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Network pricing problems: complexity, polyhedral study and solution approaches

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Network pricing problems: complexity, polyhedral study and solution approaches

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RÉSUMÉ

Considérons le problème qui consiste à maximiser les profits issus de la tarification d'un sous-ensemble d'arcs d'un réseau de transport, où les flots origine-destination (produits) sont affectés aux plus courts chemins par rapport aux tarifs et aux coûts initiaux. Cette thèse porte sur une structure de réseau particulière du problème ci-dessus, dans laquelle tous les arcs tarifables sont connectés et forment un chemin, comme c'est le cas sur une autoroute. Étant donné que les tarifs sont habituellement déterminés selon les points d'entrée et de sortie sur l'autoroute, nous considérons un sous-graphe tarifable complet, où chaque arc correspond en réalité à un sous-chemin. Deux variantes de ce problème sont étudiées, avec ou sans contraintes spécifiques reliant les niveaux de tarifs sur les arcs.

Ce problème peut être modélisé comme un programme linéaire mixte entier. Nous prouvons qu'il est $\mathcal{NP}$-difficile. Plusieurs familles d'inégalités valides sont ensuite proposées, celles-ci renforçant certaines contraintes du modèle initial. Leur efficacité est d'abord démontrée de manière théorique, puisqu'il s'agit de facettes des problèmes restreints à un ou deux produits. Certaines des inégalités valides proposées, ainsi que plusieurs contraintes du modèle initial, permettent aussi de donner une description complète de l'enveloppe convexe des solutions réalisables d'un problème restreint à un seul produit. Des tests numériques ont également été menés, et mettent en évidence l'efficacité réelle des inégalités valides pour le problème général à plusieurs produits. Enfin, nous soulignons les liens entre le problème de tarification de réseau étudié dans cette thèse et un problème plus classique de tarification de produits en gestion.

Mots clés: Tarification de réseaux, programmation mixte entière, optimisation combinatoire.

ABSTRACT

Consider the problem of maximizing the revenue generated by tolls set on a subset of arcs of a transportation network, where origin-destination flows (commodities) are assigned to shortest paths with respect to the sum of tolls and initial costs. This thesis is concerned with a particular case of the above problem, in which all toll arcs are connected and constitute a path, as occurs on highways. Further, as toll levels are usually computed using the highway entry and exit points, a complete toll subgraph is considered, where each toll arc corresponds to a toll subpath. Two variants of the problem are studied, with or without specific constraints linking together the tolls on the arcs.

The problem is modelled as a linear mixed integer program, and proved to be $\mathcal{NP}$-hard. Next, several classes of valid inequalities are proposed, which strengthen important constraints of the initial model. Their efficiency is first shown theoretically, as these are facet defining for the restricted one and two commodity problems. Also, we prove that some of the valid inequalities proposed, together with several constraints of the linear program, provide a complete description of the convex hull of feasible solutions for a single commodity problem. Numerical tests have also been conducted, and highlight the practical efficiency of the valid inequalities for the multi-commodity case. Finally, we point out the links between the problem studied in the thesis and a more classical design and pricing problem in economics.

Keywords: Network pricing, mixed-integer programming, combinatorial optimization.

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LIST OF ABBREVIATIONS

$3 - SAT$	3-Satisfiability
NPP	Network Pricing Problem
GCT-NPP	General Complete Toll Network Pricing Problem
CCT-NPP	Constrained Complete Toll Network Pricing Problem

NOTATION

(HP3)	linear mixed integer model for the CCT-NPP
(HP3*)	linear mixed integer model for the GCT-NPP
$\mathcal{N}$	node set
$\mathcal{A}$	toll arc set
$\mathcal{B}$	toll free arc set
$t(a) : a \in \mathcal{A}$	tail of the toll arc a
$h(a) : a \in \mathcal{A}$	head of the toll arc a
$\mathcal{K}$	set of commodities
$\{(o^k, d^k) : k \in \mathcal{K}\}$	set of origin destination pairs for commodities
η^k	demand for commodities
$c_a^k : k \in \mathcal{K}, a \in \mathcal{A}$	fixed costs
$t_a : a \in \mathcal{A}$	toll variables
$x_a^k : k \in \mathcal{K}, a \in \mathcal{A}$	flow variables
$p_a^k : k \in \mathcal{K}, a \in \mathcal{A}$	actual unit profit variables
$M_a^k : k \in \mathcal{K}, a \in \mathcal{A}$	upper bounds on the actual unit profit variables
$N_a : a \in \mathcal{A}$	upper bounds on the toll variables

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CHAPTER 1

INTRODUCTION

In a current context of deregulation, companies need to apply a good tarification to their products or services. Indeed, overcapacity, increased competition and higher costs have strengthened price competition in many industries. However, pricing is one of the most complex decisions facing any company.

First, customers play an important part in a price decision, because they react to prices by purchasing - or not - the products. They are looking for good products at lowest prices. But the reaction of competitors is also important. Indeed, as they influence customer choice, they impose practical limitations on pricing alternatives. Hence, companies have to find the best possible prices, low enough so that a large number of customers buy their products, and at the same time high enough to generate large revenues.

Focusing on the operational research literature, several classes of pricing problems have been considered. These can differ in the objective functions, as well as in the category of products or services considered. The main objective functions deal with the maximization of revenues, social welfare, or a combination of both criteria. In what concerns the category of products or services considered, apart from papers that address the problem of pricing a generic product, other categories of products are, for example, financial assets or transportation routes.

We deal with a particular case of a pricing problem that involves a transportation network. Let us define a transportation network as a set of nodes (cities) and a set of arcs (routes) linking some of these nodes together. Further, a fixed cost is assigned to each arc of the network. Now consider two classes of economic agents. The first, a manager, owns a subset of arcs of the network on which he/she imposes tolls so as to maximize revenues. The second category of agents are network users, which travel from one node to another of the network while minimizing their costs.

The Network Pricing Problem consists of devising the toll levels that should be imposed by the manager on the subset of toll arcs such as to maximize its revenues. Then, reacting to the tolls, the network users travel on shortest paths from their origins to their respective destinations, with respect to a cost equal to the sum of tolls and initial costs.

This thesis is concerned with a particular case of the above problem in which all toll arcs are connected and constitute a path, as occurs on highways. As toll levels are usually computed using the highway entry and exit nodes, a complete toll subgraph is considered, where each toll arc corresponds to a toll subpath. Two variants of the problem are studied, with or without specific constraints linking together the tolls on the arcs.

As the manager and the users seek to maximize revenues and to minimize costs respectively, the problem belongs to a class of hierarchical, sequential and non co-operative optimization programs. As in the Stackelberg version of the duopolistic equilibrium (see Stackelberg [63]), a leader (the manager) integrates in its optimization process the reaction of a follower (the network users) to its own decisions. More specifically, it is a bilevel problem, i.e., a hierarchical optimization problem involving two levels of decision.

This class of problems has many applications: hierarchical structures can be found in the field of transportation (network design, airline revenue management, transportation of hazardous materials, ...), management (location of schools, allotment of funds, ...), and planning (agricultural, electrical or environmental policies, ...).

As we will see later, the problem considered in the thesis is very generic. Hence the purpose of this study is to better understand the very heart of a network pricing structure, and to develop tools that could be transposed to more realistic or complex problems. More precisely, the thesis provides a first study of the polyhedral structure of a Network Pricing Problem. Hence models, valid inequalities and proofs of facets are the core of our research.

The thesis is organized as follows. In Chapter 2, we present the Network Pricing Problem. As it can be modelled as a bilinear/bilinear bilevel program, we first formulate a bilevel program. Then the Network Pricing Problem is introduced. We summarize the main contributions to this topic from the literature.

The particular Network Pricing Problem addressed in the thesis, whose network structure can represent features specific to a real highway network, is presented in Chapter 3. Mathematically, it is formulated as a linear mixed integer program with a single level. Then we prove that this problem is $\mathcal{NP}$-hard using a reduction from $3-SAT$.

In Chapter 4, we propose valid inequalities for the problem. These exploit the underlying network structure and strengthen important constraints of the model. Next, we explore the strength and efficiency of the valid inequalities.

Chapter 5 provides proofs that the valid inequalities, as well as several constraints of the initial model, are facet defining for the convex hull of feasible solutions for a restricted problem involving two origin-destination pairs. Also, we prove that some of the valid inequalities proposed, together with other constraints of the linear program, provide a complete description of the convex hull of feasible solutions for a single commodity problem.

The practical efficiency of the valid inequalities is then confirmed in Chapter 6 by numerical results. Most of the valid inequalities proposed are very efficient, at least to decrease the gap or number of nodes in the branch and cut algorithm. They also allow to decrease the computing time for one variant of the problem.

Finally, the aim of Chapter 7 is to link the specific problems studied in the thesis with a more standard design and pricing problem in economics. A description of these problems, together with an overview of the main contributions from the literature, are provided. Then we point out the strong relationships between both families of problems.

CHAPTER 2

THE NETWORK PRICING PROBLEM

The aim of this chapter is to present the Network Pricing Problem. As its initial formulation is a bilinear/bilinear bilevel program, we first give an introduction to bilevel programming. Next, we focus on (bi)linear/(bi)linear bilevel problems, i.e., problems in which both constraints and objective function are (bi)linear. We also present a more precise bilinear/bilinear bilevel pricing problem. The Network Pricing Problem is next introduced. First modelled as a bilinear/bilinear bilevel program, we show that it can be reformulated as a single level linear mixed integer model. Then we summarize the main contributions on this topic in the literature.

2.1 Bilevel programming

Consider a sequential game with two players, where a leader plays first, taking into account the possible reactions of the second player, called the follower. If vectors x and y denote the leader and follower decision variables respectively, this situation can be described mathematically by a **bilevel program**[1]:

$$
\begin{aligned}
\text{(BP)} \quad & \min_{x,y} F(x,y) \\
& \text{s.t. } G(x,y) \leq 0, \\
& \qquad y \in \arg\min_{y} f(x,y) \\
& \qquad\qquad \text{s.t. } g(x,y) \leq 0.
\end{aligned}
$$

[1]Slightly abusing notation, we use y for denoting both the optimal solution and the argument of the lower level problem.

The mathematical bilevel formulation first appears in 1973, in a document by Bracken and McGill (1973, [8]). These authors publish several articles (1973, [8]; 1974, [9]; 1978, [10]) dealing with military, production and marketing applications. The bilevel and multilevel terms come from Candler and Norton (1977, [13]), who do not consider upper level constraints involving both x and y variables in their models. The more general formulation, involving a constraint of type $G(x, y) \leq 0$ at the upper level, appears for the first time in an article by Shimizu and Aiyoshi (1981, [59]).

Also, formulation (BP) ensures that, if there are multiple optimal solutions for the lower level problem, the leader most profitable solution is selected. This is an **optimistic approach**, by opposition to a **pessimistic approach**. In the latter, the leader chooses the solution which protects himself against the follower worst possible reaction. Such situations have been studied by Loridan and Morgan (1989, [45, 46]) or Ishizuka and Aiyoshi (1992, [36]).

Note that the bilevel problems described here are very close to mathematical problems with equilibrium constraints (MPECS). In the latter, the lower level represents an equilibrium problem, often described by a variational inequality. The interested reader could refer to books by Shimizu et al. (1997, [60]), Outrata et al. (1998, [56]) or Luo et al. (1996, [47]).

Generically non differentiable and non convex, bilevel problems are, by nature, hard. Even the linear bilevel problem, where the objective functions and the constraints are linear, has shown to be $\mathcal{NP}$-hard by Jeroslow (1985, [37]). Hansen et al. (1992, [34]) prove strong $\mathcal{NP}$-hardness. Vicente et al. (1994, [68]) strengthen these results and prove that merely checking strict or local optimality is strongly $\mathcal{NP}$-hard.

Several authors have presented optimality conditions for bilevel problems. Among these ones, let us name Chen and Florian (1991, [15]), Dempe (1992, [21]) or Tuy et al. (1993, [64]) who use non linear analysis techniques, as well as Savard and Gauvin (1994, [58]) or Vicente and Calamai (1995, [67]) who take into account the geometry of the induced region. Liu et al. (1994, [44]) describe geometric features of solutions. Unfortunately, because of the difficulty of handling the mathematical objects involved in all these optimality conditions, they are quite useless in practice and do not provide any sufficient stopping criterion for numerical algorithms.

Let us now briefly summarize the algorithmic contributions to bilevel programming in the literature. Note that most algorithmic research has focused on problems involving linear, quadratic or convex constraints and/or objective function. In all these classes of problems, the lower level problem admits extremal solutions, which allows the development of methods with a guarantee of global optimality. In contrast, research on nonlinear bilevel problems has mainly focused on algorithms with a guarantee of local optimality.

One of the first method that has been proposed is based on vertex enumeration. It has been used by Candler and Townsley (1982, [14]), Bialas and Karwan (1984, [6]) or Tuy et al. (1993, [64]) to solve linear bilevel programs.

Next, when the lower level is convex and regular, it can be replaced by its Karush-Kuhn-Tucker conditions. The bilevel problem is then reformulated as a single level problem, which contains the primal-dual constraints and complementarity conditions. However, the single level problem stays very difficult to solve, mainly due to the complementarity constraints. Several algorithms based on branch and bound on these constraints have been proposed to solve different classes of bilevel programs, among which linear (Bard and Falk (1982, [30]), Fortuny-Amat and McCarl (1981, [29])), linear-quadratic (Bard and Moore (1990, [5])) and quadratic (Al-Khayal et al. (1992, [3]), Edmunds and Bard (1991, [27])). Combining branch and bound, monotonicity principles and penalties as in mixed integer programming, Hansen et al. (1992, [34]) have been able to solve linear bilevel medium size instances.

Descent methods have also been used to solve bilevel programs. These methods assume that the lower level problem has a unique optimal solution for any x, and consider y as an implicit function $y(x)$ of x, hence obtaining upper level descent directions. Such algorithms have been proposed by Savard and Gauvin (1994, [58]) or Vicente et al. (1994, [68]).

Further, penalty function methods have also been proposed to solve bilevel programs. Aiyoshi and Shimizu (1981, [59]; 1984, [1]) replace the lower level problem by a penalized problem. Ishizuka and Aiyoshi (1992, [36]) use a double penalty method in which both objective functions are penalized, the lower level penalized problem being replaced by its stationarity condition.

Finally, trust region methods have also been used for solving nonsmooth bilevel programs (see Kočvara and Outrata (1997, [38]), Fukushima and Pang (1999, [4]), Marcotte et al. (2001, [50]) or Colson et al. (2005, [17])).

Motivated by Stackelberg game theory, several authors have studied bilevel programming. For a more complete bibliography about bilevel or multilevel programming, the interested readers could refer to Vicente and Calamai (1994, [66]), Migdalas et al. (1997, [53]) or, for more recent references, to Dempe (2002, [20]), Marcotte and Savard (2005, [49]) or Colson et al. (2007, [18]).

2.2 (Bi)linear bilevel programming

As global optimality algorithms are restricted to subclasses of problems involving specific mathematical properties, we focus on bilevel programs with linear or bilinear objectives. The **linear/linear bilevel problem** takes the form:

$$
\begin{aligned}
\text{(LBP)} \quad & \max_{x,y} c_1 x + d_1 y \\
& \text{s.t. } A_1 x + B_1 y \le b_1 \\
& \quad x \ge 0 \\
& \quad y \in \arg\max_{y} d_2 y \\
& \qquad \text{s.t. } A_2 x + B_2 y \le b_2 \\
& \qquad\quad y \ge 0,
\end{aligned}
$$

where $c_1 \in \mathbb{R}^{n_x}$, $d_1, d_2 \in \mathbb{R}^{n_y}$, $A_1 \in \mathbb{R}^{n_u \times n_x}$, $A_2 \in \mathbb{R}^{n_l \times n_x}$, $b_1 \in \mathbb{R}^{n_u}$, $B_1 \in \mathbb{R}^{n_u \times n_y}$, $B_2 \in \mathbb{R}^{n_l \times n_y}$, $b_2 \in \mathbb{R}^{n_l}$.

The constraints $A_1 x + B_1 y \le b_1$ (resp. $A_2 x + B_2 y \le b_2$) are the **upper (resp. lower) level constraints**. The linear term $c_1 x + d_1 y$ (resp. $d_2 y$) is the **upper (resp. lower) level objective function**, while x (resp. y) is the vector of upper (resp. lower) level variables.

In order to characterize the solution of such a problem, the following definitions are required.

Definition 1 *The set of feasible solutions for (LBP) is defined as:*

$$\Omega = \{(x, y) : x \ge 0, y \ge 0, A_1 x + B_1 y \le b_1, A_2 x + B_2 y \le b_2\}.$$

Definition 2 *For every $x \ge 0$, the lower level feasible set is:*

$$\Omega(x) = \{y : y \ge 0, B_2 y \le b_2 - A_2 x\}.$$

Definition 3 *The trace of the lower level problem with respect to the upper level variables is:*

$$\Omega^2 = \{x : x \geq 0, \Omega(x) \neq \emptyset\}.$$

Definition 4 *For a given vector $x \in \Omega_x^2$, the lower level optimal set is:*

$$S(x) = \{y : y \in \arg\max\{d_2 y : y \in \Omega(x)\}\}.$$

Definition 5 *The induced region is defined as the set of feasible solutions for the upper level problem, i.e.,*

$$\Gamma = \{(x, y) : x \geq 0, A_1 x + B_1 y \leq b_1, y \in S(x)\}.$$

These definitions highlight the polyhedral nature of the induced region and allow to characterize the set of optimal solutions for (LBP).

Definition 6 *A point (x^*, y^*) is optimal for (LBP) if:*

- $(x^*, y^*) \in \Gamma$;
- $c_1 x^* + d_1 y^* \geq c_1 x + d_1 y$ *for all* $(x, y) \in \Gamma$.

Hence, a direct consequence of the polyhedral nature of the induced region Γ is that, if (LBP) has a solution, an optimal solution is attained at an extreme point of Ω.

Although much attention has been paid to linear/linear bilevel programming, it appears that bilinear/bilinear bilevel programs better fit real life situations. Indeed, this allows to model interactions between the leader and the follower in the objective function. An interesting class of bilinear/bilinear bilevel problems is the class of pricing problems where a firm (leader) imposes taxes on activities while consumers (follower) choose minimal cost activities.

Consider a vector of activities (x_1, x_2), a firm and a set of consumers. At the upper level, we assume that the firm seeks to maximize its revenues by imposing taxes on the activities corresponding to vector x_1. At the lower level, consumers react to the taxes by choosing minimal cost activities. Let (c, d) be the vector of initial prices for (x_1, x_2), and t be a tax vector linked with the activity vector x_1. Note that this model can cover various situations. Indeed, the tax vector t can represent taxes as well as subsidies. Also, x_1 and x_2 vectors can be consumption as well as production levels. One obtains the **bilinear/bilinear bilevel pricing model**:

$$
\begin{aligned}
\text{(BPP)} \quad & \max_{t, x_1, x_2} tx_1 \\
& \text{s.t.} \ (x_1, x_2) \in \arg\min_{x_1, x_2} (c + t)x_1 + dx_2 \\
& \qquad\qquad\qquad\qquad \text{s.t.} \ Ax_1 + Bx_2 = b \\
& \qquad\qquad\qquad\qquad\qquad x_1, x_2 \geq 0.
\end{aligned}
$$

We assume that the polyhedron $\{(x_1, x_2) : Ax_1 + Bx_2 = b, x_1, x_2 \geq 0\}$ is bounded and non empty, while $\{x_2 : Bx_2 = b, x_2 \geq 0\}$ is non empty. Hence the lower level problem has a finite optimal solution for every value of the tax vector t. These conditions also ensure that the objective function of (BPP) is finite.

Note that, for a given lower level vector (x_1, x_2), (BPP) reduces to an **inverse optimization problem** where one must select a tax vector t such that (i) (x_1, x_2) is optimal with respect to this tax vector and (ii) the revenue tx_1 is maximal.

From the leader's perspective, the objective function tx_1 is piecewise linear and discontinuous at points t that induce a change of optimal basis in the lower level problem. We illustrate the evolution of the objective function tx_1 with respect to the tax t in Figure 2.1, where (x_1^i, x_2^i) is the optimal solution of the lower level problem corresponding to a tax t with values between t^i and t^{i+1}.

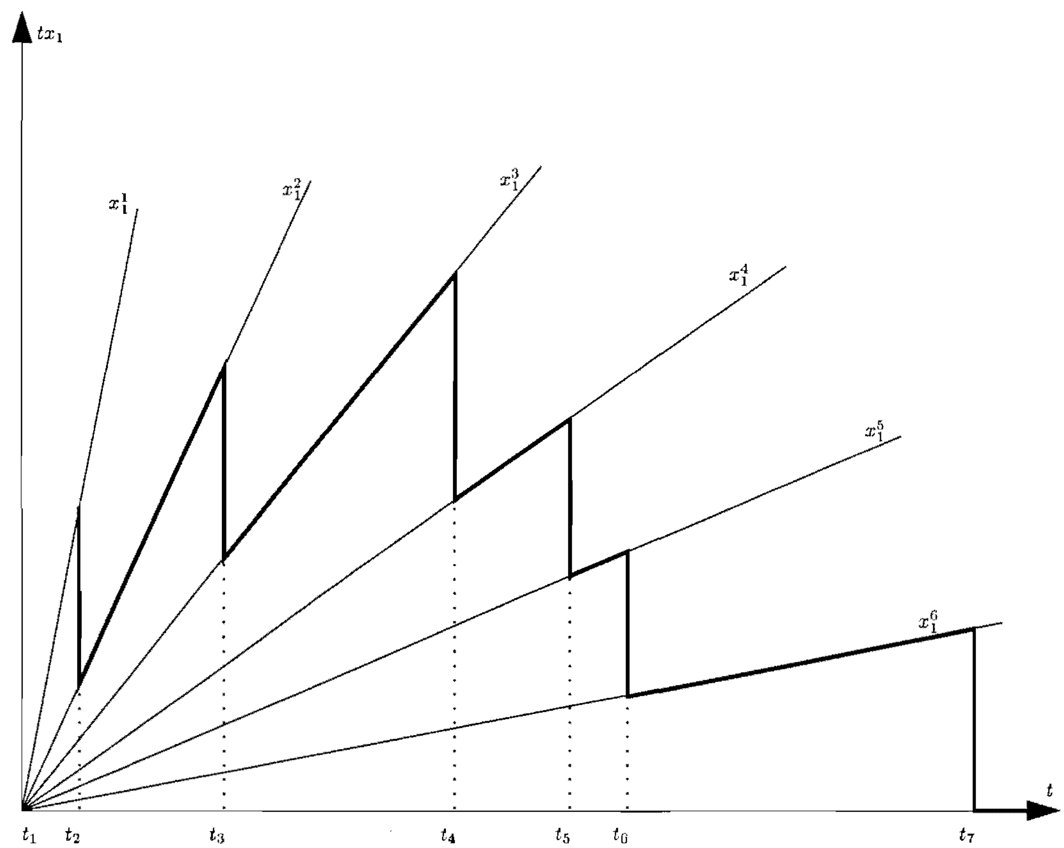

Figure 2.1: Evolution of the objective function tx_1 with respect to tax t

An optimal pricing policy consists in setting t high enough to generate large revenues for the leader but, at the same time, low enough to promote the use of taxed activities corresponding to x_1 by consumers.

