(Minimally) ϵ-Incentive Compatible Competitive Equilibria in Economies with Indivisibilities*

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Abstract

We consider competitive and budget-balanced allocation rules for problems where a number of indivisible objects and a fixed amount of money is allocated among a group of agents. In "small" economies, we identify under classical preferences each agent's maximal gain from manipulation. Using this result we find the competitive and budget-balanced allocation rules which are minimally manipulable for each preference profile in terms of any agent's maximal gain. If preferences are quasi-linear, then we can find a competitive and budget-balanced allocation rule such that for any problem, the maximal utility gain from manipulation is equalized among all agents.

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1 Introduction

Several seminal papers have investigated the manipulability of competitive mechanisms in classical exchange economies. Hurwicz (1972) has shown that in "small" finite economies any competitive mechanism is manipulable, i.e. for some economies some agents' profit from mispresenting their true preferences may be substantial. Roberts and Postlewaite (1976) have shown as when a small finite economy is replicated, then under certain assumptions, any competitive mechanism becomes limiting incentive compatible. More precisely, for any given $\varepsilon > 0$, there is a large enough economy such that the gains from manipulation do not exceed ε. Several subsequent papers have examined different qualifications of the result by Roberts and Postlewaite (1976).[1]

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[1] Among others, Jackson (1992), Manelli and Jackson (1997), Cordoba and Hammond (1998) and Kovalenkov (2002).

In this paper we consider economies with indivisible objects. Any agent's consumption bundle consists of an object and a monetary consumption. Such problems arise in rent division, job allocation, land distribution, and heritage division.[2] Specifically we are interested in investigating the manipulability of competitive (or fair) and budget-balanced allocation rules. From Green and Laffont (1979) it is known that any such rule is manipulable. Not only this, considering replica of economies with indivisible objects will neither change the set of competitive and budget-balanced allocations nor alter the amount by which any agent is able to manipulate any competitive and budget-balanced allocation rule. Therefore, we search for the rules which are minimally manipulable in the class of competitive and budget-balanced allocation rules in small finite economies.

Specifically, we determine by how much any agent can profit from manipulation for any competitive and budget-balanced rule. Namely, for any economy and any agent there exist competitive and budget-balanced allocations which maximize his utility in this set. Then this agent's gain from (optimal) manipulation is equal to the utility difference between this maximizing allocation and the allocation chosen by the rule for this economy. This result then allows us to show the existence of competitive and budget-balanced rules which are minimally manipulable in the following sense: for each preference profile the amount by which any agent can manipulate is minimal among all profitable manipulations of all competitive and budget-balanced rules. Under quasi-linear utilities, we show that there exists a competitive and budget-balanced allocation rule which for each utility profile equalizes the maximal utility gain from manipulation among all agents.

The paper is organized as follows. Section 2 introduces economies with indivisibilities and competitive and budget-balanced allocations. Section 3 characterizes any individual's maximizing competitive and budget-balanced allocations. Section 4 contains all our results regarding manipulation of competitive and budget-balanced allocation rules. Section 5 discusses our results.

2 Agents, Preferences and Allocations

Let $N = \{1, ..., n\}$ denote the finite set of agents and $M = \{1, ..., m\}$ denote the finite set of objects. Throughout we assume $|M| = |N|$.[3] Each agent $i \in N$ is assigned exactly one object $j \in M$. Each object $j \in M$ has a price denoted by p_j. Let $p \in \mathbb{R}^M$ denote the price vector for all objects in M. We call a price vector $p \in \mathbb{R}^M$ *budget-balanced* if $\sum_{j \in M} p_j = 0$.[4]

A consumption bundle is a pair $(j, p_j) \in M \times \mathbb{R}$ (which stands for consuming object j and paying price p_j (or receiving monetary compensation $-p_j$)).[5] Agent i's preference over consumption bundles are supposed to be represented by a continuous utility function $u_i : M \times \mathbb{R}^M \to \mathbb{R}$. Let $u_{ij}(p)$ denote the utility of agent i when consuming object j at price p_j (under the price vector p). The utility function is supposed to be strictly decreasing in prices, i.e. $u_{ij}(p) > u_{ij}(p')$ whenever $p_j < p'_j$. Moreover, for each agent $i \in N$ and for any two bundles (j, p_j) and (k, p_k), there exists a number $\beta \in \mathbb{R}$ such that agent i is indifferent between the bundles (j, p_j) and $(k, p_k + \beta)$, i.e. $u_{ij}(p) = u_{ik}(p')$ whenever $p'_k = p_k + \beta$. This means

[2] See e.g. Aragones (1995), Dufton and Larson (2011), Haake, Raith and Su (2000), Klijn (2000) and Jaramillo, Kayı and Klijn (2012).

[3] If $|N| > |M|$, then adding $|N| - |M|$ null objects does not alter our results.

[4] All our results remain valid when the budget constraint is replaced by $\sum_{j \in M} p_j = \alpha$ for some exogenously given number $\alpha \in \mathbb{R}$.

[5] When the budget constraint is $\sum_{j \in M} p_j = \alpha$, then (j, p_j) stands for consuming j and paying $-\frac{\alpha}{n} + p_j$.

that no object is infinitely desirable or undesirable for any agent. A list $u = (u_1, ..., u_n)$ of individual utility functions is a (utility) profile. We adopt the notational convention of writing $u = (u_i, u_{-i})$ for any $i \in N$. The set of utility profiles having the above properties is denoted by $\mathcal{U}$.

A feasible assignment $x : N \to M$ assigns every agent $i \in N$ an object $j \in M$ such that no object is assigned to more than one agent (or buyer). Let x_i denote the object assigned to agent i. Note that any feasible assignment assigns every agent one object.

