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Clones minimaux et opérations majorité

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Introduction

Soit S un ensemble; pour la plus part du mémoire un ensemble fini. Un clone sur S est un ensemble d'opérations finitaires sur S fermé par rapport à la composition et contenant toutes les projections. Un clone C est minimal si le clone des projections est le seul clone proprement inclus dans C. Un clone sur S est un clone majorité s'il contient une operation majorité; c'est-à-dire une opération ternaire m sur S satisfaisant $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$ pour tous $x, y \in S$. Dans ce mémoire nous nous concentrons sur les clones majorité minimaux.

Aux Chapitres 2 et 3 nous introduisons les faits nécessaires sur les clones et les clones minimaux, produisons des techniques de base pour déterminer si un clone est minimal ou non, et présentons un vol d'horizon sur les connaîssances sur ce sujet. En grande partie ces deux chapitres sont basés sur les tours d'horizon de B. Csákany [4] et de R.W. Quackenbush [15].

En 1941 E.L.Post a complètement décrit tous les clones sur S à 2 éléments [16] et sa liste contient tous les clones majorité minimaux. En 1983 B. Csákany [4] a determiné les clones minimaux sur S à 3 éléments qui aussi renferment les clones majorité minimaux. Trois ans plus tard en [5] il a donné la description des clones majorité minimaux engendrés par une opération majorité conservatrice (c'est-à-dire m, telle que $m(x, y, z) \in x, y, z$ pour tous $x, y, z \in S$). Finalement en 2000 T. Waldhauser [22] a trouvé tous les clones majorité minimaux sur S à 4 éléments. Nous présentons ces résultats au Chapitre 4.

De plus, pour une opération majorité sur S qui engendre un clone minimal nous présentons deux techniques différentes pour obtenir une telle opération sur tout S' de cardinalité plus grande que $|S|$. Les idées sont basées sur des arguments utilisés dans une de ses preuves.

Le fait que tout clone majorité C peut être décrit comme l'ensemble d'opérations sur S préservant une relation finitaire sur S est une conséquence d'un théorème bien connu de Baker-Pixley. Au Chapitre 5 nous trouvons telles relations pour les clones majorité minimaux si S a au plus 3 éléments. Il s'avère que à part une seule exception, ces relations peuvent être choisies comme les 3^{eme} graphiques des clones en question (voir 5.8 pour la définition du k^{eme} graphique d'un clone). Pour le cas exceptionnel nous montrons que le 4^{eme} graphique suffit.

Ce fait implique que pour $|S| = 3$, il existe un clone non minimal qui ne contient qu'un seul clone minimal; un fait qui n'était pas connu et qui n'est pas vrai pour $|S| = 2$. Finalement nous montrons que tout clone C sur S qui contient une opération majorité conservatrice est le clone des opérations qui préservent le $2|S|$-graphique de C.

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Minimal clones and majority operations

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Contents

Chapter 1

Introduction

A clone is a set of operations on a set S that is closed under composition and contains the projections. If not stated otherwise, we always assume the set S on which the clones are defined to be finite. A minimal clone is a clone whose only proper subclone is the clone that contains only the projections (the smallest of all clones). In our work, we focus on a special kind of minimal clones, namely minimal clones generated by majority operations (i.e. a ternary operation m s.t. $m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x$). In the sequel, we call clones that contain a majority operation *majority clones.*

In chapter two and three, we give an introduction to the fundamentals of clones and minimal clones. We provide ourselves with the basic techniques of determining whether a clone is minimal and give a brief overview over the current state of knowledge. Large parts of the chapters are based on the surveys of Csákány [4] and Quackenbush [16] and the results, if not stated otherwise, are common knowledge.

In 1941, E.L. Post described the set of clones on a two-element set completely [17]. Thus, the minimal majority clones on a two-element set are known. In 1983, Béla Csákány determined all minimal majority clones on the three-element set in [4]. Three years later, he gave a description of all conservative minimal majority clones in [5] (a clone is conservative if, for any operation f in the clone, we have $f(x_1, ..., x_n) \in \{x_1, ..., x_n\}$ for all $x_1, ..., x_n \in S$). It was then Tamás Waldhauser who determined all minimal majority clones on a four-element set in 2000 [23]. In chapter four, we present all the results in detail.

Furthermore, for a given minimal majority operation on S, we describe two different techniques to obtain a minimal majority operation on an arbitrarily larger set S', $\infty > |S'| > |S|$. The ideas are based on arguments that Waldhauser uses in one of his proofs.

It is a consequence of a well-known theorem of Baker and Pixley [1] that any majority clone on S can be described as the set of operations preserving a single relation on the set S. In chapter five, we determine such relations for all minimal majority clones on a two- and three-element set. It turns out that for all except one of these minimal clones, the relation can be chosen to be the third graphic of the clone. For the remaining clone, we show that the fourth graphic is sufficient (see 5.8 for a definition of the k-th graphic of a clone).

That we need the fourth graphic for one minimal clone implies that, for $|S| = 3$, there is a non-minimal clone C that contains no minimal clone except one majority clone (something that is not the case for $|S| = 2$).

We also give a bound for the number k such that the k-th graphic is enough to generate a majority clone on an n-element set S if one of its majority operations is conservative. Under these conditions, the $2n$-th graphic is enough to generate C. We also show that, under these conditions, we have $C = [C^{(2n)}]$ (i.e. C is generated by its $2n$-ary operations).

Chapter 2

Clones

2.1 The basics of clones

We begin by giving some definitions.

Definition 2.1: a) Let S be a finite fixed universe with $|S| \geq 2$. Denote by $\mathbb{N}$ the set $\{1, 2, ..\}$ of natural numbers. For $n \in \mathbb{N}$, a map $f : S^n \to S$ is an *n-ary operation* on S. Denote by $O_S^{(n)}$ the set of all n-ary operations on S. Furthermore, $O_S := \bigcup_{n\in\mathbb{N}} O_S^{(n)}$.

b) Let f be an n-ary operation on S. The ith variable is called *essential* (or *relevant*) if there are $a_1, ..., a_n, b \in S$ such that $f(a_1, ..., a_n) \neq f(a_1, ...a_{i-1}, b, a_{i+1}, ..., a_n)$. Call f *essentially k-ary* if it has exactly k essential variables.

c) For $k, n \in \mathbb{N}$, the *composition* of $f \in O_S^{(k)}$ and $g_1, g_2, ..., g_k \in O_S^{(n)}$ is the n-ary operation $f(g_1, ..., g_k)$ on S defined by setting

$$f(g_1, ..., g_k)(a_1, ..., a_n) := f(g_1(a_1, ..., a_n), ..., g_k(a_1, ..., a_n))$$

for all $a_i \in S$.

d) Denote by e_i^n ($1 \leq i \leq n$, $n \in \mathbb{N}$) the n-ary operation on S that maps any $(a_1, ...a_n) \in S^n$ onto a_i. The operations e_i^n are called *projections.* In the following, we often refer to projections as *trivial operations.* Denote by J_S the set of projections on S.

In the following, we will use $\approx$ to indicate that an identity holds for all variables ranging over S (e.g. $e_1^3(x, y, z) \approx x$).

We are now ready to define a clone. The notion of a clone generalizes a monoid to a set of operations (by a monoid we mean the set of selfmaps on a set S that is closed under composition and contains the identity mapping).

Definition 2.2: A *clone* on S is a subset of O_S that is closed under composition and contains all the projections.

Examples: Each of the following sets is a clone:

(1) The set O_S of all operations on the set S (the *full clone*).

(2) The set J_S of all projections on the set S (the *trivial clone*).

(3) All continuous operations on a topological space.

(4) All idempotent operations on the set S (f is called *idempotent* if $f(x, ..., x) \approx x$).

(5) All operations monotone in each variable on a partially ordered set $(S, \leq)$.

(6) All conservative operations on the set S (f is called *conservative* if every subset of S is closed under f, i.e. $f(a_1, ...a_m) \in \{a_1, ..., a_m\}$ for all $(a_1, ..., a_m) \in S^m$).

Lemma 2.3: A clone C is closed under identification and permutation of variables.

Proof: Let C be clone on S and let $f \in O_S$ be a k-ary operation. Set $K := \{1, ..., k\}$. Let $p, q \in K$. Without loss of generality, we can assume $p < q$. Furthermore, let ϕ be a permutation on $\{1, ..., k\}$. Define

$$f_{pq}(x_1, ..., x_{k-1}) := f(x_1, ..., x_{q-1}, x_p, x_q, x_{q+1}, ..., x_{k-1}),$$
$$f_\phi(x_1, ..., x_k) := f(x_{\phi(1)}, ..., x_{\phi(k)}).$$

But now

$$f_{pq}(x_1, ..., x_{k-1}) = f(e_1^k, ..., e_{q-1}^k, e_p^k, e_q^k, e_{q+1}^k, ..., e_{k-1}^k)(x_1, ..., x_k),$$
$$f_\phi(x_1, ..., x_n) = f(e_{\phi(1)}^k, ..., e_{\phi(k)}^k)(x_1, ..., x_k)$$

and thus $f_{pq}, f_\phi \in C$. □

Definition 2.4: Let $\sigma \subseteq S^k$ be a k-ary relation on S and $m \in \mathbb{N}$. We say that $f \in O_S^{(n)}$ *preserves* σ if for all $(\nu_{11}, ...\nu_{k1}), (\nu_{12}, ..., \nu_{k2}).\because, (\nu_{1n}, ..., \nu_{kn}) \in \sigma$.

$$(f(\nu_{11}, ..., \nu_{1n}), f(\nu_{21}, ...\nu_{2n}), ..., f(\nu_{k1}, ..., \nu_{kn})) \in \sigma.$$

To illustrate this definition it helps to think of a $k \times n$-matrix whose columns (interpreted as k-tuples) are elements of the relation σ. If we then apply f to the rows of the matrix and interpret the k resulting values as a k-tuple, this has to be in σ as well.

$$\begin{array}{cccccc} \nu_{11} & \nu_{12} & \dots & \nu_{1n} & & f(\nu_{11},\nu_{12},\dots,\nu_{1n}) \\ \nu_{21} & \nu_{22} & \dots & \nu_{2n} & & f(\nu_{21},\nu_{22},\dots,\nu_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \Rightarrow & \vdots \\ \nu_{k1} & \nu_{k2} & \dots & \nu_{kn} & & f(\nu_{k1},\nu_{k2},\dots,\nu_{kn}) \\ \in\sigma & \in\sigma & \in\sigma & \in\sigma & & \in\sigma \end{array}$$

Definition 2.5 Denote by $Pol\ \sigma$ the set of operations on S preserving σ.

Recall the example (5) from above. Clearly, (5) is the set of all operations preserving the relation $\leq$.

Lemma 2.6: *a) For a k-ary relation σ on S the set $Pol\ \sigma$ is a clone.*
b) The intersection of a set of clones on S is a clone.

Proof: a) Let $\begin{pmatrix}\nu_{11}\\ \vdots \\ \nu_{m1}\end{pmatrix}, \begin{pmatrix}\nu_{12}\\ \vdots \\ \nu_{m2}\end{pmatrix}, \dots, \begin{pmatrix}\nu_{1n}\\ \vdots \\ \nu_{mn}\end{pmatrix} \in \sigma$. Then $\begin{pmatrix} e_i^n(\nu_{11},\nu_{12},\dots,\nu_{1n}) \\ \vdots \\ e_i^n(\nu_{m1},\nu_{m2},\dots,\nu_{mn}))\end{pmatrix} = \begin{pmatrix}\nu_{1i}\\ \vdots \\ \nu_{mi}\end{pmatrix} \in \sigma$ for all $1 \leq i \leq n$. Thus, any projection preserves σ. It remains to show that the composition of operations preserving σ also preserves σ. Let $f \in O_S^{(k)}, g_1, \dots, g_k \in O_S^{(n)}$ be such operations. Then, clearly, $\begin{pmatrix} g_i(\nu_{11},\dots,\nu_{1n}) \\ g_i(\nu_{21},\dots,\nu_{2n}) \\ \vdots \\ g_i(\nu_{m1},\dots,\nu_{mn})\end{pmatrix} \in \sigma$ for all $1 \leq i \leq k$ since g_i preserves σ for all i. But now, the assumption that f preserves σ implies that we must have $\begin{pmatrix} f(g_1(\nu_{11},\dots,\nu_{1n}),\dots,g_k(\nu_{11},\dots,\nu_{1n})) \\ f(g_1(\nu_{21},\dots,\nu_{2n}),\dots,g_k(\nu_{21},\dots,\nu_{2n})) \\ \vdots \\ f(g_1(\nu_{m1},\dots,\nu_{mn}),\dots,g_k(\nu_{m1},\dots,\nu_{mn}))\end{pmatrix} \in \sigma$, as required.
b) Let C be the clone obtained by the intersection of a set of clones $\mathcal{C}$. Clearly, C contains the projections. Furthermore, if $f \in C^{(k)}$, $g_1, \dots g_k \in C^{(n)}$, then they have to be in any clone in $\mathcal{C}$. Thus, $f(g_1, \dots, g_k)$ has to be in any clone in $\mathcal{C}$ and hence in C. □

Definition 2.7: For a set of relations R on S, set $Pol\ R = \bigcap\{Pol\ \sigma \mid \sigma \in R\}$. By Lemma 2.6, this is a clone.

Recall the example (6) from above. This is $\bigcap\{Pol\ \sigma \mid \emptyset \neq \sigma \subsetneq S\}$. We can generalize the examples (5) and (6) as follows:

(7) All operations preserving a set R of (finitary) relations on a set S.

In fact, all clones are of this form. This may be formulated as follows.

Theorem 2.8: *Preserving a relation induces a Galois correspondence between operations and relations, in which the closed classes of operations are exactly the* *clones (see [2],[10])*

We learn more about the correspondence between operations and relations in the fifth chapter.

Alternatively to describing clones as sets of operations containing the projections and being closed under composition, there exists another way to define clones. The following description was done by Mal'tsev in [14]. We define three unary operations $\xi, \tau, \triangle$ and one binary operation $*$ on O_S as follows:

Let $f \in O_S^{(n)}$. For all $a_1, ..., a_n \in S$ set

- $(\xi f)(a_1, ...a_n) := f(a_2, ..., a_n, a_1)$ $(\xi f := f$ if $n = 1)$
- $(\tau f)(a_1, ..., a_n) := f(a_2, a_1, a_3..., a_n)$ $(\tau f := f$ if $n = 1)$
- $(\triangle f)(a_1, ..., a_{n-1}) := f(a_1, a_1, a_2..., a_n)$ $(\triangle f := f$ if $n = 1)$
- For $g \in O_S^{(m)}$ and all $a_1, ..., a_{m+n-1} \in S$

$$(f * g)(a_1, ..., a_{m+n-1}) := f(g(a_1, ..., a_m), a_{m+1}, ..., a_{m+n-1})$$

A set of operations is a clone if and only if it is closed under these procedures and contains e_1^2.

2.2 The clone lattices

Definition 2.9: Let $f, g \in O_S$. We say that f *generates* g if g can be obtained from f and the projections by (finitely many) compositions. Obviously, all operations generated by f form a clone. Call it the *clone generated by* f and denote it by $[f]$. Extend the terminology by writing $[F]$ for the clone generated by a set of functions F. Notice that $[F]$ is the least clone on S containing F.

It is easy to see that $F \mapsto [F]$ is a closure operator on the partially ordered set $(O_S, \subseteq)$ (i.e. $F \subseteq [F]$, $F \subseteq G \Rightarrow [F] \subseteq [G]$, $[[F]] = [F]$ for all $F, G \subseteq O_S$). Thus, the clones on S form a lattice.

Definition 2.10: Denote by L_S the set of clones on S. Call $\mathcal{L}_S := (L_S, \subseteq)$ the *clone lattice on* S with its greatest element O_S and its least element J_S.

Post described the clone lattice on the two-element set completely in [17] (they are countably many). However, for $|S| \geq 3$ it is known that there are continuum many clones and a full description of these lattices seems to be hopeless, even for $|S| = 3$. Nevertheless, a few results are known.

Definition 2.11: Call the atoms of $\mathcal{L}_S$ (F is an atom of $\mathcal{L}_S$ if its only proper subclone is the trivial clone J_S) *minimal clones* and the dual atoms (F is a dual atom of $\mathcal{L}_S$ if it is properly contained only in the full clone O_S) *maximal clones.*

Note that it is not obvious that minimal clones or maximal clones have to exist in a given clone lattice. For all we know at this point, there could be a clone $C \neq J_S$ containing no minimal clone.

A description of the maximal clones was given by Iablonskiî [11] for the three element case and finally, for every set, by Rosenberg in 1965 [18]. Kuznetsov gave a short proof for the fact that each clone distinct from O_S is contained in a maximal clone and that there are only finitely many maximal clones [13].

One may ask the same questions about minimal clones: Are they fully described? Does any clone contain a minimal clone? Are there only finitely many? We provide an overview of our knowledge about minimal clones in the next chapter.

Chapter 3

Minimal Clones

In this chapter, we explain some standard techniques to determine whether a clone is minimal and we answer the following questions:

- What is known about the operations generating a minimal clone?
- Are there finitely or infinitely many minimal clones on a given set S?
- Does every clone contain a minimal clone?

For the whole chapter, set $S = \{0, ..., n-1\}$. We start by making an easy but very useful observation:

Lemma 3.1: *A clone C is a minimal clone if and only if each nontrivial $f \in C$ (i.e. f is not a projection) generates any $g \in C$; in other words, $C = [f]$ for all nontrivial $f \in C$.*

Proof: "$\Rightarrow$" Let C be minimal and $f \in C$ nontrivial. Suppose $[f] \neq C$. This implies $J_S \subsetneq [f] \subsetneq C$, a contradiction to the minimality of C.
"$\Leftarrow$" By contraposition let C be not minimal. Then there exists a nontrivial clone C', $C' \subsetneq C$. But now, for any nontrivial $f \in C'$, $[f] \neq C$. $\square$

This lemma gives a standard technique to prove that an operation f does not generate a minimal clone on S: Show that there exists an operation g such that f generates g but g does not generate f. To show that an operation g does not generate f, it suffices to find a relation σ such that g preserves σ while f does not (since we know by Lemma 2.5 that if g preserves σ, then any composition obtained from g and the projections has to preserve σ, too). Many times, we choose the respective relation σ to be a subset of S. In the following, we use this technique very often.

Attributes of an operation $f \in O_S$ can be extended to the algebra $\langle S; f \rangle$. Call this algebra *(essentially) k-ary* if f is (essentially) k-ary and *conservative* if f is conservative. Furthermore, $\langle S; f \rangle$ is *termed minimal* (i.e. its term algebra is minimal) if and only if $[f]$ is a minimal clone. This observation is useful because it allows us to look at (minimal) clones as term algebras which is helpful in some situations (to learn more about the correspondence between clones and term algebras see for example [8]). In terms of algebras, we can formulate another technique to show that an operation g does not generate an operation f.

Lemma 3.2: *Let $f, g \in O_S$ and let $\langle A; g \rangle, \langle B; g \rangle$ be isomorphic subalgebras of $\langle S; g \rangle$. Then g does not generate f if one of the following conditions hold:*

(i) One of $\langle A; f \rangle$, $\langle B; f \rangle$ is not a subalgebra of $\langle S; f \rangle$

(ii) $\langle A; f \rangle$ and $\langle B; f \rangle$ are non-isomorphic subalgebras of $\langle S; f \rangle$.

Proof: By the way of contraposition, suppose $f \in [g]$.

(i) $f \in [g]$ implies that for a subalgebra $\langle D; g \rangle$ of $\langle S; g \rangle$, $\langle D; f \rangle$ must be a subalgebra of $\langle S; f \rangle$. In particular, this implies that $\langle A; f \rangle$ and $\langle B; f \rangle$ are subalgebras of $\langle S; f \rangle$, a contradiction.

(ii) $f \in [g]$ implies that the isomorphism between $\langle A; g \rangle$ and $\langle B; g \rangle$ is also an isomorphism between $\langle A; f \rangle$ and $\langle B; f \rangle$. □

Definition 3.3: A k-ary $f \in O_S$ such that $[f]$ is a minimal clone while every nontrivial $g \in [f]$ is of arity at least k is called *minimal operation.*

Clearly, every minimal clone is generated by a minimal operation (since, by Lemma 3.1, every nontrivial operation in a minimal clone C generates C). Thus, the set of all minimal operations determines the set of all minimal clones. This is useful, because we will see later that minimal operations can only be of certain types. The following lemma gives another technique to show that an operation is not minimal.

Lemma 3.4: *Let f be a minimal operation on S. If f preserves a subset $A \subseteq S$, then $f|_{A^n}$ (the operation obtained by restricting f to the set A^n) is either trivial or minimal on A. In particular, for a conservative minimal operation f, the operation $f|_{A^n}$ is trivial or minimal on A for all $A \subseteq S$.*

Proof: This can be deduced from the fact that composing functions and restricting functions commute. Another proof can be found in [19]. □

Definition 3.5: a) A ternary operation f on S is called a *majority operation* if

$$f(x,x,y) \approx f(x,y,x) \approx f(y,x,x) \approx x.$$

b) A ternary operation f on S is called a *minority operation* if

$$f(x,x,y) \approx f(x,y,x) \approx f(y,x,x) \approx y.$$

c) A k-ary operation f is called a *semiprojection* if, for a fixed $i \in \{1,..,k\}$,

$f(x_1,...,x_k) = x_i$ whenever $x_1,...,x_k \in S$ are not pairwise distinct.

Lemma 3.6 (The Swierczkowski Lemma): [20] *Given an at least quaternary operation f, if every operation arising from f by identification of two variables is a projection, then f is a semiprojection.*

Proof: This proof is sketched in [16]. Other proofs can be found in [20], [4]. Let $k \geq 4$ and let $f(x_1,...,x_k)$ be a k-ary operation. We distinguish two cases:

Case 1: For some $i \in \{1,...,k\}$, if we equate all but the i-th variable, then $f(\underbrace{y,...y}_{i-1},x_i,\underbrace{y,...,y}_{k-i}) \approx x_i$. Without loss of generality, we can assume $i=1$ so that $f(x_1,y,...,y) \approx x_1$. But now, equating any two of $x_2,...,x_k$ yields x_1. Furthermore, this implies $f(x_1,x_2,y,y,x_5,...,x_k) \approx x_1$ so that $f(x_1,x_1,y,y,x_5,...,x_k) \approx x_1$. Now set $x_1 = x_2$ and we must have $f(x_1,x_1,x_3,...,x_k) \approx x-1$. $f(x_1,x_2,x_1,x_4,...,x_k) \approx f(x_1,x_2,x_3,x_1,x_5...,x_k) \approx ... \approx f(x_1,x_2,x_3,x_4,...,x_{k-1},x_1) \approx x_1$ follows in the same way. Thus, f is a semiprojection since it coincides with e_1^k whenever $x_1,...,x_k$ are not pairwise distinct.

Case 2: $f(x_1,y,y,...,y) \approx f(y,x_2,y,...,y) \approx ... \approx f(y,...,y,x_k) \approx y$. But then, for any $i \neq j$, setting $x_i = x_j$ forces f to reduce to $x_i = x_j$ (since if f would reduce to x_l, $l \notin \{i,j\}$, we could equate all but the l-th variable to obtain $f(y,...,y,x_l,y...,y) \approx x_l$). However, this leaves no variable for $f(x,x,y,y,x_5,...,x_k)$ to equate to. Thus, this case cannot occur. □

Definition 3.7 A ternary operation f on S is *Mal'tsev* if

$$f(x,y,y) \approx f(y,y,x) \approx x$$

Lemma 3.8: *Let f be a Mal'tsev operation on S. If $[f]$ is a minimal clone, then there exists a prime p and an elementary Abelian p-group $\langle S;+\rangle$ such that $f(x,y,z) \approx x-y+z$.*

Proof: See [21]. □

We are now ready to prove a very important result about minimal operations, namely Rosenberg's Classification Theorem [19].

Theorem 3.9 Rosenberg's Classification Theorem (RCT): *Every minimal operation f on S is of one of the following types:*

(1) a unary operation that is either a retraction (i.e. $f^2 = f$) or a cyclic permutation of prime order, — explain

(2) a binary idempotent operation,

(3) a majority operation,

(4) the minority operation $f(x, y, z) = x + y + z$ where $\langle S; + \rangle$ is an elementary 2-group,

(5) a k-ary semiprojection for some $3 \leq k \leq n$.

Proof: Let f be a minimal operation.

If f is unary but neither a retraction or a permutation, then we have $f^2(S) \subsetneq f(S)$ which implies $f \notin [f^2]$, hence f is not a minimal operation. Let f be a permutation of order q (i.e. $f^q = e_1^1$ while $f^i \neq e_1^1$ for $i = 1, ..., q-1$). Suppose that q is not prime and let $q = rs$ where r is a prime divisor of q. Let $h := f^s$. Now $h \in [f]$ but $f \notin [h]$, i.e. $[f]$ is not minimal. Thus, f is permutation of prime order or a retraction.

Now suppose that f is at least binary. f has to be idempotent since otherwise it generates (by identification of variables) the nonidentical unary operation $f(x, ..., x)$, which cannot generate f.

Now let us suppose that f is ternary. Since f is a minimal operation and hence of minimal arity, we must obtain a projection by any identification of two variables. This leaves us with eight possible cases:

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$f(x,x,y) =$	x	x	x	x	y	y	y	y
$f(x,y,x) =$	x	x	y	y	x	x	y	y
$f(y,x,x) =$	x	y	x	y	x	y	x	y

In the cases (1) and (8), f is a majority, resp., minority operation. In the cases (2), (3) and (5), f is a ternary semiprojection. To show that the cases (4), (6) and (7) cannot occur we define

$$f_4(x, y, z) := f(x, y, f(x, y, z))$$
$$f_6(x, y, z) := f(x, f(x, y, z), z)$$
$$f_7(x, y, z) := f(f(x, y, z), y, z)$$

Clearly, $f_4, f_6, f_7 \in [f]$. Furthermore, f_4, f_6, f_7 are majority operations in the cases (4),(6),(7), respectively. But now $f \notin [f_4], [f_5], [f_6]$, respectively, because we will see later that any nontrivial ternary operation generated by a majority operation has to be a majority operation again (Lemma 4.4). This contradicts the minimality of f.
It remains to show that if f is a minority operation, then it is necessarily $x+y+z$ in an elementary 2-group. Since every minority operation is Mal'tsev, we can apply Lemma 3.8. Thus, there exists an elementary Abelian p-group $\langle S; +\rangle$ (p prime) with $f(x,y,z) = x-y+z$. But $x-y+z$ is a minority operation only for $p=2$ (otherwise, we have $x, y \in S$ such that $0 \neq 2y-2x$ and hence $x \neq 2y-x = f(y,x,y)$).
Finally, let f be at least quaternary. By the same argument as above, f turns into a projection by identification of any two variables. Then, by the Swierczkowski Lemma, these projection have to coincide. Thus, f is a semiprojection. □

Note that any operation that falls under the cases (1)-(5) (even if it is not minimal) can only generate nontrivial operations of equal or greater arity. This is implied by the fact that the operations listed in the RCT become trivial by all identification of variables (except case (1) in which the claim is trivial). In particular, the nontrivial operations generated by a majority operation are at least ternary (a fact we use several times in the sequel).

It can be shown that the classes (1) - (5) of minimal operations are disjoint. In the cases (1) and (4) the conditions ensure the minimality of f, while in the other cases they do not. However, one can find examples to show that the classes (2), (3) and (5) are also non-empty:

- For (2) consider any semilattice operation (e.g. $max(x,y)$).
- For (4), consider the dual discriminator d of Fried and Pixley [9], defined by
$$d(x,y,z) = \begin{cases} x, & \text{if } x = y \\ z, & \text{if } x \neq y \end{cases}$$
(see 4.6 for a proof of the minimality).
- For (5), consider any nearprojection s (a nearprojection is a semiprojection defined by $s(x_1, x_2, ..., x_n) = x_1$ if $x_1, ..., x_n$ are not pairwise distinct).

This means that in order to determine all minimal clones on an n-element set S it is enough to look at the operations of cases (1) to (5) and to determine which of them generate minimal clones. This is trivial for the case (1) and (4), but for the other cases, only partial results are known:

- For $n = 2$, all the minimal clones are known since - as mentioned before - the clone lattice was completely described by Post [17].

- For $n = 3$, the set of minimal clones was determined by Csákány [4].
- For $n = 4$, the binary case was settled by Szczepara in his Ph.D. thesis [22] (there are 2.182 binary minimal clones and 120 of them are essentially distinct) and all minimal clones generated by majority operations were determined by Waldhauser [23], see next chapter). Furthermore, some minimal conservative semiprojections were determined by Ježek and Quackenbush [12].

For $n \geq 5$ only very few results are known.

We now have the knowledge to answer the questions raised at the beginning of the chapter:

Corollary 3.10: *There are only finitely many minimal clones on S.*

Proof: The cases (1)-(4) of the RCT can only give us finitely many minimal operations (and thus only finitely many minimal clones). For case (5), consider a k-ary semiprojection s. If $k > n$, the number of variables exceeds the number of elements in S and hence the variables cannot be distinct and s is a projection. Thus, it cannot generate a minimal clone. But, for $k \leq n$, the set of k-ary semiprojections is clearly finite. Thus, the set of minimal clones on S is finite. □

Corollary 3.11: *Every nontrivial clone on S contains a minimal clone.*

Proof: Following the approach in [12], we call a nontrivial clone B *k-special* if every nontrivial operation in B is at least k-ary and there exists a k-ary operation in B that generates B. Any nontrivial clone C on a finite set contains k-special clones for some k (any nontrivial operation of minimal arity in C generates such a clone). Let C' be a k-special subclone of C. Since the $(k-1)$-ary operations in C' have to be trivial, any identification of variables of a k-ary operation in C' gives us a projection. Hence, by the Swierczkowski Lemma, the k-ary operations in C' are semiprojections or $k < 4$. This implies that k cannot exceed $max(4, n)$, because otherwise any k-ary operation in this clone would be a projection. This means that C contains only finitely many special clones. They are partially ordered by inclusion and a clone which is minimal with respect to this ordering is a minimal clone. □

Note that Corollary 3.10 and 3.11 do not hold for infinite sets. In this case, it is obvious that there are infinitely many minimal clones (alone the number of all retractions on an infinite set is infinite). Furthermore, an example for a clone not containing a minimal clone is the clone generated by the (unary) successor function on $\mathbb{N}$.