The Network Pricing Problem is a particular case of a bilinear/bilinear pricing problem, which involves a transportation network and considers the arcs of the network as activities. We present this problem in the next section.

2.3 The Network Pricing Problem

Let us define a transportation network as a set of nodes (cities) and a set of arcs (routes) linking some of these nodes together. At the upper and lower level, consider an authority and a set of network users respectively. We also define a commodity as a set of network users travelling from the same origin to the same destination.

In addition to a fixed cost associated with every arc, tolls are imposed by the authority on a specified subset of arcs of the network. Hence the **Network Pricing Problem** consists of devising toll levels on the specified subset of toll arcs in order to maximize the authority's revenues. Then, reacting to the tolls, each commodity travels on the shortest path from its origin to its destination, with respect to a cost equal to the sum of tolls and initial costs.

Let us now introduce additional assumptions. First, in order to avoid trivial solutions leading to infinite revenues for the authority, we assume that there exists a toll free path for each commodity. Further, we restrict our study to non negative toll vectors.

However, note that there exist models (see Labbé et al. (1998, [43]), Cirinei (2007, [16]) or Brotcorne et al. (2001, [12])) which also allow negative tolls. The latter yield compensations with other (positive) tolls, when the corresponding arcs are used by multiple commodities. Even if such situations will not be considered in the thesis, the reader should know that there exist more realistic (but also more complex) models, which consider congestion effects (see for example Fortin (2005, [28])) and/or a non uniform distribution of the fixed cost perception in a population (see for example Marcotte et al. (2007, [51])).

The small network example depicted in Figure 2.2 illustrates the Network Pricing Problem. Assume that a commodity composed by a single user travels from node 1 to node 5, the bolded arcs $(2,3)$ and $(4,5)$ being the toll arcs.

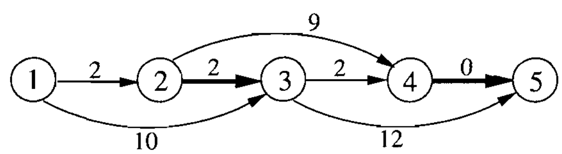

Figure 2.2: Network example

If we look closely at that network, we conclude that the user will never pay more that 22, which is the cost of the toll free path $1 \to 3 \to 5$. In contrast, if the authority sets all tolls to zero, the user will choose the path $1 \to 2 \to 3 \to 4 \to 5$ with cost 6. It means that an upper bound on the authority revenues is $22-6 = 16$.

However, this bound is not always reached, as in the example. Whatever the tolls imposed by the manager, its revenue will never exceed 15. Indeed, the toll arc $(2,3)$ can only be selected by the network user if the toll on this arc is less or equal to 5, because of the arc $(2,4)$ $((2+x)+2 \leq 9)$. In the same manner, the toll arc $(4,5)$ can only be used if the corresponding toll does not exceed 10, because of the arc $(3,5)$ $(2+x \leq 12)$. An optimal solution for this example consists in setting tolls of 5 on the arc $(2,3)$ and 10 on the arc $(3,5)$.

The bilinear/bilinear bilevel Network Pricing Problem was first introduced by Labbé et al. (1998, [43]). Consider a multi-commodity network defined by a node set $\mathcal{N}$, an arc set $\mathcal{A} \cup \mathcal{B}$ and a set of origin-destination pairs $\{(o^k, d^k) : k \in \mathcal{K}\}$, called commodities, each one endowed with a demand η^k. Let $\mathcal{A}$ be a subset of arcs a upon which tolls t_a can be added to the original fixed cost vector c and $\mathcal{B}$ the complementary subset of toll free arcs, for which the cost vector c is also given. Assuming that, for a given toll policy $t = (t_a)_{a \in \mathcal{A}}$, the network users travel on shortest paths with respect to the tolls and fixed costs on arcs, the Network Pricing Problem consists of devising a revenue maximizing toll policy. Upon the introduction of vectors $x^k = (x_a^k)_{k\in\mathcal{K},a\in\mathcal{A}}$ that specify the flows on commodities $k \in \mathcal{K}$ (i.e., $x_a^k = 1$ if commodity k travels on the toll arc a and $x_a^k = 0$ otherwize), the Network Pricing Problem can be formulated as the bilevel program (Labbé et al. (1998, [43])):

$$\text{(TP)} \quad \max_{t,x} \sum_{k\in\mathcal{K}} \sum_{a\in\mathcal{A}} \eta^k t_a x_a^k$$

subject to:

$$t_a \geq 0 \qquad \forall a \in \mathcal{A} \qquad (2.1)$$

$$x \in \arg\min_x \sum_{k\in\mathcal{K}} \left(\sum_{a\in\mathcal{A}} (c_a + t_a) x_a^k + \sum_{a\in\mathcal{B}} c_a x_a^k \right) \qquad (2.2)$$

subject to:

$$\sum_{a\in i^-\cap\mathcal{A}} x_a^k + \sum_{a\in i^-\cap\mathcal{B}} x_a^k - \sum_{a\in i^+\cap\mathcal{A}} x_a^k - \sum_{a\in i^+\cap\mathcal{B}} x_a^k = \begin{cases} -1 & \text{if } i = o^k \\ 1 & \text{if } i = d^k \\ 0 & \text{otherwise} \end{cases}$$

$$\forall k \in \mathcal{K}, \forall i \in \mathcal{N} \qquad (2.3)$$

$$x_a^k \in \{0, 1\} \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A}, \qquad (2.4)$$

where i^- (resp. i^+) denotes the set of arcs having node i as its head (resp. tail).

Note that the characterization of lower level solutions as origin-destination paths carrying either no flow or the total origin-destination flow allows to obtain an integer programming formulation of (TP) that involves binary variables. Now, in view of the unimodularity of the constraint matrix associated with the shortest path problem at the lower level, one may drop the integrality requirements for the flow variables x. It follows that the lower level problem can be replaced by its primal dual constraints and primal-dual optimality conditions, yielding a single-level program involving complementarity (i.e., disjunctive) constraints.

Through the introduction of auxiliary variables

$$p_a^k = \begin{cases} t_a & \text{if commodity } k \text{ uses arc } a \in \mathcal{A}, \\ 0 & \text{otherwise} \end{cases}$$

corresponding to the actual unit revenue associated with arc $a \in \mathcal{A}$ and commodity $k \in \mathcal{K}$, Labbé et al. (1998, [43]) derive a mixed integer linear formulation for this problem, namely

$$\text{(TP2)} \quad \max \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \eta^k p_a^k$$

subject to:

$$\sum_{a \in i^- \cap \mathcal{A}} x_a^k + \sum_{a \in i^- \cap \mathcal{B}} x_a^k - \sum_{a \in i^+ \cap \mathcal{A}} x_a^k - \sum_{a \in i^+ \cap \mathcal{B}} x_a^k = \begin{cases} -1 & \text{if } i = o^k \\ 1 & \text{if } i = d^k \\ 0 & \text{otherwise} \end{cases} \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{N} \quad (2.5)$$

$$\lambda_{h(a)}^k - \lambda_{t(a)}^k \leq c_a + t_a \qquad \forall k \in K, \forall a \in A \quad (2.6)$$

$$\lambda_{h(a)}^k - \lambda_{t(a)}^k \leq c_a \qquad \forall k \in K, \forall a \in B \quad (2.7)$$

$$\sum_{a \in A} (c_a x_a^k + p_a^k) + \sum_{a \in B} c_a x_a^k = \lambda_{d_k}^k - \lambda_{o_k}^k \qquad \forall k \in K \quad (2.8)$$

$$p_a^k \leq M_a^k x_a^k \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (2.9)$$

$$t_a - p_a^k \leq N_a(1 - x_a^k) \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (2.10)$$

$$p_a^k \leq t_a \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (2.11)$$

$$p_a^k \geq 0 \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (2.12)$$

$$x_a^k \in \{0, 1\} \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (2.13)$$

$$x_a^k \geq 0 \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{B}, \quad (2.14)$$

where $h(a)$, $t(a)$ correspond to the head and tail of the toll arc $a \in \mathcal{A}$, while M_a^k and N_a are sufficiently large constants.

Constraints (2.5) describe flows on commodities. (2.6), (2.7) and (2.8) are the primal dual constraints and optimality conditions of the lower level problem. Constraints (2.9), (2.10) and (2.11) come from the model linearization, and ensure that $p_a^k = t_a x_a^k$ for all $k \in \mathcal{K}, a \in \mathcal{A}$.

Roch et al. (2005, [57]) and Grigoriev et al. (2005, [32]) prove the $\mathcal{NP}$-hardness of this problem, even under restrictive conditions such as a single commodity or lower bounded tolls (see Labbé et al. (1998, [43])). However, several particular cases are polynomially solvable, such as the Network Pricing Problem with a single toll arc (see Brotcorne et al. (2000, [11])). Van Hoesel et al. (2003, [65]) prove that, when the number of toll arcs is upper bounded, the optimal solution of the Network Pricing Problem can be obtained by solving a polynomial number of linear programs. The latter also present other particular polynomial cases of the problem.

In contrast with (TP2) formulation, in which the paths chosen by commodities are described by flows on arcs (latter called 'arc formulation'), Bouhtou et al. (2003, [7]) and Didi et al. (1999, [24]) propose formulations involving directly flows on paths for commodities. Bouhtou et al. also propose a standard graph representation of a network together with reduction methods for this last one, which often lead to a significant reduction of the network graph. This allows obtaining good numerical results for medium size instances. Tests on randomly generated problems involving 15 to 80 commodities and 20 to 100 toll arcs (in networks with 75 or 100 nodes and 2000 or 4000 arcs) show that an optimal solution can be identified within a couple of seconds. However, note that these instances lead to only 2 or 3 non dominated paths on average for each commodity, and thus are rather easy to solve.

Unfortunately, a commercial solver for linear programs such as Xpress cannot solve large size instances, neither of the (TP2) arc formulation presented above nor of the path formulation. This is mainly due to the bad quality of the linear relaxation in variables x (i.e., (2.13) are replaced by $0 \leq x_a^k \leq 1$ for all $k \in \mathcal{K}, a \in \mathcal{A}$). To overcome this problem, several approaches are considered.

Dewez et al. (2007, [23]) set values for constants $M_a^k, N_a : k \in \mathcal{K}, a \in \mathcal{A}$ of (TP2) formulation by computing upper bounds on the tolls on arcs, and propose valid inequalities for the various models (arc formulation and path formulation). Numerical tests have been carried out on randomly generated problems involving 20 to 40 commodities and 5% to 20% toll arcs, in networks with 60 nodes and 208 arcs, latter called 'grid graphs'. The results show that the adjustment of constants makes it possible to divide by two the value of the duality gap at the root of the branch and bound tree, whereas the valid cuts allow a reduction of the explored nodes as well as the computing time.

Cirinei (2007, [16]) proposes a column generation algorithm for the inverse optimization problem, which consists of devising the tolls that should be imposed on the network, considering that the reaction of the network users is known and maximizing the authority's revenue. Tests on randomly generated problems involving 10 to 40 commodities and 15% toll arcs in grid graphs show that the method performs well in terms of computing time. All instances can be solved in a couple of seconds. The column generation algorithm also allows to solve the largest instances much faster than without the algorithm. Further, the author proposes an exact resolution algorithm based on an intelligent enumeration of the solutions of the lower level problem. This resolution method allows to define improved upper bounds on the authority's revenue.

As solving large size problems to optimality is hard, several authors propose heuristic methods for the Network Pricing Problem. Brotcorne et al. (2001, [12]) present two heuristics for the problem: the first consists in setting tolls sequentially over the arcs, while the second is based on a primal-dual approach. Tests on problems involving 10 to 20 commodities and 5% to 20% toll arcs in grid graphs show that heuristic solutions are on average within 1.5% and 7% of optimality respectively. Both heuristics are much faster than an exact resolution. The latter (2000, [11]) also examine a very similar problem, in which commodities have to be routed from several locations to customers according to their respective demands.

Also, Roch et al. (2005, [57]) propose an approximation algorithm for the single commodity Network Pricing Problem, with a guaranteed performance of $\frac{1}{2}\log n + 1$, where n is the number of toll arcs in the network.

Finally, Cirinei (2007, [16]) presents a tabu based local search algorithm, which exploits the underlying network structure of the lower level problem. This last method is very efficient, both in terms of solution quality and computing time, producing heuristic solutions within 1% of optimality for instances involving 10 to 100 commodities and 5% to 20% toll arcs in grid graphs.

Dewez (2004, [22]) also studies a particular case of the Network Pricing Problem that deals with specific network structures similar to highways. Indeed, the model considered involves a path of toll arcs as well as Triangle inequalities on the toll variables. She proves that, when it reduces to a single commodity, the problem is polynomially solvable. She presents an exact resolution algorithm for the multi-commodity problem, based on an enumeration of the solutions of the lower level problem. Unfortunately, due to the enumeration at the lower level, the time needed to solve the problem to optimality grows exponentially with the number of commodities and the number of nodes in the network.

The author also proposes several heuristics to set the flow variables for this problem. Then the inverse problem allows to determine the tolls yielding the best revenue for the authority, once flows are fixed. We briefly describe the idea behind the three best heuristics.

1) For each commodity $k \in \mathcal{K}$, set $x_a^k = 1$ for the toll arc a with the largest upper bound $M_a^k : a \in \mathcal{A}$. Then solve the inverse optimization problem to find the tolls leading to a maximal revenue for the authority.

2) For each commodity $k \in \mathcal{K}$, set $x_a^k = 1$ for the toll arc a with the largest upper bound $M_a^k : a \in \mathcal{A}$. Then observe that, if two commodities use the same toll arc $a \in \mathcal{A}$, the leader could take advantage to force the use of another toll arc $b \in \mathcal{A} \setminus \{a\}$ (i.e., $x_b^k = 1$) for one of both commodities (with respect to the demand η^k and the upper bounds M_a^k). Next, solve the inverse optimization problem to find the tolls leading to a maximal revenue.

3) For each commodity $k \in \mathcal{K}$ and for each toll arc $a \in \mathcal{A}$, set $x_a^k = 0$ if $M_a^k < \alpha \max_{k \in \mathcal{K}} M_a^k$ $(0 \leq \alpha \leq 1)$, i.e., if the upper bound on the revenue M_a^k is too small with respect to the upper bound on the same arc a for other commodities. Then solve the remaining problem.

When tested on grid graph instances involving 21 to 36 commodities and 10 to 20 toll nodes in the highway, the best heuristics produce solutions within 5% of optimality in a couple of seconds.

Grigoriev et al. (2005, [32]) consider another particular case of the Network Pricing Problem, where each commodity chooses at most one toll arc from its origin to its destination. As this specific network structure looks like a town divided by a river with crossing bridges or tunnels, this problem is called the **Cross River Network Pricing Problem**. The authors prove that this particular problem is $\mathcal{NP}$-hard.

Further, they also show that the Uniform Network Pricing Problem, in which tolls on the arcs are all equal, constitutes an $O(n)$-approximation algorithm (where n is the number of toll arcs in the network) for the Cross River Network Pricing Problem. Under some particular assumptions, the Uniform Network Pricing Problem provides an $O(\log n)$-approximation algorithm for the same problem.

We conclude this chapter with a summary (see Figure 2.3) of the main contributions to the Network Pricing Problem in literature.

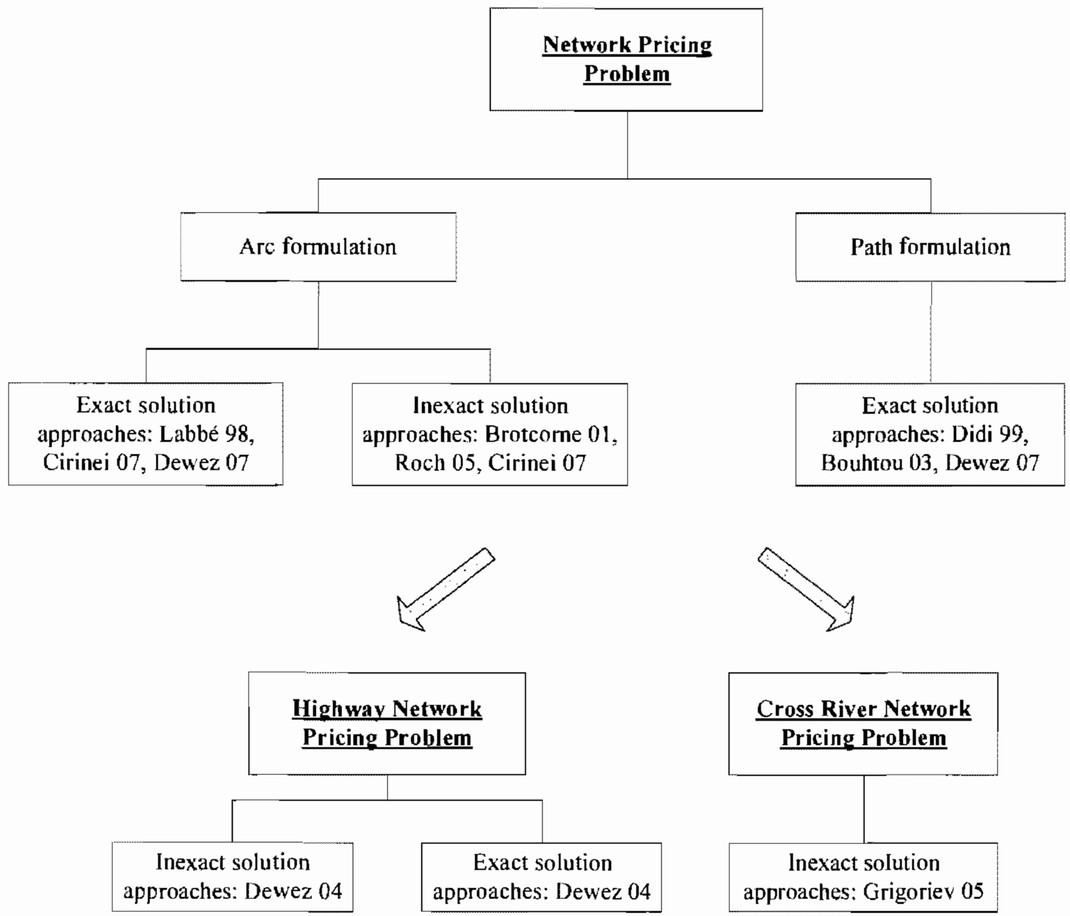

Figure 2.3: Main contributions to the Network Pricing Problem

CHAPTER 3

NETWORK PRICING WITH CONNECTED TOLL ARCS

In this chapter, we present the specific Network Pricing Problem addressed in the thesis. First modelled as a bilinear/bilinear bilevel pricing problem, it is reformulated as a single level linear mixed integer model. Next, we propose a new linear mixed integer formulation for the problem, together with settings of constants and a preprocessing of the network. Finally, the complexity of this specific Network Pricing Problem is studied.

3.1 Network Pricing Problems with Connected Toll Arcs

We now focus on a particular Network Pricing Problem dealing with structured networks in which all toll arcs must be connected and constitute a path. As these structures can represent features specific to a real highway topology and for the sake of clarity, we define a highway as the path of toll arcs in the network. The first variant of this problem, called Basic NPP, is directly derived from the classical Network Pricing Problem. However, the tolls are additive in this network structure, while toll levels are usually determined with respect to given entry and exit points on the highway. Hence, a second variant is considered, that involves a complete toll subgraph, i.e., each toll arc represents a toll subpath between two entry and exit points. It is called General Complete Toll NPP. Finally, a third variant, called Constrained Complete Toll NPP, involves a complete toll subgraph together with specific constraints that link tolls on several paths.

The first variant is directly derived from the Network Pricing Problem presented in Chapter 2. Let us define a commodity as a set of users with the same origin and destination nodes. A commodity can either take the shortest toll free path from its origin to its destination, or follow the highway, using shortest toll free paths to and from the highway. We assume that users who have left the highway are not allowed to reenter, which implies that paths are uniquely determined by their respective entry and exit nodes.

This problem is called the Basic Network Pricing Problem with Connected Toll Arcs, for short **"Basic NPP"**. It is illustrated in Figure 3.1, where toll arcs are dashed. Toll free arcs are inserted between origin and destination nodes, as well as from/to the origin and destination nodes to/from the highway. These arcs represent shortest toll free paths between the corresponding nodes. We also assume that a fixed cost is set on each arc, and provides a measure of the distance, time or gas consumed on the arc. The fixed cost set on a toll free arc corresponds to the smallest fixed cost of a path between its nodes.

