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Trois Essais en Économie des Ressources Naturelles

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RÉSUMÉ

Cette thèse est composée de trois articles en économie des ressources naturelles non-renouvelables. Nous considérons tour à tour les questions suivantes : le prix in-situ des ressources naturelles non-renouvelables ; le taux d'extraction optimal et le prix des ressources non-renouvelables et durables.

Dans le premier article, nous estimons le prix in-situ des ressources naturelles non-renouvelables en utilisant les données sur le coût moyen d'extraction pour obtenir une approximation du coût marginal. En utilisant la Méthode des Moments Généralisés, une dynamique du prix de marché derivée des conditions d'optimalité du modèle d'Hotelling est estimée avec des données de panel de 14 ressources naturelles non-renouvelables. Nous trouvons des résultats qui tendent à soutenir le modèle. Premièrement, le modèle d'Hotelling exhibe un bon pouvoir explicatif du prix de marché observé. Deuxièmement, bien que le prix estimé présente un changement structurel dans le temps, ceci semble n'avoir aucun impact significatif sur le pouvoir explicatif du modèle. Troisièmement, on ne peut pas rejeter l'hypothèse que le coût marginal d'extraction puisse être approximé par les données sur le coût moyen. Quatrièmement, le prix in-situ estimé en prenant en compte les changements structurels décroît ou exhibe une forme en U inversé dans le temps et semble être corrélé positivement avec le prix de marché. Cinquièmement, pour neuf des quatorze ressources, la différence entre le prix in-situ estimé avec changements structurels et celui estimé en négligeant les changements structurels est un processus de moyenne nulle.

Dans le deuxième article, nous testons l'existence d'un équilibre dans lequel le taux d'extraction optimal des ressources non-renouvelables est linéaire par rapport au stock de ressource en terre. Tout d'abord, nous considérons un modèle d'Hotelling avec une fonction de demande variant dans le temps caractérisée par une élasticité prix constante et une fonction de coût d'extraction variant dans le temps caractérisée par des élasticités constantes par rapport au taux d'extraction et au stock de ressource. Ensuite, nous montrons qu'il existe un équilibre dans lequel le taux d'extraction optimal est proportionnel au stock de ressource si et seulement si le taux d'actualisation et les paramètres des fonc-

tions de demande et de coût d'extraction satisfont une relation bien précise. Enfin, nous utilisons les données de panel de quatorze ressources non-renouvelables pour vérifier empiriquement cette relation. Dans le cas où les paramètres du modèle sont supposés invariants dans le temps, nous trouvons qu'on ne peut rejeter la relation que pour six des quatorze ressources. Cependant, ce résultat change lorsque nous prenons en compte le changement structurel dans le temps des prix des ressources. En fait, dans ce cas nous trouvons que la relation est rejetée pour toutes les quatorze ressources.

Dans le troisième article, nous étudions l'évolution du prix d'une ressource naturelle non-renouvelable dans le cas où cette ressource est durable, c'est-à-dire qu'une fois extraite elle devient un actif productif détenu hors terre. On emprunte à la théorie de la détermination du prix des actifs pour ce faire. Le choix de portefeuille porte alors sur les actifs suivant : un stock de ressource non-renouvelable détenu en terre, qui ne procure aucun service productif ; un stock de ressource détenu hors terre, qui procure un flux de services productifs ; un stock d'un bien composite, qui peut être détenu soit sous forme de capital productif, soit sous forme d'une obligation dont le rendement est donné. Les productivités du secteur de production du bien composite et du secteur de l'extraction de la ressource évoluent de façon stochastique. On montre que la prédiction que l'on peut tirer quant au sentier de prix de la ressource diffère considérablement de celle qui découle de la règle d'Hotelling élémentaire et qu'aucune prédiction non ambiguë quant au comportement du sentier de prix ne peut être obtenue de façon analytique.

Mots clés: Ressources naturelles non renouvelables, Prix in-situ, Prix des ressources naturelles, Modèle à changement d'état, Analyse MMG, Données de Panel, Ressources durables, Modèle inter-temporel d'évaluation d'actifs, MEDAF-C, Taux d'extraction

ABSTRACT

This thesis consists of three articles on the economics of nonrenewable natural resources. We consider in turn the following questions : the in-situ price of nonrenewable natural resources, the optimal extraction rate and the price of nonrenewable and durable resources.

The purpose of the first article is to estimate the in-situ price of nonrenewable natural resources using average extraction cost data as proxy for marginal cost. Using the regime switching Generalized Method of Moments (GMM) estimation technique, a dynamic of the market price derived from the first-order conditions of a Hotelling model is estimated with panel data for fourteen nonrenewable resources. I find results that tend to support the model. First, it appears that the Hotelling model has a good explanatory power of the observed market prices. Second, although the fitted prices seem to be subject to structural breaks over time, this does not have a significant impact on the explanatory power of the model. Third, there is evidence that marginal extraction cost can be approximated by average extraction cost data. Fourth, when allowing for structural breaks, estimates of the in-situ price decrease or exhibit an inverted U-shape over time and appear to be positively correlated with the market price. Fifth, for nine of the fourteen minerals, the difference between the estimates of the in-situ price with and without allowing for structural breaks is a zero-mean process.

The second article tests whether an equilibrium under which the rates of extraction of nonrenewable resources are linear in the stock of the resource is consistent with observed data. I first show that with a time varying demand characterized by a constant price elasticity and a time varying extraction cost function characterized by constant elasticities with respect to the rate of extraction and to the remaining stock, there exists an equilibrium in which the extraction rate is proportional to the stock of resource if and only if the discount rate and the demand and cost parameters satisfy a very specific relationship. I then use panel data on fourteen nonrenewable natural resources to test whether this relationship is satisfied empirically. I find that if the parameters are assumed time invariant, then for six of the fourteen resources I cannot reject the hypothesis. This changes

however if I account for structural changes over time, in which case the hypothesis is rejected for all fourteen resources.

In the last article, we take a capital asset pricing approach to the determination of the price of a nonrenewable natural resource in the case where the resource is *durable*, in the sense that once extracted it becomes a productive asset held above ground. The portfolio choice is then made up of the following assets : a stock of nonrenewable resource held in the ground that yields no dividend, a stock of resources held above ground that yields a dividend in the form of a flow of productive services, and a stock of composite good that can be held either in the form of productive capital or of a bond whose return is given. There is a stochastic element to the rate of change of productivity in both the production of the composite good and in the extraction of the resource. It is shown that the resulting prediction for the price path of the resource differs considerably from the one that follows from the more basic Hotelling model and that no unambiguous prediction can be drawn analytically about the pattern of behavior of that price path.

Keywords : Nonrenewable natural resources, In-situ prices, Natural resource prices, Switching model, GMM analysis, Panel data, Durable resources, Intertemporal asset pricing model, C-CAPM, Taux d'extraction

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INTRODUCTION GÉNÉRALE

La dépendance de l'économie mondiale des ressources naturelles non-renouvelables est importante. Ces ressources naturelles, y compris les forêts, le pétrole, le charbon et l'or, fournissent des services vitaux de base. Les combustibles à base des ressources naturelles non-renouvelables sont encore les principales sources d'énergie générée dans le monde en raison de leur accessibilité et leur grand contenu énergétique. Malheureusement, leur utilisation n'est pas soutenable en raison du fait que ces ressources existent souvent en quantité limitée, ou sont consommées beaucoup plus rapidement que la nature ne peut assumer le renouvellement nécessaire. Dans la plupart des cas, une hausse non anticipée du prix des ressources naturelles non-renouvelables pourrait endommager certaines activités économiques et poser de nombreux préjudices. Les chocs énergiques ont été, le plus souvent, des signes précurseurs des récessions économiques (par exemple les chocs pétroliers des années 1970 et 2008). Chaque fois, les mêmes préoccupations majeures émergent : Comment améliorer la gestion des ressources naturelles ? Comment se forme le prix de ces ressources ?

Hotelling (1931) a proposé un cadre formel d'analyse de l'impact de la disponibilité limitée des ressources naturelles non-renouvelables sur l'évolution de leur prix et taux d'extraction. Cet auteur a montré que pour une entreprise minière cherchant à maximiser la valeur actualisée des flux de bénéfices nets provenant de l'extraction, sous la contrainte que le stock de la ressource s'épuise avec sa décision d'extraction actuelle, la valeur marginale du stock de ressource en terre, appellée également le *prix in-situ*, devrait croître au taux d'intérêt. Cet énoncé, qui est au cœur du développement théorique de l'économie des ressources naturelles non-renouvelables, est souvent désigné comme la *règle d'Hotelling* de l'exploitation des ressources naturelles non-renouvelables. Ce modèle de base a été l'objet de nombreuses analyses théoriques du comportement des marchés des ressources naturelles. Il existe cependant très peu de preuves empiriques soutenant le fait que le prix des ressources se comporte effectivement tel que prédit par

Hotelling (1931). [1] Cela ne devrait pas être trop surprenant, puisque ce modèle parcimonieux de base néglige un certain nombre de facteurs importants qui jouent également un rôle déterminant dans la formation du prix des ressources naturelles. [2]

Bien qu'il y ait une abondante contribution théorique au modèle d'Hotelling de l'exploitation des ressources naturelles non-renouvelables, les études empiriques sont plus rares. À date, les chercheurs n'ont pas encore apporté une preuve empirique selon laquelle le comportement du prix in-situ serait un déterminant significatif de l'évolution du prix des ressources naturelles, la plupart du temps parce que le prix in-situ n'est pas observable. [3] Notons que, pour des cas particuliers où les données de marché du prix in-situ ont été observées, le test de la règle d'Hotelling a conduit à des résultats plus ou moins acceptables (Miller and Upton, 1985b,a; Livernois et al., 2006). Cependant, il est difficile, voire impossible d'estimer le vrai prix in-situ.

Cette thèse dévelope une nouvelle approche pour l'évaluation du modèle d'Hotelling. Cette approche consiste à construire des équations de la demande et des coûts d'extraction et à les utiliser pour résoudre le modèle d'équilibre du marché des ressources naturelles résultant d'une maximisation inter-temporelle des profits de l'extraction. Elle est composée de trois articles, chacun constituant un chapitre différent de la thèse. Le premier article propose une estimation du prix in-situ des ressources naturelles non-renouvelables en utilisant les données sur le coût moyen d'extraction pour obtenir une approximation du coût marginal. Le deuxième article teste empiriquement l'existence d'un équilibre dans lequel le taux d'extraction optimal est linéaire par rapport au stock de ressource en terre. Le troisième article adopte une approche inter-temporelle d'évaluation des actifs financiers pour déterminer le prix d'une ressource non-renouvelable et durable.

Comme l'a souligné Livernois (2009), si la rente de rareté (prix du marché moins le coût marginal d'extraction) est seulement un des nombreux facteurs qui influencent

1. Voir Slade and Thille (2009) et Livernois (2009) pour une revue de littérature sur des analyses empiriques de la règle d'Hotelling.

2. Voir Krautkraemer (1998) et Gaudet (2007) pour une revue de littérature sur des extensions théoriques du modèle de base d'Hotelling.

3. Voir, parmi d'autres, Farrow (1985), Halvorsen and Smith (1991) et Young (1992).

le comportement du prix des ressources naturelles, toutes sortes de trajectoire de ce prix sont possibles. En d'autres termes, en plus d'être testée, une question importante serait de déterminer si oui ou non la règle d'Hotelling peut être utilisée pour expliquer empiriquement l'évolution du prix de marché des ressources. Plusieurs auteurs ont déjà abordé cette question avec des résultats insatisfaisants (Livernois, 2009). Toutefois, un point doit être souligné. L'approche communément utilisée par ces auteurs, consiste à faire des hypothèses sur le comportement du coût marginal d'extraction et d'utiliser leurs implications pour la règle d'Hotelling pour expliquer l'évolution du prix de marché des ressources naturelles comme une série chronologique (Slade, 1982; Berck and Roberts, 1995; Lee et al., 2006). Bien que l'étude des propriétés des séries chronologiques du prix de marché des ressources naturelles peut fournir des informations utiles sur la règle d'Hotelling, certaines informations sont perdues en n'incluant pas de l'information sur le prix in-situ et le coût marginal d'extraction dans les prévisions du prix de marché. Mais, ces informations ne sont généralement pas disponibles parce que ni le prix in-situ, ni le coût marginal d'extraction n'est observable. Par conséquent, si l'objectif est de prédire l'évolution du prix de marché des ressources non-renouvelables à l'aide de la règle d'Hotelling, il faut obtenir au préalable des approximations appropriées pour ces deux variables. Young (1992) a essayé d'incorporer des informations sur le coût marginal d'extraction en spécifiant et en testant une fonction de coût appropriée. Cependant, le prix estimé explique au plus un pour-cent du prix de marché observé.

Même si le coût marginal n'est pas observable, le coût moyen l'est souvent, comme le coût total divisé par la production. Le but du premier article est donc de construire un cadre pour évaluer le modèle d'Hotelling et de prédire le comportement du prix des ressources en utilisant le coût moyen d'extraction pour obtenir une approximation du coût marginal. La méthodologie est simple. D'abord, nous supposons une forme fonctionnelle pour le coût d'extraction avec progrès technologique et effet de stock, ce qui nous permet de spécifier comment le coût moyen d'extraction peut être utilisé comme une approximation du coût marginal. Deuxièmement, nous utilisons le comportement théorique du prix in-situ pour estimer le prix du marché qui est cohérent avec le modèle d'Hotelling. Enfin, nous dérivons une estimation du prix in-situ ex-post. En utilisant la

Méthode des Moments Généralisés (MMG) avec changement de régime, nous estimons une dynamique du prix de la ressource, derivée des conditions d'optimalité du modèle d'Hotelling, à partir des données de panel de quatorze ressources non renouvelables. [4] Nous trouvons que le modèle d'Hotelling a un bon pouvoir explicatif. Le prix in-situ estimé semble être corrélé positivement au prix de marché pour toutes les ressources étudiées.

Bien que les résultats obtenus dans le premier article fournissent des informations empiriques importantes sur le comportement du prix des ressources non-renouvelables, il reste que le modèle empirique considéré ne nous permet pas de caractériser le taux d'extraction optimal. Il existe très peu de littérature empirique sur le comportement du taux d'extraction optimal, bien que les études théoriques, argumentent qu'il devrait décroître dans le temps dû à l'épuisement du stock de ressource en terre. Certains auteurs comme Lin and Wagner (2007) ont, pour des fins empiriques, supposé que le taux d'extraction optimal est une fonction linéaire du stock de ressource en terre ou du stock cumulé de la ressource extraite. Un taux d'extraction linéaire par rapport au stock de ressource a l'avantage d'être simple et facile à manipuler analytiquement. Il reste, cependant, à fournir un test empirique de la valideté d'un tel équilibre. C'est le but principal du deuxième article de cette thèse. Nous montrons, en utilisant les données de panel de quatorze ressources non-renouvelables, qu'on ne peut rejeter l'hypothèse d'un taux d'extraction linéaire par rapport au stock de ressource que pour six des ressources. Cependant, en prenant en compte le changement structurel dans le temps du prix des ressources, nous trouvons que cette hypothèse est rejetée pour chacune des quatorze ressources.

De nombreuses implications du modèle de base d'Hotelling sont modifiées lorsque des caractéristiques plus complexes et plus réalistes de l'exploitation des ressources non-renouvelables sont prises en compte. [5] Parmi ces caractéristiques figurent l'incertitude quant aux perspectives futures et le fait que de nombreuses ressources naturelles non-

4. Ces ressources sont le bauxite, le cuivre, l'or, le charbon, le fer, le plomb, le gaz naturel ainsi que le nickel, le pétrole, le phosphate, le charbon brun, l'argent, l'étain et le zinc.

5. Voir Krautkraemer (1998) et Gaudet (2007) pour une importante littérature sur des extensions théoriques du modèle d'Hotelling de base

renouvelables sont *durables*, contrairement à ce qui est le plus souvent supposé dans la modélisation théorique. L'objectif du troisième article est d'explorer l'impact de la prise en compte simultanée de ces deux facteurs sur l'évolution du prix d'équilibre des ressources naturelles.

Il existe une vaste littérature sur la présence de l'incertitude, sous diverses formes, dans les marchés des ressources naturelles. Ce troisième article suit étroitement la modélisation stochastique des perspectives futures proposées dans Gaudet and Khadr (1991), qui a étudié le cas des ressources naturelles *non-durables*. Ce qui distingue Gaudet and Khadr (1991) des autres auteurs est qu'il prend une approche inter-temporelle d'évaluation des actifs. [6] Ainsi, la règle d'Hotelling s'interprète comme une condition d'équilibre du marché des actifs, l'actif étant bien sûr le stock de la ressource en terre. Cet actif, contrairement aux actifs reproductibles tel que le capital physique conventionnel, a la particularité qu'il ne peut être augmenté, avec pour résultat que les décisions de désinvestissement sont irréversibles. La question devient alors : quel est le taux de rendement approprié pour détenir une unité de la ressource en terre ? Dans le cadre déterministe de base proposé par Hotelling, la réponse est tout simplement le gain en capital qui peut être obtenu en maintenant l'unité de la ressource en terre. Lorsque les opportunités d'investissement sont stochastiques, il est montré dans Gaudet and Khadr (1991) que son rendement d'équilibre espéré dépendra également du degré d'aversion au risque et de la façon dont son rendement est corrélé avec la performance de l'économie en termes de consommation.

Les ressources non-renouvelables font le plus souvent penser aux combustibles fossiles, comme le pétrole, le gaz naturel ou le charbon. Bien que ces ressources naturelles soient stockables, elles ne sont pas durables, car elles sont consommées entièrement en une seule utilisation. Cependant, de nombreuses ressources non-renouvelables, telles que les métaux, sont durables : [7] une fois extraites et détenues en surface, elles deviennent

6. Plusieurs auteurs ont introduit l'incertitude dans le modèle d'Hotelling . Entre autres, nous pouvons citer Gilbert (1979), Glenn (1978), Dasgupta and Stiglitz (1981), Pindyck (1980), Deshmukh and Pliska (1980), Levhari and Pindyck (1981), Deshmukh and Pliska (1983), Lasserre (1984), and Gaudet and Howitt (1989).

7. Entre autres, les ressources suivantes sont durables : l'or, le diamant, le bauxite, le cuivre, le fer, le plomb, aussi bien que le nickel, l'argent, l'étain et le zinc.

des actifs capables de donner un flux continu de services utilisés comme intrants dans des processus de production différents. Nous sommes alors en présence de deux actifs en ressources : un stock détenu en terre qui ne donne aucun flux de services et un stock détenu en surface qui en donne. Ces actifs ont la particularité que celui détenu en surface ne peut être augmenté qu'en réduisant celui détenu en terre. Levhari and Pindyck (1981) est la référence la plus importante sur le comportement des marchés des ressources durables, un thème sur lequel il existe, étonnamment, très peu de littérature malgré le fait que beaucoup de ressources non-renouvelables soient en réalité durables. Il a considéré la fixation du prix de ces ressources dans un contexte d'équilibre partiel.

Dans ce troisième article, nous combinons une approche similaire à celle de Levhari and Pindyck (1981) pour la modélisation de la durabilité de la ressource avec le cadre d'évaluation des ressources naturelles non-renouvelables de Gaudet and Khadr (1991) à deux produits multi-actifs. Nos résultats montrent que l'on devrait faire preuve de beaucoup de prudence lorsqu'on élabore des prédictions analytiques sur le comportement du prix des ressources dans un tel contexte. Ils soulignent l'importance pour les études empiriques sur le prix des ressources de prendre en compte l'incertitude quant aux opportunités d'investissements futures, tout particulièrement dans le cas des ressources durables.

CHAPITRE 1

ESTIMATES OF THE IN-SITU PRICE OF NON-RENEWABLE NATURAL RESOURCES

Abstract

The purpose of this paper is to estimate the in-situ price of nonrenewable natural resources using average extraction cost data as proxy for marginal cost. Using the regime switching Generalized Method of Moments (GMM) estimation technique, a dynamic of the market price obtained from the first-order conditions of a Hotelling model is estimated with panel data for fourteen nonrenewable resources. I find results that tend to support the model. First, it appears that the Hotelling model has a good explanatory power of the observed market prices. Second, although the fitted prices seem to be time inconsistent, the time inconsistency does not have a significant impact on the explanatory power of the model. Third, there is evidence that marginal extraction cost can be approximated by average extraction cost data. Fourth, the time consistent estimates of the in-situ price decreases or exhibits an inverted U-shape over time and appear to be positively correlated with the market price. Fifth, for nine of the fourteen minerals, the difference between the time-consistent and the time-inconsistent estimates of the in-situ price is a zero-mean process.

1.1 Introduction

The purpose of this paper is to assess the Hotelling model, the core theory of nonrenewable resource economics. The Hotelling model predicts that in the absence of stock effects the in-situ price, which is equal to the market price less the marginal extraction cost, should grow at the rate of interest, (Hotelling, 1931). This statement known as the Hotelling rule has been tested with unsatisfactory results [1]. In addition to being tested, an

1. See, among others, Farrow (1985), Halvorsen and Smith (1991) and Young (1992)

important question is whether or not the Hotelling model can be used to explain empirically the evolution of non-renewable resource market prices. Many authors have already addressed this issue with weak results (Livernois, 2009). However, one point has to be underlined. The common approach used by those authors is to make assumptions on the behaviour of the marginal extraction cost and use their implications for the Hotelling rule to explain the evolution of the nonrenewable natural resource market price as time series (Slade, 1982; Berck and Roberts, 1995; Lee et al., 2006). Although the properties of the time series on the market price of the resource provide useful information about the implications of the Hotelling rule, some information is lost by not including information about the in-situ price and the marginal extraction cost in predictions of the market price. But this information is usually not available, because neither the in-situ price nor the marginal extraction cost are observable. Therefore, if the goal is to predict the evolution of the market price of a nonrenewable resource using Hotelling rule, it is useful to first obtain suitable proxies for those two important variables. Young (1992) did attempt to incorporate information about marginal extraction cost by specifying and testing a suitable cost function. However, the estimated price explains at most 1 percent of the observed market price.

Even though marginal cost is not observable, average cost often is, as the total cost divided by the production. The aim of this paper is to build a framework to evaluate the Hotelling model and predict the behaviour of nonrenewable resource prices using average extraction cost to obtain a proxy for marginal extraction cost. The methodology is straightforward : first, I assume a functional form for extraction cost which allows me to specify how average extraction can be used as a proxy for marginal extraction cost ; second, I use the theoretical behaviour of the in-situ price to estimate the market price which is consistent with the Hotelling model ; finally, I derive an estimated in-situ price ex-post. I find that the Hotelling model has a good explanatory power of the observed market price.

The remainder of the paper is organized as follows. In Section 1.2, I present the theoretical model of resource extraction. In particular, I assume an increasing average extraction cost and focus on its impacts on the equilibrium price path. In Section 1.3, I propose a methodology to estimate a reduce form of first-order conditions of the Hotelling model. This empirical methodology is then used in Section 1.4 to analyse fourteen natural resource market prices. I conclude in Section 1.5.

1.2 Theoretical model of the resource extraction

In this section, I present a theoretical model of the optimal nonrenewable resource extraction in a competitive market. [2]

1.2.1 The Basic Hotelling model of resource extraction

Consider a competitive firm extracting a known and finite stock of a non-renewable resource. The firm chooses a time path of resource extraction to maximize the present value of the stream of net benefits subject to the constraint that the cumulative extraction be not greater than the initial resource endowment. The extraction cost, the principal characteristic of the firm, is subject to technological progress, which causes the cost to decrease, and to a stock effect, which causes the cost to increase.

The notation follows closely that used by Lin and Wagner (2007) and Krautkraemer (1998). At time $t \in [0, +\infty]$, the supply of the mineral is given by $q(t)$, the extraction flow at time t. The cost of extracting $q(t)$ at time t is denoted by $C(z(t), q(t), S(t))$, where $S(t)$ is the remaining stock of the resource at time t and $z(t)$ is a productivity index indicating the state of extraction technology. I assume that $C_z < 0$, $C_s < 0$, $C_{ss} > 0$ $C_q > 0$ and $C_{qq} > 0$, where C_q denotes the partial derivative $\partial C / \partial q$, etc. To simplify, I assume that the average extraction cost is an increasing function of the extraction rate, that is :

2. Assuming competitive market is a simplification of the market structure of nonrenewable natural resources. In principle, there may be a market power. Lin and Zhang (2011) found that countries supplying hard coal, lead, and oil behaved as oligopolists during the period of study, while the market for other nonrenewable resources could be characterized as perfectly competitive.

$$C_q(z,q,S) \geq \frac{C(z,q,S)}{q}, \qquad \forall z,q,S. \tag{1.1}$$

While making its extraction decision, the firm takes as given the market price $p(t)$ of the mineral. The optimal control problem it faces is to choose a time path of resource extraction $q(t)$ to maximize :

$$\int_0^{+\infty} e^{-\delta t}[p(t)q(t) - C(z(t),q(t),S(t))]dt, \tag{1.2}$$

subject to

$$\dot{S}(t) = -q(t), \qquad \forall t \tag{1.3}$$

$$q(t) \geq 0, \quad S(t) \geq 0 \quad \forall t \tag{1.4}$$

$$S(0) = S_0 \tag{1.5}$$

and where δ denotes the rate of discount.

Let $\lambda(t)$ denote the co-state variable associated to the resource stock. The current value Hamiltonian of the problem is then :

$$H(t,q(t),S(t),\lambda(t)) = p(t)q(t) - C(z(t),q(t),S(t)) - \lambda(t)q(t), \tag{1.6}$$

The static efficiency condition is :

$$\frac{\partial H}{\partial q} = p - C_q(z,q,S) - \lambda = 0, \tag{1.7}$$

where the time argument is implicit. Let $\dot{\lambda}$ denote the time derivative of λ. The dynamic efficiency condition is then given by :

$$\dot{\lambda} = \delta\lambda - \frac{\partial H}{\partial S} = \delta\lambda + C_s(z,q,S). \tag{1.8}$$

First-order conditions (1.7) and (1.8) of the Hotelling model state that, the in-situ price λ corrected by the stock effect C_S should grow at a constant rate. With this form, the behaviour of the resource price cannot be estimated because the marginal cost C_q and the stock effect C_s are not observable. However, the implications for the resource price path can be derived under specific assumptions about the cost function. When marginal extraction cost, C_q, is a constant or a decreasing function of time due to technology progress, an increasing function of extraction rate and a decreasing function of resource stock, one can find a U-shape price path,(Slade, 1982). This behaviour of the market price has been testing unsuccessful, particularly because time series data for prices exhibit stochastic trends (Berck and Roberts, 1995). In the next section, I propose an approach to obtain a useful dynamic of the market price from the optimal conditions (1.7) and (1.8) which uses the average extraction cost as a proxy for marginal extraction cost.

1.2.2 Cost specification and the dynamic of the market price

To make the optimal conditions (1.7) and (1.8) of the Hotelling model empirically tractable, I will assume the following extraction cost function, which is close to the one used in Lin and Wagner (2007).

$$C(z,q,S) = c_0 e^{-\gamma t} q^{\alpha} S^{-b} \tag{1.9}$$

with a nonnegative productivity growth rate $\gamma \geq 0$, stock elasticity $b \geq 0$ and constant $c_0 \geq 0$. Technological progress causes cost to decrease and other things being equal a one percent increase in technology causes costs to decrease by 1 percent. Costs rise as more of the resource has been extracted and other things being equal a one percent decrease in resource stock causes costs to increase by b percent.

