Université de Montréal

Dégénérescence et problèmes extrémaux pour les valeurs propres du laplacien sur les surfaces

par

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Dégénérescence et problèmes extrémaux pour les valeurs propres du laplacien sur les surfaces

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RÉSUMÉ

Le sujet principal de cette thèse est la géométrie spectrale des surfaces. Le spectre d'une surface riemannienne fermée (Σ, g) est une suite de nombres $0 = \lambda_0 < \lambda_1(g) \leq \lambda_2(g) \leq \cdots \nearrow \infty$ représentant les modes de vibrations purs de cette surface. On étudie l'influence de la géométrie sur le spectre. Ce sujet est classique, il fut initié par Lord Rayleigh **[51]**, Faber **[17]**, Krahn **[32, 33]**, Pólya **[49, 48]**, Szegö **[54]**, Hersch **[27]** et plusieurs autres mathématiciens.

Cette thèse est composée de trois articles. Le premier **[23]**, intitulé *Fundamental tone, concentration of density to points and conformal degeneration on surfaces*, est présenté au Chapitre 1. L'influence sur le ton fondamental (c'est-à-dire la première valeur propre positive du laplacien $\lambda_1(g) > 0$) de deux types de dégénérescence y est étudiée : la concentration vers un point et la dégénérescence conforme sur le tore et la bouteille de Klein. Pour ces deux types de dégénérescence, j'ai montré que si une suite de métriques (g_n) d'aire fixée est dégénérée, le ton fondamental sera asymptotiquement borné supérieurement par le ton fondamental d'une sphère ronde de même aire. C'est-à-dire

$$\limsup_{n\to\infty} \lambda_1(g_n)\text{aire}(\Sigma, g_n) \leq 8\pi.$$

Le deuxième article **[24]** de cette thèse est le fruit d'une collaboration avec Iosif Polterovich et Nikolai Nadirashvili. Son titre est *Maximization of the second positive Neumann eigenvalue for planar domains.* Le spectre de Neumann d'un domaine planaire $\Omega \subset \mathbb{R}^2$ est aussi une suite $0 = \mu_0 < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \nearrow \infty$. Un résultat classique de G. Szegö affirme pour chaque domaine planaire simplement connexe régulier que $\mu_1(\Omega)\,\text{aire}(\Omega) \leq \mu_1(\mathbb{D})\pi$ où $\mathbb{D}$ est le disque unité. Le résultat principal de cet article est une borne supérieure sur la deuxième valeur

propre :

$$\mu_2(\Omega)\,\mathrm{aire}(\Omega) \leq 2\,\mu_1(\mathbb{D})\,\pi.$$

Cette borne est atteinte par une famille de domaines dégénérant vers l'union disjointe de deux disques identiques. Ce résultat confirme la conjecture de Pólya pour μ_2. La preuve de ce théorème repose sur un argument topologique permettant de garantir l'existence d'une famille de fonctions tests appropriée. Par une méthode très similaire, nous avons obtenu une borne supérieure sur la deuxième valeur propre conforme de la classe conforme standard sur des sphères de dimension impaire.

Le troisième article **[22]** présenté s'intitule *Relative Homological Linking in Critical Point Theory*. Son sujet n'est pas directement lié à la géométrie spectrale. Il s'agit d'une extension du travail entrepris lors de ma maîtrise, sous la direction de Marlène Frigon. J'y ai introduit un outil, l'enlacement homologique relatif, permettant de détecter les points critiques d'une fonction à l'aide de la topologie de ses ensembles de niveaux. J'y montre en particulier que l'enlacement homologique implique l'enlacement homotopique.

Mots clés :
Géométrie spectrale, dégénérescence, problèmes extrémaux, valeurs propres du laplacien, surfaces riemanniennes, énergie de Dirichlet, invariance conforme, concentration de mesures, condition de Neumann, enlacement, théorie de Morse.

SUMMARY

The main topic of the present thesis is spectral geometry of surfaces. The spectrum of a closed surface (Σ, g) is a sequence of numbers $0 = \lambda_0 < \lambda_1(g) \leq \lambda_2(g) \leq \cdots \nearrow \infty$ called the eigenvalues. From the viewpoint of the theory of sound, each eigenvalue represents a frequency of vibration of the surface. We study the dependence of the eigenvalues on the geometric properties of the surface. This is a classical subject in spectral geometry, originated in the works of Lord Rayleigh [**51**], Faber [**17**], Krahn [**32, 33**], Pólya [**49, 48**], Szegö [**54**], Hersch [**27**], and many others.

This thesis is a collection of three papers. The first one [**23**], *Fundamental tone, concentration of density to points and conformal degeneration on surfaces*, is presented in Chapter 1. The influence on the fundamental tone (i.e., the first positive eigenvalue of the Laplacian) of two types of degeneration is studied : concentration of mass on any surface, and conformal degeneration on the torus and on the Klein bottle. In both cases, I prove that in the limit the fundamental tone is bounded above by the fundamental tone of a round sphere of the same area. That is,

$$\limsup_{n\to\infty} \lambda_1(g_n)\mathrm{Area}(\Sigma, g_n) \leq 8\pi.$$

The second paper [**24**] presented in the thesis is a joint work with Nikolai Nadirashvili and Iosif Polterovich. It is entitled *Maximization of the second positive Neumann eigenvalue for planar domains.* The Neumann spectrum of a planar domain $\Omega \subset \mathbb{R}^2$ is a sequence of numbers $0 = \mu_0 < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \nearrow \infty$. A classical result due to G. Szegö states that for each simply connected regular planar domain Ω, $\mu_1(\Omega)\,\mathrm{Area}(\Omega) \leq \mu_1(\mathbb{D})\pi$ where $\mathbb{D}$ is the unit disk. The main

result of the paper is a sharp upper bound on the second eigenvalue :

$$\mu_2(\Omega)\,\text{Area}(\Omega) \leq 2\,\mu_1(\mathbb{D})\,\pi.$$

This bound is attained in the limit by a family of domains degenerating to a disjoint union of two identical disks. In particular, this result implies the Pólya conjecture for μ_2. Our approach is based on a combination of analytic and topological arguments. A similar method leads to an upper bound on the second eigenvalue for conformally round spheres of odd dimension.

The subject of the third paper *Relative Homological Linking in Critical Point Theory* **[22]** is not directly related to spectral geometry. It is an extension of my M.Sc. thesis written under the direction of Marlène Frigon. A homological linking for a pair of subspaces is introduced. It is used in combination with elementary Morse theory to detect the critical points of a functional. In particular, it is proved that homological linking implies homotopical linking.

Key words :

Spectral geometry, degeneration, extremal problems, eigenvalue of the Laplacian, riemanniann surfaces, Dirichlet energy, conformal invariance, measure concentration, Neumann boundary conditions, linking, Morse theory.

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INTRODUCTION

Géométrie spectrale

La géométrie spectrale porte sur les liens entre les vibrations pouvant être émises par une surface et sa géométrie. Les surfaces considérées seront lisses et fermées, c'est-à-dire compactes, sans bord et connexes. À une métrique riemannienne g sur une surface Σ, des quantités géométriques comme la courbure, l'aire et le diamètre sont associées. L'aire par exemple est déterminée par la mesure dg associée à la métrique g :

$$\mathcal{A}(\Sigma, g) = \int_\Sigma dg.$$

Les modes de vibrations purs sont décrits par les fonctions propres et les valeurs propres de l'opérateur laplacien $\Delta_g : C^\infty(\Sigma) \to C^\infty(\Sigma)$ défini par $\Delta_g u = \mathrm{div}_g \nabla_g u$. En coordonnées locales, le laplacien s'exprime de la façon suivante :

$$\Delta_g u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{2} \partial_i \left(\sqrt{g}\, g^{ij} \partial_j u \right).$$

Un nombre $\lambda \in \mathbb{R}$ est une *valeur propre* de l'opérateur Δ_g si l'*espace propre* associé

$$E_\lambda = \left\{ u \in C^\infty(\Sigma) \mid \Delta_g u + \lambda u = 0 \right\}$$

est non nul. La *multiplicité* d'une valeur propre λ est la dimension de l'espace propre E_λ correspondant, elle est toujours finie. La compacité de la surface Σ implique que le spectre de l'opérateur laplacien Δ_g est la suite de ses valeurs propres :

$$0 = \lambda_0 < \lambda_1(g) \leq \lambda_2(g) \leq \cdots \leq \lambda_k(g) \leq \cdots \nearrow \infty$$

où, par convention, chaque valeur propre est répétée selon sa multiplicité. Cette suite représente les modes de vibrations purs de la surface étudiée. La géométrie spectrale est l'étude des liens entre le spectre de l'opérateur laplacien Δ_g et les quantités géométriques déterminées par la métrique riemannienne g sur la surface Σ.

Le livre de M. Berger, P. Gauduchon et E. Mazet **[7]** est l'un des premiers ouvrages présentant la géométrie spectrale des variétés riemanniennes. D'autres références utiles sont les livres de I. Chavel [**10**], P. Bérard [**5**], P. Buser [**8**] et R. Schoen et S.T. Yau [**53**].

Outils fondamentaux

Il est bien connu que l'espace $L^2(\Sigma, g)$ se décompose en terme des espaces propres de Δ_g :

$$L^2(\Sigma, g) = \bigoplus_{k=0}^{\infty} E_{\lambda_k(g)}. \tag{0.1}$$

En d'autres mots, il existe une base orthonormale $(u_k)_{k=0}^{\infty}$ de $L^2(\Sigma, g)$ formée de fonctions propres de Δ_g :

$$\Delta_g u_k + \lambda_k(g) u_k = 0.$$

L'*énergie de Dirichlet* d'une fonction lisse $u \in C^\infty(\Sigma)$ est

$$D_g(u) = \int_\Sigma |\nabla_g u|_g^2 \, dg.$$

Si $\phi : (\Sigma', g') \to (\Sigma, g)$ est une équivalence conforme entre deux surfaces, alors pour chaque fonction lisse $u \in C^\infty(\Sigma)$

$$D_g(u) = D_{g'}(u \circ \phi).$$

Cette propriété d'invariance conforme n'est pas vérifiée sur des variétés de dimension plus grande que deux.

Le *quotient de Rayleigh* d'une fonction lisse non identiquement nulle $u \in C^\infty(\Sigma)$ est

$$R_g(u) = \frac{D_g(u)}{\|u\|_{L^2(g)}^2}.$$

De la décomposition (0.1) on déduit la caractérisation variationnelle suivante des valeurs propres :

$$\lambda_k(g) = \inf_E \sup_{0 \neq u \in E} R_g(u) \tag{0.2}$$

où l'infimum est pris sur tous les sous-espaces vectoriels E de dimension k de l'espace des fonctions lisses $C^\infty(\Sigma)$.

Remarque 0.1. L'*espace de Sobolev* $H^1(\Sigma, g) \subset L^2(\Sigma, g)$ est l'espace de Hilbert obtenu par complétion de l'espace des fonctions lisses $C^\infty(\Sigma)$ muni du produit scalaire

$$\langle u, v \rangle_1 = \langle u, v \rangle_{L^2} + \langle \nabla_g u, \nabla_g v \rangle_{L^2}.$$

La théorie spectrale est souvent développée dans ce cadre. Je me suis permis de donner les définitions et propriétés principales en terme de l'espace des fonctions lisses par souci de simplicité. Ce point de vue est justifié par la densité de l'espace des fonctions lisses $C^\infty(\Sigma)$ dans l'espace de Sobolev $H^1(\Sigma, g)$ et par la continuité du quotient de Rayleigh $R_g : H^1(\Sigma, g) \to \mathbb{R}$.

Ton fondamental et dégénérescence

Le *ton fondamental* d'une surface riemannienne fermée (Σ, g) est sa première valeur propre positive $\lambda_1(g) > 0$. Considérons l'espace

$$\mathcal{R}(\Sigma) = \{g \mid \mathcal{A}(\Sigma, g) = 1\}$$

des métriques riemanniennes dont l'aire totale est un. Dans l'article **[23]** j'ai étudié le comportement asymptotique de la fonctionnelle

$$\lambda_1 : \mathcal{R}(\Sigma) \to \mathbb{R}$$

pour deux types de suites dégénérées de métriques $(g_n) \subset \mathcal{R}(\Sigma)$.

Concentration vers un point

On dit d'une suite de métriques $(g_n) \subset \mathcal{R}(\Sigma)$ qu'elle se concentre vers le point $p \in \Sigma$ si la suite des mesures (dg_n) converge vers la mesure de Dirac au point p, c'est-à-dire que pour chaque voisinage $\mathcal{O}$ du point p,

$$\lim_{n\to\infty}\int_{\mathcal{O}} dg_n = 1.$$

Question. La concentration d'une suite de métriques d'aire un $(g_n) \subset \mathcal{R}(\Sigma)$ vers un point $p \in \Sigma$ impose-t-elle des contraintes asymptotiques à $\lambda_1(g_n)$ lorsque $n \to \infty$?

La proposition suivante, dont la preuve sera présentée au Chapitre 1, montre que la réponse est négative.

Proposition 0.2. *Pour chaque métrique g d'aire un sur une surface fermée Σ et chaque point $p \in \Sigma$, il existe une suite de difféomorphismes $\phi_n : \Sigma \to \Sigma$ tels que les métriques $g_n = \phi_n^* g$ se concentrent vers p. En particulier, les spectres des opérateurs Δ_{g_n} sont tous identiques.*

Ce phénomène est un peu artificiel : les surfaces riemanniennes (Σ, g_n) sont toutes isométriques par construction. Pour rendre la situation plus rigide, j'ai décidé de restreindre les métriques à une classe conforme fixée. La classe conforme associée à une métrique riemannienne g est $[g] = \{\alpha g \mid 0 < \alpha \in C^\infty(\Sigma)\}$. L'ensemble des métriques d'aire un dans la classe de g est noté

$$\mathcal{R}_g(\Sigma) = \{\tilde{g} \in [g] \mid \mathcal{A}(\Sigma, \tilde{g}) = 1\}.$$

La démonstration du théorème suivant sera présentée au Chapitre 1.

Théorème 0.3 (A. Girouard [**23**]). *Soit $[g]$ une classe conforme sur une surface fermée Σ.*

a) Pour chaque suite de métriques $(g_n) \subset \mathcal{R}_g(\Sigma)$ se concentrant vers un point,

$$\limsup_{n\to\infty} \lambda_1(g_n) \leq 8\pi.$$

b) Pour chaque point $p \in \Sigma$, il existe une suite de métriques $(g_n) \subset \mathcal{R}_g(\Sigma)$ se concentrant vers p et telle que $\lim_{n\to\infty} \lambda_1(g_n) = 8\pi$.

Dégénérescence conforme sur le tore

Dans leur article [**12**] B. Colbois et A. El Soufi ont introduit la notion de spectre conforme d'une classe conforme sur une variété riemannienne fermée. En particulier, la *première valeur propre conforme* d'une classe conforme $[g]$ sur le tore $\mathbb{T}^2$ est

$$\nu([g]) = \sup_{\tilde{g}\in\mathcal{R}_g(\mathbb{T}^2)} \lambda_1(\tilde{g}).$$

Définissons

$$\mathcal{M} = \{a + ib \in \mathbb{C} \mid 0 \leq a \leq 1/2, a^2 + b^2 \geq 1, b > 0\}.$$

Il est bien connu que pour chaque métrique g sur le tore $\mathbb{T}^2$ il existe un réseau Γ engendré par 1 et par $a + ib \in \mathcal{M}$ tel que $(\mathbb{T}^2, g)$ est conformément équivalent au tore plat $\mathbb{C}/\Gamma$ (voir [**7**]). En d'autres mots, $\mathcal{M}$ représente l'espace des classes conformes sur le tore.

Définition 0.4. *On dit d'une suite de métriques* (g_n) *qu'elle est* conformément dégénérée *si la suite* $(a_n + ib_n) \subset \mathcal{M}$ *associée vérifie* $\lim_{n\to\infty} b_n = \infty$.

Intuitivement, la dégénérescence conforme dit que les tores $(\mathbb{T}^2, g_n)$ sont conformément équivalents à des tores plats de plus en plus longs et fins. Le théorème suivant, qui avait été annoncé par N. Nadirashvili [**43**], sera démontré au Chapitre 1.

Théorème 0.5 (A. Girouard [**23**]). *Si une suite* (g_n) *de métriques riemanniennes d'aire un sur le tore* $\mathbb{T}^2$ *est conformément dégénérée, alors*

$$\lim_{n\to\infty} \nu([g_n]) = 8\pi.$$

En particulier

$$\limsup_{n\to\infty} \lambda_1(g_n) \leq 8\pi.$$

Remarque 0.6. Il existe des métriques $g \in \mathcal{R}(\mathbb{T}^2)$ vérifiant $\lambda_1(g) > 8\pi$. Par exemple, la métrique équilatérale plate g_{eq}, correspondant au réseau engendré par 1 et $e^{i\pi/3}$, vérifie $\lambda_1(g_{eq}) = \frac{8\pi^2}{\sqrt{3}}$.

Maximisation du ton fondamental

Il est bien connu que pour chaque surface fermée Σ, la quantité

$$\Lambda(\Sigma) = \sup_{g\in\mathcal{R}(\Sigma)} \lambda_1(g)$$

est finie [**57, 37**] mais on ne sait la déterminer explicitement que pour quelques surfaces. Le premier résultat de ce type [**27**] est dû à J. Hersch qui, en 1970, a montré que sur la sphère $\mathbb{S}^2$, la métrique ronde $g_{\mathbb{S}^2} \in \mathcal{R}(\mathbb{S}^2)$ est un maximum unique : pour chaque métrique $g \in \mathcal{R}(\mathbb{S}^2)$, $\lambda_1(g) \leq \lambda_1(g_{\mathbb{S}^2}) = 8\pi$. Dans l'article [**43**] N. Nadirashvili a montré que pour chaque métrique $g \in \mathcal{R}(\mathbb{T}^2)$,

$$\lambda_1(g) \leq \lambda_1(g_{eq}) = \frac{8\pi^2}{\sqrt{3}}$$

où g_{eq} est la métrique plate équilatérale d'aire un sur le tore $\mathbb{T}^2$. M. Berger avait déjà montré [**6**] cette inégalité pour chaque métrique plate $g \in \mathcal{R}(\mathbb{T}^2)$ et il avait conjecturé le résultat général. La partie la plus difficile de la preuve du théorème de N. Nadirashvili est de montrer l'existence d'une métrique $\overline{g} \in \mathcal{R}(\Sigma)$ telle que $\lambda_1(\overline{g}) = \Lambda(\mathbb{T}^2)$. N. Nadirashvili utilise, pour ce faire, une idée souvent utilisée en théorie des points critiques : étant donnée une suite de métriques $(g_n) \subset \mathcal{R}(\mathbb{T}^2)$ telle que $\lim_{n\to\infty} \lambda_1(g_n) = \Lambda(\mathbb{T}^2)$, il montre que cette suite admet une sous-suite convergente. Un premier pas dans cette direction consiste à montrer que la suite des classes conformes associées $([g_n])$ est bornée. Puisque $\lambda_1(g_{eq}) = \frac{8\pi^2}{\sqrt{3}} > 8\pi$, ceci découle directement du Théorème 0.5. Dans son article [**43**], le Théorème 0.5 avait été énoncé, mais une preuve complète n'en avait pas été donnée.

Remarque 0.7. Pour le plan projectif $\Sigma = \mathbb{R}P^2$, J.P. Bourguignon a montré que $\Lambda(\Sigma) = 12\pi$ est réalisé par le quotient de la métrique ronde $g_{\mathbb{S}^2}$, voir [**37**]. Dans l'article [**31**] D. Jakobson, N. Nadirashvili et I. Polterovich exhibent une métrique sur la bouteille de Klein $\mathbb{K}$ dont ils conjecturent la maximalité. Cette conjecture à été en partie résolue par A. El Soufi, H. Giacomini et M. Jazar [**14**]. Une conjecture a aussi été proposée [**30**] pour la surface orientable de genre 2. Pour toutes les autres surfaces, on ne sait ni calculer $\Lambda(\Sigma)$ ni montrer l'existence d'une métrique g réalisant $\Lambda(\Sigma) = \lambda_1(g)$.

Dégénérescence conforme sur la bouteille de Klein

Pour chaque métrique riemannienne g sur la bouteille de Klein, il existe un nombre $b > 0$ tel que la métrique g soit conformément équivalente à la métrique plate induite par le recollement illustré à la Figure 0.1. Sur la bouteille de Klein,

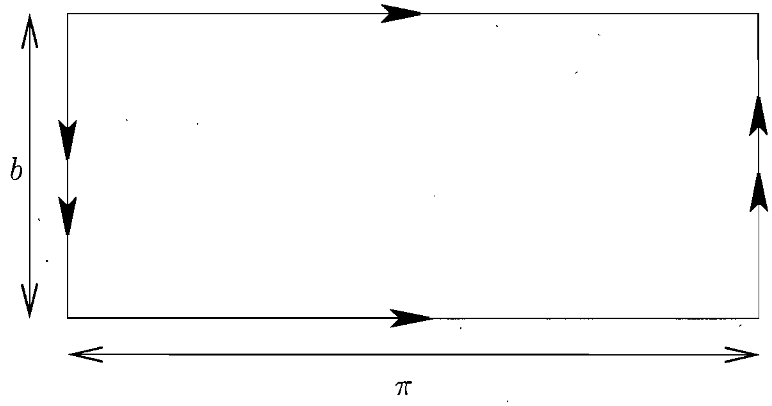

FIG. 0.1. La bouteille de Klein

l'espace de module des classes conformes est donc identifié à la demi-droite $\mathbb{R}_+$. Le théorème suivant sera prouvé au Chapitre 1.

Théorème 0.8 (A. Girouard [**23**]). *Soit $(g_n) \subset \mathcal{R}(\mathbb{K})$ une suite de métriques sur la bouteille de Klein et $(b_n) \subset \mathbb{R}_+$ la suite correspondante.*

1. *Si $\lim_{n\to\infty} b_n = 0$, alors $\limsup_{n\to\infty} \lambda_1(g_n) \leq 8\pi$.*
2. *Si $\lim_{n\to\infty} b_n = \infty$, alors $\limsup_{n\to\infty} \lambda_1(g_n) \leq 12\pi$.*

La preuve du point 1 est très similaire à la preuve du Théorème 0.5. Celle du point deux est plus subtile, elle repose sur la minimalité du plongement de Véronèse du plan projectif dans la sphère $\mathbb{S}^4$.