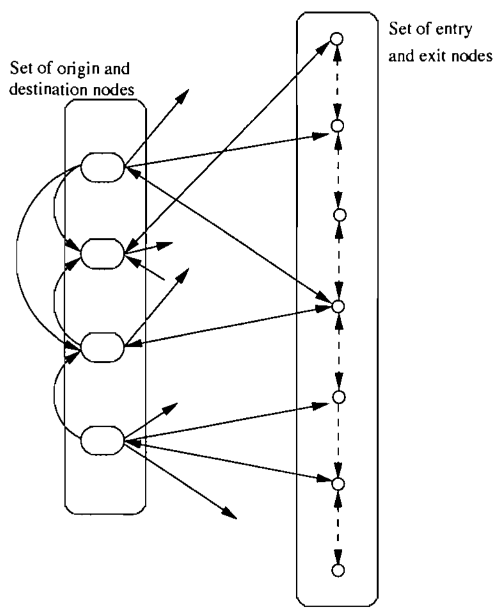

Figure 3.1: Basic NPP

The mathematical formulation (TP2) presented in Chapter 2 applies to this situation. However, additional constraints must be appended to (TP2) in order to ensure that a commodity which leaves the highway at some exit node does not reenter the highway at another entry node. Let us define the set $\widetilde{\mathcal{N}} \subseteq \mathcal{N}$ of all possible origin and destination nodes, i.e., $\widetilde{\mathcal{N}} = \{o^k, d^k : k \in \mathcal{K}\}$. Assuming that each shortest toll free path is represented by a single arc, the Basic NPP is described by model (TP2), with the additional constraints

$$\sum_{a \in i^- \cap \mathcal{B}} x_a^k + \sum_{a \in i^+ \cap \mathcal{B}} x_a^k = 0 \qquad \forall k \in \mathcal{K}, \forall i \in \widetilde{\mathcal{N}} : i \neq o^k, d^k. \qquad (3.1)$$

However, note that the tolls are additive in this network structure, i.e., a commodity must pay the sum of the tolls on all arcs that belong to its path. As toll levels are usually determined with respect to given entry and exit points on the highway, we consider the Network Pricing Problem with Connected Toll Arcs involving a complete toll subgraph. Hence, as we assume that users who have left the highway are not allowed to reenter, each toll subpath is represented by a single toll arc. This problem is depicted in Figure 3.2 and called **"General Complete Toll NPP"**.

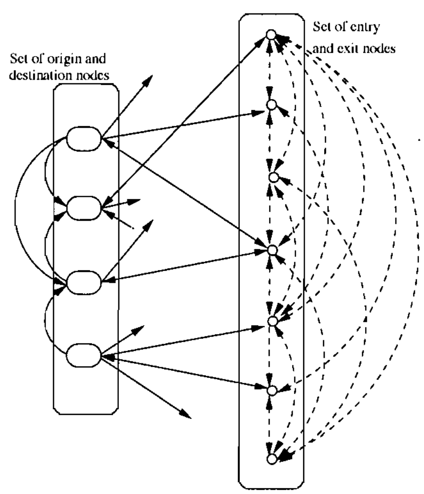

Figure 3.2: Complete Toll NPP

Let us now introduce some notation. For each arc $a \in \mathcal{A}$, let $t(a), h(a) \in \mathcal{N}$ be its tail and head nodes respectively. For each commodity $k \in \mathcal{K}$ and for each toll arc $a \in \mathcal{A}$, let c_a^k denote the fixed cost on the corresponding path $o^k \to t(a) \to h(a) \to d^k$, where $t(a), h(a) \in \mathcal{N}$ are the entry and exit nodes on the highway. The fixed cost on the toll free path $o^k \to d^k$ is denoted by c_{od}^k, while the corresponding flow variable is x_{od}^k. For each commodity $k \in \mathcal{K}$ and for each toll arc $a \in \mathcal{A}$, variable x_a^k represents the flow on the corresponding path $o^k \to t(a) \to h(a) \to d^k$, while variable t_a is the toll on the arc a (i.e., toll subpath a). Further, we consider that nodes are labelled by the index 1 to m, leading to $|A| = n = m(m-1)$ toll arcs. One obtains the following bilevel formulation (2004, Dewez [22]):

$$\text{(HP1)} \quad \max_{t,x} \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \eta^k t_a x_a^k$$

subject to:

$$t_a \geq 0 \qquad \forall a \in \mathcal{A} \qquad (3.2)$$

$$x \in \arg\min_x \sum_{k \in \mathcal{K}} \left(\sum_{a \in \mathcal{A}} (c_a^k + t_a) x_a^k + c_{od}^k x_{od}^k \right) \qquad (3.3)$$

subject to:

$$\sum_{a \in \mathcal{A}} x_a^k + x_{od}^k = 1 \qquad \forall k \in \mathcal{K} \qquad (3.4)$$

$$x_a^k \in \{0,1\} \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (3.5)$$

$$x_{od}^k \in \{0,1\} \qquad \forall k \in \mathcal{K} \qquad (3.6)$$

Note that, as each toll subpath is now represented by a single toll arc, the flow constraints (3.4) ensure that each commodity chooses either a toll path a ($x_a^k = 1$) or the toll free path ($x_{od}^k = 1$).

As for the classical Network Pricing Problem defined in Chapter 2, the constraint matrix associated with the lower level problem is unimodular. As a consequence, the lower level problem can be replaced by its primal dual constraints and optimality conditions, yielding a single level program involving complementarity (i.e., disjunctive) constraints. Further, in order to obtain a linear model, variables

$$p_a^k = \begin{cases} t_a & \text{if commodity } k \text{ uses arc } a \in \mathcal{A}, \\ 0 & \text{otherwise} \end{cases}$$

are introduced, corresponding to the actual unit profit associated with arc $a \in \mathcal{A}$ and commodity $k \in \mathcal{K}$. This yields (2004, Dewez [22]):

$$\text{(HP2)} \quad \max \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \eta^k p_a^k$$

subject to:

$$\sum_{a \in \mathcal{A}} x_a^k + x_{od}^k = 1 \qquad \forall k \in \mathcal{K} \qquad (3.7)$$

$$\lambda^k \leq c_a^k + t_a \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (3.8)$$

$$\lambda^k \leq c_{od}^k \qquad \forall k \in \mathcal{K} \qquad (3.9)$$

$$\sum_{a \in \mathcal{A}} \left(c_a^k x_a^k + p_a^k \right) + c_{od}^k x_{od}^k = \lambda^k \qquad \forall k \in \mathcal{K} \qquad (3.10)$$

$$p_a^k \leq M_a^k x_a^k \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (3.11)$$

$$t_a - p_a^k \leq N_a (1 - x_a^k) \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (3.12)$$

$$p_a^k \leq t_a \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (3.13)$$

$$p_a^k \geq 0 \qquad \forall a \in \mathcal{A} \qquad (3.14)$$

$$x_{od}^k \geq 0 \qquad \forall k \in \mathcal{K} \qquad (3.15)$$

$$x_a^k \in \{0, 1\} \qquad \forall a \in \mathcal{A}, \forall k \in \mathcal{K}, \qquad (3.16)$$

where M_a^k and N_a are suitably large constants. For now, let us assume $M_a^k = \max_{k \in \mathcal{K}} \{c_{od}^k - c_a^k\}$ and $N_a = N = \max_{k \in \mathcal{K}, a \in \mathcal{A}} M_a^k$.

Now, consider a network composed of three entry/exit nodes (labelled $1, 2, 3$) on the highway and two commodities $k_1, k_2 \in \mathcal{K}$ with respective demands $\eta^{k_1} = 80$, $\eta^{k_2} = 10$. The fixed costs on paths $c_a^k : k \in \mathcal{K}, a \in \mathcal{A}$ are described in Table 3.1, while $c_{od}^{k_1} = 20$ and $c_{od}^{k_2} = 21$. The corresponding optimal tolls, according to model (HP2), are given in Figure 3.3.

Toll arc a	$c_a^{k_1}$	$c_a^{k_2}$
$(1, 2)$	12	11
$(1, 3)$	15	14
$(2, 1)$	13	9
$(2, 3)$	17	15
$(3, 1)$	11	10
$(3, 2)$	12	10

Table 3.1: Fixed costs $c_a^k : k = k_1, k_2, a \in \mathcal{A}$ for a network example with three entry/exit nodes on the highway

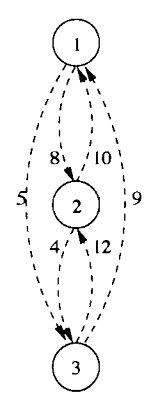

Figure 3.3: Optimal tolls $t_a : a \in \mathcal{A}$ for a network example with three entry/exit nodes on the highway

At optimality, commodity k_1 travels on the path $o^{k_1} \to 3 \to 1 \to d^{k_1}$, while commodity k_2 travels on the path $o^{k_2} \to 2 \to 1 \to d^{k_2}$. One can observe that $t_{21} = 10 < t_{31} = 9$, i.e., the toll imposed on the path $o^{k_2} \to 2 \to 1 \to d^{k_2}$ is less than the toll imposed on the path $o^{k_2} \to 3 \to 1 \to d^{k_2}$. While this can make sense in the airline industry, where tickets correspond to specific origin-destination pairs, this is unrealistic in a highway.

In order to prevent such situations, the Triangle and Monotonicity inequalities (3.17), (3.18) can be introduced, and the corresponding problem is called **"Constrained Complete Toll NPP"**.

$$t_a \leq t_b + t_c \qquad \forall a, b, c \in \mathcal{A} : \quad t(a) = t(b),\ h(b) = t(c),\ h(c) = h(a) \tag{3.17}$$

$$t_a \geq t_b \qquad \forall a, b \in \mathcal{A} : \quad t(a) = t(b) < h(a) = h(b) + 1 \text{ or } t(a) = t(b) - 1 < h(a) = h(b) \text{ or } t(a) = t(b) > h(a) = h(b) - 1 \text{ or } t(a) = t(b) + 1 > h(a) = h(b). \tag{3.18}$$

Triangle constraints ensure that between two given entry and exit nodes of the highway, a commodity would not take benefit from leaving the highway upstream and then reentering downstream latter. The Monotonicity constraints imply that the toll on a path cannot be less than the toll of any subpath. Subnetworks on which these inequalities apply are illustrated in Figure 3.4.

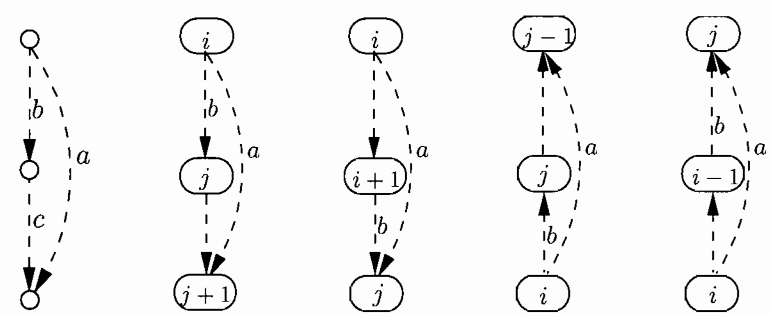

Figure 3.4: Subnetworks on which Triangle and Monotonicity constraints apply

In the next section, we propose an alternative formulation for the problem. The latter does not involve dual variables and allows to express the optimality of the lower level problem more explicitly in the constraints.

3.2 Model reformulation

While the lower level optimality conditions in (HP2) involve arc flow variables, an alternative is to express the optimality of the lower level problem in terms of path flows, without resorting to dual variables. The primal dual constraints and optimality conditions (3.8), (3.9) and (3.10) of (HP2) are then replaced by the equivalent

$$\sum_{a \in \mathcal{A}} (c_a^k x_a^k + p_a^k) + c_{od}^k x_{od}^k \leq c_b^k + t_b \qquad \forall k \in \mathcal{K}, \forall b \in \mathcal{A} \tag{3.19}$$

$$\sum_{a \in \mathcal{A}} (c_a^k x_a^k + p_a^k) + c_{od}^k x_{od}^k \leq c_{od}^k \qquad \forall k \in \mathcal{K}. \tag{3.20}$$

Indeed, these constraints ensure that the cost of the path chosen by commodity $k \in \mathcal{K}$ at optimality is smaller than (or equal to) the cost of any other path for this commodity.

However, the second family of constraints (3.20) is obviously redundant due to constraints (3.7), (3.11) and the definition of constants $M_a^k : k \in \mathcal{K}, a \in \mathcal{A}$. Next, based on constraint (3.7), variables x_{od}^k can be removed, yielding the more compact model:

$$\text{(HP3)} \quad \max \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \eta^k p_a^k$$

subject to:

$$\sum_{a \in \mathcal{A}} x_a^k \leq 1 \qquad \forall k \in \mathcal{K} \qquad (3.21)$$

$$\sum_{b \in \mathcal{A}} (p_b^k + c_b^k x_b^k) + c_{od}^k (1 - \sum_{b \in \mathcal{A}} x_b^k) \leq t_a + c_a^k \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (3.22)$$

$$p_a^k \leq M_a^k x_a^k \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (3.23)$$

$$t_a - p_a^k \leq N_a (1 - x_a^k) \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (3.24)$$

$$p_a^k \leq t_a \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \qquad (3.25)$$

$$p_a^k \geq 0 \qquad \forall a \in \mathcal{A} \qquad (3.26)$$

$$x_a^k \in \{0, 1\} \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A}, \qquad (3.27)$$

In the sequel, we consider two variants of this program. In the **General Complete Toll NPP (GCT-NPP)**, tolls are independent, while the **Constrained Complete Toll NPP (CCT-NPP)** imposes Triangle and Monotonicity constraints (3.17) and (3.18). The corresponding models are labelled (HP3) and (HP3*) respectively.

Unfortunately, these models contain a large set of variables, especially for describing flows on paths $x_a^k : k \in \mathcal{K}, a \in \mathcal{A}$. The next section provide suggestions to reduce the size of the problem, i.e., to set several flow variables to zero before solving the problem.

3.3 Preprocessing

Thanks to the complete toll subgraph structure, each feasible path from an origin to a destination contains a single toll arc, and there exists a bijection between the toll arc set for a commodity and the corresponding path set. Further, the paths that are never used by a given commodity can be deleted, i.e., the corresponding flow variables are set to zero.

Property For each commodity $k \in \mathcal{K}$, the toll arcs $a \in \mathcal{A}$ such that $c_a^k > c_{od}^k$ are never used, i.e., one can set $x_a^k = 0$.

For the Constrained Complete Toll NPP, an improved preprocessing can be applied according to the Monotonicity constraints. Let us introduce the following definition. An illustration of this definition is provided in Figure 3.5.

Definition 7 *For all a in $\mathcal{A}$, the following set is defined:*

$$\underline{C}_a = \{b \in \mathcal{A} : t(a) \leq t(b) < h(b) \leq h(a) \text{ or } t(a) \geq t(b) > h(b) \geq h(a)\}.$$

In the CCT-NPP, the toll variables must be such that $t_a \geq t_b$ for all b in $\underline{C}_a$. According to the following proposition, for each commodity, several additional paths (toll arcs) are never used, and the corresponding flow variables can be set to zero.

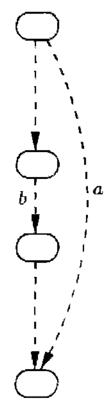

Figure 3.5: Example of a toll arc $b \in \underline{C}_a$

Proposition 1 *(Constrained Complete Toll NPP) Let* $a, b \in \mathcal{A}$ *be two toll arcs of the network such that* $b \in \underline{C}_a$. *If the fixed costs are such that* $c_b^k < c_a^k$ *for a given commodity* $k \in \mathcal{K}$, *then one can set* $x_a^k = 0$ *since the corresponding path is never used.*

Proof

The cost of the path containing $b \in \mathcal{A}$ is $c_b^k + t_b$ for commodity k, while the cost of the path containing $a \in \mathcal{A}$ is $c_a^k + t_a$. As the Monotonicity constraints impose that $t_a \geq t_b$, the cost of the path containing $a \in \mathcal{A}$ is always larger than the cost of the path containing $b \in \mathcal{A}$ for commodity k, and one can set $x_a^k = 0$. □

Finally, in order to complete the models (HP3) and (HP3*), the next section provides settings for the constants $M_a^k, N_a : k \in \mathcal{K}, a \in \mathcal{A}$.

3.4 Setting the constants M and N

For each commodity $k \in \mathcal{K}$ and each toll arc $a \in \mathcal{A}$, the constants M_a^k, N_a that appear in models (HP3) and (HP3*) represent upper bounds for p_a^k and t_a variables.

Constant M_a^k represents the largest toll that can be imposed on the toll arc $a \in \mathcal{A}$ for commodity $k \in \mathcal{K}$. Dewez (2004, [22]) shows that, for all $k \in \mathcal{K}$ and for all $a \in \mathcal{A}$, constants $M_a^k = \max\{0, c_{od}^k - c_a^k\}$ are valid for (HP3*), thus also for (HP3). Indeed, the network users travel on shortest paths with respect to a cost equal to the sum of tolls and initial costs. Hence it is clear that the largest toll that can be imposed on a toll arc a for commodity k is equal to the difference between the cost of the toll free path and the cost of the toll path a for commodity k.

Next, constant N_a represents the largest toll that can be set on the toll arc $a \in \mathcal{A}$ among all commodities. Depending on whether the Constrained or General Complete Toll NPP is considered, different settings are applied for these constants.

In what concerns the General Complete Toll NPP, the constants $N_a = \max_{k \in \mathcal{K}} M_a^k$ for all $a \in \mathcal{A}$ are clearly valid for (HP3). Unfortunately, when adding Triangle and Monotonicity constraints to the problem, the previous settings for $N_a : a \in \mathcal{A}$ are no longer valid. By contradiction, assume $N_a = \max_{k \in \mathcal{K}} M_a^k$ for all $a \in \mathcal{A}$, and consider $b \in \underline{C}_a$. If $\max_{k \in \mathcal{K}} M_a^k < \max_{k \in \mathcal{K}} M_b^k$, there does not exist any feasible solution of the problem which satisfies the Monotonicity constraint $t_a \geq t_b$. As a consequence, for the Constrained Complete Toll NPP, the constants $N_a = N = \max_{k \in \mathcal{K}, a \in \mathcal{A}} M_a^k$ are valid for (HP3*).

With these settings, models (HP3) and (HP3*) can be implemented efficiently. We refer the reader to Chapter 6 for numerical results. In the next section, we prove that the Constrained and General Complete Toll NPP are $\mathcal{NP}$-hard.

3.5 Complexity

As a highway network can take several specific formats, we distinguish three cases. First, we consider a single directional highway network, i.e., all toll arcs are oriented in the same direction: $t(a) < h(a)$ for all $a \in \mathcal{A}$ or $t(a) > h(a)$ for all $a \in \mathcal{A}$. Next, we consider a more general bi-directional highway network, in which toll arcs appear in both directions of the highway. Finally, we also consider a third case, in which the bi-directional highway network contains feasible access from all origins to all entry nodes of the highway and from all exit nodes of the highway to all destinations.

It has been shown by Dewez (2004, [22]) that the Constrained Complete Toll Network Pricing Problem is polynomially solvable when either a single commodity or a single toll arc is involved. In contrast, we prove that the Constrained Complete Toll NPP is $\mathcal{NP}$-hard, using a reduction from $3-SAT$. Note that similar reductions have been used by Roch et al. (2005, [57]) for the Network Pricing Problem or Grigoriev et al. (2005, [32]) for the Cross River Network Pricing Problem, which are also based on a reduction from $3 - SAT$.

Definition 8 *Let $x_1, ..., x_n$ be n Boolean variables, and $F = \wedge_{i=1}^{m} (l_{i1} \vee l_{i2} \vee l_{i3})$ be a conjunctive normal form of m clauses with literals $l_{ij} : j = 1, 2, 3$ that represent a variable x_i or its negation. Given a such conjunctive normal form, $3 - SAT$ consists in finding an assignment of value TRUE or FALSE to the variables such that the formula is TRUE.*

Proposition 2 *The single directional Constrained Complete Toll NPP is $\mathcal{NP}$-hard.*

Proof

Any conjunctive normal form $F = \wedge_{i=1}^{m} (l_{i1} \vee l_{i2} \vee l_{i3})$, where $l_{ij} : j = 1, 2, 3$, represents a variable $x_i : i \in \{1, ..., n\}$ or its negation, can be polynomially converted to an instance of the Constrained Complete Toll NPP, in its decision form.

For each variable $x_i : i \in \{1, ..., n\}$, a subnetwork is constructed as shown in Figure 3.6.

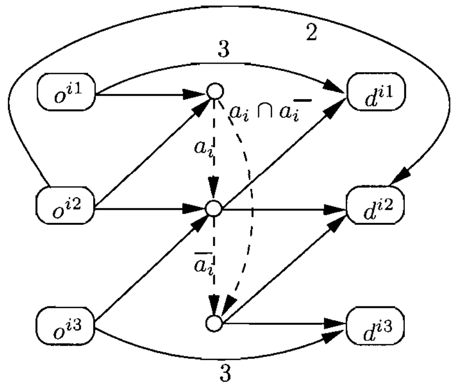

Figure 3.6: Subnetwork for variable x_i (single directional Constrained Complete Toll NPP).

Each of these subnetworks contains three commodities $(o^{ij}, d^{ij}) : j \in \{1, 2, 3\}$ with unit demand, and two toll arcs $a_i, \overline{a}_i$ of zero fixed cost corresponding to the truth and false assignment for variable x_i respectively. Further, in any subnetwork, an arc is added from the tail node of arc a_i to the head node of arc $\overline{a}_i$, which corresponds to $a_i \cap \overline{a}_i$. Toll free arcs of cost zero connect o^{i1} (resp. o^{i3}) to the tail node of arc a_i (resp. $\overline{a}_i$), the head node of arc a_i (resp. $\overline{a}_i$) to d^{i1} (resp. d^{i3}), o^{i2} to both tail nodes of toll arcs, and both head nodes of toll arcs to d^{i2}. Toll free arcs (o^{i1}, d^{i1}) of cost 3, (o^{i3}, d^{i3}) of cost 3 and (o^{i2}, d^{i2}) of cost 2 are also added.

Hence an upper bound on the revenue for each subnetwork is 7, obtained by setting to 2 the toll on either a_i or $\overline{a}_i$ and the other ones to 3. In all other cases, the revenue cannot exceed 6. Then, the subnetworks are linked together so that the single directional highway corresponds to the set of all connected subnetworks.

Note that the toll of 3 on arc $a_i \cap \overline{a}_i$ ensures that this arc is never taken. Indeed, suppose that commodity $i2$ chooses this arc $a_i \cap \overline{a}_i$ (there is no path using this arc for commodities $i1$ and $i3$). As the revenue on $i2$ is bounded by a toll free arc of cost 2, the toll on the arc $a_i \cap \overline{a}_i$ must be smaller or equal to 2. But then, due to the Monotonicity constraints added to the problem, tolls on the other two arcs of the subnetworks cannot exceed 2 and the maximal revenue of 7 cannot be reached, which is a contradiction. Also note that the toll free arcs that do not appear from some origins to tail nodes of toll arcs (resp. from head nodes of toll arcs to some destinations) are supposed to be so expensive that they can never be used and they are not depicted in the network graph.

Further, for each clause k, a clause-commodity (o^k, d^k) with unit demand is constructed as depicted in Figure 3.7.

