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Quaternionic Kähler Manifolds, Constrained Instantons, and the Magic Square

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DEDICATION

To the most extraordinary people I know: my parents, my sister and my brother.

We are lost in space and no longer follow certain paths.
So be it! When we fail, it is best to fall from the skies.
Girodet

SUMMARY

We define the concepts of real, Kähler, and quaternionic manifolds from a mathematical and physical point of view. We then introduce the results of previous classifications of these spaces and present the new technique we developed to classify these manifolds. Our method relies on the existence of constrained instantons, Seiberg-Witten curves, and the use of Freudenthal, Rosenfeld, and Tits' magic square. We conclude by arguing that our classification method reproduces the results of the previous classifications and show how it also leads to the discovery of a new set of Kähler manifolds.

Keywords: Mathematical physics, manifolds, instantons, quaternionic, Kähler, magic square, Seiberg-Witten curves.

SOMMAIRE

Nous définissons les notions de variétés différentiables réelles, de Kähler et quaternioniques d'un point de vue mathématique ainsi que physique. Nous introduisons par la suite les résultats des classifications antérieures de ces espaces et présentons la nouvelle technique que nous avons développée pour classifier ces variétés. Notre méthode est basée sur l'existence d'instantons contraints, des courbes de Seiberg-Witten et utilise le carré magique de Freudenthal, Rosenfeld et Tits. Nous concluons en montrant que notre méthode de classification reproduit les résultats des classifications précédentes et permet la découverte d'un nouvel ensemble de variétés de Kähler.

Mots-Clés: Physique mathématique, variétés différentiables, instantons, quaternionique, Kähler, carré magique, courbes de Seiberg-Witten.

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INTRODUCTION

Real, Kähler, and quaternionic manifolds play an important role in physics. It is known today that they appear as moduli space of sigma models for $N = 2$ supergravity theories in five, four and three space-time dimensions respectively. In fact, in the framework of $N = 1$ supersymmetry in four dimensions with global susy, the target manifold of a non-linear sigma model can be any Kähler manifold [**1**]. In $N = 2$ supersymmetry in four space-time dimensions with local supersymmetry, the target manifolds of a non-linear sigma model coupled to supergravity can only be quaternionic Kähler manifolds [**21, 43**].

The classification of quaternionic manifolds was started in [**18, 44**] using transitive solvable groups of isometries and finally completed in [**25**] through the use of supergravity arguments. In this thesis, we classify non-compact symmetric quaternionic manifolds using a different technique than what has been done previously. In particular, we look for gauge theory with certain global symmetries and show that all the symmetric quaternionic manifolds can be succintly classified by constrained semilocal instantons. One can show that the low momentum dynamics of this theory gives a sigma model with quaternionic target space. Such an approach was first discussed in [**45**] and later elaborated in [**46, 47**]. In this thesis, we complete the analysis by detailing the corresponding gauge theory construction.

Our theory resembles a sector of Seiberg-Witten theory in certain parametrizations but is not asymptotically free. More precisely, the action for our model is given by the following generic form (see Section 3.1 of [**24**]):

$$S = \int d^4x \left[\frac{1}{4} tr_{SU(2)}(F_{\mu\nu}F^{\mu\nu}) + tr(D_\mu q^\dagger \cdot D^\mu q) + V(tr(q^\dagger \cdot q)) + fermions \right] \tag{0.0.1}$$

where q is a generic quaternion written as a 2×2 matrix, $F^{\mu\nu}$ is the field strength, and D_μ is the covariant derivative. This is not quite a Seiberg Witten theory as it stands but it suffices to modify this equation a little for the action to resemble a part of the standard $N = 2$ action with a potential V. It is then possible to use Seiberg-Witten curves to determine the global properties of this model.

Our goal is to study instantons in (0.0.1). We perform this analysis in two ways: from a group theory perspective by reinterpreting Freudenthal, Rosenfeld, and Tits' magic square [**31**] and from the Seiberg-Witten theory point of view [**30, 38**]. The latter leads to the concept of fibration of semilocal defects over quaternionic spaces. This technique is ideally suited to study several types of manifolds in the magic square. Also, it proves to be convenient for theories that may not have a good Lagrangian description and for which the existence of instantons might be questioned.

Our new method of studying the magic square and classifying the quaternionic manifolds through Seiberg-Witten curves allows us to reproduce the results of previous classifications and to discover a new set of Kähler manifolds. In addition, we study the sigma model description of our quaternionic manifolds by deriving for the first time in detail the prepotential functions for the relevant cases. These functions determine Kähler metrics and potentials. Using a given map, we can find the metrics of the associated quaternionic manifolds.

The article presented in this thesis is self-contained. However, we take the time, in the following first chapters, to introduce mathematical and physical notions that might not be familiar to the reader. We also introduce subjects that were not addressed in detail in the article such as Alekseevskii and De Wit-Van Proeyen's previous classifications of real, Kähler and quaternionic manifolds. We finally summarize the main results found in the article.

In Chapter 1 of this thesis, we introduce many notions of Riemannian geometry required to understand our work. We also define in detail the several types of manifolds with which we work. Finally, we present the first classification of quaternionic manifolds which used the concept of isometry groups. This will allow the reader to see the results that our classification method has to reproduce.

In Chapter 2, we introduce the notions of moduli spaces, target manifolds, and sigma models. This gives a better understanding of the importance of real, Kähler, and quaternionic manifolds in physics. We also present in detail the functions generating these manifolds and introduce briefly the concepts of supersymmetry and supergravity. These are the basis to understand the complete classification of De Wit and Van Proeyen. We conclude this chapter by presenting their results.

Chapter 3 presents the physical and mathematical concepts required to understand our classification: gauge theory, Seiberg-Witten theory, and constrained semilocal instantons. We discuss about the construction of these instantons and of the associated quaternionic manifolds. We also introduce the magic square, explain how to construct several manifolds from it using the technique of sequential gauging, and summarize the results obtained. We conclude this section by presenting the new set of Kähler manifolds we found and summarizing the technique used to find the prepotential functions.

Chapter 4 of the thesis contains the article that I wrote in collaboration with Keshav Dasgupta from McGill University and Véronique Hussin from Université de Montréal. The article presents a detailed classification of symmetric quaternionic manifolds. It was published in Nuclear Physics B in April 2008.

During the preparation of our article, I studied in detail several previous classifications of real, Kähler, and quaternionic manifolds as well as the properties of these spaces and the maps that link them. I participated in the study of how quaternionic manifolds appear from string theory and their realisation of the quotient spaces. I analyzed the construction of the magic square as well as the manifolds associated to it. My contribution in the classification of quaternionic manifolds was to address spaces in string theory that were not realised directly in the magic square. Finally, I wrote the entire Section 4.6 of [**24**] where I derived all prepotential functions associated to the Kähler manifolds realised in the magic square. These functions allow one to derive the metric of the corresponding quaternionic Kähler manifolds.

Chapter 1

MATHEMATICAL DEFINITIONS

1.1. Riemannian geometry

In this chapter, we introduce several definitions required to fully understand the mathematical concepts forming the background of the article presented in this thesis.

1.1.1. Riemannian manifold

A real (complex) n-dimensional manifold M is a space which looks like an Euclidean space $\mathbb{R}^n$ ($\mathbb{C}^n$) around each point. More precisely, a manifold is defined by introducing a set of neighborhoods U_i covering M and coordinates which maps these neighborhoods onto open subsets of $\mathbb{R}^n$ ($\mathbb{C}^n$). A manifold is said to be smooth or differentiable if the coordinate maps are differentiable functions. The spaces we classify are Riemannian manifolds (M, g). They are smooth manifolds M with a Riemannian metric g. The Riemannian metric is a natural generalization of the inner product between two vectors in $\mathbb{R}^n$ defined at each tangent space T_pM. Such a tangent space is defined as the vector space spanned by the tangents at point $p \in M$ to all curves passing through p in the manifolds [**2**].

1.1.2. Topology

A Riemannian manifold M is said to be connected if and only if the only subsets which are both open and closed are the void set and the space M itself. It is said to be orientable if every closed path is orientation preserving [**9**]. A collection of open subsets of M is called a covering if the union of these elements generates M. A Riemannian manifold is compact if every covering of M has a finite subcovering [**11**] and is said to be locally reducible if it resembles locally to a product space of submanifolds.

1.1.3. Lie groups as manifolds

A Lie group G is a manifold together with differentiable maps that constitute group product and inversion, all of which turn the smooth manifold into a group [**15**]. In addition, the Lie algebra of a Lie group is the tangent space T_eG at the group identity e. Let G be a Lie group and H any subgroup of G. The coset space G/H admits a differential structure and G/H becomes a manifold, called a homogeneous space. Note that dim (G/H) =dim $G-$dim H [**14**]. The homogeneous space G/H provided with an invariant Riemannian metric is called a Riemannian homogeneous space [**13**].

1.1.4. Curvature tensor

There exist intrinsic objects whose geometrical meaning is a measure of how much a manifold is curved, namely the torsion tensor and the curvature tensor which is also called the Riemann curvature tensor. The components of the curvature tensor are represented by the Riemann tensor denoted R_{ijkl}. The Ricci tensor is a contraction of the curvature tensor. Its components are by definition $R_{ik} = R_{i\ kj}^{\ j}$. The Riemann curvature scalar or Ricci scalar is given by $R = g^{ij}R_{ij}$ [**11**]. We say that a Riemannian manifold is Einstein if its Ricci tensor is proportional to the metric tensor i.e., $R_{ij} = \lambda g_{ij}$ with some constant $\lambda \in \mathbb{R}$. A Riemannian manifold is called Ricci flat if $R_{ij} = 0$ [**12, 8**].

1.1.5. Symmetric spaces

A Riemannian manifold is locally symmetric if and only if it has a constant Riemann curvature [**12**] and is said to be nonsymmetric if it is not locally symmetric. An equivalent definition involves geodesic symmetries. A function defined on a neighborhood of $p \in M$ is called a geodesic symmetry if it fixes the point p and reverses geodesics passing through that point. In particular, this function acts as minus the identity mapping on the tangent space of p. So a Riemannian locally symmetric manifold is such that for each $p \in M$ there exists a certain neighborhood of p on which the geodesic symmetry with respect to p is an isometry. We say that the manifold is globally symmetric (symmetric for short) if the geodesic symmetry extends to a global isometry [**7**].

1.1.6. Complex manifold

Let M be a real manifold of even dimension. An almost complex structure on M is a tensor J_i^j satisfying $J^2 = -1$. J can be viewed as a matrix acting on tangent vectors. In particular it gives each tangent space T_pM the structure of a complex vector space. We can associate a tensor $N \equiv N^i_{jk}$ to J called the Nijenhuis tensor

$$N_{ij}^k = J_i^l(\partial_l J_j^k - \partial_j J_l^k) - J_j^l(\partial_l J_i^k - \partial_i J_l^k). \tag{1.1.1}$$

An almost complex structure is integrable if and only if $N = 0$. In this case, it is called a complex structure. A complex manifold (M, J) is a manifold M with a complex structure J [**12**].

We pause for a moment to discuss an analogy with general relativity that would allow us to better understand the implication of a vanishing Nijenhuis tensor [**3**].

Suppose we are given a manifold K with a symmetric tensor field g_{ij} which could be considered, for instance, as the metric tensor. By linear algebra, one can show that given any point $p \in M$ there is a coordinate system such that the metric tensor g takes the standard form $g_{ij} = \delta_{ij}$ at p. Now, we would like to ask whether we can find coordinates which will put g in the standard form not just at one point p but in a whole neighborhood of p. Such a coordinate system is called a flat coordinate system. A necessary condition for the existence of a flat coordinate system in a whole neighborhood of a point p is that the Riemann tensor R_{ijkl}, which is made of g and its derivatives, should vanish in this neighborhood.

We would like to carry out an analogous argument in the case of an almost complex structure. Given a manifold K with an almost complex structure J and any point $y \in K$, one can find a suitable basis of complex coordinates and their complex conjugates in which J takes the form $J^i_j = i\delta^i_j$, $J^{\bar{i}}_{\bar{j}} = -i\delta^{\bar{i}}_{\bar{j}}$ at that one point y. Note that the bar symbol specifies that we are working with complex conjugate coordinates. We will call these expressions the canonical form of J. We would like to know if we can choose complex coordinates to put J in the canonical form not just at the point y but in a whole open set containing y. Coordinates with this property are called local holomorphic coordinates. If such coordinates exist in a neighborhood of each point $y \in K$ then, the almost complex structure is said to be integrable. There is essentially only one tensor field constructed from J^i_j and its derivatives namely, the Nijenhuis tensor. The condition $N = 0$ reflects the fact that J will be in the canonical form not only in one point but in a neighborhood of that point, in the same way that a zero Riemann tensor assures that the Riemann metric is flat not only at one point but in its neighborhood.

1.2. Manifolds

1.2.1. Holonomy

Holonomy is the process of assigning to each closed curve the linear transformation measuring the rotation which results when a vector is parallel transported around the given curve. From these linear transformations we get a set of holonomy matrices. This set forms a group called the holonomy group $\Gamma = \Gamma(g)$ where g is the Riemannian metric [**2**].

1.2.2. Berger's classification

In 1955, Berger classified in [**5**] the possible holonomy groups associated to n-dimensional simply-connected, locally irreducible, and nonsymmetric Riemannian spaces. The possible holonomy groups for these manifolds are compact Lie subgroups of $SO(n)$ and are listed in the following Berger's classification where exactly one of the following class holds [**12, 4**]:

(1) $\Gamma = SO(n)$

(2) $\Gamma = U(m) \subset SO(2m)$ with $n = 2m, m \geq 2$

(3) $\Gamma = SU(m) \subset SO(2m)$ with $n = 2m, m \geq 2$

(4) $\Gamma = Sp(m) \subset SO(4m)$ with $n = 4m, m \geq 2$

(5) $\Gamma = Sp(m) \times Sp(1) \subset SO(4m)$ with $n = 4m, m \geq 2$

(6) $\Gamma = G_2 \subset SO(7)$ with $n = 7$

(7) $\Gamma = Spin(7) \subset SO(8)$ with $n = 8$ or

(8) $\Gamma = Spin(9) \subset SO(16)$ with $n = 16$

Recall that $SO(k)$ is the special orthogonal group. For generic Riemannian metrics, $\Gamma = SO(n)$. The unitary group and special unitary group are represented by $U(k)$ and $SU(k)$ respectively. The symplectic group is denoted $Sp(k)$ with $Sp(1) \equiv SU(2)$ in complex dimension 2. G_2 is one of the exceptional Lie groups. Spin(n) is the Spin group which is the universal cover of $SO(n)$ [**15**]. The holonomy groups G_2 and Spin(7) are called, in [**12**], the exceptional holonomy groups. Metrics with these holonomy groups are Ricci-flat. Berger's original classification took the case $\Gamma = Spin(9)$ into account. However, it has been proved in [**4**] that there exists no locally nonsymmetric Riemannian space with this holonomy group. The figure below summarizes the links between the manifolds we will use.

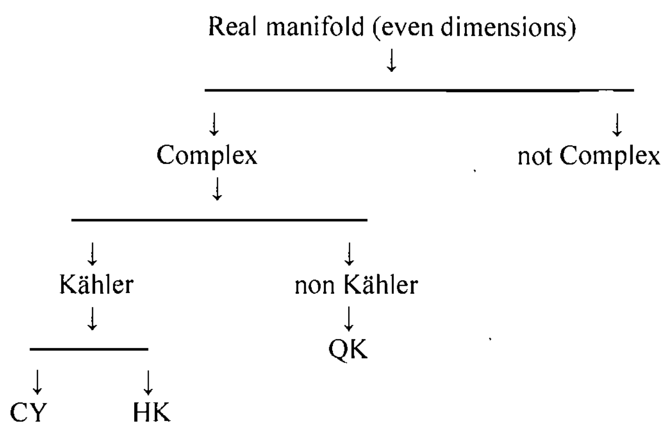

FIG. 1.1. Links between manifolds. Notation: Quaternionic Kähler (QK), HyperKähler (HK), Calabi-Yau (CY).

1.2.3. Kähler manifolds

We call g a Hermitian metric if $g_{ij} = J_i^k J_j^l g_{kl}$. A Hermitian metric g on a complex manifold (M, J) is called a Kähler metric if J is a constant tensor on M i.e., if its covariant derivative is equal to zero. Riemannian metrics g with $\Gamma \subseteq U(m)$ are also Kähler metrics. Kähler metrics are a natural class of metrics on complex manifolds and generic Kähler metrics on a given complex manifold have $\Gamma = U(m)$. A manifold that admits such a metric is called a Kähler manifold [**3**, **12**].

1.2.4. Calabi-Yau Manifold

Metrics g with $\Gamma = SU(m)$ are called Calabi-Yau metrics. Calabi-Yau metrics are locally the same as Ricci-flat Kähler metrics. Thus, a Calabi-Yau m-fold is a Ricci-flat Kähler manifold with holonomy $SU(m)$ [**12**]. Some authors also define a Calabi-Yau manifold as a Kähler manifold with vanishing first Chern class [**3**].

1.2.5. Quaternionic manifold

A Riemannian manifold with holonomy $\Gamma \subset Sp(1) \times Sp(m)$ is called a quaternionic space. These spaces are Einstein if $4m = n$. Quaternionic manifolds have non-zero Ricci curvature. If the Ricci curvature is equal to zero, the holonomy groups of these spaces reduces to $Sp(m)$: these are the hyperKähler manifolds. Quaternionic Kähler manifolds are manifolds with $\Gamma = Sp(1) \times Sp(m)$. They are in fact not Kähler: they are Einstein, but not Ricci-flat. They are not locally symmetric spaces. The quaternionic Kähler manifolds we will be working with have negative curvature. Compact, homogeneous, globally symmetric quaternionic manifolds are called Wolf spaces [**4**, **10**].

1.2.6. HyperKähler manifold

A Riemannian $4m$-manifold (M, g) is called hyperKähler if $\Gamma = Sp(m)$. These manifolds are Ricci-flat, Kähler and thus complex manifolds.

1.2.7. Quaternions

The quaternions are an extension of complex numbers. Quaternions form a 4-dimensional associative algebra denoted by $\mathbb{H} = \langle 1, i_1, i_2, i_3 \rangle \cong \mathbb{R}^4$ where objects take the form $q = (a+bi_1+ci_2+di_3, a, b, c, d \in \mathbb{R})$. Addition is given by $q_1+q_2 = q_3$ with $q_3 = (a_1+a_2)+(b_1+b_2)i_1+(c_1+c_2)i_2+(d_1+d_2)i_3$. Multiplication respects:

$$i_1i_2 = -i_2i_1 = i_3,\ i_2i_3 = -i_3i_2 = i_1,\ i_3i_1 = -i_1i_3 = i_2,\ i_1^2 = i_2^2 = i_3^2 = -1.$$

1.2.8. Maps

There exists two maps connecting certain manifolds together, namely the r-map, which links real manifolds to Kähler ones, and the c-map, which connects Kähler manifolds to quaternionic Kähler manifolds:

$$\mathbb{R}_{n-1} \xrightarrow{r} \mathbb{C}_n \xrightarrow{c} \mathbb{H}_{n+1}$$

where $n-1, n$ and $n+1$ denote the real, complex and quaternionic dimensions of the real, Kähler and quaternionic spaces respectively [**17**]. Important physical results regarding the c-maps can be found in [**16**].

1.3. Previous classifications

Several classifications of real, Kähler and quaternionic Kähler manifolds were made over the years using either mathematical or physical approaches. In the next section, we will introduce the concept used by Alekseevskii to generate the first classification of these manifolds. We will then expose his results. This will allow us to better understand the results that we should recover through our classification.

1.3.1. Isometry Group

A map between two manifolds is called a homeomorphism if it is continuous and has an inverse which is also continuous. To illustrate this, suppose we have two manifolds made of ideal rubber that we can deform at our will. These manifolds are homeomorphic to each other if we can deform one into the other continuously, that is, without tearing it apart and pasting it. If a homeomorphism and its inverse are differentiable the function is called a diffeomorphism and the two manifolds are said to be diffeomorphic. Two diffeomorphic spaces are regarded as the same space. A diffeomorphism is an isometry if it preserves the metric [**14**]. An isometry group of a manifold is the set of all isometries from the manifold onto itself, with the function composition as group operation. Its identity element is the identity function. As we will see in the next chapter, isometry groups were used to create the first classification of quaternionic manifolds.

1.3.2. Alekseevskii's classification

Alekseevskii made the first classification of homogeneous quaternionic spaces. He conjectured that all homogeneous quaternionic spaces were exhausted by compact symmetric quaternionic spaces and non-compact normal quaternionic space. Normal quaternionic spaces are quaternionic spaces which admit completely solvable transitive groups of motions $\mathcal{I}$. The rank of the group $\mathcal{I}$ that is, the dimension of its Cartan subgroup, is called the rank R of the normal quaternionic manifold. Alekseevskii classified in [**18**] normal quaternionic manifolds. He found that the rank of these spaces does not exceed four and all spaces of rank smaller than four are symmetric. He also found that there exists two series of non symmetric quaternionic spaces of rank four which were denoted $W(p,q)$ and $V(p,q)$ with p,q integers. Among these, there exists symmetric exceptions: $W(p,q)$ with $p=0$ and $V(p,q)$ when $p=1$. See the table on the next page for Alekseevskii's classification of normal non-compact symmetric quaternionic spaces.

Notation	Quotient-space representation	Rank (R)	Quaternionic dimension
C_1^m	$Sp(1,m)/Sp(1)\otimes Sp(m)$	1	m
A_1^1	$SU(2,1)/U(2)$	1	1
$A_2^m, (m>1)$	$SU(2,m)/S(U(2)\otimes U(m))$	2	m
G_2^2	$G_2/SU(2)\otimes SU(2)$	2	2
B_3^3	$SO(4,3)/SO(4)\otimes SO(3)$	3	3
$W(0,0)$	$SO(4,4)/SO(4)\otimes SO(4)$	4	4
$W(0,m-4), (m>4)$	$SO(4,m)/SO(4)\otimes SO(m)$	4	m
$V(1,1)$	$F_4/SU(2)\otimes Sp(3)$	4	7
$V(1,2)$	$E_6/SU(2)\otimes SU(6)$	4	10
$V(1,4)$	$E_7/SU(2)\otimes SO(12)$	4	16
$V(1,8)$	$E_8/SU(2)\otimes E_7$	4	28

TAB. 1.1. Classification of normal quaternionic spaces.

Chapter 2

PHYSICAL DESCRIPTIONS

One can better understand the relevance of studying real, Kähler and quaternionic Kähler manifolds if we analyze these spaces in the framework of supergravity. In this section, we will first of all introduce briefly the concepts of supersymmetry and supergravity. We will then explain how our manifolds can be thought of in this language. This will allow us to better understand another very important classification of these manifolds that was made a few years ago.

2.1. Supersymmetry and Supergravity

Supersymmetry is a continuous symmetry that mixes up fermions (matter) with bosons (the carriers of force), either in flat space (supersymmetry) or in curved space-time (supergravity). A model which possesses local (gauged) supersymmetry is called supergravity. A supersymmetric theory comes with an algebra which indicates how the various symmetry transformations affects each other. The possible systems on which the supersymmetry transformations act are multiplets of particles or quantum fields involving bosons and fermions. Supersymmetry transformations are generated by quantum operators Q which transform different members of a multiplet into each other i.e., change fermionic states into bosonic states and vice versa. These operators are in fact spinor operators, and in four space-time dimensions have at least four real components. Often called supercharges, they are denoted $Q_{\alpha i}$ with $\alpha = 1, \ldots, 4$ and $i = 1, \ldots, N$ where α is the spinor index and i is an internal index which indicates how many supersymmetry there is. The simplest supersymmetric theory is called $N = 1$ supersymmetry. It is invariant under the transformations generated by just the four independent components of a single spinor operator. In the case of additional supersymmetry, there will be several spinor generators with four components each: these theories are called N-extended supersymmetries. For a flat-space renormalisable field theory, the allowed values of N are 1, 2, and 4. $N = 3$ has the same algebra than $N = 4$. N can equal 8 for supergravity theories [**27, 28, 29**]. In this note, we will be concerned with $N = 2$ supersymmetry in four dimensions. As it is described in details in Section 2.2 of [**24**], this theory has several generic multiplets. We will be concerned with two of them: the first one being the vector multiplet that contains a (massless gauge) vector field A_μ, two real scalar fields (or one complex scalar field) and two fermions all in the adjoint representations of the gauge groups. The second multiplet will be the hypermultiplet, which has four real scalar fields (or two complex ones) and two fermions [**30**].

2.2. Moduli space, target space, and sigma models

The set of zero-mode solutions of any quantum field theory forms a moduli space. Moduli are the parameters labeling a space of degenerate and, usually, physically inequivalent vacua in quantum field theory. The moduli space is the space of geometries or vacua, whose coordinates are the moduli [**48**]. Thus, a moduli space can also be thought of as the space spanned by the scalar fields of a multiplet. A sigma model is a model whose Lagrangian is given by $\partial_\mu x^i \partial_\nu x^j h^{\mu\nu} g_{ij}$ where x^i, x^j are the coordinates of the target space (the space-time), g_{ij} is the metric on the target space, and $h^{\mu\nu}$ is the metric of the sigma model. The sigma model metric is the metric appearing in the string world-sheet action. Also known as the string metric, this differs from the Einstein metric be a dilaton-dependent Weyl transformation [**48**]. A target space is the space-time as seen from the sigma model point of view: it is the moduli space of the sigma model. The target space is the space in which a function takes its values. This is usually applied to the nonlinear sigma model on the string world-sheet, where the target space is itself the spacetime. Recall that the world-sheet is the two-dimensional surface in spacetime swept out by the motion of a string [**48**].

