Université de Montréal

Les processus additifs markoviens et leurs applications en finance mathématique

par

Romuald Hervé MOMEYA OUABO

Département de mathématiques et de statistique
Faculté des arts et des sciences

Thèse présentée à la Faculté des études supérieures
en vue de l'obtention du grade de
Philosophiæ Doctor (Ph.D.)
en Statistique

juin 2012

Université de Montréal
Faculté des études supérieures

Cette thèse intitulée

Les processus additifs markoviens et leurs applications en finance mathématique

présentée par

Romuald Hervé MOMEYA OUABO

a été évaluée par un jury composé des personnes suivantes :

François Perron
(président-rapporteur)

Manuel Morales
(directeur de recherche)

Louis Doray
(co-directeur)

Ghislain Léveillé
(membre du jury)

Tak Kuen Siu
(examinateur externe)

Yoshua Bengio
(représentant du doyen de la FES)

Thèse acceptée le:
29 Mai 2012

SOMMAIRE

Cette thèse porte sur les questions d'évaluation et de couverture des options dans un modèle exponentiel-Lévy avec changements de régime. Un tel modèle est construit sur un processus additif markovien un peu comme le modèle de Black-Scholes est basé sur un mouvement Brownien. Du fait de l'existence de plusieurs sources d'aléa, nous sommes en présence d'un marché incomplet et ce fait rend inopérant les développements théoriques initiés par Black et Scholes et Merton dans le cadre d'un marché complet.

Nous montrons dans cette thèse que l'utilisation de certains résultats de la théorie des processus additifs markoviens permet d'apporter des solutions aux problèmes d'évaluation et de couverture des options . Notamment, nous arrivons à caractériser la mesure martingale qui minimise l'entropie relative à la mesure de probabilité historique ; aussi nous dérivons explicitement sous certaines conditions, le portefeuille optimal qui permet à un agent de minimiser localement le risque quadratique associé. Par ailleurs, dans une perspective plus pratique nous caractérisons le prix d'une option Européenne comme l'unique solution de viscosité d'un système d'équations intégro-différentielles non-linéaires. Il s'agit là d'un premier pas pour la construction des schémas numériques pour approcher ledit prix.

Cette thèse est composée principalement de quatre articles soumis à différentes revues scientifiques. L'un a été publié et deux autres ont été revisés et resoumis. Plus précisément :

(1) *On the Price of Risk of the Underlying Markov Chain in a Regime-Switching Exponential Lévy Model* [**101**], revisé et resoumis à la revue *Methodology and Computing in Applied probability* ;

(2) *The Minimal entropy martingale measure for a Markov-modulated exponential Lévy model* [**102**], accepté dans la revue *Asia-Pacific Financial Markets* et disponible en ligne ;

(3) *Local risk-minimization under a partially observed Markov-modulated exponential Lévy model* [**97**], révisé et resoumis à la revue *Applied Mathematical Finance* ;

(4) *Viscosity Solutions and the pricing of European-style options in a Markov-modulated exponential Lévy model* soumis dans la revue *Stochastic Analysis and Applications.*

Cette forme de présentation induit inévitablement de nombreuses répétitions notamment au niveau de la présentation du modèle, de sa motivation et aussi au niveau des concepts de base. Nous nous en excusons auprès du lecteur.

Mots-Clés

Processus additif markovien, Incomplétude du marché, minimisation du risque local, mesure martingale, solution de viscosité, calibration de modèle.

SUMMARY

This thesis focuses on the pricing and hedging problems of financial derivatives in a Markov-modulated exponential-Lévy model. Such model is built on a Markov additive process as much as the Black-Scholes model is based on Brownian motion. Since there exist many sources of randomness, we are dealing with an incomplete market and this makes inoperative techniques initiated by Black, Scholes and Merton in the context of a complete market.

We show that, by using some results of the theory of Markov additive processes it is possible to provide solutions to the previous problems. In particular, we characterize the martingale measure which minimizes the relative entropy with respect to the physical probability measure. Also under some conditions, we derive explicitly the optimal portfolio which allows an agent to minimize the local quadratic risk associated. Furthermore, in a more practical perspective we characterize the price of a European type option as the unique viscosity solution of a system of nonlinear integro-differential equations. This is a first step towards the construction of effective numerical schemes to approximate options price.

This thesis is mainly composed of four papers, one accepted and two revised and resubmitted. More specifically, we have :

(1) *On the Price of Risk of the Underlying Markov Chain in a Regime-Switching Exponential Lévy* Model [**101**], revised and resubmitted in the journal *Methodology and Computing in Applied Probability* ;

(2) *The Minimal entropy martingale measure for a Markov-modulated exponential Lévy* model [**102**], forthcoming in the journal *Asia-Pacific Financial Markets* and available online ;

(3) *Local risk-minimization Partially Observed Under a Markov-modulated exponential Lévy* model [**97**], revised and resubmitted in the journal *Applied Mathematical Finance* ;

(4) *Viscosity Solutions and the pricing of European-style options in a Markov-modulated exponential Lévy* model submitted in the journal *Stoachastic Analysis and Applications.*

This form of presentation leads ineluctably to many repetitions : in the description of the model, motivation and also in the presentation of basic concepts. We apologize to the reader.

Keywords

Markov additive process, Incompleteness of the market, local-risk minimization, martingale measure, viscosity solution, model calibration.

TABLE DES MATIÈRES

LISTE DES TABLEAUX

LISTE DES FIGURES

à Claire, Shanice et Celui qui s'en vient
à ma mère et à feu mon père.

REMERCIEMENTS

Arrivé au terme d'un périple qui a quand même duré cinq bonnes années, je voudrais profiter de l'occasion pour exprimer toute ma gratitude à tous ceux qui m'en ont facilité le parcours. Me montrant tel panneau signalétique par-ci, m'indiquant les écueils à éviter par-là et surtout m'encourageant lors de mes passages à vide. Ah oui, il y en a eu !

En premier lieu, je voudrais dire toute ma reconnaissance à Manuel Morales, mon directeur de recherche qui m'a laissé toute la latitude de choisir mes chemins escarpés, tout en gardant un oeil à ce que je ne me perde pas. Manuel a été un mentor pour moi, toujours disponible pour discuter de quelque aspect que ce soit. Le stage Mitacs qu'il a trouvé pour moi à Desjardins GIA m'a permis de voir le côté pratique de la finance mathématique. En second lieu, je remercie Louis Doray, mon codirecteur pour ses précieux conseils et les ressources financières qu'il a obtenues pour moi.

Aussi, je remercie très humblement le professeur Tak Kuen Siu de l'Université MacQuarie qui a bien voulu examiner cette thèse. Associés à ce même élan, les professeurs François Perron de l'Université de Montréal et Ghislain Leveillé de l'école d'actuariat de l'Université Laval m'ont aidé par leurs remarques à améliorer la qualité de ce document.

Qu'il me soit permis de remercier tout le personnel du département de Mathématiques et de Statistique pour leur accueil et leur sympathie. J'y ai passé de bons moments durant ces dernières années. Je remercie l'Institut des Sciences Mathématiques (ISM) et le Centre de Recherches Mathématiques (CRM) pour leur soutien financier sans lequel cette thèse n'aurait pu être menée à terme.

Je voudrais remercier mes nombreux amis et connaissances, en particulier Mohammed Haddou qui m'a pris sous son aile dès mon arrivée au département, Zied Ben Salah, Serge Barbeau, Blasche Akpoue, Valéry Dongmo, Foutse Khomh, Aimé Kuiteng, Olivier Pamen-Menoukeu ainsi que tout le groupe des Statisticiens/économistes venus d'Abidjan, je ne citerai aucun nom ici de peur d'en oublier certains.

Enfin, je dis toute ma gratitude à toute ma famille et ma belle-famille au Cameroun, à M. et Mme Kom ainsi qu'au Dr Tamo ; sans vos prières et votre soutien constant je n'aurais pu achever ce travail.

Je rends grâce au *Très Haut*, l'Eternel des armées, d'avoir gardé ma droite forte pour accomplir cette oeuvre.

INTRODUCTION GÉNÉRALE

Motivation

Les changements brusques et imprévisibles de la tendance générale des cours boursiers sont une réalité structurelle des marchés financiers notamment sur le moyen et le long terme. Ces changements font alterner des périodes de croissance (rapide ou lente) et de décroissance comme l'illustre la figure 0.1 tirée de [**120**] et représentant l'évolution historique de l'indice boursier $SP500$ coté à la Bourse de New-York.

Fig. 0.1. évolution historique de l'indice boursier SP500.

Ces ruptures de tendance trouvent souvent leur origine dans les évènements extérieurs au marché à l'instar des changements dans la politique économique, des changements dans l'environnement socio-politique ou encore des modifications de la structure d'information des agents économiques. Hamilton [**75**] en 1989 fut l'un des premiers à rendre compte de la présence d'une telle non-linéarité dans les séries macroéconomiques à travers un modèle dont les paramètres sont des réalisations d'une chaîne de Markov à espace d'états discret.
De faccon générale, la modélisation des systèmes évoluant dans un environnement susceptible de connaître des changements est commune à de nombreux domaines comme par exemple, les phénomènes de files d'attente (voir Asmussen [**7**]) ; les télécommunications (voir Breuer [**21**] et les références incluses) et la bio-informatique (voir Hansen [**76**]). Elle a par ailleurs connu au cours des deux dernières décennies un regain d'intérêt dans le champ de la finance mathématique ([**18, 24, 25, 30, 39, 50, 74, 87, 103**], etc.) comme une alternative à la modélisation de Black-Scholes, capable de générer des modèles qui prennent mieux en compte la réalité des données boursières. Car en effet, de nombreux travaux empiriques à l'instar de Mandelbrot [**96**] et Fama [**61**] parmi les plus anciens, Jorion [**84**] et Eberlein et Keller [**47**] pour les plus récents, ont mis en évidence la présence des queues épaisses dans la distribution des cours boursiers et un caractère totalement discontinu pour la dynamique de ces prix. Ces faits discréditent les hypothèses d'une trajectoire continue des prix et d'une distribution des rendements normale avec une volatilité constante telles que prévues par le modèle de

Black-Scholes. D'autre part, en confrontant les prix d'une option (d'achat) Européenne prédits par le modèle de Black-Scholes à ceux réellement observés sur le marché, MacBeth et Merville [**94**] ont mis en évidence une différence structurelle ayant l'allure d'une courbe en forme de U en fonction du prix d'exercice et de la maturité au lieu d'être constante selon Black et Scholes : ce phénomène est connu sous le nom de *"smile"* de la volatilité (voir [**46**] pour une recension).

Le sujet central de cette thèse est l'étude de la famille des modèles *exponentiel-Lévy avec changements de régime markoviens* dans la perspective de leur utilisation en ingénierie financière. Ces modèles supposent une représentation du prix S d'un actif boursier basée sur un *processus additif markovien* (X, Y), concrètement

$$S_t = S_0 \exp(Y_t^{(X)}), \quad t \geq 0 \tag{0.1}$$

où $Y^{(X)}$ est un processus conditionnellement additif dont les caractéristiques dépendent d'une chaîne de Markov X.

Ce type de modélisation associe les avantages des modèles *exponentiel-Lévy* basés sur les processus de Lévy à une dépendance stochastique dans le temps. Elle permet notamment une flexibilité capable de saisir les principaux faits observés dans les cours boursiers à l'instar de la variabilité (stochastique) temporelle de la volatilité et des moments d'ordre supérieur, de la distribution (des rendements) asymétrique avec des queues épaisses, du "volatility clustering", c-à-d., les grandes variations de rendements ont tendance à être suivies par d'autres grandes variations, de signe opposé, et de faccon analogue pour les petites variations.

Le point de départ de notre recherche est qu'une meilleure compréhension de la *théorie des processus additifs markoviens* élaborée dans les années 70 par Ezhov et Skorohod ([**59, 60**]), Cinlar ([**?, ?**]) et Grigelionis [**73**] permet de donner des solutions aux principaux problèmes inhérents à l'utilisation de ces modèles pour l'évaluation et la couverture des produits dérivés. A cet effet, deux types de problèmes nous intéressent. Sur un plan théorique d'abord, il s'agit d'étudier le problème de l'*incomplétude du modèle de marché* basé sur un processus additif markovien, et sur un plan pratique de voir comment les outils de l'ingénierie financière peuvent être adaptés pour la détermination de prix des produits dérivés[1] et des paramètres du modèle qui soient conformes avec les prix observés sur le marché.

Le problème de l'incomplétude du (modèle de) marché

L'une des innovations des travaux de Black et Scholes [**15**] et Merton [**98**] a été de montrer que, sous l'hypothèse d'absence d'opportunités d'arbitrage (c-à-d., l'impossibilité de faire du profit sans prendre de risque), en supposant une dynamique

[1] c-à-d., des produits dont la valeur dépend de l'évolution des cours du marché d'un actif appelé sous-jacent.

de l'actif boursier S dirigée par un mouvement Brownien géométrique et un marché parfait[2], la valeur d'une option Européenne ne dépend pas des préférences de l'agent investisseur et s'obtient comme solution d'une équation aux dérivées partielles-l'*équation de Black-Scholes*. Ces travaux ont posé le premier jalon de la finance mathématique et leurs reformulations par Harrison et Kreps [**78**] et Harrison et Pliska ([**79, 80**]) dans le langage de la théorie des martingales et de l'intégration stochastique ont donné lieu à une théorie générale de l'évaluation des options par arbitrage.
Cette théorie stipule notamment que l'hypothèse d'absence d'opportunités d'arbitrage équivaut à l'existence d'une mesure de probabilité équivalente à la mesure historique sous laquelle tous les (processus de) prix actualisés d'actifs sont des martingales (*premier théorème fondamental de l'évaluation par arbitrage*) et de plus si le marché est *complet*, c'est-à-dire pour tout produit dérivé de payoff H il est toujours possible de construire une stratégie de portefeuille dont la valeur terminale coÃ¯ncide avec H, il y a unicité de la mesure de probabilité équivalente précédente. En particulier, Harrison et Kreps [**78**] et Harrison et Pliska ([**79, 80**]) montrent que le modèle de Black-Scholes conduit à un marché complet et par conséquent, le "juste" prix d'une option Européenne d'achat de payoff H est le coût initial du portefeuille le répliquant et s'obtient comme l'espérance de H sous l'unique mesure de probabilité martingale équivalente.

L'un des traits caractéristiques des modèles qui généralisent le modèle de Black-Scholes, à l'instar des modèles *exponentiel-Lévy avec changements de régime*, est l'*incomplétude du marché* à laquelle ils conduisent, c-à-d., littéralement, l'impossibilité pour un investisseur dans un tel marché de se couvrir parfaitement contre le risque dû aux fluctuations des actifs. Dès lors, les problèmatiques de l'évaluation et de la couverture des produits dérivés dans un tel marché se posent avec une certaine acuité. Notamment, l'on cherche à répondre aux questions suivantes :

- Comment choisir un opérateur d'évaluation parmi la multitude pour déterminer le "juste" prix pour un produit dérivé, dès lors que l'unicité de la mesure de probabilité martingale équivalente n'est plus garantie malgré l'hypothèse d'absence d'opportunités d'arbitrage ?

- Comment construire une stratégie de portefeuille qui assure à l'investisseur le minimum de risque maintenant qu'une couverture parfaite n'est plus possible ?

Ces questions ont donné lieu à une vaste littérature ([**63, 64, 67, 100, 109, 115**]) désormais classique et continuent encore aujourd'hui à occuper une place de choix en finance mathématique avec la théorie des mesures de risque developpée par Artzner, et *al.* [**6**] et Delbaen [**40**] entre autres. Cette thèse explore quelques-unes des approches classiques de solution au problème de couverture et d'évaluation des options dans le cadre spécifique des modèles exponentiel-Lévy avec changements de régime.

[2]absence de frictions et de coûts de transaction, absence de contraintes sur les stratégies d'investissement.

Les problèmes numériques liés à l'évaluation des options et la calibration du modèle

L'une des principales raisons d'être d'un modèle financier est de fournir un algorithme pour déterminer le "juste" prix d'un produit dérivé. Un tel prix fournit à l'investisseur une base rationnelle pour juger si les prix observés réellement sur le marché sont raisonnables et, donc de pourvoir décider de l'opportunité d'acheter ou pas.
Les modèles de marché basés sur des processus de Markov ont la particularité que les prix des actifs sont solutions d'équations aux dérivées partielles. Ainsi la résolution de celles-ci fournit des approximations du prix de ces actifs. De faccon spécifique, le modèle *exponentiel-Lévy avec changements de régime* est construit sur un processus additif markovien et la détermination du prix d'une option Européenne dans un tel modèle donne lieu à la résolution d'un système d'équations-intégro-différentielles qui est l'analogue de la célèbre *équation de Black-Scholes* pour le modèle éponyme. Un tel système n'admet généralement pas de solution au sens classique notamment en raison de la possible dégénérescence du coefficient de diffusion. Ceci rend sa résolution complexe et requiert alors l'utilisation des méthodes numériques. Ainsi donc, le défi consiste à trouver un cadre fonctionnel adéquat qui assure l'existence et l'unicité de la solution et par suite de fournir des algorithmes numériques efficaces.
Par ailleurs, comme signalé précédemment le modèle de marché basé sur un processus additif markovien est incomplet ce qui implique l'existence d'une multiplicité de mesures martingales. Dans la pratique de l'ingénierie financière, il est courant de prendre comme mesure martingale d'évaluation celle "choisie" par le marché. Concrètement, il s'agit de prendre comme paramètres du modèle ceux qui permettent de répliquer ou du moins de se rapprocher le plus possible des prix observés sur le marché pour les produits dérivés liquides : Ce qui est l'objet de la calibration du modèle.

Contributions et Structure de la thèse

Après avoir présenté les notions et concepts utiles à la compréhension de cette thèse dans le chapitre **??**, la discussion des deux problèmatiques évoquées plus haut constitue l'essence des chapitres suivants. De faccon spécifique,

- La question de l'incomplétude du marché associé à un modèle exponentiel-Lévy avec changements de régime est discutée dans les chapitres **??**, **??** et 3. Chacun de ces chapitres constitue en soi un article écrit en collaboration et soumis pour publication ;

- Les deux derniers chapitres **??** et **??** élaborent sur les questions relatives à l'utilisation pratique du modèle.

Nous présentons dans la suite le résumé de nos contributions.

Le *Chapitre* **??** est basé sur l'article Momeya et Morales [**101**] révisé et resoumis à la revue *Methodology and Computing in Applied probability.*

Dans ce chapitre, nous illustrons le caractère incomplet du modèle de marché en dérivant deux opérateurs d'évaluation grâce à la transformation d'Esscher qui dans ce cadre prend une forme particulière en raison de la présence de la chaîne de Markov X. Ce développement théorique s'inspire de Siu et Yang [**123**] et nous sert à discuter de l'hypothèse de la non-prise en compte du risque lié aux changements de régime généralement invoquée dans la littérature (voir par exemple [**18, 20**]). A l'aide des simulations de Monte Carlo, nous montrons que ce risque est substantiel pour le prix d'une option Européenne. En particulier, pour le modèle de Black-Scholes avec changements de régime nos simulations confirment a posteriori certains résultats de la littérature (Naik [**103**] et Boyle et Draviam [**20**]). Dans un second temps, nous étudions l'influence des sauts sur ce risque à travers les modèles diffusion-sauts et Variance-Gamma avec changements de régime. Là aussi, nous notons une influence significative du risque de régime sur le prix de l'option.

Dans le *Chapitre* **??** basé sur l'article Momeya et Ben Salah [**102**], publié dans la revue *Asia-Pacific Financial Markets* et disponible en ligne, nous abordons plus directement le problème de l'incomplétude à travers le choix d'une mesure équivalente martingale pour évaluer un produit dérivé. Le critère (de choix) retenu du minimum de l'entropie relative est couramment utilisé dans la littérature notamment en raison de sa relation (par dualité) avec le problème de couverture du risque pour un agent ayant une utilité exponentielle. Notre principale contribution est la caractèrisation de la mesure équivalente martingale qui minimise l'entropie relative dans le cadre d'un modèle exponentiel-Lévy avec changements de régime. Ce résultat est obtenu en travaillant conditionnellement à la trajectoire entière de la chaîne, c-à-d., en supposant connue la filtration $\mathcal{F}_T^X := \sigma(X_u : 0 \leq u \leq T)$. En procédant ainsi, nous ramenons le problème à la situation d'un modèle exponentiel-additif pour lequel un travail récent de Fujiwara [**70**] donne une solution. Des exemples de calcul sont donnés pour illustrer la méthodologie proposée.

Le *Chapitre* **??** discute comme le précédent des conséquences de l'incomplétude du modèle de marché à la différence qu'ici, il est moins question de l'évaluation du prix d'un produit dérivé que de la détermination d'une stratégie de portefeuille qui minimise le risque associé suivant un critère quadratique. Il existe dans la littérature deux approches pour ce problème selon que la contrainte d'autofinancement ou de reproductibilité est satisfaite :

- l'approche *mean-variance hedging* (voir [**19, 45, 117**]) consistant à minimiser sur l'ensemble des portefeuilles autofinancés le risque quadratique global interpreté comme la distance L^2 entre le gain G_T associé à la gestion du portefeuille et le payoff H du produit dérivé que l'on cherche à couvrir ;
- l'approche *(local) risk-minimization* (voir [**64, 114, 115**]) qui consiste à minimiser, sur l'ensemble des portefeuilles répliquant H et non nécessairement autofinancés, le risque quadratique local défini comme la variance conditionnelle à l'information disponible à la date t des incréments ΔC_t du coût du portefeuille.

C'est cette dernière approche que nous avons suivie. Partant du fait que sous une filtration élargie qui suppose la connaissance de la trajectoire entière de X le processus S est une semimartingale, nous montrons que le problème se ramène à celui de la minimisation du risque local sous information partielle. La stratégie de solution consiste alors de résoudre le problème sous information totale en suivant une méthodologie due à Colwell et Elliott [**33**]. Pour ce faire, nous établissons un théorème de représentation martingale pour un produit dérivé de type Européen. Ce qui permet d'obtenir sous une forme explicite le portefeuille optimal en cas d'information totale. Par la suite, nous obtenons la solution sous la filtration disponible é l'agent par projection. Ce travail fait l'objet de l'article [**97**], coécrit avec Menoukeu-Pamen, révisé et resoumis.

Dans le *Chapitre* **??**, nous dérivons formellement le système d'équations intégro-differentielles non-linéaires vérifié par le prix d'une option Européenne. Ensuite, nous caractérisons ce prix comme étant l'unique *solution de viscosité* d'un tel système. Le choix de ce cadre de solutions dites généralisées est qu'il impose peu de contraintes de régularité, ce qui est particulièrement intéressant car, lorsque le modèle fait intervenir des sauts, il est en général difficile d'assurer la régularité de la solution. Ce travail est l'objet d'un article soumis à la revue *Stochastic Analysis and Applications*.

Le *Chapitre* **??** porte sur des problématiques numériques associées à l'ingénierie des modèles de Lévy avec changements de régime à travers d'une part l'évaluation numérique des options et l'exercice de calibration qui consiste à ajuster les paramètres du modèle afin de retrouver les prix d'options effectivement observés sur le marché. Le défi consiste à développer des algorithmes de minimisation qui fournissent des résultats stables dans le temps. Notre approche se veut plus exploratoire, en particulier nous présentons les méthodes de Fourier d'évaluation qui semblent appropriées dans la perspective de la calibration.

Chapitre 1

PRELIMINARIES

This chapter summarizes the basic terminology and notions of Stochastic Calculus and the Mathematical Finance. We focus our attention on definitions and properties we will need in the rest of the thesis.

1.1. Review of Stochastic Calculus

The main purpose of this section is to recall the basic concepts of stochastic calculus needed in this thesis. These elements are taken from various sources but we refer the interested reader to the monographs of Jacod and Shiryaev [**83**], Protter [**111**] and Sato [**113**] to deepen various aspects discussed here.
In particular, we present the main classes of stochastic processes which are foreground objects in the modeling of financial markets. In subsection 1.1.1, we define some concepts and notation used in the sequel. In subsection 1.1.2 we define the important class of semimartingales processes whereas subsection 1.1.3 deals with additive, class of processes which include the Lévy process. Subsection 1.1.4 describes and presents some results of the literature on Markov additive processes which are the main building block focus of our modeling.

1.1.1. Stochastic Notation and Definitions

We start by fixing

- a set $\mathbb{T}$ which represents the time paramater set. This general set can be $\mathbb{T} = \{0, 1, 2, ...\}$, or $\mathbb{T} = [0, \infty)$ or also $\mathbb{T} = [0, T]$. In this thesis, we will make clear which set we are working on when needed ;
- a measurable space $(\mathbb{K}, \mathcal{K})$;
- a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ where the filtration $\mathbf{F} = (\mathcal{F}_t)_{\{t \in \mathbb{T}\}}$ is supposed to be right-continuous, i.e., $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$.

The filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ is called ***stochastic basis***. $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ is said ***complete***, or equivalently, is said to satisfy ***the usual conditions*** if the σ-algebra $\mathcal{F}$ is $\mathbb{P}$-complete and if each $\mathcal{F}_t$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. By convention, we denote $\mathcal{F} = \mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$.

Definition 1.1. *A **stochastic process** (or, a $\mathbb{K}$-valued process) is a family $Y = (Y_t)_{\{t\in\mathbb{T}\}}$ of mappings $Y_t : \Omega \longrightarrow \mathbb{K}$. When $\mathbb{T} = \{0, 1, 2, ...\}$, Y is called **discrete-time** process and when $\mathbb{T} = [0, \infty)$ or $\mathbb{T} = [0, T]$, Y is called **continuous-time** process. $(\mathbb{K}, \mathcal{K})$ is called the state space of Y.*

In this thesis, we consider unless otherwise stated that $\mathbb{K} = \mathbb{R}$ and then all processes used in the sequel will be real-valued.

The process Y may be considered as a map from $\Omega \times \mathbb{T}$ into $\mathbb{R}$ via

$$(\omega, t) \mapsto Y(\omega, t) = Y_t(\omega).$$

In this case, $t \mapsto Y_t(\omega)$ for $\omega \in \Omega$ fixed is called ***sample path*** or ***trajectory*** of the process Y.

Definition 1.2. *A process Y is **càdlàg** if all of the trajectories are right-continuous (in other words, for almost all $\omega \in \Omega$ the mapping $t \mapsto Y_t(\omega)$ is right-continuous, i.e., $\lim_{s\to t^+} Y_s = Y_t$) and admits left limits (i.e., $\exists \lim_{s\to t^-} Y_s =: Y_{t^-}$).*
*Similarly, a process is **càglàd** if all his trajectories are left-continuous and admit right limits.*

Definition 1.3. *The jump of a càdlàg process Y at time t is defined as $\Delta Y_t := Y_t - Y_{t^-}$.*

Definition 1.4. *A process Y is **adapted** to the filtration $\mathbf{F}$ if Y_t is $\mathcal{F}_t$-measurable, for every $t \in \mathbb{T}$.*

Definition 1.5. *A **stopping time** is a mapping $T : \Omega \to [0, \infty]$ such that $\{T \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{T}$.*
*For a stopping time T, the process Y^T defined as $Y^T_t = Y_{T\wedge t}$, is called **process stopped at time T**.*

Definition 1.6.
1) The σ-algebra on $\Omega \times \mathbb{T}$ generated by all càg adapted processes, namely

$$\mathcal{P} := \sigma\{Y : \Omega \times \mathbb{T} \to \mathbb{R} | Y \textit{ is càg}\}$$

*is called **predictable σ-algebra**.*
*2) The σ-algebra on $\Omega \times \mathbb{T}$ generated by all the càdlàg adapted processes is called **optional σ-algebra** and denoted by $\mathcal{O}$.*
*3) A process is said to be **predictable** or **optional** if it is measurable with respect to $\mathcal{P}$ or to $\mathcal{O}$, respectively.*

1.1.2. Basics on Semimartingales

The class of semimartingales is probably the most important in the theory of stochastic calculus since it can provide in all its generality the theory of stochastic integral. Also, because it remains stable with respect to various operations like, for example, change of measure, change of filtration and stochastic change of time. It includes in particular the class of martingales and that of processes with finite or bounded variation.

Before defining the class of semimartingales we need a few definitions.

Definition 1.7. *Let $p \geq 1$. The family of random variables $Y : \Omega \to [0, \infty]$, such that*

$$||Y||_{L^p} := (E^{\mathbb{P}}[|Y|^p])^{\frac{1}{p}} = \Big(\int_{\Omega} |Y|^p d\mathbb{P}\Big)^{\frac{1}{p}} < \infty,$$

is denoted by $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$.

A random variable Y is said to be ***integrable*** *(resp.* ***square-integrable****) if $Y \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ (resp. if $Y \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. We define an equivalence relation on $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$, by setting*

$$X \sim Y \text{ iff } X = Y \ \mathbb{P}\text{-a.s.},$$

for all $X, Y \in \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$. Then $L^p(\Omega, \mathcal{F}, \mathbb{P})$ is defined as the corresponding family of equivalence classes.

Definition 1.8. *A process Y is said to be* ***uniformly integrable(UI)*** *if it satisfies the condition*

$$\lim_{n\to\infty} \sup_{t\in\mathbb{T}} \int_{\{|Y_t|\geq n\}} |Y_t| d\mathbb{P} = 0.$$

Now, we define the notion of martingale.

Definition 1.9. *A process M is called a* ***martingale*** *if*

- *M is adapted with respect to* **F** *;*
- *M_t is integrable, for all $t \in \mathbb{T}$;*
- *$E^{\mathbb{P}}[M_t|\mathcal{F}_s] = M_s$ $\mathbb{P}$-a.s., for all $s \leq t \in \mathbb{T}$ (known as Martingale property).*

We denote by $\mathcal{M}$ the family of all uniformly integrable martingales and by $\mathcal{H}^2$ the sub-class of $\mathcal{M}$ whose elements are square-integrable martingales (i.e., $M \in \mathcal{M}$ and $\sup_{t\in\mathbb{T}} E^{\mathbb{P}}[M_t^2] < \infty$).

Definition 1.10. *A process M is a* ***local martingale*** *if and only if there exists an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times (depending on M) such that $\lim_{n\to\infty} T_n = \infty$ a.s.(almost surely) and that each stopped process M^{T_n} is a martingale. The sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times is called a* ***reducing*** *or* ***localizing sequence****.*

The family of all local martingales (resp. square-integrable local martingales) is denoted by $\mathcal{M}_{loc}$ (resp. $\mathcal{H}^2_{loc}$). The following result gives a necessary condition for a local martingale to be a UI martingale.

Theorem 1.1. *(Protter* **[111]***, Thm I.51) Let Y be a local martingale such that $E^{\mathbb{P}}[\sup_{0\le s\le t}|Y_s|] < \infty$ for every $t \in \mathbb{T}$. Then Y is a martingale.*
If $E^{\mathbb{P}}[\sup_{s\in\mathbb{T}}|Y_s|] < \infty$, then Y is an uniformly integrable martingale.

Another important subclass of family of semimartingales is that of processes with finite or bounded variation which we now define .

Definition 1.11. *A càdlàg and adapted process A such that $A_0 = 0$ is called a* ***bounded variation*** *process if almost all of its sample paths $t \to A_t(\omega)$ are functions with bounded variation over each compact interval $[0, t]$. In other words, for every $t \in \mathbb{T}$ we require*

$$\int_0^t |dA_s| := \lim_{n\to\infty} \sum_{i=0}^{m_n-1} |A_{t^n_{i+1}} - A_{t^n_i}| < \infty, \ \ a.s. \tag{1.1}$$

for all partitions $\pi^n = \{t^n_0, t^n_1, ..., t^n_{m_n}\}$ of $[0, t]$ such that $\lim_{n\to\infty} ||\pi^n|| = 0$ with

$$||\pi^n|| := \sup_{0\le i<m_n} |t^n_{i+1} - t^n_i|,$$

denoting the mesh of the partition.

The family of all bounded variation processes is denoted by $\mathcal{V}$. The sub-class of $\mathcal{V}$ whose elements are integrable (resp. locally integrable) (i.e., $E^{\mathbb{P}}[\int_0^\infty |dA_s|] < \infty$) is denoted $\mathcal{A}$ (resp. $\mathcal{A}_{loc}$).

Now, we can give the definition of a semimartingale.

Definition 1.12. *A* ***semimartingale*** *is a process Y of the form*

$$Y = Y_0 + M + A \tag{1.2}$$

where Y_0 is a finite-valued random variable $\mathcal{F}_0$-measurable, M is a local martingale beginning at 0, and where A a bounded variation process.

A ***special semimartingale*** *is a semimartingale Y which admits a decomposition $Y = Y_0 + M + A$ as above, with a process A that is predictable.*

The space of all semimartingales (resp. special semimartingales) is denoted by $\mathcal{S}$ (resp.$\mathcal{S}_p$).

Remark 1.1.1. *If Y is a special semimartingale then its decomposition $Y = Y_0 + M + A$ with A that is predictable, is unique and is called the* ***canonical decomposition*** *of Y.*

In this thesis, we will use the notion of quadratic covariation of semimartingales. We introduce it in the following.

Definition 1.13.
Let Y a semimartingale. The ***quadratic variation*** *process of Y, denoted by $[Y]$, defined by setting,*

$$[Y]_t := Y_t^2 - Y_0^2 - 2\int_0^t Y_{s-}dY_s,$$

for all $t \in \mathbb{T}$.

Definition 1.14. *Let $Y, Z \in \mathcal{S}$. By using the polarization identity, we define the* ***quadratic covariation*** *process of Y and Y as*

$$[Y, Z] := \frac{1}{2}([Y + Z] - [Y] - [Z]).$$

Remark 1.1.2. *Let $Y, Z \in \mathcal{S}$. By ItÃ´'s differentiation rule, we have that*

$$[Y, Z]_t = Y_t Z_t - Y_0 Z_0 + \int_0^t Z_{s-}dY_s + \int_0^t Y_{s-}dZ_s, \tag{1.3}$$

for every $t \in \mathbb{T}$.

Also, we introduce the notion of compensator or dual predictable projection for a finite variation process.

Proposition 1.1 (Jacod et Shiryaev [**83**], Prop. 3.18)**.**
Let $A \in \mathcal{A}_{loc}$. There exists a process, called the ***compensator*** *ou the* ***dual predictable projection*** *of A and denoted by $\widetilde{A}$, which is unique up to indistinguishability, and which is characterized by being a predictable process of $\mathcal{A}_{loc}$ such that $A - \widetilde{A} \in \mathcal{M}_{loc}$.*

Proposition 1.2 (Jacod et Shiryaev [**83**], Prop. 4.50)**.**
Let $M, N \in \mathcal{M}^2_{loc}$. Then $[M, N] \in \mathcal{A}_{loc}$ and its compensator is $\langle M, N\rangle$. If $M, N \in \mathcal{M}^2$ $MN - [M, N] \in \mathcal{M}$.

With the above, we generalize the concept of compensator to the semimartingales with locally integrable variation.

The notion of orthogonality of two semimartingales is defined as

Definition 1.15. *Two $\mathbb{P}$-semimartingales Y and Z are called* ***orthogonal*** *under a measure $\mathbb{P}$ if $[Y, Z]$ is a local martingale under $\mathbb{P}$.*

In this case, we have $\langle Y, Z\rangle = 0$ and some authors used this last property as the definition of orthogonality.

We now introduce the notion of random measures, namely the random measures associated with the jumps of a semimartingale, and their compensators.

Definition 1.16.

1) A ***random measure*** *on* $\mathbb{T}\times\mathbb{R}$ *is a family* $\pi = \{\pi(\omega; dt, dx) : \omega \in \Omega\}$ *of non-negative measures on* $(\mathbb{T}\times\mathbb{R}, \mathcal{B}(\mathbb{T})\otimes\mathcal{B})$ *satisfying* $\pi(\omega; \{0\}, dx) = 0$ *for all* $\omega \in \Omega$.

2) Let π *be a random measure and let* U *be an optional function, i.e.,* $\mathcal{O}\otimes\mathcal{B}$*-measurable. The* ***integral process*** $U * \pi$ *is defined as*

$$U * \pi_t(\omega) = \begin{cases} \int_{[0,t]\times\mathbb{R}} U(\omega, s, x)\pi(\omega; ds, dx) & \text{if the integral converges} \\ +\infty & \text{otherwise.} \end{cases} \tag{1.4}$$

3) A random measure is said to be ***optional*** *(resp.* ***predictable****) if the integral process* $U * \pi$ *is optional (resp. predictable) for any optional(resp. predictable) function* U.

Definition 1.17. *Let* Y *be a semimartingale. The random measure* N *associated to the jumps or* ***jump measure*** *of* Y *is defined as*

$$N(dt, dx) = \sum_{s>0, s\in\mathbb{T}} 1_{\{\Delta Y_s \neq 0\}} \delta_{\{s, \Delta Y_s\}}(ds, dx), \tag{1.5}$$

where δ_a *denotes the Dirac measure at point* a.

Remark 1.1.3 (Jacod and Shiryaev [**83**], Prop II.1.14, II.1.16)**.**

$N(\cdot,\cdot)$ *is integer-valued and optional. For any nonnegative optional function* U*, we have*

$$U * N(t, \cdot) = \sum_{0<s\leq t} U(s, \Delta Y_s) 1_{\Delta Y_s \neq 0}. \tag{1.6}$$

Theorem 1.2 (Jacod and Shiryaev [**83**], Prop II.1.8)**.**

Let $N(\cdot,\cdot)$ *be the jump measure of* Y*. The dual* ***predictable compensator*** *under* $\mathbb{P}$ *of* $N(\cdot,\cdot)$*, denoted* $\nu^{\mathbb{P}}$ *(which is unique up to a* $\mathbb{P}$*-null set) is the predictable random measure which satisfies one the following equivalent properties :*

(i) $E^{\mathbb{P}}(U * \nu^{\mathbb{P}}_\infty) = E^{\mathbb{P}}(U * N_\infty)$ *for every nonnegative predictable function* U *;*

(ii) For every predictable function U *such that* $|U| * N$ *is finite-valued and locally* $\mathbb{P}$*-integrable (which is equivalent to* $|U| * \nu^{\mathbb{P}}$ *being finite-valued and locally* $\mathbb{P}$*-integrable),* $U * N - U * \nu^{\mathbb{P}}$ *is a local* $\mathbb{P}$*-martingale.*

A key result in the theory of stochastic integration is ItÃ´'s lemma. This is a tool that is often used in applications in finance. We give here a general version valid for the semimartingales.

Theorem 1.3 (Generalized ItÃ´'s Formula, see [**111**]).
Let Y be a semimartingale and let f be a real-valued function twice continuously differentiable, i.e.$f \in \mathcal{C}^2((R))$. Then $f(Y)$ is again a semimartingale, and the following formula holds :

$$\begin{aligned} f(Y_t) - f(Y_0) &= \int_{(0,t]} f'(Y_{s^-})dY_s + \frac{1}{2}\int_{(0,t]} f''(Y_{s^-})d[Y,Y]^c_s \\ &+ \sum_{0<s\leq t} \{f(Y_s) - f(Y_{s^-}) - f'(Y_{s^-})\Delta Y_s\}. \end{aligned} \tag{1.7}$$

1.1.3. Additive Processes (Lévy Processes)

The most used semimartingales in mathematical finance are Lévy processes or more generally additive processes (see Cont and Tankov [**34**]). The main reason of this comes from the property of independence of increments which allows numerical calculations. The definition given below is taken from Sato [**113**].

Definition 1.18.
i) A stochastic process $L = \{L_t\}_{t\geq 0}$ on $\mathbb{R}$ is called a ***Lévy process*** *if the following conditions are satisfied :*

(1) it has independent increments, that is, for any choice of $n \geq 1$ any partition $0 \leq t_0 < t_1 < ... < t_n$, the random variables $L_{t_0}, L_{t_1} - L_{t_0}, ..., L_{t_n} - L_{t_{n-1}}$ are independent ;

(2) it starts at the origin, $\mathbb{P}(L_0 = 0) = 1$, or $L_0 = 0$ a.s. ;

(3) it is time homogeneous or stationary, that is, the distribution of $\{L_{t+s} - L_s : t \geq \epsilon\}$ does not depend on s ;

(4) it is stochastically continuous, that is, for any $\varepsilon > 0$,

$$\lim_{h\to 0} \mathbb{P}(|L_{t+h} - L_t| \geq 0) = 0;$$

(5) as a function of t, $L_t(\omega)$ is càdlàg a.s.

ii) A stochastic process $L = \{L_t\}_{t\geq 0}$ is called an ***additive process*** *if it satisfies (1),(2),(4) et (5).*

Additive processes are intimately related to the infinitely divisible distributions and therefore can be characterized by

Theorem 1.4 (Lévy-Khintchine representation).
Let Y be a real-valued additive process. Then, there is a unique continuous function $\psi : (u,t) \mapsto \psi_t(a)$ defined from $\mathbb{R} \times \mathbb{T}$ to $\mathbb{C}$ such that $\psi_0(u) = 0$ and

$$E^{\mathbb{P}}\left[e^{\mathbf{i}u(Y_t - Y_0)}\right] = e^{\psi_t(u)}, \tag{1.8}$$

for all $u \in \mathbb{R}$ and $t \in \mathbb{T}$.
Also, $\psi_t(u)$ can be written as

$$\psi_t(u) = \mathbf{i}u\gamma_t - \frac{1}{2}u^2\Sigma_t + \int_{\mathbb{R}\times[0,t]} \left(e^{\mathbf{i}ux} - 1 - \mathbf{i}u\ x1_{|x|<1}\right)\nu(ds,dx) \tag{1.9}$$

where Σ_t, γ_t, and ν are uniquely determined and satisfy the following

(1) $t \mapsto \Sigma_t$ is a continuous function from $\mathbb{T}$ to $\mathbb{R}_+$ such that $\Sigma_0 = 0$ and $\Sigma_t - \Sigma_s \geq 0$ for all $s \leq t \in \mathbb{T}$.

(2) $t \mapsto \gamma_t$ is a continuous function from $\mathbb{T}$ to $\mathbb{R}$ such that $\gamma_0 = 0$.

(3) ν is a Borel measure on $\mathbb{T} \times \mathbb{R}$ with $\nu(\mathbb{T}, \{0\}) = 0$, $\nu(\{t\}, \mathbb{R}) = 0$ for all $t \in \mathbb{T}$ and,

$$\int_{\mathbb{R}\times[0,t]} (|x|^2 \wedge 1)\ \nu(ds,dx) < \infty. \tag{1.10}$$

Furthermore, $\{(\Sigma_t, \gamma_t, \nu) : t \in \mathbb{T}\}$ uniquely determines all finite distributions of the process $Y - Y_0$.
Conversely, if $\{(\Sigma_t, \gamma_t, \nu) : t \in \mathbb{T}\}$ is any triplet satisfying the three conditions above, then there exists an additive process satisfying (1.8) and.(1.9).

This extension of the classic Lévy-Khintchine formula to additive process is taken from Lowther [**93**] where the interested readers can find the proof.
The function ψ_t is called the *characteristic exponent* of process Y. Another important characteristic property of additive process is obtained by studying its sample paths.

Proposition 1.3 (Lévy-ItÃ´ decomposition, Sato [**113**]).
Let $Y = \{Y_t\}_{t\in\mathbb{T}}$ be a real-valued additive process with the system of triplets $\{(\Sigma_t, \gamma_t, \nu(\cdot, t))\}$.
For any $G \in \mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathbb{R})$, let $N(G) = N(\omega; G)$ be the number of jumps at time s with height $Y_s(\omega) - Y_{s^-}(\omega) \in G$. Then $N(G)$ has a Poisson distribution with mean $\nu(G)$.
If $G_1, ..., G_n$ are disjoint, then $N(G_1), N(G_2), ..., N(G_n)$ are independent. We can define, for any $t \in \mathbb{T}$ and $\mathbb{P}$-a.s for every ω

$$Y_t^1(\omega) = \lim_{\varepsilon\downarrow 0} \int_{\varepsilon\leq|x|<1, s\in(0,t]} x\{N(\omega; ds,dx) - \nu(ds,dx)\} + \int_{|x|\geq 1, s\in(0,t]} xN(\omega; ds,dx), \tag{1.11}$$

where the convergence in the right-hand is uniform in t for any finite time interval of $\mathbb{T}$ a.s. The process $\{Y_t^1\}$ is a real-valued additive process with the system of

triplets $\{(0, 0, \nu(\cdot, t))\}$.
Let Y^2 *the process defined as*

$$Y_t^2(\omega) := Y_t(\omega) - Y_t^1(\omega). \tag{1.12}$$

Then $Y^2 = \{Y_t^2\}_{t\in\mathbb{T}}$ *is a real-valued additive process continuous in* t *(a.s.) with the system of triplets* $\{(\Sigma_t, \gamma_t, 0)\}$. *The two processes* Y^1 *and* Y^2 *are independent.*

The proof of this can result can be found in Sato[**113**].

1.1.4. Markov Additive Processes

In this section, we introduce the notion of Markov additive process (MAP) as discussed in the seminal papers of Ezhov and Skorohod ([**59**],[**60**]) and Çinlar ([**?**]),[**?**]). For making this presentation clearly, we recall the basic concepts from the theory of Markov processes theory as found in Blumenthal and Getoor [**17**].

We follow the same notation as in subsection 1.1.1 and we consider a measurable space $(E, \mathcal{E})$ where E is a locally compact separable metric space and $(F, \mathcal{F}) = (\mathbb{R}^m, B(R^m)$ the Euclidean space of dimension $m \geq 1$ equipped with its Borel σ-algebra.

Basics about Markov Processes

Definition 1.19. *A function* $P_{s,t}(x, A)$ *defined for* $s \leq t \in \mathbb{T}$, $x \in E$, $A \in \mathcal{E}$ *and taking its values in* $[0, 1]$ *is a* ***transition probability measure*** *on* $(E, \mathcal{E})$ *if*

- $A \to P_{s,t}(x, A)$ *is a probability measure on* $\mathcal{E}$, *for any* $(s, t, x) \in \mathbb{T} \times \mathbb{T} \times E$ *fixed ;*
- $(t, x) \to P_{s,t}(x, A)$ *is a measurable function, for each* $A \in \mathcal{E}$ *and* $(s, t) \in \mathbb{T} \times \mathbb{T}$ *fixed ;*
- $P_{s,s}(x, A) = \delta_x(A)$ *for* $s \in \mathbb{T}$ *;*
- *for* $s \leq u \leq t$ *in* $\mathbb{T}$ *and for* $x \in E$, $A \in \mathcal{E}$

$$P_{s,t}(x, A) = \int P_{s,u}(x, dy) P_{u,t}(y, A). \tag{1.13}$$

A transition probability measure $P_{s,t}(x, A)$ *on* $(E, \mathcal{E})$ *is* ***temporally homogeneous*** *if there exists a measurable function* $P_t(x, A)$ *defined for* $t > 0$, $x \in E$, *and* $A \in \mathcal{E}$ *such that*

$$P_{s,t}(x, A) = P_{t-s}(x, A) \quad \textit{for any } s, t, x \textit{ and } A. \tag{1.14}$$

In this case $P_t(x, A)$ is called a **temporally homogeneous transition probability measure** on $(E, \mathcal{E})$.

Definition 1.20. *Let $\mathcal{F}_t^X := \sigma(X_s : 0 \leq s \leq t)$ be the natural filtration of X augmented with $\mathbb{P}-$null sets of Ω.*
*1) X is a **Markov process** if*

$$\mathbb{P}\Big[X_t \in A\Big|\mathcal{F}_s^X\Big] = \mathbb{P}\Big[X_t \in A\Big|\sigma(X_s)\Big], \quad \textit{for all } s < t \in \mathbb{T} \textit{ and } A \in \mathcal{E}. \tag{1.15}$$

2) If $\{\mathcal{G}_t : t \in \mathbb{T}\}$ is a filtration with $\mathcal{F}_t^X \subset \mathcal{G}_t, \forall t \in \mathbb{T}$, X is a Markov process with respect to $\{\mathcal{G}_t : t \in \mathbb{T}\}$ if (1.15) holds with $\mathcal{F}_t^X$ replaced by $\mathcal{G}_t$.

Remark 1.1.4. *The property 1.15 is generally known as the* **Markov property**.

Definition 1.21. *X is a Markov process with transition probability measure $P_{s,t}(x, A)$ if*

$$E^{\mathbb{P}}\Big[f \circ X_t\Big|\mathcal{F}_s^X\Big] = \int f(y) P_{s,t}(X_s, dy) \tag{1.16}$$

for any $s < t \in \mathbb{T}$ and f a bounded test *function defined on E.*

Definition of a Markov additive process (MAP)

Let $(X, Y) = \{(X_t, Y_t), t \in \mathbb{T}\}$ be a bivariate Markov process on $(E \times F, \mathcal{E} \otimes \mathcal{F})$ with respect to the filtration $\{\mathcal{F}_t, t \in \mathbb{T}\}$ with transition probability measure $P_{s,t}(x, y; A \times B)$. Let $\{Q_{s,t} : s < t; s, t \in \mathbb{T}\}$ be a family of transition probability measures defined from $(E, \mathcal{E})$ into $(E \times F, \mathcal{E} \otimes \mathcal{F})$ and satisfying

$$Q_{s,t}(x, A \times B) = \int_{E \times F} Q_{s,u}(x, dy \times dz) Q_{u,t}(y, A \times (B - z)) \tag{1.17}$$

for any $s < u < t; s, t, u \in \mathbb{T}$, $x \in E$, $A \in \mathcal{E}, B \in \mathcal{F}$ where $B + a = \{b + a : b \in B\}$ for any $a \in F$.

Definition 1.22. *(Çinlar [?])*
*Let $(X, Y) = \{(X_t, Y_t), t \in \mathbb{T}\}$. (X, Y) is a **Markov additive process** with respect to the filtration $\{\mathcal{F}_t, t \in \mathbb{T}\}$ and with semi-Markov transition function $Q_{s,t}$ if*

$$P_{s,t}(x, y; A \times B) = Q_{s,t}(x, A \times (B - y)), \tag{1.18}$$

The above condition means that :

$$P_{s,t}(x, y; A \times B) = P_{s,t}(x, 0; A \times (B - y)). \tag{1.19}$$

Equation (1.19) justifies the name **Markov processes with homogeneous second component** used by Ezhov et Skorohod [**59**] for (X, Y).

Definition 1.23. *(Grigelionis* [**73**]*)*
Let $(X,Y) = \{(X_t,Y_t), t \in \mathbb{T}\}$. (X,Y) *is a* ***Markov additive process*** *with respect to the filtration* $\{\mathcal{F}_t, t \in \mathbb{T}\}$ *if*

$$\mathbb{P}\Big[X_t \in A, Y_t - Y_s \in B \Big| \mathcal{F}_s\Big] = \mathbb{P}\Big[X_t \in A, Y_t - Y_s \in B \Big| X_s\Big] \quad \mathbb{P}\text{-}a.s. \tag{1.20}$$

for all $0 \leq s \leq t \in \mathbb{T}$ *and* $A \in \mathcal{E}$, $B \in \mathcal{F}$.