Invariant de Friedlander-Nadirashvili

B. Colbois et J. Dodziuk ont montré [**11**] que si on remplace la surface Σ par une variété fermée M de dimension supérieure à deux, le ton fondamental n'est plus borné :

$$\sup_{g\in\mathcal{R}(M)} \lambda_1(g) = \infty.$$

Par contre, en restreignant les métriques à une classe conforme $[g]$ le supremum est fini :

$$\nu([g]) = \sup_{g\in\mathcal{R}_g(M)} \lambda_1(g) < \infty.$$

Pour obtenir un invariant ne dépendant que de la structure différentielle de M, L. Friedlander et N. Nadirashvili [**19**] ont défini

$$I(M) = \inf_{g\in\mathcal{R}(M)} \nu([g]).$$

Le calcul de cet invariant est très difficile, même pour les surfaces. Puisque toute métrique g sur la sphère $\mathbb{S}^2$ est conformément équivalente à la métrique ronde $g_{\mathbb{S}^2}$, il est clair que $I(\mathbb{S}^2) = 8\pi$. Similairement, $I(\mathbb{R}P^2) = 12\pi$. L. Friedlander et N. Nadirashvili ont montré ([**19**], voir aussi [**12**]) que pour chaque classe conforme $[g]$ sur une surface fermée, $\nu([g]) \geq 8\pi$ et donc que $I(\Sigma) \geq 8\pi$. En fait, ils ont conjecturé que pour chaque surface fermée Σ sauf le plan projectif $\mathbb{R}P^2$, $I(\Sigma) = 8\pi$. Il découle de mon travail (Théorème 0.5 et Théorème 0.8) que cette conjecture est vérifiée pour le tore $\mathbb{T}^2$ et la bouteille de Klein $\mathbb{K}$.

Maximisation des valeurs propres neumanniennes pour des domaines planaires

Soit $\Omega \subset \mathbb{R}^2$ un domaine planaire. La dérivée normale d'une fonction $u \in C^\infty(\overline{\Omega})$ est notée $\frac{\partial u}{\partial n}$. Le domaine Ω est dit *régulier* si l'ensemble des valeurs propres μ du problème

$$\begin{cases} \Delta u + \mu u = 0 \text{ sur } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ sur } \partial\Omega, \end{cases}$$

est discret. Un domaine dont la frontière est lisse est régulier, mais des conditions plus faibles sont suffisantes, la condition du cône par exemple [**45**]. Le *spectre neumannien* d'un domaine régulier Ω est la suite de ses valeurs propres, répétées selon leurs multiplicités : $0 = \mu_0 < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \nearrow \infty$. Rappelons que le *quotient de Rayleigh* d'une fonction lisse $0 \neq u \in C^\infty(\overline{\Omega})$ est

$$R(u) = \frac{\int_\Omega |\nabla u(z)|^2 \, dz}{\int_\Omega u^2(z) \, dz}.$$

Tout comme dans le cas des surfaces fermées, les fonctions propres neumanniennes forment une base de l'espace $L^2(\Omega)$ et les valeurs propres sont caractérisées par

$$\mu_k(\Omega) = \inf_E \sup_{0 \neq u \in E} R(u) \tag{0.3}$$

où l'infimum est pris sur tous les sous-espaces vectoriels E de dimension k de l'espace des fonctions lisses $C^\infty(\overline{\Omega})$.

Soit Ω un domaine planaire simplement connexe. G. Szegö a montré [**54**] que la première valeur propre neumannienne positive de Ω n'excède pas celle d'un disque de même aire, ce qui s'exprime aussi de la manière suivante :

$$\mu_1(\Omega)\text{aire}(\Omega) \leq \mu_1(\mathbb{D})\pi \tag{0.4}$$

où $\mathbb{D}$ est le disque unité. Weinberger a par la suite montré [**56**] que la condition de simple connexité n'est pas nécessaire. La maximisation des valeurs propres suivantes est beaucoup plus difficile. Dans son livre [**48**], Pólya a conjecturé que pour chaque $k \geq 1$,

$$\mu_k(\Omega)\,\text{aire}(\Omega) \leq 4k\,\pi. \tag{0.5}$$

Pólya a démontré quelques années plus tard que sa conjecture est vérifiée par tous les domaines réguliers qui pavent le plan, par exemple les triangles et les quadrilatères [**49**]. Comme la première valeur propre positive du disque unité vaut approximativement $3,39$, la conjecture de Pólya est aussi vérifiée par tous les domaines réguliers dans le cas de la première valeur propre positive $\mu_1(\Omega)$.

Au Chapitre 2, une étude de la maximisation de la deuxième valeur propre neumannienne est présentée. Le théorème suivant est le résultat principal de ce chapitre.

Théorème 0.9 (A. Girouard, N. Nadirashvili, I. Polterovich [**24**]). *Soit Ω un domaine planaire régulier simplement connexe. Alors*

$$\mu_2(\Omega)\,\textit{aire}(\Omega) \leq 2\,\mu_1(\mathbb{D})\,\pi. \tag{0.6}$$

De plus, cette borne est optimale : une famille de domaines dégénérant vers l'union disjointe de deux disques identiques réalise l'égalité à la limite.

Comme $2\,\mu_1(\mathbb{D}) \approx 6,78 < 8$, on obtient ainsi une vérification de la conjecture de Pólya pour la deuxième valeur propre positive $\mu_2(\Omega)$. Pour prouver le Théorème 0.9 une méthode de *pliage et réarrangement* similaire à celle introduite par N. Nadirashvili [**44**] est utilisée. Nous avons combiné cette méthode avec un argument topologique et quelques propriétés des fonctions sous-harmoniques qui sont similaires à celles utilisées par G. Szegö.

Deuxième valeur propre positive dans la classe conforme ronde d'une sphère

Une argument similaire permet de donner une borne supérieure sur la deuxième valeur propre conforme de la classe conforme ronde $[g_{\mathbb{S}^n}]$ sur des sphères de dimensions impaires.

Théorème 0.10 (A. Girouard, N. Nadirashvili, I. Polterovich [**24**]). *Soit $n \in \mathbb{N}$ un nombre impair. Soit $g \in [g_{\mathbb{S}^n}]$ une métrique conformément ronde sur la sphère $\mathbb{S}^n$. Alors*

$$\lambda_2(\mathbb{S}^n, g)\ Vol(\mathbb{S}^n, g)^{\frac{2}{n}} < (n+1)\left(\frac{4\pi^{\frac{n+1}{2}}\Gamma(n)}{\Gamma(\frac{n}{2})\Gamma(n+\frac{1}{2})}\right)^{2/n}. \tag{0.7}$$

L'énergie de Dirichlet n'est pas un invariant conforme sur une variété de dimension supérieure à deux. Pour cette raison, dans la preuve de ce théorème on a dû remplacer le quotient de Rayleigh par une version modifiée [**19**] de celui-ci :

$$R'(u) = \frac{\left(\int_{\mathbb{S}^n} |\nabla_g u|_g^n \, dg\right)^{2/n}}{\int_{\mathbb{S}^n} u^2 \, dg}.$$

De l'inégalité de Hölder, on déduit que $R(u) \leq R'(u)$.

Enlacement homologique relatif et théorie des points critiques

Soit Σ une surface fermée munie d'une métrique riemannienne g. Rappelons que l'*espace de Sobolev* $H^1(\Sigma, g)$ est l'espace de Hilbert obtenu par complétion de l'espace des fonctions lisses $C^\infty(\Sigma)$ muni du produit scalaire

$$\langle u, v\rangle_1 = \langle u, v\rangle_{L^2} + \langle \nabla_g u, \nabla_g v\rangle_{L^2}.$$

L'énergie de Dirichlet

$$D_g(u) = \int_\Sigma |\nabla_g u|_g^2 \, dg$$

définit une fonction continûment différentiable sur l'espace de Sobolev $H^1(\Sigma, g)$. La restriction de l'énergie de Dirichlet à l'ensemble

$$\mathcal{S} = \{u \in H^1(\Sigma, g) \mid \|u\|_{L^2} = 1\}$$

sera notée $E_g : \mathcal{S} \to \mathbb{R}$. En utilisant la méthode des multiplicateurs de Lagrange il est facile de se convaincre qu'une fonction lisse $u \in C^\infty(\Sigma) \cap \mathcal{S}$ est un point critique de E_g si et seulement si u est une fonction propre de l'opérateur laplacien Δ_g dont la valeur propre est $\lambda = E_g(u)$. Les valeurs critiques de E_g sont donc les valeurs propres de Δ_g :

$$\left\{ \begin{array}{c} \text{Valeurs critiques de} \\ E_g : \mathcal{S} \to \mathbb{R} \end{array} \right\} = \left\{ \begin{array}{c} \text{Valeurs propres de} \\ \Delta_g : C^\infty(\Sigma) \to C^\infty(\Sigma) \end{array} \right\}.$$

Les problèmes différentiels dont les solutions sont les points critiques d'une fonctionnelle (souvent appelée action) sont très nombreux. Ce point de vue est omniprésent en physique. Il est particulièrement utile dans l'étude des équations aux dérivées partielles elliptiques non linéaires.

Une méthode permettant de détecter les points critiques d'une fonctionnelle en terme de l'homologie relative de ses ensembles de niveaux est présentée au Chapitre 3 de cette thèse. Il s'agit d'une extension du travail que j'avais entrepris sous la direction de Marlène Frigon lors de ma maîtrise [**21**].

M. Frigon [**20**] a défini un *enlacement homotopique relatif.*

Definition 0.11 (A. Girouard [**22**]). Soit (B, A) et (Q, P) des paires de sous-espaces dans un espace topologique X vérifiant $B \cap P = \emptyset$ et $A \cap Q = \emptyset$. Aors (B, A) *enlace* (Q, P) *homotopiquement* si pour chaque déformation $\eta : [0,1] \times B \to X$ fixant chaque point de A, $\eta(1, B) \cap Q = \emptyset \Rightarrow \exists t \in]0,1], \eta(t, B) \cap P \neq \emptyset$.

La définition suivante généralise une notion d'enlacement homologique utilisée par K. Perera [**47**].

Definition 0.12 (A. Girouard [**22**]). Soit (B, A) et (Q, P) des paires de sous-espaces dans un espace topologique X. Alors (B, A) *enlace* (Q, P) *homologiquement dans* X si $(B, A) \subset (X \setminus P, X \setminus Q)$ et si cette inclusion induit un homomorphisme non trivial en homologie réduite. Étant donnés des entiers $q, \beta \geq 0$, on dira que (B, A) (q, β)*-enlace* (Q, P) *dans* X si l'inclusion précédente induit un homomorphisme de rang β sur le q-ième groupe d'homologie réduite.

L'enlacement homologique est plus faible que l'enlacement homotopique.

Théorème 0.13 (A. Girouard [**22**]). *L'enlacement homologique implique l'enlacement homotopique.*

Soit H un espace de Hilbert et $f : H \longrightarrow \mathbb{R}$ une fonction deux fois continûment différentiable. Étant donné un nombre $c \in \mathbb{R}$,

$$K(f) = \{p \in H \big| f'(p) = 0\}$$

est l'ensemble critique de f et $K_c(f) = K(f) \cap f^{-1}(c)$. On suppose que la fonction f vérifie les hypothèses suivantes :

(H1) La fonction f vérifie la condition de Palais-Smale : chaque suite $(x_n) \subset H$ telle que la suite $(f(x_n))$ est bornée et $f'(x_n) \longrightarrow 0$ admet une sous-suite convergente.

(H2) L'ensemble critique $K(f)$ est discret.

Le résultat principal du Chapitre 3 est le suivant.

Théorème 0.14 (A. Girouard, [**22**]). *Soit (B, A) et (Q, P) des paires de sous-espaces de H. Soit $a < b$ des valeurs régulières de f telles que*

$$(B, A) \subset (f_b, f_a) \subset (H \setminus P, H \setminus Q).$$

Si (B, A) (q, β)-enlace (Q, P) dans H pour $\beta \geq 1$ alors f a un point critique p tel que $a < f(p) < b$ dont le groupe critique $C_q(f, p)$ est non-trivial. De plus, si f est une fonction de Morse, elle admet au moins β tels points critiques.

La preuve de ce théorème est un exercice de théorie de Morse élémentaire [**41, 39**]. Elle sera présentée au Chapitre 3.

Chapitre 1

FUNDAMENTAL TONE, CONCENTRATION OF DENSITY TO POINTS AND CONFORMAL DEGENERATION ON SURFACES

Author : Alexandre Girouard

Abstract : We study the effect of two types of degeneration of the Riemannian metric on the first eigenvalue of the Laplace operator on surfaces. In both cases we prove that the first eigenvalue of the round sphere is an optimal asymptotic upper bound. The first type of degeneration is concentration of the density to a point within a conformal class. The second is degeneration of the conformal class to the boundary of the moduli space on the torus and on the Klein bottle. In the latter, we follow the outline proposed by N. Nadirashvili in 1996.

1.1. Introduction

Given a Riemannian metric g on a closed surface Σ, let the spectrum of the Laplace operator Δ_g acting on smooth functions be the sequence

$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \cdots \leq \lambda_k(g) \leq \cdots \nearrow \infty$$

where each eigenvalue is repeated according to its multiplicity. The first nonzero eigenvalue $\lambda_1(g)$ is called the *fundamental tone* of (Σ, g). Let $\mathcal{R}(\Sigma)$ be the space of Riemannian metrics on Σ with total area one. We are interested in the asymptotic behavior of the functional

$$\lambda_1 : \mathcal{R}(\Sigma) \to]0, \infty[$$

under two types of degeneration of the Riemannian metric derscribed below.

1.1.1. Concentration to points

It is expected that a metric maximizing $\lambda_1 : \mathcal{R}(\Sigma) \to]0, \infty[$ has lots of symmetries. For example, on the sphere, the torus and the projective plane, the λ_1-maximizing metrics are the standard homogeneous ones. Here we consider the opposite situation where the distribution of mass of a sequence of metrics concentrates to a point, developping a δ-like singularity.

Definition 1.1. A sequence $(g_n) \subset \mathcal{R}(\Sigma)$ is said to *concentrate to the point* $p \in \Sigma$ if for each neighborhood $\mathcal{O}$ of p

$$\lim_{n\to\infty} \int_{\mathcal{O}} dg_n = 1.$$

Question. Does concentration to a point impose any restriction on the asymptotic behavior of the eigenvalues of the Laplace operator Δ_{g_n} on the surface Σ?

Without any further constraints, the answer is no.

Proposition 1.2. *For any metric g_0 and any point $p \in \Sigma$, there exists a sequence (g_n) of pairwise isometric metrics concentrating to p. In particular the metrics (g_n) are isospectral.*

Under the additional assumption that the metrics g_n are conformally equivalent, we obtain an optimal asymptotic upper bound on the fundamental tone.

Theorem 1.3. *Let $[g] = \{\alpha g \mid \alpha \in C^\infty(\Sigma), \alpha > 0\}$ be a conformal class on a closed surface Σ.*

a) Let (g_n) be a sequence of metrics of unit area in the conformal class $[g]$. If (g_n) concentrates to a point then

$$\limsup_{n\to\infty} \lambda_1(g_n) \le 8\pi.$$

b) For any point $p \in \Sigma$, there exists a sequence (g_n) of metrics of unit area in the conformal class $[g]$ concentrating to p such that

$$\lim_{n\to\infty} \lambda_1(g_n) = 8\pi.$$

Proposition 1.2 and Theorem 1.3 will be proved in section 1.5.

1.1.2. Conformal degeneration

Given a conformal class $[g]$ on the torus T^2, define

$$\nu([g]) := \sup_{\tilde{g} \in \mathcal{R}(T^2) \cap [g]} \lambda_1(\tilde{g}).$$

This corresponds to the first conformal eigenvalue of Colbois and El Soufi **[12]**. Let

$$\mathcal{M} := \{a + ib \in \mathbb{C} \mid 0 \leq a \leq 1/2, a^2 + b^2 \geq 1, b > 0\}.$$

Any metric on T^2 is conformally equivalent to a flat torus $\mathbb{C}/\Gamma$ for some lattice Γ of $\mathbb{C}$ generated by $1 \in \mathbb{C}$ and $a + ib \in \mathcal{M}$. It follows that $\mathcal{M}$ is a natural

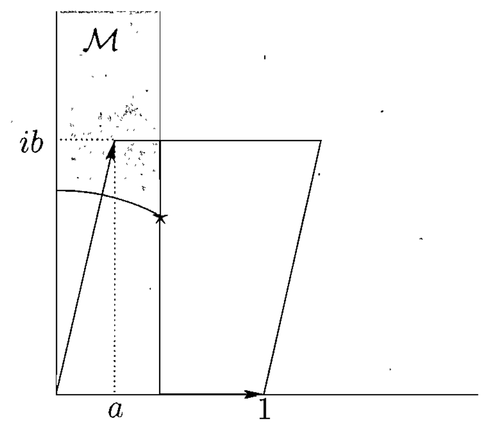

FIG. 1.1. Moduli space of conformal structures on $\mathbb{T}^2$

representation of the moduli space $\mathcal{M}(T^2)$ of conformal classes on the torus (See Figure 1.1).

Definition 1.4. A sequence of metrics on the torus T^2 is *degenerate* if the corresponding sequence $(a_n + ib_n) \subset \mathcal{M}$ satisfies $\lim_{n\to\infty} b_n = \infty$.

Theorem 1.5. *If a sequence (g_n) of Riemannian metrics of unit area on the torus is degenerate, then*

$$\lim_{n\to\infty} \nu([g_n]) = 8\pi.$$

In particular,

$$\limsup_{n\to\infty} \lambda_1(g_n) \leq 8\pi.$$

The proof will be presented in section 1.3. Using a detailed version of a concentration lemma (Lemma 1.16) implicitely used in [**43**] and an estimate on the Dirichlet energy of harmonic functions on long cylinders (Lemma 1.17 and Lemma 1.12), we complete the outline proposed by Nadirashvili in [**43**].

1.1.3. Maximization of λ_1 on surfaces

One motivation for Theorem 1.5 is the role it plays in λ_1-maximization on the torus. More generally, it is natural to ask which metric (if any) on a closed surface Σ, maximizes the fundamental tone λ_1 in the space $\mathcal{R}(\Sigma)$ of Riemannian metrics of unit area. It is well known that

$$\Lambda(\Sigma) := \sup_{g \in \mathcal{R}(\Sigma)} \lambda_1(g)$$

is finite [**27**], [**57**], [**37**] for any closed surface Σ, but the explicit value of $\Lambda(\Sigma)$ has only been computed for a few surfaces.

The first such result was obtained by Hersch [**27**] in 1970. He proved that the round metric g_{S^2} of unit area is the unique maximum of λ_1 on $\mathcal{R}(S^2)$. The proof relies on the fact that (by Riemann's uniformization theorem) any two metrics on the sphere are conformally equivalent.

In 1973, Berger [**6**] proved that among flat metrics on the torus, λ_1 is maximized by the flat equilateral metric g_{eq}. That is, the metric induced from the quotient of $\mathbb{C}$ by the lattice generated by 1 and $e^{i\pi/3}$(indicated by the $\star$ in Figure 1.1). He conjectured that this metric is a global maximum of λ_1 over all Riemannian metrics of unit area. In 1996, Nadirashvili [**43**] proposed a method of proof.

Nadirashvili's approach. The idea is to start with a maximising sequence (g_n) (i.e. such that $\lambda_1(g_n) \to \Lambda(T^2)$) and show that it admits a subsequence converging to a real analytic metric $\overline{g}$. Nadirashvili [**43**] proved that a metric maximizing λ_1 on a surface Σ is also λ_1-minimal. This means that $(\Sigma, \overline{g})$ is minimally immersed in a round sphere by its first eigenfunctions. It was proved by El Soufi and Ilias [**16**] that for any closed manifold other than the sphere (in particular for the torus), the isometry group of a λ_1-minimal metric $\overline{g}$ coincides with its group of conformal transformations (See also [**42**]). Since the group of conformal transformations of a torus acts transitively, this implies that the curvature of $\overline{g}$ is constant. By Gauss-Bonnet theorem the curvature of $\overline{g}$ must therefore be zero. The result of Berger [**6**] stated above completes the proof.

The first step in showing that (g_n) admits a convergent subsequence is to prove that the associated sequence $([g_n])$ of conformal classes admits a converging subsequence. Since $\lambda_1(g_{eq}) = 8\pi^2/\sqrt{3} > 8\pi$, Theorem 1.5 implies that the corresponding sequence $(a_n + ib_n)$ is bounded and therefore admits a convergent subsequence.

Remark 1.6.

- Explicit λ_1-maximal metrics are also known for the projective plane [**37**] and the Klein bottle [**31**], [**14**]. There is a conjecture for surfaces of genus two [**30**].
- The existence of analytic λ_1-maximal metrics has recently been used in [**31**] and [**14**].

1.1.4. Conformal degeneration on the Klein bottle

Define two affine transformations t_b and τ of $\mathbb{C}$ by

$$t_b(x+iy) = x + i(y+b),$$

$$\tau(x+iy) = x + \pi - iy.$$

Let G_b be the group of transformations generated by t_b and τ.

Lemma 1.7. *Any Riemannian metric g on the Klein bottle $\mathbb{K}$ is conformally equivalent to one of the standard flat models*

$$K_b := \mathbb{C}/G_b.$$

In other words, there exists a smooth function $\alpha : K_b \to]0,\infty[$ such that $(\mathbb{K}, g)$ is isometric to $(K_b, \alpha(dx^2 + dy^2))$.

It follows that the moduli space of conformal classes on the Klein bottle is identified with the set of positive real numbers.