With this functional form, marginal cost can be expressed as a function of the average extraction cost.

$$C_q(z,q,S) = \alpha \frac{C(z,q,S)}{q} \tag{1.10}$$

The parameter α appears as an adjustment coefficient between the average and marginal extraction costs. The elasticity of marginal extraction cost is given by :

$$\varepsilon(z,q,S) = \frac{qC_{qq}(z,q,S)}{C_q(z,q,S)} = \alpha - 1, \tag{1.11}$$

which must be nonnegative to guarantee that the supply curve will be non decreasing. It will therefore be assumed that $\alpha \geq 1$. If $\alpha = 1$, the average extraction cost is equal to the marginal extraction cost. If $\alpha > 1$, both marginal and average extraction costs are increasing. The stock effect C_s is not observable, but, with the cost function (1.9), it can be expressed as a function of observable variables. Denoting by $X(t)$ the cumulative resource extracted at time t, the stock effect can be written as follows :

$$C_s(z,q,S) = -b \frac{q}{S_0 - X} \frac{C(z,q,S)}{q}. \tag{1.12}$$

The function $f(q,X) = -b\frac{q}{S_0 - X}$ is an adjustment factor between the stock effect and the average extraction cost. Note that, in the absence of the stock effect the parameter b is zero and the adjustment factor $f(q,X)$ is zero as well. Substituting from equations (1.10) and (1.12) into the efficiency conditions (1.7) and (1.8), we get :

$$\lambda(t) = p(t) - \alpha AC(t) \tag{1.13}$$

$$\dot{\lambda}(t) = \delta\lambda(t) - b\frac{q(t)}{S_0 - X(t)} AC(t). \tag{1.14}$$

As the observed data is in discontinuous time, it will be necessary to discretize the above equations. The discrete form associated to equation (1.14) is given by :

$$\lambda_{t+1} - \lambda_t = \delta\lambda_t - b\frac{q_{t+1}}{S_0 - X_{t+1}} AC_{t+1}. \tag{1.15}$$

Substituting from equation (1.13) into equation (1.15) and rearranging, I obtain the following expression of the market price :

$$p_t = (1+\delta)p_{t-1} + \alpha AC_t - \alpha(1+\delta)AC_{t-1} - b\frac{q_t}{S_0 - X_t}AC_t. \tag{1.16}$$

In this form, the optimal conditions of the Hotelling model can be estimated to predict the behaviour of the market price, using average cost data as a proxy for marginal cost. Note that, an estimation of the above equation provides the marginal extraction cost elasticity α which can be used, ex post, to estimate the in-situ price given by the static efficiency condition (1.13).

1.3 Empirical analysis of the Hotelling model

In this section, I develop an empirical model to estimate and evaluate the dynamic of the market price derived from the Hotelling model's optimal conditions obtained in the previous section.

1.3.1 The data and the empirical model

I use the same data as in Lin and Wagner (2007).[3] This database is a compilation of the data on several countries producing a nonrenewable natural resource. It contains average annual world price, country's average cost and country's current stock data for 14 ores from previously unpublished World Bank data. The commodities are bauxite, copper, gold, hard coal, iron, lead, natural gas as well as nickel, oil, phosphate, brown coal, silver, tin. and zinc. The data cover 35 years from 1970 to 2004. The summary statistics on the ores which are analysed in this empirical section are presented in the Appendix. The market price is common to all countries, while the extraction costs are particular to each country. The best way to account for the unspecified characteristics of

3. The database was downloaded directly from http ://gwagner.com/research/hotelling/, provided in their paper by Lin and Wagner. It is described in detail in their Appendix B.

each country on which those cost depend will be to introduce country effects, which are captured by a constant term.

Assume that countries have the same evaluation of future cash flows and also the same extraction technology. In other words, parameters δ, α, b and γ are free of country effects. If future market prices are well anticipated by each country, it follows that countries choose different extraction path only because they have different resource stocks. Therefore, the initial stock S_{i0} is an appropriate parameter to describe the country's characteristics. I will use this initial stock to capture the country effects.

An estimate of the initial stocks, S_{i0}, should satisfy the following condition :

$$S_{i0} \geq \sum_{t=0}^{T_i} q_{it} \tag{1.17}$$

where T_i is the sample size. To guarantee that this condition is satisfied, I can decompose the initial stock in two components :

$$S_{i0} = \beta_i + \sum_{t=0}^{T_i} q_{it}, \tag{1.18}$$

where $\beta_i \geq 0$ is a parameter which captures the last period stock. Now define the following residual :

$$R_{it} = \sum_{\tau=0}^{T_i} q_{i\tau} - X_{it} = \sum_{\tau=t+1}^{T_i} q_{i\tau}. \tag{1.19}$$

Substituting the above expression in (1.16), the empirical model to be estimated becomes :

$$p_t = (1+\delta)p_{t-1} + \alpha AC_{it} - \alpha(1+\delta)AC_{it-1} - b\frac{q_{it}}{\beta_i + R_{it}}AC_{it} + \varepsilon_{it}, \tag{1.20}$$

subject to the constraints

$$b \geq 0; \qquad \alpha \geq 1; \qquad \beta_i \geq 0; \qquad \delta \geq 0, \tag{1.21}$$

where ε_{it} is an error term which can be correlated within countries.

Since the discount rate δ is positive the empirical model exhibits a non stationary trend for the market price. This trend can lead to spurious regressions. However, a close look at the empirical model to be estimated shows that the market price trend is offset in the right-hand side of the model. To see this clearly, rewrite the empirical model as follows :

$$p_t - p_{t-1} = \delta(p_{t-1} - \alpha AC_{it-1}) + \alpha(AC_{it} - AC_{it-1}) - b\frac{q_{it}}{\beta_i + R_{it}}AC_{it} + \varepsilon_{it}. \tag{1.22}$$

In this form, the theoretical trend of the market price disappears and the error term seems to be the sum of stationary components.

This alternative empirical formulation emphasizes the fact that the model can serve to explain the variation in the market flow price by the lagged in-situ price (the first term on the right-hand side), the variation in the average extraction cost (the second term), and the marginal stock effect on costs (see equation (1.12)). The discount rate δ is seen to capture the speed of adjustment of the market flow price to the value of the lagged in-situ price given by $p_{t-1} - \alpha AC_{it-1}$, the difference between price and marginal extraction cost. The parameter δ being restricted to nonnegative values, the in-situ price has a nonnegative effect on the market flow price p_t, and, everything else the same, the adjustment will be faster the larger if δ.

The main goal is to get an estimate of the in-situ price, which is not observable. By estimating equation (1.22) we obtain estimates of the parameters α and β_i, which can then be used to derive an estimate of the in-situ price λ_{it}. The in-situ price is in fact the price of the asset held in the ground and we therefore want its estimate to take "almost surely" nonnegative values. Since $\lambda_{it}(\alpha) = p_t - \alpha AC_{it}$, this means that we must impose an upper bound $\bar{\alpha}$ on the parameter α for estimation purposes. For all values of the market price p_t and the average extraction cost AC_{it}, it exists a real α_{it} just that, for all $\alpha > \alpha_{it}$, we have $\lambda_{it}(\alpha) \leq 0$. Let $\bar{\alpha} = Inf_{it}(\alpha_{it})$. Then if $\alpha > \bar{\alpha}$, we have $prob(\lambda_{it}(\alpha) < 0 | \alpha \geq \bar{\alpha}) > 0$. The question remains as to how to determine the appropriate value of $\bar{\alpha}$.

To characterize this upper bound on α, I will assume that :

$$prob(\lambda_{it}(\alpha) < 0 | \alpha = \bar{\alpha}) \leq prob(\lambda_{it}(\alpha) \geq 0 | \alpha = \bar{\alpha}). \tag{1.23}$$

This says that when α is equal to its upper bound, the probability of obtaining a nonnegative in-situ price is greater than the probability of obtaining a negative in-situ price. Rewriting the above condition in terms of the stochastic variable p_t/AC_{it}, we get :

$$0 < prob\left(\frac{p_t}{AC_{it}} < \bar{\alpha}\right) \leq prob\left(\frac{p_t}{AC_{it}} \geq \bar{\alpha}\right). \tag{1.24}$$

Information about the distribution of p_t/AC_{it} will be helpful to fix an approriate upper bound on α. The boundary condition (1.24) suggests that, in the probabilistic sense [4], almost surely :

$$Inf\left(\frac{p_t}{AC_{it}}\right) < \bar{\alpha} \leq Median\left(\frac{p_t}{AC_{it}}\right). \tag{1.25}$$

Table 1.I summarizes some characteristics of the data on p_t/AC_{it}. It appears that the mean is greater than the median for all ores. Therefore the stochastic variable p_t/AC_{it} is asymmetric. As the median and the mean are closer to the minimum than the maximum, the distribution of p_t/AC_{it} has a long right tail.

Notice that, from condition (1.25) and the properties of the distribution of p_t/AC_{it}, the lower bound $\alpha = 1$ should result in a high probability of positive in-situ prices. This is an interesting property, since $\alpha = 1$ corresponds to the special case where marginal extraction cost is equal to average extraction cost, and hence is independent of the extraction rate. In such a case we have constant returns to scale and the data on average

4. We consider the probability that generates the data

Tableau 1.I – Summary characteristics of the stochastic variable p_t/AC_{it}

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN	TIN
Min	1.00	1.00	0.99	1.00	1.00	0.99	1.00	1.00	1.02	1.00	1.00	1.00	0.99	1.00
Max	86.10	5606.66	3.47	4.65	5.23	6.85	7.24	20.60	11.92	4.57	3.78	3.42	3.21	1.00
Mean	8.05	93.62	1.37	1.66	1.67	1.51	2.39	2.50	2.97	1.58	1.48	1.38	1.31	1.00
Median	4.26	2.24	1.20	1.44	1.39	1.25	1.87	1.88	2.38	1.41	1.34	1.39	1.25	1.00
Std	11.36	578.96	0.36	0.65	0.67	0.64	1.24	2.28	1.91	0.51	0.45	0.18	0.25	0.00

OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; BAU, bauxite ; COP' copper ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc.

extraction cost can be safely used to represent marginal extraction cost.

1.3.2 Market value of a unit of the resource in ground

Notice that the above empirical model generates a different estimate of λ_{it} for each country, which follows the stochastic equation :

$$\lambda_{it} = (1+\delta)\lambda_{it-1} - b\frac{q_{it}AC_{it}}{S_{it}} + \varepsilon_{it}, \qquad with \qquad E(\varepsilon_{it}/S_{it}, p_t, p_{t+1}, ...) = 0, \quad (1.26)$$

where ε_{it} is the same error term considered in the equation (1.20). But there is a single market and hence a single in-situ price λ_t. The difference between the idiosyncratic λ_{it} and the unique market λ_t can be assumed to satisfy the following stochastic equation :

$$\lambda_t - \lambda_{it} = u_{it}, \qquad \text{with} \qquad E(u_{it}/S_{it}, p_t, p_{t+1}, ...) = 0 \qquad \forall i, t, \qquad (1.27)$$

where u_{it} is the error term which may be correlated among countries. This states that the difference between λ_{it} and λ_t is due to the fact that country i has information only about its own stock of the resource in the ground, S_{it}. As shown in the Appendix, from the stochastic equations (1.26) and (1.27) it follows that the real value of a unit of resource in the ground and its dynamics can be expressed as :

$$\lambda_t = p_t - \alpha \frac{\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} S_{it} AC_{it}}{\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} S_{it}} \tag{1.28}$$

$$\lambda_t = (1+\delta)\lambda_{t-1} - b \frac{\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} q_{it} AC_{it}}{\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} S_{it}}. \tag{1.29}$$

An interpretation is that when there are many producers in a mineral market, the value of a unit of the resource in ground is an aggregation of the information about this value over all countries. It follows that in a resource market with m competitive producers, the estimate of the market in-situ price is given by :

$$\lambda_t = p_t - \alpha \sum_{i=1}^{m} \frac{S_{it}}{\sum_{k=1}^{m} S_{kt}} AC_{it}. \tag{1.30}$$

1.3.3 Regime Switching Nonlinear GMM estimation

The empirical model developed in this paper is nonlinear in b, α, δ and β_i. The method of estimation depends on the nature of the explanatory variables and the further assumptions made on the behaviour of the error terms. In the simplest case, if the explanatory variables are exogenous and the error term follows a normal distribution, an appropriate estimation method is the Maximum Likelihood. The Maximum Likelihood Estimator has important properties of efficiency, but its well known limitation is the normality assumption of the error terms. The more general and frequently used approach is to assume that the behaviour of the error terms is unknown. In that situation, if the explanatory variables are strictly exogenous, a convenient estimation method is the Nonlinear Least Squares (NLS) method. But the average extraction cost is a function of the extraction rate, which is an endogenous variable. Furthermore, the residual stock R_{it} is a function of the future resource extraction rates. Therefore it will be appropriate to treat the average extraction cost and the residual stocks as endogenous variables. This implies that the NLS method may no longer be an appropriate estimation method. It is important to take into account this endogeneity property to obtain a consistent estimator of the parameters. The Generalized Method of Moments (GMM), which includes the NLS as a special case, provides a solution (see Matyas (1999)). Note that, the GMM estimator is a M.estimator, which is asymptotically normal under no restriction on the distribution of the error terms. I will now discuss the way to use this estimation method to fit the dynamic of the market price (1.22) derived from the Hotelling model's optimal conditions.

1.3.3.1 The GMM estimator

Denote the parameters to be estimated by $\theta = (b, \alpha, \delta, \beta_1, ..., \beta_n) \in \Theta$ and the observable variables by $y_{it} = (p_t, AC_{it}, AC_{it-1}, q_{it}, R_{it})$, and let $\theta_0 \in \Theta$ be the true parameter value. Assume

$$f(y_{it};\theta_0) = \varepsilon_{it}, \qquad E(\varepsilon_{it}/S_{it}, p_t, p_{t+1}, ...) = 0, \qquad \theta_0 \in \Theta, \tag{1.31}$$

where $f(y_{it};\theta) = p_t - (1+\delta)p_{t-1} - \alpha AC_{it} + \alpha(1+\delta)AC_{it-1} + b\frac{q_{it}}{\beta_i + R_{it}}AC_{it}$ is the elementary function, or the residual. Grouping all these residuals in a $T \times 1$ vector $f(y;\theta)$, I assume :

$$E\left[f(y;\theta_0)' f(y;\theta_0)\right] = \Omega, \qquad \theta_0 \in \Theta, \tag{1.32}$$

where Ω is a positive definite matrix, assumed to be unknown and $T = \sum_{i=1}^{T_i}$.

Now denote by w_{it} the instrumental variable for country i. Assuming that variables AC_{it-1}, q_{it-1} and X_{it-1} are predetermined[5] for all countries, the instrumental variable w_{it} is constituted with variables $(AC_{it-1}, q_{it-1}, X_{it-1})$ and $(AC_{jt-1}, q_{jt-1}, X_{jt-1})$ where country j produces the mineral during the same period as country i. This technique to construct instrumental variables allows me to obtain at least as many instruments as there

5. This assumption is carefully tested in the appendix using the J-test of the over identification. Our results (table I.I in appendix) shows that for almost all resource markets, there is no evidence against our choice of instrumental variables.

are parameters. Let W be a $T \times k$ matrix of instruments, assumed to be predetermined, where k is the number of instruments. The theoretical moment condition is given by :

$$E(W_{it}' f(y_{it};\theta_0)) = 0 \qquad \theta_0 \in \Theta, \tag{1.33}$$

where W_{it} is the it^{th} row of W.

Let $g_T(y;\theta) = \frac{1}{T}\sum_{t=0}^{T} W_{it}' f(y_{it};\theta)$ and $G(y;\theta) = \partial g(y;\theta)/\partial\theta$. Then the GMM estimator $\hat{\theta}$ is the value of θ which minimizes the following criterion function :

$$Q_T(\theta) = g_T(y,\theta)'\Sigma^{-1} g_T(y,\theta) \tag{1.34}$$

where Σ is the optimal covariance matrix of the moment variable $W_{it}' f(y_{it},\theta)$, $\Sigma = W'\Omega W$. It can be shown that an estimate of the optimal covariance matrix of $\hat{\theta}$ is given by :

$$\widehat{Var}(\hat{\theta}) = T^{-1}\left[G(y;\hat{\theta})'\hat{\Sigma}^{-1} G(y;\hat{\theta})\right]^{-1}, \tag{1.35}$$

where $\hat{\Sigma}$ is a HAC estimator of the covariance matrix $\Sigma = W'\Omega W$.

Since the model has many parameters (3 + the number of producing countries of each the resources), the number of instruments used to compute the GMM estimator is very large. With many instruments, the estimate of the covariance matrix with the usual procedure, the Newey-West HAC estimator of the covariance matrix $\hat{\Sigma}$, is generally not well conditioned. To obtain a well conditioned HAC estimator, I regularize the Newey-West HAC estimator with the regularization procedure of Ledoit and Wolf (2004) [6].(see the Appendix for more details).

6. These authors introduce an estimator of the covariance matrix that is both well-conditioned and more accurate than the sample covariance matrix asymptotically. Their estimator is distribution-free and has a simple explicit formula that is easy to compute and interpret.

1.3.3.2 Computation of the GMM estimator $\hat{\theta}$

To compute the GMM estimator $\hat{\theta}$ described above, I use Newton's method for constrained nonlinear minimization (see the Appendix for a description of Newton's method). Before running Newton's method to minimize the objective function $Q_T(\theta)$, I should replace the covariance matrix Σ by the Ledoit and Wolf (2004) HAC estimator obtained from an initial estimation of the model by the NLS method. Like other procedures that start from preliminary estimates, this one is iterated. Indeed, the GMM estimator residuals are used to calculate a new estimate of the covariance matrix Σ, which is then used to obtain a second GMM estimator, which is then used to obtain another GMM estimator, until the procedure converges relative to a given criterion. This iterative procedure is called the iterated GMM and was investigated by Hansen et al. (1996).

1.3.3.3 Analysis of the effect of jumps in prices

As will be discussed in the next section, the estimate of the in-situ price presents many conjunctural breaks or jumps. One important theoretical assumption about the behaviour of the resource price is continuity over time. The presence of jumps can be explained by the existence of some exogenous shocks which are not captured in the basic model. Denote by $\Delta\lambda_t = \lambda_t - \lambda_{t-1}$ the difference between two consecutive values of the in-situ price. The continuity of the in-situ price λ_t can be interpreted as the regularity of the distribution of $\Delta\lambda_t$. Therefore, any outlier of $\Delta\lambda_t$ can be interpreted as a jump for the in-situ price λ_t. Let assume that $\Delta\lambda_t$ is normally distributed.

$$\Delta\lambda_t \sim \mathcal{N}\left(\mu_\Delta, \sigma_\Delta\right) \tag{1.36}$$

It follows that an observation $\Delta\lambda_t$ is an outlier of level x if it satisfies the inequality

$$\frac{\Delta\lambda_t - \mu_\Delta}{\sigma_\Delta} > Z_x \qquad or \qquad \frac{\Delta\lambda_t - \mu_\Delta}{\sigma_\Delta} < -Z_x \tag{1.37}$$

Where $1-\Phi(Z_x)=x/2$, $\Phi(.)$ is the cumulative distribution of $\mathcal{N}(0,1)$. From the inequality (1.37), a value λ_t of the in-situ price is a jump if it satisfies the following inequality

$$\lambda_t > \lambda_{t-1} + \mu_\Delta + Z_x\sigma_\Delta \qquad or \qquad \lambda_t < \lambda_{t-1} + \mu_\Delta - Z_x\sigma_\Delta \tag{1.38}$$

To take the conjunctural breaks or jumps into account, I create a dummy variable which takes a value of 1 at a jumping date obtained exogenously from the inequality (1.38). Denote by J_{it} this dummy variable. Replacing the elementary function $f(y_{it};\theta)$ of the theoretical moment condition (1.33) with the new elementary function $f(y_{it};\theta)-\eta J_{it}$, one obtains a moment condition with jump effects. In the next section, such a model with jump effects will also be estimated and the impact of these jumps will be verified with the null hypothesis $H_0:\eta=0$. If this null hypothesis is rejected, then the appropriate specification of the model which must be used to estimate the in-situ price will be the model with jump effects.

1.3.3.4 Goodness of fit and analysis of the time dimension in parameters

An important goal of this paper is to determine the consistency of the Hotelling model with the observed data. For this purpose I use two main approaches. First, I check the explanatory power of the model by plotting the observed market prices versus the fitted market prices. If the model performs well, the curve obtained will be close to the 45 degree line. Second, I check the structural stability or the time consistency of the model.

As will be discussed in the next section, the estimate of the in-situ prices shows two regimes. In the first regime, the in-situ price increases and in the second regime, it decreases. My purpose will be to verify whether or not the two regimes are consistent with the model. To test the structural stability with the GMM estimation method, one should make a distinction between the identifying and the over identifying stability restrictions.

While the first restriction concerns the variation of the parameters between the two regimes, the second restriction focuses on the predictive stability. In this paper, I will deal particularly with the parameter stability. The idea is to compare two estimators which are assumed to be consistent under different regimes.

The stability investigation needs a special treatment because countries enter and exit the market at different dates. Indeed, it is not easy to control the parameter β capturing country fixed effects. Notice that to each country i corresponds a particular parameter β_i. As the dates at which countries enter or exit the market are independent, there may exist a sub-sample where some countries will not be represented. This situation implies that the vector of parameters β may be different from one sub-sample to another. Therefore, to test the parameter stability of the model, one should restrict the analysis to countries active in the two regimes. To overcome this situation, one could drop from the sample countries which are not represented in two sub-samples. However, I prefer to solve this problem differently. I restrict the analysis of the stability to the fixed part (b, α, δ) of the vector of parameters θ.

Let A be a $k \times k$ matrix of rank 3, which satisfies $(b, \alpha, \delta) = A\theta$, with $k = dim(\theta)$. Denote by L_i, $i \in \{1,2\}$, the sub-samples designed to test the stability of the Hotelling model. L_i is an interval of $\mathbb{N}$ and $L_1 \cap L_2 = \varnothing$. Let $\phi_0 = (\theta_1, \theta_2) \in \Phi$ be the true parameter value. The theoretical moment condition (1.33) becomes :

$$E(W_{it}' f(y_{it}; \phi_0)) = E\left[d_t(L_1) W_{it}' f(y_{it}; \theta_1) + (1 - d_t(L_1)) W_{it}' f(y_{it}; \theta 2)\right] = 0 \qquad \phi_0 \in \Phi, \tag{1.39}$$

where W_{it} is the it^{th} row of W and $d_t(L_1)$ is a dummy variable which equals one when $t \in L_1$. I will call the above moment condition, the " Switching Hotelling model" to distinguish it from the "Basic Hotelling model" characterized by the moment condition (1.33). Note that the moment condition (1.39) generalizes the moment condition (1.33) by incorporating the time dimension in the parameters of the model. This new specification will be useful to analyse the effect of time on the behaviour of the in-situ price.

As in Andrews and Fair (1988), I derive the Wald statistic :

$$W_{stat} = T(A_1\hat{\theta}_1 - A_2\hat{\theta}_2)' \left[\frac{1}{\pi}A_1\widehat{Var}(\hat{\theta}_1)A_1' + \frac{1}{1-\pi}A_2\widehat{Var}(\hat{\theta}_2)A_2'\right]^{-1} (A_1\hat{\theta}_1 - A_2\hat{\theta}_2), \tag{1.40}$$

where $\pi T = \#L_1$, $\hat{\theta}_i$ is the GMM estimator based on the sub-sample L_i, and $\widehat{Var}(\hat{\theta}_i)$ is a Ledoit-Wolf HAC estimator of Σ based on the L_i. This statistic has a limiting $\chi^2(3)$ distribution under the identifying restrictions of the two sub-samples and will be used to test the time consistency of the Hotelling model.

The structural stability of the model developed above must be interpreted with caution. Indeed, if there is no evidence to reject the null assumption of the structural stability, this does not mean that the model is time consistent. It simply means that the vector of parameters $(b,\alpha,\delta) = A\theta$ is time consistent, but not necessarily the vector θ. The second bloc β of θ can change over the two sub-periods. It is useful to note that even though a country can extract a resource in the two sub-periods, the parameters β_{1i} and β_{2i} capturing the last period resource stock are conceptually different. Indeed, by definition, $\beta_{1i} = S_{i[\pi T_i]}$ and $\beta_{2i} = S_{iT_i}$, and in the case of time consistency with respect to the last period resource stock, the following relation must be satisfied :

$$\beta_{1i} = \sum_{s=[\pi T_i]}^{T_i} q_{is} + \beta_{2i}. \tag{1.41}$$

Therefore, to verify the time consistency of the model with respect to the last period resource stock β_i, one needs an additional test. However, if the null hypothesis of structural stability is rejected then the model is really time inconsistent.

1.4 Application and results

This section presents results obtained by implementing the above methodology to analyse data for fourteen nonrenewable natural resources : bauxite, copper, gold, hard coal, iron, lead, natural gas as well as nickel, oil, phosphate, brown coal, silver, tin. and zinc. In the first subsection, I present the results of what I call the Basic Hotelling model, where parameters are assumed to be constant over time. These first results are used as a benchmark to study the structural stability of the model. The following results are based on the iterated GMM with the convergence criteria fixed on the objective function.

1.4.1 The Basic Hotelling model

To obtain the benchmark model I assume that parameters are constant over time and that the theoretical moment condition is given by equation (1.33). Two models are estimated. The second model is an extension of the first model in which I take into account jump effects in the behaviour of the in-situ price. As I have already mentioned in the previous section, I create a dummy variable to capture the jump effects. This dummy variable is associated to an additional parameter η introduced in the Basic Hotelling model and used to check the conjunctural stability.

I focus particularly on the explanatory power of the model. Since the model is estimated with a method other than the Nonlinear Least Square (NLS), the usual R^2 which measures the goodness of fit is no longer valid. Generally the R^2 is less than 0 or greater than 1, depending on the use of the residuals or the fitted data for calculation. However, in the present work, even with the GMM method used, most values are between 0.88 and 0.99. One can conclude that the model is probably doing a good job of approximating the observed data.