The first result we can deduce from the definition above is
Proposition 1.4. *(Ezhov and Skorohod* [**59**]*)*
Let $(X,Y) = \{(X_t,Y_t), t \in \mathbb{T}\}$ *be a Markov additive process with respect to the filtration* $\{\mathcal{F}_t, t \in \mathbb{T}\}$. *Then* X *is a Markov process with respect to* $\{\mathcal{F}_t, t \in \mathbb{T}\}$.
Proof. Indeed, for any $s < t \in \mathbb{T}$ and $A \in \mathcal{E}$

$$\begin{aligned} \mathbb{P}\Big[X_t \in A \Big| \mathcal{F}_s\Big] &= \mathbb{P}\Big[X_t \in A, Y_t \in F \Big| \mathcal{F}_s\Big] \\ &= \mathbb{P}\Big[X_t \in A, Y_t \in F \Big| (X_s, Y_s)\Big] \\ &= \mathbb{P}\Big[X_t \in A, Y_t - Y_s \in F \Big| X_s\Big] \\ &= \mathbb{P}\Big[X_t \in A \Big| X_s\Big], \end{aligned} \tag{1.21}$$

where we have successively used the Markov property of (X,Y), the fact that the transition probability measure associated to (X,Y) is translation invariant in Y and the definition of Markov additive process.

Remarque 1.1.1. *From the definition 1.22, if* $\{Q_{s,t} : s < t; s,t \in \mathbb{T}\}$ *is the semi-Markov transition function associated to* (X,Y) *then the component* X *is a Markov process with state space* $(E,\mathcal{E})$ *and transition probability measure* $\{P_{s,t} : s < t; s,t \in \mathbb{T}\}$ *defined for all* $x \in E$, $A \in \mathcal{E}$ $P_{s,t}(x,A) := Q_{s,t}(x, A \times F)$.

Now, we introduce the notation $\mathcal{F}^X_{s,t} := \sigma(X_u : s \leq u \leq t)$, $\mathcal{F}^X_T := \mathcal{F}^X_{0,T}$ and

$$\alpha^t_s(z) = E^{\mathbb{P}}\Big[e^{\mathbf{i}\langle z, Y_t - Y_s\rangle} \Big| \mathcal{F}^X_{s,t}\Big], \quad \text{for } 0 \leq s < t \text{ and } z \in \mathbb{R}^m,$$

which represents the conditional characteristic function of $Y_t - Y_s$ given $\mathcal{F}^X_{s,t}$.

We give here the characteristic properties of a Markov additive process (X,Y) without proofs, the interested reader can consult the references [**59**] and [**73**].
Proposition 1.5 (Grigelionis [**73**]).
For all $0 \leq s < t$ *and* $z \in \mathbb{R}^m$,

$$E^{\mathbb{P}}\Big[e^{\mathbf{i}\langle z, Y_t - Y_s\rangle} \Big| \mathcal{F}_s \vee \mathcal{F}^X_T\Big] = \alpha^t_s(z), \quad \mathbb{P}\text{-}a.s. \tag{1.22}$$

Proposition 1.6 (Ezhov and Skorohod [**59**], Grigelionis [**73**]).
The component Y *is a process with independent increment conditional on the*

σ-algebra $\mathcal{F}_T^X$ generated by all the trajectories of X. In other words, for any $0 \leq s < t \in \mathbb{T}$ and $B \in \mathcal{F}$ we have

$$\mathbb{P}[Y_t - Y_s \in B | \mathcal{F}_s \vee \mathcal{F}_T^X] = \mathbb{P}[Y_t - Y_s \in B | \mathcal{F}_T^X], \quad \mathbb{P}\text{-}a.s. \tag{1.23}$$

or equivalently,
for any integer $n \geq 1$ and subdivision $0 \leq t_1 < t_2 < ... < t_n$ of $\mathbb{T}$, if $(h_i)_1^n$ are $\mathcal{F}$-measurable bounded functions then

$$E^{\mathbb{P}}\Big[\prod_{i=1}^{n} h_i(Y_{t_i} - Y_{t_{i-1}})\Big|\mathcal{F}_T^X\Big] = \prod_{i=1}^{n} E^{\mathbb{P}}\Big[h_i(Y_{t_i} - Y_{t_{i-1}})\Big|\mathcal{F}_T^X\Big], \quad \mathbb{P}\text{-}a.s. \tag{1.24}$$

From these definitions, we can say that a Markov additive process is a bivariate Markov process (X, Y) such that

- X is also a Markov process;
- the future of Y or any measurable function of Y will be independent from its past given the present state of X.

Therefore, X is called the *Markov component* and Y is the *additive component* for the MAP (X, Y).

Examples of Markov Additive Processes

We give here some examples of Markov additive processes. These examples are defined by specifying some conditions on one or the other component of a MAP. Most of the subject of this section are based on the book of Pacheco et *al.*[**106**].

Example 1 : The state space E is a singleton $\{e\}$

For any $0 \leq s < t \in \mathbb{T}$,

$$\begin{aligned} \mathbb{P}[Y_t \in B | Y_s = y] &= \mathbb{P}[X_t = e, Y_t \in B | X_s = e, Y_s = y] \\ &= \mathbb{P}[X_t = e, Y_t - Y_s \in B - y | X_s = e] \\ &= \mathbb{P}[Y_t - Y_s \in B - y], \end{aligned} \tag{1.25}$$

where we have used successively the fact that (X, Y) is a MAP and the fact that Y is conditionally independent. In this case, Y is a process with independent increments or an *additive process*. This allows us to see that the family of Markov additive processes is an extension of that of additive processes. However, it must be said that in general, the additive component Y of a MAP does not have independent increments.

Example 2 : $\mathbb{T} = \mathbb{N}$ and E is discrete

In this case, the Markov additive process (X, Y) is a *Markov Random Walk* on space state $E \times F$ with transition probability measure satisfying, for any $1 \leq m <$

n integers, $B \in \mathcal{F}$; $j, k \in E$ and $y \in F$

$$\begin{aligned}\mathbb{P}[X_n = k, Y_n \in B|\mathcal{F}_m] &= \mathbb{P}[X_n = k, Y_n \in B|X_m = j, Y_m = y] \\ &= \mathbb{P}[X_n = k, Y_n - Y_m \in B - y|X_m = j] \\ &= \mathbb{P}[Y_n - Y_m \in B - y|X_n = k, X_m = j] \\ &\quad \times \mathbb{P}[X_n = k|X_m = j], \end{aligned} \tag{1.26}$$

where we have used the Markov property and the fact that (X, Y) is MAP. Noting that the additive component Y can be written as $Y_n = \sum_{l=1}^{n}(Y_l - Y_{l-1})$, which is like a random walk in the usual sense. When E is finite, the Markov additive process can be characterized by a matrix of transition probabilities

$$F(dx) = (F_{ij}(dx))_{i,j \in E}.$$

Indeed, by setting

- $Z_n = Y_n - Y_{n-1}$;
- $F_{ij}(dx) = P_{i,0}(X_1 = j, Z_1 \in dx)$;
- $p_{ij} = F_{ij}(F)$ the $(i, j)-$element, $i,\ j \in E$, of the matrix of transition probabilities of X, we have

$$F_{ij}(dx) = \mathbb{P}[Z_1 \in dx|X_0 = i, X_1 = j] \times p_{ij}. \tag{1.27}$$

Therefore, given $H_{ij}(dx) := \mathbb{P}[Z_1 \in dx|X_0 = i, X_1 = j]$ we can determine entirely the trajectory of Y once a sample path of X is known. This is how we proceed to simulate a sample path of (X, Y) or (X, S) (see Figure 1.1) as we will see later in chapter 2.

It is worth noting that the example of MAP described above is used to modelling the regime switches in Econometrics (see, e.g., [**75**]) and in the switching ARCH[1] modelling of volatility in the stock market.

Example 3 : $\mathbb{T}$ is continuous time parameter set and E is finite

Here, the component X is specified by an intensity matrix $\Lambda = (\lambda_{ij})_{i,j \in E}$, and

- On an interval $[s, s + t)$ where $X_s \equiv i$, Y_s evolves like a Lévy process with characteristic triplet $(\sigma_i^2, \gamma_i, \nu_i(dx))$ depending on i ;
- A transition of X from i to j, $j \neq i$ has a probability q_{ij} of giving rise to a jump of Y at the same time, the distribution of which has some distribution B_{ij}.

For the last two examples, as the state space E of the component X is finite, the Markov additive process can be completely characterized by its moment generating function which here can be expressed as a matrix. Thus, for a MAP (X, Y) if $\widehat{F}_t[\alpha]$ is the matrix with $(i, j)-$element, $i,\ j \in E$ $E^{\mathbb{P}}[e^{\alpha Y_t} 1_{\{X_t = j\}}|X_0 = i]$ then

[1]AutoRegressive Conditional Heteroscedascity

Theorem 1.5. *(Asmussen [**7**])*
1) If $\mathbb{T}$ is discrete, then $\widehat{F}_n[\alpha] = (\widehat{F}[\alpha])^n$ where

$$\begin{aligned}\widehat{F}[\alpha] &= \widehat{F}_1[\alpha] \\ &= (E^{\mathbb{P}}[e^{\alpha Y_1}\{X_1 = j\}|X_0 = i])_{i,j\in E} \\ &= (\widehat{F}_{ij}[\alpha])_{i,j\in E} \\ &= (p_{ij}\widehat{H}_{ij}[\alpha])_{i,j\in E}. \qquad (1.28)\end{aligned}$$

2) if $\mathbb{T}$ is continuous, then the matrix $\widehat{F}_t[\alpha]$ is given by $e^{tK[\alpha]}$, where

$$K[\alpha] = \Lambda + \mathbf{Diag}(\kappa^{(i)}(\alpha)) + (\lambda_{ij}q_{ij}(\widehat{B}_{ij}[\alpha] - 1)),$$

with $\kappa^{(i)}(\alpha) = \alpha\gamma_i + \frac{1}{2}\alpha^2\sigma_i^2 + \int_{\mathbb{R}\setminus\{0\}}(e^{\alpha x} - 1 - \alpha x 1_{|x|\leq 1})\nu_i(dx)$ the characteristic exponent of Lévy process $Y^{(i)}$. In this case, $\Lambda, \gamma_i, \sigma_i^2, \nu_i(dx), q_{ij}, B_{ij}$ are called parameters of the Markov additive process (X, Y).

FIG. 1.1. Sample path of a regime switching VG process.

1.2. Review of Mathematical Finance

The literature goes back to 1990, with Bachelier **[8]** as the first attempt to use probabilistic models to describe financial markets. However, it is the work of Black and Scholes **[15]** and Merton **[98]** that has paved the way of mathematical finance whose main purpose is the pricing and hedging of *derivatives*, i.e., contract which provides its holder a future payment that depends on the price of one or several primitive security(ies). The asset underlying a derivative may be a stock, a stock index, an interest rate, a foreign currency, or a commodity.

In this section, we will review main concepts of mathematical finance such as *contingent claim*, *arbitrage*, etc. and relate them to the main result named the *Fundamental theorem of asset pricing*. The main reference of this section is Cherny **[29]**.

1.2.1. Basic Notions

We start by fixing a finite time-horizon T and a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$. All processes are defined on the above stochastic basis (in particular, they are defined over the time interval $[0, T]$ and are adapted to the filtration $\mathbf{F} = (\mathcal{F}_t)_{\{t\in\mathbb{T}\}}$, which we shall assume obeys the usual conditions).

We consider a *frictionless market*[2] with a single (for sake of simplicity) risky asset available for hedging, denoted by S, and a bank account, denoted by B. The process Y will represent the discounted asset, i.e., $Y = S/B$, and we suppose that it is a special semimartingale.

Let $\mathcal{L}^2(Y)$ be the space of all $\mathbb{R}$-valued predictable process ϕ such that

$$||\phi||_{\mathcal{L}^2(Y)} := \Big(E^{\mathbb{P}}\Big[\int_0^T \phi_u^2 d[Y,Y]_u\Big]\Big)^{\frac{1}{2}} < \infty.$$

Definition 1.24. *A* ***trading strategy*** *or* ***dynamic portfolio*** *is a pair of processes $\Phi = (\phi, \psi)$ where ψ is an $\mathbf{F}$-adapted process and $\phi \in \mathcal{L}^2(Y)$ is an $\mathbf{F}$-predictable process, such that the (discounted) value process*[3] *$V := \phi Y + \psi$ has right-continuous sample paths and $E^{\mathbb{P}}[V_t^2] < \infty$ for every $t \in \mathbb{T}$ (i.e., $V_t \in \mathcal{L}^2(\Omega, \mathbb{P})$ for every $t \in \mathbb{T}$).*

For a trading strategy $\Phi = (\phi, \psi)$, $\phi = (\phi_t)_{t\in\mathbb{T}}$ denotes the number of units of the stock held at time t and $\psi = (\psi_t)_{t\in\mathbb{T}}$ the discounted amount invested in the money market account at time t.

[2] i.e., there are no transaction costs, assets are divisible and it is possible to borrow from bank and sell short without restriction.

[3] The concept of stochastic integration is related to the idea of calculating the change in value of the portfolio over time. So, we need to impose some conditions for its existence.

Definition 1.25. *A trading strategy* Φ *such that*

$$V_t(\Phi) = V_0(\Phi) + \int_0^t \phi_s dY_s,$$

for all $t \in \mathbb{T}$ *is called* ***self-financing****.*

Definition 1.26. *A* ***contingent claim*** *or* ***financial derivative*** *is any positive random variable* H *of the form* $H = F(S_T)$*, where* F *is a measurable function and* S *the stock price process above. In other words, a contingent claim is a* $\mathcal{F}_T$*-measurable function. It is characterized by its payoff* H *and its maturity* T*.*

Definition 1.27. *Let* H *be a contingent claim which is* $\mathcal{F}_T$*-measurable and square-integrable.* H *is said to be* ***attainable*** *or* ***replicable*** *if there exists a trading strategy* Φ *such that*

$$V_T = H, \ \ \mathbb{P}\text{-a.s.}$$

In that case, we say that Φ *is a* ***hedging*** *or a* ***replicating*** *strategy for* H*.*

Definition 1.28. *A trading strategy* $\Phi = (\phi, \psi)$ *is said to be* ***admissible*** *if it is self-financing and if there exists some* $\alpha \in \mathbb{R}_+$ *such that* $\int_0^t \phi_s dY_s > -\alpha$*, a.s. for all* $t \in \mathbb{T}$*. Roughly speaking, the amount of money one can borrow and invest, and the amount of stock one can short are limited.*

Definition 1.29. *A market model is* ***complete*** *if for each bounded* $\mathcal{F}_T$*-measurable* H*, there exists an admissible trading strategy* Φ *such that*

(1) there exists constants a *and* b *such that*

$$\mathbb{P}\left(\forall t \in [0,T], a \leq V_t(\Phi) \leq b\right) = 1;$$

(2) $H = V_T(\Phi)$*.*

1.2.2. Equivalent Probabilities and Change of Measure

We discuss in this section a main tool in mathematical finance, the change of measure, which shows how martingales and more generally semimartingales are changed when a new absolutely continuous probability measure is introduced. Historically, it is the work of Harrison and Pliska [**79, 80**] who, by reformulating the results of Black and Scholes, showed the importance of the change of measure through the notion of an *equivalent martingale measure*.

Definition 1.30. *A probability measure* $\mathbb{Q}$ *on* $(\Omega, \mathcal{F})$ *is said to be* ***absolutely continuous with respect to*** $\mathbb{P}$*, indicated by* $\mathbb{Q} \ll \mathbb{P}$*, if*

$$\mathbb{P}(A) = 0 \Longrightarrow \mathbb{Q}(A) = 0.$$

If $\mathbb{P} \ll \mathbb{Q}$ *and* $\mathbb{Q} \ll \mathbb{P}$*, then the measures are said to be* ***equivalent****, indicated by*

$$\mathbb{Q} \sim \mathbb{P}.$$

Theorem 1.6 (Radon-Nikodym, see [**83**]).
Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$. If $\mathbb{Q} \ll \mathbb{P}$, then there exists an $\mathbb{P}$-a.s. unique random variable $Z \geq 0$ satisfying, $E^{\mathbb{P}}[Z] = 1$, such that

$$\mathbb{Q}(A) = E^{\mathbb{P}}[Z 1_A],$$

for all $A \in \mathcal{F}$. If $\mathbb{Q} \sim \mathbb{P}$, then $Z > 0 \quad \mathbb{P}$-a.s..

The random variable Z defined in Theorem 1.6 is called the **Radon-Nikodym derivative** of $\mathbb{Q}$ with respect to $\mathbb{P}$ and is often written as $\frac{d\mathbb{Q}}{d\mathbb{P}} =: Z$.

We now give a result which relates conditional expectations under two different measures.

Theorem 1.7 (Conditional Bayes' Theorem or Bayes' rule, see [**2**]).
Suppose $\mathcal{G} \subset \mathcal{F}$ is a sub-σ-field. Suppose $\mathbb{Q}$ is another probability measure absolutely continuous with respect to $\mathbb{P}(\mathbb{Q} \ll \mathbb{P})$ and with a Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z.$$

Then if U is any integrable $\mathcal{F}$-measurable random variable,

$$E^{\mathbb{Q}}[U|\mathcal{G}] = \begin{cases} \frac{E^{\mathbb{P}}[ZU|\mathcal{G}]}{E^{\mathbb{P}}[Z|\mathcal{G}]} & \text{if } E^{\mathbb{P}}[Z|\mathcal{G}] > 0, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{1.29}$$

Theorem 1.8 (Girsanov-Meyer theorem, see [**111**]).
Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent. Let Y be a semimartingale under $\mathbb{P}$ with decomposition $Y = M + A$. Then Y is also a semimartingale under $\mathbb{Q}$ and has a decomposition $Y = L + C$, where

$$L_t = M_t + \int_0^t \frac{1}{Z_s} d[Z, M]_s$$

is a $\mathbb{Q}$-local martingale, and $C = Y - L$ is a finite variation process under $\mathbb{Q}$.

1.2.3. Fundamental Theorems of Asset Pricing

A fundamental concept in Finance is the *absence of arbitrage* which states that there is no opportunity to make an instantaneous profit without risk. This economic principle is the foundation of many results in Mathematical Finance, notably the *Fundamental theorem of asset pricing.* But this notion is "mathematically" too weak to provide the generality of this theorem as proved by Delbaen and Schachermayer [**41**]. It is replaced by the notion of *No free lunch with vanishing risk (NFLVR).*

Definition 1.31. *An **arbitrage** possibility is a self-financing strategy Φ with the properties*

a) $V_0 = 0$;

b) $\mathbb{P}(V_T(\Phi) \geq 0) = 1$ *;*

c) $\mathbb{P}(V_T(\Phi) > 0) > 0$.

Definition 1.32 (Cherny [**29**])**.**
A sequence of strategies $\Phi_k = (\phi_k, \psi_k)$ *realizes* ***free lunch with vanishing risk*** *if*

i) For each k, $\psi_k = 0$ *;*

ii) For each k*, there exists* a_k *such that*

$$\mathbb{P}\left(\forall t \in [0,T], V_t(\Phi_k) \geq a_k\right) = 1;$$

iii) For each k, $V_t(\Phi_k) \geq -\frac{1}{k}, \quad \mathbb{P}$*-a.s. ;*

iv) There exist $\delta_1 > 0, \delta_2 > 0$ *such that, for each* k*,*

$$\mathbb{P}\left(V_T(\Phi_k) > \delta_1\right) > \delta_2.$$

A model satisfies the NFLRV condition if such sequence of strategies does not exist.

Before stating the result which links the concept of NFLVR to the existence of risk-neutral measures, we recall the following definitions.

Definition 1.33. *A probability measure* $\mathbb{Q}$ *on* $(\Omega, \mathcal{F})$ *is called an* ***equivalent (local) martingale measure*** *or a* ***risk-neutral measure*** *if* $\mathbb{Q} \sim \mathbb{P}$ *and the discounted price process* Y *is a (local) martingale under* $\mathbb{Q}$*.*

Definition 1.34. *A process* Y *is an* $(\mathcal{F}_t, \mathbb{P})$ σ*-martingale if there exists an* $(\mathcal{F}_t, \mathbb{P})$ *local martingale* M *and a* $\mathbf{F}$*-predictable* η *such that*

$$Y_t = Y_0 + \int_0^t \eta_u dM_u, \quad \forall t \in [0,T] \tag{1.30}$$

Theorem 1.9 (The First fundamental theorem of asset pricing)**.**
A semimartingale market model (B, Y) *as defined above, admits* ***no free lunch with vanishing risk*** *if and only if there is a probability measure* $\mathbb{Q}$ *equivalent to* $\mathbb{P}$ *such that* Y *is a* σ*-martingale (or martingale transform) under* $\mathbb{Q}$*.*
If Y *is bounded (resp. locally bounded) the term* σ*-martingale may equivalently be replaced by the term martingale (resp. local martingale).*

We can also formulate the *Second fundamental theorem of asset pricing* which can be viewed as the counterpart of the previous result. We begin by define the *predictable Representation property.*

Definition 1.35. *A process Y has the* ***predictable representation property (PRP)*** *with respect to the probability measure $\mathbb{Q}$ if for any $(\mathcal{F}_t, \mathbb{Q})$- local martingale $M = (M_t)_{t\in\mathbb{T}}$, there exists a predictable, Y-integrable, process ξ such that*

$$M_t = M_0 + \int_0^t \xi_s dY_s.$$

Theorem 1.10 (The Second fundamental theorem of asset pricing)**.**
Suppose a semimartingale market model (B, Y) satisfies the NFLVR condition. Then, the following conditions are equivalent

i) the model is complete ;

ii) the equivalent σ-martingale measure $\mathbb{Q}$ is unique ;

iii) there exists an equivalent σ-martingale measure $\mathbb{Q}$ such that Y has the predictable representation property with respect to $\mathbb{Q}$.

Completeness is an ideal property for a market model. There are few examples of complete markets in continuous-time setting. We can nevertheless mention the models based on a Brownian motion or on a Poisson process. Indeed, we have the following result.

Theorem 1.11 (Cherny[**29**])**.**
Consider a model where the discounted stock price is given by

$$S_t^* = S_0 \exp(Y_t), \; S_0 > 0$$

and which satisfies the NFLVR condition. Moreover, we impose that $\mathcal{F}_t = \mathcal{F}_t^S$. Then the model is complete only in the following cases :

(i) $Y_t = \mu t + \sigma W_t$, where W is a standard Brownian motion and $\sigma > 0$;

(ii) $Y_t = bt + \delta N_t$, where N is a Poisson process with intensity λ and $\delta b < 0$.

The classical Black-Scholes model is then complete as shown in Harrison and Pliska [**79**]. Also, we deduce from this result that the exponential-Lévy models and the Regime-switching exponential-Lévy models are incomplete.

Chapitre 2

ON THE PRICE OF RISK OF THE UNDERLYING MARKOV CHAIN IN A REGIME-SWITCHING EXPONENTIAL LÉVY MODEL

Ce chapitre est basé sur l'article de Momeya et Morales [**101**], qui a été revisé et resoumis à la revue *Methodology and Computing in Applied probability*. Le premier auteur est Romuald Momeya et le second auteur est le directeur de recherche Manuel Morales.

Résumé

Dans ce chapitre, à la suite d'un article de Siu et Yang [**123**], nous nous interrogeons sur l'impact sur le prix d'une option Européenne que peut avoir la prise en compte ou non du risque lié aux changements de régime. L'intérÃªt de cette question vient de l'hypothèse généralement admise dans la littérature de la non-prise en compte de ce risque. Pour ce faire, nous construisons deux noyaux d'évaluation basés sur la transformation d'Esscher dont l'un prend en compte le risque de régime et l'autre non. Ceci est rendu possible par la substitution dans la transformation d'Esscher de l'espérance inconditionnelle habituelle par une espérance conditionnelle sachant l'information disponible sur la chaÃ®ne de Markov responsable des changements de régime. Les résultats numériques obtenus par simulation montrent que l'impact de la prise en compte du risque de régime est assez significatif et semble Ãªtre mitigé par la présence des sauts (du processus additif). En particulier, nous comparons les résultats obtenus avec ceux de la littérature, notamment Naik [**103**] Boyle et Draviam [**20**].

Abstract

Regime-switching models (RSM) have been recently used in the literature as alternatives to the Black-Scholes model. Several authors favor RSM as being more realistic since, by construction, they model exogenous macroeconomic cycles against which asset prices evolve. In the context of derivatives pricing, these models lead

to incomplete markets and therefore there exist multiple Equivalent Martingale Measures (EMM) yielding different pricing rules. A fair amount of literature (Buffington and Elliott[**24**], Elliott *et al.* [**50**]) focuses on conveniently choosing a family of EMM leading to closed-form formulas for option prices. These studies often make the assumption that the risk attributed to transitions in the state of the Markov chain is not priced. Recently, Siu and Yang [**123**], proposed an EMM kernel that takes into account all risk components of a regime-switching Black-Scholes model. In this chapter, we extend the setting in Siu and Yang [**123**], into a more general Lévy regime-switching model that allows us to assess the influence on the price of risk of jumps in the price process . We specialize this general framework to Regime-switching Jump-Diffusion and Variance-Gamma models and carry out a comparative analysis of the resulting option price formulas with existing regime-switching models such as Naik [**103**] and Boyle and Draviam [**20**].

2.1. Introduction

Regime-switching models (RSM) were originally introduced in order to model the canvas of macroeconomic cycles against which asset prices evolve (Hamilton [**75**]). These cycles are modeled by an underlying Markov chain that drives the asset prices through structurally different market scenarios against which asset prices evolve with different characteristics. Since nineties, many papers concerning option pricing in a regime-switching setup have been published with a view towards applications. In particular, it is interesting to mention recent applications in insurance where regime-switching processes are suitable models for equity-linked insurance products [see Hardy [**77**], Siu [**121**]]. This suitability is due to the long term maturity of these products where economic cycles need to be taken into account. Regardless of the domain of application, most of the available literature assume that the risk due to the underlying Markov chain is not priced (or is diversifiable), i.e., no premium is paid for such a risk. For example, Kijima and Yoshida [**87**] in a discrete-time model with Markov volatilities made some assumptions which lead to the price of volatility risk to be zero. Bollen [**18**] in his regime-switching lattice model supposes explicitly that the "regime-risk" is not priced in the market. This assumption allowed him to work with the risk-neutrality argument. Boyle and Draviam [**20**] study a Black-Scholes model where the volatility parameter depends upon a hidden Markov chain and they also assume, for the sake of simplicity, that the volatility risk is not priced. Recently, in Lin *et al.* [**91**] we find an insurance application where equity-linked insurance are studied under a regime-switching model.

Among the above mentioned articles, to the best of our knowledge, only Naik [**103**] discussed the situation where the risk due to the jumps in volatility is nondiversifiable (or priced). In his article, he makes a risk adjustment to the persistence parameters of his model. Recently Siu and Yang [**123**], Elliott and Siu [**56**] and Elliott et *al.* [**57**] have explicitly addressed the issue of pricing the risk due to the underlying Markov chain for a Regime-switching Black-Scholes model.

The aim of this chapter is two-fold. First, we generalize the framework and results first introduced in Siu and Yang [**123**] to a more general setting. We study option prices under a Regime-switching exponential Lévy model. In particular, we derive expressions for option prices in the special cases of a regime-switching jump-diffusion and variance-gamma models. Second, we carry out numerical analyses in order to assess the impact of pricing versus not pricing the risk associated with the underlying Markov chain. In these analyses, we compare two conceptually different derivative pricing rules : one that takes into account the risk associated with the Markov chain and a second one that does not. We find a significant gap between the prices given by these two pricing rules which illustrates the potential error that can be made when the risk of the underlying Markov chain is not taken into account.

The chapter is organized as follows. Section 2.2, presents a general Regime-switching Exponential Lévy model and lays down the main concepts and notation used throughout this chapter.

In Section 2.3, we discuss how we can define two EMM kernels leading to two conceptually different derivative pricing rules. These two kernels correspond to two assumptions : pricing and not pricing the risk associated with the Markov chain. Some special cases are studied in more depth in Section 2.4. In particular the Black-Scholes regime switching model of Boyle and Draviam [**20**] and Naik [**103**]. Finally, in Section 2.5, we present a simulation analysis of our model and a comparative study of the impact of pricing versus not pricing the risk embedded in the Markov chain. Section 2.6 concludes our work and discusses some future work.

2.2. A General Regime-Switching Exponential Lévy Model

2.2.1. Description of the Model

We consider a financial market with two primary securities, namely a money market account B and a stock S which are traded continuously over the time horizon $\mathbb{T} := [0, T]$ where $0 < T < \infty$.

In order to formally define this market, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ is the real-world probability.

Let $X := (X_t)_{t\in\mathbb{T}}$ denote an irreducible homogeneous continuous-time Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ with a finite state space $\mathbb{S}$. Let $M \in \mathbb{N}$ be the size of the state space $\mathbb{S}$, then X is characterized by a rate (or intensity) matrix $\mathbb{A} := \{a_{ij} : 1 \leq i, j \leq M, t \in \mathbb{T}\}$.

For simplicity sake, we follow the notation of Elliott [**49**] and we identify $\mathbb{S}$ with the canonical basis of the linear space $\mathbb{R}^M$. Let us denote the i^{th} element of the canonical basis by $\mathbf{e}_i$, i.e., $\mathbf{e}_i = (0, 0, \ldots, \underbrace{1}_{\text{i-th}}, \ldots, 0)$. In this chapter, we set the state space to be $\mathbb{S} = \{\mathbf{e}_1, \ldots, \mathbf{e}_M\}$. This implies that the process X is a vector-valued Markov chain taking values in $\mathbb{R}^M$, i.e., $X \in \mathbb{S} = \{\mathbf{e}_1, \ldots, \mathbf{e}_M\} \subset \mathbb{R}^M$.

Now, we can model the stochastic evolution of the instantaneous interest rate $r = (r_t)_{t\in\mathbb{T}}$ of the money market account B at time t as follows

$$r_t := \langle \underline{r} | X_t \rangle = \sum_{i=1}^{M} r_i \langle \mathbf{e}_i | X_t \rangle, \tag{2.1}$$

where $\langle \cdot | \cdot \rangle$ is the usual scalar product in $\mathbb{R}^M$ and $\underline{r} = (r_1, \ldots, r_M) \in \mathbb{R}_+^M$. Here the value r_i, the i^{th} entry of the vector $\underline{r}$, represents the value of the interest rate when the Markov chain is in the space state $\mathbf{e}_i$, i.e., when $X_t = \mathbf{e}_i$. The price dynamics of B can now be written as

$$B_t = B_0 \exp\Big(\int_0^t r_s ds \Big), \; B_0 = 1, \quad t \in \mathbb{T} \tag{2.2}$$

Moreover, let μ_t and σ_t denote respectively the mean return and the volatility of the stock S at time t. Using the same convention, we define the following processes

$$\begin{aligned} \mu_t &= \langle \underline{\mu} | X_t \rangle = \sum_{i=1}^{M} \mu_i \langle \mathbf{e}_i | X_t \rangle \;, \\ \sigma_t &= \langle \underline{\sigma} | X_t \rangle = \sum_{i=1}^{M} \sigma_i \langle \mathbf{e}_i | X_t \rangle \;, \end{aligned}$$

where

$$\underline{\mu} = (\mu_1, \mu_2, ..., \mu_M) \in \mathbb{R}^M \;,$$

and

$$\underline{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_M) \in {\mathbb{R}_+}^M \;.$$

In a similar way, μ_i and σ_i represent respectively the mean value and volatility of S when the Markov chain is in state $\mathbf{e}_i$, i.e., when $X_t = \mathbf{e}_i$.

The price dynamics of the stock S can now be described by the following exponential Markov modulated Lévy process :

$$S_t = S_0 \exp(Y_t), \quad S_0 > 0 \tag{2.3}$$

with

$$\begin{aligned} Y_t &= \int_0^t \Big(\mu_s - \frac{1}{2}\sigma_s^2 \Big) ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z \tilde{N}^X(ds, dz) \\ &\quad - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z) \rho^X(ds, dz). \end{aligned} \tag{2.4}$$

Where $\widetilde{N}^X(dt, dz) := N^X(dt, dz) - \rho^X(dt, dz)$ denote the compensated random measure associated to $N^X(dt, dz)$. $N^X(dt, dz)$ denotes the differential form of a Markov-modulated random measure on $\mathbb{T} \times \mathbb{R}\setminus\{0\}$. We recall from Elliott and Osakwe **[50]** and Elliott and Royal **[55]** that a Markov-modulated random measure on $\mathbb{T} \times \mathbb{R}\setminus\{0\}$ is a family $\{N^X(\omega; dt, dz) : \omega \in \Omega\}$ of non-negative measures on the measurable space $(\mathbb{T} \times \mathbb{R}\setminus\{0\}, \mathcal{B}(\mathbb{T}) \otimes \mathcal{B}(\mathbb{R}\setminus\{0\}))$, which satisfy

$N^X(\omega;\{0\},\mathbb{R}\backslash\{0\}) = 0$ and has the following compensator, or dual predictable projection :

$$\rho^X(dt,dz) := \sum_{i=1}^{M} \langle \mathbf{e}_i | X_{t^-} \rangle \ \rho_i(dz)dt \tag{2.5}$$

where $\rho_i(dz)$ is the density for the jump size when the Markov chain X is in state $\mathbf{e}_i$. We suppose that

$$\sum_{i=1}^{M} \int_B \int_{\mathbb{R}\backslash\{0\}} \min(z^2,1)\rho_i(dz)dt < \infty \tag{2.6}$$

for each Borel set $B \in \mathcal{B}(\mathbb{T})$.
Let $W := (W_t)_{t\in[0,\infty)}$ be a standard Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$ which is supposed independent of X and N^X.
We refer to the model defined by (2.3) as a Regime-Switching exponential Lévy model. In such a model, the evolution of asset prices is influenced by the Markov chain X representing a macroeconomic regime of the market. One feature of this model is that during the time spent by the Markov chain in any given regime, the asset price evolves as an exponential Lévy process. And as the background environment moves from one regime to another, the stock prices are also modeled by different exponential Lévy processes representing the asset dynamics under the current state of the market. The model in (2.3) is a generalization of existing models and it has been constructed using the readily available theory of Markov Additive Processes [See, e.g., Cinlar ([**?, ?**]), Grigelionis [**73**], Asmussen [**7**] and Pacheco *et al.*[**106**]]. Despite being a natural extension of previous regime-switching models, it has been little studied in the literature in a general form [See Chourdakis[**30**]]. Interestingly, equation (2.3) contains, as particular cases, several models that have appeared in the literature. In the following subsection we briefly discuss particular cases that are of interest to our discussion.

2.2.2. Some Particular Models

The Regime-Switching Black-Scholes Model

The simplest regime switching model in continuous time that can be found in the literature is the *Black-Scholes regime-switching* model. Introduced in Kijima and Yoshida[**87**], it assumes a number M known, of states or regimes and considers that within each possible market regime, asset prices evolve according to a geometric Brownian motion. Each regime specifies different model parameters accounting for different price dynamics. We can recuperate the Black-Scholes regime-switching model from equations (2.3) by setting $N^X(.;.) \equiv 0$. In such a case, there are no jumps and the equation (2.3)-(2.4) reduce to a regime-switching geometric Brownian motion. We refer the reader to Di Masi *et al.*[**39**], Guo [**74**] and Boyle and Draviam [**20**] for comprehensive studies regarding this model. This model is the object of our attention in Section 2.5 where we provide numerical illustrations of the results of this chapter.

The Naik Model

A second model that takes into account possible shifts in macroeconomic market environments through regime switching is the one in Naik [**103**]. He generalizes the previous ones by introducing some jumps in the dynamic of stock price. In Naik [**103**], he assumes two distinct market regimes where in each the risky asset dynamics is a jump-diffusion process. The jumps have only two sizes and occur only whenever there is a regime shift. From equations (2.2)-(2.3), we can also recuperate this model by making the following assumptions,

(1) Two regimes, i.e., $M = 2$,
(2) One interest rate and one mean return across regimes, i.e., $r_1 = r_2$ and $\mu_1 = \mu_2$;
(3) Within each Markov state, $\mathbf{e}_1$ or $\mathbf{e}_2$, the Poisson random measure N^i (for $i = 1, 2$) has as a compensator λ_i ;
(4) The size of jump in the stock price level takes only one value y_i in each Markov state, $\mathbf{e}_i$ ($i = 1, 2$).

In Section 2.5, we look for a slightly modified version of the Naik model ; in particular we suppose that the size of jumps is gaussian.

Other Models

In Elliott and Osakwe [**50**], the authors have introduced a model for asset prices in terms of the exponential of a pure-jump process with an M-state Markov switching compensator. Their model extended that of Konikov and Madan [**88**] in which a two-state Markov chain modulates two Variance-Gamma processes. In this chapter, the model defined in equation (2.3) contains in fact a family of models that evolve as a Lévy process within each possible market regime very much like those in Elliott and Osakwe [**50**]. Well known examples that could be used are Normal Inverse Gaussian, Hyperbolic and Variance-Gamma Lévy processes. We could also remark that if we set $M = 1$, i.e., the model have just one regime, then we retrieve the well-known family of exponential-Lévy models. In Section 2.5, we work out a numerical illustration of such a model based on the Variance-Gamma Lévy process.

2.3. Two Conceptually Different Pricing Kernels

One of the main features of the Regime-switching exponential Lévy model is that it leads to an incomplete market ; that means that there exist infinitely many equivalent martingale measures describing the risk-neutral price evolution. Each of these measures gives rise to a set of derivatives prices compatible with no arbitrage requirement, hence the problem of selecting one of them is crucial. A popular choice for finding an equivalent martingale measure is based on the Esscher transform [See, e.g., Gerber and Shiu [**71**], Kallsen and Shiryaev[**86**]].

In the framework of Regime-switching model, Elliott and his coauthors [See Elliott *et al.* [**50**], Siu and Yang [**123**]] introduced the notion of *Regime-switching*

Esscher transform in order to price contingent claims. This change of measure is an adaptation of the concept of *conditional Esscher transform* introduced by Bühlmann *et al.*[**23**]. The form of this new change of measure includes the underlying Markov chain which causes the regime shift. The difference with the standard Esscher transform change of measure lies in the expectation operator used in its definition. It is standard to define the Esscher transform through the moment generating function (mgf) of the random variable or stochastic process at hand. In the Regime-switching extension of Elliott *et al.*[**50**], due to the particular role of the Markov Chain X, the calculation of the mgf is conditional to a subset of the information available on X. This gives rise to two *a priori* different pricing kernels based on the conditional Esscher transform. In the following Section we discuss these concepts further.

2.3.1. Pricing Kernel that Ignores the Risk Associated with the Markov Chain

Here, we present the construction of an Esscher change of measure that produces a pricing kernel that does not take into account the risk associated with the Markov chain. In other words, this change of measure is based on a conditioning argument that assumes knowledge of the whole history (past and future) of the underlying Markov chain. In the literature, this is often referred to as ignoring the risk associated with the Markov chain [See, for example Naik [**103**], Boyle and Draviam [**20**], Siu and Yang [**123**] and Lin *et al.*[**91**]]. Following Elliott *et al.*[**50**], we start by introducing some notation.
Let $\mathcal{F}^X := \{\mathcal{F}_t^X\}_{t\in\mathbb{T}}$ and $\mathcal{F}^Y := \{\mathcal{F}_t^Y\}_{t\in\mathbb{T}}$ denote the right-continuous, $\mathbb{P}$-completed filtrations generated by X and Y respectively. Moreover, we define for $t \in \mathbb{T}$, $\overline{\mathcal{G}}_t := \mathcal{F}_T^X \vee \mathcal{F}_t^Y$ and $\overline{\mathbf{G}} := \{\overline{\mathcal{G}}_t : t \in \mathbb{T}\}$.
We also make use of the following definition of Siu and Yang [**123**]. We set

$$\Theta := \Bigg\{(\theta_t)_{t\in\mathbb{T}} \mid \theta_t := \sum_{i=1}^{M} \theta_i \langle X_{t-}|\mathbf{e}_i\rangle \text{ with } (\theta_1, \theta_2, ..., \theta_M) \in \mathbb{R}^M \\ \text{such that } E^{\mathbb{P}}\left[e^{-\int_0^t \theta_r dY_r}\middle|\mathcal{F}_T^X\right] < \infty\Bigg\}.$$

And for $\theta := (\theta_t)_{t\in\mathbb{T}} \in \Theta$, we define the generalized Laplace transform of a $\overline{\mathbb{G}}$-adapted process Y as

$$\mathcal{M}_Y(\theta)_t := E^{\mathbb{P}}\left[e^{-\int_0^t \theta_s dY_s}\middle|\mathcal{F}_T^X\right]. \tag{2.7}$$

Notice that the expectation is taken conditional on the information of all the future values of the Markov chain X. With this extended definition of a Laplace transform, we can now define the kernel of a generalized Esscher transform (with respect to the parameter θ, thus called *Esscher parameter*).

Let $\Lambda = \{\Lambda_t\}_{t\in\mathbb{T}}$ denote a $\overline{\mathbf{G}}$-adapted stochastic process defined as

$$\Lambda_t := \frac{e^{-\int_0^t \theta_s dY_s}}{\mathcal{M}_Y(\theta)_t}, \quad t \in \mathbb{T};\ \theta \in \Theta. \tag{2.8}$$

It can be shown that (See e.g., Elliott *et al.* [**50**])

$$\begin{aligned} \Lambda_t &= \exp\Bigg[-\int_0^t \theta_s \sigma_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 \sigma_s^2 ds - \int_0^t \int_{\mathbb{R}\setminus\{0\}} \theta_{s^-} z \widetilde{N}^X(ds;dz) \\ &\quad - \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Big(e^{-z\theta_s} - 1 + \theta_s z \Big) \rho^X(dz)ds \Bigg]. \end{aligned} \tag{2.9}$$

It is a straightforward exercise to show that the process in Equation (2.8) is a density process inducing a change of measure in the probability space $(\Omega, \overline{\mathcal{G}})$. This fact is a direct consequence of the following result.

Proposition 2.1 (Siu and Yang [**123**])**.**
The stochastic process $\Lambda = \{\Lambda_t\}_{t\in\mathbb{T}}$ defined by (2.8) is a positive $(\overline{\mathbf{G}}, \mathbb{P})$-martingale and

$$E^{\mathbb{P}}[\Lambda_t] = 1, \quad \forall t \in \mathbb{T}. \tag{2.10}$$

Proposition 2.1 immediately implies that Λ is a density process and it is possible to define for each process θ in Θ a new probability measure $\mathbb{Q}^\theta$ equivalent to $\mathbb{P}$ by setting

$$\frac{d\mathbb{Q}^\theta}{d\mathbb{P}}\bigg|_{\overline{\mathcal{G}}_t} = \Lambda_t\ , \quad t \in \mathbb{T}. \tag{2.11}$$

We derive the pricing kernel associated to this equivalent probability measure by imposing some conditions on θ. We shall discuss this issue in subsection 2.3.3

2.3.2. Pricing Kernel that takes into Account the Risk Associated with the Markov chain

In this subsection, we present the construction of an Esscher change of measure that produces a pricing kernel that takes into account the risk associated with the Markov chain. In other words, this change of measure is based on a conditioning argument that assumes knowledge of only the starting state of the underlying Markov chain. Unlike the first change of measure of the previous subsection, the denominator of the kernel presented here does not assume knowledge of the whole path of the Markov chain but only its initial state. In order to construct the second pricing kernel, we give an alternative definition of the generalized Laplace transform of X by conditioning on the initial value of the Markov chain X instead of conditioning on the whole path of X. Let us first introduce a new filtration in our space namely $\mathbf{G} := \{\mathcal{G}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^Y : t \in \mathbb{T}\}$ which denotes the right-continuous, $\mathbb{P}$-complete filtration generated by the bivariate process (X, Y). We introduce also the set

$$\Theta^* := \Bigg\{ (\theta^*_t)_{t\in\mathbb{T}} \mid \theta^*_t := \sum_{i=1}^{M} \theta^*_i \langle X_{t-} | \mathbf{e}_i \rangle \text{ with } (\theta^*_1, \theta^*_2, ..., \theta^*_M) \in \mathbb{R}^M$$

$$\text{such that } E^{\mathbb{P}}\Big[e^{-\int_0^t \theta^*_r dY_r} \Big| X_0 \Big] < \infty \Bigg\}.$$

and define following Siu and Yang [**123**] the new kernel $\Lambda^* = \{\Lambda^*_t\}_{t\in\mathbb{T}}$ as a **G**-adapted stochastic process as follows

$$\begin{cases} \Lambda^*_0 & := 1 \\ \Lambda^*_t & := \quad E^{\mathbb{P}}\left[\frac{e^{-\int_0^T \theta^*_s dY_s}}{E^{\mathbb{P}}[e^{-\int_0^T \theta^*_s dY_s} | X_0]} \middle| \mathcal{G}_t \right], \qquad t \in (0, T];\ \theta^* \in \Theta^*. \end{cases} \tag{2.12}$$

By construction, $\{\Lambda^*_t\}_{t\in\mathbb{T}}$ is a positive **G**-martingale which verifies

$$E^{\mathbb{P}}[\Lambda^*_t] = 1, \quad \forall t \in \mathbb{T}\ .$$

It is clear that we can now define a family of probability measures $\{\mathbb{Q}_{\theta^*} : \theta^* \in \Theta^*\}$ equivalent to $\mathbb{P}$ through

$$\frac{d\mathbb{Q}_{\theta^*}}{d\mathbb{P}}\bigg|_{\mathcal{G}_t} = \Lambda^*_t\ , \quad t \in \mathbb{T}\ . \tag{2.13}$$

As for the family $\{\mathbb{Q}^\theta : \theta \in \Theta\}$ introduced above, the family $\{\mathbb{Q}_{\theta^*} : \theta^* \in \Theta^*\}$ define a pricing kernel under some conditions on θ^*. These conditions are discussed in the next subsection.

Remark 2.3.1. *The first kernel can be viewed as the one resulting from assuming knowledge of the whole sample path (past and future) of the Markov chain, so one can suppose that in this case an agent has much information for hedging himself from the risk due to the regime change thus he doesn't need a premium for this risk. Contrarily to the first, the second kernel (2.12) is defined only on information of the initial state X_0 of the Markov chain thus we can think that an agent will require some premium for taking into account the risk associated with regime shifts.*

Despite the resemblance, the two kernels defined above have some fundamental differences [See Appendix 2.7.1]. In particular, they lead to different measure changes which in turn imply different pricing rules. The purpose of this note is to numerically illustrate these differences. At this point, we have two density processes (2.8) and (2.12) inducing two different families of equivalent measure changes $\{\mathbb{Q}^\theta : \theta \in \Theta\}$ and $\{\mathbb{Q}_{\theta^*} : \theta^* \in \Theta^*\}$. From the fundamental theorem of asset pricing we know that any contingent claim in any financial market would be priced under an equivalent martingale measure and, in order to derive such pricing rule, we need to impose the so-called martingale condition to the kernels defined above. This is carried out in the next subsection.

2.3.3. Martingale Condition

Consider the market composed of two assets B and S as defined in equations (2.2) and (2.3). We denote by $\{S_t^* := \frac{S_t}{B_t} : t \in \mathbb{T}\}$ the discounted price process. A key element in the theory of option pricing is that of Equivalent Martingale Measure (EMM). Indeed, a milestone in mathematical finance is the fundamental theorem of asset pricing [see Harrison and Pliska ([**79**], [**80**])] which states that a no-arbitrage price of a contingent claim in this market is given in terms of an equivalent measure satisfying

$$E^{\mathbb{Q}}\Big[S_t^*|\mathcal{G}_0\Big] = S_0^* \,, \tag{2.14}$$

with $\mathbb{Q} \in \{\mathbb{Q}^\theta, \mathbb{Q}_{\theta^*}\}$. This is known as the martingale condition and implicitly gives the condition on the process θ (resp. θ^*) that determines an EMM within the families $\{\mathbb{Q}^\theta : \theta \in \Theta\}$ and $\{\mathbb{Q}_{\theta^*} : \theta^* \in \Theta^*\}$.

The family of equivalent measures $\{\mathbb{Q}^\theta : \theta \in \Theta\}$ and $\{\mathbb{Q}_{\theta^*} : \theta^* \in \Theta^*\}$ in (2.11) and (2.13) were introduced and studied in the context of derivative pricing [See Elliott *et al.* [**50**] and Siu and Yang [**123**]]. In both cases, there exist results that identify explicitly the martingale condition [See, for example Elliott *et al.* [**50**] and Siu and Yang [**123**] for regime-switching Black-Scholes model]. Here we present a more general condition that identifies the equivalent martingale measure within the families $\{\mathbb{Q}^\theta : \theta \in \Theta\}$ and $\{\mathbb{Q}_{\theta^*} : \theta^* \in \Theta^*\}$ under the general regime-switching exponential Lévy model defined in (2.2) and (2.3).

The necessary and sufficient condition for $\mathbb{Q}^\theta$ to be an equivalent martingale measure is a somewhat straightforward result and we present it here in the form of the following proposition.

Proposition 2.2.

Consider the Lévy regime-switching market defined in (2.2) and (2.3). An equivalent probability measure $\mathbb{Q}^\theta$ defined through (2.11) is an equivalent martingale measure on $(\Omega, \overline{\mathcal{G}}_T)$, i.e., it satisfies condition (2.14), if and only if θ satisfies the following equation

$$\mu_i - r_i - \theta_i \sigma_i^2 + \int_{\mathbb{R}} (e^z - 1)(e^{-z\theta_i} - 1)\rho_i(z)dz = 0 \,, \tag{2.15}$$

for $i = 1, 2, ..., M$.

Proof. The proof is a straightforward adaptation of that of Proposition 2.2 in Elliott *et al.* [**50**]. The main ingredient is an explicit computation of the generalized Laplace transform defined as (2.7).□

As for the necessary and sufficient condition for $\mathbb{Q}_{\theta^*}$ to be an equivalent martingale measure on $(\Omega, \overline{\mathcal{G}}_T)$, the result is more complicated and we need to lay down some preliminary results before. First we recall the following result for the occupation times of a Markov chain which is adapted from Elliott and Osakwe [**54**].

Lemma 2.1.

Consider an irreducible homogeneous continuous-time Markov chain $X := (X_t)_{t\in\mathbb{T}}$ on $(\Omega, \overline{\mathcal{G}}_T, \mathbf{G}, \mathbb{P})$ with a finite state space $\mathbb{S}$ of size $M \in \mathbb{N}$ and with an intensity *matrix* $\mathbb{A} := \{a_{ij} : 1 \leq i, j \leq M\}$. *Let*

$$\underline{\mathbf{J}}(u,v) := (J_1(u,v), J_2(u,v), ..., J_M(u,v)) \tag{2.16}$$

denote the vector of the occupation times of X *during a period of time* $[u,v] \subset \mathbb{T}$. *The conditional moment generating function of* $\underline{\mathbf{J}}(u,v)$ *is given by*

$$\begin{aligned} \mathcal{M}_{\underline{\mathbf{J}}}(u,v)(\underline{\zeta}) &:= \mathbb{E}^{\mathbb{P}}\left[e^{\sum_{k=1}^{M}\zeta_k J_k(u,v)}\Big|\mathcal{G}_u\right] \\ &= \left\langle e^{\left(\mathbb{A}+\boldsymbol{Diag}(\underline{\zeta})\right)(v-u)}X_u\Big|\underline{\mathbf{1}}\right\rangle, \quad \underline{\zeta} \in \mathbb{R}^M \end{aligned} \tag{2.17}$$

where $\underline{\mathbf{1}} = (1,1,...,1)' \in \mathbb{R}^M$, $\langle \cdot | \cdot \rangle$ *is the usual scalar product in* $\mathbb{R}^M$ *and* $\boldsymbol{Diag}(\underline{\zeta})$ *is a* $M \times M$ *diagonal matrix of the form*

$$\boldsymbol{Diag}(\underline{\zeta}) = \begin{pmatrix} \zeta_1 & 0 & \cdots & 0 & 0 \\ 0 & \zeta_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \zeta_{N-1} & 0 \\ 0 & \cdots & 0 & 0 & \zeta_M \end{pmatrix}.$$

.