Theorem 1.8. *Let $(g_n) \subset \mathcal{R}(\mathbb{K})$ be a sequence of metrics on the Klein bottle.*

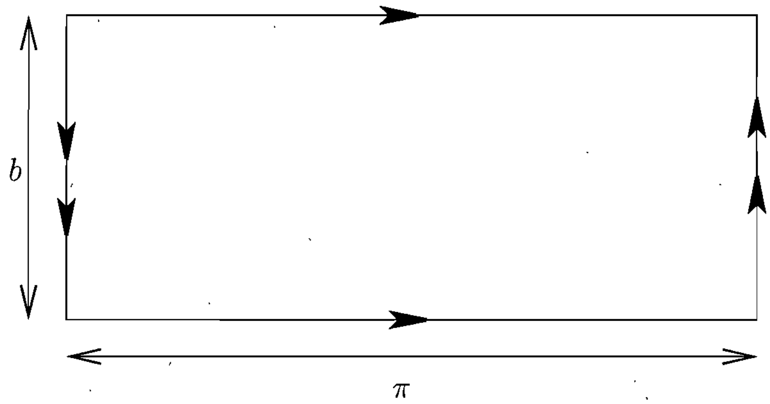

FIG. 1.2. Moduli space of conformal structures on $\mathbb{K}$

1. *If* $\lim_{n\to\infty} b_n = 0$, *then* $\limsup_{n\to\infty} \lambda_1(g_n) = 8\pi$.
2. *If* $\lim_{n\to\infty} b_n = \infty$, *then* $\limsup_{n\to\infty} \lambda_1(g_n) \leq 12\pi$.

The proof will be presented in section 1.4, we follow the outline proposed by Nadirashvili in **[43]** and in a private communication. The first case is very similar to the corresponding result for the torus (Theorem 1.5). The second case uses the fact that the standard metric on $\mathbb{R}P^2$ is minimally embedded in S^4 by its first eigenfunctions. A theorem of Li and Yau **[37]** on conformal area of minimal surfaces is then used to obtain an estimate on the Dirichlet energy of a test function.

1.1.5. Friedlander and Nadirashvili invariant

For a closed manifold M of dimension at least 3, Colbois and Dodziuk **[11]** proved that the first eigenvalue is unbounded on the set of Riemannian metrics of unit area, that is $\Lambda(M) = +\infty$. On the other hand, it is known that the supremum $\nu(C)$ of λ_1 restricted to metrics of unit area in any fixed conformal class C is finite **[15]**. Friedlander and Nadirashvili **[19]** introduced the differential invariant

$$I(M) := \inf \left\{ \nu(C) \;\middle|\; C \text{ is a conformal class on } M \right\}$$

and proved that it satisfies

$$I(M) \geq \lambda_1(S^n, g_{S^n})$$

where g_{S^n} is the round metric of unit area on the sphere S^n. It is unknown if this invariant distinguishes nonequivalent differential structures. In fact, it is very difficult to compute $I(\Sigma)$ explicitely, even for surfaces. Since any two metrics on S^2 are conformally equivalent, it is obvious that $I(S^2) = 8\pi$. For similar reasons, $I(\mathbb{R}P^2) = 12\pi$. The invariant for the torus and for the Klein bottle are obtained as corollaries to Theorem 1.5 and Theorem 1.8.

Corollary 1.9. *The Friedlander-Nadirashvili invariants of the torus and of the Klein bottle are* 8π.

This result is in agreement with the following.

Conjecture 1.10 (Friedlander-Nadirashvili). *For any closed surface* Σ *other than the projective plane* $\mathbb{R}P^2$, $I(\Sigma) = 8\pi$.

Remark 1.11. A recent paper by P. Jammes shows that $I(\Sigma) \leq 16\pi$.

1.2. Analytic background

Let Σ be a closed surface. In dimension two the Dirichlet energy of a function $u \in C^\infty(\Sigma)$

$$D(u) = \int_\Sigma |\nabla_g u|^2 \, dg$$

is invariant under conformal diffeomorphisms. In order to estimate the first eigenvalue of the Laplace operator Δ_g, the following variational characterization will be used :

$$\lambda_1(g) = \inf \left\{ \frac{D(u)}{\int_\Sigma u^2 \, dg} \;\middle|\; u \in C^\infty(\Sigma), u \neq 0, \int_\Sigma u \, dg = 0 \right\}. \qquad (1.1)$$

1.2.1. Dirichlet energy estimate on thin cylinders

The main technical tool that we use is an estimate on the Dirichlet energy of harmonic functions on long cylinders in terms of their restrictions to the boundary circles.

Lemma 1.12. *Let* $\Omega = (0, L) \times S^1$ *with* $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. *Consider* $f \in C^\infty(\overline{\Omega})$ *such that* $f(0, \theta) = 0$ *and* $|f(L, \theta)| \leq 1$. *Let* $u(\theta) = f(L, \theta)$. *If* f *is harmonic, then*

$$\int_\Omega |\nabla f|^2 \leq \frac{2\pi}{L} + \coth(L) \int_{\theta=0}^{2\pi} |u'(\theta)|^2 d\theta.$$

The idea is to express the Fourier series of the function f in terms of the Fourier series of u. This is similar to Hurwitz's proof of the isoperimetric inequality [**28**].

PROOF. Let the Fourier representation of u be

$$u(\theta) = K + \sum_{k=1}^{\infty}\big(a_k\cos(k\theta) + b_k\sin(k\theta)\big).$$

Direct computation shows that f admits the following representation :

$$f(x,\theta) = \frac{Kx}{L} + \sum_{k=1}^{\infty}\frac{\sinh(kx)}{\sinh(kL)}\Big(a_k\cos(k\theta) + b_k\sin(k\theta)\Big)$$

and integration by parts leads to

$$\begin{aligned}
\int_\Omega |\nabla f|^2 &= -\int_\Omega f\Delta f + \int_{\partial\Omega} f\frac{\partial f}{\partial\nu} \\
&= \int_{\theta=0}^{2\pi} u(\theta)\partial_x f(x,\theta)d\theta\Big|_{x=L} \\
&= \int_{\theta=0}^{2\pi}\Big(K + \sum_{k=1}^{\infty} a_k\cos(k\theta) + b_k\sin(k\theta)\Big)\times \\
&\qquad \Big(\frac{K}{L} + \sum_{k=1}^{\infty} k\coth(kL)\,(a_k\cos(k\theta) + b_k\sin(k\theta))\Big)d\theta \\
&= 2\pi\frac{K^2}{L} + \pi\sum_{k=1}^{\infty} k\coth(kL)(a_k^2 + b_k^2)
\end{aligned}$$

For $x > 0$,

$$\frac{d}{dx}\coth(x) = -\frac{4}{(e^x - e^{-x})^2} < 0$$

so that for each $k \geq 1$,

$$\coth(kL) \leq \coth(L).$$

It follows that

$$\begin{aligned}
\int_\Omega |\nabla f|^2 &\leq 2\pi\frac{K^2}{L} + \pi\coth(L)\sum_{k=1}^{\infty} k(a_k^2 + b_k^2) \\
&\leq 2\pi\frac{K^2}{L} + \pi\coth(L)\sum_{k=1}^{\infty} k^2(a_k^2 + b_k^2) \\
&= 2\pi\frac{K^2}{L} + \coth(L)\int_{\theta=0}^{2\pi}|u'(\theta)|^2 d\theta
\end{aligned}$$

where $|K| = \frac{1}{2\pi}|\int_{\theta=0}^{2\pi} u(\theta)d\theta| \leq 1$ since $u(\theta) \in [-1,1]$. □

A simple conformal change of coordinates is used to extend Lemma 1.12 to the situation where the boundary circle has arbitrary length.

Corollary 1.13. *Suppose the hypothesis of Lemma 1.12 holds with the circle* $\mathbb{R}/2\pi\mathbb{Z}$ *replaced by* $\mathbb{R}/r\mathbb{Z}$. *For any* $L > 0$

$$\int_\Omega |\nabla f|^2 \leq \frac{r}{L} + \frac{r}{2\pi}\coth(\frac{2\pi L}{r})\int_{x=0}^{r} |u'(x)|^2 dx. \tag{1.2}$$

1.2.2. Conformal renormalization of centers of mass

It is possible to conformally move any nonsingular distribution of mass on the sphere $S^n \subset \mathbb{R}^{n+1}$ in such a way that its center of mass becomes the origin of $\mathbb{R}^{n+1}$.

Lemma 1.14 (Hersch Lemma). *Let* μ *be a measure on the sphere* S^n. *If the support of* μ *is not a point, then there exists a conformal transformation* τ *of the sphere* S^n *such that*

$$\int_{S^n} \pi \circ \tau \; d\mu = 0 \tag{1.3}$$

where $S^n \overset{\pi}{\hookrightarrow} \mathbb{R}^{n+1}$ *is the standard embedding.*

This lemma was obtained by Hersch [**27**] in 1970, see also [**53**]. It is proved using a topological argument similar to the proof of Brouwer fixed point theorem.

Corollary 1.15. *Let* μ *be a measure on a surface* Σ. *Consider an embedding* $\phi : \Sigma \to S^n$. *If the support of* μ *is not a point, then there exists a conformal transformation* τ *of* S^n *such that*

$$\int_\Sigma \pi \circ \tau \circ \phi \, d\mu = 0.$$

Proof. The result follows from application of Hersch Lemma to the push-forward measure $\phi_*\mu$ since $\int_{S^n} f \, d(\phi_*\mu) = \int_\Sigma f \circ \phi \, d\mu$ for any smooth function f. □

1.3. Conformal degeneration on the torus

The goal of this section is to prove Theorem 1.5.

1.3.1. Moduli space of tori

For any $b > 0$, let

$$T_b := \left\{ [x+iy] \in \mathbb{C}/\mathbb{Z} \;\middle|\; -b/2 < y < b/2 \right\}$$

be a flat cylinder of length b. Given $a+ib \in \mathcal{M}$, let $\Gamma_{a,b}$ be the lattice of $\mathbb{C}$ generated by 1 and $a+ib$. Consider the group $G_{a,b}$ of transformations of the cylinder T_∞ generated by

$$[x+iy] \mapsto [x+a+i(y+b)].$$

The cylinder T_b is a fundamental domain of this action and the torus $\mathbb{C}/\Gamma_{a,b}$ can also be obtained as $T_\infty/G_{a,b}$.

1.3.2. Concentration on thin cylinders

In order to make notation less cumbersome, consider a sequence (g_n) of metrics such that $b_n = n$. The first eigenvalue of the flat torus corresponding to g_n,

$$\lambda_1(T_\infty/G_{a_n,n}) = \frac{4\pi^2}{n^2}$$

tends to zero with n going to infinity [**7**]. Imposing a uniform positive lower bound $\lambda_1(g_n) \geq K > 0$ on the first eigenvalues for g_n should therefore imply that g_n is "far from being flat". The next lemma makes this intuitive idea precise by showing that the Riemannian measures dg_n concentrate on relatively thin cylindrical parts in (T^2, g_n).

Lemma 1.16. *If* $\liminf_{n\to\infty} \lambda_1(g_n) > 0$, *then for any* $\epsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that for any* $n \geq N$,

$$\max\{\int_{T_{3n/4}} dg_n, \int_{T_n \setminus T_{n/4}} dg_n\} \geq 1-\epsilon.$$

Let A_n^ϵ be the maximizing cylinder : either $T_{3n/4}$ or $T_n \setminus T_{n/4}$. This lemma says that most of the mass (i.e. $1-\epsilon$) is concentrated on a cylinder whose lenth is $3/4$ of the total length. Without loss of generality, we will suppose $A_n^\epsilon = T_{3n/4}$.

PROOF. The function

$$\gamma_n([x+iy]) = \cos(\frac{2\pi y}{n})$$

is a first eigenfunction on the flat torus $T_n/G_{a_n,n}$ corresponding to g_n. Let

$$c_n = \int_{T_n} \gamma_n(x+iy)\, dg_n$$

and define $h_n : T_n \to \mathbb{R}$ by

$$h_n = \gamma_n - c_n,$$

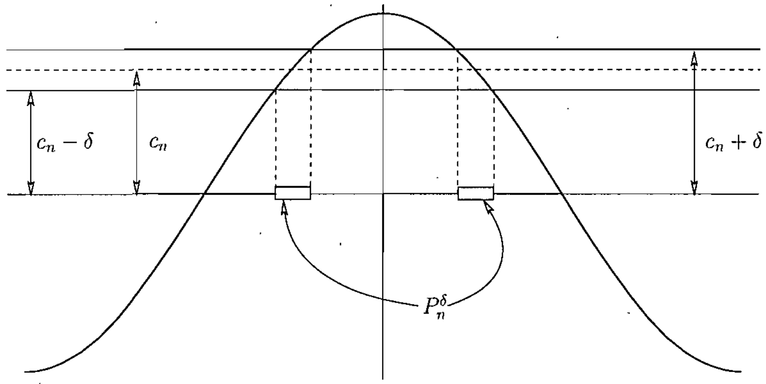

FIG. 1.3. Concentration I.

which satisfies

$$\int_{T_n} h_n \, dg_n = 0.$$

Given $\delta > 0$, let $P_n^\delta = h_n^{-1}([-\delta, \delta])$ and $Q_n^\delta = T_n \setminus P_n^\delta$. Since $\delta^2 \leq h_n^2$ on Q_n^δ,

$$\delta^2 \int_{Q_n^\delta} dg_n \leq \int_{Q_n^\delta} h_n^2 \, dg_n \leq \int_{T_n} h_n^2 \, dg_n.$$

Let $K = \liminf_{n\to\infty} \lambda_1(g_n) > 0$. Using the variational characterization of $\lambda_1(g_n)$ and the conformal invariance of the Dirichlet energy,

$$\delta^2 \int_{Q_n^\delta} dg_n \leq \frac{\int_{T_n} |\nabla h_n|^2 \, dg_n}{\lambda_1(g_n)} \leq \frac{\int_{T_n} |\nabla \gamma_n|^2 \, dg_n}{K} = \frac{4\pi^2}{Kn^2} \to 0.$$

So that, for each $\delta > 0$,

$$\lim_{n\to\infty} \int_{Q_n^\delta} dg_n = 0$$

and

$$\lim_{n\to\infty} \int_{P_n^\delta} dg_n = 1.$$

Observe that

$$P_n^\delta = \left\{ [x + iy] \;\middle|\; c_n - \delta \leq \cos(2\pi y/n) \leq c_n + \delta \right\}.$$

For $\delta > 0$ small enough, P_n^δ is the union of two intervals around $\pm \arccos(c_n)$ whose lengths are at most $\frac{n}{10}$. The set P_n^δ will either be included in $T_{3n/4}$ or in $T_n \setminus T_{n/4}$ (see Figure 1.3). $\square$

1.3.3. Transplantation to the sphere

Let $\sigma : \mathbb{C} \longrightarrow S^2$ be the stereographic parametrization of the sphere by its equatorial plane

$$\sigma(u+iv) = \frac{1}{1+u^2+v^2}(2u, 2v, u^2+v^2-1) \tag{1.4}$$

and define the conformal equivalence $\phi : \mathbb{C}/\mathbb{Z} \longrightarrow S^2 \setminus \{\text{poles}\}$ by

$$\phi([z]) = \sigma(e^{-2\pi i z}).$$

1.3.4. Renormalization of the centers of mass

It follows from Corollary 1.15 that there exists conformal transformations τ_n such that

$$\int_{A_n^\epsilon} \pi \circ \tau_n \circ \phi \, dg_n = 0 \in \mathbb{R}^3 \tag{1.5}$$

where $S^2 \overset{\pi}{\hookrightarrow} \mathbb{R}^3$ is the standard embedding. For each $n \in \mathbb{N}$, since $\pi_1^2+\pi_2^2+\pi_3^2 = 1$ on S^2, there exists an indice $i = i(n) \in \{1,2,3\}$ such that the function

$$u_n = \pi_i \circ \tau_n \circ \phi$$

satisfies

$$\int_{A_n^\epsilon} u_n^2 \, dg_n \geq \frac{1}{3}\int_{A_n^\epsilon} dg_n \geq \frac{1-\epsilon}{3}. \tag{1.6}$$

1.3.5. Test functions

The function u_n will be extended and perturbed to a function f_n defined on T_n and admissible for the variational characterization (1.1) of $\lambda_1(g_n)$.

Let $I_n^- = [-7n/16, -6n/16]$ and $I_n^+ = [6n/16, 7n/16]$.

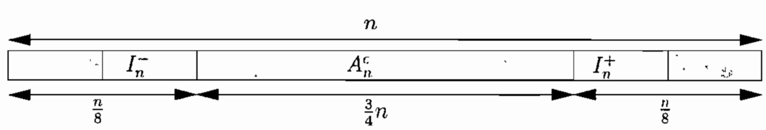

FIG. 1.4. Concentration II

Given $\alpha_n^- \in I_n^-$ and $\alpha_n^+ \in I_n^+$, define cylinders

$$B(\alpha_n^-) = \{[x+iy] \in T_n \mid y \leq \alpha_n^-\},$$
$$B(\alpha_n^+) = \{[x+iy] \in T_n \mid \alpha_n^+ \leq y\}.$$

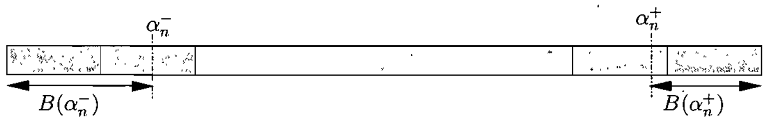

FIG. 1.5. Concentration III

Their lengths are at least $n/16$. Define $w_n : T_n \to \mathbb{R}$ by the following differential problem :

$$\begin{cases} \Delta w_n = 0 & \text{on } B(\alpha_n^-) \cup B(\alpha_n^+), \\ w_n = 0 & \text{on } \partial T_n = \left\{[x+iy] \in \mathbb{C}/\mathbb{Z} \;\Big|\; |y| = n/2\right\}, \\ w_n = u_n & \text{on } T_n \setminus (B(\alpha_n^-) \cup B(\alpha_n^+)). \end{cases}$$

Since the continuous function w_n is piecewise smooth and satisfies $w_n = 0$ on the boundary of T_n, it is compatible with the identification of the boundary and induces a piecewise smooth function on the torus.

Let $\delta_n = \int_{T_n} w_n \, dg_n$. Since $\int_{A_n^\epsilon} w_n \, dg_n = 0$, it follows from concentration of the measures dg_n on A_n^ϵ and from the maximum principle that

$$|\delta_n| = \left| \int_{T_n \setminus A_n^\epsilon} w_n \, dg_n \right| \leq \max_{x \in T_n \setminus A_n^\epsilon} |w_n(x)| \int_{T_n \setminus A_n^\epsilon} dg_n \leq \varepsilon.$$

This means that w_n is almost admissible for the variational characterization of $\lambda_1(g_n)$. Define $f_n : T_n \to \mathbb{R}$ by

$$f_n = w_n - \delta_n \tag{1.7}$$

so that $\int_{T_n} f_n \, dg_n = 0$. From (1.5) and (1.6) it follows that

$$\begin{aligned} \int_{T_n} f_n^2 \, dg_n &\geq \int_{A_n^\epsilon} (u_n - \delta_n)^2 \, dg_n \\ &= \int_{A_n^\epsilon} u_n^2 \, dg_n + \delta_n^2 \int_{A_n^\epsilon} dg_n \geq \frac{1-\epsilon}{3} \end{aligned} \tag{1.8}$$

Using the variational characterization (1.1) of $\lambda_1(g_n)$ and conformal invariance of the Dirichlet energy leads to

$$\begin{aligned}
\lambda_1(g_n) &\leq \frac{3\int_{T_n} |\nabla f_n|^2\, dg_n}{1-\varepsilon} \qquad (1.9)\\
&= \frac{3}{1-\varepsilon}\left(\int_{T_n\setminus(B(\alpha_n^-)\cup B(\alpha_n^+))} |\nabla w_n|^2\, dg_n \right.\\
&\qquad\qquad \left. + \int_{B(\alpha_n^-)\cup B(\alpha_n^+)} |\nabla w_n|^2\, dg_n\right)\\
&\leq \frac{3}{1-\varepsilon}\left(\int_{S^2} |\nabla \pi_i|^2\, dg_{S^2} + \int_{B(\alpha_n^-)\cup B(\alpha_n^+)} |\nabla w_n|^2\, dg_n\right)\\
&= \frac{8\pi}{1-\varepsilon} + \frac{3}{1-\varepsilon}\int_{B(\alpha_n^-)\cup B(\alpha_n^+)} |\nabla w_n|^2\, dg_n.
\end{aligned}$$

1.3.6. Energy estimate

On a long flat cylinder like $B(\alpha_n^\pm)$, the Dirichlet energy of a harmonic function is controled by the Dirichlet energy of its restriction to the boundary circles. Corollary 1.13 is applied to the function $f = w_n$ on the cylinders $\Omega = B(\alpha_n^\pm)$, their lengths are at least $n/16$. For any x, $u(x) := f(x, \alpha_n^\pm) \in [-1,1]$ since it is a coordinate function on the sphere.

The next lemma shows that the numbers $\alpha_n^\pm \in I_n^\pm$ can be chosen to make the Dirichlet energy of u_n on the boundary of $B(\alpha_n^\pm)$ small.