To see this clearly, I plot the observed data versus the fitted data. The results are presented in Figure 1.1. When the dots are around the 45 degree line, this suggests a good prediction of the model. From this figure, one can conclude that the Hotelling model has

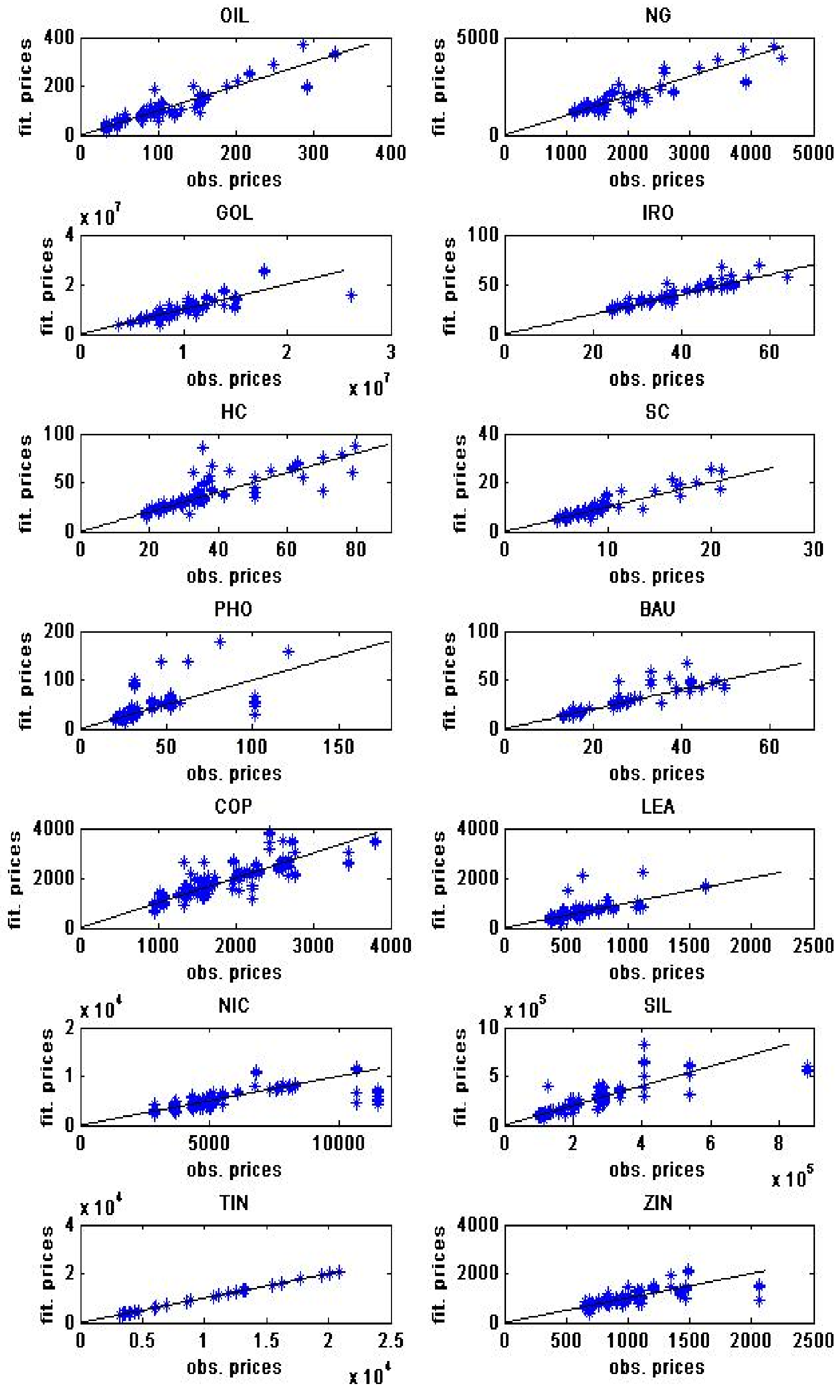

Figure 1.1 – Measure of the goodness of fit

a good explanatory power regarding natural resource market prices. The purpose of this paper being to estimate the in-situ prices, which are not observable, the result that the fitted prices are close to the observed prices suggests that the outcome for the in-situ prices will be close to their true values.

Table 1.II presents results based on an implementation of the GMM to estimate the model with and without jump effects. The empirical model has many parameters. For brevity, I present results only for the key ones. The parameter β, which is not reported in the table, is summarized further on a box plot, in Figure 1.2.

Note that the results of the model with jumps and without jumps are only slightly different. This suggests that the exogenous shocks which affect the natural resource market do not have a significant impact on the behaviour of the resource prices. To formally justify this point, I compute the test of the null hypothesis that the parameter η associated to the jumps is equal to 0. The results of this test are reported in the Table 1.II. Except for gold, hard coal and iron, I find that there is no evidence against $\eta = 0$. Therefore, the Basic Hotelling model captures resource price fluctuations. This can be explained by the fact that the Basic Hotelling model is a particular form of an error correction model (see equation (1.22)). That is, the empirical model is designed to explain the fluctuations in the market price instead of the market price itself. Given this result, we can use the Basic Hotelling model to estimate the in-situ price without too much concern about the conjunctural breaks or discontinuities. However, to be as general as possible, my preference goes to the model with jump effects, because it captures the conjunctural breaks found in the case of gold and hard coal.

The following analysis is therefore based on the model with jump effects. It appears that the stock elasticity b is different from zero except for hard coal and nickel. Thus, the market prices of hard coal and nickel are not significantly affected by the variation of the resource stock.

The marginal extraction cost parameter α is close to one for all ores. Parameter α gives the extent to which marginal extraction cost differs from average extraction cost.

Tableau 1.II – Estimated results

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN	TIN
Basic model : jump effects not included														
b	0.06	0.00	1.12	0.00	0.18	0.27	0.58	1.00	0.29	0.43	0.00	33.1	0.05	0.01
α	2.63	1.00	1.00	1.44	1.00	1.11	1.52	1.00	1.05	1.10	1.00	1.00	1.05	1.00
δ	0.00	0.02	0.00	0.04	0.00	0.04	0.04	0.00	0.00	0.08	0.00	0.00	0.02	0.00
Basic model : jump effects included														
b	0.30	0.18	1.24	0.00	0.20	0.25	0.44	0.36	0.15	0.32	0.00	0.14	0.24	0.00
α	1.00	1.00	1.00	1.00	1.00	1.12	1.13	1.38	1.00	1.00	1.00	1.00	1.08	1.00
δ	0.01	0.00	0.10	0.00	0.00	0.04	0.00	0.00	0.00	0.10	0.00	0.01	0.00	0.00
Test of the conjunctural breaks ($H_0 : \eta = 0$)														
η	-0.5	687.3	-828.6	3.6	0.7	-0.4	-1.5	50.0	3.5	-168.2	-9.5	-90.8	72.6	0.0
χ^2	0.00	2.69	179	47.3	0.03	0.00	0.01	0.01	7.6	0.00	0.00	0.00	0.06	0.00
$p.v.$	0.96	0.10	0.00	0.00	0.85	0.98	0.90	0.89	0.00	0.94	0.99	0.99	0.97	0.99

NB : the standard errors of the estimated parameters have not be reported because boundary solutions are admissible. There is no evidence against H_0 at level 1% (resp. 5%) when the p.value $(p.v)$ is greater than 0.01 (resp. 0.05). OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; BAU, bauxite ; COP, copper ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc.

With a value close to one, the parameter α indicates that the marginal extraction cost and the average extraction cost do not differ much. In such a case it should be acceptable to use the average extraction cost data as a proxy for marginal extraction cost. It is therefore useful to perform a statistical test of whether or not marginal extraction cost is equal to average extraction cost. The results are reported in Table 1.III. [7] It indeed appears that the evidence against a constant marginal extraction cost is weak for all minerals. Although there is a strong support for a value of 1 for the parameter α, it remains important to compute its exact value in order to obtain an adequate estimate of the in-situ price.

Tableau 1.III – Test of a constant marginal extraction cost

$\alpha = 1$	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN	TIN
χ^2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.00	0.00
$\chi^2 p.v.$	0.99	0.99	1.00	1.00	0.99	0.93	0.98	0.86	0.99	0.99	0.99	0.99	0.97	1.00

There is no evidence against H_0 at level 1% (resp. 5%) when the p.value $(p.v)$ is greater than 0.01 (resp. 0.05) OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; COP' copper ; BAU, bauxite ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc.

The discount rate δ is an essential element of the analysis. The results indicate that the discount rate is between 0 and 0.10, with most values close to zero. This suggests that the same valuation is given to present and future cash flow from the extraction of the

7. The results of the Wald statistic reported in this table should be interpreted with caution because some of the parameter estimates fall on the boundary of the parameter space (for example $\alpha = 1$). In that case, the Wald statistic does not have a limiting χ^2 distribution (see Andrews (2001)).

stock of a nonrenewable natural resource. As noted earlier, the parameter δ also captures the speed at which the market price adjusts to the in-situ price. A δ close to 0 suggests that the transmission of the information from the in-situ price to the market price is slow. That is, as the resource becomes scarce, the market price takes time to incorporate this information.

I did not reported the last period stock β_i in Table 1.II. The distribution of β_i is very important because it provides information about the depletion of the resource and the share of the remaining stock among countries. Figure 1.2 presents the box plot and the histogram for the end of period resource stock, β_i, of oil. The box plot contains seven handles corresponding to : the smallest observation, the lower quartile, median, upper quartile, the largest observation and the outliers.

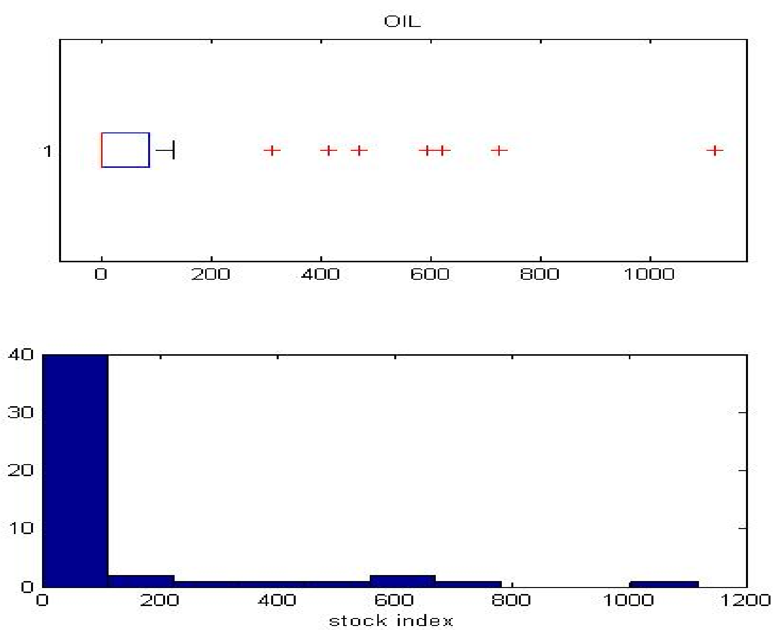

Figure 1.2 – Distribution of the resource stock, β_i, of oil

The long right tail and plus signs show the lack of symmetry in the sample values. This indicates that the large resource stock belongs to few countries. The same patterns are observed for other ores. A good estimate of the last period resource stock β_i is useful to obtain an estimate of the current resource stock, S_{it} given by :

$$S_{it} = \beta_i + R_{it} \tag{1.42}$$

Since the estimated model has a high explanatory power, we can expect that the resulting in-situ price will be close to its real value. The in-situ price λ_t is the price at which a unit of the current resource stock in the ground S_{it} could be sold. With an estimate of the parameter α and the current resource stock S_{it}, the value of the in-situ price can be obtained as follows ;

$$\lambda_t = \sum_{i=1}^{m} \frac{S_{it}}{\sum_{k=1}^{m} S_{kt}} (P_t - \alpha AC_{it}) \tag{1.43}$$

Figure 1.3 plots the paths of the estimated in-situ prices and of the actual market flow prices. These figures show that the evolution of the in-situ prices of oil and natural gas follow similar patterns. For these ores, the in-situ price increases during the period 1970-1980 and reaches a peak in 1980. From 1981 to 1999, it decreases. Starting in 2000, the in-situ price exhibits a slight increase for both of those resources. A similar behaviour is observed for hard coal and brown coal except that the in-situ price continues to decrease after 2000. As in the case of the market prices, the similarity between the in-situ price of oil and natural gas as well as the similarity of the in-situ price of brown coal and hard coal can be explained by the fact that these ores are substitutes. The in-situ price of all other resources shows a declining trend in recent years, some for quite a long period.

Figure 1.3 also shows the evolution of the observed market flow prices, so as to compare it with the evolution of the in-situ prices. The two prices are highly correlated. This correlation suggests that, the market price is a projection of the in-situ price. Therefore, even though the in-situ price is not observable, it would seem that the observed market flow price can be used as an indicator of its evolution and hence of the evolution of the scarcity of the nonrenewable resource. However, as already noted, the speed at which the information is transmitted from the in-situ prices to market flow prices is low. Because of this it remains important to get an estimate of the real in-situ price.

For all resources the in-situ price has been decreasing since 1980. In theoretical work, such a decrease of the in-situ price is sometimes explained by technological progress which reduces extraction cost. For this fall in the in-situ prices to continue in the long-run, this technological progress effect should continue to dominate the depletion effect. It is likely that at some point the depletion effect will begin to offset the technological progress effect, with the result that the in-situ prices will begin to increase again. Unfortunately the model estimated in this paper cannot predict whether and when this might happen.

1.4.2 The Regime Switching Hotelling model

The results obtained in the previous section support the use of the Hotelling model in empirical studies of nonrenewable natural resource markets. However, the resulting evolution of the in-situ prices shows the existence of two regimes in each case : an increasing and a decreasing regime for in-situ prices. There remains to determine whether or not the regression parameters are significantly affected by these structural breaks. In other words, is the model time-consistent. I take as breakpoint the date at which the trend changes in the in-situ price estimated from the Basic Hotelling. These dates are reported in Table 1.IV. As can be seen, most of the structural breaks occur between 1975 and 1980. This period is characterized by the world recession and by important adjustments, particularly in the energy importing countries.

Tableau 1.IV – Test of the structural stability

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN
Breaks	1980	1981	1980	1976	1976	1975	1977	1974	1975	1979	1988	1980	1974
$W(10^3)$	36.2	6.7	4.3	0.2	0.09	2.5	1.2	15.7	32.9	9.8	1.2	3.1	2.5
$W\,p.v.$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

There is no evidence against H_0 at level 1% (resp. 5%) when the p.value ($p.v$) is greater than 0.01 (resp. 0.05)
OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; BAU, bauxite ; COP, copper ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc.

To capture these regimes switches, I build a model which I call the " Switching

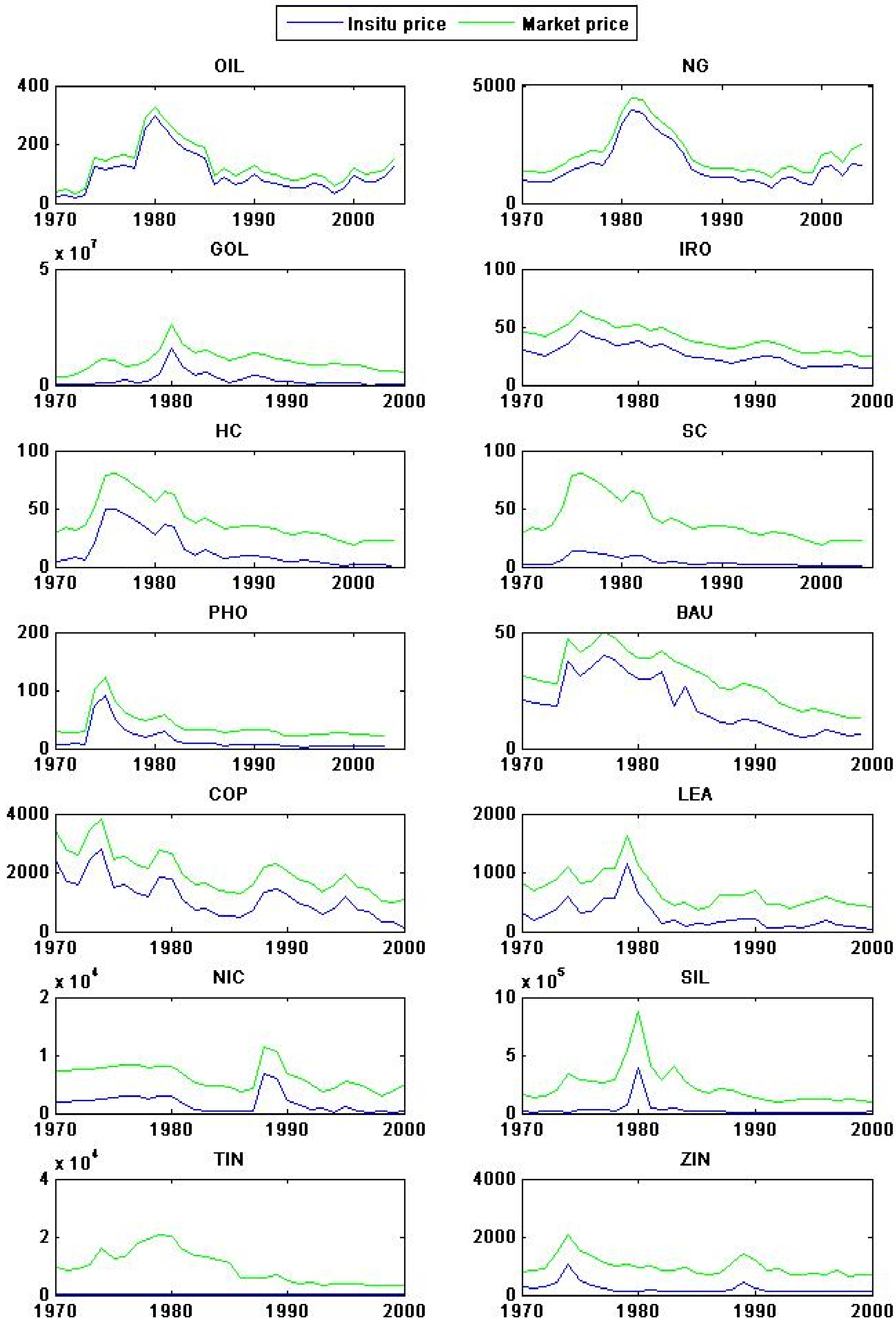

Figure 1.3 – Evolution of the estimated in-situ prices and the market flow prices

Hotelling model" to distinguish it from the "Basic Hotelling model" discussed in the previous section. This Switching Hotelling model assumes that the moment condition given in equation (1.39) holds. This moment condition allows the parameters to differ between the two regimes. The Basic Hotelling model is a particular case, where the parameters are the same ($\theta_1 = \theta_2 = \theta$). With this new specification, it becomes easy to verify the time consistency of the Hotelling model using the Wald statistic described in equation (1.40). The results of such a test are reported in Table 1.IV. The stability of the Basic Hotelling model is rejected for all the resources. Therefore the results obtained in the previous section for the Basic Hotelling model are not time consistent.

An appropriate specification of the Hotelling model will be to allow the parameters to change between the two regimes. Table 1.V presents the results obtained from the Switching Hotelling model. As in the case of the Basic model, the Switching Hotelling model performs well in approximating the observed market prices, as shown in Figure 1.4. It also appears that α is still not significantly different from one, so that it is not unreasonable to assume a constant marginal extraction cost, regardless of the regime. This is tested explicitly and the results are reported in Table 1.VI.[8] Except for bauxite and lead, it is apparent that the marginal extraction cost elasticity α may be considered time consistent for all resources. While the parameter α is approximately the same over the two regimes, the discount rate appears to vary significantly.

Note from Table 1.V that when the in-situ price is increasing the discount rate is between 0.10 and 0.50, and it shifts to 0 and 0.10, with most value around zero, when the in-situ price is decreasing. The shift in discount rate could be one explanation of the lack of time consistency of the Basic Hotelling model. The fact that the discount rate is greater in the regime where in-situ price is increasing than in the regime where in-situ price is decreasing suggests that agents give less value to the future cash flow when their marginal benefit is increasing and more value to the future cash flow when their marginal

8. The results of the Wald statistic reported in this table should be interpreted with caution because some of the parameter estimates fall on the boundary of the parameter space (for example $\alpha = 1$). In that case, the Wald statistic does not have a limiting χ^2 distribution (see Andrews (2001)).

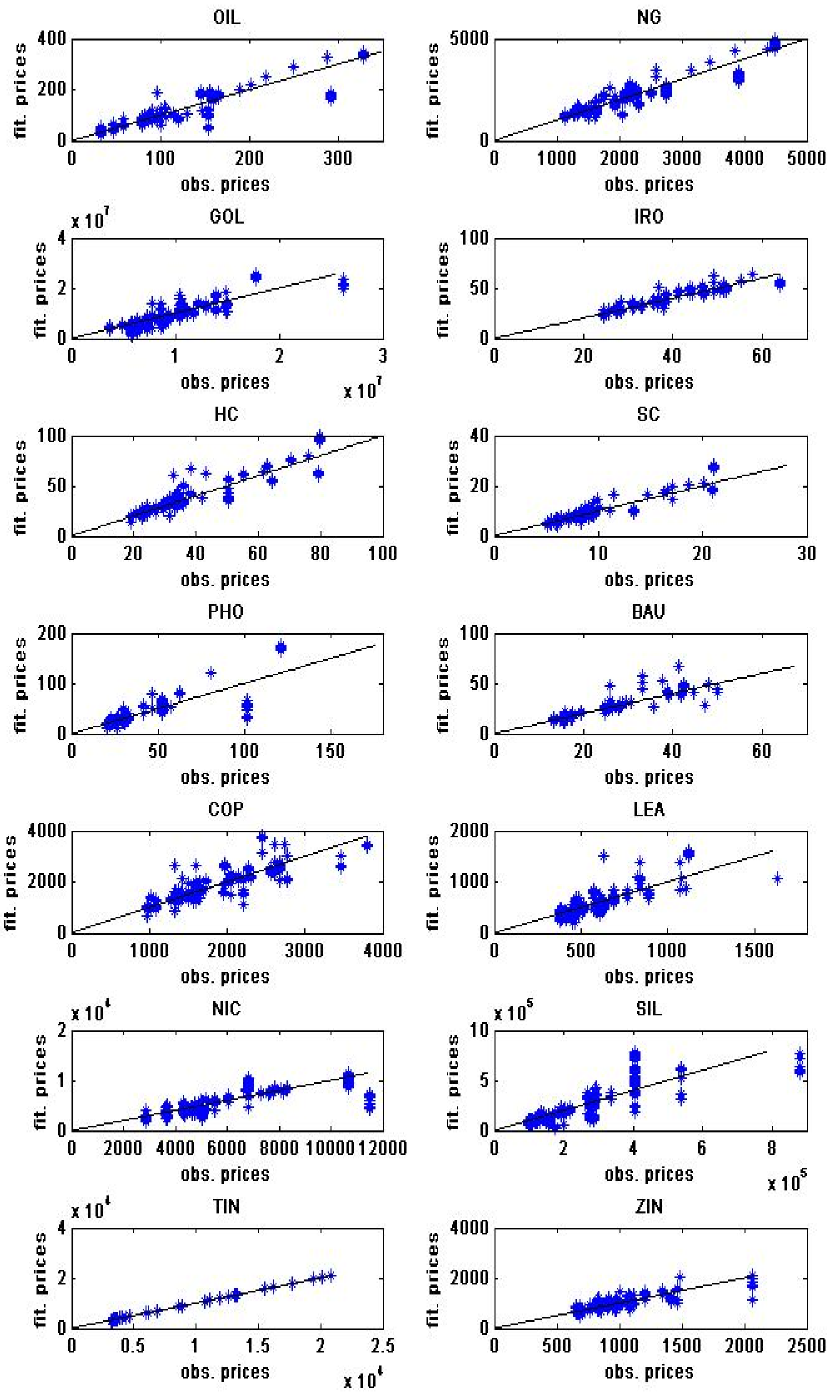

Figure 1.4 – Goodness of fit in the switching model

Tableau 1.V – Results of the switching model

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN
Sub-period where the in-situ price is increasing													
b	0.00	0.00	0.00	0.05	0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
α	1.00	1.00	1.00	1.00	1.00	1.00	2.26	1.00	1.00	1.34	1.00	1.00	1.00
δ	0.21	0.10	0.36	0.00	0.10	0.53	0.25	0.24	0.24	0.58	0.27	0.48	1.20
Sub-period where the in-situ price is decreasing													
b	0.00	0.00	20.1	0.01	0.05	0.02	0.23	0.19	0.18	0.08	0.46	5.84	0.02
α	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.10	1.00	1.10	1.00	1.00	1.14
δ	0.00	0.03	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.00

NB : the standard errors of the estimated parameters have not be reported because boundary solutions are admissible. OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; COP' copper ; BAU, bauxite ;IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc.

benefit is decreasing.

As already noted, the parameter δ also captures the speed of adjustment of the market flow price to the in-situ asset price. The fact that its estimated values is larger in the increasing in-situ price regime than in the decreasing in-situ price regime suggests that changes in the in-situ price are transmitted faster to the market flow price when the in-situ price is increasing than when it is decreasing.

As in the case of the Basic Hotelling model, I have not reported the result of the last period resource stock β.

Tableau 1.VI – Test of a constant marginal extraction cost in the switching model

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN
$\mathbf{H_0} : \alpha_1 = \alpha_2$													
$\chi^2(10^{-2})$	0.04	0.0	0.02	1.31	0.00	0.00	13629	0.01	0.0	2560.5	0.7	0.0	203.6
$\chi^2 p.v.$	0.98	1.00	0.98	0.90	0.99	0.99	0.00	0.99	0.99	0.00	0.92	1.00	0.15
$\mathbf{H_0} : \alpha_1 = 1$													
χ^2	0.00	0.00	0.00	0.00	0.00	0.00	0.35	0.00	0.00	0.00	8.35	0.00	0.00
$\chi^2 p.v.$	1.00	1.00	1.00	0.99	0.99	0.99	0.55	1.00	1.00	0.00	0.99	0.99	1.00
$\mathbf{H_0} : \alpha_2 = 1$													
χ^2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.001	0.009	0.00	0.02
$\chi^2 p.v.$	0.98	1.00	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.92	1.00	0.88

There is no evidence against H_0 at level 1% (resp. 5%) when the p.value $(p.v)$ is greater than 0.01 (resp. 0.05)
OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; COP' copper ; BAU, bauxite ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc ; The value between ( ) is the standard deviation.

However, it is difficult to obtain a snapshot of its distribution in the switching model because this distribution is not stationary.

A good estimate of this last period resource stock β_{it} is useful to obtain an estimate of the current resource stock, S_{it}, given by

$$S_{it} = \beta_{it} + R_{it}; \qquad \beta_{it} = d_t(L_1)\left(\beta_{1i} - \sum_{s=[\pi T_i]}^{T_i} q_{is}\right) + (1 - d_t(L_1))\beta_{2i}. \tag{1.44}$$

The above equation shows that the current resource stock S_{it} is the sum of a continuous process R_{it} and a jump process β_{it}. Therefore, the lack of time consistency of the Basic Hotelling model may also be explained by a jump of the state variable S_{it}.

