all sequences  $\langle x^i \rangle_{i \in \mathbb{N}}$  with  $x^i \in \varphi(C^i)$  for all  $i \in \mathbb{N}$ , if  $\lim_{i \rightarrow \infty} x^i = x$ , then  $x \in \varphi(C)$ .

**Lower semicontinuity.** For all  $C \in \mathcal{C}$ , for all sequences  $\langle C^i \rangle_{i \in \mathbb{N}}$  with  $C^i \in \mathcal{C}$  for all  $i \in \mathbb{N}$  and for all  $x \in \varphi(C)$ , if  $\lim_{i \rightarrow \infty} C^i = C$ , then there exists a sequence  $\langle x^i \rangle_{i \in \mathbb{N}}$  with  $x^i \in \varphi(C^i)$  for all  $i \in \mathbb{N}$  such that  $\lim_{i \rightarrow \infty} x^i = x$ .

The conjunction of independence of irrelevant alternatives and collinear interval continuity is not sufficient to imply continuity. The following example illustrates this observation.

**Example 1** Define a relation  $R$  on  $\mathbb{R}^2$  by

$$xRy \Leftrightarrow [|x_1| < |y_1|] \quad \text{or} \quad [|x_1| = |y_1| \quad \text{and} \quad |x_2| \leq |y_2|]$$

for all  $x, y \in \mathbb{R}^2$ . Define a single-plateaued (and single-valued) choice correspondence  $\varphi$  by letting  $\varphi(C)$  be the (singleton) set of  $R$ -best elements in  $C$  for all  $C \in \mathcal{C}$ . Then  $\varphi$  satisfies independence of irrelevant alternatives and collinear interval continuity but it does not satisfy continuity.

This example raises the question of which additional properties need to be satisfied if an independent choice correspondence is to satisfy continuity rather than merely collinear interval continuity. The following theorem provides an answer.

**Theorem 3** Let the choice correspondence  $\varphi: \mathcal{C} \rightarrow \mathbb{R}^n$  satisfy independence of irrelevant alternatives and continuity. Then there exists a closed and strictly convex set  $P \subseteq \mathbb{R}^n$  such that, for all  $C \in \mathcal{C}$ ,

- (i)  $C \cap P \neq \emptyset \Rightarrow \varphi(C) = C \cap P$ ;
- (ii)  $C \cap P = \emptyset \Rightarrow [\varphi(C) \subseteq bd(C) \quad \text{and} \quad |\varphi(C)| = 1]$ .

**Proof.** Define  $P$  as in the proof of Theorem 1. In view of Theorem 1, it remains to be shown that  $P$  is strictly convex and that choices are single-valued if case (ii) of the theorem statement applies.

By Theorem 1,  $P$  is convex. If  $P$  is not strictly convex, the boundary  $bd(P)$  contains a line segment  $[x, y]$  with  $x \neq y$ . We can take a sequence  $\langle y^i \rangle_{i \in \mathbb{N}}$  such that  $[y^i, x] \cap P = \{x\}$  for all  $i \in \mathbb{N}$  and  $\lim_{i \rightarrow \infty} y^i = y$ . By (i), it follows that  $\varphi([x, y^i]) = \{x\}$  for all  $i \in \mathbb{N}$ . But  $\lim_{i \rightarrow \infty} [x, y^i] = [x, y]$  and, again using (i),  $\varphi([x, y]) = [x, y]$ , a contradiction to continuity.

If  $C \in \mathcal{C}$  is such that  $C \cap P = \emptyset$ , Theorem 1 implies  $\varphi(C) \subseteq bd(C)$ . By way of contradiction, suppose there exist  $x, y \in \varphi(C)$  such that  $x \neq y$ . Because  $\varphi$  is convex-valued,  $[x, y] \subseteq \varphi(C)$ . Independence of irrelevant alternatives implies  $\varphi([x, y]) = [x, y]$ . Let a sequence  $\langle D^i \rangle_{i \in \mathbb{N}}$  be such that  $D^i \in \mathcal{C}$ ,  $D^i$  is strictly convex,  $x, y \in bd(D^i)$  and  $D^i \cap P = \emptyset$  for all  $i \in \mathbb{N}$  and, furthermore,  $\lim_{i \rightarrow \infty} D^i = [x, y]$ . By part (ii) of Theorem 1,  $\varphi(D^i) \subseteq bd(D^i)$  and, because  $\varphi$  is convex-valued,  $|\varphi(D^i)| = 1$  for all  $i \in \mathbb{N}$ . Continuity requires that  $\varphi([x, y])$  is single-valued but we have  $\varphi([x, y]) = [x, y]$ , a contradiction. ■

It is easy to see that in the second and third paragraphs of the proof of Theorem 3 only lower semicontinuity of  $\varphi$  is used. Hence, the theorem remains valid if in its premise continuity is

replaced by collinear interval continuity together with lower semicontinuity. From there, it can be proved – by using the conclusion of the theorem – that  $\varphi$  is actually continuous. On the other hand, the following example shows that we cannot replace continuity by collinear interval continuity together with upper semicontinuity.