Chapter 4

Minimal majority clones

In this chapter, we focus on the minimal clones on the n-element set $S = \{0, ..., n-1\}$ of case (3) in the RCT, i.e. the clones generated by majority operations. For the sake of brevity, we call clones that contain a majority operation *majority clones* in the sequel. Note that a minimal clone that contains a majority operation has to be generated by this operation by Lemma 3.1. First, we follow Csákány's work [4] to determine all minimal majority clones for $n = 3$. Then, we follow another paper of Csákány [5] to determine all conservative minimal majority clones for any $n \in \mathbb{N}$. We call a clone C conservative if all operations in C are conservative. Finally, we follow Waldhauser's approach [23] to determine all minimal majority clones for $n = 4$. However, before we do so, we have to observe some facts that will be useful in the sequel.

Recall that in order to prove the non-minimality of a clone $[f]$ it is sufficient to find an operation $g \in [f]$ such that $f \notin [g]$. To create $g \in [f]$ we will sometimes give terms t and apply them to f.

Definition 4.1: Let f_1 and f_2 be k-ary resp. l-ary operations on S such that $f_1 \in [f_2]$. Then there exists a k-ary term (i.e. polynomial symbol) t of type $\langle l \rangle$ such that f_1 is the result of substituting f_2 for the l-ary operation symbol in t, in sign: $f_1 = t(f_2)$. In this case we say that we *apply* t *to* f_2. The result of successive application of two terms t_1, t_2 to an operation f is denoted by $t_2t_1(f)$. Furthermore, denote by $t^i(f)$ the i-times successive application of t to f.

When we give terms t of type $\langle 3 \rangle$ we may omit the sign of the ternary operation symbol. For example, we write $t = (x, y, (xyz))$ and for a 3-ary operation f we obtain $t(f) = f(x, y, f(x, y, z))$ and $t^2(f) = f(x, y, f(x, y, f(x, y, f(x, y, z))))$.

Definition 4.2: Let f be a k-ary operation on S. Let ϕ_1 be a permutation on S and let phi_2 be a permutation on $\{1, ..., k\}$. Set

$$f^{\phi_1}(x_1, ..., x_k) := \phi_1(f(\phi_1^{-1}(x_1), ..., \phi_1^{-1}(x_k))) \qquad f_{\phi_2}(x_1, ..., x_k) := f(x_{\phi_2(1)}, ...x_{\phi_2(k)}).$$

Call f^{ϕ_1} an *isomer of* f and f_{ϕ_2} a *permutation of* f. For $F \subseteq O_S$, extend definition by setting $F^{\phi_1} = \{f^{\phi_1} | f \in F\}$. The map $f \mapsto f^{\phi_1}$ carries each clone C onto the clone we denote by C^{ϕ_1}. The algebra $\langle S; F^{\phi_1} \rangle$ is the *isomorphic image of* $\langle S; F \rangle$ *under* ϕ_1. Furthermore, note that $g \in [f]$ implies $g^{\phi_1} \in [f^{\phi_1}]$ and, since clones are invariant under permutation, $[f] = [f_{\phi_2}]$. Isomers and permutations of variables generate a permutation group T_k of order $3!k!$ on the set of all k-ary operations on S. Two operations are said to be *essentially distinct* if they have different arities, or belong to distinct orbits of T_k (where k is the arity of the two operations). Two clones C_1, C_2 are said to be *essentially distinct* if $C_1 \neq C_2^{\phi}$ for all nontrivial permutations ϕ on S.

Definition 4.3: Let $k \geq 3$. A k-ary operation f is called a *near-unamity operation* if

$$f(y, x, ..., x) \approx f(x, y, x, ..., x) \approx ... \approx f(x, ..., x, y) \approx x.$$

Lemma 4.4: *If* m *is a majority operation on* S*, then any nontrivial* $f \in [m]$ *is a near-unamity operation.*

Proof: [5] We call a term a term *regular* if it is nontrivial and no occurrence of the operation symbol in it has two graphically equal arguments. As mentioned above, every nontrivial $f \in [m]$ can be expressed as $t(m)$ where t is a term generated by a ternary operation symbol g. We prove the claim by induction on the length $l(t)$ of t (i.e. the number of occurrences of g in t) For $l(t) = 1$, a nontrivial f clearly equals $t(m) = m_\phi$ where ϕ is a permutation on $\{1, 2, 3\}$ and f is a near-unamity operation. Now suppose $k \geq 1$ and assume that the claim is true for all regular terms of length at most k. Let a nontrivial p-ary f equal $t(m)$ where $l(t) = k + 1$. We can express $t(m)$ as $m(t_1(m), t_2(m), t_3(m))$ where $l(t_i) \leq k$ $(i = 1, 2, 3)$. By the way of contraposition, suppose that f is not a near-unamity operation. Then there exists $1 \leq r \leq p$ such that

$$f(\underbrace{x, ..., x}_{r-1}, y, \underbrace{x, ..., x}_{p-r}) = y.$$

Now, by the induction hypothesis, each $t_i(m)$ is either trivial or a near-unamity operation. In the latter case we have $t_i(m)(x, ..., x, y, x, ..., x) = x$ and this can happen only for at most one $i \in \{1, 2, 3\}$. Were all the $t_i(m)$ trivial, then $t(m)$ would be of length 1 and $f(x, ..., x, y, x, ..., x) = x$. Thus, exactly for one i, say $i = 3$, we get $t_3(m)(x, ..., x, y, x, ..., x) = x$ while $t_1(m)$ and $t_2(m)$ are trivial, say $t_1(m) = e^p_{j_1}$ and $t_2(m) = e^p_{j_2}$. Were $j_1 \neq j_2$ then we had $t_i(m)(x, ..., x, y, x, ..., x) = x$ for at least one $i \in \{1, 2\}$, say $i = 1$, which would implies $f(x, ..., x, y, x, ..., x) = m(x, x_{j_2}, x) = x$. Thus, $j_1 = j_2$ and $f(x, ..., x, y, x..., x) = m(x_{j_1}, x_{j_1}, x)$ which makes f trivial. □

Corollary 4.5: *Let m be a majority operation on S. Any minimal subclone of $[m]$ is again generated by a majority operation.*

Proof: Let $f \in [m]$ be a nontrivial, k-ary operation such that $[f]$ is a minimal subclone of $[m]$. Since f is generated by m, it has to be at least ternary. From the RCT we can conclude that f is therefore a majority operation, a semiprojection or $x + y + z$ in a boolean group. Furthermore, Lemma 4.4 implies that f is a near-unamity operation. A semiprojection that becomes e^k_i by any identification of variables maps $(\underbrace{x, ..., x}_{i-1}, y, \underbrace{x..., x}_{k-i})$ to y which implies that it cannot be a near-unamity-operation. Similarly, $x+y+z$ in a boolean group cannot be a near-unamity operation since it maps (x, x, y) to y for all $x, y \in S$. Thus, f is a majority operation, as required. □

This means that in order to prove the minimality of clone C generated by a majority operation, it suffices to show that any two majority operations in C generate each other, because if C has a proper minimal subclone then this must be generated by a majority operation.

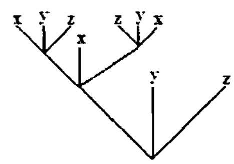
Recall that we have already seen a minimal majority clone in the last chapter. The clone generated by the dual discriminator d of Fried and Pixley (recall that d is defined by $d(x, y, z) = x$ if $x = y$ and $d(x, y, z) = z$ if $x \neq y$). The minimality of d can be seen at follows: d is a homogeneous operation (that is, an operation that preserves all permutations), so the same must be true for any nontrivial majority operation $f \in [d]$. Thus, f and d are both homogeneous majority operations and hence have to coincide up to ordering of variables (see [3]). Thus, $d \in [f]$. Here, we present a different proof, not relying on the result in [3].

Theorem 4.6: *The dual discriminator d of Fried and Pixley is minimal.*

Proof: Let f be a nontrivial ternary operation in $[d]$. We have to show $d \in [f]$. By Lemma 4.4, f is a near-unamity and hence a majority operation. Clearly, f can be represented by a rooted ternary tree whose leaves are labelled by the variables hence have to coincide up to ordering of variables (see [3]). Thus, $d \in [f]$. Here, we present a different proof, not relying on the result in [3].

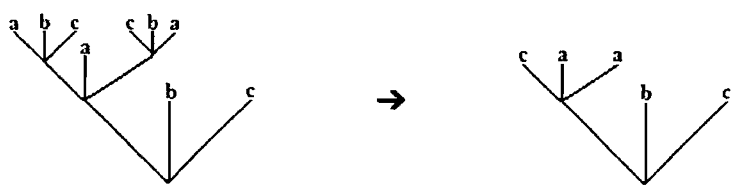
Theorem 4.6: *The dual discriminator d of Fried and Pixley is minimal.*

Proof: Let $f \in [d]$ be a majority operation. We have to show $d \in [f]$. Clearly, f can be represented by a rooted ternary tree whose leaves are labelled by the variables x, y, z; e.g.

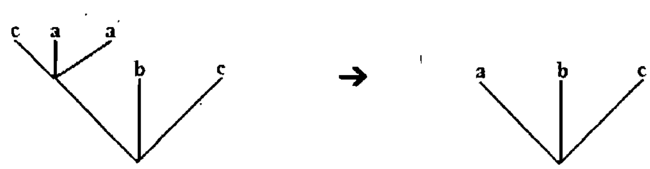


represents the operation defined by $f(x, y, z) := d(d(d(x, y, z), x, d(z, y, x)), y, z)$.

Set $x = a$, $y = b$, $z = c$. Take a subtree with three leaves. If two leaves have the same value, say a, we can replace this subtree by a leaf with a. If the three leaves have all three variables, then we can replace the subtrees by a leaf carrying the value of the rightmost leave. For the above tree, we get



We can repeat the procedure until we obtain a result for $f(a, b, c)$.



Clearly, the result ($f(a, b, c) = c$ in our example) is independent of the ordered triplet (a, b, c) of distinct elements of S and thus f is one of the three operations obtained from d by coordinate exchange. Thus, $d \in [f]$. □

Definition 4.7: Let f be a majority operation on S. Set

$$\omega := \{(a, b, c) \in S^3 \mid a \neq b \neq c \neq a\} \text{ and } \iota := S^3 \setminus \omega.$$

Call the set $f(\omega)$ the *range* of f.

Lemma 4.8: *Let f be a majority operation and let $g \in [f]$ be a nontrivial ternary operation. Then $g(\omega) \subseteq f(\omega)$*

Proof: [4] Suppose to the contrary that there exists $a \in g(\omega) \setminus f(\omega)$. Now $a = g(u)$ for some $u \in \omega$. As $g \in [f]$, we have $g = t(f)$ where t is a term of minimal length. Clearly, $g = f(g_1, g_2, g_3)$ where $g_i = t_i(f) \in [f]$ is ternary such that the length of t_i is less than k ($i = 1, 2, 3$, $k \in \mathbb{N}$). Now

$$a = g(u) = f(g_1(u), g_2(u), g_3(u)).$$

As $a \notin f(\omega)$, we must have $(g_1(u), g_2(u), g_3(u)) \in \iota$ and hence two of the coordinates are equal, say $g_1(u) = g_2(u)$. Then $a = g_1(u)$ since f is a majority operation. Now $g_1 = t_1(f)$ with $l(t_1) < k$. Continuing this argument we arrive at $a = g_i(u)$ where $g_i = t_i(f)$ with $l(t_1) = 1$. Now $t_i(f) = f_\phi$ where ϕ is a permutation on $\{1, 2, 3\}$. This implies $a \in f(w)$, a contradiction. □

This lemma gives us a minimal majority operation on S. Let $c \in S$ and define the majority operation m_c by setting $m_c(x, y, z) = c$ for all $(x, y, z) \in \omega$. By Corollary 4.5, we know that, in order the prove the minimality of m_c, it is sufficient to show that any majority operation $g \in [m_c]$ generates m_c. By Lemma 4.8, $g(\omega) \subseteq m_c(\omega) = \{c\}$ and hence $g = m_c$. Thus, $[m_c]$ is minimal and another example for the non-emptiness of class (3) of the RCT.

We can formulate an even stronger result if f is minimal.

Corollary 4.9: *Let f be a majority operation and let $g \in [f]$ be a nontrivial ternary operation. Then $g(\omega) = f(\omega)$.*

Proof: $g(\omega) \subseteq f(\omega)$ by Lemma 4.8. By the minimality of f, we also have $f \in [g]$ and hence $f(\omega) \subseteq g(\omega)$. □

4.1 Minimal majority clones on a two-element set

In the case $S = \{0, 1\}$ the values of a majority operation are all uniquely determined by the majority property. Thus, we have only one majority operation. It is the dual discriminator d of Fried and Pixley [9] that we have already seen as an example for the non-emptiness of the corresponding class in the RCT. It is therefore the only minimal majority operation on S. Note that it coincides with the operations m_c, $c \in S$, that we mentioned after Lemma 4.8. Thus, $[d]$ is the only minimal majority clone on S.

$$d(x,y,z) = \begin{cases} x, & \text{if } x = y \\ z, & \text{if } x \neq y \end{cases}$$

4.2 Minimal majority clones on a three-element set

In the following, fix $n = 3$ and denote by S the set $\{0, 1, 2\}$. If not stated otherwise, all results presented in this and the following section 4.3 are due to Béla Csákány [4],[5].

Definition 4.10: Let f be a majority operation on S. Call the number $\mu(f)$ defined by

$$\mu(f) = 3^5 f(0,1,2) + 3^4 f(0,2,1) + 3^3 f(1,0,2) + 3^2 f(1,2,0) + 3f(2,0,1) + f(2,1,0)$$

the *mantissa* of f.

Note that a majority operation is uniquely determined by its values on ω which are, in turn, uniquely determined by $\mu(f)$. Thus, a majority operation f is uniquely determined by $\mu(f)$. From now on, we denote by m_i the majority operation on S with the mantissa i (e.g. m_{364} is the operation whose value is always 1 on ω).

To determine all minimal majority operations f, we split the problem in two cases:

- Case 1: $|f(\omega)| = 3$

We have already seen a minimal operation with that property in the last chapter: The dual discriminator d of Fried and Pixley [9]. Note that, using our notation above, $d = m_{624}$. Now we show that, up to permutation of variables, any minimal majority operation f with $|f(\omega)| = 3$ is, in fact, d.

Lemma 4.11: *Let f be a majority operation on S with range S. The binary relations preserved by f are p-rectangular relations (a relation σ is* p-rectangular *if for every pair $\binom{i}{j} \notin \sigma$ the set $\{\binom{x}{y} \in \sigma \mid x = i$ or $y = j\}$ has at most two elements).*

Proof: Suppose to the contrary that there is a binary relation σ that is not p-rectangular and preserved by f. Without loss of generality (interchanging the two coordinates of σ if necessary), we can assume $\binom{i}{j} \notin \sigma$ but $\binom{i}{a}, \binom{i}{b}, \binom{x}{j} \in \sigma$ for some $a, b, x \in S$ where $j \neq a \neq b \neq j$, $x \neq i$. We assumed $|f(\omega)| = 3$, so we have $j \in f(\omega)$, and we may permute the variables of f so that $f(a, b, j) = j$. But then, we have $\binom{i}{j} = \binom{f(i,i,x)}{f(a,b,j)} \in \sigma$, a contradiction. □

Lemma 4.12: *The dual discriminator d of Fried and Pixley defined on an arbitrary set A preserves any p-rectangular relation on A.*

Proof: Suppose that d does not preserve a p-rectangular σ relation on A. This means that there exist $\binom{x_1}{y_1}, \binom{x_2}{y_2}, \binom{x_3}{y_3} \in \sigma$ such that $\binom{d(x_1,x_2,x_3)}{d(y_1,y_2,y_3)} \notin \sigma$. We have four cases:

(1) $d(x_1,x_2,x_3) = x_3$ and $d(y_1,y_2,y_3) = y_3$
(2) $d(x_1,x_2,x_3) = x_1 = x_2$ and $d(y_1,y_2,y_3) = y_3$
(3) $d(x_1,x_2,x_3) = x_3$ and $d(y_1,y_2,y_3) = y_1 = y_2$
(4) $d(x_1,x_2,x_3) = x_1 = x_2$ and $d(y_1,y_2,y_3) = y_1 = y_2$

Case (1) is a contradiction, because $\binom{x_3}{y_3} \in \sigma$ by the assumption. In case (2) we must have $y_1 \neq y_2$ and hence $\binom{x_1}{y_3} \notin \sigma$ but $\binom{x_1}{y_1}, \binom{x_1}{y_2}, \binom{x_3}{y_3} \in \sigma$, a contradiction to the p-rectangularity of σ. The same argument contradicts case (3) and, finally, case (4) is contradicted by the assumption $\binom{x_1}{y_1} \in \sigma$. □

Combining the two results, we obtain that d preserves all binary relations preserved by f. We use this fact to apply the Baker-Pixley Theorem [1]:

Theorem 4.13 (The Baker Pixley Theorem): *If C is a majority clone on a set A, then every operation that preserves the set of all binary relations on A that are preserved by C is in C.*

Proof: We prove this theorem when we restate it in chapter 5 (see 5.4). For now, the proof may be omitted. □

If we apply the theorem to the clone $[f]$, it follows that d is in $[f]$. Since we assumed that f is a minimal operation, we also have $f \in [d]$. This implies $[f] = [d]$ and, since we have already seen that $[d]$ is a minimal clone, we can conclude the following theorem:

Theorem 4.14: *All minimal majority operations f on S with $|f(\omega)| = 3$ generate the same clone $[d]$ ($d = m_{624}$).*

- Case 2: $|f(\omega)| < 3$

There are $3 * 2^6 - 3 = 189$ majority operations with that property and they belong to 10 distinct orbits of T_3 (see Definition 4.2). The following table gives each orbit represented by the operation having the least index (the number of operations in the respective orbit is given by the number in brackets)

Table 4.1

m_0 (3)	m_{13} (18)	m_{109} (6)
m_1 (36)	m_{28} (36)	m_{120} (18)
m_4 (18)	m_{39} (18)	
m_{10} (18)	m_{85} (18)	

Now, we prove that all minimal clones generated by majority operations f with $|f(\omega)| < 3$ are generated by the three operations in the orbit of m_0, namely m_0, $m_{364}(= (m_0)^{(01)})$, $m_{728}(= (m_0)^{(02)})$ and the six operations in the orbit of m_{109}. However, to determine all distinct minimal clones among them, it is enough to look at three operations in the orbit of m_{109}, namely m_{109}, $m_{473}(= (m_{109})^{(02)})$ and $m_{510}(= (m_{109})^{(12)})$, since the other operations in the orbit of m_{109} can be obtained from the listed ones by permutation of variables and hence do not generate further clones.

Theorem 4.15: *The clones $[m_0]$, $[m_{364}]$, $[m_{728}]$, $[m_{109}]$, $[m_{473}]$, $[m_{510}]$ on S are exactly the clones generated by minimal majority operations f with $|f(\omega)| < 3$.*

Table 4.2

(x, y, z)	$m_0(x, y, z)$	$m_{364}(x, y, z)$	$m_{728}(x, y, z)$	$m_{109}(x, y, z)$	$m_{473}(x, y, z)$	m_{510}(x,y,z)
$(0, 1, 2)$	0	1	2	0	1	2
$(1, 2, 0)$	0	1	2	0	1	2
$(2, 0, 1)$	0	1	2	0	1	2
$(1, 0, 2)$	0	1	2	1	2	0
$(0, 2, 1)$	0	1	2	1	2	0
$(2, 1, 0)$	0	1	2	1	2	0

To prove the theorem, we first prove a simple but useful observation.

Lemma 4.16: *Let $H \subsetneq G \subsetneq I$ where I is the set of all nontrivial special operations (i.e. the operations of the classes (1)-(5) in the RCT). Suppose that the following conditions hold:*

(I) $[g] \cap I \subseteq G$ for all $g \in G$

(II) $[g] \cap H \neq \emptyset$ for all $g \in G$

(III) $h_1, h_2 \in H$, $h_1 \neq h_2 \Rightarrow h_1 \notin [h_2]$

Then the clones generated by the operations in H are exactly the set of pairwise distinct minimal clones generated by $g \in G$.

Proof: By $H \subseteq G$, $\{[h] | h \in H\} \subseteq \{[g] | g \in G\}$. The distinctness of the minimal clones generated by the operations in H follows trivially from (III). It remains to show that $[h]$ is minimal for all $h \in H$. Let us suppose that there exists a $h \in H$ such that $[h]$ is not minimal. Then, $[h]$ has to contain a minimal clone C s.t. $C \subsetneq [h]$. This implies that there has to exist a special operation $f \in I$ such that $[f] \subsetneq [h]$. By $H \subseteq G$, $h \in G$ and by (I), h does not generate a special operation in $I \setminus G$. It follows $f \in G$. But now (II) makes sure that there exists $h' \in H$ such that $h' \in [f] \cap H \subsetneq [h]$. It follows $h' \neq h$ and $h' \in [h]$, a contradiction to (III). □

We use this lemma to prove our claim. This means that we have to show (I)-(III) for the set G containing all 189 majority operations f with $|f(\omega)| < 3$ and the set $H = \{m_i \mid i = 0, 364, 728, 109, 473, 510\}$. We conclude (I) with the help of two Lemmas we have already proved:

Let $f \in [g] \cap I$ where $g \in G$. Since f is generated by g, it has to be at least ternary. Since f is also in I, we can conclude that f is exactly ternary, because semi-projections of arity greater than 3 are trivial on a three-element set. But now, by Lemma 4.4, any nontrivial ternary operation that is generated by a majority operation is again a majority operation. Thus, f is a majority operation. It also follows by Lemma 4.8 that we have $|f(\omega)| < 3$ and hence $f \in G$. This proves (I).

Now, we prove (II). Recall that we have noted $g \in [f] \Rightarrow g^\phi \in [f^\phi]$ at the beginning of this chapter. In the virtue of this observation it is enough to show that, for each operation m_i in Table 4.1, there is an $m_j \in H$ such that $m_j \in [m_i]$. Set

$$\tilde{f} := f(f, f_{(123)}, f_{(132)}) = f(f(x,y,z), f(y,z,x), f(z,x,y)).$$

It is easy to see that if two of the three values $f(a,b,c)$, $f(b,c,a)$, $f(c,a,b)$ equal d, then $\tilde{f}(a,b,c) = \tilde{f}(b,c,a) = \tilde{f}(c,a,b) = d$. This implies

$$m_0 = \tilde{m}_1 = \tilde{m}_4 = \tilde{m}_{10} = \tilde{m}_{120} \Rightarrow m_0 \in [m_1], [m_4], [m_{10}], [m_{120}]$$

and

$$m_{109} = (\tilde{m}_{13})_{(12)} = \tilde{m}_{28} = (\tilde{m}_{39})_{(12)} = \tilde{m}_{85} \Rightarrow m_{109} \in [m_{13}], [m_{28}], [m_{39}], [m_{85}],$$

as required.

Finally, to prove (III), it remains to show that none of the operations in H is contained in the clone generated by another operation in H. There are $6 * 5 = 30$ pairs to be checked. However, we can reduce the problem by noting that the unique nontrivial permutation on S preserved by m_{109} and m_{728} is (01), that preserved by m_{364} and m_{510} is (02) and that preserved by m_0 and m_{473} is (12). Hence, no operation of one of these three pairs is contained in the clone generated by an operation of another pair. Furthermore, we can use Table 4.2 to check that the ranges of m_{109} and m_{728}, m_{364} and m_{510}, m_0 and m_{473} have no common entry, respectively. Thus, by Lemma 4.8, each operation of one of the three pairs is not contained in the clone generated by the other operation of the pair in question. This proves (III).

We apply Lemma 4.16 and obtain the claim of Theorem 4.15. Combined with the result for case 1 (stated in Theorem 4.14), we can summarize the minimal majority clones on the three-element set S in the table below. The clone standing at the meet of the row starting with $[f]$ and the column marked by the permutation ϕ is $[f]^\phi$. The place of $[f]^\phi$ is empty if $[f]^\phi$ is equal to some clone appearing earlier in the table. We have seven minimal majority clones, three of which are essentially distinct.

Table 4.3
The minimal majority clones on $\{0, 1, 2\}$

	(01)	(02)	(12)
$[m_0]$	$[m_{364}]$	$[m_{728}]$	
$[m_{109}]$		$[m_{473}]$	$[m_{510}]$
$[m_{624}](= [d])$			

Note that this table gives us all distinct minimal majority clones on S but not all minimal majority operations on this set. To determine all minimal majority operations, we need another lemma.

Lemma 4.17: *Set*

$M_0 := \{m_0\}$,

$M_{109} := \{m_{109}, m_{255}, m_{325}, m_{39}, m_{253}, m_{327}, m_{111}, m_{37}\}$,

$M_{624} := \{m_{44}, m_{424}, m_{624}\}$.

For a clone C denote by $C*$ the set of majority operations from C. Then

a) $[m_0]* = M_0$,

b) $[m_{109}]* = M_{109}$.

c) $[m_{624}]* = M_{624}$,

Proof: a) The range of m_0 consists of 0 only, hence, by Lemma 4.8, the same holds for each nontrivial ternary operation in $[m_0]$. Thus, the set of nontrivial ternary operations in $[m_0]$ is exactly $\{m_0\} = M_0$.

b) One one hand, $\{m_{109}, m_{255}, m_{325}, m_{39}, m_{253}, m_{327}, m_{111}, m_{37}\} = M_{109} \subseteq [m_{109}]$ since we have $m_{255} = (m_{109})_{(12)}$, $m_{325} = \tilde{m_{109}}$, $m_{39} = (m_{325})_{(12)}$, $m_{253} = (m_{39})_{(23)}$, $m_{327} = (m_{39})_{(132)}$, $m_{111} = (m_{39})_{(13)}$, $m_{37} = (m_{39})_{(23)}$. On the other hand, the range of m_{109} is $\{0, 1\}$. Since these operations are all minimal, we can apply Corollary 4.9 and we obtain that the range of every nontrivial ternary operation in $[m_{109}]$ is also $\{0, 1\}$. At the same time, the operations in $[m_{109}]$ have to share the property of m_{109} to be invariant under the transposition (01) of S. One can check that under all 189 majority operations on S, only the majority operations in M_{109} have these two properties. Thus, $[m_{109}]* = M_{109}$.
c) $\{m_{44}, m_{424}, m_{624}\} = M_{624} \subseteq [m_{624}]$ since we have $m_{44} = (m_{624})_{(13)}$ and $m_{424} = (m_{624})_{(23)}$. Furthermore, by the proof of Theorem 4.6, any majority operation in $[m_{624}]$ is one of the three operations obtained from m_{624} by coordinate exchange and this are exactly the operations in M_{624}. □

It follows directly that we have

$[m_{364}]* = \{m^{(01)} | m \in M_0\} = \{m_{364}\}$,
$[m_{728}]* = \{m^{(02)} | m \in M_0\} = \{m_{728}\}$,
$[m_{473}]* = \{m^{(02)} | m \in M_{109}\} = \{m_{437}, m_{619}, m_{481}, m_{611}, m_{457}, m_{643}, m_{635}, m_{449}\}$,
$[m_{510}]* = \{m^{(12)} | m \in M_{109}\} = \{m_{510}, m_{218}, m_{666}, m_{62}, m_{182}, m_{702}, m_{546}, m_{26}\}$.

Analog to the notation above, denote these sets by M_{364}, M_{728}, M_{473}, M_{510}, respectively. So the clones $[m_0]$, $[m_{364}]$ and $[m_{728}]$ contain only one (minimal) majority operation each, while the clones $[m_{109}]$, $[m_{473}]$ and $[m_{510}]$ contain eight (minimal) majority operations each. $[m_{624}]$ contains three (minimal) majority operations. This means that there are $3 * 1 + 3 * 8 + 3 = 30$ distinct minimal majority operations. We can now list the minimal majority operations of the three essentially distinct minimal clone $[m_0]$, $[m_{510}]$, $[m_{44}]$ (the reason for choosing $[m_{510}]$ instead of $[m_{109}]$ as a representative will be seen later).

Table 4.4
The minimal majority operations on $\{0, 1, 2\}$ (up to isomorphism)

	$[m_0]$	$[m_{510}]$								$[m_{624}]$		
(x, y, z)	m_0	m_{510}	m_{546}	m_{26}	m_{666}	m_{218}	m_{182}	m_{702}	m_{62}	m_{624}	m_{44}	m_{424}
$(0, 1, 2)$	0	2	2	0	2	0	0	2	0	2	0	1
$(1, 2, 0)$	0	2	0	2	2	0	2	0	0	0	1	2
$(2, 0, 1)$	0	2	2	2	0	0	0	0	2	1	2	0
$(1, 0, 2)$	0	0	2	0	0	2	0	2	2	2	1	0
$(0, 2, 1)$	0	0	0	0	2	2	2	2	0	1	0	2
$(2, 1, 0)$	0	0	0	2	0	2	2	0	2	0	2	1

4.3 Conservative minimal majority clones

The operations we consider in this section may be defined on $S = \{0, ..., n-1\}$. Note that a clone C is conservative if and only if C is generated by a set of conservative operations (this follows because an operation on S is conservative if and only if it preserves all unary relations on S). Hence, all clones generated by conservative operations are conservative. In particular, in a minimal clone, either all or none of the nontrivial operations are conservative.

Note that a majority operation on S is necessarily conservative for $n \leq 3$. Thus, for $n \leq 3$, any minimal majority clone is conservative.

Definition 4.18: Denote the set of all k-element subsets of S by $P_k(S)$.

For a majority operation f on {0,1,2} we use the notation of the last section (see Definition 4.10 and 4.2). In this section, we state our results in terms of algebras (see chapter 3 for a justification). We call an algebra $\langle A, f\rangle$ minimal if its term algebra is minimal, which, as noted in chapter 3, is the case if and only if $[f]$ is a minimal clone. Furthermore, we will call $\langle A, f\rangle$ *majority algebra* if f is a majority operation.

Clearly, a k-ary conservative algebra $\langle S; f\rangle$ is uniquely determined by the set of its k-element subalgebras $\{\langle A; f|_{A^k}\rangle \mid A \in P_k(S)\}$.