The Switching Hotelling model is designed to obtain a time consistent estimate of the in-situ price. A formula similar to (1.43) obtained in the Basic Hotelling model is now given by :

$$\lambda_t = \sum_{i=1}^{m} \frac{S_{it}}{\sum_{k=1}^{m} S_{kt}} (P_t - \alpha_t AC_{it}); \qquad \alpha_t = d_t(L_1)\alpha_1 + (1 - d_t(L_1))\alpha_2. \tag{1.45}$$

This formula allows different distribution for the in-situ price in each regime and, consequently, provides a time consistent estimate of the in-situ price. Theoretically, this formula is more general than the one obtained in the Basic Hotelling model. However, comparing the two outcomes by plotting the evolution of the in-situ price (see Figure 1.5), it appears that the behaviour of the in-situ price is only slightly different in the two models. In order to test whether this difference is statistically significant, denote by λ^b (respectively λ^s) the in-situ price obtained from the Basic Hotelling model (respectively from the Switching Hotelling model). If the two distributions of the in-situ price are not statistically different then it must have the same mean. Therefore, one can compare the two outcomes by testing the following null hypothesis :

$$H_0 : E\left(\lambda_t^s - \lambda_t^b\right) = 0, \qquad \forall t \tag{1.46}$$

Note that, the above null hypothesis could be tested using the Student's t-test. Howe-

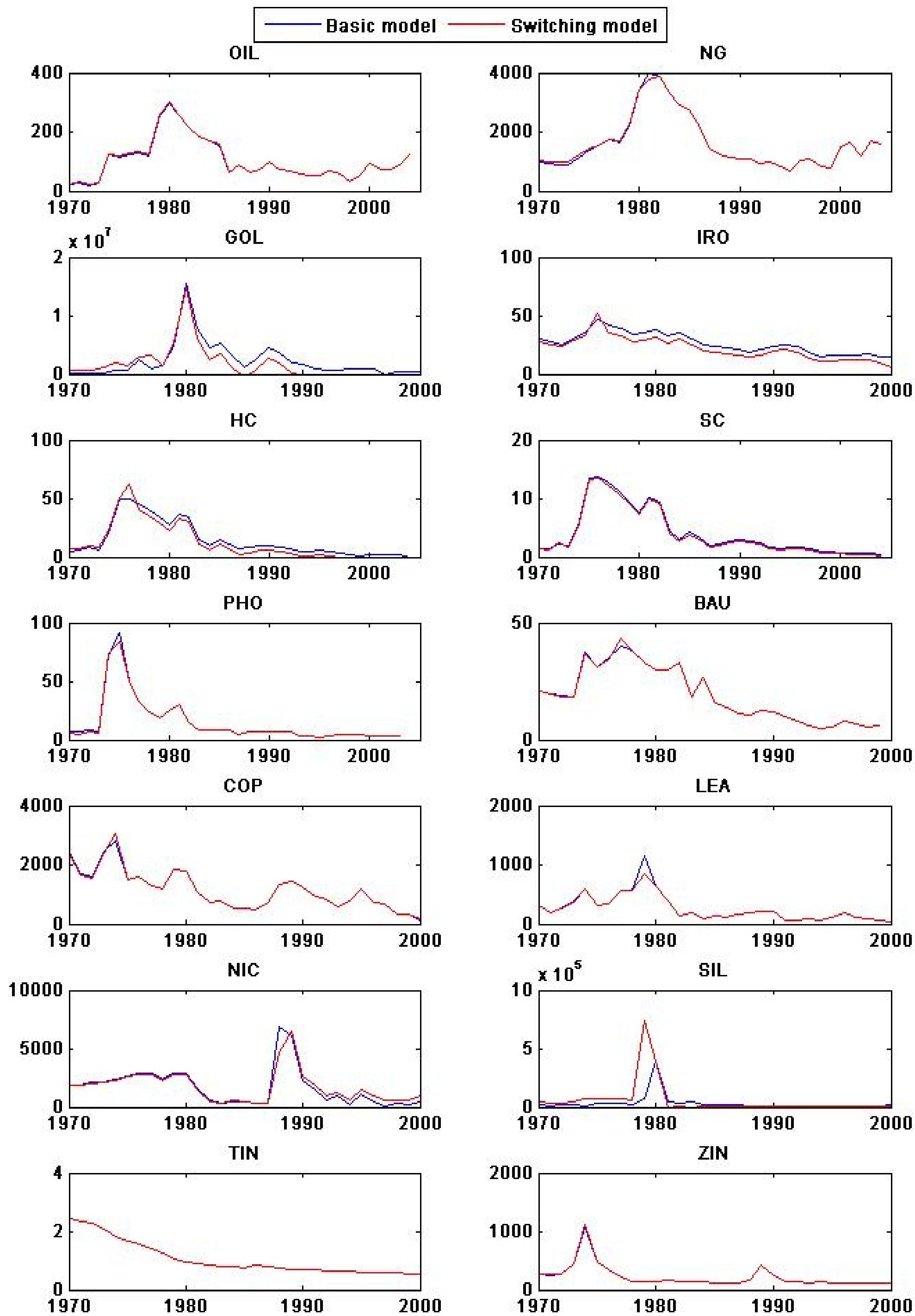

Figure 1.5 – Evolution of the in-situ prices in the switching model

ver, this requires that the stochastic process $\lambda_t^s - \lambda_t^b$ be normally distributed. This seems like a very strong assumption in the present context, given the GMM method used. To be as general as possible, assume that the stochastic variable $\lambda_t^s - \lambda_t^b$ is stationary and ergodic. Using the Lyapunov condition of the central limit theorem, one can show that the Wald statistic, given by :

$$W_{stat} = m\frac{\left[\frac{1}{m}\sum_{t=1}^{m}\left(\lambda_t^s - \lambda_t^b\right) - E\left(\lambda_t^s - \lambda_t^b\right)\right]^2}{Var\left(\lambda_t^s - \lambda_t^b\right)}, \tag{1.47}$$

follows the $\chi^2(1)$ distribution. This statistic can be used to test if the differences between the in-situ price of Basic Hotelling model and the Switching Hotelling model are statistically significant.

Table 1.VII gives the results of this test. It appears that for 8 of the 13 minerals, there is no evidence at the one percent level to reject the null hypothesis of equal means.

Tableau 1.VII – Test of equal means

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN
$\mathbf{H_0 : E(\lambda_t^s) = E(\lambda_t^b)}$													
χ^2	13.4	1.50	13.4	16.4	195.0	0.09	0.21	0.35	135.0	1.26	0.29	1.15	2.07
$\chi 2 p.v.$	0.00	0.22	0.00	0.00	0.00	0.99	0.64	0.55	0.00	0.26	0.58	0.28	0.15
$\mathbf{H_0 : E(\lambda_{it}^s) = E(\lambda_{it}^b)}$													
$\chi^2(\times 10^3)$	3.10	0.62	2.99	7.03	4.93	4.93	1.74	4.52	2.66	1.20	1.27	1.12	4.75
$\chi 2 p.v.$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\mathbf{H_0 : E(\lambda_{it}^s) = E(\lambda_{it}^0)}$													
$\chi^2(\times 10^3)$	5.62	0.74	0.01	2.33	4.93	4.93	1.54	1.74	2.66	2.01	0.38	0.01	4.75
$\chi 2 p.v.$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

There is no evidence against H_0 at level 1% (resp. 5%) when the p.value $(p.v)$ is greater than 0.01 (resp. 0.05)
OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; COP' copper ; BAU, bauxite ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc.

Therefore, for most resources, the time consistent estimate of the in-situ price λ_t^s has the same mean as the time inconsistent estimate λ_t^b. However, this does not mean that the two in-situ prices have the same distribution. It simply implies that the stochastic relation between λ_t^s and λ_t^b can be described by the following stochastic equation :

$$\lambda_t^s = \lambda_t^b + u_t; \qquad E(u_t) = 0 \tag{1.48}$$

The above relation states that the difference between the time-consistent and the time-inconsistent estimates of the in-situ price is a zero-mean process.

1.5 Conclusion

Hotelling (1931) proposed a framework to explain theoretically the optimal extraction of nonrenewable natural resource. One fundamental result was that the in-situ price, which is an economic measure of the resource scarcity should grow at the rate of interest. This principle, known as the Hotelling rule has been unsuccessfully tested, in good part because the in-situ price is not observable. The fundamental question however is whether or not the Hotelling principle can be used to explain the observed behaviour of the market price and, if possible, to estimate the unobserved behaviour of the in-situ price (Livernois, 2009).

This paper presents a framework to evaluate the Hotelling model and estimate the unobserved in-situ price. The paper offers three main contributions. Instead of using the property of time series on the market price or an estimate of the in-situ price to evaluate the Hotelling model, I combine the first-order conditions for optimal resource extraction to estimate a market price relation which is consistent with the Hotelling model. Instead of an econometric approximation of the marginal extraction cost, I use average extraction cost data as a proxy for marginal cost to estimate the unobserved in-situ price. Those two contributions can be summarized in three steps. First, I assume a functional form for extraction cost in order to use average extraction cost as a proxy for marginal cost. Second, I combine equilibrium conditions to estimate a market price which is consistent with the Hotelling model. Finally, I derive an estimate of the corresponded in-situ price. The other contribution of this paper is the empirical technique used to investigate the Hotelling model of a nonrenewable resource extraction. I use the Regime Switching GMM estimation with panel data. This robust estimation technique seems to be very useful to handle the endogenous property of the average extraction cost and discuss the time consistency of the Hotelling model.

The methodology is applied to analyse fourteen non-renewable natural resource prices. I find results that strongly support the use of the Hotelling model as a framework to analyze the behaviour of natural resource prices. It appears that the Hotelling model explains 88 to 99 per cent of the observed market prices.

Using appropriate breakpoints to evaluate the stability of my estimations I find that there are two regimes. The first regime is characterized by an increasing in-situ price and the second regime by a decreasing in-situ price. Using a regime-switching model to reconcile the Hotelling model with the change of regime, I find that this time inconsistency does not have a significant impact on the explanatory power of the model. However, this new specification of the model provides useful information on the behaviour of the natural resource markets. It appears that the rate at which agents discount the future is high in the regime where the in-situ price is increasing and low when the in-situ price is decreasing. This suggests that agents given less valuations to future cash flows when their marginal benefit is increasing and more when their marginal benefit is decreasing. Furthermore, I find that the average extraction cost data is a good proxy for the marginal extraction cost : marginal extraction cost does not seem to be significantly different from average extraction cost regardless of the regime. The time-consistent estimates of in-situ prices either decrease or exhibit an inverted U-shape form over time and it is highly correlated to the market price. Furthermore, the difference between the time-consistent and the time-inconsistent estimate of the in-situ price is simply a zero-mean process.

Some encouraging results have been obtained in this paper about the use of the Hotelling model as a framework to analyze nonrenewable natural resource markets. Unfortunately, the dynamic of the market price derived from the optimal conditions of the Hotelling model estimated in this paper has some limitations. In particular, it does not permit any predictions as to whether, and much less when, the in-situ price will start increasing in the long run, as the basic theoretical Hotelling framework predicts must eventually happen.

CHAPITRE 2

TESTING EMPIRICALLY FOR LINEARITY IN THEIR STOCK OF THE EXTRACTION RATES OF NONRENEWABLE RESOURCES

Abstract

The purpose of this paper is to test whether an equilibrium under which the rates of extraction of nonrenewable resources are linear in the stock of the resource is consistent with observed data. I first show that with a time varying demand characterized by a constant price elasticity and a time varying extraction cost function characterized by constant elasticities with respect to the rate of extraction and to the remaining stock, there exists an equilibrium in which the extraction rate is proportional to the stock of resource if and only if the discount rate and the demand and cost parameters satisfy a very specific relationship. I then use panel data on fourteen nonrenewable natural resources to test whether this relationship is satisfied empirically. I find that if the parameters are assumed time invariant, then for six of the fourteen resources I cannot reject the hypothesis. This changes however if I account for structural changes over time, in which case the hypothesis is rejected for all fourteen resources.

2.1 Introduction

The Hotelling model of nonrenewable resource exploitation has dominated the economics of exhaustible resources for decades. Since the famous publication of Hotelling (1931), this model has been used to understand the long-run behaviour of prices and optimal extraction rates of nonrenewable natural resources under different market structures. The basic Hotelling model says that the in-situ price (the price of a unit of the resource in the ground) should increase at the rate of interest when the stock of the resource is optimally extracted. [1] Although there is an important empirical literature on the behaviour of resource prices (such as Livernois (2009) or Slade and Thille (2009)), there is little information about the behaviour or the functional form of the optimal resource extraction rate. The theoretical literature argues that the optimal extraction rate should decrease over time as the resource stock gets depleted. Some authors, such as Lin and Wagner (2007), have, for empirical purposes, taken the optimal extraction rate to be a linear function of the resource stock or of the cumulative resource extracted. A rate of extraction that is proportional to the remaining resource stock is attractive for its simplicity and its tractability. There remains to see, however, whether such an equilibrium is consistent with the observed data. That is the purpose of this paper.

It is found that for six of the fourteen natural resource markets considered, the hypothesis that there is a linear relationship between the rate of extraction and the remaining stock of the resource cannot be rejected. However, if the possibility of structural breaks are taken into account, the hypothesis is rejected for all fourteen resources.

The remaining of the paper is organized as follow. In Section 2.2, I derive theoretically, given specific functional forms for demand and extraction cost, a necessary condition for the existence of an equilibrium that has the property that the rate of extraction is proportional to the stock of the resource. The functional forms assumed are an iso-elastic time varying demand and a time varying extraction cost that depends on both the rate of extraction and remaining resource stock, also with constant elasticities. An econometric specification is proposed in Section 2.3, which suggests an approach to testing for the linearity in the stock of the extraction rate. This approach is then used in Section 2.4 to analyze data of fourteen nonrenewable natural resource markets. In Section 2.5 I offer some concluding remarks.

1. There are important theoretical contributions which have extended the basic Hotelling model to accommodate more realistic economic features in the prediction of resource prices. See Krautkraemer (1998) or Gaudet (2007) for surveys of those contributions.

2.2 The basic Hotelling model of resource extraction

Consider a representative price-taking firm extracting a known and finite stock of a nonrenewable natural resource. The representative firm chooses a time path of resource extraction to maximize the present value of the stream of net benefits subject to the constraint that the cumulative extraction be not greater than the initial resource endowment. The extraction cost, the principal characteristic of the firm, is subject to technological progress, which causes the cost to decrease, and to a stock effect, which causes the cost to increase.

The notation follows closely that used in Atewamba (2011) and Lin and Wagner (2007). At time $t \in [0, +\infty]$, the supply of the mineral is given by $q(t)$, the extraction flow at time t. The cost of extracting $q(t)$ at time t is denoted by $C(z(t), q(t), S(t))$, where $S(t)$ is the remaining stock of the resource at time t and $z(t)$ is a productivity index indicating the state of extraction technology. It will be assumed that $C_z < 0$, $C_s < 0$, $C_{ss} > 0$ $C_q > 0$ and $C_{qq} \geq 0$, where C_q denotes the partial derivative $\partial C / \partial q$, etc. It will also be assumed that the average extraction cost is an increasing function of the extraction rate, so that

$$C_q(z, q, S) \geq \frac{C(z, q, S)}{q}, \qquad \forall q \geq 0. \tag{2.1}$$

Denote by $p(t)$ the endogenous market price of the resource at time t. The corresponding demand is given by $D(\theta(t), p(t))$, which is assumed to be a decreasing and convex function of the market price p ($\partial D / \partial p < 0$ and $\partial^2 D / \partial p^2 > 0$). As for the exogenous parameter $\theta(t)$, it can be viewed as capturing changes which occur in the economy and affect the demand for the resource. I will assume that demand is an increasing function of θ ($D_\theta \geq 0$).

At each time t, the market price $p(t)$ adjusts to equate the aggregate supply $q(t)$ to the aggregate demand $D(\theta(t), p(t))$, so that

$$q(t) = D(\theta(t), p(t)), \qquad \forall t \in [0, T], \tag{2.2}$$

where T is the date of exhaustion of the resource stock, which may be infinite.

In making its extraction decision, the firm takes as given the demand price of the resource $p(t) = D^{-1}(q(t), \theta(t))$. The optimal control problem it faces is to choose a time path of resource extraction, $q(t)$, to maximize

$$\int_0^T e^{-\delta t}[p(t)q(t) - C(z(t), q(t), S(t))]dt \tag{2.3}$$

subject to

$$\dot{S}(t) = -q(t); \quad q(t) \geq 0 \quad \forall t \tag{2.4}$$

$$\lim_{t \to T} S(t) \geq 0; \quad T \textit{ is free} \tag{2.5}$$

$$S(0) = S_0 \quad \textit{is given}, \tag{2.6}$$

where δ denotes the rate of discount.

Letting $\lambda(t)$ denote the in-situ price of the resource stock, the current value of the Hamiltonian for this problem is then

$$H(t, q(t), S(t), \lambda(t)) = p(t)q(t) - C(z(t), q(t), S(t)) - \lambda(t)q(t). \tag{2.7}$$

The economic efficiency requires that

$$\frac{\partial H}{\partial q} \quad = \quad p - C_q(z, q, S) - \lambda = 0 \tag{2.8}$$

$$\dot{\lambda} \quad = \quad \delta\lambda - \frac{\partial H}{\partial S} = \delta\lambda + Cs(z, q, S), \tag{2.9}$$

where $\dot{\lambda}$ denotes the time derivative of λ and where the time argument is implicit.

Solving the differential equation (2.8) and rearranging the solution, I obtain the expression for the in-situ price of the resource, namely

$$\lambda(t) = \lambda_0 e^{\delta t} + \int_0^t C_s(z(\tau), q(\tau), S(\tau)) e^{\delta(t-\tau)} d\tau. \tag{2.10}$$

Substituting the in-situ price (2.10) into the static efficiency condition (2.8), I obtain the following formula for the market price of the nonrenewable natural resource :

$$p(t) = \lambda_0 e^{\delta t} + \int_0^t C_s(z(\tau), q(\tau), S(\tau)) e^{\delta(t-\tau)} d\tau + C_q(z(t), q(t), S(t)). \tag{2.11}$$

In order to make this expression empirically tractable, specific functional forms will be assumed for the demand $D(\theta, p)$ and the extraction cost $C(z, q, S)$. More precisely, I will assume an iso-elastic time-varying demand given by

$$D(\theta, p) = \theta^{\frac{1}{\eta}} p^{-\frac{1}{\eta}}; \qquad \dot{\theta} = a\theta, \tag{2.12}$$

where η is the absolute value of the inverse of the demand elasticity. The demand curve is growing over time if a is positive and is decreasing if a is negative. This demand function satisfies all the properties assumed above. As for the extraction cost function, it will be assumed to take the following form :

$$C(z, q, S) = z^{-1} q^{\alpha} S^{-b}; \qquad \dot{z} = \gamma z, \qquad z_0 \geq 0 \tag{2.13}$$

where $b \geq 0$ is the non-negative stock effect elasticity, $\gamma \geq 0$ is the non-negative growth rate of technology, and $\alpha \geq 1$ is the output elasticity of extraction cost. Cost rises as the resource is depleted and, all else equal, a one-percent decrease in the resource stock causes extraction cost to increase by b percent. Technological progress causes cost to decrease and, all else equal, a one-percent improvement in technology causes cost to decrease by 1 percent. Notice that if $\alpha = 1$, marginal extraction cost is equal to average extraction cost, whereas if $\alpha > 1$, the average extraction is increasing in the extraction rate and marginal cost is greater than average cost for any positive rate of extraction.

Given those functional forms, we can prove the following proposition :

Proposition 1. *: If and only if the discount rate satisfies*

$$\delta = a + \eta g \tag{2.14}$$

where the constant g is

$$g = \frac{\alpha(a+\gamma)}{\alpha(b-\eta-\alpha+1)-b} \tag{2.15}$$

then there exists an equilibrium such that

$$q(t) = gS(t), \tag{2.16}$$

with the constant g being positive if (i) $\alpha < b$*, (ii)* $-\gamma \leq a$ *and (iii)* $\eta \leq \alpha^{-1}(\alpha-1)(b-\alpha)$

For the proof of this proposition see Appendix II.2.

The three conditions for the constant g to be positive are derived from the fact that g is positive if and only if the numerator and the denominator of (2.15) are of the same sign, or in other words $(a+\gamma)[\alpha(b-\eta-\alpha+1)-b] \geq 0$. To see this, notice that there are two cases. First, suppose that $a+\gamma \geq 0$ and $\alpha(b-\eta-\alpha+1)-b \geq 0$. It follows that $a \geq -\gamma$ and $\eta \leq \alpha^{-1}(\alpha-1)(b-\alpha)$. Second, suppose that $a+\gamma \leq 0$ and $\alpha(b-\eta-\alpha+1)-b \leq 0$. It follows that $a \leq -\gamma$ and $\eta \geq \alpha^{-1}(\alpha-1)(b-\alpha)$. As $a \leq 0$, the demand always decreases over time. This is not realistic. So I impose $\eta \leq \alpha^{-1}(\alpha-1)(b-\alpha)$ to allow for the possibility that demand be increasing over time.

Condition (i) of the Proposition 1 ensures that the inverse of the absolute value of the demand elasticity η is positive. Conditions (ii) and (iii) ensure that the extraction growth rate g is positive even if the demand growth rate a is negative. Note that this equilibrium is unique if the demand for the resource is elastic, $0 \leq \eta \leq 1$.[2] Furthermore, if average

2. The proof of this statement is in the Appendix

extraction cost is equal to marginal extraction cost ($\alpha = 1$), then, from condition (iii), demand for the resource is infinitely elastic $(\eta = 0)$; it then follows from expression (2.15) that $g = \infty$. Hence the rate of change of the resource stock is minus infinity, which means that the stock is depleted instantaneously. To rule out this uninteresting case it will be assumed that $\alpha > 1$, and therefore both marginal and average cost are increasing, with the latter everywhere greater than average cost for all positive rates of extraction.

We have so far neglected the terminal date T, which is endogenous. Its determination is given by the following proposition :

Proposition 2. *: With iso-elastic time-varying demand and increasing marginal and average extraction cost (*$\alpha > 1$*), if the equilibrium extraction rate is of the form* $q = gS$ *then,*

i - *it will not be efficient to leave any stock in ground so that is* $S(T) = 0$ *;*

ii- *the terminal date is* $T = \infty$*.*

The proof of this proposition is provided in Appendix II.3.

The requirement that the discount rate satisfies (2.14) in order for the equilibrium described in Proposition 1 to exist is of course a very stringent one. Our task is now, in what follows, to estimate the various parameters of the model that enter in expression (2.15) for g so as to be able to test statistically whether the constraint (2.14) is satisfied for some of the resources.

2.3 Empirical analysis of the equilibrium

In this section, I propose an empirical specification that can be used to estimate the relevant parameters and perform statistical test on whether condition (2.14) holds empirically.

2.3.1 Data and the empirical model

I use the same data as in Atewamba (2011).[3] This database is a compilation of the data on several countries producing a nonrenewable natural resource. It contains average annual world price, country's average cost and country's current stock data for 14 ores from previously unpublished World Bank data. The commodities are bauxite, copper, gold, hard coal, iron, lead, natural gas as well as nickel, oil, phosphate, brown coal, silver, tin. and zinc. The data cover 35 years from 1970 to 2004. The summary statistics on the ores which are analyzed in this empirical section are presented in Appendix II.1. The market price is common to all countries, while the extraction costs are particular to each country. An appropriate way to account for the unspecified characteristics of each country on which those cost depend will be to introduce country effects, which may be captured by a constant term. This will be discussed in the continuation.

A discrete form associated to the first-order condition (2.8) of the Hotelling model is given by

$$p_t - p_{t-1} = \delta(p_{t-1} - \alpha AC_{it-1}) + \alpha(AC_{it} - AC_{it-1}) - b\frac{q_{it}}{S_{it}}AC_{it} \qquad \forall i,t \qquad (2.17)$$

where i is the country index and t is the time index. The only unobservable variable of the price dynamic (2.17) is the current resource stock S_{it}. Extracting the current stock S_{it} from equation (2.16) of Proposition 1 and substituting into (2.17), the dynamic of price becomes

$$p_t - p_{t-1} = \delta(p_{t-1} - \alpha AC_{it-1}) + \alpha(AC_{it} - AC_{it-1}) - bgAC_{it} \qquad \forall i,t. \qquad (2.18)$$

This equation can allow us to estimate the parameters δ, α, b and the stock growth rate g. However, to verify if the equilibrium condition (2.14) holds, I need also to estimate parameters η and a.

3. The database was downloaded directly from http ://gwagner.com/research/hotelling/, provided by Lin and Wagner (2007). It is described in detail in their Appendix B.

Substituting expression (2.12) for the demand function into (2.2) and taking the logarithm of the resulting equation, I obtain

$$log(q_{it}) = d_i + \frac{a}{\eta}t - \frac{1}{\eta}log(p_t) \qquad \forall i,t, \tag{2.19}$$

where d_i is a constant term that captures the country effect i. This equation can allow us to estimate the additional parameters η and a. It follows from (2.18) and (2.19) that the empirical model to be estimated is given by

$$\left.\begin{array}{l} p_t - p_{t-1} = \delta(p_{t-1} - \alpha AC_{it-1}) + \alpha(AC_{it} - AC_{it-1}) - bgAC_{it} + u_{it} \\ log(q_{it}) = d_i + \frac{a}{\eta}t - \frac{1}{\eta}Log(p_t) + v_{it} \end{array}\right\} \tag{2.20}$$

subject to the constraints

$$b \geq \alpha > 1; \qquad \delta \geq 0; \qquad \eta \geq 0; \qquad \gamma \geq 0; \qquad -\gamma \leq a; \qquad \eta \leq \alpha^{-1}(\alpha-1)(b-\alpha), \tag{2.21}$$

where u_{it} and v_{it} are error terms which can be correlated within countries.

Note that the main purpose of the empirical model (2.20) is to determine whether or not the equilibrium condition (2.14) is satisfied for some natural resources. From equation (2.14) of Proposition 1, the null hypothesis of the existence of an equilibrium that is linear in the stock is formulated as follow :

$$H_0 \;\; : \;\; \delta = a + \eta g; \qquad with \qquad g = \frac{\alpha(a+\gamma)}{\alpha(b-\eta-\alpha+1)-b} \tag{2.22}$$

In what follows, I will specify an appropriated econometric model to estimate the empirical model (2.20) and test condition (2.22).

2.3.2 Econometric specification

The empirical model (2.20) developed in the previous subsection is nonlinear in b, α, γ, δ, η, a and d_i. The difference between this empirical model and the one estimated in Atewamba (2011) is the existence of a second equation, which captures the equilibrium in the market for the resource. The method of estimation depends on the nature of the explanatory variables and the further assumptions made on the behaviour of the error terms. In the simplest case, if the explanatory variables are exogenous and the error term follows a normal distribution, an appropriate estimation method is the Maximum Likelihood. The Maximum Likelihood Estimator has important properties of efficiency, but its well known limitation is the normality assumption of the error terms. A more general and frequently used approach is to assume that the behaviour of the error terms is unknown. In that situation, if the explanatory variables are strictly exogenous, a convenient estimation method is the Nonlinear Least Squares (NLS) method. But the average extraction cost is a function of the extraction rate, which is an endogenous variable. Therefore, in addition to the market price p_t and the extraction rate q_{it}, it will be appropriate to treat the average extraction cost as an endogenous variable. This implies that the NLS method may no longer be an appropriate estimation method. It is important to take into account this endogeneity property to obtain a consistent estimator of the parameters. The Generalized Method of Moments (GMM), which includes the NLS as a special case, provides a solution (see Matyas (1999)). Note that the GMM estimator is a M.estimator, which is asymptotically normal under minimal restrictions on the distribution of the error terms. [4] Although, the GMM estimation method is technically more complex with multiple equations than a single equation [5], its theoretical framework is the same. I will now discuss

4. The method requires that a certain number of moment conditions be specified by the model. These moment conditions are functions of the parameters of the model and of the data, such that their expectation is zero at the true values of the parameters. GMM estimators are known to be consistent, asymptotically normal, and efficient in the class of all estimators that don't use any extra information aside from that contained in the moment conditions.

5. In Atewamba (2011), the GMM estimation method was based on a single equation. Technically, a GMM method applied to a single equation is less complex than a GMM method applied to a multiple equations. However, the theory remains the same for both.

the way this estimation method can be used to fit the Hotelling model (2.20).

2.3.2.1 Switching GMM estimation

As observed in Atewamba (2011), the Hotelling model exhibits two significant regimes. The first regime is characterized by an increase of the resource price and the second regime is characterized by a decrease of the resource price. As a result, the Hotelling model has been verified to be subject to a structural shift (Atewamba, 2011). Therefore, it will be appropriate to take into account those two regimes in the estimation of the empirical model (2.20). A Switching GMM estimation technique provides a solution.