Lemma 1.17. *For each n, there exist $\alpha_n^- \in I_n^-$ and $\alpha_n^+ \in I_n^+$ such that*

$$\int_{x=0}^{1} |\partial_x u_n(x + i\alpha_n^\pm)|^2 dx \leq \frac{128\pi}{3n}.$$

Proof. Let $E_n^\pm = \left\{[x+iy] \in \mathbb{C}/\mathbb{Z} \;\middle|\; y \in I_n^\pm\right\}$. Since the width of $I_n^\pm$ is at least $n/16$, the mean-value theorem implies

$$\iint_{E_n^\pm} |\nabla u_n|^2\, dxdy \geq \frac{n}{16}\min\left\{\int_{x=0}^{1} |\nabla u_n(x+i\alpha_n^\pm)|^2 dx \;\middle|\; \alpha_n^\pm \in I_n^\pm\right\}$$

In particular, there exists $\alpha_n^\pm \in I_n^\pm$ such that

$$\int_{x=0}^{1} |\partial_x u_n(x + i\alpha_n^\pm)|^2 dx \leq \frac{16}{n}\iint_{E_n^\pm} |\nabla u_n|^2\, dxdy.$$

By conformal invariance of the Dirichlet energy, this last quantity is bounded above by

$$\frac{16}{n}\int_{\tau_n\circ\phi(E_n^{\pm})}|\nabla\pi_i|^2\,dg_{S^2}\leq\frac{16}{n}\int_{S^2}|\nabla\pi_i|^2\,dg_{S^2}=\frac{16}{n}.\frac{8\pi}{3}=\frac{128\pi}{3n}.$$

□

Proof of Theorem 1.5. Let f_n be the function given by (1.7). Using the estimate on boundary derivative (Lemma 1.17), the Dirichlet energy estimate on cylinders (Lemma 1.12) implies

$$\lim_{n\to\infty}\int_{B(\alpha_n^{\pm})}|\nabla f_n|^2\,dg_n=0. \tag{1.10}$$

Using inequality (1.9), obtained by the variational characterization of $\lambda_1(g_n)$, it follows that $\limsup_n \lambda_1(g_n)\leq\frac{8\pi}{1-\epsilon}$. Since $\epsilon>0$ is arbitrary, it follows that

$$\limsup_{n\to\infty}\lambda_1(g_n)\leq 8\pi.$$

Finally, the lower bound follows from the result of Friedlander and Nadirashvili **[19]** stated in the introduction : for any conformal class C on a closed surface, $\nu(C)\geq 8\pi$. □

1.4. Conformal degeneration on the Klein bottle

The goal of this section is to prove Theorem 1.8. Let $S^k(r)$ be the k-dimentional sphere of radius r with its standard metric $g_{S^k(r)}$. Let $\mathbb{R}P^k(r)$ be the associated projective space with standard metric $g_{\mathbb{R}P^k(r)}$. Recall from the introduction that any Klein bottle is conformally equivalent to a unique $K_b=\mathbb{C}/G_b$ where G_b is the group of transformations of $\mathbb{C}$ generated by $t_b(x+iy)=x+i(y+b)$ and $\tau(x+iy)=x+\pi-iy$. The rectangle $[0,\pi)\times[-b/2,b/2)$ is a fundamental domain for the action of G_b on $\mathbb{C}$. Reversing and identifying the opposite vertical sides this rectangle, we obtain a Möbius strip

$$M_b=\Big([0,\pi]\times[-b/2,b/2]\Big)/\tau.$$

1.4.1. Transplantation to the sphere S^4 via projective space

In this paragraph we exhibit a conformal embedding of the infinite Möbius strip M_∞ in the sphere S^4. We start with a lemma which will be used to embed Möbius strip conformally in $\mathbb{R}P^2$.

Lemma 1.18. *The conformal application* $\phi : \mathbb{C} \to S^2 \subset \mathbb{R}^3$ *defined by*

$$\phi(x+iy) = \frac{1}{e^{2y}+1}(2e^y\cos(x), 2e^y\sin(x), e^{2y}-1)$$

satisfies

$$\phi(x+2\pi+iy) = \phi(x+iy),$$

$$\phi(\tau(x+iy)) = -\phi(x+iy).$$

It induces a conformal equivalence $\phi : M_\infty \to \mathbb{R}P^2 \setminus \{[0:0:1]\}$.

The Veronese map v is a well known minimal isometric embedding of $\mathbb{R}P^2(\sqrt{3})$ in the sphere S^4 by its first eigenfunctions. This means that the components of v are eigenfunctions for $\lambda_1(g_{\mathbb{R}P^2(\sqrt{3})}) = 2$. For details, see **[7]** and **[35]**.

It follows that the composition $v \circ \phi$ is a conformal embedding of the Möbius strip M_∞ in S^4.

1.4.2. Concentration on Möbius strips

Without loss of generality, consider a sequence $b_n = n$.

Lemma 1.19. *If* $\liminf_{n\to\infty} \lambda_1(g_n) > 0$, *then for any* $\epsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that for any* $n \geq N$,

$$\max\{\int_{M_{3n/4}} dg_n, \int_{M_n \setminus M_{3n/4}} dg_n\} \geq 1-\epsilon.$$

PROOF. Since the function

$$u_n([x+iy]) = \cos(\frac{2\pi y}{n})$$

used in the proof of Lemma 1.16 is even, it induces a first eigenfunction on the flat Klein bottle K_n. The cylinders constructed in this proof are compatible with the identification on the Möbius strip, also because the function u_n is even. ☐

Notation : Without loss of generality, we suppose the maximum is reached by $A_n^\epsilon := M_{3n/4}$. See Lemma 1.16 for details.

1.4.3. Renormalization of the centers of mass

It follows from Corollary 1.15 that there exists conformal transformations τ_n of S^4 such that

$$\int_{A_n^\epsilon} \pi \circ \tau_n \circ v \circ \phi \, dg_n = 0$$

where $\pi : S^4 \hookrightarrow \mathbb{R}^5$ is the standard embedding. For $1 \leq i \leq 5$, let

$$u_n^i = \pi_i \circ \tau_n \circ v \circ \phi. \tag{1.11}$$

1.4.4. Test functions

The numbers

$$\alpha_n \in I_n := [\frac{6n}{16}, \frac{7n}{16}]$$

will be chosen later (see Lemma 1.17). For each $1 \leq i \leq 5$, define $w_n^i : M_n \to \mathbb{R}$ by the following differential problem :

$$\begin{cases} \Delta w_n^i = 0 & \text{on } M_n \setminus M_{\alpha_n}, \\ w_n^i = 0 & \text{on } \partial M_n = \left\{[(x,y)] \in M_n \;\middle|\; |y| = n/2\right\}, \\ w_n^i = u_n^i & \text{on } M_{\alpha_n} \end{cases}$$

where u_n^i is defined by (1.11). Since the continuous function w_n^i is piecewise smooth and satisfies $w_n^i = 0$ on the boundary of M_n, it is compatible with the identification of the boundary and induces a piecewise smooth function on the Klein bottle K_n.

Since $\int_{A_n^\epsilon} w_n^i \, dg_n = 0$, it follows from concentration of the measures dg_n on A_n^ϵ and from the maximum principle that $\delta_n^i := \int_{M_n} w_n^i \, dg_n$ satisfies $|\delta_n^i| \leq \epsilon$. This means that w_n^i is almost admissible for the variational characterization of $\lambda_1(g_n)$. Thus, it is natural to define $f_n^i : M_n \to \mathbb{R}$ by

$$f_n^i = w_n^i - \delta_n^i \tag{1.12}$$

so that for each i

$$\int_{M_n} f_n^i \, dg_n = 0$$

and similarly to inequality (1.8)

$$\sum_{i=1}^{5} \int_{M_n} (f_n^i)^2 \, dg_n \geq \int_{A_n^\epsilon} dg_n \geq 1 - \epsilon. \tag{1.13}$$

It follows from the variational characterization of $\lambda_1(g_n)$ and from inequality (1.13) that

$$\lambda_1(g_n)\,(1-\epsilon) \leq \lambda_1(g_n) \left(\sum_{i=1}^{5} \int_{M_n} (f_n^i)^2 \, dg_n \right) \leq \sum_{i=1}^{5} \int_{M_n} |\nabla f_n^i|^2 \, dg_n$$
$$= \sum_{i=1}^{5} \left(\int_{M_{\alpha n}} |\nabla w_n^i|^2 \, dg_n + \int_{M_n \setminus M_{\alpha n}} |\nabla w_n^i|^2 \, dg_n \right). \tag{1.14}$$

1.4.5. Energy estimate

First step : Bounding $\sum_{i=1}^{5} \int_{M_{\alpha n}} |\nabla w_n^i|^2 \, dg_n$.

Recall that the Veronese embedding $v : \mathbb{R}P^2(\sqrt{3}) \to S^4$ is isometric and minimal. On $\Sigma_n := \tau_n \circ v(\mathbb{R}P^2(\sqrt{3}))$ we consider the metric induced by g_{S^4}. Proposition 1 of [**37**] says that if a compact surface is minimally immersed in a sphere, then its area cannot be increased by conformal transformations of the sphere. In our particular case this leads to the following proposition.

Proposition 1.20. *For each* $n \in \mathbb{N}$,

$$\int_{\Sigma_n} dg_{S^4} \leq \textit{Area of } \mathbb{R}P^2(\sqrt{3}) = 6\pi.$$

It follows from conformal invariance of the Dirichlet energy that

$$\sum_{i=1}^{5} \int_{M_{\alpha n}} |\nabla w_n^i|^2 \, dg_n \leq \sum_{i=1}^{5} \int_{\Sigma_n} |\nabla \pi_i|^2 \, dg_{S^4}$$

It is proved on page 146 of [**53**] that Σ_n being isometrically immersed in S^4 implies the pointwise identity $\sum_{i=1}^{5} |\nabla \pi_i|^2 = 2$. Whence

$$\sum_{i=1}^{5} \int_{M_{\alpha n}} |\nabla w_n^i|^2 \, dg_n \leq 2 \int_{\Sigma_n} dg_{S^4} \leq 12\pi. \tag{1.15}$$

Second step : Bounding $\sum_{i=1}^{5} \int_{M_n \setminus M_{\alpha n}} |\nabla w_n^i|^2 \, dg_n$.

The function w_n is harmonic on the set $M_n \setminus M_{\alpha_n}$. This is a cylinder of length $L_n := n - \alpha_n \geq 9n/16$ and of width 2π. The next lemma shows that the Dirichlet energy of w_n on the boundary of these cylinders can be controlled by appropriate choice of α_n.

Lemma 1.21. *The number $\alpha_n \in I_n = [6n/16, 7n/16]$ can be chosen such that*

$$\sum_{i=1}^{5} \int_{x=0}^{\pi} |\partial_x w_n^i(x \pm i\alpha_n)|^2 dx \leq 192\pi/n.$$

PROOF. We argue as in Lemma 1.17. Let $E_n^\pm = \{[x+iy] \mid y \in I_n\}$. It follows from the mean-value theorem that

$$\begin{aligned} \frac{n}{16} \min_{\alpha_n^\pm \in I_n^\pm} \sum_{i=1}^{5} \int_{x=0}^{2\pi} |\nabla w_n^i(x, \alpha_n^\pm)|^2 dx &\leq \sum_{i=1}^{5} \iint_{E_n^\pm} |\nabla w_n^i(x+iy)|^2 \, dxdy \\ &\leq 12\pi. \end{aligned}$$

□

Since $M_n \setminus M_{\alpha_n}$ has length L_n at least $n/16$ and is of width 2π, Lemma 1.12 implies

$$\begin{aligned} &\sum_{i=1}^{5} \int_{M_n \setminus M_{\alpha_n}} |\nabla w_n^i|^2 \, dg_n \qquad (1.16) \\ &\quad \leq \sum_{i=1}^{5} \Big(\frac{2\pi}{L_n} + \coth(L_n) \int_{x=0}^{2\pi} |\partial_x w_n^i(x \pm i\alpha_n)|^2 \, dx \Big) \\ &\quad\quad \leq \frac{\pi}{n} \left(160 + 5 \times 192 \coth(\frac{n}{16}) \right). \end{aligned}$$

PROOF OF THEOREM 1.8. Substitution of inequality (1.15) and inequality (1.16) in the variational characterization (1.14) leads to

$$\limsup_{n\to\infty} \lambda_1(g_n)(1-\epsilon) \leq 12\pi + \limsup_{n\to\infty} \frac{\pi}{n} (160 + 960 \coth(n/16)) = 12\pi.$$

Since $\epsilon > 0$ is arbitrary, this completes the proof of Theorem 1.8. □

1.5. CONCENTRATION TO POINTS

The main goal of this section is to prove Theorem 1.3. We start by proving that concentration to a point has no influence on spectrum.

PROOF OF PROPOSITION 1.2. There exists a sequence of diffeomorphisms ϕ_n such that $\lim_{n\to\infty}\phi_n(x) = p \quad dg_0$-almost everywhere. Indeed, let $f : M \to \mathbb{R}$ be any Morse function having p as its unique local minimum and consider $\phi : \mathbb{R}\times M \to M$ be its negative gradient flow with respect to g_0. Since the stable manifolds of any critical point other than p are of codimension strictly greater than 1, they are dg_0-negligible, hence the sequence ϕ_n satisfies the required property and $g_n = \phi_n^* g_0$ concentrates to p. □

We know proceed with the proof of Theorem 1.3.

1.5.1. Construction of a neighborhood system

Let $(g_n) \subset \mathcal{R}(\Sigma)$ be a sequence of metrics concentrating to $p \in \Sigma$. Let $\mathbb{D}$ be the unit open disk in $\mathbb{C}$. Let $\eta : D \to \mathbb{D}$ a conformal chart around p such that $\eta(p) = 0$. Observe that since the metrics g_n are all in the same conformal class, the same chart η will be conformal for each of them.

Lemma 1.22. *There exists a conformal equivalence*

$$\psi : D \setminus \{p\} \to (0,\infty)\times S^1$$

and a family $\mathcal{U}_n \subset D$ of neighborhood of p such that

$$\lim_{n\to\infty}\int_{\mathcal{U}_n} dg_n = 1$$

and

$$\psi(D\setminus\mathcal{U}_n) = (0, L_n)\times S^1$$

with $L_n \to \infty$ and $\psi(\partial D) = \{0\}\times S^1$.

PROOF. For $0 \le \epsilon \le 1$, let $\mathcal{U}(\epsilon) = \eta^{-1}B(0,\epsilon)$. Define ϵ_n by

$$\int_{\mathcal{U}(\epsilon_n)} dg_n = 1-\epsilon_n.$$

It follows from concentration that $\lim_{n\to\infty}\epsilon_n = 0$ so that

$$\lim_{n\to\infty}\int_{\mathcal{U}(\epsilon_n)} dg_n = 1$$

Define the conformal equivalence $\alpha : \mathbb{R}\times S^1 \to \mathbb{C}^*$ by

$$\alpha(x,y) = e^{-x}\big(\cos(y),\sin(y)\big).$$

The composition $\psi = \alpha^{-1} \circ \eta$ has the required property since

$$\psi(D \setminus \mathcal{U}(\epsilon_n)) = \big(0, -\ln(\epsilon_n)\big) \times S^1.$$

□

1.5.2. Renormalization of the centers of mass

Recall that σ, as defined in (1.4), is the stereographic parameterization of the sphere S^2 by its equatorial plane. Let $H \subset S^2$ be the southern hemisphere of S^2. The map

$$\phi = \sigma \circ \eta : D \to H$$

is a conformal equivalence such that $\phi(p)$ is the south pole. It follows from Corollary 1.15 that there exist conformal transformations τ_n of the sphere such that

$$\int_{\mathcal{U}_n} \pi \circ \tau_n \circ \phi \, dg_n = 0$$

where $\pi : S^2 \hookrightarrow \mathbb{R}^3$ is the standard embedding. For each $n \in \mathbb{N}$, since $\pi_1^2 + \pi_2^2 + \pi_3^2 = 1$ on S^2, there exists an indice $i = i(n) \in \{1, 2, 3\}$ such that the function $u_n = \pi_i \circ \tau_n \circ \phi$ satisfies

$$\int_{\mathcal{U}_n} u_n^2 \, dg_n \geq \frac{1}{3} \int_{\mathcal{U}_n} dg_n.$$

1.5.3. Test functions

Consider $\alpha_n \in [L_n/2, L_n]$ to be chosen later. Define $w_n : \Sigma \to \mathbb{R}$ as the unique solution of

$$\begin{cases} w_n = 0 & \text{on } \Sigma \setminus D, \\ w_n = \pi_i \circ \tau_n \circ \phi & \text{on } \mathcal{U}_n \cup \psi^{-1}\big((\alpha_n, L_n) \times S^1\big), \\ \Delta w_n = 0 & \text{on } \psi^{-1}\big((0, \alpha_n) \times S^1\big). \end{cases}$$

By the maximum principle, $\delta_n := \int_\Sigma w_n \, dg_n$ is such that $|\delta_n| \leq \int_{\Sigma \setminus \mathcal{U}_n} dg_n$. Define $f_n = w_n - \delta_n$ so that $\int_\Sigma f_n = 0$ and f_n is admissible for the variational characterisation of $\lambda_1(g_n)$:

$$\lambda_1(g_n) \leq \frac{\int_\Sigma |\nabla f_n|^2 \, dg_n}{\int_\Sigma f_n^2 \, dg_n} \leq \frac{\int_D |\nabla w_n|^2 \, dg_{S^2}}{\int_\Sigma (w_n - \delta_n)^2 \, dg_n}.$$

The denominator satisfies

$$\int_{\Sigma}(w_n-\delta_n)^2\,dg_n \geq \int_{\mathcal{U}_n}(w_n^2-2\delta_n w_n+\delta_n^2)\,dg_n$$
$$\geq \int_{\mathcal{U}_n} w_n^2\,dg_n$$
$$\geq \frac{1}{3}\int_{\mathcal{U}_n} dg_n.$$

Whence,

$$\frac{\lambda_1(g_n)}{3}\int_{\mathcal{U}_n} dg_n \leq \int_D |\nabla w_n|^2\,dg_n$$
$$\leq \int_{\mathcal{U}_n\cup\psi_n^{-1}(0,\alpha_n)\times S^1}|\nabla w_n|^2\,dg_n+\int_{\psi_n^{-1}(\alpha_n,L_n)\times S^1}|\nabla w_n|^2\,dg_n$$
$$\leq \frac{8\pi}{3}+\int_{\psi_n^{-1}(\alpha_n,L_n)}|\nabla w_n|^2\,dg_n.$$

Proof of Theorem 1.3. The set $\psi^{-1}\big((0,\alpha_n)\times S^1\big)$ where w_n is harmonic is conformally equivalent to a cylinder of length $\alpha_n \geq \frac{L_n}{2}$ which becomes infinite as n goes to infinity. The proof is completed by choosing appropriate α_n as it was done in Lemma 1.17 and Lemma 1.21 and then applying Lemma 1.12 to bound the Dirichlet energy. □

1.6. Acknowledgments

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Chapitre 2

MAXIMIZATION OF THE SECOND POSITIVE NEUMANN EIGENVALUE FOR PLANAR DOMAINS

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Abstract : We prove that the second positive Neumann eigenvalue of a bounded simply-connected planar domain of a given area does not exceed the first positive Neumann eigenvalue on a disk of a twice smaller area. This estimate is sharp and attained by a sequence of domains degenerating to a union of two identical disks. In particular, this result implies Polya conjecture for the second Neumann eigenvalue. The proof is based on a combination of analytic and topological arguments. As a by-product of our method we obtain an upper bound on the second eigenvalue for conformally round metrics on odd-dimensional spheres.

2.1. Introduction and main results

2.1.1. Neumann eigenvalues of planar domains

Let Ω be a bounded planar domain. The domain Ω is said to be *regular* if the spectrum of the Neumann boundary value problem on Ω is discrete. This is true, for instance, if Ω satisfies the cone condition, that is there are no outward pointing cusps (see **[45]** for more refined conditions and a detailed discussion).

Let $0 = \mu_0 < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \nearrow \infty$ be the Neumann eigenvalues of a regular domain Ω. According to a classical result of Szegö (**[54]**, see also **[53**, p.137], [**26**, section 7.1]), for any regular simply-connected domain Ω

$$\mu_1(\Omega)\,\text{Area}(\Omega) \le \mu_1(\mathbb{D})\pi \approx 3.39\,\pi, \tag{2.1}$$

where $\mathbb{D}$ is the unit disk, and $\mu_1(\mathbb{D})$ is the square of the first zero of the derivative $J_1'(x)$ of the first Bessel function of the first type. The proof of Szegö's theorem relies on the Riemann mapping theorem and hence works only if Ω is simply-connected. However, inequality (2.1) holds without this assumption, as was later shown by Weinberger [**56**].

The Pólya conjecture for Neumann eigenvalues [**48**] (see also [**53**, p. 139]) states that for any regular bounded domain Ω

$$\mu_k(\Omega)\,\text{Area}(\Omega) \le 4k\,\pi \tag{2.2}$$

for all $k \ge 1$. This inequality is true for all domains that tile the plane, e.g., for any triangle and any quadrilateral [**49**]. It follows from the two-term asymptotics for the eigenvalue counting function ([**29**], [**40**]) that for any domain there exists a number K such that (2.2) holds for all $k > K$.

Inequality (2.1) implies that (2.2) is true for μ_1. The best one could show for $k \ge 2$ was $\mu_k \le 8\pi k$ ([**34**]). In the present paper we consider the case $k = 2$. Our main result is

Theorem 2.1. *Let Ω be a regular simply-connected planar domain. Then*

$$\mu_2(\Omega)\,\text{Area}(\Omega) \le 2\,\mu_1(\mathbb{D})\,\pi \approx 6.78\,\pi, \tag{2.3}$$

with the equality attained in the limit by a family of domains degenerating to a disjoint union of two identical disks.

The second part of the theorem immediately follows from (2.3). Indeed, if Ω is a disjoint union of two identical disks then (2.3) is an equality. Joining the two disks by a passage of width ϵ we can construct a family of simply-connected domains such that the left-hand side in (2.3) tends to $2\mu_1(\mathbb{D})\pi$ as $\epsilon \to 0$.

Theorem 2.1 gives a positive answer to a question of Parnovski (private communication), motivated by an analogous result proved in [**44**] for the second eigenvalue on a sphere. Note that (2.3) immediately implies (2.2) for $k = 2$ for any regular simply-connected planar domain.

Remark 2.2. It would be interesting to check the bound (2.3) for non-simply connected domains. We believe it remains true in this case as well.

Remark 2.3. All estimates discussed in this section have analogues in the Dirichlet case. For example, (2.1) is the Neumann counterpart of the celebrated Faber-Krahn inequality ([**17, 32**] , see also [**26**, section 3.2]), which states that among all bounded planar domains of a given area, the first Dirichlet eigenvalue is minimal on a disk. Similarly, Theorem 2.1 can be viewed as an analogue of the result due to Krahn and Szegö ([**33**], [**26**, Theorem 4.1.1]), who proved that among bounded planar domains of a given area, the second Dirichlet eigenvalue is minimized by the union of two identical disks.