2.2.1. Manifolds' description in supergravity theories.

We can now better appreciate the role of real, Kähler and quaternionic Kähler manifolds in physics. In $N = 2$ supersymmetry, a real manifold is a moduli space parametrised by the scalar fields in the vector multiplets for five dimensional supergravity. Dimensionally reducing this to four dimensions yields a Kähler moduli space for the vector multiplets and further dimensional reduction to three dimensions yields a quaternionic Kähler manifold which is a moduli space spanned by the scalar fields of the hypermultiplets. This last fact is used in Section 2.2 of [**24**] to show in detail how a quaternionic sigma model appears in string theory by compactifying Type II strings on a Calabi-Yau three-fold.

The physics literature denotes the spaces mentioned above as special manifolds [**25**]. Special Kähler manifolds are those in the image of an r-map whereas special quaternionic manifolds are in the image of a c-map. As we will see in the next subsection, these spaces are generated by certain functions. Although we will now drop the term special, these are the manifolds we will be concerned with for the rest of this note.

2.3. Previous Classifications

In this section, we introduce in details the notion of generating functions. This is done for two reasons. First of all, generating functions combined with supergravity are the pillars on which De Wit and Van Proeyen's classification relies. Secondly, we studied the generating functions in some details in [**24**] and will come back to them in later chapters.

2.3.1. Generating functions

The structure of Kähler manifolds is encoded in a single holomorphic and homogeneous function $F(X^I)$ of degree two [**19**]. Under (an inverse) r-map, these functions correspond to cubic polynomials $C(h)$ which characterize real spaces.

These cubic functions were classified by De Wit and Van Proeyen in [**25**] where they were parametrized by:

$$C(h) = d_{ABC} h^A h^B h^C, (A, B, C = 1, \ldots, n) \tag{2.3.1}$$

where d_{ABC} is a symmetric tensor and $h^K, K = A, B, C$ represent scalar fields with certain restrictions. It was shown in [**21**] that the F functions could be written in the form

$$F(X^I) = i\frac{d_{ABC} X^A X^B X^C}{X^0} \tag{2.3.2}$$

where $X^I, I = 1, \ldots, n+1$ are complex scalar fields corresponding to certain $N = 2$ vector multiplets. As discussed in [**22**], these F functions give us important physical informations such as the Kähler potential $K(X, \bar{X})$ and metric ds^2 of a Kähler manifold:

$$K(X, \bar{X}) = i(\bar{X}^I F_I - X^I \bar{F}_I),\ ds^2 = N_{IJ} dX^I d\bar{X}^J,\ N_{IJ} = i(F_{IJ} - \bar{F}_{IJ}) \tag{2.3.3}$$

where $F_I = \partial F / \partial X^I$. These F functions are referred to as prepotential functions in the physics literature and determine the Kähler spaces associated to certain quaternionic spaces. By using the c-map, one can find out the metric of the associated quaternionic space.

2.3.2. Cecotti's classification

Cecotti classified in [**20**] normal homogeneous Kähler spaces. The classification of symmetric Kähler manifolds was already solved some time ago by Cremmer and Van Proeyen [**26**]. Cecotti's main result was that there exists two infinite families of homogeneous non-symmetric Kähler manifolds allowed in $N = 2$ supergravity: $K(p, q)$, and $H(p, q)$ with some symmetric exceptions. These spaces have rank 3 and are in one-to-one correspondence with the homogeneous quaternionic spaces found by Alekseevskii [**18**]. See the table at the end of this section for a complete list of the symmetric cases.

2.3.3. De Wit and Van Proeyen's classification

Alekseevskii and Cecotti's classifications of homogeneous spaces was completed a few years ago by De Wit and Van Proeyen in [**25**] where they used

$N = 2$ supergravity arguments. They derived a classification of all homogeneous quaternionic spaces that were in the image of $c \circ r$ map. Their analysis was performed completely at the level of special real spaces and amounted to classifying all the cubic polynomials $C(h)$ generating these spaces. As a result, they found, in addition to the previously classified spaces, a new class of rank-3 spaces of quaternionic dimension larger than 3 and specified in more details some of Alekseevskii's rank-4 spaces. The table at the end of the section shows the complete classification as it is accepted today. Note that the star symbol corresponds to spaces first discussed by De Wit and Van Proeyen[1] and that SG stands for pure $N = 2$ supergravity theory in five, four and three dimensions for real, Kähler and quaternionic spaces respectively. This table, which is a summary of the results presented in [**25, 17**], describes all the homogeneous real, quaternionic and Kähler spaces known before our classification.

[1]We do not know yet if these spaces are described by a quotient space structure.

C(h)	real	Kähler	quaternionic	R
			SG	0
			$\frac{USp(2n+2,2)}{USp(2n+2)\otimes SU(2)}$	1
		SG	$\frac{U(1,2)}{U(1)\otimes U(2)}$	1
		$\frac{U(n,1)}{U(n)\otimes U(1)}$	$\frac{U(n+1,2)}{U(n+1)\otimes U(2)}$	2
	SG	$\frac{SU(1,1)}{U(1)}$	$\frac{G_{2(+2)}}{SU(2)\otimes SU(2)}$	2
$L(-1,0)$	$SO(1,1)$	$\left[\frac{SU(1,1)}{U(1)}\right]^3$	$\frac{SO(3,4)}{(SU(2))^3}$	3
$L(-1,P)$	$\frac{SO(P+1,1)}{SO(P+1)}$	$\star$	$\star$	3
$L(0,0)$	$[SO(1,1)]^2$	$\left[\frac{SU(1,1)}{U(1)}\right]^3$	$\frac{SO(4,4)}{SO(4)\otimes SO(4)}$	4
$L(0,P)$	$\frac{SO(P+1,1)}{SO(P+1)} \otimes SO(1,1)$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(P+2,2)}{SO(P+2)\otimes SO(2)}$	$\frac{SO(P+4,4)}{SO(P+4)\otimes SO(4)}$	4
$L(0,P,\dot{P})$	$Y(P,\dot{P})$	$K(P,\dot{P})$	$W(P,\dot{P})$	4
$L(q,P)$	$X(P,q)$	$H(P,q)$	$V(P,q)$	4
$L(4m,P,\dot{P})$	$\star$	$\star$	$\star$	4
$L(1,1)$	$\frac{Sl(3,\mathbb{R})}{SO(3)}$	$\frac{Sp(6)}{U(3)}$	$\frac{F_4}{USp(6)\otimes SU(2)}$	4
$L(2,1)$	$\frac{Sl(3,\mathbb{C})}{SU(3)}$	$\frac{SU(3,3)}{SU(3)\otimes SU(3)\otimes U(1)}$	$\frac{E_6}{SU(6)\otimes SU(2)}$	4
$L(4,1)$	$\frac{SU^*(6)}{Sp(3)}$	$\frac{SO^*(12)}{SU(6)\otimes U(1)}$	$\frac{E_7}{SO(12)\otimes SU(2)}$	4
$L(8,1)$	$\frac{E_6}{F_4}$	$\frac{E_7}{E_6\otimes U(1)}$	$\frac{E_8}{E_7\otimes SU(2)}$	4

TAB. 2.1. Homogeneous special real, Kähler, and quaternionic spaces. Rank (R) of quaternionic spaces indicated. The rank of the corresponding real and Kähler spaces is found by decreasing R by 2 or 1 respectively. Integers $P, \dot{P}, q$ and m can take values ≥ 1.

Chapter 3

OUR CLASSIFICATION

We are now ready to discuss the classification method presented in our paper. In [**24**], we tried to understand the classification of non-compact symmetric quaternionic Kähler manifolds using a different technique than what has been done previously. More precisely, our classification does not rely on isometry groups nor on supergravity. Instead, we describe the whole system via $SU(2)$ gauge theories with global symmetries $\mathcal{G}$ and exploit the resemblance of this theory with a sector of $N = 2$ Seiberg-Witten theory to classify all quaternionic manifolds. We start this section by introducing some preliminary concepts. We then walk the reader through the different steps of our paper, leaving aside the heavy details and focusing on the premises.

3.1. Preliminary notions

3.1.1. Gauge theory

A lagrangian invariant under a continuous symmetry of a certain group G is said to be globally gauge-invariant if the transformation does not depend on space-time coordinates. More precisely, a transformation $\phi_i(x) \rightarrow U_{ij}\phi_j(x)$ which acts on scalar fields and which leaves a particular lagrangian invariant is called a global symmetry transformation. The symmetry group $U_{ij} \in G$ is independent of the space-time label x and is called a gauge group. A lagrangian invariant under a transformation $\phi_i(x) \rightarrow U_{ij}(x)\phi_j(x)$ where the gauge group depends on space-time coordinates is said to be locally gauge-invariant [**27**].

In the rest of this note, we will be concerned with an $SU(2)$ gauge theory with global symmetry $\mathcal{G}$ where $\mathcal{G} = \{Sp(n+1), G_2, F_4, E_6, E_7, E_8\}$. In other words, we will have a lagrangian locally symmetric under $SU(2)$ and globally symmetric[1] under the group $\mathcal{G}$.

3.1.2. Seiberg-Witten theory

In 1994, Seiberg and Witten studied the vacuum structure of $N = 2$ supersymmetric gauge theory in four dimensions with gauge group $SU(2)$. The theory is remarkably rich and has physical properties which can be described precisely; exact formulas can be obtained, for instance, for the metric on the moduli space of the vacua. This theory allows one to obtain information about the strong coupling behavior of $N = 2$ theories in the case of $SU(2)$ gauge theory without matter multiplets [**30**] and with matter multiplets [**38**]. Global properties of these theories are contained in what are known as Seiberg-Witten curves [**39, 40**]. The model we study in [**24**] is a sector of this larger framework.

A table of precise connections between elements of the full Seiberg-Witten theory and our theory is provided in Section 3.1 of [**24**]. Our theory being a small sub-sector of the full theory, the complicacies of the original Seiberg-Witten theory do not affect our analysis. We will come back to this in later subsections.

3.1.3. Instantons

The term instanton has come to refer to localised finite-action solutions of the classical Euclidean field equations of a theory. In particular, they are finite-action solutions of the Euclidean $SU(2)$ Yang-Mills gauge theory. Instantons are by definition gauge field configurations. They are associated to self-dual or anti-dual field strength and carry non-vanishing topological quantum number [**2**]. Instantons also play an important role in quantum theory where they lead

[1]There is a subtlety here since there might not be a lagrangian description of our model for high global symmetry such as E_n and F_4. This is discussed in detail in [**24**].

to vacuum tunneling and related phenomena [**37**]. In our paper, we will be using Euclidean $SU(2)$ Yang-Mills instantons in the framework of gauge theory. Although a lagrangian description of our model might not be possible for high global symmetries, this does not affect the existence of these instantons since we find alternative ways to construct them such as embedding the system in F-theory. These details are presented in Section 3.1 of [**24**].

3.2. ANALYSIS OF INSTANTONS

The key point we will be using to classify non-compact symmetric quaternionic Kähler manifolds is the following: we look for $SU(2)$ gauge theories with global symmetry $\mathcal{G}$ and find what are called semilocal constrained instantons configurations. We know that the low momentum dynamics of these theories are sigma models with quaternionic target spaces.

Thus, our goal is to study instantons in (0.0.1). The analysis of instanton in this theory can be done two ways, both of which will be discussed here and lead to the same results. First, observe that a theory like (0.0.1) will not allow any non-trivial instantons if

$$\pi_3\left(\frac{\mathcal{G}}{\mathcal{H}}\right) = 1 \tag{3.2.1}$$

where π_3 is the third homotopy, $\mathcal{G}$ is the global symmetry of our theory, and $\mathcal{H}$ is an ungauged subgroup of $\mathcal{G}$. We call the coset space $\left(\frac{\mathcal{G}}{\mathcal{H}}\right)$ the vacuum manifold $\mathcal{M}_1$. A certain type of instantons called constrained instantons will be possible when a subgroup of $\mathcal{G}$ is gauged. We will describe this in detail in the next section.

The second way one can analyze instantons in (0.0.1) is through Seiberg-Witten theory. Indeed, when one adds a few terms to (0.0.1) and rewrites the equation in complex coordinates (see Section 3 of [**24**]), the theory describes a sector called the Higgs branch of the full Seiberg-Witten theory. The vacuum manifold $\mathcal{M}_1$ becomes the moduli space of one-instanton. These instantons are described by embedding the $SU(2)$ group inside the global symmetry group. The different $SU(2)$ orientations describe the moduli space of the theory. These

$SU(2)$ orientations form an S^3. In this language, the instanton moduli space will be fibered over a quaternionic Kähler space. Thus, from a mathematical point of view, we have an $SU(2)$ instanton fibered over a base space $\left(\frac{G}{H}\right)$. The target space is $\left(\frac{G}{H}\right) \times SU(2)$.

Now that we have laid down the elementary criteria to construct quaternionic manifolds with global symmetry $\mathcal{G}$ from instantons configurations, there are some important points to analyze.

First, we have to verify if it is possible to construct a Seiberg-Witten-like theory with this global symmetry $\mathcal{G}$. This will be confirmed by the existence of the corresponding Seiberg-Witten curve for the system. A generic curve has the form: $y^2 - x^3 - a_2x^2k(z) + a_1xyl(z) + a_3yh(z) - a_4xf(z) - a_6g(z) = 0$ where a_i are constants and $k(z), l(z), h(z), f(z)$ and $g(z)$ are polynomials in z. The right choice of k, l, h, f and g can generate a curve that reflects the global symmetry (see Section 3.1 of [**24**]).

The next step would be to check the existence of instantons in this full Seiberg-Witten theory with global symmetry $\mathcal{G}$. Note that to get a Seiberg-Witten curve for our model, we had to sum over all the instantons contributions. Thus the existence of instantons is verified by construction. In the next section, we will construct explicitly these instantons as well as the quaternionic spaces on which they are fibered.

3.3. Construction of instantons and quaternionic manifolds

The construction of instantons for our model is subtle. First because based on the global symmetries we are using, $\mathcal{G} = \{Sp(n+1), G_2, E_6, E_7, E_8, F_4\}$, instantons are not allowed i.e., $\pi_3(\mathcal{M}_1) = 1$ for all cases[2]. The only allowed instanton

[2] $\mathcal{H} = \{Sp(n) \times SU(2), SU(2) \times SU(2), SU(6) \times SU(2), SO(12) \times SU(2), E_7 \times SU(2), Sp(3) \times SU(2)\}$ respectively for non-compact quaternionic manifolds.

configuration in our system is the semilocal instanton obtained by gauging an $SU(2)$ part of the global symmetry. The second subtlety comes from the presence of $V(tr(q^\dagger \cdot q))$ in the action (0.0.1). Namely, the presence of a mass term in the potential makes all the instantons squeeze to zero size. Instantons with this property are called constrained instantons. They resemble the standard instantons at short distances but decay exponentially in the infrared limit. Thus, our semilocal instantons are also constrained instantons.

To construct constrained instantons, all we require is for the maximal subalgebra of the extended Dynkin diagram of $\mathcal{G}$ to be expressible as a product of two subalgebras and demand that one of these subalgebra be $sp(1)$. Constrained instantons are exactly of the gauged $SU(2) \equiv Sp(1)$ group.

We move on by giving an example which illustrates how to construct the quaternionic manifold which is associated to these instantons. Take for instance the global symmetry group $\mathcal{G} = Sp(n+1)$ which is studied in detail in Section 3.1 of **[24]**. The decomposition of the maximal subalgebra into a sum of subalgebras leads to $sp(n) \oplus sp(1)$. Knowing this decomposition, the quaternionic space associated with the global symmetry group $Sp(n+1)$ is built in the following way:

$$\frac{Sp(n+1)}{Sp(n) \times Sp(1)} = \mathbb{HP}^n. \tag{3.3.1}$$

Therefore, the constrained $SU(2)$ instantons are non trivially fibered over the quaternionic base $\mathbb{HP}^n$. To conclude the argument, we show in **[24]** that the vacuum manifold associated to a global symmetry $Sp(n+1)$ is

$$\mathcal{M}_1 = \frac{\mathcal{G}}{\mathcal{H}} = \frac{Sp(n+1)}{Sp(n)} \approx S^{4n+3} \tag{3.3.2}$$

Now recall that any $4n+3$ sphere is equivalent to a S^3 fibration over a quaternionic base $\mathbb{HP}^n$ which confirms the validity of our construction.

A few remarks: in order for the reader to see clearly that our classification of quaternionic manifolds is consistent with that of Alekseevskii and De Wit-Van

Proeyen, we will work with the non-compact version of these manifolds. Thus, the compact version of the quaternionic projective space we found earlier i.e. $\frac{Sp(n+1)}{Sp(n)\times Sp(1)}$ becomes $\frac{Sp(n,1)}{Sp(n)\times Sp(1)}$. Quaternionic manifolds are written in this way in previous classifications. As we just saw, our method allows us to construct spaces such as $\mathbb{HP}^n$ even though these quaternionic manifolds are not in the image of a c-map. In Section 4 of [**24**], we show in detail that all quaternionic symmetric spaces can be studied using the technique of constrained semilocal instantons.

3.4. Realisation of the quotient space

From the steps described in the previous section, the structure of quotient spaces such as $\frac{Sp(n+1)}{Sp(n)\times Sp(1)}$ should be clear. Namely, the maximal subgroup of $Sp(n+1)$ expressed in terms of a product of two subgroups such that one of those subgroup is $SU(2)$ gives us $Sp(n)\times SU(2)$. What remains to study is the precise embedding of the $SU(2)$ group inside $\mathcal{G} = Sp(n+1)$.

Section 3.2 of our paper presents, as an example, the realisation of the quotient space $\frac{G_2}{Sp(1)\times Sp(1)}$ which has been already given in [**41, 42**]. For the case of $\mathcal{G} = G_2$, we use the embedding of the exceptional complex Lie group $G_2(\mathbb{C})$ into the complex orthogonal Lie group $SO(7,\mathbb{C})$. This allows us to identify the maximal subalgebra $so(4) = su(2)\oplus su(2)$ in G_2.

3.5. Magic square

The magic square is a mathematical construction which was first developed by Freudenthal, Rosenfeld and Tits in the mid 20^{th} century and was introduced in string theory a few years later by Gunaydin, Sierra and Townsend [**31, 32**]. The magic square is used to show the relation between division algebras, Jordan algebras, and Lie algebras. It consists of a 4×4 square with entries given by elements of Lie algebras. The columns of the magic square are defined by the Jordan algebras whereas the rows are defined by the division algebras. The division algebras are the real ($\mathbb{R}$), complex ($\mathbb{C}$), quaternion ($\mathbb{H}$), and the octonion

($\mathbb{O}$). The columns are labeled from left to right by $J^3(\mathbb{R}), J^3(\mathbb{C}), J^3(\mathbb{Q}), J^3(\mathbb{O})$ where $J^3(\mathbb{K})$ is the algebra of 3×3 Hermitian matrices over $\mathbb{K}$. The magic square can be represented as follow:

A_1	A_2	C_3	F_4
A_2	A_2^2	A_5	E_6
C_3	A_5	D_6	E_7
F_4	E_6	E_7	E_8

TAB. 3.1. Magic square

where A_i, C_i, D_i, E_i, F_4 are the usual $su(i+1), sp(i), so(2i), E_{6,7,8}$ and F_4 Lie algebras respectively. Through out this note, we will be concerned with the version of the square associated to the corresponding Lie groups.

3.5.1. Classification of manifolds from the magic square

To construct quaternionic, Kähler, and real manifolds from the elements of the magic square, we developed a technique called sequential gauging. This method consists of gauging various subgroups of a gauge group $\mathcal{G}$ as we move along the magic square. We argue in Section 4 of **[24]** that the existence of constrained instantons indicates that we have to gauge by an $SU(2)$ subgroup to construct quaternionic manifolds whereas Kähler manifolds are built by gauging a $U(1)$ subgroup associated to semilocal strings. As indicated in our paper, real manifolds do not require any gauging.

The magic square turns out to be a very useful tool to classify these manifolds since we can read off from it the global symmetry groups as well as the gauged maximal subgroups needed to build the quotient space which defines these manifolds. This construction is analyzed in great details in Section 4 of our paper. Applying the method of sequential gauging through the entire magic square we find the following results where columns from right to left represent quaternionic Kahler, Kahler, and two real manifolds respectively[3].

$SO(3)$	$\frac{SL(3,\mathbb{R})}{SO(3)}$	$\frac{Sp(3,\mathbb{R})}{U(3)}$	$\frac{F_{4(+4)}}{Sp(3)\times SU(2)}$
$SU^*(3)$	$\frac{SL(3,\mathbb{C})}{SU(3)}$	$\frac{SU(3,3)}{SU(3)\times U(3)}$	$\frac{E_{6(+2)}}{SU(6)\times SU(2)}$
$Sp(3)$	$\frac{SU^*(6)}{Sp(3)}$	$\frac{SO^*(12)}{SU(6)\times U(1)}$	$\frac{E_{7(-5)}}{SO(12)\times SU(2)}$
F_4	$\frac{E_{6(-26)}}{F_4}$	$\frac{E_{7(-25)}}{E_6\times U(1)}$	$\frac{E_{8(-24)}}{E_7\times SU(2)}$

TAB. 3.2. Sequential gauging method applied to the entire magic square.

3.5.2. Beyond the magic square

After having described the complete magic square in terms of constrained instantons and other semilocal defects, we use the same procedure to study other coset spaces in string theory. We analyze, in Section 4.5 of our paper, spaces generated by $U(p)$ local symmetry with $SU(n+p)$ global symmetry and $SU(2)$ local symmetry with $SO(p+q)$ global symmetry. With these coset spaces, we exhaust Alekseevskii and De Wit-Van Proeyen's classifications.

In addition of reproducing the classification of existing manifolds with our new technique, we also describe, in Section 4.5 of our paper, the construction of a new sequence of Kähler manifolds not realized directly in the magic square. To construct these manifolds, we built coset spaces out of elements of the magic square and their unused subgroups. By unused we mean here subgroups that our technique of sequential gauging did not already use to construct real, Kähler and quaternionic Kähler manifolds. This new sequencing of the magic square follows rather straightforwardly from our original prescription. Applying this technique to the elements of the second column of the magic square we found the following new set of Kähler manifolds:
This construction has recently appeared also in [**35**].

[3]Notation: the number in the bracket of the global symmetry group i.e. $E_6(+2)$ denotes the difference between the number of compact and non-compact generators whereas the * symbol represents complex matrices with some constraints (see [**23**]).

$\frac{SU(2,1)}{U(2)}$	$\frac{SU(2,1)\times SU(2,1)}{SU(2)\times U(2)}$	$\frac{SU(4,2)}{SU(4)\times U(2)}$	$\frac{E_{6(-14)}}{SO(10)\times U(1)}$

TAB. 3.3. New set of Kähler manifolds

3.6. Generating functions revisited

In previous sections, we discussed the issue of F-functions that could be used to determine the metric on the quaternionic Kähler manifolds. In this section, we complete the analysis by postulating the procedure to determine the F-functions for the magic cases. Manifolds with global symmetry $\mathcal{G} = \{F_4, E_6, E_7, E_8\}$ are referred to as the magic cases because they appear in the magic square.

Cecotti constructed in [**20**] each F-functions corresponding to the normal quaternionic spaces that appeared in Alekseevskii's classification. However, his construction of those functions was not explicitly given for the magic cases and was written in a parametrization that did not allow the use of (2.3.3). In [**23**], the authors wrote down the F-functions corresponding to the magic cases but again in a parametrization that made the computation of the metric in the format proposed in (2.3.3) not possible. This is why we decided in [**24**] to start from the beginning and write down these F-functions explicitly in the canonical parametrization for the magic cases. This form allows one to compute classical moduli space metrics and Kähler potentials. Note that a general form of the F-functions which resembles ours was given in [**17**] but where the authors used an arbitrary linear redefinition of the complex variables instead of the canonical parametrization.

We start this construction with the cubic functions $C(h) = d_{ABC}h^A h^B h^C$ associated to real manifolds. These were classified in [**25**]. Imposing certain conditions [**21, 25**] on the generic form of $C(h)$, we find a construction for the holomorphic function $F(X^I)$ for the magic cases. These functions can be expressed in terms of complex variables X^I and gamma matrices which generates certain Clifford algebras. A classification of Clifford algebras in terms of gamma matrices already exits [**36**]. To classify the several F-functions, we needed to find the right gamma matrices with the knowledge of the Clifford algebras present.

In Section 4.6 of our paper, we derive all the F-functions associated to the magic Kähler manifolds which allows us to determine the Kähler metric of those spaces. One can then derive the associated quaternionic metric.

Chapter 4

QUATERNIONIC KÄHLER MANIFOLDS, CONSTRAINED INSTANTONS, AND THE MAGIC SQUARE

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Quaternionic Kähler manifolds, constrained instantons and the magic square: I

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Abstract

The classification of homogeneous quaternionic manifolds has been done by Alekseevskii, Wolf et al. using transitive solvable group of isometries. These manifolds are not generically symmetric, but there is a subset of quaternionic manifolds that are symmetric and Einstein. A further subset of these manifolds are the magic square manifolds. We show that all the symmetric quaternionic manifolds including the magic square can be succinctly classified by constrained instantons. These instantons are mostly semilocal, and their constructions for the magic square can be done from the corresponding Seiberg–Witten curves for certain $\mathcal{N} = 2$ gauge theories that are in general not asymptotically free. Using these, we give possible constructions, such as the classical moduli space metrics, of constrained instantons with exceptional global symmetries. We also discuss the possibility of realising the Kähler manifolds in the magic square using other solitonic configurations in the theory, and point out an interesting new sequence of these manifolds in the magic square.