Denote the parameters to be estimated by $\theta = (b, \alpha, \gamma, \delta, \eta, a, d_1, ..., d_n) \in \Theta$ and the observable variables by $y_{it} = (p_t, p_{t-1}, AC_{it}, AC_{it-1}, q_{it})$, and let $\theta_0 \in \Theta$ be the true parameter value. Let us assume

$$\begin{aligned} f_1(y_{it};\theta_0) &= u_{it}, \qquad E(u_{it}) = 0, \qquad \theta_0 \in \Theta, \\ f_2(y_{it};\theta_0) &= v_{it}, \qquad E(v_{it}) = 0 \end{aligned} \tag{2.23}$$

where $f_1(y_{it};\theta) = p_t - (1+\delta)p_{t-1} - \alpha AC_{it} + \alpha(1+\delta)AC_{it-1} + bgAC_{it}$ and $f_2(y_{it};\theta) = Log(q_{it}) - d_i - \frac{a}{\eta}t + \frac{1}{\eta}Log(p_t)$ are elementary functions, or residuals. Grouping all these residuals in a $2T \times 1$ vector $f(y;\theta)$, I assume

$$E\left[f(y;\theta_0)f(y;\theta_0)'\right] = \Omega, \qquad \theta_0 \in \Theta, \tag{2.24}$$

where Ω is an unknown positive definite matrix and $T = \sum_{i=1}^{n} T_i$.

Now denote by w_{it} the instrumental variables for country i. The variable w_{it} is constituted with variables $(AC_{it-1}, q_{it-1}, X_{it-1})$ and $(AC_{jt-1}, q_{jt-1}, X_{jt-1})$, where country j produces the mineral during the same period as country i and X_{it} is the cumulative resource stock extracted in country i at time t. This technique for constructing instrumental variables allows me to obtain at least as many instruments as there are parameters.[6] Let W_l be a $T \times k_l$ matrix of instruments, $l = 1,2$, assumed to be predetermined, where $k = k_1 + k_2$ is the number of instruments, with k_l greater or equal to the number of parameters in the function f_l. Denote by W the diagonal block matrix of W_1 and W_2.

Now, let us determine the moment condition of the model. Denote by R_l, $l \in \{1,2\}$, the sub-samples corresponding to the two regimes of the model. R_l is a subset of $\mathbb{N}$ and $R_1 \cap R_2 = \varnothing$. Let $\phi_0 = (\theta_1, \theta_2) \in \Phi$ be the true parameter value, where θ_l is the true parameter value of the regime $l = 1,2$. The theoretical moment condition is given by

$$E(W'_{it} f(y_{it}; \phi_0)) = E\left[d_t(R_1) W'_{it} f(y_{it}; \theta_1) + (1 - d_t(R_1)) W'_{it} f(y_{it}; \theta_2)\right] = 0 \qquad \phi_0 \in \Phi, \tag{2.25}$$

where W_{it} is the it^{th} row of W and $d_t(R_1)$ is a dummy variable which equals one when $t \in R_1$. I will call the above moment condition the " Switching Hotelling model" to distinguish it from the "Basic Hotelling model", where no regime is considered. If the two regimes R_1 and R_2 are statistically significant then the moment condition (2.25) allows us to incorporate their effect into our estimation. As in Atewamba (2011), to verify the statistical significance of the Switching Hotelling model, I restrict the analysis of the time varying parameters to the fixed part of the parameter θ, that is $b, \alpha, \gamma, \delta, \eta$ and a. Let A be a $k \times k$ matrix of rank 6, which satisfies $(b, \alpha, \gamma, \delta, \eta, a) = A\theta$, with $k = dim(\theta)$. As in Andrews and Fair (1988), I derive the Wald statistic as follows :

$$W_{stat} = T(A_1\hat{\theta}_1 - A_2\hat{\theta}_2)' \left[\frac{1}{\pi} A_1 \widehat{Var}(\hat{\theta}_1) A'_1 + \frac{1}{1-\pi} A_2 \widehat{Var}(\hat{\theta}_2) A'_2\right]^{-1} (A_1\hat{\theta}_1 - A_2\hat{\theta}_2), \tag{2.26}$$

where $\pi T = \#R_1$, $\hat{\theta}_l$ is the GMM estimator[7] based on the sub-sample R_l, and $\widehat{Var}(\hat{\theta}_l)$ is a Ledoit-Wolf HAC estimator of Σ based on the R_l.[8] This statistic has a limiting $\chi^2(6)$

6. As reported in the Table II.I in the Appendix H, the J-test shows that those choices of instruments match the data very well for almost all resources.

7. See Appendix II.6 for more details on the GMM estimator

8. Since the model has many parameters (6 + the number of producing countries of each the resources), the number of instruments used to compute the GMM estimator is very large. With many ins-

distribution under the identifying restrictions of the two sub-samples and will be used to test the structural stability of the Hotelling model with a constant stock growth rate and an iso-elastic time varying demand [9]

2.3.2.2 Computation of the GMM estimator $\hat{\theta}$

For each regime a GMM estimator is obtained by minimizing an objective function $Q_T(\theta)$ obtained from the moment condition (2.25). [10] To compute the GMM estimator $\hat{\theta}$ for each regime, I use Newton's method for constrained nonlinear minimization (See Appendix II.7 for a description of Newton's method). Before running Newton's method to minimize the objective function $Q_T(\theta)$, I should replace the covariance matrix Σ in $Q_T(\theta)$ by the Ledoit and Wolf (2004) HAC estimator obtained from an initial estimation of the model by the NLS method. Like other procedures that start from preliminary estimates, this one is iterated. Indeed, the GMM estimator residuals are used to calculate a new estimate of the covariance matrix Σ, which is then used to obtain a second GMM estimator, which is then used to obtain another GMM estimator, until the procedure converges relative to a given criterion. This iterative procedure is called the iterated GMM and was investigated by Hansen et al. (1996).

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truments, the estimate of the covariance matrix with the usual procedure, the Newey-West HAC estimator of the covariance matrix $\hat{\Sigma}$ (see Appendix II.4 is generally not well conditioned. To obtain a well conditioned HAC estimator, I regularize the Newey-West HAC estimator with the regularization procedure of Ledoit and Wolf (2004). These authors introduce an estimator of the covariance matrix that is both well-conditioned and more accurate than the sample covariance matrix asymptotically. Their estimator is distribution-free and has a simple explicit formula that is easy to compute and interpret.

9. The structural stability of the model developed here must be interpreted with caution. Indeed, if there is no evidence to reject the null hypothesis of structural stability, this does not mean that parameters of the model are constant over time. It simply means that the vector of parameters $(b, \alpha, \gamma, \delta, \eta, a) = A\theta$ is constant over time, but not necessarily the vector θ. The second block $d = (d_1, d_2, ..., d_n)$ of θ can change over the two sub-periods.

10. See Davidson and Mackinnon (2003) for more details on the construction of the objective function $Q_T(\theta)$ and the Newton method of optimization.

2.3.2.3 Test of the existence of a linear equilibrium

The main purpose of this paper is to determine whether or not condition (2.22) is consistent with the data. This condition is equivalent to

$$H_0 : h(\theta) = \delta - a - \frac{\eta \alpha (a+\gamma)}{\alpha(b-\eta-\alpha+1)-b} = 0. \tag{2.27}$$

Under the null hypothesis (2.27), one can show that the Wald statistic

$$Wstat = h^{'}(\hat{\theta}) \left[H(\hat{\theta}) \widehat{Var}(\hat{\theta}) H^{'}(\hat{\theta}) \right]^{-1} h(\hat{\theta}) \tag{2.28}$$

is asymptotically distributed as $\chi^2(r)$, where $H(\hat{\theta})$ is an $r \times k$ matrix with typical element $\partial h_j / \partial \theta_i$.[11] The Wald statistic (2.28) will be used to determine, for each of the resource markets under consideration, whether or not condition (2.22) is satisfied. Note that if the null hypothesis (2.27) is rejected, it will not mean that there is no equilibrium to Hotelling model of resource extraction. It would mean that assumptions made for the demand and cost functions are not appropriate for observed data.

2.4 Empirical results

This section presents results obtained by implementing the above methodology to analyze data for fourteen nonrenewable natural resources : bauxite, copper, gold, hard coal, iron, lead, natural gas as well as nickel, oil, phosphate, brown coal, silver, tin. and zinc. General results of the estimations are presented in the first subsection, while the existence of an equilibrium is discussed in the second subsection.

2.4.1 General Results

Recall that the estimation results are based on the GMM estimation method. GMM estimators are known to be consistent, asymptotically normal, and efficient in the class

11. As r=1, I can also use the t-statistic, the square of the Wald statistic (2.28), to test (2.27).

of all estimators that do not use any extra information aside from that contained in the moment conditions. Therefore, it is important to first determine whether the moment conditions (2.25) match the data well or not. The over-identification test or the J-test provides the answer.[12] Results of this J-test are reported in the Table II.I, in Appendix II.8. It appears that for almost all resources the moment conditions match the data very well for all regimes.[13]

Aside from the GMM specification test, there is a question of whether the Structural or Switching GMM considered is acceptable or not. Table 2.I reports results from the implementation of the Wald statistic (2.26) for the structural break test.[14] As in Atewamba (2011), it appears that the parameters of the Hotelling model vary over time. Therefore, the Switching GMM estimation technique is appropriate to take the time varying property of parameters into account for a suitable estimation of parameters and test statistics.

Tableau 2.I – Test of the structural stability

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN
Breaks	1980	1981	1980	1976	1976	1975	1977	1974	1975	1979	1988	1980	1974
$W(10^3)$	242.8	270.2	3.3	151.5	26.3	22.5	1.9	0.7	9.6	0.5	2.25	5.1	29.1
$W\,p.v.$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

There is no evidence against H_0 at level 1% (resp. 5%) when the p.value ($p.v$) is greater than 0.01 (resp. 0.05)
OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; BAU, bauxite ; COP, copper ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc.

The results reported in Table 2.II are based on a Switching GMM estimation technique. For the sake of brevity, only results on key parameters are reported. It appears that for almost all resources, the parameter α, which captures the gap between average extraction cost and marginal extraction cost is close to 1 in both regimes. Although this result is similar to the one obtained in Atewamba (2011), it does not suggest that the

12. See Matyas (1999) for more information on the J-test

13. In fact, a sufficient condition for the moment conditions (2.25) to be verified is that it is verified in each regime. As a result, a Switching GMM estimation can be reduced to applying a GMM in each regime (see Matyas (1999)).

14. For the structural stability test, I use the same break points as in Atewamba (2011). They correspond to the date at which the trend changes in prices of resources. These dates are reported in Table 2.I. As it appears, most of the structural breaks occur between 1975 and 1980. This period is characterized by the world recession and by important adjustments, particularly in the energy importing countries.

average extraction cost could be used as a proxy for marginal extraction cost. Formally, the marginal extraction cost elasticity should be greater than one for an equilibrium to exist (see Proposition 1).

Tableau 2.II – Estimation results

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN	TIN
Sub-period where the price is increasing														
α	1.00	1.00	1.01	1.00	1.00	2.16	1.03	1.00	1.00	1.00	2.27	1.00	1.00	.
	(0.00)	(0.00)	(0.74)	(0.00)	(1.36)	(2.09)	(0.38)	(0.00)	(1.11)	(1.10)	(1.72)	(0.29)	(0.38)	.
b^*	1.39	4.41	0.00	1.66	6.28	0.00	31.7	2.84	0.71	7.21	0.00	0.00	4.31	.
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	.
γ	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	.
δ	0.20	0.24	1.02	0.22	0.23	0.00	0.07	0.02	0.13	0.18	0.00	0.69	0.78	.
η	4.92	5.19	0.01	3.00	9.60	7.23	84.0	1.51	2.57	2.45	1.19	0.00	4.80	.
a	0.00	0.00	-0.00	0.00	-0.00	0.00	0.00	-0.00	0.00	-0.00	0.00	0.00	0.00	.
	(0.34)	(0.00)	(0.00)	(0.00)	(0.17)	(1.06)	(0.50)	(0.00)	(1.30)	(0.29)	(1.80)	(1.69)	(0.00)	.
Sub-period where the price is decreasing														
α	1.00	1.00	3.03	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.01	1.60	1.00	1.00
	(6.51)	(1.83)	(1.41)	(2.12)	(5.50)	(0.62)	(1.47)	(2.09)	(2.10)	(0.57)	(0.75)	(0.14)	(0.34)	(0.00)
b^*	0.26	0.16	0.00	0.74	0.08	0.90	1.24	2.11	0.14	1.68	0.00	0.00	2.59	0.16
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.03)	(0.00)	(0.00)
γ	0.07	0.00	0.00	0.25	0.12	0.08	0.08	0.04	0.15	0.05	0.00	0.00	0.08	0.00
δ	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
η	3.95	0.56	0.04	8.03	1.78	2.73	1.06	2.24	1.42	1.67	0.24	0.00	1.97	3.56
a	-0.07	0.00	-0.00	-0.25	-0.11	-0.08	-0.07	-0.04	-0.15	-0.05	0.00	-0.00	-0.07	-0.00

NB : The values between parentheses are the standard errors. The standard errors of some estimated parameters have not be reported because boundary solutions are admissible. The sign (*) means that the value of the parameter should be multiplied. by 10^4. OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO phosphate ; COP, copper ; BAU, bauxite ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc.

The discount rate δ is close to zero in the regime where the resource price is decreasing over time. Furthermore, the discount rate is larger in the regime where the resource price is increasing than in the regime where the resource price is decreasing. This would suggest that, at least under the specification of the model retained here, agents are likely to give more value to their future incomes from resource extraction when the price of the resource is decreasing than when the price of the resource is increasing.

The value of b, which measures the stock effect on extraction costs, is very large compared to the one obtained in Atewamba (2011). This result may be explained by the additional constraint $b \geq \alpha$ imposed on the parameter b in order to obtaining a positive value for g. In order to understand how this result may affect the test statistic (2.28) for the existence of a linear equilibrium, I relax the constraint $b \geq \alpha$ and reestimate the empirical model (2.20). However, to ensure that the resource stock growth rate g

is non-negative, I assume that the demand for the resource decreases over time, that is the parameter a is negative. Results of this new estimation is reported in Table II.II of Appendix II.9. It appears that the value of b is close to zero for almost all natural resources, as found in Atewamba (2011), and g is positive, as required. As discussed in Appendix II.9, assuming that the demand of the resource decreases over time does not have a significant impact on the test statistic (2.28) for the existence of a linear equilibrium.

One of the results of the estimations is the value of γ, the rate of technological progress in extraction. As reported in Table 2.II, it is close to zero when the resource price is increasing and smaller than when the resource price is decreasing. This suggests that the technological progress may partially explain the decrease of resource prices.

The value of a, the exogenous rate of growth of demand, is negative for all resources. This observation can justify the assumption $a \leq 0$ used for the reestimation of the empirical model (2.20) (see Appendix II.9 for more details).

Estimates of the parameter g obtained under the constraints (i), (ii) and (iii) of the Proposition 1 are reported in the Table 2.III. It appears to be close to zero in all cases, but larger in the regime where the resource price is decreasing than in the regime where the resource price is increasing. This result suggests that agents extract at a larger rate when the price is decreasing than when the price is increasing.

Tableau 2.III – Estimates of the parameter g

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN	TIN
All data														
$g(10^{-6})$	0.00	0.00	374	77.9	1.27	550	4.67	9.43	111	0.87	72.1	0.00	1.21	0.02
Regime where the price increases														
$g(10^{-12})$	9.25	1.03	577	74.3	0.00	0.00	0.61	7.60	166	2.48	0.00	8.48	0.46	.
Regime where the price decreases														
$g(10^{-6})$	63.2	55.8	0.00	10.1	89.3	15.0	11.8	4.58	50.9	9.79	725	932	0.25	0.02

OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; COP' copper ; BAU, bauxite ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc.

2.4.2 Existence of an equilibrium of the Hotelling model

The main purpose of this paper is to test whether the condition (2.14), which is required for the existence of the linear equilibrium described in Proposition 1 to exist, is consistent with the observed data. To this effect, the results of the implementation of the Wald statistic (2.26) is reported in Table 2.IV. To ensure the robustness of my results, I implement this test for each identified subperiods as well as for the entire period.

Tableau 2.IV – Test of the existence of a linear equilibrium

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN	TIN
Entire period														
χ^2	0.00	0.01	0.02	487	124	235	188	128	889	588	0.02	0.00	217	0.95
$\chi^2 p.v.$	0.99	0.93	0.86	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.89	0.99	0.00	0.33
Sub-period where the price is increasing														
χ^2	321	280	635	431	344	0.00	191	531	423	476	0.00	239	510	.
$\chi^2 p.v.$	0.00	0.00	0.00	0.00	0.00	0.99	0.00	0.00	0.00	0.00	1.00	0.00	0.00	.
Sub-period where the price is decreasing														
χ^2	561	749	0.00	130	363	198	433	281	627	522	377	89.0	810	0.95
$\chi^2 p.v.$	0.00	0.00	0.99	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.33

There is no evidence against H_0 at level 1% (resp. 5%) when the p.value $(p.v)$ is greater than 0.01 (resp. 0.05)
OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; COP' copper ; BAU, bauxite ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc ; The value between ( ) is the standard deviation.

Consider first the two sub-periods. We may conclude that the equilibrium proposed in Proposition 1 is consistent with the data for a given resource if there is no evidence to reject the null hypothesis (2.27) for both sub-periods. From Table 2.IV, it appears that, except for phosphate and nickel markets, the null hypothesis is rejected when the resource price is increasing, and, except for gold and tin, it is rejected when the resource price is decreasing. In other words, for all resources the null hypothesis is rejected in at least one of the sub-periods. Therefore, the equilibrium proposed in Proposition 1 appears not to be consistent with the data.

If instead we consider the estimates using the data for the entire period, it appears that there is no evidence to reject the null hypothesis (2.27) for six of fourteen natural resource markets. As reported in Table II.IV of the Appendix II.9, this result remains unchanged if we impose that the demand for the resource decreases over time and that the stock effect (b) be close to zero.

2.5 Conclusion

The purpose of this paper has been to test whether an equilibrium under which the rates of extraction of nonrenewable resources are linear in the stock of the resource is consistent with observed data. To do this, iso-elastic time-varying demand and cost functions have been assumed. With such demand and cost functions, an equilibrium characterized by a linear relation between the rate of extraction of the resource and its remaining stock will exist if and only if the discount rate and the parameters of the problem together satisfy a specific testable condition. Panel data for fourteen nonrenewable resources were used to test statistically whether this condition can be viewed as empirically valid. It appears that if the possibility of structural breaks in the data are ignored, the null hypothesis that the condition holds cannot be rejected for six of the fourteen resources being considered. The data shows however that, for each resource, the market price is characterized by a sub-period where price is increasing and one where it is decreasing. If the possibility of a structural break is allowed for each resource at the date at which price goes from increasing to decreasing, then it appears that the null hypothesis is rejected for all resources. This could be taken to suggest that in future research it would be more appropriate to consider a model in which the proportion of the stock being extracted each period is allowed to vary smoothly over time.

CHAPITRE 3

PRICING OF DURABLE EXHAUSTIBLE RESOURCES UNDER STOCHASTIC INVESTMENT OPPORTUNITIES

Abstract

We take a capital asset pricing approach to the determination of the price of a nonrenewable natural resource in the case where the resource is *durable*, in the sense that once extracted it becomes a productive asset held above ground. The portfolio choice is then made up of the following assets : a stock of nonrenewable resource held in the ground that yields no dividend, a stock of resources held above ground that yields a dividend in the form of a flow of productive services, and a stock of composite good that can be held either in the form of productive capital or of a bond whose return is given. There is a stochastic element to the rate of change of productivity in both the production of the composite good and in the extraction of the resource. It is shown that the resulting prediction for the price path of the resource differs considerably from the one that follows from the more basic Hotelling model and that no unambiguous prediction can be drawn analytically about the pattern of behavior of that price path.

3.1 Introduction

The basic Hotelling model of the exploitation of a nonrenewable natural resource (Hotelling, 1931) predicts that the *in situ* price of the resource (its flow price minus the marginal cost of extracting it, often called the Hotelling rent) will, in equilibrium, grow at the rate of interest. This means that the rate of growth of the flow price will be a weighted average of the rate of interest and the rate of change of the cost of extraction, with the weights being respectively the share of rent and of cost in the price. Therefore, although the price may at first decline if the cost of extraction is decreasing, it must eventually follow an increasing path since the share of rent in the price is increasing with time and that of cost is decreasing. This basic model, known as the Hotelling rule, has been the source of much theoretical insights into the behavior of natural resource markets. There is however very little evidence that resource prices do indeed behave as predicted. [1] This should not be too surprising, since this parsimonious basic model neglects a number of important factors which will also play a role in determining the real world behavior of resource prices. [2] Among those factors are uncertainty about future prospects and the fact that many nonrenewable natural resources are *durable*, contrary to what is most often assumed in theoretical modeling. The purpose of this paper is to explore the impact on the equilibrium pricing of natural resources of simultaneously taking into account those two factors.

There is an extensive literature on the presence of uncertainty in various forms in natural resource markets. [3] This paper follows closely the modeling of stochastic future prospects proposed in Gaudet and Khadr (1991), which studied the case of *non durable* natural resources. What distinguishes Gaudet and Khadr (1991) from the previous literature is that it takes an intertemporal asset pricing approach to the problem. Thus the Hotelling rule is interpreted as an equilibrium asset pricing condition, the asset being of course the stock of the resource held in the ground. This asset, contrary to a reproducible asset such as conventional physical capital, has the particularity that it cannot be increased, with the result that disinvestment decisions are irreversible. The question then becomes : what is the appropriate rate of return on holding a unit of the resource in the ground. In the basic deterministic framework assumed by Hotelling the answer is simply the capital gains that can be obtained from holding it *in situ*. When investment opportunities are stochastic, it is shown in Gaudet and Khadr (1991) that its equilibrium expected return will depend also on the degree of risk aversion and on how its return

1. See Slade and Thille (2009) and Livernois (2009) for excellent recent surveys of empirical analyses of the Hotelling rule.
2. Gaudet (2007) discusses in more details some of those factors.
3. See Gaudet and Khadr (1991) and the references cited therein.

happens to be correlated with the performance of the economy in terms of consumption.

Nonrenewable resources most often bring to mind fossil fuels, such as petroleum, natural gas and coal. Although those natural resources are storable, they are not durable, since they are consumed in a single usage. But many nonrenewable resources, such as metals, are durable : once extracted they become above ground assets capable of yielding a continuous flow of services used as input into various production processes. We are then in presence of two resource assets : a stock held below ground that yields no flow of services, and a stock held above ground that does. Those assets have the particularity that the one held above ground can only be increased by depleting the one below ground. Levhari and Pindyck (1981) is the most important reference on the behavior of markets for durable resources, a topic on which there is surprisingly very little literature given that many important nonrenewable resources are in fact durable. It considered the pricing of durable resources in a partial equilibrium context. In this paper we combine an approach similar to that of Levhari and Pindyck (1981) for modeling the durability of the resource with the two-goods multi-assets stochastic pricing framework of Gaudet and Khadr (1991).

The next section will present the model ; it follows closely that of Gaudet and Khadr (1991), into which we integrate the durability of the resource à la Levhari-Pindyck. We then characterize in succession the efficient extraction of the resource in Section 3.3 and, in Section 3.4, the efficient production of a composite good that uses the services of the above ground stock of the resource as an input and that can be either consumed or accumulated. The efficiency conditions thus derived serve to determine the consumer's opportunity set subject to which he makes his consumption and portfolio decisions, solved for in Section 3.5. This enables us to characterize in Section 3.6 the expected equilibrium behavior of both the asset price and the flow price of the resource, and to highlight the effect of durability on the expected price path as compared to non durable resources, as well as to a deterministic context. Brief concluding remarks follow in Section 3.7.

3.2 The model

Consider an economy in which there are two goods : a composite good and a durable nonrenewable natural resource. The composite good can be either consumed or accumulated. Its accumulated stock is held either in the form of physical capital, the stock of which at time t is denoted $K(t)$, or in the form of a "bond", the stock of which is $B(t)$. This bond is assumed to reproduce itself at the given exogenous and riskless rate r, which will represent the force of interest in the economy. [4] The accumulated stock of capital is used as an input both in the production of the composite good and in the extraction of the nonrenewable resource.

The ultimate stock of the resource available is assumed given and known. The reserves left in the ground at time t will be denoted $X(t)$. Being durable, the resource, once extracted, accumulates above ground in the form of a stock which depreciates at the constant rate $\delta \geq 0$. This durable above ground stock, $Q(t)$, yields a flow of services which enters the production of the composite good along with the services of physical capital. There are therefore four assets in which the wealth of this economy can be held, at any given time : bonds, capital, reserves of the natural resource, and the above ground resource stock. The latter two assets have the particularity that the above ground stock can only be increased by reducing the in ground reserves. The resource being nonrenewable, its reserves cannot be increased and the decision to extract is irreversible. As for the stock of the composite good, it will be assumed costlessly transferable between its three uses. For simplicity, the stock of capital will be assumed not to depreciate.

The production process and the extraction process are both assumed to have a stochastic element. More precisely, if $y(t)$ denotes the production of the composite good

4. For notational convenience we will treat r as time-invariant. Doing so does not affect our results. It will become clear that it can just as well be thought of as an exogenous time path, which could also be stochastic.

and $x(t)$ the extraction of the resource, then :

$$y(t) = F(K_y(t), Q(t), \theta_1(t)) \tag{3.1}$$

$$x(t) = G(K_x(t), \theta_2(t)), \tag{3.2}$$

where $K_y(t) + K_x(t) = K(t)$. The state variables $\theta_1(t)$ and $\theta_2(t)$ can be viewed as stochastic productivity indices. They will be assumed to evolve exogenously according to the following Itô processes :

$$\frac{d\theta_i(t)}{\theta_i(t)} = \mu_i dt + \sigma_i dZ_i(t), \qquad i = 1,2; \qquad \forall t, \tag{3.3}$$

with $dZ_i(t) = \xi_i\sqrt{dt}$, $\xi_i \sim N(0,1)$, $cov(d\theta_1, d\theta_2) = \sigma_{12}dt + o(dt)$ and $\sigma_{12} = \sigma_1\sigma_2 cov(\xi_1, \xi_2)$. The drifts μ_i and the variance σ_i could depend on time t and the state variables.

We will assume $F_1 > 0$, F_{1Q} and $F_{1K} > 0$, where the subscript 1 denotes the partial derivative of $F(\cdot)$ with respect to θ_1. We will also assume $F_K > 0$, $F_Q > 0$, $F_{KK} < 0$, $F_{QQ} < 0$, $F_{QK} > 0$, and $\lim_{K\to 0} F_K(K,Q) = \infty$, $\lim_{Q\to 0} F_K(K,Q) = \infty$. Under those conditions it will take an infinite time to exhaust the reserves.

As for the extraction process, it will be assumed, for simplicity, to be given by :

$$G(K_x, \theta_2) = \frac{K_x}{\gamma(\theta_2)}. \tag{3.4}$$

The function $\gamma(\theta_2)$ tells us how many units of capital is required to extract a unit of the resource. Hence the cost of extraction will be $r\gamma(\theta_2)x(t)$, r being the opportunity cost of capital, and $\gamma(\theta_2)x(t)$ the quantity of capital in use as input. It is assumed that $\gamma'(\theta_2) < 0$, $\lim_{\theta_2\to 0}\gamma(\theta_2) = \infty$, and $\lim_{\theta_2\to +\infty}\gamma(\theta_2) = 0$.