2.1.2. Eigenvalue estimates on spheres

Let $(\mathbb{S}^n, g)$ be a sphere of dimension $n \geq 2$ with a Riemannian metric g. Let

$$0 < \lambda_1(\mathbb{S}^n, g) \leq \lambda_2(\mathbb{S}^n, g) \leq \cdots \nearrow \infty$$

be the eigenvalues of the Laplacian on $(\mathbb{S}^n, g)$. Hersch [**27**] adapted the approach of Szegö to prove that $\lambda_1(\mathbb{S}^2, g)\,\mathrm{Area}(\mathbb{S}^2, g) \leq 8\pi$ for any Riemannian metric g, with the equality attained on a sphere with the standard round metric g_0. In order to obtain a similar estimate in higher dimensions, one needs to restrict the Riemannian metrics to a fixed conformal class [**15**]. Indeed, in dimension ≥ 3, if one only restricts the volume, λ_1 is unbounded [**11**]. In particular, it was shown in [**15**] (see also [**38**]) that for any metric g in the class $[g_0]$ of conformally round metrics,

$$\lambda_1(\mathbb{S}^n, g)\,\mathrm{Vol}(\mathbb{S}^n, g)^{\frac{2}{n}} \leq n\,\omega_n^{2/n}, \tag{2.4}$$

where

$$\omega_n = \frac{2\,\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$$

is the volume of the unit round n-dimensional sphere. This result can be viewed as a generalization of Hersch's inequality, since all metrics on $\mathbb{S}^2$ are conformally equivalent to the round metric g_0.

A similar problem for higher eigenvalues is much more complicated. It was proved in [**12**, Corollary 1] that

$$\lambda_k^c(\mathbb{S}^n,[g_0]) := \sup_{g\in[g_0]} \lambda_k(\mathbb{S}^n,g)\,\mathrm{Vol}(\mathbb{S}^n,g)^{\frac{2}{n}} \geq n\,(k\,\omega_n)^{2/n}, \tag{2.5}$$

The number $\lambda_k^c(\mathbb{S}^n,[g_0])$ is called the k-th conformal eigenvalue of $(\mathbb{S}^n,[g_0])$. It was shown in [**43**] that for $k=2$ and $n=2$ the inequality in (2.5) is an equality, and the supremum is attained by a sequence of surfaces tending to a union of two identical round spheres. We conjecture that the same is true in all dimensions :

Conjecture 2.4. *The second conformal eigenvalue of* $(\mathbb{S}^n,[g_0])$ *equals*

$$\lambda_2^c(\mathbb{S}^n,[g_0]) = n\,(2\,\omega_n)^{2/n} \tag{2.6}$$

for all $n\geq 2$.

As a by-product of the method developed for the proof of Theorem 2.1, we prove un upper bound for $\lambda_2^c(\mathbb{S}^n,[g_0])$ when the dimension n is *odd* (this condition is explained in Remark 2.29). Our result is in good agreement with Conjecture 2.4.

Theorem 2.5. *Let* $n\in\mathbb{N}$ *be odd and let* $(\mathbb{S}^n,g)$ *be a* n*-dimensional sphere with a conformally round metric* $g\in[g_0]$. *Then*

$$\lambda_2(\mathbb{S}^n,g).\mathrm{Vol}(\mathbb{S}^n,g)^{\frac{2}{n}} < (n+1)\left(\frac{4\pi^{\frac{n+1}{2}}\Gamma(n)}{\Gamma(\frac{n}{2})\Gamma(n+\frac{1}{2})}\right)^{2/n} \tag{2.7}$$

Remark 2.6. Note that the Dirichlet energy is not conformally invariant in dimensions $n\geq 3$ and therefore to prove Theorem 2.5 we have to work with the modified Rayleigh quotient (cf. [**19**]). This is in fact the reason why we do not get a sharp bound (see Remark 2.32). At the same time, the estimate (2.7) is just a little bit weaker than the conjectured bound (2.6) : one can check numerically that the ratio of the constants at the right-hand sides of (2.7) and (2.6) is contained in the interval $(1,1.04)$ for all n. Moreover, the difference between the two constants tends to 0 as the dimension $n\to\infty$, and hence (2.7) is "asymptotically sharp" as follows from (2.5).

Remark 2.7. It was conjectured in [**44**] that if $n=2$ then (2.5) is an equality for all $k\geq 1$, with the maximizer given by the union of k identical round spheres.

One could view it as an analogue of the Pólya conjecture (2.2) for the sphere. Note that a similar "naive" guess about the maximizer of the k-th Neumann eigenvalue of a planar domain is false : a union of k equal disks can not maximize μ_k for all $k \geq 1$, because, as one could easily check, this would contradict Weyl's law. For the same reason, (2.5) can not be an equality for all $k \geq 1$ in dimensions $n \geq 5$.

2.1.3. Plan of the paper

The paper is organized as follows. In sections 2.1–2.5 we develop the "folding and rearrangement" technique based on the ideas of [**44**] and apply it to planar domains. The topological argument used in the proof of Theorem 2.1 is presented in section 2.6. In section 2.7 we complete the proof of the main theorem using some facts about subharmonic functions. In sections 3.1 and 3.2 we prove the auxiliary lemmas used in the proof of Theorem 2.1. In section 4.1 we present a somewhat stronger version of the classical Hersch's lemma ([**27**]). In sections 4.2 and 4.3 we adapt the approach developed in sections 2.1-2.7 for the case of the sphere. In section 4.4 we use the modified Rayleigh quotient to complete the proof of Theorem 2.5.

Acknowledgments

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2.2. Proof of Theorem 2.1

2.2.1. Standard eigenfunctions for μ_1 on the disk

Let

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$$

be the open unit disk. Let J_1 be the first Bessel function of the first kind, and let $\zeta \approx 1.84$ be the smallest positive zero of the derivative J_1'. Set

$$f(r) = J_1(\zeta r).$$

Given $R \geq 0$ and $s = (R\cos\alpha, R\sin\alpha) \in \mathbb{R}^2$, define $X_s : \mathbb{D} \to \mathbb{R}$ by

$$X_s(z) = f(|z|)\frac{z \cdot s}{|z|} = Rf(r)\cos(\theta - \alpha), \tag{2.8}$$

where $r = |z|$, $\theta = \arg z$, and $z \cdot s$ denotes the scalar product in $\mathbb{R}^2$. The functions X_s are the Neumann eigenfunctions corresponding to the double eigenvalue

$$\mu_1(\mathbb{D}) = \mu_2(\mathbb{D}) = \zeta^2 \approx 3.39.$$

The functions X_{e_1} and X_{e_2} form a basis for this space of eigenfunctions (where the vectors $\{e_1, e_2\}$ form the standard basis of $\mathbb{R}^2$).

2.2.2. Renormalization of measure

We say that a conformal transformation T of the disk *renormalizes* a measure $d\nu$ if for each $s \in \mathbb{R}^2$,

$$\int_{\mathbb{D}} X_s \circ T \, d\nu = 0. \tag{2.9}$$

Finite signed measures on $\mathbb{D}$ can be seen as elements of the dual of the space $C(\overline{\mathbb{D}})$ of continuous functions. As such, the norm of a measure $d\nu$ is

$$\|d\nu\| = \sup_{f \in C(\overline{\mathbb{D}}), |f| \leq 1} \left| \int_{\mathbb{D}} f \, d\nu \right| \tag{2.10}$$

The following result is an analogue of Hersch's lemma (see [**27**], [**53**]).

Lemma 2.8. *For any finite measure $d\nu$ on $\mathbb{D}$ there exists a point $\xi \in \mathbb{D}$ such that $d\nu$ is renormalized by the automorphism $d_\xi : \mathbb{D} \to \mathbb{D}$ defined by*

$$d_\xi(z) = \frac{z + \xi}{\bar{\xi} z + 1}.$$

PROOF. Set $M = \int_{\mathbb{D}} d\nu$ and define the continuous map $C : \mathbb{D} \to \mathbb{D}$ by

$$C(\xi) = \frac{1}{M\, f(1)} \int_{\mathbb{D}} (X_{e_1}, X_{e_2}) \; (d_\xi)_* d\nu = \frac{1}{M\, f(1)} \int_{\mathbb{D}} (X_{e_1} \circ d_\xi, X_{e_2} \circ d_\xi) \, d\nu$$

Let $e^{i\theta} \in S^1 = \partial\mathbb{D}$. For any $z \in \mathbb{D}$,

$$\lim_{\xi \to e^{i\theta}} d_\xi(z) = e^{i\theta}.$$

This means that the map C can be continuously extended to the closure $\overline{\mathbb{D}}$ by $C = \mathrm{id}$ on $\partial\mathbb{D}$. By the same topological argument as in Hersch's lemma (and as in the proof of the Brouwer fixed point theorem), a continuous map $C : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$

such that $C(\xi) = \xi$ for $\xi \in \partial\mathbb{D}$ must be onto. Hence, there exists some $\xi \in \mathbb{D}$ such that $C(\xi) = 0 \in \mathbb{D}$. □

Lemma 2.9. *For any finite measure $d\nu$ the renormalizing point ξ is unique.*

Proof. First, let us show that if the measure $d\nu$ is already renormalized then $\xi = 0$. Suppose that $\mathbb{D} \ni \eta \neq 0$ renormalizes $d\nu$. Without loss of generality assume that η is real and positive (if not, apply a rotation). Setting $s = 1$, it will be proved in Lemma 2.22 that $X_s(d_\eta(z)) > X_s(z)$ for all $z \in \mathbb{D}$ and hence

$$\int_{\mathbb{D}} X_s \circ d_\eta \, d\nu > \int_{\mathbb{D}} X_s \, d\nu = 0,$$

which contradicts the hypothesis that η renormalizes $d\nu$.

Now let $d\nu$ be an arbitrary finite measure which is renormalized by $\xi \in \mathbb{D}$. Assume $\eta \in \mathbb{D}$ also renormalizes $d\nu$. Let us show that $\eta = \xi$. Taking into account that $d_{-\xi} \circ d_\xi = d_0 = \mathrm{id}$, we can write

$$(d_\eta)_* d\nu = (d_\eta \circ d_{-\xi})_* (d_\xi)_* \, d\nu.$$

A straightforward computation shows that

$$d_\eta \circ d_{-\xi} = \frac{1 - \eta\bar{\xi}}{1 - \bar{\eta}\xi} d_\alpha,$$

where $\alpha = d_{-\xi}(\eta)$ and $\left|\frac{1-\eta\bar{\xi}}{1-\bar{\eta}\xi}\right| = 1$. This implies that d_α renormalizes $(d_\xi)_* \, d\nu$ which is already renormalized. Hence, as we have shown above, $\alpha = d_{-\xi}(\eta) = 0$, and therefore $\xi = \eta$. □

Given a finite measure, we write $\Gamma(d\nu) \in \mathbb{D}$ for its unique renormalizing point $\xi \in \mathbb{D}$.

Corollary 2.10. *The renormalizing point $\Gamma(d\nu) \in \mathbb{D}$ depends continuously on the measure $d\nu$.*

Proof. Let $(d\nu_n)$ be a sequence of measures converging to the measure $d\nu$ in the norm (2.10). Without loss of generality suppose that $d\nu$ is renormalized. Let $\xi_n \in \mathbb{D} \subset \overline{\mathbb{D}}$ be the unique element such that d_{ξ_n} renormalizes $d\nu_n$. Let (ξ_{n_k}) be a convergent subsequence, say to $\xi \in \overline{\mathbb{D}}$. Now, by definition of ξ_n there holds

$$0 = \lim_{k\to\infty} \left| \int_{\mathbb{D}} X_s \, (d_{\xi_{n_k}})_* d\nu_{n_k} \right| = \left| \int_{\mathbb{D}} X_s \, (d_\xi)_* d\nu \right|,$$

and hence d_ξ renormalizes $d\nu$. Since we assumed that $d\nu$ is normalized, by uniqueness we get $\xi = 0$. Therefore, 0 is the unique accumulation point of the set $\xi_n \in \mathbb{D}$ and hence by compactness we get $\xi_n \to 0$. This completes the proof of the lemma. □

Corollary 2.10 will be used in the proof of Lemma 2.11, see section 2.3.2.

2.2.3. Variational characterization of μ_2

It follows from the Riemann mapping theorem and Lemma 2.8 that for any simply-connected domain Ω there exists a conformal equivalence $\phi : \mathbb{D} \to \Omega$, such that the pullback measure

$$d\mu(z) = \phi^*(dz) = |\phi'(z)|^2\, dz$$

satisfies for any $s \in S^1$

$$\int_{\mathbb{D}} X_s(z)\, d\mu(z) = 0. \tag{2.11}$$

Using a rotation if necessary, we may also assume that

$$\int_{\mathbb{D}} X_{e_1}^2(z)\, d\mu(z) \geq \int_{\mathbb{D}} X_s^2(z)\, d\mu(z). \tag{2.12}$$

for any $s \in S^1$. The proof of Theorem 2.1 is based on the following variational characterization of $\mu_2(\Omega)$:

$$\mu_2(\Omega) = \inf_E \sup_{0 \neq u \in E} \frac{\int_{\mathbb{D}} |\nabla u|^2\, dz}{\int_{\mathbb{D}} u^2\, d\mu} \tag{2.13}$$

where E varies among all two-dimensional subspaces of the Sobolev space $H^1(\mathbb{D})$ that are orthogonal to constants, that is for each $f \in E$, $\int_{\mathbb{D}} f\, d\mu = 0$. Note that the Dirichlet energy is conformally invariant in two dimensions, and hence the numerator in (2.13) can be written using the standard Euclidean gradient and the Lebesgue measure.

2.2.4. Folding of hyperbolic caps

It is well-known that the group of automorphisms of the disk coincides with the isometry group of the Poincaré disk model of the hyperbolic plane [**2**, section

7.4]. Therefore, for any $\xi \in \mathbb{D}$, the automorphism

$$d_\xi(z) = \frac{z+\xi}{\bar{\xi}z+1}$$

is an isometry. Note that we have $d_0 = \mathrm{id}$ and $d_\xi(0) = \xi$ for any ξ.

Let γ be a geodesic in the Poincaré disk model, that is a diameter or the intersection of the disk with a circle which is orthogonal to $\partial\mathbb{D}$. Each connected component of $\mathbb{D} \setminus \gamma$ is called a *hyperbolic cap* on $\mathbb{D}$. The space of hyperbolic caps is parametrized as follows. Given $(r,p) \in (-1,1) \times S^1$ let

$$a_{r,p} = d_{rp}(a_{0,p}),$$

where

$$a_{0,p} = \{x \in \mathbb{D} \ : \ x \cdot p > 0\}$$

is the half-disk such that p is the center of its boundary half-circle. The limit $r \to 1$ corresponds to a cap degenerating to a point on the boundary $\partial\mathbb{D}$ (that is, $a \to p$), while the limit $r \to -1$ corresponds to degeneration to the full disk $\mathbb{D}$ (that is, $a \to \mathbb{D}$). Given $p \in \mathbb{D}$, we define the automorphism $R_p(z) = -p^2\bar{z}$.

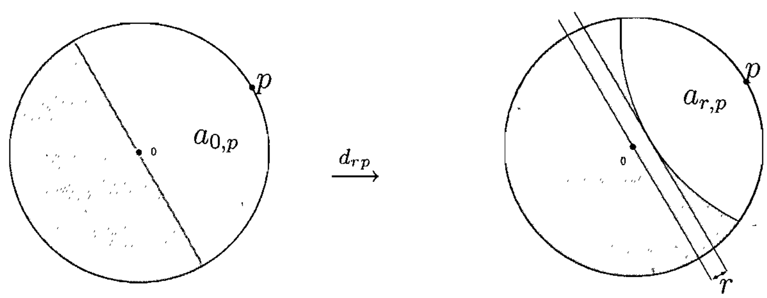

FIG. 2.1. Hyperbolic caps

It is the reflection with respect to the line going through 0 and orthogonal to the segment joining 0 and p. For each cap $a = a_{r,p}$, let us define a conformal automorphism

$$\tau_a = d_{rp} \circ R_p \circ d_{-rp}.$$

One can check that this is the reflection with respect to the hyperbolic geodesic $\partial a_{r,p}$. In particular, $\tau_a(a) = \mathbb{D} \setminus \bar{a}$ and τ_a is the identity on ∂a.

2.2.5. Folding and rearrangement of measure

Given a measure $d\mu$ on $\mathbb{D}$ and a hyperbolic cap $a \subset \mathbb{D}$, the *folded measure* $d\mu_a$ is defined by

$$d\mu_a = \begin{cases} d\mu + \tau_a^* d\mu & \text{on } a, \\ 0 & \text{on } \mathbb{D} \setminus \overline{a}. \end{cases}$$

Clearly, the measure $d\mu_a$ depends continuously in the norm (2.10) on the cap $a \subset \mathbb{D}$. For each cap $a \in \mathbb{D}$ let us construct the following conformal equivalence $\psi_a : \mathbb{D} \longrightarrow a$. First, observe that it follows from the proof of the Riemann mapping

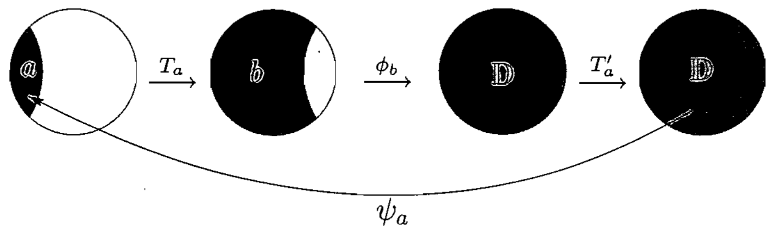

FIG. 2.2. Conformal equivalences from the disk

theorem [**55**, p.342] that there exists a family $\phi_a : a \longrightarrow \mathbb{D}$ of conformal equivalences depending continuously on the cap a such that $\lim_{a \to \mathbb{D}} \phi_a = \text{id}$ pointwise. Let $\xi(a) = \Gamma(d\mu_a)$ be the normalizing point for the measure $d\mu_a$ and set $T_a = d_{\xi(a)}$. The measure $(T_a)_* d\mu_a$ is supported in the cap $b = T_a(a)$. Pushing this measure to the full disk using $\phi_b : b \longrightarrow \mathbb{D}$ leads to the measure

$$(\phi_b \circ T_a)_* d\mu_a.$$

Let $\eta(a) = \Gamma((\phi_b \circ T)_* d\mu_a)$ and set

$$T'_a := d_{\eta(a)} : \mathbb{D} \longrightarrow \mathbb{D}$$

The conformal equivalence $\psi_a : \mathbb{D} \longrightarrow a$ is defined by

$$\psi_a = (T'_a \circ \phi_b \circ T_a)^{-1}.$$

The pull-back by ψ_a of the folded measure is

$$d\nu_a = \psi_a^* d\mu_a \tag{2.14}$$

It is clear from the above construction that $d\nu_a$ is a normalized measure on the whole disk. We call $d\nu_a$ the *rearranged measure*. It also follows from the construction that the conformal transformations $\psi_a : \mathbb{D} \to a$ depend continuously on a and

$$\lim_{a \to \mathbb{D}} \psi_a = \text{id} : \mathbb{D} \to \mathbb{D} \tag{2.15}$$

in the sense of the pointwise convergence. We will make use of the following lemma.

Lemma 2.11. *If a sequence of hyperbolic caps $a \in \mathbb{D}$ degenerates to a point $p \in \partial\mathbb{D}$, the limiting rearranged measure is a "flip-flop" of the original measure $d\mu$:*

$$\lim_{a \to p} d\nu_a = R_p^* d\mu. \tag{F}$$

We call (F) the flip-flop property. The proof of Lemma 2.11 will be presented at the end of the paper.

2.2.6. Maximizing directions

Given a finite measure $d\nu$ on $\mathbb{D}$, consider the function $V : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$V(s) = \int_{\mathbb{D}} X_s^2 \, d\nu.$$

This function is a quadratic form since the mapping $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$(s,t) \mapsto \int_{\mathbb{D}} X_s X_t \, d\nu$$

is symmetric and bilinear (the latter easily follows from (2.8)). In particular, $V(s) = V(-s)$ for any s.

Let $\mathbb{R}P^1 = S^1/\mathbb{Z}_2$ be the projective line. We denote by $[s] \in \mathbb{R}P^1$ the element of the projective line corresponding to the pair of points $\pm s \in S^1$. We say that $[s] \in \mathbb{R}P^1$ is a *maximizing direction* for the measure $d\nu$ if $V(s) \geq V(t)$ for any $t \in S^1$. The measure $d\nu$ is called *simple* if there is a unique maximizing direction. Otherwise, the measure $d\nu$ is said to be *multiple.* We have the following

Lemma 2.12. *A measure $d\nu$ is multiple if and only if $V(s)$ does not depend on $s \in S^1$.*

PROOF. Since $V(s)$ is a symmetric quadratic form, it can be diagonalized. This means that there exists an orthonormal basis (v_1, v_2) of $\mathbb{R}^2$, such that for any $s = \alpha v_1 + \beta v_2 \in \mathbb{D}$ we have $V(s) = M\alpha^2 + m\beta^2$. for some numbers $0 < m \leq M$. It is clear now that the measure $d\nu$ is multiple if and only if $M = m$, and therefore $V(s)$ takes the same value for all $s \in S^1$. □

Note that by (2.12), $[e_1]$ is a maximizing direction for the measure $d\mu$.

Proposition 2.13. *If the measure $d\mu$ is simple, then there exists cap $a \subset \mathbb{D}$ such that the folded measure $d\nu_a$ is multiple.*

The proof of this proposition is based on a topological argument, somewhat more subtle than the one used in the proof of Lemma 2.8. This is a proof by contradiction. We assume the measure $d\mu$ as well as the measures $d\nu_a$ to be simple. Given a cap $a \subset \mathbb{D}$, let $[s(a)] \in \mathbb{R}P^1$ be the unique maximizing direction for $d\nu_a$. Since $d\nu_a$ depends continuously on a and X_s depends continuously on s, it follows that the map $a \mapsto [s(a)]$ is continuous. Let us understand the behavior of the maximizing directions as the cap a degenerates to the full disk and to a point.