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Keywords: Strings; Branes; Quaternionic manifolds; Gauge theories

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1. Introduction

A Riemannian manifold $(\mathcal{M}, g)$ is a smooth manifold $\mathcal{M}$ endowed with a metric g defined in $T^*\mathcal{M}$. The holonomy of such a connected oriented Riemannian manifold belongs to the following list:

- $SO(n)$: generic case.
- $SU(n), U(n) \subset SO(2n)$: Calabi–Yau and Kähler cases.
- $Sp(n), Sp(n) \times Sp(1) \subset SO(4n)$: Hyper-Kähler and quaternionic Kähler cases.
- $G_2 \subset SO(7)$, $\mathrm{Spin}(7) \subset SO(8)$.

The above is the so-called Berger's classification theorem [1]. We will be mainly concerned with the following two holonomies: $Sp(n)$ and $Sp(n) \times Sp(1)$. Both these groups act on $\mathbf{H}^n = \mathbf{R}^{4n}$ where $\mathbf{H}^n$ is the right vector space over the quaternions $\mathbf{H}$. The $Sp(1) \equiv SU(2)$ factor in $Sp(n) \times Sp(1)$ is the group of unit quaternions acting from the right.

The quaternionic Kähler manifolds are always Einstein[1] for $n \geqslant 2$ and are self-dual Einstein for $n = 1$. They are considered positive if their metrics are complete and have positive scalar curvatures. When the scalar curvatures are zero, then the holonomies of these manifolds reduce to $Sp(n)$ and are called the hyper-Kähler manifolds. Thus clearly quaternionic Kähler manifolds are not Ricci flat.

Examples of quaternionic Kähler manifolds with positive scalar curvatures are given by compact symmetric spaces classified by Wolf [2] and Alekseevskii [3] and are known as the Wolf spaces. They are classified by taking centerless Lie group $\mathbf{G}$ which form the isometry group of quaternionic Kähler spaces given as conjugacy classes of $Sp(1)$ in $\mathbf{G}$ determined by the highest root of $\mathbf{G}$. These spaces are:

$$\begin{aligned}
&\mathbf{HP}^n = \frac{Sp(n+1)}{Sp(n) \times Sp(1)}, \qquad \mathbf{Gr}_2(\mathbf{C}^{n+2}) = \frac{SU(n+2)}{S(U(n) \times U(2))},\\
&\mathbf{Gr}_4(\mathbf{R}^{n+4}) = \frac{SO(n+4)}{SO(n) \times SO(4)},\\
&\frac{E_6}{SU(6) \times Sp(1)}, \qquad \frac{E_7}{\mathrm{Spin}(12) \times Sp(1)}, \qquad \frac{E_8}{E_7 \times Sp(1)},\\
&\frac{F_4}{Sp(3) \times Sp(1)}, \qquad \frac{G_2}{SO(4)}.
\end{aligned} \tag{1.1}$$

Observe that all these spaces are modded by a $Sp(1)$ group as expected. This will be useful later when we will map our configurations to semi-local defects.

The above examples are all compact. The non-compact duals are symmetric examples of quaternionic Kähler manifolds with *negative* scalar curvatures. The non-symmetric, non-compact examples with negative scalar curvatures are also known. However no concrete examples of non-compact non-symmetric positive curvature manifolds are presently known.

In Section 2 we will give some examples of symmetric quaternionic Kähler manifolds that appear in string theory. We will study few representative cases—in Sections 2.1 and 2.2—and discuss possible quantum corrections to these spaces. Although most of this is well known, we will present it in a way so as to connect to latter parts of the paper. Important concepts like c, s

[1] By this we mean that the Ricci tensor is proportional to the metric.

and r-maps will be introduced in Section 2.2. The connection between c and r-maps, as we will discuss soon, is the following:

$$\text{Real manifold} \xrightarrow{r} \text{Kähler manifold} \xrightarrow{c} \text{Quaternionic Kähler manifold},$$

which in the language of supergravity means the following: the moduli space of the scalar fields in the vector multiplets for a five-dimensional supergravity is a real manifold. Dimensionally reducing this to four dimension yields a Kähler moduli space for the vector multiplets and further dimensional reduction to three dimensions yields a quaternionic Kähler manifold for the hypermultiplets. This way of viewing the connection was described by various authors, for example [4–6], which also led to the connection to the *magic square* of Freudenthal, Rosenfeld and Tits [7] that we describe at the beginning of Section 4.

Our method of studying the magic square and classifying the quaternionic manifolds is *different* from what has been attempted so far. We will not analyse using supergravities at all, instead we will describe the whole system via $SU(2)$ gauge theories with global symmetries $\mathcal{G}$ that resemble sectors of $\mathcal{N}=2$ Seiberg–Witten theories [8] in certain parametrisations, but are not asymptotically free. Most of these theories that we analyse are at strong couplings, and in certain cases simple Yang–Mills description may not suffice. Nevertheless we will show that one-instanton moduli spaces could be studied in all these cases, and the corresponding Seiberg–Witten curves could be used to classify the quaternionic spaces. The instantons that we study are not only constrained instantons [9], but are also semilocal [10].[2] The Kähler[3] and the real spaces could then be classified by other semilocal defects in the theory for certain choices of global symmetries that we analyse using the so-called *sequential gauging*. These aspects will be described in Sections 3 and 4. In Sections 4.1 to 4.4, we will give strong evidence that all the elements of the magic square [7] can be reproduced starting from certain sectors of $\mathcal{N}=2$ $SU(2)$ gauge theories with E_6, E_7, E_8 and F_4 global symmetries. The case with G_2 global symmetry is interesting, and we study this in Section 3.2 by detailing an explicit construction of the associated quaternionic space. Normally one would not attach G_2 to the magic square, but we show that there is a way to incorporate the G_2 group sequence in the magic square too by adding one extra column.

In Section 4.5 we study another example that has not been discussed in the physics literature in details. This new sequencing of the magic square follows rather straightforwardly from our arguments of sequential gauging and could also be added to the magic square by a different choice of the underlying Jordan algebras [12].

In Section 4.5.2 we discuss the sigma model descriptions of these quaternionic spaces by analysing the F-functions [5] for all the relevant cases. These F-functions are the prepotential that determine the Kähler spaces associated to the quaternionic spaces. We then use the c-map to determine metrics of all the quaternionic spaces. Finally, in Section 5 we conclude with a brief discussion and point out some future directions.

We now begin with the very basics of quaternionic spaces: their role in string theory and gauge theories.

2. Quaternionic manifolds and string theory

Our first question would be to ask where does the quaternionic manifolds fit in the whole paradigm of string compactifications. One of the place where these manifolds appear is well

[2] In mathematical terminology therefore these instantons are constrained instanton bundles.

[3] These Kähler spaces have been originally classified in [11].

known: the moduli space of sigma models for $\mathcal{N}=2$ supergravity in four space–time dimensions. Imposing only global $\mathcal{N}=2$ supersymmetry in four dimensions would lead to sigma models with hyper-Kähler target spaces [13]. The $\mathcal{N}=2$ multiplets on the other hand can be written in terms of $\mathcal{N}=1$ multiplets. This should tell us the moduli space structure for the corresponding $\mathcal{N}=1$ case also. In fact one can now make the following classifications for $\mathcal{N}=1$ supersymmetry in four dimensions:

- With global supersymmetry the target manifold of a non-linear sigma model can be *any* Kähler manifold [14].
- With local supersymmetry the target manifold of a non-linear sigma model (which is coupled to supergravity) can only be a restricted Kähler type, also known as a Hodge manifold [15].

The second point is easy to show [15]. We can define a Kähler potential K in terms of the chiral superfield Φ^i and $\bar{\Phi}^i$. The terms appearing in the $\mathcal{N}=1$ Lagrangian can be expanded from $-3e^{-\frac{K}{3}}$. The first two relevant terms are

$$S=\int d^4x\,\sqrt{g}\left[-\frac{R}{2}-g_{i\bar{j}}\partial_\mu\phi^i\partial^\mu\bar{\phi}^j+\text{fermions}\right], \tag{2.1}$$

where $g_{i\bar{j}}$ (not to be confused with g) is the metric on the moduli space parametrised by the ϕ^i—the scalar component in the chiral multiplet Φ^i.

The Lagrangian (2.1) possesses Kähler invariance under a Kähler transformation. On a local patch it is easy to demonstrate. However to demonstrate this globally one has to show how this transformation can be defined from one patch to another. This gives rise to the consistency condition on triple junctions. From here one can argue the condition required on the elements of the second cohomology group of the target manifold H^2: they have to be even integers [15]. Quantization of Newton's constant also follows directly from here [15].

On the other hand, the classification for $\mathcal{N}=2$ supersymmetry is more interesting. We discussed this briefly at the beginning of this section. We will now elaborate this in some details. As before, global and local supersymmetry will have distinct properties:

- With global supersymmetry the target manifold of a non-linear sigma model can be *any* hyper-Kähler manifold [13]. These are $4n$-dimensional real Riemannian manifolds with holonomy group lying in $Sp(n)$.
- With local supersymmetry the target manifolds of a non-linear sigma model coupled to supergravity can only be quaternionic Kähler manifolds [16]. These manifolds are oriented $4n$ real dimensional manifolds with holonomy groups lying in $Sp(n)\times Sp(1)$. These manifolds have negative curvatures given by [16]:

  $$R=-64\pi n(n+2)G_N, \tag{2.2}$$

  where G_N is the Newton's constant and n is an integer. This means that the Newton's constant is fixed for a given manifold and not quantised like the earlier $\mathcal{N}=1$ cases. It also means that the global susy couplings are no longer compatible for the local susy case. Only in the limit $G_N\to 0$ the local and global cases could be identified.

2.1. An example in detail

Let us consider one concrete example where quaternionic target space can be illustrated. As mentioned above, a sigma model with quaternionic target space has to be coupled to supergravity to make sense. Global supersymmetry cannot yield a quaternionic target space. Therefore our four-dimensional Lagrangian can be taken as:

$$S = \Lambda^2 \int d^4x \sqrt{g}\left[-\frac{R}{2} - \frac{1}{\bar{z}^f z^f}\left(\partial_\mu z^a - \frac{z^a \bar{z}^b \partial_\mu z^b}{\bar{z}^c z^c}\right)\left(\partial^\mu \bar{z}^a - \frac{\bar{z}^a z^d \partial^\mu \bar{z}^d}{\bar{z}^e z^e}\right)\right], \tag{2.3}$$

which is a Fubini–Study metric on the target space. In fact the way we wrote the Lagrangian only implies a $\mathbf{CP}^N$ target because the coordinates z^a go from $a = 1$ to $a = N + 1$. This is a Kähler metric, but still not quaternionic because the Kähler potential K is

$$K = \log(1 + z^a \bar{z}^a), \tag{2.4}$$

where z^a are summed from $a = 1$ to $a = N$ because we are in a patch with $z^{N+1} = 1$. To convert (2.3) to quaternionic case, we will first replace all $z^a \leftrightarrow q^a$, where q_a is a 2×2 matrix given as:

$$q^a = \begin{pmatrix} q_0^a + iq_3^a & q_2^a + iq_1^a \\ -q_2^a + iq_1^a & q_0^a - iq_3^a \end{pmatrix}, \tag{2.5}$$

where $a = 1, \ldots, N$. This would then convert (2.3) to the following quaternionic analogue:

$$S = \Lambda^2 \int d^4x \sqrt{g}\left[-\frac{R}{2} - \frac{1}{\mathrm{tr}(q^\dagger \cdot q)}\left(\mathrm{tr}(\partial_\mu q^\dagger \cdot \partial^\mu q) - \frac{\mathrm{tr}(q^\dagger \cdot \partial_\mu q)\,\mathrm{tr}(\partial^\mu q^\dagger \cdot q)}{\mathrm{tr}(q^\dagger \cdot q)}\right)\right], \tag{2.6}$$

where we have defined $\mathrm{tr}(q^\dagger \cdot q)$ as $\sum_a \mathrm{tr}(q^{a\dagger} q^a)$ and similarly the other terms. Such a redefinition to convert (2.3) to (2.6) changes $\mathbf{CP}^N$ to $\mathbf{HP}^N$ where

$$\mathbf{HP}^N = \frac{Sp(N+1)}{Sp(N) \times Sp(1)}. \tag{2.7}$$

The quaternionic analogue of $\mathbf{CP}^N$ i.e., $\mathbf{HP}^N$ in fact shares the same properties as $\mathbf{CP}^N$: the q^a vectors are defined upto a scaling by a quaternion (recall z^a are only defined upto a complex scaling). It is also important to note that any $4N + 3$ sphere is equivalent to a S^3 fibration over a quaternionic base $\mathbf{HP}^N$. This will be useful soon.

2.2. Structure of the multiplets

The quaternionic sigma model that we discussed above can be shown to appear in string theory by compactifying type II strings on a Calabi–Yau three-fold. This leads to $\mathcal{N} = 2$ supersymmetry in four-dimensional space–time with the following generic multiplets:

- Vector multiplet: $(A_\mu, 2\phi, 2\psi)$.
- Hypermultiplet: $(4\phi, 2\psi)$.
- Tensor multiplet: $(B_{\mu\nu}, 3\phi, 2\psi)$.
- Double tensor multiplet: $(2B_{\mu\nu}, 2\phi, 2\psi)$.
- Vector tensor multiplet: $(B_{\mu\nu}, A_\mu, \phi, 2\psi)$.
- Gravity multiplet: $(g_{\mu\nu}, A_\mu, 2\psi_\mu)$.

Where ϕ appearing in all these multiplets are real scalars, ψ are Weyl fermions in four dimensions and ψ_μ are four-dimensional gravitinis. Observe that both the double tensor multiplet as well as the tensor multiplet are dual to the hypermultiplet. Similarly the vector tensor multiplet is dual to the vector multiplet. Thus the non-trivial four-dimensional $\mathcal{N}=2$ multiplets are the vector, hyper and the gravity multiplets. Compactifying type IIB theory on a Calabi–Yau three-fold gives rise to the following multiplets:

$$(g_{\mu\nu}, A_\mu) \oplus h_{12}(A_\mu, 2\phi) \oplus h_{11}(B_{\mu\nu}, 3\phi) \oplus (2B_{\mu\nu}, 2\phi), \tag{2.8}$$

where we have ignored the fermionic degrees of freedom. From ten-dimensional type IIB point of view, the metric fluctuations give rise to $(2h_{21}+h_{11})$ scalars in four dimensions, the NS and RR antisymmetric tensors both contribute h_{11} scalars in four dimensions along with the axio-dilaton contributing two more scalars. Thus the scalars in the vector multiplets all come from the metric fluctuations whereas the scalars in the tensor multiplets come partly from the metric fluctuations and partly from the zero mode fluctuations of the NS and RR two form tensors. Finally the axio-dilaton go to the double tensor multiplet. On the other hand, the vectors in the gravity as well as vector multiplets all come from the zero mode fluctuations of the four-form field. The four-form fluctuations also contribute h_{11} antisymmetric tensors that go to the tensor multiplets whereas the NS and RR two forms both go to the double tensor multiplet. It is also easy to see that once we dualise the tensor and the double tensor multiplets, we will have one gravity multiplet, h_{12} number of vector multiplets and $(1+h_{11})$ number of hypermultiplets. On the other hand, type IIA theory when compactified on the *same* Calabi–Yau will give us the following four-dimensional multiplets:

$$(g_{\mu\nu}, A_\mu) \oplus h_{11}(A_\mu, 2\phi) \oplus h_{21}(4\phi) \oplus (B_{\mu\nu}, 3\phi), \tag{2.9}$$

where again we have ignored the fermions. To keep track of the scalars: the hypermultiplet scalars come from both the metric fluctuations and a zero mode fluctuations of the three-form field. The vector multiplet scalars come partially from the zero mode fluctuations of the $B_{\rm NS}$ field and partially from the fluctuations of the metric. The dilaton however goes to the tensor multiplet this time. On the other hand, the vectors in the vector multiplets do not come from the IIA vectors but from the zero mode fluctuations of the three form field. In fact the type IIA vector go to the gravity multiplet. The antisymmetric tensor in the tensor multiplet is the type IIA $B_{\rm NS}$ field. Observe also that in the dual picture (i.e., dualising the antisymmetric $B_{\mu\nu}$ field) we have one gravity multiplet, h_{11} number of vector multiplets and $(1+h_{21})$ number of hypermultiplets. This would be exactly the same as the type IIB multiplets if

$$h_{11}(\mathtt{IIA}) = h_{21}(\mathtt{IIB}), \quad \text{and} \quad h_{21}(\mathtt{IIA}) = h_{11}(\mathtt{IIB}), \tag{2.10}$$

which is of course the statement of mirror symmetry at perturbative tree level.

At this point we should also note that the structures of quaternionic manifolds in string theory are restricted in string compactification. This is easy to see from the fact that some of the scalars in the hypermultiplets come from the zero mode fluctations of the metric. The moduli space of these scalars are Kähler manifolds and therefore the full quaternionic structure of the hypermultiplet moduli space [6,16]—that come from adding RR scalars to the metric fluctuations—should have a submanifold that is a Kähler manifold. This mapping of a Kähler submanifold to the full quaternionic manifold is called as a c-map[4] [5]. Thus, for example, in type IIB on a Calabi–Yau ?

[4] Or sometime as the s-map [5].

manifold the quaternionic space is of real dimension $4(1+h_{11})$ with a subspace given by

$$\frac{SU(1,1)}{U(1)} \times \mathcal{M}_k, \tag{2.11}$$

where the first part is parametrised by four-dimensional axion-dilaton i.e., the double tensor multiplet, and the second part is the Kähler submanifold. On the other hand, in type IIA theory the first part of (2.11) comes from the four-dimensional tensor multiplet. Thus clearly the hypermultiplet target space cannot be a generic quaternionic manifold because of the c-map constraint [5]. Furthermore since the dilaton resides in the hypermultiplets, the tree level picture is not correct. Details of these have been worked out various authors (see for example [17–19] and references therein). In particular, the perturbative corrections are now fully understood, and not just for the universal hypermultiplet—as shown by [18] there are no quantum corrections beyond 1-loop due to a nonrenormalization theorem. Moreover, the complete worldsheet, D1 and D(−1) instanton corrections in IIB as well as half of the D2 instanton effects in IIA have been determined by [18] together with [19]. The resulting modified moduli spaces are quaternionic in agreement with unbroken $\mathcal{N}=2$ supersymmetry.[5]

2.3. Few more examples

The restriction that we mentioned regarding construction of quaternionic manifolds may pose a difficulty in having explicit examples. However string theory gives us a very simple way to construct quaternionic manifolds that are consistent with the c-map:

- Construct a vector multiplet Lagrangian in four dimensions. The multiplet is $(A_\mu, 2\phi, 2\psi)$ with the real-scalars forming a Kähler target space. Such a Lagrangian coupled to gravity is well known [16].
- Dimensionally reduce this Lagrangian to three space–time dimensions. The vector multiplet will give us $(A_\mu, 3\phi, 2\psi)$ in three dimensions.
- Dualise the vector to another scalar φ via $d\varphi = *dA$ to convert the vector multiplet to a hypermultiplet $(4\phi, 2\psi)$. The metric on the moduli space of these scalars is exactly quaternionic [6].
- The quaternionic metric is also consistent with the c-map because we derived this from the vector multiplet with a Kähler target. Thus the quaternionic manifold will have a submanifold that is Kähler, as one would have expected [6].

In fact the above set of steps can be put into a more concrete setting. Consider a simple $\mathcal{N}=2$ Lagrangian with complex scalars coupled to one forms and gravity. A typical set up is

$$S_4 = \int d^4x\,\sqrt{g}\big[R + G_{a\bar{b}}\partial_\mu\phi^a\partial^\mu\bar{\phi}^b + c_{ij}F^i \wedge *F^j\big] + d_{ij}F^i \wedge F^j, \tag{2.12}$$

where $G_{a\bar{b}}$ is the metric on the moduli space—which will be a Kähler metric as we discussed above—and c_{ij} and d_{ij} are some coefficients which are functions of the moduli ϕ^a. The subscript i, j signify the number of vector multiplets that we couple to gravity.

In this form the Lagrangian (2.12) is almost like a D3-brane action coupled to gravity. However the resulting configuration should not be viewed as a D3 located at a point on a Calabi–Yau

[5] We thank Ulrich Theis for pointing this out to us.

because the supersymmetry will not be $\mathcal{N}=2$ and the dimension of the Kähler moduli space will be fixed. Furthermore the instanton coefficient d_{ij} is not quite related to the ten-dimensional axion. We will however relate a slight variant of this configuration to a D3-brane metric soon.

After a dimensional reduction and subsequent duality, we will get a three-dimensional action for the hypermultiplets. This is given by:

$$S_3 = \int d^3x \sqrt{g}\left[R + G_{a\bar{b}}\partial_\mu\phi^a\partial^\mu\bar{\phi}^b + \mathcal{G}_{c\bar{d}}\mathcal{D}_\mu\varphi^c\mathcal{D}^\mu\bar{\varphi}^d\right], \tag{2.13}$$

where $(\phi, \varphi, \bar{\phi}, \bar{\varphi})$ form the coordinates of a quaternionic space with a metric $\mathcal{G}_{c\bar{d}}$ spanning the submanifold specified by the coordinate φ^c. The covariant derivatives $\mathcal{D}_\mu\varphi^c$ are with respect to some connection. This structure of the moduli space can be easily connected to the ones studied by [17,20].

We can try to make this a bit more precise using the previous form of our action (2.6). Let us consider the following choice of the quaternion:

$$q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \tag{2.14}$$

where both B and C are complex numbers (not necessarily independent). The scalar target space parametrised by the quaternion then will have the following structure:

$$\mathcal{L} = \frac{|\partial_\mu C|^2 + |\partial_\mu B|^2}{|C|^2 + |B|^2} - \frac{|C\partial_\mu C^* + B\partial_\mu B^*|^2}{(|C|^2 + |B|^2)^2}, \tag{2.15}$$

where we have suppressed the gravity part. Consider now the scenario where B and C appearing above are complex numbers, but are not independent. They are related by

$$B = -C^* \tag{2.16}$$

as is clear from the quaternionic structure of the q coordinate. Such a choice of B, C would imply that the Lagrangian (2.15) can be recast as

$$\mathcal{L} = \frac{2|\partial_\mu C|^2}{\mathcal{S} + \mathcal{S}^*} - \frac{|\partial_\mu \mathcal{S} - 2C\partial_\mu C^*|^2}{(\mathcal{S} + \mathcal{S}^*)^2}, \tag{2.17}$$

where, in our notation, $\mathcal{S}$ is not quite an independent variable as it stands. It is given by

$$\mathcal{S} = |C|^2. \tag{2.18}$$

The reason for writing (2.17) in the present form is to allude to the subsequent structure that we will be inferring from string theory.

The string theory examples that have been studied earlier are all non-compact symmetric spaces with negative curvatures. In fact string theory tells us precisely how S defined above (2.18) should be modified so as not to change the underlying quaternionic structure. The resulting metric will be consistent with the target space metric of a tensor multiplet $(B_{\mu\nu}, 3\phi, 2\psi)$ when dualised to a hypermultiplet in four dimensions. Although this is no way the most generic method to derive the metric, it does help us to see the subsequent structure. In type IIA this is therefore a compactification on a Calabi–Yau three-fold that has no complex structure deformations (more on this later). Furthermore since dilaton sits precisely in such a multiplet, quantum corrections are expected to affect the target space metric. After the dust settles, the final answer is a slight modification of our simple calculation above. The quantity S changes from (2.18) to

$$\mathcal{S} = |C|^2 + e^{-2\phi} + i\varphi, \tag{2.19}$$

where ϕ is the dilaton sitting in the tensor multiplet, φ is the corresponding axion (dualised from the $B_{\mu\nu}$ field in four dimensions) and C, C^* are the other two scalars in the tensor multiplet. These are the two scalars that come from type IIA three form in ten dimensions. Similarly the Kähler potential is changed to

$$\mathcal{K} = -\ln\left(\mathcal{S} + \mathcal{S}^* - 2|C|^2 + \text{quantum corrections}\right), \tag{2.20}$$

which implies that the resulting manifold is also Kähler (see [17] for some details). Without quantum corrections the tree level moduli space for the universal hypermultiplet is given by

$$\mathcal{M}_{\mathrm{H}} = \frac{SU(1,2)}{U(2)}, \tag{2.21}$$

which is the non-compact analogue of $\mathbf{Gr}_2(\mathbf{C}^3)$ because of the negative curvature. Under tree level quantum corrections the Kähler structure of the moduli space is broken [21]. Further corrections to the moduli space come from the two- and five-brane instantons. These and others have been addressed in [18,19,22] as we discussed briefly before, although a full treatment is far from complete.

Let us consider another example. This time we compactify type IIA theory on a Calabi–Yau threefold with no complex structure deformations (i.e., $h_{21} = 0$). Thus in four dimension we will have the following multiplet structure:

$$(g_{\mu\nu}, A_\mu) \oplus h_{11}(A_\mu, 2\phi) \oplus (B_{\mu\nu}, 3\phi), \tag{2.22}$$

which is a slight modification of (2.9). As we can see, the universal hypermultiplet is always there. The moduli space therefore is from the vector multiplet Kähler space as well as the universal hypermultiplet, as is given by

$$\mathcal{M} = \mathcal{G}^{h_{11}}_{\text{Kähler}} \otimes \frac{SU(2,1)}{U(2)}, \tag{2.23}$$

where $\mathcal{G}$ is the Kähler manifold of dimension h_{11}. Observe also the fact that there are $(1 + h_{11})$ vectors in this setup (extra one coming from the gravi-photon).