An *allocation* consists of a price vector p and a feasible assignment x, denoted by (p, x) for short. An allocation (p, x) is *budget-balanced* if $\sum_{j\in M} p_j = 0$. An allocation rule is a function φ choosing for each utility profile $u \in \mathcal{U}$ a non-empty set of allocations $\varphi(u)$ such that for all $(p, x), (q, y) \in \varphi(u)$, we have $u_{ix_i}(p) = u_{iy_i}(q)$ for all $i \in N$, i.e. any two chosen allocations are utility-equivalent for all agents. Such a correspondence is called *essentially single-valued.*

Definition 1. At a given profile $u \in \mathcal{U}$, a competitive allocation (or a competitive equilibrium) is a price vector p and a feasible assignment x such that:

$$u_{ix_i}(p) \geq u_{ij}(p) \text{ for all } i \in N \text{ and all } j \in M.$$

If (p, x) is a competitive allocation, then p is a competitive price vector and x is a competitive assignment.

Let $\hat{\mathcal{F}}(u)$ denote the set of competitive allocations at a given profile $u \in \mathcal{U}$. It is well-known that $\hat{\mathcal{F}}(u)$ is a non-empty set for each profile $u \in \mathcal{U}$. In the remaining part of the paper we will only consider competitive and budget-balanced allocations. These allocations are a (non-empty) subset of $\hat{\mathcal{F}}(u)$ denoted by $\mathcal{F}(u)$. For convenience, in the following "competitive allocation(s)" implicitly stands for "competitive and budget-balanced allocation(s)".

A competitive (allocation) rule is a non-empty correspondence φ choosing for each profile $u \in \mathcal{U}$ a non-empty set of competitive allocations $\varphi(u) \subseteq \mathcal{F}(u)$ such that $u_{ix_i}(p) = u_{iy_i}(q)$ for all $i \in N$ and all $(p, x), (q, y) \in \varphi(u)$. Note that if a competitive rule chooses a unique allocation for each utility profile, then this rule is essentially single-valued.

Given $u \in \mathcal{U}$, allocation (p, x) is *efficient* if (p, x) is budget-balanced and there does not exist a budget-balanced allocation (q, y) such that $u_{iy_i}(q) \geq u_{ix_i}(p)$ for all $i \in N$ with strict inequality holding for some $k \in N$. By Svensson (1983), all competitive allocations are efficient.

3 Individual Utility Maximizing Competitive Allocations

For our purposes, it will turn out to be useful to characterize the utility maximizing competitive allocations for any (individual) agent $k \in N$. Obviously, for any agent $k \in N$ and for each profile $u \in \mathcal{U}$, there exists an allocation in $\mathcal{F}(u)$ which maximizes the utility of agent k. This follows simply from the fact that the set $\mathcal{F}(u)$ is compact under our assumptions. For any utility profile u, let $\phi^k(u)$ denote the set of competitive allocations which maximize the utility for agent k. In the remaining part of the paper, let (p^k, x^k) stand for some element in $\phi^k(u)$ unless otherwise stated.

For any $i, j \in N$ we write $i \to_{(p,x)} j$ if $u_{ix_i}(p) = u_{ix_j}(p)$, i.e., if agent i is indifferent between his consumption bundle and agent j's consumption bundle at (p, x). Now, to characterize allocation (p^k, x^k) in more detail, the following concepts from Andersson, Ehlers and Svensson (2010) will be useful.

Definition 2. Let (p,x) be a competitive allocation.

(i) An indifference chain at (p,x) consists of a tuple of distinct agents $g=(i_0,...,i_t)$ such that $i_0 \to_{(p,x)} i_1 \to_{(p,x)} \cdots \to_{(p,x)} i_t$.

(ii) Agent $i \in N$ is linked to agent $k \in N$ at (p,x) if there exists an indifference chain of type $(i_0,...,i_t)$ at (p,x) with $i=i_0$ and $i_t=k$.

(iii) (p,x) is agent k-linked if each agent $i \in N$ is linked to agent k.

An indifference chain is simply a sequence of agents such that any agent in the sequence is indifferent between his bundle and the bundle of the agent following him in the sequence. Indifference chains indirectly link agents via indifference in a sequence of linked agents. At agent k-linked competitive allocations each agent is linked to agent k through some indifference chain.

The following result establishes that for any utility profile, agent k-linked competitive allocations coincide with the set of equilibria which maximize agent k's utility among all competitive allocations.

Theorem 1. For each profile $u \in \mathcal{U}$, each $k \in N$ and each $(p,x) \in \mathcal{F}(u)$, we have:

$(p,x) \in \phi^k(u)$ if and only if (p,x) is agent k-linked.

Proof. Let $u \in \mathcal{U}$, $k \in N$ and $(p,x) \in \phi^k(u)$. First, we demonstrate that (p,x) is k-linked. To obtain a contradiction, suppose that (p,x) is not k-linked, i.e., that there is an agent $l \in N$ that is not linked to agent k. Let:

$$\mathcal{G} = \{i \in N : \ i \text{ is linked to } k \text{ at } (p,x)\} \cup \{k\}.$$

Because $k \in \mathcal{G}$ and $l \in N \setminus \mathcal{G}$, both $\mathcal{G}$ and $N \setminus \mathcal{G}$ are non-empty. It follows by construction that $u_{ix_i}(p) > u_{ix_j}(p)$ if $i \in N \setminus \mathcal{G}$ and $j \in \mathcal{G}$. From the Perturbation Lemma in Alkan, Demange and Gale (1991) it then follows that there exists another equilibrium $(q,y) \in \mathcal{F}(u)$ such that $q_{x_i} < p_{x_i}$ for all $i \in \mathcal{G}$. Then by Definition 1 and monotonicity it follows that:

$$u_{iy_i}(q) \geq u_{ix_i}(q) > u_{ix_i}(p) \text{ for all } i \in \mathcal{G}.$$

Because $k \in \mathcal{G}$ it follows that $u_{ky_k}(q) > u_{kx_k}(p)$, which contradicts the fact that $(p,x) \in \phi^k(u)$ and (p,x) maximizes k's utility among all competitive allocations. Hence, if $(p,x) \in \phi^k(u)$, then (p,x) is agent k-linked.