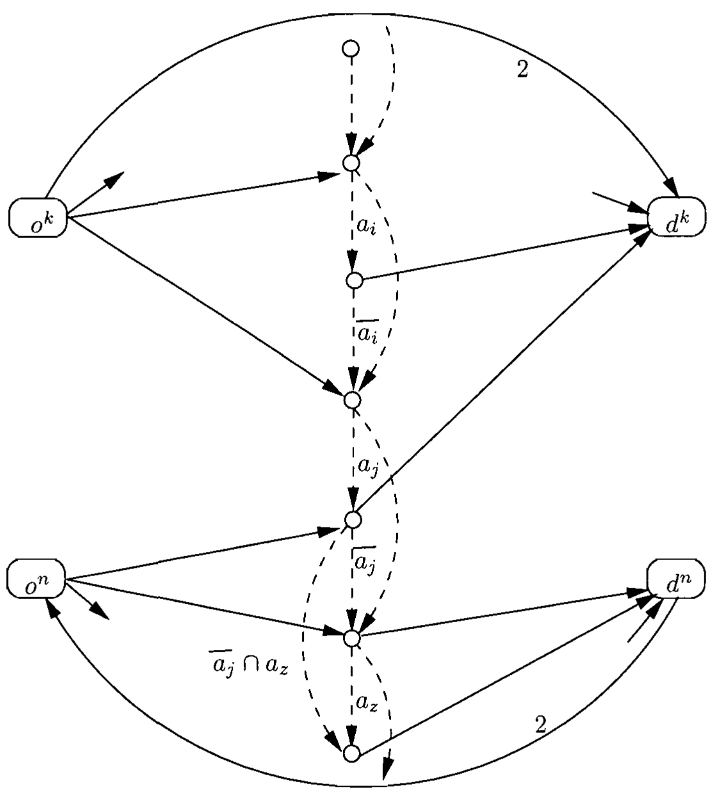

Figure 3.7: Part of network for $F = (\ldots \vee x_i \vee x_j) \wedge (\overline{x_j} \vee x_z \vee \ldots) \wedge \ldots$ (single directional Constrained Complete Toll NPP).

If variable x_i (resp. $\overline{x}_i$) is a literal of clause k, toll free arcs of cost 0 are added from

o^k to the tail node of a_i (resp. $\overline{a}_i$) and from the head node of a_i (resp. $\overline{a}_i$) to d^k, which means that toll arc a_i belongs to the clause-commodity k. Further, if two or three toll arcs are connected, in the same direction (in the highway graph) and belong to the same clause-commodity, toll arcs are added between the corresponding entry and exit nodes of the network. For the part of network example depicted in Figure 3.7, a single toll arc is added between the tail node of $\overline{a}_j$ and the head node of a_z. Those additional toll arcs correspond to the intersection of corresponding variables, i.e., for the example the toll arc is $\overline{a}_j \cap a_z$. Note that there is no arc linking tail and head nodes of toll arcs belonging to different clause-commodities, because there does not exist any path which could include them both. Tolls on the new arcs $\overline{a}_j \cap a_z$ are set to 2 if $\overline{a}_j$ and a_z are both set to 2, and to 3 otherwise. Finally, an additional toll free arc (o^k, d^k) of cost 2 is added, which defines an upper bound of 2 on the revenue from each clause-commodity.

Now let us show that a satisfying truth assignment for F exists if and only if the revenue for the Constrained Complete Toll NPP is equal to $2m + 7n$, where m is the number of clauses and n is the number of variables.

Assume there exists a satisfying truth assignment, which means that at least one literal is true in each clause. We set the corresponding tolls to 2, and the other tolls (in the corresponding subnetworks) to 3. Then the total revenue from all clause-commodities is $2m$. For all remaining subnetworks, if any (i.e., this situation only happens if a variable x_i does not appear in any clause), the toll arcs are set arbitrarily to 2 and 3 for a variable and its negation respectively. Hence the revenue from all subnetworks is $7n$, which means that the total revenue is $2m + 7n$. Conversely, suppose there exists tolls such that the total revenue is $2m + 7n$. The maximal possible revenue from all subnetworks is $7n$, only achievable by setting one toll per subnetwork to 2 and the other ones to 3. On the other hand, the maximal possible revenue from all clause-commodities is $2m$. We set to true the literals corresponding to arcs with toll 2, and to false the others. This corresponds to a well-defined assignment for F, since there is exactly one toll of 2 in each subnetwork. Further, each clause-commodity contributes to the total revenue with a toll of 2, which means that at least one literal per clause is true, and there exists a truth assignment for F. □

Next, we extend the previous result to a more general highway network, which contains toll arcs in both directions of the highway.

Proposition 3 *The bi-directional Constrained Complete Toll NPP is $\mathcal{NP}$-hard.*

Proof

As toll arcs can now appear in both direction of the highway, subnetworks for variables $x_i : i \in \{1, ..., n\}$ are constructed in a slightly different way, as shown in Figure 3.8.

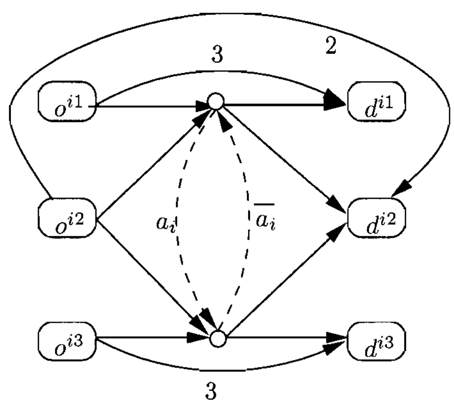

Figure 3.8: Subnetwork for variable x_i.

We consider that toll arcs $a_i : i \in \{1, ..., n\}$ are in one direction of the highway, while toll arcs $\overline{a}_i : i \in \{1, ..., n\}$ are in the other direction.

Further, for each clause-commodity and as for the single directional graph, if two or three connected and with same direction (in the highway graph) toll arcs belong to the same clause-commodity, toll arcs are added between the corresponding entry and exit nodes of the network. Such a network is depicted in Figure 3.9.

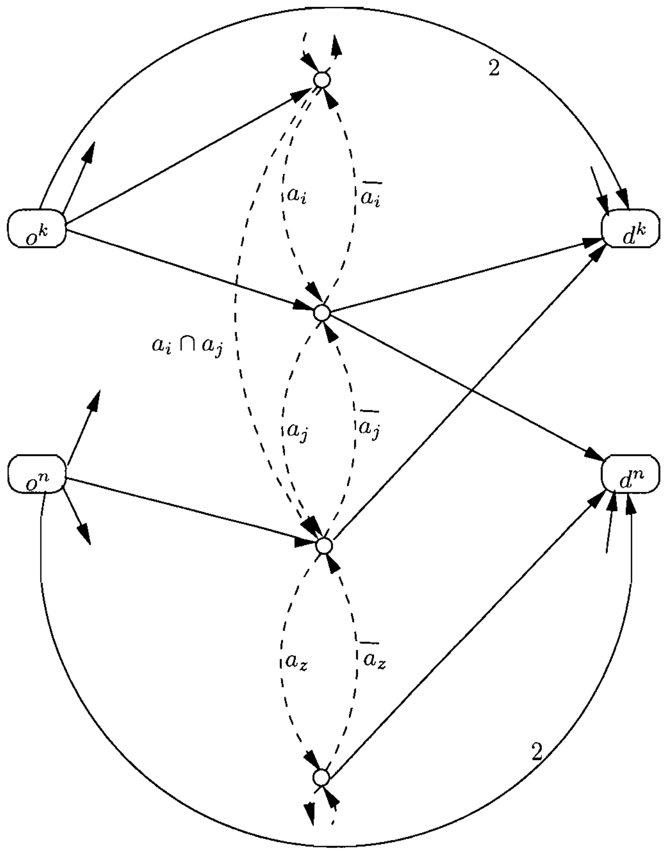

Figure 3.9: Subnetwork for $F = (... \vee x_i \vee x_j) \wedge (\overline{x_j} \vee x_z \vee ...) \wedge ...$ (bi-directional Constrained Complete Toll NPP).

As the users are supposed to choose one direction or the other of the highway, no toll arc is added between the tail node of a toll arc in one direction of the highway network to the head node of another toll arc in the other direction of the highway network. Then, the same argument as before can be applied. □

Note that the problem instances constructed in both preceding proofs contain non feasible access from several origins to the highway, or from the highway to several destinations. It means that the corresponding path are so expensive that they could never be taken. Hence we consider a third case, in which the bi-directional highway network contains feasible access from all origins to all entry nodes of the highway and from all exit nodes of the highway to all destinations.

Proposition 4 *The bi-directional Constrained Complete Toll NPP is $\mathcal{NP}$-hard, even if access to all entry nodes (resp. from all exit nodes) is feasible from all origins (resp. to all destinations).*

Proof

This additional condition means that no path is so expensive that it could never be taken, which is slightly different from the situation described before.

Subnetworks are constructed as before, except that several additional toll free arcs (the ones that were too expensive) are added so that there is one toll free arc from any origin to any tail node of a toll arc, and from any head node of a toll arc to any destination. For each commodity k and for each toll arc a_i, the cost on the arcs $(o^k, t(a_i))$ and $(h(a_i), d^k)$ are set such that the sum of both fixed costs is equal to the cost of the toll free arc (o^k, d^k). Such a subnetwork is depicted in Figure 3.10.

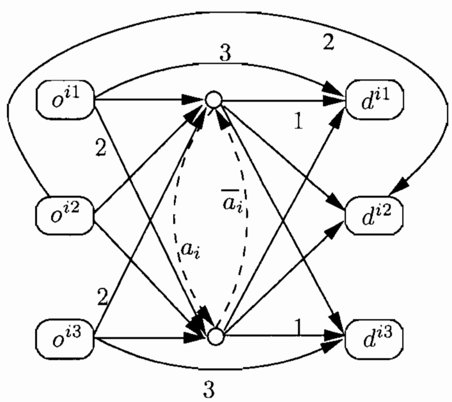

Figure 3.10: Subnetwork for variable x_i (All feasible access Constrained Complete Toll NPP)

Hence the new arcs can only be used if tolls are set to zero on the corresponding arcs, which does not lead to a maximal revenue for the leader. Then, tolls are set as before and the same argument can be applied. □

It is clear that very similar proofs can be derived for the General Complete Toll NPP. For space considerations, we omit the proof that the Basic NPP is also $\mathcal{NP}$-hard (see the Appendix for a detailed proof).

CHAPTER 4

VALID INEQUALITIES

Solving large mixed integer linear programs is hard. Hence, a common practice consists of appending valid inequalities to the initial model. These can, sometimes but not ever, help to solve 'faster or better' the initial problem. By 'faster or better', we mean to decrease (i) the computing time to solve the problem, (ii) the number of nodes in the branch and cut algorithm, or (iii) the gap between the optimal solution value of the linear relaxation (i.e., $0 \leq x_a^k \leq 1$ in our problem) and the optimal solution value of the integer problem.

In this chapter, we propose several new valid inequalities for the Constrained Complete Toll NPP and General Complete Toll NPP. We used the open source software Porta (see http://www.zib.de/Optimization/Software/Porta/ for more details), which offers a collection of routines for analysing polyhedra. More specifically, given a set of inequalities, the software returns the set of vertices and extreme rays of the corresponding polyhedron. Conversely, given a set of vertices and extreme rays of a polyhedron, it returns the corresponding facet equations. Using this specific routine, we obtain the facet equations for very small instances[1] of the CCT-NPP and GCT-NPP. Next, we generalize some of these facet equations, in order to obtain valid inequalities for a multi-commodity problem. The efficiency of the valid inequalities will be examined, both theoretically and numerically, in Chapters 6 and 7.

[1]By very small instances, we mean a single commodity and at most five entry and exit nodes on the highway for the CCT-NPP, and at most two commodities and five entry and exit nodes for the GCT-NPP. Indeed, as the number of commodities or toll arcs increase, so does the number of variables and constraints of the problem.

In the next sections, we propose valid inequalities that strengthen the 'Shortest Path' constraints (3.22) and 'Profit Upper Bound' constraints (3.23) of (HP3) and (HP3*).

4.1 Strengthening the Shortest Path Inequalities

The valid inequalities presented in this section involve a pair $k_1, k_2 \in \mathcal{K}$ of commodities, and strengthen the 'Shortest Path' constraints (3.22) of (HP3). Recall that these constraints ensure that the cost of the path chosen by commodity k_1 at optimality is smaller than (or equal to) the cost of any other path for this commodity. The valid inequalities proposed state that the cost of the path chosen by commodity k_1 also depends on the choice of a path b for commodity k_2.

Proposition 5 *Inequalities*

$$\sum_{b\in\mathcal{A}} (p_b^{k_1} + c_b^{k_1} x_b^{k_1}) + c_{od}^{k_1}(1 - \sum_{b\in\mathcal{A}} x_b^{k_1}) \leq t_a + c_a^{k_1} + \sum_{b\in\mathcal{A}\setminus(\mathcal{S}\cup\{a\})} (p_b^{k_2} + (c_b^{k_1} - c_a^{k_1}) x_b^{k_2}) \tag{4.1}$$

$$\sum_{b\in\mathcal{A}} (p_b^{k_1} + c_b^{k_1} x_b^{k_1}) + c_{od}^{k_1}(1 - \sum_{b\in\mathcal{A}} x_b^{k_1}) \leq c_{od}^{k_1} + \sum_{b\in\mathcal{A}\setminus\mathcal{S}} (p_b^{k_2} + (c_b^{k_1} - c_{od}^{k_1}) x_b^{k_2}) \tag{4.2}$$

where $k_1, k_2 \in \mathcal{K}, a \in \mathcal{A}$ and $\mathcal{S}$ is any subset of $\mathcal{A}$ (possibly the empty set), are valid for CCT-NPP and GCT-NPP.

Proof

Let $k_1, k_2 \in \mathcal{K}$ and $\mathcal{S} \subseteq \mathcal{A}$. If $x_b^{k_1} = 0$ for all $b \in \mathcal{A}$, then:

- If there exists $b \in \mathcal{A} \setminus (\mathcal{S} \cup \{a\})$ such that $x_b^{k_2} = 1$, (4.1)-(4.2) yield $c_{od}^{k_1} \leq t_a + p_b^{k_2} + c_b^{k_1}$ for all $a \in \mathcal{A}$ and $c_{od}^{k_1} \leq p_b^{k_2} + c_b^{k_1}$ respectively. As $p_b^{k_2} = t_b$ by (3.24) and (3.25), the inequalities imply that the cost of the path containing

$b \in \mathcal{A}$ must be larger than the cost of the toll free path for commodity k_1, and are valid by (3.22) and (3.26).

- In all other cases, (4.1)-(4.2) yield $c_{od}^{k_1} \leq t_a + c_a^{k_1}$ for all $a \in \mathcal{A}$ and $c_{od}^{k_1} \leq c_{od}^{k_1}$ respectively, which are valid by (3.22).

Now assume that there exists $b \in \mathcal{A}$ such that $x_b^{k_1} = 1$.

- If there exists $d \in \mathcal{A} \setminus (\mathcal{S} \cup \{a\})$ such that $x_d^{k_2} = 1$, (4.1)-(4.2) yield $p_b^{k_1} + c_b^{k_1} \leq t_a + p_d^{k_2} + c_d^{k_1}$ for all $a \in \mathcal{A}$ and $p_b^{k_1} + c_b^{k_1} \leq p_d^{k_2} + c_d^{k_1}$ respectively. As $p_b^{k_1} = t_b$ and $p_d^{k_2} = t_d$ by (3.24) and (3.25), the inequalities mean that the path containing $b \in \mathcal{A}$ must be cheaper than the path containing $d \in \mathcal{A}$ for commodity k_1, and are valid by (3.22) and (3.26).
- In all other cases (i.e., if there does not exist any $d \in \mathcal{A} \setminus (\mathcal{S} \cup \{a\})$ such that $x_d^{k_2} = 1$), (4.1)-(4.2) become $p_b^{k_1} + c_b^{k_1} \leq t_a + c_a^{k_1}$ for all $a \in \mathcal{A}$ and $p_b^{k_1} + c_b^{k_1} \leq c_{od}^{k_1}$ respectively. This means that the path containing $b \in \mathcal{A}$ must be cheaper than any other path for commodity k_1, and are valid by (3.22). □

Note that the Strengthened Shortest Path inequalities (4.1)-(4.2) still hold when $k_1 = k_2 = k$. In this case, they become

$$\sum_{b \in \mathcal{A}} (p_b^k + c_b^k x_b^k) + c_{od}^k (1 - \sum_{b \in \mathcal{A}} x_b^k) \leq t_a + c_a^k + \sum_{b \in \mathcal{A} \setminus (\mathcal{S} \cup \{a\})} (p_b^k + (c_b^k - c_a^k) x_b^k) \quad (4.3)$$

$$\sum_{b \in \mathcal{A}} (p_b^k + c_b^k x_b^k) + c_{od}^k (1 - \sum_{b \in \mathcal{A}} x_b^k) \leq c_{od}^k + \sum_{b \in \mathcal{A} \setminus \mathcal{S}} (p_b^k + (c_b^k - c_{od}^k) x_b^k) \quad (4.4)$$

for all $k \in \mathcal{K}$, for all $a \in \mathcal{A}$ and for all $\mathcal{S} \subseteq \mathcal{A}$. However, (4.4) can be equivalently rewritten as:

$$\sum_{b \in \mathcal{S}} p_b^k \leq \sum_{b \in \mathcal{S}} (c_{od}^k - c_b^k) x_b^k \qquad \forall k \in \mathcal{K}, \forall \mathcal{S} \subseteq \mathcal{A}$$

which are redundant by constraints (3.23).

Any choice for the set $\mathcal{S}$ is valid. Therefore the number of inequalities (4.1)-(4.2) is exponential and an efficient separation procedure is required. Let $(\bar{\mathbf{t}}, \bar{\mathbf{p}}, \bar{\mathbf{x}})$ be a current fractional solution of (HP3). For each commodity $k_1 \in \mathcal{K}$, the separation problem consists in determining a commodity $k_2 \in \mathcal{K}$, a toll arc $a \in \mathcal{A}$ and a subset $\mathcal{S}$ of $\mathcal{A}$ such that the corresponding inequalities (4.1)-(4.2) are most violated, i.e., minimizing the respective right hand sides of these inequalities. For (4.1), we only consider inequalities such that $c_b^{k_1} \leq c_a^{k_1}$ for all $b \in \mathcal{A} \setminus (\mathcal{S} \cup \{a\})$, as only these are stronger than the Shortest Path constraints (3.22). Further, as we would like to test the efficiency of both inequalities (4.1) and (4.2), we devise a separation procedure for each family.

For each commodity $k_1 \in \mathcal{K}$, the separation procedure is the following one. First, sort the toll arcs so that $c_1^k \leq \ldots \leq c_n^k \leq c_{od}^k$, where n is the number of toll arcs in the network. Our goal is to construct the right hand sides of (4.1) and (4.2) as small as possible. Hence, for a given commodity k_2 and a given toll arc a, the toll arcs in $\mathcal{A} \setminus (\mathcal{S} \cup \{a\})$ (resp. $\mathcal{A} \setminus \mathcal{S}$) are these for which:

$$\bar{p}_b^{k_2} + (c_b^{k_1} - c_a^{k_1})\bar{x}_b^{k_2} \leq 0 \quad \text{i.e.,} \quad \frac{\bar{p}_b^{k_2} + c_b^{k_1}\bar{x}_b^{k_2}}{\bar{x}_b^{k_2}} \leq c_a^{k_1} \quad \left(\text{resp. } \frac{\bar{p}_b^{k_2} + c_b^{k_1}\bar{x}_b^{k_2}}{\bar{x}_b^{k_2}} \leq c_{od}^{k_1}\right). \tag{4.5}$$

Hence, for any commodity $k_2 \in \mathcal{K}$, each toll arc $b \in \mathcal{A}$ is assigned to a node of a singly-linked list so that the corresponding values $\frac{\bar{p}_b^{k_2} + c_b^{k_1}\bar{x}_b^{k_2}}{\bar{x}_b^{k_2}}$ are sorted in increasing order, i.e., each node of this list contains a toll arc as data and points to the node which contains the toll arc $b \in \mathcal{A}$ with the next largest term $\frac{\bar{p}_b^{k_2} + c_b^{k_1}\bar{x}_b^{k_2}}{\bar{x}_b^{k_2}}$.

Then, for each commodity k_2 and for toll arcs a going from 1 to n (i.e., sorted in increasing order), the sets $\mathcal{A} \setminus \mathcal{S}$ are constructed sequentially in order to obtain the smallest values for the respective right hand sides of inequalities (4.1)-(4.2) corresponding to commodity k_1. Note that, for a given commodity k_2, the smallest right hand side values of (4.1) do not differ very much from a given toll arc a to the next one $a+1$ (in terms of increasing order of the corresponding fixed costs), as if $a \in \mathcal{A} \setminus \mathcal{S}$ for a given toll arc, then $a+1 \in \mathcal{A} \setminus \mathcal{S}$ by (4.5). Finally, we deduce the smallest values for the right hand sides of inequalities (4.1) and (4.2) among all $k_2 \in \mathcal{K}$. For each commodity $k_1 \in \mathcal{K}$, this separation procedure can be computed in $O(kn \log n)$, where k and n are the number of commodities and toll arcs respectively.

We also derive other valid inequalities for the CCT-NPP and GCT-NPP, that tighten the 'Profit Upper Bound' constraints (3.23). These are presented in the next section.

4.2 Strengthening the Profit Upper Bound Inequalities

In this section, we propose valid inequalities for the CCT-NPP and GCT-NPP that strengthen the 'Profit Upper Bound' constraints (3.23). For any pair of commodities $k_1, k_2 \in \mathcal{K}$ and any toll arc $a \in \mathcal{A}$, these ones link the profit variables $p_a^{k_1}$, $p_a^{k_2}$ with flows on both commodities x^{k_1} and x^{k_2}.

First note that, due to the fixed costs $c_a^k : k \in \mathcal{K}, a \in \mathcal{A}$, any toll arc for a given commodity $k_1 \in \mathcal{K}$ cannot be chosen alongside with any other toll arc for another commodity $k_2 \in \mathcal{K}$. More formally, we have the following definition.

Definition 9 *For any pair of commodities $k_1, k_2 \in \mathcal{K}$, two toll arcs $a, b \in \mathcal{A}$ are said to be compatible for k_1 and k_2 respectively if there exists a feasible solution of (HP3) (resp. (HP3*)) such that $x_b^{k_1} = 1$ and $x_a^{k_2} = 1$. For clarity of notation, this is labelled $(b, k_1) \sim (a, k_2)$ (and $(b, k_1) \not\sim (a, k_2)$ otherwise).*

The main constraints of (HP3) (resp. (HP3*)) that influence compatibility are the Shortest Path constraints (3.22). The following lemma states a necessary condition such that two toll arcs are compatible for a pair of commodities.