Proof. The proof is easily adapted from that of Proposition 2 in Elliott and Osakwe [**54**]. □

Lemma 2.1 gives the explicit form of the moment generating function of a Markov chain in terms of the occupation times. This is useful when we study the kernel (2.12) which is defined in terms of one such moment generating function.

Using Lemma 2.1 we have the following proposition which is an extension of lemma 3.1 of Siu and Yang [**123**].

Proposition 2.3.

Let $\{S^*_t := \frac{S_t}{B_t} : t \in \mathbb{T}\}$ *be the discounted price process in the market defined in equations (2.2) and (3.4) and let* $\mathbb{Q}_{\theta^*}$ *be the family of equivalent measures defined through (2.12) and (2.13) on* $(\Omega, \overline{\mathcal{G}}_T)$. *Then, for all* $u, v \in \mathbb{T}$ *such that* $u \leq v$

$$\mathbb{E}^{\mathbb{Q}_{\theta^*}}[S^*_v|\mathcal{G}_u] = S^*_u \frac{\langle e^{\left(\mathbb{A}+\boldsymbol{Diag}(\tilde{\underline{\xi}}(\theta^*))\right)(v-u)}X_u|\underline{\mathbf{1}}\rangle}{\langle e^{\left(\mathbb{A}+\boldsymbol{Diag}(\underline{\xi}(\theta^*))\right)(v-u)}X_u|\underline{\mathbf{1}}\rangle}, \tag{2.18}$$

where

$$\underline{\xi}(\theta^*) = (\xi_1(\theta^*_1), \xi_2(\theta^*_2), ..., \xi_M(\theta^*_M)),$$
$$\tilde{\underline{\xi}}(\theta^*) = (\tilde{\xi}_1(\theta^*_1), \tilde{\xi}_2(\theta^*_2), ..., \tilde{\xi}_M(\theta^*_M)),$$

with

$$\xi_i(\theta^*_i) = -\theta^*_i(\mu_i - \frac{1}{2}\sigma_i^2) + \frac{1}{2}(\theta^*_i)^2\sigma_i^2 + \int_{\mathbb{R}}\left(e^{-z\theta^*_i} - 1 + \theta^*_i(e^z - 1)\right)\rho_i(z)dz,$$

$$\tilde{\xi}_i(\theta_i^*) = -r_i - (\theta_i^* - 1)(\mu_i - \frac{1}{2}\sigma_i^2) + \frac{1}{2}(\theta_i^* - 1)^2\sigma_i^2 + \int_{\mathbb{R}} \Big(e^{-z(\theta_i^*-1)} - 1 + (\theta_i^* - 1)(e^z - 1)\Big)\rho_i(z)dz \ ,$$

for $i = 1, 2, ..., M$.

Proof.

For all $u, v \in \mathbb{T}$ so that $u \leq v$ and by a version of the Bayes' rule [See, Aggoun and Elliott [**2**]]

$$\begin{aligned} E^{\mathbb{Q}_{\theta^*}}[S_v^*|\mathcal{G}_u] &= \frac{E^{\mathbb{P}}[\Lambda_v^* S_v^*|\mathcal{G}_u]}{E^{\mathbb{P}}[\Lambda_v^*|\mathcal{G}_u]} \\ &= E^{\mathbb{P}}\left[\frac{\Lambda_v^*}{\Lambda_u^*} S_v^* \middle| \mathcal{G}_u\right] \quad \text{(since } \Lambda^* \text{ is a } \mathbf{G}\text{-martingale).} \end{aligned} \tag{2.19}$$

Using the expressions in (3.4) and (2.12) we can explicitly write

$$S_v^* = S_u^* e^{-\int_u^v r_s ds} e^{\int_u^v dY_s} \ , \tag{2.20}$$

$$\frac{\Lambda_v^*}{\Lambda_u^*} = e^{-\int_u^v \theta_s^* dY_s} \frac{E^{\mathbb{P}}[e^{-\int_v^T \theta_s^* dY_s}|\mathcal{G}_v]}{E^{\mathbb{P}}[e^{-\int_u^T \theta_s^* dY_s}|\mathcal{G}_u]} \ . \tag{2.21}$$

A simple substitution of equations (2.20) and (2.21) into (2.19) yields,

$$E^{\mathbb{Q}_\Theta}[S_v^*|\mathcal{G}_u] = S_u^* \frac{E^{\mathbb{P}}\Big[e^{-\int_u^v r_s ds} e^{-\int_u^v (\theta_s^* - 1)dY_s} E^{\mathbb{P}}[e^{-\int_v^T \theta_s^* dY_s}|\mathcal{G}_v]\Big|\mathcal{G}_u\Big]}{E^{\mathbb{P}}[e^{-\int_u^T \theta_s^* dY_s}|\mathcal{G}_u]} \ . \tag{2.22}$$

If we use the fact that X is a homogeneous Markov chain, we can write (2.22) in terms of the vector of occupation times $\underline{\mathbf{J}}$ defined in (2.16)

$$E^{\mathbb{Q}_\Theta}[S_v^*|\mathcal{G}_u] = \frac{E^{\mathbb{P}}\left[e^{\sum_{i=1}^M \tilde{\xi}_i(\theta_i^*)J_i(u,v)} E^{\mathbb{P}}\left[e^{\sum_{i=1}^M \xi_i(\theta_i^*)J_i(v,T)}\middle|\mathcal{G}_v\right]\middle|\mathcal{G}_u\right]}{E^{\mathbb{P}}\left[e^{\sum_{i=1}^M \xi_i(\theta_i^*)J_i(u,v)} E^{\mathbb{P}}\left[e^{\sum_{i=1}^M \xi_i(\theta_i^*)J_i(v,T)}\middle|\mathcal{G}_v\right]\middle|\mathcal{G}_u\right]} \ . \tag{2.23}$$

This last equation can be simplified if we use the following property of homogeneous Markov chains (See, for example Norris [**104**]),

$$\begin{aligned} Law^1\left(J_1(v,T), ..., J_M(v,T)\middle|\mathcal{G}_v\right) &= Law\left(J_1(v,T), ..., J_M(v,T)\middle|X_v\right) \qquad (2.24) \\ &= Law\left(J_1(0,T-v), ..., J_M(0,T-v)\middle|X_0\right) \ . \end{aligned}$$

By using property (2.24) in (2.23), we have

$$\begin{aligned} E^{\mathbb{Q}_\Theta}[S_v^*|\mathcal{G}_u] &= S_u^* \frac{E^{\mathbb{P}}\left[e^{\sum_{i=1}^M \xi_i(\theta_i^*)J_i(0,T-v)}\Big|X_0\right] E^{\mathbb{P}}\left[e^{\sum_{i=1}^M \tilde{\xi}_i(\theta_i^*)J_i(u,v)}\Big|\mathcal{G}_u\right]}{E^{\mathbb{P}}\left[e^{\sum_{i=1}^M \xi_i(\theta_i^*)J_i(0,T-v)}\Big|X_0\right] E^{\mathbb{P}}\left[e^{\sum_{i=1}^M \xi_i(\theta_i^*)J_i(u,v)}\Big|\mathcal{G}_u\right]} \\ &= S_u^* \frac{E^{\mathbb{P}}\left[e^{\sum_{i=1}^M \tilde{\xi}_i(\theta_i^*)J_i(u,v)}\Big|\mathcal{G}_u\right]}{E^{\mathbb{P}}\left[e^{\sum_{i=1}^M \xi_i(\theta_i^*)J_i(u,v)}\Big|\mathcal{G}_u\right]} . \end{aligned} \tag{2.25}$$

If we now use Lemma 2.1, we finally obtain

$$E^{\mathbb{Q}_\Theta}[S_v^*|\mathcal{G}_u] = S_u^* \frac{\langle e^{\left(A+\mathbf{Diag}(\underline{\tilde{\xi}}(\theta^*))\right)(v-u)} X_u|\underline{\mathbf{1}}\rangle}{\langle e^{\left(A+\mathbf{Diag}(\underline{\xi}(\theta^*))\right)(v-u)} X_u|\underline{\mathbf{1}}\rangle} . \tag{2.26}$$

□

Proposition 2.3 gives a particular form for the martingale condition (2.14) for an equivalent probability measure $\mathbb{Q}_{\theta^*}$ defined on $(\Omega, \overline{\mathcal{G}}_T)$. It immediately yields a characterization of the martingale measures in $\{\mathbb{Q}_{\theta^*} : \theta^* \in \Theta^*\}$ as stated in the following result which is adapted from Siu and Yang **[123]**.

Theorem 2.1. *Consider the Lévy regime-switching market defined in (2.2) and (2.3). An equivalent measure $\mathbb{Q}_{\theta^*}$ defined through (2.13) is an equivalent martingale measure on $(\Omega, \overline{\mathcal{G}}_T)$, i.e., it satisfies condition (2.14), if and only if θ^* satisfies the following equation*

$$\langle e^{\left(\mathbb{A}+\boldsymbol{Diag}(\underline{\tilde{\xi}}(\theta^*))\right)t} X_0|\underline{\mathbf{1}}\rangle - \langle e^{\left(\mathbb{A}+\boldsymbol{Diag}(\underline{\xi}(\theta^*))\right)t} X_0|\underline{\mathbf{1}}\rangle = 0 \, , \tag{2.27}$$

where we use the same notation as in Proposition 2.3.

Proof. By set $v = t$ and $u = 0$ in equation (2.18) we have immediately that the martingale condition

$$E^{\mathbb{Q}_\Theta}\left[S_t^*|\mathcal{G}_0\right] = S_0^*, \tag{2.28}$$

is equivalent to equation (2.27). □

2.3.4. Some Approximations

The importance of Proposition 2.2 and Theorem 2.1 is that they characterize the equivalent measures within the families $\{\mathbb{Q}^\theta : \theta \in \Theta\}$ and $\{\mathbb{Q}_{\theta^*} : \theta^* \in \Theta^*\}$ under which the discounted price process S^* is a martingale. Under non-arbitrage assumptions, so-called equivalent martingale measures are needed in order to price any contingent claim on the underlying process S. These results endow us with a means to evaluate derivative products under two conceptually different assumptions regarding the underlying market regimes. In view of this, it is of uttermost importance to explicitly determine the martingale conditions in both

Proposition 2.2 and Theorem 2.1. We focus our attention to the condition for the family $\{\mathbb{Q}_{\theta^*} : \theta^* \in \Theta^*\}$ since it is a more complicated object. In fact, the presence of a matrix exponential in equation (2.27) makes it very cumbersome for the determination of Esscher parameters $(\theta_1^*, \theta_2^*, ..., \theta_M^*)$. It is often the case that approximations are needed in order to find the solution of equation (2.27). It turns out that a standard approximation for the matrix exponential in (2.27) leads to interesting insight about the difference between the two families of equivalent measures $\mathbb{Q}_\Theta$ and $\mathbb{Q}_\Theta^*$ and their underlying assumptions. In this Section we carry out a comparative analysis (in line with Siu and Yang [**123**])of these two families via certain types of approximations for the martingale condition in (2.27). Recall that a matrix exponential of a square matrix C is defined as

$$\exp(C) := \sum_{k=0}^{\infty} \frac{C^k}{k!}. \tag{2.29}$$

A Comparison between $\mathbb{Q}^\theta$ *and* $\mathbb{Q}_{\theta^*}$

By replacing the expression of matrix exponential $\exp(C)$ in the equation (2.27) by its first-order approximation, i.e.,

$$\exp(C) \approx I + C \tag{2.30}$$

where I denoted the identity matrix. We have that :

$$\left\langle \left(I + \left(\mathbb{A} + (\underline{\tilde{\xi}}(\theta))\right) t\right) X_0 \middle| \underline{\mathbf{1}} \right\rangle - \left\langle \left(I + \left(\mathbb{A} + \mathbf{Diag}(\underline{\xi}(\theta))\right) t\right) X_0 \middle| \underline{\mathbf{1}} \right\rangle = 0. \tag{2.31}$$

And by taking $X_0 = \mathbf{e}_i;\ i = 1, ..., M$ the last equation is equivalent to this system of N equations

$$\left(\sum_{k=1,k\neq i}^{M} a_{ki} + 1 + \left(a_{ii} + \tilde{\xi}_i(\theta_i)\right) t \right) - \left(\sum_{k=1,k\neq i}^{M} a_{ki} + 1 + \left(a_{ii} + \xi_i(\theta_i)\right) t \right) = 0. \tag{2.32}$$

Thus, for $i = 1, ..., M$

$$\tilde{\xi}_i(\theta_i) - \xi_i(\theta_i) = 0 \quad \text{because} \quad t > 0. \tag{2.33}$$

or, for $i = 1, ..., M$

$$\mu_i - r_i - \theta_i \sigma_i^2 + \int_{\mathbb{R}} (e^z - 1)(e^{-z\theta_i} - 1)\rho_i(z) dz = 0. \tag{2.34}$$

Notice that this last expression is exactly the martingale condition for the family $\{\mathbb{Q}^\theta : \theta \in \Theta\}$ as given in equation (2.2). In fact, this shows that the martingale condition for the family $\mathbb{Q}^\Theta$ in (2.2) is the first-order approximation of the martingale condition for $\{\mathbb{Q}_{\theta^*} : \theta^* \in \Theta^*\}$ in (2.27). This has an interesting interpretation with respect to the underlying assumptions behind these two families, we can think of the second pricing kernel Λ^* as having more information than the first one Λ. Indeed, the second kernel Λ^* is obtained under the assumption that only the initial state of the market regime is known, whereas the first kernel Λ operates under the less realistic assumption that the whole path of the market regimes are known. Apparently, under kernel Λ^*, a first-order approximation to identify the martingale condition is equivalent to assuming that the whole path of the Markov chain is known, which in turn reduces to using kernel Λ. This result had been previously observed in Siu and Yang [**123**] for a Regime-switching Black-Scholes model, here we have established the same result for a slightly more general regime-switching model.

When working with the second kernel Λ^*, there are higher-order approximations that can be used to compute the martingale condition (2.27) through the series (2.29). In view of the previous discussion, we can see higher-order approximations as giving new degrees of information that allow us to move away from an unlikely assumption where the whole path of the Markov chain is known towards a more seemingly assumption where only the initial state of the Markov chain is known. In the following subsection, we discuss a second-order approximation in a particular example.

Further Approximation

Here, we derive explicitly the martingale condition for $\mathbb{Q}_{\theta^*}$ by taking a two-order approximation for matrix exponential. For simplifications we take $N = 2$.
By setting $a_{11} = -a_{12} = -a_1; a_{22} = -a_{21} = -a_2; a_1, a_2 \geq 0$ and for $t > 0$,

$$M = \Big(\mathbb{A} + (\underline{\tilde{\xi}}(\theta^*))\Big)\, t \tag{2.35}$$

explicitly we have

$$M = \begin{pmatrix} -a_1 + \tilde{\xi}_1(\theta_1^*) & a_1 \\ a_2 & -a_2 + \tilde{\xi}_2(\theta_2^*) \end{pmatrix} t \tag{2.36}$$

hence

$$\exp(M) \approx I + M + \frac{M^2}{2} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where

$$\begin{aligned} A_{11} &= 1 - a_1 t + \tilde{\xi}_1 t + \frac{1}{2} a_1 a_2 t^2 + \frac{1}{2}(a_1 - \tilde{\xi}_1)^2 t^2 \\ A_{12} &= a_1 t + \frac{1}{2} a_1 (\tilde{\xi}_1 - a_1 - a_2 + \tilde{\xi}_2) t^2 \\ A_{21} &= a_2 t + \frac{1}{2} a_2 (\tilde{\xi}_1 - a_1 - a_2 + \tilde{\xi}_2) t^2 \\ A_{22} &= 1 - a_2 t + \tilde{\xi}_2 t + \frac{1}{2} a_1 a_2 t^2 + \frac{1}{2}(a_2 - \tilde{\xi}_2)^2 t^2 \end{aligned} \tag{2.37}$$

So the martingale condition (2.27) gives for $X_0 = \mathbf{e}_1 = (1,0)$:

$$\Big(\tilde{\xi}_1(\theta_1^*) - \xi_1(\theta_1^*)\Big)t + \frac{1}{2}t^2\bigg[\Big(\tilde{\xi}_1(\theta_1^*) - \xi_1(\theta_1^*)\Big)\Big(\tilde{\xi}_1(\theta_1^*) + \xi_1(\theta_1^*)\Big) + \Big(a_2 - 2a_1\Big)\Big(\tilde{\xi}_1(\theta_1^*) - \xi_1(\theta_1^*)\Big) + a_2\Big(\tilde{\xi}_2(\theta_2^*) - \xi_2(\theta_2^*)\Big)\bigg] = 0. \quad (2.38)$$

and for $X_0 = \mathbf{e}_2 = (0,1)$:

$$\Big(\tilde{\xi}_2(\theta_2^*) - \xi_2(\theta_2^*)\Big)t + \frac{1}{2}t^2\bigg[\Big(\tilde{\xi}_2(\theta_2^*) - \xi_2(\theta_2^*)\Big)\Big(\tilde{\xi}_2(\theta_2^*) + \xi_2(\theta_2^*)\Big) + \Big(a_1 - 2a_2\Big)\Big(\tilde{\xi}_2(\theta_2^*) - \xi_2(\theta_2^*)\Big) + a_1\Big(\tilde{\xi}_1(\theta_1^*) - \xi_1(\theta_1^*)\Big)\bigg] = 0. \quad (2.39)$$

Equations (2.38) and (2.39) are examples of the type of equations that need to be solved in order to identify the martingale condition. These equations are somewhat more tractable than (2.27) and they will be used to determine the EMM parameters for the numerical illustrations. In the following Section we carry out numerical examples for certain particular cases using equations (2.38) and (2.39).

2.4. Particular Cases

In this Section we present in detail the developments made above for particular models. In the sequel we take $M = 2$, i.e., the Markov chain X moves only between the two states $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$.

2.4.1. The Regime-switching Black-Scholes model

By taking into account the assumptions of no jumps in equation (2.2), we obtain the martingale condition for $\mathbb{Q}^\theta$ as

$$\mu_i - r_i - \theta_i \sigma_i^2 = 0, \quad (2.40)$$

for $i = 1,2$ and we deduce easily the Esscher parameter

$$\theta_i = \frac{\mu_i - r_i}{\sigma_i^2}, \quad i = 1,2. \quad (2.41)$$

The martingale condition associated to $\mathbb{Q}_{\theta^*}$ leads to this system of equations in (θ_1, θ_2) :

$$\frac{\sigma_1^4 t^2}{2}\theta_1^3 - \frac{(3\mu_1 - r_1)\sigma_1^2 t^2}{2}\theta_1^2 + \left(\sigma_1^2 t + \frac{(\mu_1 - r_1)(\sigma_1^2 + 2\mu_1)t^2 + \sigma_1^2(a_2 - 2a_1)t^2}{2}\right)\theta_1 + \frac{a_2\sigma_2^2 t^2}{2}\theta_2 - \left(\frac{(\mu_1 - r_1)^2 + (a_2 - 2a_1)(\mu_1 - r_1) + a_2(\mu_2 - r_2)}{2}\right)t^2 - (\mu_1 - r_1)t = 0,$$

$$\text{for} \quad 0 < t \leq T. \quad (2.42)$$

$$\frac{\sigma_2^4 t^2}{2}\theta_2^3 - \frac{(3\mu_2 - r_2)\sigma_2^2 t^2}{2}\theta_2^2 + \left(\sigma_2^2 t + \frac{(\mu_2 - r_2)(\sigma_2^2 + 2\mu_2)t^2 + \sigma_2^2(a_1 - 2a_2)t^2}{2}\right)\theta_2$$
$$+\frac{a_1\sigma_1^2 t^2}{2}\theta_1 - \left(\frac{(\mu_2 - r_2)^2 + (a_1 - 2a_2)(\mu_2 - r_2) + a_1(\mu_1 - r_1)}{2}\right)t^2 - (\mu_2 - r_2)t = 0,$$
$$\text{for} \quad 0 < t \leq T. \quad (2.43)$$

2.4.2. The Regime-switching Merton Jump-Diffusion Model

This model is obtained by supposing that all the parameters of the classic Merton Jump-Diffusion are modulated by a Markov chain X. So, we have this dynamics for the risk asset

$$S_t = S_0 \exp\left[\int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right)ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\widetilde{N}_{JD}^X(ds, dz) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z)\rho_{JD}^X(dz)ds\right]$$

where

- the jump process $N_{JD}^X(t, dz)$ is compound Poisson with a stochastic intensity $\lambda_t^X = \lambda(t, X_t) = \langle \underline{\lambda} | X_t \rangle$, where $\underline{\lambda} = (\lambda_1, \lambda_2) \in {\mathbb{R}_+}^2$;
- the Lévy measure in this case is given by :

$$\rho_{JD}^X(dz) = \frac{\lambda^X}{(\delta^X)\sqrt{2\pi}} \exp\left\{\frac{-\left(dz - \mu_J^X\right)^2}{2(\delta^X)^2}\right\} \quad (2.44)$$

with $\mu_J^X = \mu_J^X(t, X_t) = \langle \underline{\mu_J} | X_t \rangle$ and $\delta^X = \delta^X(t, X_t) = \langle \underline{\delta} | X_t \rangle$ where $\underline{\mu_J} = (\mu_J^1, \mu_J^2) \in \mathbb{R}^2$ and $\underline{\delta} = (\delta_1, \delta_2) \in \mathbb{R}_+^2$.

The martingale condition for $\mathbb{Q}^\theta$ is given by :

$$\mu_i - r_i - \theta_i\sigma_i^2 + \lambda_i e^{-\theta_i\mu_J^X + \frac{1}{2}\theta_i^2\delta_i^2}\left(e^{-\theta_i\delta_i^2 + \mu_J^X + \frac{1}{2}\delta_i^2} - 1\right) - \lambda_i\left(e^{\mu_J^X + \frac{1}{2}\delta_i^2} - 1\right) = 0 \quad (2.45)$$

for $i = 1, 2$.

In the other side, the martingale condition for $\mathbb{Q}_{\theta^*}$ includes complex expressions. Indeed, we have to replace in Equations (2.38)-(2.39) for $i = 1, 2$, the expressions

$$\tilde{\xi}_i(\theta_i^*) - \xi_i(\theta_i^*) = \mu_i - r_i - \theta_i^*\sigma_i^2 + \lambda_i e^{-\theta_i^*\mu_J^X + \frac{1}{2}(\theta_i^*)^2\delta_i^2}\left(e^{-\theta_i^*\delta_i^2 + \mu_J^X + \frac{1}{2}\delta_i^2} - 1\right) - \lambda_i(e^{\mu_J^X + \frac{1}{2}\delta_i^2} - 1) \quad (2.46)$$

and

$$\tilde{\xi}_i(\theta_i^*) + \xi_i(\theta_i^*) = \mu_i - r_i - 2\theta_i\mu_i + (\theta_i^*)^2\sigma_i^2 + (2\theta_i^* + 1)\lambda_i + (2\theta_i^* - 1)\lambda_i e^{\mu_J^X + \frac{1}{2}\delta_i^2} + \lambda_i e^{\frac{1}{2}(\theta_i^*)^2\delta_i^2 - \theta_i^*\mu_J^X}\Big(e^{-\theta_i^*\delta_i^2 + \mu_J^X + \frac{1}{2}\delta_i^2} + 1\Big). \quad (2.47)$$

2.4.3. The Regime-switching Variance-Gamma Model

This model is obtained from the general case by setting the dynamics of risk process S as

$$S_t = S_0 \exp\left[\int_0^t \mu_s ds + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\widetilde{N}_{VG}^X(ds,dz) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z)\nu_{VG}^X(dz)ds\right], \quad (2.48)$$

where the jump process $N_{VG}^X(t,.)$ has the compensator or dual predictable projection

$$\nu_{VG}^X(dz)dt = \sum_{i=1}^{2}\langle \mathbf{e}_i | X_{t^-}\rangle \nu_i^{VG}(z)dt \quad (2.49)$$

with

$$\nu_i^{VG} = C_i \frac{e^{-G_i x}}{x} 1_{\{x>0\}} + C_i \frac{e^{-M_i|x|}}{|x|} 1_{\{x<0\}},$$

the Lévy measure associated to the Variance-Gamma process $VG(C_i, G_i, M_i)$.

By calculating explicitly integrals involved in Equation(2.2), we obtain the following system of equations from the martingale condition associated to $\mathbb{Q}^\theta$:

$$\mu_i - r_i - C_i \log\left(\frac{G_i M_i}{(G_i+1)(M_i-1)}\right) + C_i \log\left(\frac{(G_i-\theta_i)(M_i+\theta_i)}{(G_i-\theta_i+1)(M_i+\theta_i-1)}\right) = 0 \quad \text{for } i = 1, 2. \quad (2.50)$$

And for $\mathbb{Q}_{\theta^*}$, the following system of non linear equations in (θ_1, θ_2) is obtained :

$$\left\{\mu_1 - r_1 + C_1 \log\left(\frac{(G_1-\theta_1)(M_1+\theta_1)}{(G_1-\theta_1+1)(M_1+\theta_1-1)}\right) - C_1 \log\left(\frac{G_1M_1}{(G_1+1)(M_1-1)}\right)\right\}$$
$$\times\left\{t + \frac{1}{2}t^2\left[\mu_1 - r_1 - 2\theta_1\mu_1 + (2\theta_1-1)C_1\log\left(\frac{G_1M_1}{(G_1+1)(M_1-1)}\right)\right.\right.$$
$$+ C_1\log\left(\frac{G_1M_1}{(G_1-\theta_1+1)(M_1+\theta_1-1)}\right) + C_1\log\left(\frac{G_1M_1}{(G_1-\theta_1)(M_1+\theta_1)}\right)$$
$$\left.\left. + (2\theta_1-1)C_1\log\left(\frac{G_1M_1}{(G_1+1)(M_1-1)}\right) + a_2 - 2a_1\right]\right\}$$
$$+ \frac{1}{2}a_2t^2\left\{\mu_2 - r_2 + C_2\log\left(\frac{(G_2-\theta_2)(M_2+\theta_2)}{(G_2-\theta_2+1)(M_2+\theta_2-1)}\right)\right.$$
$$\left. - C_2\log\left(\frac{G_2M_2}{(G_2+1)(M_2-1)}\right)\right\} = 0. \quad (2.51)$$

$$\left\{\mu_2 - r_2 + C_2 \log\left(\frac{(G_2-\theta_2)(M_2+\theta_2)}{(G_2-\theta_2+1)(M_2+\theta_2-1)}\right) - C_2 \log\left(\frac{G_2M_2}{(G_2+1)(M_2-1)}\right)\right\}$$
$$\times\left\{t + \frac{1}{2}t^2\left[\mu_2 - r_2 - 2\theta_2\mu_2 + (2\theta_2-1)C_2\log\left(\frac{G_2M_2}{(G_2+1)(M_2-1)}\right)\right.\right.$$
$$+ C_2\log\left(\frac{G_2M_2}{(G_2-\theta_2+1)(M_2+\theta_2-1)}\right) + C_2\log\left(\frac{G_2M_2}{(G_2-\theta_2)(M_2+\theta_2)}\right)$$
$$\left.\left. + (2\theta_2-1)C_2\log\left(\frac{G_2M_2}{(G_2+1)(M_2-1)}\right) + a_1 - 2a_2\right]\right\}$$
$$+ \frac{1}{2}a_1t^2\left\{\mu_1 - r_1 + C_1\log\left(\frac{(G_1-\theta_1)(M_1+\theta_1)}{(G_1-\theta_1+1)(M_1+\theta_1-1)}\right)\right.$$
$$\left. - C_1\log\left(\frac{G_1M_1}{(G_1+1)(M_1-1)}\right)\right\} = 0. \quad (2.52)$$

In general, solving the system of equations obtained from the martingale condition for $\mathbb{Q}_{\theta^*}$ is quite involved. Thus, the only way to solve it is by numerical techniques. The solutions are not unique therefore we need to use some criterion to select the final Esscher parameters. We discuss this issue in the next Section.

2.4.4. Criterion for Selecting Esscher Parameters

In many cases, the system of equations in (θ_1, θ_2) resulting to the martingale condition for $\mathbb{Q}_{\theta^*}$ has more than one solution. Following the idea of the two-stage pricing method proposed by Siu and Yang [**123**], we impose a criterion for selecting one set of (θ_1, θ_2) for pricing purposes.
So we choose $\theta = (\theta_1, \theta_2)$ as solution of the following minimization problem

$$\min_{\theta \in \Gamma} I(\mathbb{Q}_{\theta^*}, \mathbb{P}) \tag{2.53}$$

with $\Gamma := \{\theta \in \mathbb{R}^2 | \ \theta \text{ solution of (2.38)-(2.39)}\}$ and

$$\begin{aligned} I(\mathbb{Q}_{\theta^*}, \mathbb{P}) &:= \max_{i=1,2} I(\mathbb{Q}_{\theta^*}, \mathbb{P} | X_0 = \mathbf{e}_i) \\ &:= \max_{i=1,2} E^{\mathbb{P}}\left[\frac{d\mathbb{Q}_{\theta^*}}{d\mathbb{P}} \ln\left(\frac{d\mathbb{Q}_{\theta^*}}{d\mathbb{P}} \right) \bigg| X_0 = \mathbf{e}_i \right] \\ &= \max_{i=1,2} E^{\mathbb{P}}\left[\Lambda_T^* \ln \Lambda_T^* \bigg| X_0 = \mathbf{e}_i \right]. \end{aligned} \tag{2.54}$$

or,

$$\begin{aligned} I(\mathbb{Q}_{\theta^*}, \mathbb{P}) = \max_{i=1,2} \Bigg\{ & \frac{E^{\mathbb{P}}\left[\left(- \int_0^T \theta_s dY_s \right) e^{-\int_0^T \theta_s dY_s} \Big| X_0 = \mathbf{e}_i \right]}{E^{\mathbb{P}}\left[e^{-\int_0^T \theta_s dY_s} \Big| X_0 = \mathbf{e}_i \right]} \\ & - \ln E^{\mathbb{P}}\left[e^{-\int_0^T \theta_s dY_s} \Big| X_0 = \mathbf{e}_i \right] \Bigg\}. \end{aligned} \tag{2.55}$$

2.5. Numerical Analysis

In this Section, many numerical experiments are conducted to illustrate the effect of pricing regime-switching risk particularly on the price of an European call. To do this, we compare the prices obtained by each of the two pricing kernels introduced in Section 2.3 and for the different models presented above.

Firstly, we present how we have proceeded.

2.5.1. Discretization

To obtain numerical approximations of the price of the European Call, we shall use Monte Carlo simulations. This assumes that the dynamic process of asset prices are given for a discrete grid. To do so, we subdivide the time horizon $[0, T]$ in J intervals of length $\Delta := T/J$ with J a positive integer. This gives rise to the family of points $\{t_j = \Delta j : j = 0, 1, ..., J\}$ where $t_0 = 0$ and $t_{J-1} = T$.
The continuous-time Markov chain X is approximated as in Yuan and Mao [**126**] and we refer to this work to see the details. Once the simulated path $\{X_{t_j}\}_{j=1}^J$ of

X is known, we deduce those of $\{\mu_{t_j}\}_{j=1}^J$, $\{r_{t_j}\}_{j=1}^J$,$\{\sigma_{t_j}\}_{j=1}^J$. Now, we can use these to construct an Euler-forward scheme for the log-return process Y as follows :

$$Y_{j+1} = Y_j + \Delta * \Big(\mu_j - \frac{1}{2}\sigma_j^2\Big) + \Delta * \int_{\mathbb{R}} (e^z - 1 - z)\rho^{X_j}(dz) + \sigma_j * \varepsilon * \sqrt{\Delta} + \widetilde{N}_j^X(t_{j+1}) - \widetilde{N}_j^X(t_j), \quad (2.56)$$

where $Y_j = Y_{t_j}; \mu_t = \mu_{t_j}; \sigma_j = \sigma_{t_j}$, $\varepsilon \sim \aleph(0,1)$ and

$$\widetilde{N}_j^X(t) = \int_{\mathbb{R}} z N_j^X(t; dz) - \int_0^t \int_{\mathbb{R}} z\rho^{X_j}(dz)dt. \quad (2.57)$$

Given $\{X_{t_j}\}_{j=1}^J$ and $Y_0 = 0$, we then sample $\{Y_{t_j}\}_{j=1}^J$ using (2.56) recursively.

2.5.2. Monte Carlo Simulations

The simulation procedure is inspired from Siu and Yang [**123**] and is summarized as follows :

Step 1
For each $l = 1, 2, ..., L$, simulate the discrete-time version of the Markov chain X and obtain $\{X_j^{(l)}\}_{j=1}^J$.

step 2
Given $\{X_j^{(l)}\}_{j=1}^J$, identify the samples paths of the processes

$$\{\mu_j^{(l)}\}_{j=1}^J, \ \{\sigma_j^{(l)}\}_{j=1}^J, \ \{\theta_j^{(l)}\}_{j=1}^J, \ \{r_j^{(l)}\}_{j=1}^J, \ for \ l = 1, 2, ..., L.$$

Step 3
for each $l = 1, 2, ..., L$, simulate the discrete-time version of the log-return process Y and obtain $\{Y_j^{(l)}\}_{j=1}^J$.

Step 4
Approximate the call price for both kernels respectively by :

$$C(0, S_0, X_0) \approx \frac{1}{L}\sum_{l=1}^{L} \left[\frac{e^{-\sum_{j=1}^J \theta_j^{(l)}(Y_j^{(l)} - Y_{j-1}^{(l)})} e^{-\Delta \sum_j^J r_j^{(l)}} \max(S_0 e^{Y_T^{(l)}} - K, 0)}{e^{-\Delta \sum_{j=1}^J \theta_j^{(l)}(\mu_j^{(l)} - \frac{1}{2}(\sigma_j^{(l)})^2) + \frac{1}{2}\Delta \sum_{j=1}^J (\theta_j^{(l)})^2 (\sigma_j^{(l)})^2 + \sum_{j=1}^J (Z^{(l)}(t_{j+1}) - Z^{(l)}(t_j))}} \right]$$

$$\text{with } Z^{(l)}(t) = \int_0^t \int_{\mathbb{R}} \Big(e^{-z\theta_s^{(l)}} - 1 + \theta_s^{(l)}(e^z - 1)\Big)\rho^{X^{(l)}}(dz)ds \quad (2.58)$$

and

$$C(0, S_0, X_0) \approx \frac{\sum_{l=1}^{L} \left[e^{-\sum_{j=1}^J \theta_j^{*(l)}(Y_j^{(l)} - Y_{j-1}^{(l)})} e^{-\Delta \sum_j^J r_j^{(l)}} \max(S_0 e^{Y_T^{(l)}} - K, 0)\right]}{\sum_{l=1}^{L} \left[e^{-\sum_{j=1}^J \theta_j^{*(l)}(Y_j^{(l)} - Y_{j-1}^{(l)})}\right]} \quad (2.59)$$

where θ (resp. θ^*) are obtained through equations (2.34) (resp. (2.38) and (2.39)) which are expressions of the Martingale condition in the general form. For the particular cases of last Section, this specializes according to the form of the dynamics of the risky asset.

2.5.3. Experiments and Results

For our experiments we work with three models, namely : Regime-switching Black-Scholes model (*model I*) ; Regime-switching Merton Jump-Diffusion model (*model II*) and Regime-switching Variance-Gamma (*model III*). For the first one which is the most common regime-switching continuous model in the literature, we conduct several numerical analysis to compare our results with those of the literature, especially with Naik [**103**] and Boyle and Draviam [**20**].

We set the parameters to be $J = 2$ and

(1) $\underline{r} = (0.05, 0.05)$;

(2) $\underline{\mu} = (0.35, 0.05)$;

(3) $\underline{\sigma} = (0.15, 0.25)$;

(4) $\mathbb{A} = \begin{pmatrix} -a_1 & a_1 \\ a_2 & -a_2 \end{pmatrix}$ with $a_1 = a_2 \in \{0, 0.25, 0.5, 0.75, 1, 1.25, 1.50, 1.75, 2\}$;

(5) $K = 100; X_0 \in \{1, 2\}$ and $S_0 \in \{94.0, 96.0, 98.0, 100.0, 102.0, 104.0\}$.

We note that the solution by Naik [**103**] is obtained from an analytical closed form, that from Boyle and Draviam [**20**] came from a numerical resolution of a system of weak-coupled PDE and ours is obtained by Monte- Carlo simulations. The tables 2.1 and 2.2 present numerical results for prices of European Call options at time zero for different values of moneyness (S_0/K). The figures of the last two columns are estimated by simulating $50,000$ trajectories of Y and this procedure is repeated independently 10 times to provide an estimate of the standard errors. No variance reduction technique is used and all computations are done in MATLAB codes.

TAB. 2.1. **Comparison with existing results for $T = 1$, $X_0 = 1$ and $a_1 = a_2 = 0.5$.**

S_0/**K**	**B-S**	**Naik**	**Boyle-D**	**Risk no priced**	**Risk priced**
0.94	5.1096	5.8620	5.8579	5.8749(.0850)	18.7601(.0850)
0.96	6.1624	6.9235	6.9178	6.9369(.0863)	20.6900(.0863)
0.98	7.3248	8.0844	8.0775	8.0961(.0600)	22.7037(.1068)
1	8.5917	9.3401	9.3324	9.3303(.0573)	24.7740(.0856)
1.02	9.9563	10.6850	10.6769	10.7061(.0622)	26.8103(.1100)
1.04	11.4110	12.1127	12.1045	12.1304(.0967)	28.9475(.0938)

TAB. 2.2. **Comparison with existing results for $T = 1$, $X_0 = 2$ and $a_1 = a_2 = 0.5$.**

S_0/**K**	**B-S**	**Naik**	**Boyle-D**	**Risk no priced**	**Risk priced**
0.94	8.8557	8.2292	8.2193	8.2503(.0550)	7.2390(.0463)
0.96	9.9510	9.3175	9.3056	9.3302(.0869)	8.1968(.0626)
0.98	11.1190	10.4775	10.4647	10.4682(.0810)	9.2208(.0571)
1	12.3360	11.7063	11.6929	11.6853(.0828)	10.2929(.0547)
1.02	13.6206	13.0008	12.9870	12.9901(.0851)	11.4228(.0821)
1.04	14.9629	14.3575	14.3436	14.2823(.1069)	12.5857(.0781)

In Figure 2.1 we have a visual representation of these results.

FIG. 2.1. European Call prices versus Moneyness

We have also made some comparisons for different maturities (T) and for different values of intensity rate that characterize the Markov chain. Figures 2.2 and 2.3 present the results.

FIG. 2.2. European Call prices versus Time to maturity

FIG. 2.3. European Call prices versus intensity rate

In light of what precede, we may draw the following conclusions for *model I* :

- the impact of risk due to regime is significant (the relative difference in the prices ranges between 0.2 % and 97%) ;
- the regime-risk is too sensitive to market parameters like volatility or intensity rates of leaving(for each state).

Turning now to *model II*, in order to outline the effect of the introduction of jumps in the previous model and to see how the regime-risk is sensitive we constraint as in Ballotta [**9**] the value of the volatility in each state to be constant equal to the instantaneous volatility of log-return in *model I.*

We set the parameters to be $J = 2$ and

(1) $\underline{r} = (0.035, 0.035)$;
(2) $\underline{\mu} = (0.35, 0.35)$, $\underline{\mu}_J = (-.0537 - .0537)$;
(3) $\underline{\sigma}_{BS} = (0.2, 0.2)$, $\underline{\sigma}_{JP} = (.01884, .1884)$
(4) $\mathbb{A} = \begin{pmatrix} -.5 & .5 \\ .5 & -.5 \end{pmatrix}$
(5) $S_0 = 100; X_0 \in \{1, 2\}$ and $K \in \{60, 70, 80, 90, 100, 110, 120, 130, 140\}$.

The following panel shows the results.

FIG. 2.4. RSBlack-Scholes vs RSJump-Diffusion : Call prices across Strikes

FIG. 2.5. RSBlack-Scholes vs RSJump-Diffusion : Call prices across Maturities

We remark firstly that there is a difference between the prices in each regime-switching model. For the small maturities, the prices given by the RS-JD model with regime risk priced are higher than those given by the RS-JP model with regime risk not priced whereas for the RS-BS model, the difference across the strike are not perceptible. For the long maturities, the difference (across the strike) between the regime-risk, priced and not, are significant in both models. Particularly, in the RS-JD model the differences(across the strike) seem constant whereas in the RS-BS model these differences seem to grow as the time.

The last model we have to look at is *model III*. We present in Figures below the results of our simulation. The parameters are set to be

(1) $\underline{r} = (0.05, 0.015)$;

(2) $\underline{\mu} = (0.35, 0.05)$;

(3) $\underline{C} = (2, 3)$, $\underline{G} = (4, 5)$, $\underline{M} = (8, 6)$

(4) $\mathbb{A} = \begin{pmatrix} -.5 & .5 \\ .5 & -.5 \end{pmatrix}$

(5) $S_0 = 100; X_0 \in \{1, 2\}$ and $K \in \{60, 70, 80, 90, 100, 110, 120, 130, 140\}$.

The panel below displays the results

FIG. 2.6. RSVariance-Gamma model : Call prices across Maturities

FIG. 2.7. RSVariance-Gamma model : Call prices across Strikes

We see that even in this case, there is a perceptible difference between the price of option when the regime-risk is priced and the case where it is not priced.

2.6. Conclusion

In this chapter, we evaluate the impact of taking into account or not of the regime-risk in a Regime-Switching Levy Model. We derive two pricing kernels to illustrate this situation. Numerical experiments made show us the significant departure of values of prices of an European Call from when the regime-risk is not priced to one when it's priced. We also look the influence of the introduction of jumps in this analysis. Although the analysis presented would effectively highlight the importance of the regime-risk in the prices, it does not however explicitly quantify it. This aspect of things will be of our attention in the future.

2.7. Appendix

2.7.1. An explicit Comparison between Λ and Λ^*

Let $\alpha \in \Theta \cap \Theta^*$, for all $t \in \mathbb{T}$ we have by using lemma 2.1

$$\begin{aligned} E^{\mathbb{P}}\left[e^{-\int_0^t \alpha_s dY_s}\middle|\mathcal{F}_T^X\right] &= E^{\mathbb{P}}\left[E^{\mathbb{P}}\left[e^{-\int_0^t \alpha_s dY_s}\middle|\mathcal{G}_T\right]\middle|\mathcal{F}_T^X\right] \\ &= E^{\mathbb{P}}\left[\langle e^{\left(\mathbb{A}+\mathbf{Diag}(\underline{\xi}(\alpha))\right)t} X_T|\underline{\mathbf{1}}\rangle\middle|\mathcal{F}_T^X\right] \\ &= \langle e^{\left(\mathbb{A}+\mathbf{Diag}(\underline{\xi}(\alpha))\right)t} X_T|\underline{\mathbf{1}}\rangle. \end{aligned} \tag{2.60}$$

Thus,

$$\Lambda_t = e^{-\int_0^t \alpha_s dY_s} \times \frac{1}{\langle e^{\left(\mathbb{A}+\mathbf{Diag}(\underline{\xi}(\alpha))\right)t} X_T|\underline{\mathbf{1}}\rangle}. \tag{2.61}$$

Also,

$$\Lambda_t^* = e^{-\int_0^t \alpha_s dY_s} \times \frac{\langle e^{\left(\mathbb{A}+\mathbf{Diag}(\underline{\xi}(\alpha))\right)(T-t)} X_t, \underline{\mathbf{1}}\rangle}{\langle e^{\left(\mathbb{A}+\mathbf{Diag}(\underline{\xi}(\alpha))\right)T} X_0, \underline{\mathbf{1}}\rangle}. \tag{2.62}$$

Therefore we have that

$$\Lambda_t^* = \Lambda_t \frac{\langle e^{\left(\mathbb{A}+\mathbf{Diag}(\underline{\xi}(\alpha))\right)(T-t)} X_t, \underline{\mathbf{1}}\rangle \langle e^{\left(\mathbb{A}+\mathbf{Diag}(\underline{\xi}(\alpha))\right)t} X_T|\underline{\mathbf{1}}\rangle}{\langle e^{\left(\mathbb{A}+\mathbf{Diag}(\underline{\xi}(\alpha))\right)T} X_0|\underline{\mathbf{1}}\rangle}. \tag{2.63}$$

Chapitre 3

LOCAL RISK-MINIMIZATION UNDER A PARTIALLY OBSERVED MARKOV-MODULATED EXPONENTIAL LÉVY MODEL

Ce chapitre résulte d'une collaboration avec Olivier Menoukeu-Pamen. Il existe sous forme de rapport de recherche [**97**] et il a été révisé et resoumis dans la revue *Applied Mathematical Finance*. Notre contribution a consisté en la formulation du problÃ¨me et à la recherche des premiÃ¨res ébauches de solution. Avec notre co-auteur, nous avons formulé les résultats du problÃ¨me sous information partielle. La rédaction de l'article a été faite en partie par nos soins.

Résumé

Cet article adresse la question de la couverture quadratique du risque local associé à une option de type Européenne dans un modÃ¨le exponentiel-Lévy avec changements de régime. Nous commenÃ§ons par observer que sous une filtration élargie, le processus de prix S est une semimartingale ce qui nous permet d'implémenter la méthodologie due à Colwell et Elliott [**33**] pour résoudre le problÃ¨me de minimisation sous information totale. Enfin, nous obtenons la solution pour la filtration accessible à l'agent par projection.

Abstract

The option hedging problem for a Markov-modulated exponential Lévy model is examined. We employ the local risk-minimization approach to study optimal hedging strategies for European-type derivatives under both full and partial information. Then, we project the hedging strategies on the observed information to obtain hedging strategies under partial information.

3.1. Introduction

Unpredictable structural changes in the trends of asset prices or stock indices on financial markets are a reality currently . They are not usually caused by internal events of the market itself but are more closely related to the global socio-economical and political environment. To account for these features, Markov-modulated (or regime-switching) models have since been widely used in econometrics and financial mathematics. See for instance, Hamilton [**75**] for exhibiting the non-stationarity of macroeconomic times series, Elliott and Van der Hoek [**52**] for asset allocation, Pliska [**110**] and Elliott, et *al.* [**48**] for short rate models, Naik [**103**], Guo [**74**] and Buffington and Elliott [**24**] for option valuation.

The Markov-modulated exponential Lévy model is very attractive as an alternative to the classical Black-Scholes model because it couples the benefit of an exponential Lévy model, i.e., the presence of jumps, with the possibility, thanks to the Markov chain, of having long-term variability of some characteristics of the return distribution. However, in the context of derivative pricing, these models lead to incomplete markets. Therefore, the question of hedging becomes a crucial one.

In this paper, we consider the problem of optimal quadratic hedging of a European derivative contract in a market driven by a Markov-modulated Lévy model. Typically, in this model full information about the modulating factor X is not available in the market and the agent has only access to the information contained in past asset prices. Consequently, we shall deal with an optimal quadratic hedging problem for a partially observed model (or partial information scenario).

This kind of problem has been extensively studied in the literature. Di Masi, Platen and Runggaldier [**39**] were the first to discuss the problem of risk-minimizing (mean-variance) hedging under restricted information when the stock price is a martingale and the prices are observed only at discrete time instants. In [**118**], Schweizer made explicit for general filtrations $\mathbf{G} := \{\mathcal{G}_t\}_{t\in\mathbb{T}} \subseteq \{\mathcal{F}_t\}_{t\in\mathbb{T}} := \mathbf{F}$ a risk-miminizing strategy based on $\mathbf{G}$-predictable projections. Pham [**109**] solved the problem of mean-variance hedging for partially observed drift processes. Frey and Runggaldier [**65**] determined a locally risk-minimizing hedging strategy when the asset price process follows a stochastic model and is observed only at discrete random times. Frey [**66**] considered risk-minimization with incomplete information in a model for high-frequency data. In the same framework, but for more general model, Ceci [**27**] computed the optimal hedge strategy under the criterion of risk-minimization. In all these papers, the method consists firstly, to determine the optimal strategy under the full information, and secondly, to determine the final solution by projecting on the filtration available to the investor. Then the natural question arises : given a Markov-modulated Lévy model, can we apply the above methods to study the problem of local risk-minimization under partial information

The aim of this paper is to give an answer to the previous question. In fact, we show that under some restrictive conditions on our model, we can apply the same techniques used by the preceding authors to obtain an optimal hedging strategy for local risk-minimization under partial information. In fact, we first derive a martingale representation for the wealth process under full information. Then we proceed, as in the classical setting, by solving a local risk minimization under full information. The optimal strategy obtained under full information is quite explicit. Finally, using the fact that our processes do not jump simultaneously, we deduce an orthogonal projection of the claim with respect to the smaller filtration and therefore the optimal strategy.

The paper is organized as follows. Section 3.2 describes in detail our model setup and constructs two different filtrations that characterize the situation where investors have full and partial information. In Section 3.3, we recall some basic results on risk-minimization. Section 3.4 contains the main results, namely the martingale representation property for the value process, and the existence of optimal strategies in our market model under full and partial information.

3.2. The Model

In this section, we introduce the setting in which we are going to solve local-risk minimizing problem. We shall construct two filtrations that characterize the situation where investors have full and partial information.

3.2.1. Framework

We consider a financial market with two primary securities, namely a money market account B, and a stock S which are traded continuously over the time horizon $\mathbb{T} := [0, T]$, where $T \in (0, \infty)$ is fixed and represents the maturity time for all economic activities. To formalize this market, we fix a (complete) filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ that satisfies the usual conditions. We suppose also that $\mathcal{F}_T = \mathcal{F}$ and that $\mathcal{F}_0$ contains only the null sets of $\mathcal{F}$ and their complements. All processes are defined on the stochastic basis above. Further, we shall add to this setup a filtration which specifies the flow of informations available for the investors.

Let $X := \{X_t : t \in \mathbb{T}\}$ be an irreducible homogeneous continuous-time Markov chain with a finite state space $\mathbb{S} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_M\} \subset \mathbb{R}^M$ characterized by a rate (or intensity) matrix $A := \{a_{ij} : 1 \leq i, j \leq M\}$. Following Dufour and Elliott [**44**], we can identify $\mathbb{S}$ with the basis set of the linear space $\mathbb{R}^M$. From now, we set $\mathbf{e}_i = (0, 0, \ldots, \underbrace{1}_{\text{i-th}}, \ldots, 0)$. It follows from Elliott [**50**] that X admits the following semimartingale representation

$$X_t = X_0 + \int_0^t AX_s + \Gamma_t, \tag{3.1}$$

where $\Gamma := \{(\Gamma_t^i)_{i=1}^M : t \in \mathbb{T}\}$ is a vector-martingale in $\mathbb{R}^M$ with respect to the filtration generated by X.

Let r_t denote the instantaneous interest rate of the money market account B at time t. If we suppose that $r_t := r(t, X_t) = \langle \underline{r} | X_t \rangle$, where $\langle \cdot | \cdot \rangle$ is the usual scalar product in $\mathbb{R}^M$ and $\underline{r} = (r_1, r_2, \ldots, r_M) \in {\mathbb{R}_+}^M$, then the price dynamics of B is given by :

$$dB_t = r_t B_t \, dt, \;\; B(0) = 1 \quad \text{for} \quad t \in \mathbb{T}. \tag{3.2}$$

The appreciation rate μ_t and the volatility σ_t of the stock S at time t are defined as

$$\begin{aligned} \mu_t &:= \mu(t, X_t) = \langle \underline{\mu} | X_t \rangle, \\ \sigma_t &:= \sigma(t, X_t) = \langle \underline{\sigma} | X_t \rangle, \quad t \in \mathbb{T} \end{aligned} \tag{3.3}$$

where $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_M) \in \mathbb{R}^M$ and $\underline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_M) \in \mathbb{R}_+^{\mathbb{M}}$.