In showing the other direction, let $u \in \mathcal{U}$, $k \in N$, and $(p,x),(q,y) \in \mathcal{F}(u)$ be two k-linked competitive allocations. By the first part of the proof, without loss of generality, we may suppose $(p,x) \in \phi^k(u)$. Obviously, if $p=q$, then for all $i \in N$, $u_{ix_i}(p) = u_{iy_i}(q)$ and $(q,y) \in \phi^k(u)$.

Suppose that $p \neq q$. Since $\sum_{i \in M} p_i = \sum_{i \in M} q_i = 0$, the set $G = \{j \in M : p_j > q_j\}$ is non-empty.

We first show for all $i \in N$, if $x_i \in G$, then $y_i \in G$. To obtain a contradiction, suppose that $x_i \in G$ and $y_i \notin G$. But then by Definition 1 and monotonicity of u_i:

$$u_{ix_i}(q) > u_{ix_i}(p) \geq u_{iy_i}(p) \geq u_{iy_i}(q). \tag{1}$$

But this is a contradiction to $(q,y) \in \mathcal{F}(u)$. Hence, $y_i \in G$.

Let $H = \{i \in N : x_i \in G\}$. By $|N| = |M|$ we have $H \neq \emptyset$. Now for $i \in H$, we have by (1), $y_i \in G$.

First, let $k \in H$. Because (p, x) is k-linked, there exist $i \in N \backslash H$ and $j \in H$ such that $i \rightarrow_{(p,x)} j$. But then we have:

$$u_{iy_i}(q) \leq u_{iy_i}(p) \leq u_{ix_i}(p) = u_{ix_j}(p) < u_{ix_j}(q),$$

where the first inequality follows from $i \in N \backslash H$ and $q_{y_i} \geq p_{y_i}$, the second inequality from competitiveness of (p, x), the equality from $i \rightarrow_{(p,x)} j$, and the last inequality from $x_j \in G$ and $p_{x_j} > q_{x_j}$. Now (q, y) is not competitive, a contradiction.

If $k \in N \backslash H$, then we obtain similarly a contradiction to competitiveness of (p, x) using the k-linked allocation (q, y). □

Remark 1. Since agent k utility maximizing competitive allocations exist for any profile $u \in \mathcal{U}$ (because $\mathcal{F}(u)$ is compact), it is clear that agent k-linked competitive allocations exist for any profile $u \in \mathcal{U}$. In addition, the proof of Theorem 1 has shown, if (p, x) and (q, y) are k-linked, then $p = q$ and all allocations in $\phi^k(u)$ are utility equivalent. Hence, the price vector at the agent k utility maximizing allocations is unique. Note, however, that competitive allocations maximizing the utility of agent k need not be unique because there may be several utility equivalent competitive assignments.

4 Manipulability

It is well-known from Green and Laffont (1979) that any efficient and budget-balanced allocation rule is manipulable for some profile $u \in \mathcal{U}$. Since competitive (and budget-balanced) allocation rules are efficient, this result implies that each competitive allocation rule is manipulable for some profile $u \in \mathcal{U}$. Manipulability (and non-manipulability) in this context refers to the following.

Definition 3. An allocation rule φ is manipulable at a profile $u \in \mathcal{U}$ by an agent $i \in N$ if there exists a profile $(\hat{u}_i, u_{-i}) \in \mathcal{U}$ and two allocations $(p, x) \in \varphi(u)$ and $(q, y) \in \varphi(\hat{u}_i, u_{-i})$ such that $u_{iy_i}(q) > u_{ix_i}(p)$. If the allocation rule φ is not manipulable by any agent at any profile $u \in \mathcal{U}$, then φ is said to be incentive compatible (or non-manipulable).[6]

A natural weakening of incentive compatibility is ε-incentive compatibility where no agent can gain by more than ε from manipulation.

Definition 4. Let $\varepsilon \geq 0$. An allocation rule φ is ε-incentive compatible at a profile $u \in \mathcal{U}$ if for all $i \in N$ and any profile $(\hat{u}_i, u_{-i}) \in \mathcal{U}$, and any $(p, x) \in \varphi(u)$ and $(q, y) \in \varphi(\hat{u}_i, u_{-i})$, we have $u_{iy_i}(q) \leq u_{ix_i}(p) + \varepsilon$. If the allocation rule φ is ε-incentive compatible at any profile $u \in \mathcal{U}$, then φ is said to be ε-incentive compatible.

Note that 0-incentive compatibility is identical with incentive compatibility.

Because each allocation rule φ that makes a selection from the set $\mathcal{F}(u)$ is manipulable it is important to characterize exactly how much agents can gain from strategic misrepresentation. The next result states that if agent $k \in N$ manipulates the competitive rule, then the agent must be assigned an object whose price has decreased.

[6]Note that for single-valued rules (which choose for each profile a unique allocation), Definition 3 may be rewritten as follows: φ is manipulable at a profile $u \in \mathcal{U}$ by an agent $i \in N$ if there exists a profile $(\hat{u}_i, u_{-i}) \in \mathcal{U}$ such that for $\{(p, x)\} = \varphi(u)$ and $\{(q, y)\} = \varphi(\hat{u}_i, u_{-i})$ we have $u_{iy_i}(q) > u_{ix_i}(p)$.

Lemma 1. For any competitive allocation rule φ, for any profile $u \in \mathcal{U}$ and for any agent $k \in N$, we have:

(i) If there exist $(p, x) \in \varphi(u)$ and $(q, y) \in \varphi(\hat{u}_k, u_{-k})$ such that $u_{ky_k}(q) > u_{kx_k}(p)$, then $q_{y_k} < p_{y_k}$.

(ii) If there exist $(p, x) \in \varphi(u)$ and $(q, y) \in \varphi(\hat{u}_k, u_{-k})$ such that $u_{ky_k}(q) \geq u_{kx_k}(p)$, then $q_{y_k} \leq p_{y_k}$.