Definition 4.19: Denote by M the set

$$M = M_0 \cup M_{624} \cup M_{510}$$
$$(M = \{m_i \mid i = 0, 44, 424, 624, 510, 218, 666, 62, 182, 702, 546, 26\}).$$

Denote by R_3 a set of representatives of isomorphism classes of all 3-ary algebras. Assume in the following that we have $\langle\{0,1,2\}, g\rangle \in R_3$ for all $g \in M$ (note that this implies $\langle\{0,1,2\}, g\rangle \notin R_3$ for all $g \in M_{364} \cup M_{728} \cup M_{109} \cup M_{473}$). For a 3-ary conservative operation f on S and an arbitrary $A \in P_3(S)$, there exists a unique 3-ary algebra $\langle\{0,1,2\}; g\rangle$ in R_3 such that $\langle A; f|_{A^3}\rangle \cong \langle\{0,1,2\}; g\rangle$. The set of these algebras $\langle\{0,1,2\}; g\rangle$ is called the *spectrum* of $\langle S; f\rangle$.

In the following, we use the spectrum to characterize conservative minimal algebras.

Lemma 4.20: *(1) Each subalgebra of a minimal algebra is either minimal or trivial.*

(2) The set of all 3-element minimal majority algebras is (up to isomorphism) $\{\langle S; m_i\rangle | m_i \in M\}$.

(3) An essentially k-ary algebra $\langle S; f\rangle$ *is minimal if and only if*

(a) For every nontrivial $g \in [f]$ *there exists an essentially k-ary* $g' \in [g]$

(b) $h \in [f]$ *and h essentially k-ary* $\Rightarrow f \in [h]$

(4) A conservative algebra $\langle S; f\rangle$ *is not minimal if there exists a nontrivial operation Let f be k-ary and* $g \in [f]$ *be l-ary. Let* $A, B \subseteq S$ *such that* $\langle A; g|_{A^l}\rangle \cong \langle B; g|_{B^l}\rangle$, *but* $\langle A; f|_{A^k}\rangle \not\cong \langle B; f|_{B^k}\rangle$

Proof: (1) This is Lemma 3.4 written in terms of algebras.

(2) We have proved in the last section, that the minimal majority clones on $\{0,1,2\}$ are exactly $[m_0], [m_{44}]$ and $[m_{510}]$ (up to isomorphism). Since $M = M_0 \cup M_{624} \cup M_{510}$, this together with Lemma 4.17 (and the remarks made shortly after) gives the claim.

(3) (a) and (b) are equivalent to the statement that any nontrivial $g \in [f]$ has to generate f. We have already noted that this is the case if and only if the clone $[f]$ is minimal (see Lemma 3.1).

(4) Note that $\langle A; g|_{A^l}\rangle$, $\langle B; g|_{B^l}\rangle$, $\langle A; f|_{A^k}\rangle$, $\langle B; f|_{B^k}\rangle$ are subalgebras of $\langle S; f\rangle$ resp. $\langle S; g\rangle$ since these two algebras are conservative. By Lemma 3.2, the second condition of the assumption implies $f \notin [g]$ and this, together with $g \in [f]$, gives the non-minimality of the algebra. □

We are now ready to prove the main result of this section.

Theorem 4.21: *Let* $n \geq 3$ *(recall* $S = \{0, ...n-1\}$*). A conservative majority algebra* $\langle S; m\rangle$ *is minimal if and only if*

$$Spec\langle S; m\rangle \subseteq \{\langle\{0,1,2\}; m_i\rangle | m_i \in M\}$$

which contains at most one algebra from each of $\{\langle\{0,1,2\}; m_i\rangle | m_i \in M_{624}\}$ *and* $\{\langle\{0,1,2\};\ m_i\rangle | m_i \in M_{510}\}$.

Proof: In this proof, we need the following five terms of type $\langle 3\rangle$ (we write (xyz) instead of $g(x,y,z)$, where g is the operation symbol in the term; see Definition 4.1)

$$p = ((xzy)(yzx)(zxy)),$$
$$q = ((xyz)zy),$$
$$r = (z(xyz)x),$$
$$s = (x(yzx)y),$$
$$u = (y(zyx)x).$$

"$\Leftarrow$" Assume that $Spec\langle S; m\rangle$ satisfies the conditions above. Then there exist sets R, U, V and operations $m_i \in M_0$, $m_j \in M_{624}$, $m_k \in M_{510}$ such that $P_3(S) = R \cup U \cup V$ and

$$\text{(I)} \quad \langle A; m|_{A^3}\rangle \cong \begin{cases} \langle\{0,1,2\}; m_i\rangle & \text{if } A \in R, \\ \langle\{0,1,2\}; m_j\rangle & \text{if } A \in U, \\ \langle\{0,1,2\}; m_k\rangle & \text{if } A \in V. \end{cases}$$

We use (3) of the Lemma above to prove that $\langle S; m\rangle$ is minimal.
First, we have to show that, for each nontrivial $g \in [m]$, there exists an essentially ternary operation in $[g]$: The nontrivial operations in $[g]$ are at least ternary (they are generated by m) and since $[g]$ has to contain a minimal clone (see Corollary 3.10) there has to exist an operation h such that $[h] \subseteq [g] \subseteq [m]$ is a minimal subclone of $[m]$. By Corollary 4.5, a minimal subclone of $[m]$ is a majority clone. Thus, h is a ternary (majority) operation.
Now consider an arbitrary nontrivial ternary operation $m' \in [m]$. Then there are $m_{i'} \in M_0$, $m_{j'} \in M_{624}$ and $m_{k'} \in M_{510}$ such that

$$\text{(II)} \quad \langle A; m'|_{A^3}\rangle \cong \begin{cases} \langle\{0,1,2\}; m_{i'}\rangle & \text{if } A \in R, \\ \langle\{0,1,2\}; m_{j'}\rangle & \text{if } A \in U, \\ \langle\{0,1,2\}; m_{k'}\rangle & \text{if } A \in V. \end{cases}$$

The isomorphism in (II) is the same as the one in (I). To show that m' generates m it is sufficient to show that there is a term t of type $\langle 3\rangle$ such that $t(m_{i'}) = m_i$, $t(m_{j'}) = m_j$ and $t(m_{k'}) = m_k$ because then it follows that

$$\langle A; t(m')|_{A^3}\rangle = \langle A; t(m'|_{A^3})\rangle \cong \begin{cases} \langle\{0,1,2\}; t(m_{i'})\rangle = \langle\{0,1,2\}; m_i\rangle & \text{if } A \in R, \\ \langle\{0,1,2\}; t(m_{j'})\rangle = \langle\{0,1,2\}; m_j\rangle & \text{if } A \in U, \\ \langle\{0,1,2\}; t(m_{k'})\rangle = \langle\{0,1,2\}; m_k\rangle & \text{if } A \in V. \end{cases}$$

and hence $t(m') = m$ and $m \in [m']$ (the isomorphism above is again the same as in (I) and (II)). Since $M_0 = \{m_0\}$, it follows $m_i = m_{i'} = m_0$ and thus $t(m_0) = m_0$ for any nontrivial ternary t. Hence, we have to take care of m_j and m_k only. We do so in two steps: First we find a term t_1 such that

$$\text{(III)} \quad t_1(m_{j'}) = m_{44},\ t_1(m_{k'}) = m_{510},$$

and secondly we find a term t_2 such that

$$\text{(IV)} \quad t_2(m_{44}) = m_j,\ t_2(m_{510}) = m_k$$

so we can finish the proof by defining $t = t_2 t_1$.
To find t_1 for (III), it suffices to find two terms t_{11} and t_{12} such that $t_{11}(m_{k'}) = m_{510}$ for all $m_{k'} \in M_{510}$, $t_{11}(m_{j'}) = m_j$ for all $m_{j'} \in M_{624}$ and also $t_{12}(m_{510}) = m_{510}$, $t_{12}(m_{424}) = m_{624}$, $t_{12}(m_{624}) = m_{44}$. We can then choose

$$t_1 = \begin{cases} t_{11} & \text{if } j = 44, \\ (t_{12})^2 t_{11} & \text{if } j = 424, \\ t_{12}t_{11} & \text{if } j = 624. \end{cases}$$

The construction of such terms is the most difficult and essential part of the proof. Here, we will skip the process of construction. It can be checked that $t_{11} = prqp$ and $t_{12} = p^2 s$ are such terms.

We can use the same idea to find t_2 for (IV): It suffices to find two terms t_{21} and t_{22} such that $t_{21}(m_{510}) = m_k$, $t_{21}(m_{44}) = m_{44}$ and $t_{22}(m_{k'}) = m_k$, $t_{22}(m_{44}) = m_{424}$, $t_{22}(m_{424}) = m_{624}$, $t_{22}(m_{624}) = m_{44}$ since we can then choose

$$t_2 = \begin{cases} t_{21} & \text{if } j = 44, \\ t_{22}t_{21} & \text{if } j = 424, \\ (t_{22})^2 t_{21} & \text{if } j = 624. \end{cases}$$

Skipping the construction part again, it can be checked that the terms t_{21}, t_{22} given in the table below are appropriate:

Table 4.5

Terms t_{21}, t_{22}

k	t_{21}	t_{22}
510		p^2s
218	pqr	p^2s
666	q	$qspqrp$
62	rs	rp^2s
182	s^2	sp^2
702	qr	qrp^2s
546	r	rp^2s
26	s	sp^2

This completes the proof of the minimality of $\langle S; m \rangle$.

"$\Rightarrow$" A subalgebra of a minimal conservative majority algebra is either minimal or trivial by (1) of Lemma 4.20. Hence, the algebras in $Spec\langle S; m \rangle$ must be minimal, because they cannot be trivial since m does not become a trivial operation if we restrict it to the triplets of a three-element set. It follows from (2) of the same lemma that the spectrum of a minimal conservative algebra is a subset of $\{\langle S; m_i \rangle | m_i \in M\}$. Now suppose that $Spec\langle S; m \rangle$ contains two distinct algebras from $\{\langle \{0,1,2\}; m_i \rangle | m_i \in M_{624}\}$. This means that there have to exist two distinct three-element subsets A, B and two distinct majority operation $m_{j_1}, m_{j_2} \in M_{624}$ such that $\langle A; m|_{A^3} \rangle \cong \langle \{0,1,2\}; m_{j_1} \rangle$ and $\langle B; m|_{B^3} \rangle \cong \langle \{0,1,2\}; m_{j_2} \rangle$. One can check that we have $u^2(m_j) = m_{424}$ for any of the three $m_j \in M_{624}$. Now we define

$g = u^2(m)$ which is an operation that clearly belongs to $[m]$. We obtain $\langle A; g|_{A^3}\rangle \cong \langle\{0,1,2\}; m_{424}\rangle \cong \langle B; g|_{B^3}\rangle$ but $\langle A; m|_{A^3}\rangle \cong \langle\{0,1,2\}; m_{j_1}\rangle \not\cong \langle\{0,1,2\}; m_{j_2}\rangle \cong \langle B; m|_{B^3}\rangle$. This is a contradiction to the minimality of $\langle S; m\rangle$ by (4) of Lemma 4.20. Similarly, we have already noted that, for any $m_k \in M_{510}$, we have $prqp(m_k) = m_{510}$. Hence, we can use the same argument to conclude that $Spec\langle S; m\rangle$ also cannot contain two distinct algebras from $\{\langle\{0,1,2\}; m_i\rangle | m_i \in M_{510}\}$. □

4.4 Minimal majority clones on a four-element set

We will now determine the minimal majority clones on a four-element set. If not stated otherwise, all results in this section are due to Tamás Waldhauser [23]. Recall $S = \{0, ..., n-1\}$.

Recall what we have noted in Lemma 3.4: If a k-ary operation f on S is minimal, then so is any nontrivial operation $f|_{A^k}$ for a subset $A \subseteq S$ providing f preserves A. This is a useful observation, because it implies that if f is a minimal majority operation on S and A is a three-element subset of S preserved by f, then the (due to the majority property of f) necessarily nontrivial $f|_{A^3}$ must be isomorphic to one of the operations we obtained in the second section in this chapter: Up to isomorphism, it must be listed in Table 4.4.

Furthermore, we have already noted that in order to prove the minimality of a clone C generated by a majority operation m, it suffices to show that any two majority operations in C generate each other.

Now, we formulate a theorem which helps us to reduce the number of operations that have to be checked in order to find all minimal majority clones.

Theorem 4.22: *Let f be a majority operation on S. Then there exists a majority operation $g \in [f]$ which satisfies the following identity*

$$(o) \quad g(g(x,y,z), g(y,z,x), g(z,x,y)) \approx g(x,y,z)$$

Proof: We define $f^{(k)}$ $(k \geq 1)$ recursively as follows:

- $f^{(1)}(x,y,z) := f(x,y,z)$
- $f^{(k+1)}(x,y,z) := f(f^{(k)}(x,y,z), f^{(k)}(y,z,x), f^{(k)}(z,x,y))$

We show by induction on k that the following identity holds for a fixed $l \geq 1$.

$$f^{(k+l)}(x,y,z) \approx f^{(k)}(f^{(l)}(x,y,z), f^{(l)}(y,z,x), f^{(l)}(z,x,y))$$

The base case ($k = 1$) is trivial by definition. Now we assume $k \geq 1$ and that the claim is true for all $1, ..., k$. We have

$$\begin{aligned}
&f^{((k+1)+l)}(x, y, z) \\
&\approx f(f^{(k+l)}(x, y, z), f^{(k+l)}(y, z, x), f^{(k+l)}(z, x, y)) \\
&\approx f(\ f^{(k)}(f^{(l)}(x, y, z), f^{(l)}(y, z, x), f^{(l)}(z, x, y)), \\
&\qquad f^{(k)}(f^{(l)}(y, z, x), f^{(l)}(z, x, y), f^{(l)}(x, y, z)), \\
&\qquad f^{(k)}(f^{(l)}(z, x, y), f^{(l)}(x, y, z), f^{(l)}(y, z, x))\ ) \\
&\approx f^{(k+1)}(f^{(l)}(x, y, z), f^{(l)}(y, z, x), f^{(l)}(z, x, y))
\end{aligned}$$

Now we define $*$ on $D = \{f^{(k)} | k \in \mathbb{N}\}$ by

$$(f^{(k)} * f^{(l)})(x, y, z) = f^{(k)}(f^{(l)}(x, y, z), f^{(l)}(y, z, x), f^{(l)}(z, x, y))$$

Our claim shown above means that $k \mapsto f^{(k)}$ is a homomorphism from $\langle \mathbb{N}; + \rangle$ to $\langle D; * \rangle$. We will finish the proof by showing that $\langle D; * \rangle$ contains a unique idempotent element, say $f^{(l)}$, because then, $f^{(l)} * f^{(l)} = f^{(l)}$ and (o) follows for $g = f^{(l)} \in [f]$. D is clearly finite so we can assume $D = \{f^{(1)}, ..., f^{(r)}\}$. Now there exists $s > 0$ such that $f^{(r+s)} = f^{(r)}$ and $f^{(r+1)}, f^{(r+2)}, ..., f^{(r+s)}$ are pairwise distinct. s must be a divisor of one of the numbers $r + 1, ..., r + s$, so one of these can be written as sq for some $q \in \mathbb{N}$. But now, $f^{(sq)} * f^{(sq)} = f^{(sq+sq)} = f^{(sq)}$. Thus, $\langle D; * \rangle$ has an idempotent element. Now suppose that there are two idempotent elements $f^{(l_1)}$ and $f^{(l_2)}$. Then

$$f^{(l_1)} = \underbrace{f^{(l_1)} * ... * f^{(l_1)}}_{l_2} = f^{(l_1 l_2)} = \underbrace{f^{(l_2)} * ... * f^{(l_2)}}_{l_1} = f^{(l_2)}.$$

Thus, $\langle D; * \rangle$ contains a unique idempotent element. □

This means that any minimal majority clone is generated by a majority operation satisfying (o). Thus, in order to find all minimal majority clones on S it suffices to check all majority operations on S satisfying (o).

Note that the uniqueness of the idempotent element in $\langle D; * \rangle$ is not needed to proof the claim. However, it is needed in the next definition.

Definition 4.23: Let f be a majority operation on S. Set

$$\langle abc \rangle := \{(a, b, c), (b, c, a),\ (c, a, b)\}.$$

Write

$$f|_{\langle abc \rangle} \equiv u \text{ whenever } f(a, b, c) = f(b, c, a) = f(c, a, b) = u$$

and

$$f|_{\langle abc\rangle} = p \text{ if } f(a,b,c) = a,\ f(b,c,a) = b,\ f(c,a,b) = c.$$

Furthermore, for a ternary operation f define $f^{(k)}$ as in the proof of Theorem 4.22 and denote by $\widehat{f}$ the unique element in $\{f^{(l)}|l \in \mathbb{N}\} \subseteq [f]$ that satisfies (o).

Now we observe some of the properties that a majority operation satisfying (o) must have. Recall that $\omega := \{(a,b,c) \in S^3 | a \neq b \neq c \neq a\}$.

Lemma 4.24: *Let f be a majority operation satisfying (o) and let $(a,b,c) \in \omega$. Let $u = f(a,b,c)$, $v = f(b,c,a)$ and $w = f(c,a,b)$. Then $|\{u,v,w\}| \neq 2$ and $f|_{\langle uvw\rangle} = p$.*

Proof: Suppose to the contrary that $|\{u,v,w\}| = 2$. Without loss of generality we can assume $u = v \neq w$. We use the property (o) to obtain

$$w = f(c,a,b) = f(f(c,a,b), f(a,b,c),\ f(b,c,a)) = f(w,u,u) = u,$$

a contradiction. For the second part we use (o) again and we obtain

$$f(u,v,w) = f(f(a,b,c), f(b,c,a), f(c,a,b)) = f(a,b,c) = u.$$

$f(v,w,u) = v$ and $f(w,u,v) = w$ follow in the same way. □

Now we also assume that f is minimal.

Lemma 4.25 *Let f be minimal majority operation satisfying (o) and let $(a,b,c) \in \omega$. Then $f|_{\langle abc\rangle} = p$ if and only if $f|_{\langle bac\rangle} = p$.*

Proof: "⇒" Let $f|_{\langle abc\rangle} = p$. Define g by setting

$$g(x,y,z) := f(f(x,y,z), f(x,z,y), x).$$

Since $f|_{\langle abc\rangle} = p$, we get $f(x,y,z) = x$ for $(x,y,z) \in \langle abc\rangle$ and $f(x,z,y) = x$ for $(x,y,z) \in \langle bac\rangle$. Due to the majority property of f, this implies $g|_{\langle abc\rangle} = p$ and hence g preserves the set $\{a,b,c\}$. Due to $g \in [f]$ and f minimal, f has to preserve $\{a,b,c\}$ as well. Thus, the restriction $f|_{\{a,b,c\}^3}$ is a minimal majority operation on the three-element set $\{a,b,c\}$. But now we can look at the Table 4.4 that gives us (up to isomorphism) all minimal majority operations on a three-element set. We note that there is only one operation, namely m_{44}, satisfying $f|_{\langle abc\rangle} = p$. However, m_{44} also satisfies $f|_{\langle bac\rangle} = p$. Thus, $f|_{\langle bac\rangle} = p$.
"⇐" Apply "⇒" to $f|_{\langle bac\rangle}$. □

The last two lemmas prove the following theorem.

Theorem 4.26: *Let f be a minimal majority operation satisfying (o) and let $(a,b,c) \in \omega$. Set $u = f(a,b,c)$, $v = f(b,c,a)$ and $w = f(c,a,b)$. Then $f|_{\langle uvw \rangle} = p$ and $f|_{\langle vuw \rangle} = p$*

Corollary 4.27 *Every conservative minimal majority clone C is generated by an operation f that has the following property:*

(oo) $f|_{\langle abc \rangle} \equiv u$ or $f|_{\langle abc \rangle} = p$ for all $(a,b,c) \in \omega$ (where u depends on a, b, c)

Proof: By Theorem 4.22, C is generated by a conservative majority operation f satisfying (o). Setting $u = f(a,b,c)$, $v = f(b,c,a)$, $w = f(c,a,b)$, we obtain $(u,v,w) \in \{a,b,c\}^3$. Hence, Theorem 4.26 gives us $f|_{\langle uvw \rangle} = p$ and $f|_{\langle vuw \rangle} = p$. Thus, $f|_{\langle abc \rangle} = p$ (if $u \neq v \neq w \neq u$) or $f|_{\langle abc \rangle} \equiv u$ (if $u = v = w$). □

Definition 4.28: Denote by Υ the set of all majority operations on S for which the property (oo) holds.

Note that the a majority operation that satisfies (oo) also satisfies (o). Hence, the majority operations in Υ form a subset of the set of majority operations satisfying (o).

By the proof of Corollary 4.27, a conservative minimal majority operation satisfying (o) is in the set Υ. Unfortunately, this is not true if the operation is not conservative. However, we can show that in the four-element case, there is essentially only one exception (this means that, up to isomorphism, there is only one minimal majority operation on S that satisfies (o) but is not in Υ). Before we start our work on proving that claim and determining the operation in question, we introduce some more notation.

From now on until the end of this section, fix $n = 4$ and $S = \{0,1,2,3\}$.

Definition 4.29: Denote by $[p,q,r;s,t,u]$ the set of majority operations on S for which

$$\begin{array}{ll} f(0,1,2) = p & f(1,0,2) = s \\ f(1,2,0) = q & f(0,2,1) = t \\ f(2,0,1) = r & f(2,1,0) = u \end{array}$$

If we do not want to specify all these six values of f, then we use $*$ to indicate an arbitrary element of S. For example, $[0,*,*;*,1,*]$ is the set of all majority

operations f such that $f(0,1,2)=0$ and $f(0,2,1)=1$.
Furthermore, let f_x, f_y, f_z stand for the composite where the first, second resp. third variable of f is replaced by f itself.

$$\begin{aligned}f_x(x,y,z) &:= f(f(x,y,z),y,z)\\ f_y(x,y,z) &:= f(x,f(x,y,z),z)\\ f_z(x,y,z) &:= f(x,y,f(x,y,z))\end{aligned}$$

Instead of $(f_x)_y$ we briefly write f_{xy}. We also use the convention that the lower indices have priority to the upper ones. This means that $f_{xy}^{(k)}$ stands for $(f_{xy})^{(k)}$ and not for $(f^{(k)})_{xy}$.

We now prove a lemma that helps us every time we have to calculate the values of f_{zy}. Since we use the operation f_{zy} very often in the sequel, this lemma is very useful for us.

Lemma 4.30: *Let f be a majority operation and a,b,c,d pairwise distinct elements of S (i.e. $\{a,b,c,d\}=\{0,1,2,3\}$). If $f(a,b,c)\neq d$, then $f_{zy}(a,b,c)=f(a,b,c)$. If this is not the case, then $f_{zy}(a,b,c)=f(a,b,d)$ if the latter does not equal d. If it does, then $f_{zy}(a,b,c)=f(a,d,c)$ if this is not b. If we have $f(a,d,c)=b$, then $f_{zy}=f(a,d,b)$.*

Proof: We have

$$f_{zy}(a,b,c)=f(a,f(a,b,f(a,b,c)),f(a,f(a,b,f(a,b,c)),c)).$$

Now

$$\begin{aligned}f(a,b,c)=a \Rightarrow f_{zy}(a,b,c) &= f(a,f(a,b,a),f(a,f(a,b,a),c))\\ &= f(a,a,f(a,a,c))=f(a,a,a)=a,\\ f(a,b,c)=b \Rightarrow f_{zy}(a,b,c) &= f(a,f(a,b,b),f(a,f(a,b,b),c))\\ &= f(a,b,f(a,b,c))=f(a,b,b)=b,\\ f(a,b,c)=c \Rightarrow f_{zy}(a,b,c) &= f(a,f(a,b,c),f(a,f(a,b,c),c))\\ &= f(a,c,f(a,c,c))=f(a,c,c)=c.\end{aligned}$$

Thus, $f_{zy}(a,b,c)=f(a,b,c)$ if $f(a,b,c)\neq d$. Now suppose $f(a,b,c)=d$. Then

$$\begin{aligned}f_{zy}(a,b,c) &= f(a,f(a,b,f(a,b,c)),f(a,f(a,b,f(a,b,c)),c))\\ &= f(a,f(a,b,d),f(a,f(a,b,d),c))\end{aligned}$$

and we have

$$\begin{aligned}f(a,b,d)=a &\Rightarrow f_{zy}(a,b,c)=f(a,a,f(a,a,c))=f(a,a,a)=a,\\ f(a,b,d)=b &\Rightarrow f_{zy}(a,b,c)=f(a,b,f(a,b,c))=f(a,b,d)=b,\end{aligned}$$

$f(a,b,d) = c \Rightarrow f_{zy}(a,b,c) = f(a,c,f(a,c,c)) = f(a,c,c) = c.$

Thus, $f_{zy}(a,b,c) = f(a,b,d)$ if the latter does not equal d. Now suppose $f(a,b,d) = d$. Then

$f_{zy}(a,b,c) = f(a, f(a,b,d), f(a, f(a,b,d), c)) = f(a,d,f(a,d,c))$

and we have

$f(a,d,c) = a \Rightarrow f_{zy}(a,b,c) = f(a,d,a) = a$
$f(a,d,c) = c \Rightarrow f_{zy}(a,b,c) = f(a,d,c) = c$
$f(a,d,c) = d \Rightarrow f_{zy}(a,b,c) = f(a,d,d) = d$

and finally

$f(a,d,c) = b \Rightarrow f_{zy}(a,b,c) = f(a,d,b)$, as required. □

Now we prove a series of lemmas that we will afterwards use to prove the first big result of this section.

Lemma 4.31: *Let f be a minimal majority operation on S satisfying (o) and let $(a,b,c) \in \omega$. If $f(\langle abc\rangle) \subseteq \{a,b,c\}$, then one of the two following conditions hold:*
(i) $f|_{\langle abc\rangle} = p$ and $f|_{\langle bac\rangle} = p$
(ii) $f|_{\langle abc\rangle} \equiv u$ and $f|_{\langle bac\rangle} \equiv v$ for some $u, v \in S$.

Proof: $f|_{\langle abc\rangle}$ has one or three elements by Lemma 4.24. If it has three elements, these three elements are necessarily $\{a,b,c\}$ and we have $f|_{\langle abc\rangle} = p$ and $f|_{\langle bac\rangle} = p$ by Theorem 4.26. Now we suppose that $f|_{\langle abc\rangle}$ contains only one element. Without loss of generality, we can assume that $f|_{\langle abc\rangle} \equiv a$. Let d be the remaining element in S after taking away a,b,c. If $d \notin f(\langle bac\rangle)$, then f preserves $\{a,b,c\}$ and we can again use Table 4.4 to check that all minimal majority operations on the three-element set $\{a,b,c\}$ that have the property $f|_{\langle abc\rangle} \equiv a$ (namely the operations m_0, m_{510} and m_{218}) also have the property $f|_{\langle bac\rangle} \equiv v$ for some $v \in \{a,b,c\}$. Now suppose $d \in f(\langle bac\rangle)$ which means that f does not preserve $\{a,b,c\}$. If $a \in f(\langle bac\rangle)$, then we can permute the variables cyclically to have $f(b,a,c) = a$. Now we can look at the superposition

$$g(x,y,z) := f(f(x,y,z), f(x,z,y), x)$$

that we have already used in the proof of Theorem 4.26. We can use the identities $f(b,a,c) = a$ and $f|_{\langle abc\rangle} \equiv a$ to obtain that $g(x,y,z) = a$ if $(x,y,z) \in \{(a,b,c),(a,c,b),(b,a,c),(b,c,a)\}$. This implies

$$g^{(2)}(x,y,z) = g(g(x,y,z), g(y,z,x), g(z,x,y) = a$$

for all (x, y, z) where $\{x, y, z\} = \{a, b, c\}$. This means that $g^{(2)} \in [f]$ preserves $\{a, b, c\}$ and hence contradicts the minimality of f (since we assumed that f does not preserve this set and hence $f \notin [g^{(2)}]$). Finally, if $a \notin f(\langle bac \rangle)$ (but $d \in f(\langle bac \rangle)$), then $f(\langle bac \rangle) = \{b, c, d\}$ or $f(\langle bac \rangle) \equiv d$ since $f(\langle bac \rangle)$ it cannot contain two elements by Lemma 4.24. The second case finishes the proof, so we suppose $f(\langle bac \rangle) = \{b, c, d\}$. We may suppose without loss of generality

(i) $f(b, a, c) = c$, $f(a, c, b) = d$, $f(c, b, a) = b$ or
(ii) $f(b, a, c) = b$, $f(a, c, b) = d$, $f(c, b, a) = c$

after a cyclic permutations of variables (if necessary). In the first case, we have (using the identities we have just listed and $f|_{\langle abc \rangle} \equiv a$)

$$g(x, y, z) = \begin{cases} a & \text{if } (x, y, z) \in \{(a, b, c), (a, c, b), (b, a, c), (c, a, b)\}, \\ c & \text{if } (x, y, z) = (c, b, a), \\ d & \text{if } (x, y, z) = (b, c, a). \end{cases}$$

which implies $g^{(2)}(x, y, z) = a$ for all (x, y, z) such that $\{x, y, z\} = \{a, b, c\}$. Hence, $g^{(2)}$ preserves $\{a, b, c\}$. In the second case we have

$$g(x, y, z) = \begin{cases} a & \text{if } (x, y, z) \in \{(a, b, c), (a, c, b)\}, \\ b & \text{if } (x, y, z) \in \{(b, c, a), (b, a, c)\}, \\ c & \text{if } (x, y, z) \in \{(c, a, b), (c, b, a)\}. \end{cases}$$

and g preserves $\{a, b, c\}$. Both cases contradict the minimality of f. This finishes the proof. □

Lemma 4.32: *Let f be a majority operation on S satisfying (o). In either of the following cases f is not minimal*

(a) $f \in [3, 1, 0; *, *, *]$
(b) $f \in [3, 0, 1; *, *, *]$
(c) $f \in [3, 0, 2; *, *, *]$
(d) $f \in [3, 2, 0; *, *, *]$.