The stock price process S is described by this following Markov modulated Lévy process :

$$dS_t = S_{t^-} \Big(\mu_t dt + \sigma_t dW_t + \int_{\mathbb{R}\backslash\{0\}} (e^z - 1) \widetilde{N}^X(dt; dz) \Big), \quad S(0) = S_0 > 0. \tag{3.4}$$

Here $W := (W_t)_{t \in \mathbb{T}}$ is a one-dimensional standard Brownian motion or Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$, independent of X and N^X, and the compensated random measure

$$\widetilde{N}^X(dt, dz) := N^X(dt, dz) - \rho^X(dz)dt, \tag{3.5}$$

where $N^X(dt, dz)$ is the differential form of a Markov-modulated random measure on $\mathbb{T} \times \mathbb{R}\backslash\{0\}$. We recall from Elliott and Osakwe **[54]** and Elliott and Royal **[55]** that a Markov-modulated random measure on $\mathbb{T} \times \mathbb{R}\backslash\{0\}$ is a family $\{N^X(dt, dz; \omega) : \omega \in \Omega\}$ of non-negative measures on the measurable space $(\mathbb{T} \times \mathbb{R}\backslash\{0\}, \mathcal{B}(\mathbb{T}) \otimes \mathcal{B}(\mathbb{R}\backslash\{0\}))$, which satisfies $N^X(\{0\}, \mathbb{R}\backslash\{0\}; \omega) = 0$ and has the following compensator, or dual predictable projection

$$\rho^X(dz)dt := \sum_{i=1}^{M} \langle X_{t^-} | \mathbf{e}_i \rangle \rho_i(dz) dt, \tag{3.6}$$

where $\rho_i(dz)$ is the density for the jump size when the Markov chain X is in state $\mathbf{e}_i$ and satisfying

$$\int_{|z| \geq 1} (e^z - 1)^2 \rho_i(dz) < \infty. \tag{3.7}$$

The general setting considered here can be seen as an extension of the exponential-Lévy model described in Cont and Tankov **[34]** where a factor of modulation is introduced. Hence, we can retrieve in a simple way most of some current models which exist in the literature (for example, the classical Black-Scholes model and the family of exponential-Lévy models.)

The subsequent assumption shall be fundamental, particularly in Section 3.4.1 to obtain a martingale representation for the value process.

Assumption 3.1. *We assume that a transition of Markov chain X from state $\mathbf{e}_j$ to state $\mathbf{e}_k$ and a jump of S do not happen simultaneously almost surely.*

Let $\xi := \{\xi_t\}_{t \in \mathbb{T}}$ denote the discounted stock price. Then,

$$\xi_t := \frac{S_t}{B_t} = e^{-\int_0^t r_u du} S_t.$$

If $R_t = e^{\int_0^t r_u du}$, for each $t \in \mathbb{T}$. Then, the discounted stock price process is given by :

$$\begin{cases} d\xi_t &= F_\mu(t, \xi_{t^-}, X_t)dt + F_\sigma(t, \xi_{t^-}, X_t)dW_t + \int_{\mathbb{R}\setminus\{0\}} F_\gamma(t, \xi_{t^-}, X_t)\widetilde{N}^X(dt; dz), \\ \xi(0) &= S_0 > 0 \ \mathbb{P} \ a.s, \end{cases} \tag{3.8}$$

where

$$\begin{cases} F_\mu(t, \xi_t, X_t) := & \Big(\mu(t, R_t\xi_t, X_t) - r(t, R_t\xi_t, X_t)\Big)\xi_t \\ F_\sigma(t, \xi_t, X_t) := & \sigma(t, R_t\xi_t, X_t)\xi_t \\ F_\gamma(t, \xi_t, X_t) := & \xi_t(e^z - 1). \end{cases} \tag{3.9}$$

The theory of stochastic flows shall also be used to identify the integrands in the stochastic integrals involved in the martingale representation property in Section 3.4.1.

Let now consider a general form of stochastic differential equation (SDE) (3.8) :

$$\begin{cases} d\xi_t &= F_\mu(t, \xi_{t^-}, X_t)dt + F_\sigma(t, \xi_{t^-}, X_t)dW_t + \int_{\mathbb{R}\setminus\{0\}} F_\gamma(t, \xi_{t^-}, X_t)\widetilde{N}^X(dt; dz), \\ \xi_s &= x > 0 \quad \mathbb{P}\text{-a.s. } for\ 0 \leq s < t \leq T. \end{cases} \tag{3.10}$$

We assume that the coefficients $F_\mu, F_\sigma, F_\gamma$ are smooth enough to guaranty the existence and uniqueness of an adapted *càdlàg* (strong) solution $\xi_{s,\ t}(x)$ (see Fujiwara and Kunita [**68**]). Furthermore, this solution forms a stochastic flow of diffeomorphisms $\Phi_{s,\ t} : (0, +\infty) \times \Omega \to (0, +\infty)$ given by

$$\Phi_{s,\ t}(x, \omega) = \xi_{s,\ t}(x)(\omega), \tag{3.11}$$

for each (s, t) such that $0 \leq s < t \leq T$, $x \in (0, +\infty)$ and $\omega \in \Omega$. $(\Phi_{s,\ t})_{s<t}$ verifies the following properties :

- $\Phi_{s,\ t} = \Phi_{0,\ t} \circ \Phi_{0,\ s}^{-1}$ for all $s < t$;
- Cocycle property : $\Phi_{s,\ u} = \Phi_{t,\ u} \circ \Phi_{s,\ t}$ for all $s < t < u$;
- Conditional independent increments : for $t_0 \leq t_1 \leq \ldots \leq t_n$, $\Phi_{t_0,\ t_1}, \Phi_{t_1,\ t_2}, \ldots, \Phi_{t_{n-1},\ t_n}$ are conditionally independent given $\mathcal{F}_T^X$.

Let $x = \xi_{0,\ t}(x_0)$, for each $t \in \mathbb{T}$. By the uniqueness of solutions of SDE and the semi-group property, we get

$$\begin{aligned} \xi_{0,\ T}(x_0) &= \xi_{t,\ T}(\xi_{0,\ t}(x_0)) \\ &= \xi_{t,\ T}(x). \end{aligned} \tag{3.12}$$

Differentiating (3.12) with respect to x_0, we obtain :

$$\frac{\partial \xi_{0,T}(x_0)}{\partial x_0} = \frac{\partial \xi_{t,\ T}(x)}{\partial x}\frac{\partial \xi_{0,\ t}(x_0)}{\partial x_0}. \tag{3.13}$$

3.2.2. Market Information

In general, the Markov-modulated Lévy model, as described by Equation (3.4), is based on the mathematical framework of Markov additive processes (MAP). These processes are widely studied in stochastic analysis (See, e.g., [**?, ?, 59, 72**].) In particular, the couple (X, S) is a Markov additive process and yields two important filtrations as we shall see below.

Let $\mathcal{F}^X := \{\mathcal{F}_t^X\}_{t\in\mathbb{T}}$ and $\mathcal{F}^S := \{\mathcal{F}_t^S\}_{t\in\mathbb{T}}$ denote the right-continuous, $\mathbb{P}$-complete filtrations generated by X and S respectively. We define for $t \in \mathbb{T}$,

$$\mathcal{G}_t := \mathcal{F}_t^S \tag{3.14}$$

and

$$\overline{\mathcal{G}}_t := \mathcal{F}_T^X \vee \mathcal{F}_t^S. \tag{3.15}$$

The filtration $\mathbf{G} := \{\mathcal{G}_t\}_{t\in\mathbb{T}}$ represents all the information up to time t gained from the observations of the price fluctuations S. The strictly larger filtration $\overline{\mathbf{G}} := \{\overline{\mathcal{G}}_t\}_{t\in\mathbb{T}}$ denotes the information about the stock price history up to time t and the information about the entire path $\mathcal{F}_T^X$ of the modulation factor process X.

We shall assume in the last section of this paper that the investors in the market only have access to the former filtration, which is thus the one used practically, whereas the latter filtration serves mainly for theoretical purposes.

It is easy to see that under $\overline{\mathbf{G}}$, the discounted price ξ is a special semimartingale and its canonical decomposition is given by

$$\begin{aligned}\xi_t = S_0 + &\underbrace{\int_0^t F_\mu(s, \xi_{s^-}, X_s)ds}_{\text{finite variation part}} \\ &+ \underbrace{\int_0^t F_\sigma(s, \xi_{s^-}, X_s)dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} F_\gamma(s, \xi_{s^-}, X_s)\widetilde{N}^X(ds; dz)}_{\text{local-martingale part}}. \end{aligned} \tag{3.16}$$

3.2.3. Esscher Transform Change of Measure

One of the main features of the Markov-modulated Lévy model is that it leads to an incomplete market. We shall therefore employ the regime-switching Esscher transform as in Elliott *et al.* [**50**] to determine an equivalent martingale measure. For doing so, we define the process Y by

$$\begin{aligned} Y_t = &\int_0^t \left(\mu_r - \frac{1}{2}\sigma_r^2\right)dr + \int_0^t \sigma_r dW_r + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\widetilde{N}^X(dr; dz) \\ &- \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z)\rho^X(dz)dr \end{aligned} \tag{3.17}$$

As in [**122**], consider the following set

$$\Theta := \left\{ (\theta_t)_{t\in\mathbb{T}} \;\middle|\; \theta_t := \sum_{i=1}^M \theta_i \langle X_{t^-} | \mathbf{e}_i\rangle \text{ with } (\theta_1, \theta_2, ..., \theta_M) \in \mathbb{R}^M \right.$$
$$\left. \text{such that } E^{\mathbb{P}}\left[e^{-\int_0^t \theta_r dY_r} \middle| \mathcal{F}_T^X \right] < \infty \right\}.$$

For $\theta := (\theta_t)_{t\in\mathbb{T}} \in \Theta$, the generalized Laplace transform of a $\overline{\mathbf{G}}$-adapted process Y is defined as

$$\mathcal{M}_Y(\theta)_t := E^{\mathbb{P}}\left[e^{-\int_0^t \theta_r dY_r}\Big|\mathcal{F}_T^X\right]. \tag{3.18}$$

Notice that contrary to the usual Esscher transform, the expectation involved here is taken conditionally on the information regarding the future of the Markov chain X. With this extended definition of a Laplace transform, we can now define the generalized Esscher transform (with respect to the parameter θ, called the *Esscher parameter*).

Let $\Lambda^\theta = \{\Lambda_t^\theta\}_{t\in\mathbb{T}}$ denote a $\overline{\mathbf{G}}$-adapted stochastic process defined as

$$\Lambda_t^\theta := \frac{e^{-\int_0^t \theta_r dY_r}}{\mathcal{M}_Y(\theta)_t}, \quad t \in \mathbb{T};\ \theta \in \Theta. \tag{3.19}$$

It can be shown that (See, e.g., [**50**])

$$\begin{aligned}\Lambda_t^\theta = \exp\Bigg[& -\int_0^t \theta_r\sigma_r dW_r - \frac{1}{2}\int_0^t \theta_r^2\sigma_r^2 dr - \int_0^t\int_{\mathbb{R}\setminus\{0\}} \theta_{r-} z\widetilde{N}^X(dr;dz) \\ & - \int_0^t\int_{\mathbb{R}\setminus\{0\}} \Big(e^{-z\theta_r} - 1 + \theta_r z\Big)\rho^X(dz)dr\Bigg].\end{aligned} \tag{3.20}$$

Moreover, as shown in [**122**], the stochastic process $\Lambda^\theta = \{\Lambda_t^\theta\}_{t\in\mathbb{T}}$ defined by (3.19) is a positive $(\overline{\mathbf{G}}, \mathbb{P})$-martingale and

$$E^{\mathbb{P}}[\Lambda_t^\theta] = 1, \quad \forall t \in \mathbb{T}. \tag{3.21}$$

From Equation 3.21, we deduce that the process $\Lambda^\theta = \{\Lambda_t^\theta\}_{t\in\mathbb{T}}$ given by Equation (3.20) is a density process inducing a change of measure in the probability space $(\Omega, \overline{\mathcal{G}}_T)$. Indeed, by setting

$$\frac{d\mathbb{Q}^\theta}{d\mathbb{P}}\bigg|_{\overline{\mathcal{G}}_t} = \Lambda_t^\theta \quad t \in \mathbb{T}, \tag{3.22}$$

we define for each process θ in Θ a new probability measure $\mathbb{Q}^\theta$ equivalent to $\mathbb{P}$. Actually, $\mathbb{Q}^\theta$ is just an equivalent probability measure. To transform it into a martingale equivalent measure, we need to impose a condition generally known as the *martingale condition*. It stipulates that the discounted stock price $\{\xi_t\}_{t\in\mathbb{T}}$ will be a $\overline{\mathbf{G}}$-martingale under $\mathbb{Q}^\theta$. Then,

$$E^{\mathbb{Q}^\theta}\left[\xi_t\Big|\overline{\mathcal{G}}_0\right] = \xi(0),\ \forall t \in \mathbb{T}. \tag{3.23}$$

Hence, we have

Proposition 3.1. *An equivalent probability measure $\mathbb{Q}^\theta$ defined by (3.22) is an equivalent martingale measure on $(\Omega, \overline{\mathcal{G}}_T)$, i.e., it satisfies condition (3.23), if and only if the process θ satisfies the following equation*

$$\mu_t - r_t - \theta_t\sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}} (e^z - 1)(e^{-z\theta_t} - 1)\rho^X(dz) = 0,\ \forall t \in \mathbb{T}. \tag{3.24}$$

Proof. The proof is a straightforward adaptation of that of Proposition 2.2 in Elliott *et al.* [**50**]. The main ingredient is an explicit computation of the generalized Laplace transform defined by (3.18). □

However, the process θ is completely determined by the vector $(\theta_1, \theta_2, \ldots, \theta_M)$ solution of the system of equations

$$\mu_i - r_i - \theta_i \sigma_i^2 + \int_{\mathbb{R}} (e^z - 1)(e^{-z\theta_i} - 1)\rho_i(z)dz = 0, \tag{3.25}$$

for $i = 1, 2, \ldots, N$.

For pricing purposes, we need to know the dynamics of the discounted stock price under the martingale probability measure $\mathbb{Q}^\theta$. The following proposition states a result in this direction.

Proposition 3.2. *Under the risk-neutral probability measure $\mathbb{Q}^\theta$, the discounted stock price process ξ is the solution to the following stochastic differential equation*

$$\begin{cases} d\xi_t & = F_\sigma(t, \xi_{t^-}, X_t)dW_t^\theta + \int_{\mathbb{R}\setminus\{0\}} F_\gamma(t, \xi_{t^-}, X_t)\underline{\widetilde{N}}^\theta(dt; dz) \\ \xi(0) & = S_0 > 0 \quad \mathbb{P}\text{-}a.s. \ for \ 0 \leq t \leq T, \end{cases} \tag{3.26}$$

where

- W^θ *defined as*

$$W_t^\theta := W_t + \int_0^t \theta_r \sigma_r dr, \tag{3.27}$$

  is the standard Brownian motion under $\mathbb{Q}^\theta$ *;*
- $\underline{\widetilde{N}}^\theta$ *defined as*

$$\underline{\widetilde{N}}^\theta(dr; dz) = N^X(dr; dz) - \underline{\rho}^{\theta X}(dz)dr, \tag{3.28}$$

  is the compensated measure of N^X under $\mathbb{Q}^\theta$ with $\underline{\rho}^{\theta X}(dz) := e^{-\theta z}\rho^X(dz)$.

Proof. This follows easily from (3.8) by the application of the Girsanov-Meyer Theorem (See Øksendal and Sulem [**105**], Protter [**111**]). □

3.3. The Locally Risk-Minimizing Hedging Problem

In this section, we recall some terminology on local risk minimization. We shall simply give essential results ; for further information, the reader is referred to the survey of Schweizer [**119**], to which our presentation owes much.

3.3.1. Review of Some Notions on The Risk-Minimization Approach

This concept has been introduced by Föllmer and Sondermann [**64**] for a non-redundant contingent claim written on a one-dimensional, square-integrable discounted risky asset ξ which is a martingale under the original measure $\mathbb{P}$. Concretely, given a stochastic basis as above, the goal consists of minimizing the conditional remaining risk : $\mathcal{R}_t := E^{\mathbb{P}}[(C_T - C_t)^2 | \mathcal{F}_t]$ for all $t \in \mathbb{T}$. Here C_t stands for the cost process and is defined as the difference between the value of the (portfolio) strategy detained by the investor at time t and the gains made from trading in the financial market up to time t. Let $\mathcal{L}^2(\xi)$ the space of all $\mathbb{R}$-valued predictable process ϕ such that

$$||\phi||_{\mathcal{L}^2(\xi)} := \left(E^{\mathbb{P}}\Big[\int_0^T \phi_u^2 d[\xi,\xi]_u\Big]\right)^{\frac{1}{2}} < \infty,$$

A trading strategy is a pair of processes $\varphi = (\phi, \psi)$ where ψ is an adapted process and $\phi \in \mathcal{L}^2(\xi)$ is a $\mathbf{F}$-predictable process, such that the value process $V := \phi\xi + \psi$ has right continuous sample paths and $E^{\mathbb{P}}[V_t^2] < \infty$ for every $t \in \mathbb{T}$(i.e., $V_t \in \mathcal{L}^2(\Omega, \mathbb{P})$ for every $t \in \mathbb{T}$).

Consider a trading strategy $\varphi = (\phi, \psi)$, where $\phi = (\phi_t)_{t\in\mathbb{T}}$ denotes at time t, the number of stocks held and $\psi = (\psi_t)_{t\in\mathbb{T}}$ is the amount invested in the money market account.

Let H be a claim which is $\mathcal{F}_T$-measurable and square-integrable. Consider strategies that replicate the contingent claim H at time T; that are the strategies which satisfy the assumption :

$$V_T = H \ \ \mathbb{P}\text{-a.s.}$$

Such strategies are called *H-attainable*[1].

A trading strategy φ such that $C_t(\varphi) = C_0(\varphi)$ for all $t \in \mathbb{T}$ is called *self-financing*. Furthermore, if the cost process $C_t(\varphi)$ is a $\mathbb{P}$-martingale then φ is said to be *mean self-financing*.

Definition 3.1. *Let (ϕ, ψ) and $(\widetilde{\phi}, \widetilde{\psi})$ be H-attainable strategies. Then $(\widetilde{\phi}, \widetilde{\psi})$ is called a H-attainable strategy continuation of (ϕ, ψ) at time $t \in [0, T)$ if $\widetilde{\phi}_s = \phi_s$ for $s \in [0, t]$ and $\widetilde{\psi}_s = \psi_s$ for $s \in [0, t)$.*

The following result obtained by Föllmer and Sondermann [**64**] is based on the Galtchouk-Kunita-Watanabe (GKW) decomposition (see Kunita-Watanabe [**90**]) of H and gives a risk-minimizing hedging strategy under full information.

Theorem 3.1. *Assume the GKW decomposition of the claim $H \in \mathcal{L}^2(\Omega, \mathbb{P})$ given by*

$$H = H_0 + \int_0^T \phi_s^H d\xi_s + L_T^H,$$

with $\phi^H \in \mathcal{L}^2(\xi)$, L^H a square-integrable $\mathbb{P}$-martingale orthogonal to ξ with $H_0 = E^{\mathbb{P}}[H]$ $\mathbb{P}$-a.s.

Then, the trading strategy $\varphi^{\otimes} = (\phi^{\otimes}, \psi^{\otimes})$ defined as

$$(\phi_t^{\otimes}, \psi_t^{\otimes}) := (\phi_t^H, H_0 + \int_0^t \phi_s^H d\xi_s - \phi_t^H \xi_t + L_t^H), \quad \forall t \in \mathbb{T} \tag{3.29}$$

is H-attainable and risk-minimizing. Its associated risk process $\mathcal{R}^{\otimes}$ is given by

$$\mathcal{R}_t^{\otimes} = E^{\mathbb{P}}[(L_T^H - L_t^H)^2 | \mathcal{F}_t], \ \ \mathbb{P}\text{-a.s.} \quad \forall t \in \mathbb{T}. \tag{3.30}$$

Furthermore, this strategy is unique.

[1] In [**64**], the authors refer to that as *H-admissible*.

From now on, we assume that the one-dimensional discounted asset ξ is no longer a martingale under the measure $\mathbb{P}$ but only a semimartingale with the following decomposition

$$\xi = \xi_0 + Z + A \tag{3.31}$$

where Z is a square-integrable martingale for which $Z_0 = 0$, and A is a predictable process of finite variation $|A|$(i.e., $\sup_\tau \sum_{i=1}^{N_\tau} |A_{t_i} - A_{t_{i-1}}| < \infty$) for every partition τ of $\mathbb{T}$. In this situation, we can no longer apply the preceding result of Föllmer and Sondermann **[64]**. To deal with such a case, Schweizer **[118, 119]** introduced the concept of Locally risk-minimizing strategy where the conditional variances are kept as small as possible but now in a local manner. Now, to adapt the definition of a trading strategy in this case we need that $\phi \in \mathcal{L}^2(Z)$ and that $\int_0^T |\phi_u dA_u| \in \mathcal{L}^2(\Omega, \mathbb{P})$.

Definition 3.2. *(small perturbation). A trading strategy $\Delta = (\delta, \epsilon)$ is called a* small perturbation *if it satisfies the following conditions :*

- *δ is bounded ;*
- *$\int_0^T |\delta_u||dA_u|$ is bounded ;*
- *$\delta_T = \epsilon_T = 0$.*

For any subinterval $(s, T] \subset \mathbb{T}$, we define the small perturbation $\Delta\big|_{(s,T]} := (\delta 1_{(s,T]}, \epsilon 1_{(s,T]})$.

Now we can define

Definition 3.3. *(locally risk-minimizing strategy). For a trading strategy φ, a small perturbation Δ and a partition τ of $\mathbb{T}$ the* risk-quotient (R-quotient) $r^\tau[\varphi, \Delta]$ *which is a sort of relative local risk is defined as*

$$r^\tau[\varphi, \Delta] := \sum_{t_i, t_{i+1} \in \tau} \frac{\mathcal{R}_{t_i}(\varphi + \Delta\big|_{(t_i, t_{i+1}]}) - \mathcal{R}_{t_i}(\varphi)}{E^{\mathbb{P}}[\langle Z \rangle_{t_{i+1}} - \langle Z \rangle_{t_i} | \mathcal{F}_{t_i}]} 1_{(t_i, t_{i+1}]}. \tag{3.32}$$

A trading strategy φ is called locally risk-minimizing if

$$\liminf_{n \to \infty} r^{\tau_n}[\varphi, \Delta] \geq 0, \quad \mathbb{P} \times \langle Z \rangle\text{-}a.s.$$

for every small perturbation Δ and every increasing sequence (τ_n) of partitions of $\mathbb{T}$ such that $||\tau_n|| \to 0$.

To present the main results, we need the following technical assumptions :

Assumption 3.2.

- **(A1)** *For $\mathbb{P}$-almost all ω the measure on $\mathbb{T}$ induced by $\langle Z \rangle(\omega)$ has the whole interval $\mathbb{T}$ as its support, i.e., $\langle Z \rangle$ should be $\mathbb{P}$-almost surely strictly increasing on the whole interval $\mathbb{T}$.*
- **(A2)** *A is continuous.*
- **(A3)** *A is absolutely continuous with respect to $\langle Z \rangle$ with a density α satisfying*

$$E^{\mathbb{P}}\Big[\int_0^T |\alpha_u| \max(\log|\alpha_u|, 0) d\langle Z \rangle_u\Big] < \infty.$$

A sufficient condition for **(A3)** is that $E^{\mathbb{P}}\Big[\int_0^T |\alpha_u|^2 d\langle Z\rangle_u\Big] < \infty$ and one refers to that by saying : ξ satisfies *the Structure Condition (SC)*. We can remark that with assumption **(A2)**, ξ is a *special* semimartingale. We can now state the optimality result.

Theorem 3.2. *A contingent claim $H \in \mathcal{L}^2(\Omega, \mathbb{P})$ admits a (pseudo-optimal) locally risk-minimizing strategy $\varphi^{\odot} = (\phi^{\odot}, \psi^{\odot})$ with $V_T(\varphi^{\odot}) = H$ $\mathbb{P}$-a.s. if and only if H can be written as*

$$H = H_0 + \int_0^T \phi_s^H d\xi_s + L_T^H \quad \mathbb{P}\text{-}a.s. \tag{3.33}$$

with $H_0 \in \mathcal{L}^2(\Omega, \mathbb{P})$, $\phi^H \in \mathcal{L}^2(\xi)$, L^H a square-integrable $\mathbb{P}$-martingale null at the origin and $\mathbb{P}$-strongly orthogonal to M. The strategy φ is then given by

$$\phi_t^{\odot} = \phi_t^H, \quad t \in \mathbb{T}$$

and

$$C_t(\varphi^{\odot}) = H_0 + L_t^H, \quad t \in \mathbb{T};$$

its value process is

$$V_t(\varphi^{\odot}) = C_t(\varphi) + \int_0^t \phi_s^{\odot} d\xi_s = H_0 + \int_0^t \phi_s^H d\xi_s + L_t^H, \quad t \in \mathbb{T}. \tag{3.34}$$

Proof. See Proposition 3.4 of Schweizer [**119**]. □

Equation (3.33) is called the *Föllmer-Schweizer decomposition (FS)* for the contingent claim H. In practice, it is very difficult to obtain this decomposition so the more natural approach introduced by Föllmer and Schweizer [**63**] consists of using a Girsanov transformation to shift the problem back to a martingale measure where standard techniques shuch as Galchouk-Kunita-Watanabe projection are available.

3.4. Main Results

This section is devoted to the main results of this paper. We shall first derive a martingale representation for the wealth process of a claim written on a risky asset whose price evolution is given by a Markov-modulated exponential Lévy process. After, we solve the problem of local-risk minimization under full and partial information.

3.4.1. A Martingale Representation Property

In this section, we give an explicit representation of a martingale which is useful for the problem of hedging in the context of a Markov-modulated Lévy model. The proof of the result is similar to the one given by Elliott *et al.* [**53**]. We give an explicit martingale representation of the wealth process which will be useful later on in the finding of an optimal strategy the proof of our main result.

First, it is easy to see that the Esscher transform change of measure Λ^θ introduced in Section 3.2.3 is the solution to the following SDE

$$\begin{cases} \Lambda_{t,\ u}(x) & = 1 + \int_t^u \Lambda_{t,\ r^-}(x)(-\theta_r\sigma_r)(r, \xi_{t,\ r^-}(x), X_r)dW_r \\ & \quad + \int_t^u \int_{\mathbb{R}\setminus\{0\}} \Lambda_{t,\ r^-}(x)(e^{-z\theta_r(r,\xi_{t,\ r^-}(x),X_r)} - 1)\widetilde{N}^X(dr; dz) \\ \Lambda_{t,\ t}(x) & = 1 \quad \mathbb{P}\text{-a.s. } for\ 0 \leq t < u \leq T. \end{cases} \tag{3.35}$$

Indeed, for all $t \in \mathbb{T}$, $\Lambda^\theta_t = \Lambda_{0,\ t}(x)$.

Now, consider a function $c(\cdot) : (0, +\infty) \to \mathbb{R}$ such that $c(\cdot)$ is twice differentiable and $c(\cdot)$ and $\frac{\partial c(\cdot)}{\partial x}$ are at most linear growth in x. We shall determine the current price at time t of a contingent claim of the form $c(S_T)$, which is the payoff of the claim at maturity $T > t$. In the sequel, we have to work with the discounted claim as function of the discounted stock price, that by :

$$\hat{c}(\xi_{0,T}) := R_T^{-1} c(R_T \xi_{0,T}(x_0)) = R_T^{-1} c(S_T). \tag{3.36}$$

So, we assume that the process θ is chosen such that $E^{\mathbb{Q}^\theta}[\hat{c}^2(\xi_{0,T}(x_0))] < \infty$ and then we define the square-integrable $(\overline{\mathbf{G}}, \mathbb{Q}^\theta)$-martingale $\{V_t\}_{t\in\mathbb{T}}$ by :

$$V_t := E^{\mathbb{Q}^\theta}[\hat{c}(\xi_{0,T}(x_0))|\overline{\mathcal{G}}_t], \quad t \in \mathbb{T}. \tag{3.37}$$

As (X, ξ) and (X, Λ) are Markov additive processes (See Çinlar [**?**]), they verify the Markov property with respect to the large filtration $\overline{\mathbf{G}}$. Hence, by using Bayes' rule, we obtain :

$$\begin{aligned} V_t &:= E^{\mathbb{Q}^\theta}[\hat{c}(\xi_{0,\ T}(x_0))|\overline{\mathcal{G}}_t] \\ &= \frac{E^{\mathbb{P}}[\Lambda_{0,\ T}(x_0)\hat{c}(\xi_{0,\ T}(x_0))|\overline{\mathcal{G}}_t]}{E^{\mathbb{P}}[\Lambda_{0,\ T}(x_0)|\overline{\mathcal{G}}_t]} \\ &= E^{\mathbb{P}}\Big[\frac{\Lambda_{0,\ t}(x_0)\Lambda_{t,\ T}(x)\hat{c}(\xi_{t,\ T}(x))}{\Lambda_{0,\ t}(x_0)}\Big|\overline{\mathcal{G}}_t\Big], \quad \text{because } E^{\mathbb{P}}[\Lambda_{t,\ T}(x)|\overline{\mathcal{G}}_t] = 1; \\ &= E^{\mathbb{P}}[\Lambda_{t,\ T}(x)\hat{c}(\xi_{t,\ T}(x))|\overline{\mathcal{G}}_t] \\ &= E^{\mathbb{P}}[\Lambda_{t,\ T}(x)\hat{c}(\xi_{t,\ T}(x))|(X_t, \xi_{0,\ t}) = (\mathbf{e}, x)]. \end{aligned} \tag{3.38}$$

Thus, we define for each $x \in (0, +\infty)$ and $\mathbf{e} \in \mathbb{S}$,

$$\begin{aligned} V(t, x, \mathbf{e}) &:= E^{\mathbb{P}}[\Lambda_{t,\ T}(x)\hat{c}(\xi_{t,\ T}(x))|(X_t, \xi_{0,\ t}) = (\mathbf{e}, x)] \\ &\quad (= E^{\mathbb{Q}^\theta}[\hat{c}(\xi_{t,\ T}(x))|(X_t, \xi_{0,\ t}) = (\mathbf{e}, x)]). \end{aligned} \tag{3.39}$$

For each (t, u) such that $0 \leq t < u \leq T$, let introduce the following processes :

(1) L defined as

$$\begin{aligned} L_{t,\ u} &:= \int_t^u \frac{\partial(-\theta_r\sigma_r)}{\partial\xi}(r,\ \xi_{t,\ r}(x), X_r) \times \frac{\partial\xi_{t,\ r}}{\partial x} dW^\theta_r \\ &\quad + \int_t^u \int_{\mathbb{R}\setminus\{0\}} \Big[e^{z\theta_r(r,\ \xi_{t,\ r^-}(x),\ X_r)} \frac{\partial e^{-z\theta_r(r,\ \xi_{t,\ r^-}(x),\ X_r)}}{\partial\xi} \times \frac{\partial\xi_{t,\ r^-}}{\partial x}(x)\Big] \underline{\widetilde{N}}^\theta(dr, dz), \end{aligned}$$

(2) K defined as

$$K_{t,\ u} := \int_t^u \frac{\Lambda_{t,\ r}(x+\zeta_t(y))}{\Lambda_{t,\ r}(x)}$$
$$\times \Big[(-\theta_r\sigma_r)(r,\ \xi_{t,\ r}(x+\zeta_t(y)), X_r) + (\theta_r\sigma_r)(r,\ \xi_{t,\ r}(x), X_r)\Big] dW_r^\theta$$
$$+ \int_t^u \int_{\mathbb{R}\setminus\{0\}} \frac{\Lambda_{t,\ r^-}(x+\zeta_t(y))}{\Lambda_{t,\ r^-}(x)}$$
$$\times \Big[\frac{e^{-z\theta_r(r,\ \xi_{t,\ r^-}(x+\zeta_t(y)),\ X_r)} - e^{-z\theta_r(r,\ \xi_{t,\ r^-}(x),\ X_r)}}{e^{-z\theta_r(r,\ \xi_{t,\ r^-}(x),\ X_r)}}\Big] \underline{\widetilde{N}}^\theta(dr, dz)$$

with $\xi_{t^-} = x,\ \zeta_t(y) := \zeta(t, x, y)$.

(3) $\mathbf{V}$ the vector process defined as

$$\mathbf{V}(t, \xi_{0,\ t}(x_0)) := \Big(V(t, \xi_{0,\ t}(x_0), \mathbf{e}_1), V(t, \xi_{0,\ t}(x_0), \mathbf{e}_2), \ldots, V(t, \xi_{0,\ t}(x_0), \mathbf{e}_M)\Big).$$

Now, we are able to give a martingale representation for the $\{V_t\}_{t\in\mathbb{T}}$.

Proposition 3.3. *The $(\overline{\mathbf{G}}, \mathbb{Q}^\theta)$-martingale $\{V_t\}_{t\in\mathbb{T}}$ has the representation*

$$V_t = V_0 + \int_0^t \phi_r^c(\xi_r, X_r) dW_r^\theta + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \phi_r^d(z, \xi_{r^-}, X_{r^-}) \underline{\widetilde{N}}^\theta(dr, dz)$$
$$+ \int_0^t \langle \alpha_r, d\Gamma_r \rangle, \qquad (3.40)$$

where ϕ^c, ϕ^d and α are such that,

- $E^{\mathbb{Q}^\theta}\Big[\int_0^T (\Phi_r^c)^2 dr\Big] < \infty,$
- $E^{\mathbb{Q}^\theta}\Big[\int_0^T ||\alpha_r||^2 dr\Big] < \infty$ *and*
- $E^{\mathbb{Q}^\theta}\Big[\int_0^T \int_{\mathbb{R}\setminus\{0\}} (\phi_r^d(z))^2 \rho^X(dz) dr\Big] < \infty,$

with the following explicit expressions

$$\phi_r^c(\xi_r, X_r) = E^{\mathbb{Q}^\theta}\Big[L_{r,\ T}\hat{c}(\xi_{r,\ T}(x))$$
$$+ \frac{\partial \hat{c}}{\partial \xi}(\xi_{r,\ T}(x)) \frac{\partial \xi_{r,\ T}}{\partial x}(x) \Big| (X_r, \xi_{0,\ r}(x_0)) = (\mathbf{e}, x)\Big] \sigma_r(r, \xi_r, X_r), \quad (3.41)$$

$$\phi_r^d(y, \xi_{r^-}, X_r) = E^{\mathbb{Q}^\theta}\Big[(K_{r,\ T}+1)\hat{c}(\xi_{r,\ T}(x_- + \zeta_r(z)))$$
$$- \hat{c}(\xi_{r,\ T}(x)) \Big| (X_r, \xi_{0,\ r}(x_0)) = (\mathbf{e}, x)\Big], \quad (3.42)$$

$$\alpha_r = \mathbf{V}(r, \xi_{0,\ r}(x_0)) \in \mathbb{R}^M. \tag{3.43}$$

with $x = \xi_{0,\ r}(x_0)$ *and* $x_- = \xi_{0,\ r^-}(x_0)$.

In order to prove Proposition 3.3, we need the subsequent result

Lemma 3.1. *The following identities hold*

$$\frac{\partial \Lambda_{t,\ T}}{\partial x}(x) = \Lambda_{t,\ T}(x) \times L_{t,\ T} \tag{3.44}$$

and

$$\Lambda_{t,\ T}(x + \zeta(z)) - \Lambda(x) = \Lambda_{t,\ T}(x) \times K_{t,\ T}. \tag{3.45}$$

Proof. See Appendix. □

Now, we give the proof of the Proposition 3.3.

Proof. Noting that

$$V(t, \xi_t, X_t) = \langle \mathbf{V}(t, \xi_t) | X_t \rangle, \tag{3.46}$$

we obtain by differentiation

$$dV(t, \xi_t, X_t) = \langle d\mathbf{V}(t, \xi_t) | X_t \rangle + \langle \mathbf{V}(t, \xi_t) | dX_t \rangle, \tag{3.47}$$

and from Itô differentiation rule

$$\begin{aligned} dV(t, \xi_t, X_t) = \left\langle \mathbf{V}(t, \xi_t) \middle| dX_t \right\rangle + \bigg\langle \frac{\partial \mathbf{V}}{\partial t} dt + \frac{\partial \mathbf{V}}{\partial \xi} d\xi_t + \frac{1}{2} \frac{\partial^2 \mathbf{V}}{\partial \xi^2} d[\xi, \xi]^c_t \\ + \int_{\mathbb{R}\setminus\{0\}} \left[\mathbf{V}(t, \xi_{t^-} e^z) - \mathbf{V}(t, \xi_{t^-}) - \Delta \xi_t \frac{\partial \mathbf{V}}{\partial \xi} \right] N^X(dt, dz) \bigg| X_t \bigg\rangle. \end{aligned} \tag{3.48}$$

From (3.31), we deduce that

$$dX_t = AX_{t^-} dt + d\Gamma_t. \tag{3.49}$$

By replacing this last expression in (3.48), we obtain

$$\begin{aligned} & dV(t, \xi_t, X_t) \\ & = \bigg\langle \bigg[\frac{\partial \mathbf{V}}{\partial t} + \frac{1}{2} \sigma_t^2 \xi_{t^-}^2 \frac{\partial^2 \mathbf{V}}{\partial \xi^2} + \int_{\mathbb{R}\setminus\{0\}} \Big[\mathbf{V}(t, \xi_{t^-} e^z) - \mathbf{V}(t, \xi_{t^-}) \\ & - \xi_{t^-}(e^z - 1) \frac{\partial \mathbf{V}}{\partial \xi} \Big] \rho^{\theta^X}(dz) \bigg] dt \bigg| X_t \bigg\rangle \\ & + \left\langle \mathbf{V}(t, \xi_t) \middle| AX_{t^-} \right\rangle dt + \left\langle \mathbf{V}(t, \xi_t) \middle| d\Gamma_t \right\rangle \\ & + \bigg\langle \sigma_t \xi_{t^-} \frac{\partial \mathbf{V}}{\partial \xi} dW_t^\theta + \int_{\mathbb{R}\setminus\{0\}} \Big[\mathbf{V}(t, \xi_{t^-} e^z) - \mathbf{V}(t, \xi_{t^-}) \Big] \underline{\tilde{N}}^\theta(dt, dz) \bigg| X_t \bigg\rangle. \end{aligned} \tag{3.50}$$

As $\{V_t = V(t,\xi_t,X_t)\}_{t\in\mathbb{T}}$ is a $(\overline{\mathbf{G}},\mathbb{Q}^\theta)$-martingale, the continuous finite variation part is identically equal to zero $\mathbb{Q}^\theta$ a.s, thus

$$\begin{aligned}&\Big\langle \frac{\partial \mathbf{V}}{\partial t} + \frac{1}{2}\sigma_t^2\xi_t^2\frac{\partial^2\mathbf{V}}{\partial \xi^2} + \int_{\mathbb{R}\setminus\{0\}}\Big[\mathbf{V}(t,\xi_{t^-}e^z)\\ &- \mathbf{V}(t,\xi_{t^-}) - \xi_{t^-}(e^z-1)\frac{\partial \mathbf{V}}{\partial \xi}\Big]\rho^{\theta^X}(dz)\Big| X_t\Big\rangle\\ &+\Big\langle \mathbf{V}(t,\xi_t)\Big| AX_{t^-}\Big\rangle = 0. \end{aligned}\tag{3.51}$$

This is equivalent with $X_t = \mathbf{e}$ to :

$$\begin{aligned}&\frac{\partial V}{\partial t}(t,\xi_t,\mathbf{e}) + \frac{1}{2}\sigma_t^2\xi_t^2\frac{\partial^2 V}{\partial \xi^2}(t,\xi_t,\mathbf{e}) + \Big\langle \mathbf{V}(t,\xi_t)\Big| AX_{t^-}\Big\rangle\\ &+\int_{\mathbb{R}\setminus\{0\}}\Big[V(t,\xi_{t^-}e^z,\mathbf{e}) - V(t,\xi_{t^-},\mathbf{e}) - \xi_{t^-}(e^z-1)\frac{\partial V}{\partial \xi}(t,\xi_t,\mathbf{e})\Big]\rho^{\theta^X}(dz) = 0.\end{aligned}\tag{3.52}$$

Hence, by going back to Equation (3.50), we deduce that

$$\begin{aligned}V(t,\xi_t,\mathbf{e}) = V(0,\xi_0,X_0) &+ \int_0^t \sigma_s\xi_s\frac{\partial V}{\partial \xi}(s,\xi_s,X_s)dW_s^\theta\\ &+\int_0^t\int_{\mathbb{R}\setminus\{0\}}\Big[V(s,\xi_{s^-}e^z,X_s) - V(s,\xi_{s^-},X_s)\Big]\underline{\tilde{N}}^\theta(ds,dz)\\ &+\int_0^t\Big\langle \mathbf{V}(s,\xi_s)\Big| d\Gamma_s\Big\rangle.\end{aligned}\tag{3.53}$$

We deduce from the uniqueness of the decomposition of the special semimartingale V that for any $t\in\mathbb{T}$

- $\Phi_t^c(\xi_t) = \sigma_t\xi_t\frac{\partial V}{\partial \xi}(t,\xi_t,\mathbf{e})$;
- $\Phi_t^d(z,\xi_{t^-}) = V(t,\xi_{t^-}e^z,\mathbf{e}) - V(t,\xi_t,\mathbf{e})$;
- $\alpha_t = \mathbf{V}(t,\xi_t)$.

To obtain more explicit expressions for these quantities, we note that $\xi_{0,\ t} = x$ and $\xi_{0,\ t^-} = x_-$ therefore for any $t\in(0,T]$

$$
\begin{aligned}
\Phi_t^c(\xi_t) &= x\sigma_t(t,x,\mathbf{e})\frac{\partial V}{\partial x}(t,x,\mathbf{e})\\
&= x\sigma_t(t,x,\mathbf{e})\frac{\partial}{\partial x}E^{\mathbb{P}}[\Lambda_{t,\ T}(x)\hat{c}(\xi_{t,\ T}(x))|(X_t,\xi_{0,\ t})=(\mathbf{e},x)] \quad \text{by (3.39)}\\
&= x\sigma_t(t,x,\mathbf{e})E^{\mathbb{P}}\Big[\frac{\partial \Lambda_{t,\ T}}{\partial x}(x)\hat{c}(\xi_{t,\ T}(x))+\Lambda_{t,\ T}(x)\frac{\partial \hat{c}}{\partial \xi}(\xi_{t,\ T}(x))\frac{\partial \xi_{t,\ T}}{\partial x}(x)\Big|(X_t,\xi_{0,\ t}(x_0))\Big]\\
&= x\sigma_t(t,x,\mathbf{e})E^{\mathbb{P}}\Big[\Lambda_{t,\ T}(x)L_{t,\ T}\hat{c}(\xi_{t,\ T}(x))\\
&\quad +\Lambda_{t,\ T}(x)\frac{\partial \hat{c}}{\partial \xi}(\xi_{t,\ T}(x))\frac{\partial \xi_{t,\ T}}{\partial x}(x)\Big|(X_t,\xi_{0,\ t})=(\mathbf{e},x)\Big] \quad \text{(Lemma 3.1)}\\
&= x\sigma_t(t,x,\mathbf{e})E^{\mathbb{Q}^\theta}\Big[L_{t,\ T}\hat{c}(\xi_{t,\ T}(x))+\frac{\partial \hat{c}}{\partial \xi}(\xi_{t,\ T}(x))\frac{\partial \xi_{t,\ T}}{\partial x}(x)\Big|(X_t,\xi_{0,\ t})=(\mathbf{e},x)\Big].
\end{aligned} \tag{3.54}
$$

In the same way,

$$
\begin{aligned}
\Phi_t^d(z,\xi_{t^-}) &= V(t,\xi_{t^-}e^z,\mathbf{e})-V(t,\xi_{t^-},\mathbf{e})\\
&= E^{\mathbb{P}}\Big[\Lambda_{t,\ T}(x_-+\zeta_r(z))\hat{c}(\xi_{t,\ T}(x_-+\zeta_r(z)))\Big|(X_t,\xi_{0,\ t})=(\mathbf{e},x)\Big]\\
&\quad -E^{\mathbb{P}}\Big[\Lambda_{t,\ T}(x)\hat{c}(\xi_{t,\ T}(x))\Big|(X_t,\xi_{0,\ t})=(\mathbf{e},x)\Big]\\
&= E^{\mathbb{P}}\Big[\Big(\Lambda_{t,\ T}(x_-+\zeta_r(z))-\Lambda_{t,\ T}(x)\Big)\hat{c}(\xi_{t,\ T}(x_-+\zeta_r(z)))\Big|(X_t,\xi_{0,\ t})=(\mathbf{e},x)\Big]\\
&\quad +E^{\mathbb{P}}\Big[\Lambda_{t,\ T}(x)\Big(\hat{c}(\xi_{t,\ T}(x_-+\zeta_r(z)))-\hat{c}(\xi_{t,\ T}(x))\Big)\Big|(X_t,\xi_{0,\ t})=(\mathbf{e},x)\Big]\\
&= E^{\mathbb{P}}\Big[\Lambda_{t,\ T}(x)K_{t,T}\Big(\hat{c}(\xi_{t,\ T}(x_-+\zeta_r(z))\Big)\\
&\quad +\Lambda_{t,\ T}(x)\Big(\hat{c}(\xi_{t,\ T}(x_-+\zeta_r(z)))-\hat{c}(\xi_{t,\ T}(x))\Big)\Big|(X_t,\xi_{0,\ t})=(\mathbf{e},x)\Big] \quad \text{(Lemma 3.1)}\\
&= E^{\mathbb{Q}^\theta}\Big[(K_{t,T}+1)\hat{c}(\xi_{t,\ T}(x_-+\zeta_r(z)))-\hat{c}(\xi_{t,\ T}(x))\Big|(X_t,\xi_{0,\ t})=(\mathbf{e},x)\Big].
\end{aligned} \tag{3.55}
$$

Finally, we have to show that the different components involved in (3.53) are mutually orthogonal $(\overline{\mathbf{G}},\mathbb{Q}^\theta)$-local martingales, that is, the different products $W^\theta\cdot\underline{\tilde{N}}^\theta(\cdot,dz)$, $W^\theta\cdot\Gamma$ and $\Gamma\cdot\underline{\tilde{N}}^\theta(\cdot,dz)$ are $(\overline{\mathbf{G}},\mathbb{Q}^\theta)$-local martingales. The claim is easily verified in the first case by noting that W^θ is a continuous $(\overline{\mathbf{G}},\mathbb{Q}^\theta)$ local-martingale such that $W_0^\theta=0$ whereas $\underline{\tilde{N}}^\theta(\cdot,dz)$ and Γ are pure jump $(\overline{\mathbf{G}},\mathbb{Q}^\theta)$ local-martingales. For the last case, we have $\forall t\in\mathbb{T}$ and $\forall i\in\{1,2,...,M\}$

$$
\begin{aligned}
[\Gamma^i,\underline{\tilde{N}}^\theta(\cdot,dz)]_t &= \sum_{0\le s\le t}\Delta\Gamma_s^i\Delta\underline{\tilde{N}}^\theta(s,dz)\\
&= 0.
\end{aligned} \tag{3.56}
$$

This result comes from Assumption 3.1 and the decomposition theorem of the (additive) component of the MAP (X,S) given in [**32**, Theorem 2.23]. □

Remark 3.4.1. *On one hand, this result can be viewed as a specification of a general result of a representation theorem for square integrable martingales. (See* [**43, 90**].*)*

On the other hand, this theorem is a generalization of the representation theorem for Lévy martingales (see [**13**]*) to the case of a Markov modulated exponential Lévy process.*

3.4.2. The Locally Risk-Minimizing Hedging Problem under Full Information for The Model (3.4)-(3.2)

In this section, we consider the problem of hedging a contingent claim H in the Markov-modulated exponential Lévy model given by (3.2)-(3.4) given that the information set is $\overline{\mathbf{G}}$. In general, in such a market the claim H cannot be perfectly hedged. Therefore, we need to take into account the market participant's attitude toward risk in the search of the viable market transactions. One way of doing this, in the literature, consists of optimizing a given criterion which may be based on the preference of the market participant. In particular, the choice of the quadratic criterion is quite natural and pertinent because it leads to a linear pricing rule which is very meaningful in financial economics.

Let B be a contingent claim with a discounted payoff $H = \hat{c}(\xi_{0,\ T}(x_0)) \in \mathcal{L}^2(\Omega, \mathbb{P})$. Following Schweizer [**115**], a locally risk-minimizing strategy $\varphi = (\phi, \psi)$ which generates $\hat{c}(\xi_{0,\ T}(x_0))$ must be such that

(1) $V_T = \hat{c}(\xi_{0,T}(x_0))$ $\mathbb{P}$-a.s.;

(2) $V_t(\varphi) = V_0(\varphi) + \int_0^t \phi_r d\xi_r + \Upsilon_t, \quad$ for all $t \in \mathbb{T}$;

(3) Υ is a martingale under $\mathbb{P}$ and Υ is orthogonal to the martingale part Z of ξ under $\mathbb{P}$.

We shall require that $(V_t(\varphi))_{0\le t\le T}$ is a $(\overline{\mathbf{G}}, \mathbb{Q}^\theta)$-martingale. With this assumption and Equation (3.39), we have

$$\begin{aligned} V_t(\varphi) &= E^{\mathbb{Q}^\theta}[V_T(\varphi)|\overline{\mathcal{G}}_t] \\ &= E^{\mathbb{Q}^\theta}[\hat{c}(\xi_{0,\ T}(x_0))|(X_t, \xi_{0,\ t}) = (\mathbf{e}, x)] \\ &= V(t, x, \mathbf{e}). \end{aligned}$$

Now we can state the main proposition if this section.