Lemma 1 *For any pair of commodities $k_1, k_2 \in \mathcal{K}$ and for any toll arcs $a, b \in \mathcal{A}$, a necessary condition so that $(b, k_1) \sim (a, k_2)$ is*

$$c_a^{k_2} - c_a^{k_1} \leq c_b^{k_2} - c_b^{k_1}.$$

Proof

$x_b^{k_1} = 1 = x_a^{k_2}$ implies that $t_b + c_b^{k_1} \leq t_a + c_a^{k_1}$ and $t_a + c_a^{k_2} \leq t_b + c_b^{k_2}$ by the Shortest Path constraints (3.22). This yields $c_b^{k_1} - c_a^{k_1} \leq t_a - t_b \leq c_b^{k_2} - c_a^{k_2}$, which proves that the condition is necessary. □

For the GCT-NPP (i.e., model (HP3)), and assuming that the network has been presolved according to the Property mentioned in Section 3.3, the necessary condition mentioned above is also sufficient to allow compatibility between two toll arcs for a pair of commodities.

Lemma 2 *For any pair of commodities $k_1, k_2 \in \mathcal{K}$ and for any toll arcs $a, b \in \mathcal{A}$ such that $c_b^{k_1} \leq c_{od}^{k_1}$ (resp. $c_b^{k_2} \leq c_{od}^{k_2}$) and $c_a^{k_2} \leq c_{od}^{k_2}$ (resp. $c_a^{k_1} \leq c_{od}^{k_1}$), a necessary and sufficient condition so that $(b, k_1) \sim (a, k_2)$ for (HP3) is*

$$c_a^{k_2} - c_a^{k_1} \leq c_b^{k_2} - c_b^{k_1}.$$

Proof

As we already know that the condition is necessary, let us proof that it is sufficient. Assuming that $c_b^{k_2} - c_a^{k_2} \geq 0$, point $x_b^{k_1} = 1 = x_a^{k_2}$, $t_b = p_b^{k_1} = 0$, $t_a = p_a^{k_2} = c_b^{k_2} - c_a^{k_2}$

and $t_d = N_d$ for all $d \in \mathcal{A} \setminus \{a, b\}$ is a feasible solution of (HP3). Indeed, the Shortest Path constraints (3.22) imply that

$$p_b^{k_1} + c_b^{k_1} \leq t_a + c_a^{k_1}$$
$$p_a^{k_2} + c_a^{k_2} \leq t_b + c_b^{k_2},$$

thus also

$$c_b^{k_1} \leq c_b^{k_2} - c_a^{k_2} + c_a^{k_1}$$
$$c_b^{k_2} - c_a^{k_2} + c_a^{k_2} \leq c_b^{k_2}$$

which are valid since $c_a^{k_2} - c_a^{k_1} \leq c_b^{k_2} - c_b^{k_1}$. The other Shortest Path constraints are clearly valid since variables t_d have been set sufficiently large for all $d \in \mathcal{A} \setminus \{a, b\}$. Further, $c_b^{k_2} \leq c_{od}^{k_2}$ ensures that $p_a^{k_2} \leq M_a^{k_2}$, i.e., constraints (3.23) are satisfied.
In the same way, if $c_b^{k_2} - c_a^{k_2} < 0$, point $x_b^{k_1} = 1 = x_a^{k_2}$, $t_a = 0$, $t_b = c_a^{k_2} - c_b^{k_2}$ and $t_d = N_d$ for all $d \in \mathcal{A} \setminus \{a, b\}$ is a feasible solution of (HP3). □

Next, given the profit upper bounds $M_a^k : k \in \mathcal{K}, a \in \mathcal{A}$, the flow values x^{k_1} for a commodity $k_1 \in \mathcal{K}$ give information about the feasible flow values for another commodity $k_2 \in \mathcal{K}$. This is expressed in the following lemma.

Lemma 3 *For any pair of commodities $k_1, k_2 \in \mathcal{K}$ and for any toll arc $a \in \mathcal{A}$ such that $M_a^{k_2} \geq M_a^{k_1}$, $x_a^{k_1} = 1$ implies that there exists $b \in \mathcal{A}$ such that $x_b^{k_2} = 1$, with total cost $t_b + c_b^{k_2} \leq t_a + c_a^{k_2}$.*

Proof

As $x_a^{k_1} = 1$, one should have $t_a = p_a^{k_1} \leq M_a^{k_1}$ by (3.23), (3.24) and (3.25). Hence $t_a \leq M_a^{k_2}$, which means that the path containing toll arc $a \in \mathcal{A}$ is cheaper than the toll free path for commodity k_2. □

Now we can present several valid inequalities that link the profit variables $p_a^{k_1}$, $p_a^{k_2}$ with flows on both commodities x^{k_1} and x^{k_2}. They exploit the notion of compatibility defined above and allow to strengthen the Profit Upper Bound constraints (3.23). These rely on the following sets.

Definition 10 *For any pair of commodities $k_1, k_2 \in \mathcal{K}$ and for any toll arc $a \in \mathcal{A}$, we define the set*

$$\mathcal{A}_a^{>} = \{b \in \mathcal{A} : c_b^{k_2} - c_b^{k_1} > c_a^{k_2} - c_a^{k_1}\} \tag{4.6}$$

which contain the toll arcs $b \in \mathcal{A}$ such that $(b, k_1) \sim (a, k_2)$ and $(b, k_2) \not\sim (a, k_1)$. The complementary subset $\mathcal{A} \setminus \mathcal{A}_a^{>}$ is denoted $\mathcal{A}_a^{\leq}$.

First we propose valid inequalities that link a given Profit Upper Bound constraint $p_a^{k_2} \leq M_a^{k_2} x_a^{k_2}$ with (i) the flow variables $x_b^{k_1} : b \in \mathcal{A}$ for another commodity k_1 and (ii) the remaining flow variables $x_b^{k_2} : b \in \mathcal{A}$ for commodity k_2.

Proposition 6 *Consider a pair of commodities $k_1, k_2 \in \mathcal{K}$, a toll arc $a \in \mathcal{A}$ such that $M_a^{k_1} \leq M_a^{k_2}$ and $b^* = \arg\min_{b \in \mathcal{A}_a^{>} : M_b^{k_2} \geq M_b^{k_1}} \{c_b^{k_1} - c_b^{k_2}\}$. Under the assumption that there does not exist any $b \in \mathcal{A} \setminus \{a\}$ such that $c_a^{k_2} - c_a^{k_1} = c_b^{k_2} - c_b^{k_1}$, the following inequalities are valid for CCT-NPP and GCT-NPP:*

$$p_a^{k_2} \leq M_a^{k_2} x_a^{k_2} + \left(M_a^{k_2} - M_a^{k_1}\right) \left(\sum_{b \in \mathcal{A}_a^{\leq} \setminus \{a\}} (x_b^{k_2} - x_b^{k_1}) - x_a^{k_1} \right) \tag{4.7}$$

$$\begin{aligned} p_a^{k_2} \leq M_a^{k_2} x_a^{k_2} &+ \left(M_a^{k_2} - M_a^{k_1}\right) \left(\sum_{b \in \mathcal{A}_a^{\leq} \setminus \{a\}} (x_b^{k_2} - x_b^{k_1}) - x_a^{k_1} \right) \\ &+ \left(M_{b^*}^{k_2} - M_{b^*}^{k_1}\right) \sum_{b \in \mathcal{A}_a^{>} : M_b^{k_2} \geq M_b^{k_1}} (x_b^{k_2} - x_b^{k_1}), \end{aligned} \tag{4.8}$$

Note that, for given $k_1, k_2 \in \mathcal{K}$ and $a \in \mathcal{A}$, the inequality (4.7) (resp. (4.8)) is non redundant if and only if $x_b^{k_1} = 1 = x_a^{k_2}$ for $b = a$ (resp. $b = a$ or $b \in \mathcal{A}_a^>$), and helps to restrain the upper bound on $p_a^{k_2}$ in this case.

Proof

Let $k_1, k_2 \in \mathcal{K}$ and $a \in \mathcal{A}$ be fixed. If $x_a^{k_2} = 0$ for all $a \in \mathcal{A}$, then $x_a^{k_1} = 0$ for all $a \in \mathcal{A}$ such that $M_a^{k_2} \geq M_a^{k_1}$ by Lemma 3. This yields $0 \leq 0$.

Now assume that $x_a^{k_2} = 1$. The case $x_b^{k_1} = 1$ with $b \in \mathcal{A}_a^{\leq} \setminus \{a\}$ is avoided by Lemma 2 and the fact that there does not exists any $b \in \mathcal{A} \setminus \{a\}$ such that $c_a^{k_2} - c_a^{k_1} = c_b^{k_2} - c_b^{k_1}$. Otherwise:

- If $x_a^{k_1} = 1$, inequalities (4.7)-(4.8) become $p_a^{k_2} \leq M_a^{k_1}$, which is valid by (3.23), (3.24), (3.25).

- If $x_b^{k_1} = 1$ with $b \in \mathcal{A}_a^>$ such that $M_b^{k_2} \geq M_b^{k_1}$, (4.7)-(4.8) yield $p_a^{k_2} \leq M_a^{k_2}$ and $p_a^{k_2} \leq M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_a^{k_2}$. The first inequality is valid by (3.23). For the second inequality, constraint (3.22) imposes that $p_a^{k_2} + c_a^{k_2} \leq t_b + c_b^{k_2}$. Further, $p_b^{k_1} \leq M_b^{k_1}$ by (3.23). As $t_b = p_b^{k_1}$ by (3.24) and (3.25), one has $p_a^{k_2} \leq M_b^{k_1} + c_b^{k_2} - c_a^{k_2}$. The result follows from the definition of b^*.

- In all other cases, one obtains $p_a^{k_2} \leq M_a^{k_2}$, which is valid by (3.23).

If $x_b^{k_2} = 1$ with $b \in \mathcal{A}_a^{\leq} \setminus \{a\}$ and $x_a^{k_1} = 0$ for all $a \in \mathcal{A}$ or $x_d^{k_1} = 1$ with $d \in \mathcal{A}_a^>$: $M_d^{k_2} < M_d^{k_1}$, inequalities (4.7)-(4.8) become $0 \leq M_a^{k_2} - M_a^{k_1}$. If $x_d^{k_1} = 1$ with $d \in \mathcal{A}_a^>$ such that $M_d^{k_2} \geq M_d^{k_1}$, inequalities (4.7)-(4.8) become $0 \leq M_a^{k_2} - M_a^{k_1}$ and $0 \leq M_a^{k_2} - M_a^{k_1} - M_{b^*}^{k_2} + M_{b^*}^{k_1}$ respectively, valid by definition of b^*. Otherwise, i.e., if $x_d^{k_1} = 1$ with $d \in \mathcal{A}_a^{\leq}$, one obtains $0 \leq 0$.

If $x_b^{k_2} = 1$ for any $b \in \mathcal{A}_a^>$ such that $M_b^{k_2} \geq M_b^{k_1}$ and $x_a^{k_1} = 0$ for all $a \in \mathcal{A}$ or $x_d^{k_1} = 1$ with $d \in \mathcal{A}_a^>$: $M_d^{k_2} < M_d^{k_1}$, inequalities (4.7)-(4.8) become $0 \leq 0$ and $0 \leq M_{b^*}^{k_2} - M_{b^*}^{k_1}$, which is valid by the definition of b^*. If $x_d^{k_1} = 1$ for any

$d \in \mathcal{A}_a^> : M_d^{k_2} \geq M_d^{k_1}$, one obtains $0 \leq 0$. Other cases, i.e., $x_d^{k_1} = 1$ for any $d \in \mathcal{A}_a^\leq$ cannot occur due to Lemma 2. Indeed, it would yield $c_b^{k_2} - c_b^{k_1} \leq c_d^{k_2} - c_d^{k_1}$, which is impossible by definition of the sets $\mathcal{A}_a^\leq$ and $\mathcal{A}_a^>$.

Finally, if $x_b^{k_2} = 1$ for any $b \in \mathcal{A}_a^>$ such that $M_b^{k_2} < M_b^{k_1}$ and $x_d^{k_1} = 1$ with $d \in \mathcal{A}_a^> : M_d^{k_2} < M_d^{k_1}$, inequalities (4.7)-(4.8) become $0 \leq 0$. The case $x_a^{k_1} = 0$ for all $a \in \mathcal{A}$ is avoided by Lemma 3. Other cases are avoided by Lemma 2. Indeed, case $x_d^{k_1} = 1$ for any $d \in \mathcal{A}_a^\leq$ would yield $c_b^{k_2} - c_b^{k_1} \leq c_d^{k_2} - c_d^{k_1}$, which is impossible by definition of the sets $\mathcal{A}_a^\leq$ and $\mathcal{A}_a^>$. Case $x_d^{k_1} = 1$ for any $d \in \mathcal{A}_a^>$ such that $M_d^{k_2} \geq M_d^{k_1}$ would yield $c_b^{k_2} - c_b^{k_1} \leq c_d^{k_2} - c_d^{k_1}$. As one knows that $c_d^{k_2} - c_d^{k_1} \leq c_{od}^{k_2} - c_{od}^{k_1}$ and $c_b^{k_2} - c_b^{k_1} > c_{od}^{k_2} - c_{od}^{k_1}$, this is a contradiction. □

Further, we also propose similar valid inequalities that link a given Profit Upper Bound constraint (3.23) $p_a^{k_2} \leq M_a^{k_2} x_a^{k_2}$ with (i) the profit variable $p_a^{k_1}$ for another commodity k_1, (ii) the flow variables $x_b^{k_1} : b \in \mathcal{K}$ for this same commodity k_1 and (iii) the remaining flow variables for commodity k_2.

Proposition 7 *Consider a pair of commodities $k_1, k_2 \in \mathcal{K}$, a toll arc $a \in \mathcal{A}$ such that $M_a^{k_1} \leq M_a^{k_2}$ and $b^* = \arg\min_{b \in \mathcal{A}_a^> : M_b^{k_2} \geq M_b^{k_1}} \{c_b^{k_1} - c_b^{k_2}\}$. Under the assumption that there does not exist any $b \in \mathcal{A} \setminus \{a\}$ such that $c_a^{k_2} - c_a^{k_1} = c_b^{k_2} - c_b^{k_1}$, the following inequalities are valid for CCT-NPP and GCT-NPP:*

$$p_a^{k_2} - p_a^{k_1} \leq M_a^{k_2} \sum_{b \in \mathcal{A}_a^\leq} (x_b^{k_2} - x_b^{k_1}) \tag{4.9}$$

$$p_a^{k_2} - p_a^{k_1} \leq M_a^{k_2} \sum_{b \in \mathcal{A}_a^\leq} (x_b^{k_2} - x_b^{k_1}) + \left(M_{b^*}^{k_2} - M_{b^*}^{k_1}\right) \sum_{b \in \mathcal{A}_a^> : M_b^{k_2} \geq M_b^{k_1}} (x_b^{k_2} - x_b^{k_1}). \tag{4.10}$$

Note that, for given $k_1, k_2 \in \mathcal{K}$ and $a \in \mathcal{A}$, (4.9)-(4.10) are non redundant either if $x_a^{k_1} = 1 = x_a^{k_2}$ or if $x_b^{k_1} = 1 = x_a^{k_2}$ for $b \in \mathcal{A}_a^>$ such that $M_b^{k_2} \geq M_b^{k_1}$.

Proof

Let $k_1, k_2 \in \mathcal{K}$ and $a \in \mathcal{A}$ be fixed.

If $x_a^{k_1} = 0$ for all $a \in \mathcal{A}$, then either

- $x_a^{k_2} = 1$, yielding $p_a^{k_2} \leq M_a^{k_2}$ which is valid by (3.23);
- $x_b^{k_2} = 1$ for any $b \in \mathcal{A}_a^{\leqq} \setminus \{a\}$, which yields $0 \leq M_a^{k_2}$;
- $x_b^{k_2} = 1$ with $b \in \mathcal{A}_a^{>}$ such that $M_b^{k_2} \geq M_b^{k_1}$ and (4.9)-(4.10) become $0 \leq 0$ and $0 \leq M_{b^*}^{k_2} - M_{b^*}^{k_1}$, which is valid by definition of b^*.
- Or, in all other cases, one obtains $0 \leq 0$.

If $x_a^{k_1} = 1$, then either:

- $x_a^{k_2} = 1$, and (4.9)-(4.10) become $p_a^{k_2} - p_a^{k_1} \leq 0$, which is valid by (3.24);
- $x_b^{k_2} = 1$ for any $b \in \mathcal{A}_a^{\leqq} \setminus \{a\}$, which yields to $-p_a^{k_1} \leq 0$, valid by (3.26);
- $x_b^{k_2} = 1$ with $b \in \mathcal{A}_a^{>}$ are avoided by Lemma 2.
- $x_b^{k_2} = 0$ for all $b \in \mathcal{A}$ cannot occur due to Lemma 3.

If $x_b^{k_1} = 1$ for any $b \in \mathcal{A}_a^{\leqq} \setminus \{a\}$ and if $x_d^{k_2} = 1$ for any $d \in \mathcal{A}_a^{\leqq} \setminus \{a\}$, one obtains $0 \leq 0$. The case $x_a^{k_2} = 0$ for all $a \in \mathcal{A}$ is avoided by Lemma 3. All other cases cannot occur due to Lemma 2. Indeed, case $x_a^{k_2} = 1$ would yield $c_a^{k_2} - c_a^{k_1} \leq c_b^{k_2} - c_b^{k_1}$, which is impossible since $b \in \mathcal{A}_a^{\leqq} \setminus \{a\}$ and there does not exists any $b \in \mathcal{A} \setminus \{a\}$ such that $c_a^{k_2} - c_a^{k_1} = c_b^{k_2} - c_b^{k_1}$. Case $x_d^{k_2} = 1$ with $d \in \mathcal{A}_a^{>}$ would yield $c_d^{k_2} - c_d^{k_1} \leq c_b^{k_2} - c_b^{k_1}$. Or, by definition of the sets $\mathcal{A}_a^{\leqq}$ and $\mathcal{A}_a^{>}$, one has $c_d^{k_2} - c_d^{k_1} > c_a^{k_2} - c_a^{k_1}$ and $c_b^{k_2} - c_b^{k_1} \leq c_a^{k_2} - c_a^{k_1}$, which is a contradiction.

If $x_b^{k_1} = 1$ for any $b \in \mathcal{A}_a^{>}$ such that $M_b^{k_2} \geq M_b^{k_1}$, case $x_a^{k_2} = 0$ for all $a \in \mathcal{A}$ is avoided by Lemma 3. Case $x_d^{k_2} = 1$ with $d \in \mathcal{A}_a^{>}$ such that $M_d^{k_2} < M_d^{k_1}$ cannot occur. Indeed, Lemma 1 yields $c_d^{k_2} - c_d^{k_1} \leq c_b^{k_2} - c_b^{k_1}$. But one also knows that $c_b^{k_2} - c_b^{k_1} \leq c_{od}^{k_2} - c_{od}^{k_1}$ and $c_d^{k_2} - c_d^{k_1} > c_{od}^{k_2} - c_{od}^{k_1}$, which is a contradiction. Otherwise,

- if $x_a^{k_2} = 1$, (4.9) becomes $p_a^{k_2} \leq M_a^{k_2}$, which is valid by (3.23). Further, constraints (3.22) and (3.23) impose $p_a^{k_2} + c_a^{k_2} \leq t_b + c_b^{k_2}$ and $p_b^{k_1} \leq M_b^{k_1}$ respectively. As $t_b = p_b^{k_1}$ by (3.24) and (3.25), this yields $p_a^{k_2} \leq M_b^{k_1} + c_b^{k_2} - c_a^{k_2} \leq M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_a^{k_2}$ by definition of b^* and (4.10) is valid.

- If $x_d^{k_2} = 1$ for any $d \in \mathcal{A}_a^{\leq} \setminus \{a\}$, one obtains $0 \leq M_a^{k_2}$ and $0 \leq M_a^{k_2} + M_{b^*}^{k_1} - M_{b^*}^{k_2}$, valid by definition of b^*.

- The last case $x_d^{k_2} = 1$ for any $d \in \mathcal{A}_a^{>}$ such that $M_d^{k_2} \geq M_d^{k_1}$ leads to $0 \leq 0$.

Finally, if $\underline{x_b^{k_1} = 1 \text{ with } b \in \mathcal{A}_a^{>} \text{ such that } M_b^{k_2} < M_b^{k_1}}$, then either

- $x_a^{k_2} = 1$, yielding $p_a^{k_2} \leq M_a^{k_2}$, valid by (3.23);

- $x_d^{k_2} = 1$ for any $d \in \mathcal{A}_a^{\leq} \setminus \{a\}$, and one obtains $0 \leq M_a^{k_2}$;

- $x_d^{k_2} = 1$ for any $d \in \mathcal{A}_a^{>}$ such that $M_d^{k_2} \geq M_d^{k_1}$, and (4.9)-(4.10) become $0 \leq 0$ and $0 \leq M_{b^*}^{k_2} - M_{b^*}^{k_1}$, which is valid by definition of b^*.

- Or, in all other cases, one obtains $0 \leq 0$. □

Finally, the following valid inequalities link the profit variables $p_a^{k_2}, p_a^{k_1}$ for a given toll arc $a \in \mathcal{A}$ and two commodities $k_1, k_2 \in \mathcal{K}$ with corresponding flow variables $x_b^{k_1}, x_b^{k_2} : b \in \mathcal{A}$. Even if they do not really strengthen the Profit Upper Bound constraints (3.23), we decide to present these here since they are very similar to the previous valid inequalities proposed.

Proposition 8 *For any pair of commodities $k_1, k_2 \in \mathcal{K}$ and for any toll arc $a \in \mathcal{A}$ such that $M_a^{k_1} \leq M_a^{k_2}$, assuming that there does not exist any $b \in \mathcal{A} \setminus \{a\}$ such that $c_a^{k_2} - c_a^{k_1} = c_b^{k_2} - c_b^{k_1}$, the following inequalities are valid for CCT-NPP and GCT-NPP:*

$$p_a^{k_1} - p_a^{k_2} \leq M_a^{k_1} \sum_{b \in \mathcal{A}_a^{\leq} \setminus \{a\}} (x_b^{k_2} - x_b^{k_1}). \tag{4.11}$$

Proof

Let $k_1, k_2 \in \mathcal{K}$ and $a \in \mathcal{A}$ be fixed. If $\underline{x_a^{k_1} = 0 \text{ for all } a \in \mathcal{A}}$, then either

- $x_a^{k_2} = 1$, yielding $-p_a^{k_2} \leq 0$ which is valid by (3.26),
- $x_b^{k_2} = 1$ for any $b \in \mathcal{A}_a^{\leq} \setminus \{a\}$, which yields to $0 \leq M_a^{k_1}$.
- Or, in all other cases, one obtains $0 \leq 0$.