Compactifying type IIB theory on the same Calabi–Yau gives us $(1 + h_{11})$ hypermultiplets coupled to gravity (and graviphoton) and no vector multiplets. The quaternionic manifold that we get here can in fact be derived from the moduli space (2.23) via the c-map. This is given by

$$\mathcal{G}^{4(h_{21}+1)}_{\text{quaternion}}, \tag{2.24}$$

from where we can easily see that the quaternionic space $\frac{SU(2,1)}{U(2)}$ forms a sub-manifold of the final *irreducible* quaternionic space $\mathcal{G}^{4(h_{21}+1)}$. This is the essence of the c-map in the presence of the universal hypermultiplet.

In the following section we will address the question of classifying quaternionic manifolds using constrained instantons and Seiberg–Witten curves, and discuss the emergence of the so-called magic square.

3. On the classification of quaternionic manifolds: standard cases

As discussed in earlier sections, the classification of quaternionic manifolds have been started in [2,3], and completed finally in [23]. Many of the cases that we studied so far (or have been addressed in the literature) can be seen to follow from the above framework. For our case we will try to understand the classification of the compact symmetric quaternionic Kähler manifolds using a different technique. Some aspects of this have been addressed earlier in [24].

3.1. $Sp(n+1)$ quaternionic space

Our first starting point will be the simplest case of $Sp(n+1)$ quaternionic space.[6] As we will discuss below, the quaternionic space associated with $Sp(n+1)$ group is special in the whole classification of quaternionic spaces. The key point that we will follow to classify these spaces is this: we look for gauge theories with certain global symmetries $\mathcal{G}$ (here, for this case, it is $Sp(n+1)$) and find semi-local instanton configurations. The low momentum dynamics of these theories (by low momenta we mean momenta lower than the masses of the Higgs and the masses of the photons) can be shown to be sigma models with quaternionic target spaces. Such an approach was first discussed in [25] (see also [26] for sigma models on Kähler target spaces) and later elaborated in [24]. Here we will try to complete the analysis by detailing the corresponding gauge theory constructions.

The gauge theory that we are looking for is an $Sp(1) \equiv SU(2)$ gauge theory with a global symmetry $\mathcal{G}$. Clearly this theory resembles closely to a sector of the corresponding Seiberg–Witten theory with global symmetries [8]. To make this precise, let us write the action for our theory. This is given by the following generic form [25]:

$$S = \int d^4x \left[\frac{1}{4} \mathrm{tr}_{SU(2)}\big(F_{\mu\nu}F^{\mu\nu}\big) + \mathrm{tr}\big(D_\mu q^\dagger \cdot D^\mu q\big) + V\big(\mathrm{tr}(q^\dagger \cdot q)\big) + \text{fermions} \right], \tag{3.1}$$

where q is a generic quaternion as described in the previous section, and the trace is over the global symmetry.[7] Obviously, as mentioned above, this is not quite a Seiberg–Witten theory as it stands. However once we write the quaternions in terms of complex fields (we show an example below), the action will resemble a part of the standard $\mathcal{N}=2$ action with a potential V (a simple case is the one worked out in [27] for an $Sp(1)_g \times Sp(1)_l$ case). In this sense, we can use the Seiberg–Witten curves to determine the global properties of this model. A recent example of semilocal defects like strings in Seiberg–Witten theory is [28]. Our goal is to study instantons in the model (3.1), i.e., a sector of, and not quite the actual, Seiberg–Witten theory. In fact the analysis of instantons in this theory can be done in two different ways, both leading to the same result. The first way is to observe that a theory like (3.1) will not allow any non-trivial instantons if

$$\pi_3\left(\frac{\mathcal{G}}{\mathcal{H}}\right) = 1, \tag{3.2}$$

where $\mathcal{H}$ is the unbroken subgroup. However instantons are possible when a subgroup of $\mathcal{G}$ is gauged.[8] Let us call the ungauged subgroup of $\mathcal{G}$ to be $\mathcal{G}_g \equiv \mathcal{H}$. Then the vacuum manifold $\mathcal{M}_1$ of this theory is rather simple. It is given by:

$$\mathcal{M}_1 = \frac{\mathcal{G}}{\mathcal{G}_g} = \frac{Sp(n+1)}{Sp(n)} \approx \mathbf{S}^{4n+3}, \tag{3.3}$$

[6] A point about notation: we will be considering $Sp(n)$ groups instead of $Sp(2n)$ groups used sometime in the literature. In our notation therefore $Sp(n)$ group is just the quaternionic unitary group $U(n, \mathbf{H})$. Its a real, compact and simply connected Lie group of dimension $n(2n+1)$. In particular $Sp(1) \equiv SU(2)$ and we will not distinguish between them in this paper.

[7] Note that q will transform as a fundamental of both the global $\mathcal{G}$ and the local $SU(2)$ groups for all choices of $\mathcal{G}$ considered henceforth unless mentioned otherwise.

[8] These are the *constrained* instantons [9] as we will explain below.

where, as should be clear from the above analysis, $\mathcal{G}_g = Sp(n)$ and we are taking the following breaking pattern:

$$\frac{Sp(n+1)_g \times Sp(1)_l}{\mathbf{Z}_2} \xrightarrow{\Phi} \frac{Sp(n)_g \times Sp(1)_g}{\mathbf{Z}_2} \tag{3.4}$$

with Φ being the Higgs field. The Higgs field is to be considered as a quaternion and *not* a complex number, although we could consider this also to be a complex matrix. The quaternion that could be used to represent the Higgs field is already pointed out above in (2.5). Thus

$$\Phi \equiv (q_a) = \begin{pmatrix} -\phi_a^1\phi_a^{2*} & |\phi_a^1|^2 \\ -|\phi_a^2|^2 & \phi_a^2\phi_a^{1*} \end{pmatrix} \tag{3.5}$$

is a good representation of the Higgs field in terms of the quaternions $(q_a) = (q_1, q_2, \ldots, q_n)$ or in terms of ϕ_a. As pointed out in [10,25] (and references therein) this is equivalent to a model with $n+1$ copies of the electroweak scalar sector with an $Sp(n+1)$ global symmetry in the $\theta_W = 0$ limit.

The second way is to view (3.1), when written in terms of complex coordinates and incorporating other terms, as describing the Higgs branch of Seiberg–Witten theory. Then the semilocal instantons can be related to the *small* instantons described by Witten [29] and Ganor–Hanany [30] and the vacuum manifold $\mathcal{M}_1$ becomes the moduli space of one-instanton. These instantons are described by embedding $SU(2)$ groups inside the global groups, and therefore the different $SU(2)$ orientations describe the moduli space of the theory.[9] These $SU(2)$ orientations form an S^3 and the moduli associated with the sizes of these instantons form the radii of the three cycles. In this language these three cycles will be fibered over quaternionic Kähler spaces. Such an approach has been used to study quaternionic Kähler manifolds associated with A_n, B_n, C_n and D_n groups [34]. The moduli space then is a 3-Sasakian spaces that are $Sp(1)$ fibrations over quaternionic Kähler spaces [35] and is given by:

$$\mathcal{M}_k = \mathbf{R}^4 \times \mathbf{R}^+ \times \left[Sp(1) \otimes_f \mathbf{Q}_k\right], \tag{3.6}$$

where $\mathbf{R}^4$ denotes the four-translation moduli, $\mathbf{R}^+$ denotes the size moduli, the subscript k denotes k-instantons, $\mathbf{Q}_k$ denotes the quaternionic space associated with k-instantons and the subscript f denotes non-trivial fibration. In the following we will give a concrete example of such fibration using mostly the first technique (although in many cases we will alternate between the two techniques[10]). This will prove convenient for theories that may not have a good

[9] Observe that, if we view the Seiberg–Witten theory to be generated by D3/D7 system a la [31], then the gauge instantons are D(−1)-branes inside D3-branes, whereas the small instantons are the bound states of D3-branes with the D7-branes [32]. If we T-dualise the system then we will have a configuration of D1, D5 and D9-branes. The moduli space of the small instantons on D9-branes i.e., D5-branes in D9-branes is given via ADHM data by a special hyper-Kähler manifold, or a quaternionic Kähler manifold when coupled to supergravity [29]. On the other hand the moduli space of D1-branes is given by a sigma model with ADHM target space [32,33]. Thus both the pictures describe the same physics.

[10] There is also a *third* way of studying the moduli spaces of these instantons that is slightly different from the above two approaches (although more related to the second one). This has to do with the fact that $\mathcal{N}=2$ supersymmetric gauge theories also have hypermultiplets in the *adjoint* representations of the gauge groups. Observe that the hypermultiplets that we considered for the above two cases are all in the fundamental representations of the gauge groups. Combining these adjoint hypermultiplets with the $\mathcal{N}=2$ vector multiplets will give us the spectrum of $\mathcal{N}=4$ gauge theories. In these theories moduli spaces of instantons will be exactly the same as for the fundamental hypermultiplets if we exchange the global symmetries with *gauge* symmetries. Thus $\mathcal{N}=4$ theories with exceptional gauge symmetries will have the same moduli spaces of instantons as we study here. Such an approach has been discussed by Stefan Vandoren in the last

Table 1

Semilocal theory	Seiberg–Witten theory
Semilocal instantons	Small instantons
G/H = Vacuum manifold	Higgs branch = Instanton moduli space = Special hyper-Kähler manifold
Gauging an $SU(2)$ subgroup of the global group $\Pi_3(G/H) = 1$, $\Pi_3(SU(2)) = Z$	Embedding $SU(2)$ instanton in the global group = Orienting $SU(2)$ group in a global group
H = Unbroken subgroup	H = Stability group of the instanton
Quaternions	$N = 1$ chiral multiplets or $N = 2$ hypermultiplets
$SU(2)$ gauge group	Microscopic $SU(2)$ group
Quaternionic Kähler manifold	Quaternionic Kähler manifold
Semilocal strings	Semilocal strings
Mass term in the potential	Massive hypers, mixed Coulomb–Higgs branch

Lagrangian description (and therefore no well defined Higgs branch) but more importantly the technique of semilocal defects is ideally suited to study other manifolds in the *magic square* as we will discuss soon. In Table 1 we show the precise connection between our semilocal theory, and the full Seiberg–Witten theory.

From above table it should be clear that although our theory (3.1) is a small sub-sector of the original Seiberg–Witten theory, it has all the necessary ingredients to understand the detailed aspects of magic square as we will demonstrate soon. The complicacies of the full Seiberg–Witten theory, for example the existence of Coulomb branch or mixed Coulomb–Higgs branch, do not effect the analysis that we are going to perform therefore we will continue with our simpler version (3.1).[11] However we will try to demonstrate, whenever possible, how to analyse the system from the full Seiberg–Witten theory.

Thus having laid down the possible criteria to construct explicit $Sp(n+1)$ quaternionic manifolds, there are a few important points to analyse now:

- We have to verify whether it is possible to construct a Seiberg–Witten like theory with $Sp(n+1)$ global symmetry. This would be confirmed by the existence of the corresponding Seiberg–Witten curve for the system. We expect, on generic ground, a curve of the form:

$$y^2 - x^3 - a_2 x^2 k(z) + a_1 x y l(z) + a_3 y h(z) - a_4 x f(z) - a_6 g(z) = 0, \tag{3.7}$$

  with a_i being constants and $k(z), l(z), h(z), f(z)$ and $g(z)$ are polynomials in z. The coordinate z specifies the complex plane in the corresponding Seiberg–Witten theory. The above equation with the right choice of k, l, h, f and g takes the following Weierstrass form that reflects an $Sp(n+1)$ global symmetry:

$$y = \pm\sqrt{x^3 + x z^{n+1} + \frac{5}{4} z^{2n+2}} - \frac{z^{n+1}}{2}. \tag{3.8}$$

reference of [34]. For most of our analysis in this paper we will not consider the adjoint hypermultiplets as we want to analyse $\mathcal{N} = 2$ gauge theories only.

[11] In fact our model (3.1) does not have a Coulomb branch. So in the corresponding Seiberg–Witten theory this is the pure Higgs branch.

Using this one can check that the curve[12] has the right singularity structure to allow an $Sp(n+1)$ global symmetry. A similar curve should then describe the global properties of our model.

- The next step to verify would be the existence of instantons in this model. Clearly existence of the corresponding curve (3.8) means that we have summed *all* the instanton contributions to get the required Seiberg–Witten curve. However it is instructive to actually construct these instantons. Out of the various different possibilities of instanton configurations in our system (because of the matter representations) we will henceforth only concentrate on the so-called *semilocal* instantons unless mentioned otherwise. These are the small instantons in the Higgs branch of the full theory. Their construction is subtle because of two reasons. Firstly the vacuum manifold being S^{4n+3} would imply

$$\pi_3(\mathcal{M}_1) = 1, \tag{3.9}$$

so would disallow instantons. The only allowed instanton configurations therefore would be the *semilocal* instantons by gauging an $Sp(1)$ part of the global symmetry.[13] We may then expect that the low momentum dynamics of the theory should be a sigma-model on a certain quaternionic space, or alternatively the moduli space of the Higgs branch instantons should be given by the quaternionic space. The structure of the corresponding quaternionic space can be determined from the following gauge field configuration:

$$A_\mu \equiv A_\mu^a \sigma^a = \frac{1}{2g_{\mathrm{YM}}^2} \cdot \frac{q^\dagger \cdot \partial_\mu q - q \cdot \partial_\mu q^\dagger}{\mathrm{tr}(q^\dagger \cdot q)}, \tag{3.10}$$

where the sum over repeated indices are implied via the dot product and σ^a are the Pauli matrices. Now due to the existence of F- and D-terms the low energy effective action will be a quaternionic manifold $\mathbf{HP}^n$ as shown in (2.6) when (3.10) is plugged in the action (3.1). The semilocal instantons in this model have the following structure (see also [25]):

$$\pi_3(S^3) = \mathbf{Z}, \qquad S^{4n+3} \xrightarrow{S^3} \mathbf{HP}^n, \tag{3.11}$$

provided certain subtleties are considered. This is the second reason. The subtlety has to do with the presence of $V(\mathrm{tr}(q^\dagger \cdot q))$ term in the action (3.1), namely, due to Derrick's theorem once the scale invariance is broken by a mass term in the potential, the instantons all squeeze to zero size. So the semilocal instantons that we are alluding to should exactly be the constrained instantons of Affleck [9]. These constrained instantons resemble the standard instanton at short distances only but decay exponentially at the IR [9] (see also [36]). In the notation of [27], when

$$S = \zeta_\pm = \zeta_3 = 0, \tag{3.12}$$

[12] Observe that this is only a genus one curve. For higher local gauge symmetry, for example $SU(N)$ with $N > 2$, we will have a genus $N-1$ curve. In this paper we will look mostly at the sector of the theory that is given by a genus one (i.e., $N = 2$) curve although in the last part of Section 4 we will give some examples of higher genus curves. Generic case of an $SU(N)$ gauge theory broken to $SU(2) \times G_{\mathrm{local}}$ gauge theory will be studied in the sequel to this paper.

[13] One might be wondering about the connection between the curve (3.8) and the contributions from the semilocal instantons. As is well known *all* possible instantons should contribute to the path integral to determine the full curve of the theory [33]. The curve (3.8) is the minimal curve with $Sp(n+1)$ global symmetry so will have contributions from the semilocal instantons (which are of course the small instantons in the Higgs branch). The situation gets tricky when the global symmetry becomes very large (for example E_n as we will encounter later). In those situations how exactly all the instantons contribute to give us the full curve will be described elsewhere.

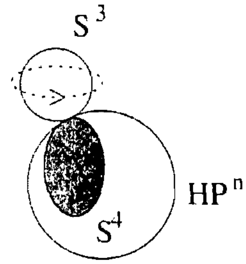


Fig. 1.

where ζ are the FI terms, then the instanton allowed are the standard instantons. For the case when the FI terms are non-zero, to construct constrained instantons all we require is the maximal subalgebra of the extended Dynkin diagram of $Sp(n+1)$:

should be expressible as a product of two subalgebras. This fixes the maximal subalgebra for our case to be $sp(n) \oplus sp(1)$. The constrained instantons are exactly of the gauged $Sp(1) \equiv SU(2)$ group. The simplest non-trivial example of such an instanton is for the global group $Sp(2)$. The quaternionic space associated with this global group is a four sphere S^4 because:

$$\frac{Sp(2)}{Sp(1) \times Sp(1)} = S^4 \equiv \mathbf{HP}^1 \tag{3.13}$$

and therefore the constrained instantons are non-trivially fibered over the four sphere (this has also been noticed for a non-stringy example in [25]). For our case when $\zeta_3 \neq 0$ and all other FI terms vanishing in $V(\mathrm{tr}(q^\dagger \cdot q))$ of (3.1), the constrained instanton can be explicitly worked out to be of the following form:

$$A_\mu = \frac{2\rho^2 \sigma^a \eta^a_{\mu\nu} x_\nu}{x^2(x^2+\rho^2)} - \frac{\zeta_3 g^2_{\mathrm{YM}}}{2} \cdot \frac{\sigma^a \eta^a_{\mu\nu} x_\nu}{x^2} + \cdots, \tag{3.14}$$

where ρ^2 is the typical size of the instanton in the scale invariant limit (which is of course the $\zeta_3 = 0$ limit). Observe that we need to also switch on non-zero expectation values for the quaternions. It can be easily shown that the background values of the quaternions are always proportional to the FI term ζ_3 so that in the scale invariant limit their expectation values have to vanish to allow the standard instantons to exist. In Fig. 1 a typical constrained instanton is shown. We see that the instanton is non-trivially fibered over the quaternionic base $\mathbf{HP}^n$ and wraps the three sphere S^3 once at infinity. Over the rest of the space it completes a non-trivial four sphere S^4 in the quaternionic space. This also means that for $\mathbf{HP}^1$ we will have a controlled theoretical way to study the instanton. This is in fact further facilated by the following group theory identities:

$$\mathbf{HP}^1 = S^4 = \frac{SO(5)}{SO(4)}, \tag{3.15}$$

which means that this special case could even be studied using real fields. This is indeed the case, and has been attempted in [25].

The above set of procedures was to construct a configuration of the simplest quaternionic space $\mathbf{HP}^n$ using constrained instantons. The relevant non-compact extension of the above space is the quaternionic space

$$\frac{Sp(n,1)}{Sp(n)\times Sp(1)}, \tag{3.16}$$

which is more useful to study the moduli spaces in type II theories. Now recall that there is a natural one-to-one correspondence between quaternionic normal Lie algebras and quaternionic simply connected normal homogeneous spaces. In fact any normal quaternionic algebra should contain a one-dimensional quaternionic subalgebra called the canonical quaternionic subalgebra. The manifold that we studied above (3.16) correspond to the following totally geodesic subalgebra:

manifold

$$C_1^1 \equiv \frac{Sp(1,1)}{Sp(1)\times Sp(1)}. \tag{3.17}$$

In fact (3.16) is the *unique* quaternionic algebra whose canonical subalgebra is isomorphic to C_1^1 [3]. However there is *no* Kähler space associated with (3.16) because there is no c-map. So (3.16) cannot appear as low energy Lagrangian in type II theories. Thus our construction of the corresponding compact $\mathbf{HP}^n$ gives the only legitimate way to study this manifold in string theory. Below we will show that all the compact versions of the symmetric quaternionic spaces can be studied using the technique of constrained instantons. In fact we will show how the magic square appears in this analysis. But first, lets go to the next non-trivial example related to the G_2 quaternionic space.

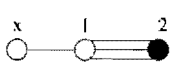
3.2. G_2 quaternionic space

The technique that we developed in the previous subsection is universal. We will use the same procedure of constrained instantons to construct quaternionic manifolds for the G_2 cases also. However instead of repeating the same constructions once again, we will give a concrete mathematical way to build the quotient space:

$$\frac{G_2}{Sp(1)\times Sp(1)}, \tag{3.18}$$

so that combining this procedure and the steps elucidated in the previous subsection we will be able to classify the magic square cases in the next section.

Before going into the details of the specific construction of (3.18) we would like to make the following comments. The quotient structure of (3.18) should be obvious from the previous analysis, namely, the maximal subalgebra of G_2 without an $U(1)$ factor from the extended Dynkin diagram:



is $so(4) \equiv su(2) \oplus su(2)$. As this is already expressed in terms of two product group (with an $su(2)$ factor) we need not go any further. In fact the $\mathbf{7}$ of G_2 then decomposes as[14]:

$$\mathbf{7} \to (\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{3}), \tag{3.19}$$

[14] We thank Tom Kephart for discussions on this point.

under $SU(2)\times SU(2)$, where once we give a VEV to $(\mathbf{2},\mathbf{2})$ one of the global $SU(2)$ (which is broken) mixes with the broken local $SU(2)$ to give us a diagonal unbroken $SU(2)$. The quotient space is then clearly (3.18). What remains to study however is the precise embedding of the $SU(2)$ groups inside a G_2. This will be addressed below.

The next issue is the existence of the corresponding Seiberg–Witten curve for a global G_2 group. We have already laid down the possible curve for any global group $\mathcal{G}$ in (3.7). For $\mathcal{G}=G_2$ we can choose certain specific functional form for k, l, h, f and g in (3.7) to give us the following curve:

$$\begin{aligned}&\left(y+\frac{12a_1zx-4a_1a_2z^2-4a_1^2z^2+12a_3z^2}{24}\right)^2\\&\quad=x^3-\frac{x}{48}\left[a_1^4z^4+8\left(a_1^2a_2-3a_1a_3-6a_4\right)z^3+16a_2^2z^2\right]\\&\qquad+\frac{1}{864}\left[a_1^8z^8+12\left(a_1^4a_2-3a_1^3a_3\right)z^5+\left(48a_1^2a_2^2+216a_3^2-72a_1^2a_4-144a_1a_2a_3\right)z^4\right.\\&\qquad\left.+\left(64a_2^3-288a_2a_4+864a_6\right)z^3\right],\end{aligned}\tag{3.20}$$

where a_i are some constants. The precise mapping of this curve to the G_2 Casimirs can be worked out but we will not do so here as our emphasis is more on the magic square. One can check that the discriminant is

$$\Delta\sim z^6+\mathcal{O}\left(z^8\right)\tag{3.21}$$

and therefore reflects a global G_2 symmetry near the point $z=0$. To see the full global symmetry for other cases one has to generalise the above curve (3.20) further. Examples of these will be discussed in the next section.

Another point is the existence of third homotopy groups for various coset spaces. For a global group $\mathcal{G}$ broken to a subgroup $\mathcal{H}\times SU(2)$ our first criteria would be to ask the value of the third homotopy from the exact sequence

$$\longrightarrow\pi_3(\mathcal{H})\longrightarrow\pi_3(\mathcal{G})\longrightarrow\pi_3(\mathcal{G}/\mathcal{H})\longrightarrow 0,\tag{3.22}$$

where both $\mathcal{G},\mathcal{H}$ are Lie groups.[15] For simple cases dealing with non-exceptional groups this is easy and well known. The interesting question comes when $\mathcal{G}$ is an exceptional group or when both $\mathcal{G}$ and $\mathcal{H}$ are exceptional groups. Three rules have been developed to address these questions [37]:

- When both $\mathcal{G}$ and $\mathcal{H}$ are simple, i.e., when both $\mathcal{G}$ and $\mathcal{H}$ do not have invariant Lie subgroups, then

$$\pi_3(\mathcal{G}/\mathcal{H})=\mathbf{Z}_M,\quad M\equiv\frac{l}{L},\tag{3.23}$$

where L is a non-negative integer called the index of a representation $\mathcal{D}_\mathcal{G}$ for the group $\mathcal{G}$. Similarly l is the index for the corresponding representation $\mathcal{D}_\mathcal{H}$ for the group $\mathcal{H}$. These

[15] This is crucial because, as mentioned earlier, our theory is only a sector of a bigger theory. Consistency requires that we evaluate the third homotopy of $\frac{\mathcal{G}}{\mathcal{H}}$ to study the instantons. On the other hand, in the full Seiberg–Witten theory, the instantons are in the Higgs branch and so we would only require to evaluate the third homotopy of the global group $\mathcal{G}$. For more details see the table of comparison given earlier.

indexes are tabulated in details for many representations in [38].[16] The idea is to look for a particular representation (say vector or tensor) for the group $\mathcal{G}$ and then look for the *same* representation for the group $\mathcal{H}$. The ratio of the corresponding indexes will give us the value for $\pi_3(\mathcal{G}/\mathcal{H})$. It is interesting to note that as long as we choose the same representations for both $\mathcal{G}$ and $\mathcal{H}$ the ratio l/L will always be the same.

- If $\mathcal{G}$ is simple but $\mathcal{H}$ is of the form of $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ with $\mathcal{H}_i$ simple, then

$$\pi_3(\mathcal{G}/\mathcal{H}) = \mathbf{Z} \text{ mod every } \frac{l_i}{L}, \tag{3.24}$$

where $(l_1, l_2, \ldots, l_i)$ are the collection of n-tuples. In fact $\mathcal{H}$ can have an additional Abelian subgroup without changing the result. Furthermore modding by a discrete subgroup also does not change the result.