Proof: (a) Suppose that f is minimal. We can use Theorem 4.26 to conclude from $f(0, 1, 2) = 3$, $f(1, 2, 0) = 1$, $f(2, 0, 1) = 0$ that we have $f|_{\langle 310 \rangle} = p$ and $f|_{\langle 130 \rangle} = p$. We now use Lemma 4.30 to show that f_{zy} preserves $\{0, 1, 2\}$ (which contradicts the minimality of f since f obviously doesn't preserve this set) apart from very few exceptions. We have $f(0, 1, 2) = 3$ and hence $f_{zy}(0, 1, 2) = f(0, 1, 3) = 0$. Next, $f(1, 2, 0) = 0$ and $f(2, 0, 1) = 2$ imply $f_{zy}(1, 2, 0) = 0$ and $f_{zy}(2, 0, 1) = 2$. Also, $f_{zy}(1, 0, 2) = 3$ is possible only if $f(1, 0, 2) = 3$ and $f(1, 0, 3) = 3$. However, this is not possible since we know $f(1, 0, 3) = 1$. Similarly, $f_{zy}(0, 2, 1) = 3$ is possible only if $f(0,2,1) = 3$, $f(0,2,3) = 3$ and $f(0,3,1) \in \{2,3\}$. Again, the latter is impossible since we have $f(0,3,1) = 0$. Finally, $f_{zy}(2,1,0) = 3$ is possible but only if

(i) $f(2,1,0) = 3, f(2,1,3) = 3, f(2,3,0) = 3$ or

(ii) $f(2,1,0) = 3, f(2,1,3) = 3, f(2,3,0) = 1, f(2,3,1) = 3$.

So f must satisfy one of these two cases. We now try to examine the set $f(\langle 102\rangle)$. We know that this set contains one or three elements by Lemma 4.24. If that set is $\{0\}$, $\{1\}$ or $\{2\}$, then we have $f|_{\langle 102\rangle} \equiv v$ for some $v \in \{0,1,2\}$ but we obviously do not have $f|_{\langle 012\rangle} \equiv u$ for any $u \in \{0,1,2\}$. This implies that f cannot be minimal by Lemma 4.31. If we have $f|_{\langle 102\rangle} \equiv 3$, then we can calculate (using all the identities above) $f_{zy} \in [0,1,0;1,u,3]$, where $u = f_{zy}(0,2,1) \neq 3$. If we have $u = 1$, then $f_{zy} \in [0,1,0;1,1,3]$ implies $f_{zy}^{(2)} \in [0,0,0;1,1,1]$ which means that $f_{zy}^{(2)}$ preserves the set $\{0,1,2\}$ and hence f cannot be minimal. For $u = 2$ we can calculate $g(x,y,z) := f_{zy}(y,z,f_{zy}(x,y,z)) \in [1,0,0;2,2,1]$ and hence g preserves $\{0,1,2\}$ and the minimality of f is contradicted. For $u = 0$ we have $f_{zy} \in [0,1,0;1,0,3]$. We can calculate $f_{zy}^{(2)} \in [0,0,0;1,0,3]$ because $f|_{\langle 103\rangle} = p$ implies $f_{zy}|_{\langle 103\rangle} = p$ by Lemma 4.30. Thus, $\widehat{f_{zy}} \in [0,0,0;1,0,3]$. This implies that $\widehat{f_{zy}}$ is not minimal by Lemma 4.31 and it follows that f is not minimal, too. Thus, the assumption that $f(\langle 102\rangle)$ contains only one element was wrong and $f(\langle 102\rangle)$ has to contain three elements. If they are $\{0,2,3\}$, then Theorem 4.26 implies $f|_{\langle 023\rangle} = p$ which contradicts our observation that $f(2,3,0)$ is either 3 or 1. In the same way, $f(\langle 102\rangle) = \{0,1,2\}$ implies $f|_{\langle 012\rangle} = p$ which cannot be true since we know $f(0,1,2) = 3$. Likewise, $f(\langle 102\rangle) = \{1,2,3\}$ is also impossible because the implicated $f|_{\langle 213\rangle} = p$ is a contradiction to our observation that $f(2,1,3) = 3$. Hence, $f(\langle 102\rangle)$ can be nothing else but $\{0,1,3\}$. Since we already know $f(2,1,0) = 3$, there are only two possibilities left: $f \in [3,1,0;0,1,3]$ or $f \in [3,1,0;1,0,3]$. In the first case we have $f_{zy} \in [0,1,0;0,1,3]$ and in the second case we have $f_{zy} \in [0,1,0;1,0,3]$. It follows that we have $\widehat{f_{zy}} \in [0,0,0;0,1,3]$ in the first and $\widehat{f_{zy}} \in [0,0,0;1,0,3]$ in the second case, because $f|_{\langle 013\rangle} = p$ implies $f_{zy}|_{\langle 013\rangle} = p$ by Lemma 4.30 and we already know $f_{zy}|_{\langle 103\rangle} = p$. This contradicts the minimality of f by Lemma 4.31 in both cases and we have finished the proof.

(b) We can use the same arguments as in (a). The only difference is that in this case we have $f_{zy} \in [1,1,2;*,*,*]$.

(c) The operation $f(x,z,y) \in [*,*,*;0,3,2]$ is isomorphic to an operation which is not minimal by case (a), because for the permutation $\phi = (12)$ we have $f(x,z,y)^{\phi} \in [3,1,0;*,*,*]$.

(d) Similar as in (c), for $f(x,z,y) \in [*,*,*;2,3,0]$ we obtain $f(x,z,y)^{\phi} \in [3,0,1;*,*,*]$ which implies that $f(x,z,y)$ is not minimal by (b). □

As already used in the proof, this lemma obviously implies that a majority operation that is isomorphic to an operation that falls under the cases (a)-(d) is not minimal. In the tables below we list majority operations f^ϕ where f falls under the cases (a)-(d) and ϕ is a permutation on S. These operations are therefore not minimal (providing they satisfy (o)).

Table 4.6

		(a)	(b)	(c)	(d)
ϕ	$f \in$	$[3,1,0;*,*,*]$	$[3,0,1;*,*,*]$	$[3,0,2;*,*,*]$	$[3,2,0;*,*,*]$
(01)	$f^\phi \in$	$[*,*,*;3,0,1]$	$[*,*,*;3,1,0]$	$[*,*,*;3,1,2]$	$[*,*,*;3,2,1]$
(02)	$f^\phi \in$	$[*,*,*;1,2,3]$	$[*,*,*;2,1,3]$	$[*,*,*;2,0,3]$	$[*,*,*;0,2,3]$
(12)	$f^\phi \in$	$[*,*,*;0,3,2]$	$[*,*,*;2,3,0]$	$[*,*,*;1,3,0]$	$[*,*,*;0,3,1]$
(012)	$f^\phi \in$	$[1,3,2;*,*,*]$	$[2,3,1;*,*,*]$	$[0,3,1;*,*,*]$	$[1,3,0;*,*,*]$
(021)	$f^\phi \in$	$[0,2,3;*,*,*]$	$[2,0,3;*,*,*]$	$[2,1,3;*,*,*]$	$[1,2,3;*,*,*]$

Lemma 4.33: *Let f be a majority operation on S satisfying (o). If $f \in [3,2,1;*,*,*]$, then f is not minimal.*

Proof: Suppose that f is minimal. We can use Theorem 4.26 again to conclude from $f(0,1,2) = 3$, $f(1,2,0) = 2$ and $f(2,0,1) = 1$ that we have $f|_{\langle 321\rangle} = p$ and $f|_{\langle 231\rangle} = p$. As in the previous lemma, we examine the set $f(\langle 102\rangle)$. Once again, Lemma 4.24 implies that it has to contain exactly one or three elements. If $f(\langle 102\rangle) \subseteq \{0,1,2\}$, then Lemma 4.31 contradicts the minimality of f; hence, $f(\langle 102\rangle)$ cannot be $\{0\}$, $\{1\}$, $\{2\}$ or $\{0,1,2\}$. Thus, it is $\{3\}$ or a three-element set containing 3.

Case 1: $f(\langle 102\rangle) = \{3\}$. Define

$$g(x,y,z) := f(z,y,f(x,y,z))$$

and we can calculate $g \in [2,2,1;u,1,v]$. Clearly, if none of u,v equals 3, then g preserves $\{0,1,2\}$ which contradicts the minimality of f (because f does not preserve that set). For $u=1$ we have $g \in [2,2,1;1,1,3]$ which implies $g^{(2)} \in [2,2,2;1,1,1]$ which means that $g^{(2)}$ preserves $\{0,1,2\}$. For $u=0$ we have $g \in [2,2,1;0,1,3]$ and it follows for $h(x,y,z) := g(g(x,y,z),z,x)$ that we have $h \in [2,1,1;1,1,*]$ and hence $h^{(2)} \in [1,1,1;1,1,1]$ preserves $\{0,1,2\}$. For $u=3$ we have $g \in [2,2,1;3,1,*]$ which (using $f|_{\langle 321\rangle} = p$ and $f|_{\langle 231\rangle} = p$) implies $h \in [2,1,1;2,2,*]$ and again $h^{(2)} \in [1,1,1;2,2,2]$ preserves $\{0,1,2\}$. It remains the case $u=2, v=3$. But then $g \in [2,2,1;2,1,3]$ implies $g^{(2)} \in [2,2,2;3,2,1]$, $g^{(3)} \in [2,2,2;1,3,2]$, $g^{(4)} \in [2,2,2;2,1,3]$ and hence $\widehat{g} \in [2,2,2;1,3,2]$. Now Lemma 4.31 implies that $\widehat{g} \in [f]$ is not a minimal operation which contradicts the minimality of f.

Case 2: $f(\langle 102\rangle) \neq \{3\}$. Hence, $f(\langle 102\rangle)$ is a three-element set containing 3. Suppose it is $\{0,1,3\}$. This implies $f \in [3,2,1,0,1,3]$ or $f \in [3,2,1,1,0,3]$ because the other cases would give an operation that is listed in Table 4.6 and hence not minimal. In both cases, we can conclude from Theorem 4.26 that $f|_{\langle 013\rangle} = p$. This, together with Lemma 4.30 and the identity $f|_{\langle 321\rangle} = p$, implies that we have $f_{zy} \in [0,2,1;0,1,2]$ in the first and $f_{zy} \in [0,2,1;1,0,2]$ in the second case. In both cases f_{zy} preserves the set $\{0,1,2\}$. Using Table 4.6 again, $f(\langle 102\rangle) = \{0,2,3\}$ implies $f \in [3,2,1;3,0,2]$ or $f \in [3,2,1;3,2,0]$. Lemma 4.30 gives us $f_{zy} \in [u,2,1;v,0,2]$ in the first and $f_{zy} \in [u,2,1,v,2,0]$ in the second case. Furthermore, Theorem 4.26 gives us $f|_{\langle 320\rangle} = p$. Using this identity and Lemma 4.30, we obtain in both cases

$$\mathrm{u} = f_{zy}(0,1,2) = \begin{cases} 0, & \text{if } f(0,1,3) \in \{0,3\}, \\ 1, & \text{if } f(0,1,3) = 1, \\ 2, & \text{if } f(0,1,3) = 2. \end{cases}$$
$$v = f_{zy}(1,0,2) = \begin{cases} 0, & \text{if } f(1,0,3) = 0, \\ 1, & \text{if } f(1,0,3) \in \{1,3\}, \\ 2, & \text{if } f(1,0,3) = 2. \end{cases}$$

and hence $u, v \neq 3$. Thus, f_{zy} preserves $\{0,1,2\}$ in both cases. Finally, if $f(\langle 102\rangle) = \{0,2,3\}$, then it follows from Table 4.6 in the same way as above that we have $f \in [3,2,1;2,3,1]$ or $f \in [3,2,1;1,3,2]$. Now we use $g(x,y,z) := f(z,y,f(x,y,z))$ again. We use the identities $f|_{\langle 213\rangle} = p$ and $f|_{\langle 123\rangle} = p$ once more to conclude that we have $g \in [2,2,1;2,1,1]$ in the first and $g \in [2,2,1;1,1,3]$ in the second case. In the first case, g preserves $\{0,1,2\}$ and in the second case $g^{(2)} \in [2,2,2;1,1,1]$ preserves $\{0,1,2\}$. Both cases contradict the minimality of f and this finishes the proof. □

Again, we can use isomers to create operations that are isomorphic to an operation $f \in [3,2,1;*,*,*]$ and therefore not minimal. This means that we can add a column to Table 4.6 and we obtain the following table of operations that are not minimal (providing they satisfy (o)).

Table 4.7

		4.32 (a)	4.32 (b)	4.32 (c)	4.32 (d)	4.33
ϕ	$f \in$	$[3,1,0;*,*,*]$	$[3,0,1;*,*,*]$	$[3,0,2;*,*,*]$	$[3,2,0;*,*,*]$	$[3,2,1;*,*,*]$
(01)	$f^\phi \in$	$[*,*,*;3,0,1]$	$[*,*,*;3,1,0]$	$[*,*,*;3,1,2]$	$[*,*,*;3,2,1]$	$[*,*,*;3,2,0]$
(02)	$f^\phi \in$	$[*,*,*;1,2,3]$	$[*,*,*;2,1,3]$	$[*,*,*;2,0,3]$	$[*,*,*;0,2,3]$	$[*,*,*;0,1,3]$
(12)	$f^\phi \in$	$[*,*,*;0,3,2]$	$[*,*,*;2,3,0]$	$[*,*,*;1,3,0]$	$[*,*,*;0,3,1]$	$[*,*,*;2,3,1]$
(012)	$f^\phi \in$	$[1,3,2;*,*,*]$	$[2,3,1;*,*,*]$	$[0,3,1;*,*,*]$	$[1,3,0;*,*,*]$	$[2,3,0;*,*,*]$
(021)	$f^\phi \in$	$[0,2,3;*,*,*]$	$[2,0,3;*,*,*]$	$[2,1,3;*,*,*]$	$[1,2,3;*,*,*]$	$[1,0,3;*,*,*]$

Looking at this table carefully shows us that whenever we have a minimal majority operation $f \in O_S$ satisfying (o) and neither $f|_{\langle 012\rangle} = p$ nor $f|_{\langle 012\rangle} \equiv u$ (implying $f(\langle 012\rangle) \nsubseteq \{0,1,2\}$ by Lemma 4.31), then we must have that on two of the three triplets of $\langle 012\rangle$ the value of f equals the first variable, while on the third one f equals 3. Any other possibility can be found in the table (recall $|f(\langle 012\rangle)| \neq 2$ by Lemma 4.24). The same holds for the triplet $\langle 102\rangle$. We can generalize this result by replacing $0,1,2$ by any three distinct elements $a,b,c \in S$: If we have a triple $(a,b,c) \in \omega$ and a minimal majority operation $f \in O_S$ satisfying (o) such that neither $f(\langle abc\rangle) = p$ nor $f|_{\langle abc\rangle} \equiv u$ holds, then we must have that on two of the three triplets of $\langle abc\rangle$ the value of f equals the first variable, while on the third one f equals $d \notin \{a,b,c\}$.

Lemma 4.34: *Let f be a majority operation on S satisfying (o). If $f \in [3,1,2;1,0,3]$ or $f \in [3,1,2;3,0,2]$, then f is not minimal.*

Proof: Suppose that f is minimal. In the first case, we can use Theorem 4.26 once again to conclude from $f(0,1,2) = 3$, $f(1,2,0) = 1$, $f(2,0,1) = 2$ that $f|_{\langle 321\rangle} = p$ and from $f(1,0,2) = 1$, $f(0,2,1) = 0$, $f(2,1,0) = 3$ that $f|_{\langle 013\rangle} = p$. With the help of this two identities we can calculate $f_z(x,y,z) = f(x,y,f(x,y,z)) \in [0,1,2,1,0,2]$. Thus, f_z preserves $\{0,1,2\}$ which contradicts the minimality of f. In the second case, Theorem 4.26 gives us $f|_{\langle 321\rangle} = p$ and $f|_{\langle 203\rangle} = p$ which allows us to calculate $f_y \in [0,1,2,1,0,2]$. This means that f_y preserves $\{0,1,2\}$ and the minimality of f is contradicted. □

Lemma 4.35: *Let f be a minimal majority operation on S satisfying (o). If $f \in [3,1,2;1,3,2]$, then $f = m_{4,44}$ where $m_{4,44}$ is the majority operation defined as shown below (the reason for naming this operation $m_{4,44}$ will be seen later).*

(x,y,z)	$m_{4,44}(x,y,z)$
$(0,1,2)$	3
$(1,2,0)$	1
$(2,0,1)$	2
$(1,0,2)$	1
$(0,2,1)$	3
$(2,1,0)$	2
$\{0,1,3\}$	3
$\{0,2,3\}$	3
$(3,1,2)$	3
$(1,2,3)$	1
$(2,3,1)$	2
$(1,3,2)$	1
$(3,2,1)$	3
$(2,1,3)$	2

The middle two rows mean that if $\{x,y,z\}$ equals $\{0,1,3\}$ or $\{0,2,3\}$, then $m_{4,44}(x,y,z)=3$.

Proof: As usual, we can apply Theorem 4.26 to conclude $f|_{\langle 312\rangle}=p$ and $f|_{\langle 132\rangle}=p$. It remains to show that $f(x,y,z)$ equals 3 whenever $\{x,y,z\}=\{0,1,3\}$ or $\{0,2,3\}$. Set $g_1:=f_z$ and we can calculate

$$g_1=f_z\in[f(0,1,3),1,2;1,f(0,2,3),2].$$

If none of $f(0,1,3),f(0,2,3)$ equals 3, then g_1 preserves $\{0,1,2\}$ and f is not minimal. Suppose that exactly one of them equals 3. If $f(0,1,3)=0$, then $g_1\in[0,1,2;1;3;2]$ and $g_1^{(2)}\in[0,1,2;1,3,2]$ since we obviously have $g_1|_{\langle 012\rangle}=p$ and $f|_{\langle 132\rangle}=p$ implies $g_1|_{\langle 132\rangle}=p$. Hence, $\widehat{g_1}\in[0,1,2;1,3,2]$ which contradicts the minimality of f because g_1 is now not minimal by Lemma 4.31. If $f(0,1,3)=1$ or $f(0,1,3)=2$, then we have $g_1\in[1,1,2;1,3,2]$ or $[1,2,2;1,3,2]$ and $\widehat{g_1}\in[1,1,1;1,3,2]$ or $[2,2,2;1,3,2]$ and in both cases Lemma 4.31 implies that g_1 is not minimal. If $f(0,2,3)=0$, then $g_1\in[3,1,2;1,0,2]$ and $g_1^{(2)}\in[3,1,2;1,0,2]$ because we obviously have $g_1|_{\langle 102\rangle}=p$ and $f|_{\langle 312\rangle}=p$ implies $g_1|_{\langle 312\rangle}=p$. Hence, $\widehat{g_1}\in[3,1,2;1,0,2]$ which contradicts the minimality of f as above. For $f(0,2,3)=1$ or $f(0,2,3)=2$ we get $g_1\in[3,1,2;1,1,2]$ or $[3,1,2;1,2,2]$ and $\widehat{g_1}\in[3,1,2;1,1,1]$ or $[3,1,2;2,2,2]$ and in both cases Lemma 4.31 implies again that $\widehat{g_1}$ is not minimal. Thus, we must have $f(0,1,3)=f(0,2,3)=3$. Now we define $g_2(x,y,z):=f(y,x,f(x,y,z))$ and we obtain in exactly the same way from $g_2\in[f(1,0,3),1,2;1,f(2,0,3),2]$ that $f(1,0,3)=f(2,0,3)=3$. For $g_3:=f_y$ we get $g_3\in[f(0,3,2),1,2;1,f(0,3,1),2]$ from which we can conclude

$f(0,3,2) = f(0,3,1) = 3$. For $g_4(x,y,z) := f(z, f(x,y,z), x)$ we obtain $g_4 \in [f(2,3,0),1,2;1,f(1,3,0),2]$ which gives us $f(2,3,0) = f(1,3,0) = 3$. For $g_5(x,y,z) := f(f(x,y,z),y,x)$ we get $g_5 \in [f(3,1,0),1,2;1,f(3,2,0),2]$ and $f(3,1,0) = f(3,2,0) = 3$. Finally, for $g_6(x,y,z) := f(f(x,y,z),x,y)$ we get $g_6 \in [f(3,0,1),1,2;1,f(3,0,2),2]$ and $f(3,0,1) = f(3,0,2) = 3$. □

Lemma 4.36: *Let f be a minimal majority operation on S satisfying (o). If $f \in [3,1,2;3,3,3]$, then f is not minimal.*

Proof: Suppose to the contrary that f is minimal. We can use Theorem 4.26 once again to obtain $f|_{\langle 312\rangle} = p$ and $f|_{\langle 132\rangle} = p$. Now let us look at the operation f_{zy}. We can calculate

$$f_{zy} \in [u,1,2;v,w,2]$$

for some $u,v,w \in S$. We have (using Lemma 4.30):

$$v = f_{zy}(1,0,2) = \begin{cases} 0, & \text{if } f(1,0,3) = 0 \\ 1, & \text{if } f(1,0,3) \in \{1,3\} \\ 2, & \text{if } f(1,0,3) = 2 \end{cases}$$

and hence $v \neq 3$. Suppose $v = 0$. Then $f_{zy} \in [u,1,2;0,w,2]$. We define

$$g(x,y,z) := f_{zy}(z,x,\ f_{zy}(x,y,z))$$

and we obtain $g \in [r,1,2;2,*,2]$ for some $r \in S$. This implies $\widehat{g} \in [*,*,*;2,2,2]$. Since $\widehat{g} \in [f]$ has to be minimal, it can't preserve $\{0,1,2\}$ and it must meet the conditions in Lemma 4.31. Only the case $\widehat{g} \in [3,3,3;2,2,2]$ remains. However, this is not the case for any $r \in S$. Now suppose $v \neq 0$ (i.e. $v \in \{1,2\}$). For $u = 0$ we have $f_{zy} \in [0,1,2;v,w,2]$. Since $v \neq 3$ we must have $w = 3$ since otherwise f_{zy} preserves $\{0,1,2\}$. Thus, $f_{zy} \in [0,1,2;v,3,2]$. For $v = 1$ we obtain $f_{zy} \in [0,1,2;1,3,2]$ and consequently $f_{zy}^{(2)} \in [0,1,2;1,3,2]$ because we obviously have $f_{zy}|_{\langle 012\rangle} = p$ and $f|_{\langle 132\rangle} = p$ implies $f_{zy}|_{\langle 132\rangle} = p$ by Lemma 4.30. Hence, $\widehat{f_{zy}} \in [0,1,2;1,3,2]$. Now Lemma 4.31 implies that $\widehat{f_{zy}}$ is not minimal. If $v = 2$ we get $f_{zy}^{(2)} \in [0,1,2;2,2,2]$ and $f_{zy}^{(2)}$ preserves $\{0,1,2\}$. For $u = 1$ we have $f_{zy} \in [1,1,2;v,w,2]$ and again we must have $w = 3$ so that f_{zy} does not preserve $\{0,1,2\}$. Similarly as above, we get $f_{zy}^{(2)} \in [1,1,1;1,3,2]$ for $v = 1$ and $f_{zy}^{(2)} \in [1,1,1;2,2,2]$ for $v = 2$. In the first case we have $\widehat{f_{zy}} \in [1,1,1;1,3,2]$ and $\widehat{f_{zy}}$ is not minimal because of Lemma 4.31 and in the second case $f_{zy}^{(2)}$ preserves $\{0,1,2\}$. The same argument can be used for $u = 2$. The only difference is that $f_{zy}^{(2)} \in [2,2,2;*,*,*]$. Only the case $u = 3$ remains. We have $f_{zy} \in [3,1,2;v,w,2]$ where $v \neq 3$. If $v = 2$ or $w = 2$, then we have

$\widehat{f_{zy}} \in [3,1,2;2,2,2]$ which means that $\widehat{f_{zy}}$ is not minimal by Lemma 4.31. Thus, $v = 1$ (recall that we suppose $v \neq 0$) and $w \in \{0,1,3\}$. If $w = 0$, then we have $f_{zy} \in [3,1,2;1,0,2]$ and $\widehat{f_{zy}} = [3,1,2;1,0,2]$ which implies that $\widehat{f_{zy}}$ is not minimal by Lemma 4.31. If $w = 1$, then we have $\widehat{f_{zy}} \in [3,1,2;1,1,1]$ and the same follows. Finally, it remains the case $v = 1$ and $w = 3$. We have $f_{zy} \in [3,1,2;1,3,2]$ and we know from the last lemma that we have $f_{zy} = m_{4,44}$. But we will see later that the clone generated by $m_{4,44}$ contains only 3 majority operations and f is not one of them. Hence, $f \notin [m_{4,44}]$ and f is not minimal. □

We are ready to prove the first big result of this section.

Theorem 4.37: *Any minimal nonconservative majority operation on S which satisfies (o) is isomorphic to $m_{4,44}$ or belongs to the set Υ.*

Proof: Let f be a minimal nonconservative majority operation on S which satisfies (o) and does not belong to Υ. This implies that there exist a triple $(a,b,c) \in \omega$ such that neither $f|_{\langle abc \rangle} = p$ nor $f|_{\langle abc \rangle} \equiv u$ holds. Recall what we have noted after Table 4.7: Having this condition, it follows that on two of the three triplets of $\langle abc \rangle$ the value of f equals the first variable, while on the third one f equals $d \notin \{a,b,c\}$. We can suppose (after an isomorphism if necessary) that we have $f(0,1,2) = 3$, $f(1,2,0) = 1$ and $f(2,0,1) = 2$, i.e. $f \in [3,1,2;*,*,*]$. Now we must have $3 \in f(\langle 102 \rangle)$ because otherwise Lemma 4.31 would contradict the minimality of f. In the case $f|_{\langle 102 \rangle} \equiv 3$ the minimality of f is contradicted by Lemma 4.36. Thus, $f(\langle 102 \rangle)$ must be a three-element set containing 3 (it cannot be a two-element set by Lemma 4.24). If we look at the Table 4.7 (or use the above argument) we can see that they are only three possible cases: $f \in [3,1,2;1,0,3]$, $f \in [3,1,2;3,0,2]$ or $f \in [3,1,2;1,3,2]$. But now Lemma 4.34 eliminates the first two cases and in the third case Lemma 4.35 yields that f equals $m_{4,44}$. □

We have succeeded in determining all minimal majority operations on S satisfying (o) that do not belong to the set Υ. Now, we have to find the minimal operations on S that are in Υ. We have already described the conservative ones in the last section, so we can additionally limit or quest on nonconservative operations. We prove several properties of such operations, until we find that only a few (essentially two) operations possess these properties and they happen to be minimal.

Definition 4.38: A ternary operation f is called *cyclically commutative* if it is invariant under cyclic permutations on the variables. In other words,

$$f(x,y,z) \approx f(y,z,x) \approx f(z,x,y).$$

Lemma 4.39: *Let $f \in \Upsilon$ be a minimal nonconservative majority operation on S. Then f is cyclically commutative.*

Proof: Recall that $f \in \Upsilon$ implies that $f|_{\langle abc\rangle} = p$ or $f|_{\langle abc\rangle} \equiv u \in S$ for all triple $(a,b,c) \in \omega$. For contradiction, let us suppose without loss of generality that $f|_{\langle 013\rangle} = p$. Then, by Lemma 4.31 we must also have $f|_{\langle 103\rangle} = p$. Since f is nonconservative, we may also suppose (without loss of generality) $f|_{\langle 012\rangle} \equiv 3$. We must have $f|_{\langle 102\rangle} \equiv u$ for some $u \in S$ or $f|_{\langle 102\rangle} = p$. But, by Lemma 4.31, it follows from $f|_{\langle 012\rangle} \equiv 3$ that the latter is impossible. First let us suppose $u \neq 3$. Then $f_{zy} \in [0,v,w;u,u,u]$ where $f_{zy}|_{\langle 102\rangle} \equiv u$ follows from Lemma 4.30. Clearly, f_{zy} preserves $\{0,1,2\}$ and hence contradicts the minimality of f if $v \neq 3$ and $w \neq 3$. We can use Lemma 4.30 to conclude that we have $v = f_{zy}(1,2,0) = 3$ if and only if

(i) $f(1,2,0) = 3,\ f(1,2,3) = 3,\ f(1,3,0) = 3$ or

(ii) $f(1,2,0) = 3,\ f(1,2,3) = 3,\ f(1,3,0) = 2,\ f(1,3,2) = 3$.

But neither of the cases is possible since we know that $f(1,3,0) = 1$. So we must have $w = 3$. Using Lemma 4.30 again, we can conclude that this is the case if and only if

a) $f(2,0,1) = 3,\ f(2,0,3) = 3,\ f(2,3,1) = 3$ or

b) $f(2,0,1) = 3,\ f(2,0,3) = 3,\ f(2,3,1) = 0,\ f(2,3,0) = 3$.

In a) it follows from $f(2,3,1) = 3$ that we have $f|_{\langle 231\rangle} \equiv 3$ since $f \in \Upsilon$. We can conclude that $v = f_{zy}(1,2,0)$ equals 1 from Lemma 4.30 by using $f|_{\langle 231\rangle} \equiv 3$ and $f|_{\langle 013\rangle} = p$. In b) we have $f|_{\langle 231\rangle} \equiv 0$ and it follows $v = 0$. Thus, we have two possible cases: $f_{zy} \in [0,1,3;u,u,u]$ or $f_{zy} \in [0,0,3;u,u,u]$. In the first case we have $\widehat{f_{zy}} \in [0,1,3;u,u,u]$ since $f|_{\langle 013\rangle} = p$ and Lemma 4.30 imply $f_{zy}|_{\langle 013\rangle} = p$. It follows that $\widehat{f_{zy}}$ is not minimal by Lemma 4.31. In the second case, $\widehat{f_{zy}} \in [0,0,0;u,u,u]$ preserves $\{0,1,2\}$. This finishes the case $u \neq 3$. Now suppose $u = 3$. By the same argument as above, we have three cases: $f_{zy}(\langle 012\rangle) \subseteq \{0,1,2\}$, $f_{zy} \in [0,0,3;*,*,*]$ or $f_{zy} \in [0,1,3;*,*,*]$. In the third case we have $\widehat{f_{zy}} \in [0,1,3;*,*,*]$ and $\widehat{f_{zy}}$ does not belong to Υ and is therefore isomorphic to $m_{4,44}$ by Theorem 4.37. But then $f \notin [\widehat{f_{zy}}]$, which contradicts the minimality of f (in fact, we will later see that the clone generated by $m_{4,44}$ contains no operation from Υ expect for the first projection). Thus, we have one of the following two cases:

$$(1)\ f_{zy}(\langle 012\rangle) \subseteq \{0,1,2\}, \quad (2)\ f_{zy} \in [0,0,3;*,*,*],$$

the latter implying $f_{zy}^{(2)} \in [0,0,0;*,*,*]$. Now we deal with $f(\langle 102\rangle)$. Suppose $f(\langle 102\rangle) \not\subseteq \{0,1,2\}$. We can conclude that $f_{zy}(1,0,2)$ equals 1 by using the identities $f(1,0,2)=3$, $f(1,0,3)=1$ and Lemma 4.30. It follows $f_{zy} \in [*,*,*;1,s,t]$. As above, we can use Lemma 4.30 to conclude that $s = f_{zy}(0,2,1) = 3$ if and only if

(i) $f(0,2,1)=3$, $f(0,2,3)=3$, $f(0,3,1)=3$ or

(ii) $f(0,2,1)=3$, $f(0,2,3)=3$, $f(0,3,1)=2$, $f(0,3,2)=3$.