If $\underline{x_a^{k_1} = 1}$, then:

- If $x_a^{k_2} = 1$, (4.11) becomes $p_a^{k_1} - p_a^{k_2} \leq 0$, which is valid by (3.24),
- If $x_b^{k_2} = 1$ for any $b \in \mathcal{A}_a^{\leq} \setminus \{a\}$, one obtains $p_a^{k_1} \leq M_a^{k_1}$, valid by (3.23).
- Other cases, i.e., $x_b^{k_2} = 1$ with $b \in \mathcal{A}_a^{>}$ and $x_a^{k_2} = 0$ for all $a \in \mathcal{A}$, cannot occur due to Lemmas 2 and 3 respectively.

If $\underline{x_b^{k_1} = 1 \text{ for any } b \in \mathcal{A}_a^{\leq} \setminus \{a\}}$, case $x_a^{k_2} = 0$ for all $a \in \mathcal{A}$ is avoided by Lemma 3. Further, cases $x_a^{k_2} = 1$ or $x_d^{k_2} = 1$ with $d \in \mathcal{A}_a^{>}$ cannot occur due to Lemmas 2. Indeed, $x_a^{k_2} = 1$ would yield $c_a^{k_2} - c_a^{k_1} \leq c_b^{k_2} - c_b^{k_1}$, which is impossible since $b \in \mathcal{A}_a^{\leq} \setminus \{a\}$ and there does not exists any $b \in \mathcal{A} \setminus \{a\}$ such that $c_a^{k_2} - c_a^{k_1} = c_b^{k_2} - c_b^{k_1}$. The case $x_d^{k_2} = 1$ with $d \in \mathcal{A}_a^{>}$ would yield $c_d^{k_2} - c_d^{k_1} \leq c_b^{k_2} - c_b^{k_1}$, which is avoided by the definitions of the sets $\mathcal{A}_a^{>}$ and $\mathcal{A}_a^{\leq}$. Otherwise, i.e., if $x_d^{k_1} = 1$ for any $d \in \mathcal{A}_a^{\leq} \setminus \{a\}$, one obtains $0 \leq 0$.

Finally, if $\underline{x_b^{k_1} = 1 \text{ for any } b \in \mathcal{A}_a^{>}}$, then:

- If $x_d^{k_2} = 1$ for any $d \in \mathcal{A}_a^{\leq} \setminus \{a\}$, one obtains $0 \leq M_a^{k_1}$.
- The case $x_a^{k_2} = 1$ yields to $-p_a^{k_2} \leq 0$, which is valid by (3.26).
- All other cases yield $0 \leq 0$. □

4.3 Extension

Upon close inspection of the facet equations returned by Porta for small instances, one can infer that the preceding inequalities, as well as constraints of model (HP3), can be combined to produce new valid inequalities.

For example, let us consider the two following inequalities

$$p_{\tilde{a}}^{k_2} - p_{\tilde{a}}^{k_1} \leq M_{\tilde{a}}^{k_2} \sum_{b \in \mathcal{A}_{\tilde{a}}^{\leq}} (x_b^{k_2} - x_b^{k_1})$$

$$p_{\tilde{b}}^{k_2} \leq M_{\tilde{b}}^{k_2} x_{\tilde{b}}^{k_2}$$

of type (4.9) and (3.23), with $\tilde{b} \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$ for the second inequality. These can be combined to produce the new valid inequality

$$\alpha p_{\tilde{b}}^{k_2} + p_{\tilde{a}}^{k_2} - p_{\tilde{a}}^{k_1} \leq \alpha M_{\tilde{b}}^{k_2} x_{\tilde{b}}^{k_2} + M_{\tilde{a}}^{k_2} \sum_{b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{b}\}} (x_b^{k_2} - x_b^{k_1}), \tag{4.12}$$

with

$$\alpha \geq \frac{M_{\tilde{a}}^{k_2}}{M_{\tilde{b}}^{k_2} - \max_{b \in \mathcal{A}_{\tilde{a}}^{\leq}} \{M_b^{k_1} + c_b^{k_2} - c_{\tilde{b}}^{k_2}, 0\}}.$$

Let us prove the validity of (4.12). If $x_{\tilde{b}}^{k_2} = 0$, one obtains (4.9). Hence $x_{\tilde{b}}^{k_2} = 1 = x_{\tilde{d}}^{k_1}$ with $d \in \mathcal{A}_{\tilde{a}}^{\leq}$ is the only non trivial case, which yields

$$\alpha p_{\tilde{b}}^{k_2} \leq \alpha M_{\tilde{b}}^{k_2} - M_{\tilde{a}}^{k_2}.$$

But constraints (3.22) ensure that $p_{\tilde{b}}^{k_2} + c_{\tilde{b}}^{k_2} \leq t_b + c_b^{k_2}$, while constraints (3.23) to (3.26) give $t_b = p_b^{k_1} \leq M_b^{k_1}$ for all $b \in \mathcal{A}_{\tilde{a}}^{\leq}$. Thus the α value must correspond to the largest possible choice for $t_b = p_b^{k_2}$, and the result follows.

However, given the apparent number of possible combinations, deriving them all and/or trying to find the better combinations would be too long for this thesis. Further, before beginning such an analysis, one should ponder its real usefulness in a larger context compared to the difficulty of performing it.

Also note that, if two toll arcs $a, b \in \mathcal{A}$ are not compatible for a pair of commodities $k_1, k_2 \in \mathcal{K}$, i.e., $(a, k_1) \not\sim (b, k_2)$, hence inequality $x_a^{k_1} + x_b^{k_2} \leq 1$ holds. Based on this compatibility notion, one could imagine to construct a conflict graph with vertices $x_a^k : k \in \mathcal{K}, a \in \mathcal{A}$, in which the arcs would represent 'non compatibility' between vertices. In this graph, any clique of vertices would be a subset of 'non compatible' vertices and yields a corresponding valid inequality. However, this class of inequalities, which only contains flow variables, does not appear in the facet equations given by Porta for small instances. Indeed, all the facet equations contain at least one toll or profit variable $t_a, p_a^k : k \in \mathcal{K}, a \in \mathcal{A}$. Hence we decided to focus on this second class of inequality, at least for the purpose of the thesis.

4.4 Conclusion

In this chapter, we proposed several families of valid inequalities for CCT-NPP and GCT-NPP. They exploit the underlying network structure, and strengthen important constraints of (HP3) and (HP3*) models. In the next chapters, we will study the efficiency of these valid inequalities, both theoretically and numerically.

CHAPTER 5

ASSESSING THE VALID INEQUALITIES

The aim of this chapter is to assess the valid inequalities presented in Chapter 4. In that purpose, we restrict our attention to problems involving one or two commodities. In Section 1, we focus on single commodity problems. Several (HP3) and (HP3*) constraints, as well as valid inequalities from the preceding chapter, are proved to be facet defining for the single commodity CCT-NPP and GCT-NPP, i.e., for the polyhedron described by the convex hull of (HP3) or (HP3*) feasible solutions. Further, for the single commodity GCT-NPP, a complete description of the convex hull of (HP3) feasible solutions can be highlighted. Next, in Section 2, we prove that several (HP3) constraints, as well as most valid inequalities from the preceding chapter, define facets for the two-commodity GCT-NPP, i.e., for the convex hull of (HP3) feasible solutions.

5.1 Single commodity Problems

While it is known that the single commodity case is polynomially solvable, yet its analysis provides some insight. This section aims to highlight several facets of the convex hull of (HP3*) solutions, i.e., for the Constrained Complete Toll NPP. For the single commodity General Complete Toll NPP, we also show that a complete description of the convex hull of (HP3) feasible solutions can be obtained.

5.1.1 Single commodity CCT-NPP

Now we focus on the Constrained Complete Toll NPP with a single commodity and, for notational simplicity, remove the index k. Let us denote by $\mathcal{P}^C$ the convex hull of feasible solutions for the Constrained Complete Toll NPP, i.e.,

$$\mathcal{P}^C = conv\left\{(\mathbf{t};\mathbf{p};\mathbf{x}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \{0,1\}^n : (3.17)-(3.18),(3.21)-(3.27)\right\},$$

where n is the number of toll arcs[1].

In what follows, we assume that the network has been preprocessed according to Section 3.3, i.e., the toll arcs that are never used are removed and the set $\mathcal{A}$ is modified accordingly.

Note that, in the single commodity case, constraints (3.24) can be removed from the model (HP3), allowing to set variables $t_a : a \in \mathcal{A}$ to arbitrarily large values when the corresponding toll arcs $a \in \mathcal{A}$ are not used, i.e., when $x^k_a = 0$ for all $k \in \mathcal{K}$. We introduce a scalar M such that $M > c_{od} - c_a$ for all $a \in \mathcal{A}$. Further, let $M_a = \max\{0, c_{od} - c_a\}$ as defined in (HP3*), and denote by $\mathbf{e}_a$ a unit vector in the direction a. We also denote by $\mathbf{1}$ a vector with all coordinates equal to 1.

We also recall the definition of the sets $\underline{C}_a$ that have been introduced in Section 3.3, i.e.,

$$\underline{C}_a = \{b \in \mathcal{A} : t(a) \le t(b) < h(b) \le h(a) \text{ or } t(a) \ge t(b) > h(b) \ge h(a)\}.$$

Further, we define similar sets $\overline{C}_a$, for all a in $\mathcal{A}$. The corresponding subnetworks are depicted in Figure 5.1.

Definition 11 *For each a in $\mathcal{A}$, the following set is defined:*

$$\overline{C}_a = \{b \in \mathcal{A} : t(b) \le t(a) < h(a) \le h(b) \text{ or } t(b) \ge t(a) > h(a) \ge h(b)\}.$$

[1]For notational simplicity, dashed letters denote vectors.

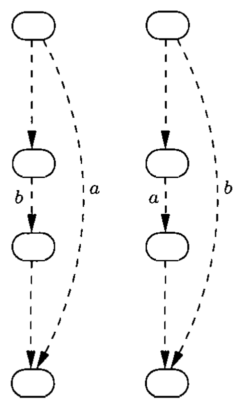

Figure 5.1: Examples of $b \in \underline{C}_a$ and $b \in \overline{C}_a$

Hence, if Triangle and Monotonicity inequalities are included in the problem, the toll variables must satisfy $t_a \geq t_b$ for all b in $\underline{C}_a$. Note that an arc b is in $\overline{C}_a$ if and only if a is in $\underline{C}_b$.

Throughout the analysis, we assume that the arcs are totally ordered (labels 1 to n) in a manner that is compatible with the partial order induced by the Monotonicity and Triangle inequalities. More specifically, the coordinates of toll arcs are such that if $b \in \underline{C}_a$ for some couple $a, b \in \mathcal{A}$, i.e., the corresponding toll variables satisfy $t_a \geq t_b$, then a has a smaller index (also denoted 'a') than b.

First, we prove that $\mathcal{P}^C$ is full dimensional.

Proposition 9 *The polyhedron $\mathcal{P}^C$ has full dimension, i.e., $Dim\left(\mathcal{P}^C\right) = 3n$.*

Proof

Suppose by contradiction that the points $\mathcal{P}^C$ lie on a generic hyperplane $\alpha\mathbf{t} + \beta\mathbf{p} + \gamma\mathbf{x} = \delta$. The points $(M\mathbf{1}; \mathbf{0}; \mathbf{0})$ and $\left(M\mathbf{1} + \sum_{b \in \mathcal{A}: b \leq a} \mathbf{e_b}; \mathbf{0}; \mathbf{0}\right)$ belong to $\mathcal{P}^C$ for all

$a \in \mathcal{A}$. It follows that

$$M \sum_{b \in \mathcal{A}} \alpha_b = \delta \quad \text{and} \quad M \sum_{b \in \mathcal{A}} \alpha_b + \sum_{b \in \mathcal{A}: b \leq a} \alpha_b = \delta,$$

thus $\alpha_a = 0$ for all $a \in \mathcal{A}$ and $\delta = 0$. Further, the points

$$\big(M(\mathbf{1} - \sum_{b \in \underline{C}_a} \mathbf{e_b}) + \sum_{b \in \mathcal{A}: b < a} \mathbf{e_b}; \mathbf{0}; \mathbf{e}_a\big)$$
$$\big(M\mathbf{1} + \sum_{b \in \mathcal{A}: b < a} \mathbf{e_b} + (M_a - M) \sum_{b \in \underline{C}_a} \mathbf{e_b}; M_a \mathbf{e}_a; \mathbf{e}_a\big)$$

are in $\mathcal{P}^{\mathcal{C}}$ for all $a \in \mathcal{A}$, thus $\gamma_a = 0 = \beta_a$ for all $a \in \mathcal{A}$ and the result follows. □

One can also prove that several constraints of model (HP3*) define facets of $\mathcal{P}^{\mathcal{C}}$. Let $(\mathbf{t}; \mathbf{p}; \mathbf{x})$ be points of $\mathcal{P}^{\mathcal{C}}$. In order to prove that a given inequality is facet defining for $\mathcal{P}^{\mathcal{C}}$, we define $\mathcal{H}$ as the hyperplane induced by a given inequality, and $\mathcal{G}$ as a generic hyperplane defined by $\alpha\mathbf{t} + \beta\mathbf{p} + \gamma\mathbf{x} = \delta$. Hence we select points of $\mathcal{P}^{\mathcal{C}} \cap \mathcal{H}$ and we deduce that $\mathcal{G} = \mathcal{H}$.

Proposition 10 *Constraints (3.21), (3.23) and (3.26) are facet defining for* $\mathcal{P}^{\mathcal{C}}$.

Proof

(i) Let $\mathcal{H} = \big\{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : \sum_{a \in \mathcal{A}} x_a = 1\big\}$.

For all $a \in \mathcal{A}$, we consider the points $\Big(M(\mathbf{1} - \sum_{b \in \underline{C}_a} \mathbf{e_b}); \mathbf{0}; \mathbf{e}_a\Big)$ and $\Big(M(\mathbf{1} - \sum_{b \in \underline{C}_a} \mathbf{e_b}) + \sum_{b \in \mathcal{A}: b < a} \mathbf{e_b}; \mathbf{0}; \mathbf{e}_a\Big)$ of $\mathcal{P}^{\mathcal{C}} \cap \mathcal{H}$. This yields

$$M \sum_{b \in \mathcal{A} \setminus \underline{C}_a} \alpha_b + \gamma_a = \delta \quad \text{and} \quad M \sum_{b \in \mathcal{A} \setminus \underline{C}_a} \alpha_b + \sum_{b \in \mathcal{A}: b < a} \alpha_b + \gamma_a = \delta.$$

Hence $\alpha_a = 0$ and $\gamma_a = \delta$ for all $a \in \mathcal{A}$. Next, points

$$\Big(M\mathbf{1} + (M_a - M)\sum_{b\in\underline{C}_a} \mathbf{e_b}; M_a\mathbf{e}_a; \mathbf{e}_a\Big)$$

are also in $\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A}$. Hence $\beta_a = 0$ for all $a \in \mathcal{A}$.

(ii) Let $\mathcal{H} = \{(\mathbf{t};\mathbf{p};\mathbf{x}) : p_{\tilde{a}} = M_{\tilde{a}}x_{\tilde{a}},\ \tilde{a} \in \mathcal{A}\}$.

Points $(M\mathbf{1};\mathbf{0};\mathbf{0})$ and $\big(M\mathbf{1} + \sum_{b\in\mathcal{A}:b\leq a} \mathbf{e_b};\mathbf{0};\mathbf{0}\big)$ are in $\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A}$, which implies that

$$M\sum_{b\in\mathcal{A}} \alpha_b = \delta \quad \text{and} \quad M\sum_{b\in\mathcal{A}} \alpha_b + \sum_{b\in\mathcal{A}:b\leq a} \alpha_b = \delta,$$

thus $\alpha_a = 0$ for all $a \in \mathcal{A}$ and $\delta = 0$. Points $\Big(M(\mathbf{1} - \sum_{b\in\underline{C}_a} \mathbf{e_b});\mathbf{0};\mathbf{e}_a\Big)$ also belong to $\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A}\setminus\{\tilde{a}\}$, and $\gamma_a = 0$ for all $a \in \mathcal{A}\setminus\{\tilde{a}\}$. As $\Big(M\mathbf{1} + (M_a - M)\sum_{b\in\underline{C}_a} \mathbf{e_b}; M_a\mathbf{e}_a; \mathbf{e}_a\Big)$ are in $\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A}$, it follows that $\beta_a = 0$ for all $a \in \mathcal{A}\setminus\{\tilde{a}\}$ and $\gamma_{\tilde{a}} = -M_{\tilde{a}}\beta_{\tilde{a}}$. The result follows.

(iii) Let $\mathcal{H} = \{(\mathbf{t};\mathbf{p};\mathbf{x}) : p_{\tilde{a}} = 0,\ \tilde{a} \in \mathcal{A}\}$.

Points $(M\mathbf{1};\mathbf{0};\mathbf{0})$ and $\big(M\mathbf{1} + \sum_{b\in\mathcal{A}:b\leq a} \mathbf{e_b};\mathbf{0};\mathbf{0}\big)$ are in $\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A}$, thus $\alpha_a = 0$ for all $a \in \mathcal{A}$ and $\delta = 0$. As $\Big(M(\mathbf{1} - \sum_{b\in\underline{C}_a} \mathbf{e_b});\mathbf{0};\mathbf{e}_a\Big)$ also belong to $\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A}$, it follows that $\gamma_a = 0$ for all $a \in \mathcal{A}$. Finally, the points $\Big(M\mathbf{1} + (M_a - M)\sum_{b\in\underline{C}_a} \mathbf{e_b}; M_a\mathbf{e}_a; \mathbf{e}_a\Big)$ are in $\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A}\setminus\{\tilde{a}\}$, thus $\beta_a = 0$ for all $a \in \mathcal{A}\setminus\{\tilde{a}\}$ and the result follows. □

Further, the Triangle and Monotonicity constraints also define facets of $\mathcal{P}^C$.

Proposition 11 *The Triangle constraints (3.17) are facet defining for* $\mathcal{P}^C$.

Proof

Let $\mathcal{H} = \Big\{(\mathbf{t};\mathbf{p};\mathbf{x}) : t_{\tilde{a}} = t_{\tilde{b}} + t_{\tilde{c}},\ \tilde{a},\tilde{b},\tilde{c} \in \mathcal{A}\ s.t.\ t(\tilde{a}) = t(\tilde{b}), h(\tilde{b}) = t(\tilde{c}), h(\tilde{c}) = h(\tilde{a})\Big\}$. Further, assume that the index are so that $\tilde{b} < \tilde{c}$. The point $\big(M(\mathbf{1} + \sum_{b\in\mathcal{A}:b\leq\tilde{a}} \mathbf{e_b}); \mathbf{0}; \mathbf{0}\big)$ belongs to $\mathcal{P}^C \cap \mathcal{H}$, which implies

$$M\sum_{b\in\mathcal{A}}\alpha_b + M\sum_{b\in\mathcal{A}:b\leq\tilde{a}}\alpha_b = \delta.$$

For all $a \in \mathcal{A}\setminus\{\tilde{a}\}$, points $\Big(M(\mathbf{1} + \sum_{b\in\mathcal{A}\setminus\{\tilde{a}\}:b\leq a} \mathbf{e_b}) + S\mathbf{e}_{\tilde{a}}; \mathbf{0}; \mathbf{0}\Big)$ are in $\mathcal{P}^C \cap \mathcal{H}$ with $S = M$ for $a < \tilde{b}$, $S = 2M$ for $\tilde{b} \leq a < \tilde{c}$, and $S = 3M$ for $a \geq \tilde{c}$. It follows that

$$M\sum_{b\in\mathcal{A}}\alpha_b + M\sum_{b\in\mathcal{A}\setminus\{\tilde{a}\}:b\leq a}\alpha_b + S\alpha_{\tilde{a}} = \delta.$$

From the first case $a < \tilde{b}$ we know that $\alpha_a = 0$ (for $a \neq \tilde{a}$) ; from the second one we conclude that $\alpha_{\tilde{a}} = -\alpha_{\tilde{b}}$ and that the other index $\alpha_a = 0$; from the last one we obtain $\alpha_{\tilde{a}} = -\alpha_{\tilde{c}}$ and $\alpha_a = 0$ for the other index. Hence $\delta = 0$.

Further, for all $a \in \mathcal{A}$ such that $\tilde{a},\tilde{b},\tilde{c} \notin \underline{C}_a$, the points

$$\big(M(\mathbf{1} + \sum_{b\in\overline{C}_{\tilde{a}}}\mathbf{e_b} - \sum_{b\in\underline{C}_a}\mathbf{e_b}); \mathbf{0}; \mathbf{e}_a\big)$$

belong to $\mathcal{P}^C \cap \mathcal{H}$. In those point coordinates, the first sum ensures that the given Triangle inequality holds at equality, while the second sum ensures that $t_a = p_a$ (as imposed by constraints (3.24) and (3.25) of (HP3*)) and that the Monotonicity inequalities hold. For all $a \in \mathcal{A}$ such that either $\tilde{a}$ or $\tilde{b}$ or $\tilde{c}$ in $\underline{C}_a$, the points $\Big(M(\mathbf{1} - \sum_{b\in\underline{C}_a}\mathbf{e_b}); \mathbf{0}; \mathbf{e}_a\Big)$ are in $\mathcal{P}^C \cap \mathcal{H}$. Hence $\gamma_a = 0$ for all $a \in \mathcal{A}$.