Proposition 3.4. *Assume that* $\sigma_t > 0$ *for any* $t \in \mathbb{T}$*. If there exists a process* θ^* *satisfying (3.24) and such that*

$$\theta_t^* = \frac{\mu_t - r_t}{\sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x - 1)^2 \rho^X(dx)}, \tag{3.57}$$

$$e^{-z\theta_t^*} - 1 = -\frac{(\mu_t - r_t)(e^z - 1)}{\sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x - 1)^2 \rho^X(dx)}, \quad \forall z \in \mathbb{R} \tag{3.58}$$

then there exists a minimal martingale measure *defined by the Esscher transform* Λ^{θ^*}*. Furthermore, the locally risk-minimizing strategy for the contingent claim H* *is given by*

$$\phi_t^* = \frac{1}{\xi_{t^-}} \times \frac{\sigma_t \phi_t^c(\xi_t, X_t) + \int_{\mathbb{R}\setminus\{0\}} (e^z - 1)\phi_t^d(y, \xi_{t^-}, X_{t^-})\rho^X(dz)}{\sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}} (e^x - 1)^2 \rho^X(dx)}, \tag{3.59}$$

and

$$\begin{aligned} \psi_t^* &:= V_t(\varphi) - \phi_t^* \xi_t \\ &= E^{\mathbb{Q}^{\theta^*}}[\hat{c}(\xi_{0,\ T}(x_0))|(X_t, \xi_{0,\ t}) = (\mathbf{e}, x)] - \phi_t^* \xi_t. \end{aligned} \tag{3.60}$$

Proof.
1- We have to show that if there exists a process θ^* satisfying the Equations (3.24), (3.57) and (3.58), then the process Λ^{θ^*} defines a minimal martingale measure in the sense of Schweizer [**114**]. Indeed, under these assumptions we have from Equation (3.35)

$$\begin{aligned} \Lambda_t^{\theta^*} &= 1 + \int_0^t \Lambda_{s^-}^{\theta^*}(-\theta_s^* \sigma_s) dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Lambda_{s^-}^{\theta^*}(e^{-\theta_s^* z} - 1)\widetilde{N}^X(ds, dz) \\ &= 1 - \int_0^t \Lambda_{s^-}^{\theta^*}\Big[\frac{\mu_s - r_s}{\sigma_s^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x - 1)^2 \rho^X(dx)}\Big]\Big[\sigma_s dW_s + \int_{\mathbb{R}\setminus\{0\}}(e^z - 1)\widetilde{N}^X(ds, dz)\Big] \\ &= 1 - \int_0^t \Lambda_{s^-}^{\theta^*} \frac{1}{\xi_{s^-}} \times \Big[\frac{\mu_s - r_s}{\sigma_s^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x - 1)^2 \rho^X(dx)}\Big] dZ_s, \end{aligned} \tag{3.61}$$

where Z denotes the martingale part of the (special) semimartingale ξ. Using Assumptions 3.57 and 3.58, it is easy to see that the process λ given for $t \in \mathbb{T}$:

$$\begin{aligned} \lambda_t &:= \frac{dA_t}{d\langle Z\rangle_t} \\ &= \frac{1}{\xi_{t^-}} \times \frac{\sigma_t^2 \theta_t^* + \int_{\mathbb{R}\setminus\{0\}}(e^z - 1)(e^{-\theta_t^* z} - 1)\rho^X(dz)}{\sigma_s^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x - 1)^2 \rho^X(dx)} \\ &= \frac{1}{\xi_{t^-}} \times \frac{\mu_t - r_t}{\sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x - 1)^2 \rho^X(dx)} \end{aligned} \tag{3.62}$$

is $\overline{\mathbf{G}}$-predictable and verifies $\int_0^t \lambda_s^2 d\langle Z\rangle_s < \infty$ $\mathbb{P}$-a.s. Hence, we see that

$$\Lambda_t^{\theta^*} = 1 - \int_0^t \Lambda_{s^-}^{\theta^*} \lambda_s dZ_s. \tag{3.63}$$

This defines precisely the minimal martingale measure according to Föllmer and Schweizer [**63**].

In the sequel we shall denote it by $\mathbb{Q}^{\theta^*}$.

2- From Föllmer and Schweizer ([**63**]) we know that once a MMM is found, the locally risk-minimizing strategy of the contingent claim is uniquely determined from the $(\overline{\mathbf{G}}, \mathbb{Q}^{\theta^*})$-projection of the Galtchouk-Kunita-Watanabe decomposition of $\hat{c}(\xi_{0,\ T}(x_0))$.

From Proposition 3.3, we have for all $t \in [0, T]$,

$$V_t = V_0 + \int_0^t \phi^c_r(\xi_r, X_r) dW^{\theta^*}_r + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \phi^d_r(z, \xi_{r^-}, X_{r^-}) \underline{\widetilde{N}}^{\theta^*}(dr, dz) + \int_0^t \langle \alpha_r | d\Gamma_r \rangle \tag{3.64}$$

where ϕ^c and ϕ^d are given by Equations (3.41) and (3.42) respectively. Therefore, we have from (2)

$$\begin{aligned} \Upsilon_t &= V_t(\varphi) - \int_0^t \phi_r d\xi_r - V_0(\varphi) \\ &= \int_0^t \Big[\phi^c_r(\xi_r, X_r) - \sigma_r \xi_r \phi_r\Big]\Big[dW_r + \theta^*_t \sigma_r dr\Big] \\ &\quad + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Big[\phi^d_r(z, \xi_{r^-}, X_{r^-}) - \xi_{r^-}(e^z - 1)\phi_r\Big]\Big[N^X(dr, dz) - e^{-\theta^*_r z})\rho^X(dz)dr\Big] \\ &\quad + \int_0^t \langle \alpha_r | d\Gamma_r \rangle. \end{aligned} \tag{3.65}$$

From (3), Υ should be a $(\overline{\mathbf{G}}, \mathbb{P})$-martingale thus the drift term in (3.65) should be zero or equivalently

$$\begin{aligned} &\xi_{t^-}\phi_t\Big[\int_{\mathbb{R}\setminus\{0\}} (e^z - 1)(e^{-\theta^*_t z} - 1)\rho^X(dz) - \theta^*_t \sigma^2_t\Big] \\ &= \int_{\mathbb{R}\setminus\{0\}} \phi^d_t(z, \xi_{t^-}, X_{t^-})(e^{-\theta^*_t z} - 1)\rho^X(dz) - \phi^c_t(\xi_t, X_t)\theta^*_t \sigma_t. \end{aligned} \tag{3.66}$$

Hence

$$\begin{aligned} \Upsilon_t &= \int_0^t [\phi^c_r(\xi_r, X_r) - \sigma_r \xi_r \phi_r] dW_r + \int_0^t \langle \alpha_r | d\Gamma_r \rangle \\ &\quad + \int_0^t \int_{\mathbb{R}\setminus\{0\}} [\phi^d_r(z, \xi_{r^-}, X_{r^-}) - \xi_{r^-}(e^z - 1)\phi_r] \widetilde{N}^X(dr, dz). \end{aligned} \tag{3.67}$$

The requirement (3) stipulates also that Υ is orthogonal to the martingale part Z of ξ under $\mathbb{P}$. This is verified if and only if ΥZ is a $(\overline{\mathbf{G}}, \mathbb{P})$-martingale, therefore

$$\begin{aligned} \xi_{t^-}\phi_t\Big[\int_{\mathbb{R}\setminus\{0\}} (e^x - 1)^2 \rho^X(dx) + \sigma^2_t\Big] &= \phi^c_t(\xi_t, X_t)\sigma_t \\ &\quad + \int_{\mathbb{R}\setminus\{0\}} \phi^d_t(z, \xi_{r^-}, X_{r^-})(e^z - 1)\rho^X(dz). \end{aligned} \tag{3.68}$$

Recalling the martingale condition (3.24) and substituting it in Equation (3.66) we obtain

$$\xi_{t^-}\phi_t(r_t - \mu_t) = \int_{\mathbb{R}\setminus\{0\}} \phi^d_t(z, \xi_{t^-}, X_{t^-})(e^{-\theta^*_t z} - 1)\rho^X(dz) - \phi^c_t(\xi_t, X_t)\theta^*_t \sigma_t, \tag{3.69}$$

and using Equation (3.68), we know that θ^* satisfies

$$\left[\theta_t^* - \frac{\mu_t - r_t}{\sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x-1)^2\rho^X(dx)}\right]\phi_t^c(\xi_t, X_t)\sigma_t$$
$$- \int_{\mathbb{R}\setminus\{0\}}\left[(e^{-\theta_t^* z}-1) + \frac{(\mu_t - r_t)(e^z-1)}{\sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x-1)^2\rho^X(dx)}\right]\phi_t^d(y, \xi_{t^-}, X_{t^-})\rho^X(dz) = 0. \tag{3.70}$$

Thus, if there exists a process θ^* verifying (3.24) and such that $\forall t \in \mathbb{T}$

$$\theta_t^* = \frac{\mu_t - r_t}{\sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x-1)^2\rho^X(dx)}$$

and

$$e^{-\theta_t^* z} - 1 = -\frac{(\mu_t - r_t)(e^z-1)}{\sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x-1)^2\rho^X(dx)}, \quad \forall z \in \mathbb{R}\setminus\{0\},$$

then a locally risk-minimizing strategy exists (independently of the claim to be hedged) and is deduced from Equations (3.68) and (3.29)

$$\begin{cases} \phi_t^* = & \frac{1}{\xi_{t^-}} \times \frac{\sigma_t\phi_t^c + \int_{\mathbb{R}\setminus\{0\}}\phi_{t^-}^d(z)(e^z-1)\rho^X(dz)}{\sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x-1)^2\rho^X(dx)} \\ \psi_t^* = & V_0 + \int_0^t \phi_s^* d\xi_s - \phi_t^*\xi_t + \Gamma_t. \end{cases} \tag{3.71}$$

The expression of ψ^* follows from the definition of the portfolio value process V. This ends the proof. □

We can derive easily the expression of the residual $\overline{\mathbf{G}}$-risk process Υ for all $t \in \mathbb{T}$ as

$$\Upsilon_t = \int_0^t \frac{1}{\sigma_r^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x-1)^2\rho^X(dx)}$$
$$\times \left[\phi_r^c \int_{\mathbb{R}\setminus\{0\}}(e^z-1)^2\rho^X(dz) - \sigma_r\int_{\mathbb{R}\setminus\{0\}}\phi_{r^-}^d(z)(e^z-1)\rho^X(dz)\right]dW_r$$
$$+ \int_0^t\int_{\mathbb{R}\setminus\{0\}}\frac{1}{\sigma_r^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x-1)^2\rho^X(dx)} \times \left[\sigma_r^2\phi_{r^-}^d(z) - (e^z-1)\sigma_r\phi_r^c\right]\widetilde{N}^X(dr, dz)$$
$$+ \int_0^t \langle\alpha_r | d\Gamma_r\rangle. \tag{3.72}$$

Remark 3.4.2.

- *It is possible that Equation (3.70) does not have a unique solution, for example if $\phi^c \equiv 0 \equiv \phi^d$.*
- *This result is an explicit representation of the hedging strategy as computed in* **[43]**.

3.4.3. The Locally Risk-Minimizing Hedging Problem under Partial Information

This section considers the problem of the local risk-minimization of the contingent claim H when the asset dynamics follows Equation (3.4) from the viewpoint of an investor/hedger who does not have at his disposal the full information as described by the filtration $\overline{\mathbf{G}} = \{\overline{\mathcal{G}}_t\}_{t\in\mathbb{T}}$, but only the information set $\mathbf{G} = \{\mathcal{G}_t\}_{t\in\mathbb{T}}$; with $\mathcal{G}_t \subset \overline{\mathcal{G}}_t$ for all $t \in \mathbb{T}$. We have that $\mathcal{G}_T = \overline{\mathcal{G}}_T$. Thus the contingent claim $H = \hat{c}(\xi_{0,T}(x_0))$, which is $\overline{\mathcal{G}}_T$-measurable, will also be $\mathcal{G}_T$-measurable.
We aim at finding a $\mathbf{G}$-locally risk-minimizing strategy. From the previous section, we have the following representation

$$V_t(\varphi^*) = V_0(\varphi^*) + \int_0^t \phi_r^* d\xi_r + \Upsilon_t, \quad \text{for all } t \in \mathbb{T}, \tag{3.73}$$

where Υ is a $(\overline{\mathbf{G}}, \mathbb{P})$-martingale which is orthogonal to the martingale part Z of ξ under $\mathbb{P}$. Since we only admit strategies $\varphi = (\phi, \psi)$ such that the process $(V)_{t\in\mathbb{T}}$ is square-integrable, has right continuous paths and satisfies $V_T = H$, we have that

$$H = \widetilde{H}_0 + \int_0^T \phi_r^* d\xi_r + \Upsilon_T, \tag{3.74}$$

where $\widetilde{H}_0 = V_0(\varphi^*)$ is $\overline{\mathcal{G}}_0$-measurable and $\phi^* = (\phi_t^*)_{t\in\mathbb{T}}$ is $\overline{\mathbf{G}}$-predictable.
In the sequel, we make the following assumption

$$E^{\mathbb{P}}\Big[\widetilde{H}_0^2 + \int_0^T (\phi_r^*)^2 d\langle\xi\rangle_r + \Big(\int_0^T |\phi_r^*| dA_r\Big)^2\Big] < \infty. \tag{3.75}$$

Let $\mathcal{P}$ (resp.$\widetilde{\mathcal{P}}$) denote the σ-field of predictable subsets on $\overline{\Omega} = \Omega \times \mathbb{T}$ associated to the filtration $(\mathcal{G}_t)_{t\in\mathbb{T}}$ (resp.$(\overline{\mathcal{G}}_t)_{t\in\mathbb{T}}$). We denote by $\overline{\mathbb{P}}$ the finite measure on $\mathcal{P}$ defined by

$$\overline{\mathbb{P}}(d\omega, dt) = \mathbb{P}(d\omega) \times d\langle\xi\rangle_t(\omega).$$

$\overline{\mathbb{Q}}^{\theta^*}$ is defined in the same way. We can now state a Föllmer-Schweizer type decomposition result. This result is adapted from Föllmer and Schweizer **[63]**

Theorem 3.3.

Giving the decomposition (3.74), H admits the following representation (Föllmer-Schweizer decomposition)

$$H = H_0 + \int_0^T \phi_r^H d\xi_r + L_T^H \tag{3.76}$$

with $H_0 := E^{\mathbb{P}}[\widetilde{H}_0|\mathcal{G}_0]$, where

$$\phi^H = E^{\overline{\mathbb{P}}}[\phi^*|\mathcal{P}] \tag{3.77}$$

is the conditional expectation of ϕ^* with respect to $\mathcal{P}$ and $\overline{\mathbb{P}}$, and where $L^H := (L_t^H)_{t\in\mathbb{T}}$ is the square-integrable $\mathbf{G}$-martingale orthogonal to Z associated to

$$L_T^H = \widetilde{H}_0 - H_0 + \int_0^T (\phi_r^* - \phi_r^H) d\xi_r + \Upsilon_T \in \mathcal{L}^2(\Omega, \mathcal{G}_T, \mathbb{P}). \tag{3.78}$$

Proof.
1- We need to show in a similar way as in Föllmer and Schweizer [**63**] that all components in (3.76) are square-integrable. From Assumption 3.75, $\phi^* \in \mathcal{L}^2(\overline{\Omega}, \mathcal{P}, \overline{\mathbb{P}})$ and thus $\phi^H \in \mathcal{L}^2(\overline{\Omega}, \mathcal{P}, \overline{\mathbb{P}})$ by Jensen's inequality. Since $\phi^H \in \mathcal{L}^2(\overline{\Omega}, \mathcal{P}, \overline{\mathbb{P}})$, by Doob's maximal inequality we have that $\int_0^T \phi_r^H dZ_r \in \mathcal{L}^2(\Omega, \mathcal{G}_T, \mathbb{P})$.
To show that $\int_0^T \phi_r^H dA_r \in \mathcal{L}^2(\Omega, \mathcal{G}_T, \mathbb{P})$, we have by the predictable projection, Assumption 3.75 and Doob's maximal inequality that the application
$\vartheta \longrightarrow E^{\mathbb{P}}\Big[\vartheta \int_0^T \phi_r^H dA_r\Big]$ defined on $\mathcal{L}^2(\Omega, \mathcal{G}_T, \mathbb{P})$ is an element of the dual of this space. This dual is exactly (up to an isomorphism) $\mathcal{L}^2(\Omega, \mathcal{G}_T, \mathbb{P})$.
2- Now, Let us show that L_T^H is orthogonal to all square-integrable stochastic integrals of Z. It is sufficient to show that for any bounded $\mathcal{P}$-measurable process $\chi = (\chi)_{t\in\mathbb{T}}$ the following holds :

$$E^{\mathbb{P}}\Big[\Big(\int_0^T (\phi_r^* - \phi_r^H) d\xi_r\Big).\Big(\int_0^T \chi_r dZ_r\Big)\Big] = 0$$

$$\Leftrightarrow E^{\mathbb{P}}\Big[\Big(\int_0^T \phi_r^* d\xi_r\Big).\Big(\int_0^T \chi_r dZ_r\Big)\Big] = E^{\mathbb{P}}\Big[\Big(\int_0^T \phi_r^H d\xi_r\Big).\Big(\int_0^T \chi_r dZ_r\Big)\Big].$$

But the left hand side can be decomposed into two components. So, by Itô-type isometry

$$E^{\mathbb{P}}\Big[\Big(\int_0^T \phi_r^* dZ_r\Big).\Big(\int_0^T \chi_r dZ_r\Big)\Big] = E^{\mathbb{P}}\Big[\int_0^T \phi_r^H \chi_r d\langle\xi\rangle_r\Big]$$

and by the predictable projection, we have

$$E^{\mathbb{P}}\Big[\Big(\int_0^T \phi_r^* dA_r\Big).\Big(\int_0^T \chi_r dZ_r\Big)\Big] = E^{\mathbb{P}}\Big[\int_0^T \phi_r^*.\Big(\int_0^r \chi_s dZ_s\Big) d\langle\xi\rangle_r\Big].$$

Now, we can replace in both parts ϕ^* by ϕ^H which finally gives the result.
3- It remains to show that Υ_T is orthogonal to all square-integrable stochastic integrals of Z. This follows from the fact that $(\Upsilon_t)_{t\in\mathbb{T}}$ is orthogonal to Z. Therefore, L is orthogonal to Z. □

Remark 3.4.3. *The last result states that the contingent claim H has an orthogonal decomposition with respect to the smaller filtration. This result follows from the fact that the same decomposition is available with respect to the larger filtration. However, as pointed by Arai* [**5**]*, it is not always true in general that the contingent claim will have an orthogonal decomposition when dealing with a discontinuous market model. Such an orthogonal decomposition holds for instance when making the restrictive assumption that jumps of processes Z, L and Λ^θ do not happen simultaneously almost surely. Our model is one of those where the orthogonal decomposition (3.76) holds, this leads to the following proposition.*

Proposition 3.5. *Under the hypothesis of Proposition 3.4 and Theorem 3.3, there exists a unique $\mathbf{G}$-locally risk-minimizing hedging strategy $({}^{\mathbf{G}}\phi^*, {}^{\mathbf{G}}\psi^*)$ given* *by*

$$\begin{array}{ll} {}^{\mathbf{G}}\phi^* & = \overline{E}^{\mathbb{Q}^{\theta^*}}[\phi^*|\mathcal{P}] \\ {}^{\mathbf{G}}\psi^* & =^{\mathbf{G}} V -^{\mathbf{G}} \phi^*.\xi \end{array} \tag{3.79}$$

with ${}^{\mathbf{G}}V_t := E^{\mathbb{Q}^{\theta^*}}[H|\mathcal{G}_t]$ *for* $t \in \mathbb{T}$.

Proof. The existence and the uniqueness of the $\mathbf{G}$-locally risk-minimizing hedging strategy follows from Theorem 3.3 and Proposition 3.2. For the explicit expression of this strategy we need to show that

$$\phi^H = \overline{E}^{\mathbb{Q}^{\theta^*}}[\phi^*|\mathcal{P}]$$

where

$$\phi^H = E^{\overline{\mathbb{P}}}[\phi^*|\mathcal{P}].$$

Without loss of generality, we can suppose that $\phi^* \geq 0$, otherwise we can decompose it into the difference of two non-negative terms. So, it is equivalent to showing that

$$E^{\mathbb{Q}^{\theta^*}}\Big[\int_0^T \vartheta_r \phi_r^* d\langle\xi\rangle_s\Big] = E^{\mathbb{Q}^{\theta^*}}\Big[\int_0^T \vartheta_r \phi_r^H d\langle\xi\rangle_s\Big]$$

for any non-negative $\mathcal{P}$-measurable process ϑ. By the definition of $\mathbb{Q}^{\theta^*}$,

$$\begin{aligned} E^{\mathbb{Q}^{\theta^*}}\Big[\int_0^T \vartheta_r \phi_r^* d\langle\xi\rangle_s\Big] &= E^{\mathbb{P}}\Big[\Lambda_T^{\theta^*}\int_0^T \vartheta_r \phi_r^* d\langle\xi\rangle_s\Big] \\ &= E^{\mathbb{P}}\Big[\int_0^T \Lambda_r^{\theta^*}\vartheta_r \phi_r^* d\langle\xi\rangle_s\Big] \quad \text{by predictable projection} \\ &= E^{\mathbb{P}}\Big[\int_0^T \Lambda_r^{\theta^*}\vartheta_r \phi_r^H d\langle\xi\rangle_s\Big] \quad \text{by definition of } \phi^H \\ &= E^{\mathbb{P}}\Big[\Lambda_T^{\theta^*}\int_0^T \vartheta_r \phi_r^H d\langle\xi\rangle_s\Big] \\ &= E^{\mathbb{Q}^{\theta^*}}\Big[\int_0^T \vartheta_r \phi_r^H d\langle\xi\rangle_s\Big] \end{aligned} \tag{3.80}$$

□

3.5. Concluding Remarks

The problem of local risk-minimization under a Markov-modulated exponential Lévy model was studied. By noting that it consists of finding a locally risk-minimizing strategy for a partially observed model (or partial information scenario), we first solve the problem in the case of full information by providing a useful explicit martingale representation for the contingent claim. After that, we give a solution to the main problem by using the predictable projection.

For practical purpose, it would be interesting to give a computational algorithm for the optimal strategy. To this end, we shall use filtering theory and this will be objective of future research.

Acknowledgements

We wish thank Giulia Di Nunno, Paul Kettler, Manuel Morales, Frank Proske, Priyanka Parbhoo and an anonymous referee for helpful comments and suggestions.

Appendix

Proof of Lemma 3.1. First, from (3.35)

$$\begin{aligned}\Lambda_{t,\ T}(x) &= 1+\int_t^T \Lambda_{t,\ r}(x)(-\theta_r\sigma_r)(r,\xi_{t,\ r}(x),X_r)dW_r \\ &\quad +\int_t^T\int_{\mathbb{R}\backslash\{0\}} \Lambda_{t,\ r^-}(x)(e^{-z\theta_r(r,\xi_{t,\ r^-}(x),X_r)}-1)\widetilde{N}^X(dr,dz)\end{aligned}$$

and by differentiation we obtain that

$$\begin{aligned}&\frac{\partial \Lambda_{t,\ T}}{\partial x}(x) \\ &= \int_t^T(-\theta_r\sigma_r)\frac{\partial\Lambda_{t,\ r}}{\partial x}(x)dW_r+\int_t^T\int_{\mathbb{R}\backslash\{0\}}\frac{\partial\Lambda_{t,\ r^-}}{\partial x}(x)(e^{-z\theta_{r^-}}-1)\widetilde{N}^X(dr,dz) \\ &+\int_t^T\Lambda_{t,\ r}\frac{\partial(-\theta_r\sigma_r)}{\partial\xi}\times\frac{\partial\xi_{t,\ r}}{\partial x}(x)dW_r \\ &+\int_t^T\int_{\mathbb{R}\backslash\{0\}}\Lambda_{t,\ r^-}\frac{\partial(e^{-z\theta_r})}{\partial\xi}\times\frac{\partial\xi_{t,\ r^-}}{\partial x}(x)\widetilde{N}^X(dr,dz). \qquad (3.81)\end{aligned}$$

Also, by applying the Itô's differentiation rule to the product $\Lambda_{t,\ T}(x)L_{t,\ T}$, we have

$$\begin{aligned}&\Lambda_{t,\ T}(x)L_{t,\ T} \\ &= \int_t^T(-\theta_r\sigma_r)\Lambda_{t,\ r}(x)L_{t,\ r}dW_r+\int_t^T\int_{\mathbb{R}\backslash\{0\}}\Lambda_{t,\ r^-}(x)L_{t,\ r^-}(e^{-z\theta_{r^-}}-1)\widetilde{N}^X(dr,dz) \\ &+\int_t^T\Lambda_{t,\ r}\frac{\partial(-\theta_r\sigma_r)}{\partial\xi}\times\frac{\partial\xi_{t,\ r}}{\partial x}(x)dW_r \\ &+\int_t^T\int_{\mathbb{R}\backslash\{0\}}\Lambda_{t,\ r^-}\frac{\partial(e^{-z\theta_r})}{\partial\xi}\times\frac{\partial\xi_{t,\ r^-}}{\partial x}(x)\widetilde{N}^X(dr,dz). \qquad (3.82)\end{aligned}$$

Comparing Equations (3.81) and (3.82), we have by the unicity of solution of SDE that

$$\frac{\partial\Lambda_{t,\ T}}{\partial x}(x)=\Lambda_{t,\ T}(x)\times L_{t,\ T}. \qquad (3.83)$$

For the second part, we remark that

$$
\begin{aligned}
&\Lambda_{t,\ T}(x_- + \zeta(z)) - \Lambda_{t,T}(x) \\
&= \int_t^T \Big[(-\theta_r\sigma_r)(r, \xi_{t,\ r}(\xi_t + \zeta(z)), X_r) + (\theta_r\sigma_r)(r, \xi_{t,\ r}(\xi_t), X_r)\Big] dW_r \\
&+ \int_t^T \int_{\mathbb{R}\backslash\{0\}} \Lambda_{t,\ r^-}(\xi_{t^-} + \zeta(z)) \\
&\times \Big[e^{-z\theta_{r^-}(r,\ \xi_{t,\ r^-}(\xi_{t^-} + \zeta(z)),\ X_r)} - e^{-z\theta_{r^-}(r,\ \xi_{t,\ r^-}(\xi_t),\ X_r)}\Big] \widetilde{N}(dr, dz) \\
&+ \int_t^T \Big[\Lambda_{t,\ r}(x + \zeta(z)) - \Lambda_{t,r}(x)\Big] \Big[(-\theta_r\sigma_r)(r, \xi_{t,\ r}(\xi_t), X_r)\Big] dW_r \\
&+ \int_t^T \int_{\mathbb{R}\backslash\{0\}} \Big[\Lambda_{t,\ r^-}(x_- + \zeta(z)) - \Lambda_{t,r^-}(x)\Big] \times \Big[e^{-z\theta_{r^-}(r,\ \xi_{t,\ r^-}(\xi_t),\ X_r)} - 1\Big] \widetilde{N}(dr, dz).
\end{aligned}
\tag{3.84}
$$

On the other hand, by applying Itô differentiation rule

$$
\begin{aligned}
\Lambda_{t,\ T} K_{t,\ T} \;=\; & \int_t^T \Big[(-\theta_r\sigma_r)(r, \xi_{t,\ r}(\xi_t + \zeta(z)), X_r) + (\theta_r\sigma_r)(r, \xi_{t,\ r}(\xi_t), X_r)\Big] dW_r \\
& + \int_t^T \int_{\mathbb{R}\backslash\{0\}} \Lambda_{t,\ r^-}(\xi_{t^-} + \zeta(z)) \\
& \times \Big[e^{-z\theta_{r^-}(r,\ \xi_{t,\ r^-}(\xi_{t^-} + \zeta(z)),\ X_r)} - e^{-z\theta_{r^-}(r,\ \xi_{t,\ r^-}(\xi_t),\ X_r)}\Big] \widetilde{N}(dr, dz) \\
& + \int_t^T \Lambda_{t,\ r^-}(x) K_{t,\ r} \Big[(-\theta_r\sigma_r)(r, \xi_{t,\ r}(\xi_t), X_r)\Big] dW_r \\
& + \int_t^T \int_{\mathbb{R}\backslash\{0\}} \Lambda_{t,\ r^-}(x) K_{t,\ r^-} \Big[e^{-z\theta_{r^-}(r,\ \xi_{t,\ r^-}(\xi_t),\ X_r)} - 1\Big] \widetilde{N}(dr, dz).
\end{aligned}
$$

As above, we deduce the second identity from the uniqueness of solution of SDE.
□

Chapitre 4

VISCOSITY SOLUTIONS AND THE PRICING OF EUROPEAN-STYLE OPTIONS IN A MARKOV-MODULATED EXPONENTIAL LÉVY MODEL

Ce travail fait l'objet d'un article dont nous sommes le seul auteur et qui est soumis à la revue *Stochastic Analysis and Applications.*

Résumé

Nous montrons dans cet chapitre que le prix d'une option Européenne dans un modéle exponentiel-Lévy avec changements de régime vérifie un systéme d'équations intégro-différentielles non-linéaires qui peut être dégénéré. Par suite, nous démontrons qu'il se caractérise comme l'unique *solution de viscosité* d'un tel système. Ce cadre fonctionnel introduit par Crandall et Lions [**38**] s'avère adéquat car il ne requiert pas d'hypothèse sur la solution autre que la continuité, ce qui permet notamment de prendre en compte une possible dégénérescence. Le résultat obtenu généralise notamment ceux de Pemy [**107**] et Voltchkova [**125**] à un modèle qui semble plus proche des réalités des données de marché.

Abstract

We address in this paper the valuation of European-style contingent claims under a Markov-modulated exponential Lévy model. This type of models has become a recurrent theme in the literature notably by the fact that they allow, by introducing a modulator factor, to take into account the empirical facts observed in asset prices dynamics such as the long-term (stochastic) variability and time inhomogeneities. Although there exists some works in the literature about the option valuation under the simple case of regime-switching Black-Scholes model, the most general case has not yet retained much attention. To fill this gap, we give in this paper a characterization of the value of an European option as the unique viscosity solution of a system of coupled linear Integro-Partial Differential Equations (IPDEs) when the payoff function satisfies a Lipschitz condition.

Keywords : Viscosity solutions, Lévy process, regime-switching model, Integro-partial differential equation, comparison principle.

4.1. Introduction

The Markov-modulated exponential Lévy models are attractive in part because they have the benefit of the Lévy models[1] with the possibility thanks to the Markov chain of having a long-term variability of some characteristics of the return distribution. Once the model is chosen, the main task is to find an efficient algorithm to determine the price of a contingent claim. Three approaches are currently used in the literature : the first consists of solving (analytically or) numerically some partial differential equations. This approach is studied by Black and Scholes [**15**]. The second approach uses integral transforms (Fourier, Laplace, Mellin) to obtain an approximated value of the price and the third one consists in using Monte-Carlo simulations to approximate the value of the expectation which gives the option price under a risk-neutral measure by no-arbitrage as shown by Harrison and Kreps [**78**] and Harrison and Pliska [**79**].
In using numerical methods to solve PDE/IPDEs related to option prices, one first needs to show the existence and the uniqueness of the solution in the adequate functional spaces. This theoretical issue is not always addressed in the literature. So, the main purpose of this paper is to give an answer to this prerequisite in the case of pricing an European option in the Markov-modulated exponential Lévy model. In particular, we characterize the value of an European option as the unique viscosity solution of a system of coupled linear Integro-partial differential equations when the payoff function satisfies a Lipschitz condition.
There are many studies in the literature concerning the viscosity solutions in financial problems. We can mention Barles and Soner [**10**], Pham [**108**], Benth, et *al.* [**12**] and Cont and Voltchkova [**36**] . The framework of viscosity solutions is particularly useful for dealing with the possibly degeneracy of the diffusion coefficient which arises in the case of pure jumps models.
The paper is organized as follows. In Section 4.2, we describe the model setup and derive the system of IPDE verified by an European-style option price. The main problem is discussed in Section 4.3 . In particular, we argue that the class of viscosity solutions is the adequate framework in which we can discuss the existence and uniqueness of solutions for such type of IPDE. We conclude with Section 4.4 by giving some possible extensions and indicating some further discussions which shall hold our attention in the future.

4.2. Preliminaries

In this Section, we first define the Markov-modulated exponential-Lévy model which can be seen as an extension of that described in Cont and Tankov ([**34**],

[1] notably the presence of jumps

p.283) where a factor of modulation is introduced. Then, we derive the system of IPDE satisfied by an European option price under the assumption of sufficient regularity.

4.2.1. The model setup

We consider a financial market with two primary securities, namely a bond B, and a stock S which are traded continuously over the time horizon $\mathbb{T} := [0, T]$, where $T \in (0, \infty)$ represents the maturity time for investment. To formalize this market, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ is the real-world probability. Further, we shall add a filtration to this setup which specifies the flow of information available for the investors.
Let $X := \{X(t) : t \in \mathbb{T}\}$ denote an irreducible homogeneous continuous-time Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ with a finite state space $\mathbb{S} = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_M\} \subset \mathbb{R}^M$ characterized by a rate (or intensity) matrix $\mathbb{A} := \{a_{ij} : 1 \leq i, j \leq M\}$. Following Elliott [**49**], we can identify $\mathbb{S}$ with the basis set of the linear space $\mathbb{R}^M$. By now, we set $\mathbf{e}_i = (0, 0, \ldots, \underbrace{1}_{i\text{-th}}, \ldots, 0)$ and often we denote it by i. From Elliott [**50**], X admits the following semimartingale representation

$$X(t) = X(0) + \int_0^t \mathbb{A}X(s)ds + \Gamma(t) \tag{4.1}$$

where $\{\Gamma(t) : t \in [0, T]\}$ is a vector-martingale in $\mathbb{R}^M$ with respect to the filtration generated by X.
Let r_t denote the instantaneous interest rate of the money market account B at time t. We suppose that $r_t := r(X(t)) = \langle \underline{r} | X(t) \rangle$ where $\langle \cdot | \cdot \rangle$ is the usual scalar product in $\mathbb{R}^M$ and $\underline{r} = (r_1, r_2, ..., r_M) \in \mathbb{R}_+^{\mathbb{M}}$, then the price dynamics of B is given by :

$$dB(t) = r_t B(t), \; B(0) = 1 \quad \text{for} \quad t \in \mathbb{T}. \tag{4.2}$$

Let μ_t and σ_t denote the appreciation rate and the volatility of the stock S at time t, we suppose respectively that

$$\begin{aligned} \mu_t &= \langle \underline{\mu} | X(t) \rangle, \\ \sigma_t &= \langle \underline{\sigma} | X(t) \rangle, \end{aligned} \tag{4.3}$$

where

- $\underline{\mu} = (\mu_1, \mu_2, ..., \mu_M) \in \mathbb{R}^M$;
- $\underline{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_M) \in \mathbb{R}_+^{\mathbb{M}}$.

The stock price process S is described by the following Markov modulated Lévy process :

$$dS(t) = S(t^-)\Big(\mu_t dt + \sigma_t dW_t + \int_{\mathbb{R}\setminus\{0\}} (e^z - 1)\widetilde{N}^X(dt; dz)\Big), \quad S(0) = S_0 > 0 \tag{4.4}$$

where we set as the compensated random measure

$$\widetilde{N}^X(dt, dz) := N^X(dt, dz) - \rho^X(dz)dt, \tag{4.5}$$

where $N^X(dt, dz)$ denotes the differential form of a Markov-modulated random measure on $\mathbb{T} \times \mathbb{R}\backslash\{0\}$. We recall from Elliott and Osakwe **[54]** and Elliott and Royal **[55]** that a Markov-modulated random measure on $\mathbb{T} \times \mathbb{R}\backslash\{0\}$ is a family $\{N^X(\omega; dt, dz) : \omega \in \Omega\}$ of non-negative measures on the measurable space $(\mathbb{T} \times \mathbb{R}\backslash\{0\}, \mathcal{B}(\mathbb{T}) \otimes \mathcal{B}(\mathbb{R}\backslash\{0\}))$, which satisfies $N^X(\omega; \{0\}, \mathbb{R}\backslash\{0\}) = 0$ and has the following compensator, or dual predictable projection :

$$\rho^X(dz)dt := \sum_{i=1}^{M} \langle X(t^-)|e_i\rangle \rho_i(dz)dt. \tag{4.6}$$

$\rho_i(dz)$ is the Lévy Measure for the jump size when the Markov chain X is in state $\mathbf{e}_i$, i.e., a σ-finite Borel measure on $\mathbb{R}\backslash\{0\}$ with the property

$$\int_{\mathbb{R}\backslash\{0\}} \min(1, z^2)\rho_i(dz) < \infty. \tag{4.7}$$

We suppose furthermore that $\rho_i(dz)$ satisfying

$$\int_{|z|\geq 1} |e^z - 1|\rho_i(dz) < \infty. \tag{4.8}$$

This additional integrability condition assures that Equation (4.4) is well defined. $W := (W_t)_{t\in\mathbb{T}}$ denotes a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ which is supposed to be independent of X and N^X.

From Equation (4.4), by using Itô's formula we obtain that :

for $0 \leq s < t \leq T$,

$$S(t) = S(s)\exp\Bigg[\int_s^t \Big(\mu_u - \frac{1}{2}\sigma_u^2\Big)du - \int_s^t \int_{\mathbb{R}\backslash\{0\}} \Big(e^z - 1 - z\Big)\rho^X(dz)du + \int_s^t \sigma_u dW_u + \int_s^t \int_{\mathbb{R}\backslash\{0\}} z\widetilde{N}^X(ds; dz)\Bigg]. \tag{4.9}$$

If we substitute μ_s, σ_s and $\rho^X(dz)ds$ in (4.9) by their expressions given in (4.3) and (4.6) respectively, we deduce that $(X(t), S(t))$ is Markovian with infinitesimal generator $\mathcal{L}$ given as follows :

$$\begin{aligned}\mathcal{L}f(t,S,k) = {}& \frac{\partial f}{\partial t}(t,S,k) + \mu_k S\frac{\partial f}{\partial S}(t,S,k) + \frac{1}{2}\sigma_k^2 S^2\frac{\partial^2 f}{\partial S^2}(t,S,k) \\ & + \mathbb{A}f(t,S,\cdot)(k) + \int_{\mathbb{R}\backslash\{0\}} \Big[f(t,Se^z,k) - f(t,S,k) \\ & - S(e^z-1)\frac{\partial f}{\partial S}(t,S,k)\Big]\rho_k(dz), \end{aligned} \tag{4.10}$$

$$\tag{4.11}$$

for any function f such that for each state $\mathbf{e}_k$ (or simply $k \in \mathbb{S}$) and $f(\cdot,\cdot,k) \in \mathcal{C}_0^{1,2}([0,\infty) \times \mathbb{R})$ where $\mathbb{A}f(t,S,\cdot)(k) := \sum_{l\neq k,\ l\in\mathbb{S}} a_{kl}\Big(f(t,S,l) - f(t,S,k)\Big)$.

4.2.2. A system of second-order coupled linear IPDE verified by an European option

In this section, we derive formally the system of coupled integro-partial differential equations that will be satisfied by the the price of an European option. The starting point is to consider an European option with terminal payoff $H(S(T), X(T))$, strike K and maturity T. We suppose that we already have chosen an equivalent martingale measure $\mathbb{Q}$. This can be done by using a generalized form of the Esscher transform [See, Momeya and Morales [**101**]]. Under this martingale measure $\mathbb{Q}$ the risk neutral dynamics for asset price S is given by the following stochastic differential equation

$$dS(t) = S(t^-)\Big(r_t dt + \sigma_t dW_t + \int_{\mathbb{R}\setminus\{0\}} (e^z - 1)\widetilde{N}^X(dt; dz)\Big), \quad S(0) = S_0 > 0 \tag{4.12}$$

or more explicitely by using Itô's formula,

$$S(t) = S_0 \exp(Y(t)), \quad t \in \mathbb{T} \tag{4.13}$$

where

$$Y(t) = \int_0^t \Big(r_s - \frac{1}{2}\sigma_s^2\Big) ds - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z)\rho^X(dz) ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\widetilde{N}^X(ds; dz). \tag{4.14}$$

Let $\widetilde{V}(t, T, \cdot, \cdot)$ the discounted European option price at time t; $t \in \mathbb{T}$, i.e.,

$$\widetilde{V}(t, T, \cdot, \cdot) = e^{-\int_0^t r_s ds} V(t, T, \cdot, \cdot)$$

We know from the fundamental theorem of asset pricing (Harrison and Pliska [**79**], [**80**]) that $\{\widetilde{V}(t, T, \cdot, \cdot) : t \in \mathbb{T}\}$ is a $(\mathbf{G}, \mathbb{Q})$-martingale, and its terminal value at time T is

$$\widetilde{V}(T, T, \cdot, \cdot) = e^{-\int_0^T r_s ds} H(S_0 e^{Y(T)}, X(T)).$$

Therefore, by the martingale property

$$\begin{aligned}
\widetilde{V}(t, T, \cdot, \cdot) &= E^{\mathbb{Q}}\Big[e^{-\int_0^T r_s ds} H(S_0 e^{Y(T)}, X(T))\Big|\mathcal{G}_t\Big] \\
&= E^{\mathbb{Q}}\Big[e^{-\int_0^T r_s ds} H(S_0 e^{Y(T)}, X(T))\Big|(X(t), S(t))\Big] \\
&=: \widetilde{V}(t, T, X(t), S(t))
\end{aligned} \tag{4.15}$$

where the second equality comes from the Markov property of (X, S). Furthermore, it is known from the property of the infinitesimal generator of a Markov process that for any smooth function $f(\cdot, \cdot)$, the process

$$\Upsilon_t := f(X(t), S(t)) - \int_0^t \mathcal{L} f(X_u, S_u) du$$

is a $(\mathcal{G}, \mathbb{Q})$-martingale. In particular, by taking f as the discounted European option price $\widetilde{V}(t,T,\cdot,\cdot)$ we have, assuming sufficient regularity of $\widetilde{V}(t,T,\cdot,\cdot)$, that the bounded variations terms of Υ which are not martingales must sum to zero. Thus, we obtain that $\widetilde{V}(t,T,\cdot,\cdot)$ satisfies

$$\frac{\partial \widetilde{V}}{\partial t}(t,T,S,k) + r_k S \frac{\partial \widetilde{V}}{\partial S}(t,T,S,k) + \frac{1}{2}\sigma_k^2 S^2 \frac{\partial^2 \widetilde{V}}{\partial S^2}(t,T,S,k) + \mathbb{A}\widetilde{V}(t,T,S,.)(k) + \int_{-\infty}^{+\infty} \Big[\widetilde{V}(t,T,Se^z) - \widetilde{V}(t,T,S) - S(e^z-1)\frac{\partial \widetilde{V}}{\partial S}\Big]\rho_k(dz) = 0. \quad (4.16)$$

Using the relation between $\widetilde{V}$ and V, we have

$$\frac{\partial \widetilde{V}}{\partial t} = e^{-\int_0^t r_s ds}\Big(-r_k V + \frac{\partial V}{\partial t}\Big), \quad \frac{\partial \widetilde{V}}{\partial S} = e^{-\int_0^t r_s ds}\frac{\partial V}{\partial S}$$

and finally the European option price satisfies the system of IPDEs

$$\begin{aligned}\frac{\partial V}{\partial t}(t,T,S,k) + r_k S \frac{\partial V}{\partial S}(t,T,S,k) + \frac{1}{2}\sigma_k^2 S^2 \frac{\partial^2 V}{\partial S^2}(t,T,S,k) \\ - r_k V(t,T,S,k) + \mathbb{A}V(t,T,S,.)(k) \\ + \int_{\mathbb{R}\setminus\{0\}} \Big[V(t,T,Se^z,k) - V(t,T,S,k) - S(e^z-1)\frac{\partial V}{\partial S}(t,T,S,k)\Big]\rho_k(dz) = 0,\end{aligned} \quad (4.17)$$

for $(t,S) \in (0,T) \times \mathbb{R}^+$ and $k = 1,2,...,M$ associated with terminal conditions

$$V(T,T,S(T),k) = H(S_0 e^{Y_T}, X(T) = k), \quad k = 1,2,...,M. \quad (4.18)$$

4.3. Viscosity Solutions for The Coupled System of IPDE (4.17)-(4.18)

In general, when we find the value of an European option in an exponential Lévy model we have to solve an IPDE. Cont and Voltchkova [**36**] evoke some difficulties that arise in this situation : the non-local character of the integral operator, nonsmoothness of initial conditions, the singularity at zero of the integral kernel and the possible degeneracy of the diffusion coefficient. All these difficulties can occur in the case of the regime-switching exponential Lévy model. Thus, we have to find solutions in the framework of viscosity solutions, introduced by Crandall and Lions [**38**] for PDEs and extended later by many authors [see, e.g., [**3**], [**11**], [**16**]].

First, we write the dynamics of stock price (4.12) in the following more general form,

$$\begin{cases} dS(t) & = \mu(t, S(t), X(t))dt + \sigma(t, S(t), X(t))dW_t \\ & \quad + \int_{\mathbb{R}\setminus\{0\}} \gamma(t, S(t), X(t), z)\widetilde{N}^X(dt; dz) \\ S(s) & = x, \quad x \in \mathbb{R};\ 0 \le s \le t \le T. \end{cases} \tag{4.19}$$

where the functions

$$\mu(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R} \times \mathbb{S} \to \mathbb{R},$$
$$\sigma(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R} \times \mathbb{S} \to \mathbb{R}$$

and

$$\gamma(\cdot,\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R} \times \mathbb{S} \times \mathbb{R} \to \mathbb{R}$$

satisfy the following properties.

- **Lipschitz continuity** : For each $i \in \mathbb{S}$ and for all $(t, x, y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, there exists a constant $\alpha > 0$ such that

$$|\mu(t,x,i) - \mu(t,y,i)|^2 + |\sigma(t,x,i) - \sigma(t,y,i)|^2 + \int_{\mathbb{R}\setminus\{0\}} |\gamma(t,x,i,z) - \gamma(t,y,i,z)|^2 \rho_i(dz)dt < \alpha|x-y|^2. \tag{4.20}$$

- **Growth condition** : There exists a constant $\beta > 0$ such that

$$|\mu(t,x,i)|^2 + |\sigma(t,x,i)|^2 + \int_{\mathbb{R}\setminus\{0\}} |\gamma(t,x,i,z)|^2 \rho_i(dz)dt < \beta(1+|x|)^2. \tag{4.21}$$

4.3.1. Characterization of the European Option Price

In this Section, we prove that the European option price satisfies certain growth and continuity properties. The proofs of these results are adapted from Pemy **[107]**.

Lemma 4.1. *For each $i \in \mathbb{S}$, the European option price at time $s \in [0,T]$ given by $(S(s), X(s)) = (x, i)$, denoted by $v(s,x,i)$, is a continuous function in (s,x) and has at most a polynomial growth.*

Proof. We only prove the lemma for the case of an European Call option, the case of an European put option is similar. So, we have

$$v(s,x,i) = E^{\mathbb{Q}}\Big[e^{-\int_s^T r_\xi d\xi}(S_0 e^{Y_T} - K, 0)^+ \Big| (S(s), X(s)) = (x,i)\Big]$$

To show the continuity of $v(s,x,i)$ in x, let S_1 and S_2 be two solutions of (P) with initial values $S_1(s) = x_1$ and $S_2(s) = x_2$ respectively. For each $t \in [s,T]$, we have

$$\begin{aligned} S_1(t) - S_2(t) &= (x_1 - x_2) + \int_s^t [\mu(\xi, S_1(\xi), X(\xi)) - \mu(\xi, S_2(\xi), X(\xi))]d\xi \\ &\quad + \int_s^t [\sigma(\xi, S_1(\xi), X(\xi)) - \sigma(\xi, S_2(\xi), X(\xi))]dW_\xi \\ &\quad + \int_s^t \int_{\mathbb{R}\setminus\{0\}} [\gamma(\xi, S_1(\xi), X(\xi), z) - \gamma(\xi, S_1(\xi), X(\xi), z)]\widetilde{N}^X(d\xi; dz). \end{aligned} \tag{4.22}$$

Then, using the elementary inequality $|\sum_{i=1}^{M} a_i|^2 \leq M \sum_{i=1}^{M} |a_i|^2$ for all real numbers a_i, it turns out

$$\begin{aligned}(S_1(t)-S_2(t))^2 \;\leq\; & 4|x_1-x_2|^2 + 4\Big(\int_s^t |\mu(\xi,S_1(\xi),X(\xi)) - \mu(\xi,S_2(\xi),X(\xi))|d\xi\Big)^2 \\ & +4\Big(\int_s^t |\sigma(\xi,S_1(\xi),X(\xi)) - \sigma(\xi,S_2(\xi),X(\xi))|dW_\xi\Big)^2 \\ & +4\Big(\int_s^t \int_{\mathbb{R}\setminus\{0\}} |\gamma(\xi,S_1(\xi),X(\xi),z) \\ & -\gamma(\xi,S_1(\xi),X(\xi),z)|\widetilde{N}^X(d\xi;dz)\Big)^2. \qquad (4.23)\end{aligned}$$

For the first part of the proof we assume that all expectations are taken under the condition that $S_1(s)=x_1, S_2(s)=x_2$ and $X(s)=i$.
Using Jensen's inequality, we have

$$\begin{aligned}&\Big(\int_s^t |\mu(\xi,S_1(\xi),X(\xi)) - \mu(\xi,S_2(\xi),X(\xi))|d\xi\Big)^2 \\ &\qquad \leq (t-s)\int_s^t [\mu(\xi,S_1(\xi),X(\xi)) - \mu(\xi,S_2(\xi),X(\xi))]^2 d\xi \quad (4.24)\end{aligned}$$

and by the Itô-Lévy isometry (see, e.g., Oksendal and Sulem [**105**])

$$\begin{aligned}E^{\mathbb{Q}}\Big(\int_s^t |\sigma(\xi,S_1(\xi),X(\xi)) \;&-\; \sigma(\xi,S_2(\xi),X(\xi))|dW_\xi + \int_s\int_{\mathbb{R}\setminus\{0\}} |\gamma(\xi,S_1(\xi),X(\xi),z) \\ &-\; \gamma(\xi,S_1(\xi),X(\xi),z)|\widetilde{N}^X(d\xi;dz)\Big)^2 \\ &=\; E^{\mathbb{Q}}\int_s^t [\sigma(\xi,S_1(\xi),X(\xi)) - \sigma(\xi,S_2(\xi),X(\xi))]^2 d\xi \\ &\quad +E^{\mathbb{Q}}\int_s^t\int_{\mathbb{R}\setminus\{0\}} [\gamma(\xi,S_1(\xi),X(\xi),z) \\ &\quad -\gamma(\xi,S_1(\xi),X(\xi),z)]^2 d\xi. \qquad (4.25)\end{aligned}$$

Therefore, the Lipschitz condition (4.20) implies that there exists a constant $C_1>0$ such that

$$E^{\mathbb{Q}}(S_1(t)-S_2(t))^2 \leq 4|x_1-x_2|^2 + C_1(T+1)\int_s^t E^{\mathbb{Q}}(S_1(\xi)-S_2(\xi))^2 d\xi. \quad (4.26)$$

Applying Gronwall's inequality, we have

$$E^{\mathbb{Q}}(S_1(t)-S_2(t))^2 \leq 4|x_1-x_2|^2 e^{C_1(T+1)}. \quad (4.27)$$

Thus,

$$\begin{aligned}|v(s,x_1,i)-v(s,x_2,i)| &= \Big|E^{\mathbb{Q}}e^{-\int_s^T r_\xi d\xi}\Big((S_1(T)-K)^+-(S_2(T)-K)^+\Big)\Big| \\ &\le E^{\mathbb{Q}}\Big[e^{-\int_s^T r_\xi d\xi}\Big|S_1(T)-S_2(T)\Big|\Big] \\ &\le \Big(E^{\mathbb{Q}}e^{-2\int_s^T r_\xi d\xi}\Big)^{\frac{1}{2}}\Big(E^{\mathbb{Q}}|S_1(t)-S_2(t)|^2\Big)^{\frac{1}{2}} \\ &\le \widetilde{C}|x_1-x_2|e^{C_1T}. \end{aligned} \tag{4.28}$$

This implies the (uniform) continuity of $v(s,x,i)$ with respect to x.

Now, we show the continuity of $s\to v(s,x,i)$.