Proof. We only show (i) since (ii) can be shown similarly. Let $k \in N$, $u \in \mathcal{U}$ and φ be an arbitrary competitive allocation rule. Suppose that $(p, x) \in \varphi(u)$, $(q, y) \in \varphi(\hat{u}_k, u_{-k})$ and $u_{ky_k}(q) > u_{kx_k}(p)$. Then by Definition 1 it follows that:

$$u_{ky_k}(q) > u_{kx_k}(p) \geq u_{ky_k}(p).$$

This and monotonicity yields $q_{y_k} < p_{y_k}$ which concludes the proof. □

The following result states that each agent $k \in N$ for each profile $u \in \mathcal{U}$ can manipulate any competitive allocation rule φ at most by receiving the utility that agent k receives from his utility maximizing competitive allocation (p^k, x^k) at profile $u \in \mathcal{U}$, i.e. by Theorem 1, the agent k-linked competitive allocation at profile $u \in \mathcal{U}$.

For any profile $u \in \mathcal{U}$ and any $(p, x) \in \varphi(u)$, let:

$$f_k(\varphi, u) = \sup_{\hat{u}_k} \min_{(q,y) \in \varphi(\hat{u}_k, u_{-k})} u_{ky_k}(q) - u_{kx_k}(p).$$

Theorem 2. For any competitive allocation rule φ, for any profile $u \in \mathcal{U}$ and for any agent $k \in N$, we have for $(p^k, x^k) \in \phi^k(u)$ and $(p, x) \in \varphi(u)$, $f_k(\varphi, u) = u_{kx^k_k}(p^k) - u_{kx_k}(p)$.

Proof. To prove the result, let $k \in N$, $u \in \mathcal{U}$, φ be a competitive rule, and $(p, x) \in \varphi(u)$.

Take some $\hat{u}_k$ and some $(q, y) \in \varphi(\hat{u}_k, u_{-k})$. If:

$$u_{ky_k}(q) - u_{kx_k}(p) > u_{kx_k}(p^k) - u_{kx_k}(p),$$

then $u_{ky_k}(q) > u_{kx_k}(p^k)$ which would mean the agent k-linked competitive rule is manipulable by agent k, which is a contradiction to Andersson, Ehlers and Svensson (2010, Corollary 1)[7]. Thus, $f_k(\varphi, u) \leq u_{kx^k_k}(p^k) - u_{kx_k}(p)$.

Suppose that $(p^k, x^k) \in \phi^k(u)$. In the remaining part of the proof, for all $z \in \mathbb{R}^M$ let $\hat{u}^\varepsilon_{kj}(z) = p^k_j - z_j$ for all $j \in M \setminus \{x^k_k\}$ and $\hat{u}^\varepsilon_{kx^k_k}(z) = p^k_{x^k_k} - z_{x^k_k} + \varepsilon$ for some "small" $\varepsilon > 0$.

Note first that $(p^k, x^k) \in \mathcal{F}(\hat{u}^\varepsilon_k, u_{-k})$. This follows since $(p^k, x^k) \in \mathcal{F}(u)$ and:

$$\hat{u}^\varepsilon_{kx^k_k}(p^k) = \varepsilon > 0 = \hat{u}^\varepsilon_{kj}(p^k) \tag{2}$$

for all $j \in M \setminus \{x^k_k\}$ by construction.

Second, by $(p^k, x^k) \in \phi^k(u)$ and Theorem 1, (p^k, x^k) is k-linked under u. But now (p^k, x^k) is k-linked under $(\hat{u}^\varepsilon_k, u_{-k})$ and again by Theorem 1, $(p^k, x^k) \in \phi^k(\hat{u}^\varepsilon_k, u_{-k})$. Thus, ε is agent k's maximal utility in $\mathcal{F}(\hat{u}^\varepsilon_k, u_{-k})$.

Let $(q, y) \in \varphi(\hat{u}^\varepsilon_k, u_{-k})$. If $p^k_{x^k_k} > q_{x^k_k}$, then $u_{ky_k}(q) > \varepsilon$, which contradicts the fact that ε is agent k's maximal utility in $\mathcal{F}(\hat{u}^\varepsilon_k, u_{-k})$.

[7] Their arguments remain valid in our setting.

Thus, $p^k_{x^k_k} \leq q_{x^k_k}$. We show that $y_k = x^k_k$. Suppose that $y_k \neq x^k_k$. If $q = p^k$, then $u_{ky_k}(q) = 0 < \varepsilon = u_{kx^k_k}(q)$, a contradiction. Thus, $q \neq p^k$. By budget-balance, now $u_{ky_k}(q) > 0 = u_{ky_k}(p^k)$. Thus, $p^k_{y_k} > q_{y_k}$. Let $j \in N$ be such that $x^k_j = y_k$. Now we have:

$$u_{jy_j}(q) \geq u_{jy_k}(q) > u_{jy_k}(p^k) \geq u_{jy_j}(p^k)$$

where the weak inequalities follow from competitiveness and the strict inequality from $p^k_{y_k} > q_{y_k}$. Thus, $p^k_{y_j} > q_{y_j}$. Now again let $l \in N$ be such that $x^k_l = y_j$. Using the same arguments it can be shown $p^k_{y_l} > q_{y_l}$. Continuing iteratively, now for some $h \in N$ we must (cycle and) have $y_h = x^k_k$. But now again $p^k_{y_h} > q_{y_h}$, or $p^k_{x^k_k} > q_{x^k_k}$, which is a contradiction to $p^k_{x^k_k} \leq q_{x^k_k}$.