Next, for all $a \in \mathcal{A}$ such that $\tilde{a}, \tilde{b}, \tilde{c} \notin \underline{C}_a$, one considers points $\left(M(1 + \sum_{b \in \overline{C}_{\tilde{a}} \cap \mathcal{A}} \mathbf{e_b}) + (M_a - M) \sum_{b \in \underline{C}_a} \mathbf{e_b}; M_a \mathbf{e}_a; \mathbf{e}_a\right)$ of $\mathcal{P^C} \cap \mathcal{H}$. Otherwise, i.e., if either $\tilde{a}$ or $\tilde{b}$ or $\tilde{c}$ is in $\underline{C}_a$, the points

$$\left(M1 + M_a \sum_{b \in \overline{C}_{\tilde{a}}} \mathbf{e_b} + (M_a - M) \sum_{b \in \underline{C}_a} \mathbf{e_b}; M_a \mathbf{e}_a; \mathbf{e}_a\right)$$

are in $\mathcal{P^C} \cap \mathcal{H}$. This yields $\beta_a = 0$ for all $a \in \mathcal{A}$ and the result follows. □

Proposition 12 *The Monotonicity constraints (3.18)* $t_{\tilde{a}} \geq t_{\tilde{b}}$ *are facet defining for* $\mathcal{P^C}$ *if and only if* $c_d \leq c_{\tilde{a}}$ *for all* $d \in \mathcal{A}$ *such that* $\tilde{b} \in \underline{C}_d$ *and* $\tilde{a} \notin \underline{C}_d$ *(see Figure 5.2).*

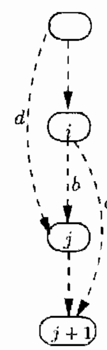

Figure 5.2: Part of network for the assumption of Proposition 29

Proof

Let $\mathcal{H} = \left\{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : t_{\tilde{a}} = t_{\tilde{b}},\ \tilde{a}, \tilde{b} \in \mathcal{A}\ s.t.\ t(\tilde{a}) = t(\tilde{b}) < h(\tilde{a}) = h(\tilde{b}) + 1\right\}$.

First, let us show that the proposition assumption is required. By contradiction, suppose that there exists $d \in \mathcal{A}$ such that $\tilde{b} \in \underline{C}_d$, $\tilde{a} \notin \underline{C}_d$ and $c_d > c_{\tilde{a}}$ (see Figure 5.2). If $\mathbf{e_d} = 1$, one must have $c_d + t_d \leq c_{\tilde{a}} + t_{\tilde{a}}$, which implies $t_d < t_{\tilde{a}}$. But, as $\tilde{b} \in \underline{C}_d$, one also has $t_d \geq t_{\tilde{b}} = t_{\tilde{a}}$ for points of $\mathcal{H}$. Hence all points of $\mathcal{P^C} \cap \mathcal{H}$ belong to the hyperplane $x_d = 0$, which is a contradiction.

Now assume that the proposition assumption holds. For all $a \in \mathcal{A} \setminus \{\tilde{a}\}$, points

$\left(M(\mathbf{1} + \sum_{b \in \mathcal{A} \setminus \{\tilde{a}\}: b \leq a} \mathbf{e_b}) + S\mathbf{e}_{\tilde{a}}; \mathbf{0}; \mathbf{0}\right)$ are in $\mathcal{P}^C \cap \mathcal{H}$ if the constant S is so that $S = 0$ for $a < \tilde{b}$ and $S = M$ for $a \geq \tilde{b}$. As $(M\mathbf{1}; \mathbf{0}; \mathbf{0})$ also belongs to $\mathcal{P}^C \cap \mathcal{H}$, it follows that

$$M \sum_{b \in \mathcal{A}} \alpha_b = \delta \quad \text{and} \quad M \sum_{b \in \mathcal{A}} \alpha_b + M \sum_{b \in \mathcal{A} \setminus \{\tilde{a}\}: b \leq a} \alpha_b + S\alpha_{\tilde{a}} = \delta.$$

From the first case $a < \tilde{b}$ we know that $\alpha_a = 0$ (for $a \neq \tilde{a}$) ; from the second one we conclude that $\alpha_{\tilde{a}} = -\alpha_{\tilde{b}}$ and that the other index $\alpha_a = 0$. Hence $\delta = 0$.
Further, for all $a \in \mathcal{A}$, points $\left(M(\mathbf{1} - \sum_{b \in \underline{C}_a \cup \underline{C}_{\tilde{a}}} \mathbf{e_b}); \mathbf{0}; \mathbf{e}_a\right)$ are in $\mathcal{P}^C \cap \mathcal{H}$. Note that imposing $t_b = 0$ for all $b \in \underline{C}_{\tilde{a}}$ allows that the given Monotonicity inequality holds at equality when a is so that $\tilde{b} \in \underline{C}_a$. This implies that $\gamma_a = 0$ for all $a \in \mathcal{A}$. Finally, for all $a \in \mathcal{A}$ such that $\tilde{a}, \tilde{b} \notin \underline{C}_a$, one considers points

$$\left(M\mathbf{1} + (M_a - M) \sum_{b \in \underline{C}_a} \mathbf{e_b}; M_a \mathbf{e}_a; \mathbf{e}_a\right)$$

of $\mathcal{P}^C \cap \mathcal{H}$. Otherwise, i.e., when $\tilde{b} \in \underline{C}_a$, points

$$\left(M\mathbf{1} + (M_{\tilde{b}} - M) \sum_{b \in \underline{C}_a \cup \underline{C}_{\tilde{a}}} \mathbf{e_b}; M_{\tilde{b}} \mathbf{e}_a; \mathbf{e}_a\right)$$

belong to $\mathcal{P}^C \cap \mathcal{H}$. Again, the small change in those point coordinates (M_a becomes $M_{\tilde{b}}$) allows that the Monotonicity inequality involving $\tilde{a}, \tilde{b} \in \mathcal{A}$ holds at equality. This yields $\beta_a = 0$ for all $a \in \mathcal{A}$, and the result follows. □

Hence most constraints of the model (HP3*) define facets of the convex hull of (HP3*) feasible solutions. Now let us focus on the valid inequalities presented in Chapter 4. For a single commodity, the Strengthened Profit Upper Bound inequalities (4.7) to (4.11) are obviously redundant, thus are not considered here.

In contrast, the Strengthened Shortest Path inequalities (4.1) and (4.2) become

$$\sum_{b\in\mathcal{A}}(p_b + c_b x_b) + c_{od}(1 - \sum_{b\in\mathcal{A}} x_b) \le t_a + c_a + \sum_{b\in\mathcal{A}\setminus(\mathcal{S}\cup\{a\})}(p_b + (c_b - c_a)x_b) \quad (5.1)$$

$$\sum_{b\in\mathcal{A}}(p_b + c_b x_b) + c_{od}(1 - \sum_{b\in\mathcal{A}} x_b) \le c_{od} + \sum_{b\in\mathcal{A}\setminus\mathcal{S}}(p_b + (c_b - c_{od})x_b) \quad (5.2)$$

for all $a \in \mathcal{A}$ and for all $\mathcal{S} \subseteq \mathcal{A}$. However, (5.2) can be equivalently rewritten as:

$$\sum_{b\in\mathcal{S}} p_b \le \sum_{b\in\mathcal{S}}(c_{od} - c_b)x_b \qquad \forall \mathcal{S} \subseteq \mathcal{A}$$

which are redundant by constraints (3.23).

Let us state the conditions in which the Strengthened Shortest Path inequalities (4.7) define facets of $\mathcal{P}^{\mathcal{C}}$.

Proposition 13 *The single commodity Strengthened Shortest Path inequalities*

$$(5.1) \quad \sum_{b\in\mathcal{A}}(p_b + c_b x_b) + c_{od}(1 - \sum_{b\in\mathcal{A}} x_b) \le t_{\tilde{a}} + c_{\tilde{a}} + \sum_{b\in\mathcal{A}\setminus(\mathcal{S}\cup\{\tilde{a}\})}(p_b + (c_b - c_{\tilde{a}})x_b)$$

are facet defining for $\mathcal{P}^{\mathcal{C}}$ *for all sets* $\mathcal{S}$ *such that* $c_{\tilde{a}} \ge c_b$ *for all* b *in* $\mathcal{A} \setminus (\mathcal{S} \cup \{\tilde{a}\})$.

Proof

Considering a Strengthened Shortest Path inequality in its generic form, let $\mathcal{H}$ be defined as

$$\mathcal{H} = \Big\{(\mathbf{t};\mathbf{p};\mathbf{x}) : t_{\tilde{a}} - \sum_{a\in\mathcal{S}\cup\{\tilde{a}\}} p_a + \sum_{a\in\mathcal{S}\cup\{\tilde{a}\}}(c_{od} - c_a)x_a + \sum_{a\in\mathcal{A}\setminus(\mathcal{S}\cup\{\tilde{a}\})}(c_{od} - c_{\tilde{a}})x_a$$
$$= c_{od} - c_{\tilde{a}}$$
$$\tilde{a} \in \mathcal{A}, \mathcal{S} \subseteq \mathcal{A} \text{ s.t. } c_{\tilde{a}} \ge c_a \ \forall a \in \mathcal{A} \setminus (\mathcal{S} \cup \{\tilde{a}\})\Big\}.$$

First, points

$$\big(M\mathbf{1} + (M_{\tilde{a}} - M)\sum_{b\in \underline{C}_{\tilde{a}}} \mathbf{e_b}; \mathbf{0}; \mathbf{0}\big)$$

$$\big(M\mathbf{1} + (M_{\tilde{a}} - M)\sum_{b\in \underline{C}_{\tilde{a}}} \mathbf{e_b} + \sum_{b\in \mathcal{A}\setminus\{\tilde{a}\}: b\leq a} \mathbf{e_b}; \mathbf{0}; \mathbf{0}\big)$$

belong to $\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A}$. This implies

$$M\sum_{b\in \mathcal{A}\setminus \underline{C}_{\tilde{a}}} \alpha_b + M_{\tilde{a}}\sum_{b\in \underline{C}_{\tilde{a}}} \alpha_b = \delta \quad \text{and} \quad M\sum_{b\in \mathcal{A}\setminus \underline{C}_{\tilde{a}}} \alpha_b + M_{\tilde{a}}\sum_{b\in \underline{C}_{\tilde{a}}} \alpha_b + \sum_{b\in \mathcal{A}\setminus\{\tilde{a}\}: b\leq a} \alpha_b = \delta,$$

thus $\alpha_b = 0$ for any $b \neq \tilde{a}$, and $\delta = M_{\tilde{a}}\alpha_{\tilde{a}}$.
The point $\Big(M\mathbf{1} + (M_{\tilde{a}} - M)\sum_{b\in \underline{C}_{\tilde{a}}} \mathbf{e_b}; M_{\tilde{a}}e_{\tilde{a}}; e_{\tilde{a}}\Big)$ of $\mathcal{P}^C \cap \mathcal{H}$ yields to $M_{\tilde{a}}\alpha_{\tilde{a}} + M_{\tilde{a}}\beta_{\tilde{a}} + \gamma_{\tilde{a}} = M_{\tilde{a}}\alpha_{\tilde{a}}$, and $\gamma_{\tilde{a}} = -M_{\tilde{a}}\beta_{\tilde{a}}$. The point $\Big(M(\mathbf{1} - \sum_{b\in \underline{C}_{\tilde{a}}} \mathbf{e_b}); \mathbf{0}; e_{\tilde{a}}\Big)$ also belongs to $\mathcal{P}^C \cap \mathcal{H}$, leading to $\gamma_{\tilde{a}} = -\alpha_{\tilde{a}}$.

Further, for all $a \in \mathcal{A} \setminus (\mathcal{S} \cup \{\tilde{a}\})$, points $\Big(M(\mathbf{1} - \sum_{b\in \underline{C}_{\tilde{a}}} \mathbf{e_b} - \sum_{b\in \underline{C}_a \setminus \underline{C}_{\tilde{a}}} \mathbf{e_b}); \mathbf{0}; \mathbf{e}_a\Big)$ belong to $\mathcal{P}^C \cap \mathcal{H}$, which implies $\gamma_a = M_{\tilde{a}}\alpha_{\tilde{a}}$. From points

$$\big(M(\mathbf{1} - \sum_{b\in \underline{C}_{\tilde{a}}} \mathbf{e_b}) + (c_{\tilde{a}} - c_a - M)\sum_{b\in \underline{C}_a \setminus \underline{C}_{\tilde{a}}} \mathbf{e_b}); (c_{\tilde{a}} - c_a)\mathbf{e}_a; \mathbf{e}_a\big)$$

of $\mathcal{P}^C \cap \mathcal{H}$, it follows that $(c_{\tilde{a}} - c_a)\beta_a + \gamma_a = M_{\tilde{a}}\alpha_{\tilde{a}}$, and $\beta_a = 0$.
For all $a \in \mathcal{S} \setminus \{\tilde{a}\}$, the points

$$\big(M\mathbf{1} + (M_{\tilde{a}} - M)\sum_{b\in \underline{C}_{\tilde{a}}} \mathbf{e_b} + (M_a - M)\sum_{b\in \underline{C}_a \setminus \underline{C}_{\tilde{a}}} \mathbf{e_b}; M_a\mathbf{e}_a; \mathbf{e}_a\big)$$

are in $\mathcal{P}^C \cap \mathcal{H}$, yielding $\gamma_a = -M_a\beta_a$. If $c_a \geq c_{\tilde{a}}$, the point

$$\big(M(\mathbf{1} - \sum_{b\in \underline{C}_a \setminus \underline{C}_{\tilde{a}}} \mathbf{e_b}) + (c_a - c_{\tilde{a}} - M)\sum_{b\in \underline{C}_{\tilde{a}}} \mathbf{e_b}; \mathbf{0}; \mathbf{e}_a\big)$$

belongs to $\mathcal{P}^{\mathcal{C}} \cap \mathcal{H}$. Hence, $(c_a - c_{\tilde{a}})\alpha_{\tilde{a}} + \gamma_a = M_{\tilde{a}}\alpha_{\tilde{a}}$ and $\gamma_a = M_a\alpha_{\tilde{a}}$. Otherwise, i.e., if $c_a < c_{\tilde{a}}$, points $\left(M(1 - \sum_{b \in \underline{C}_{\tilde{a}}} \mathbf{e_b}) + (c_{\tilde{a}} - c_a - M) \sum_{b \in \underline{C}_a \setminus \underline{C}_{\tilde{a}}} \mathbf{e_b}; (c_{\tilde{a}} - c_a)\mathbf{e}_a; \mathbf{e}_a\right)$ are in $\mathcal{P}^{\mathcal{C}} \cap \mathcal{H}$ and $(c_{\tilde{a}} - c_a)\beta_a + \gamma_a = M_{\tilde{a}}\alpha_{\tilde{a}}$. As $\gamma_a = -M_a\beta_a$, we obtain $\beta_a = -\alpha_{\tilde{a}}$ for all $a \in \mathcal{S} \setminus \{\tilde{a}\}$ and the result follows. □

In the next subsection, we restrict our attention to the GCT-NPP. In this case, the Strengthened Shortest Path inequalities allow to present a complete description of the convex hull of feasible solutions for the corresponding single commodity problem.

5.1.2 Single commodity GCT-NPP

Now consider the single commodity General Complete Toll NPP, and let us define $\mathcal{P}$ as the convex hull of feasible solutions for this problem, i.e.,

$$\mathcal{P} = conv\left\{(\mathbf{t}; \mathbf{p}; \mathbf{x}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \{0,1\}^n : (3.21) - (3.27)\right\}.$$

As $\mathcal{P}^{\mathcal{C}} \subseteq \mathcal{P}$, we have the following results.

Corollary 1 *$Dim(\mathcal{P}) = 3n$.*

Corollary 2 *Inequalities*

$$\sum_{a \in \mathcal{A}} x_a \leq 1$$

$$p_a \leq M_a x_a \qquad \forall a \in \mathcal{A}$$

$$p_a \geq 0 \qquad \forall a \in \mathcal{A}$$

$$\sum_{b \in \mathcal{A}} (p_b + c_b x_b) + c_{od}(1 - \sum_{b \in \mathcal{A}} x_b) \leq t_a + c_a + \sum_{b \in \mathcal{A} \setminus (\mathcal{S} \cup \{a\})} (p_b + (c_b - c_a)x_b)$$

$$\forall a \in \mathcal{A}, \forall \mathcal{S} \subseteq \mathcal{A} \ s.t. \ c_a \geq c_b \ \forall b \in \mathcal{A} \setminus (\mathcal{S} \cup \{a\})$$

with $M_a = \max\{0, c_{od} - c_a\}$, *are facet defining for* $\mathcal{P}$.

Further, this allows us to present a minimal complete description of $\mathcal{P}$.

Proposition 14 *Let* $\tilde{\mathcal{P}} = \{(\mathbf{t}; \mathbf{p}; \mathbf{x}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ :$

$$\sum_{a \in \mathcal{A}} x_a \leq 1 \tag{5.3}$$

$$p_a \leq M_a x_a \qquad \forall a \in \mathcal{A} \tag{5.4}$$

$$p_a \geq 0 \qquad \forall a \in \mathcal{A} \tag{5.5}$$

$$\sum_{b \in \mathcal{A}} (p_b + c_b x_b) + c_{od}(1 - \sum_{b \in \mathcal{A}} x_b) \leq t_a + c_a + \sum_{b \in \mathcal{A} \setminus (\mathcal{S} \cup \{a\})} (p_b + (c_b - c_a) x_b)$$
$$\forall a \in \mathcal{A}, \forall \mathcal{S} \subseteq \mathcal{A} : c_a \geq c_b \ \forall b \in \mathcal{A} \setminus (\mathcal{S} \cup \{a\}) \tag{5.6}$$

with $M_a = \max\{0, c_{od} - c_a\}$ $\}$. *Then* $\tilde{\mathcal{P}} = \mathcal{P}$.

Proof

Let $\mathcal{A} = \{1, ..., n\}$ be the toll arcs of the network. We define a fractional point of $\tilde{\mathcal{P}}$ as a point of $\tilde{\mathcal{P}}$ with a fractional component $\mathbf{x}$, i.e., there exists i in $\{1, ..., n\}$ such that $0 < x_i < 1$. Similarly, an integer point of $\tilde{\mathcal{P}}$ is defined as a point of $\tilde{\mathcal{P}}$ with an integer component $\mathbf{x}$, i.e., $x_i \in \{0, 1\}$ for all i in $\{1, ..., n\}$.

We have proved that (5.3)-(5.6) define facets of $\mathcal{P}$. Now one can show that any fractional point of $\tilde{\mathcal{P}}$ is not extremal. Indeed, let $(\bar{\mathbf{t}}; \bar{\mathbf{p}}; \bar{\mathbf{x}})$ be a fractional point of $\tilde{\mathcal{P}}$, where $0 \leq \bar{x}_i \leq 1$ for all $i \in \{1, ..., n\}$ and there exists at least one $\bar{x}_i$ such that $0 < \bar{x}_i < 1$. This point can be presented as a convex combination of integer points of $\tilde{\mathcal{P}}$:

$$\begin{aligned}(\bar{\mathbf{t}};\bar{\mathbf{p}};\bar{\mathbf{x}}) &= \sum_{i\in\{1,...,n\}:\bar{x}_i\neq 0} \bar{x}_i(\mathbf{t^i};\frac{1}{\bar{x}_i}\bar{p}_i e_i;e_i) + (1-\sum_{i=1}^{n}\bar{x}_i)\left(\mathbf{t^{n+1}};\mathbf{0};\mathbf{0}\right)\\ &= \left(\sum_{i\in\{1,...,n\}:\bar{x}_i\neq 0} \bar{x}_i\mathbf{t^i} + (1-\sum_{i=1}^{n}\bar{x}_i)\mathbf{t^{n+1}};\bar{\mathbf{p}};\bar{\mathbf{x}}\right),\end{aligned}$$

with $0 \leq \bar{x}_i \leq 1\ \forall i \in \{1,...,n\}$, $\exists\ i : 0 < \bar{x}_i < 1$, $\mathbf{t^i} = (t_j^i)_{j=1,...,n+1}$ and

$$t_j^i = \max\left\{0, \frac{\bar{p}_i}{\bar{x}_i} + c_i - c_j\right\} \qquad \forall i,j \in \{1,...,n\} : \bar{x}_i \neq 0 \tag{5.7}$$

$$t_j^{n+1} = c_{od} - c_j + s_j \qquad \forall j \in \{1,...,n\}, \tag{5.8}$$

with $s_j \in \mathbb{R}_+$.

These integer points belong to $\tilde{\mathcal{P}}$. Indeed, (5.3)-(5.5) are clearly satisfied. Next, for the integer points corresponding to $i \in \{1,...,n\}$, (5.6) yields either $t_j^i \geq 0$ or $t_j^i \geq \frac{\bar{p}_i}{\bar{x}_i} + c_i - c_j$ following that $i \in \mathcal{A}\backslash(\mathcal{S}\cup\{j\})$ or not, which is ensured by (5.7). For the last integer point $n+1$, the null components $\mathbf{x}$ and $\mathbf{p}$ force taxes $t_j^{n+1} : j \in \mathcal{A}$ to be larger that $c_{od} - c_j$ in (5.6), which is also ensured by the definition (5.7) of these taxes.

Next, for each $j \in \mathcal{A}$, we introduce the set $\mathcal{B}(j) = \left\{i \in \mathcal{A} : \bar{x}_i \neq 0, \frac{\bar{p}_i}{\bar{x}_i} + c_i - c_j > 0\right\}$ and let s_j be such that

$$s_j = \frac{1}{1-\sum_{i=1}^{n}\bar{x}_i}\left[\bar{t}_j - \sum_{i\in\mathcal{B}(j)}(\bar{p}_i + c_i\bar{x}_i - c_j\bar{x}_i) - (1-\sum_{i=1}^{n}\bar{x}_i)(c_{od}-c_j)\right].$$

Since the fractional point $(\bar{\mathbf{t}},\bar{\mathbf{p}},\bar{\mathbf{x}})$ satisfies (5.6) with $S\cup\{j\} = \mathcal{B}(j)$, it follows that $s_j \geq 0$.

Hence we have:

$$\sum_{i\in\{1,\dots,n\}:\bar{x}_i\neq 0} \bar{x}_i t_j^i + (1-\sum_{i=1}^{n}\bar{x}_i)t_j^{n+1} \tag{5.9}$$

$$= \sum_{i\in\{1,\dots,n\}:\bar{x}_i\neq 0} \bar{x}_i \max\left\{0, \frac{1}{\bar{x}_i}\bar{p}_i + c_i - c_j\right\} + (1-\sum_{i=1}^{n}\bar{x}_i)\,(c_{od}-c_j+s_j) \tag{5.10}$$

$$= (c_{od}-c_j)(1-\sum_{i=1}^{n}\bar{x}_i) + \sum_{i\in\{1,\dots,n\}:\bar{x}_i\neq 0} \bar{x}_i \max\left\{0, \frac{1}{\bar{x}_i}\bar{p}_i + c_i - c_j\right\} \tag{5.11}$$

$$+ \bar{t}_j - \sum_{i\in\mathcal{B}(j)} (\bar{p}_i + c_i\bar{x}_i - c_j\bar{x}_i) - (1-\sum_{i=1}^{n}\bar{x}_i)(c_{od}-c_j), \tag{5.12}$$

which is equal to $\bar{t}_j$ by definition of $\mathcal{B}(j)$. □

In the next section, we focus on two-commodity instances. We prove that most valid inequalities and (HP3) constraints are facet defining for the two-commodity GCT-NPP, i.e., for the polyhedron described by the convex hull of (HP3) feasible solutions.