**Example 2** Let  $n = 2$  and let  $P = \{x \in \mathbb{R}^2 \mid -1 \leq |x_1| \leq 1, -1 \leq |x_2| \leq 1\}$ . For  $\lambda \in \mathbb{R}$  denote  $\lambda P = \{(\lambda x_1, \lambda x_2) \mid (x_1, x_2) \in P\}$ . Define the choice correspondence  $\varphi$  as follows. If  $C \in \mathcal{C}$  with  $C \cap P \neq \emptyset$ , then let  $\varphi(C) = C \cap P$ . If  $C \in \mathcal{C}$  with  $C \cap P = \emptyset$ , then let  $\lambda_C = \min\{\lambda \in \mathbb{R} \mid \lambda > 1, \lambda P \cap C \neq \emptyset\}$  and let  $\varphi(C) = C \cap \lambda_C P$ . Then  $\varphi$  satisfies independence of irrelevant alternatives, collinear interval continuity and upper semicontinuity but not continuity.

We conclude the section with examples of choice correspondences showing that the combination of independence of irrelevant alternatives with either upper or lower semicontinuity is not sufficient to guarantee single-plateauedness or – equivalently, in view of Theorem 2 – collinear interval continuity.

**Example 3** Let  $n = 1$ , so  $\mathcal{C}$  is the set of all intervals  $[a, b]$  with  $a \leq b$ . We define the choice correspondences  $\varphi^1$  and  $\varphi^2$  as follows. For all  $[a, b] \in \mathcal{C}$ ,

$$\varphi^1([a, b]) = \begin{cases} [a, b] & \text{if } b \leq 0 \\ [\max\{a, 0\}, b] & \text{if } b > 0 \end{cases}$$

and

$$\varphi^2([a, b]) = \begin{cases} [a, b] & \text{if } b < 0 \\ [\max\{a, 0\}, b] & \text{if } b \geq 0. \end{cases}$$

Then both  $\varphi^1$  and  $\varphi^2$  satisfy independence of irrelevant alternatives;  $\varphi^1$  satisfies upper but not lower semicontinuity and  $\varphi^2$  satisfies lower but not upper semicontinuity. Both violate single-plateauedness and collinear interval continuity.

## 5 Closedness and convexity

The assumptions that  $\varphi$  is closed-valued and that  $\varphi$  is convex-valued cannot be dispensed with in our results – neither of them follows from our axioms. The following examples show that closed-valuedness is required even in the presence of convex-valuedness and that convex-valuedness is required even in the presence of closed-valuedness. For simplicity, we state the examples for the one-dimensional case but they can be embedded in spaces of higher dimension.

**Example 4** Let  $n = 1$  and define a choice correspondence  $\varphi$  as follows. For all  $x, y \in \mathbb{R}$ ,

$$\varphi([x, y]) = \begin{cases} [x, y] \cap (0, 1) & \text{if } [x, y] \cap (0, 1) \neq \emptyset; \\ \{y\} & \text{if } y \leq 0; \\ \{x\} & \text{if } x \geq 1. \end{cases}$$

*This choice correspondence is convex-valued and satisfies independence of irrelevant alternatives and single-plateauedness. Closed-valuedness is violated.*

**Example 5** Let  $n = 1$  and define a choice correspondence  $\varphi$  as follows. For all  $x, y \in \mathbb{R}$ ,

$$\varphi([x, y]) = \begin{cases} \{y\} & \text{if } y \leq 0; \\ \{0, y\} & \text{if } x \leq 0 \text{ and } y \in (0, 1); \\ \{x, y\} & \text{if } x > 0 \text{ and } y \in (0, 1); \\ \{0, 1\} & \text{if } x \leq 0 \text{ and } y \geq 1; \\ \{x, 1\} & \text{if } x \in (0, 1) \text{ and } y \geq 1; \\ \{x\} & \text{if } x \geq 1. \end{cases}$$

*This choice correspondence is closed-valued and satisfies independence of irrelevant alternatives and single-plateauedness. Convex-valuedness is violated.*

## 6 Concluding remarks

Our focus in this paper is on the description and characterization of single-plateauedness in a general choice-theoretic setting. In the case of single-peaked choice, we also examined notions of rationalizability and representability; see Bossert and Peters (2009). Because the arguments involved are entirely parallel, we do not include a formal treatment of these issues here.

In addition to our characterization which is of interest in its own right by clarifying some links between single-plateauedness and a continuity property in the multi-valued framework (Theorem 2), our results on the consequences of the definition of single-plateauedness (Theorems 1,3) show that this is the natural way of formulating the notion of single-plateauedness. Thus, our results provide a foundation for the assumption of ‘single-plateaued’ preferences or utility functions in various applications.

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