- When both $\mathcal{G}$ and $\mathcal{H}$ are not simple and $\mathcal{G}$ is of the form $\mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \cdots \otimes \mathcal{G}_n$ where $\mathcal{G}_i$ are simple,[17] then $\pi_3(\mathcal{G}/\mathcal{H})$ consists of n-tuples of the form

$$(\sigma_1, \sigma_2, \ldots, \sigma_n) \text{ mod every } \left[\frac{l_i^{(1)}}{L_1}, \frac{l_i^{(2)}}{L_2}, \ldots, \frac{l_i^{(n)}}{L_n}\right], \tag{3.25}$$

where $(l_i^{(1)}, l_i^{(2)}, l_i^{(2)}, \ldots, l_i^{(n)})$ are the n-tuples associated with the simple groups $\mathcal{H}_i^{(1)}, \mathcal{H}_i^{(2)}, \ldots$, etc., where the Lie algebras g, g_i, h_i associated with the Lie groups $\mathcal{G}, \mathcal{G}_i, \mathcal{H}_i$, respectively have the decomposition $h_i = \bigoplus_j h_i^{(j)}$ with the condition $h_i^{(j)} \subseteq g_j$. The Lie algebras $h_i^{(j)}$ are either isomorphic to h_i or $\{0\}$. For more details the readers may want to refer to [37,38].

Therefore the upshot of all these discussions is that the third homotopy groups for coset spaces can either be 1 or $\mathbf{Z}_p$. For exceptional groups the third homotopy groups are all $\mathbf{Z}$. In fact generically $\pi_3(SU(n))|_{n\geqslant 2} = \mathbf{Z}$. Similarly $\pi_3(SO(n))|_{n\geqslant 3, n\neq 4} = \mathbf{Z}$ and $\pi_3(SO(4)) = \pi_3(SU(2) \times SU(2)) = \mathbf{Z} \oplus \mathbf{Z}$. This would mean that $\pi_3(G_2/SU(2)) = 1$ i.e., the third homotopy group is trivial,[18] although this does not mean much because with G_2 global symmetry a Lagrangian description of the system like (3.1) discussed previously is not possible.[19] Therefore to study the constrained instantons in the system we gauge the $SU(2)$ subgroup of the maximal

16 These indexes are represented as $\frac{l(2)}{\text{rank}}$ in [38].

17 Additionally allowing Abelian groups as well as discrete moddings.

18 It turns out there are other possible embeddings of an $SU(2)$ group in G_2, namely that the $\mathbf{7}$ of G_2 goes to $\mathbf{3}+\mathbf{2}+\mathbf{2}$ of $SU(2)$ or the $\mathbf{7}$ of G_2 goes to $\mathbf{7}$ of $SU(2)$. For these two cases $\pi_3(G_2/SU(2)) = \mathbf{Z}_3$ or $\mathbf{Z}_{28}$ respectively. We thank V.P. Nair for pointing this out to us.

19 It is an issue—and we will discuss this again later—for all theories with exceptional global symmetries. One can see this from the D3/D7-brane construction of these theories. The fundamental hypermultiplets appear from the strings connecting the D3-branes with the D7-branes. The gauge symmetries of the seven brane theories appear as global symmetries of the underlying D3-brane theories. For classical Lie groups as gauge or global symmetries, the seven branes are all D7-branes. However when we have exceptional Lie groups, not all seven branes are D7-branes. Some of them are $SL(2,\mathbf{Z})$ transform of the D7-branes. Because of that strings connecting the D3 and the seven branes may take *non-trivial* paths in the u-planes of corresponding Seiberg–Witten theories [39]. For such strings simple Born–Infeld action may not be easy to write down. Nevertheless such theories exist as can be easily shown from the corresponding F-theory, or the Seiberg–Witten curves. Since the curves are constructed by summing up all the instantons, we also know that these instantons exist. Therefore in this paper we will try to give as much information as possible, for these instantons, that do not rely on explicit Lagrangian formulations. In the sequel to this paper we will attempt more explicit constructions.

$SU(2) \times SU(2)$ group, or alternatively—viewing this in the Higgs branch—we study the orientations of $SU(2)$ inside G_2. Thus effectively we are studying $SU(2)$ constrained instantons in a theory with the maximal group. These instantons are non-trivially fibered over the base (3.18).

As we discussed for the $Sp(n+1)$ case in the previous section we can now describe a possible quaternionic geometry associated with the constrained instantons. In fact, as before, we need the sigma model on the non-compact version of the geometry namely, on

$$\frac{G_{2(+2)}}{SU(2) \times SU(2)}. \tag{3.26}$$

To determine this we can use the trick of the c-map, that uses the metric of the Kähler manifold to determine the quaternionic manifold. The Kähler manifold and the associated F function in question are [5,6,16]:

$$\mathcal{M}_{\text{Kähler}} = \frac{SU(1,1)}{U(1)}, \qquad F(X^I) = \frac{i(X^2)^3}{X^1}, \tag{3.27}$$

where X^I, $I = 1, 2, \ldots, n+1$ are the scalar fields corresponding to certain other $\mathcal{N} = 2$ vector multiplets (including the gravi-photon) and we have introduced the F function to determine the Kähler metric of the manifold $\mathcal{M}_{\text{Kähler}}$. This F function can be used to determine the Kähler potential $\mathcal{K}$ and the metric $G_{A\bar{B}} \equiv -\mathcal{K}_{A\bar{B}} = -\partial_A \bar{\partial}_B \mathcal{K}$ in the following way [6,40]:

$$\mathcal{K}(Z, \bar{Z}) = \ln(Z^I N_{IJ} \bar{Z}^J) \quad \text{with} \quad N_{IJ} = i(\partial_I \partial_J F - \bar{\partial}_I \bar{\partial}_J F), \tag{3.28}$$

where $Z^I = \frac{X^I}{X^1} \equiv \{1, Z^A\}$ and the Kähler metric therefore is the usual form $ds^2 = -\mathcal{K}_{A\bar{B}} dZ^A d\bar{Z}^B$. Observe that the metric is only positive definite in the region where $Z^I N_{IJ} \bar{Z}^J$ is positive definite. Therefore $\mathcal{K}_{A\bar{B}}$ is *negative* definite [6,40].

It is now time to use the power of the c-map to determine the quaternionic metric for our case. To build the quaternionic manifold we need $4(n+1)$ coordinates. The Z^A, $\bar{Z}^A$ contribute $2n$ coordinates. The other $2n$ coordinates are denoted as A^I, B_I, along with two more complex coordinates ϕ, φ. The c-map then defines the quaternionic metric in the following way [6][20]:

$$ds^2 = |d\phi|^2 - 2e^{-\phi}(\text{Re}\mathcal{N})_{I\bar{J}} W^I \bar{W}^J + e^{-2\phi} \left| d\varphi - \frac{A \cdot dB - B \cdot dA}{2} \right|^2 - 4\mathcal{K}_{A\bar{B}} dZ^A d\bar{Z}^B, \tag{3.29}$$

where it should be clear that the Kähler geometry (3.27) forms a submanifold in the quaternionic space as expected. The structure of the universal hypermultiplet can also be extracted from (3.29). The components of the matrix $\mathcal{N}$, and W^I are defined as:

$$\begin{aligned} \mathcal{N}_{IJ} &= -i\partial_{\bar{I}}\partial_{\bar{J}}\bar{F} - \frac{N_{IK}N_{JL}X^K X^L}{X^I N_{IJ} X^J}, \\ W^I &= \left[(\text{Re}\mathcal{N})^{-1}\right]^{IJ} \left(2\bar{\mathcal{N}}_{JK} dA^K - i dB_J\right), \end{aligned} \tag{3.30}$$

where Re$\mathcal{N}$ is negative definite. For other details about the properties of $\mathcal{N}$ etc the readers may want to refer to [6,16,40,41]. In the remaining part of this section we will give an explicit realisation of the quotient space (3.18).

[20] We are using the notations of [40].

3.2.1. Realisation of the quotient space

To give an explicit realization of the homogeneous space (3.18), i.e., $\frac{G_2}{Sp(1)\times Sp(1)} \equiv \frac{G_2}{SO(4)}$, we use the embedding of the exceptional complex Lie group $G_2(\mathbf{C})$ into the complex orthogonal Lie group $SO(7,\mathbf{C})$. Similar embeddings are valid for the two real forms of G_2, since the compact group $G_2^C(\mathbf{R})$ is included in $SO(7,\mathbf{R})$ and the non-compact real group $G_2^{NC}(\mathbf{R})$ in the real Lie group $SO(4,3)$. In the following, we will consider only the complex case and so we will omit the presence of $\mathbf{C}$ in the definition of our Lie groups.

The group G_2 has been shown [42,43] to be isomorphic to the group of orthogonal transformations $SO(7)$ acting on the vector space $\mathbf{C}^7$ and leaving invariant a third-order completely antisymmetric tensor T. It is completely characterized by the following:

$$T_{127} = T_{154} = T_{163} = T_{235} = T_{264} = T_{374} = T_{576} = 1. \tag{3.31}$$

Choosing to realize the group $SO(7)$ by matrices $G \equiv \{g_{ab}\} \in \mathbf{C}^{7\times 7}$ with determinant equal to 1 that satisfy the orthogonality relation:

$$G^\top G = I \quad \Longleftrightarrow \quad g_{ab} g_{ac} = \delta_{bc}, \tag{3.32}$$

we know that G will thus be characterized by 21 independent parameters. The invariance of the tensor T under such transformations may be written as

$$G^\top T_a G = g_{ab} T_b \quad \Longleftrightarrow \quad T_{aef} g_{ec} g_{fd} = g_{ab} T_{bcd}, \tag{3.33}$$

where T_a is the 7×7 matrix which elements are given by $(T_a)_{bc} = T_{abc}$. It gives rise to 7 additional constraints on the elements of G and G thus contains the 14 independent parameters that leads to G_2.

A simple realization of these conditions could be easily seen when we consider the algebra g_2. It can indeed be realized as the set of orthogonal matrices $M \in o(7)$ such that $M^\top = -M$ and satisfying the invariance condition

$$[T_i, M] = a_{ij} T_i, \tag{3.34}$$

which can be easily obtained from the relation (3.33) using the usual derivation of the exponential map which relates the group and algebra elements. We thus find an explicit form of $M \in G_2$ in terms of 14 independent parameters as:

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} & -a_{15}-a_{26} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{27}-a_{36} & -a_{17}+a_{35} & a_{16}-a_{25} \\ -a_{15} & -a_{25} & -a_{35} & -a_{27}+a_{36} & 0 & a_{12}-a_{34} & a_{13}+a_{24} \\ -a_{16} & -a_{26} & -a_{36} & a_{17}-a_{35} & -a_{12}+a_{34} & 0 & -a_{14}+a_{23} \\ -a_{17} & -a_{27} & a_{15}+a_{26} & -a_{16}+a_{25} & -a_{13}-a_{24} & a_{14}-a_{23} & 0 \end{pmatrix}. \tag{3.35}$$

Let us mention that the maximal subalgebra $so(4) = su(2) \oplus su(2)$ is easily identified. Indeed, we first take $a_{i5} = a_{i6} = a_{i7} = 0$ for $i = 1, 2, 3$ to reduce the matrix to the form

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 \\ -a_{12} & 0 & a_{23} & a_{24} & 0 & 0 & 0 \\ -a_{13} & -a_{23} & 0 & a_{34} & 0 & 0 & 0 \\ -a_{14} & -a_{24} & -a_{34} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{12} - a_{34} & a_{13} + a_{24} \\ 0 & 0 & 0 & 0 & -a_{12} + a_{34} & 0 & -a_{14} + a_{23} \\ 0 & 0 & 0 & 0 & -a_{13} - a_{24} & a_{14} - a_{23} & 0 \end{pmatrix} \tag{3.36}$$

and then take the six remaining independent parameters as $a_{34} \pm a_{12} = 2x_{\pm 3}, a_{24} \mp a_{13} = 2x_{\pm 2}, a_{14} \pm a_{23} = 2x_{\pm 1}$ to get the direct sum decomposition as $\mathbf{A} \oplus \mathbf{B}$, where:

$$\mathbf{A} = \begin{pmatrix} 0 & x_{+3} & -x_{+2} & x_{+1} & 0 & 0 & 0 \\ -x_{+3} & 0 & x_{+1} & x_{+2} & 0 & 0 & 0 \\ x_{+2} & -x_{+1} & 0 & x_{+3} & 0 & 0 & 0 \\ -x_{+1} & -x_{+2} & -x_{+3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 0 & -x_{-3} & x_{-2} & x_{-1} & 0 & 0 & 0 \\ x_{-3} & 0 & -x_{-1} & x_{-2} & 0 & 0 & 0 \\ -x_{-2} & x_{-1} & 0 & x_{-3} & 0 & 0 & 0 \\ -x_{-1} & -x_{-2} & -x_{-3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2x_{-3} & 2x_{-2} \\ 0 & 0 & 0 & 0 & 2x_{-3} & 0 & -2x_{-1} \\ 0 & 0 & 0 & 0 & -2x_{-2} & 2x_{-1} & 0 \end{pmatrix}. \tag{3.37}$$

We also see the inclusion of the preceding subalgebra $so(4)$ of G_2 in the algebra $so(4) \oplus so(3)$ as a subalgebra of $so(7)$.

3.2.2. Coordinates of the quotient space

We start with the well-known realization of the Grassmannian of nondegenerate three-planes $\mathrm{Gr}_4(\mathbf{C}^7)$ which is isomorphic to $SL(7)/\mathrm{Aff}(4,3)$ where $\mathrm{Aff}(4,3)$ is realized by matrices of the form

$$G_0 = \begin{pmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{pmatrix}, \qquad G_{11} \in \mathbf{C}^{4\times 4}, \qquad G_{22} \in \mathbf{C}^{3\times 3},$$
$$G_{21} \in \mathbf{C}^{3\times 4}, \qquad \det G_{11} \cdot \det G_{22} = 1. \tag{3.38}$$

We then define homogeneous coordinates on $Gr_4(\mathbf{C}^7)$ as

$$\mathcal{X} = \begin{pmatrix} X \\ z^{\top} \\ Y \end{pmatrix}, \qquad X, Y \in \mathbf{C}^{3\times 3},\ z \in \mathbf{C}^3, \tag{3.39}$$

so that $SL(7)$ acts from the left as $\mathcal{X}' = G\mathcal{X}$ with $G \in SL(7)$ and $\mathrm{Aff}(4,3)$ is thus the isotropy group of the origin chosen as $\mathcal{X}_0 = (0, 0, I_3)^{\top}$ and $\mathcal{X} = G\mathcal{X}_0$. The restriction to $SO(7)$ leads to the isotropy group $SO(4) \otimes SO(3)$ since G_0 being orthogonal, it implies $G_{21} = 0$. The homogeneous coordinates $\mathcal{X} = G\mathcal{X}_0$ of $SO(7)/(SO(4) \times SO(3))$ satisfy the orthogonality condition:

$$X^{\top}X + zz^{\top} + Y^{\top}Y = 1, \tag{3.40}$$

which represents a set of 6 independent equations between the 21 parameters characterizing $\mathcal{X}$. Since we have

$$\dim\left[\frac{SL(7)}{\mathrm{Aff}(4,3)}\right] = \dim\left[\frac{SO(7)}{SO(4)\times SO(3)}\right] = 12, \tag{3.41}$$

the usual way to reduce further the independent quantities is to use the affine coordinates defined as

$$W = XY^{-1}, \qquad w = z^{\mathsf{T}}Y^{-1}, \qquad \det Y \neq 0. \tag{3.42}$$

Let us now consider the quotient space $G_2/SO(4)$. We have

$$\dim\left[\frac{G_2}{SO(4)}\right] = 14 - 6 = 8. \tag{3.43}$$

This space can be characterized by the homogeneous coordinates $\mathcal{X} = G\mathcal{X}_0$ where now $G \in G_2 \subset SO(7)$ and thus satisfies the relations (3.33). They give rise to supplementary conditions on the 21 parameters characterizing $\mathcal{X}$. Indeed, we can write

$$\mathcal{X} = G\mathcal{X}_0 = G\begin{pmatrix} 0 \\ 0 \\ I_3 \end{pmatrix} = \begin{pmatrix} g_{15} & g_{16} & g_{17} \\ g_{25} & g_{26} & g_{27} \\ g_{35} & g_{36} & g_{37} \\ g_{45} & g_{46} & g_{47} \\ g_{55} & g_{56} & g_{57} \\ g_{65} & g_{66} & g_{67} \\ g_{75} & g_{76} & g_{77} \end{pmatrix}. \tag{3.44}$$

The relations (3.33) imply, together with (3.31), that:

$$g_{ab}(T_b)_{76} = g_{a5} = T_{aef}\, g_{e7} g_{f6}, \tag{3.45}$$

so the 7 parameters of the first column of $\mathcal{X}$ are expressed in terms those of the other columns. Moreover, we have the orthogonality condition (3.40) which implies 3 more relations between the remaining parameters:

$$g_{a6}g_{a6} = g_{a7}g_{a7} = 1, \qquad g_{a6}g_{a7} = 0. \tag{3.46}$$

So, the number of independent parameter has been reduced to 11 at this stage. As before, the last step to reduce further the number of parameters is to use the affine coordinates. The conditions on W and w that leads a characterization of the quotient $G_2/SO(4)$ are explicitly given in [44].

With this we are now ready to discuss the magic square. We will also show how some of the aspects that we studied here can be elucidated from the properties of the magic square.

4. On the classification of quaternionic manifolds: The magic square

The magic square in mathematics is used to show the relation between division algebras, Jordan algebras [12] and Lie algebras. The idea was first developed by Freudenthal, Rozenfeld and Tits [7] and is introduced to string theory by Gunaydin–Sierra–Townsend [4]. The magic square in mathematics is a 4×4 square with the entries given by elements of the Lie algebras. The columns of the magic square are defined by the Jordan algebras, whereas the rows are defined by the division algebras [45]. The division algebras are the real (**R**), complex (**C**), quaternion (**Q**) and the octonion (**O**). The columns are labelled by: $J^3(\mathbf{R})$, $J^3(\mathbf{C})$, $J^3(\mathbf{Q})$, $J^3(\mathbf{O})$ where $J^3(\mathbf{K})$

is the algebra of 3×3 Hermitian matrices over $\mathbf{K}$. The magic square is then given by:

A_1	A_2	C_3	F_4
A_2	A_2^2	A_5	E_6
C_3	A_5	D_6	E_7
F_4	E_6	E_7	E_8

where A_i, C_i, D_i, E_i, F_4 are the usual $SU, Sp, SO, E_{6,7,8}$ and F_4 Lie groups respectively (a similar square can be drawn for the corresponding algebras also). The rules for filling up the entry L of the magic square can be given by the relation (see for example [45]):

$$L = \mathrm{Der}\, A \oplus (A_0 \otimes J_0) \oplus \mathrm{Der}\, J, \tag{4.1}$$

where Der A and Der J are the generators of the automorphism group of the Hurwitz (division) algebra A and of the algebra J, A_0 are the pure imaginary elements of $\mathbf{R}_0 = S^0$, $\mathbf{C}_0 = S^1$, $\mathbf{Q}_0 = S^3$ and $\mathbf{O}_0 = S^7$ and J_0 are the elements of trace zero of the Jordan algebra J. To make this clear, we can write the magic square in terms of the dimensions of the Lie algebras in the following way:

3	8	21	52
8	16	35	78
21	35	66	133
52	78	133	248

The reason for the *magical* property of the square can be made clear from the entry-rule given in (4.1). In terms of last to the first row, we can write the elements of the magic square in the following way:

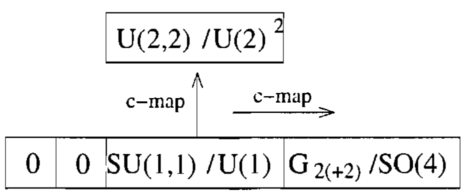
52	78 = 52 + 1 x 26	133 = 52 + 3 x 26 + 3	248 = 52 + 7 x 26 + 14
21	35 = 21 + 1 x 14	66 = 21 + 3 x 14 + 3	133 = 21 + 7 x 14 + 14
8	16 = 8 + 1 x 8	35 = 8 + 3 x 8 + 3	78 = 8 + 7 x 8 + 14
3	8 = 3 + 1 x 5	21 = 3 + 3 x 5 + 3	52 = 3 + 7 x 5 + 14

For more details see for example [46] (and references therein). The interesting feature of the magic square is that its symmetric and four of the five exceptional Lie algebras occur in the last row. In fact one could also add G_2 to the magic square by adding an extra column (therefore some literature also refers the magic square as a 4×5 rectangle). The extra column corresponds to the Jordan algebra **R** (see figure below):

0	
0	
A_1	C_3
G_2	F_4

where other elements of the square are to be filled in the dotted parts. Once we have the Lie groups, we should ask how to accomodate the quaternionic spaces or Kähler spaces in the magic square. To describe this let us use the column containing G_2 and A_1 Lie groups as this is the simplest. In the language of constrained instantons, observe that in the maximal subgroup of G_2 i.e., $SU(2) \times SU(2)$ one of the $SU(2)$ is gauged. This leaves one free $SU(2)$ and the quaternionic manifold is (3.18) (or (3.26) in the non-compact limit). For the next element of the magic square i.e., A_1 here, we look at the $U(1)$ subgroup of the ungauged $SU(2)$ and gauge it. The resulting space is $\frac{SU(2)}{U(1)}$ or $\frac{SU(1,1)}{U(1)}$ in the non-compact limit. This reproduces the next element of the magic square. Finally since we have gauged the remaining $U(1)$ we have nothing else to gauge, so the other two remaining elements of the magic square are 0 and 0 (see figure above).

Observe however that in the above figure we have ignored a subtlety regarding the c-map for the G_2 case. This has to do with the existence of *two* different non-trivial F functions for the corresponding $SU(1,1)/U(1)$ Kähler space [5,6]. This can be illustrated in the following way:



where we see that the same Kähler space can give rise to *two* different quaternionic space. One of the quaternionic space does not lie in the magic square and is generated by a F function given by:

$$F(X^I) = (X^1)^2 - (X^2)^2. \tag{4.2}$$

The fact that this is no contradiction is explained in [6]. What we are looking for is the c-map related to Jordan algebra and this is given by the horizontal arrow.

Thus for the generic case our procedure should now be clear. We are gauging various subgroups as we move along the magic square. We call this *sequential gauging*. Let us consider a part of magic square represented by *non-compact* group elements A, B, C and D in the following way:

D	C	B	A

Question now is whether we can determine the corresponding manifolds associated with these elements of the magic square using the arguments of constrained instantons. The manifold associated with group A is easy. This has to be a quaternionic manifold in such a way that a $SU(2)$

subgroup of the maximal group is gauged. What is the maximal subgroup of A here? This is exactly given by the next element B of the magic square. Let B_c be the compact version of the group B. Then the maximal subgroup of A is clearly $B_c \times SU(2)$ giving rise to the quaternionic manifold:

$$\frac{A}{B_c \times SU(2)}. \tag{4.3}$$

Now question is whether we can determine the next manifold that should be Kähler (recall the c-map constraint). Looking at the next element we find the group C whose compact version is C_c. What we need now is that the ungauged group B should decompose into C_c and another subgroup. This is easy to determine from the list of subgroups given in [38]. Let the subgroup be H_1. This therefore gives us the Kähler manifold:

$$\frac{B}{C_c \times H_1}, \tag{4.4}$$

whose c-map therefore will be (4.3). Going in this way we can reproduce all the manifolds associated with the elements of the magic square in the following way:

$\dfrac{D}{H_3}$	$\dfrac{C}{D_c \times H_2}$	$\dfrac{B}{C_c \times H_1}$	$\dfrac{A}{B_c \times SU(2)}$

where the subgroups H_i could in principle be determined from [38]; and the dotted lines are used to show the connection between the ungauged groups. But the story does not end here because it turns out that the subgroups themselves are not arbitrary. The quaternionic space was determined by gauging the $SU(2)$ subgroup. This was related to the constrained instantons. Now what could be the next subgroup that we can gauge? Clearly this has to be a $U(1)$ subgroup related to semilocal strings. Similarly we can ask about the next to next subgroup. Since we gauged $SU(2)$ as well as $U(1)$ we cannot gauge any other group! So our prediction for the magic square will be

$$H_1 = U(1), \qquad H_2 = 1, \qquad H_3 = 1. \tag{4.5}$$

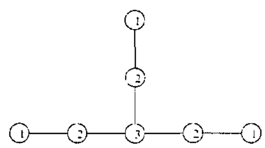
Observe however that there are some subtleties related to these identifications because the third manifold associated with the group C in the magic square should be a *real* manifold, so we might have to consider appropriate complex conjugates of the relevant groups. The final picture that emerges from all the above consideration is:

D	$\dfrac{C}{D_c}$	$\dfrac{B}{C_c \times U(1)}$	$\dfrac{A}{B_c \times SU(2)}$

which we would verify in the next few examples. A more detailed analysis of the manifolds other than the quaternionic ones will be presented in the sequel. In the following sections we will mainly study the quaternionic manifolds associated with E_n and F_4 groups.