The representative consumer derives utility $U(c(t))$ from consuming the composite good at the rate $c(t)$. This utility function satisfies $U(c) > 0$, $U'(c) > 0$, $U''(c) < 0$, and $\lim_{c\to 0} U'(c) = \infty$. The consumer discounts the utility flows at the constant rate α.

It will be assumed that all agents in this economy behave as price takers, both in the goods and in the assets markets. Consumers are assumed to be the owners of the assets in the economy. In deciding on their consumption and on their portfolio, they transmit demand prices to the composite good producers and the resource extractors, who take them as given in making their decisions. Their production and extraction decisions then enter the determination of the rates of return on the assets, which the consumers take as given in making their own decisions. These prices and returns are taken to be those that equilibrate the markets when production, extraction and consumption take place simultaneously.

In the next two sections we derive necessary efficiency conditions for the extraction of the resource and for the production of the composite good. These will generate the rates of return on assets that will enter the wealth constraint to the consumer's intertemporal optimization problem.

3.3 Efficient resource extraction

The typical price-taking resource extraction firm chooses its rate of extraction so as to maximize the expected present value of the flow of profits over time. Those profits are measured in monetary units (we will call them "utils") and are discounted at the constant rate α. Let $p(t)$ denote the market flow price of a unit of the resource, measured in units of the composite good, and let $q(t)$ denote the demand price (in utils) of the composite good. The extraction firm takes those prices as given in making its extraction decision. We will assume for now that those prices evolve as Itô processes of the same form as the productivity indices. It will be shown in the Appendix that this is indeed the case of the equilibrium outcomes for $p(t)$ and $q(t)$. Therefore :

$$\frac{dp(t)}{p(t)} = \mu_p(t)dt + \sigma_p(t)dZ_p(t) \tag{3.5}$$

$$\frac{dq(t)}{q(t)} = \mu_q(t)dt + \sigma_q(t)dZ_q(t), \tag{3.6}$$

where again $dZ_i(t) = \xi_i\sqrt{dt}$ and $\xi_i \sim N(0, s_i)$, $i = p, q$.

The current value function for this problem is :

$$V(X(t), p(t), q(t), \theta_2(t)) = \max_{\{x(s)|s\in[t,\infty)\}} E_t \int_t^\infty e^{-\alpha(s-t)} q(s)\,[p(s) - r\gamma(\theta_2(s))]\,x(s)ds, \tag{3.7}$$

where the maximization is subject to $X(t)$, (3.3), (3.5) and (3.6), as well as to the resource constraint

$$dX(t) = -x(t)dt, \qquad X(0) = X_0.$$

Notice that since the demand price $q(t)$ is taken to be the marginal utility of consumption ($U'(c)$), and the discount rate α is taken to be that of the representative consumer, this value function can be interpreted as measuring the present value of the dividend stream accruing to the representative consumer as the ultimate owner of the reserves (see Duffie, 1988, chap. 25). In maximizing this present value, the extraction firm therefore takes into account the preferences of the consumer-owner, including his attitude towards risk. [5] It is therefore as if the resource firm (i.e. the manager) were acting on behalf of the consumer (i.e. the owner) to maximize the present value of the dividend stream, thus resulting in a consumption-efficient outcome.

Let V_i denote the derivative with respect to argument i, for $i, j = p, q, \theta_2$, and let $\Delta_i = i$. Then the Bellman equation associated to this time-autonomous problem is :

$$\alpha V = \max_x \left[q[p - r\gamma]x - V_X x + \sum_i \Delta_i \mu_i V_i + \frac{1}{2}\sum_{i,j} \Delta_i \Delta_j \sigma_i \sigma_j V_{ij} \right]. \tag{3.8}$$

5. Notice that if consumers were risk neutral, then $q(t)$ would be independent of $c(t)$ and whether the dividend stream is valued in terms of utils or in terms of the composite good would be irrelevant. That is not the case however if the consumer is risk averse, as it will be generally assumed here.

An interior solution for x will satisfy the following necessary condition :

$$V_X = q(p - r\gamma). \tag{3.9}$$

Making use of (3.9) in differentiating both sides of (3.8) with respect to X, we get

$$\begin{aligned} \alpha V_X &= -V_{XX}x + \sum_i \Delta_i \mu_i V_{Xi} + \frac{1}{2}\sum_{i,j} \Delta_i \Delta_j \sigma_i \sigma_j V_{Xij} \\ &= \frac{1}{dt} E_t(dV_X), \end{aligned} \tag{3.10}$$

where the second line is obtained using Itô's lemma, $(1/dt)E_t(\cdot)$ being Itô's differential operator. From (3.9) and (3.10) we therefore derive the following condition for efficient extraction of the natural resource :

$$\frac{1}{\pi}\frac{1}{dt}E_t(d\pi) = \alpha, \tag{3.11}$$

where

$$\pi = q[p - r\gamma] \tag{3.12}$$

is the marginal profit, or net price of the resource in the ground (also called the *in situ* price, the asset price, or the resource rent). It is expressed in utils, as is condition (3.11). The condition therefore says that the discounted marginal profit from extraction, expressed in utils, must be constant over time, thus assuring indifference between extracting the marginal unit of the resource and leaving it in the ground. It can be viewed as a *partial equilibrium* stochastic version of Hotelling's rule.

This partial equilibrium stochastic arbitrage condition is the same as that found by Gaudet and Khadr (1991) in the case of a non durable resource. The durability property of the resource will however intervene, since, in the "general" equilibrium, $p(t)$ and $q(t)$ will depend on $Q(t)$, the above ground stock of the resource which is used as input in the production of the composite good, to which we now turn.

3.4 Efficient production of the composite good

As already noted, the production of the composite good is a function of the services of capital and the services of the above ground stock of the durable resource. We will assume that there are no costs of adjustment. The stock of capital will therefore simply be adjusted instantaneously to its desired level. As for the above ground stock of the resource, it is adjusted by purchasing the flow $x(t)$ extracted by the resource sector, but it is subject to depreciation at the rate δ. It therefore evolves over time according to

$$dQ(t) = [x(t) - \delta Q(t)]dt. \tag{3.13}$$

The representative firm producing the composite good acts as a price-taker in choosing its level of capital, $K_y(t)$, and its rate of investment in the above stock of the resource, $x(t)$. The associated current value function, expressed in utils, is

$$\Gamma(Q(t), p(t), q(t), \theta_1(t)) = \max_{\{K_y(s), x(s) | s \in [t,\infty)\}} E_t \int_t^{\infty} e^{-\alpha(s-t)} q(s) \left[F(K_y(s), Q(s), \theta_1(s)) - rK_y(s) - p(s)x(s)\right] ds, \tag{3.14}$$

where the maximization is subject to (3.13), (3.5) and (3.6). For the reasons already mentioned in the previous section in the case of the resource extraction firm, this representative firm can be thought of as managing the production of the composite good on behalf of the consumer-owner so as to maximize the present value of the resulting dividend stream, hence generating a consumption-efficient outcome.

The Bellman equation associated to this optimization problem is

$$\alpha\Gamma = \max_{K_y, x} \left[q[F - rK_y - px] + \Gamma_Q(x - \delta Q) + \sum_i \Delta_i \mu_i \Gamma_i + \frac{1}{2} \sum_{i,j} \Delta_i \Delta_j \sigma_i \sigma_j \Gamma_{ij} \right], \tag{3.15}$$

for $i, j = p, q, \theta_1$ and $\Delta_i = i$. The first-order necessary conditions for the maximization of the right-hand side are

$$F_K = r \tag{3.16}$$

$$\Gamma_Q = qp. \tag{3.17}$$

Differentiating both sides of (3.15) with respect to Q, making use of (3.16) and (3.17) and of Itô's lemma, we find that :

$$\begin{aligned} \alpha\Gamma_Q &= qF_Q - \Gamma_Q\delta + \Gamma_{QQ}(x - \delta Q) + \sum_i \mu_i \Delta_i \Gamma_{Qi} + \frac{1}{2}\sum_{i,j} \Delta_i \Delta_j \sigma_i \sigma_j \Gamma_{Qij} \\ &= qF_Q - \Gamma_Q\delta + \frac{1}{dt}E_t(d\Gamma_Q). \end{aligned} \tag{3.18}$$

From (3.17) and (3.18) it follows that the optimal holding of the resource stock as input in the production of the composite good must satisfy

$$\frac{F_Q}{p} - \delta + \frac{1}{qp}\frac{1}{dt}E_t d(qp) = \alpha. \tag{3.19}$$

Recall that qp is the gross market price of the resource expressed in utils. The left-hand side of this arbitrage condition represents the marginal return at date t from holding the stock of resource $Q(t)$ above ground : its marginal product, corrected for the rate of depreciation and for the expected capital gains to be made from holding it. Since the right-hand side is constant, so must be the left-hand side : the marginal return must be the same at each date, leaving the owner indifferent between adding another unit to the stock above ground or leaving it in the ground to be exploited at a future date. This condition is of course specific to the fact that the resource is durable.

The efficiency conditions derived in this section and the previous one serve to determine the rates of return on assets that will be used to define the intertemporal stochastic opportunity set of the consumer. We now turn to the consumer's optimization problem.

3.5 The consumer's consumption and portfolio decisions

Consumers, as owners of the assets in the economy, decide both on how much to consume at each date and how to allocate their wealth between capital, bonds, *in situ* resources, and above ground resources. Those consumption and asset demands serve to generate the price signals that the producers take into account in making their own decisions.

Except for the bond, whose instantaneous rate of return r is riskless, the asset returns can be expected to evolve as a stochastic process of the following form :

$$dR_i(t) = \mu_i(t)dt + \sigma_i(t)dZ_i(t), \qquad i = K, X, Q. \tag{3.20}$$

As in the case of the prices of the previous two sections, it will be shown in the Appendix that those stochastic processes are indeed compatible with equilibrium. As for the riskless asset, its return will be denoted

$$dR_B(t) = rdt. \tag{3.21}$$

We know however that $\mu_k = F_K$ and that, from the efficiency condition (3.16) for the production of the composite good, one of the equilibrium conditions will be $F_K = r$. It follows that in equilibrium we must have $\sigma_K = 0$. Therefore we may write

$$dR_K(t) = dR_B(t) = dR(t) = rdt. \tag{3.22}$$

In other words, the return on the accumulated stock of the composite good must be the same at all times in both of its uses.

Denote by $\lambda(t)$ the asset price of a unit of reserves expressed in terms of the composite good. The consumers total wealth at time t, $W(t)$, will therefore be given by

$$W(t) = K(t) + B(t) + \lambda(t)X(t) + p(t)Q(t). \tag{3.23}$$

The first three elements of wealth are the same as in Gaudet and Khadr (1991). Because of the durability property of the resource, a fourth element now appears, namely $Q(t)$, which is valued at the gross market price of the resource in term of the composite good, $p(t)$.

Differentiating (3.23) totally with respect to time we obtain the consumer's stochastic wealth constraint,

$$dW(t) = -c(t)dt + W(t)\left[\omega_X(t)dR_X(t) + \omega_Q(t)dR_Q(t) + (1 - \omega_X(t) - \omega_Q(t))dR(t)\right], \tag{3.24}$$

where $\omega_X(t)$ and $\omega_Q(t)$ are respectively the share of the representative consumer's wealth invested in the stock of resource below ground and above ground, and $c(t)$ is consumption.

The representative consumer's current value function is then

$$J(W(t), \theta_1(t), \theta_2(t)) = \max_{\{c(s), \omega_X(s), \omega_Q(s) | s \in [t, \infty)\}} \int_t^{\infty} e^{-\alpha(s-t)} U(c(s)) ds, \tag{3.25}$$

where the maximization is subject to (3.24), (3.20), and (3.22), as well as to the state $(W(t), \theta_1(t), \theta_2(t))$ inherited at date t.

The corresponding Bellman equation is given by

$$\begin{aligned} \alpha J &= \max_{c, \omega_X, \omega_Q} \Big[U(c) + \{W(\omega_X \mu_X + \omega_Q \mu_Q + (1 - \omega_X - \omega_Q) r) - c\} J_W \\ &+ \frac{1}{2} \sum_{k,l} \omega_k \omega_l \sigma_{kl} W^2 J_{WW} + \frac{1}{2} \sum_i \sum_l \omega_l \Delta_i \sigma_{li} W J_{Wi} + \sum_i \Delta_i \mu_i J_i + \frac{1}{2} \sum_{i,j} \Delta_i \Delta_j \sigma_{ij} J_{ij} \Big], \end{aligned} \tag{3.26}$$

for $i, j = \theta_1, \theta_2$, $l, k = X, Q$, and $\Delta_i = i$. Note that $\sigma_{Xi} = cov(dR_X, d\theta_i)$ and

$\sigma_{Qi} = cov(dR_Q, d\theta_i)$.

The following conditions must hold for an interior solution for c, ω_X, and ω_Q :

$$J_W = U'(c) \tag{3.27}$$

$$J_W(\mu_X - r) + J_{WW}W\left(\omega_X\sigma_X^2 + \omega_Q\sigma_{XQ}\right) + \sum_i \Delta_i\sigma_{Xi}J_{Wi} = 0 \tag{3.28}$$

$$J_W(\mu_Q - r) + J_{WW}W\left(\omega_Q\sigma_Q^2 + \omega_X\sigma_{XQ}\right) + \sum_i \Delta_i\sigma_{Qi}J_{Wi} = 0. \tag{3.29}$$

Condition (3.27) is the usual envelope condition, while (3.28) and (3.29) jointly relate the shares ω_X and ω_Q of the consumer's wealth held in the risky assets $X(t)$ and $Q(t)$ to their excess returns over the riskless rate, their variances and covariances.

3.6 Evolution of the asset and the flow prices of the resource

In the case of a non-durable resource, the Hotelling rule is the sole condition that determines the evolution of the *in situ* value of the resource and, as a result, of the market flow price of the resource. As shown in Gaudet and Khadr (1991), this intertemporal arbitrage condition can be viewed as an equilibrium asset-pricing condition. In the case of a durable resource, the Hotelling rule must still hold, but it is not anymore the only assets market equilibrium condition, since the resource can also be held above ground as a productive asset once extracted. Those two equilibrium conditions will now simultaneously play a role in determining the evolution of the equilibrium resource price.

The two conditions have already been encountered in a partial equilibrium form as efficiency conditions (3.11) and (3.19). We will now use those two conditions along with the consumer's optimality conditions just derived to establish their interpretation as equilibrium asset-pricing rules. To do this, first differentiate both sides of the Bellman equation (3.26) with respect to W, to get

$$
\begin{aligned}
\alpha J_W &= \Big[\big(W\big(\omega_X\mu_X+\omega_Q\mu_Q+(1-\omega_X-\omega_Q)r\big)-c\big)J_{WW} \\
&+ \frac{1}{2}\sum_{k,l}\omega_k\omega_l\sigma_{kl}W^2J_{WWW}+\sum_i\sum_l\omega_l\Delta_i\sigma_{li}WJ_{WWi} \\
&+ \sum_i\Delta_i\mu_iJ_{Wi}+\frac{1}{2}\sum_{i,j}\Delta_i\Delta_j\sigma_{ij}J_{Wij}\Big] \\
&+ \Big[\big(\omega_X\mu_X+\omega_Q\mu_Q+(1-\omega_X-\omega_Q)r\big)J_W \\
&+ \sum_{k,l}\omega_k\omega_l\sigma_{kl}WJ_{WW}+\sum_i\sum_l\omega_l\Delta_i\sigma_{li}J_{Wi}\Big],
\end{aligned} \tag{3.30}
$$

for $i,j=\theta_1,\theta_2$; $l,k=X,Q$ and $\Delta_i=i$. Using Itô's lemma, we verify that the first three lines of the right-hand side are simply $(1/dt)E_tdJ_W$. Condition (3.30) can therefore be rewritten

$$
\begin{aligned}
\alpha J_W &= \frac{1}{dt}E_tdJ_W \\
&+ \Big[\big(\omega_X\mu_X+\omega_Q\mu_Q+(1-\omega_X-\omega_Q)r\big)J_W+\sum_{k,l}\omega_k\omega_l\sigma_{kl}WJ_{WW}+\sum_i\sum_l\omega_l\Delta_i\sigma_{li}J_{Wi}\Big] \\
&= \frac{1}{dt}E_tdJ_W \\
&+ rJ_W \\
&+ \omega_XJ_W(\mu_X-r)+J_{WW}W\left(\omega_X^2\sigma_X^2+\omega_X\omega_Q\sigma_{XQ}\right)+\sum_i\omega_X\Delta_i\sigma_{Xi}J_{Wi} \\
&+ \omega_QJ_W(\mu_Q-r)+J_{WW}W\left(\omega_Q^2\sigma_Q^2+\omega_Q\omega_X\sigma_{XQ}\right)+\sum_i\omega_Q\Delta_i\sigma_{Qi}J_{Wi}
\end{aligned} \tag{3.31}
$$

Substituting for the necessary conditions (3.27), (3.28) and (3.29) for an optimal consumption-portfolio choice, this reduces to

$$
\frac{1}{U'}\frac{1}{dt}E_tdU'=\alpha-r. \tag{3.32}
$$

Notice next that since $q=U'(c)$ and $\lambda=p-r\gamma$, (3.11) and (3.19) can be rewritten respectively as

$$\frac{1}{U'\lambda}\frac{1}{dt}E_t d(U'\lambda) = \alpha \tag{3.33}$$

and

$$\frac{F_Q}{p} - \delta + \frac{1}{U'p}\frac{1}{dt}E_t d(U'p) = \alpha. \tag{3.34}$$

Using (3.32) to eliminate α, we find that the asset-pricing equilibrium requires that the following two conditions be satisfied simultaneously :

$$\frac{1}{U'\lambda}\frac{1}{dt}E_t d(U'\lambda) - \frac{1}{U'}\frac{1}{dt}E_t dU' = r \tag{3.35}$$

and

$$\frac{F_Q}{p} - \delta + \frac{1}{U'p}\frac{1}{dt}E_t d(U'p) - \frac{1}{U'}\frac{1}{dt}E_t dU' = r. \tag{3.36}$$

Condition (3.35) is the equilibrium asset-pricing formulation of the stochastic Hotelling rule. It is the same as that found in Gaudet and Khadr (1991) for a non durable resource ; it applies to durable as well as non-durable resources. The left-hand side measures the expected rate of return on holding the marginal unit of the resource in the ground : the rate of growth of the value of the marginal unit of *in situ* resource measured in utility terms, corrected for the rate of change of the marginal utility of consumption. This expected rate of return must equal the "rate of interest" r, which is the return that can be obtained by holding wealth in the form of the composite good instead of in the form of resources in the ground. Notice that if (and only if) the representative consumer were risk neutral, so that U' was constant, then the condition reduces to simply equating the expected rate of growth of the *in situ* price, $(1/\lambda)(1/dt)E_t(d\lambda)$, to the rate of interest. But in the case of risk averse consumers, this is not sufficient : account must then be taken of the rate of change in the marginal utility of consumption. Notice also that even with a non-linear utility function, if there is no uncertainty in the investment prospects, then the condition reduces to $(d\lambda/dt)/\lambda = r$, which is the usual formulation of the basic Hotelling rule in a deterministic context.

In the case of a durable resource, the Hotelling rule (3.35) is not sufficient to characterize the evolution of the resource price. In that case, condition (3.36), which is specific to durable resources, must hold simultaneously with the Hotelling rule. The condition expresses the fact that the return on the stock of the resource accumulated above ground as a productive asset must, at the margin, be equal to the return that can be obtained by accumulating the composite good instead, either in the form of capital of bond, which in equilibrium both yield the rate of return r. Indeed, the left-hand side of (3.36) is the return on the marginal unit of the resource accumulated above ground : the marginal product of its services in the production of the composite good, minus the rate of depreciation of the stock, plus the expected rate of change in the resource price valued in utils, corrected for the rate of change in the marginal utility of consumption.

Notice that since the right-hand sides of (3.35) and (3.36) are the same, both left-hand sides must be equal : there must, in equilibrium, be indifference between holding the resource below ground or above ground.

It is useful to rewrite those conditions with the rate of change of the *in situ* price and the flow price expressed directly in terms of the composite good rather than in utility terms. To do this, we first use Itô's lemma to obtain

$$\frac{1}{U'\lambda}\frac{1}{dt}E_t d(U'\lambda) = \frac{1}{\lambda}\frac{1}{dt}E_t d\lambda + \frac{U''}{U'}\frac{1}{dt}E_t dc + \frac{1}{2}\frac{U'''}{U'}\frac{1}{dt}E_t(dc)^2 + \frac{U''}{U'\lambda}\frac{1}{dt}E_t(d\lambda, dc) \tag{3.37}$$

$$\frac{1}{U'p}\frac{1}{dt}E_t d(U'p) = \frac{1}{p}\frac{1}{dt}E_t dp + \frac{U'''}{U'}\frac{1}{dt}E_t dc + \frac{1}{2}\frac{U'''}{U'}\frac{1}{dt}E_t(dc)^2 + \frac{U''}{U'p}\frac{1}{dt}E_t(dp, dc) \tag{3.38}$$

$$\frac{1}{U'}\frac{1}{dt}E_t dU' = \frac{U''}{U'}\frac{1}{dt}E_t dc + \frac{1}{2}\frac{U'''}{U'}\frac{1}{dt}E_t(dc)^2. \tag{3.39}$$

Substituting from (3.37), (3.38) and (3.39) into (3.35) and (3.36), we find that

$$\frac{1}{\lambda}\frac{1}{dt}E_t d\lambda = r + A(c)\sigma_{\lambda c} \tag{3.40}$$

$$\frac{F_Q}{p} - \delta + \frac{1}{p}\frac{1}{dt}E_t dp = r + A(c)\sigma_{pc}, \tag{3.41}$$

where $A(c) = -U''c/U'$ is the measure of relative risk aversion and $\sigma_{\lambda c} = (1/dt)E_t(d\lambda/\lambda, dc/c)$ and $\sigma_{pc} = (1/dt)E_t(dp/p, dc/c)$ are respectively the covariances of the rate of growth of consumption with the rate of growth of the *in situ* price and the rate of growth of the flow price of the resource (measured in terms of the composite good). Thus the assets market equilibrium requires that the expected rate of change of the *in situ* price, which is the expected return on the below ground reserves, $X(t)$, must be equal to the rate of interest corrected for the consumer's degree of risk aversion multiplied by the covariance between the rate of growth of the *in situ* price and the rate growth of consumption. Since the above ground resource is durable, it also requires that the rate of return on its stock, $Q(t)$, be equal to the same rate of interest corrected for the measure of the consumer's risk aversion multiplied by the covariance between the rate of growth of the market flow price and the rate of growth of consumption.

Notice that if U'' is negative, as is being assumed, the measure of relative risk aversion is positive, the consumer being risk averse. This means that the second term on the right-hand side of each equation will take on the sign of the relevant covariance. For instance, if $\sigma_{\lambda c}$ is positive, so that a high (low) return on holding the resource stocks in the ground tends to be associated with a high (low) rate of growth in consumption, then holding reserves in the ground is a relatively risky investment and requires a return that exceeds the riskless rate r. The same can be said for holding resource stocks above ground if σ_{pc} is positive. On the other hand, if $\sigma_{\lambda c}$ is negative, then holding resources in the ground constitutes a form of insurance against adverse results concerning the growth of consumption. The rate of return on those reserves will then be lower than the riskless rate r. It may in fact be negative if, for any given degree of risk aversion, the covariance is sufficiently negative, or if, for any given negative covariance, the consumer is sufficiently risk averse. The same can be said of investment in above ground stocks of the resource when σ_{pc} is negative.

The covariances σ_{pc} and $\sigma_{\lambda c}$ are of course related. Indeed, since $p = \lambda + r\gamma$, we will have

$$p\sigma_{pc} = \lambda\sigma_{\lambda c} + r\gamma\sigma_{\gamma c} \tag{3.42}$$

or, written differently,

$$\sigma_{pc} = \left(1 - \frac{r\gamma}{p}\right)\sigma_{\lambda c} + \frac{r\gamma}{p}\sigma_{\gamma c}. \tag{3.43}$$

Hence σ_{pc} is the weighted sum of $\sigma_{\lambda c}$ and $\sigma_{\gamma c}$, with the weights being respectively the share in the price (p) of the rent (λ) and of marginal extraction cost ($r\gamma$). Thus σ_{pc} and $\sigma_{\lambda c}$ can be of different signs only if $\sigma_{\lambda c}$ and $\sigma_{\gamma c}$ are of different signs. Furthermore, with positive extraction cost, we will have $\sigma_{pc} = \sigma_{\lambda c}$ if and only if $\sigma_{\lambda c} = \sigma_{\gamma c}$, and hence $\sigma_{pc} = \sigma_{\gamma c}$.

Eliminating $A(c)$ from (3.40) and (3.41) and using (3.42) and the fact that

$$\frac{1}{\lambda}\frac{1}{dt}E_t d\lambda = \frac{p}{\lambda}\frac{1}{p}\frac{1}{dt}E_t dp - \frac{r\gamma}{\lambda}\frac{1}{\gamma}\frac{1}{dt}E_t d\gamma,$$

we find that

$$\frac{1}{p}\frac{1}{dt}E_t dp = \frac{\lambda}{r\gamma}\left\{\left(\frac{\sigma_{pc} - \sigma_{\lambda c}}{\sigma_{\gamma c}}\right)r + \frac{\sigma_{\lambda c}}{\sigma_{\gamma c}}\left(\frac{F_Q}{p} - \delta\right)\right\} + \frac{\sigma_{pc}}{\sigma_{\gamma c}}\frac{1}{\gamma}\frac{1}{dt}E_t d\gamma, \tag{3.44}$$

which is the expression for the expected rate of change of the flow price of the resource. [6] It is interesting to compare this expression for the expected rate of price change to its

6. This expression could also be written as :

$$\frac{1}{p}\frac{1}{dt}E_t dp = \left(1 - \frac{r\gamma}{p}\right)\left\{\left(1 - \frac{\sigma_{\lambda c}}{\sigma_{\gamma c}}\right)r + \frac{\sigma_{\lambda c}}{\sigma_{\gamma c}}\left(\frac{F_Q - \delta p}{r\gamma} + \frac{1}{\gamma}\frac{1}{dt}E_t d\gamma\right)\right\} + \frac{r\gamma}{p}\frac{1}{\gamma}\frac{1}{dt}E_t d\gamma. \tag{3.45}$$

On this form, we can remark that the long run behaviour of the resource price will depend on three factors : the interest rate, r ; the productivity per marginal extraction cost of a unit of resource, $F_Q - \delta p/r\gamma$ and the expected marginal extraction cost growth rate $\frac{1}{\gamma}\frac{1}{dt}E_t d\gamma$. It is interesting to compare this expression for the expected rate of price change to its equivalent for non durable resource reported in 3.47, where the long run behaviour of the resource price depends only on the interest rate and the consumer risk aversion rate.

equivalent in the deterministic case, which is

$$\frac{1}{p}\frac{dp}{dt} = \frac{\lambda}{r\gamma}\left(\frac{F_Q}{p} - \delta\right) + \frac{1}{\gamma}\frac{d\gamma}{dt}. \tag{3.46}$$

If and only if $\sigma_{pc} = \sigma_{\lambda c} = \sigma_{\gamma c}$ will (3.44) yield a somewhat similar result, with, however, the important distinction that the deterministic rates of change of price and of cost being replaced by their *expected values*, since the uncertainty remains. It is, however, highly unlikely that all three covariances with consumption will take the exact same value.