Thus, $y_k = x^k_k$ and $p^k_{x^k_k} \leq q_{x^k_k}$. If $p^k_{x^k_k} = q_{x^k_k}$, then $f_k(\varphi, u) \geq u_{kx^k_k}(p^k) - u_{kx_k}(p)$, the desired conclusion. Let $p^k_{x^k_k} < q_{x^k_k}$ But now we have:

$$\hat{u}^\varepsilon_{kx^k_k}(q) = p^k_{x^k_k} - q_{x^k_k} + \varepsilon \geq \max_{j \in M} p^k_j - q_j > 0,$$

where the first inequality follows from construction of $\hat{u}^\varepsilon_k$ and competitiveness of (q, y), and the last inequality follows from $p^k_{x^k_k} < q_{x^k_k}$ and $\sum_{j \in M} q_j = 0 = \sum_{j \in M} p^k_j$. Now as $\varepsilon \to 0$, we must have $q_{x^k_k} \to_{\varepsilon \to 0} p_{x^k_k}$. Thus, by $y_k = x^k_k$,

$$\lim_{\varepsilon \to 0} u_{ky_k}(q) = \lim_{\varepsilon \to 0} u_{kx^k_k}(q) = u_{kx^k_k}(p^k),$$

and $f_k(\varphi, u) \geq u_{kx^k_k}(p^k) - u_{kx_k}(p)$, the desired conclusion. □

Theorem 2 yields as corollary that if some agent i's profit from manipulation is greater for one rule than for a second rule, then there is another agent j whose profit from manipulation is smaller for the first rule than for the second one.

Corollary 1. For each profile $u \in \mathcal{U}$ and for any two competitive allocation rules φ and ψ, it holds that: if $f_i(\varphi, u) > f_i(\psi, u)$ for some $i \in N$, then $f_j(\varphi, u) < f_j(\psi, u)$ for some $j \in N$.

Proof. Let $(p, x) \in \varphi(u)$ and $(q, y) \in \psi(u)$. Suppose that the statement is not true, i.e. that for all $l \in N$, $f_l(\varphi, u) \geq f_l(\psi, u)$. By Theorem 2, we have then for all $l \in N$, $u_{lx_l}(p) \leq u_{ly_l}(q)$, and $u_{ix_i}(p) < u_{iy_i}(q)$. But now (x, p) is not efficient, which contradicts the fact that all competitive and budget-balanced allocations are efficient (Svensson, 1983). □

Theorem 2 characterizes the exact amount by which an agent may manipulate an arbitrary competitive rule. In applications, we may want to minimize the gains from manipulation for all agents in the spirit of Definition 4, i.e. identifying a smallest global bound and a competitive rule such that for any given profile no agent can manipulate by more than this bound (and we cannot find another competitive rule with a smaller bound). Of course, by Theorem 2 this approach is fruitless because utilities are arbitrary and for any $\varepsilon \geq 0$ there does not exist a competitive allocation rule which is ε-incentive compatible at any given profile. Instead we follow below a local bound approach, i.e. where the bound is dependent on the given profile and we minimize this bound for any profile.

4.1 Existence

Because each agent can manipulate an arbitrary competitive allocation rule it is natural to ask if there is an allocation rule that is better than other from the viewpoint of manipulability. In some recent papers this issue has been investigated by minimizing the number of profiles in $\mathcal{U}$ for which the rule is manipulable (see Aleskerov and Kurbanov (1999) and Maus, Peters and Storcken (2007a,b)), by minimizing the domain (with respect to inclusion) on which the rule is manipulable (Pathak and Sönmez, 2011), and by finding rules that prevent the most agents and coalitions of agents by gaining from misrepresentation (Andersson, Ehlers and Svensson, 2010). Here, we have a somewhat different approach and instead search for rules that minimize the maximal gain that any agent can obtain by strategic misrepresentation. This maximal gain is given by the functions of type f is given in Theorem 2. Hence, the aim is to identify a rule satisfying the following:

Definition 5. Let $\epsilon : \mathcal{U} \to \mathbb{R}_+$.

(i) A rule φ is ϵ-incentive compatible if for any profile $u \in \mathcal{U}$ we have $\max_{i \in N} f_i(\varphi, u) \leq \epsilon(u)$.

(ii) A competitive rule ψ is minimally ϵ-incentive compatible if for any $\epsilon' : \mathcal{U} \to \mathbb{R}_+$ and any competitive rule φ which is ϵ'-incentive compatible, we have for any $u \in \mathcal{U}$, $\epsilon(u) \leq \epsilon'(u)$.

Alternatively, Definition 5 means finding a competitive rule ψ such that for all $u \in \mathcal{U}$,

$$\psi = \arg\min_{\varphi} \max_{i \in N} f_i(\varphi, u) \tag{3}$$

The following theorem establishes the existence of such rule for each profile in $\mathcal{U}$. We state the result without a proof since the result follows directly from the fact that the set $\mathcal{F}(u)$ is compact.

Theorem 3. There exists a competitive allocation rule ψ solving (3) for each $u \in \mathcal{U}$.

4.2 Quasi-Linear Utilities

To obtain more specific results, we shall consider the subclass of quasi-linear utility functions $\mathcal{U}^q \subset \mathcal{U}$: $u \in \mathcal{U}^q$ if and only if for each $i \in N$ there exists $v_i \in \mathbb{R}^M$ such that for all $p \in \mathbb{R}^M$ and all $j \in M$,

$$u_{ij}(p) = v_{ij} - p_j.$$

Under quasi-linear utility functions, the following result from Svensson (2009, Proposition 2) will be useful.[8]

Lemma 2. For each profile $u \in \mathcal{U}$ and all $(p, x), (q, y) \in \mathcal{F}(u)$, we have $(p, y), (q, x) \in \mathcal{F}(u)$.