But neither of the cases is possible since we have $f(0,3,1)=0$. Similarly, we obtain $t = f_{zy}(2,1,0) = 3$ if and only if

a) $f(2,1,0)=3$, $f(2,1,3)=3$, $f(2,3,0)=3$ or

b) $f(2,1,0)=3$, $f(2,1,3)=3$, $f(2,3,0)=1$, $f(2,3,1)=3$.

In a) we obtain from $f(2,3,0)=3$ that we have $f|_{\langle 230\rangle} \equiv 3$ and we can conclude (using the same method as above) that s equals 0. In b) we have $f|_{\langle 230\rangle} \equiv 1$ and s equals 1. So we have $f_{zy}(\langle 102\rangle) \subseteq \{0,1,2\}$ or $f_{zy} \in [*,*,*;1,1,3]$ or $f_{zy} \in [*,*,*;1,0,3]$. In the third case we have $\widehat{f_{zy}} \in [*,*,*;1,0,3]$ and $\widehat{f_{zy}}$ does not belong to Υ, is therefore isomorphic to $m_{4,44}$ by Theorem 4.37 and the minimality of f is contradicted as above. Thus, we must have one of the first two cases:

$$\text{(a) } f_{zy}(\langle 102\rangle) \subseteq \{0,1,2\} \qquad \text{(b) } f_{zy} \in [*,*,*;1,1,3],$$

the latter implying $f_{zy}^{(2)} \in [*,*,*;1,1,1]$. But no matter how we combine the cases (1), (2) and (a), (b), we always obtain that $f_{zy}^{(2)}$ preserves $\{0,1,2\}$. □

We have now limited the set of possible nonconservative minimal majority operations to the set of nonconservative cyclically commutative majority operations. In [7] these are determined by computer, here we do so by hand.

Lemma 4.40: *Let $f \in \Upsilon$ be a minimal nonconservative majority operation on S. Suppose $f|_{\langle 012\rangle} \equiv 3$ (and hence $f_{\langle 102\rangle} \equiv u \in S$). Then $\{0,1,2\}$ is the only subset of S that is not preserved by f.*

Proof: By Lemma 4.39, f is cyclically commutative.
Case 1: Suppose $u \neq 3$. For contradiction suppose without loss of generality that f does not preserve $\{0,1,3\}$. Now $f \in \Upsilon$ implies $f|_{\langle 103\rangle} \equiv 2$ or $f|_{\langle 013\rangle} \equiv 2$. Suppose $f|_{\langle 103\rangle} \equiv 2$. We define

$$g(x,y,z) := f(y,x,f(x,y,z))$$

and we examine the values of g on the sets $\{0,1,2\}$ and $\{0,1,3\}$. We deal with the case $f|_{\langle 013\rangle} \equiv 2$ later, so we can assume for now that $f|_{\langle 013\rangle} \equiv v$ where v is not 2.

(x,y,z)	$g(x,y,z)$
$(0,1,2)$	2
$(1,2,0)$	$f(2,1,3)$
$(2,0,1)$	$f(0,2,3)$
$(1,0,2)$	$f(0,1,u)$
$(0,2,1)$	$f(2,0,u)$
$(2,1,0)$	$f(1,2,u)$

(x,y,z)	$g(x,y,z)$
$(0,1,3)$	$f(1,0,v)$
$(1,3,0)$	$f(3,1,v)$
$(3,0,1)$	$f(0,3,v)$
$(1,0,3)$	3
$(0,3,1)$	$f(3,0,2)$
$(3,1,0)$	$f(1,3,2)$

Now g equals v on two of the three triplets in $\langle 013\rangle$. This implies $g^{(2)}|_{\langle 013\rangle} \equiv v \neq 2$. Similarly, g equals u on two of the three triplets in $\langle 102\rangle$ since we have $u \neq 3$ and hence $g^{(2)}|_{\langle 102\rangle} \equiv u \neq 3$. If one of the two values $f(2,1,3)(= f(1,3,2))$ and $f(0,2,3)(= f(3,0,2))$ equals 2, then we have $g^{(2)}|_{\langle 012\rangle} \equiv 2$ and g preserves $\{0,1,2\}$. Similarly, if one of the two values is 3, then we have $g^{(2)}|_{\langle 103\rangle} \equiv 3$ and g preserves $\{0,1,3\}$. The same follows if both values are 0 or both values are 1. Two cases remain:

$$(1)\ f(2,1,3)=0,\ f(0,2,3)=1 \qquad (2)\ f(2,1,3)=1,\ f(0,2,3)=0.$$

In the first case we have

$$\begin{aligned} g^{(2)}(0,1,2) &= g(2,0,1) = 1 \\ g^{(2)}(1,2,0) &= g(0,1,2) = 2 \\ g^{(2)}(2,0,1) &= g(1,2,0) = 0 \end{aligned}$$

which means that $g^{(2)}(\langle 012\rangle) \subseteq \{0,1,2\}$ and thus $g^{(2)}$ preserves $\{0,1,2\}$. In the second case we have

$$\begin{aligned} g^{(2)}(0,1,2) &= g(2,1,0) \\ g^{(2)}(1,2,0) &= g(1,0,2) \\ g^{(2)}(2,0,1) &= g(0,2,1). \end{aligned}$$

But we know that two of these three values are u which implies $g^{(3)} \in [u,u,u;u,u,u]$ and hence $g^{(3)}$ preserves $\{0,1,2\}$. This finishes the case $f|_{\langle 013\rangle} \equiv v$ where $v \neq 2$. Now suppose $v = 2$. We can calculate $f_z \in [2, f(1,2,3), f(2,0,3), u,u,u]$ (recall $u \neq 3$). If neither $f(1,2,3)$ nor $f(2,0,3)$ equals 3, then f_z preserves $\{0,1,2\}$. Otherwise we define $h(x,y,z) := f(x, f_z(x,y,z), z)$ and we obtain

$$h \in [2, f(1, f(1,2,3), 0), f(2, f(2,0,3), 1); u,u,u].$$

If $f(1,2,3) = 3$, then $h \in [2,2,*;u,u,u]$ and thus $h^{(2)} \in [2,2,2;u,u,u]$ and $h^{(2)}$ preserves $\{0,1,2\}$. If $f(2,0,3) = 3$, then

$$h \in [2, f(1, f(1,2,3), 0), f(2,3,1); u,u,u].$$

Note that $f(2,3,1) = f(1,2,3)$. We obtain

$$h \in \begin{cases} [2,0,0;u,u,u], & \text{if } f(2,3,1) = 0, \\ [2,1,1;u,u,u], & \text{if } f(2,3,1) = 1, \\ [2,3,2;u,u,u], & \text{if } f(2,3,1) = 2, \\ [2,2,3;u,u,u], & \text{if } f(2,3,1) = 3. \end{cases}$$

In all cases it follows that $h^{(2)}$ preserves $\{0,1,2\}$. This finishes the proof for the case $u \neq 3$.

Case 2: Suppose $f|_{\langle 102\rangle} \equiv 3$. For contradiction, let us again suppose that f does not preserve $\{0,1,3\}$ i.e. $f|_{\langle 013\rangle} \equiv 2$ or $f|_{\langle 103\rangle} \equiv 2$. Let us abstract what we have just proved in case 1: If $f|_{\langle abc\rangle} \equiv d$ (a,b,c,d pairwise distinct) and $f|_{\langle bac\rangle} \equiv e \in \{a,b,c\}$, then f preserves the other three subsets of S (namely $\{a,b,d\}$, $\{a,c,d\}$ and $\{b,c,d\}$). But now, for $\{a,b,c\} = \{0,1,3\}$ and $d = 2$ it follows that both of the identities $f|_{\langle 013\rangle} \equiv 2$ and $f|_{\langle 103\rangle} \equiv 2$ must be true (otherwise it would follow that f preserves all other three subsets including $\{0,1,2\}$ which is obviously a contradiction). Furthermore, we can use the same argument to conclude that whenever $f|_{\langle 123\rangle} \equiv 0$ or $f|_{\langle 213\rangle} \equiv 0$ (i.e. f does not preserve $\{1,2,3\}$) it follows that both identities have to be true. In other words: $f|_{\langle 123\rangle} \equiv 0$ if and only if $f|_{\langle 213\rangle} \equiv 0$. In the same way we can also conclude that $f|_{\langle 023\rangle} \equiv 1$ if and only if $f|_{\langle 203\rangle} \equiv 1$. Now, we look at the operation f_z and examine its values on the sets $\{0,1,2\}$ and $\{0,1,3\}$.

(x,y,z)	$f_z(x,y,z)$
$(0,1,2)$	2
$(1,2,0)$	$v = f(1,2,3)$
$(2,0,1)$	$w = f(2,0,3)$
$(1,0,2)$	2
$(0,2,1)$	$r = f(0,2,3)$
$(2,1,0)$	$s = f(2,1,3)$

(x,y,z)	$f_z(x,y,z)$
$(0,1,3)$	3
$(1,3,0)$	$s = f(1,3,2)$
$(3,0,1)$	$r = f(3,0,2)$
$(1,0,3)$	3
$(0,3,1)$	$w = f(0,3,2)$
$(3,1,0)$	$v = f(3,1,2)$

Additionally, we know from above that v equals 0 if and only if s equals 0. Similarly $r = 1$ if and only if $w = 1$. Clearly, f_z must not preserve $\{0,1,2\}$ and $\{0,1,3\}$ hence $3,2 \in \{v,w,r,s\}$. Furthermore, it can be seen that $f_z^{(2)}$ preserves $\{0,1,2\}$ if $2 \in \{v,w\} \cap \{r,s\}$, because in this case we have $f_z^{(2)} \in [2,2,2;2,2,2]$. Similarly $f_z^{(2)}$ preserves $\{0,1,3\}$ if $3 \in \{v,w\} \cap \{r,s\}$. This leaves us with 26 cases.

(1) $f_z \in [2,3,2;2,0,1]$	(10) $f_z \in [2,2,2;2,3,3]$	(19) $f_z \in [2,1,3;2,2,2]$
(2) $f_z \in [2,2,3;2,0,1]$	(11) $f_z \in [2,3,0;2,0,2]$	(20) $f_z \in [2,0,2;2,3,0]$
(3) $f_z \in [2,3,0;2,2,1]$	(12) $f_z \in [2,3,1;2,1,2]$	(21) $f_z \in [2,1,2;2,3,1]$
(4) $f_z \in [2,3,0;2,2,2]$	(13) $f_z \in [2,3,3;2,0,2]$	(22) $f_z \in [2,1,2;2,3,3]$
(5) $f_z \in [2,3,3;2,2,1]$	(14) $f_z \in [2,2,0;2,0,3]$	(23) $f_z \in [2,1,3;2,0,2]$
(6) $f_z \in [2,3,3;2,2,2]$	(15) $f_z \in [2,2,1;2,1,3]$	(24) $f_z \in [2,1,2;2,0,3]$
(7) $f_z \in [2,2,0;2,3,1]$	(16) $f_z \in [2,2,2;2,0,3]$	(25) $f_z \in [2,1,0;2,3,2]$
(8) $f_z \in [2,2,0;2,3,3]$	(17) $f_z \in [2,0,3;2,2,0]$	(26) $f_z \in [2,1,0;2,2,3]$
(9) $f_z \in [2,2,2;2,3,1]$	(18) $f_z \in [2,1,3;2,2,1]$	

For each of this 26 cases we have one of the following eight cases:

(a) $f_z^{(2)} \in [2,2,2;2,2,3]$	(d) $f_z^{(2)} \in [3,3,3;2,2,2]$	(g) $f_z^{(2)} \in [2,2,2;0,0,0]$
(b) $f_z^{(2)} \in [2,2,2;3,2,2]$	(e) $f_z^{(2)} \in [2,2,2;3,3,3]$	(h) $f_z^{(2)} \in [1,1,1;2,2,2]$
(c) $f_z^{(2)} \in [2,2,2;2,2,2]$	(f) $f_z^{(2)} \in [0,0,0;2,2,2]$	

In the cases (a) and (b) we have $f_z^{(3)} \in [2,2,2;2,2,2]$, hence $f_z^{(3)} \in [f]$ preserves $\{0,1,2\}$. In the cases (c),(f),(g),(h) we have that $f_z^{(2)}$ preserves $\{0,1,2\}$. Only the cases (d) and (e) remain. In case (d), the operation $\widehat{f_z} \in [3,3,3;2,2,2]$. $\widehat{f_z}$ is not conservative and satisfies (o) but is clearly not isomorphic to $m_{4,44}$. Hence, by Theorem 4.37, it belongs to Υ. But now, it falls under case 1 of this proof and hence it preserves $\{0,1,3\}$. We suppose that f does not preserve this set, so $f \notin [\widehat{f_z}]$. The same argument can be used for (e) after a permutation of variables. Thus, the minimality of f is contradicted in all 26 cases. This finishes the proof. □

Let us summarize our latest results. We have seen that any minimal nonconservative majority operation $f \in \Upsilon$ is cyclically commutative and preserves all except one of the four three-element subsets of S. We supposed that this set is $\{0,1,2\}$ and that we have $f|_{\langle 012 \rangle} \equiv 3$ which implies $f|_{\langle 102 \rangle} \equiv u$. We will now show that there are essentially two operations satisfying this properties, depending on whether u equals 3 or not.

Lemma 4.41: *Let $f \in \Upsilon$ be a minimal nonconservative majority operation such that $f|_{\langle 012 \rangle} \equiv 3$. If $f|_{\langle 102 \rangle} \equiv u \neq 3$, then $[f] \cong [m_{4,218}]$ where $m_{4,218}$ is the majority operation defined as shown below.*

(x,y,z)	$m_{4,218}(x,y,z)$
$\langle 012\rangle$	3
$\langle 102\rangle$	2
$\{0,1,3\}$	3
$\{0,2,3\}$	3
$\langle 123\rangle$	3
$\langle 213\rangle$	2

Proof: Without loss of generality, we can suppose $f|_{\langle 102\rangle} \equiv 2$. With the help of Lemma 4.30, we can calculate $f_{zy} \in [r, s, t; 2, 2, 2]$ where

$$r = \begin{cases} f(0,1,3), & \text{if } f(0,1,3) \in \{0,1\} \\ f(0,3,2), & \text{if } f(0,1,3) = 3 \end{cases}$$

$$s = \begin{cases} f(1,2,3), & \text{if } f(1,2,3) \in \{1,2\} \\ f(1,3,0), & \text{if } f(1,2,3) = 3 \end{cases}$$

$$t = \begin{cases} f(2,0,3), & \text{if } f(2,0,3) \in \{0,2\} \\ f(2,3,1), & \text{if } f(2,0,3) = 3 \end{cases}$$

Note that $f(0,1,3) = 2$, $f(1,2,3) = 0$ and $f(2,0,3) = 1$ are not possible since f preserves the sets $\{0,1,3\}$, $\{1,2,3\}$ and $\{0,2,3\}$ by Lemma 4.40. Since f is cyclically commutative, we have $f(0,1,3) = f(1,3,0)$, $f(1,2,3) = f(2,3,1)$ and $f(2,0,3) = f(0,3,2)$. We can see that if at most one of the three values $f(0,1,3)$, $f(1,2,3)$, $f(2,0,3)$ equals 3, then f_{zy} preserves $\{0,1,2\}$ and f is not minimal. If two of the three values equal 3 while the third equals $v \in \{0,1,2\}$, then $f_{zy} \in [3,v,v;2,2,2] \cup [v,3,v;2,2,2] \cup [v,v,3;2,2,2]$ and it follows $f_{zy}^{(2)} \in [v,v,v;2,2,2]$ and $f_{zy}^{(2)}$ preserves $\{0,1,2\}$. Only the case $f|_{\langle 013\rangle} \equiv 3$, $f|_{\langle 123\rangle} \equiv 3$, $f|_{\langle 203\rangle} \equiv 3$ remains possible. Now we define

$$g(x,y,z) := f(y,x,f(x,y,z))$$

and we obtain $g \in [f(1,0,3), f(2,1,3), f(0,2,3); 3, 2, 2]$. Suppose that none of the first three values equals 3. Then, by Lemma 4.40, we have $f(1,0,3) \in \{0,1\}$, $f(2,1,3) \in \{1,2\}$ and $f(0,2,3) \in \{0,2\}$. If the three values are not pairwise distinct, then $g^{(2)} \in [0,0,0;2,2,2] \cup [1,1,1;2,2,2] \cup [2,2,2;2,2,2]$ and $g^{(2)}$ preserves $\{0,1,2\}$. If the three values are pairwise distinct, then $(f(1,0,3), f(2,1,3), f(0,2,3)) \in \langle 012\rangle$. Now $3 \notin g(\langle 012\rangle)$ implies that $g^{(2)}$ preserves $\{0,1,2\}$. Thus, $3 \in \{f(1,0,3), f(2,1,3), f(0,2,3)\}$. To eliminate other cases define

$$h(x,y,z) := f(g(x,y,z), g(z,y,x), g(x,z,y)).$$

We examine the range of h completely by determining the range of g.

(x,y,z)	$g(x,y,z)$
$(0,1,2)$	$f(1,0,3)$
$(1,2,0)$	$f(2,1,3)$
$(2,0,1)$	$f(0,2,3)$
$(1,0,2)$	3
$(0,2,1)$	2
$(2,1,0)$	2
$(0,1,3)$	$f(1,0,3)$
$(1,3,0)$	3
$(3,0,1)$	3
$(1,0,3)$	$f(1,0,3)$
$(0,3,1)$	$\begin{cases} 0, & \text{if } f(1,0,3)=0 \\ 3, & \text{if } f(1,0,3)\in\{1,3\} \end{cases}$
$(3,1,0)$	$\begin{cases} 1, & \text{if } f(1,0,3)=1 \\ 3, & \text{if } f(1,0,3)\in\{0,3\} \end{cases}$
$(0,2,3)$	$f(0,2,3)$
$(2,3,0)$	$\begin{cases} 2, & \text{if } f(0,2,3)=2 \\ 3, & \text{if } f(0,2,3)\in\{0,3\} \end{cases}$
$(3,0,2)$	$\begin{cases} 0, & \text{if } f(0,2,3)=0 \\ 3, & \text{if } f(0,2,3)\in\{2,3\} \end{cases}$
$(2,0,3)$	$f(0,2,3)$
$(0,3,2)$	3
$(3,2,0)$	3
$(1,2,3)$	$f(2,1,3)$
$(2,3,1)$	3
$(3,1,2)$	3
$(2,1,3)$	$f(2,1,3)$
$(1,3,2)$	$\begin{cases} 1, & \text{if } f(2,1,3)=1 \\ 3, & \text{if } f(2,1,3)\in\{2,3\} \end{cases}$
$(3,2,1)$	$\begin{cases} 2, & \text{if } f(2,1,3)=2 \\ 3, & \text{if } f(2,1,3)\in\{1,3\} \end{cases}$

$\Longrightarrow$

(x,y,z)	$h(x,y,z)$
$(0,1,2)$	2
$(1,2,0)$	$\begin{cases} 2, & \text{if } f(2,1,3)=2 \\ 3, & \text{if } f(2,1,3)\in\{1,3\} \end{cases}$
$(2,0,1)$	$\begin{cases} 2, & \text{if } f(0,2,3)=2 \\ 3, & \text{if } f(0,2,3)\in\{0,3\} \end{cases}$
$(1,0,2)$	$f(3,f(0,2,3),f(2,1,3))$
$(0,2,1)$	$f(2,f(2,1,3),f(1,0,3))$
$(2,1,0)$	$f(2,f(1,0,3),f(0,2,3))$
$(0,1,3)$	$f(1,0,3)$
$(1,3,0)$	$\begin{cases} f(1,0,3), & \text{if } f(1,0,3)\in\{0,3\} \\ 3, & \text{if } f(1,0,3)=1 \end{cases}$
$(3,0,1)$	$\begin{cases} f(1,0,3), & \text{if } f(1,0,3)\in\{1,3\} \\ 3, & \text{if } f(1,0,3)=0 \end{cases}$
$(1,0,3)$	3
$(0,3,1)$	$\begin{cases} f(1,0,3), & \text{if } f(1,0,3)=0 \\ 3, & \text{if } f(1,0,3)\in\{1,3\} \end{cases}$
$(3,1,0)$	$\begin{cases} f(1,0,3), & \text{if } f(1,0,3)=1 \\ 3, & \text{if } f(1,0,3)\in\{0,3\} \end{cases}$
$(0,2,3)$	3
$(2,3,0)$	$\begin{cases} f(0,2,3), & \text{if } f(0,2,3)=2 \\ 3, & \text{if } f(0,2,3)\in\{0,3\} \end{cases}$
$(3,0,2)$	$\begin{cases} f(0,2,3), & \text{if } f(0,2,3)=0 \\ 3, & \text{if } f(0,2,3)\in\{2,3\} \end{cases}$
$(2,0,3)$	$f(0,2,3)$
$(0,3,2)$	$\begin{cases} f(0,2,3), & \text{if } f(0,2,3)\in\{2,3\} \\ 3, & \text{if } f(0,2,3)=0 \end{cases}$
$(3,2,0)$	$\begin{cases} f(0,2,3), & \text{if } f(0,2,3)\in\{0,3\} \\ 3, & \text{if } f(0,2,3)=2 \end{cases}$
$(1,2,3)$	$f(2,1,3)$
$(2,3,1)$	$\begin{cases} f(2,1,3), & \text{if } f(2,1,3)\in\{1,3\} \\ 3, & \text{if } f(2,1,3)=2 \end{cases}$
$(3,1,2)$	$\begin{cases} f(2,1,3), & \text{if } f(2,1,3)\in\{2,3\} \\ 3, & \text{if } f(2,1,3)=1 \end{cases}$
$(2,1,3)$	3
$(1,3,2)$	$\begin{cases} f(2,1,3), & \text{if } f(2,1,3)=1 \\ 3, & \text{if } f(2,1,3)\in\{2,3\} \end{cases}$
$(3,2,1)$	$\begin{cases} f(2,1,3), & \text{if } f(2,1,3)=2 \\ 3, & \text{if } f(2,1,3)\in\{1,3\} \end{cases}$

Again, note that $f(0,1,3) \neq 2$, $f(1,2,3) \neq 0$ and $f(2,0,3) \neq 1$ because the sets $\{0,1,3\}$, $\{1,2,3\}$ and $\{0,2,3\}$ are preserved by f. We show that at least one of the two values $f(2,1,3), f(0,2,3)$ has to equal 2. Let us suppose otherwise. We then have $h(1,0,2) = f(3, f(0,2,3), f(2,1,3)) = 3$, because we either have that one of the two values $f(0,2,3), f(2,1,3)$ equals 3 or $h(1,0,2) = f(3,0,1) = 3$. We can also observe that $h(0,2,1)$ can only equal 2 if $f(2,1,3) = 1$ and $f(1,0,3) = 0$ and, furthermore, $h(2,1,0)$ can only equal 2 if $f(1,0,3) = 1$ and $f(0,2,3) = 0$. It is obvious that both conditions cannot be true at the same time, so we cannot have $h(0,2,1) = h(2,1,3) = 2$. This means that $h^{(2)}(\langle 102 \rangle) = h(\langle 3rs \rangle)$ for some $r, s \in S$ where r and s cannot both equal 2. But we can see in the table above that

$h(r,s,t) \neq 2$ whenever $3 \in \{r,s,t\}$ and at most one variable of r,s,t equals 2 since $2 \notin \{f(1,0,3), f(2,1,3), f(0,2,3)\}$. Hence, $2 \notin h^{(2)}(\langle 102 \rangle)$. Furthermore, we have $h^{(2)}|_{\langle 012 \rangle} \equiv 3$ because $h(1,2,0) = 3$ and $h(2,0,1) = 3$ since $f(2,1,3), f(0,2,3) \neq 2$. The remaining values of $h^{(2)}$ can be calculated easily by using the table of h above (e.g. we have $h^{(2)}|_{\langle 013 \rangle} \equiv f(1,0,3)$ because it can be observed that h equals $f(1,0,3)$ on at least two of the three triplets in $\langle 013 \rangle$). We obtain

(x,y,z)	$h^{(2)}(x,y,z)$
$\langle 012 \rangle$	3
$(1,0,2)$	$\neq 2$
$(0,2,1)$	$\neq 2$
$(2,1,0)$	$\neq 2$
$\langle 013 \rangle$	$f(1,0,3) \neq 2$
$\langle 103 \rangle$	3
$\langle 023 \rangle$	3
$\langle 032 \rangle$	$f(0,2,3) \neq 2$
$\langle 123 \rangle$	3
$\langle 213 \rangle$	$f(2,1,3) \neq 2$

But now the fact that 2 is not in the range of $h^{(2)} \in [f]$ is a contradiction to the fact that f is minimal by Lemma 4.8. Thus, $2 \in \{f(2,1,3), f(0,2,3)\}$. Recall that we already know $3 \in \{f(1,0,3), f(2,1,3), f(0,2,3)\}$ This leaves us with the following nine cases

(1) $g \in [0,2,3;3,2,2]$ (4) $g \in [1,2,3;3,2,2]$ (7) $g \in [3,2,2;3,2,2]$
(2) $g \in [0,3,2;3,2,2]$ (5) $g \in [3,1,2;3,2,2]$ (8) $g \in [3,2,3;3,2,2]$
(3) $g \in [1,3,2;3,2,2]$ (6) $g \in [3,2,0;3,2,2]$ (9) $g \in [3,3,2;3,2,2]$

We define $l(x,y,z) := f(g(x,y,z), g(z,x,y), g(y,z,x)$ and we obtain for the cases (2),(4),(5),(6),(7):

(2) $l \in [2,2,2;2,2,2]$ (4) $l \in [2,2,2;2,2,2]$ (5) $l \in [1,1,1;2,2,2]$
(6) $l \in [0,0,0;2,2,2]$ (7) $l \in [2,2,2;2,2,2]$

Clearly, $l \in [f]$ preserves $\{0,1,2\}$ in the listed cases and hence contradicts the minimality of f. For the case (1), define $k(x,y,z) := g(g(x,y,z), y, g(y,z,x)$ and we obtain $k \in [0,2,0;3,2,2]$. This implies $k^{(2)} \in [0,0,0;2,2,2]$ and hence $k^{(2)} \in [f]$ preserves $\{0,1,2\}$. In the case (3), define $k(x,y,z) := g(y, g(y,z,x), g(x,y,z))$ and we obtain $k \in [1,2,1;2,2,3]$. Hence, $k^{(2)} \in [1,1,1;2,2,2]$ and again $k^{(2)} \in [f]$ preserves $\{0,1,2\}$. Only the cases (8) and (9) are left. We determine f for these cases and also f^{ϕ}, $\phi = (01)$, for the case (9).

(x,y,z)	(8) $f(x,y,z)$	(9) $f(x,y,z)$	(9) $f^\phi(x,y,z)$
$\langle 012\rangle$	3	3	2
$\langle 102\rangle$	2	2	3
$\langle 013\rangle$	3	3	3
$\langle 103\rangle$	3	3	3
$\langle 023\rangle$	3	2	3
$\langle 203\rangle$	3	3	3
$\langle 123\rangle$	3	3	2
$\langle 213\rangle$	2	3	3

In case (8) we have $f = m_{4,218}$ and the claim follows trivially. In case (9) we have $[f^\phi] = [m_{4,218}]$ because we can obtain f^ϕ from $m_{4,218}$ (and vice versa) by interchanging the first and the second variable. Thus, $[f] \cong [m_{4,218}]$. This finishes the proof. □

Lemma 4.42: *Let $f \in \Upsilon$ be a minimal nonconservative majority operation such that $f|_{\langle 012\rangle} \equiv 3$, $f|_{\langle 102\rangle} \equiv 3$. Then $f = m_{4,0}$ where $m_{4,0}$ is the majority operation that equals* 3 *on triplets of distinct elements.*

Proof: Note that f is cyclically commutative by Lemma 4.39. We use f_{zy} to prove the claim. Let us determine the values of f_{zy} on the set $\{0,1,2\}$ as far as we can by using the identities $f|_{\langle 012\rangle} \equiv f|_{\langle 102\rangle} \equiv 3$ and Lemma 4.30.