Let $S(t)$ be the solution of the SDE (P) that starts at $t=s$ with $S(s)=x$ and $X(s)=i$. Let $s'\in[s,T]$, we define

$$\begin{cases} S'(t) &= S(t-(s'-s)), \\ X^{'}(t) &= X(t-(s'-s)). \end{cases} \tag{4.29}$$

Let us consider the change of variable $u=t-(s'-s)$, thus we obtain $dt=du$, $dW_t=dW_u$ and $\widetilde{N}^X(dt;dz)=\widetilde{N}^X(du;dz)$. Moreover,

$$\begin{aligned}S(t)=x+\int_s^t\mu(\xi,S(\xi),X(\xi))d\xi+\int_s^t\sigma(\xi,S(\xi),X(\xi))dW_\xi \\ +\int_s^t\int_{\mathbb{R}\backslash\{0\}}\gamma(\xi,S(\xi),X(\xi),z)\widetilde{N}^X(d\xi;dz),\end{aligned}$$

$$\begin{aligned}S'(t)=x+\int_{s'}^t\mu(\xi,s'(\xi),X^{'}(\xi))d\xi+\int_s^t\sigma(\xi,s'(\xi),X^{'}(\xi))dW_\xi \\ +\int_s^t\int_{\mathbb{R}\backslash\{0\}}\gamma(\xi,s'(\xi),X^{'}(\xi),z)\widetilde{N}^{X^{'}}(d\xi;dz).\end{aligned}$$

With this in mind, we have

$$\begin{aligned}(S(t)-S'(t))=\int_{t-(s'-s)}^t\mu(\xi,S(\xi),X(\xi))d\xi+\int_{t-(s'-s)}^t\sigma(\xi,S(\xi),X(\xi))dW_\xi \\ +\int_{t-(s'-s)}^t\int_{\mathbb{R}\backslash\{0\}}\gamma(\xi,S(\xi),X(\xi),z)\widetilde{N}^X(d\xi;dz).\end{aligned} \tag{4.30}$$

For the second part of the proof, we assume that all expectations are taken under the conditions that $S(s)=x=S'(s')$ and $X(s)=i=X^{'}(s')$. Thus for any random variable η, we denote

$$E^{\mathbb{Q}}[\eta]=E^{\mathbb{Q}}[\eta|S(s)=x=S'(s'),X(s)=i=X^{'}(s')].$$

Consequently, we have

$$\begin{aligned} E^{\mathbb{Q}}(S(t)-S'(t))^2 &\leq 3E^{\mathbb{Q}}\Big(\int_{t-(s'-s)}^{t}\mu(\xi,S(\xi),X(\xi))d\xi\Big)^2 \\ &\quad +3E^{\mathbb{Q}}\Big(\int_{t-(s'-s)}^{t}\sigma(\xi,S(\xi),X(\xi))dW_\xi\Big)^2 \\ &\quad +3E^{\mathbb{Q}}\Big(\int_{t-(s'-s)}^{t}\int_{\mathbb{R}\setminus\{0\}}\gamma(\xi,S(\xi),X(\xi),z)\widetilde{N}^X(d\xi;dz)\Big)^2. \end{aligned}$$

Using the Itô-Lévy isometry and the linear growth condition (4.21) we obtain

$$E^{\mathbb{Q}}(S(t)-S'(t))^2 \leq 3\beta\Big(s'-s+2\Big)\Big(s'-s+\int_{t-(s'-s)}^{t}E^{\mathbb{Q}}|S(\xi)|^2d\xi\Big). \tag{4.31}$$

In addition, by the existence and uniqueness theorem of solution of stochastic differential equation (P) [see, e.g., Applebaum [**4**]] there exists a constant $\delta_1 > 0$ such that $E^{\mathbb{Q}}|S(\xi)|^2 < \delta_1$ almost everywhere in the interval $[0,T]$. Therefore

$$\int_{t-(s'-s)}^{t}E^{\mathbb{Q}}|S(\xi)|^2d\xi < \delta_1(s'-s),$$

which implies from (4.31) that

$$E^{\mathbb{Q}}(S(t)-S'(t))^2 \leq \delta_2(s'-s). \tag{4.32}$$

where $\delta_2 = 3\beta(\delta_1+1)(s'-s+2)$.
Moreover, we have

$$\begin{aligned} |v(s,x,i)-v(s',x,i)| &\leq \Big|E^{\mathbb{Q}}\Big(e^{-\int_s^T r_\xi d\xi}(S(T)-K)^+ - e^{-\int_{s'}^T r_\xi d\xi}(s'(T)-K)^+\Big)\Big| \\ &\leq \Big|E^{\mathbb{Q}}\Big(e^{-\int_s^T r_\xi d\xi}\Big((S(T)-K)^+ - (s'(T)-K)^+\Big)\Big| \\ &\quad +\Big|E^{\mathbb{Q}}\Big((s'(T)-K)^+(e^{-\int_s^T r_\xi d\xi} - e^{-\int_{s'}^T r_\xi d\xi})\Big)\Big| \\ &\leq \Big(E^{\mathbb{Q}}e^{-2\int_s^T r_\xi d\xi}\Big)^{\frac{1}{2}}\Big(\delta_2(s'-s)\Big)^{\frac{1}{2}} \\ &\quad +(s'-s)\Big(E^{\mathbb{Q}}e^{-2\int_{s_0}^T r_\xi d\xi}\Big)^{\frac{1}{2}}\Big(E^{\mathbb{Q}}|(s'(T)|^2\Big)^{\frac{1}{2}}, \end{aligned} \tag{4.33}$$

where we have used the mean value theorem for the last inequality.
Finally, we have

$$|v(s,x,i)-v(s',x,i)| \leq \sqrt{(s'-s)}\Bigg(\sqrt{E^{\mathbb{Q}}e^{-2\int_s^T r_\xi d\xi}} + \sqrt{\delta_1(s'-s)}\sqrt{E^{\mathbb{Q}}e^{-2\int_{s_0}^T r_\xi d\xi}}\Bigg). \tag{4.34}$$

which implies the (uniform) continuity of $s \to v(s,x,i)$.

Global continuity follows from the following inequality

$$|v(s,x,i)-v(s',y,i)| \leq |v(s,x,i)-v(s,y,i)| + |v(s,y,i)-v(s',y,i)|$$

and the fact that the first bound is independent of s.

Now let us prove that $v(s,x,i)$ has at most a polynomial growth.
Let $(s_1,x_1),(s_2,x_2)\in[0,T]\times\mathbb{R}$, we have that

$$\begin{aligned}
|v(s_1,x_1,i)-v(s_2,x_2,i)| &\le \Big|E^{\mathbb{Q}}\Big(e^{\int_{s_1}^T r_\xi d\xi}(S_1(T)-K)^+ - e^{\int_{s_2}^T r_\xi d\xi}(S_2(T)-K)^+\Big)\Big| \\
&\le \Big|E^{\mathbb{Q}}\Big(e^{\int_{s_1}^T r_\xi d\xi}\Big((S_1(T)-K)^+ - (S_2(T)-K)^+\Big)\Big| \\
&\quad +\Big|E^{\mathbb{Q}}\Big((S_2(T)-K)^+(e^{-\int_{s_1}^T r_u du} - e^{-\int_{s_2}^T r_u du})\Big)\Big| \\
&\le E^{\mathbb{Q}}\Big[e^{-\int_s^T r_\xi d\xi}\Big|S_1(T)-S_2(T)\Big|\Big] \\
&\quad +\Big|E^{\mathbb{Q}}\Big((S_2(T)-K)^+(e^{-\int_{s_1}^T r_\xi d\xi} - e^{-\int_{s_2}^T r_\xi d\xi})\Big)\Big| \\
&\le \widetilde{C}|x_1-x_2|e^{CT} \\
&\quad +|s_1-s_2|\Big(E^{\mathbb{Q}}e^{-2\int_{s_0}^T r_\xi d\xi}\Big)^{\frac{1}{2}}\Big(E^{\mathbb{Q}}|(S_2(T)|^2\Big)^{\frac{1}{2}}. \qquad (4.35)
\end{aligned}$$

We have that $E^{\mathbb{Q}}|S_2(T)|^2<\delta_1$, thus there exists a constant $C_1>0$ such that

$$|v(s,0,i)| = |E^{\mathbb{Q}}e^{-\int_s^T r_\xi d\xi}\Big((S_2(T)-K)^+\Big)| \le E^{\mathbb{Q}}e^{-2\int_s^T r_\xi d\xi}E^{\mathbb{Q}}|S_2(T)|^2 < C_1.$$

Setting $x_1=x, s_1=s=s_2$ and $x_2=0$ in (4.35), we have

$$|v(s,x,i)| \le \widetilde{C}|x| + |v(s,0,i)| \le \max(C_1,\widetilde{C}e^{CT})(1+|x|). \qquad (4.36)$$

This completes the proof of the lemma. □

4.3.2. Viscosity Solution

In this section, we characterize the European option price as the unique viscosity solution of the coupled system of IPDE (4.17)-(4.18). To simplify the presentation, we introduce the following notations

For $i\in\mathbb{S},\ s\in(0,T),\ x\in\mathbb{R}$, let

$$\begin{aligned}
&\mathcal{H}\Big(i,s,x,f(s,x,i),f(s,x,\cdot),\frac{\partial f(s,x,i)}{\partial s},\frac{\partial f(s,x,i)}{\partial x},\frac{\partial^2 f(s,x,i)}{\partial x^2},\mathcal{I}(s,x,i,f)\Big) \\
&\quad := \frac{\partial f(s,x,i)}{\partial s} + r(i)x\frac{\partial f(s,x,i)}{\partial x} + \frac{1}{2}\sigma^2(i)x^2\frac{\partial^2 f(s,x,i)}{\partial x^2} - r(i)f(s,x,i) \\
&\qquad + \mathbb{A}f(s,x,\cdot)(i) + \mathcal{I}(s,x,i,f), \quad (4.37)
\end{aligned}$$

where

$$\mathcal{I}(s,x,i,f) := \int_{\mathbb{R}\backslash\{0\}}\Big[f(s,xe^z,i)-f(s,x,i)-x(e^z-1)\frac{\partial f(s,x,i)}{\partial x}\Big]\rho_i(dz).$$

Equation (4.17) takes the following form

$$\mathcal{H}\Big(i,s,x,f(s,x,i),f(s,x,\cdot),\frac{\partial f(s,x,i)}{\partial s},\frac{\partial f(s,x,i)}{\partial x},\frac{\partial^2 f(s,x,i)}{\partial x^2},\mathcal{I}(s,x,i,f)\Big)=0 \qquad (4.38)$$

for $i \in \mathbb{S}$.
Due to the fact that the Lévy measure $\rho_i(dz)$ for $i \in \mathbb{S}$ could present a possible singularity in zero, we need to be more specific about the meaning of integro-differential operator $\mathcal{I}(s,x,i,f)$.
To do this, let $\mathcal{C}_{Lip}([0,T]\times\mathbb{R}\times\mathbb{S})$ be the set of functions $f(s,x,i)$ defined on $[0,T]\times\mathbb{R}\times\mathbb{S}$ which are Lipschitz continuous with respect to the variable x.
We define for any $\kappa \in (0,1)$, $(s,x,i) \in (0,T]\times\mathbb{R}\times\mathbb{S}$, $g \in \mathcal{C}_{Lip}([0,T]\times\mathbb{R}\times\mathbb{S})$ and $p \in \mathcal{C}^1(\mathbb{R})$,

$$\mathcal{I}^\kappa(s,x,i,g,p) := \int_{|z|>\kappa} \Big[g(s,xe^z,i) - g(s,x,i) - x(e^z-1)p\Big]\rho_i(dz).$$

The integral of $\mathcal{I}^\kappa(s,x,i,g,p)$ is bounded by $\text{Const}(x,p,\kappa,i)\times(1+|e^z-1|)$ and thanks to (4.8) this integral is convergent for every $\kappa > 0$.
Also, for $\kappa \in (0,1)$, $(s,x,i) \in [0,T]\times\mathbb{R}\times\mathbb{S}$ and $h : [0,T]\times\mathbb{R}\times\mathbb{S} \to \mathbb{R}$ such that $h(s,.,i) \in \mathcal{C}^2(\mathbb{R})$ we define

$$\mathcal{I}_\kappa(s,x,i,h) := \int_{|z|\le\kappa} \Big[h(s,xe^z,i) - h(s,x,i) - x(e^z-1)\frac{\partial h(s,x,i)}{\partial x}\Big]\rho_i(dz).$$

By Taylor's formula

$$h(s,xe^z,i) = h(s,x,i) + x(e^z-1)\frac{\partial h(s,x,i)}{\partial x} + (x(e^z-1))^2\frac{\partial^2 h(s,\xi_0,i)}{\partial x^2},$$

where ξ_0 lies in the segment (x,xe^z). Hence, the integrand of $\mathcal{I}_\kappa(s,x,i,h)$ is bounded by $\text{Const}(x,\kappa,i)\times|e^z-1|^2$ and the integral is convergent since every Lévy measure integrates z^2 in a neighborhood of zero. Also note that

$$\lim_{\kappa\downarrow 0}\mathcal{I}_\kappa(s,x,i,h) = 0. \tag{4.39}$$

Now, we can define for $g,h \in \mathcal{C}_{Lip}([0,T]\times\mathbb{R}\times\mathbb{S})$ such that $h(s,\cdot,i) \in \mathcal{C}^2(\mathbb{R})$ the integro-differential operator $\underline{\mathcal{I}}$ by

$$\underline{\mathcal{I}}(s,x,i,g,h) := \mathcal{I}^\kappa(s,x,i,g,\frac{\partial h}{\partial x}) + \mathcal{I}_\kappa(s,x,i,h). \tag{4.40}$$

It follows that the system of IPDE (4.17) with terminal condition (4.18) verified by the European option price is well-defined for all $v \in \mathcal{C}_{Lip}([0,T]\times\mathbb{R}\times\mathbb{S})$ such that $v(s,\cdot,i) \in \mathcal{C}^2(\mathbb{R})$.

In general, for a financial model which involves Lévy process as dynamics of the asset price, the European option price is not a regular function of the underlying's price. We refer to Voltchoka [**125**] for a deep discussion of this question. Consequently, the system of IPDEs (4.17) should be interpreted in a weaker sense, notably in the framework of viscosity solutions. The precise definition goes as :

Definition 4.1. *We say that $f(s,x,i)$ is a viscosity solution of the system*

$$(\mathbf{P})\begin{cases} \mathcal{H}\Big(i,s,x,f(s,x,i),f(s,x,\cdot),\ldots \\ \frac{\partial f(s,x,i)}{\partial s},\frac{\partial f(s,x,i)}{\partial x},\frac{\partial^2 f(s,x,i)}{\partial x^2},\underline{\mathcal{I}}(s,x,i,f(\cdot),h)\Big) = 0 \\ f(T,x,i) = H(x,i) \\ \\ for\ s\in(0,T),\ i\in\mathbb{S},\ x\in\mathbb{R}\ and\ h(s,\cdot,i)\in\mathcal{C}^2(\mathbb{R}). \end{cases} \tag{4.41}$$

if

(1) For each $i\in\mathbb{S}$, $f(s,x,i)$ is continuous in $(s,x)\in[0,T)\times\mathbb{R}$ and there exist positive constants C_1 and m such that

$$f(s,x,i)\le C_1(1+|x|^m).$$

(2) For each $i\in\mathbb{S}$, $f(T,x,i)\le H(x,i)$ and

$$\mathcal{H}\Big(i,s_0,x_0,f(s_0,x_0,i),f(s_0,x_0,\cdot),\frac{\partial\varphi(s_0,x_0)}{\partial s},\ldots \frac{\partial\varphi(s_0,x_0)}{\partial x},\frac{\partial^2\varphi(s_0,x_0)}{\partial x^2},\underline{\mathcal{I}}(s_0,x_0,i,f(\cdot),\varphi)\Big)\le 0, \tag{4.42}$$

whenever $\varphi(s,x)\in\mathcal{C}_{Lip}([0,T]\times\mathbb{R})\cap\mathcal{C}^{1,2}((0,T]\times\mathbb{R})$ such that $f(s,x,i)-\varphi(s,x)$ has local maximum at $(s_0,x_0)\in(0,T)\times\mathbb{R}$.

(3) For each $i\in\mathbb{S}$, $f(T,x,i)\ge H(x,i)$ and

$$\mathcal{H}\Big(i,s_0,x_0,f(s_0,x_0,i),f(s_0,x_0,\cdot),\frac{\partial\phi(s_0,x_0)}{\partial s},\ldots \frac{\partial\phi(s_0,x_0)}{\partial x},\frac{\partial^2\phi(s_0,x_0)}{\partial x^2},\underline{\mathcal{I}}(s_0,x_0,i,f(\cdot),\phi,\frac{\partial\phi}{\partial x})\Big)\ge 0, \tag{4.43}$$

whenever $\phi(s,x)\in\mathcal{C}_{Lip}([0,T]\times\mathbb{R})\cap\mathcal{C}^{1,2}((0,T]\times\mathbb{R})$ such that $f(s,x,i)-\phi(s,x)$ has local minimum at $(s_0,x_0)\in(0,T)\times\mathbb{R}$.

Let f be a function that satisfies (1) and (2) [resp. (1) and (3)]. Then f is said to be a viscosity subsolution (resp. supersolution).

Remark 4.3.1. *The notation $f(\cdot)$ in the integro-differential operator $\underline{\mathcal{I}}$ indicates that nonlocal terms are used on $f(\cdot)$, not only from x.*

We are going to state the main result of this section.

Theorem 4.1. *The price of the European option is a viscosity solution to the system* $(\mathbf{P})$.

Proof. Note that for $t=T$,

$$v(T,x,i):=E^{\mathbb{Q}}[H(S(T),X(T))|(X(T),S(T))=(i,x)]=h(x,i).$$

In the sequel, we denote $E^{\mathbb{Q}}[\eta(\cdot)|X(s)=i,S(s)=x]$ by $E^{\mathbb{Q}}_{s,x,i}[\eta(\cdot)]$.

Following Definition 4.4, it suffices to show that $v(\cdot,\cdot,i)$ is a viscosity subsolution and supersolution.
Let $X_s\in\mathbb{S}$. First, we want to show that $v(\cdot,\cdot,X_s)$ is a viscosity supersolution.
Let $\phi\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ and $(t,x)\in[0,T]\times\mathbb{R}$ such that $v(t,x,X_s)-\phi(t,x)$ attains its local minimum on a neighborhood $\mathcal{V}_{(s,x_s)}$ at $(s,x)\in(0,T]\times\mathbb{R}$. We define a new function Ψ as follows :

$$\Psi(t,x,i)=\begin{cases}\phi(t,x)+[v(s,x_s,X_s)-\phi(s,x_s)], & \text{if } i=X_s,\\ v(t,x,i), & \text{if } i\neq X_s.\end{cases}\tag{4.44}$$

Let $\tau=\inf\{t\geq s: X(t)\neq X_s\}$ the first jump time of $X(\cdot)$ from the state X_s. From the hypothesis of irreducibility of $X(\cdot)$, we have $\mathbb{Q}(\tau<\infty)=1$.

Now, consider $\theta\in[s,\tau]$ to be such that $(t,X(t))$ starts at (s,X_s) and stays in $\mathcal{V}_{(s,x_s)}$ for $s\leq\theta$. Since $\theta\leq\tau$ we have $X(t)=X_s$, for $s\leq t\leq\theta$. Recalling that $v(\cdot,\cdot,X_s)-\phi(\cdot,\cdot)$ attains its minimum in $\mathcal{V}_{(s,x_s)}$ at (s,x_s), for $s\leq t\leq\theta$ we have

$$v(t,S(t),X_s)\geq\phi(t,S(t))+v(s,x_s,X_s)-\phi(s,x_s)=\Psi(t,S(t),X_s).\tag{4.45}$$

Hence,

$$E^{\mathbb{Q}}_{s,x_s,X_S}\Big[e^{-\int_s^\theta r_\xi d\xi}v(\theta,S(\theta),X_s)\Big]\geq E^{\mathbb{Q}}_{s,x_s,X_S}\Big[e^{-\int_s^\theta r_\xi d\xi}\Psi(\theta,S(\theta),X_\theta)\Big].$$

By Dynkin's formula for jump-diffusion processes [see, e.g., Oksendal and Sulem [**105**]], we have

$$\begin{aligned}&E^{\mathbb{Q}}_{s,x_s,X_s}\Big[e^{-\int_s^\theta r_\xi d\xi}\Psi(\theta,S(\theta),X_s)\Big]-\Psi(s,x_s,X_s)\\&=E^{\mathbb{Q}}_{s,x_s,X_s}\int_s^\theta\Bigg[e^{-\int_s^t r_\xi d\xi}\Big(-r(X_s)\Psi(t,S(t),X_s)+\mathcal{L}\Psi(t,S(t),X_s)\Big)\Bigg]dt.\end{aligned}\tag{4.46}$$

and thus,

$$\begin{aligned}E^{\mathbb{Q}}_{s,x_s,X_S}\Big[e^{-\int_s^\theta r_\xi d\xi}v(\theta,S(\theta),X_s)\Big]\ &-\ v(s,x_s,X_s)\\ &\geq\ E^{\mathbb{Q}}_{s,x_s,X_S}\int_s^\theta\Bigg[e^{-\int_s^t r_\xi d\xi}\Bigg\{-r(X_s)\Psi(t,S(t),X_s)\\&+\frac{\partial\phi(t,S(t))}{\partial t}+r(X_s)S(t)\frac{\partial\phi(t,S(t))}{\partial x}\\&+\frac{1}{2}\sigma(X_s)^2(S(t))^2\frac{\partial^2\phi(t,S(t))}{\partial x^2}+\mathbb{A}\Psi(t,S(t),.)(X_s)\\&+\int_{\mathbb{R}\setminus\{0\}}\Big[\Psi(t,S(t)e^z,X_s)-\Psi(t,S(t),X_s)\\&-S(t)(e^z-1)\frac{\partial\Psi(t,S(t),X_s)}{\partial x}\Big]\rho_{X_s}(dz)\Bigg\}\Bigg]dt.\end{aligned}\tag{4.47}$$

Moreover, we have

$$\begin{aligned}\mathbb{A}\Psi(t,S(t),.)(X_s) &= \sum_{l\neq X_s,\ l\in\mathbb{S}} a_{X_s l}\Big(\Psi(t,S(t),l)-\Psi(t,S(t),X_s)\Big)\\ &\geq \sum_{l\neq X_s,\ l\in\mathbb{S}} a_{X_s l}\Big(v(t,S(t),l)-v(t,S(t),X_s)\Big)\\ &\geq \mathbb{A}v(t,S(t),.)(X_s). \end{aligned} \tag{4.48}$$

Combining the inequalities (4.47) and (4.48), we have

$$\begin{aligned} E^{\mathbb{Q}}_{s,x_s,X_S}\Big[e^{-\int_s^\theta r_\xi d\xi}v(\theta,S(\theta),X_\theta)\Big] &- v(s,x_s,X_s)\\ &\geq E^{\mathbb{Q}}_{s,x_s,X_S}\int_s^\theta\Bigg[e^{-\int_s^t r_\xi d\xi}\Bigg\{-r(X_s)v(t,S(t),X_s)\\ &+\frac{\partial\phi(t,S(t))}{\partial t}+r(X_s)S(t)\frac{\partial\phi(t,S(t))}{\partial x}\\ &+\frac{1}{2}\sigma(X_s)^2(S(t))^2\frac{\partial^2\phi(t,S(t))}{\partial x^2}+\mathbb{A}v(t,S(t),.)(X_s)\\ &+\int_{\mathbb{R}\setminus\{0\}}\Big[\Psi(t,S(t)e^z,X_s)-\Psi(t,S(t),X_s)\\ &-S(t)(e^z-1)\frac{\partial\Psi(t,S(t),X_s)}{\partial x}\Big]\rho_{X_s}(dz)\Bigg\}\Bigg]dt. \end{aligned} \tag{4.49}$$

By using the fact that

$$E^{\mathbb{Q}}_{s,x_s,X_S}\Big[e^{-\int_s^\theta r_\xi d\xi}v(\theta,S(\theta),X_\theta)\Big]-v(s,x_s,X_s)=0,$$

due to the martingale property of the discounted European option price $\{e^{-\int_s^t r_\xi d\xi}v(t,S(t),X(t)) : t\in[0,T]\}$ for a fixed $s\in[0,T]$, we obtain

$$\begin{aligned} &E^{\mathbb{Q}}_{s,x_s,X_S}\int_s^\theta\Bigg[e^{-\int_s^t r_\xi d\xi}\Bigg\{-r(X_s)v(t,S(t),X_s)\\ &+\frac{\partial\phi(t,S(t))}{\partial t}+r(X_s)S(t)\frac{\partial\phi(t,S(t))}{\partial x}+\frac{1}{2}\sigma(X_s)^2(S(t))^2\frac{\partial^2\phi(t,S(t))}{\partial x^2}+\mathbb{A}v(t,S(t),.)(X_s)\\ &+\int_{\mathbb{R}\setminus\{0\}}\Big[\Psi(t,S(t)e^z,X_s)-\Psi(t,S(t),X_s)-S(t)(e^z-1)\frac{\partial\Psi(t,S(t),X_s)}{\partial x}\Big]\rho_{X_s}(dz)\Bigg\}\Bigg]dt\leq 0. \end{aligned} \tag{4.50}$$

Multiplying both sides by $\frac{1}{(\theta-s)} > 0$ and letting $\theta \downarrow s$, it turns out that

$$\begin{aligned}
&- r(X_s)v(s,S(s),X_s) + \frac{\partial \phi(s,S(s))}{\partial s} + r(X_s)S(s)\frac{\partial \phi(s,S(s))}{\partial x} \\
&\qquad + \frac{1}{2}\sigma(X_s)^2(S(s))^2\frac{\partial^2 \phi(s,S(s))}{\partial x^2} + \mathbb{A}v(s,S(s),.)(X_s) \\
&+ \int_{\mathbb{R}\setminus\{0\}} \Big[\Psi(s,S(s)e^z,X_s) - \Psi(s,S(s),X_s) - S(s)(e^z-1)\frac{\partial \Psi(s,S(s),X_s)}{\partial x}\Big]\rho_{X_s}(dz) \le 0.
\end{aligned} \tag{4.51}$$

Hence, recalling that $S(s) = x_s$ and

$$\begin{aligned}
\phi(s,x_se^z,X_s) - \phi(s,x_s),X_s) &= \phi(s,x_se^z) - \phi(s,x_s) \\
&\ge v(s,x_se^z,X_s) - v(s,x_s,X_s)
\end{aligned} \tag{4.52}$$

by the local minimum property of $v(\cdot,\cdot,X_s) - \phi(\cdot,\cdot)$ at (s,x_s), we obtain finally

$$\begin{aligned}
&r(X_s)\Big(v(s,x_s,X_s) - x\frac{\partial \phi(s,x_s)}{\partial x}\Big) - \frac{\partial \phi(s,x_s)}{\partial s} - \frac{1}{2}\sigma(X_s)^2(x)^2\frac{\partial^2 \phi(s,x_s)}{\partial x^2} - \mathbb{A}v(s,x_s,\cdot)(X_s) \\
&\quad - \int_{\mathbb{R}\setminus\{0\}} \Big[v(s,x_se^z,X_s) - v(s,x_s,X_s) - x(e^z-1)\frac{\partial \phi(s,x_s,X_s)}{\partial x}\Big]\rho_{X_s}(dz) \ge 0
\end{aligned} \tag{4.53}$$

which is the desired inequality.

Now, let us prove the subsolution inequality.
Let $\varphi \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ and $(t,x) \in [0,T]\times\mathbb{R}$ such that $v(t,x,X_s) - \varphi(t,x)$ has a local maximum on a neighborhood $\mathcal{V}_{(s,x_s)}$ at (s,x_s). We define a function Φ as follows :

$$\Phi(t,x,i) = \begin{cases} \varphi(t,x) + [v(s,x_s,X_s) - \varphi(s,x_s)], & if\ i = X_s, \\ v(t,x,i), & if\ i \neq X_s. \end{cases} \tag{4.54}$$

Let τ be defined as above and let $\theta_0 \in [s,\tau]$ be such that $(\theta, X(\theta))$ starts at (s,X_s) and stays in $\mathcal{V}_{(s,x_s)}$ for $s \le \theta \le \theta_0$. Note that $S(\theta) = X_s$ for $s \le \theta \le \theta_0$. Moreover, recall that $v(t,x,X_s) - \varphi(t,x)$ attains its maximum at (s,x_s) in $\mathcal{V}_{(s,x_s)}$; it follows that for any $\theta \in [s,\theta_0]$, $v(\theta,S(\theta),X_\theta) - \varphi(\theta,X_\theta) \le v(s,x_s,X_s) - \varphi(s,x_s)$.
In view of the definition of Φ in (4.54), we have for any $\theta \in [s,\theta_0]$

$$v(\theta,S(\theta),X_\theta) \le \Phi(\theta,S(\theta),X_\theta). \tag{4.55}$$

This implies,

$$E^{\mathbb{Q}}_{s,x_s,X_S}\Big[e^{-\int_s^\theta r_\xi d\xi}v(\theta,S(\theta),X_s)\Big] \le E^{\mathbb{Q}}_{s,x_s,X_S}\Big[e^{-\int_s^\theta r_\xi d\xi}\Phi(\theta,S(\theta),X_\theta)\Big].$$

And by using Dynkin's formula, we have

$$\begin{aligned}
E^{\mathbb{Q}}_{s,x_s,X_S}\Big[e^{-\int_s^\theta r_\xi d\xi}v(\theta,S(\theta),X_s)\Big] \;\leq\; & v(s,x_s,X_s) \\
& +E^{\mathbb{Q}}_{s,x_s,X_S}\int_s^\theta\Bigg[e^{-\int_s^t r_\xi d\xi}\Bigg\{-r(X_s)v(t,S(t),X_s) \\
& +\frac{\partial\varphi(t,S(t))}{\partial t}+r(X_s)S(t)\frac{\partial\varphi(t,S(t))}{\partial x} \\
& +\frac{1}{2}\sigma(X_s)^2(S(t))^2\frac{\partial^2\varphi(t,S(t))}{\partial x^2}+\mathbb{A}\Phi(t,S(t),.)(X_s) \\
& +\int_{\mathbb{R}\setminus\{0\}}\Big[\Phi(t,S(t)e^z,X_s)-\Phi(t,S(t),X_s) \\
& -S(t)(e^z-1)\frac{\partial\Phi(t,S(t),X_s)}{\partial x}\Big]\rho_{X_s}(dz)\Bigg\}\Bigg]dt.
\end{aligned} \tag{4.56}$$

Moreover, we have

$$\begin{aligned}
\mathbb{A}\Phi(t,S(t),.)(X_s) &= \sum_{l\neq X_s,\ l\in\mathbb{S}} a_{X_s l}\Big(\Phi(t,S(t),l)-\Phi(t,S(t),X_s)\Big) \\
&\leq \sum_{l\neq X_s,\ l\in\mathbb{S}} a_{X_s l}\Big(v(t,S(t),l)-v(t,S(t),X_s)\Big) \\
&=: \mathbb{A}v(t,S(t),.)(X_s).
\end{aligned} \tag{4.57}$$

Combining (4.56) and (4.57), and the fact
$v(\theta,S(\theta),X_s)=v(\theta,S(\theta),X_\theta)$ we have

$$\begin{aligned}
E^{\mathbb{Q}}_{s,x_s,X_S}\Big[e^{-\int_s^\theta r_\xi d\xi}v(\theta,S(\theta),X_\theta)\Big] \leq\; & v(s,x_s,X_s) \\
& + E^{\mathbb{Q}}_{s,x_s,X_S}\int_s^\theta\Bigg[e^{-\int_s^t r_\xi d\xi}\Bigg\{-r(X_s)v(t,S(t),X_s) \\
& + \frac{\partial\phi(t,S(t))}{\partial t}+r(X_s)S(t)\frac{\partial\phi(t,S(t))}{\partial x} \\
& + \frac{1}{2}\sigma(X_s)^2(S(t))^2\frac{\partial^2\phi(t,S(t))}{\partial x^2}+\mathbb{A}v(t,S(t),.)(X_s) \\
& + \int_{\mathbb{R}\setminus\{0\}}\Big[\Phi(t,S(t)e^z,X_s)-\Phi(t,S(t),X_s) \\
& - S(t)(e^z-1)\frac{\partial\Phi(t,S(t),X_s)}{\partial x}\Big]\rho_{X_s}(dz)\Bigg\}\Bigg]dt.
\end{aligned} \tag{4.58}$$

Now, using the fact that

$$E^{\mathbb{Q}}_{s,x_s,X_S}\Big[e^{-\int_s^\theta r_\xi d\xi}v(\theta,S(\theta),X_\theta)\Big]-v(s,x_s,X_s)=0$$

due the martingale property of the discounted European option price $\{e^{-\int_s^t r_\xi d\xi} v(t, S(t), X(t)) : t \in [0,T]\}$ for a fixed $s \in [0,T]$, we deduce that

$$
\begin{aligned}
&E^{\mathbb{Q}}_{s,x_s,X_S} \int_s^\theta \Bigg[e^{-\int_s^t r_\xi d\xi} \Bigg\{ - r(X_s) v(t, S(t), X_s) \\
&+ \frac{\partial \phi(t, S(t))}{\partial t} + r(X_s) S(t) \frac{\partial \varphi(t, S(t))}{\partial x} + \frac{1}{2}\sigma(X_s)^2 (S(t))^2 \frac{\partial^2 \varphi(t, S(t))}{\partial x^2} + \mathbb{A} v(t, S(t), .)(X_s) \\
&\qquad + \int_{\mathbb{R}\setminus\{0\}} \Big[\Phi(t, S(t)e^z, X_s) - \Phi(t, S(t), X_s) \\
&\qquad\qquad - S(t)(e^z - 1) \frac{\partial \Phi(t, S(t), X_s)}{\partial x} \Big] \rho_{X_s}(dz) \Bigg\} \Bigg] dt \geq 0. \quad (4.59)
\end{aligned}
$$

Dividing both sides by $(\theta - s) > 0$ and taking the limit as $\theta \downarrow s$

$$
\begin{aligned}
&- r(X_s) v(s, S(s), X_s) + \frac{\partial \varphi(s, S(s))}{\partial s} + r(X_s) S(s) \frac{\partial \varphi(s, S(s))}{\partial x} \\
&\qquad + \frac{1}{2}\sigma(X_s)^2 (S(s))^2 \frac{\partial^2 \varphi(s, S(s))}{\partial x^2} + \mathbb{A} v(s, S(s), .)(X_s) \\
&\qquad + \int_{\mathbb{R}\setminus\{0\}} \Big[\Phi(s, S(s)e^z, X_s) - \Phi(s, S(s), X_s) \\
&\qquad\qquad - S(s)(e^z - 1) \frac{\partial \Phi(s, S(s), X_s)}{\partial x} \Big] \rho_{X_s}(dz) \geq 0. \quad (4.60)
\end{aligned}
$$

Hence, recalling that $S(s) = x_s$ and

$$
\begin{aligned}
\Phi(s, x_s e^z, X_s) - \Phi(s, x_s, X_s) &= \varphi(s, x_s e^z) - \varphi(s, x_s) \\
&\geq v(s, x_s e^z, X_s) - v(s, x_s, X_s) \qquad (4.61)
\end{aligned}
$$

by the local maximum property of $v(\cdot, \cdot, X_s) - \phi(\cdot, \cdot)$ at (s, x_s), we obtain finally

$$
\begin{aligned}
&r(X_s)\Big(v(s, x_s, X_s) - x_s \frac{\partial \varphi(s, x_s)}{\partial x} \Big) - \frac{\partial \varphi(s, x_s)}{\partial s} - \frac{1}{2}\sigma(X_s)^2 x_s^2 \frac{\partial^2 \varphi(s, x_s)}{\partial x^2} - \mathbb{A} v(s, x_s, \cdot)(X_s) \\
&- \int_{\mathbb{R}\setminus\{0\}} \Big[v(s, x_s e^z, X_s) - v(s, x_s, X_s) - x_s(e^z - 1) \frac{\partial \varphi(s, x_s)}{\partial x} \Big] \rho_{X_s}(dz) \leq 0 \quad (4.62)
\end{aligned}
$$

This last inequality is what we want to prove. □

4.3.3. Comparison Principle : Existence and Uniqueness of the Viscosity Solution

In this section, we prove a comparison result from which we obtain the existence and uniqueness of the solution of the coupled system IPDE (**P**). In proving comparison results for viscosity solutions, the notion of parabolic superjet and subjet as defined in Crandall, Ishii and Lions [**38**] is particularly useful.

Definition 4.2. *Let* $f(s,x,i):[0,T]\times\mathbb{R}\times\mathbb{S}\to\mathbb{R}$ *be a function. We define* $f^*(s,x,i):[0,T]\times\mathbb{R}\times\mathbb{S}\to\mathbb{R}$ *by*

$$f^*(s,x,i):=\limsup_{r\downarrow 0}\{f(t,y,i:(t,y)\in B_r(s,x)\}.$$

$f^*(s,x,i)$ *is called the upper semicontinuous envelop of* $f(s,x,i)$.
Similarly we define $f_*(s,x,i):[0,T]\times\mathbb{R}\times\mathbb{S}\to\mathbb{R}$ *the lower semicontinuous envelop of* $f(s,x,i)$ *as follows*

$$f_*(s,x,i):=\liminf_{r\downarrow 0}\{f(t,y,i:(t,y)\in B_r(s,x)\}.$$

Definition 4.3. *Let* $f(s,x,i):[0,T]\times\mathbb{R}\times\mathbb{S}\to\mathbb{R}$. *Define the parabolic superjet by*

$$\begin{aligned}\mathcal{P}^{2,+}f(s,x,i)=\{(p,q,L)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}:f(t,y,i)-f(s,x,i)-p(t-s)-q(y-x)\\ +\frac{1}{2}L.(y-x)^2\le o(|y-x|^2)\ as\quad (t,y)\to(s,x)\},\end{aligned}$$

and its closure as

$$\begin{aligned}\overline{\mathcal{P}}^{2,+}f(s,x,i)=\{(p,q,L)=\lim_{n\to\infty}(p_n,q_n,L_n) with\ (p_n,q_n,L_n)\in\mathcal{P}^{2,+}f(s_n,x_n,i)\\ and\ \lim_{n\to\infty}(s_n,x_n,f(s_n,x_n,i))=(s,x,f(s,x,i))\}.\end{aligned}$$

Similarly, we define the parabolic subjet $\mathcal{P}^{2,-}f(s,x,i)=-\mathcal{P}^{2,+}(-f)(s,x,i)$ *and its closure* $\overline{\mathcal{P}}^{2,-}f(s,x,i)=\overline{\mathcal{P}}^{2,+}(-f)(s,x,i)$.
We have the following result for which we can find a proof in Fleming and Soner [**62**].

Lemma 4.2. $\mathcal{P}^{2,+}f(s,x,i)(resp.\mathcal{P}^{2,-}f(s,x,i))$ *consist of the set of* $\left(\frac{\partial\phi(s,x)}{\partial s},\frac{\partial\phi(s,x)}{\partial x},\frac{\partial^2\phi(s,x)}{\partial x^2}\right)$ *where* $\phi\in\mathcal{C}^2([0,T]\times\mathbb{R})$ *and* $f-\phi$ *has a global maximum(resp. minimum) at* (s,x).
With this in mind, we have this equivalent formulation of the notion of viscosity solution

Definition 4.4. *A function* $u(s,x,i)$ *continuous in* (s,x) *satisfying the polynomial growth condition is a viscosity solution of*

$$\begin{aligned}\mathcal{H}\Big(i,s,x,f(s,x,i),f(s,x,\cdot),\frac{\partial f(s,x,i)}{\partial s},\\ \frac{\partial f(s,x,i)}{\partial x},\frac{\partial^2 f(s,x,i)}{\partial x^2},\underline{\mathcal{I}}(s,x,i,f(\cdot),h)\Big)=0\end{aligned}$$

if

(1) *for each* $i\in\mathbb{S}$, $u(\cdot,\cdot,i)$ *is upper semicontinuous and for all* $(s,x)\in[0,T]\times\mathbb{R}$, $\mathbf{u}=(u(\cdot,\cdot,i):i\in\mathbb{S})$, $A_1\in\mathbb{R}$ *and* $(a,p,L_1)\in\mathcal{P}^{2,+}u(s,x,i)$

$$\mathcal{H}(i,s,x,u,\mathbf{u},a,p,L_1,A_1)\le 0,$$

in this case u is a viscosity subsolution,

(2) for each $i\in\mathbb{S}$, $u(\cdot,\cdot,i)$ is lower semicontinuous and for all $(s,x)\in[0,T]\times\mathbb{R}$, $\mathbf{u}=(u(\cdot,\cdot,i):i\in\mathbb{S})$, $A_2\in\mathbb{R}$ and $(b,q,L_2)\in\mathcal{P}^{2,+}u(s,x,i)$

$$\mathcal{H}(i,s,x,u,\mathbf{u},b,q,L_2,A_2)\ge 0,$$

in this case u is a viscosity supersolution.

To prove our comparison result, we need the following result from Crandall, Ishii and Lions [**37**].

Theorem 4.2 (Crandall, Ishii, Lions [**37**])**.**
For $i=1,2$, let Ω_i be locally compact subsets of $\mathbb{R}$, and $\Omega=\Omega_1\times\Omega_2$, let u_i be upper semicontinuous in $[0,T]\times\Omega_i$, and $\overline{\mathcal{P}}^{2,+}_{\Omega_i}u_i(t,x)$ the closure of the parabolic superjet of $u_i(t,x)$, and Φ be twice continuously differentiable in a neighborhood of $[0,T]\times\Omega$.
Set

$$w(t,x_1,x_2)=u_1(t,x_1)+u(t,x_2),$$

for $(t,x_1,x_2)\in[0,T]\times\Omega$, and suppose $(\hat{t},\hat{x}_1,\hat{x}_2)\in[0,T]\times\Omega$ is a local maximum of $w-\Phi$ relative to $[0,T]\times\Omega$.
Moreover, let us assume that there is an $r>0$ such that for every $\Delta>0$ there exists a $C=C(\Delta)$ such that for $i=1,2$

$$b_i\le C$$

whenever the following condition is satisfied :

$$|x_i-\hat{x}_i|+|t-\hat{t}|\le r \text{ and } |u_i(t,x_i)|+|q_i|+|L_i|\le\Delta. \tag{4.63}$$

for $(b_i,q_i,L_i)\in\overline{\mathcal{P}}^{2,+}_{\Omega_i}u_i(t,x_i)$.
Then for each $\zeta>0$, there exists $M_i\in\mathcal{S}(1)=\mathbb{R}$ such that

(1) $(b_i,D_{x_i}\Phi(\hat{t},\hat{x}),M_i)\in\overline{\mathcal{P}}^{2,+}_{\Omega_i}u_i(\hat{t},\hat{x})$ for $i=1,2$.

(2)

$$-\Big(\frac{1}{\zeta}+||D^2_{(x_1,x_2)}\Phi(\hat{t},\hat{x})||\Big)I_2\le\begin{pmatrix}M_1 & 0\\ 0 & M_2\end{pmatrix}\le D^2_{(x_1,x_2)}\Phi(\hat{t},\hat{x})+\zeta(D^2_{(x_1,x_2)}\Phi(\hat{t},\hat{x}))^2. \tag{4.64}$$

(3)

$$b_1+b_2=\frac{\partial\Phi(\hat{t},\hat{x})}{\partial t}. \tag{4.65}$$

Now we can state our comparison result.

Theorem 4.3 (Comparison Principle)**.**
If $v_1(t,x,i)$ and $v_2(t,x,i)$ are continuous in (t,x) and are respectively viscosity subsolution and supersolution of the system $(\mathbf{P})$ *with at most linear growth, then*

$$v_1(t,x,i) \le v_2(t,x,i) \quad \textit{for all } (t,x,i) \in [0,T]\times\mathbb{R}\times\mathbb{S}. \tag{4.66}$$

Proof. For $\beta,\ \epsilon,\ \delta,\ \lambda > 0$, we define the auxiliary functions $\varphi : (0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\Xi : (0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{S}\to\mathbb{R}$ by :

$$\varphi(t,x,y) = \frac{\beta}{t} + \frac{1}{2\epsilon}|x-y|^2 + \delta e^{\lambda(T-t)}(|x|^2+|y|^2), \tag{4.67}$$

and

$$\Xi(t,x,y,i) = v_1(t,x,i) - v_2(t,x,i) - \varphi(t,x,y). \tag{4.68}$$

By using the linear growth of v_1 and v_2, we have for each $i \in \mathbb{S}$

$$\lim_{|x|+|y|\to\infty} \Xi(t,x,y,i) = -\infty. \tag{4.69}$$

Then, since v_1 and v_2 are uniformly continuous with respect to (t,x) on each compact subset of $[0,T]\times\mathbb{R}$ and that $\mathbb{S}$ is a finite set we have that Ξ attains its (global) maximum at some finite point belonging to a compact $K \subset (0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{S}$, say, $(t_\epsilon, x_\epsilon, y_\epsilon, X_\epsilon)$.

Writing that $2\Xi(t_\epsilon,x_\epsilon,y_\epsilon,X_\epsilon) \ge \Xi(t_\epsilon,x_\epsilon,x_\epsilon,X_\epsilon) + \Xi(t_\epsilon,y_\epsilon,y_\epsilon,X_\epsilon)$ and using the uniform continuity of v_1, v_2 on K we have

$$\begin{aligned}\frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2 &\le v_1(t_\epsilon,x_\epsilon,i) - v_1(t_\epsilon,y_\epsilon,i) + v_2(t_\epsilon,x_\epsilon,i) - v_2(t_\epsilon,y_\epsilon i)\\ &\le 2C|x_\epsilon - y_\epsilon|.\end{aligned} \tag{4.70}$$

Thus

$$|x_\epsilon - y_\epsilon| \le 2C\epsilon, \tag{4.71}$$

where C is a positive constant independent of $\beta,\epsilon,\delta,\lambda$. From the inequality $\Xi(T,0,0,X_\epsilon) \le \Xi(t_\epsilon,x_\epsilon,y_\epsilon,X_\epsilon)$ and in view of the linear growth for v_1 and v_2, we have :

$$\begin{aligned}\delta(|x_\epsilon|^2+|y_\epsilon|^2) &\le e^{-\lambda(T-t_\epsilon)}\Big(v_1(t_\epsilon,x_\epsilon,i) - v_1(T,0,i) + v_2(T,0,i) - v_2(t_\epsilon,x_\epsilon,i)\Big)\\ &\le e^{-\lambda(T-t_\epsilon)}C_2(1+|x_\epsilon|+|y_\epsilon|).\end{aligned} \tag{4.72}$$

It follows that

$$\frac{\delta(|x_\epsilon|^2+|y_\epsilon|^2)}{(1+|x_\epsilon|+|y_\epsilon|)} \le C_2.$$

Consequently, there exists $C_\lambda > 0$ such that

$$|x_\epsilon| + |y_\epsilon| \le C_\lambda. \tag{4.73}$$

This inequality implies that the sets $\{x_\epsilon,\ \epsilon > 0\}$ and $\{y_\epsilon,\ \epsilon > 0\}$ are bounded by C_λ independent of ϵ.
It follows from inequalities (4.71) and (4.73) that, along a subsequence, $(t_\epsilon, x_\epsilon, y_\epsilon, X_\epsilon)$ converges to $(t_0, x_0, x_0, X_\epsilon) \in (0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}$, as $\epsilon \downarrow 0$.

Now, if $t_\epsilon = T$ then writing that $\Xi(t,x,x,X_\epsilon) \le \Xi(T, x_\epsilon, y_\epsilon, X_\epsilon)$, we have

$$\begin{aligned} v_1(t,x,i) - v_2(t,x,i) - \frac{\beta}{t} - 2\delta e^{\lambda(T-t)}|x|^2 &\le v_1(T,x_\epsilon,i) - v_2(T,y_\epsilon,i) - \frac{\beta}{T} \\ &\quad -\frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^2 - (|x_\epsilon|^2 + |y_\epsilon|^2) \\ &\le v_1(T,x_\epsilon,i) - v_2(T,y_\epsilon,i) \\ &= \Big(v_1(T,x_\epsilon,i) - v_2(T,x_\epsilon,i)\Big) \\ &\quad +\Big(v_2(T,x_\epsilon,i) - v_2(T,y_\epsilon,i)\Big) \\ &\le C_1|x_\epsilon - y_\epsilon|, \end{aligned} \tag{4.74}$$

where the last inequality follows from the uniform continuity of v_2 and by assumption that $v_1(T,x_\epsilon,i) = H(x,i) = v_2(T,x_\epsilon,i)$.
Sending $\beta,\ \epsilon,\ \delta \downarrow 0$ and using estimate (4.71), we have : $v_1(t,x,i) \le v_2(t,x,i)$.

Assume now that $t_\epsilon < T$.
Applying Theorem 4.2 to the function $\varphi(t,x,y)$ at the point $(t_\epsilon, x_\epsilon, y_\epsilon) \in (0,T) \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}$, we can find $p_0, M_\epsilon, N_\epsilon \in \mathbb{R}$ such that :

$$\begin{aligned} &\Big(p_0 - \frac{\beta}{t^2} - \lambda\delta e^{\lambda(T-t_\epsilon)}(|x_\epsilon|^2 + |y_\epsilon|^2), \\ &\qquad\qquad \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) + 2\delta e^{\lambda(T-t_\epsilon)}x_\epsilon, M_\epsilon + 2\delta e^{\lambda(T-t_\epsilon)}\Big) \\ &\qquad\qquad \in \overline{\mathcal{P}}^{2,+}_{\Omega_i} v_1(t,x,i), \\ &\Big(p_0, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) - 2\delta e^{\lambda(T-t_\epsilon)}y_\epsilon, N_\epsilon - 2\delta e^{\lambda(T-t_\epsilon)}\Big) \in \overline{\mathcal{P}}^{2,-}_{\Omega_i} v_2(t,x,i) \end{aligned}$$

and

$$\begin{aligned} -\Big(\frac{1}{\zeta} + ||D^2_{(x,y)}\varphi(t_\epsilon,x_\epsilon,y_\epsilon)||\Big) I_2 \le \begin{pmatrix} M_\epsilon & 0 \\ 0 & -N_\epsilon \end{pmatrix} &\le D^2_{(x,y)}\varphi(t_\epsilon,x_\epsilon,y_\epsilon) \\ &\quad + \zeta(D^2_{(x,y)}\varphi(t_\epsilon,x_\epsilon,y_\epsilon))^2. \end{aligned} \tag{4.75}$$

Note that,

$$M_\epsilon x_\epsilon^2 - N_\epsilon y_\epsilon^2 = (x_\epsilon, y_\epsilon) \begin{pmatrix} M_\epsilon & 0 \\ 0 & -N_\epsilon \end{pmatrix} \begin{pmatrix} x_\epsilon \\ y_\epsilon \end{pmatrix}. \tag{4.76}$$

Thus,

$$M_\epsilon x_\epsilon^2 - N_\epsilon y_\epsilon^2 \leq (x_\epsilon, y_\epsilon)\Bigg[\frac{\epsilon + \zeta(2+4\delta\epsilon e^{\lambda(T-t_\epsilon)})}{\epsilon^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \Big(2\delta + 4\zeta\delta^2\epsilon e^{\lambda(T-t_\epsilon)}\Big)e^{\lambda(T-t_\epsilon)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Bigg]\begin{pmatrix} x_\epsilon \\ y_\epsilon \end{pmatrix}. \tag{4.77}$$

Letting $\delta \downarrow 0$ and taking $\zeta = \frac{\epsilon}{2}$, we obtain

$$\begin{aligned} M_\epsilon x_\epsilon^2 - N_\epsilon y_\epsilon^2 &\leq (x_\epsilon, y_\epsilon)\Bigg[\frac{2}{\epsilon}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\Bigg]\begin{pmatrix} x_\epsilon \\ y_\epsilon \end{pmatrix} \\ &\leq \frac{2}{\epsilon}|x_\epsilon - y_\epsilon|^2. \end{aligned} \tag{4.78}$$

Furthermore, the definition of the viscosity subsolution v_1 and supersolution v_2 implies that

$$\begin{aligned} & r(X_\epsilon)v_1(t_\epsilon, x_\epsilon, X_\epsilon) - \Big(p_0 - \frac{\beta}{t^2} - \lambda\delta e^{\lambda(T-t_\epsilon)}(|x_\epsilon|^2 + |y_\epsilon|^2)\Big) \\ & -x_\epsilon r(X_\epsilon)\Big(\frac{1}{\epsilon}(x_\epsilon - y_\epsilon) + 2x_\epsilon\delta e^{\lambda(T-t_\epsilon)}\Big) - \frac{1}{2}\sigma^2(X_\epsilon)x_\epsilon^2\Big(M_\epsilon + 2\delta e^{\lambda(T-t_\epsilon)}\Big) - \mathbb{A}v_1(t_\epsilon, x_\epsilon, \cdot)(X_\epsilon) \\ & - \mathcal{I}_\eta(t_\epsilon, x_\epsilon, X_\epsilon, \phi_1, \frac{\partial\phi_1}{\partial x}) - \mathcal{I}^\eta(t_\epsilon, x_\epsilon, X_\epsilon, v_1, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) + 2x_\epsilon\delta e^{\lambda(T-t_\epsilon)}) \leq 0, \end{aligned} \tag{4.79}$$

and

$$\begin{aligned} & r(X_\epsilon)v_2(t_\epsilon, y_\epsilon, X_\epsilon) - p_0 - y_\epsilon r(X_\epsilon)\Big(\frac{1}{\epsilon}(x_\epsilon - y_\epsilon) - 2y_\epsilon\delta e^{\lambda(T-t_\epsilon)}\Big) \\ & \quad - \frac{1}{2}\sigma^2(X_\epsilon)y_\epsilon^2\Big(N_\epsilon - 2\delta e^{\lambda(T-t_\epsilon)}\Big) - \mathbb{A}v_2(t_\epsilon, y_\epsilon, \cdot)(X_\epsilon) - \mathcal{I}_\eta(t_\epsilon, y_\epsilon, X_\epsilon, \phi_2, \frac{\partial\phi_2}{\partial x}) \\ & \qquad - \mathcal{I}^\eta(t_\epsilon, y_\epsilon, X_\epsilon, v_2, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) - 2y_\epsilon\delta e^{\lambda(T-t_\epsilon)}) \geq 0, \end{aligned} \tag{4.80}$$

for some $\phi_1, \phi_2 \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$.