Using Theorem 2 and Lemma 2, we can simplify the maximal manipulation possibility of agent k under quasi-linear utilities as follows: for any arbitrary competitive and budget-balanced allocation rule φ, any agent $k \in N$, and any $u \in \mathcal{U}^q$, let $(p, x) \in \varphi(u)$ and $(p^k, x^k) \in$

[8]Furthermore, by Svensson (2009, Proposition 3), Definition 3 may be rewritten as follows on the domain of quasi-linear utilities: φ is manipulable at a profile $u \in \mathcal{U}^q$ by an agent $i \in N$ if there exists a profile $(\hat{u}_i, u_{-i}) \in \mathcal{U}^q$ such that for all $(p, x) \in \varphi(u)$ and all $(q, y) \in \varphi(\hat{u}_i, u_{-i})$ we have $u_{iy_i}(q) > u_{ix_i}(p)$.

$\phi^k(u)$. Now by Lemma 2 we have $(p, x^k) \in \mathcal{F}(u)$ and $u_{kx_k}(p) = u_{kx_k^k}(p)$. Now we obtain (using quasi-linearity):

$$f_k(\varphi, u) = p_{x_k^k} - p^k_{x_k^k}.$$

The following result establishes that if the manipulation possibilities (defined by the function f) for one agent decreases, then the manipulation possibilities for some other agent (again defined by the function f) must increase. Or put differently, the sum of manipulation possibilities, at a given profile, is always constant independently of which competitive allocation rules that are considered.

Theorem 4. Let φ and ψ be two competitive and budget-balanced allocation rules. Then for each profile $u \in \mathcal{U}^q$ it holds that:

$$\sum_{i\in N} f_i(\varphi, u) = \sum_{i\in N} f_i(\psi, u). \tag{4}$$

Proof. Let $i, j \in N$, $(p^i, x^i) \in \phi^i(u)$ and $(p^j, x^j) \in \phi^j(u)$. By Lemma 2, we have $(p^j, x^i) \in \mathcal{F}(u)$ and obviously $(p^j, x^i) \in \phi^j(u)$. Thus, without loss of generality, we may assume $x^i = x^j$ for all $i, j \in N$. From the definition of the function f we obtain:

$$\sum_{i\in N} f_i(\varphi, u) = \sum_{i\in N}(p_{x_i^i} - p^i_{x_i^i}) = \sum_{i\in N} p_{x_i^i} - \sum_{i\in N} p^i_{x_i^i},$$
$$\sum_{i\in N} f_i(\psi, u) = \sum_{i\in N}(q_{x_i^i} - p^i_{x_i^i}) = \sum_{i\in N} q_{x_i^i} - \sum_{i\in N} p^i_{x_i^i}.$$

Thus, (4) holds by essentially single-valuedness and Remark 1 if:

$$\textstyle\sum_{i\in N} p_{x_i^i} = \sum_{i\in N} q_{x_i^i}. \tag{5}$$

We have $x_i^i \neq x_j^j$ for any $j \in N$ where $j \neq i$. Hence, by Definition 1, feasibility and budget-balance:

$$\textstyle\sum_{i\in N} p_{x_i^i} = \sum_{i\in N} q_{x_i^i} = 0. \tag{6}$$

This together with (5) yields the desired conclusion. □

Under quasi-linearity there exists a competitive allocation rule where for each profile $u \in \mathcal{U}^q$ the manipulation possibilities are equal for all agents.

Theorem 5. For each profile $u \in \mathcal{U}^q$ and there exists a competitive allocation rule φ where:

$$f_i(\varphi, u) = f_j(\varphi, u) \text{ for all } i, j \in N. \tag{7}$$

Proof. Let $u \in \mathcal{U}^q$. By Lemma 2, if $(p, x) \in \mathcal{F}(u)$ and $(q, y) \in \mathcal{F}(u)$, then $(p, y) \in \mathcal{F}(u)$ and $(q, x) \in \mathcal{F}(u)$. For this reason we shall assume in the remaining part of the proof, without loss of generality, that the feasible assignment is constant and given by x for all competitive allocations in $\mathcal{F}(u)$.

We need to show that there exists a competitive allocation $(p, x) \in \mathcal{F}(u)$ such that:

$$p_{x_i} - p^i_{x_i} = p_{x_j} - p^j_{x_j} \text{ for all } i, j \in N.$$

Consider now the utility maximizing price vectors $p^1_{x_1}, ..., p^n_{x_n}$ for agents $1, ..., n$, respectively, at profile $u \in \mathcal{U}^q$ and note that they are unique by Remark 1. Since preferences are quasi-linear, we now have for all $i \in N$ and all $(p, x) \in \mathcal{F}(u)$,

$$p^i_{x_i} \leq p_{x_i}. \tag{8}$$

Thus, for any $(p, x) \in \mathcal{F}(u)$, we have $\sum_{i \in N} p^i_{x_i} \leq \sum_{i \in N} p_{x_i} = 0$. Now, obviously there exists $\varepsilon \geq 0$ such that:

$$\sum\nolimits_{i \in N} (p^i_{x_i} + \varepsilon) = 0. \tag{9}$$

Let $p^\varepsilon \in \mathbb{R}^M$ be the price vector where $p^\varepsilon_{x_i} = p^i_{x_i} + \varepsilon$ for each $i \in N$ such that (9) holds. To complete the proof we need to demonstrate that $(p^\varepsilon, x) \in \mathcal{F}(u)$. To obtain a contradiction, suppose that $(p^\varepsilon, x) \notin \mathcal{F}(u)$. Then:

$$v_{ix_i} - p^\varepsilon_{x_i} < v_{jx_j} - p^\varepsilon_{x_j} \text{ for some } i, j \in N. \tag{10}$$

From the definition of p^ε we obtain that:

$$v_{ix_i} - p^i_{x_i} - \varepsilon < v_{ix_j} - p^j_{x_j} - \varepsilon,$$

i.e. (using $-p^j_{x_i} \leq -p^i_{x_i}$ from (8)),

$$v_{ix_i} - p^j_{x_i} \leq v_{ix_i} - p^i_{x_i} < v_{ix_j} - p^j_{x_j},$$

which contradicts $(p^j, x) \in \mathcal{F}(u)$. Hence, $(p^\varepsilon, x) \in \mathcal{F}(u)$, the desired conclusion. □