4.1. E_6 quaternionic space

Our first case is to look for a theory with global symmetry $\mathcal{G} = E_6$. To extract the quaternionic space associated with this group we should study the maximal subalgebra.[21] The maximal regular subalgebra of E_6 can be extracted from the extended Dynkin diagram:



and is given by $\mathcal{H} = su(6) \oplus su(2)$. This immediately tells us two things: One, we are dealing with a gauge theory with $\mathcal{H}_1 = SU(2) = Sp(1)$ gauge group, and two, the manifold $\mathcal{M}_{E_6}$ is

$$\mathcal{M}_{E_6} = \frac{E_6}{SU(6) \times Sp(1)}. \tag{4.8}$$

From the analysis that we presented in the previous section and using [37], one can verify that $\pi_3(\frac{E_6}{SU(6)}) = 1$, so we need to gauge an $SU(2)$ subgroup. Indeed, as like the previous cases, one can find the following decomposition:

$$\mathbf{27} \to (\bar{\mathbf{6}}, \mathbf{2}) + (\mathbf{15}, \mathbf{1}), \tag{4.9}$$

under $SU(6) \times SU(2)$ subgroup. The $SU(2)$ subgroup that we want to gauge is slightly different. This subgroup is the diagonal subgroup of the $SU(2)_g \times SU(2)_l$ where g, l stand for the global and local groups respectively. Both the global and the local groups are broken by Higgs expectation value—once we give a VEV to $(\bar{\mathbf{6}}, \mathbf{2})$—and therefore an $SU(2)'_g$ group survives (which we will call $SU(2)$ henceforth). Since $SU(2) \sim S^3$, the homotopy classification will tell us that $\pi_3(S^3) = \mathbf{Z}$. These are the constrained instantons, and therefore should have a construction via the quaternion as we discussed before. These instantons are again non-trivially fibered over the space (4.8) and therefore exist only as semi-local defects.

Thus we seem to get our required exceptional semilocal defect in this model. However in the process of deriving this we have ignored a subtlety. This subtlety cannot be seen at the level of group structure, in the sector of Seiberg–Witten theory that we study, but is visible when we look at the corresponding Seiberg–Witten curve associated to our manifold. Therefore let us construct the corresponding curve by modifying the G_2 curve that we discussed in (3.20). The reason why

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[21] Notice that in addition to the choice of *maximal* subalgebras, we also ask for *symmetric* subalgebras of the groups. The symmetric subalgebras for various groups have been listed in [38]. For the A_n, B_n, C_n, D_n cases, they are

$$\begin{aligned} &su(p+q) \to su(p) \oplus su(q) \oplus u(1), \qquad so(p+q) \to so(p) \oplus so(q), \\ &sp(2p+2q) \to sp(2p) \oplus sp(2q), \end{aligned} \tag{4.6}$$

where p and q form the various distribution (as even or odd integers). For the E_n cases one would have

$$\begin{aligned} &e_8 \to so(16), \quad su(2) \oplus e_7, \\ &e_7 \to su(8), \quad su(2) \oplus so(12), \quad e_6 \oplus u(1), \\ &e_6 \to sp(8), \quad su(2) \oplus su(6), \quad so(10) \oplus u(1), \quad f_4. \end{aligned} \tag{4.7}$$

From the list one has to extract out the relevant algebras that we would require for our case.

we want to start from G_2 and go all the way to E_8 is because of the last row of the magic square

G_2	F_4	E_6	E_7	E_8

which is expressed as a part of the 4×5 rectangle. Since the magic square elements are related, we will then take (3.20) and add changes so that it eventually becomes the curve for E_6, and then subsequently for other cases (we have ignored the F_4 case for the time being because it will be shown later to be very close to the E_6 case).

Our first modification would be to change the powers of z in (3.20). This modifies the curve to the following:

$$\begin{aligned}&\left(y+\frac{12a_1zx-4a_1a_2z^3-4a_1^2z^2+12a_3z^2}{24}\right)^2\\&\quad=x^3-\frac{x}{48}\left[\left(a_1^4+8a_1^2a_2+16a_2^2\right)z^4-24(a_1a_3+2a_4)z^3\right]\\&\qquad+\frac{1}{864}\left[a_1^8z^8+\left(12a_1^4a_2+48a_1^2a_2^2+64a_2^3\right)z^6+216a_3^2z^4\right.\\&\qquad\left.-\left(36a_1^3a_3+72a_1^2a_4+144a_1a_2a_3+288a_2a_4-864a_6\right)z^5\right],\end{aligned}\tag{4.10}$$

with a_i arbitrary. To fix the values of a_i we have to study the singularity structures carefully. The discriminant locus of this equation near the points $z=0$ can be easily worked out. For us this will be given by

$$\Delta\sim z^8+\mathcal{O}\left(z^9\right),\tag{4.11}$$

up to an overall numerical factor. To study the singularities at $z\neq 0$ the curve (4.10) is not generic enough. To derive the actual curve we need to manipulate (4.10) further. We will do this in few steps. First observe that (4.10) can be re-written as:

$$Y^2=x^3-xz^3(Az+B)+\frac{z^4}{864}\left(Cz^4+Dz^3+Ez+F\right),\tag{4.12}$$

where the new coefficients $A,\dots,F$ and Y are defined from (4.10) in the following way:

$$\begin{aligned}&Y=y+\frac{12a_1zx-4a_1a_2z^3-4a_1^2z^2+12a_3z^2}{24}; \qquad C=a_1^8, \qquad F=216a_3^2,\\&A=a_1^4+16a_2^2+8a_1^2a_2, \qquad D=12a_1^4+48a_1^2a_2^2+64a_2^3,\\&B=-24(a_1a_3+2a_4), \qquad E=36a_1^3a_3+72a_1^2a_4+144a_1a_2a_3+288a_2a_4-864a_6.\end{aligned}\tag{4.13}$$

Secondly, that the curve (4.12) does not fully capture the E_6 singularities completely can be easily demonstrated (see also [47]). The dimensionality of x. Y, z, etc. can be worked out from the equation

$$\frac{d\lambda_{\mathrm{SW}}}{dz}=\frac{dx}{Y},\tag{4.14}$$

where λ_{SW} is the Seiberg–Witten differential. We can then break the E_6 global symmetry to $SO(10)\times U(1)$ such that the fundamental **27** decomposes as

$$\mathbf{27}=\mathbf{16}_{+1}+\mathbf{10}_{-2}+\mathbf{1}_{+4},\tag{4.15}$$

where the subscripts denote the $U(1)$ charges. This would then imply that the coefficient of x in (4.12) should have a z^2 term [47]. Similar conclusion can be extracted by further breaking the global symmetry to $D_4 \equiv SO(8)$ where we know that z^2 should exist (see Eq. (2.16) in [48]). Therefore if we redefine x, Y to $\tilde{x}, \tilde{y}$ as:

$$\tilde{x} = xz^{-\frac{3}{2}}, \qquad \tilde{y} = Yz^{-2}, \tag{4.16}$$

where the redefinition makes sense because we are not analysing the $z = 0$ points, then (4.12) can be written as

$$\tilde{y}^2 = \tilde{x}^3 - \tilde{x}(Gz^2 + A'z + B) + \frac{1}{864}(Cz^4 + Dz^3 + Ez + F)\left(1 - \frac{1}{2}\log z + \cdots\right), \tag{4.17}$$

where A' and G are the minimal changes to (4.12). Observe that we can assume $A' \propto A$ without a loss of generality.

The new curve (4.17) is almost the one discussed in [47] with the exception of the additional $\log z$ terms. These terms could be ignored for our case as we want to realise the pure E_6 global symmetry.[22] To complete the picture we need to derive the explicit form for G, A' and a_i ($i = 1, 2, 3, 4, 6$). These are given in terms of E_6 Casimirs defined in the following way [50]:

$$p_n(x_j) = \sum_{\{n_i\}} \mathcal{C}_{\{n_i\}} x_1^{n_1} x_2^{n_2} x_4^{n_3} x_5^{n_4} x_6^{n_5} x_8^{n_6}, \tag{4.18}$$

where the operators x_i are defined in terms of the Cartan subalgebra of E_6 and n, n_i are integers satisfying the following algebraic equation:

$$n \equiv \{2, 5, 6, 8, 9, 12\} = n_1 + 2n_2 + 4n_3 + 5n_4 + 6n_5 + 8n_6 \tag{4.19}$$

and $\mathcal{C}_{\{n_i\}}$ are integers. The sum is over all possible integer solutions of the above Eq. (4.19). As an example the Casimir p_6 will be defined via the following values of the coeffcients $\mathcal{C}_{\{n_i\}}$:

$$\begin{aligned} &\mathcal{C}_{000010} = -1, \quad &&\mathcal{C}_{410000} = -1062, \quad &&\mathcal{C}_{011000} = \frac{5}{4}, \quad &&\mathcal{C}_{030000} = -\frac{23}{8}, \\ &\mathcal{C}_{201000} = -15, \quad &&\mathcal{C}_{220000} = -\frac{177}{2}, \quad &&\mathcal{C}_{100100} = -60, \quad &&\mathcal{C}_{600000} = -4680, \end{aligned} \tag{4.20}$$

where one can get the full list in [50]. Using these Casimirs one can easily determine the coefficients G, A' and a_i by comparing the curve (4.17) with the one given in [47]. They are given by:

$$\begin{aligned} &G = -\frac{p_2}{3}, \qquad A' = \frac{2p_5}{3}, \qquad a_1 = 2^{5/8}3^{3/8} \approx 2.328, \\ &\frac{a_3^2}{4} = \frac{32}{135}p_{12} - \frac{298}{18225}p_2^2 p_8 - \frac{101}{218700}p_2^3 p_6 + \frac{13}{405}p_6^2 - \frac{49}{1049700}p_2^6 - \frac{19}{3645}p_2 p_5^2, \\ &a_4 = \frac{1}{2}\left[\frac{7}{10368}p_2^4 - \frac{11}{1080}p_2 p_6 + \frac{p_8}{45} - 2^{5/8}3^{3/8}a_3\right], \\ &a_2 = \omega + \frac{b^2}{9a^2}\cdot\frac{1}{\omega} - \frac{b}{3a} \approx \omega + \frac{1.837}{\omega} - 1.355, \end{aligned}$$

[22] It is not clear to us what singularities would the additional $\log z$ dependent terms would imply. Of course additional singularities besides E_6 have been observed for certain F-theory curves in [49], but there the singularities were simple.

$$a_6 = \frac{a_1^3 a_3}{24} + \frac{a_1^2 a_4}{12} + \frac{a_1 a_2 a_3}{6} + \frac{a_2 a_4}{3} - \frac{p_2^2 p_5}{18} - \frac{8}{21} p_9, \tag{4.21}$$

where using

$$c = 576 p_6 - 56 p_2^3 - 144\sqrt{6}, \qquad a = 64, \qquad b = 2^{21/4} 3^{7/4} \approx 260.237, \tag{4.22}$$

we can define ω appearing in the definition of a_2 above as

$$\omega^3 = -\frac{1}{2}\left(\frac{2}{27}\frac{b^3}{a^3} - \frac{c}{a}\right) \pm \frac{1}{2}\sqrt{\left(\frac{2}{27}\frac{b^3}{a^3} - \frac{c}{a}\right)^2 - 4\left(\frac{b}{3a}\right)^6}, \tag{4.23}$$

which would imply that a_2 is a negative definite quantity. From the above we can also determine the proportionality constant between A and A'. This is given by

$$\frac{2p_5}{48a_2^2 + 2^{17/4}3^{7/4}a_2 + 36\sqrt{6}} \approx \frac{2p_5}{48a_2^2 + 302.8a_2 + 29.37}, \tag{4.24}$$

where a_2 can be extracted from above. This therefore completes the full analysis of the Seiberg–Witten curve for the system.

The subtlety that we were alluding to earlier lies in the realisation of the subalgebra (or the subgroup (4.9)) associated with the E_6 symmetry that would be used to determine the quaternionic manifold *directly* from the curve (4.17). Knowing the discriminant we can in principle extract the corresponding subalgebra associated with the global group $\mathcal{G} = E_6$ provided the background space is specified. However the issue is more intricate because:

- There is *no* Lagrangian description of the system with exceptional global symmetry. In fact existence of the curve does not guarantee that the system is a SYM theory in some limit.
- Even if there exist some suitable description, the system is at strong coupling [49] where a controlled analytical calculation cannot be done. Furthermore due to large number of flavors the theory is not asymptotically free.

All these issues might still be resolved if we embed our gauge theory in some stringy set-up. There are various possibilites here. We might embed it in a F-theory set-up much like the one discussed in [24,27,31,49,51,52], etc., or in a M-theory set-up like [53].[23] Using any of these cases, all we need is that the eight singularities decompose into a bunch of six and two singularities giving rise to the discriminant and subgroup

$$\Delta \sim (z-a)^6(z^2+b) \quad \Rightarrow \quad E_6 \subset SU(6) \times SU(2), \tag{4.25}$$

which is of course the maximal subgroup for our case. Once the global symmetry is broken, a Lagrangian description is possible when the system is embedded in a F-theory set-up. In F-theory, analysing the curve however leads to the following subalgebra:

$$su(5) \oplus su(2) \oplus u(1) \tag{4.26}$$

[23] In fact, since our theory is just a sector of the Seiberg–Witten theory, all the subtleties afflicting the original theory will not have much effect on our analysis. Furthermore the Seiberg–Witten curve is the only output that we will be using for our case.

instead of the subalgebra associated with the decomposition (4.9). This is almost the maximal subalgebra that we wanted, but not quite.[24] In fact $su(6)$ is broken to $su(5) \oplus u(1)$. Thus this is the closest we come to getting the full structure of the coset space directly from type IIB string theory (or F-theory).[25] In fact what we need is that the **6** of $SU(6)$ should decompose under $SU(5) \times U(1)$ as:

$$\mathbf{6} \to \mathbf{5}_1 + \mathbf{1}_{-5}, \tag{4.28}$$

which would form the ungauged maximal subgroup. The associated monodromy matrix is then clearly

$$\begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}, \tag{4.29}$$

which leaves one of the dyonic point in the monodromy matrix and determines the rest of the $SU(6)$ generators non-perturbatively. The surviving diagonal $SU(2)$ is now gauged according to our earlier discussion.[26]

The above construction therefore gives us the constrained instanton configurations associated with global symmetry E_6 that are fibered over the quaternionic base $\mathcal{M}_{E_6}$ (4.8). However, as in the previous sections, this is *not* quite the manifold that we are looking for. We should aim for the non-compact version of (4.8), i.e.,

$$V(1,2) \equiv \frac{E_{6(+2)}}{SU(6) \times SU(2)}, \tag{4.30}$$

where +2 in the bracket denote the difference between the number of compact and non-compact generators. The corresponding Kähler space associated with (4.30) can be constructed by gauging subgroups of $SU(6)$ according to our scheme. The relevant subgroup of $SU(6)$ for us is $SU(3) \times SU(3) \times U(1)$ under which **6** decomposes as:

$$\mathbf{6} = (\mathbf{1},\mathbf{3})_{-1} + (\mathbf{3},\mathbf{1})_{+1}, \tag{4.31}$$

where by modding A_5 by the corresponding subgroup gives rise to the following Kähler space:

$$\frac{SU(3,3)}{SU(3) \times SU(3) \times U(1)}, \tag{4.32}$$

---

[24] When our theory is embedded in the full Seiberg–Witten theory the same subtlety should show up in determining the Higgs branch. However in the absence of a proper Lagrangian description this may not be easy to implement.

[25] The full configuration on the other hand can be determined in the following way: First we decompose the E_6 adjoint in terms of the subalgebra (4.26) as

$$\mathbf{78} = (\mathbf{24},\mathbf{1})_0 + (\mathbf{1},\mathbf{1})_0 + (\mathbf{1},\mathbf{3})_0 + (\mathbf{10},\mathbf{2})_{-3} + (\mathbf{5},\mathbf{1})_6 + \text{c.c.}, \tag{4.27}$$

where the subscripts refer to the $U(1)$ charges and the c.c are associated with $\bar{10}$ and $\bar{5}$ with $U(1)$ charges 3 and −6, respectively. Secondly, having given the decomposition, the rest of the discussion now should follow the familiar line developed in the series of papers [39,49]. We will not elaborate on this aspect as the readers can look up the details in those papers. It will simply suffice to mention that the non-trivial configuration required to get the full group structure lies in the process of brane creation via the Hanany–Witten effect [54] leading to strings with multiple prongs [55–57] that fill out the rest of the group generators [39].

[26] Recall that before combining the $SU(2)$ part of the unbroken global group with the local $SU(2)$ gauge symmetry we expect a monodromy matrix of the form $\begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$.

where $SU(3,3)$ is the non-compact version of $SU(6)$. Observe also that (4.32) is exactly of the form (4.4) with $H_1 = U(1)$ and $C_c = SU(3) \times SU(3)$. Furthermore, under a c-map (4.31) does give us (4.30) once the F-function is specified. We will specify the F-function a bit later. Looking now into the magic square for the E_6 sequence:

			G_2	
		map	F_4	
A_2	A_2^2	A_5	E_6	curve
			E_7	
			E_8	

where the vertical sequence is shown to emphasise how the curves were constructed, and the horizontal sequence is constructed by various maps: c, r etc., we can easily argue the various manifolds associated with the horizontal elements of the magic square using the technique of partial gauging of the subgroups discussed in the previous section. This will give us the following sequence:

$SU^*(3)$	$\dfrac{SL(3;C)}{SU(3)}$	$\dfrac{SU(3,3)}{SU(3) \times U(3)}$	$\dfrac{E_{6(+2)}}{SU(6) \times SU(2)}$

where the third term in the sequence has $H_2 = 1$ and the fourth term has $H_3 = 1$ as predicted in (4.5). With this sequencing structure we can now determine the sigma-model metric associated with the constrained instantons fibered over the quaternionic base (4.10) (or (4.30) in the non-compact limit). The quaternionic metric is always of the form (3.29) which is derived from the corresponding Kähler metric (3.28). All we need to complete the picture for the E_6 case would be the F-value. We will present a detailed analysis of this in Section 4.5.2 including a generic derivation for all possible cases.

Before we end this section, notice that we have not yet checked whether there is some semi-local soliton that could be fibered over the space (4.32) much like the quaternionic examples studied so far. For this we have to study the associated vacuum structure. Whether this theory could be studied in the same moduli space as the present ones needs to be investigated. It is of course highly suggestive that there are semilocal string like defects because $\pi_1(U(1)) = \mathbf{Z}$ and using the exact sequence for Lie group $\mathcal{G}$ and its subgroup $\mathcal{H}$:

$$0 \longrightarrow \pi_2\left(\frac{\mathcal{G}}{\mathcal{H}}\right) \longrightarrow \pi_1(\mathcal{H}) \longrightarrow \pi_1(\mathcal{G}) \longrightarrow \pi_1\left(\frac{\mathcal{G}}{\mathcal{H}}\right) \longrightarrow 0, \tag{4.33}$$

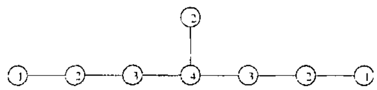
one can easily argue that for $\mathcal{G} = SU(n) = SU(6)$ and $\mathcal{H} = SU(3) \times SU(3)$ (or in fact for any generic Lie subgroups [58]):

$$\pi_1\left(\frac{SU(6)}{SU(3) \times SU(3)}\right) = 0 = \pi_2\left(\frac{SU(6)}{SU(3) \times SU(3)}\right), \tag{4.34}$$

showing that there could only be semilocal defects. We will however leave a detailed study of this for future investigations.

4.2. E_7 quaternionic space

Let us now turn towards the next group $\mathcal{G} = E_7$. The extended Dynkin diagram of E_7:



can be *cut* in different ways to give rise to various maximal regular subalgebras of E_7. They are given by

$$su(8), \qquad \text{spin}(12) \oplus su(2), \qquad su(6) \oplus su(3), \qquad e_6 \oplus u(1), \tag{4.35}$$

where spin(12) actually comes from $so(12)$ with some identification between the generators. From the set of steps that we mentioned earlier, we can immediately ignore the subalgebras $su(8), su(6) \oplus su(3)$ and $e_6 \oplus u(1)$ and therefore the associated groups $SU(8), SU(6) \times SU(3)$, $E_6 \times U(1)$ as they cannot be realised in the present scenario (recall that the gauge group is $SU(2)$).[27] The above consideration immediately gives us the corresponding unique coset manifold for the global symmetry E_7 as

$$\mathcal{M}_{E_7} = \frac{E_7}{\text{Spin}(12) \times Sp(1)}. \tag{4.37}$$

Our previous consideration will require us to view this as a homogeneous quaternionic Kähler manifold. The $SU(2)$ constrained instantons are fibered over this manifold because the third homotopy group of the vacuum manifold is trivial i.e., $\pi_3(\frac{E_7}{SO(12)}) = 1$. But then again such a big global symmetry will not allow a Lagrangian description of the system, so to make any concrete statements we have to analyse the maximal subgroup $SO(12) \times SU(2)$ associated with the system.

However as before, analysing the corresponding Seiberg–Witten curve will tell us that the actual subgroup realised perturbatively is different from $SO(12) \times SU(2)$ or $\text{Spin}(12) \times Sp(1)$. To see this we will study the theory in few steps. Firstly, the breaking pattern for the **56** of E_7 is:

$$\mathbf{56} = (\mathbf{12}, \mathbf{2}) + (\mathbf{32}, \mathbf{1}) \tag{4.38}$$

under $SO(12) \times SU(2)$. Giving a VEV to $(\mathbf{12}, \mathbf{2})$ the broken global $SU(2)$ can combine with the broken local $SU(2)$ and give us the unbroken global group $SU(2) \equiv Sp(1)$. This is the $Sp(1)$ that appears in (4.37). Furthermore once we have the coset space (4.37) we have to analyse the rest of the coset spaces from the magic square column:

C_3	A_5	D_6	E_7

To analyse the coset space (4.37) let us determine the curve associated with E_7 by deforming the E_6 curve (4.17) that we determined earlier. Our first attempt to determine the curve using the

[27] Observe however that the third homotopy groups of $SU(2)$ and $SU(3)$ are both given by

$$\pi_3(SU(2)) = \pi_3(SU(3)) = \mathbf{Z} \tag{4.36}$$

and therefore $SU(3)$ theory can also allow non-trivial constrained instantons. It would be interesting to study the manifold associated with this setup.

following values of the variables in (3.7):

$$\{l, k, h, f, g\} = \{z, z^2, z^3, z^3, z^5\}, \tag{4.39}$$

can only tell us the discriminant behavior at $z = 0$. To determine the curve at any generic point $z \neq 0$ we can deform (4.17) to the following curve:

$$\begin{aligned}\tilde{y}^2 &= \tilde{x}^3 - \tilde{x}(2z^3 + Mz^2 + Nz + P) \\ &\quad + \frac{1}{864}(Qz^4 + Rz^3 + Sz + T)\left(1 - \frac{1}{2}\log z + \cdots\right),\end{aligned} \tag{4.40}$$

where $M, N, \ldots$, etc., are written in terms of $SO(12)$ Casimirs (see [47] for details). The discriminant locus that we can realise here will be:

$$\Delta \sim z^9 + \mathcal{O}(z^{10}) \tag{4.41}$$

and therefore would show an E_7 singularity. On the other hand, we would not be able to realise the maximal $SO(12) \times SU(2)$ subgroup here. The curve (4.40) will reflect the following subalgebra:

$$su(6) \oplus su(2) \oplus u(1) \tag{4.42}$$

where the $SU(2)$ is the same $SU(2)$ symmetry that gets broken completely to give us an unbroken global $SU(2)$ in (4.37). Also as expected the **12** and **32** of $SO(12)$ decomposes as:

$$\mathbf{12} = \mathbf{6}_{+1} + \mathbf{6}_{-1}, \qquad \mathbf{32} = \mathbf{1}_{+3} + \mathbf{1}_{-3} + \mathbf{15}_{-1} + \bar{\mathbf{15}}_{+1}, \tag{4.43}$$

under $SU(6) \times U(1)$. The monodromy matrix is now different from (4.29) that we had earlier for the E_6 case. It is given by

$$\begin{pmatrix} -2 & -3 \\ -1 & -2 \end{pmatrix}, \tag{4.44}$$

although the same dyonic point is enclosed. The two monodromy matrices (4.29) and (4.44) differ by the monodromy matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ as expected.

As before the manifold (4.37) is not quite the quaternionic manifold that we are looking for. Our aim is to get the non-compact version of this. Therefore using the compact and non-compact generators of E_7 we can construct the following manifold:

$$\frac{E_{7(-5)}}{SO(12) \times SU(2)}, \tag{4.45}$$

which is the required quaternionic manifold falling in the classification of Alekseevskii [3]. In this classification the manifold (4.45) is known as $V(1, 4)$ manifold, and should be generated from a Kähler space via the c-map. So the question is: can we derive the Kähler space associated with (4.45) using our argument of partial gauging? From the technique developed in earlier sections, we have to look for the $U(1)$ subgroup of the ungauged group in the global symmetry. Here the ungauged group is $SO(12)$ whose subgroup is clearly $SU(6) \times U(1)$. Therefore from the sequencing of the magic square, we can predict the Kähler space to be:

$$\frac{SO^*(12)}{SU(6) \times U(1)}, \tag{4.46}$$

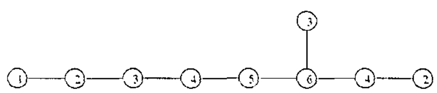
which when acted by the c-map will generate (4.45). The other coset spaces associated with the magic square can also be easily generated using the arguments of the previous sections. The final magic square sequence for E_7 will be given by:

Sp(3)	$\frac{SU^*(6)}{Sp(3)}$	$\frac{SO^*(12)}{SU(6) \times U(1)}$	$\frac{E_{7(-5)}}{SO(12) \times SU(2)}$

which is consistent with the classification [3]. Observe that to go from the second element from the left of the sequence to the third element we use the r-map. This is universal for the whole magic square.