In the general case, the behavior of the expected price path will therefore depend critically on the relative values of the three covariances and will be highly unpredictable. To illustrate, suppose that the rate of depreciation of the above ground stock is sufficiently small so that $F_Q > \delta p$, and that the expected rate of growth of cost is negative (through technological progress).[7] Assume also that $\sigma_{\lambda c} > 0$, so that the return on holding reserves in the ground is positively correlated with the rate of change of consumption, and that the reverse is true of the rate of change of costs, so that $\sigma_{\gamma c} < 0$. Note that the latter assumption implies that positive technological change in resource extraction tends to occur when the economy is performing well in terms of consumption, since $\sigma_{\gamma c}$ is negatively related to σ_{2c}.[8] Under those assumptions, it can be seen from (3.42) that $\sigma_{pc} - \sigma_{\lambda c} < 0$, so that both terms on the right-hand side of (3.44) are negative and so is the expected rate of change of the resource. If we assume instead that $\sigma_{\lambda c} < 0$, so that holding resources in the ground tends to be viewed as insurance against unfavorable performances in consumption, and that $\sigma_{2c} < 0$ and hence $\sigma_{\gamma c} > 0$, then the sign of the first term is ambiguous and so is that of the second term (since σ_{pc} may well be negative). In such a case, the sign of the expected rate of price change cannot be determined analytically. Analytical indeterminacy will obviously persist if $\sigma_{\lambda c}$ and $\sigma_{\gamma c}$ happen to be

7. Note that the reverse assumption is also plausible if there is an important depletion effect on extraction cost that dominates any effect of technological progress.

8. In fact, $\sigma_{\gamma c} = \frac{\gamma'}{\gamma}\theta_2\sigma_{2c}$, and $\gamma' < 0$.

of the same sign, whether positive or negative.

The expression in (3.44) for the expected rate of price change is also quite different from the one that arises when the resource is a non durable, analyzed in Gaudet and Khadr (1991), namely

$$\frac{1}{p}\frac{1}{dt}E_t dp = \left(1 - \frac{r\gamma}{p}\right)\left(r + A(c)\sigma_{\lambda c}\right) + \frac{r\gamma}{p}\frac{1}{\gamma}\frac{1}{dt}E_t d\gamma. \tag{3.47}$$

In that case the expected price change is simply a weighted average of the rate of interest adjusted for the risk aversion factor and the rate of change of the cost of extraction, where the weights are respectively the share of the rent in price and the share of costs in price. This is to be compared to the well known basic pricing equation that arises from the Hotelling rule in the deterministic case, namely

$$\frac{1}{p}\frac{dp}{dt} = \left(1 - \frac{r\gamma}{p}\right) r + \frac{r\gamma}{p}\frac{1}{\gamma}\frac{d\gamma}{dt} \tag{3.48}$$

Thus, even in the case of a non durable resource, stochasticity in the production processes has an important impact on the equilibrium behavior of the resource price. Let us assume the rate of change of the cost of extraction to be negative. Then, in the deterministic case, the price may be declining at first since the second term may dominate the first one for low levels of rent, but it must eventually be increasing as the share of the rent in the price increases and that of cost decreases. Thus the resource price path will be either continuously increasing or be U-shaped, and therefore will necessarily end up increasing. Things are different in the stochastic case. Indeed, if $\sigma_{\lambda c}$ is negative, which means that favorable returns on the *in situ* resource stock tend to be associated to unfavorable performances of the economy (as captured by the growth in consumption), then the return expected from holding the resource stock in the ground will be smaller than the rate of interest, since holding the resource stock then appears as a form of insurance against bad prospects for consumption. In fact there is nothing to prevent the first term from being negative, since $r + A(c)\sigma_{\lambda c}$ may well be negative if $A(c)\sigma_{\lambda c}$ is sufficiently large, in which case, if the expected rate of change of extraction cost is also negative, the expected rate of change in the price will be negative independently of the share of the rent in the price. As is clear from (3.44), the durability of the resource further highlights the need to take into account uncertainty when attempting to characterize empirically the evolution of resource prices.

Note finally that since, by definition, $\mu_X = (1/\lambda)(1/dt)E_t d\lambda$ and $\mu_Q = (F_Q/p) - \delta + (1/p)(1/dt)E_t dp$, respectively the expected rates of return on holding resource stocks respectively below ground and above ground, then from (3.40) and (3.41) we have that the expected instantaneous excess returns on holding those assets are written

$$\mu_X - r = A(c)\sigma_{\lambda c} \tag{3.49}$$

and

$$\mu_Q - r = A(c)\sigma_{pc}. \tag{3.50}$$

Furthermore, if there exists a reference market portfolio (denote it M) with the property that $\sigma_{Mc} \neq 0$, then we will also have $\mu_M - r = A(c)\sigma_{Mc}$. By substitution into (3.49) and (3.50) we then get

$$\mu_X - r = \beta_X(\mu_M - r) \tag{3.51}$$

and

$$\mu_Q - r = \beta_Q(\mu_M - r) \tag{3.52}$$

where $\beta_i = \sigma_{ic}/\sigma_{Mc}$, $i = \lambda, p$ are the well known “beta-coefficients”. A positive β_i implies that holding the resource in question constitutes a relatively risky investment, whereas the reverse is true if it is negative. Such specifications suggest that an asset pricing formulation of the nonrenewable resource exploitation problem can offer an interesting approach to estimating the temporal behavior of resource prices. [9]

9. There have been a few attempts, based at least in part on Gaudet and Khadr (1991), at using such an approach to estimate the Hotelling rule for *non durable* resources or, if durable, by treating it implicitly as non durable. See in particular Slade and Thille (1997) and Young and Ryan (1996), and more recently

3.7 Conclusion

The Hotelling rule is best viewed as an equilibrium condition in the assets market rather than simply an equilibrium condition in the flow market, as it very often is. Assets market equilibrium requires that holding a unit of the resource yield no more and no less than extracting it in order to invest in some other asset, thus irreversibly depleting the resource stock. Establishing the intertemporal assets market equilibrium in such a context requires that careful thought be given to what enters the return on holding the resource stock. As mentioned at the outset, there are a number of real world factors that make this task more complicated than it appears from the seminal paper of Hotelling, where the return could only be the capital gains it generates by holding it in the ground. As was shown in Gaudet and Khadr (1991) for non durable resources, not the least of those factors is uncertainty about future investment prospects. If in addition the resource is durable, depleting the *in situ* resource stock creates an above ground asset which, contrary to the *in situ* stock, yields a dividend in the form of productive services. This paper has shown that caution should be used in drawing analytical predictions about resource pricing behavior in the context of durable resources and stochastic investment opportunities. It is certainly too simplistic to imply from the most basic formulation of the Hotelling rule that the net price of the resource should be growing at the rate of interest, and it should be no surprise that observed resource prices do not behave in such a fashion. Our results highlight the importance for empirical studies of resource prices of taking account uncertainty about future investment prospects, and especially so in the case of durable resources. Of course, other factors, such as depletion effects on extraction costs and the structure of the resource markets are also very important in explaining the departure of the observed price behavior from the simple *r%* rule.

Kakeu (2010) who makes use of stock market data and financial econometric methods to estimate the beta coefficient for oil and gas. Empirical studies of resource price behavior that explicitly take into account the *durability* of the resource are to our knowledge still nonexistent.

CONCLUSION

Cette thèse a traité de questions liées au prix et au taux d'extraction optimal des ressources naturelles non-renouvelables dans divers contextes.

Le premier chapitre présente une approche novatrice d'estimation du prix in-situ des ressources naturelles non-renouvelable à partir du modèle d'Hotelling. Cette approche offre trois contributions principales. Au lieu d'utiliser les propriétés des séries chronologiques du prix du marché ou une estimation du prix in-situ pour évaluer le modèle d'Hotelling, nous combinons les conditions de premier ordre pour l'extraction optimale des ressources pour estimer une relation du prix du marché qui est cohérente avec le modèle d'Hotelling. Au lieu d'une approximation économétrique du coût marginal d'extraction, nous utilisons les données sur le coût moyen d'extraction pour obtenir une approximation du coût marginal nécessaire pour l'estimation du prix in-situ. Ces deux contributions peuvent être résumées en trois étapes. Tout d'abord, nous supposons une forme fonctionnelle pour le coût d'extraction qui montre comment utiliser le coût moyen d'extraction pour obtenir une approximation du coût marginal. Ensuite, nous combinons les conditions d'équilibre du modèle d'Hotelling pour estimer un prix de marché qui est cohérent avec ce modèle. Enfin, nous dérivons une estimation du prix in-situ correspondant. L'autre contribution de ce premier article est la technique économétrique utilisée pour évaluer le modèle d'Hotelling. Nous utilisons la Méthode des Moments Généralisés avec changement de régime appliquée aux données de panel. Cette technique d'estimation semble robuste pour tenir compte de l'endogénéité des coûts moyens d'extraction et de l'effet des changements structurels.

La méthodologie est utilisée pour analyser le prix de quatorze ressources non-renouvelables. Nous trouvons des résultats qui soutiennent fortement l'utilisation du modèle d'Hotelling comme un cadre d'analyse du comportement des marchés des ressources naturelles. Il ressort que le modèle d'Hotelling a un bon pouvoir explicatif des prix du marché observés. En utilisant des points de rupture appropriés pour évaluer la stabilité de nos estimations, nous identifions deux régimes : le premier est caractérisé par une croissance du prix in-situ et le second par une décroissance du prix in-situ. En utilisant un modèle à changement de régime pour réconcilier le modèle d'Hotelling avec le changement structurelle, nous trouvons que celui-ci n'a aucun impact significatif sur le pouvoir explicatif du modèle. Cependant, cette nouvelle spécification du modèle fournit des informations utiles sur le comportement des marchés des ressources naturelles. Il apparaît que le taux auquel les agents actualisent le futur est élevé dans le régime où le prix in-situ croît et faible dans le régime où le prix in-situ décroît. Ceci suggère que les agents donnent moins de valeur à des flux de trésorerie futurs lorsque leur bénéfice marginal est croissant et plus lorsque leur bénéfice marginal est décroissant. Par ailleurs, nous trouvons que le coût moyen d'extraction est une bonne approximation pour le coût marginal d'extraction : le coût marginal d'extraction ne semble pas être significativement différent du coût moyen d'extraction quelque soit le régime. Les estimations du prix in-situ, avec cohérence temporelle, décroissent ou présentent une forme en U inversé dans le temps et sont corrélées positivement au prix de marché. Par ailleurs, la différence entre le prix in-situ estimé sans changement de régime et celui estimé avec changement de régime est un processus de moyenne nulle.

Dans ce premier chapitre, des résultats encourageants ont été obtenus sur l'évolution du prix des ressources. Cependant, son cadre d'analyse ne nous permet pas de tirer des conclusions empiriques sur l'évolution du taux d'extraction optimale. Cette question est analysée dans le deuxième chapitre, dont le but principal est de tester empiriquement l'existence d'un équilibre dans lequel le taux d'extraction optimal est linéaire par rapport au stock de ressource en terre. Pour parvenir à cette fin, nous spécifions des formes fonctionnelles pour la demande et le coût d'extraction de la ressource. Nous montrons alors qu'il existe un équilibre dans lequel le taux d'extraction optimale est proportionnel au stock de ressource en terre si et seulement si le taux d'actualisation et les autres paramètres du modèle satisfont une relation bien spécifique. En utilisant les données de panel de quatorze ressources pour tester cette relation, nous trouvons qu'il n'y a pas lieu de rejeter l'hypothèse pour seulement six des ressources considérées. Cependant, en prenant en compte le changement structurelle dans le temps du prix des ressources dans nos estimations, nous trouvons que cette hypothèse est rejetée pour toutes les ressources. Ces résultats suggèrent qu'il serait approprié, dans les recherches futures, de considérer un modèle dans lequel la fraction de la ressource extraite à chaque période varie de manière continue dans le temps.

Le troisième chapitre caractérise l'évolution du prix des ressources *durables* et non-renouvelables dans une économie où les opportunités d'investissement sont stochastiques. Il faut noter que la règle d'Hotelling est mieux appréhendée comme une condition d'équilibre d'un marché des actifs plutôt que simplement une condition d'équilibre d'un marché des flux, comme c'est très souvent le cas. L'équilibre du marché des actifs exige que la détention d'une unité de la ressource ne donne pas plus et pas moins que ce qu'on obtiendrait si on l'avait extraite et investie dans d'autres actifs, diminuant ainsi de manière irréversible le stock de ressource en terre. L'établissement de l'équilibre intertemporel du marché des actifs dans un tel contexte nécessite que l'on prête attention aux déterminants du rendement d'une unité du stock de ressource détenue en terre. Tel que mentionné dans l'introduction, il y a un certain nombre de facteurs du monde réel qui rendent cette tâche plus compliquée qu'il n'en paraît dans l'article fondateur d'Hotelling , où le rendement ne peut être que le gain en capital que l'unité du stock de ressource génère lorsqu'elle est maintenue en terre. Tel que l'a montré Gaudet and Khadr (1991) pour les ressources non durables, l'un de ces facteurs est l'incertitude quant aux perspectives d'investissements futurs. Si en plus la ressource est durable, épuiser la ressource *in-situ* crée un actif sur le marché qui, contrairement au stock *in-situ*, génère un dividende sous la forme de services productifs.

Ce troisième chapitre a montré que l'on devrait faire preuve de beaucoup de prudence lorsqu'on élabore des prédictions analytiques sur le comportement du prix des ressources dans un contexte où la ressource est durable et les opportunités d'investissement stochastiques. Il est certainement trop simpliste d'affirmer à partir de la formulation la plus élémentaire de la règle d'Hotelling que le prix net de la ressource devrait croître au taux d'intérêt. Il n'est donc pas surprenant de constater que les prix des ressources observées ne se comportent pas tel que prédit. Nos résultats soulignent l'importance pour les études empiriques sur le prix des ressources de prendre en compte l'incertitude quant aux perspectives d'investissements futurs et tout particulièrement dans le cas des ressources durables. D'autres facteurs, tels que l'effet de l'épuisement du stock de la ressource sur les coûts d'extraction ainsi que la structure des marchés peuvent aussi expliquer l'écart entre le comportement observé du prix des ressources et la règle de r% d'Hotelling.

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Annexe I

Appendix to Chapter 1

I.1 Summary statistics for price, average cost and extraction rate data

Ores	OIL	NG	GOL	HC	SC	PHO	BAU
Price (1982-1984 US$ per ton)							
n.id.	87	85	58	72	41	33	32
n.obs.	2587	2570	1584	2008	1082	894	745
Mean	130.9	2073.6	$1.0*10^7$	39.0	10.0	36.9	30.6
Std.Dev.	69.6	900.5	$4.4*10^6$	17.0	4.3	21.3	10.9
Min	33.5	1138.8	$3.2*10^6$	19.1	5.0	20.6	13.2
Max	327.7	4482.2	$2.6*10^7$	79.6	21.0	120.7	49.9
Average extraction by country (Million tons)							
n.id.	87	85	58	72	41	33	32
n.obs.	2587	2570	1584	2008	1082	894	745
Mean	131.8	2102.9	$7.4*10^6$	39.8	10.5	38.2	14.0
Std.Dev.	71.8	920.7	$2.0*10^6$	17.4	17.4	22.6	4.3
Min	33.4	1138.8	$2.2*10^6$	19.1	5.0	20.6	5.1
Max	327.7	4482.2	$1.1*10^7$	79.6	21.0	120.7	27.0
Extraction rate by country (Million tons)							
n.id.	87	85	58	130.90	41	33	32
n.obs.	2587	2570	1584	130.90	1082	894	745
Mean	$3.8*10^7$	$7.8*10^5$	30.1	$2.3*10^7$	$4.7*10^7$	$4.1*10^6$	$3.4*10^6$
Std.Dev.	$1.0*10^8$	$3.4*10^6$	150.0	$5.4*10^7$	$1.5*10^8$	$1.0*10^7$	$6.8*10^6$
Min	307.6	1.2	$6.1*10^{-4}$	1275	0.5	4912	96.2
Max	$1.37*10^9$	$5.8*10^7$	$2.5*10^3$	$5.7*10^8$	$1.4*10^9$	$9.0*10^7$	$4.3*10^7$

Ores	COP	IRO	LEA	NIC	SIL	TIN	ZIN
Price (1982-1984 US$ per ton)							
n.id.	63	51	50	48	52	37	52
n.obs.	1437	1305	1184	789	1337	780	1234
Mean	$2.0*10^3$	40.4	697.6	$6.2*10^3$	$2.2*10^5$	$9.8*10^3$	981.8
Std.Dev.	724.6	10.4	282.1	$2.2*10^3$	$1.6*10^5$	$5.5*10^3$	316.6
Min	964.5	24.4	376.9	$2.8*10^3$	$9.8*10^4$	$3.1*10^3$	659.9
Max	$3.7*10^3$	63.90	$1.6*10^3$	$1.1*10^4$	$8.8*10^5$	$2.0*10^4$	$2.0*10^3$
Average extraction by country (Million tons)							
n.id.	63	51	50	48	52	37	52
n.obs.	1437	1305	1184	789	1337	780	1234
Mean	$1.0*10^3$	17.2	441.5	$4.2*10^3$	$1.6*10^5$	$9.8*10^3$	747.2
Std.Dev.	353.3	7.7	114.0	$1.0*10^3$	$1.0*10^5$	$5.5*10^3$	184.5
Min	152.0	4.4	220.50	$1.9*10^3$	$6.6*10^4$	$3.1*10^3$	463.0
Max	$2.6*10^3$	48.6	832.1	$6.9*10^3$	$4.8*10^5$	$2.0*10^4$	$1.0*10^3$
Extraction rate by country (Million tons)							
n.id.	63	51	50	48	52	37	52
n.obs.	1437	1305	1184	789	1337	780	1234
Mean	$1.8*10^5$	$1.9*10^7$	$8.5*10^4$	$4.6*10^4$	283.0	$9.7*10^3$	$1.5*10^5$
Std.Dev.	$4.0*10^5$	$2.1*10^7$	$1.6*10^5$	$8.6*10^4$	568.7	$2.0*10^4$	$3.1*10^5$
Min	1.9	65.62	99.6	42.0	0.4	0.6	22.6
Max	$4.0*10^6$	$1.3*10^8$	$1.4*10^6$	$7.1*10^5$	$4.2*10^3$	$1.9*10^5$	$3.2*10^6$

OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ;PHO, phosphate ;
COP, copper ; BAU, bauxite ; IRO, iron ; LEA, lead ;NIC nickel ; SIL, silver ; TIN, tin ; ZIN, zinc
Source : Unpublished World Bank data

I.2 Characterization of λ_t

From the equation (1.27), we derive :

$$E(S_{it}u_{it}) = E\left(E(S_{it}u_{it}/S_{it}, p_t, p_{t+1}, ...)\right) = E\left(S_{it}E(u_{it}/S_{it}, p_t, p_{t+1}, ...)\right) = 0. \quad \text{(I.1)}$$

Substituting in (I.1) for the error term u_{it} from (1.27), we get :

$$E(S_{it}u_{it}) = E(S_{it}(\lambda_t - \lambda_{it})) = 0. \quad \text{(I.2)}$$

Using the law of large number, it follows that :

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} S_{it}(\lambda_t - \lambda_{it}) = E(S_{it}(\lambda_t - \lambda_{it})) = 0 \quad \text{(I.3)}$$

Extracting the value of λ_t from above equation, the expression for the value of a unit of resource in ground is therefore :

$$\lambda_t = \frac{\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} S_{it}\lambda_{it}}{\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} S_{it}} = p_t - \alpha \frac{\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} S_{it}AC_{it}}{\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} S_{it}}, \quad \text{(I.4)}$$

as stated in (1.28). To obtain the dynamics of λ_t, extract the expression for λ_{it} from the static equation (1.27) and substitute it into the dynamic equation (1.26). Using the same approach as above, we get (1.30).

I.3 Newey West HAC estimator of matrix Σ

The asymptotic covariance matrix of the variable $W'_{it}f(y_{it};\theta_0)$ is given by :

$$\Sigma = \lim_{T\to\infty} \left(\Gamma(0) + \sum_{j=1}^{T-1} \left(\Gamma(j) + \Gamma'(j)\right)\right). \quad \text{(I.5)}$$

The Newey-West estimator of this covariance matrix takes the form :

$$\widehat{\Sigma}_{NW} = \widehat{\Gamma}(0) + \sum_{j=1}^{p}\left(1-\frac{j}{p+1}\right)\left(\widehat{\Gamma}(j)+\widehat{\Gamma}'(j)\right) \tag{I.6}$$

$$\widehat{\Gamma}(j) = \frac{1}{T}\sum_{t=j+1}^{T}\widehat{\varepsilon}_t\widehat{\varepsilon}_{t-j}W_t^{'}W_{t-j}. \tag{I.7}$$

Note that the optimal cutoff p is given by $T^{1/4}$.

I.4 Ledoit-Wolf well conditioned HAC estimator of Σ

Consider the Frobenius norm $||X|| = \sqrt{(tr\,(XX')\,/k}$, X being a $k \times T$ matrix whose associated inner product is $\langle X_1 X_2 \rangle = tr\,(X_1 X_2')\,/k$. The Ledoit-Wolf HAC estimator is given by :

$$\widehat{\Sigma}_{LW} = \frac{\hat{b}^2}{\hat{d}^2}\hat{m}I + \frac{\hat{a}^2}{\hat{d}^2}\widehat{\Sigma}_{NW}, \tag{I.8}$$

where $\widehat{\Sigma}_{NW}$ is a Newey West HAC estimator of Σ, I is the $k \times k$ identity matrix and the coefficients are given by :

$$\hat{m} = \langle \widehat{\Sigma}_{NW}, I \rangle \tag{I.9}$$

$$\hat{d}^2 = ||\widehat{\Sigma}_{NW} - \hat{m}I|| \tag{I.10}$$

$$\bar{b}^2 = \frac{1}{p}\sum_{j}^{p}\left|\left|\left(1-\frac{j}{p+1}\right)\left(\widehat{\Gamma}(j)+\widehat{\Gamma}'(j)\right) - \widehat{\Sigma}_{NW}\right|\right| \tag{I.11}$$

$$\hat{b}^2 = min\left(\bar{b}^2, \hat{d}^2\right) \tag{I.12}$$

$$\hat{a}^2 = \hat{d}^2 - \hat{b}^2 \tag{I.13}$$

I.5 Quasi-Gauss-Newton- Algorithm

Denote by $Q(\theta)$ the objective function. Since $Q(\theta)$ is twice continuously differentiable for M-estimators, there exists a second-order Taylor expansion :

$$Q_T(\theta) \simeq Q_T(\hat{\theta}_j) + s_T(\hat{\theta}_j)'(\theta - \hat{\theta}_j) + \frac{1}{2}(\theta - \hat{\theta}_j)'H_T(\hat{\theta}_j)(\theta - \hat{\theta}_j), \tag{I.14}$$

where $\hat{\theta}_j$ is the estimate in the j^{th} round of the iterative procedure to be described in a moment, and s_T and H_T are the gradient and the Hessian of the objective function :

$$s_T(\theta) = \frac{\partial Q_T(\theta)}{\partial \theta'} \qquad ; \qquad H_T(\theta) = \frac{\partial^2 Q_T(\theta)}{\partial \theta \partial \theta'}. \tag{I.15}$$

The (j+1)-round estimator $\hat{\theta}(j+1)$ is the maximizer of the quadratic function on the right-hand side of (II.10). It is given by :

$$\hat{\theta}_{j+1} = \hat{\theta}_j - [H_T(\hat{\theta}_j]^{-1} s_T(\hat{\theta}_j). \tag{I.16}$$

This iterative procedure is called the Newton-Raphson algorithm. If the objective function is concave, the algorithm often converges to the global minimum. This algorithm works well if the matrix $H_T(\hat{\theta}(j)$ is positive definite.

$$\hat{\theta}_{j+1} = \hat{\theta}_j + \alpha_j [D_T(\hat{\theta}_j]^{-1} s_T(\hat{\theta}_j), \qquad \theta_0 \qquad \text{given}, \tag{I.17}$$

where α_j is a scalar which is determined at each step to be minimized. $Q_T(\theta_j + \alpha_j[D_T(\hat{\theta}_j]^{-1} s_T(\hat{\theta}_j))$, $D_T(\hat{\theta}_j)$ is a matrix which approximates the Hessian (matrix of second derivatives of the objective function) near the maximum, but it is constructed so that it is always positive definite ; $s_T(\hat{\theta}(j))$ is the gradient (vector of first derivatives of the objective function).

For the objective function given in (II.9), an approximation of the Hessian matrix is given by :

$$D_T(\hat{\theta}_j) = \frac{1}{T} G'(\hat{\theta}_j) G(\hat{\theta}_j) \tag{I.18}$$

and the second term of the quasi-Newton algorithm becomes :

$$[D_T(\hat{\theta}_j]^{-1} s_T(\hat{\theta}_j) = \left[\sum_i G'(\hat{\theta}_j) G(\hat{\theta}_j)\right]^{-1} G'(\hat{\theta}_j) g(\hat{\theta}_j) \ . \tag{I.19}$$

I.6 GMM Specification Test

Tableau I.I – Test of the over identification

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN	TIN
All data														
χ^2	0.00	0.00	0.00	0.03	0.00	0.02	0.00	0.00	0.00	0.00	0.00	0.04	0.00	0.01
$\chi^2 p.v.$	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
Regime where the price increases														
χ^2	0.00	0.00	0.00	0.00	226	148	0.00	362	221	110	337	0.00	0.00	.
$\chi^2 p.v.$	0.99	0.99	0.99	0.99	0.00	0.00	0.99	0.00	0.00	0.00	0.00	0.99	0.99	.
Regime where the price decreases														
χ^2	138	0.00	0.00	0.00	121	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\chi^2 p.v.$	0.00	0.99	0.99	0.99	0.00	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99

There is no evidence against H_0 at level 1% (resp. 5%) when the p.value $(p.v)$ is greater than 0.01 (resp. 0.05)
OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; COP' copper ; BAU, bauxite ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc ; The value between ( ) is the standard deviation.

Table I.I reports results of an implementation of the J-test. Note that the J-test allows us to assess moment conditions (1.33) and (1.39). It appears that for almost all models the moment conditions match the data very well for all regimes. Therefore, there is not evidence against the GMM specification used in this paper.