Lemma 2.14. *Assume the measures $d\mu$ as well as each $d\nu_a$ to be simple. Then*

$$\lim_{a \to \mathbb{D}} [s(a)] = [e_1] \tag{2.16}$$

$$\lim_{a \to e^{i\theta}} [s(a)] = [e^{2i\theta}]. \tag{2.17}$$

PROOF. First, note that formula (2.16) immediately follows from (2.15) and (2.12). Let us prove (2.17). Set $p = e^{i\theta}$. Lemma 2.11 implies

$$\lim_{a \to p} \int_{\mathbb{D}} X_s^2 \, d\nu_a = \int_{\mathbb{D}} X_s^2 \, R_p^* d\mu = \int_{\mathbb{D}} X_s^2 \circ R_p \, d\mu = \int_{\mathbb{D}} X_{R_p s}^2 \, d\mu. \tag{2.18}$$

Since $[e_1]$ is the unique maximizing direction for $\mathbb{D}$, the right hand side of (2.18) is maximal for $R_p s = \pm e_1$. Applying R_p on both sides we get $s = \pm e^{2i\theta}$ and hence $[s] = [e^{2i\theta}]$. □

PROOF OF PROPOSITION 2.13. Suppose that for each cap $a \subset \mathbb{D}$ the measure $d\nu_a$ is simple. Recall that the space of caps is identified with $(-1, 1) \times S^1$. Define

$h : (-1,1) \times S^1 \to \mathbb{R}P^1$ by by $h(r,p) = [s(a_{r,p})]$. It follows from Lemma 2.14) that h extends to a continuous map on $[-1,1] \times S^1$ such that

$$h(-1, e^{i\theta}) = [e_1], h(1, e^{i\theta}) = [e^{2i\theta}].$$

This means that h is a homotopy between a trivial loop and a non-contractible loop on $\mathbb{R}P^1$. This is a contradiction. □

2.2.7. Test functions

Assume that $d\mu$ is simple. By Proposition 2.13 and Lemma 2.12 there exists a cap $a \subset \mathbb{D}$ such that

$$\int_{\mathbb{D}} X_s^2 \, d\nu_a(z)$$

does not depend on the choice of $s \in S^1$. Let $a^* = \mathbb{D} \setminus \overline{a}$.

Definition 2.15. Given a function $u : a \to \mathbb{R}$, the *lift* of u, $\tilde{u} : \mathbb{D} \to \mathbb{R}$ is given by

$$\tilde{u}(z) = \begin{cases} u(z) & \text{if } z \in a, \\ u(\tau_a z) & \text{if } z \in a^*. \end{cases}$$

Given $u : a \to \mathbb{R}$ we have

$$\int_a u \, d\mu_a = \int_a u \, d\mu + \int_{a^*} u \circ \tau_a \, d\mu = \int_{\mathbb{D}} \tilde{u} \, d\mu,$$

For every $s \in \mathbb{R}^2$, set

$$u_a^s = X_s \circ \psi_a^{-1} : a \to \mathbb{R}.$$

We will use the two-dimensional space

$$E = \left\{ \tilde{u}_a^s \,\middle|\, s \in \mathbb{R}^2 \right\}$$

of test functions in the variational characterization (2.13) of μ_2.

Proposition 2.16. *For each* $s \in \mathbb{R}^2$

$$\frac{\int_{\mathbb{D}} |\nabla \tilde{u}_a^s|^2 \, dz}{\int_{\mathbb{D}} (\tilde{u}_a^s)^2 \, d\mu} \leq 2\mu_1(\mathbb{D}). \tag{2.19}$$

We split the proof of Proposition 2.16 in two parts.

Lemma 2.17. *For any hyperbolic cap $a \subset \mathbb{D}$,*

$$\int_{\mathbb{D}} |\nabla \tilde{u}_a^s|^2 \, dz = \left(2\pi \int_{r=0}^1 f^2(r) r \, dr \right) \mu_1(\mathbb{D}).$$

Lemma 2.18.

$$\int_{\mathbb{D}} (\tilde{u}_a^s)^2 \, d\mu \geq \pi \left(\int_{r=0}^1 f^2(r) r \, dr \right). \tag{2.20}$$

PROOF OF LEMMA 2.17. It follows from the definition of the lift that

$$\int_{\mathbb{D}} |\nabla \tilde{u}_a^s|^2 \, dz = \int_a |\nabla u_a^s|^2 \, dz + \int_{a^*} |\nabla (u_a^s \circ \tau_a)|^2 \, dz.$$

By conformal invariance of the Dirichlet energy, the two terms on the right hand side are equal, so that

$$\begin{aligned} \int_{\mathbb{D}} |\nabla \tilde{u}_a^s|^2 \, dz &= 2 \int_a |\nabla u_a^s|^2 \, dz = 2 \int_a |\nabla (X_s \circ \psi_a^{-1})|^2 \, dz \\ &= 2 \int_{\mathbb{D}} |\nabla X_s|^2 \, dz \quad \longleftarrow \text{(by conformal invariance)} \\ &= 2\mu_1(\mathbb{D}) \int_{\mathbb{D}} X_s^2 \, dz \quad \longleftarrow \text{(since } X_s \text{ is the first eigenfunction on a disk)} \end{aligned} \tag{2.21}$$

It follows from (2.8) that given two orthogonal directions $s, t \in S^1$ we have

$$\int_{\mathbb{D}} (X_s^2 + X_t^2) \, dz = \int_{\mathbb{D}} f^2(|z|) \, dz.$$

Therefore, by symmetry we get

$$\int_{\mathbb{D}} X_s^2 \, dz = \frac{1}{2} \int_{\mathbb{D}} f^2(|z|) \, dz = \pi \int_{r=0}^1 f^2(r) r \, dr.$$

This completes the proof of the lemma. □

To prove Lemma 2.18 we use the following result.

Lemma 2.19. *The rearranged measure $d\nu_a$ on $\mathbb{D}$ can be represented as $d\nu_a = \delta(z)dz$, where $\delta : \mathbb{D} \to \mathbb{R}$ is a subharmonic function.*

PROOF. Indeed, $d\nu_a = \psi_a^* d\mu_a$, where the measure $d\mu_a$ on the cap a is obtained as the sum of measures $d\mu$ and $\tau_a^* d\mu$. Both measures $d\mu$ and $\tau_a^* d\mu$ correspond to flat Riemannian metrics on a, because $d\mu$ is the pullback of the Euclidean measure dz on the domain Ω by the conformal map $\phi : \mathbb{D} \to \Omega$ (see section 2.2.3). Since the maps ψ_a and τ_a are also conformal, one has $\psi_a^* d\mu = \alpha(z)dz$

and $\psi_a^*(\tau_a^* d\mu) = \beta(z)dz$ for some *subharmonic* functions $\alpha(z)$, $\beta(z)$. Indeed, the metrics corresponding to these measures are flat (they are pullbacks by ψ_a of flat metrics on a that we mentioned above), and the well-known formula for the Gaussian curvature in isothermal coordinates yields $\Delta \log \alpha(z) = \Delta \log \beta(z) = 0$ (cf. [**2**, p. 663]). Therefore, $\alpha(z)$ and $\beta(z)$ are subharmonic as exponentials of harmonic functions [**36**, p. 45]. Finally, $d\nu_a = \delta(z)dz$, where $\delta(z) = \alpha(z) + \beta(z)$ is subharmonic as a sum of subharmonic functions. This completes the proof of the lemma. □

PROOF OF LEMMA 2.18. Set

$$G(r) = \int_{B(0,r)} \delta(z)\, dz = \int_0^r \int_0^{2\pi} \delta(\rho\, e^{i\phi})\rho\, d\rho\, d\phi.$$

By Lemma 2.19 the function δ is subharmonic. The function

$$W(\rho) = \int_0^{2\pi} \delta(\rho\, e^{i\phi}) d\phi$$

is 2π times the average of δ over the circle of radius ρ, hence it is monotone non-decreasing in ρ ([**36**, p. 46]). Therefore, since $r \leq 1$, we get as in [**53**, p.138] that

$$G(r) = \int_0^r W(\rho)\rho\, d\rho = \\ r^2 \int_0^1 W(r\,\rho)\rho\, d\rho \leq r^2 \int_0^1 W(\rho)\, \rho\, d\rho = r^2 G(1) = \pi r^2. \quad (2.22)$$

Now, because $\tilde{u}_a^s$ is the lift of $u_a^s = X_s \circ \psi_a$, we have

$$\int_{\mathbb{D}} (\tilde{u}_a^s)^2\, d\mu = \int_a (u_a^s)^2\, d\mu_a = \int_{\mathbb{D}} X_s^2\, d\nu_a.$$

Moreover since $V_a(s)$ doesn't depend on $s \in S^1$,

$$\begin{aligned} V_a(s) = \int_{\mathbb{D}} X_s^2\, d\nu_a &= \frac{1}{2} \int_{\mathbb{D}} \left(X_{e_1}^2 + X_{e_2}^2\right)\, d\nu_a \\ &= \frac{1}{2} \int_{\mathbb{D}} f^2(|z|)\, \delta(z)\, dz = \frac{1}{2} \int_{r=0}^1 f^2(r) G'(r)\, dr \end{aligned} \quad (2.23)$$

Integrating by parts and taking into account that $G(r) \le \pi r^2$ due to (2.22), we get

$$\int_{r=0}^{1} f^2(r)G'(r)\,dr = f^2(1)G(1) - \int_0^1 \frac{d}{dr}\big(f^2(r)\big)G(r)\,dr \ge f^2(1)G(1) - \pi\int_0^1 \frac{d}{dr}\big(f^2(r)\big)r^2\,dr = 2\pi\int_0^1 f^2(r)r\,dr \tag{2.24}$$

This completes the proof of Lemma 2.18 and Proposition 2.16. □

Remark 2.20. The proof of Lemma 2.18 is quite similar to the proof of (2.1), see [**54**, p. 348] and [**53**, p. 138]. Our approach is somewhat more direct since it explicitly uses the subharmonic properties of the measure.

Proof of Theorem 2.1. Assume that $d\mu$ is simple. Then (2.3) immediately follows from Proposition 2.16 and the variational characterization (2.13) of μ_2.

Suppose now that $d\mu$ is multiple. In fact, the proof is simpler in this case. Indeed, it follows from Lemma 2.12, that any direction $[s] \in \mathbb{R}P^1$ is maximizing for $d\mu$ so that we can use the space

$$E = \{X_s \,|\, s \in \mathbb{R}^2\}$$

of test functions in the variational characterization (2.13) of μ_2. Inspecting the proof of Proposition 2.16 we notice that the factor 2 disappears in (2.21) and hence in (2.19) as well. Therefore, in this case we get using (2.13) that $\mu_2(\Omega) \le \mu_1(\mathbb{D})$. This completes the proof of the theorem. □

Remark 2.21. When $d\mu$ is multiple, we get a stronger estimate

$$\mu_2(\Omega) \le \mu_1(\mathbb{D}).$$

To illustrate this case, consider $\Omega = \mathbb{D}$. Then indeed $\mu_2(\mathbb{D}) = \mu_1(\mathbb{D})$.

2.3. Proofs of auxiliary lemmas

2.3.1. Uniqueness of the renormalizing point

The following lemma is used in the proof Lemma 2.9.

Lemma 2.22. *Let $r \in (0,1)$ and $s = 1$. Then $X_s(d_r(z)) > X_s(z)$ for all $z \in \mathbb{D}$.*

PROOF. We have $X_s(z) = f(|z|)\cos\theta_1$ and $X_s(d_r(z)) = f(|d_r(z)|)\cos\theta_2$, where $\theta_1 = \arg z$ and $\theta_2 = \arg d_r(z)$. We need to show

$$f(|d_r(z)|)\cos\theta_2 > f(|z|)\cos\theta_1 \tag{2.25}$$

for all $z \in \mathbb{D}$. Note that the function f is monotone increasing, positive on the interval $(0,1]$, and $f(0) = 0$. Set $z = a + ib$. It is easy to check that for $|z| = 0$ the inequality in question is satisfied and therefore in the sequel we assume that $a^2 + b^2 > 0$.

Let us compare $|z|$ and $|d_r(z)|$. We note that $|z| = |\bar{z}|$. Since

$$|d_r(z)| = \frac{|z+r|}{|rz+1|},$$

we need to compare $|z+r|$ and $|r|z|^2+\bar{z}|$. This boils down to comparing $(a+r)^2+b^2$ and $((r(a^2+b^2)+a)^2+b^2$, or, equivalently, $(a+r)^2$ and $((r(a^2+b^2)+a)^2$. Note that $a^2+b^2 < 1$ since $z \in \mathbb{D}$. We have three cases :

(i) $a \geq 0$. Then $|d_r(z)| > |z|$.
(ii) $a < 0$ and $a + r \leq 0$. Then $|d_r(z)| < |z|$.
(iii) $a < 0$ and $a + r > 0$.

Let us now study the arguments θ_1 and θ_2.

We have :

$$d_r(z) = \frac{z+r}{rz+1} = \frac{(a+r)+ib}{(ar+1)+ibr} = \frac{(a+r)(ar+1)+b^2r+ib(1-r^2)}{(ar+1)^2+b^2r^2}$$

Taking into account that $ar+1 > 0$, we obtain from this formula that in case (iii) $\cos\theta_2 > 0$. On the other hand, $\cos\theta_1 < 0$ in this case, and therefore the inequality (2.25) is satisfied since $f > 0$.

Consider now case (i). Using the formula above we get that

$$\tan\theta_2 = \frac{b(1-r^2)}{(a+r)(ar+1)+b^2r}.$$

If $a = 0$ then (2.25) is true since $\cos\theta_1 = 0$ and one may easily check that $\cos\theta_2 > 0$. So let us assume that $a \neq 0$. Then $\tan\theta_1 = b/a$. Note that the tangent is a monotone increasing function. If $b = 0$ then $\theta_1 = \theta_2 = 0$ and (2.25) is satisfied since $|d_r(z)| > |z|$. If $b \neq 0$, dividing by b and taking into account that $a > 0$, $r > 0$ we easily get :

$$\frac{1}{a} > \frac{1-r^2}{(a+r)(ar+1)+b^2r}.$$

Therefore, if $b > 0$ we get that $\tan\theta_1 > \tan\theta_2$ implying $0 < \theta_2 < \theta_1 < \pi/2$, and if $b < 0$ we get that $\tan\theta_1 < \tan\theta_2$ implying that $3\pi/2 < \theta_1 < \theta_2 < 2\pi$. At the same time, in the first case the cosine is monotonely decreasing, and in the second case the cosine is monotonely increasing. Therefore, for any $b \neq 0$ we get $0 < \cos\theta_1 < \cos\theta_2$, which implies (2.25).

Finally, consider the case (ii). If $(a+r)(ar+1)+b^2r \geq 0$ then we immediately get (2.25) since in this case $\cos\theta_2 \geq 0$ and $\cos\theta_1 < 0$. So let us assume $(a+r)(ar+1)+b^2r < 0$. If $b = 0$ then $\theta_1 = \theta_2 = \pi$, hence $\cos\theta_1 = \cos\theta_2 = -1$ and (2.25) is satisfied because $|d_r(z)| < |z|$. If $b \neq 0$, as in case (ii) we compare $\tan\theta_1$ and $\tan\theta_2$. We claim that again

$$\frac{1}{a} > \frac{1-r^2}{(a+r)(ar+1)+b^2r}.$$

Since by our hypothesis the denominators in both cases are negative, it is equivalent to $a - ar^2 < a^2r + ar^2 + a + r + b^2r$. After obvious transformations we see that this reduces to $a^2 + 2ar + 1 + b^2 = (a+r)^2 + (1-r^2) + b^2 > 0$ which is true.

Therefore, taking into account that tangent is monotone increasing, we get that if $b > 0$ then $\pi/2 < \theta_2 < \theta_1 < \pi$, and if $b < 0$ then $\pi < \theta_1 < \theta_2 < 3\pi/2$. This implies that in either case $\cos\theta_1 < \cos\theta_2 < 0$. Together with the inequality $|d_r(z)| < |z|$ this gives (2.25) in case (ii). This completes the proof of the lemma. □

2.3.2. Proof of Lemma 2.11

Let $\mathcal{M}$ be the space of signed finite measures on $\mathbb{D}$ endowed with the norm (2.10). Recall that the map $\Gamma : \mathcal{M} \to \mathbb{D}$ is defined by $\Gamma(d\nu) = \xi$ in such a way that $d_\xi : \mathbb{D} \to \mathbb{D}$ renormalizes $d\nu$. It is continuous by Corollary 2.10. The key idea of the proof of the "flip-flop" lemma is to replace the folded measure $d\mu_a$ by

$$d\hat{\mu}_a := (\tau_a)_* d\mu.$$

It is clear that

$$||d\mu_a - d\hat{\mu}_a|| \to 0 \tag{2.26}$$

in the norm (2.10) as a degenerates to a point $p \in \partial\mathbb{D}$. At the same time, the next lemma shows that the "flip-flop"property is true for *each* cap when the rearranged measure $d\nu_a$ is replaced by $(d\zeta_a)_* d\hat{\mu}_a$, where $\zeta_a = \Gamma(d\hat{\mu}_a)$.

Lemma 2.23. *Let $a = a_{r,p}$ be a hyperbolic cap. Then*

$$(d\zeta_a)_* d\hat{\mu}_a = (d_{\zeta_a})_*(\tau_a)_* d\mu = R_p^* d\mu.$$

PROOF. Let us show that $\zeta_a = -\frac{2r}{r^2+1}p$. Recall that $\tau_a(z) = d_{rp} \circ R_p \circ d_{-rp}$. A simple explicit computation then leads to

$$d_{\zeta_a} \circ \tau_a = R_p.$$

This implies

$$\begin{aligned} \int_{\mathbb{D}} X_s \circ d_{\zeta_a}\, d\hat{\mu}_a &= \int_{\mathbb{D}} X_s \circ d_{\zeta_a} \circ \tau_a\, d\mu \\ &= \int_{\mathbb{D}} X_s \circ R_p\, d\mu = \int_{\mathbb{D}} X_{R_p s}\, d\mu = 0 \end{aligned}$$

which proves the claim. ☐

Let $\eta_a := \Gamma((d_{\zeta_a})_* d\mu_a)$ be the renormalizing vector for the measure $(d_{\zeta_a})_* d\mu_a$.

Lemma 2.24. *As the cap a degenerates to a point $p \in \partial\mathbb{D}$, $\eta_a \to 0$.*

PROOF. Since d_{ζ_a} is a diffeomorphism, $(d_{\zeta_a})_* : \mathcal{M} \to \mathcal{M}$ is an isometry so that

$$\begin{aligned} (d_{\zeta_a})_* d\mu_a &= (d_{\zeta_a})_*(d\mu_a - d\hat{\mu}_a) + (d_{\zeta_a})_* d\hat{\mu}_a \\ &= \underbrace{(d_{\zeta_a})_*(d\mu_a - d\hat{\mu}_a)}_{\to 0} + \underbrace{(d_{\zeta_a} \circ \tau_a)}_{R_p}{}_* d\mu \to (R_p)_* d\mu. \end{aligned}$$

Here we have used (2.26). Continuity of Γ leads to

$$0 = \Gamma((R_p)_* d\mu) = \lim_{a \to p} \Gamma((d_{\zeta_a})_* d\mu_a) = \lim_{a \to p} \eta_a.$$

Note that the first equality follows from (2.11) and the identity $X_s \circ R_p = X_{R_p s}$ that we used earlier. ☐

Set

$$q(a)=\frac{\overline{\zeta_a}\eta_a+1}{\zeta_a\overline{\eta_a}+1},\quad \xi(a)=d_{\zeta_a}(\eta_a)=\left(\frac{\eta_a+\zeta_a}{\overline{\zeta_a}\eta_a+1}\right). \tag{2.27}$$

A direct computation (cf. the proof of Lemma 2.9) leads to

$$\tilde{T}_a(z):=d_{\eta_a}\circ d_{\zeta_a}=q(a)d_{\xi(a)}(z).$$

It follows from its definition that $\tilde{T}_a$ renormalizes $d\mu_a$. Hence, $\Gamma(d\mu_a)=\xi(a)$ and $d_{\xi(a)}=T_a$, where the transformation T_a was defined in section 2.2.5. We have

$$\begin{aligned} T_{a*}d\mu_a &= (\frac{1}{q(a)}d_{\eta_a})_*(d_{\zeta_a})_*d\mu_a \\ &= (\frac{1}{q(a)}d_{\eta_a})_*(d_{\zeta_a})_*\left(d\hat{\mu}_a+(d\mu_a-d\hat{\mu}_a)\right). \end{aligned}$$

Now, it follows from Lemma 2.24 that $\lim_{a\to p}q(a)=1$ and $\lim_{a\to p}d_{\eta_a}=\text{id}$, because $\eta_a\to 0$. Therefore, taking into account (2.26) we get

$$\lim_{a\to p}T_{a*}d\mu_a=\lim_{a\to p}(d_{\xi_a})_*d\hat{\mu}_a=R_p^*d\mu.$$

To complete the proof of Lemma 2.11 it remains to show that as the cap a degenerates to p, $||T_{a*}d\mu_a-d\nu_a||\to 0$. By definition $d\nu_a=\psi_a^*d\mu$, where $\psi_a=(T_a'\circ\phi_b\circ T_a)^{-1}$ (see section 2.2.5). Let us show that $b=T_a(a)\to\mathbb{D}$ as $a\to p$. Indeed,

$$T_a=d_{\xi(a)}=d_{\zeta_a}\circ(d_{-\zeta_a}\circ d_{\xi(a)})=R_p\circ\tau_a\circ(d_{-\zeta_a}\circ d_{\xi(a)}).$$

Since $\eta_a\to 0$ when $a\to p$, it follows from (2.27) that the composition $d_{-\zeta_a}\circ d_{\xi(a)}$ tends to identity. Therefore, the cap $T_a(a)$ gets closer to $\mathbb{D}\setminus R_p(a)$ when a goes to p and thus $\lim_{a\to p}T_a(a)=\mathbb{D}$. This implies $\lim_{a\to p}\phi_{T_a(a)}=\text{id}$ and $\lim_{a\to p}T_a'=\text{id}$, and hence $\lim_{a\to p}||T_{a*}d\mu_a-d\nu_a||=0$. □

2.4. Proof of Theorem 2.5

2.4.1. Hersch's lemma and uniqueness of the renormalizing map

The proof of Theorem 2.5 is quite similar to the proof of Theorem 2.1. We use the following notation

$$\mathbb{B}^{n+1} = \{x \in \mathbb{R}^{n+1}, |x| < 1\}$$
$$\mathbb{S}^n = \partial \mathbb{B}^{n+1}.$$

The standard round metric on $\mathbb{S}^n$ is g_0. Given a conformally round metric $g \in [g_0]$ we write dg for its induced measure. Given $s \in \mathbb{R}^{n+1}$, define $X_s : \mathbb{S}^n \to \mathbb{R}$ by

$$X_s(x) = (x, s).$$

Similarly to (2.11) and (2.12), we assume that for each $s \in \mathbb{S}^n$:

$$\int_{\mathbb{S}^n} X_s \, dg = 0. \tag{2.28}$$

$$\int_{\mathbb{S}^n} X_{e_1}^2 \, dg \geq \int_{\mathbb{S}^n} X_s^2 \, dg. \tag{2.29}$$

Given $p \in \mathbb{S}^n$, $R_p : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the reflection with respect to the hyperplane going through 0 and orthogonal to the segment joining 0 and p, that is

$$R_p(x) = x - 2(p, x)p.$$

Given $\xi \in \mathbb{B}^{n+1}$. define $d_\xi : \overline{\mathbb{B}}^{n+1} \to \overline{\mathbb{B}}^{n+1}$ by

$$d_\xi(x) = \frac{(1 - |\xi|^2)x + (1 + 2(\xi, x) + |x|^2)\xi}{1 + 2(\xi, x) + |\xi|^2 |x|^2}. \tag{2.30}$$

Note that $d_\xi(0) = \xi$ and $d_\xi \circ d_{-\xi} = \mathrm{id}$. The map d_ξ is a conformal (Möbius) transformation of $\mathbb{S}^n$ [**53**, p. 142]. Indeed, one can check that for $\xi \neq 0$,

$$d_\xi = \gamma_\xi \circ R_{\frac{\xi}{|\xi|}}$$

where γ_ξ is the spherical inversion with center $\frac{\xi}{|\xi|^2}$ and radius $\frac{1-|\xi|^2}{|\xi|^2}$. Note that for $n = 1$, the map d_ξ coincides with the one introduced in Lemma 2.8, where complex notation was used for convenience.