The rule described in the previous proof can be related to the constrained equal losses rule (Aumann and Maschler, 1985; Hokari and Thomson, 2003; Thomson, 2003) in bankruptcy problems. To see this connection, suppose that all agents in N are asked by the mechanism designer to select a competitive allocation at profile $u \in \mathcal{U}^q$. Obviously, each agent $k \in N$ would suggest (or claim) an allocation $(p^k, x^k) \in \phi^k(u)$, i.e., a competitive allocation that maximizes agent k's utility. Again, as above we can fix an assignment x. Now obviously $(p^k, x) \in \phi^k(u)$ and without loss of generality, we may suppose $x^k = x$ for all $k \in N$. Now we simply set for all $k \in N$,

$$p_{x_k} = p^k_{x_k} + \lambda(u) \text{ and } \lambda(u) \text{ is chosen so that } \sum_{k \in N} (p^k_{x_k} + \lambda(u)) = 0.$$

In this sense each agent incurs an equal loss of $\lambda(u)$ between the chosen competitive allocation and the allocations that maximize his utility among all competitive allocations. Setting $\epsilon(u) = \lambda(u)$ for any $u \in \mathcal{U}^q$, Theorem 4 implies that the above rule is minimally ϵ-incentive compatible in the class of competitive rules on the domain of quasi-linear preferences.

5 Discussion

5.1 Replication of Economies

One of our motivations for our paper was that replicating the economy does not alter the gains from manipulation of competitive and budget-balanced allocation rules. In other words, even as the economy becomes large leaves the manipulation possibilities unchanged and in determining the minimally manipulable competitive and budget-balanced allocation rules we need to do this for small economies. To formalize this point, let $E = (N, M, u)$ denote the original economy. Let $E^{<t>}$ denote the t-replica of E with tN agents (each agent $i \in N$ is replicated t times), with tM objects (each object $j \in M$ is replicated t times) and each replica of agent i has i's utility function u_i (where agents have identical utilities for an object j and its replicas). Similarly, for an allocation (p, x) of E let $(p, x)^{<t>}$ stand for the allocation of $E^{<t>}$ where any replica of agent $i \in N$ receives the replica of i's consumption bundle (x_i, p_{x_i}) in (p, x) (and i receives (x_i, p_{x_i})).

The following observations are straightforward: if (p, x) is a competitive and budget-balanced allocation in E, then $(p, x)^{<t>}$ is a competitive and budget-balanced allocation in $E^{<t>}$. Thus, for any agent k, the utility of his maximizing competitive and budget-balanced allocations in E is smaller than or equal to the utility of his maximizing competitive and budget-balanced allocations in $E^{<k>}$. In fact, these utilities must be equal as the following argument shows[9]:

Consider E and $E^{<2>}$ and suppose that some allocation (p, x) in $E^{<2>}$ maximizes agent k's utility among all competitive and budget-balanced allocations in $E^{<2>}$. Note that (p, x) does not need to be a 2-replica of some allocation in E. Since (p, x) is competitive, now any two agents who receive the replica of the same object must receive identical prices. Setting $2M = M \cup \{j' : j \in M\}$, we have $p_j = p_{j'}$ for all $j \in M$. But then by budget-balance of (p, x) we must have $\sum_{j \in M} p_j = 0$. Now we construct from (p, x) an allocation for E as follows (again setting $2N = N \cup \{i' : i \in N\}$): for any agent $i \in N$, if his replica receives the same consumption bundle as i, then just drop i' and his consumption bundle; otherwise choose the agent (l or l') who receives the same consumption bundle as i' and assign to l the consumption bundle of i' and drop l' and the one consumption bundle of i'; now l or l' received an object different than i' and we repeat the procedure for this object; at some point there will be a cycle (going back to i) and we simply keep i's consumption bundle unchanged. Now this gives us an allocation for E which is competitive and budget-balanced. Since we chose an allocation with maximal utility of agent k in the set of competitive and budget-balanced allocations in $E^{<2>}$, now this utility must be identical as in $\phi^k(u)$.

Of course, the above argument is true for E, $E^{<2>}$, $E^{<4>}$, $E^{<8>}, \ldots, E^{<2^t>}, \ldots$, i.e. using the first fact, in E and in all replicas $E^{<t>}$ the maximal utility of agent k among all competitive and budget-balanced allocations is identical. Hence, Theorem 2 applies and the gains from manipulation remain unchanged for competitive and budget-balanced allocation rules in E and $E^{<t>}$.

5.2 Competitive and Minimal Rules

Disregarding budget-balance, there is an interesting connection between Theorem 1 and a general non-manipulability result for competitive and minimal (or fair and maximal) allocation rules. More explicitly, as demonstrated by Andersson and Svensson (2008) and Svensson (2009), the competitive and minimal allocation rule (described below), introduced by Sun and Yang (2003), completely characterizes the class of competitive and non-manipulable allocation rules. The main difference between this rule and a competitive and budget-balanced rule is that the former always selects the utility maximizing allocation for each agent $k \in N$ at any profile $u \in \mathcal{U}$ in sharp contrast to the latter rule. This is also the underlying reason why

[9] We omit the formal details which are available from the authors upon request.

competitive and minimal allocation rules are globally non-manipulable whereas competitive and budget-balanced rules are manipulable.

To adopt the competitive and minimal allocation rule, it is required that each object $j \in M$ has an exogenously given minimum price limit $\underline{p}_j$ such that $\sum_{j \in M} \underline{p}_j = 0$. These minimal prices are gathered in the vector $\underline{p} \in \mathbb{R}^M$. Then, for a given profile $u \in \mathcal{U}$, the allocation $(p,x) \in \hat{\mathcal{F}}(u)$ is competitive and minimal with respect to $\underline{p}$ if $\sum_{j \in M} p_j$ is minimal subject to $p \geq \underline{p}$. A competitive and minimal allocation rule φ is a rule that always selects competitive allocations that are minimal with respect $\underline{p}$. Note that the allocations selected by this rule are typically not budget-balanced.