Annexe II

Appendix to Chapter 2

II.1 Summary statistics for price, average cost and extraction rate data

Ores	OIL	NG	GOL	HC	SC	PHO	BAU
Price (1982-1984 US$ per ton)							
n.id.	87	85	58	72	41	33	32
n.obs.	2587	2570	1584	2008	1082	894	745
Mean	130.9	2073.6	$1.0*10^7$	39.0	10.0	36.9	30.6
Std.Dev.	69.6	900.5	$4.4*10^6$	17.0	4.3	21.3	10.9
Min	33.5	1138.8	$3.2*10^6$	19.1	5.0	20.6	13.2
Max	327.7	4482.2	$2.6*10^7$	79.6	21.0	120.7	49.9
Average extraction by country (Million tons)							
n.id.	87	85	58	72	41	33	32
n.obs.	2587	2570	1584	2008	1082	894	745
Mean	131.8	2102.9	$7.4*10^6$	39.8	10.5	38.2	14.0
Std.Dev.	71.8	920.7	$2.0*10^6$	17.4	17.4	22.6	4.3
Min	33.4	1138.8	$2.2*10^6$	19.1	5.0	20.6	5.1
Max	327.7	4482.2	$1.1*10^7$	79.6	21.0	120.7	27.0
Extraction rate by country (Million tons)							
n.id.	87	85	58	130.90	41	33	32
n.obs.	2587	2570	1584	130.90	1082	894	745
Mean	$3.8*10^7$	$7.8*10^5$	30.1	$2.3*10^7$	$4.7*10^7$	$4.1*10^6$	$3.4*10^6$
Std.Dev.	$1.0*10^8$	$3.4*10^6$	150.0	$5.4*10^7$	$1.5*10^8$	$1.0*10^7$	$6.8*10^6$
Min	307.6	1.2	$6.1*10^{-4}$	1275	0.5	4912	96.2
Max	$1.37*10^9$	$5.8*10^7$	$2.5*10^3$	$5.7*10^8$	$1.4*10^9$	$9.0*10^7$	$4.3*10^7$

Ores	COP	IRO	LEA	NIC	SIL	TIN	ZIN
Price (1982-1984 US$ per ton)							
n.id.	63	51	50	48	52	37	52
n.obs.	1437	1305	1184	789	1337	780	1234
Mean	$2.0*10^3$	40.4	697.6	$6.2*10^3$	$2.2*10^5$	$9.8*10^3$	981.8
Std.Dev.	724.6	10.4	282.1	$2.2*10^3$	$1.6*10^5$	$5.5*10^3$	316.6
Min	964.5	24.4	376.9	$2.8*10^3$	$9.8*10^4$	$3.1*10^3$	659.9
Max	$3.7*10^3$	63.90	$1.6*10^3$	$1.1*10^4$	$8.8*10^5$	$2.0*10^4$	$2.0*10^3$
Average extraction by country (Million tons)							
n.id.	63	51	50	48	52	37	52
n.obs.	1437	1305	1184	789	1337	780	1234
Mean	$1.0*10^3$	17.2	441.5	$4.2*10^3$	$1.6*10^5$	$9.8*10^3$	747.2
Std.Dev.	353.3	7.7	114.0	$1.0*10^3$	$1.0*10^5$	$5.5*10^3$	184.5
Min	152.0	4.4	220.50	$1.9*10^3$	$6.6*10^4$	$3.1*10^3$	463.0
Max	$2.6*10^3$	48.6	832.1	$6.9*10^3$	$4.8*10^5$	$2.0*10^4$	$1.0*10^3$
Extraction rate by country (Million tons)							
n.id.	63	51	50	48	52	37	52
n.obs.	1437	1305	1184	789	1337	780	1234
Mean	$1.8*10^5$	$1.9*10^7$	$8.5*10^4$	$4.6*10^4$	283.0	$9.7*10^3$	$1.5*10^5$
Std.Dev.	$4.0*10^5$	$2.1*10^7$	$1.6*10^5$	$8.6*10^4$	568.7	$2.0*10^4$	$3.1*10^5$
Min	1.9	65.62	99.6	42.0	0.4	0.6	22.6
Max	$4.0*10^6$	$1.3*10^8$	$1.4*10^6$	$7.1*10^5$	$4.2*10^3$	$1.9*10^5$	$3.2*10^6$

OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ;PHO, phosphate ;
COP, copper ; BAU, bauxite ; IRO, iron ; LEA, lead ;NIC nickel ; SIL, silver ; TIN, tin ; ZIN, zinc
Source : Unpublished World Bank data

II.2 Proof of Proposition 1

Assume that $q = gS$ with g constant. The inverse demand function is given by $p = \theta q^{-\eta}$, with $\dot{\theta} = a\theta$. The cost function is given by $C(z,q,S) = z^{-1}q^{\alpha}S^{-b}$, with $\dot{z} = \gamma z$. Let $\Psi(z,S) = z^{-1}S^{-b}$ and $\phi(q) = q^{-\eta}$. Then, since $\dot{S} = -q$, if we differentiate with respect to time we get

$$\dot{q} = -gq; \qquad \dot{S} = -gS; \qquad \frac{d\phi^{'}(q)}{dt} = \eta g\phi^{'}(q)$$

$$\dot{p} = (\eta g + a)\theta\phi^{'}(q); \qquad \frac{d\Psi(z,S)}{dt} = (-\gamma + bg)\Psi(z,S).$$

From the static efficiency condition (2.8) we have that

$$\begin{aligned} \lambda &= p - C_q(z,q,S) \\ &= \theta\phi^{'}(q) - \alpha q^{\alpha-1}\Psi(z,S). \end{aligned} \tag{II.1}$$

Differentiating (II.1) with respect to time, we get

$$\dot{\lambda} = (\eta g + a)\theta\phi^{'}(q) - \alpha[(b - \alpha + 1)g - \gamma]q^{\alpha-1}\Psi(z,S). \tag{II.2}$$

From the dynamic efficiency condition (2.8) we have

$$\dot{\lambda} = \delta\lambda - bgq^{\alpha-1}\Psi(z,S). \tag{II.3}$$

Substituting (II.1) and (II.2) into (II.3) and simplifying, we get the following necessary condition :

$$(\eta g + a - \delta)\theta\phi^{'}(q) - [\alpha((b - \alpha + 1)g - \gamma - \delta) - bg]q^{\alpha-1}\Psi(z,S) = 0.$$

This condition is satisfied if δ and g satisfy the following system of two linear equations :

$$\delta = a + \eta g \tag{II.4}$$

$$\alpha((b-\alpha+1)g-\gamma-\delta)-bg = 0, \tag{II.5}$$

from which we get that if (and only if) $\delta = a + \eta g$ with g a constant given by

$$g = \frac{\alpha(a+\gamma)}{\alpha(b-\eta-\alpha+1)-b}$$

then $q(t) = gS(t)$ constitutes an equilibrium.

The second derivative of Hamiltonian with respect to the stock S, evaluated at $q^* = gS$, is

$$H_{ss}(t,q^*,S,\lambda) = -\eta(1-\eta)g^{1-\eta}S^{-\eta-1}\theta - (\alpha-b)(\alpha-b-1)g^{\alpha}S^{\alpha-2}\Psi(z,S),$$

which is negative, thus guaranteing sufficiency.

II.3 Proof of Proposition 2

The Hamiltonian at t is given by.

$$\begin{aligned} H &= pq - C(z,q,S) - \lambda q \\ &= pq - q^{\alpha}\Psi(z,S) - (p - C_q)q \\ &= -q^{\alpha}\Psi(z,S) + \alpha q^{\alpha-1}\Psi(z,S))q \\ &= -(1-\alpha)q^{\alpha}\Psi(z,S). \end{aligned}$$

Differentiating with respect to time, we get that

$$\dot{H} = (-\alpha g - \gamma + bg)H,$$

and it follows that

$$H(t, q(t), S(t), \lambda(t)) = H(0, q(0), S(0), \lambda(0))e^{(-\gamma + g(b-\alpha))t}$$

If there is extraction at time 0, then $H(0, q(0), S(0), \lambda(0)) > 0$ and there will be extraction at all dates, so that

$$H(t, q(t), S(t), \lambda(t)) > 0 \qquad \forall t. \tag{II.6}$$

As a consequence, the stock of resource will be fully depleted. Therefore, at the terminal date T, $S(T) = 0$. In other words, it is not optimal to leave any stock in the ground.

Since $S(T) = 0$, the cumulative extraction will be equal to the initial resource endowment at T, namely

$$\int_0^T q(\tau)d\tau = S(0). \tag{II.7}$$

Substituting the extraction rate $q(\tau) = q(0)e^{-g\tau}$ into the resource constraint (II.7), we obtain that

$$\begin{aligned} S(0) &= \frac{q(0)}{g}(1 - e^{-gT}) \\ &= S(0)(1 - e^{-gT}). \end{aligned} \tag{II.8}$$

It follows from (II.8) that

$$e^{-gT} = 0$$

and thus $T = \infty$, as stated.

II.4 Newey West HAC estimator of matrix Σ

The asymptotic covariance matrix of the variable $W_{it}' f(y_{it};\theta_0)$ is given by

$$\Sigma = \lim_{T\to\infty}\left(\Gamma(0)+\sum_{j=1}^{T-1}\left(\Gamma(j)+\Gamma^{'}(j)\right)\right),$$

where $f(y_{it};\theta_0)$ is the zero function defined in (2.24).

The Newey-West estimator of this covariance matrix takes the form :

$$\begin{aligned}\widehat{\Sigma}_{NW} &= \widehat{\Gamma}(0)+\sum_{j=1}^{p}\left(1-\frac{j}{p+1}\right)\left(\widehat{\Gamma}(j)+\widehat{\Gamma}^{'}(j)\right)\\ \widehat{\Gamma}(j) &= \frac{1}{2T}\sum_{t=j+1}^{2T}\widehat{\varepsilon}_t\widehat{\varepsilon}_{t-j}W_t^{'}W_{t-j}.\end{aligned}$$

Note that the optimal cutoff p is given by $2T^{1/4}$.

II.5 Ledoit-Wolf well conditioned HAC estimator of Σ

Consider the Frobenius norm $||X|| = \sqrt{(tr\,(XX')\,/k}$, X being a $k\times T$ matrix whose associated inner product is $\langle X_1X_2\rangle = tr\,(X_1X_2')\,/k$. The Ledoit-Wolf HAC estimator is given by

$$\widehat{\Sigma}_{LW} = \frac{\hat{b}^2}{\hat{d}^2}\hat{m}I+\frac{\hat{a}^2}{\hat{d}^2}\widehat{\Sigma}_{NW},$$

where $\widehat{\Sigma}_{NW}$ is a Newey West HAC estimator of Σ, I is the $k \times k$ identity matrix and the coefficients are given by

$$
\begin{aligned}
\hat{m} &= \langle \widehat{\Sigma}_{NW}, I \rangle \\
\hat{d}^2 &= ||\widehat{\Sigma}_{NW} - \hat{m}I|| \\
\bar{b}^2 &= \frac{1}{p}\sum_{j}^{p}\left|\left|\left(1 - \frac{j}{p+1}\right)\left(\widehat{\Gamma}(j) + \widehat{\Gamma}'(j)\right) - \widehat{\Sigma}_{NW}\right|\right| \\
\hat{b}^2 &= min\left(\bar{b}^2, \hat{d}^2\right) \\
\hat{a}^2 &= \hat{d}^2 - \hat{b}^2.
\end{aligned}
$$

II.6 The GMM estimator

Let $g_T(y;\theta) = \frac{1}{T}\sum_{t=0}^{T} W_{it}' f(y_{it};\theta)$ and $G(y;\theta) = \partial g(y;\theta)/\partial\theta$. Then the GMM estimator $\hat{\theta}$ is the value of θ which minimizes the following criterion function :

$$Q_T(\theta) = g_T(y,\theta)'\Sigma^{-1}g_T(y,\theta) \tag{II.9}$$

where Σ is the optimal covariance matrix of the moment variable $W_{it}' f(y_{it},\theta)$, $\Sigma = W'\Omega W$. It can be shown that an estimate of the optimal covariance matrix of $\hat{\theta}$ is given by :

$$\widehat{Var}(\hat{\theta}) = T^{-1}\left[G(y;\hat{\theta})'\hat{\Sigma}^{-1}G(y;\hat{\theta})\right]^{-1},$$

where $\hat{\Sigma}$ is a HAC estimator of the covariance matrix $\Sigma = W'\Omega W$.

II.7 Quasi-Gauss-Newton Algorithm

Denote by $Q(\theta)$ the objective function. Since $Q(\theta)$ is twice continuously differentiable for M-estimators, there exists a second-order Taylor expansion

$$Q_T(\theta) \simeq Q_T(\hat{\theta}_j) + s_T(\hat{\theta}_j)'(\theta - \hat{\theta}_j) + \frac{1}{2}(\theta - \hat{\theta}_j)'H_T(\hat{\theta}_j)(\theta - \hat{\theta}_j), \tag{II.10}$$

where $\hat{\theta}_j$ is the estimate in the j^{th} round of the iterative procedure to be described in a moment, and s_T and H_T are the gradient and the Hessian of the objective function, namely

$$s_T(\theta) = \frac{\partial Q_T(\theta)}{\partial \theta'} \qquad ; \qquad H_T(\theta) = \frac{\partial^2 Q_T(\theta)}{\partial \theta \partial \theta'}.$$

The (j+1)-round estimator $\hat{\theta}(j+1)$ is the maximizer of the quadratic function on the right-hand side of (II.10). It is given by

$$\hat{\theta}_{j+1} = \hat{\theta}_j - [H_T(\hat{\theta}_j]^{-1} s_T(\hat{\theta}_j).$$

This iterative procedure is called the Newton-Raphson algorithm. If the objective function is concave, the algorithm often converges to the global minimum. This algorithm works well if the matrix $H_T(\hat{\theta}(j)$ is positive definite. We have

$$\hat{\theta}_{j+1} = \hat{\theta}_j + \alpha_j [D_T(\hat{\theta}_j]^{-1} s_T(\hat{\theta}_j), \qquad \theta_0 \qquad \text{given},$$

where α_j is a scalar which is determined at each step to be minimized, $Q_T(\theta_j + \alpha_j [D_T(\hat{\theta}_j]^{-1} s_T(\hat{\theta}_j))$. $D_T(\hat{\theta}_j)$ is a matrix which approximates the Hessian (matrix of second derivatives of the objective function) near the maximum, but it is constructed so that it is always positive definite ; $s_T(\hat{\theta}(j))$ is the gradient (vector of first derivatives of the objective function).

For the objective function given in (II.9), an approximation of the Hessian matrix is given by

$$D_T(\hat{\theta}_j) = \frac{1}{T} G'(\hat{\theta}_j) G(\hat{\theta}_j)$$

and the second term of the quasi-Newton algorithm becomes

$$[D_T(\hat{\theta}_j]^{-1} s_T(\hat{\theta}_j) = \left[\sum_i G'(\hat{\theta}_j) G(\hat{\theta}_j)\right]^{-1} G'(\hat{\theta}_j) g(\hat{\theta}_j). \ .$$

II.8 GMM Specification Test

Tableau II.I – Test of the over identification

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN	TIN
All data														
χ^2	0.00	0.00	0.00	0.03	0.00	0.02	0.00	0.00	0.00	0.00	0.00	0.04	0.00	0.01
$\chi^2 p.v.$	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
Regime where the price increases														
χ^2	0.00	0.00	0.00	0.00	226	148	0.00	362	221	110	337	0.00	0.00	.
$\chi^2 p.v.$	0.99	0.99	0.99	0.99	0.00	0.00	0.99	0.00	0.00	0.00	0.00	0.99	0.99	.
Regime where the price decreases														
χ^2	138	0.00	0.00	0.00	121	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\chi^2 p.v.$	0.00	0.99	0.99	0.99	0.00	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99

There is no evidence against H_0 at level 1% (resp. 5%) when the p.value $(p.v)$ is greater than 0.01 (resp. 0.05)
OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; COP' copper ; BAU, bauxite ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc ; The value between ( ) is the standard deviation.

Table II.I reports results of an implementation of the J-test. Note that the J-test allows us to assess the moment conditions (2.25). It appears that for almost all resources the moment conditions match the data very well for all regimes. Therefore, there is no evidence against the GMM specification used in this paper.

II.9 Estimations under the constraint that demand decreases over time

The results presented in this paper are obtained under the constraints (i) $\alpha < b$, (ii) $-\gamma \leq a$ and (iii) $\eta \leq \alpha^{-1}(\alpha-1)(b-\alpha)$ of Proposition 1. Those constraints come from the fact that that the demand for the resource is assumed not to always decreases over time. From the estimations reported in Table 2.II, it appears that demand does decrease over time for almost all resources. Therefore, it may be acceptable to impose that the demand for the nonrenewable natural resource decreases at equilibrium. Following this observation, if I allow the parameter b to be close to zero, then it appears that the condition $a \leq -\gamma$ is sufficient to obtain a positive value for g. This sufficiency comes from the fact that the non constrained value of b is smaller than the value of the extraction cost elasticity α. In order words, under the condition $a \leq -\gamma$, estimates of the parameters of the model satisfy $b \leq \alpha$, $\eta \geq 0$ and $\alpha^{-1}(\alpha-1)(b-\alpha) \leq 0$. Hence the addition constraint $\eta \geq \alpha^{-1}(\alpha-1)(b-\alpha)$ for a positive extraction growth rate is satisfied automatically.

The results of the reestimation under this different specification of the parameter constraints ($a \leq -\gamma$ and $b \geq 0$) are presented in Table II.IV. The conclusions as to the existence of linear equilibria do not differ from those that can be drawn from the original estimation results reported in Table 2.IV.

Tableau II.II – Estimation results

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN	TIN
Sub-period where the price is increasing														
α	1.01	1.00	1.04	1.00	1.00	1.00	1.00	1.00	1.00	1.08	2.22	1.05	1.13	.
b	0.97	6.76	0.59	8.30	0.51	2.92	1.29	0.40	0.00	8.38	0.33	7.36	2.82	.
γ	0.04	0.16	6.54	0.10	0.58	0.79	0.10	2.48	2.11	2.37	0.22	2.04	5.12	.
δ	0.12	0.21	1.30	0.00	0.01	0.00	0.00	0.00	0.10	0.25	0.00	1.17	1.38	.
η	0.73	13.0	19.1	3.64	2.54	7.66	1.03	11.3	4.01	3.70	4.42	3.68	1.95	.
a	-0.04	-0.16	-7.91	-0.10	-0.58	-0.79	-0.10	-2.48	-2.24	-2.46	-0.22	-2.41	-6.10	.
Sub-period where the price is decreasing														
α	1.00	1.00	3.03	1.12	1.02	1.08	1.27	1.00	1.00	1.13	1.21	1.61	1.09	1.00
b	0.00	0.00	0.00	3.73	1.32	7.43	4.85	2.94	1.88	3.29	3.98	0.54	2.29	0.21
γ	0.09	0.00	0.00	3.07	0.25	1.50	0.25	0.06	0.13	5.74	1.18	0.00	2.36	0.59
δ	0.00	0.00	0.00	0.11	0.00	0.00	0.05	0.11	0.00	0.59	0.27	0.00	0.13	0.00
η	2.89	1.58	1.48	1.28	2.40	3.80	9.48	1.08	1.97	1.76	1.82	0.71	2.48	5.73
a	-0.09	-0.00	-0.00	-4.70	-0.25	-1.53	-0.32	-0.06	-0.13	-6.21	-1.21	-0.00	-2.37	-0.59

NB : The standard errors of estimated parameters have not be reported because boundary solutions are admissible. OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; COP' copper ; BAU, bauxite ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc.

Tableau II.III – Estimates of the parameter g

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN	TIN
Regime where the price is increasing														
$g(10^{-5})$	0.13	0.00	714	0.03	90.2	0.01	0.09	0.00	341	261	0.00	100	507	.
Regime where the price is decreasing														
$g(10^{-5})$	0.34	0.00	0.02	126	21.7	80.4	811	0.38	0.05	266	160	0.08	58.8	0.01

OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; COP' copper ; BAU, bauxite ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc.

Tableau II.IV – Test of the existence of a linear equilibrium

	OIL	NG	GOL	HC	SC	PHO	BAU	COP	IRO	LEA	NIC	SIL	ZIN	TIN
All data														
χ^2	128	182	0.00	0.03	507	1.42	91.6	0.09	331	580	0.00	0.00	584	138
$\chi^2 p.v.$	0.00	0.00	0.99	0.85	0.00	0.22	0.00	0.75	0.00	0.00	0.99	0.99	0.00	0.00
Regime where the price increases														
χ^2	640	397	249	191	249	113	834	598	152	476	124	152	694	.
$\chi^2 p.v.$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	.
Regime where the price decreases														
χ^2	226	0.00	0.00	105	118	245	273	211	678	142	141	0.13	136	138
$\chi^2 p.v.$	0.00	1.00	0.99	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.70	0.00	0.00

There is no evidence against H_0 at level 1% (resp. 5%) when the p.value $(p.v)$ is greater than 0.01 (resp. 0.05)
OIL, oil ; NG, natural gas ; GOL, gold ; HC, hard coal ; SC, brown coal ; PHO, phosphate ; COP' copper ; BAU, bauxite ; IRO, iron ; LEA, lead ; NIC, nickel ; SIL, silver ; TIN, tin ; ZIN, zinc.

Annexe III

Appendix to Chapter 3

III.1 Characterization of the equilibrium prices and returns

The demand prices and asset returns have been taken to be those that equilibrate the markets when extraction, production and consumption take place simultaneously. They have furthermore been assumed to evolve in equilibrium as stochastic processes, given that the exogenous productivity indices evolve stochastically. In this Appendix we show that such assumptions are indeed compatible with the equilibrium and illustrate how the respective equilibrium drifts and variances can be calculated as functions of the primitives.

It has already been argued in the last section that we must have in equilibrium $dR_K(t) = dR_B(t) = rdt$. There remains to characterize the prices $p(t)$ and $q(t)$, and the returns $dR_X(t)$ and $dR_Q(t)$. Each of those prices and returns will at any given date be a function of the state of the economy, which is given by the vector $(K(t)+B(t),\, X(t),\, Q(t),\, \theta_1(t),\, \theta_2(t))$.

Consider first the utility price of the composite good. It is given by $q = U'(c)$. If we replace the decision variable $c(t)$ by its equilibrium value, then $q(t)$ can be expressed as

$$q(t) = \mathbf{Q}(K(t)+B(t), X(t), Q(t), \theta_1(t), \theta_2(t)). \qquad \text{(III.1)}$$

Similarly, we know from condition (3.17) that $q(t)p(t) = \Gamma_Q(Q(t), p(t), q(t), \theta_1(t))$. Substituting for the utility price $q(t)$ from (III.1), we see that the implicit solution for $p(t)$ will take the form

$$p(t) = \mathbf{P}(K(t)+B(t), X(t), Q(t), \theta_1(t), \theta_2(t)). \qquad \text{(III.2)}$$

As for the asset price (*in situ* price) of the resource, $\lambda(t)$, it is given by $\lambda(t) = p(t) - r\gamma(\theta_2)$, namely the gross market price of a unit or the resource minus the cost of taking it out of the ground, expressed in terms of the composite good. Hence, substituting for $p(t)$ from (III.2), the equilibrium value of $\lambda(t)$ can be written

$$\begin{aligned}
\lambda(t) &= \Lambda(K(t)+B(t), X(t), Q(t), \theta_1(t), \theta_2(t)) \\
&= \mathbf{P}(K(t)+B(t), X(t), Q(t), \theta_1(t), \theta_2(t)) - r\gamma(\theta_2(t)).
\end{aligned} \tag{III.3}$$

Given that the productivity indices θ_1 and θ_2 evolve as Itô processes, then, as assumed in (3.6), (3.5) and (3.20), so will the equilibrium values of p, q, dR_X and dR_Q. To verify this, consider the case of the equilibrium gross price p. Denote by k and b the instantaneous rates of change of K and B respectively, so that $dK = kdt$ and $dB = bdt$, and recall that $dX = -xdt$ and $dQ = x - \delta Q$. Then, using Itô's lemma, we get that

$$\begin{aligned}
dp &= \mathbf{P}_{K+B}(k+b) - \mathbf{P}_X x + \mathbf{P}_Q(x - \delta Q) \\
&+ \mathbf{P}_{\theta_1} d\theta_1 + \mathbf{P}_{\theta_2} d\theta_2 + \mathbf{P}_{\theta_1\theta_2} + \frac{1}{2}\mathbf{P}_{\theta_1\theta_1}(d\theta_1)^2 + \frac{1}{2}\mathbf{P}_{\theta_2\theta_2}(d\theta_2)^2
\end{aligned} \tag{III.4}$$

Substituting for $d\theta_1$ and $d\theta_2$ from (3.3), we get

$$\begin{aligned}
\frac{dp}{p} &= \left[\frac{\mathbf{P}_{K+B}}{\mathbf{P}} - \frac{\mathbf{P}_X}{\mathbf{P}}x + \frac{\mathbf{P}_Q}{\mathbf{P}}(x - \delta Q) + \frac{\mathbf{P}_{\theta_1}}{\mathbf{P}}\mu_1 + \frac{\mathbf{P}_{\theta_2}}{\mathbf{P}}\mu_2 + \frac{\mathbf{P}_{\theta_1\theta_2}}{\mathbf{P}}\sigma_1\sigma_2\xi_1\xi_2 \right. \\
&+ \left. \frac{1}{2}\left(\frac{\mathbf{P}_{\theta_1\theta_1}}{\mathbf{P}}\sigma_1^2\xi_1^2 + \frac{\mathbf{P}_{\theta_2\theta_2}}{\mathbf{P}}\sigma_2^2\xi_2^2\right)\right] dt \\
&+ \left[\frac{\mathbf{P}_{\theta_1}}{\mathbf{P}}\sigma_1\xi_1 + \frac{\mathbf{P}_{\theta_2}}{\mathbf{P}}\sigma_2\xi_2\right]\sqrt{dt},
\end{aligned} \tag{III.5}$$

which yields (3.5) as assumed, where

$$\begin{aligned}
\mu_p &= \frac{\mathbf{P}_{K+B}}{\mathbf{P}} - \frac{\mathbf{P}_X}{\mathbf{P}}x + \frac{\mathbf{P}_Q}{\mathbf{P}}(x-\delta Q) + \frac{\mathbf{P}_{\theta_1}}{\mathbf{P}}\mu_1 + \frac{\mathbf{P}_{\theta_2}}{\mathbf{P}}\mu_2 + \frac{\mathbf{P}_{\theta_1\theta_2}}{\mathbf{P}}\sigma_1\sigma_2\xi_1\xi_2 \\
&+ \frac{1}{2}\left(\frac{\mathbf{P}_{\theta_1\theta_1}}{\mathbf{P}}\sigma_1^2\xi_1^2 + \frac{\mathbf{P}_{\theta_2\theta_2}}{\mathbf{P}}\sigma_2^2\xi_2^2\right) \\
&= \frac{1}{\mathbf{P}(\cdot)}\frac{1}{dt}E_t(d\mathbf{P}(\cdot))
\end{aligned}$$

$$\sigma_p = \frac{\left[\frac{\mathbf{P}_{\theta_1}}{\mathbf{P}}\sigma_1\xi_1 + \frac{\mathbf{P}_{\theta_2}}{\mathbf{P}}\sigma_2\xi_2\right]}{\xi_2} = \left(\frac{1}{dt}var\left(\frac{d\mathbf{P}(\cdot)}{\mathbf{P}(\cdot)}\right)\right)^{¡}$$

and

$$\xi_p = \xi_2.$$

Using Itô's lemma we can derive in the same way $\mu_q = (1/\mathbf{Q}(\cdot))(1/dt)E_t(d\mathbf{Q}(\cdot))$ and $\sigma_q = ((1/dt)varE_t(d\mathbf{Q}(\cdot)/\mathbf{Q}(\cdot))^{¡}$, as well as $\mu_\lambda = (1/\Lambda(\cdot))(1/dt)E_t(d\Lambda(\cdot))$ and $\sigma_\lambda = ((1/dt)varE_t(d\Lambda(\cdot)/\Lambda(\cdot))^{¡}$. As for the rates of returns on X and on Q, given by

$$dR_X = \frac{d\lambda}{\lambda} \qquad \text{and} \qquad dR_Q = \left(\frac{F_Q}{p} - \delta\right)dt + \frac{dp}{p}$$

their equilibrium drifts, μ_X and μ_Q, and volatility, σ_X and σ_Q, can be obtained using the above, hence verifying the appropriateness of the assumption made in (3.20).