4.3. *E_8 quaternionic space*

Our next exceptional global symmetry that we want to study here is E_8. This is straightforward (modulo some subtlety that we mention below) from all the considerations of the previous sections. The extended Dynkin diagram is now given by:



From here the relevant allowed maximal subalgebras are

$$e_7 \oplus su(2), \qquad so(16), \qquad su(5) \oplus su(5), \qquad su(3) \oplus e_6, \qquad su(9), \tag{4.47}$$

out of which only two of them, namely, $so(16)$ and $e_7 \oplus su(2)$ are also symmetric subalgebras. We can now easily ignore the $SO(16)$ subgroup because we are looking for constrained instantons associated with the $SU(2)$ group. Again constrained instantons exist because $\pi_3(\frac{E_8}{E_7}) = 1$. The **248** of E_8 then decomposes as[28]:

$$\mathbf{248} = (\mathbf{1}, \mathbf{3}) + (\mathbf{133}, \mathbf{1}) + (\mathbf{56}, \mathbf{2}), \tag{4.48}$$

under $E_7 \times SU(2)$ subgroup. Once we give an expectation value to $(\mathbf{56}, \mathbf{2})$ we can break both the local and global $SU(2)$s to give us an unbroken global $SU(2)$. Therefore the final symmetry group $E_7 \times SU(2)$ is completely global and we can now gauge the $SU(2)$ subgroup. Constrained instantons can exist for the $SU(2)$ theory, and they are fibered over the base manifold:

$$\frac{E_8}{E_7 \times SU(2)}, \tag{4.49}$$

which gives us the quaternionic Kähler manifold associated with E_8 global symmetry.

There are few other details we could consider parallel to the details associated with other E_n groups studied above. First is the existence of Seiberg–Witten curve for E_8 global symmetry that could be described here by deforming the E_7 curve (4.40). This deformation is simple and is

[28] Observe that **248** is the dimension of the *adjoint* representation of E_8. This is the smallest representation of E_8. There is no smaller fundamental representation. This would mean—from our earlier analysis of the potential in (3.1)—we can no longer use the argument of the quaternion q being in fundamental of the global E_8. However since there is no simple Lagrangian formulation of this theory, an absence of fundamental representation may not pose an issue in constructing the vacuum manifold. Indeed as we will show below, there is a possible alternative way to verify that the moduli space of these instantons do not change when we work with the adjoint representation of E_8. We will deal with this issue in more details in the sequel to this paper.

explained in [47]. The curve therefore is:

$$\tilde{y}^2 = \tilde{x}^3 - \left(z^2 T_2 + \mathcal{O}(z^2)\right)\tilde{x} - \left[2z^5 + z^4\left(T_6 + \frac{T_2 T_4}{6} + \cdots\right) + \mathcal{O}(z^3)\right]. \tag{4.50}$$

where T_i are $SO(16)$ Casimirs. For more details the readers can refer to [47]. The manifest subalgebra that one gets from analysing the curve is neither $so(16)$ not $e_7 \oplus su(2)$ rather it is:

$$su(7) \oplus su(2) \oplus u(1), \tag{4.51}$$

which in turn means that the breaking pattern of E_7 global symmetry is not directly to (4.51) but through an intermediate $su(8)$ subalgebra. In terms of the corresponding groups this is:

$$E_7 \to SU(8) \to SU(7) \times U(1), \tag{4.52}$$

under which **56** and **133** should be decomposed. The associated monodromy matrix containing the *same* dyonic point is:

$$\begin{pmatrix} -3 & -5 \\ -1 & -2 \end{pmatrix}, \tag{4.53}$$

under the decomposition (4.52). Using this monodromy matrix one can construct the other generators of E_7 non-perturbatively.

As before the quaternionic manifold of interest is not quite (4.49). We have to look for the non-compact version of this. This is given by:

$$\frac{E_{8(-24)}}{E_7 \times SU(2)} \tag{4.54}$$

and is known as $V(1,8)$ manifold in the classification of Alekseevskii [3]. The associated Kähler manifold should have the necessary $U(1)$ coset as predicted in (4.5). Gauging the $U(1)$ will correspond to the semilocal strings. The Kähler manifold therefore is:

$$\frac{E_{7(-25)}}{E_6 \times U(1)}, \tag{4.55}$$

which under c-map will reproduce (4.54). Similarly (4.55) should come from the r-map of a real coset space according to (4.5). The final sequence therefore should be:

F_4	$\frac{E_{6(-26)}}{F_4}$	$\frac{E_{7(-25)}}{E_6 \times U(1)}$	$\frac{E_{8(-24)}}{E_7 \times SU(2)}$

which is again consistent with the existing classification [3]. In addition to the above scheme, observe that the generators of the E_n exceptional groups appearing in the magic square can be alternatively formulated in the following way [45]:

$$\begin{aligned} E_6 &= SO(8) + SU(3) + 6 \times 7 = 28 + 8 + 6 \times 7 = 78, \\ E_7 &= SO(8) + Sp(3) + 12 \times 7 = 28 + 21 + 12 \times 7 = 133, \\ E_8 &= SO(8) + F_4 + 24 \times 7 = 28 + 52 + 24 \times 7 = 248, \end{aligned} \tag{4.56}$$

where the existence of $SO(8) = \mathrm{Spin}(8)$ has to do with the underlying triality symmetry [45] and the Lie groups in (4.56) are precisely the F_4, C_3 and A_2 groups appearing in the magic square.

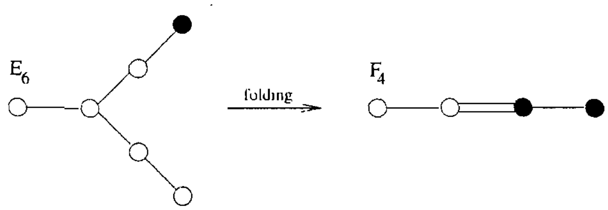
Finally, before ending this section, let us come back to the issue of E_8 representation that we discussed briefly at the beginning. An alternative way to verify that we have the correct one-instanton moduli space is to use the *adjoint* hypermultiplets of $\mathcal{N}=2$ gauge theory. The E_8 global symmetry can be enhanced to E_8 gauge symmetry by changing the Seiberg–Witten curve (4.50) to a new one. The curve for this case takes the following general form [59]:

$$y+\frac{\mu^2}{y}+\mathcal{P}_{\mathcal{R}}(x;u_j)=0, \tag{4.57}$$

where $\mathcal{P}_{\mathcal{R}}$ is a polynomial in x of order dim $(\mathcal{R})$, and $\mathcal{R}$ is the adjoint representation of E_8; $\tilde{y}$ in (4.50) and y differ at most by the polynomial $\mathcal{P}_{\mathcal{R}}$. The other terms occuring in (4.57) are defined as follows: $\mu \equiv \Lambda^h$ where h is the dual Coxeter number of E_8 and Λ is the Pauli–Villars scale. The functions u_j, $j=1,2,\ldots,8$ are the fundamental Casimirs of E_8 with the top Casimir u_8 has degree h. By changing (4.50) to (4.57) we have actually enhanced the susy to $\mathcal{N}=4$. Now it is well known that for E_8 small instantons in $\mathcal{N}=4$ gauge theory the moduli space is indeed given by (4.49), thus confirming our above analysis.

4.4. F_4 quaternionic space

The final example of exceptional global symmetry is F_4 whose properties are not very different from all the other E_n examples that we have been studying so far. In fact F_4 symmetry is very close to the exceptional E_6 symmetry. One hint comes from the folding relation between the Dynkin diagrams of E_6 and F_4:



Such similarity between the Dynkin diagrams is also reflected in the corresponding Seiberg–Witten curves near $z=0$ point. The curves for F_4 and E_6 have the following structures:

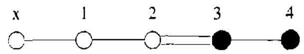
group	k(z)	l(z)	h(z)	f(z)	g(z)
F_4	z^2	z	z^2	z^3	z^4
E_6	z^2	z	z^2	z^3	z^5

where we have referred to only the highest order polynomials for a given coefficient. Clearly the singularity structures of both the curves would then be very similar. Indeed the discriminant of F_4 curve is given by:

$$\Delta \sim z^8+\mathcal{O}(z^9), \tag{4.58}$$

which is identical to the E_6 curve (4.11). The distinction between the two curves come from analysing points $z \neq 0$. The fundamental representation of F_4 is **26**-dimensional whereas the fundamental representation of E_6 is **27**, so they differ by a singlet. The maximal subalgebras of

F_4 can be extracted from the extended Dynkin diagram of F_4:



by cutting the diagram at various points. This will give rise to the following subalgebras:

$$so(9), \qquad su(3) \oplus su(3), \qquad su(2), \qquad sp(3) \oplus su(2), \qquad g_2 \oplus su(2), \tag{4.59}$$

out of which we will only keep $sp(3) \oplus su(2)$ subalgebra because we want to keep the symmetric subgroups. Clearly the group $G_2 \times SU(2)$ corresponding to the maximal subalgebra $g_2 \oplus su(2)$ is not symmetric, and therefore we will not quotient F_4 by this subgroup. Under $Sp(3) \times SU(2)$ subgroup the **26** of F_4 decomposes as

$$\mathbf{26} = (\mathbf{6}, \mathbf{2}) + (\mathbf{14}, \mathbf{1}). \tag{4.60}$$

Giving VEV to $(\mathbf{6}, \mathbf{2})$ we can break the global and local $SU(2)$s to have an unbroken $SU(2)$. Since $\pi_3(\frac{F_4}{Sp(3)}) = 1$, the constrained instantons will be fibered over the following quaternionic manifold:

$$\frac{F_4}{Sp(3) \times SU(2)}, \tag{4.61}$$

which is a compact manifold by construction. The manifold that we are concerned about is the non-compact version of (4.61). This is given by:

$$\frac{F_{4(+4)}}{Sp(3) \times SU(2)}, \tag{4.62}$$

which is also known as $V(1,1)$ manifold in the classification of Alekseevskii [3]. The $Sp(3)$ part of the subgroup $Sp(3) \times SU(2)$ used for quotienting F_4 is ungauged. To construct the relevant Kähler manifold associated with (4.62) we need the symmetric subgroup of $Sp(3)$. From [38] we see that there is one unique subgroup: $SU(3) \times U(1) \equiv U(3)$ containing a $U(1)$. This means that for a theory with $Sp(3)$ global symmetry semilocal strings can exist by gauging the $U(1)$ subgroup. This immediately gives us the corresponding Kähler manifold associated with (4.62):

$$\frac{Sp(3, \mathbf{R})}{SU(3) \times U(1)} \tag{4.63}$$

from which (4.62) can be generated by a c-map. The real manifold associated with (4.63) can be similarly constructed by looking into the symmetric subgroup of $SU(3)$ that does not have an $U(1)$ factor. This subgroup is $SO(3)$ [38], and therefore the magic square sequence for F_4 symmetry will be:

SO(3)	$\frac{\mathrm{SL(3, R)}}{\mathrm{SO(3)}}$	$\frac{\mathrm{Sp(3, R)}}{\mathrm{U(3)}}$	$\frac{\mathrm{F_{4(+4)}}}{\mathrm{Sp(3) \times SU(2)}}$

where $SL(3, \mathbf{R})$ is the non-compact group associated with the compact group $SU(3)$. The Kähler manifold (4.62) is the real image of the second coset from the left of the magic square. It is also interesting to note that the **52** of F_4 can be connected to spin(8) $\equiv SO(8)$ in the following way:

$$F_4 = SO(8) + SO(3) + 3 \times 7 = 28 + 3 + 3 \times 7 = 52, \tag{4.64}$$

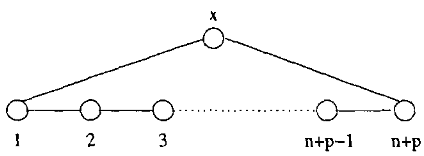
which is much like (4.56) described earlier. Finally, to determine the sigma-model description of the quaternionic manifold (4.61) or (4.62) we will need the F function that describes the metric of the Kähler manifold (4.63). This will be determined in Section 4.5.2.

4.5. *Other examples of quaternionic spaces*

After describing the complete magic square in terms of constrained instantons and possible other semilocal solitons, let us now use the same procedure to study other coset spaces in string theory.

4.5.1. *Example 1: $U(p)$ local symmetry and $SU(n+p)$ global symmetry*

Our first example is for a $U(p)$ gauge theory with a global symmetry $SU(n+p)$. The extended Dynkin diagram for such a symmetry is



which will give us a symmetric subgroup of $SU(n) \times SU(p) \times U(1)$ [38]. The existence of the extra $U(1)$ factor commuting with $SU(n)$ group can be directly explained from the corresponding gauge theory dynamics (see [34] for details).

The above theory can also be realised in the Seiberg–Witten setup by slightly modifying the present scenario. First of all we need a genus $g = p - 1$ curve instead of genus one curves that we have been studying so far. The construction of such a curve is very well known [60] so we will be brief. The curve for $\mathcal{N} = 2$ $U(p)$ gauge theory with $SU(n+p)$ global symmetry is [60]:

$$y^2 = \left[x^p + \sum_{i=2}^{p} s_i x^{p-i} + \Lambda^{p-n} \sum_{i=0}^{n} g_i x^{n-i}\right]^2 - \Lambda^{p-n} x^{n+p}, \tag{4.65}$$

where Λ is the Pauli–Villars scale and (s_i, g_i) are some constants that depend on the parameters of the theory. The exponent of Λ is evaluated as:

$$\Lambda^{\mathcal{I}(\mathbf{R}_A) - \mathcal{I}(\mathbf{R}_M)}, \tag{4.66}$$

where $\mathcal{I}(\mathbf{R}_A), \mathcal{I}(\mathbf{R}_M)$ are the Dynkin indices of the adjoint representations of vector multiplet and representations of matter hypermultiplets respectively [61].

The vacuum manifold of this theory will be a Stiefel manifold $\mathbf{V}_{n+p,p}$ [25] which is a space of p-frames in $\mathbf{C}^{n+p}$. This is isomorphic to $\frac{SU(n+p)}{SU(n)}$. At low energy the sigma model target space therefore will be given by the following manifold:

$$\mathbb{C}G(n, p) \equiv \frac{SU(n+p)}{SU(n) \times SU(p) \times U(1)}, \tag{4.67}$$

which is nothing but the manifold constructed by modding out $U(p)$ gauge orbits from the Stiefel manifold. This immediately implies:

$$\mathbf{V}_{n+p,p} \approx U(p) \otimes_f \mathbb{C}G(n, p), \tag{4.68}$$

where the subscript f implies non-trivial fibration. Thus the Stiefel manifold is a $U(p)$ bundle over a Grassmanian manifold. The quaternionic extension of the above case is to consider the complex Grassman manifold $\mathbb{C}G(n,2)$. This is denoted as $\mathbf{Gr}_2(\mathbf{C}^{n+2})$ in (1.1). For our purpose, however, we need the non-compact version of this space. This is given by:

$$\frac{SU(n,2)}{SU(n)\times SU(2)\times U(1)}. \tag{4.69}$$

The constrained instantons will be non-trivially fibered over (4.69) in the theory. The manifold (4.69) can be mapped to the corresponding Kähler space by gauging a $U(1)$ subgroup of the unbroken group. The Kähler space corresponding to (4.69) is:

$$\frac{U(n-1,1)}{U(n-1)\times U(1)}, \tag{4.70}$$

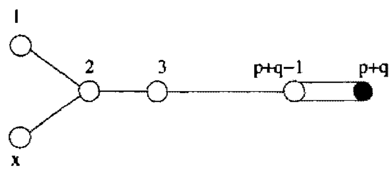
where (4.70) and (4.69) are related by a c-map as expected. Observe that the unbroken subgroup in (4.70) is $U(n-1)\equiv SU(n-1)\times U(1)$. To get the corresponding real manifold—that could be related to (4.70) by an inverse r-map—we need a subgroup of $U(n-1)$ that *does not* have a $SU(2)$ or an $U(1)$ factor. This is not possible, so our simple rule tells us that there could be no non-zero-dimensional real space associated with (4.70). This can be confirmed (see for example [23]). The sequence therefore is:

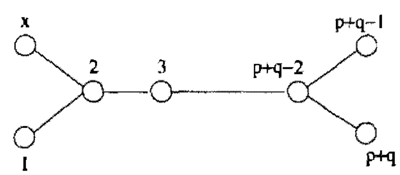
0	0	$\frac{U(n-1,1)}{U(n-1)\times U(1)}$	$\frac{SU(n,2)}{SU(n)\times U(2)}$

which fits into Alekseevskii classification [3] as well as the recent completion [23]. Notice that for $n=1$ there is no Kähler space.

4.5.2. Example 2: $SU(2)$ local symmetry and $SO(p+q)$ global symmetry

Our next example is almost self-explanatory. This is a $SU(2)$ Seiberg–Witten theory with $SO(p+q)$ global symmetry. The symmetric subgroup of $SO(p+q)$ from any of the two extended Dynkin diagrams (related to B_n and D_n):





is $SO(p)\times SO(q)$. Therefore taking $SO(7)$ global symmetry, or more appropriately, $SO(3,4)$ global symmetry we can easily find constrained instantons in the theory that are fibered over the following quaternionic space:

$$\frac{SO(3,4)}{SU(2)\times SU(2)\times SU(2)}. \tag{4.71}$$

The steps to generate Kähler space associated with semilocal strings is also evident: we have to mod the non-compact version of $SO(4)$ global symmetry by $U(1)\times U(1)$ symmetry. The manifold therefore is

$$\left[\frac{SU(1,1)}{U(1)}\right]^2, \tag{4.72}$$

so that gauging one of the $U(1)$ we can get semilocal strings in our theory. Manifolds (4.72) and (4.71) are related by a c-map. The real manifold associated with (4.72) is clearly $SO(1,1)$. The sequence therefore is:

0	SO(1,1)	$\left[\frac{SU(1,1)}{U(1)}\right]^2$	$\frac{SO(3,4)}{[SU(2)]^3}$

which fits consistently with the de Wit–Van Proeyen completion [23] of Alekseevskii's classification [3].

Let us consider one more example that is in the same vein as our previous example. For this case we take $p = q = 4$ so that our non-compact global symmetry is $SO(4,4)$. Clearly the maximal (and symmetric) subgroup is $SO(4) \times SO(4) \equiv [SU(2)]^4$, so that the constrained instantons are fibered over the following quaternionic manifold:

$$\frac{SO(4,4)}{SO(4) \times SO(4)} \tag{4.73}$$

where we have, as usual, gauged a $SU(2)$ subgroup of the maximal group. The ungauged subgroup therefore is $SO(4) \times SU(2) \equiv [SU(2)]^3$ whose non-compact version would be $[SU(1,1)]^3$. To determine the Kähler manifold we have to gauge an $U(1)$ subgroup of $[SU(1,1)]^3$ so that we are studying semilocal strings. The Kähler manifold will have more or less the same coset structure as (4.72) discussed above because the ungauged subgroups are of the same form as above. Following this trend, the sequence of manifolds that we now expect are:

0	$[SO(1,1)]^2$	$\left[\frac{SU(1,1)}{U(1)}\right]^3$	$\frac{SO(4,4)}{SO(4) \times SO(4)}$

which again fits perfectly with the classification of [23]. The zero-dimensional manifold in the last box of the sequence is expected because the real manifold does not have a coset structure. In fact so long as $p \leqslant 4, q \leqslant 4$ we do not expect to get a non-zero-dimensional manifold. This should give us a hint that if we choose a more generic global symmetry from the start, then maybe we could get a non-trivial manifold in the last box of the corresponding sequence. This is indeed the case if we choose $p = P + 4, q = 4$ with P any integer. The sequence of manifolds are rather straightforward to determine and they are of the following form:

SO(P,1)	$\frac{SO(P+1,1)}{SO(P+1)} \times SO(1,1)$	$\frac{SU(1,1)}{U(1)} \times \frac{SO(P+2,2)}{SO(P+2) \times SO(2)}$	$\frac{SO(P+4,4)}{SO(P+4) \times SO(4)}$

where we see that we do get a manifold in the last box from which we can get the real, Kähler and quaternionic manifolds by various possible mappings. Needless to say, the above sequence fits with the classifications of [3,23].

4.5.3. Example 3: New sequence of Kähler manifolds in the magic square

Our final example is a rather curious one. Let us look at the third row of the magic square containing the elements associated with E_7, etc.:

$\frac{Sp(3,R)}{SU(3) \times U(1)}$	$\frac{SU(3,3)}{SU(3) \times U(3)}$	$\frac{SO^*(12)}{SU(6) \times U(1)}$	$\frac{E_{7(-25)}}{E_6 \times U(1)}$

By construction these are all Kähler manifolds that are related to the corresponding semilocal strings (observe the $U(1)$ quotients). An inverse r-map to each of these cosets will give us the corresponding real manifolds that we studied in the earlier sections. For example for the unbroken E_6 subgroup of (4.55) has the following symmetric subgroups:

$$F_4, \qquad SU(6)\times SU(2), \qquad SO(10)\times U(1), \qquad Sp(4), \tag{4.74}$$

out of which we have used F_4 to construct the real manifold $\frac{E_{6(-26)}}{F_4}$. The other subgroup $SU(6)\times SU(2)$ was used in a different example to construct a quaternionic manifold (which is of course unrelated to this sequence of magic square). So we can ask the following question: what if instead of (4.55) we want to construct coset space associated with $SO(10)\times U(1)$ symmetry? This would mean that we are again looking for semilocal strings for a $U(1)$ gauge theory with E_6 global symmetry. For such a case the associated coset space will be:

$$\frac{E_{6(-14)}}{SO(10)\times U(1)}, \tag{4.75}$$

which was first conjectured by [4]. Here we see that there is a natural way to justify[29] the existence of such coset space! But this is not the end of the story. Let us look at the next element in the above row of the magic square. The symmetric subgroups of $SU(6)$ are:

$$Sp(3), \qquad SU(4)\times SU(2)\times U(1), \qquad SU(4), \qquad SU(3)\times U(3) \tag{4.76}$$

where $Sp(3)$ was used earlier to build a real space $\frac{SU^*(6)}{Sp(3)}$ whereas $SU(3)\times U(3)$ was used in a different sequence of the magic square to construct a Kähler manifold (4.32). Out of the remaining ones we can build a new non-compact coset space:

$$\frac{SU(4,2)}{SU(4)\times SU(2)\times U(1)}, \tag{4.77}$$

which in fact does exist in supergravity literature as target space of some sigma model of $\mathcal{N}=2$ supergravity. Thus a *new* sequence, not realised directly in the magic square, will be:

$\frac{SU(2,1)}{U(2)}$	$\frac{SU(2,1)\times SU(2,1)}{SU(2)\times U(2)}$	$\frac{SU(4,2)}{SU(4)\times U(2)}$	$\frac{E_{6(-14)}}{SO(10)\times U(1)}$

which could in principle be embedded in the magic square using the Rozenfeld-Tits constructions [7]. For some more details about these $U(1)$ quotients the readers may want to look up [26].

4.6. A note on holomorphic F-functions

In the previous subsections we discussed the issue of F-functions that could be used to determine the metric on the quaternionic Kähler manifolds. In this section we will complete the

29 One can view the coset (4.75) as a *plane* in the sense of projective geometry. The elements of this plane belong to certain Jordan pair such that one can define points and lines along with an incidence relation among them. It turns out that the group $E_{6(-14)}$ acts transitively on points and the stability group of a fixed point is $SO(10)\times U(1)$, thus realising the correspondence between the plane and the coset space (4.75) (see [62] for details).

Table 2
Rank 2 homogeneous special real spaces and their corresponding rank 3 and rank 4 Kähler and quaternionic spaces respectively associated with the magic square

$X(P,q)$	Real	$H(P,q)$	Kähler	$V(P,q)$	Quaternionic
$X(1,1)$	$SL(3,R)/SO(3)$	$H(1,1)$	$Sp(3)/U(3)$	$V(1,1)$	$F_4/Sp(3)\times SU(2)$
$X(1,2)$	$SL(3,C)/SU(3)$	$H(1,2)$	$SU(3,3)/SU(3)\times SU(3)\times U(1)$	$V(1,2)$	$E_6/SU(6)\times SU(2)$
$X(1,4)$	$SU^*(6)/Sp(3)$	$H(1,4)$	$SO^*(12)/SU(6)\times U(1)$	$V(1,4)$	$E_7/SO(12)\times SU(2)$
$X(1,8)$	E_6/F_4	$H(1,8)$	$E_7/E_6\times U(1)$	$V(1,8)$	$E_8/E_7\times SU(2)$

analysis by postulating the procedure to determine the F-function for any given Kähler manifolds. Although the following analysis is standard (see for example [5,23]) the F-functions for E_n and F_4 cases have not been explicitly presented anywhere.[30]

Throughout this section, we use the canonical parametrization introduced by [23] and the third reference of [16] but where all indices are shifted by one unit in order to fit our notation. The indices $A,B,C=2,\dots,n+1$ have been decomposed into indices $2,3,\mu$ and m, where μ and m take respectively $q+1$ and r values.