Similarly to the disk case, the transformation d_ξ is said to *renormalize* a measure $d\nu$ on the sphere $\mathbb{S}^n$ if for each $s \in \mathbb{R}^{n+1}$,

$$\int_{\mathbb{S}^n} X_s \circ d_\xi \, d\nu = 0. \tag{2.31}$$

This condition is clearly equivalent to

$$\int_{\mathbb{S}^n} x_i \circ d_\xi \, d\nu = 0, \quad i = 1, 2, \ldots, n+1,$$

which means that the center of mass of the measure $(d_\xi)_* d\nu$ on $\mathbb{S}^n$ is at the origin. The following result is a combination of Hersch's lemma [**27**] and a uniqueness result announced in [**44**].

Proposition 2.25. *For any finite measure $d\nu$ on $\mathbb{S}^n$, there exists a unique point $\xi \in \mathbb{B}^{n+1}$ such that d_ξ renormalizes $d\nu$. Moreover, the dependence of the point $\xi \in \mathbb{B}^{n+1}$ on the measure $d\nu$ is continuous.*

PROOF. The existence of ξ is precisely Hersch's lemma (see [**27**], [**53**, p. 144], [**37**, p. 274]).

Let us prove uniqueness. First, let us show that if $d\nu$ is a renormalized measure then $\xi = 0$. It follows from (2.30) by a straightforward computation that if $\mathbb{B}^{n+1} \ni \xi \neq 0$ then $X_\xi(x) < X_\xi(d_\xi(x))$ for any $x \in \mathbb{S}^n$. Assume that d_ξ renormalizes $d\nu$ for some $\xi \neq 0$. Then

$$0 = \int_{\mathbb{S}^n} X_\xi \, d\nu < \int_{\mathbb{S}^n} X_\xi \circ d_\xi \, d\nu = 0,$$

and we get a contradiction.

Now, let $d\nu$ be an arbitrary finite measure and suppose that it is renormalized by d_ξ and d_η. Writing $d_\eta = d_\eta \circ d_{-\xi} \circ d_\xi$ we get

$$\int_{\mathbb{S}^n} X_s \circ d_\eta \circ d_{-\xi} \, d\tilde{\sigma} = 0 \tag{2.32}$$

where the measure $d\tilde{\sigma} = (d_\xi)_* d\sigma$ is renormalized. At the same time, it easy to check that $d_\eta \circ d_{-\xi} = R \circ d_{d_{-\xi}(\eta)}$, where R is an orthogonal transformation. Indeed, since $-d_{-\xi}(\eta) = d_\xi(-\eta)$ we have

$$d_\eta \circ d_{-\xi} \circ d_{d_\xi(-\eta)}(0) = d_\eta(-\eta) = 0,$$

and it is well known that any Möbius transformation of the unit ball preserving the origin is orthogonal [**2**, Theorem 3.4.1]. Since R preserves the center of mass at zero, it follows from (2.32) that $d_{d_{-\xi}(\eta)}$ renormalizes the measure $d\tilde{\sigma}$, which is already renormalized. Therefore, as we have shown above, $d_{-\xi}(\eta) = 0$ and hence $\xi = \eta$.

Similarly to Corollary 2.10, uniqueness of the renormalizing point implies that its dependence on the measure is continuous. □

2.4.2. Spherical caps, folding and rearrangement

The set $\mathcal{C}$ of all spherical caps is parametrized as follows : given $p \in \mathbb{S}^n$ let

$$a_{0,p} = \{x \in \mathbb{S}^n \ : (x,p) > 0\}$$

be the half-sphere centered at p. Given $-1 < r < 1$, let

$$a_{r,p} = d_{rp}(a_{0,p}).$$

To every spherical cap $a \in \mathcal{C}$ we associate a *folded* measure :

$$d\mu_a = \begin{cases} dg + \tau_a^* dg & \text{on } a, \\ 0 & \text{on } a^*, \end{cases}$$

where $a^* = \mathbb{S}^n \setminus \overline{a} \in \mathcal{C}$ is the cap adjacent to a, and τ_a is the conformal reflection with respect to the boundary circle of a. That is, for $a = a_{r,p}$

$$\tau_a = d_{rp} \circ R_p \circ d_{-rp}.$$

Let $\xi(a) \in \mathbb{B}^{n+1}$ be the unique point such that $d_{\xi(a)}$ renormalizes $d\mu_a$. We obtain a *rearranged folded measure*

$$d\nu_a = (d_{\xi(a)})_* d\mu_a. \tag{2.33}$$

2.4.3. Maximizing directions

Given a finite measure $d\nu$ on $\mathbb{S}^n$, define

$$V(s) = \int_{\mathbb{S}^n} X_s^2 d\nu.$$

Let $\mathbb{R}P^n$ be the projective space and let $[s] \in \mathbb{R}P^n$ be the point corresponding to $\pm s \in \mathbb{S}^n$. We say that $[s] \in \mathbb{R}P^n$ is a *maximizing direction* for $d\nu$ if $V(s) \geq V(t)$

for all $t \in \mathbb{S}^n$. We say that the spherical cap is *simple* if the maximizing direction is unique. Otherwise, similarly to Lemma 2.12, there exists a two-dimensional subspace $W \subset \mathbb{R}^{n+1}$ such that any $s \in W \cap \mathbb{S}^n$ is a maximizing direction for $d\nu$. In particular for each $s, t \in W$, $V(s) = V(t)$. In this case the measure $d\nu$ is called *multiple.*

Proposition 2.26. *Let $g \in [g_0]$ be a conformally round metric on a sphere $\mathbb{S}^n$ of odd dimension. If the measure dg is simple then there exists a spherical cap such that the rearranged folded measure $d\nu_a$ is multiple.*

The proof of Proposition 2.26 is similar to the proof of Proposition 2.13. We assume the measures dg as well as each $d\nu_a$ to be simple. Given a cap $a \subset \mathbb{S}^n$ let $[s(a)] \in \mathbb{R}P^1$ be the unique maximizing direction for $d\nu_a$. The map $a \mapsto [s(a)]$ is continuous. The following spherical version of the "flip-flop" property is proved exactly as Lemma 2.11.

Lemma 2.27. *If a sequence of spherical caps $a \in C$ degenerates to a point $p \in \mathbb{S}^n$, the limiting rearranged measure is a "flip-flop" of the original measure dg :*

$$\lim_{a \to p} d\nu_a = R_p^* dg. \tag{2.34}$$

Similarly to Lemma 2.14 we study the maximizing directions for degenerating caps.

Lemma 2.28. *Suppose the measures dg as well as each $d\nu_a$ are simple. Then*

$$\lim_{a \to \mathbb{S}^n} [s(a)] = [e_1] \tag{2.35}$$

$$\lim_{a \to p} [s(a)] = [R_p e_1]. \tag{2.36}$$

Proof of Proposition 2.26. By convention (2.29), $[e_1]$ is the unique maximizing direction for dg. Recall that the space of caps has been identified with $(-1, 1) \times \mathbb{S}^n$. The continuous map

$$h : [-1, 1] \times \mathbb{S}^n \to \mathbb{R}P^n$$

is defined by

$$h(r,p) = \begin{cases} [e_1] & \text{for } r = -1, \\ [s(a_{r,p})] & \text{for } -1 < r < 1, \\ [R_p e_1] & \text{for } r = 1. \end{cases}$$

That is, h is an homotopy between a constant map and the map

$$\phi : \mathbb{S}^n \longrightarrow \mathbb{R}P^n$$

defined by $\phi(p) = [R_p e_1]$. We will show that this is impossible when n is odd by computing its degree. The map ϕ lifts to the map $\psi : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ defined by

$$\psi(p) = -R_p e_1 = 2(e_1, p)p - e_1. \tag{2.37}$$

The two solutions of $\psi(p) = e_1$ are e_1 and $-e_1$. It is easy to check that since the dimension n is odd, both differentials

$$D_{e_1}\psi : T_{e_1}\mathbb{S}^n \longrightarrow T_{e_1}\mathbb{S}^n$$
$$D_{-e_1}\psi : T_{-e_1}\mathbb{S}^n \longrightarrow T_{e_1}\mathbb{S}^n$$

preserve the orientation. This implies $\deg(\psi) = 2$. Moreover, the quotient map $\pi : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ has degree 2 for n odd. It follows that

$$\begin{aligned} \deg(\phi) &= \deg(\pi \circ \psi) \\ &= \deg(\pi)\deg(\psi) = 4. \end{aligned}$$

Since the degree of a map is invariant under homotopy, this is a contradiction. □

Remark 2.29. In even dimensions one of the differentials $D_{\pm e_1}$ preserves the orientation and the other reverses it. Therefore, $\deg(\phi) = 0$ and the proof of Proposition 2.26 does not work in this case. In dimension two the existence of a multiple cap was proved in **[44]** using a more sophisticated topological argument.

2.4.4. Test functions and the modified Rayleigh quotient

Let g_0 be the standard round metric on the sphere $\mathbb{S}^n$, so that

$$\omega_n := \int_{\mathbb{S}^n} dg_0 = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}. \tag{2.38}$$

Let $g \in [g_0]$ be a conformally round Riemannian metric of volume one, that is $\int_{\mathbb{S}^n} dg = 1$. The Rayleigh quotient of a non-zero function $u \in H^1(\mathbb{S}^n)$ is

$$R(u) = \frac{\int_{\mathbb{S}^n} |\nabla_g u|_g^2 \, dg}{\int_{\mathbb{S}^n} u^2 \, dg}.$$

We use the following variational characterization of $\lambda_2(g)$:

$$\lambda_2(g) = \inf_E \sup_{0 \neq u \in E} R(u) \tag{2.39}$$

where E varies among all two-dimensional subspaces of the Sobolev space $H^1(\mathbb{S}^n)$ that are orthogonal to constants, in the sense that for each $f \in E$, $\int_{\mathbb{S}^n} f \, dg = 0$. Following **[19]**, we use a *modified Rayleigh quotient* :

$$R'(u) = \frac{\left(\int_{\mathbb{S}^n} |\nabla_g u|_g^n \, dg\right)^{2/n}}{\int_{\mathbb{S}^n} u^2 \, dg}.$$

It follows from Holder inequality that $R(u) \leq R'(u)$ for each $0 \neq u \in H^1(\mathbb{S}^n)$. It is easy to check that $\int_{\mathbb{S}^n} |\nabla_g u|_g^n \, dg$ is conformally invariant for each dimension n so that we can rewrite the modified Rayleigh quotient as follows :

$$R'(u) = \frac{\left(\int_{\mathbb{S}^n} |\nabla u|^n \, dg_0\right)^{2/n}}{\int_{\mathbb{S}^n} u^2 \, dg}$$

where the gradient and it's norm are with respect to the round metric g_0.

Assume that dg is simple and let $a \subset \mathbb{S}^n$ be a spherical cap such that $d\nu_a$ is multiple. Let $W \subset \mathbb{R}^{n+1}$ be the corresponding two dimensional subspace of maximizing directions. Given a function $u : a \to \mathbb{R}$, the *lift* of u, $\tilde{u} : \mathbb{S}^n \to \mathbb{R}$ is defined exactly as in Definition 2.15.

Proposition 2.30. *Given $s \in W \subset \mathbb{R}^{n+1}$, the function $u_a^s = X_s \circ d_{\xi(a)} : a \to \mathbb{R}$ is such that*

$$R'(\tilde{u}_a^s) < (n+1)\left(4\frac{\pi^{\frac{n+1}{2}}\Gamma(n)}{\Gamma(\frac{n}{2})\Gamma(n+\frac{1}{2})}\right)^{2/n}.$$

PROOF. The conformal invariance of the numerator in $R'(u)$ implies

$$\left(\int_{\mathbb{S}^n} |\nabla_g \tilde{u}_a^s|_g^n \, dg\right)^{2/n} = \left(\int_a |\nabla_g u_a^s|_g^n \, dg\right)^{2/n} +$$

$$\left(\int_{a^*} |\nabla_g (u_a^s \circ \tau_a)|_g^n \, dg\right)^{2/n} = \left(2 \cdot \int_a |\nabla_g u_a^s|_g^n \, dg\right)^{2/n}$$

$$= \left(2 \int_{d_{\xi(a)}(a)} |\nabla_g X_s|_g^n \, dg\right)^{2/n} < \left(2 \int_{\mathbb{S}^n} |\nabla_{g_0} X_s|_{g_0}^n \, dg_0\right)^{2/n} \quad (2.40)$$

Here the second equality follows from conformal invariance. To obtain the inequality at the end we again use the conformal invariance as well as the fact that $d_{\xi(a)}(a) \subsetneq \mathbb{S}^n$. To estimate the denominator in the modified Rayleigh quotient we first note that for any $x = (x_1, \dots x_{n+1}) \in \mathbb{S}^n$,

$$\sum_{j=1}^{n+1} \tilde{u}_a^{e_j}(x)^2 = \sum_{j=1}^{n+1} x_j^2 = 1.$$

Therefore, given that $\int_{\mathbb{S}^n} dg = 1$ we obtain :

$$\sum_{j=1}^{n+1} \int_{\mathbb{S}^n} (\tilde{u}_a^{e_j})^2 \, dg = 1$$

Now, since W is a subspace of maximizing directions for the measure $d\nu_a$ defined by (2.33), for each $s \in W$ we have

$$\int_{\mathbb{S}^n} (\tilde{u}_a^s)^2 \, dg \geq \frac{1}{n+1}. \quad (2.41)$$

Set

$$K_n := \int_{\mathbb{S}^n} |\nabla_{g_0} X_s|_{g_0}^n \, dg_0.$$

Combining (2.40) and (2.41) we get

$$R'(\tilde{u}_a^s) \leq (n+1)\,(2K_n)^{2/n}. \quad (2.42)$$

Proposition 2.30 then follows from the lemma below.

Lemma 2.31. *The constant K_n is given by*

$$K_n = \frac{2\pi^{\frac{n+1}{2}}\Gamma(n)}{\Gamma(\frac{n}{2})\Gamma(n+\frac{1}{2})}.$$

Proof. Recall that g_0 is the standard round metric on the unit sphere $\mathbb{S}^n$. If we consider $X_s(x) = (x, s)$ as a function on $\mathbb{R}^{n+1}$ then its gradient is just the constant vector s :

$$\mathrm{grad}_{\mathbb{R}^{n+1}} X_s = s.$$

This means that for any point $p \in \mathbb{S}^n$ the gradient of the function $X_s : \mathbb{S}^n \to \mathbb{R}$ at p is the projection of s on the tangent space $T_p\mathbb{S}^n$:

$$\nabla X_s(p) = s - (s,p)p.$$

Therefore, taking into account that $|s| = |p| = 1$, we get

$$|\nabla X_s(p)|^n = (|s - (s,p)p|^2)^{n/2} = (1 - (s,p)^2)^{n/2},$$

and hence

$$K_n = \int_{\mathbb{S}^n} (1 - (s,p)^2)^{n/2} dg_0.$$

Let θ be the angle between the vectors p and s. Making a change of variables we obtain

$$K_n = \omega_{n-1} \int_0^{\pi} (1 - \cos^2\theta)^{n/2} (\sin\theta)^{n-1}\, d\theta = \omega_{n-1} \int_0^{\pi} \sin^{2n-1}\theta\, d\theta,$$

where ω_{n-1} is the volume of the standard round sphere $\mathbb{S}^{n-1}$ given by (2.38).

The calculation of a table integral [**25**, 3.621(4)]

$$\int_0^{\pi} \sin^{2n-1}\theta\, d\theta = \frac{\sqrt{\pi}\,\Gamma(n)}{\Gamma(n+\frac{1}{2})}$$

completes the proofs of Lemma 2.31 and Proposition 2.30. □

Remark 2.32. It follows from Hölder inequality that $R(u) = R'(u)$ if and only if u is a constant function. Since $\nabla_{g_0} X_s \neq \mathrm{const}$ we get a *strict* inequality $R'(\tilde{u}_a^s) > R(\tilde{u}_a^s)$. This is why the estimate (2.7) is not sharp. In the context of the first eigenvalue, a similar difficulty was encountered in [**6**, Lemma 4.15]) and overcame in [**15**]. To apply the approach of [**15**] we need to have a spherical cap of multiplicity $n+1$; existence of a cap of multiplicity two proved in Proposition 2.26 is not enough for this purpose.

PROOF OF THEOREM 2.5. If the measure dg is simple, then (2.7) follows from Proposition 2.30 and the variational principle (2.39). If dg is multiple, then, as in the proof of Theorem 2.1 at the end of section 2.2.7, one can work directly with this measure without any folding and rearrangement. Inspecting the proof of Proposition 2.30 we notice that the factor $2^{2/n}$ disappears in (2.40) and hence also in (2.42). Therefore, in this case we get an even better bound than (2.7). This completes the proof of Theorem 2.5. □

Chapitre 3

RELATIVE HOMOLOGICAL LINKING IN CRITICAL POINT THEORY

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Abstract : A relative homological linking of pairs is proposed. It is shown to imply homotopical linking, as well as earlier non-relative notion of homological linkings. Using Morse theory we prove a simple "homological linking principle", thereby generalizing and simplifying many well known results in critical point theory.

INTRODUCTION

The use of linking methods in critical point theory is rather new. It was implicitely present in the work of Ambrosetti and Rabinowitz [**1**] in the early 70's as well as in the work of Benci and Rabinowitz [**4**]. The first explicit definition was given by Ni in 1980 [**46**].

Definition 3.1 (Classical Homotopical Linking)**.** Let $A \subset B$ and Q be subspaces of a topological space X such that the pair (B, A) is homeomorphic to (D^n, S^{n-1}). Then A *homotopically links* Q if for each deformation $\eta : [0,1] \times B \to X$ fixing A, $\eta(1, B) \cap Q \neq \emptyset$.

In the early 80's, homological linking was introduced in critical point theory (see Fadell [**18**], Benci [**3**] and Chang [**9**] for instance).

Definition 3.2 (Classical Homological Linking). Let A and S be non-empty disjoint subspaces in a topological space X. Then A *homologically links* S if the inclusion of A in $X \setminus S$ induces a non-trivial homomorphism in reduced homology.

In her 1999's article **[20]**, Frigon generalized homotopical linking to pairs of subspaces.

Definition 3.3 (Relative Homotopical Linking). Let (B, A) and (Q, P) be two pairs of subspaces in a topological space X such that $B \cap P = \emptyset$ and $A \cap Q = \emptyset$. Then (B, A) *homotopically links* (Q, P) if for each deformation $\eta : [0, 1] \times B \to X$ fixing A pointwise, $\eta(1, B) \cap Q = \emptyset \Rightarrow \exists t \in]0, 1], \eta(t, B) \cap P \neq \emptyset$.

The classical definition corresponds to the case where $(B, A) \cong (D^n, S^{n-1})$ and $P = \emptyset$.

The goal of this article is to propose a similar generalization for homological linking. In section 1.1 we explore the properties of this new homological linking and in 1.2 we give some detailed examples. In section 2 we interpret homotopical linking as an obstruction to factoring certain homotopy through homotopically trivial pairs. It becomes clear from this point of view that homological linking is stronger than homotopical linking. Our definition of homological linking fits very nicely with Morse theory. We exploit this in section 3 to derive a new linking principle (see 3.27) for detecting and locating critical points. Despite its simplicity, the idea is quite fruitful. Close analog to the Mountain Pass Theorem of Ambrosetti and Rabinowitz [1] as well as to the Saddle Point Theorem of Rabinowitz **[50]** are easy corollaries. In Proposition 3.31, we also obtain a homological version of the generalized saddle point theorem of Frigon **[20]**. In section 4, some multiplicity results are studied.

Our approach has many advantages : each critical point is detected by a different linking, stability type is directly available (i.e. critical groups are known) and last but not least, the proofs are easy. However, it also has a disadvantage : working with Morse theory requires more regularity than using a "min-max" method for example. It might appear as if the content of this paper is extremely easy. We agree with this point of view. In fact, it is rather surprising to see that so many of the classical results of critical point theory are straightforward consequences of this new definition of homological linking.

3.1. Homological linking

3.1.1. Definition and properties

The principal contribution of this article is the following definition.

Definition 3.4 (Relative Homological Linking). Let (B, A) and (Q, P) be pairs of subspaces in a topological space X. Then (B, A) *homologically links* (Q, P) *in* X if $(B, A) \subset (X \setminus P, X \setminus Q)$ and if this inclusion induces a non-trivial homomorphism in reduced homology. Given integers $q, \beta \geq 0$, we say that

$$(B, A)\ (q, \beta)\text{-}links\ (Q, P)\ in\ X$$

if the above inclusion induces a homomorphism of rank β on the q-th reduced homology groups.

Remark 3.5. For notational convenience, a topological pair $(B, \emptyset)$ will be identified with the space B.