Let us define the operator $\mathcal{F}(x, v, \phi, \Xi, Z)$ by

$$\begin{aligned} \mathcal{F}\Big(x, v, \phi, \Xi, Z\Big) \;&:=\; -xr(X_\epsilon)\Xi - \frac{1}{2}\sigma^2(X_\epsilon)Zx^2 - \mathbb{A}v(t_\epsilon, x, \cdot)(X_\epsilon) \\ & \quad -\mathcal{I}_\eta(t_\epsilon, x, X_\epsilon, \phi, \frac{\partial\phi}{\partial x}) - \mathcal{I}^\eta(t_\epsilon, x, X_\epsilon, v, \Xi). \end{aligned} \tag{4.81}$$

By using this operator, the inequalities (4.79) and (4.80) become respectively

$$\begin{aligned} & r(X_\epsilon)v_1(t_\epsilon, x_\epsilon, X_\epsilon) - \Big(p_0 - \frac{\beta}{t^2} - \lambda\delta e^{\lambda(T-t_\epsilon)}(|x_\epsilon^2| + |y_\epsilon^2|)\Big) \\ & \qquad + \mathcal{F}\Big(x_\epsilon, v_1, \phi_1, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) + 2x_\epsilon\delta e^{\lambda(T-t_\epsilon)}, M_\epsilon + 2\delta e^{\lambda(T-t_\epsilon)}\Big) \leq 0, \end{aligned} \tag{4.82}$$

and

$$r(X_\epsilon)v_2(t_\epsilon, y_\epsilon, X_\epsilon) - p_0 + \mathcal{F}\Big(y_\epsilon, v_2, \phi_2, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) - 2y_\epsilon \delta e^{\lambda(T-t_\epsilon)}, N_\epsilon - 2\delta e^{\lambda(T-t_\epsilon)}\Big) \geq 0. \quad (4.83)$$

Subtracting these last two inequalities, we have :

$$\begin{aligned} r(X_\epsilon)\Big[v_1(t_\epsilon, x_\epsilon, X_\epsilon) - v_2(t_\epsilon, y_\epsilon, X_\epsilon)\Big] + \frac{\beta}{t^2} + \lambda\delta e^{\lambda(T-t_\epsilon)}(|x_\epsilon|^2 + |y_\epsilon|^2) \\ \leq -\mathcal{F}\Big(x_\epsilon, v_1, \phi_1, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) + 2x_\epsilon \delta e^{\lambda(T-t_\epsilon)}, M_\epsilon + 2\delta e^{\lambda(T-t_\epsilon)}\Big) \\ + \mathcal{F}\Big(y_\epsilon, v_2, \phi_2, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) - 2y_\epsilon \delta e^{\lambda(T-t_\epsilon)}, N_\epsilon - 2\delta e^{\lambda(T-t_\epsilon)}\Big). \quad (4.84) \end{aligned}$$

However, the right hand side of the inequality (4.84) is equal to

$$\begin{aligned} RHS &= \frac{1}{2}\sigma^2(X_\epsilon)\Big[M_\epsilon x_\epsilon^2 - N_\epsilon y_\epsilon^2 + 2\delta e^{\lambda(T-t_\epsilon)}(|x_\epsilon|^2 + |y_\epsilon|^2)\Big] \quad (A_1) \\ &+ r(X_\epsilon)\Big[\frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2 + 2\delta e^{\lambda(T-t_\epsilon)}(|x_\epsilon|^2 + |y_\epsilon|^2)\Big] \quad (A_2) \\ &+ \Big[\mathbb{A}v_1(t_\epsilon, x_\epsilon, \cdot)(X_\epsilon) - \mathbb{A}v_2(t_\epsilon, y_\epsilon, \cdot)(X_\epsilon)\Big] \quad (A_3) \\ &+ \Big[\mathcal{I}_\eta(t_\epsilon, x_\epsilon, X_\epsilon, \phi_1, \frac{\partial \phi_1}{\partial x}) - \mathcal{I}_\eta(t_\epsilon, y_\epsilon, X_\epsilon, \phi_2, \frac{\partial \phi_2}{\partial x})\Big] \quad (A_4) \\ &+ \Big[\mathcal{I}^\eta(t_\epsilon, x_\epsilon, X_\epsilon, v_1, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) + 2x_\epsilon \delta e^{\lambda(T-t_\epsilon)}) \\ &- \mathcal{I}^\eta(t_\epsilon, y_\epsilon, X_\epsilon, v_2, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) - 2y_\epsilon \delta e^{\lambda(T-t_\epsilon)})\Big]. \quad (A_5) \end{aligned} \quad (4.85)$$

Thus, (4.84) becomes

$$r(X_\epsilon)\Big[v_1(t_\epsilon, x_\epsilon, X_\epsilon) - v_2(t_\epsilon, y_\epsilon, X_\epsilon)\Big] + \frac{\beta}{t^2} + \lambda\delta e^{\lambda(T-t_\epsilon)}(|x_\epsilon|^2 + |y_\epsilon|^2) \leq A_1 + A_2 + A_3 + A_4 + A_5. \quad (4.86)$$

From (4.78),

$$A_1 + A_2 \leq \Big(\frac{2}{\epsilon}|x_\epsilon - y_\epsilon|^2 + 2\delta e^{\lambda(T-t_\epsilon)}(|x_\epsilon|^2 + |y_\epsilon|^2)\Big)\Big(\frac{1}{2}\sigma^2(X_\epsilon) + r(X_\epsilon)\Big). \quad (4.87)$$

Using the Lipschitz condition for v_1 and v_2, we have

$$A_3 \leq 2|x_\epsilon - y_\epsilon|\Big(\sum_{l \in \mathbb{S},\ l \neq X_\epsilon} a_{X_\epsilon l}\Big). \quad (4.88)$$

Also, by remark 4.39

$$\limsup_{\eta \downarrow 0} A_4 = 0.$$

By easy calculations, we have that

$$\begin{aligned} A_5 &= \int_{|z|>\eta} \Big[\Xi(t_\epsilon, x_\epsilon e^z, y_\epsilon e^z, X_\epsilon) - \Xi(t_\epsilon, x_\epsilon, y_\epsilon, X_\epsilon) + (e^z-1)^2\Big(\frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^2 \\ &\quad + \delta e^{\lambda(T-t_\epsilon)}(|x_\epsilon|^2 + |y_\epsilon|^2)\Big)\Big]\rho_{X_\epsilon}(dz) \\ &\leq \Big(\frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^2 + \delta e^{\lambda(T-t_\epsilon)}(|x_\epsilon|^2 + |y_\epsilon|^2)\Big)\int_{|z|>\eta}(e^z-1)^2\rho_{X_\epsilon}(dz) \qquad (4.89) \end{aligned}$$

where for the last inequality, we use the fact that $(t_\epsilon, x_\epsilon, y_\epsilon, X_\epsilon)$ is a maximum point of $\Xi \in (0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}$.
Writing that $\Xi(t,x,x,i) \leq \Xi(t_\epsilon, x_\epsilon, x_\epsilon, X_\epsilon)$ for $i \in \mathbb{S}$ and using the inequality (4.84), we have by noting that $r(X_\epsilon)$ and β are greater than 0 :

$$\begin{aligned} v_1(t,x,i) - v_2(t,x,i) - \frac{\beta}{t} - 2\delta e^{\lambda(T-t)}|x|^2 &\leq v_1(t_\epsilon, x_\epsilon, X_\epsilon) - v_2(t_\epsilon, y_\epsilon, X_\epsilon) - \frac{\beta}{t} \\ &\quad - \frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^2 \delta e^{\lambda(T-t_\epsilon)}(|x_\epsilon|^2 + |y_\epsilon|^2) \\ &\leq \frac{1}{r(X_\epsilon)}\Bigg[\Big(A_1 + A_2 + A_3 + A_4 + A_5\Big) \\ &\quad - \frac{\beta}{t^2} - \lambda\delta e^{\lambda(T-t_\epsilon)}(|x_\epsilon|^2 + |y_\epsilon|^2)\Bigg] \\ &\leq \frac{1}{r(X_\epsilon)}\Big(A_1 + A_2 + A_3 + A_4 + A_5\Big) \\ &\quad - \frac{\lambda\delta}{r(X_\epsilon)} e^{\lambda(T-t_\epsilon)}(|x_\epsilon|^2 + |y_\epsilon|^2) \qquad (4.90) \end{aligned}$$

Sending $\epsilon,\ \eta \downarrow 0$, with the above estimates of (A_1)-(A_2)-(A_3)-(A_4)-(A_5), we obtain :

$$v_1(t,x,i) - v_2(t,x,i) - \frac{\beta}{t} - 2\delta e^{\lambda(T-t)}|x|^2 \leq \frac{2\delta}{r(X_0)}|x_0|^2 e^{\lambda(T-t_0)}(2 + c - \lambda) \quad (4.91)$$

Choose λ sufficiently large positive ($\lambda \geq 2+c$) and send $\beta,\ \delta \downarrow 0$ to conclude that

$$v_1(t,x,i) \leq v_2(t,x,i).$$

This completes the proof. □

The uniqueness of the viscosity solution of the system (**P**) follows immediately from this theorem because any viscosity solution is both a viscosity subsolution and supersolution.

4.3.4. Existence of a viscosity solution for the system (P)

In this last section, we give a result for the existence of the viscosity solution of the system (**P**) based on the Perron's method. We make the remark that, by using the change of variables $(t, \xi) = (T - s, \ln x)$ the terminal value problem

(**P**) can be converted to an initial value problem (or Cauchy problem) ($\mathbf{P_0}$) [see Appendix]. Thus, we can see our problem as a particular case of that discussed in Biswa *et al.*[**16**]. Before giving the main proposition of this section, we need some preparatory results.

Proposition 4.1. *Let $\mathcal{S}$ be a non-empty family of subsolutions of* ($\mathbf{P_0}$) *and for* $i \in \mathbb{S}$,

$$v(t,\xi,i) := \sup\{u(t,\xi,i) \ : \ u(\cdot,\cdot,i) \in \mathcal{S}\} \ for \ (t,\xi) \in (0,T] \times \mathbb{R}.$$

Suppose $v^*(t,\xi,i) < \infty$ for $(t,\xi,i) \in (0,T] \times \mathbb{R} \times \mathbb{S}$, then $v^*(\cdot,\cdot,i)$ is a subsolution of ($\mathbf{P_0}$).

Proof. We recall that v^* denote the upper semicontinuous envelope of v. For $i \in \mathbb{S}$, we want to prove that $v^*(\cdot,\cdot,i)$ is a subsolution of ($\mathbf{P_0}$). First, we check that the initial condition is satisfied. To do so, we use a barrier argument.
For every $z \in \mathbb{R}$ and $\epsilon > 0$, define $\phi_{z,\epsilon}(\xi,i) = \widetilde{H}(\xi,i) + \kappa^i(|\xi - z|^2 + \epsilon)^{1/2}$, where κ^i is the Lipschitz constant of $\widetilde{H}(\xi,i)$. It follows that

$$\phi_{z,\epsilon}(\xi,i) \geq \widetilde{H}(z,i),$$

for all $\xi, \ z \in \mathbb{R}, \ i \in \mathbb{S}, \ \epsilon > 0$.
For a some constant $A_\epsilon \geq 0$, it is easy to verifies that $U_{z,\epsilon}(t,\xi,i) := A_\epsilon t + \phi_{z,\epsilon}(\xi,i)$ is a supersolution of ($\mathbf{P_0}$). Therefore, by using the comparison principle and the definition of v,

$$v(t,\xi,i) \leq U_{z,\epsilon}(t,\xi,i) \ for \ all \ \xi, z \in \mathbb{R}, \ i \in \mathbb{S}, \ \epsilon > 0,$$

and hence $v^*(t,\xi,i) \leq (U_{z,\epsilon})^*(\xi,i) = U_{z,\epsilon}(t,\xi,i)$.
So, by setting $t = 0$ and minimizing with respect to z, ϵ, the initial condition follows :

$$v^*(0,\xi,i) \leq \inf_{z,\epsilon}(U_{z,\epsilon})^*(0,\xi,i) = \inf_{z,\epsilon} \phi_{z,\epsilon}(\xi,i) = \widetilde{H}(\xi,i).$$

Next, we need to prove that for all function $\phi \in \mathcal{C}^{1,2}((0,T]\times\mathbb{R})$, if $\max_{(0,T]\times\mathbb{R}}[v^*(\cdot,\cdot,i) - \phi] = [v^*(\cdot,\cdot,i) - \phi](t,\overline{\xi})$
then at $(t,\overline{\xi})$,

$$\mathcal{H}\Big(i,t,\overline{\xi},v^*(t,\overline{\xi},i),v^*(t,\overline{\xi},\cdot),\frac{\partial\phi(t,\overline{\xi},i)}{\partial t},\frac{\partial\phi(t,\overline{\xi},i)}{\partial \xi},\frac{\partial^2\phi(t,\overline{\xi},i)}{\partial \xi^2},\underline{\mathcal{I}}(t,\overline{\xi},i,v^*(\cdot),\phi)\Big) \leq 0.$$

Without the loss of generality, assume that $[v^*(\cdot,\cdot,i) - \phi](t,\overline{\xi}) = 0$, we can always replace ϕ with $\phi + [v^*(\cdot,\cdot,i) - \phi](t,\overline{\xi})$ to achieve this ;
Set now $\psi(t,\xi) = \phi(t,\xi) + |\xi - \overline{\xi}|^4 + |t - t|^2$, then $[v^*(\cdot,\cdot,i) - \psi]$ attains its strict maximum in $(0,T] \times \mathbb{R}$ at $(t,\overline{\xi})$ so

$$(v^*(\cdot,\cdot,i) - \psi)(t,\xi) + |\xi - \overline{\xi}|^4 + |t - t|^2 = [v^*(\cdot,\cdot,i) - \psi](t,\xi) \leq 0 = [v^*(\cdot,\cdot,i) - \psi](t,\overline{\xi})$$

and

$$[v^*(\cdot,\cdot,i) - \psi](t,\xi) \leq -|\xi - \overline{\xi}|^4 - |t - t|^2.$$

By the definition of v^*, there is a sequence of $(t_k, \xi_k) \in [0,T] \times \mathbb{R}$, $(t_k, \xi_k) \to (t, \overline{\xi})$ such that

$$\lim_{k\to\infty} a_k = [v^*(\cdot,\cdot,i) - \phi](t, \overline{\xi}),$$

with $a_k = [v^*(\cdot,\cdot,i) - \phi](t_k, \xi_k)$.

For $(t,\xi) \in [0,T) \times \mathbb{R}$, let $v(t,\xi,i) := \sup\{u(t,\xi,i) : u(\cdot,\cdot,i)$ is a subsolution of problem $(\mathbf{P_0})\}$ then there is u_k subsolution of problem $(\mathbf{P_0})$ such that

$$v(t_k, \xi_k, i) - \frac{1}{k} < u_k(t_k, \xi_k, i)$$

and

$$a_k - \frac{1}{k} \leq [u_k(\cdot,\cdot,i) - \phi](t_k, \xi_k, i) \leq [u_k^*(\cdot,\cdot,i) - \phi](t_k, \xi_k, i).$$

But $u_k(\cdot,\cdot,i) \leq v(\cdot,\cdot,i)$ so

$$[u_k^*(\cdot,\cdot,i) - \psi](t,\xi) \leq -|\xi - \overline{\xi}|^4 - |t - t|^2.$$

Noting that $(0,T] \times \mathbb{R}$ is locally compact, there is a compact neighborhood $\mathcal{V}$ of $(t, \overline{\xi})$ such that $[u_k^*(\cdot,\cdot,i) - \psi]$ is upper semicontinuous on $(0,T] \times \mathbb{R}$ and has upper bound, then its attains maximum on $\mathcal{V}$ at $(s_k, \zeta_k) \in \mathcal{V}$, so

$$a_k - \frac{1}{k} < [u_k^*(\cdot,\cdot,i) - \psi](t_k, \xi_k) \leq [u_k^*(\cdot,\cdot,i) - \psi](s_k, \zeta_k) \leq -|\zeta_k - \overline{\xi}|^4 - |s_k - t|^2,$$

and we have by noting that $\lim_{k\to\infty} a_k = 0$, $(s_k, \zeta_k) \to (t, \overline{\xi})$ by considering the extreme terms in the last inequality. Therefore,

$$\lim_{k\to\infty} [u_k^*(\cdot,\cdot,i) - \psi](s_k, \zeta_k) = 0,$$

thus,

$$\lim_{k\to\infty} u_k^*(s_k, \zeta_k, i) = \lim_{k\to\infty} \psi(s_k, \zeta_k),$$

and finally,

$$\lim_{k\to\infty} \psi(s_k, \zeta_k) = \psi(t, \overline{\xi}) = v^*(t, \overline{\xi}, i).$$

Since that $u_k(\cdot,\cdot,i)$ is a subsolution of $(\mathbf{P_0})$, we have at $(t, \overline{\xi})$

$$\mathcal{H}\Big(i, t, \overline{\xi}, u_k(t, \overline{\xi}, i), u_k(t, \overline{\xi}, \cdot), \ldots$$
$$\frac{\partial \psi(t, \overline{\xi}, i)}{\partial t}, \frac{\partial \psi(t, \overline{\xi}, i)}{\partial \xi}, \frac{\partial^2 \psi(t, \overline{\xi}, i)}{\partial \xi^2}, \underline{\mathcal{I}}(t, \overline{\xi}, i, u_k(\cdot), \psi)\Big) \leq 0,$$

we then get by letting $k \to \infty$ and using the continuity of operator $\mathcal{H}\Big(i,\cdot\Big)$ that

$$\mathcal{H}\Big(i,t,\overline{\xi},v^*(t,\overline{\xi},i),v^*(t,\overline{\xi},\cdot),\ldots \\ \frac{\partial\psi(t,\overline{\xi},i)}{\partial t},\frac{\partial\psi(t,\overline{\xi},i)}{\partial \xi},\frac{\partial^2\psi(t,\overline{\xi},i)}{\partial \xi^2},\underline{\mathcal{I}}(t,\overline{\xi},i,v^*(\cdot),\psi)\Big) \leq 0.$$

But at $(t,\overline{\xi})$, $\frac{\partial\psi}{\partial t}=\frac{\partial\phi}{\partial t}$, $\frac{\partial\psi}{\partial\xi}=\frac{\partial\phi}{\partial\xi}$,$\frac{\partial^2\psi}{\partial\xi^2}=\frac{\partial^2\phi}{\partial\xi^2}$, so

$$\mathcal{H}\Big(i,t,\overline{\xi},v^*(t,\overline{\xi},i),v^*(t,\overline{\xi},\cdot),\ldots \\ \frac{\partial\phi(t,\overline{\xi},i)}{\partial t},\frac{\partial\phi(t,\overline{\xi},i)}{\partial \xi},\frac{\partial^2\phi(t,\overline{\xi},i)}{\partial \xi^2},\underline{\mathcal{I}}(t,\overline{\xi},i,v^*(\cdot),\phi)\Big) \leq 0.$$

This completes the proof that $v^*(\cdot,\cdot,i)$ is a subsolution.□

Lemma 4.3. *Let $\overline{v}:(0,T]\times\mathbb{R}\to\mathbb{R}$ be a supersolution of $(\mathbf{P_0})$. For $i\in\mathbb{S}$, let*

$$\mathcal{S}_{\overline{v},i} := \{v : v(\cdot,\cdot,i) \textit{ is a subsolution of } (\mathbf{P_0}) \textit{ and } v(\cdot,\cdot,i)\leq\overline{v}\}$$

If $v(\cdot,\cdot,i)\in\mathcal{S}_{\overline{v},i}$ and $v(\cdot,\cdot,i)\geq w(\cdot,\cdot)$ for all $w(\cdot,\cdot)\in\mathcal{S}$, then $v_*(\cdot,\cdot,i)$ is a supersolution of $(\mathbf{P_0})$.

Proof. For $i\in\mathbb{S}$. Let $v_*(\cdot,\cdot,i)$ be the lower semicontinuous envelope of $v(\cdot,\cdot,i)$, where $v(\cdot,\cdot,i)\in\mathcal{S}_{\overline{v},i}$ and such that $v(t,\xi,i)\geq w(t,\xi)$, $(t,\xi)\in(0,T]\times\mathbb{R}$ for all $w(\cdot,\cdot)\in\mathcal{S}$.

We claim that $v_*(\cdot,\cdot,i)$ is a supersolution of $(\mathbf{P_0})$.

We start by checking the initial condition. For $z\in\mathbb{R}$ and $\epsilon>0$, let $\Phi_{z,\epsilon}(\xi,i)=\widetilde{H}(\xi,i)-\kappa(|\xi-z|^2+\epsilon)^{1/2}$ and $V_{z,\epsilon}(t,\xi,i)=-A^\epsilon t+\Phi_{z,\epsilon}(\xi,i)$ where $\kappa=\max_i\kappa^i$ and A^ϵ is a constant. Note that $\Phi_{z,\epsilon}(\xi,i)\leq\widetilde{H}(\xi,i)$ for all $\xi,\ z,\ \epsilon$.

It is straightforward to verify that there is a constant A^ϵ such that $V_{z,\epsilon}(\cdot,\cdot,i)$ is a subsolution of $(\mathbf{P_0})$. Therefore, $V_{z,\epsilon}(t,\xi,i)\leq v(t,\xi,i)$. It follows that $V_{z,\epsilon}(t,\xi,i)\leq v_*(t,\xi,i)$ and hence the initial condition holds :

$$v_*(0,\xi,i)\geq\sup_{\epsilon,z}V_{z,\epsilon}(0,\xi,i)=\sup_{\epsilon,z}\Phi_{z,\epsilon}(\xi,i)=\widetilde{H}(\xi,i).$$

We continue with proving that for a function-test $\psi\in\mathcal{C}^{1,2}((0,T]\times\mathbb{R})$ the inequality

$$\mathcal{H}\Big(i,\overline{t},\overline{\xi},v_*(\overline{t},\overline{\xi},i),v_*(\overline{t},\overline{\xi},\cdot),\frac{\partial\psi(\overline{t},\overline{\xi})}{\partial t},\frac{\partial\psi(\overline{t},\overline{\xi})}{\partial\xi},\frac{\partial^2\psi(\overline{t},\overline{\xi})}{\partial\xi^2},\underline{\mathcal{I}}(\overline{t},\overline{\xi},i,v_*(\cdot),\psi)\Big)\geq 0,$$

holds at some point $(\overline{t},\overline{\xi})\in(0,T]\times\mathbb{R}$.

By contradiction.

We suppose that there exist a point $(t_0, \xi_0) \in (0,T] \times \mathbb{R}$ and a function-test $\phi \in \mathcal{C}^{1,2}((0,T] \times \mathbb{R})$ such that

$$\min_{(0,T]\times\mathbb{R}} [v_*(\cdot,\cdot,i) - \phi] = [v_*(\cdot,\cdot,i) - \phi](t_0,\xi_0),$$

and

$$\mathcal{H}\Big(i, t_0, \xi_0, v_*(t_0,\xi_0,i), v_*(t_0,\xi_0,\cdot), \frac{\partial\phi(t_0,\xi_0)}{\partial t}, \frac{\partial\phi(t_0,\xi_0)}{\partial \xi}, \frac{\partial^2\phi(t_0,\xi_0)}{\partial \xi^2}, \underline{\mathcal{I}}(t_0,\xi_0,i,v_*(\cdot),\phi)\Big) < 0. \quad (4.92)$$

We may assume $[v_*(\cdot,\cdot,i) - \phi](t_0,\xi_0) = 0$, since the function ϕ can be modified as $\phi + [v_*(\cdot,\cdot,i) - \phi](t_0,\xi_0)$.

Let us prove that $v_*(t_0,\xi_0,i) < \overline{v}(t_0,\xi_0)$. It is the case, otherwise as $v_*(t_0,\xi_0,i) \leq \overline{v}(t_0,\xi_0)$ we will have $v_*(t_0,\xi_0,i) = \overline{v}(t_0,\xi_0)$ and thus $\overline{v} - \phi$ has a local minimum at (t_0,ξ_0). But, Inequality (4.92) would contradict the fact $\overline{v}$ is a supersolution of $(\mathbf{P_0})$. Then $v_*(t_0,\xi_0,i) < \overline{v}(t_0,\xi_0)$.

Now, for $r > 0$, $\epsilon > 0$ such that $\overline{B}_r(t_0,\xi_0) \subset (0,T] \times \mathbb{R}$ (closed ball centered at (t_0,ξ_0) and

i) $\mathcal{H}\Big(i, t, \xi, \phi(t,\xi) + \epsilon, \phi(t,\xi) + \epsilon, \frac{\partial\phi(t,\xi)}{\partial t}, \frac{\partial\phi(t,\xi)}{\partial \xi}, \frac{\partial^2\phi(t,\xi)}{\partial \xi^2}, \underline{\mathcal{I}}(t,\xi,\phi(t,\cdot)+\epsilon,\phi)\Big) \leq 0$, $\forall (t,\xi) \in B_r(t_0,\xi_0)$.
(This is possible, thanks 4.92 and by continuity of ϕ) ;

ii) $[v_*(\cdot,\cdot,i) - \phi](t,\xi) > \epsilon \; \forall (t,\xi) \in \partial B_r((t_0,\xi_0))$.
(This is possible because $[v_*(\cdot,\cdot,i) - \phi]$ has a strict local minimum at (t_0,ξ_0) and $v_*(t_0,\xi_0,i) = \phi(t_0,\xi_0)$) ;

iii) $\phi(t,\xi) < \overline{v}(t,\xi) - \epsilon \; \forall (t,\xi) \in B_r(t_0,\xi_0)$.
(This is possible because $\overline{v}$ is lower semicontinuous and $\phi(t_0,\xi_0) = v_*(t_0,\xi_0,i) < \overline{v}(t_0,\xi_0)$).

Let

$$z(t,\xi,i) := \begin{cases} v^*(t,\xi,i) & \text{if } (t,\xi) \in (0,T] \times \mathbb{R} \backslash B_r(t_0,\xi_0) \\ \max\{v^*(t,\xi,i); \phi(t,\xi) + \epsilon\} & \text{if } (t,\xi) \in B_r(t_0,\xi_0). \end{cases} \quad (4.93)$$

We claim that $z(\cdot,\cdot,i)$ is a subsolution of $(\mathbf{P_0})$.

Let ψ a test-function such that $z(\cdot,\cdot,i) - \psi$ has a local maximum at $(t_1,\xi_1) \in (0,T] \times \mathbb{R}$. If $z(t_1,\xi_1,i) = v^*(t_1,\xi_1,i)$ then $v^*(\cdot,\cdot,i) - \psi$ has also a local maximum at (t_1,ξ_1) and thus,

$$\mathcal{H}\Big(i, t_1, \xi_1, z(t_1,\xi_1,i), z(t_1,\xi_1,\cdot), \frac{\partial\phi(t_1,\xi_1)}{\partial t}, \frac{\partial\phi(t_1,\xi_1)}{\partial \xi}, \frac{\partial^2\phi(t_1,\xi_1)}{\partial \xi^2}, \underline{\mathcal{I}}(t_1,\xi_1,i,z(\cdot),\phi)\Big) \leq 0. \quad (4.94)$$

because $v^*(\cdot,\cdot,i)$ is a subsolution of $(\mathbf{P_0})$ as proven in Proposition 4.1.

Suppose now that $z(t_1,\xi_1,i) > v^*(t_1,\xi_1,i)$. Then, from the definition of $z(\cdot,\cdot,i)$, we have that $(t_1,\xi_1) \in \overline{B}_r(t_0,\xi_0)$.

Note that $z(\cdot,\cdot,i) = \max\{v^*(\cdot,\cdot,i);\phi+\epsilon\}$ on $\overline{B}_r(t_0,\xi_0)$, thus $z(t_1,\xi_1,i) = \phi(t_1,\xi_1)+\epsilon$, and $ii)$ implies that $(t_1,\xi_1) \in B_r(t_0,\xi_0)$.
Therefore, $\phi+\epsilon-\psi$ has a local maximum at (t_1,ξ_1) and this implies

$$\frac{\partial\psi}{\partial t} = \frac{\partial\phi}{\partial t},\ \frac{\partial\psi}{\partial\xi} = \frac{\partial\phi}{\partial\xi},\ \frac{\partial^2\psi}{\partial\xi^2} \geq \frac{\partial^2\phi}{\partial\xi^2},$$

$$\phi(t_1,\xi_1+y) - \phi(t_1,\xi_1) \leq \psi(t_1,\xi_1+y) - \psi(t_1,\xi_1),$$

for y such that $(t_1,\xi_1+y) \in B_r(t_0,\xi_0)$. Hence, by $i)$

$$\frac{\partial\psi(t_1,\xi_1)}{\partial t} - \Big(r(i) - \frac{1}{2}\sigma^2(i)\Big)\frac{\partial\psi(t_1,\xi_1)}{\partial\xi} - \frac{1}{2}\sigma^2(i)\frac{\partial^2\psi(t_1,\xi_1)}{\partial\xi^2} + r(i)\psi(t_1,\xi_1) - \underline{\mathcal{I}}(t_1,\xi_1,i,\psi(\cdot),\psi) \leq 0. \quad (4.95)$$

This complete the proof that $z(\cdot,\cdot,i)$ is a subsolution of $(\mathbf{P_0})$.
We can now provide the proof of the lemma since $z(\cdot,\cdot,i)$ is a subsolution satisfying

$$z_*(t,\xi,i) \geq \sup\{\phi(t,\xi)+\epsilon, v_*(t,\xi,i)\} = \phi(t,\xi)+\epsilon \geq v_*(t,\xi,i)+\epsilon$$

where the last inequality comes from the definition of $v_*(\cdot,\cdot,i)$.
Thus, $z(s,y,i) > v(s,t,i)$ for some $(s,y) \in (0,T] \times \mathbb{R}$. This contradicts the definition of $v(\cdot,\cdot,i)$, so $v_*(\cdot,\cdot,i)$ is a supersolution of $(\mathbf{P_0})$. □
Now, we can state the main result on the existence of viscosity solution of problem $(\mathbf{P_0})$.

Theorem 4.4 (Existence)**.**
Assume conditions 4.20-4.21. Let $\overline{u},\overline{v} : [0,T) \times \mathbb{R} \to \mathbb{R}$*, two bounded continuous functions which are respectively sub- and supersolutions of the system* $(\mathbf{P_0})$ *and such that* $\overline{u} \leq \overline{v}$*. Then there exists a bounded continuous viscosity solution* u *to the the system* $(\mathbf{P_0})$ *satisfying* $\overline{u} \leq u \leq \overline{v}$*.*

Proof. We will use Perron's method. As in Lemma 4.3, for $i \in \mathbb{S}$ we set

$$\mathcal{S}_{\overline{v},i} := \{v : v(\cdot,\cdot,i) \textit{ is a subsolution of } (\mathbf{P_0}) \textit{ and } v(\cdot,\cdot,i) \leq \overline{v}\}.$$

We have that $\overline{u} \in \mathcal{S}_{\overline{v},i}$, so $\mathcal{S}_{\overline{v},i} \neq \emptyset$.
Define

$$u(\cdot,\cdot,i) = \sup\{v : v(\cdot,\cdot,i) \in \mathcal{S}_{\overline{v},i}\},$$

by Proposition 4.1, $u^*(\cdot,\cdot,i)$ is a subsolution of $(\mathbf{P_0})$, so $u^*(\cdot,\cdot,i) \in \mathcal{S}_{\overline{v},i}$ since $u^*(\cdot,\cdot,i) \leq \overline{v}$. Then by Lemma 4.3, $u_*(\cdot,\cdot,i)$ is a supersolution of $(\mathbf{P_0})$ and we have $\overline{u} \leq u_*(\cdot,\cdot,i) \leq u^*(\cdot,\cdot,i) \leq \overline{v}$. Since,by the comparison principle, $u^*(\cdot,\cdot,i) \leq u_*(\cdot,\cdot,i)$ we have that $u^*(\cdot,\cdot,i) = u_*(\cdot,\cdot,i)$ This concludes the proof. □

4.4. Concluding Remarks

We have characterized in this paper the value of an European option under the Markov-modulated exponential Lévy model. In particular, we have shown that it is the unique viscosity solution of a system of coupled linear integro-partial differential equations. The present work is a prerequisite for finding the numerical procedures to approximate the option price. Our future work looks to this direction with in mind the model calibration problem.

Acknowledgment

We wish to thank here my PhD advisor Manuel Morales and Mustapha Pemy for helpful comments and suggestions.

Appendix

$$(\mathbf{P_0})\begin{cases} \mathcal{H}\Big(i,t,\xi,f(t,\xi,i),f(t,\xi,\cdot),\frac{\partial f(t,\xi,i)}{\partial t},\frac{\partial f(t,\xi,i)}{\partial \xi},\frac{\partial^2 f(t,\xi,i)}{\partial \xi^2},\underline{\mathcal{I}}(t,\xi,i,f(\cdot),h)\Big) & = 0 \\ f(0,\xi,i) & = \widetilde{H}(\xi,i) \\ \quad \textit{for } t\in(0,T],\ i\in\mathbb{S},\ \xi\in\mathbb{R} \textit{ and } h(t,\cdot,i)\in\mathcal{C}^2(\mathbb{R}). \end{cases} \tag{4.96}$$

Here,

$$\begin{aligned} &\mathcal{H}\Big(i,t,\xi,f(t,\xi,i),f(t,\xi,\cdot),\frac{\partial f(t,\xi,i)}{\partial t},\frac{\partial f(t,\xi,i)}{\partial \xi},\frac{\partial^2 f(t,\xi,i)}{\partial \xi^2},\underline{\mathcal{I}}(t,\xi,i,f(\cdot),h)\Big) \\ &:= \frac{\partial f(t,\xi,i)}{\partial t} - (r(i)-\frac{1}{2}\sigma^2(i))\frac{\partial f(t,\xi,i)}{\partial \xi} - \frac{1}{2}\sigma^2(i)\frac{\partial^2 f(t,\xi,i)}{\partial \xi^2} + r(i)f(t,\xi,i) \\ &\qquad - \mathbb{A}f(t,\xi,\cdot)(i) - \underline{\mathcal{I}}(t,\xi,i,f(\cdot),h). \end{aligned} \tag{4.97}$$

Chapitre 5

NUMERICAL ISSUES AROUND THE REGIME-SWITCHING EXPONENTIAL-LÉVY MODEL : PRICING AND CALIBRATION

Dans ce chapitre, il est question de la calibration d'un modèle financier. Ce problème consiste à la détermination des paramètres du modèle qui le rendent consistant avec les observations du marché. Il s'agit de la démarche inverse à celle de l'évaluation des prix d'options. Notre étude procède au préalable à la recension des techniques de Fourier utilisées pour déterminer le prix d'une option Européenne dans un modèle exponentiel-Lévy avec changements de régime. Une application est présentée pour illustrer l'applicabilité des résultats théoriques.

5.1. Introduction

In this chapter, we deal with the numerical problems arising in the practical implementation of a regime-switching exponential-Lévy model. These problems mainly concern the numerical valuation of an option when the model parameters are known, and its reverse, the calibration of the model parameters consistent with option prices observed in the market. More particularly, this exploratory study aims at exposing the different (Fourier) approaches of solution found in the literature.

Determining the theoretical price of a liquid option [1], i.e., the price obtained from the model, is necessary to calibrate the model to market data. In Chapter 4, we derived a system of integro-differential equations satisfied by the price of an European option in a regime-switching exponential-Lévy model. In general, such a system does not admit analytical solutions, therefore we often resort to numerical methods in order to obtain approximate solutions (See, e.g., [**20**]). In this regard, numerical methods commonly used in financial engineering are finite differences schemes (see [**125**] and [**22**] and references therein) and finite elements methods (see [**1**]). However, these techniques become tricky in the case of this model, partly because of the presence of an integral term. Thus, methods

[1] e.g., a European option.

based on direct calculation of the pricing operator given as the expectation of a random quantity have proved to be suitable. These methods include Monte Carlo techniques and methods based on Fourier transform. The Monte Carlo techniques (see chapter 2) have the advantage that they can be applied to most types of situation (different models and payoffs), but their major drawback is that they are time-machine consuming and increase the computational complexity. This is a handicap especially in regard to the calibration to market data and in this case, Fourier methods provide a comparative advantage. At this point, we should remark that the literature which addresses these issues is quite limited. One can cite Chourdakis [**30**] which in a framework similar to ours proposes to use the QUAD algorithm to determine prices of exotic options and Liu et *al.* [**92**] who adapt the method of Carr and Madan to determine the price of a European option in a regime switching Black-Scholes model. Recently, Jackson et *al.* [**82**] proposed an algorithm based on the Fourier transform of the system of equations (4.17)-(4.18).

In Section 5.2, we present the algorithm developed in [**82**] and we also describe how the method of Carr and Madan can be applied in our context. In Section 5.3, we address the calibration problem with a few examples.

5.2. Fourier Methods for an European Option Valuation

In this section, we describe the state of art on Fourier methods for option pricing in a regime-switching exponential-Lévy model. Firstly, we begin by recalling some basics on the Fourier transform as we can see in any calculus book.

5.2.1. Basics on the Fourier Transform

Definition 5.1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be an* L^1*-integrable function. The* Fourier transform *of* f *denoted by* $\hat{f}$ *is the function defined by*

$$\widehat{f}(\omega) = \Im[f](\omega) := \int_{-\infty}^{+\infty} f(x)e^{\mathbf{i}\omega x}dx, \quad \omega \in \mathbb{R}. \tag{5.1}$$

When $\hat{f}$ *is* L^1*-integrable then one can retrieve the original by taking the inverse Fourier transform as*

$$f(x) = \Im^{-1}[\hat{f}](x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)e^{-\mathbf{i}\omega x}d\omega \quad a.s.\ x \in \mathbb{R}. \tag{5.2}$$

The main properties of the Fourier transform are given below.

(1) $\hat{f}$ *is a bounded and continuous function.*

(2) $\widehat{\frac{\partial^n f}{\partial x^n}}(\omega) = (\mathbf{i}\omega)^n \hat{f}(\omega)$

(3) $\widehat{f * g}(\omega) = \hat{f}(\omega)\hat{g}(\omega)$

5.2.2. Algorithm of Jackson, et *al.* [82]

The aim is to solve the system of nonlinear integro-differential equations satisfied by the European option

$$\begin{aligned}\frac{\partial V}{\partial t}(t,T,S(t^-),k) &+ r_k S(t^-)\frac{\partial V}{\partial S}(t,T,S(t^-),k)+\frac{1}{2}\sigma_k^2 S^2(t^-)\frac{\partial^2 V}{\partial S^2}(t,T,S(t^-),k)\\ &- r_k V(t,T,S(t^-),k)+\mathbb{A}V(t,T,S(t^-),.)(k)\\ &+ \int_{\mathbb{R}\setminus\{0\}}\Big[V(t,T,S(t^-)e^z,k)-V(t,T,S(t^-),k)\\ &- S(t^-)(e^z-1)\frac{\partial V}{\partial S}(t,T,S(t^-),k)\Big]\rho_k(dz)\\ &= 0, \end{aligned} \tag{5.3}$$

for $k=1,2,...,M$ associated to the terminal condition

$$V(T,T,S(T),k)=H(S_0e^{Y_T},X(T)=k),\quad k=1,2,...,M. \tag{5.4}$$

By supposing the L^1-integrability of $S\mapsto V(t,T,S,k)$ we have that its Fourier transform $\omega\mapsto \widehat{V}_k(t,\omega)$ satisfies

$$\begin{aligned}\frac{\partial \widehat{V}_k}{\partial t}(t,\omega) &+ \mathbf{i}\omega r_k\widehat{V}_k(t,\omega)+\frac{1}{2}\sigma_k^2\omega^2\widehat{V}_k(t,\omega)\\ &- r_k\widehat{V}_k(t,\omega)+\mathbb{A}\widehat{V}_k(t,\omega)\\ &+ \int_{\mathbb{R}\setminus\{0\}}\Big[e^{\mathbf{i}\omega z}-1-\mathbf{i}\omega(e^z-1)\Big]\widehat{V}_k(t,\omega)\rho_k(dz)=0\end{aligned} \tag{5.5}$$

and

$$\widehat{V}_k(T,\omega)=\widehat{H}(\omega),\quad k=1,2,...,M. \tag{5.6}$$

Setting

$$\underline{\widehat{\mathbf{V}}}(t,\omega)=(\widehat{V}_1(t,\omega),\widehat{V}_2(t,\omega),...,\widehat{V}_M(t,\omega))'$$

and

$$\underline{\mathbf{\Psi}}(\omega)=(\Psi^{(1)}(\omega),\Psi^{(2)}(\omega),...,\Psi^{(M)}(\omega))'$$

where

$$\Psi^{(k)}(\omega)=\mathbf{i}\omega r_k-\frac{1}{2}\sigma_k^2\omega^2+\int_{-\infty}^{+\infty}\Big[e^{\mathbf{i}\omega z}-1-\mathbf{i}\omega(e^z-1)\Big]\rho_k(dz) \tag{5.7}$$

denotes the characteristic exponent (under $\mathbb{Q}$) of the Lévy process $Y^{(k)}$ when the Markov chain state is $\mathbf{e}_k$.

Equations (5.5)-(5.6) can be written as

$$\begin{cases}\partial_t\ \underline{\widehat{\mathbf{V}}}(t,\omega)+(\mathbb{A}+\mathbf{Diag}(\underline{\mathbf{\Psi}}-\underline{\mathbf{r}}))\underline{\widehat{\mathbf{V}}}(t,\omega) = 0\\ \underline{\widehat{\mathbf{V}}}(T,\omega)=\widehat{H}(\omega)\cdot\underline{\mathbf{1}}\end{cases} \tag{5.8}$$

Thus, we have a linear dynamical system in $\underline{\widehat{\mathbf{V}}}(\cdot,\omega)$ with terminal condition which can be solved easily. Indeed, for any $t \in \mathbb{T}$, one can verify that the solution at time t of (5.8) is given by

$$\underline{\widehat{\mathbf{V}}}(t,\omega) = \exp\left\{\Big(\mathbb{A} + \mathbf{Diag}(\underline{\mathbf{\Psi}}(\omega) - \underline{\mathbf{r}})\Big)(T-t)\right\} \cdot \widehat{H}(\omega) \cdot \underline{\mathbf{1}}. \tag{5.9}$$

In general, we have by the *flow property*

$$\underline{\widehat{\mathbf{V}}}(s,\omega) = \exp\left\{\Big(\mathbb{A} + \mathbf{Diag}(\underline{\mathbf{\Psi}}(\omega) - \underline{\mathbf{r}})\Big)(t-s)\right\} \cdot \underline{\widehat{\mathbf{V}}}(t,\omega), \quad 0 \le s < t \in \mathbb{T}. \tag{5.10}$$

Taking the inverse Fourier transform in Equation 5.10, we obtain

$$\underline{\mathbf{V}}(s,T,Y) = \Im^{-1}\Big[\exp\left\{\Big(\mathbb{A} + \mathbf{Diag}(\underline{\mathbf{\Psi}}(\omega) - \underline{\mathbf{r}})\Big)(t-s)\right\} \cdot \underline{\widehat{\mathbf{V}}}(t,\omega)\Big](Y), \quad 0 \le s < t \in \mathbb{T} \tag{5.11}$$

with $\underline{\mathbf{V}}(s,T,Y) = \Big(V(s,T,Y,\mathbf{e}_1), V(s,T,Y,\mathbf{e}_2), ..., V(s,T,Y,\mathbf{e}_M)\Big)^{'}$.

This last result is the basis of the algorithm developed in **[82]** which we describe in the following.

• *"Fourier Space Time(FST)" Algorithm*

We begin by discretizing the continuous-time Markov chain X. To do so, we define a partition of the time domain $\mathbb{T} = [0,T]$ in intervals $\{(t_n, t_{n+1}] : t_n = n\Delta t,\ n \in \mathbb{N}\}$ with mesh size Δt such that X is held constant in each interval and switch from one to another state with transition probabilities

$$P_{kl} := \begin{cases} 1 + a_{ll}\Delta t, & k = l \\ a_{kl}\Delta t, & \textit{otherwise.} \end{cases} \tag{5.12}$$

Using the martingale property of $\{\widetilde{V}(t,T,Y_t,X_t) : t \in \mathbb{T}\}$ and the fact that $V(t_n,T,Y_{t_n},X_{t_n}) = e^{-r(X_{t_n})}\widetilde{V}(t_n,T,Y_{t_n},X_{t_n})$ on $(t_n, t_{n+1}]$, we obtain

$$\begin{aligned} V(t_n,T,Y_{t_n},X_{t_n}) &= E^{\mathbb{Q}}\Big[V(t_{n+1},T,Y_{t_{n+1}},X_{t_{n+1}})\Big|\mathcal{F}^X_{t_n} \vee \mathcal{F}^Y_{t_n}\Big] \\ &= E^{\mathbb{Q}}\Big[E^{\mathbb{Q}}\Big[V(t_{n+1},T,Y_{t_{n+1}},X_{t_{n+1}})\Big|\mathcal{F}^X_{t_{n+1}} \vee \mathcal{F}^Y_{t_n}\Big]\Big|\mathcal{F}^X_{t_n} \vee \mathcal{F}^Y_{t_n}\Big] \\ &= E^{\mathbb{Q}}\Big[E^{\mathbb{Q}}\Big[V(t_{n+1},T,Y_{t_{n+1}},X_{t_{n+1}})\Big|\sigma(X_{t_{n+1}}) \vee \mathcal{F}^Y_{t_n}\Big]\Big|\sigma(X_{t_n}) \vee \mathcal{F}^Y_{t_n}\Big] \end{aligned} \tag{5.13}$$

because $\mathcal{F}^X_{t_n} \vee \mathcal{F}^Y_{t_n} \subset \mathcal{F}^X_{t_{n+1}} \vee \mathcal{F}^Y_{t_n}$ and X is markovian.

Furthermore, by the fact that Y is a Lévy process with characteristic exponent $\Psi^{(X_{t_{n+1}})}(\omega)$ on $(t_n, t_{n+1}]$ we have

$$\begin{aligned} V(t_n, T, Y_{t_n}, X_{t_n}) &= E^{\mathbb{Q}}\Big[E^{\mathbb{Q}}\Big[V(t_{n+1}, T, Y_{t_{n+1}}, X_{t_{n+1}})\Big|\sigma(X_{t_{n+1}}) \vee \mathcal{F}^Y_{t_n}\Big]\Big|\sigma(X_{t_n}) \vee \mathcal{F}^Y_{t_n}\Big] \\ &= E^{\mathbb{Q}}\Big[\Im^{-1}\Big[\exp\Big\{\Big(\Psi^{(X_{t_{n+1}})}(\omega) - r(X_{t_{n+1}})\Big)\Delta t\Big\} \\ &\quad \times \underline{\widehat{\mathbf{V}}}_{X_{t_{n+1}}}(t_{n+1}, \omega)\Big]\Big|(Y_{t_n}, X_{t_n})\Big], \end{aligned} \tag{5.14}$$

where the last equality follows from (5.11). Finally, for $X_{t_n} = \mathbf{e}_j$ we deduce

$$V(t_n, T, Y_{t_n}, j) = \sum_{k=1}^{M} P_{jk}\Im^{-1}\Big[\exp\Big\{\Big(\Psi^{(X_{t_{n+1}})}(\omega) - r(X_{t_{n+1}})\Big)\Delta t\Big\} \cdot \underline{\widehat{\mathbf{V}}}_{X_{t_{n+1}}}(t_{n+1}, \omega)\Big](Y_{t_n}) \tag{5.15}$$

This recursive procedure allows to compute the price of a European option in every state of the chain X at the present assuming that we know the price of the previous period.

5.2.3. Algorithm of Carr and Madan ([**26**])

The method of valuing a European option due to Carr and Madan [**26**] has become a reference in financial engineering over time . It makes possible to compute, with good accuracy and minimum time, option prices for a family of strike prices K. This is interesting in view of the calibration model's performance. This technique was first used for a regime-switching model by Liu, et *al.* [**92**]. In this section, we show that it can be adapted to the broader context of regime-switching exponential-Lévy models.

•*Method of Carr and Madan (CM)*

We aim at finding the numerical value C at time 0 of a European call written on an underlying S with strike $K > 0$ and maturity $T > 0$, knowing that the state of the Markov chain X is $X_0 = \mathbf{e}_j$.
By the fundamental theorem of asset pricing (see [**79, 80**]) and Equation 4.15, we know that this value is given by

$$\begin{aligned} C(K, T, j) &= E^{\mathbb{Q}}\Big[e^{-\int_0^T r_s ds}\Big(S_0 e^{Y_T} - K\Big)^+\Big|X_0 = j, S_0\Big] \\ &= S_0 E^{\mathbb{Q}}\Big[e^{-\int_0^T r_s ds}\Big(e^{Y_T} - e^k\Big)^+\Big|X_0 = \mathbf{e}_j, S_0\Big] \\ &=: \widetilde{C}(k, T, j), \end{aligned} \tag{5.16}$$

where we have set $k = \log(K/S_0)$.