(x,y,z)	$f_{zy}(x,y,z)$
$(0,1,2)$	$\begin{cases} f(0,1,3) & \text{if } f(0,1,3) \in \{0,1\}, \\ f(2,0,3) & \text{if } f(0,1,3) = 3. \end{cases}$
$(1,2,0)$	$\begin{cases} f(1,2,3) & \text{if } f(1,2,3) \in \{1,2\}, \\ f(0,1,3) & \text{if } f(1,2,3) = 3. \end{cases}$
$(2,0,1)$	$\begin{cases} f(2,0,3) & \text{if } f(2,0,3) \in \{0,2\}, \\ f(1,2,3) & \text{if } f(2,0,3) = 3 \end{cases}$
$(1,0,2)$	$\begin{cases} f(1,0,3) & \text{if } f(1,0,3) \in \{0,1\} \\ f(2,1,3) & \text{if } f(1,0,3) = 3 \end{cases}$
$(0,2,1)$	$\begin{cases} f(0,2,3) & \text{if } f(0,2,3) \in \{0,2\} \\ f(1,0,3) & \text{if } f(0,2,3) = 3 \end{cases}$
$(2,1,0)$	$\begin{cases} f(2,1,3) & \text{if } f(2,1,3) \in \{1,2\} \\ f(0,2,3) & \text{if } f(2,1,3) = 3 \end{cases}$

$f(0,1,3), f(1,0,3) \neq 2$, $f(0,2,3), f(2,0,3) \neq 1$ and $f(1,2,3), f(2,1,3) \neq 0$ because f preserves the sets $\{0,1,3\}$, $\{0,2,3\}$, $\{1,2,3\}$ by Lemma 4.39. Let us distinguish cases by using the following sets:

$$U = \{f(0,1,3), f(2,0,3), f(1,2,3)\},$$

$$V = \{f(1,0,3), f(0,2,3), f(2,1,3)\}.$$

If $U = \{3\}$, then $f_{zy} \in [3,3,3;*,*,*]$ and $\widehat{f_{zy}} \in [3,3,3;*,*,*]$. If $3 \notin U$, then we have $f_{zy} \in [0,1,2;*,*,*]$ (and $\widehat{f_{zy}} \in [0,1,2;*,*,*]$) or $f_{zy} \in [1,2,0;*,*,*]$ (and again $\widehat{f_{zy}} \in [0,1,2;*,*,*]$) or two of the three values in U have to coincide (say they equal r) and we have $\widehat{f_{zy}} \in [r,r,r;*,*,*]$. If exactly one of the three values in U equals 3, say $f(0,1,3)$, then $r = f_{zy}(0,1,2) = f_{zy}(2,0,1) = f(2,0,3) \neq 3$ and we have $f_{zy} \in [r,*,r;*,*,*]$ and hence $\widehat{f_{zy}} \in [r,r,r;*,*,*]$. If two of three values in U equal 3, then the remaining value appears twice in $f_{zy}(\langle 012\rangle)$ and again $\widehat{f_{zy}} \in [r,r,r;*,*,*]$ for some $r \neq 3$. Thus,

$$\widehat{f_{zy}}|_{\langle 012\rangle} \begin{cases} \equiv 3 & \text{if } U = \{3\}, \\ = p & \text{if } U = \{0,1,2\}, \\ \equiv r \neq 3 & \text{otherwise.} \end{cases}$$

The same arguments can be used to show that

$$\widehat{f_{zy}}|_{\langle 102\rangle} \begin{cases} \equiv 3 & \text{if } V = \{3\}, \\ = p & \text{if } V = \{0,1,2\}, \\ \equiv s \neq 3 & \text{otherwise.} \end{cases}$$

Let us combine possible cases. If $U \neq \{3\}$ and $V \neq \{3\}$, then $\widehat{f_{zy}}$ preserves $\{0,1,2\}$ and f is not minimal. If $U = \{3\}$ and $V = \{0,1,2\}$, then we have $\widehat{f_{zy}}|_{\langle 012\rangle} \equiv 3$ and $\widehat{f_{zy}}|_{\langle 102\rangle} = p$. Now Lemma 4.31 implies that $\widehat{f_{zy}}$ is not minimal which contradicts the minimality of f. The same follows for $U = \{0,1,2\}$ and $V = \{3\}$. If $U = \{3\}$ and V falls under the third case, then $\widehat{f_{zy}}|_{\langle 012\rangle} \equiv 3$ and $\widehat{f_{zy}}|_{\langle 102\rangle} \equiv u \neq 3$. But now, $\widehat{f_{zy}}$ falls under the case handled in the last lemma. Thus, $[\widehat{f_{zy}}] \cong [m_{4,218}]$. But, as we will see later, the clone $[m_{4,218}]$ does not contain an operation isomorphic to f, hence $f \notin [\widehat{f_{zy}}]$, a contradiction. The same follows if U falls under the third case and $V = \{3\}$. Only the case $U = V = \{3\}$ remains. But then we have $f(x,y,z) = 3$ for all $(x,y,z) \in \omega$. Thus, $f = m_{4,0}$. □

Let us summarize what we have done in this section. We have shown that a nonconservative minimal majority operation satisfying (o) on the four-element set S must be (up to isomorphism and permutation of variables) $m_{4,44}$ (if $f \notin \Upsilon$) or one of the two operations $m_{4,0}$, $m_{4,218}$ (if $f \in \Upsilon$). Now, we show that these three operations are indeed minimal operations.

Lemma 4.43: *If we restrict the operations $m_{4,0}$, $m_{4,218}$, $m_{4,44}$ on the set $\{1,2,3\}^3$, then they are isomorphic to the operations m_0, m_{510} and m_{44}, respectively (see Table 4.4).*

Proof: We have already seen that all three operations preserve the set $\{1,2,3\}$. Hence, $m_{4,0}|_{\{1,2,3\}^3}$, $m_{4,218}|_{\{1,2,3\}^3}$, $m_{4,44}|_{\{1,2,3\}^3}$ can be considered as majority operations on the three-element set $\{1,2,3\}$. Now $m_{4,0}|_{\{1,2,3\}^3} \cong m_0$, $m_{4,44}|_{\{1,2,3\}^3} \cong m_{44}$ and $m_{4,218}|_{\{1,2,3\}^3} \cong m_{510}$ can be seen by renaming the elements 1,2,3 to 1,2,0. □

Lemma 4.44: *$m_{4,0}$, $m_{4,218}$, $m_{4,44}$ are minimal (majority) operations on S.*

Proof: The proof is the same for all three operations, so let f be any of them. Let $g \in [f]$ be an arbitrary majority operation. We have to show $f \in [g]$. f preserves the equivalence relation σ whose blocks are $\{0,3\}$, $\{1\}$, $\{2\}$ and its range does not contain the element 0. By Lemma 2.5 and Lemma 4.8, this must also be true for g. We show that these two properties determine $g|_{\{0,1,2\}^3}$ whenever $g|_{\{1,2,3\}^3}$ is given: Let $(r,s,t) \in \{0,1,2\}^3$. We can assume that r,s,t are pairwise distinct, because otherwise $g(r,s,t)$ is determined by the majority rule. We can chose $(u,v,w) \in \{1,2,3\}^3$ such that $\begin{pmatrix} r \\ u \end{pmatrix}, \begin{pmatrix} s \\ v \end{pmatrix}, \begin{pmatrix} t \\ w \end{pmatrix} \in \sigma$. Since g has to preserve σ, we must have $\begin{pmatrix} g(r,s,t) \\ g(u,v,w) \end{pmatrix} \in \sigma$. But now, $0 \notin \{g(r,s,t), g(u,v,w)\}$ implies $g(r,s,t) = g(u,v,w)$. Thus, $g|_{\{1,2,3\}^3}$ determines $g|_{\{0,1,2\}^3}$. Since f preserves $\{1,2,3\}$, we can consider $f|_{\{1,2,3\}^3}$ as an operation on the set $\{1,2,3\}$. Now this operation is minimal since it is isomorphic to one of the operations m_0, m_{510}, m_{44} by the last lemma and we have already seen that these are minimal. This implies that there has to exist $h \in [g]$ such that $h|_{\{1,2,3\}^3} = f|_{\{1,2,3\}^3}$. But now, h also has the two properties described above and hence $h|_{\{1,2,3\}^3} = f|_{\{1,2,3\}^3}$ determines $h|_{\{0,1,2\}^3}$ uniquely: It can be nothing else but $f|_{\{0,1,2\}^3}$. On the remaining two three-element subsets of S, namely $\{0,1,3\}$ and $\{0,2,3\}$, f is always 3 which means that we must have the same thing for h. Thus, $h = f$ and $f \in [g]$. □

Combining this lemma with our previous results gives us the main theorem of this section.

Theorem 4.45: *Up to isomorphism, we have exactly three nonconservative minimal majority clones on S: $[m_{4,0}]$, $[m_{4,218}]$ and $[m_{4,44}]$.*

Lemma 4.43 clearly implies that the restriction to the set $\{1,2,3\}^3$ gives us a one-to-one correspondence between the majority operations in $m_{4,i}$ on S and m_i on the three-element set $\{0,1,2\}$. Hence, our clones $[m_{4,0}]$, $[m_{4,218}]$ and $[m_{4,44}]$ contain one, eight and three majority operations, respectively. We can easily determine all of them by using Table 4.4. They can be seen in the tables on the following page.

Table 4.8

The nonconservative minimal majority operations on $\{0,1,2,3\}$ *(up to isomorphism)*

(x,y,z)	$[m_{4,0}]$ $m_{4,0}$
$(0,1,2)$	3
$(1,2,0)$	3
$(2,0,1)$	3
$(1,0,2)$	3
$(0,2,1)$	3
$(2,1,0)$	3
$\{0,1,3\}$	3
$\{0,2,3\}$	3
$(3,1,2)$	3
$(1,2,3)$	3
$(2,3,1)$	3
$(1,3,2)$	3
$(3,2,1)$	3
$(2,1,3)$	3

		$[m_{4,44}]$	
(x,y,z)	$m_{4,624}$	$m_{4,44}$	$m_{4,424}$
$(0,1,2)$	2	3	2
$(1,2,0)$	3	1	3
$(2,0,1)$	1	2	1
$(1,0,2)$	2	1	3
$(0,2,1)$	1	3	2
$(2,1,0)$	3	2	1
$\{0,1,3\}$	3	3	3
$\{0,2,3\}$	3	3	3
$(3,1,2)$	2	3	1
$(1,2,3)$	3	1	2
$(2,3,1)$	1	2	3
$(1,3,2)$	2	1	3
$(3,2,1)$	1	3	2
$(2,1,3)$	3	2	1

				$[m_{4,218}]$				
(x,y,z)	$m_{4,510}$	$m_{4,546}$	$m_{4,26}$	$m_{4,666}$	$m_{4,218}$	$m_{4,182}$	$m_{4,702}$	$m_{4,62}$
$(0,1,2)$	2	2	3	2	3	3	2	3
$(1,2,0)$	2	3	2	2	3	2	3	3
$(2,0,1)$	2	2	2	3	3	3	3	2
$(1,0,2)$	3	2	3	3	2	3	2	2
$(0,2,1)$	3	3	3	2	2	2	2	3
$(2,1,0)$	3	3	2	3	2	2	3	2
$\{0,1,3\}$	3	3	3	3	3	3	3	3
$\{0,2,3\}$	3	3	3	3	3	3	3	3
$(3,1,2)$	2	2	3	2	3	3	2	3
$(1,2,3)$	2	3	2	2	3	2	3	3
$(2,3,1)$	2	2	2	3	3	3	3	2
$(1,3,2)$	3	2	3	3	2	3	2	2
$(3,2,1)$	3	3	3	2	2	2	2	3
$(2,1,3)$	3	3	2	3	2	2	3	2

4.5 Extending minimal majority operations

Arguments similar to the ones that Waldhauser used in Lemma 4.44 can be used to construct, for a given minimal majority operation f on $S = \{0, ..., n-1\}$, a minimal majority operation f' on $S \cup \{n\}$. The following two theorems give two different ways to do so:

Theorem 4.46: *Let f be a minimal majority operation on $S = \{0, ..., n-1\}$. Set $S' := S \cup \{n\}$ and let $a \in S$. For $x \in S'$ set $\widetilde{x} = \begin{cases} x, & \text{if } x \in S \\ a, & \text{if } x = n \end{cases}$. Define $f'(x, y, z) := f(\widetilde{x}, \widetilde{y}, \widetilde{z})$. Then f' is a minimal majority operation.*

Proof: For notational simplicity let $a = 0$. It is easy to see that f' is a majority operation. Suppose $g' \in [f']$ where g' is a majority operation. As usual, we have to show $f' \in [g']$. Clearly, n is not in the range of f' (and hence f' preserves the set S) and one can check that f' preserves the equivalence relation σ on S' whose blocks are $\{0, n\}, \{1\}, ..., \{n-1\}$; hence, these two properties have to hold for g' as well. But now, similarly as in the proof of Lemma 4.44, the restriction $g'|_{S^3}$ determines g': Let $(r, s, t) \in S'^3$. We can assume that r, s, t are pairwise distinct, because otherwise $g(r, s, t)$ is determined by the majority rule. We can chose $(u, v, w) \in S^3$ such that $\binom{r}{u}, \binom{s}{v}, \binom{t}{w} \in \sigma$. Since g' has to preserve σ we must have $\binom{g'(r,s,t)}{g'(u,v,w)} \in \sigma$. But now $n \notin \{g'(r, s, t), g'(u, v, w)\}$ implies $g'(r, s, t) = g'(u, v, w)$. Thus, $g'|_{S^3}$ determines g' uniquely. Since f' preserves S, we can consider $f'|_{S^3}$ as an operation on the set S. Now $f'|_{S^3} = f$ implies that $f'|_{S^3}$ is minimal on S. This implies that there has to exist $h' \in [g']$ such that $h'|_{S^3} = f'|_{S^3}$. But now, h' also has the two properties described above and hence $h'|_{S^3} = f'|_{S^3}$ determines h uniquely: It can be nothing else but f'. Thus, $h' = f'$ and $f' \in [g']$, implying that f' is minimal. □

Theorem 4.47: *Let f be a minimal majority operation on $S = \{0, ..., n-1\}$. Set $S' := S \cup \{n\}$. Define the operation f' on S' as follows: For all pairwise distinct $x, y, z \in S'$*

$$f'(x, y, z) = \begin{cases} f(x, y, z), & \text{if } (x, y, z) \in S^3 \\ n, & \text{otherwise} \end{cases}$$

The remaining values may be defined by the majority rule. Then f' is a minimal majority operation on S'.

Proof: Suppose $g' \in [f']$ where g' is a majority operation. Again, we have to show $f' \in [g']$. For all pairwise distinct x, y, z where $(x, y, z) \in S'^3 \setminus S^3$ (i.e. $n \in \{x, y, z\}$), we have $f'(x,y,z) = n$ which means that f' preserves $\{x,y,z\}$ and that the range of the restriction $f'|_{\{x,y,z\}^3}$ consists of n only. Hence, g' also preserves $\{x,y,z\}$ and the range of the restriction $g'|_{\{x,y,z\}^3}$ also consists of $\{n\}$ only since it is generated by $f'|_{\{x,y,z\}^3}$ (see Lemma 4.8). As above, $f'|_{S^3}$ is a minimal operation on S. This implies that there has to exist $h' \in [g']$ such that $h'|_{S^3} = f'|_{S^3}$. Furthermore, $h' \in [g']$ implies that the property described above must hold for h'. It follows $h'(x,y,z) = f'(x,y,z)$ for all $(x,y,z) \in S'^3 \setminus S^3$. Hence, $h'(x,y,z) = f'(x,y,z)$ for all pairwise distinct $(x,y,z) \in S'^3$. On the remaining triplets, h' has to coincide with f' due to the majority property. Thus, $h' = f'$ and $f' \in [g']$. □

These two techniques allow us to extend a given minimal majority operation to an arbitrarily larger universe. In particular, they allow us to find nonconservative minimal majority operation on any finite set S. Note that both techniques can be generalized to work for all minimal operations.

Example: We will use the two techniques given in the theorems above to extend the minimal operation $m_{4,44}$ (see Lemma 4.41) on $\{0,1,2,3\}$ to the minimal operation $m'_{4,44}$ on $\{0,1,2,3,4\}$. The result is stated in the table on the following page.

(x,y,z)	$m'_{4,44}(x,y,z)$ as in Theorem 4.46 (setting $a=0$)	$m'_{4,44}(x,y,z)$ as in Theorem 4.47
$(0,1,2)$	3	3
$(1,2,0)$	1	1
$(2,0,1)$	2	2
$(1,0,2)$	1	1
$(0,2,1)$	3	3
$(2,1,0)$	2	2
$(0,1,3)$	3	3
$(1,3,0)$	3	3
$(3,0,1)$	3	3
$(1,0,3)$	3	3
$(0,3,1)$	3	3
$(3,1,0)$	3	3
$(0,2,3)$	3	3
$(2,3,0)$	3	3
$(3,0,2)$	3	3
$(2,0,3)$	3	3
$(0,3,2)$	3	3
$(3,2,0)$	3	3
$(1,2,3)$	1	1
$(2,3,1)$	2	2
$(3,1,2)$	3	3
$(2,1,3)$	2	2
$(1,3,2)$	1	1
$(3,2,1)$	3	3
$(0,1,4)$	0	4
$(1,4,0)$	0	4
$(4,0,1)$	0	4
$(1,0,4)$	0	4
$(0,4,1)$	0	4
$(4,1,0)$	0	4
$(0,2,4)$	0	4
$(2,4,0)$	0	4
$(4,0,2)$	0	4
$(2,0,4)$	0	4
$(0,4,2)$	0	4
$(4,2,0)$	0	4
$(1,2,4)$	1	4
$(2,4,1)$	2	4
$(4,1,2)$	3	4
$(2,1,4)$	2	4
$(1,4,2)$	1	4
$(4,2,1)$	3	4
$(2,3,4)$	3	4
$(3,4,2)$	3	4
$(4,2,3)$	3	4
$(3,2,4)$	3	4
$(2,4,3)$	3	4
$(4,3,2)$	3	4

Chapter 5

Majority clones and relations

We have already noted that for any clone C (on a finite set S) there exists a set of relations R such that the clone C is the set of all operations on S preserving all relations in R (i.e. $C = Pol\ R$). In fact, in this chapter, we will see that if C is a majority clone, then we can chose R to contain only one relation. After reproducing this well-known fact, our first goal will be to find such relations for the minimal majority clones on $\{0,1\}$ and $\{0,1,2\}$.

Assume S to be the n-element set $\{0,...,n-1\}$.

Definition 5.1: Denote by R_S the set of finitary relations on S. For a positive integer m, let $R_S^{(m)}$ denote the set of m-ary relations on S. For a set F of operations on S define $Inv\ F$ to be the set of relations on S that are preserved by all $f \in F$, i.e.

$$Inv\ F = \{\sigma \in R_S \mid f \text{ preserves } \sigma \text{ for all } f \in F\}.$$

For a single operation f, we write $Inf\ f$ instead of $Inv\ \{f\}$. Furthermore, set $Inv^{(m)}F = (Inv\ F) \cap O_S^{(m)}$.

Recall that, for $R \in R_S$, $Pol\ R$ is the set of operations on S preserving all relations in R (by Lemma 2.6, they form a clone). Set $Pol^{(m)}R = (Pol\ R) \cap R_S^{(m)}$.

It is easy to see that, for two relations $\sigma_1, \sigma_2 \in Inv\ F^{(m)}$, we must have $\sigma_1 \cap \sigma_2 \in Inv\ F$. Furthermore, for an arbitrary $m \in \mathbb{N}$, the relation S^m is obviously preserved by any set of operations. This justifies the following definition.

Definition 5.2: Let C be a clone, $\sigma \in R_S^{(m)}$ and $m \in \mathbb{N}$. Denote by $\Gamma_C(\sigma)$ the smallest relation in $Inv^{(m)}C$ (with respect to $\subseteq$) that contains σ, i.e.

$$\Gamma_C(\sigma) = \bigcap\{\sigma' | \sigma \subseteq \sigma', \sigma' \in Inv^{(m)}C\}).$$

Clearly, Γ_C is a closure operator on the partially ordered set $(R_S^{(m)}, \subseteq)$; i.e. for all σ_1, σ_2: $\sigma_1 \subseteq \Gamma_C(\sigma_1)$, $\sigma_1 \subseteq \sigma_2 \Rightarrow \Gamma_C(\sigma_1) \subseteq \Gamma_C(\sigma_2)$ and $\Gamma_C(\Gamma_C(\sigma_1)) = \Gamma_C(\sigma_1)$. Furthermore, we introduce the following notation. For a k-ary operation f and m-tuples $\nu_i = \begin{pmatrix} \nu_{i1} \\ \vdots \\ \nu_{im} \end{pmatrix} \in S^m$ $(i = 1, ..., m)$ we write $[\nu_1, ..., \nu_k]$ for the $m \times k$-matrix whose columns are (in this order) $\nu_1, ..., \nu_k$ and $f[\nu_1, ..., \nu_k]$ for the m-tuple $\begin{pmatrix} f(\nu_{11}, \nu_{21}, ..., \nu_{k1}) \\ \vdots \\ f(\nu_{1m}, \nu_{2m}, ..., \nu_{km}) \end{pmatrix}$.

For example, if $k = m = 2$, $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then

$$f[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}] = \begin{pmatrix} f(1,0) \\ f(2,1) \end{pmatrix}$$

.

Lemma 5.3: *Let C be a clone, $m \in \mathbb{N}$ and $\sigma \in R_S^{(m)}$. Then*
(a) $\Gamma_C(\sigma) = \{f[\nu_1, ..., \nu_k] \mid \nu_1, ..., \nu_k \in \sigma, f \in C^{(k)}, k \in \mathbb{N}\}$.
(b) If $\sigma = \{\nu_1, ...\nu_q\}$, then $\Gamma_C(\sigma) = \{g[\nu_1, ..., \nu_q] \mid g \in C^{(q)}\}$.

Proof: [15] (a) Denote by δ for the right hand side of the equation.
First, we show $\Gamma_C(\sigma) \subseteq \delta$. For all $\nu \in \sigma$ we have $e_1^1[\nu] = \nu$, thus $\sigma \subseteq \delta$. Furthermore, we show that $\delta \in Inv\ C$. Let $l \in \mathbb{N}$, $f \in C$ be l-ary and let $\tau_1, ..., \tau_l \in \delta$ be arbitrary. We have to show $f[\tau_1, ..., \tau_l] \in \delta$. As $\tau_i \in \delta$, we have $\tau_i = f_i[\nu_{i,1}, ..., \nu_{i,k_i}]$ for some $k_i \in \mathbb{N}$, $f_i \in C^{(k_i)}$, $\nu_{i,j} \in \sigma$ $(i \in \{1, ..., l\}$, $j \in \{1, ..., k_i\})$. Set $k := \sum_{i=1}^{l} k_i$ and define the k-ary operation h by

$$h := f(f_1(e_1^k, ..., e_{k_1}^k), f_2(e_{k_1+1}^k, ..., e_{k_2}^k), ..., f_l(e_{(\sum_{i=1}^{l-1} k_i)+1}^k, ..., e_k^k))$$

(see Definition 2.1). Clearly, $h \in C$. Now

$$\begin{aligned} f[\tau_1, ..., \tau_l] &= f[f_1[\nu_{1,1}, ..., \nu_{1,k_1}], ..., f_l[\nu_{l,1}, ..., \nu_{l,k_l}]] \\ &= h[\nu_{1,1}, ..., \nu_{1,k_1}, \nu_{2,1}, ..., \nu_{l,k_l}] \in \delta. \end{aligned}$$

Thus, $\delta \in Inv\ C$. Now the minimality of $\Gamma_C(\sigma)$ together with $\sigma \subseteq \delta \in Inv\ C$ implies $\Gamma_C(\sigma) \subseteq \delta$.
Conversely, to show $\delta \subseteq \Gamma_C(\sigma)$, let $\tau \in \delta$. This means that there exist $f \in C^{(k)}$ and $\nu_1, ..., \nu_k \in \sigma$ such that $\tau = f[\nu_1, ..., \nu_k]$. But now $\nu_1, ..., \nu_k \in \Gamma_C(\sigma) \in Inv\ C$ implies $\tau = f[\nu_1, ..., \nu_k] \in \Gamma_C(\sigma)$.
(b) First, $\{g[\nu_1, ..., \nu_q] \mid g \in C^{(q)}\} \subseteq \Gamma_C(\sigma)$ is obvious by (a). It remains to show that $\{g[\nu_1, ..., \nu_q] \mid g \in C^{(q)}\} \supseteq \Gamma_C(\sigma)$. By (a), the latter equals

$$\{f[\theta_1, ..., \theta_k] \mid \theta_1, ..., \theta_k \in \sigma, f \in C^{(k)}, k \in \mathbb{N}\}.$$

Let $k \in \mathbb{N}$, $f \in C^{(k)}$ and $\theta_1, ..., \theta_k \in \sigma = \{\nu_1, ..., \nu_q\}$. Now, there exists a map $\phi : \{1, .., k\} \to \{1, ..., q\}$ such that $\theta_i = \nu_{\phi(i)}$. Hence,

$$f[\theta_1, ..., \theta_k] = f[e^q_{\phi(1)}[\nu_1, ...\nu_q], ..., e^q_{\phi(k)}[\nu_1, ...\nu_q]] = f(e^q_{\phi(1)}, ..., e^q_{\phi(k)})[\nu_1, ..., \nu_q]$$

This, together with $f(e^q_{\phi(1)}, ..., e^q_{\phi(k)}) \in C^{(q)}$, finishes the proof. □

Example: Let C be the minimal majority clone $[d]$ on $\{0, 1\}$ where d is the dual discriminator of Fried and Pixley. We know that the set of ternary operations in $[d]$ is $\{e^3_1, e^3_2, e^3_3, d\}$.

$$\text{Let } \sigma = \left\{ \begin{pmatrix} 0\\0\\0\\0\\1\\1\\1\\1 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\1\\0\\0\\1\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\1\\0\\1\\0\\1 \end{pmatrix} \right\}.$$

$$\text{Then } \Gamma_C(\sigma) = \left\{ \begin{pmatrix} e^3_1(0,0,0)\\e^3_1(0,0,1)\\e^3_1(0,1,0)\\e^3_1(0,1,1)\\e^3_1(1,0,0)\\e^3_1(1,0,1)\\e^3_1(1,1,0)\\e^3_1(1,1,1) \end{pmatrix}, \begin{pmatrix} e^3_2(0,0,0)\\e^3_2(0,0,1)\\e^3_2(0,1,0)\\e^3_2(0,1,1)\\e^3_2(1,0,0)\\e^3_2(1,0,1)\\e^3_2(1,1,0)\\e^3_2(1,1,1) \end{pmatrix}, \begin{pmatrix} e^3_3(0,0,0)\\e^3_3(0,0,1)\\e^3_3(0,1,0)\\e^3_3(0,1,1)\\e^3_3(1,0,0)\\e^3_3(1,0,1)\\e^3_3(1,1,0)\\e^3_3(1,1,1) \end{pmatrix}, \begin{pmatrix} d(0,0,0)\\d(0,0,1)\\d(0,1,0)\\d(0,1,1)\\d(1,0,0)\\d(1,0,1)\\d(1,1,0)\\d(1,1,1) \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} 0\\0\\0\\0\\1\\1\\1\\1 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\1\\0\\0\\1\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\1\\0\\1\\0\\1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1\\0\\1\\1\\1 \end{pmatrix} \right\}.$$

We are now ready to prove that any majority clone can be written as $Pol\ \sigma$ for a single relation σ. We start with the Baker-Pixley Theorem that we have already used in the last chapter (see 4.13). We now have the tools to prove it.

Theorem 5.4: (**The Baker-Pixley Theorem**) *If C is a majority clone on S, then $C = Pol\ Inv^{(2)}C$.*

Proof: [15] Let $m \in C$ be a majority operation and $f \in Pol\ Inv^{(2)}C$. We have to show $f \in C$. Let l be the arity of f. Set

$$\mathcal{M} = \{B \subseteq S^l \mid f|_B = g|_B \text{ for some } g \in C^{(l)}\}.$$

It suffices to show that $S^l \in \mathcal{M}$ (because then $f = g \in C$). To do so, we prove $B \in \mathcal{M}$ for all $B \subseteq S^l$ by induction over $k = |B|$.
For the base case, let $k = 2$, i.e. $B = \{(\tau_1, ..., \tau_l), (\tau_1', ..., \tau_l')\}$. Set $\sigma_B = \{\begin{pmatrix}\tau_1\\ \tau_1'\end{pmatrix}, ..., \begin{pmatrix}\tau_l\\ \tau_l'\end{pmatrix}\}$.
Then $\Gamma_C(\sigma_B) \in Inv^{(2)}C$ implies $\begin{pmatrix}f(\tau_1, ..., \tau_l)\\ f(\tau_1', ..., \tau_l')\end{pmatrix} \in \Gamma_C(\sigma_B)$ and, by Lemma 5.3, $\Gamma_C(\sigma_B) = \{g[\begin{pmatrix}\tau_1\\ \tau_1'\end{pmatrix}, ..., \begin{pmatrix}\tau_l\\ \tau_l'\end{pmatrix}] \mid g \in C^{(l)}\}$. Thus, there exists $g \in C$ such that $f|_B = g|_B$. It follows $B \in \mathcal{M}$, as required.
Now suppose $k \geq 2$ and that the claim is true for all $B' \subseteq S^l$ with $|B'| \leq k$. Let $|B| = k + 1$ and let $a_1, a_2, a_3 \in B$ be pairwise different (possible since $k \geq 2$). We define $B_i = B \setminus \{a_i\}$ $(i = 1, 2, 3)$. By induction hypothesis, there exist $g_1, g_2, g_3 \in C$ such that $f|_{B_i} = g_i|_{B_i}$. Set $g = m(g_1, g_2, g_3)$. We show that $f|_B = g|_B$. Let $x \in B$. Due to $|B| = k + 1$ and $B_i \subseteq B$, $B_i = |k|$, clearly x belongs to at least two sets among B_1, B_2, B_3, say B_1 and B_2. Hence, by the majority rule,

$$g(x) = m(g_1(x), g_2(x), g_3(x)) = m(f(x), f(x), g_3(x)) = f(x).$$

Thus, $B \in \mathcal{M}$. This finishes the induction and for $k = |S|^l$ it follows $S^l \in \mathcal{M}$, as required. □

Clearly, this theorem implies that for any majority clone C, we have a set of binary relations R such that $C = Pol\ R$. Since the set of binary relations on a finite set is finite, R is finite. However, this implies that C can be written as $Pol\ \sigma$ for a single relation σ as the following lemma shows.

Corollary 5.5 *Let C be a clone and let R be a finite set of (finitary) relations. If $C = Pol\ R$, then $C = Pol\ \sigma$ for a single relation σ.*

Proof: If suffices to show this for $|R| = 2$. Let $R = \{\xi_1, \xi_2\}$ where ξ_i is j_i-ary $(i = 1, 2)$. Set

$$\sigma := \{\begin{pmatrix}a_1\\ \vdots\\ a_{j_1+j_2}\end{pmatrix} \mid \begin{pmatrix}a_1\\ \vdots\\ a_{j_1}\end{pmatrix} \in \xi_1, \begin{pmatrix}a_{j_1+1}\\ \vdots\\ a_{j_1+j_2}\end{pmatrix} \in \xi_2\}.$$

It can be checked directly that on one hand $Pol\ R \subseteq Pol\ \sigma$. On the other hand $Pol\ \sigma \subseteq Pol\ \xi_1$ since ξ_1 is the projection of σ on the first j_1 coordinates. Similarly $Pol\ \sigma \subseteq Pol\ \xi_2$ and so $Pol\ \sigma \subseteq Pol\ R$. □

This implies that any majority clone can be written as $Pol\ \sigma$ for some $\sigma \in R_S$.