Let $\hat{\phi}^k(u)$ be the set of utility maximizing allocations of agent k that are competitive and minimal with respect to $\underline{p}$ at profile $u \in \mathcal{U}$. The following shows that competitive and minimal allocation rules always select for each agent a utility maximizing allocation in the set of competitive and minimal allocations. This gives a deeper reason why these rules are (globally) non-manipulable.

Theorem 6. Let φ be competitive and minimal with respect to $\underline{p}$. Then for all $u \in \mathcal{U}$ and all $(p,x) \in \varphi(u)$, we have $(p,x) \in \hat{\phi}^k(u)$ for all $k \in N$.

Proof. Suppose that $(p,x) \notin \hat{\phi}^k(u)$. This means that there is an allocation $(q,y) \in \hat{\mathcal{F}}(u)$ that is competitive and minimal with respect to $\underline{p}$ such that $u_{iy_i}(q) > u_{ix_i}(p)$. Since (p,x) is competitive, we have:

$$u_{iy_i}(q) > u_{ix_i}(p) \geq u_{iy_i}(p), \tag{11}$$

implying $\underline{p}_{y_i} \leq q_{y_i} < p_{y_i}$. Let $G = \{j \in M : q_j < p_j\}$ and $H = \{i \in N : u_{ij}(p) \geq u_{il}(p)$ for all $k \in M$ and some $j \in G\}$.

By (11), $y_i \in G$. Hence, $G \neq \emptyset$. Moreover, $H \neq \emptyset$ by competitiveness of (p,x). Suppose now that $k \in H$. Then by competitiveness and monotonicity:

$$u_{ky_k}(q) \geq u_{kj}(q) > u_{kj}(p) = u_{kx_k}(p) \geq u_{ky_k}(p) \text{ for some } j \in G.$$

Hence, $y_k \in G$. Consequently, if $k \in H$, then $y_k \in G$, implying that $|H| \leq |G|$. But this is a contradiction to (p,x) being competitive and minimal with respect to $\underline{p}$ because in this case $|H| > |G|$ by Lemma 4 in Andersson and Svensson (2008). Hence, $(p,x) \in \hat{\phi}^k(u)$. □

References

Aleskerov, F., Kurbanov, E., 1999. Degree of manipulability of social choice procedures. In: Proceedings of the Third International Meeting of the Society for the Advancement of Economic Theory. Springer: Berlin.

Alkan, A., Demange, G., Gale, D., 1991. Fair allocation of indivisible objects and criteria of justice. Econometrica 59, 1023–1039.

Andersson, T., Svensson, L-G., 2008. Non-manipulable Assignment of Individuals to Positions Revisited. Math. Soc. Sciences 56, 350–354.

Andersson, T., Ehlers, L., Svensson, L.-G., 2010. Budget-balance, fairness and minimal manipulability. Working Paper, CIREQ Cahier 18-2010 (revised October 2011).

Aragones, E., 1995. A derivation of the money Rawlsian solution. Soc. Choice Welfare 12, 267-276.

Aumann, R., Maschler, M., 1985. Game theoretic analysis of a bankruptcy problem from the Talmud. J. Econ. Theory 36, 195–213.

Dufton, L., Larson, K., 2011. Randomized room assignment-rent division. Working Paper.

Green, J., Laffont, J.-J., 1979. Incentives in Public Decision Making. North-Holland: Amsterdam.

Fujinaka, Y., Wakayama, T., May 2011. Maximal manipulation in fair allocation. Working Paper.

Haake, C.-J., Raith, M., Su, F., 2003. Bidding for envy-freeness: a procedural approach to n-player fair division problems. Soc. Choice Welfare 19, 723–749.

Hokari, T., Thomson, W., 2003. Claims problems and weighted generalizations of the Talmud rule. Econ. Theory 21, 241–261.

Hurwicz, L., 1972. On informationally decentralized systems. In Decision and Organisation: A Volume in Honor of Jacob Marschak, ed. by C.B. McGuire and Roy Radnes. Amsterdam: North Holland.

Jackson, M.O., 1992. Incentive compatibility and competitive allocations, Econ. Letters 40, 299–302.

Jackson, M.O., Manelli, A.M., 1997. Approximately competitive equilibria in large finite economies, J. Econ. Theory 77, 354–376.

Jaramillo, P., Kayı, Ç., Klijn, F., 2012. Asymmetrically fair rules for an indivisible good problem with a budget constraint. Working paper.

Klijn, F., 2000. An algorithm for envy-free allocations in an economy with indivisible objects and money. Soc. Choice Welfare 17, 201–216.

Kovalenkov, A., 2002. Simple strategy-proof approximately Walrasian mechanisms, J. Econ. Theory 103, 475–487.

Maus, S., Peters, H., Storcken, T., 2007a. Minimal manipulability: anonymity and unanimity. Soc. Choice Welfare 29, 247–269.

Maus, S., Peters, H., Storcken, T., 2007b. Anonymous voting and minimal manipulability. J. Econ. Theory 135, 533–544.

Pathak, P.A., Sönmez, T. 2011., School Admissions Reform in Chicago and England: Comparing Mechanisms by their Vulnerability to Manipulation. Amer. Econ. Rev., forthcoming.

Roberts, D.J., Postlewaite, A., 1976. The incentives for price-taking behavior in large exchange economies. Econometrica 44, 115–127.

Sun, N., Yang, Z., 2003. A general strategy proof fair allocation mechanism. Econ. Letters 81, 73–79.

Svensson, L.-G., 1983. Large indivisibilities: an analysis with respect to price equilibrium and fairness. Econometrica 51, 939–954.

Svensson, L.-G. 2009. Coalitional strategy-proofness and fairness. Econ. Theory 40, 227–245.

Thomson, W., 2003, Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Math. Soc. Sciences 45, 249–297.