From [23], we know the form of the cubic functions $C(h)$ in terms of scalar fields h^A associated to the real manifolds of rank 1 and 2:

$$\begin{aligned}C(h)=d_{ABC}h^Ah^Bh^C&=\left(h^2\right)^3-\frac{1}{2}h^2\left(h^\alpha\right)^2\\&+\frac{1}{\sqrt{2}}\left\{\left(h^3\right)^3-3h^3\left(\left(h^\mu\right)^2-\frac{1}{2}\left(h^m\right)^2\right)+\frac{3}{2}\sqrt{3}(\gamma_\mu)_{mn}h^\mu h^m h^n\right\},\end{aligned}\tag{4.78}$$

with $\alpha\in\{3,\dots,n+1\}$ and where the gamma matrices $(\gamma_\mu)_{mn}$ are viewed as $r\times r$ matrices generating a $(q+1)$-dimensional Clifford algebra denoted $\mathbf{C}(q+1,0)$.

The coefficients d_{ABC} can also be used to describe Kähler manifolds.[31] By imposing the following conditions on the symmetric tensor d_{ABC} [23]:

$$d_{333}=\frac{1}{\sqrt{2}},\qquad d_{3\mu\mu}=d_{3mm}=0,\qquad d_{\mu mm}=0,\tag{4.79}$$

we construct the holomorphic functions $F(X^I)$, in terms of complex variables X^I, associated to Kähler manifolds that are in the image of an r-map:

$$\begin{aligned}F(X^I)&=id_{ABC}\frac{X^AX^BX^C}{X^1}\\&=\frac{i}{X^1}\left\{\left(X^2\right)^3-\frac{1}{2}X^2\left(X^\alpha\right)^2+\frac{1}{\sqrt{2}}\left(X^3\right)^3+3(\gamma_\mu)_{mn}X^\mu X^m X^n\right\}.\end{aligned}\tag{4.80}$$

As explained in the third reference of [16], these conditions constrain the allowed values of q to $1,2,4$ and 8. Since $r=2q$ and $n=3(q+1)$ for Kähler manifolds, these are exactly the spaces corresponding to the magic square with $n=6,9,15,27$ [4].

These Kähler manifolds are respectively associated to the Jordan algebras $J^3(\mathbf{R})$, $J^3(\mathbf{C})$, $J^3(\mathbf{H})$, and $J^3(\mathbf{O})$. They were classified in [5]: the Kähler $H(P,q)$ spaces generate quaternionic

[30] See however Eqs. (3.38) to (3.42) in the recent paper [63]. We thank Sergio Ferrara for pointing this to us. It will be interesting to relate these values to the ones that we determine here.

[31] Note that the use of the canonical parametrisation defines the tensor d_{ABC} up to arbitrary $O(n-1)$ rotations.

Table 3
$C(q+1,0)$ represents real Clifford algebras, $K(n)$ are $n \times n$ matrices with entries over the field K and D_{q+1} represents the real dimension of an irreducible representation of the Clifford algebra

q	$C(q+1,0)$	D_{q+1}
1	$R(2)$	2
2	$C(2)$	4
4	$H(2) \times H(2)$	8
8	$R(16) \times \{R \times R\}$	16

Table 4
Classification of generating matrices of Clifford algebras

Algebra	Matrix dimensions	Generating matrices
C_2	2×2	$i\sigma_2, i\sigma_3$
C_3	2×2	$\sigma_1, \sigma_2, \sigma_3$
C_4	4×4	$\gamma_1, \gamma_2, \gamma_3, \gamma_4$
C_5	4×4	$i\gamma_1, i\gamma_2, i\gamma_3, i\gamma_4$
C_6	8×8	$i\phi_j, j = 1, 2, 3, \ldots, 6$
C_7	8×8	$\phi_j, j = 1, 2, 3, \ldots, 7$
C_8	16×16	$\varpi_j, j = 1, 2, 3, \ldots, 8$
C_9	16×16	$i\varpi_j, j = 1, 2, 3, \ldots, 9$

$V(P,q)$ spaces [3] under c-map. This in turn emerge from the real $X(P,q)$ manifolds under the r-map [23] (see Table 2 for a list of relevant coset spaces).

The trivial case $q = 0$ with $n = 3$, which is also generated by the above restriction, is part of the Kähler $K(P, \dot{P})$ space and is associated with $W(P, \dot{P})$ quaternionic manifolds. P and $\dot{P}$ represents the multiplicity of each irreducible representations of the Clifford algebras which are listed in Table 3:

We restrict our study to $q > 0$ cases. In order to classify all F-functions associated to $H(P,q)$, one needs to consider all gamma matrices generating a $(q+1)$-dimensional real Clifford algebra with positive metric. This classification was done in [64], see Table 4 above for the relevant cases.

Solutions are characterised by specifying the multiplicities P and $\dot{P}$ of each irreducible representations of the Clifford algebras. In all cases we will discuss, we will consider $\dot{P} = 0$ and $P = 1$. The generating matrices σ_i used in the above table are simply the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.81}$$

The γ_i matrices are the Dirac Gamma matrices made out of the sigma matrices in the standard way. The ϕ_i matrices are in turn made of the γ_i matrices in the following way:

$$\phi_j = \begin{pmatrix} 0 & i\gamma_j \\ -i\gamma_j & 0 \end{pmatrix}, \qquad \phi_6 = \begin{pmatrix} \mathbf{I}_4 & 0 \\ 0 & -\mathbf{I}_4 \end{pmatrix}, \qquad \phi_7 = \begin{pmatrix} 0 & \mathbf{I}_4 \\ \mathbf{I}_4 & 0 \end{pmatrix}, \quad j = 1, 2, 3, 4, 5. \tag{4.82}$$

Finally the ϖ_j are similarly constructed using the ϕ_j matrices in exactly the same way as above with j running from $j = 1, 2, \ldots, 7$. The other two matrices ϖ_8 and ϖ_9 are constructed by $\mathbf{I}_8$ like ϕ_6 and ϕ_7, respectively.

We associate C_n with $C_{(q+2)}$. Hence, C_3 is associated to $\mathbf{R}(2) \in C(q+1,0)_{q=1}$, C_4 to $\mathbf{C}(2)$, etc. This association allows us to generate a $(q+1)$-dimensional Clifford algebra with $r \times r$

basis that satisfy simultaneously the condition imposed in (4.79) on the gamma matrices, i.e., $(\gamma_\mu)_{mm} = 0$. Thus, say we have $\sigma_1, \sigma_2, \sigma_3$ and we impose the condition $(\sigma_\mu)_{mm} = 0$, the term $(\sigma_\mu)_{mn}$ will therefore be equal to zero when it comes to σ_3 and we will be left with two (2×2) matrices, i.e., σ_1, σ_2 to span $\mathbf{R}(2)$ as required.

We are now ready to construct the $F(X^I)$ holomorphic functions for each Kähler spaces associated to the magic square. For the Kähler space associated with G_2 coset we already gave the F-function in (3.27), and for the coset associated with $Sp(n+1)$ we know that there is no Kähler space (see Section 3.1 for details).

- Kähler space $H(1,1)$:
  For $H(1,1)$, $q = 1$, $r = 2$ and $n = 6$. Hence $A, B, C = 2, \dots, 7$. In addition, $\mu \in \{4, \dots, 7\}$ and takes exactly $q+1$ values say $\mu = 4, 5$ whereas $m \in \{4, \dots, 7\}$ and takes r values for instance $m = 5, 6$. The quantity α takes all values in $\{3, \dots, 7\}$. The matrices generating the Clifford algebra $\mathbf{R}(2) \in \mathbf{C}_3$ would be σ_1, σ_2 according to the previous argument and we shall rename them σ_μ. Furthermore $\mathcal{F}(X^I) \equiv -iX^1 F(X^I)$:

$$\mathcal{F}(X^I) = (X^2)^3 - \frac{1}{2}X^2(X^3)^2 - \dots - \frac{1}{2}X^2(X^7)^2 + \frac{1}{\sqrt{2}}(X^3)^3 + 3(\sigma_4)_{mn}X^4X^mX^n + 3(\sigma_5)_{mn}X^5X^mX^n. \tag{4.83}$$

- Kähler space $H(1,2)$:
  For $H(1,2)$, $q = 2$, $r = 4$, $n = 9$, $\mu \in \{4, \dots, 10\}$ and takes 3 values say 4, 5, 6 and $m \in \{4, \dots, 10\}$ takes 4 values say 7, 8, 9, 10. $\alpha \in \{3, \dots, 10\}$ and the Clifford algebra would be generated by $\gamma_2, \gamma_3, \gamma_4$ which we rename γ_μ:

$$\mathcal{F}(X^I) = (X^2)^3 - \frac{1}{2}X^2(X^3)^2 - \dots - \frac{1}{2}X^2(X^{10})^2 + \frac{1}{\sqrt{2}}(X^3)^3 + 3(\gamma_4)_{mn}X^4X^mX^n + 3(\gamma_5)_{mn}X^5X^mX^n + 3(\gamma_6)_{mn}X^6X^mX^n. \tag{4.84}$$

- Kähler space $H(1,4)$:
  For $H(1,4)$, $q = 4$, $r = 8$, $n = 15$, $\mu \in \{4, \dots, 16\}$ and takes 5 values say 4, 5, 6, 7, 8 and $m \in \{4, \dots, 16\}$ takes 8 values say 9, 10, 11, 12, 13, 14, 15, 16. $\alpha \in \{3, \dots, 16\}$ and the Clifford algebra would be generated by five (8×8) elements of C_6, i.e., $i\phi_j$ with $j = 1, \dots, 5$ which we rename $i\phi_\mu$:

$$\mathcal{F}(X^I) = (X^2)^3 - \frac{1}{2}X^2(X^3)^2 - \dots - \frac{1}{2}X^2(X^{16})^2 + \frac{1}{\sqrt{2}}(X^3)^3 + 3i(\phi_4)_{mn}X^4X^mX^n + \dots + 3i(\phi_8)_{mn}X^8X^mX^n. \tag{4.85}$$

- Kähler space $H(1,8)$:
  For $H(1,8)$, $q = 8$, $r = 16$, $n = 27$, $\mu \in \{4, \dots, 28\}$ and takes 9 values and $m \in \{4, \dots, 28\}$ takes 16 values. $\alpha \in \{3, \dots, 28\}$ and the Clifford algebra would be generated by nine (16×16) elements of C_9, i.e., ϖ_j with $j = 1, \dots, 9$ which we rename ϖ_μ:

$$\mathcal{F}(X^I) = (X^2)^3 - \frac{1}{2}X^2(X^3)^2 - \dots - \frac{1}{2}X^2(X^{28})^2 + \frac{1}{\sqrt{2}}(X^3)^3 + 3(\varpi_4)_{mn}X^4X^mX^n + \dots + 3(\varpi_{12})_{mn}X^{12}X^mX^n. \tag{4.86}$$

The above analysis therefore summarises all the F-functions that we need to determine the Kähler spaces. To get the corresponding quaternionic spaces, we use the metric given in (3.29) for each of the four cases. With this therefore we have the complete picture of all the quaternionic and the Kähler manifolds in the magic square.

5. Summary, discussions and future directions

In this paper we hopefully gave a new way to study the magic square in mathematics and string theory that is not based on the dimensional reduction of supergravity theories. Our method relies on the existence of constrained instantons in certain $\mathcal{N}=2$ gauge theories with exceptional global symmetries. These theories are *not* asymptotically free and are at strong coupling. This means that a simple Yang–Mills description may not suffice and we might even lack a Lagrangian description of these theories. Nevertheless we have ample evidence that these theories exist: via Seiberg–Witten curves, F-theory and possible quaternionic formulations of low energy descriptions.

Viewing them as sectors of Seiberg–Witten theories, the exceptional global symmetries form non-trivial fixed points of renormalisation group flows. This is well known and they lead to the following sequence of theories:

$$E_8 \longrightarrow E_7 \longrightarrow E_6 \longrightarrow D_4 \longrightarrow A_2 \longrightarrow A_1 \longrightarrow \{0\}. \tag{5.1}$$

Our idea of sequential gauging is partially motivated by the above sequence. The $SU(2)$ constrained instanton which is also a semilocal instanton for our case is constructed by gauging an $SU(2)$ subgroup of the global group. The $U(1)$ part of ungauged global group—that also contains the monodromy associated with a dyonic point—is then further gauged to construct semilocal strings in the model. These two process give us quaternionic and Kähler spaces that are related by a c-map. Once we have these spaces, the real space associated to the Kähler space can be easily constructed.

Our whole analysis therefore depends on the existence of *one* instanton moduli space in these theories. In the absence of a proper Lagrangian description we cannot give a concrete construction of these instantons solutions of course, but moduli space can still be constructed. Existence of Seiberg–Witten curves also means that we have added all the instanton contributions in the path-integral. Recall that the instanton contributions to the Seiberg–Witten prepotential $\mathcal{F}_{\mathrm{SW}}$ can be written as:

$$\mathcal{F}_{\mathrm{SW}} = \mathcal{F}_{\text{classical}} + \mathcal{F}_{\text{one-loop}} + \frac{1}{2\pi i}\sum_{k=1}^{\infty}\int_{\mathcal{M}_k} \omega e^{-S}\Lambda^{k(4-N_F)}, \tag{5.2}$$

where $\mathcal{M}_k$ is the moduli space of k-instantons, ω is the volume form, S is the instanton action, N_F is the number of flavors and Λ is the same Pauli–Villars scale that we used earlier. It is therefore an interesting question to ask how instantons in these gauge theories with E_n, F_4 global symmetries give us the right Seiberg–Witten curves. Note however that if one breaks the E_n symmetry by giving masses to quarks and keeping the gauge coupling finite, one may hope to get a convergent expression for the instanton partition function. However, to show that an analytic continuation to the E_n symmetric point would make sense, requires more work.[32] We leave this aspect for future work.

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[32] We thank Nikita Nekrasov for comments on this.

Another interesting direction is to look for theories with exceptional *gauge* symmetries. Incidentally one instanton moduli spaces will be the same for these theories—its just an embedding of $SU(2)$ in exceptional gauge groups[33]—but the corresponding curves will be different. We gave one example before. Another example would be a theory with F_4 gauge symmetry. Such a theory with one massless hypermultiplet has the following Seiberg–Witten curve[34] [61]:

$$y^2 = \left[\left(x^2 - b_1^2\right)\left(x^2 - b_2^2\right)\left(x^2 - b_3^2\right)\left(x^2 - b_4^2\right)\right]^2 - x^4 \Lambda^{12}, \tag{5.3}$$

where b_i are the projections of the weights: (1000), (-1100), $(0-111)$ and $(00-11)$. It would be interesting to study these theories with more than one massless hypermultiplets.

One final issue is the classification of de-Wit and Van-Proeyen [23] that completes Alekseevskii's classification of quaternionic manifolds [3]. We have shown that we can reproduce all of Alekseevskii's symmetric manifolds and few more of de-Wit and Van-Proeyen also. However we have not investigated enough to see whether we could reproduce all other manifolds in the classification of [23]. In fact its an interesting question to ask whether these manifolds have a coset structure like the other manifolds in the classification. We leave this for future work.

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[33] Recall $\pi_3(E_n) = \pi_3(F_4) = \mathbf{Z}$.

[34] This could also be derived from (4.57) with appropriate polynomial $\mathcal{P}_{\mathcal{R}}$.

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Chapter 5

CONCLUSION

In this work, we classified all symmetric non-compact quaternionic manifolds using $SU(2)$ gauge theory with global symmetries. We reproduced the results of previous classifications and found a new set of Kähler manifolds. Future directions involve a detailed analysis of how the group $SU(2)$ is embedded in $\mathcal{G} = Sp(n+1)$ and in the exceptional global symmetry groups. In a sequel to this paper, we will also be studying the fibration of semilocal defects on Kähler manifolds. In addition to confirming the new set of Kähler manifolds we proposed, Ferrara's paper [**35**] also suggested several other new sequences of manifolds beyond the magic square that we would like to reproduce using our method. The new sequence of Kähler manifolds that both Ferrara and us found has different physical meanings. In Ferrara's case, these spaces represent moduli spaces of non-BPS attractors with vanishing central charge whereas in our case they are related to the moduli spaces of semilocal defects. We are looking forward to explore if a possible gauge/gravity correspondence exists between the two models.

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Appendix A

COMMENTS ON THE DERIVATION OF HOLOMORPHIC FUNCTIONS FOR MAGIC KÄHLER MANIFOLDS

Non-linear sigma models corresponding to real (n-1) dimensional spaces in five space time dimensions are characterized by cubic polynomials $C(h)$ in n real variables

$$C(h) = d_{ABC} h^A h^B h^C \tag{A.0.1}$$

where $A, B, C = 2, \ldots, n+1$ and the scalar fields h^A are restricted by $C(h) = 1$. One can dimensionally reduce such a theory to four space-time dimensions and find non-linear sigma models corresponding to Kähler spaces of complex dimensions n, characterized by a homogeneous holomorphic function $F(X)$ of second degree, depending on $n+1$ complex variables

$$F(X) = i d_{ABC} \frac{X^A X^B X^C}{X^1} \tag{A.0.2}$$

where X^1 and X^A are complex variables. In three space-time dimensions, one finds a non-linear sigma model corresponding to a quaternionic manifold of quaternionic dimension $n+1$. There exists a class of quaternionic manifolds whose structure is encoded in the homogeneous holomorphic function of the special Kähler manifold. The map between special Kähler manifolds of complex dimension n and certain quaternionic manifolds of quaternionic dimension $n+1$ is called the c-map. The map between Kähler manifolds and real manifolds is denoted the r-map [**25**].

In [25], De Wit and Van Proeyen classify all homogeneous quaternionic spaces that are in the image of the $c \circ r$ map. Their analysis if performed completely at the level of the real spaces, and amounts to classifying all the cubic polynomials $\mathcal{C}(h)$. To do so, they first classify all d_{ABC} tensors for real manifolds. These tensors are completely symmetric and satisfy the relation

$$\Gamma_{abcd} = D_{abc;d} \tag{A.0.3}$$

where $a, b, \ldots$ take $n-1$ values $\{3, \ldots, n+1\}$ and where we have the following definitions

$$\begin{aligned} \Gamma_{abcd} &\equiv d_{e(ab}d_{cd)e} - \frac{1}{2}\delta_{(ab}\delta_{cd)} && \text{(A.0.4)} \\ D_{abc;d} &= d_{e(ab}A_{c)e;d} && \text{(A.0.5)} \end{aligned}$$

We define the canonical parametrization as

$$\begin{aligned} d_{222} &= 1 && \text{(A.0.6)} \\ d_{22a} &= 0 && \text{(A.0.7)} \\ d_{2ab} &= -\frac{1}{2}\, d_{222}\, \delta_{ab}, && \text{(A.0.8)} \end{aligned}$$

In the case of Kähler manifolds, the tensors d_{ABC} satisfy a similar relation than the one for real spaces, namely

$$\begin{aligned} \Gamma_{abcd} &\equiv d_{e(ab}d_{cd)e} - \frac{1}{2}\delta_{(ab}d_{cd)} = 0 && \text{(A.0.9)} \\ d_{abc}\delta^{bc} &= 0 && \text{(A.0.10)} \end{aligned}$$

where the canonical parametrization is still applied.

For the real and Kähler manifolds, the authors of [25, 21] argue that it is always possible to bring the tensors d_{abc} into a form such that

$$d_{333} = \frac{1}{\sqrt{2}}, \qquad d_{33a} = \frac{1}{\sqrt{2}}\delta_{a3} \tag{A.0.11}$$

$$d_{3\mu\nu} = -\frac{1}{\sqrt{2}}\delta_{\mu\nu}, \qquad d_{3mn} = \frac{1}{2\sqrt{2}}\delta_{mn} \tag{A.0.12}$$

where the indices run over $\mu \in \{4, \ldots, n+1\}$ and $m \in \{5, \ldots, n+1\}$ where $\mu \neq m$. We let μ take $q+1$ indices and m take r indices. The Kähler case needs to satisfy the additional property $d_{abb} = 0$. In particular, it differs of the real case by the fact that

$$d_{3bb} = \frac{1}{2\sqrt{2}}(r - 2q), \quad d_{\mu bb} = d_{\mu mm} \tag{A.0.13}$$

with $r = 2q$, so that $n = 3(q+1)$, and $d_{\mu mm} = 0$ [**25**]. Having split the indices $\{3, \ldots, n+1\}$ into $\{3, \mu, m\}$, we find that $b \neq 3$ runs over μ and m, thus we get the conditions $d_{3\mu\mu} = 0$ and $d_{3mm} \doteq 0$.

In the case of real manifolds, De Wit and Van Proeyen find that the only non vanishing value of d_{abc} is $d_{\mu mn}$. Using the fact that the tensors d_{abc} can be concisely summarized by the cubic polynomial [**25, 21**]

$$\mathcal{Y}(x) = d_{abc} x_a x_b x_c \tag{A.0.14}$$

we can collect all the non vanishing values of the real manifold tensor we find

$$\mathcal{Y}(x) = \frac{1}{\sqrt{2}} \left(x_3^3 - 3x_3 \left(x_\mu^2 - \frac{1}{2} x_m^2 \right) \right) + 3 d_{\mu mn} x_\mu x_m x_n \tag{A.0.15}$$

where $d_{\mu mn}$ satisfies the defining relation of the generators of a Clifford algebra and can thus be expressed as gamma matrices according to

$$d_{\mu mn} = \sqrt{\frac{3}{8}} (\gamma_\mu)_{mn} \tag{A.0.16}$$

Considering the above cubic polynomial and adding the canonical parametrization, one can construct in a straightforward way the $\mathcal{C}(h)$ functions characterizing the real manifolds

$$\begin{aligned} \mathcal{C}(h) &= (h^2)^3 - \frac{1}{2} h^2 (h^a)^2 + && \text{(A.0.17)} \\ &+ \frac{1}{\sqrt{2}} \left\{ (h^3)^3 - 3h^3 \left((h^\mu)^2 - \frac{1}{2} (h^m)^2 \right) + \frac{3}{2} \sqrt{3} (\gamma_\mu)_{mn} h^\mu h^m h^n \right\} && \text{(A.0.18)} \end{aligned}$$

where the gamma matrices $(\gamma_\mu)_{mn}$ are viewed as $r \times r$ matrices generating a $(q+1)$-dimensional Clifford algebra [**25**].

Playing the same game for Kähler manifolds, we can construct the associated $\mathcal{Y}(x)$ polynomial from the conditions listed above. We find

$$\mathcal{Y}(x) = \frac{1}{\sqrt{2}} x_3^3 + 3 d_{\mu mn} x_\mu x_m x_n \tag{A.0.19}$$

Adding to this the canonical parametrization, the $F(X)$ functions take the form

$$F(X^I) = \frac{i}{X^1}\left\{(X^2)^3 - \frac{1}{2}X^2(X^a)^2 + \frac{1}{\sqrt{2}}(X^3)^3 + 3(\gamma_\mu)_{mn}X^\mu X^m X^n\right\} \quad \text{(A.0.20)}$$

The conditions imposed for Kähler manifolds severely constrain the dimensions of the Clifford algebra. In fact, [**25, 21**] show that the only allowed values are $q = 1, 2, 4, 8$. Since this theory has the relation $n = 3(q+1)$, this leads to $n = 6, 9, 15, 27$ which are exactly the dimensions of a special kind of Kähler manifolds called magic Kähler manifolds. These manifolds are found in the magic square.

Hence, what is left in order to complete our classification of the $F(X)$ functions of magic Kähler manifolds is to consider $r \times r$ gamma matrices generating a $(q+1)$-dimensional Clifford algebra. Such a classification already exists and can be found in [**49, 50**]. Recall however that for the Kähler cases we also need to respect the additional property that $r = 2q$.

Lets consider an example in detail. Take for instance $H(1,1) = Sp(3)/U(3)$. We know in this case that $q = 1$ since $H(P,q) \equiv H(1,1)$. Recall from Section 4.6 of [**24**] that the index P indicates the multiplicity of the irreducible representation of the Clifford algebra. Since $q = 1$ we use the relation $r = 2q$ and $n \equiv 3(q+1)$ to find that $r = 2$ and $n = 6$. Since $a \in \{3, \ldots, n+1\}$, a runs from 3 to 7. Implementing this information into (A.0.20) we find

$$\begin{aligned} \mathcal{F}(X^I) &= (X^2)^3 - \frac{1}{2}X^2(X^3)^2 - \frac{1}{2}X^2(X^4)^2 - \frac{1}{2}X^2(X^5)^2 && \text{(A.0.21)} \\ &- \frac{1}{2}X^2(X^6)^2 - \frac{1}{2}X^2(X^7)^2 + 3(\gamma_\mu)_{mn}X^\mu X^m X^n && \text{(A.0.22)} \end{aligned}$$

where $\mathcal{F}(X^I) \equiv -iF(X^I)$. Next we look for 2×2 matrices generating a certain 2-dimensional Clifford algebra. We find them in a higher dimensional algebra, namely C_3 (see Section 4.6 of [**24**]) and show that we can set one of the matrix to zero so that it respects the conditions (A.0.13). Thus, we get two Pauli matrices and find the final form of $\mathcal{F}(X^I)$

$$\begin{aligned}
\mathcal{F}(X^I) &= (X^2)^3 - \frac{1}{2}X^2(X^3)^2 - \frac{1}{2}X^2(X^4)^2 - \frac{1}{2}X^2(X^5)^2 && \text{(A.0.23)}\\
&- \frac{1}{2}X^2(X^6)^2 - \frac{1}{2}X^2(X^7)^2 + && \text{(A.0.24)}\\
&+ 3(\sigma_4)_{mn}X^4X^mX^n + 3(\sigma_5)_{mn}X^5X^mX^n && \text{(A.0.25)}
\end{aligned}$$

where μ takes $q+1$ values i.e. 2, amongst $\{4,\ldots,7\}$ that were chosen here to be 4 and 5.

THE END