We can deduce more from the Baker-Pixley Theorem.

Corollary 5.6: *Recall that L_S denotes the set of clones on S (see Definition 2.10). Let m be a majority operation on S. Set $M_m = \{C \in L_S \mid m \in C\}$. Then*

a) M_m is finite.

b) Each $C \in M_m$ is finitely generated (i.e. there exist $f_1, ..., f_k$ such that $C = [\{f_1, ..., f_k\}]$).

Proof: a) $|S| = n$ implies that there are 2^{n^2} binary relations on S. According to the Baker-Pixley Theorem, each $C \in M_m$ is determined by a set of binary relations and so $|M_m| \leq 2^{2^{n^2}}$.
b) Let $C \in M_m$. Suppose to the contrary that C is not finitely generated. For $k \geq 3$, denote by C_k the clone generated by $C^{(k)}$. Clearly $m \in C_3 \subsetneq C_4 \subsetneq ... \subseteq C$ while $\bigcup_{k\geq 3} = C$. Thus $C_k \in M_m$ for all $k \geq 3$, contradicting a). $\square$

Before we start dealing with the minimal majority clones, we introduce some more definitions. Recall that $S = \{0, ..., n-1\}$

Definition 5.7: Let $m \in \mathbb{N}$. Denote by χ_m the $n^m \times m$ matrix over S whose i-th row is $(a_{m-1}, ..., a_0)$ where $a_{m-1}, ..., a_0$ are the unique elements of S such that

$$i - 1 = a_{m-1}n^{m-1} + ... + a_1 n + a_0$$

(i.e. the rows are the elements of S^m listed in the increasing lexicographic order). Call χ_m the *m-th abscissa of S*.

Example:

$$\chi_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ & \vdots & \\ 0 & 0 & n-1 \\ 0 & 1 & 0 \\ & \vdots & \\ n-1 & n-1 & n-1 \end{pmatrix}.$$

Definition 5.8: Denote by $\kappa_1, ...\kappa_m$ the columns of χ_m. For a clone C on S, the *m-th graphic* of C is the n^m-ary relation $\Gamma_C(\{\kappa_1, ..., \kappa_m\})$. Denote it by Γ_C^m.

Using Lemma 5.3 (b), this means $\Gamma_C^m = \{f[\kappa_1, ..., \kappa_m] \mid f \in C^{(m)}\}$. If we look back at the example given after Lemma 5.3, we may note that it determines $\Gamma_{[d]}^3$ for $n = 2$, $S = \{0, 1\}$. We may also note that any tuple $f[\kappa_1, ..., \kappa_m]$ determines f uniquely (and vice versa).

Lemma 5.9: *Let C be a clone on S. Then $C \subseteq Pol\ \Gamma_C^m$ and $C^{(k)} = Pol^{(k)}\Gamma_C^m$ for all $k \leq m$.*

Proof: [2], [15] Clearly, $\Gamma_C^m \in Inv\ C$ implies $C \subseteq Pol\ \Gamma_C^m$. This proves the first part and $C^{(k)} \subseteq Pol^{(k)}\Gamma_C^m$. It remains to show that $Pol^{(k)}\Gamma_C^m$ is contained in $C^{(k)}$ for all $k \leq m$. Let $f \in Pol^{(k)}\Gamma_C^m$. We add $m-k$ non-relevant variables to f obtaining $f' \in O_S^{(m)}$. Clearly, $f \in Pol^{(k)}\Gamma_C^m$ implies $f' \in Pol^{(m)}\Gamma_C^m$. Let $\kappa_1, ..., \kappa_m$ be the columns of χ_m. Now the projections imply $\kappa_1, ..., \kappa_m \in \Gamma_C^m$ and hence $f'[\kappa_1, ..., \kappa_m] \in \Gamma_C^m$. By Lemma 5.3 there has to exist $g' \in C^{(m)}$ such that $f'[\kappa_1, ..., \kappa_m] = g'[\kappa_1, ..., \kappa_m]$ which implies $f' = g'$. This means that g' is essentially the k-ary operation f. Thus, $f \in Pol^{(k)}\Gamma_C^m$. $\square$

In fact, Γ_C^m is the largest clone D such that $D^{(m)} = C^{(m)}$.

Definition 5.10: Let C be a clone on S. Define

$$\gamma(C) = min\{i \in \mathbb{N} \mid C = Pol\ \Gamma_C^i\}$$

and set $\gamma(C) = \infty$ if $C \neq Pol\ \Gamma_C^i$ for all $i \in \mathbb{N}$.

We will see that $\gamma(C) < \infty$ holds for all majority clones C.

Definition 5.11: Let C be a clone and let σ, σ' be relations of the same arity. We say that σ' *C-generates* σ if $\Gamma_C(\sigma') = \sigma$. Call σ a *C-independent* relation if $\nu \notin \Gamma_C(\sigma \setminus \{\nu\})$ for all $\nu \in \sigma$.

Example: Let $C = [m]$ where m is a majority operation with $m(0,1,2) = 0$. The relation $\sigma = \{\binom{0}{0}, \binom{0}{1}, \binom{1}{1}, \binom{2}{1}\}$ is not C-independent, because $\binom{0}{1} \in \Gamma_C(\sigma \setminus \{\binom{0}{1}\}) = \Gamma_C(\{\binom{0}{0}, \binom{1}{1}, \binom{2}{1}\})$.

Note that any relation σ contains a C-independent relation σ' such that $\sigma \subseteq \Gamma_C(\sigma')$. In particular, any relation that is preserved by C is C-generated by a C-independent relation.

Lemma 5.12: *Let C be a clone on S. Then $Pol\ \Gamma_C^k$ preserves any relation $\sigma \in R_S$ that is C-generated by a relation $\sigma' \in R_S$ where $|\sigma'| \leq k$.*

Proof: Note that the fact that σ is C-generated by σ' implies that C preserves σ. Let $\kappa_1, ...\kappa_k$ be the columns of χ_k. Let f be an arbitrary operation in $Pol\ \Gamma_C^k$ and let l be the arity of f. We have to show that $f[\tau_1, ..., \tau_l] \in \sigma$ for all $\tau_i \in \sigma$

$(i = 1, .., l)$. By the assumption of the claim, there exists a C-independent relation $\{\nu_1, ..., \nu_k\} \subseteq \sigma$ such that $\Gamma_C(\{\nu_1, ..., \nu_k\}) = \sigma$ (note that $\nu_1, ...\nu_k$ do not have to be distinct). Let q be the arity of the relation σ. We can construct the $q \times k$ matrix $[\nu_1, ..., \nu_k]$ by taking q (not necessarily distinct) rows of the matrix $\chi_k = [\kappa_1, ..., \kappa_k]$. Let $r_1, ..., r_q$ be the indices of these rows. Now $\sigma = \Gamma_C(\{\nu_1, ..., \nu_k\})$ implies that the remaining elements of σ can be obtained by taking the positions $r_1, ...r_q$ of some other tuples $\kappa'_1, ..., \kappa'_{|\sigma|-|\sigma'|}$ that are also in Γ^k_C. This means that we can construct $[\tau_1, ..., \tau_l]$ as the matrix we obtain by taking the rows number $r_1, ..., r_q$ of a matrix $[\kappa_{i_1}, ..., \kappa_{i_l}]$ where $\kappa_{i_j} \in \{\kappa_1, ...\kappa_k, \kappa'_1, ..., \kappa'_{|\sigma|-|\sigma'|}\} \subseteq \Gamma^k_C$ $(j = 1, ..., l)$. Since f has to preserve Γ^k_C, we must have a tuple $\delta \in \Gamma^k_C$ such that $f[\tau_1, ..., \tau_l] = \begin{pmatrix} \delta_{r_1} \\ \vdots \\ \delta_{r_q} \end{pmatrix}$ where δ_{r_i} is the r_i-th position of δ. But now the fact that C preserves σ implies that for any tuple $\delta \in \Gamma^k_C = \{g[\kappa_1, ..., \kappa_k] \mid g \in C^{(k)}\}$ we must have $\begin{pmatrix} \delta_{r_1} \\ \vdots \\ \delta_{r_q} \end{pmatrix} \in \sigma$, because the tuples obtained by taking the positions $r_1, ..., r_q$ of $\kappa_1, ..., \kappa_k$ give us $\nu_1, ...\nu_k$ and these elements form a subset of σ. Thus, $f[\tau_1, ..., \tau_l] \in \sigma$, as required. □

The following statement is an immediate consequence.

Corollary 5.13: *Let C be a clone on S. Then, for $k \geq n$, Pol Γ^k_C preserves any subset that is preserved by C. In particular, if C is conservative, then so is Pol Γ^k_C.*

Proof: Let $A \subseteq S$ be a unary relation that is preserved by C. Obviously, $|A| \leq n \leq k$. By Lemma 5.12, the relation A is preserved by *Pol* Γ^k_C. □

We can deduce the following lemma.

Lemma 5.14: *Let C be a clone on S containing a majority operation m. Set*

$$k = max\{|\sigma| \mid \sigma \in R^{(2)}_S \text{ is } C\text{-independent and } \Gamma_C(\sigma) \neq S^2\}.$$

Then $C = Pol\ \Gamma^k_C$.

Proof: Let $\kappa_1, \kappa_2, ..., \kappa_k$ be the k columns of χ_k. For the sake of brevity, denote by C' the clone *Pol* Γ^k_C. Notice that $C \subseteq C'$ by Lemma 5.9. We have to show that $C' \subseteq C$ which is equivalent to *Inv* $C \subseteq$ *Inv* C'. Since $m \in C \cap C'$, we can apply the Baker-Pixley Theorem (see 5.4) and it suffices to show that any binary relation on S that is preserved by C is also preserved by C'. The full binary relation is trivially preserved by C', so it remains to show that any binary relation $\sigma \neq S^2$ that is preserved by C is also preserved by C'. But now, by the assumption of the claim, σ is C-generated by a (C-independent) relation σ' such that $|\sigma'| \leq k$. Thus, by Lemma 5.12, σ is preserved by C'. □

Now, the following corollary is immediate.

Corollary 5.15: *Let C be a majority clone on S. Then $C = Pol\ \Gamma_C^{n^2-1}$. In other words, $\gamma(C) \leq n^2 - 1$.*

In particular, this implies that any clone C on $S = \{0, 1\}$ that contains a majority operation can be written as $Pol\ \Gamma_C^3$. Thus, we have solved the problem for our minimal majority clone $[d]$ on $\{0, 1\}$ and hence for the case $n = 2$. Recall that d is the dual discriminator of Fried and Pixley and the only majority operation on $S = \{0, 1\}$.

Theorem 5.16: *Let C be the minimal majority clone on $S = \{0, 1\}$. Then $C = Pol\ \Gamma_{[d]}^3$.*

We have already calculated $\Gamma_{[d]}^3$ in the example after Lemma 5.3.

Table 5.1

The two-element case

$Pol\ \sigma$	σ
$[d]$	$e_1^3(\kappa_1, \kappa_2, \kappa_3) = (0,0,0,0,1,1,1,1)$
	$e_2^3(\kappa_1, \kappa_2, \kappa_3) = (0,0,1,1,0,0,1,1)$
	$e_3^3(\kappa_1, \kappa_2, \kappa_3) = (0,1,0,1,0,1,0,1)$
	$d(\kappa_1, \kappa_2, \kappa_3) = (0,0,0,1,0,1,1,1)$

Back to the case $S = \{0, ..., n-1\}$

Lemma 5.17: *If C is a minimal majority clone on S and $k \geq max(3, n)$, then C is the single minimal clone contained in $Pol\ \Gamma_C^k$.*

Proof: Suppose C_{min} is a subclone of $Pol\ \Gamma_C^k$ generated by a minimal operation f. Since the arity of any minimal operation on an n-element set cannot exceed $max(3, n)$, we have $f \in C_{min}^{(l)}$ for some $l \leq max(3, n) \leq k$. But now, $C^{(l)} = Pol^{(l)}$ Γ_C^k (see Lemma 5.9) implies $f \in C$. Thus, $C_{min} = C$. □

This Lemma gives us a correspondence between the clone lattice on S and the value of $\gamma(C)$. If $\gamma(C) > max(3, n)$, then there must be a non-minimal clone in the clone lattice that contains exactly one minimal majority clone. There is no such clone in the clone lattice for $n = 2$ which is another argument to obtain the result stated in Theorem 5.16. To my knowledge, it is not known whether there exists such a clone in the clone lattice on a three-element set.

5.1 The three-element case

In this section, fix $n = 3$ and denote by S the set $\{0, 1, 2\}$. We will determine $\gamma(C)$ for the three (up to isomorphism) minimal majority clones $[m_0]$, $[m_{44}]$, $[m_{510}]$ on S.

We can use Lemma 5.14 to calculate bounds for $\gamma(C)$ for $C = [m_0], [m_{510}], [m_{44}]$. To do so, it is necessary to determine all C-independent subsets of S^2. For $|S| = 3$, this can be done by a straight-forward computer-calculation within a few seconds. We obtain the following results.

Table 5.2

$\gamma([m_0])$	≤ 6	(i.e. $[m_0] = Pol\ \Gamma^6_{[m_0]}$)
$\gamma([m_{510}])$	≤ 5	(i.e. $[m_{510}] = Pol\ \Gamma^5_{[m_{510}]}$)
$\gamma([m_{44}])$	≤ 4	(i.e. $[m_{44}] = Pol\ \Gamma^4_{[m_{44}]}$)

Note that, for a given clone C, the calculation of Γ^k_C is equivalent to the problem of determining all k-ary operations in the clone C. Even for clones generated by only one operation (such as minimal clones) and for very small k, it can be practically impossible to do so by a straight-forward calculation since the number of cases to be checked can be enormous. However, these bounds a merely for information. We do not rely on them in any of the upcoming results.

Theorem 5.18: $[m_0] = Pol\ \Gamma^3_{[m_0]}$.

Proof: Suppose $[m_0] \neq Pol\ \Gamma^3_{[m_0]}$. By Lemma 5.9, this is equivalent to $Pol\ \Gamma^3_{[m_0]} \not\subseteq [m_0]$. This implies that there is a nontrivial operation f in $Pol\ \Gamma^3_{[m_0]}$ that is not generated by m_0; i.e. $[f] \not\subseteq [m_0]$. Since $[f]$ has to contain a minimal clone, by Lemma 5.17, this minimal clone has to be $[m_0]$. Thus, $m_0 \in [f]$. This means that we can apply the Baker-Pixley Theorem and we obtain that $[f] \not\subseteq [m_0]$ implies $Inv^{(2)} m_0 \not\subseteq Inv^{(2)} f$. Thus, it follows from our assumption that there is a binary relation σ on S that is preserved by m_0 but not by f. This means $f[\binom{x_1}{y_1}, \binom{x_2}{y_2}, \ldots, \binom{x_k}{y_k}] = \binom{a}{b}$ where $\binom{x_1}{y_1}, \binom{x_2}{y_2}, \ldots, \binom{x_k}{y_k} \in \sigma$ and $\binom{a}{b} \notin \sigma$. By Corollary 5.13, f has to be conservative, hence $a \in \{x_1, ..., x_k\}$ and $b \in \{y_1, ..., y_k\}$. Without loss of generality, we can assume $x_1 = ... = x_{i_a} = a$ and $x_{i_a+1}, ..., x_k \neq a$ and also $y_{i_a+1} = ... = y_{i_b} = b$ and $y_1, ..., y_{i_a}, y_{i_b+1}, ..., y_k \neq b$ for some $0 < i_a < k$, $0 < i_b \leq k$. For an element $x \in S$ ($x \neq 0$) we define $\overline{x}$ to be the remaining element in S after taking away 0 and x. For $i \in \{1, ..., k\}$, we also define $r_i(x)$ and $s_i(x)$ as follows.

$$r_i(x) = \begin{cases} 1, & x_i \in \{0, \overline{x}\} \\ 0, & x_i = x \end{cases} \qquad s_i(x) = \begin{cases} 1, & y_i \in \{0, \overline{x}\} \\ 0, & y_i = x \end{cases}$$

Furthermore, note that f has to preserve any relation that is $[m_0]$-generated by a relation of cardinality at most 3 by Lemma 5.12. In particular, this means that f has to preserve any three-element binary relation on S that is preserved by m_0. We distinguish three cases.

Case 1: $a \neq 0$, $b \neq 0$. The fact that m_0 preserves the relation $\{\binom{a}{0}, \binom{0}{1}, \binom{\overline{a}}{1}\}$ implies that f preserves this relation as well. This, together with $f(x_1, ..., x_k) = a$, implies $f[\binom{x_1}{0}, ..., \binom{x_{i_a}}{0}, \binom{x_{i_a+1}}{1}, ..., \binom{x_k}{1}] = f[\binom{a}{0}, ..., \binom{a}{0}, \binom{x_{i_a+1} \neq a}{1}, ..., \binom{x_k \neq a}{1}] = \binom{a}{0}$. Hence,

$$\text{(a)} \quad f(\underbrace{0, ..., 0}_{i_a}, \underbrace{1, ..., 1}_{k-i_a}) = 0.$$

In the same way, it follows from $f(y_1, ..., y_k) = b$ and the fact that m_0 preserves $\{\binom{b}{0}, \binom{0}{1}, \binom{\overline{b}}{1}\}$ that we have $f[\binom{y_1}{1}, ..., \binom{y_{i_a}}{1}, \binom{y_{i_a+1}}{0}, ..., \binom{y_{i_b}}{0}, \binom{y_{i_b+1}}{1}, ..., \binom{y_k}{1}] = \binom{b}{0}$ and hence

$$\text{(b)} \quad f(\underbrace{1, ..., 1}_{i_a}, \underbrace{0, ..., 0}_{i_b-i_a}, \underbrace{1, ..., 1}_{k-i_b}) = 0.$$

Combining (a) and (b), we can deduce

$$f[\underbrace{\binom{0}{1}, ..., \binom{0}{1}}_{i_a}, \underbrace{\binom{1}{0}, ..., \binom{1}{0}}_{i_b-i_a}, \underbrace{\binom{1}{1}, ..., \binom{1}{1}}_{k-i_b}] = \binom{0}{0}$$

which contradicts that f preserves $\{\binom{1}{0}, \binom{0}{1}, \binom{1}{1}\}$ which must be the case since m_0 preserves this three-element relation.

Case 2: $a = 0$, $b \neq 0$. Suppose that we have $\binom{1}{b} \in \sigma$ and $\binom{2}{b} \in \sigma$ simultaneously. Then, the fact $m_0[\binom{0}{y_1}, \binom{1}{b}, \binom{2}{b}] = \binom{0}{b}$ implies $\binom{0}{b} \in \sigma$, a contradiction. Thus, there is a unique element $x \in S \setminus \{0\}$ such that $\binom{x}{b} \in \sigma$. This implies

$x_{i_a+1} = ... = x_{i_b} = x$. Similar as above, m_0 and consequently f have to preserves the relation $\{\binom{x}{0}, \binom{0}{1}, \binom{\overline{x}}{1}\}$, which, together with $f(x_1, ..., x_k) = 0$, implies that we must have $f[\binom{x_1}{1}, ..., \binom{x_{i_a}}{1}, \binom{x_{i_a+1}}{0}, ..., \binom{x_{i_b}}{0}, \binom{x_{i_b+1}}{r_{i_b+1}(x)}, ..., \binom{x_k}{r_k(x)}] = f[\binom{0}{1}, ..., \binom{0}{1}, \binom{x}{0}, ..., \binom{x}{0}, \binom{x_{i_b+1} \neq 0}{r_{i_b+1}(x)}, ..., \binom{x_k \neq 0}{r_k(x)}] = \binom{0}{1}$ and consequently

$$\text{(a)} \quad f(\underbrace{1, ..., 1}_{i_a}, \underbrace{0, ..., 0}_{i_b - i_a}, r_{i_b+1}(x), ..., r_k(x)) = 1.$$

As in case 1, we also have

$$\text{(b)} \quad f(\underbrace{1, ..., 1}_{i_a}, \underbrace{0, ..., 0}_{i_b - i_a}, \underbrace{1, ..., 1}_{k - i_b}) = 0.$$

Thus, combining (a) and (b), we obtain

$$f[\underbrace{\binom{1}{1}, ..., \binom{1}{1}}_{i_a}, \underbrace{\binom{0}{0}, ..., \binom{0}{0}}_{i_b - i_a}, \binom{r_{i_b+1}(x)}{1}, ..., \binom{r_k(x)}{1}] = \binom{1}{0}$$

which means that f does not preserve $\{\binom{1}{1}, \binom{0}{0}, \binom{0}{1}\}$, which is a contradiction since this three-element relation is preserved by m_0. The case $a \neq 0$, $b = 0$ is clearly analogue.

Case 3: $a = 0, b = 0$. Suppose that we have $\binom{1}{0} \in \sigma$ and $\binom{2}{0} \in \sigma$. Then, $m_0[\binom{0}{y_1}, \binom{1}{0}, \binom{2}{0}] = \binom{0}{0}$ implies that we have $\binom{0}{0} \in \sigma$, a contradiction. In the same way, we can conclude that we cannot have $\binom{0}{1} \in \sigma$ and $\binom{0}{2} \in \sigma$ at the same time. Thus, there exist two unique elements $x, y \in S \setminus \{0\}$ such that $\binom{x}{0}, \binom{0}{y} \in \sigma$. Hence, $x_{i_a+1} = ... = x_{i_b} = x$ and $y_1 = ... = y_{i_a} = y$. Let us suppose that we also have $\binom{\overline{x}}{\overline{y}} \in \sigma$. But then $m_0[\binom{0}{y}, \binom{x}{0}, \binom{\overline{x}}{\overline{y}}] = \binom{0}{0}$ and hence $\binom{0}{0} \in \sigma$ which is a contradiction. As in case 2, we can conclude

$$\text{(a)} \quad f(\underbrace{1, ..., 1}_{i_a}, \underbrace{0, ..., 0}_{i_b - i_a}, r_{i_b+1}(x), ..., r_k(x)) = 1.$$

Furthermore, m_0 preserves the relation $\{\binom{y}{0}, \binom{0}{1}, \binom{\overline{y}}{1}\}$ and so does f. This, together with $f(y_1, ..., y_k) = 0$, implies $f[\binom{y_1}{0}, ..., \binom{y_{i_a}}{0}, \binom{y_{i_a+1}}{1}, ..., \binom{y_{i_b}}{1}, \binom{y_{i_b+1}}{s_{i_b+1}(y)}, ..., \binom{y_k}{s_k(y)}] = f[\binom{y}{0}, ..., \binom{y}{0}, \binom{0}{1}, ..., \binom{0}{1}, \binom{y_{i_b+1} \neq 0}{s_{i_b+1}(y)}, ..., \binom{y_k \neq 0}{s_k(y)}] = \binom{0}{1}$ and hence

$$\text{(b)} \quad f(\underbrace{0,...,0}_{i_a}, \underbrace{1,...,1}_{i_b-i_a}, s_{i_b+1}(y), ..., s_k(y)) = 1.$$

Combining (a) and (b), it follows that we have

$$\text{(I)} \quad f[\underbrace{\binom{1}{0},...,\binom{1}{0}}_{i_a}, \underbrace{\binom{0}{1},...,\binom{0}{1}}_{i_b-i_a}, \binom{r_{i_b+1}(x)}{s_{i_b+1}(y)},...,\binom{r_k(x)}{s_k(y)}] = \binom{1}{1}.$$

Note that, for $i \in \{1,...,k\}$, $\binom{r_i(x)}{s_i(y)} = \binom{1}{1}$ if and only if $\binom{x_i}{y_i} \in \{\binom{0}{0}, \binom{0}{\overline{y}}, \binom{\overline{x}}{0}, \binom{\overline{x}}{\overline{y}}\}$. But now, this cannot be true since we have shown that none of these elements belongs to σ. Thus, $\binom{r_i(x)}{s_i(y)} \in \{\binom{1}{0}, \binom{0}{1}, \binom{0}{0}\}$ $(i = i_b+1,..,i_k)$ and the equation (I) is contradicted by the fact that m_0 and hence f have to preserve the relation $\{\binom{1}{0}, \binom{0}{1}, \binom{0}{0}\}$.

This contradicts the assumption and finishes the proof. □

Theorem 5.19: $[m_{44}] = Pol\ \Gamma^3_{[m_{44}]}$.

Proof: Again, we suppose $[m_{44}] \neq Pol\ \Gamma^3_{[m_{44}]}$, implying that there is an operation f in $Pol\ \Gamma^3_{[m_{44}]}$ that is not generated by m_{44}. As explained in the proof of Theorem 5.18, this implies that there is a binary relation $\sigma = \{\binom{r_1}{s_1}, ..., \binom{r_l}{s_l}\}$ that is preserved by m_{44} but not by f. In the following we denote by $\sigma_{(1)}$ the set of elements $\{r_1, ..., r_l\}$ and by $\sigma_{(2)}$ the set of elements $\{s_1, ..., s_l\}$. m_{44} is conservative, so Corollary 5.13 implies that the same is true for f. This also implies that any relation of the form $A \times B$ for two subsets $A, B \subseteq S$ is preserved by f. In the following, we denote by a, b, c and x, y, z three distinct elements in S. Again, note that f has to preserve any relation that is $[m_{44}]$-generated by a relation of cardinality at most 3 by Lemma 5.12. This contradicts the case $|\sigma| \leq 3$. Now, let us suppose that we have $|\sigma| \geq 7$. Suppose $\binom{a}{x} \notin \sigma$. Since $|\sigma| \geq 7$, at least one of the four sets $\{\binom{a}{y}, \binom{b}{x}, \binom{c}{x}\}$, $\{\binom{a}{z}, \binom{b}{x}, \binom{c}{x}\}$, $\{\binom{c}{x}, \binom{a}{y}, \binom{a}{z}\}$, $\{\binom{b}{x}, \binom{a}{y}, \binom{a}{z}\}$ is a subset of σ. But now, each of these subsets $[m_{44}]$-generates a relation containing $\binom{a}{x}$. Thus, $\binom{a}{x} \in \sigma$, a contradiction.

Now, let us suppose that we have $|\sigma| = 6$. This implies $S = \sigma_{(1)}$, because otherwise σ can be nothing else except $A \times S$ for a two-element set $A \subseteq S$, which we ruled out above. Similarly, $S = \sigma_{(2)}$. Denote by d_1, d_2 and w any (not necessarily distinct) elements in S that can also equal a, b, c resp. x, y, z. Without loss of generality, we have three cases for our relation σ:

(1) $\{\binom{a}{x},\binom{b}{x},\binom{c}{x}\}\subseteq\sigma$ (2) $\{\binom{a}{x},\binom{b}{x},\binom{c}{y}\}\subseteq\sigma$ (3) $\{\binom{a}{x},\binom{b}{y},\binom{c}{z}\}\subseteq\sigma$

In the first case, we must also have $\{\binom{d_1}{y},\binom{d_2}{z}\}\subseteq\sigma$ because $\sigma_{(2)}=S$. Thus, $\{\binom{a}{x},\binom{b}{x},\binom{c}{x},\binom{d_1}{y},\binom{d_2}{z}\}\subseteq\sigma$. Since σ has to contain more than five elements, we can assume $d_1\neq d_2$. But now, this subrelation $[m_{44}]$-generates the full binary relation, so it follows that $|\sigma|$ exceeds 6 and the case (1) is contradicted. In the case (2), we have $\Gamma_{[m_{44}]}(\{\binom{a}{x},\binom{b}{x},\binom{c}{y}\})=\{\binom{a}{x},\binom{b}{x},\binom{c}{y},\binom{c}{x}\}$. Thus, $\{\binom{a}{x},\binom{b}{x},\binom{c}{x}\}\subseteq\sigma$ and we are in case (1). In case (3), σ also has to contain another element $\binom{d_1}{w}$. But now, $\{\binom{a}{x},\binom{b}{y},\binom{c}{z},\binom{d_1}{w}\}$ $[m_{44}]$-generates a relation containing the set $\{\binom{a}{w},\binom{b}{w},\binom{c}{w}\}$, implying that we are in case (1). Thus, the case $|\sigma|=6$ is contradicted.

Now suppose $|\sigma|=4$. We must have $S=\sigma_{(1)}$ or $S=\sigma_{(2)}$, because otherwise σ can be nothing else but $A\times B$ for two two-element sets $A,B\subseteq S$, which is impossible. Without loss of generality we may suppose $S=\sigma_{(1)}$. Again, we have the cases (1), (2), (3) listed above. The same arguments as above show that the cases (2) and (3) can be reduced to case (1), so we can assume $\{\binom{a}{x},\binom{b}{x},\binom{c}{x}\}\subseteq\sigma$. Since σ has to contain four elements by our assumption we must also have another element $\binom{d_1}{y}\in\sigma$. Without loss of generality, we can assume $d_1=a$. But now, σ is $[m_{44}]$-generated by its three-element subrelation $\{\binom{a}{y},\binom{b}{x},\binom{c}{x}\}$ and therefore preserved by f, a contradiction. Thus, $|\sigma|=4$ is not possible.

It remains the case $|\sigma|=5$. By the same argument as above, we can suppose (without loss of generality) $S=\sigma_{(1)}$. Once again, we have the cases (1)-(3), but again we can show that they are essentially the same case. So we can suppose $\{\binom{a}{x},\binom{b}{x},\binom{c}{x}\}\subseteq\sigma$. Without loss of generality, we can suppose that we also have $\binom{a}{y}\in\sigma$. If the fifth element of σ is $\binom{b}{y}$ or $\binom{c}{z}$, then we must also have $\binom{c}{y}\in\sigma$ (since $\binom{c}{y}\in\Gamma_{[m_{44}]}(\sigma)$) and hence $|\sigma|>5$. Similarly, if the fifth element is $\binom{c}{y}$ or $\binom{b}{z}$, then we also have $\binom{b}{y}\in\sigma$ and again $|\sigma|>5$. This means that the fifth element of σ has to be $\binom{a}{z}$ and we have $\sigma=\{\binom{a}{x},\binom{b}{x},\binom{c}{x},\binom{a}{y},\binom{a}{z}\}$.