Assume the initial state $X_0 = \mathbf{e}_j$ is fixed, $C(k, T, j)$ is not rapidly decreasing because $\widetilde{C}(k, T, j) = S_0$ as $k \to -\infty$. Therefore, $\widetilde{C}(k, T, j)$ is not Fourier-integrable. To overcome this problem, Carr and Madan [**26**] have defined a modified price as :

$$C_{mod}(k, T, j) := e^{\alpha k}\widetilde{C}(k, T, j), \tag{5.17}$$

where $\alpha > 0$ is chosen such that $C_{mod}(k,T,j)$ satisfies the integrability condition

$$\int_{-\infty}^{\infty} |C_{mod}(k,T,j)|dk < \infty. \tag{5.18}$$

Thus, the Fourier transform of the modified price is

$$\psi_T(\omega,j) := \int_{-\infty}^{\infty} e^{\mathbf{i}\omega k} C_{mod}(k,T,j)dk, \tag{5.19}$$

and we deduce the original call price $\widetilde{C}(k,T,j)$ by taking the inverse Fourier transform of $\psi_T(\omega,j)$ multiplied by a correction term

$$\begin{aligned}\widetilde{C}(k,T,j) &= \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-\mathbf{i}\omega k}\psi_T(\omega,j)d\omega \\ &= \frac{e^{-\alpha k}}{\pi}\int_{0}^{\infty} e^{-\mathbf{i}\omega k}\psi_T(\omega,j)d\omega, \end{aligned} \tag{5.20}$$

where the last equality follows from the fact that the function $\omega \to e^{-\mathbf{i}\omega k}\psi_T(\omega,j)$ is even.

To obtain a numerical value of $\widetilde{C}$, we only need an analytical expression of $\psi_T(\omega,j)$. This is quite simple because conditionally on filtration $\mathcal{F}_T^X$, Y is a Lévy process which has a characteristic function given in a closed form in many cases. If $f_T(x)$ denotes the conditional density function of Y_T[2] given the filtration $\mathcal{F}_T^X$ then, by double expectation law and Fubini theorem

$$\begin{aligned}\psi_T(\omega,j) &= S_0\int_{-\infty}^{\infty} e^{\mathbf{i}\omega k}e^{\alpha k}E^{\mathbb{Q}}\Big[e^{-\int_0^T r_s ds}\Big(e^{Y_T}-e^k\Big)^+\Big|X_0=\mathbf{e}_j,S_0\Big]dk \\ &= S_0\int_{-\infty}^{\infty} e^{\mathbf{i}\omega k}e^{\alpha k}E^{\mathbb{Q}}\Big[e^{-\int_0^T r_s ds}\int_k^{\infty}\Big(e^x-e^k\Big)f_T(x)dx\Big|X_0=\mathbf{e}_j,S_0\Big]dk \\ &= S_0E^{\mathbb{Q}}\Big[e^{-\int_0^T r_s ds}\Big\{\int_{-\infty}^{x} e^{\mathbf{i}\omega k}\Big(e^{x+\alpha k}-e^{(1+\alpha)k}\Big)dk\Big\}f_T(x)dx\Big|X_0=\mathbf{e}_j,S_0\Big] \\ &= S_0\frac{E^{\mathbb{Q}}\Big[e^{-\int_0^T r_s ds}\phi_T(\omega-\mathbf{i}(1+\alpha))\Big|X_0=\mathbf{e}_j,S_0\Big]}{\alpha^2+\alpha-\omega^2+\mathbf{i}(2\alpha+1)\omega}, \end{aligned} \tag{5.21}$$

where $\phi_T(\omega)$ denotes the conditional characteristic function of Y_T given the filtration $\mathcal{F}_T^X$. By noting that

$$\begin{aligned} Y_T = & \underbrace{\int_0^T(\mu_s-\frac{1}{2}\sigma_s^2)ds+\int_0^t\sigma_s dW_s}_{Y_T^{(1)}} \\ & + \underbrace{\int_0^T\int_{-\infty}^{+\infty} z\widehat{N}^X(ds;dz)-\int_0^T\int_{-\infty}^{+\infty}(e^z-1-z)\rho^X(dz)ds}_{Y_T^{(2)}}, \end{aligned} \tag{5.22}$$

[2]We suppose that Y_T is independent of S_0.

we obtain

$$\begin{aligned}\phi_T(\omega) &= E^{\mathbb{Q}}\Big[e^{\mathbf{i}\omega Y_T}\Big|\mathcal{F}_T^X\Big]\\ &= E^{\mathbb{Q}}\Big[e^{\mathbf{i}\omega Y_T^{(1)}}\Big|\mathcal{F}_T^X\Big]E^{\mathbb{Q}}\Big[e^{\mathbf{i}\omega Y_T^{(2)}}\Big|\mathcal{F}_T^X\Big]\\ &:= \phi_T^{(1)}(\omega)\phi_T^{(2)}(\omega). \end{aligned} \tag{5.23}$$

If we set $R_T := \int_0^T r_s ds$ and $V_T := \int_0^T \sigma_s^2 ds$ then $Y_T^{(1)}$ has a normal distribution with mean $(R_T - \frac{1}{2}V_T)$ and variance V_T. Therefore,

$$\phi_T^{(1)}(\omega) = \exp\Big(\mathbf{i}\omega(R_T - \frac{1}{2}V_T) - \frac{1}{2}\omega^2 V_T\Big). \tag{5.24}$$

Also, one can easily check that (see [**51**])

$$\begin{aligned}\phi_T^{(2)}(\omega) &= E^{\mathbb{Q}}\Big[e^{\mathbf{i}\omega Y_T^{(2)}}\Big|\mathcal{F}_T^X\Big]\\ &= \exp\Big[\sum_{j=1}^M \int_0^T \langle X_{s^-}|\mathbf{e}_j\rangle ds \int_{-\infty}^{\infty}\Big(e^{\mathbf{i}\omega z} - 1 - \mathbf{i}\omega(e^z - 1)\Big)\rho_j(dz)\Big]. \end{aligned} \tag{5.25}$$

The general formula admits more simpler expressions depending on the model used. Indeed, we have :

TAB. 5.1. **Expression of $\int_{-\infty}^{\infty}\left(e^{\mathbf{i}\omega x}-1-\mathbf{i}\omega(e^{x}-1)\right)\rho_j(dx)$ depending on the type of model.**

regime-switching model	**Lévy measure ρ_j**	$\int_{-\infty}^{\infty}\left(e^{\mathbf{i}ux}-1-\mathbf{i}u(e^{x}-1)\right)\rho_j(dx)$
Black-Scholes	NA	NA
Merton Jump-Diffusion	$\frac{\lambda_j}{\delta_j\sqrt{2\pi}}\exp\left[\frac{-(x-\mu_j^J)^2}{2\delta_j^2}\right]$	$\lambda_i\left[\left(e^{\mathbf{i}u\mu_j^J-\frac{1}{2}\delta_j^2u^2}-1\right)-\mathbf{i}u\left(e^{\mu_j^J+\frac{1}{2}\delta_j^2}-1\right)\right].$
Kou Jump-Diffusion	$p\eta_j^+e^{-\eta_j^+x}1_{\{x>0\}}+(1-p)\eta_j^-e^{-\eta_j^-\lvert x\rvert}1_{\{x<0\}}$	$p\mathbf{i}u\left[\frac{1}{\eta_j^+-\mathbf{i}u}-\frac{1}{\eta_j^+-1}\right]-(1-p)\mathbf{i}u\left[\frac{1}{\eta_j^-+\mathbf{i}u}+\frac{1}{\eta_j^-+1}\right].$
Variance-Gamma (C,G,M)	$C_j\frac{e^{-G_jx}}{x}1_{\{x>0\}}+C_j\frac{e^{-M_j\lvert x\rvert}}{\lvert x\rvert}1_{\{x<0\}}$	$C_j\log\left(\frac{G_jM_j}{G_jM_j+(M_j-G_j)\mathbf{i}u+u^2}\right)-\mathbf{i}uC_j\log\left(\frac{G_jM_j}{G_jM_j+(M_j-G_j)-1}\right)$
CGMY	$C_j\frac{e^{-G_j\lvert x\rvert}}{\lvert x\rvert^{1+Y_j}}1_{\{x>0\}}+C_j\frac{e^{-M_j\lvert x\rvert}}{\lvert x\rvert^{1+Y_j}}1_{\{x<0\}}$	$C_j\Gamma(-Y_j)\left[G_j^{Y_j}(G_j-\mathbf{i}u)^{Y_j}-G_j^{Y_j}-\mathbf{i}u\left((G_j-1)^{Y_j}-G_j^{Y_j}\right)\right]$ $+C_j\Gamma(-Y_j)\left[M_j^{Y_j}(M_j+\mathbf{i}u)^{Y_j}-M_j^{Y_j}-\mathbf{i}u\left((M_j+1)^{Y_j}-M_j^{Y_j}\right)\right]$

Expressions (5.24) and (5.25) can be made explicit by using the sojourn time in different states of X. Indeed, by letting

$$T_j = \int_0^T \langle X_{s^-} | \mathbf{e}_j \rangle ds, \quad j = 1, 2, ..., M \tag{5.26}$$

the sojourn time of X in the state $\mathbf{e}_j$ during the time interval $[0, T]$ we have $\sum_{j=1}^{M} T_j = T$ and

$$\begin{aligned} R_T &= \sum_{j=1}^{M-1} (r_j - r_M) T_j + r_M T, \\ V_T &= \sum_{j=1}^{M-1} (\sigma_j^2 - \sigma_M^2) T_j + \sigma_M^2 T. \end{aligned} \tag{5.27}$$

If we set for $j = 1, 2, ..., M-1$

$$\begin{aligned} A_1(\omega, j) &= \Big[(r_j - r_M) + (\frac{1}{2} + \alpha)(\sigma_j^2 - \sigma_M^2)\Big]\omega + \frac{1}{2}\mathbf{i}(\sigma_j^2 - \sigma_M^2)\omega^2 - \mathbf{i}\Big[\alpha(r_j - r_M) \\ &\quad + \frac{1}{2}\alpha(1+\alpha)(\sigma_j^2 - \sigma_M^2)\Big], \\ B_1(\omega, M) &= \mathbf{i}\omega\Big[r_M + (\frac{1}{2} + \alpha)\sigma_M^2\Big] - \frac{1}{2}\omega^2\sigma_M^2 + \alpha r_M + \frac{1}{2}\alpha(1+\alpha)\sigma_M^2. \end{aligned} \tag{5.28}$$

then

$$e^{-\int_0^t r_s ds} \phi_T^{(1)}(\omega - \mathbf{i}(1+\alpha)) = \exp\Big(B_1(\omega, M)T\Big) \exp\Big(\mathbf{i} \sum_{j=1}^{M-1} A_1(\omega, j) T_j\Big). \tag{5.29}$$

In the same line, by letting for $j = 1, 2, ..., M-1$

$$\varphi(\omega, j) = \int_{-\infty}^{\infty} \Big(e^{(\mathbf{i}\omega + 1 + \alpha)z} - 1 - (\mathbf{i}\omega + 1 + \alpha)(e^z - 1)\Big) \rho_i(dz) \tag{5.30}$$

we obtain

$$\phi_T^{(2)}(\omega - \mathbf{i}(1+\alpha)) = \exp\Big(\varphi(\omega, M)T\Big) \exp\Big(\sum_{k=1}^{M-1} (\varphi(\omega, k) - \varphi(\omega, M)) T_k\Big). \tag{5.31}$$

Therefore, we deduce from the previous calculations an explicit expression of Fourier transform of the European call price :

$$\psi_T(\omega, j) = S_0 \frac{\exp\Big(B(\omega, M)\Big) E^{\mathbb{Q}}\Big[\exp\Big(\mathbf{i}\sum_{k=1}^{M-1} A(\omega, k) T_k\Big) \Big| X_0 = \mathbf{e}_j, S_0\Big]}{\alpha^2 + \alpha - \omega^2 + \mathbf{i}(2\alpha + 1)\omega} \tag{5.32}$$

where

$$\begin{aligned} A(\omega, k) &= A_1(\omega, k) - \mathbf{i}(\varphi(\omega, k) - \varphi(\omega, M)), \ k = 1, 2, ..., M-1 \\ B(\omega, M) &= B_1(\omega, M) + \varphi(\omega, M). \end{aligned}$$

In conclusion, the determination of $\psi_T(\omega, .)$ is reduced to calculating the characteristic function of the random vector of sojourn time $(T_1, T_2, ..., T_{M-1})$ for which we have an explicit expression given by lemma A.1 of Buffington and Elliott [**24**],

$$E^{\mathbb{Q}}\Big[\exp\Big(\mathbf{i}\sum_{j=1}^{M-1}\theta_j T_j\Big)\Big|X_0\Big] = \Big\langle \exp[(A+\mathbf{i}\ \mathbf{Diag}(\theta_1,\theta_2,...,\theta_{M-1},0))]X_0\Big|\underline{\mathbf{1}}\Big\rangle \tag{5.33}$$

where $\underline{\mathbf{1}} = (1,1,...,1) \in \mathbb{R}^M$.

5.2.4. Comparison of the Algorithms "Fourier-Stepping Time(FST)" and "Carr-Madan(CM)"

Both algorithms presented above have many similarities but also some differences.

- The algorithm of Carr and Madan is based on the transformation of strike variable K, while the FST algorithm uses a transformation of the terminal price S_T. This has important implications : firstly, for the calibration procedure we need to provide the theoretical prices of options for a wide range of strikes K in a short time and in this case the method of Carr-Madan is the most competitive ; secondly, when the number of states M of the Markov chain X is greater than 2, the algorithm of Carr and Madan involves an exponential matrix whose estimation needs long calculations, thus it is more time-consuming. Contrarily, the FST algorithm performs better in this case because the number of states is already involved in the variable S_T.
- Both algorithms provide quite similar results : we will check this by calculating the price of a European call option in a regime-switching Black-Scholes model and a regime-switching Jump-Diffusion model with two states. The results are presented in the following.

For our calculations, we considered a two-state RS-Black-Scholes model with parameters : $\underline{r} = (0.05, 0.1)$, $\underline{\sigma} = (0.5, 0.3)$, $a_{12} = 20$, $a_{21} = 30$, $S = 100$ and $T = 1$ and a two-state RS-Jump-Diffusion model with parameters $\underline{r} = (0.05, 0.1)$, $\underline{\sigma} = (0.5, 0.3)$, $\underline{\mu} = (-0.3, -0.5)$, $\underline{\delta} = (0.1, 0.5)$, $\underline{\lambda} = (5, 1)$, $a_{12} = 20$, $a_{21} = 30$, $S = 100$ and $T = 1$.

FIG. 5.1. Comparison of the two algorithms "FST" and "Carr-Madan" by the implied volatility generated

Tab. 5.2. ***FST*** **versus** ***Carr-Madan*** **: European call prices in a two-state RS-Black-Scholes model.**

log(K/S0)	regime 1			regime 2		
	FST	**CM**	**Rel. Err.**	**FST**	**CM**	**Rel. Err.**
-0.3	34.7666	34.7736	0.02%	34.7507	34.7417	0.03%
-0.2	29.6855	29.6958	0.03%	29.6588	29.6423	0.06%
-0.1	24.7501	24.7635	0.05%	24.7127	24.6886	0.10%
0	20.1000	20.1160	0.08%	20.0533	20.0224	0.15%
0.1	15.8625	15.8806	0.11%	15.8091	15.7735	0.23%
0.2	12.1377	12.1569	0.16%	12.0810	12.0433	0.31%
0.3	8.9865	9.0059	0.22%	8.9302	8.8932	0.42%

Tab. 5.3. ***FST*** **versus** ***Carr-Madan*** **: European call prices in a two-state RS-Jump-Diffusion model.**

log(K/S0)	regime 1			regime 2		
	FST	**CM**	**Rel. Err.**	**FST**	**CM**	**Rel. Err.**
-0.3	39.5131	39.4952	0.05%	39.5606	39.5902	0.07%
-0.2	35.0067	34.9904	0.05%	35.0517	35.0803	0.08%
-0.1	30.5208	30.5064	0.05%	30.5612	30.5872	0.09%
0	26.1425	26.1306	0.05%	26.1766	26.1987	0.08%
0.1	21.9625	21.9532	0.04%	21.9890	22.0062	0.08%
0.2	18.0676	18.0609	0.04%	18.0860	18.0976	0.06%
0.3	14.5328	14.5285	0.03%	14.5433	14.5493	0.04%

5.3. Calibration of a Family of Regime-Switching Exponential Lévy Models

The calibration model can be defined as the statistical estimation of the parameters of that model from real data. This data is then viewed as realizations or samples of the underlying stochastic process. In financial engineering, we work particularly with a probability measure called *risk-neutral* $\mathbb{Q}$ which is equivalent to the historical probability measure $\mathbb{P}$ with the property that it does not lead to arbitrage opportunity. Then, determining the model parameters that are consistent with observed prices of liquid options is useful to evaluate and to construct optimal hedging strategies for non-traded or over-the-counter derivatives.
In this section, we discuss the calibration problem through a few examples.

5.3.1. Problem Setting

In general, if the set of parameters of a regime-switching exponential-Lévy model is given by a vector $\underline{\theta} = (\theta_1, ..., \theta_N)$, then the calibration problem consists in :

Finding $\underline{\theta}$ *such that for a set of observed option prices*

$$\Big\{C^*_{ij}(T_i, K_j); \quad (i,j) \in \{1, ..., N_T\} \times \{1, ..., M_K\}\Big\}$$

with maturities and strikes (T_i, K_j) *we have :*

$$C^*_t(T_i, K_j, \underline{\theta}) = E^{\mathbb{Q}^{\underline{\theta}}}[e^{-\int_t^{T_i} r_s ds} H(S_{T_i}, K_j)|\mathcal{F}_t], \quad (i,j) \in \{1, ..., N_T\} \times \{1, ..., M_K\} \tag{5.34}$$

where H denotes the payoff of the European option of maturity T and strike K.

The problem is the reverse of that of option pricing. Because of possible over-determination of the model, it is possible to find several sets of parameters $\underline{\theta}$ that are compatible with the option prices observed on the market with the event that the solution is not specific to the problem. Thus, the problem 5.34 is in the class of problems called *ill-posed*. To avoid this, the calibration problem is formulated in terms of minimizing a certain distance between the theoretical prices derived from the model and the prices actually observed, namely :

Finding $\underline{\theta}$ *such that for a set of observed option prices*

$$\Big\{C^*_{ij}(T_i, K_j); \quad (i,j) \in \{1, ..., N_T\} \times \{1, ..., M_K\}\Big\}$$

with maturities and strikes (T_i, K_j) *we have :*

$$\underline{\theta} = \arg\min \ \mathcal{L}(\theta) \tag{5.35}$$

where

$$\mathcal{L}(\theta) = \sum_{i=1}^{N_T} \sum_{j=1}^{M_K} \omega_{ij} \left(C^{\theta}_t(S_t, T_i, K_j) - C^*_t(T_i, K_j)\right)^2 .$$

$\omega_{ij} > 0$ denoted a weight optimally chosen as the inverse of the variance of the residuals and $C^{\theta}(\cdot,\cdot,\cdot)$ the option price given by the model. In practice, the weights $\Big\{\omega_{ij} : \ (i,j) \in \{1,...,N_T\} \times \{1,...,M_K\}\Big\}$ are chosen as the inverse of square of *bid-ask spread* , i.e.,

$$\omega_{ij} \propto \frac{1}{(C_t^{Ask}(t_i,K_j) - C^{Bid_t}(t_i,K_j))^2}.$$

The idea is to estimate the parameters of the model which assure that the distance between the prices given by the model and those actually observed in the market is as small as possible.
Besides the choice of weighted least squares as loss function, there are other examples that are used in the literature as shown in the Table below. 5.4.

TAB. 5.4. **Usual loss functions used for model calibration.**

Root Mean Square Error (RMSE)	$\mathcal{L}(\theta) = \sqrt{\sum_{i=1}^{N_T}\sum_{j=1}^{M_K} \frac{(C_t^{\theta}(S_t,T_i,K_j) - C_t^*(T_i,K_j))^2}{\text{no. of options}}}$
Average Relative Percentage Error(ARPE)	$\mathcal{L}(\theta) = \frac{1}{\text{no. of options}} \sum_{i=1}^{N_T}\sum_{j=1}^{M_K} \left\lvert \frac{C_t^{\theta}(S_t,T_i,K_j) - C_t^*(T_i,K_j)}{C_t^*(T_i,K_j)} \right\rvert$
Average Absolute Error(AAE)	$\mathcal{L}(\theta) = \frac{1}{\text{no. of options}} \sum_{i=1}^{N_T}\sum_{j=1}^{M_K} \left\lvert C_t^{\theta}(S_t,T_i,K_j) - C_t^*(T_i,K_j) \right\rvert$

In the following, we shall discuss the calibration problem through three examples of regime-switching exponential Lévy model.

Problem 1 : Case of the Regime-Switching Black-Scholes

In this model, the parameters r, μ et σ depend on a continuous-time Markov chain X. Then, the dynamics of the stock S under the risk-neutral measure is given by :

$$S_t = S_0 \exp\Big[\int_0^t r(X_s)ds + \int_0^t \sigma(X_s)dW_s\Big] \tag{5.36}$$

Here, we suppose that X has only two states, 0 and 1, and its intensity-matrix is :

$$Q = \begin{pmatrix} -\lambda_{0,1} & \lambda_{0,1} \\ \lambda_{1,0} & -\lambda_{1,0} \end{pmatrix}. \tag{5.37}$$

The set of parameters we have to calibrate is

$$\Theta^{cal} = \{\sigma_0, \sigma_1, \lambda_{0,1}, \lambda_{1,0}\} \tag{5.38}$$

where $\sigma_i, \lambda_{0,1}, \lambda_{1,0} \geq 0$.

Problem 2 : Case of the Regime-Switching Jump-Diffusion Model

This model extends the previous one by adding a jump component. So, in addition to r, μ et σ, we have the parameters corresponding to the jump α, μ^J and δ which also depend on X. Precisely, the risk-neutral dynamics of the stock is given by

$$S_t = S_0 \exp\Big[\int_0^t r(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\widetilde{N}_{JD}^X(ds,dz) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z)\rho_{JD}^X(dz)ds\Big]. \quad (5.39)$$

where the jump process $N_{JD}^X(t,.)$ admits as predictable compensator

$$\rho_{JD}^X(dz)dt = \sum_{i=0}^{1} \langle \mathbf{e}_i | X_{t^-}\rangle \rho_i^{JD}(z)dt. \quad (5.40)$$

The Lévy measure of Y when X is in state $\mathbf{e_i}$ is

$$\rho_i^{JD}(z) = \alpha_i \frac{1}{\delta_i\sqrt{2\pi}} \exp\left\{ -\frac{(z-\mu_i^J)^2}{\delta_i^2}\right\}.$$

where α_i, δ_i are positive constant. The Markov chain X has the same characteristic as above. Then, the set of parameters to be calibrated is

$$\Theta^{cal} = \big\{\alpha_0, \alpha_1, \mu_0^J, \mu_1^J, \delta_0, \delta_1, \sigma_0, \sigma_1, \lambda_{0,1}, \lambda_{1,0}\big\} \quad (5.41)$$

where α_i,δ_i,σ_i, $\lambda_{0,1}$, $\lambda_{1,0} \geq 0$.

Problem 3 : Case of the Regime-Switching Variance-Gamma Model

The details of this are given in chapter **??**. We recall the risk-neutral dynamics of the stock :

$$S_t = S_0 \exp\left[\int_0^t r(X_s)ds + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\widetilde{N}_{VG}^X(ds,dz) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z)\rho_{VG}^X(dz)ds\right] \quad (5.42)$$

where the jump process $N_{VG}^X(t,.)$ has as predictable compensator

$$\rho_{VG}^X(dz)dt = \sum_{i=0}^{1} \langle \mathbf{e}_i | X_{t^-}\rangle \rho_i^{VG}(z)dt. \quad (5.43)$$

The Lévy measure of Y when X is in state $\mathbf{e_i}$ is

$$\rho_i^{VG} = C_i \frac{e^{-G_i x}}{x} 1_{\{x>0\}} + C_i \frac{e^{-M_i |x|}}{|x|} 1_{\{x<0\}}.$$

where C_i, G_i and M_i are positive constants. Then, the set of parameters to be calibrated is :

$$\Theta = \{C_0, C_1, G_0, G_1, M_0, M_1, \lambda_{0,1}, \lambda_{1,0}\} \tag{5.44}$$

where $\lambda_{0,1}, \lambda_{1,0} \geq 0$.

5.3.2. Numerical Results

In this final section, we worked out the previous calibration problems by using a sample data set consisting of 155 observed European call prices on the Dow Jones Industrial average Index. Our data are recorded on March 3th, 2008 and are given in Table 5.8 (see Appendix). The spot price at this moment is US$ 122.5 The risk-free rate is taken piecewise constant along each of the five intervals of maturities and the dividend rate is taken to be zero. We have supposed that these two quantities are the same in the two states of our model.
We have made a sequential calibration, i.e., for each $T_i \in \{47, 75, 110, 201, 292\}$ we solve the problem

$$\underline{\theta}(T_i) = \arg\min_{\underline{\theta}} \sum_{j=1}^{M_K} \left(C_t^{\theta}(S_t, T_i, K_j) - C_t^{*}(T_i, K_j)\right)^2 \tag{5.45}$$

where $K_j \in \{98, 99, ..., 127, 128\}$. All codes for minimization are in MATLAB.

For problem 1, the calibrated parameters are shown in Table 5.5 and it also includes the calibration error (RMSE) defined in Table 5.4.

TAB. 5.5. **Calibrated parameters for problem 1.**

interval of maturities	$\widehat{\sigma}_0$	$\widehat{\sigma}_1$	$\widehat{\lambda}_{0,1}$	$\widehat{\lambda}_{1,0}$	RMSE
(0-47]	0.2458	0.2463	0.1203	0.1525	0.0039
(47-75]	0.2401	0.2408	0.2841	0.3599	0.0043
(75-110]	0.2379	0.2387	0.3807	0.4824	0.0051
(110-201]	0.2298	0.2307	0.1294	0.1640	0.0055
(201-292]	0.2209	0.2220	0.1072	0.1358	0.0055

FIG. 5.2. RSBlack-Scholes model : Comparison between the market prices 'o' and model prices '*'

For problem 2, the calibrated parameters are given in Table 5.6.

Tab. 5.6. **Calibrated parameters for problem 2.**

interval of maturities	$\widehat{\mu}_0$	$\widehat{\sigma}_0$	$\widehat{\alpha}_0$	$\widehat{\delta}_0$	$\widehat{\mu}_1$	$\widehat{\sigma}_1$	$\widehat{\alpha}_1$	$\widehat{\delta}_1$	$\widehat{\lambda}_{0,1}$	$\widehat{\lambda}_{1,0}$	RMSE
(0-47]	-0.0129	0.2457	0.2419	0.0304	-0.1056	0.2444	0.0585	0.0153	1.2088	0.0266	0.0038
(47-75]	-0.0019	0.2407	0.0315	0.0045	-0.0142	0.2409	0.0080	0.0020	0.1571	0.0034	0.0043
(75-110]	-0.0281	0.2371	0.0603	0.0591	-0.1414	0.0007	1.9952	0.1041	0.5042	4.2449	0.0033
(110-201]	-0.0030	0.2296	0.0052	0.0064	-1.5708	0.1633	0.0694	0.9753	0.5044	2.5482	0.0037
(201-292]	-0.0014	0.2209	0.0023	0.0028	-0.7015	0.1614	0.0947	0.43772	0.23301	1.1468	0.0037

FIG. 5.3. RSJump-Diffusion model : Comparison between the market prices 'o' and model prices '*'

The calibrated parameters for problem 3 are given in Table 5.7.

Tab. 5.7. **Calibrated parameters for problem 3.**

interval of maturities	$\widehat{C}_0$	$\widehat{G}_0$	$\widehat{M}_0$	$\widehat{C}_1$	$\widehat{G}_1$	$\widehat{M}_1$	$\widehat{\lambda}_{0,1}$	$\widehat{\lambda}_{1,0}$	RMSE
(0-47]	7.5957	8.1103	0.4907	0.8701	0.8429	0.2453	6.2510	0.5885	0.0038
(47-75]	10.6480	11.4777	0.6980	1.0059	0.8099	0.3490	8.7628	1.3056	0.0037
(75-110]	9.0202	17.3724	0.0760	9.0683	15.0383	0.0608	0.6564	1.4792	0.0010
(110-201]	7.6622	14.0319	0.0612	7.5966	12.1641	0.0489	0.5197	1.1973	0.0015
(201-292]	9.8108	0.0426	1.2748	5.5107	8.5315	0.0341	0.3442	0.8586	0.0014

Fig. 5.4. RSVariance-gamma model : Comparison between the market prices 'o' and model prices '*'

Looking from the tables 5.6 and 5.7, we see that the calibration results are more accurate when we add jumps to the first model. Indeed, the RMSE decreases when we move from regime-switching Black-Scholes to regime-switching Variance-gamma model. Figure 5.2, Figure 5.3 and Figure 5.4 show the result of the calibration to Dow Jones options for the three models. When we move from the first to the third model, the fit is practically indistinguishable and the calibration performs well with slightly less success on the shortest maturity.

Tab. 5.8. **DJX options prices.**

Strike	S/K	Time to maturity (in days)				
		47	75	110	201	292
98	1.25	24.43	24.60	24.95	25.88	26.63
99	1.23	23.40	23.63	24.10	25.03	25.83
100	1.22	22.50	22.73	23.20	24.18	25.03
101	1.21	21.55	21.80	22.30	23.35	24.20
102	1.20	20.63	20.88	21.40	22.53	23.40
103	1.19	19.68	20.08	20.55	21.70	22.68
104	1.18	18.75	19.15	19.70	20.90	21.88
105	1.16	17.83	18.25	18.85	20.10	21.13
106	1.15	16.90	17.30	18.00	19.30	20.38
107	1.14	15.98	16.53	17.15	18.53	19.63
108	1.13	15.10	15.60	16.33	17.75	18.90
109	1.12	14.23	14.75	15.50	17.03	18.18
110	1.11	13.33	13.98	14.70	16.28	17.35
111	1.10	12.45	13.10	13.93	15.53	16.65
112	1.09	11.63	12.30	13.18	14.80	16.00
113	1.08	10.78	11.50	12.40	14.10	15.35
114	1.07	9.95	10.80	11.65	13.40	14.63
115	1.06	9.18	9.98	10.93	12.70	13.98
116	1.05	8.40	9.30	10.23	12.03	13.33
117	1.04	7.68	8.58	9.53	11.35	12.73
118	1.04	6.93	7.85	8.83	10.73	12.05
119	1.03	6.23	7.20	8.18	10.10	11.45
120	1.02	5.58	6.58	7.55	9.50	10.85
121	1.01	4.95	5.93	6.93	8.90	10.33
122	1.00	4.35	5.35	6.35	8.33	9.75
123	0.99	3.80	4.80	5.78	7.78	9.23
124	0.99	3.25	4.23	5.23	7.23	8.63
125	0.98	2.74	3.75	4.73	6.73	8.15
126	0.97	2.28	3.25	4.23	6.23	7.63
127	0.96	1.90	2.79	3.78	5.73	7.15
128	0.95	1.52	2.37	3.35	5.28	6.70

BIBLIOGRAPHIE

[1] Achdou Y. , Pironneau O. (2005). *Computational Methods for Option Pricing.* SIAM series in Applied Math, Philadelphia.

[2] Aggoun L., Elliott R.J. (2004). *Measure Theory and Filtering : Introduction with Applications.* Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press.

[3] Alvarez O., Tourin A. (1996). Viscosity solutions of nonlinear integro-differential equations. *Ann. Inst. poincaré Anal. Non Linéaire* **13(3)**, 290-317.

[4] Applebaum D. (2004). *Lévy processes and Stochastic Calculus.* Cambridge studies in advanced mathematics, Cambridge University Press.

[5] Arai T. (2005). An extension of mean-variance hedging to the discontinuous case. *Finance and Stochastics* **9, no. 1**, 129-139.

[6] Artzner P., Delbaen F., Eber J.M., Heath D. (1999). Coherent measures of risk. *Mathematical Finance*, **9(3)**, 203-228.

[7] Asmussen S. (2003). *Applied Probability and Queues, 2nd edition.* Springer-Verlag, Berlin, Heidelberg, New York.

[8] Bachelier L. (1900). Théorie de la Spéculation. *Annales Scientifiques de l'École Normale Supérieure* **3(17)**, 21-86.

[9] Ballotta L. (2005). A Lévy process-based framework for the fair valuation of participating life insurance contracts. *Insurance : Mathematics and Economics* **vol. 37, no. 2**, 173-196.

[10] Barles G.,Soner M. (1998). Option pricing with transaction costs and a nonlinear Black-Scholes equation. *Finance Stochastics* **2**, 369-397.

[11] Barles G., Buckdahn R., Pardoux E. (1997). backward stochastic differential equations and integral-partial differential equations. *Stochastics and Stochastics Reports* **60(1-20**, 57-83.

[12] Benth F.E, Karlsen K.H, Reikvam K. (2001). Optimal portfolio management rules in a non-Gaussian market with durability and intertempral substitution. *Finance and Stochastics* **5(4)**, 447-467.

[13] Benth F.E, Di Nunno G., Lokka A.,Oksendal B., Proske F. (2003). Explicit Representations of the Minimal variance Portfolio in Markets driven by Lévy processes, *Mathematical Finance*, **Vol. 13, No. 1**, 55-72.

[14] Björk T. (1998). *Arbitrage Theory in Continuous time.* Oxford University Press Inc.,New York.

[15] Black F., Scholes M. (1973). The pricing of Options and Corporate Liabilities. *Journal of Political Economy* **81**, 637-659.

[16] Biswas I. H., Espen R. J., Karlsen K.H. (2010). Viscosity Solutions for a System of Integro-PDEs and Connections to Optimal Switching and Control of Jump-Diffusion Processes. *Applied Mathematics and Optimization* **62(1)**, 47-80.

[17] Blumenthal R. M., Getoor R.K. (1968). *Markov processes and Potential Theory.* Pure and Applied mathematics, vol. 29, Academic Press, Inc (London, New-York).

[18] Bollen N.P.B. (1998). Valuing options in regime-switching models. *Journal of Derivatives* **6**, 38-49.

[19] Bouleau N., Lamberton D. (1989). Residual Risk and Hedging Strategies in Markovian Markets. *Stochastic Processes and Their Applications* **33**, 131-150.

[20] Boyle P., Draviam T. (2007). Pricing exotic options under regime switching. *Insurance : Mathematics and Economics* **40**, 267-282.

[21] Breuer L. (2000). Spatial Queues. PhD Dissertation, University of Trier. 128p.

[22] Briani M., La Chioma C., Natalini R. (2004). Convergence of numerical schemes for viscosity solutions to integro-differential degenerate parabolic problems arising in financial theory, *Numer. Math.* **98, no. 4**, 607-646.

[23] Bühlmann H., Delbaean F., Embrechts P. (1996). No-arbitrage, Change of Measure and Conditional Esscher Transforms. *CWI Quarterly* **9, No 4**, 291-317.

[24] Buffington J., Elliott R.J. (2002a). Regime switching and European options. *Stochastic Theory and Control, Proceedings of a Worshop, Springer Verlag*, 73-81.

[25] Buffington J., Elliott R.J. (2002b). American options with regime switching. *International Journal of Theoretical and Applied Finance* **5**, 497-514.

[26] Carr P., Madan D. (1998). Option valuation using the fast Fourier transform. *Journal of Computational Finance*, **2**, 61-73.

[27] Ceci C. (2006). Risk minimizing hedging for a partially observed high frequency data model. *Stochastics : An International Journal of Probability and Stochastics Processes.* **vol. 78, no.1**, 13-31.

[28] Chan T. (1999). Pricing contingent claims on stocks driven by Lévy processes. *The Annals of Applied probability* **vol 9(2)**, 504-528.

[29] Cherny A. S. (2001). No-arbitrage and Completeness for the Linear and Exponential Models based on Lévy Processes. *Working Paper* available at http ://www.maphysto.dk/cgi-bin/gp.cgi ?publ=325.

[30] Chourdakis K. (2005). Regime Lévy models in continuous time : Finite distributions and Option pricing. *Working Paper*

[31] Çinlar E. (1972a). Markov additive processes : I. *Wahrscheinlichkeitstheorie u. Verw. Geb.* **24, No 2**, 85-93.

[32] Çinlar E. (1972b). Markov additive processes : II. *Wahrscheinlichkeitstheorie u. Verw. Geb.* **24, No 2**, 95-121.

[33] Colwell B. D., Elliott R. J. (1993). Discontinuous asset prices and nonattainable contingent claims. *Mathematical Finance.* **3**, 295-308.

[34] Cont R., Tankov P. (2004). *Financial Modeling With Jump processes.* Chapman & Hall/CRC Financial mathematics series :London, UK.

[35] Cont R. (2001). Empirical properties of asset returns : stylized facts and statistical issues. *Quantitative Finance.* **Vol 1**, 223-236.

[36] Cont R. Voltchkova E. (2005). Integrodifferential equations for option prices in exponential Lévy models.*Finance and Stochastics***9(3)**, 299-325.

[37] Crandall M. J., Ishii H., Lions P. L. (1992). User's guide to viscosity solutions of second order Partial Differential Equations. *Bull. Amer. Math. Soc.*, **27**, 1-67.

[38] Crandall M. J., Lions P. L. (1983). Viscosity solutions to Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.* **273, No 1**, 1-42.

[39] Di Masi G.B., Platen E., Runggaldier W.J. (1995). Hedging of options under discrete observation on assets with stochastic volatility. In : *Bolthausen E, Dozzi M, Russo F (eds.) Seminar on stochastic analysis, random fields and applications, Progress in Probability* **Vol. 36**, Birkhaüser Verlag, pp. 359-364.

[40] Delbaen F. (2002). *Coherent risk measures on general probability spaces.* Essays in Honour of Dieter Sondermann. Springer Verlag.

[41] Delbaen F., Schachermayer W. (1994). A General version of the Fundamental Theorem of Asset Pricing. *Math. Annalen*, **vol. 300**, 463-520.

[42] Dellacherie C., Meyer P.A (1980). *Probabilités et Potentiel, Chap V à $VIII$. Théorie des Martingales.* Hermann, Paris.

[43] Di Nunno G. (2002). Stochastic integrals Representations, Stochastic Derivatives and Minimal Variance Hedging. *Stochastics and Stochastics Reports*, **vol.73, No. 1-2**, 181-198

[44] Dufour F., Elliott R.J. (1999). Filtering with Discrete State Observations. *Applied Mathematics & Optimization*, **40**, 259-272.

[45] Duffie D., Richardson H. L. (1991). Mean-variance hedging in continuous time. *Annals of Applied Probability*, **1**, 1-15.

[46] Dumas B.,Fleming I., Whaley R. (1998). Implied volatility functions : empirical tests. *The journal of finance* **vol.53, No.6**.

[47] Eberlein E., Keller U. (1995). Hyperbolic distribution in finance. *Bernouilli* **1(3)**, 281-299.

[48] Elliott R.J., Hunter W. C., Jamieson B. M. (2001). Financial signal processing. *International Journal of Theoretical and Applied Finance* **4**, 497-514.

[49] Elliott R.J. (1993). New Finite-Dimensional Filters and Smoothers for Noisily Observed Markov Chains. *IEEE Transactions on Informations theory* **39-1**, 265-271.

[50] Elliott R.J., Chan L., Siu T.K. (2005). Option Pricing and Esscher Transform Under Regime Switching. *Annals of Finance* **1(4)**, 23-432.

[51] Elliott R.J., Chan L., Siu T.K. (2006). Risk measures for derivatives with Markov-modulated pure jump processes. *Asia-Pacific Financial Markets* **13**, 129-149.

[52] Elliott R.J.,van der Hoek J. (1997). An application of hidden Markov models to asset allocation problems. *Finance and Stochastics* **1(3)**, 229-238.

[53] Elliott R. J., Siu T.K, Yang H. (2007). Martingale Representation for Contingent Claims With Regime Switching.*Communications on Stochastic Analysis*,**1(2)**, 279-292.

[54] Elliott R.J.,Osakwe C.J (2006). Option Pricing for pure jump processes with Markov switching compensators. *Finance and Stochastics* **10**, 250-275.

[55] Elliott R. J., Royal A. J. (2007). Asset prices with regime switching variance gamma dynamics. *in Handbook on Mathematical Finance, A. Bensoussan and Q. Zhang, Eds.*, Elsevier, Amsterdam, The Netherlands.

[56] Elliott R. J., Siu T. K. (2011). Pricing and Hedging Contingent Claims With Regime Switchings Risk. *Communications in Mathematical Sciences* **vol. 9, No. 2**, 477-498.

[57] Elliott R. J., Siu T. K., Badescu A. (2011). On Pricing and Hedging Options Under Double Markov-Modulated Models With Feedback Effect. *Journal of Economics Dynamics and Control*, Accepted.

[58] Ethier S.N., Kurtz T.G. (1986). Markov processes. (Wiley series in probability and mathematical statistics, John Wiley & Sons. New York)

[59] Ezhov I. I., Skorohod A.V (1969a). Markov processes with homogeneous second component :I. *Theory of Probability and its Applications*, **vol.14, No 1**, 1-13.

[60] Ezhov I. I., Skorohod A.V. (1969b). Markov processes with homogeneous second component :II. *Theory of Probability and its Applications*, **vol.14, No 4**, 652-667.

[61] Fama E. (1965). The Behavior of Stock prices. *Journal of Business* **37**, 34-105.

[62] Fleming W.H., Soner H.M. (1993).*Controlled Markov Processes and Viscosity solutions.* (Springer-Verlag, New-York.)

[63] Föllmer H., Schweizer M. (1991). Hedging of Contingent Claims under Incomplete Information. *in : M. H. A. Davis and R. J. Elliott (eds.), "Applied Stochastic Analysis", Stochastics Monographs*, **vol. 5**, Gordon and Breach, London/New York, 389-414.

[64] Föllmer H., Sondermann D. (1986). Hedging of Non-redundant Contingent Claims. *Contributions to Mathematical Economics. In Honor of G. Debreu (Eds. W. Hildenbrand and A. Mas-Colell)*, Elsevier Science Publ. North-Holland, 205-223.

[65] Frey R., Runggaldier W. J. (1999). Risk-minimizing hedging strategies under restricted information : the case of stochastic volatility models observable only at discrete random times. *Math. Methods Oper. Res.* **50, no. 2**, 339-350.

[66] Frey R. (2000). Risk Minimization with Incomplete Information in a Model for High-Frequency Data. *Mathematical finance*, **vol. 10, no. 2**, 215-255.

[67] Frittelli M. (2000). The Minimal Entropy Martingale Measure and the Valuation Problem in Incomplete Markets. *Mathematical Finance* **10, No 1**, 39-52

[68] Fujiwara T., Kunita H. (1985). Stochastic Different equations of Jump type and Lévy process in the diffeomorphism group. *Journal of mathematics of Kyoto University*, **vol. 25, no. 1**, 71-106.

[69] Fujiwara T., Miyahara Y. (2003). The minimal entropy martingale measure for geometric Lévy processes. *Finance and Stochastics*, **7**, 509-531.

[70] Fujiwara T. (2009). The minimal entropy martingale measure for exponential additive processes.*Asia-Pacific Financial Markets*, **16**, 65-95.

[71] Gerber H.U., Shiu E.S.W. (1994). Option pricing by Esscher transforms.*Trans. Soc. Actuaries*, **46**, 99-191.

[72] Grigelionis B. (1975). Characterization of random processes with conditionally independent increments. *Liet. Mat. Rinkinys*, **15, No 1**, 53-60.

[73] Grigelionis B. (1978). Additive Markov processes. *Liet. Mat. Rinkinys* **18, No 3**, 43-47.

[74] Guo X. (2001). Information and option pricing. *Quantitative Finance* **1**, 38-44.

[75] Hamilton J.D. (1989). A new approach to the economics analysis of non-stationary time series. *Econometrica* **57**, 357-384.

[76] Hansen R. N. (2003). Markov Controlled Excursions , Local Alignment and Structure. PhD Thesis, University of Copenhagen, 139p.

[77] Hardy, M. (2003). *Investment Guarantees : Modeling and Risk Management for Equity-linked Insurance Products.* Wiley & Sons, New Jersey.

[78] Harrison J.M., Kreps D.M. (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* **20**, 381-408.

[79] Harrison J.M.,Pliska S.R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and Their Applications* **11**, 215-260.

[80] Harrison J.M.,Pliska S.R. (1983). A Stochastic Calculus Model of Continuous Trading : Complete Markets. *Stochastic Processes and Their Applications* **15**, 313-316.

[81] Jacobsen E.R., Karlsen K.H. (2005) Continuous dependence estimates for viscosity solutions of integro-PDEs equations. *J. Differential Equations***212(2)**, 278-318.

[82] Jackson R. K., Jaimungal S., Surkov V. (2008) Fourier Space Time Stepping for Option Pricing with Levy Models. *Journal of Computational Finance* **vol. 12, Issue 2**, 1-29.

[83] Jacod J., Shiryaev A. N. (1987). *Limit theorems for stochastic processes*. Grundlehren der Mathematischen Wissenschaften, vol. 288, Springer-Verlag, Berlin, Heidelberg, New York.

[84] Jorion P. (1988). On Jump Processes in the The Foreign Exchange and Stock market. *Review of Financial Studies* **1(4)**, 427-445.

[85] Kallenberg O. (1997). *Foundations of Modern Probability*. Springer : New York.

[86] Kallsen J., Shiryaev A. (2002). The cumulant process and Esscher's change of measure. *Finance & Stochastics* **6**, 397-428.

[87] Kijima M., Yoshida T. (1993). A simple option pricing model with markovian volatilities. *Journal of the Operations Research, Society of Japan* **Vol.36, No 3**, 149-166.

[88] Konikov M., Madan D. B. (2002). Option Pricing using Variance-Gamma Markov Chains. *Review of Derivatives Research* **5**, 81-115.

[89] Kunita H. (1990). *Stochastic flows and Stochastic differential equations*. Cambridge Studies in advanced mathematics. vol 24, Cambridge University Press. Cambridge, New-York, Melbourne.

[90] Kunita H., Watanabe S. (1967). On the square integrables martingales. *Nagoya Mathematics Journal*, **vol. 30**, 209-245.

[91] Lin S. X.,Siu T.K., Yang H. (2009). Pricing Annuity Guarantees Under A Regime-Switching Model. *North American Actuarial Journal* **vol 13(2)**, 333- 337.

[92] Liu R.H., Zhang Q., Yin G. (2006). Option pricing in a regime-switching model using the fast Fourier transform, *Journal of Applied Mathematics and Stochastic Analysis*,**vol. 2006**, Article ID 18109doi :10.1155/JAMSA/2006/18109, 22 pages.

[93] Lowther G. (2010). 15 Septembre. In *almostsure.wordpress* Online ligne.<http ://almostsure.wordpress.com/2010/09/15/processes-with-independent-increments/>. visited at february, 11th 2011.

[94] MacBeth, J., and Merville, L. (1980). Tests of the Black-Scholes and Cox-Call Option Valuation Models.*Journal of Finance*, pp.285-301.

[95] McWalter T. A. (2006). *Quadratic Criteria for Optimal Martingales Measures in Incomplete Markets*. Master Thesis, University of Witwatersrand, 122p.

[96] Mandelbrot B. (1963). The variations of certain speculative price. *Journal of Business* **36**, 394-419.

[97] Menoukeu-Pamen O., Momeya R. (2011). Local Risk-Mimimization under a partially observed Markov-modulated Exponential Lévy Model. *Preprint series. Pure mathematics*, University of Oslo, 24p.

[98] Merton R. C. (1973). Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science* **vol.4, no 1**, 141-183.

[99] Merville L.J., Pieptea D.R. (1989). Stock-price volatility, mean-reverting diffusion, and noise.*Journal of Financial Economics* **242**, 193-214.

[100] Miyahara Y. (1999). Minimal Entropy Martingale Measures of Jump Type Price Processes in Incomplete Assets Markets. *Asia-Pacific Financial Markets*, **Vol. 6, No. 2**, 97-113.

[101] Momeya R., Morales M (2010). On the Price of Risk of the Underlying Markov Chain in a Regime-Switching Exponential Lévy Model. *Working Paper CRM* 3303, 37p.

[102] Momeya R., Ben Salah Z. (2012). The Minimal Entropy Martingale Measure (MEMM) for a Markov-Modulated Exponential Lévy Model. *Asia Pacific Financial Markets*, **Vol. 19, No. 1**, 63-98.

[103] Naik V. (1993). Option valuation and hedging strategies with jumps in the volatility of asset returns. *Journal of Finance* **48**, 1969-1984.

[104] Norris J.R. (1997). *Markov Chains.*Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press.

[105] Oksendal B., Sulem A. (2005). *Applied Stochasic Control of Jump Diffusions.* (Springer)

[106] Pacheco A., Tang L. C., Prabhu, N. U. (2009) *Markov-modulated processes and semigenerative phenomena.* Hackensack, NJ : World Scientific. xi, 224p.

[107] Pemy M. (2005). *Regime switching market models and applications.* PhD Dissertation, University of Georgia-USA, 118p.

[108] Pham H. (1998). Optimal stopping of controled jump diffusion processes : A viscosity solution approach. *J. Math. Systems. Estim. Control* **8(1)**,

[109] Pham H. (2001). Mean-variance hedging for partially observed drift processes. *Int. J. Theor. Appl. Finance* **4, no. 2**, 263-284.

[110] Pliska S. (1997) *Introduction to Mathematical Finance : Discrete Time Models.* New-York : Blackwell.

[111] Protter M. (2003). *Stochastic Integration and Differential Equations, Second Edition, Version 2.1.* (Stochastic Modelling and Applied Probability, vol 21. Springer).

[112] Rasmus S. (2006). Derivative Pricing for models using Lévy processes and Markov switching. *PhD Dissertation*, Lund University.

[113] Sato, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press : Cambridge, UK.

[114] Schweizer M. (1988). Hedging of Options in a General Semimartingale Model.*Diss. ETHZ no. 8615, Zurich.*

[115] Schweizer M. (1991). Options Hedging for semimartingales.*Stochastics and Stochastic Reports* **30**, 123-131.

[116] Schweizer M. (1992). Martingales Densities for general Asset Prices.*Journal of mathematical Economics* **21**, 339-378.

[117] Schweizer M. (1992). Mean-Variance Hedging for Generals Claims. *Annals of Applied Probability*, **2**, 1536-1575.

[118] Schweizer M. (1994). Risk-Minimizing Hedging Strategies under Restricted Information. *Mathematical Finance* **4**, 327-342.

[119] Schweizer M. (1999). A Guided Tour through Quadratic Hedging approaches. *In : Jouini E.J.C and Musiela M (Eds), Option pricing, Interest rates and Risk management.* Cambridge University Press, pp. 538-574.

[120] Shilley R.J. (2010). Online.<www.econ.yale.edu/ shiller/data/$ie_{d}ata.xls$>. Visited at July 04th 2011.

[121] Siu T.K. (2005). Fair Valuation of Participating Policies With Surrender Options and Regime Switching. *Insurance : Mathematics and Economics* **vol. 37, No 3**, 533-552.

[122] Siu T. K., Lau W.J., Yang H. (2008). Pricing participating products under a generalized jump-diffusion model. *Journal of Applied Mathematics and stochastic analysis* **vol. 2008**, 30 p.

[123] Siu T. K., Yang H. (2009). Option Pricing when the Regime-switching Risk is priced. *Acta Mathematicae Applicatae Sinica, English Series* **vol.25, No 3**, 339-388.

[124] Vandaele N., Vanmaele M. (2008). A locally risk-minimizing hedging strategy for unit-linked life insurance contracts in a Levy process financial market. *Insurance : Mathematics and Economics* **42**, 1128-1137.

[125] Voltchkova E. (2005). *Équations intégro-différentielles d'évolution : méthodes numériques et appication en finance.* PhD Dissertation, École Polytechnique-France, 188p.

[126] Yuan C., Mao X. (2004). Convergence of the Euler-Maruyama method for stochastic differential equations with Markovian switching. *Mathematics and Computers in simulation*, **64**, 223-235.