Remark 3.6. The classical definition corresponds to the case A (q, β)-links (X, Q) and $\beta > 0$.

Remark 3.7. For any space X, X $(q, b_q(X))$-links X in X, where $b_q(X)$ is the q-th reduced Betti number of X. Thus our linking contains as much information as Betti numbers.

The next proposition and it's corollary shows that in many situations, it suffices to consider linking locally to deduce a global linking situation.

Proposition 3.8. *Let $\mathcal{O}$ be an open subset of X. If $A, B, P, Q \subset \mathcal{O}$ with Q closed, then*

$$(B, A)\ (q, \beta)\text{-}links\ (Q, P)\ in\ X$$

$$\Leftrightarrow$$

$$(B, A)\ (q, \beta)\text{-}links\ (Q, P)\ in\ \mathcal{O}.$$

PROOF. Since $\mathcal{O}^c$ is closed and $X \setminus Q$ is open in $X \setminus P$, the excision axiom applies to

$$\mathcal{O}^c \subset X \setminus Q \subset X \setminus P.$$

It follows that the the bottom line of the following commutative diagram is an isomorphism.

$$\begin{array}{ccc} \tilde{H}_q(B,A) & & \\ \downarrow i & \overset{j}{\searrow} & \\ \tilde{H}_q(\mathcal{O} \setminus P, \mathcal{O} \setminus Q) & \xrightarrow{\cong} & \tilde{H}_q(X \setminus P, X \setminus Q) \end{array}$$

Hence, rank $j =$ rank i. □

Corollary 3.9. *Let $\mathcal{O}$ be the domain of a chart on a manifold M. If the pair (B, A) links the pair (Q, P) in $\mathcal{O}$, with Q closed, then (B, A) also links the pair (Q, P) in M.*

The two following theorems show how some simple linking situations lead to new linkings.

Theorem 3.10. *If A (q, β)-links (X, Q) and A (q, δ)-links $(X, X \setminus B)$ in X for some $\delta < \beta$ then (B, A) $(q + 1, \mu)$-links Q in X for some $\mu \geq \beta - \delta$.*

PROOF. It follows from the commutativity of

$$\begin{array}{ccccc} \tilde{H}_{q+1}(B,A) & \xrightarrow{\Delta_1} & \tilde{H}_q(A) & \xrightarrow{k} & \tilde{H}_q(B) \\ \downarrow \alpha & & \downarrow i & & \downarrow \\ \tilde{H}_{q+1}(X, X \setminus Q) & \xrightarrow{\Delta_2} & \tilde{H}_q(X \setminus Q) & \longrightarrow & \tilde{H}_q(X) \end{array}$$

that

$$\begin{aligned} \mu := \text{rank } \alpha \geq \text{rank } \Delta_2 \circ \alpha &= \text{rank } i \circ \Delta_1 \\ &\geq \text{rank } \Delta_1 - \dim(\ker i) \\ &= \text{rang } \Delta_1 - (\dim \tilde{H}_q(A) - \text{rank } i) \\ &= \text{rank } i + \text{rank } \Delta_1 - \dim \tilde{H}_q(A) \\ &= \text{rank } i + \text{rank } \Delta_1 - (\text{rank } k + \dim(\ker k)). \end{aligned}$$

By exactness, rank $\Delta_1 = \dim(\ker k)$, thus

$$\mu \geq \text{rank } i - \text{rank } k = \beta - \delta.$$

$\square$

Theorem 3.11. *If B (q,β)-links (X,P) and $X \setminus Q$ (q,δ)-links (X,P) for some $\delta < \beta$, then B (q,μ)-links (Q,P) in X for some $\mu \geq \beta - \delta$.*

PROOF. From the commutativity of

$$\begin{array}{ccccc} & & \tilde{H}_q(B) & \xrightarrow{\cong} & \tilde{H}_q(B,\emptyset) \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \alpha} \\ \tilde{H}_q(X \setminus Q) & \xrightarrow{k} & \tilde{H}_q(X \setminus P) & \xrightarrow{j} & \tilde{H}_q(X \setminus P, X \setminus Q) \end{array}$$

it follows that

$$\begin{aligned} \mu = \text{rank } \alpha &= \text{rank } j \circ i \\ &\geq \text{rank } i - \dim(\ker j) \\ &= \text{rank } i - \text{rank } k \\ &= \beta - \delta. \end{aligned}$$

$\square$

3.1.2. Examples of linking

Our definition permits to obtain new situations of linking and to recover others already known. In particular, in Propositions 3.12, 3.13 and 3.14 we present linking situations equivalent to those already studied by Perera in **[47]** using a non relative definition of homological linking.

Let E be a Banach space. Given a direct sum decomposition $E = E_1 \oplus E_2$, B_i denotes the closed ball in E_i and S_i its relative boundary $(i = 1, 2)$.

Proposition 3.12. *Let $e \in E$, $\|e\| > 1$. Then $\{0, e\}$ $(0,1)$-links (E, S) in E.*

PROOF. The map $r : E \setminus S \to \{0, e\}$ defined by

$$r(x) = \begin{cases} 0 & \text{if } \|x\| < 1, \\ e & \text{if } \|x\| > 1. \end{cases}$$

is a retraction. That is, the following diagram commutes

$$\begin{array}{ccc} E \setminus S & \xrightarrow{\ r\ } & \{0, e\} \\ \uparrow & \nearrow_{\text{id}} & \\ \{0, e\} & & \end{array}$$

It follows that the inclusion of $\{0, e\}$ in $E \setminus S$ is of rank 1 in reduced homology. □

Proposition 3.13. *Let $E = E_1 \oplus E_2$ with $k = \dim E_1 \in]0, \infty[$. Then*

$$S_1 \ (k-1, 1)\text{-}\textit{links}\ (E, E_2)$$

in E.

PROOF. The long exact sequence induced by $S_1 \subset E \setminus E_2$ is

$$\cdots \to \tilde{H}_k(E \setminus E_2, S_1) \to \tilde{H}_{k-1}(S_1) \xrightarrow{i} \tilde{H}_{k-1}(E \setminus E_2) \to \cdots$$

Because $E \setminus E_2$ strongly retract on S_1, $H_k(E \setminus E_2, S_1) = 0$. It follows that rank $i = \dim \tilde{H}_{k-1}(S_1) = 1$. □

Proposition 3.14. *Let $E = E_1 \oplus E_2$ with $k = \dim E_1 \in]0, \infty[$ and let $e \in E_2$ be of unit length. Let $A = \partial(B_1 \oplus [0, 2]e)$ in $E_1 \oplus \mathbb{R}e$. Then A $(k, 1)$-links (E, S_2) in E.*

PROOF. Let $P : E \to E_1$ be the projection on E_1 and $r : E \setminus S_2 \to (E_1 \oplus \mathbb{R}e) \setminus \{e\}$ be defined by $r(x) = P(x) + \|x - P(x)\|e$. Let's make sure $\{e\}$ really is omitted by r. Suppose $x \in E$ is such that $P(x) + \|x - P(x)\|e = e$. Then $P(x) = 0$ and $1 = \|x - P(x)\| = \|x\|$. In other words, $x \in E_2$ and $\|x\| = 1$ wich is impossible for x in the domain of r. Let i be the inclusion of A in $E \setminus S_2$. If $i_k : \tilde{H}_k(A) \to \tilde{H}_k(E \setminus S_2)$ is null, then so is

$$r_k \circ i_k : \tilde{H}_k(A) \to \tilde{H}_k((E_1 \oplus \mathbb{R}e) \setminus \{e\}).$$

However, $r \circ i$ is the inclusion of A in $(E_1 \oplus \mathbb{R}e) \setminus \{e\}$ and $(E_1 \oplus \mathbb{R}e) \setminus \{e\}$ strongly retract on A. Thus $\tilde{H}_*((E_1 \oplus \mathbb{R}e) \setminus \{e\}, A) \cong 0$. It then follows from the long exact sequence induced by the inclusion $r \circ i$ of A in $(E_1 \oplus \mathbb{R}e) \setminus \{e\}$

$$0 = \tilde{H}_{k+1}((E_1 \oplus \mathbb{R}e) \setminus \{e\}, A) \to \tilde{H}_k(A) \xrightarrow{r_k \circ i_k} \tilde{H}_k(E_1 \oplus \mathbb{R}e \setminus \{e\})$$

that $r_k \circ i_k$ is not trivial because $\tilde{H}_k(A) \cong \mathbb{K}$. Consequently A $(k,1)$-links (E, S_2) in E, as was to be proved. □

Theorem 3.10 and the previous linking situations give rise to other linkings which are in fact the classical situations treated in the litterature. Observe that, in these classical situations, the pair (Q, P) is always of the form $(Q, \emptyset)$ and the pair (B, A) always has $A \neq \emptyset$.

Corollary 3.15. *Let $e \in E$ with $\|e\| > 1$. Then $([0,e], \{0,e\})$ $(1,1)$-links S in E.*

Corollary 3.16. *Let $E = E_1 \oplus E_2$ with $k = \dim E_1 \in\,]0, \infty[$. Then*

$$(B_1, S_1)\ (k,1)\text{-links } E_2$$

in E.

Corollary 3.17. *Let $E = E_1 \oplus E_2$ with $k = \dim E_1 \in\,]0, \infty[$ and let $e \in E_2$ be of unit length. Let $B = B_1 \oplus [0,2]e$ and $A = \partial B$ in $E_1 \oplus \mathbb{R}e$. Then (B, A) $(k+1,1)$-links S_2 in E.*

By combining the linking situations of proposition 3.12, 3.13 and 3.14 with theorem 3.11, we get a new familly of linking situations. These linking situation will be particularyly useful in applications to critical point theory since they will allow us to relax the a priori estimates on f. For these linking, the pair (B, A) is always of the form $(B, \emptyset)$ and the pair (Q, P) always has $P \neq \emptyset$.

Corollary 3.18. *Let $e \in E$, $\|e\| > 1$. Then*

$$\{0, e\}\ (0,1)\text{-links } (B, S)$$

in E.

Corollary 3.19. *Let $E = E_1 \oplus E_2$ with $k = \dim E_1 \in\,]0, \infty[$ and let $e \in E_1$ be of unit length. Let $B = S_1, Q = E_2 + [0, \infty[e$ and $P = E_2$. Then*

$$B\ (k-1,1)\text{-links } (Q, P)$$

in E.

Corollary 3.20. *Let $E = E_1 \oplus E_2$ with $k = \dim E_1 \in\,]0, \infty[$ and let $e \in E_2$ be of unit length. Let $A = \partial(B_1 \oplus [0,2]e)$ in $E_1 \oplus \mathbb{R}e$. Then A $(k,1)$-links (B_2, S_2) in E.*

The two following propositions exhibit new homological linking situations. From a homotopical point of view, they where studied by Frigon [**20**]. These linking fully deserve to be called "linking of pairs" since for both of them we have $A \neq \emptyset$ and $P \neq \emptyset$. A more geometrical argument is also possible, but it is longuer.

Proposition 3.21. *Let* $E = E_1 \oplus E_2 \oplus \mathbb{R}e$ *with* $e \in E$ *of unit length and* $k = \dim E_1 \in]0, \infty[$. *Let* $B = B_1 + e$, $A = S_1 + e$, $Q = E_2 + [0, \infty[e$ *et* $P = E_2$ *Then* (B, A) $(k, 1)$*-links* (Q, P) *in* E.

PROOF. Let $\epsilon \in]0, 1[$ and

$$\hat{B} = B \cup (\epsilon B_1 +]0, \infty[e + E_2),$$

$$\hat{A} = \hat{B} \setminus (]0, \infty[e + E_2).$$

Since B (resp. A) is a strong deformation retract of $\hat{B}$ (resp. $\hat{A}$), the inclusion $(B, A) \to (\hat{B}, \hat{A})$ induces an isomorphism $H_k(B, A) \cong H_k(\hat{B}, \hat{A})$. Let

$$U = (E \setminus P) \setminus \hat{B} \subset E \setminus Q \subset E \setminus P,$$

and observe that $\overline{U} \subset \text{int}\,(E \setminus Q)$ in $E \setminus P$, $\hat{B} = (E \setminus P) \setminus U$ and $\hat{A} = (E \setminus Q) \setminus U$. Hence, by excision, the inclusion $(\hat{B}, \hat{A}) \to (E \setminus P, E \setminus Q)$ induces an isomorphism $H_k(\hat{B}, \hat{A}) \cong H_k(E \setminus P, E \setminus Q)$. The result follows from $H_k(B, A) \cong \mathbb{K}$. □

A similar argument leads to the following proposition.

Proposition 3.22. *Let* $E = E_1 \oplus E_2$ *with* $k = \dim E_1 \in]0, \infty[$.
Then (B_1, S_1) $(k, 1)$*-links* (B_2, S_2) *in* E.

3.2. HOMOTOPICAL CONSEQUENCES OF HOMOLOGICAL LINKING

Let (B, A) and (Q, P) be pairs of subspaces in a topological space X such that $B \cap P = \emptyset$ and $A \cap Q = \emptyset$. The following lemma shows that relative homotopical linking is an obstruction to extension factoring through a homotopically trivial pair.

Lemma 3.23. *The following statements are equivalent.*

(1) The pair (B, A) *homotopicaly links* (Q, P),

(2) There exists no homotopy $\eta : [0,1] \times (B, A) \to (X \setminus P, X \setminus Q)$ *such that* $\eta = id$ *on* $\{0\} \times B \cup [0,1] \times A$ *making the following diagram commutative*

$$\begin{array}{ccc} (B, A) & \xrightarrow{\eta_1} & (X \setminus P, X \setminus Q) \\ & \searrow^{\eta_1} & \uparrow \\ & & (X \setminus Q, X \setminus Q) \end{array}$$

Corollary 3.24. *Homological linking implies homotopical linking.*

Remark 3.25. To see that homotopical linking doesn't imply homological linking, it is sufficient to consider $X = B = Q$ to be a singleton and $A = P = \emptyset$.

3.3. HOMOLOGICAL LINKING PRINCIPLE

Let H be a Hilbert space and let $f \in C^2(H, \mathbb{R})$. The following notation is standard. Given $c \in \mathbb{R}$, $f_c = \{p \in H | f(p) \leq c\}$ is a level set of f, $K(f) = \{p \in H | f'(p) = 0\}$ is the critical set of f, $K_c(f) = K(f) \cap f^{-1}(c)$.

Throughout this section, the following hypothesis are assumed,

(H1) the Palais-Smale condition for f holds. That is, each sequence $(x_n)_{n \in \mathbb{N}}$ such that $(f(x_n))$ is bounded and $f'(x_n) \to 0$ admits a convergent subsequence,

(H2) the set $K(f)$ of critical point of f is discrete.

In particular, $f(K)$ is discrete and for each bounded interval I, $K \cap I$ is compact.

Under these assumptions, there is a suitable Morse theory which is well behaved (see [**39**] for instance). We shall use the following standard notation. Given $p \in K_c(f)$,

$$C_q(f, p) := H_q(f_c, f_c \setminus \{p\})$$

is the q-th critical group of f at p. Let $a < b$ be two regular values of f,

$$\mu_q(f_b, f_a) := \sum_{p \in K(f) \cap f^{-1}[a,b]} \dim C_q(f, p)$$

is the Morse number of the pair (f_b, f_a). The function f is said to be a Morse function if its critical points are all non-degenerate.

Remark 3.26. Most of our results depend only on the Morse inequalities. It is thus possible to use any other setting where they hold. For example, in

[**13**] a Morse theory for continuous functions on metric spaces is presented. In applications to PDE, it may be necessary to use the Finsler structure approach of Chang [**9**] to apply the results in suitable Sobolev spaces.

The following theorem is an easy exercise and was probably first observed by Marston Morse himself.

Theorem 3.27 (homological linking principle). *Let (B, A) and (Q, P) be pairs of subspaces in H and let $a < b$ be regular values of f such that $(B, A) \subset (f_b, f_a) \subset (H \setminus P, H \setminus Q)$. If (B, A) (q, β)-links (Q, P) in H for some $\beta \geq 1$ then f admits a critical point p such that $a < f(p) < b$ and $C_q(f, p) \neq 0$. Moreover, if f is a Morse function then it admits at least β such points.*

PROOF. It follows from commutativity of

$$\begin{array}{ccc} \tilde{H}_q(B, A) & \longrightarrow & \tilde{H}_q(H \setminus P, H \setminus Q) \\ \downarrow & \nearrow & \\ \tilde{H}_q(f_b, f_a) & & \end{array}$$

that $\dim \tilde{H}_q(f_b, f_a) \geq \beta$. Application of the weak Morse inequalities leads to $\mu_q(f_b, f_a) \geq \beta$ and to the first conclusion. The non-degeneracy condition leads to the second one. □

Remark 3.28. From Remark 3.7 and our linking principle we recover the weak Morse inequalities. This shows that our homological linking contains nearly as much information as classical Morse theory.

Lemma 3.29. *Let (B, A) and (Q, P) be pairs of subspaces in H such that*

$$\sup f(B) < \inf f(P),$$

$$\sup f(A) \leq \inf f(Q).$$

If (B, A) (q, β)-links (Q, P) in H for some $\beta \geq 1$ then $\inf f(Q) \leq \sup f(B)$.

PROOF. Let the opposite be supposed : $\sup f(B) < \inf f(Q)$. For each $n \in \mathbb{N}$, there exist regular values $a_n < b_n$ in $]\sup f(B), \sup f(B) + 1/n[$. If n is big enough, $\sup f(B) + 1/n < \inf f(Q) \leq \inf f(P)$ so that

$$(B, A) \subset (f_{b_n}, f_{a_n}) \subset (X \setminus P, X \setminus Q).$$

It follows from the homological linking principle that f admits a critical value $c_n \in]a_n, b_n[$. The infinite sequence (c_n) converges to $c = \sup f(B)$ which must therefore be critical because the set of all critical values of f is closed. This contradicts the fact that critical values must be isolated. □

The next theorem will be usefull for applications. In the next section, it will be used to prove some multiplicity results.

Theorem 3.30. *Let (B, A) and (Q, P) be pairs of subspaces in H such that*

$$\sup f(B) < \inf f(P),$$

$$\sup f(A) < \inf f(Q).$$

If (B, A) (q, β)-links (Q, P) in H for some $\beta \geq 1$ then f admits a critical point p such that

$$\inf f(Q) \leq f(p) \leq \sup f(B)$$

and $C_q(f, p) \neq 0$. Moreover if f is a Morse function then it admits at least β such points.

PROOF. By the preceding lemma,

$$\sup f(A) < \inf f(Q) \leq \sup f(B) < \inf f(P).$$

There exist regular values $a_n < b_n$ $(n \in \mathbb{N})$ such that

$$\sup f(A) < a_n < \inf f(Q) \leq \sup f(B) < b_n < \inf f(P)$$

and $a_n \longrightarrow \inf f(Q)$, $b_n \longrightarrow \sup f(B)$. By the linking principle, there must exist a sequence (p_n) of critical points such that $C_q(f, p_n) \neq 0$ and such that the sequence $(c_n) = (f(p_n))$ satisfies $a_n < c_n < b_n$. Because critical values are isolated, $c_n \in [\inf f(Q), \sup f(B)]$ for n big enough. □

The following result follows directly from Propositions 3.22 and Theorem 3.30. As far as we know, this result is new.

Theorem 3.31. *Let $H = H_1 \oplus H_2$ with $k = \dim H_1 < \infty$. If*

$$\sup f(S_1) < \inf f(B_2),$$

$$\sup f(B_1) < \inf f(S_2)$$

then f admits a critical point p such that

$$\inf f(S_2) \leq f(p) \leq \sup f(S_1)$$

and $C_k(f,p) \neq 0$.

3.3.1. Multiplicity results

By combining Corollaries 3.17 and 3.20 with Thorem 3.30, we get a version of a well known multiplicity result (see [**52**] for instance). As before, we get extra information about the critical groups.

Proposition 3.32. *Let $H = H_1 \oplus H_2$ with $k = \dim H_1 \in]0,\infty[$ and $e \in H_2$ be of unit length. Let $B = B_1 \oplus [0,2]e$ and $A = \partial B$ in in $H_1 \oplus \mathbb{R}e$. If f is bounded below on B_2 and if*

$$\sup f(A) < \inf f(S_2)$$

then f admits two critical points $p_0 \neq p_1$ such that

$$\inf(f(B_2) \leq f(p_0) \leq \sup f(A),$$

$$\inf f(S_2) \leq f(p_1) \leq \sup f(B)$$

and $C_k(f,p_0) \neq 0, C_{k+1}(f,p_1 \neq 0)$.

PROOF. Because

$$\sup f(A) < \inf f(S_2)$$

$$\sup f(\emptyset) = -\infty < \inf f(B_2)$$

and A $(k,1)$-links (B_2,S_2), it follows from Theorem 3.30 that f admits a critical point p_0 such that $\inf f(B_2) \leq f(p_0) \leq \sup f(A)$ and $C_k(f,p_0) \neq 0$. Also, Corrolary 3.17 says that (B,A) $(k+1,1)$-links S_2. Since

$$\sup f(B) < \infty = \inf f(\emptyset)$$

$$\sup f(A) < \inf f(S_2)$$

it follows from Theorem 3.30 that f admits a critical point p_1 such that $\inf f(S_2) \leq f(p_1) \leq \sup f(B)$ and $C_{k+1}(f,p_1) \neq 0$. The inequality

$$f(p_0) \leq \sup f(A) < \inf f(S_2) \leq f(p_1)$$

insure that p_0 and p_1 are distinct. □

A similar argument using Corollaries 3.16 and 3.19 leads to the next theorem. This result was already known to Perera [**47**].

Theorem 3.33. *Let* $H = H_1 \oplus H_2$ *with* $k = \dim H_1 \in]0, \infty[$ *and let* $e \in H_1$ *be of unit length. If* f *is bounded below on* $H_1 + [0, \infty[e$ *and if*

$$\sup f(S_1) < \inf f(H_2)$$

then f *admits two critical points* $p_0 \neq p_1$ *such that*

$$\inf(f(H_1 + [0, \infty[e)) \leq f(p_0) \leq \max f(S_1),$$

$$\inf f(H_2) \leq f(p_1) \leq \max f(B(0, 1))$$

and $C_{k-1}(f, p_0) \neq 0, C_k(f, p_1 \neq 0)$.

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