Our assumption was that f does not preserve σ. This means that, without loss of generality, we can assume that f is a 5-ary operation s.t.

$$f[\binom{a}{x},\binom{b}{x},\binom{c}{x},\binom{a}{y},\binom{a}{z}] \in \{\binom{b}{y},\binom{b}{z},\binom{c}{y},\binom{c}{z}\}$$

Suppose $f[\binom{a}{x},\binom{b}{x},\binom{c}{x},\binom{a}{y},\binom{a}{z}] = \binom{b}{y}$. We define a ternary operation g on S by $g(u_1, u_2, u_3) := f(u_1, u_2, u_3, u_1, u_1)$. Clearly, $g \in [f]^{(3)} \subseteq Pol^{(3)}\Gamma^3_{[m_{44}]}$ and, by Lemma 5.9, $g \in [m_{44}]^{(3)}$. This means that g is either a majority operation or a projection. Furthermore, $g(a, b, c) = f(a, b, c, a, a) = b$ implies that if g is a projection, then it is necessarily e^3_2. In both cases it follows $g(0, 1, 1) = 1$ and hence $f(0, 1, 1, 0, 0) = 1$. This means that we have $f[\binom{0}{x},\binom{1}{x},\binom{1}{x},\binom{0}{y},\binom{0}{z}] = \binom{1}{y}$ but the fact that the relation $\{\binom{0}{x},\binom{1}{x},\binom{0}{y},\binom{0}{z}\}$ is $[m_{44}]$-generated by the three-element relation $\{\binom{1}{x},\binom{0}{y},\binom{0}{z}\}$ implies that this relation is preserved by f, so we have a contradiction. The other possibilities can be handled in the same way. This finishes the proof. □

We have seen that $C = Pol\ \Gamma^3_C$ (i.e. $\gamma(C) = 3$) holds for two out of the three classes of minimal majority clones on S. One might hope that the same holds for the minimal majority clone $[m_{510}]$, but we now show that this is not the case and that $\gamma([m_{510}])$ is in fact 4.

Lemma 5.20: *$[m_{510}] \neq Pol\ \Gamma^3_{[m_{510}]}$ (i.e. $\gamma([m_{510}]) \geq 4$).*

Proof: $[m_{510}] \neq Pol\ \Gamma^3_{[m_{510}]}$ is equivalent to $Pol\ \Gamma^3_{[m_{510}]} \nsubseteq [m_{510}]$ by Lemma 5.9. Hence, it is sufficient to find a nontrivial operation f such that $f \in Pol\ \Gamma^3_{[m_{510}]}$ but m_{510} does not generate f. $[f]$ has to contain a minimal clone and, by Lemma 5.17, this clone can be nothing else but $[m_{510}]$. Thus, $[m_{510}] \subseteq [f]$. This means that our assumption that m_{510} does not generate f is equivalent to $[f] \nsubseteq [m_{510}]$, which, in turn, is equivalent to $Inv^{(2)}m_{510} \nsubseteq Inv^{(2)}f$ by the Baker-Pixley Theorem. Thus, it is sufficient to show that there exist an operation $f \in Pol\ \Gamma^3_{[m_{510}]}$ and a binary relation σ on S such that m_{510} preserves σ while f does not. Now, we give such a relation and operation.

$$f(w,x,y,z) = \begin{cases} e^3_1(w,y,z), & \text{if } w = x \\ e^3_1(w,x,z), & \text{if } w = y \\ m_{218}(w,x,y), & \text{if } w = z \\ e^3_2(w,x,z), & \text{if } x = y \\ m_{510}(w,x,y), & \text{if } x = z \\ m_{510}(w,x,y), & \text{if } y = z \end{cases}$$

Note that f is well defined since, on a three-element set, at least two variables of a quaternary operation have to coincide and the cases do not contradict each other if more than two variables coincide. It is the operation given in the following table.

(w,x,y,z)	$f(w,x,y,z)$	(w,x,y,z)	$f(w,x,y,z)$	(w,x,y,z)	$f(w,x,y,z)$
$(0,0,0,0)$	0	$(1,0,0,0)$	0	$(2,0,0,0)$	0
$(0,0,0,1)$	0	$(1,0,0,1)$	0	$(2,0,0,1)$	0
$(0,0,0,2)$	0	$(1,0,0,2)$	0	$(2,0,0,2)$	0
$(0,0,1,0)$	0	$(1,0,1,0)$	1	$(2,0,1,0)$	2
$(0,0,1,1)$	0	$(1,0,1,1)$	1	$(2,0,1,1)$	2
$(0,0,1,2)$	0	$(1,0,1,2)$	1	$(2,0,1,2)$	0
$(0,0,2,0)$	0	$(1,0,2,0)$	0	$(2,0,2,0)$	2
$(0,0,2,1)$	0	$(1,0,2,1)$	2	$(2,0,2,1)$	2
$(0,0,2,2)$	0	$(1,0,2,2)$	0	$(2,0,2,2)$	2
$(0,1,0,0)$	0	$(1,1,0,0)$	1	$(2,1,0,0)$	0
$(0,1,0,1)$	0	$(1,1,0,1)$	1	$(2,1,0,1)$	0
$(0,1,0,2)$	0	$(1,1,0,2)$	1	$(2,1,0,2)$	2
$(0,1,1,0)$	1	$(1,1,1,0)$	1	$(2,1,1,0)$	1
$(0,1,1,1)$	1	$(1,1,1,1)$	1	$(2,1,1,1)$	1
$(0,1,1,2)$	1	$(1,1,1,2)$	1	$(2,1,1,2)$	1
$(0,1,2,0)$	0	$(1,1,2,0)$	1	$(2,1,2,0)$	2
$(0,1,2,1)$	2	$(1,1,2,1)$	1	$(2,1,2,1)$	2
$(0,1,2,2)$	2	$(1,1,2,2)$	1	$(2,1,2,2)$	2
$(0,2,0,0)$	0	$(1,2,0,0)$	2	$(2,2,0,0)$	2
$(0,2,0,1)$	0	$(1,2,0,1)$	0	$(2,2,0,1)$	2
$(0,2,0,2)$	0	$(1,2,0,2)$	2	$(2,2,0,2)$	2
$(0,2,1,0)$	2	$(1,2,1,0)$	1	$(2,2,1,0)$	2
$(0,2,1,1)$	0	$(1,2,1,1)$	1	$(2,2,1,1)$	2
$(0,2,1,2)$	0	$(1,2,1,2)$	1	$(2,2,1,2)$	2
$(0,2,2,0)$	2	$(1,2,2,0)$	2	$(2,2,2,0)$	2
$(0,2,2,1)$	2	$(1,2,2,1)$	2	$(2,2,2,1)$	2
$(0,2,2,2)$	2	$(1,2,2,2)$	2	$(2,2,2,2)$	2

Now, we have $f[\binom{0}{2},\binom{2}{0},\binom{1}{1},\binom{2}{2}] = \binom{0}{0}$ hence f does not preserve the relation $\{\binom{0}{2},\binom{2}{0},\binom{1}{1},\binom{2}{2}\}$ which is preserved by m_{510}. This is our σ. Furthermore, it can be checked that f preserves the eleven-element relation $\Gamma^3_{[m_{510}]}$ (i.e. $f \in Pol\ \Gamma^3_{[m_{510}]}$). Thus, we have found an operation f and a relation σ as required. $\square$

Lemma 5.21: $[m_{510}] = Pol\ \Gamma^4_{[m_{510}]}$ *(i.e.* $\gamma([m_{510}]) \leq 4$*).*

Proof: Suppose $[m_{510}] \neq Pol\ \Gamma^4_{[m_{510}]}$. By Lemma 5.9, this is equivalent to $Pol\ \Gamma^4_{[m_{510}]} \nsubseteq [m_{510}]$. This implies that there is a nontrivial operation f in $Pol\ \Gamma^4_{[m_{510}]}$ that is not generated by m_{510}. As seen in the proof of Theorem 5.18, it follows that there is a binary relation σ on S that is preserved by m_{510} but not by f. This means $f[\binom{x_1}{y_1},\binom{x_2}{y_2},...,\binom{x_k}{y_k}] = \binom{a}{b}$ where $\binom{x_1}{y_1},\binom{x_2}{y_2},...,\binom{x_k}{y_k} \in \sigma$ and $\binom{a}{b} \notin \sigma$. By Corollary 5.13, f has to be conservative, hence $a \in \{x_1,...,x_k\}$ and $b \in \{y_1,...,y_k\}$. As above, we assume without loss of generality that we have $x_1 = ... = x_{i_a} = a$

and $x_{i_a+1}, ..., x_k \neq a$ and also $y_{i_a+1} = ... = y_{i_b} = b$ and $y_1, ..., y_{i_a}, y_{i_b+1}, ..., y_k \neq b$ for some $0 < i_a < k$, $0 < i_b \leq k$. Here, for $x \in S$ $(x \neq 1)$ we denote by $\overline{x}$ the remaining element in S after taking away 1 and x. Furthermore, we define r_i and s_i $(i = 1, ..., k)$ as follows.

$$r_i = \begin{cases} 1, & x_i = 1 \\ 0, & \text{otherwise} \end{cases} \qquad s_i = \begin{cases} 1, & y_i = 1 \\ 0, & \text{otherwise} \end{cases}$$

Note that f has to preserve any relation that is $[m_{510}]$-generated by a relation of cardinality at most 4 by Lemma 5.12. We distinguish three cases.

Case 1: $a = 1$, $b = 1$. Note that m_{510} preserves the relation $\{\binom{0}{0}, \binom{1}{1}, \binom{2}{0}\}$ and hence f has to preserve this relation as well. Now $f(x_1, ..., x_k) = 1$ implies $f[\binom{x_1}{1}, ..., \binom{x_{i_a}}{1}, \binom{x_{i_a+1}}{0}, ..., \binom{x_k}{0}] = f[\binom{1}{1}, ..., \binom{1}{1}, \binom{x_{i_a+1} \neq 1}{0}, ..., \binom{x_k \neq 1}{0}] = \binom{1}{1}$ and $f(y_1, ..., y_k) = 1$ implies $f[\binom{y_1}{0}, ..., \binom{y_{i_a}}{0}, \binom{y_{i_a+1}}{1}, ..., \binom{y_{i_b}}{1}, \binom{y_{i_b+1}}{0}, ..., \binom{y_k}{0}] = \binom{1}{1}$ in the same way. This implies $f(\underbrace{1, ..., 1}_{i_a}, \underbrace{0, ..., 0}_{k-i_a}) = 1$ and $f(\underbrace{0, ..., 0}_{i_a}, \underbrace{1, ..., 1}_{i_b-i_a}, \underbrace{0, ..., 0}_{k-i_b}) = 1$. It follows that we must have

$$f[\underbrace{\binom{1}{0}, ..., \binom{1}{0}}_{i_a}, \underbrace{\binom{0}{1}, ..., \binom{0}{1}}_{i_b-i_a}, \underbrace{\binom{0}{0}, ..., \binom{0}{0}}_{k-i_b}] = \binom{1}{1}$$

which implies that f does not preserve $\{\binom{1}{0}, \binom{0}{1}, \binom{0}{0}\}$. This is a contradiction, because m_{510} preserves this three-element relation.

Case 2: $a = 1$, $b \neq 1$. Suppose that we have $\binom{1}{1} \in \sigma$ and $\binom{1}{\overline{b}} \in \sigma$ simultaneously. Then, $m_{510}[\binom{1}{1}, \binom{1}{\overline{b}}, \binom{x_{i_b}}{b}] = \binom{1}{b}$ or $m_{510}[\binom{1}{1}, \binom{x_{i_b}}{b}, \binom{1}{\overline{b}}] = \binom{1}{b}$. This implies $\binom{1}{b} \in \sigma$, a contradiction. Hence, there is a unique $x \in S$ such that $\binom{1}{x} \in \sigma$ (i.e. $y_1 = ... = y_{i_a} = x$). As in case 1, we have $f(\underbrace{1, ..., 1}_{i_a}, \underbrace{0, ..., 0}_{k-i_a}) = 1$. Thus,

$$f[\underbrace{\binom{1}{x}, ..., \binom{1}{x}}_{i_a}, \underbrace{\binom{0}{b}, ..., \binom{0}{b}}_{i_b-i_a}, \underbrace{\binom{0}{y_{i_b+1}}, ..., \binom{0}{y_k}}_{k-i_b}] = \binom{1}{b}.$$

This means that f does not preserve the relation $\{\binom{1}{x}, \binom{0}{b}, \binom{0}{1}, \binom{0}{\overline{b}}\}$. This is a contradiction, because this four-element relation is preserved by m_{510}. This finishes the case $a = 1, b \neq 1$. Clearly, the case $a \neq 1$, $b = 1$ is analogue.

Case 3: $a \neq 1$, $b \neq 1$. Suppose that we have $\binom{a}{1} \in \sigma$ and $\binom{a}{\bar{b}} \in \sigma$ simultaneously. Then, one of the two tuples $m_{510}[\binom{a}{1}, \binom{a}{\bar{b}}, \binom{x_{i_b}}{b}]$, $m_{510}[\binom{a}{1}, \binom{x_{i_b}}{b}, \binom{a}{\bar{b}}]$ has to equal $\binom{a}{b}$. This implies $\binom{a}{b} \in \sigma$, a contradiction. Similarly, we can show that we cannot have $\binom{1}{b} \in \sigma$ and $\binom{\bar{a}}{b} \in \sigma$ at the same time. This leaves us with four cases.

(a) $\binom{a}{1}$, $\binom{1}{b} \in \sigma$ (i.e. $x_{i_a+1} = \ldots = x_{i_b} = 1$ and $y_1 = \ldots = y_{i_a} = 1$)

(b) $\binom{a}{1}$, $\binom{\bar{a}}{b} \in \sigma$ (i.e. $x_{i_a+1} = \ldots = x_{i_b} = \bar{a}$ and $y_1 = \ldots = y_{i_a} = 1$)

(c) $\binom{a}{\bar{b}}$, $\binom{1}{b} \in \sigma$ (i.e. $x_{i_a+1} = \ldots = x_{i_b} = 1$ and $y_1 = \ldots = y_{i_a} = \bar{b}$)

(d) $\binom{a}{\bar{b}}$, $\binom{\bar{a}}{b} \in \sigma$ (i.e. $x_{i_a+1} = \ldots = x_{i_b} = \bar{a}$ and $y_1 = \ldots = y_{i_a} = \bar{b}$)

(a) Suppose $\binom{\bar{a}}{\bar{b}} \in \sigma$. We have $m_{510}[\binom{a}{1}, \binom{1}{b}, \binom{\bar{a}}{\bar{b}}] \in \{\binom{a}{b}, \binom{a}{\bar{b}}\}$ or $m_{510}[\binom{a}{1}, \binom{\bar{a}}{\bar{b}}, \binom{1}{b}] \in \{\binom{a}{b}, \binom{a}{\bar{b}}\}$ which implies that we then have $\binom{a}{b} \in \sigma$ or $\binom{a}{\bar{b}} \in \sigma$, a contradiction. Thus, $\sigma \subseteq \{\binom{a}{1}, \binom{1}{b}, \binom{1}{1}, \binom{1}{\bar{b}}, \binom{\bar{a}}{1}\}$. f has to preserve the relations $\{\binom{1}{1}, \binom{a}{0}, \binom{\bar{a}}{0}\}$ and $\{\binom{1}{1}, \binom{b}{0}, \binom{\bar{b}}{0}\}$ because they are preserved by m_{510}. Hence, we must have $f[\binom{x_1}{0}, \ldots, \binom{x_{i_a}}{0}, \binom{x_{i_a+1}}{1}, \ldots, \binom{x_{i_b}}{1}, \binom{x_{i_b+1}}{r_{i_b+1}}, \ldots, \binom{x_k}{r_k}] = f[\binom{a}{0}, \ldots, \binom{a}{0}, \binom{1}{1}, \ldots, \binom{1}{1}, \binom{x_{i_b+1}}{r_{i_b+1}}, \ldots, \binom{x_k}{r_k}] = \binom{a}{0}$ and also $f[\binom{y_1}{1}, \ldots, \binom{y_{i_a}}{1}, \binom{y_{i_a+1}}{0}, \ldots, \binom{y_{i_b}}{0}, \binom{y_{i_b+1}}{s_{i_b+1}}, \ldots, \binom{y_k}{s_k}] = f[\binom{1}{1}, \ldots, \binom{1}{1}, \binom{b}{0}, \ldots, \binom{b}{0}, \binom{y_{i_b+1}}{s_{i_b+1}}, \ldots, \binom{y_k}{s_k}] = \binom{b}{0}$. Thus,

$$f[\underbrace{\binom{0}{1}, \ldots, \binom{0}{1}}_{i_a}, \underbrace{\binom{1}{0}, \ldots, \binom{1}{0}}_{i_b - i_a}, \underbrace{\binom{r_{i_b+1}}{s_{i_b+1}}, \ldots, \binom{r_k}{s_k}}_{k - i_b}] = \binom{0}{0}.$$

But now, $\binom{r_i}{s_i} = \binom{0}{0}$ if and only if $\binom{x_i}{y_i} \in \{\binom{a}{b}, \binom{a}{\bar{b}}, \binom{\bar{a}}{b}, \binom{\bar{a}}{\bar{b}}\}$ which cannot be true. Thus, f does not preserve the relation $\{\binom{1}{0}, \binom{0}{1}, \binom{1}{1}\}$ which is a contradiction since m_{510} preserves this three-element relation.

(b) We have $\binom{1}{\bar{b}} \notin \sigma$ because otherwise we would have $m_{510}[\binom{a}{1}, \binom{\bar{a}}{b}, \binom{1}{\bar{b}}] \in \{\binom{a}{b}, \binom{a}{\bar{b}}\}$ or $m_{510}[\binom{a}{1}, \binom{1}{\bar{b}}, \binom{\bar{a}}{b}] \in \{\binom{a}{b}, \binom{a}{\bar{b}}\}$ which would be a contradiction to the fact that neither $\binom{a}{b}$ nor $\binom{a}{\bar{b}}$ belong to σ. As in case (a), we obtain $f(\underbrace{1, \ldots, 1}_{i_a}, \underbrace{0, \ldots 0}_{i_b - i_a},$

$s_{i_b+1}, ..., s_k) = 0$ and it follows that $f[\binom{x_1}{1}, ..., \binom{x_{i_a}}{1}, \binom{x_{i_a+1}}{0}, ..., \binom{x_{i_b}}{0}, \binom{x_{i_b+1}}{s_{i_b+1}}, ..., \binom{x_k}{s_k}] = f[\binom{a}{1}, ..., \binom{a}{1}, \binom{\overline{a}}{0}, ..., \binom{\overline{a}}{0}, \binom{x_{i_b+1}}{s_{i_b+1}}, ..., \binom{x_k}{s_k}] = \binom{a}{0}$. Note that $\binom{x_i}{s_i} = \binom{1}{0}$ if and only if $\binom{x_i}{y_i} \in \{\binom{1}{b}, \binom{1}{\overline{b}}\}$ which is impossible. Thus, the equation implies that f does not preserve the relation $\{\binom{a}{1}, \binom{\overline{a}}{0}, \binom{1}{1}, \binom{\overline{a}}{1}\}$. This is a contradiction since m_{510} preserves this four-element relation.

(c) This case is clearly analogue to case (b).

(d) Suppose that we have $\binom{\overline{a}}{1} \in \sigma$ and $\binom{1}{1} \in \sigma$ at the same time. Then, $m_{510}[\binom{a}{\overline{b}}, \binom{1}{1}, \binom{\overline{a}}{1}] = \binom{a}{1}$ or $m_{510}[\binom{a}{\overline{b}}, \binom{\overline{a}}{1}, \binom{1}{1}] = \binom{a}{1}$ and hence $\binom{a}{1} \in \sigma$, a contradiction. In the same way we can conclude that we cannot have $\binom{1}{\overline{b}} \in \sigma$ and $\binom{1}{1} \in \sigma$ at the same time. This leaves us with two cases:

$$\sigma \subseteq \{\binom{a}{\overline{b}}, \binom{\overline{a}}{b}, \binom{1}{1}, \binom{\overline{a}}{\overline{b}}\} \quad \text{or} \quad \sigma \subseteq \{\binom{a}{\overline{b}}, \binom{\overline{a}}{b}, \binom{\overline{a}}{1}, \binom{1}{\overline{b}}, \binom{\overline{a}}{\overline{b}}\}.$$

Hence, σ has at most four elements or it is $[m_{510}]$-generated by the four-element relation $\{\binom{a}{\overline{b}}, \binom{\overline{a}}{b}, \binom{\overline{a}}{1}, \binom{1}{\overline{b}}\}$. This means that our assumption that m_{510} preserves σ implies that f preserves σ as well, a contradiction.

This finishes the proof. □

Note that σ from Lemma 5.20 falls under the case (3)(d) in the proof of Lemma 5.21. In fact, all the other cases in the proof could have been handled by supposing only $f \in \Gamma^3_{[m_{510}]}$ since all the four-element relations in *Pol* $[m_{510}]$ that we used for contradiction are $[m_{510}]$-generated by three-element relations.

We have proved the following theorem.

Theorem 5.22: $\gamma([m_{510}]) = 4$.

We have also answered the question whether there exists a clone C and a minimal majority clone $[m] \neq C$ in $\mathcal{L}_S$ such that $[m]$ is the single minimal clone in C.

Corollary 5.23: *It exists a non-minimal clone C and a minimal majority clone $[m]$ on S such that $[m]$ is the only minimal clone in C.*

Proof: By the proof of Lemma 5.20, we obtain the claim for $m = m_{510}$ and $C = [f]$ where f is defined as in the proof of the lemma. □

Analog to the two-element case, Theorem 5.18 and Theorem 5.19 provide us with a very simple way to determine σ_i such that $Pol\ \sigma_i = [m_i]$ for $i = 0, 44$.

$Pol\ \sigma$	σ
$[m_0]$	$e_1^3(\kappa_1, \kappa_2, \kappa_3) = (0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2)$
	$e_2^3(\kappa_1, \kappa_2, \kappa_3) = (0,0,0,1,1,1,2,2,2,0,0,0,1,1,1,2,2,2,0,0,0,1,1,1,2,2,2)$
	$e_3^3(\kappa_1, \kappa_2, \kappa_3) = (0,1,2,0,1,2,0,1,2,0,1,2,0,1,2,0,1,2,0,1,2,0,1,2,0,1,2)$
	$m_0(\kappa_1, \kappa_2, \kappa_3) = (0,0,0,0,1,0,0,0,2,0,1,0,1,1,1,0,1,2,0,0,2,0,1,2,2,2,2)$
$[m_{44}]$	$e_1^3(\kappa_1, \kappa_2, \kappa_3) = (0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2)$
	$e_2^3(\kappa_1, \kappa_2, \kappa_3) = (0,0,0,1,1,1,2,2,2,0,0,0,1,1,1,2,2,2,0,0,0,1,1,1,2,2,2)$
	$e_3^3(\kappa_1, \kappa_2, \kappa_3) = (0,1,2,0,1,2,0,1,2,0,1,2,0,1,2,0,1,2,0,1,2,0,1,2,0,1,2)$
	$m_{624}(\kappa_1, \kappa_2, \kappa_3) = (0,0,0,0,1,2,0,1,2,0,1,2,1,1,1,0,1,2,0,1,2,0,1,2,2,2,2)$
	$m_{44}(\kappa_1, \kappa_2, \kappa_3) = (0,0,0,0,1,0,0,0,2,0,1,1,1,1,1,1,2,0,2,2,2,1,2,2,2,2)$
	$m_{424}(\kappa_1, \kappa_2, \kappa_3) = (0,0,0,0,1,1,0,2,2,0,1,0,1,1,1,2,1,2,0,0,2,1,1,2,2,2,2)$

Note that it is not similarly easy to obtain the tuples in $\sigma = \Gamma^4_{[m_{510}]}$. Determining them is equivalent to determining all quaternary operation in $[m_{510}]$. Even by computer, this is a somewhat time-consuming calculation. However, it turns out that $[m_{510}]$ contains 1892 quaternary operations and hence $|\sigma| = 1892$.

Question: Denote by $\mathcal{M}_n$ the set of minimal majority clones on $S = \{0, ..., n-1\}$. We have seen $\gamma(C) \leq 4$ for all $C \in \mathcal{M}_n$, $n \leq 3$. For a given $n \geq 4$, what is the sharp bound for $\{\gamma(C) \mid C \in \mathcal{M}_n\}$? It must exist since we know $\gamma(C) \leq n^2 - 1$.

5.2 The conservative case

For a majority clone C on the n-element set $S = \{0, ..., n-1\}$, we have seen the bound $\gamma(C) \leq n^2 - 1$ in Corollary 5.15. Now, we show that we can improve this bound significantly, namely to $2n$, if one of the majority operations in the clone is conservative.

Lemma 5.24: *Let C be a clone on S containing a conservative majority operation m. Let σ be a C-independent relation on S. Suppose that there exist pairwise distinct $a, b, c \in S$ such that $\binom{a}{x}, \binom{b}{x}, \binom{c}{x} \in \sigma$ for some $x \in S$. Then $\binom{m(a,b,c)}{y} \notin \sigma$ for all $y \neq x$.*

Proof: Since m is conservative, we may assume $m(a, b, c) = a$ without loss of generality. Let us suppose that there exists $y \in S$, $x \neq y$ such that $\binom{a}{y} \in \sigma$. Then

$m(\binom{a}{y}, \binom{b}{x}, \binom{c}{x}) = \binom{a}{x} \in \sigma$ which is a contradiction to our assumption that σ is C-independent. □

Now we are ready to prove the theorem that gives us the improved bound for clones containing a conservative majority operation.

Theorem 5.25: *Let C be a clone on S containing a conservative majority operation m. Then $\gamma(C) \leq 2n$.*

Proof: By Lemma 5.14, it suffices to show that any C-independent binary relation $\sigma \subseteq S \times S$ can have at most $2n$ elements. Let us visualize a binary relation σ on S by an $n \times n$ matrix M with entries 0 and 1 where $M(i,j) = 1$ if and only if $\binom{i-1}{j-1} \in \sigma$. For example,

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

stands for the relation $\{\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \binom{0}{1}, \binom{2}{2}, \binom{1}{3}\}$. Suppose that we have three pairwise distinct elements $a, b, c \in S$ such that $\binom{a}{x}, \binom{b}{x}, \binom{c}{x} \in \sigma$ for some $x \in S$. It follows from Lemma 5.24 that $\binom{m(a,b,c)}{y} \notin \sigma$ for all $y \neq x$. Since $m(a,b,c) \in \{a,b,c\}$, this means that whenever we take three 1's from the same column, at least one of them has to be the only one in its row. Similarly, we can conclude that whenever we take three 1's from the same row, at least one of them has to be the only one in its column (since, clearly, Lemma 5.24 also holds dually). For example, the diagram above does not satisfy these conditions since we have 1 at the places $M(1,1)$, $M(2,1)$ and $M(3,1)$ but none of these 1's stands alone in its row. The following example satisfies these conditions:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We prove that the number of 1's in this matrix is maximal i.e. the number of 1's in an $n \times n$ matrix satisfying these conditions cannot exceed $2n$. For that, we show

that the number of 1's in an $n \times k$ matrix satisfying the two conditions above can be at most $n+k$ by induction on k.
The claim is obviously true for $k = 1$. Let $r \geq 2$. We can assume that the claim is true for all $n \times k$ matrices where $k \leq r-1$. Suppose that we have a $n \times r$ matrix satisfying the above conditions such that the number of 1's is at least $r+n+1$. There has to be at least one row or column that contains three 1's (otherwise the number of 1's can be at most $2 * min(r, n) \leq n + r$). Without loss of generality, we can assume that this is a row. By the conditions mentioned above, at least one of these three 1's must be the only one in its column. We remove this column from the matrix (this is possible since we have $r \geq 2$) and we obtain a $n \times (r-1)$ matrix that contains at least $n+r+1-1 = n+r$ 1's. This is a contradiction to the induction hypothesis.
For $k = n$ it follows that the matrix representing σ can have at most $2n$ elements. This finishes the proof. □

There is another property of such clones C that we can prove. In order to do so, we need another Lemma.

Lemma 5.26: *Let C be a clone on S containing a majority operation m. Set*

$$k = max\{|\sigma| \mid \sigma \in R_S^{(2)} \text{ is } [m]\text{-independent and } \Gamma_{[m]}(\sigma) \neq S^2\}.$$

Then $[C^{(max(3,k))}] = C$.

Proof: For notational simplicity, denote by C_k the clone $[C^{(max(3,k))}]$. $C_k \subseteq C$ is trivial. It remains to show $C \subseteq C_k$. Clearly, $m \in C_k \subseteq C$. Hence, by the Baker-Pixley Theorem, it is sufficient to show that any binary relation on S preserved by C_k is also preserved by C. Suppose $\sigma \in Inv^{(2)}C_k$. Since C trivially preserves the full binary relation, we can assume $\sigma \neq S^2$. For some C_k-independent relation σ', we have $\sigma = \Gamma_{C_k}(\sigma')$. Since $[m] \subseteq C_k$, σ' is also $[m]$-independent and $\Gamma_{[m]}(\sigma') \subseteq \sigma \neq S^2$. But now, by the assumption, we have $|\sigma'| \leq k$. Hence, we can write σ' as $\{\nu_1, ..., \nu_k\}$. By Lemma 5.3, this implies

$$\sigma = \Gamma_{C_k}(\sigma') = \{g[\nu_1, ..., \nu_k] \mid g \in C_k^{(k)}\} = \{g[\nu_1, ..., \nu_k] \mid g \in C^{(k)}\} = \Gamma_C(\sigma')$$

and hence $\sigma \in Inv\ C$. □

Note that the number k in this lemma is not the same as the one in Lemma 5.14. In fact, the value given in this lemma can be significantly higher. We are now ready to prove our last result:

Theorem 5.27: *Let C be a clone on S containing a conservative majority operation m. Then $[C^{(2n)}] = C$.*

Proof: We can suppose $n > 1$. In the proof of Theorem 5.25, we concluded that any C-independent binary relation σ on S can have at most $2n$ elements. Note that we only used m of all the operations in C, so we have actually shown that any $[m]$-independent binary relation can have at most $2n$ elements (which is a somewhat stronger result). Thus, by Lemma 5.26, $[C^{(max(3,2n))}] = [C^{(2n)}] = C$. $\square$

Note that the results in this section hold for conservative minimal majority clones in particular, but they are in fact more general: We have only required the clone C to contain a conservative majority operation. The whole clone does not have to be conservative and it does not have to be generated by this majority operation (and, in particular, it does not have to be minimal).

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