5.2 Two-commodity problems

We now focus on two-commodity problems. Further, as Triangle and Monotonicity constraints are not involved in the valid inequalities presented in Chapter 4, we restrict the theoretical analysis of two-commodity problems to the GCT-NPP. This section aims to point out several facets of the convex hull of (HP3) feasible solutions, i.e., for the GCT-NPP.

Let $\mathcal{P}$ be the convex hull of feasible solutions for the two-commodity GCT-NPP, i.e.,

$$\mathcal{P} = conv\left\{(\mathbf{t};\mathbf{p^{k_1}};\mathbf{p^{k_2}};\mathbf{x^{k_1}};\mathbf{x^{k_2}}) \in \mathbb{R}^n_+ \times \mathbb{R}^{2n}_+ \times \{0,1\}^{2n} : (3.21) - (3.27)\right\},$$

where n is the number of toll arcs.

Let us denote by $\mathbf{e_a}$ a unit vector in the direction a, and by ϵ a very small non negative scalar. Further, let $M^k_a = \max\{0, c^k_{od} - c^k_a\}$ as defined in (HP3), and $N_a = \max_{k\in\mathcal{K}} M^k_a + \epsilon$. This last choice, which is not exactly the same as in Chapter 3, is motivated by the fact that we need some flexibility to be able to vary the $t_a : a \in \mathcal{A}$ variables when the corresponding toll arcs are not used by any commodity. However, note that this choice does not change the set of optimal solutions for the problem.

Throughout the analysis, we also make the following assumptions.

Assumption 1: *For all $k \in \mathcal{K}$ and for all $a \in \mathcal{A}$, $M^k_a > 0$.*

By contrast, if there exists $k \in \mathcal{K}$ and $a \in \mathcal{A}$ with $M^k_a = 0$, all points of $\mathcal{P}$ lie on an hyperplane $p^k_a = 0$. Hence $\mathcal{P}$ is not full dimensional and the proofs presented further need to be transformed accordingly.

Assumption 2: *For all $a \in \mathcal{A}$, either $M^{k_1}_a \neq M^{k_2}_a$, or there exists $b \in \mathcal{A} \setminus \{a\}$ such that $c^{k_2}_a - c^{k_1}_a \neq c^{k_2}_b - c^{k_1}_b$.*

Note that this assumption is not very restrictive. Indeed, it excludes the particular case in which $c^{k_2}_a = c^{k_1}_a + K$ ($K \in \mathbb{R}$) for all $a \in \mathcal{A} \cup \{od\}$. In this case, all points of $\mathcal{P}$ lie on the following hyperplane:

$$\sum_{a\in\mathcal{A}}(p^{k_1}_a + c^{k_1}_a x^{k_1}_a) + c^{k_1}_{od}(1 - \sum_{a\in\mathcal{A}} x^{k_1}_a) + K = \sum_{a\in\mathcal{A}}(p^{k_2}_a + c^{k_2}_a x^{k_2}_a) + c^{k_2}_{od}(1 - \sum_{a\in\mathcal{A}} x^{k_2}_a).$$

Hence the cost structure is the same for commodities k_1 and k_2, and it becomes a single commodity problem.

Let $\mathcal{H} = \big\{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : \mu\mathbf{t} + \nu^{k_1}\mathbf{p}^{k_1} + \nu^{k_2}\mathbf{p}^{k_2} + \xi^{k_1}\mathbf{x}^{k_1} + \xi^{k_2}\mathbf{x}^{k_2} = \phi\big\}$ and assume that all points of $\mathcal{P} \cap \mathcal{H}$ lie on a generic hyperplane $\mathcal{G} = \big\{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : \alpha\mathbf{t} + \beta^{k_1}\mathbf{p}^{k_1} + \beta^{k_2}\mathbf{p}^{k_2} + \gamma^{k_1}\mathbf{x}^{k_1} + \gamma^{k_2}\mathbf{x}^{k_2} = \delta\big\}$. The following lemmas show that properties on the coefficients of $\mathcal{H}$ impose conditions on the coefficients of $\mathcal{G}$. Note that the second and third statements of the following lemmas are variants of the corresponding first statements, and will be used for the proofs of facets of constraints (3.21) and (3.24).

Lemma 4 *Consider that all points of $\mathcal{P} \cap \mathcal{H}$ lie on the generic hyperplane $\mathcal{G}$.*

1. *If the coefficients of $\mathcal{H}$ are such that $\sum_{a\in\mathcal{A}} N_a\mu_a = \phi$ and $\mu_b = 0$ for $b \in \mathcal{A}$, then $\alpha_b = 0$ and $\sum_{a\in\mathcal{A}} N_a\alpha_a = \delta$. Further, if $\phi = 0$ and $\mu_a = 0$ for all $a \in \mathcal{A}$, then $\alpha_a = 0$ for all $a \in \mathcal{A}$ and $\delta = 0$.*

2. *If the coefficients of $\mathcal{H}$ are such that $\mu_a = 0$, $\xi_a^{k_1} = 0$ and $\xi_a^{k_2} = \phi$ for all $a \in \mathcal{A}$, then $\alpha_a = 0$ and $\gamma_a^{k_1} + \gamma_a^{k_2} = \delta$ for all $a \in \mathcal{A}$.*

Proof

1. The points

$$\Big(\sum_{a\in\mathcal{A}} N_a\mathbf{e}_a; 0; 0; 0; 0\Big) \tag{5.13}$$

$$\Big(\sum_{a\in\mathcal{A}} N_a\mathbf{e}_a - \epsilon\mathbf{e}_b; 0; 0; 0; 0\Big) \tag{5.14}$$

are in $\mathcal{P} \cap \mathcal{H}$. As they also belong to the generic hyperplane $\mathcal{G}$, it follows that

$$\sum_{a\in\mathcal{A}} N_a\alpha_a = \delta$$

$$\sum_{a\in\mathcal{A}} N_a\alpha_a - \epsilon\alpha_b = \delta,$$

thus $\alpha_b = 0$ and $\sum_{a\in\mathcal{A}} N_a\alpha_a = \delta$. Further, if $\phi = 0$ and $\mu_a = 0$ for all $a \in \mathcal{A}$, then the points (5.13), (5.14) are in $\mathcal{P} \cap \mathcal{H}$ for all $b \in \mathcal{A}$. Hence $\alpha_a = 0$ for all $a \in \mathcal{A}$ and $\delta = 0$.

2. The points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a; \mathbf{0}; \mathbf{0}; \mathbf{e}_b; \mathbf{e}_b\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a - \epsilon\mathbf{e}_f; \mathbf{0}; \mathbf{0}; \mathbf{e}_b; \mathbf{e}_b\Big)$$

are in $\mathcal{P} \cap \mathcal{H}$ for all $b \in \mathcal{A}$ and for all $f \in \mathcal{A} \setminus \{b\}$. As they also belong to the generic hyperplane $\mathcal{G}$, it follows that

$$\sum_{a\in\mathcal{A}} N_a\alpha_a + \gamma_b^{k_1} + \gamma_b^{k_2} = \delta$$
$$\sum_{a\in\mathcal{A}} N_a\alpha_a - \epsilon\alpha_f + \gamma_b^{k_1} + \gamma_b^{k_2} = \delta.$$

Hence $\alpha_a = 0$ and $\gamma_a^{k_1} + \gamma_a^{k_2} = \delta$ for all $a \in \mathcal{A}$. □

Lemma 5 *Consider that all points of $\mathcal{P} \cap \mathcal{H}$ lie on the generic hyperplane $\mathcal{G}$.*

1. *If the coefficients of $\mathcal{H}$ are such that $\sum_{a\in\mathcal{A}} N_a\mu_a = \phi$, $\mu_b = 0$ and $\xi_b^{k_1} = -\xi_b^{k_2}$ for $b \in \mathcal{A}$, then $\gamma_b^{k_1} = -\gamma_b^{k_2}$. Further, if $\nu_b^{k_1} = -\nu_b^{k_2}$, then $\beta_b^{k_1} = -\beta_b^{k_2}$.*

2. *If the coefficients of $\mathcal{H}$ are such that $\mu_a = 0$, $\xi_a^{k_1} = 0$, $\xi_a^{k_2} = \phi$ for all $a \in \mathcal{A}$ and $\nu_b^{k_1} = -\nu_b^{k_2}$ for $b \in \mathcal{A}$, then $\beta_b^{k_1} = -\beta_b^{k_2}$.*

3. *If the coefficients of $\mathcal{H}$ are such that $\mu_a = 0$ for all $a \in \mathcal{A} \setminus \{b\}$, $\mu_b = -\nu_b^{k_2}$, $\nu_b^{k_1} = 0 = \xi_b^{k_1}$ and $\xi_b^{k_2} = \phi = N_b\mu_b$, then $\alpha_b + \beta_b^{k_1} + \beta_b^{k_2} = 0$ and $\gamma_b^{k_1} + \gamma_b^{k_2} = N_b\alpha_b$.*

Proof

1. As $\mu_b = 0$ and $\sum_{a\in\mathcal{A}} N_a\mu_a = \phi$, one knows that $\alpha_b = 0$ and $\sum_{a\in\mathcal{A}} N_a\alpha_a = \delta$ by Lemma 4.1. Then, the point $\left(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a; \mathbf{0}; \mathbf{0}; \mathbf{e}_b; \mathbf{e}_b\right)$ is in $\mathcal{P}\cap\mathcal{H}$, and one obtains $\gamma_b^{k_1} + \gamma_b^{k_2} = 0$. Further, if $\nu_b^{k_1} = -\nu_b^{k_2}$, then the point $\left(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a + \epsilon\mathbf{e}_b; \epsilon\mathbf{e}_b; \epsilon\mathbf{e}_b; \mathbf{e}_b; \mathbf{e}_b\right)$ also belongs to $\mathcal{P}\cap\mathcal{H}$, which yields $\beta_b^{k_1} = -\beta_b^{k_2}$.

2. The point $\left(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a + \epsilon\mathbf{e}_b; \epsilon\mathbf{e}_b; \epsilon\mathbf{e}_b; \mathbf{e}_b; \mathbf{e}_b\right)$ belongs to $\mathcal{P}\cap\mathcal{H}$. As $\mu_a = 0$, $\xi_a^{k_1} = 0$ and $\xi_a^{k_2} = \phi$ for all $a \in \mathcal{A}$ imply $\alpha_a = 0$ and $\gamma_a^{k_1} + \gamma_a^{k_2} = \delta$ for all $a \in \mathcal{A}$ by Lemma 4.2, one obtains $\beta_b^{k_1} = -\beta_b^{k_2}$.

3. As $\mu_a = 0$ for all $a \in \mathcal{A}\setminus\{b\}$ and $\phi = N_b\mu_b$, one knows that $\alpha_a = 0$ for all $a \in \mathcal{A}\setminus\{b\}$ and $N_b\alpha_b = \delta$ by Lemma 4.1. Further, the points $\left(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a; \mathbf{0}; \mathbf{0}; \mathbf{e}_b; \mathbf{e}_b\right)$ and $\left(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a + \epsilon\mathbf{e}_b; \epsilon\mathbf{e}_b; \epsilon\mathbf{e}_b; \mathbf{e}_b; \mathbf{e}_b\right)$ are in $\mathcal{P}\cap\mathcal{H}$, thus one obtains

$$\begin{aligned}
&\gamma_b^{k_1} + \gamma_b^{k_2} = N_b\alpha_b \\
&\epsilon(\alpha_b + \beta_b^{k_1} + \beta_b^{k_2}) + \gamma_b^{k_1} + \gamma_b^{k_2} = N_b\alpha_b.
\end{aligned}$$

   and the result follows. □

Lemma 6 *Consider that all points of $\mathcal{P}\cap\mathcal{H}$ lie on the generic hyperplane $\mathcal{G}$ and let $b \in \mathcal{A}$ such that $M_b^{k_1} < M_b^{k_2}$.*

1. *If the coefficients of $\mathcal{H}$ are such that $\sum_{a\in\mathcal{A}} N_a\mu_a = \phi$, $\mu_b = 0$ and $\nu_b^{k_2} = 0 = \xi_b^{k_2}$, then $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$.*

2. *If the coefficients of $\mathcal{H}$ are such that $\mu_a = 0$, $\xi_a^{k_1} = 0$, $\xi_a^{k_2} = \phi$ for all $a \in \mathcal{A}$ and $\nu_b^{k_2} = 0$, then $\beta_b^{k_2} = 0$ and $\gamma_b^{k_2} = \delta$.*

3. If the coefficients of $\mathcal{H}$ are such that $\mu_a = 0$ for all $a \in \mathcal{A} \setminus \{b\}$, $\mu_b = -\nu_b^{k_2}$ and $\xi_b^{k_2} = \phi = N_b\mu_b$, then $\alpha_b = -\beta_b^{k_2}$ and $\gamma_b^{k_2} = N_b\alpha_b$.

Proof

The points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a + M_b^{k_1}\mathbf{e}_b; \mathbf{0}; M_b^{k_1}\mathbf{e}_b; \mathbf{0}; \mathbf{e}_b\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a + (M_b^{k_1}+\epsilon)\mathbf{e}_b; \mathbf{0}; (M_b^{k_1}+\epsilon)\mathbf{e}_b; \mathbf{0}; \mathbf{e}_b\Big)$$

are in $\mathcal{P} \cap \mathcal{H}$.

1. As $\mu_b = 0$, $\sum_{a\in\mathcal{A}} N_a\mu_a = \phi$ imply $\alpha_b = 0$ and $\sum_{a\in\mathcal{A}} N_a\alpha_a = \delta$ by Lemma 4.1, one obtains

$$M_b^{k_1}\beta_b^{k_2} + \gamma_b^{k_2} = 0$$
$$(M_b^{k_1}+\epsilon)\beta_b^{k_2} + \gamma_b^{k_2} = 0,$$

thus $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$.

2. As $\mu_a = 0$, $\xi_a^{k_1} = 0$ and $\xi_a^{k_2} = \phi$ for all $a \in \mathcal{A}$ imply $\alpha_a = 0$ for all $a \in \mathcal{A}$ by Lemma 4.2, one obtains

$$M_b^{k_1}\beta_b^{k_2} + \gamma_b^{k_2} = \delta$$
$$(M_b^{k_1}+\epsilon)\beta_b^{k_2} + \gamma_b^{k_2} = \delta,$$

thus $\beta_b^{k_2} = 0$ and $\gamma_b^{k_2} = \delta$.

3. As $\mu_a = 0$ for all $a \in \mathcal{A} \setminus \{b\}$ and $\phi = N_b\mu_b$, one knows that $\alpha_a = 0$ for all

$a \in \mathcal{A} \setminus \{b\}$ and $N_b\alpha_b = \delta$ by Lemma 4.1. It follows that

$$M_b^{k_1}\alpha_b + M_b^{k_1}\beta_b^{k_2} + \gamma_b^{k_2} = \delta$$
$$(M_b^{k_1} + \epsilon)\alpha_b + (M_b^{k_1} + \epsilon)\beta_b^{k_2} + \gamma_b^{k_2} = \delta,$$

thus $\alpha_b = -\beta_b^{k_2}$ and $\gamma_b^{k_2} = N_b\alpha_b$. □

Lemma 7 *Consider that all points of $\mathcal{P} \cap \mathcal{H}$ lie on the generic hyperplane $\mathcal{G}$ and let $b, d \in \mathcal{A}$ such that $c_d^{k_2} - c_d^{k_1} \leq c_b^{k_2} - c_b^{k_1}$ and $M_d^{k_1} \leq M_d^{k_2}$ (resp. $M_d^{k_1} \geq M_d^{k_2}$).*

1. *If the coefficients of $\mathcal{H}$ are such that $\sum_{a\in\mathcal{A}} N_a\mu_a = \phi$, $\mu_b = 0 = \mu_d$, $\xi_b^{k_1} = -\xi_d^{k_2}$ and $\nu_b^{k_1} = 0 = \nu_d^{k_2}$, then $\beta_b^{k_1} = -\beta_d^{k_2}$ and $(M_b^{k_1} - M_d^{k_1})\beta_b^{k_1} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$ (resp. $(M_b^{k_2} - M_d^{k_2})\beta_b^{k_1} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$).*

2. *If the coefficients of $\mathcal{H}$ are such that $\mu_a = 0$ for all $a \in \mathcal{A} \setminus \{d\}$, $\mu_d = -\nu_d^{k_2}$, $\xi_d^{k_2} = \phi = N_d\mu_d$ and $\nu_b^{k_1} = 0 = \xi_b^{k_1}$, then $\alpha_d + \beta_b^{k_1} + \beta_d^{k_2} = 0$ and $(M_b^{k_1} - M_d^{k_1})\beta_b^{k_1} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$ (resp. $(M_b^{k_2} - M_d^{k_2})\beta_b^{k_1} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$).*

Proof

If $M_d^{k_1} \leq M_d^{k_2}$, the points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + M_b^{k_1}\mathbf{e_b} + M_d^{k_1}\mathbf{e}_d; M_b^{k_1}\mathbf{e}_b; M_d^{k_1}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + (M_b^{k_1} - \epsilon)\mathbf{e}_b + (M_d^{k_1} - \epsilon)\mathbf{e}_d; (M_b^{k_1} - \epsilon)\mathbf{e}_b; (M_d^{k_1} - \epsilon)\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$

are in $\mathcal{P} \cap \mathcal{H}$. If $M_d^{k_1} \geq M_d^{k_2}$, the points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + M_b^{k_2}\mathbf{e_b} + M_d^{k_2}\mathbf{e}_d; M_b^{k_2}\mathbf{e}_b; M_d^{k_2}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + (M_b^{k_2} - \epsilon)\mathbf{e}_b + (M_d^{k_2} - \epsilon)\mathbf{e}_d; (M_b^{k_2} - \epsilon)\mathbf{e}_b; (M_d^{k_2} - \epsilon)\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$

are in $\mathcal{P} \cap \mathcal{H}$. Let us assume that $M_d^{k_1} \leq M_d^{k_2}$ (the case $M_d^{k_1} \geq M_d^{k_2}$ is similar).

1. As $\mu_b = 0 = \mu_d$ and $\sum_{a \in \mathcal{A}} N_a \mu_a = \phi$ imply $\alpha_b = 0 = \alpha_d$ and $\sum_{a \in \mathcal{A}} N_a \alpha_a = \delta$ by Lemma 4.1, one obtains

$$M_b^{k_1} \beta_b^{k_1} + M_d^{k_1} \beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$$
$$(M_b^{k_1} - \epsilon)\beta_b^{k_1} + (M_d^{k_1} - \epsilon)\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0,$$

thus $\beta_b^{k_1} = -\beta_d^{k_2}$ and $(M_b^{k_1} - M_d^{k_1})\beta_b^{k_1} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$.

2. As $\mu_a = 0$ for all $a \in \mathcal{A} \setminus \{d\}$ and $\phi = N_d \mu_d$, one knows that $\alpha_a = 0$ for all $a \in \mathcal{A} \setminus \{d\}$ by Lemma 4.1. It follows that

$$M_d^{k_1} \alpha_d + M_b^{k_1} \beta_b^{k_1} + M_d^{k_1} \beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$$
$$(M_d^{k_1} - \epsilon)\alpha_d + (M_b^{k_1} - \epsilon)\beta_b^{k_1} + (M_d^{k_1} - \epsilon)\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta,$$

thus $\alpha_d + \beta_b^{k_1} + \beta_d^{k_2} = 0$ and $(M_b^{k_1} - M_d^{k_1})\beta_b^{k_1} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$. □

Lemma 8 *Consider that all points of $\mathcal{P} \cap \mathcal{H}$ lie on the generic hyperplane $\mathcal{G}$ and let $b, d \in \mathcal{A}$ such that $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$.*

1. *If the coefficients of $\mathcal{H}$ are such that $\sum_{a \in \mathcal{A}} N_a \mu_a = \phi$, $\mu_b = 0 = \mu_d$, $\xi_b^{k_1} = -\xi_d^{k_2}$ and $\nu_b^{k_1} = 0 = \nu_d^{k_2}$, then $\beta_b^{k_1} = 0$ and $\min\{M_d^{k_1}, M_d^{k_2}\}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$.*

2. *If the coefficients of $\mathcal{H}$ are such that $\mu_a = 0$, $\xi_a^{k_1} = 0$, $\xi_a^{k_2} = \phi$ for all $a \in \mathcal{A}$ and $\nu_b^{k_1} = 0 = \nu_d^{k_2}$, then $\beta_b^{k_1} = 0$ and $\min\{M_d^{k_1}, M_d^{k_2}\}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$.*

3. *If the coefficients of $\mathcal{H}$ are such that $\mu_a = 0$ for all $a \in \mathcal{A} \setminus \{d\}$, $\mu_d = -\nu_d^{k_2}$, $\xi_d^{k_2} = \phi = N_d \mu_d$ and $\nu_b^{k_1} = 0 = \xi_b^{k_1}$, then $\beta_b^{k_1} = 0$ and $\min\{M_d^{k_1}, M_d^{k_2}\}(\alpha_d + \beta_d^{k_2}) + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$.*

Proof

If $M_d^{k_1} \leq M_d^{k_2}$, the points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + M_b^{k_1}\mathbf{e}_b + M_d^{k_1}\mathbf{e}_d; M_b^{k_1}\mathbf{e}_b; M_d^{k_1}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + (M_b^{k_1}-\epsilon)\mathbf{e}_b + M_d^{k_1}\mathbf{e}_d; (M_b^{k_1}-\epsilon)\mathbf{e}_b; M_d^{k_1}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$

are in $\mathcal{P}\cap\mathcal{H}$. Otherwise, i.e., if $M_d^{k_1} > M_d^{k_2}$, the points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + M_b^{k_2}\mathbf{e}_b + M_d^{k_2}\mathbf{e}_d; M_b^{k_2}\mathbf{e}_b; M_d^{k_2}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + (M_b^{k_2}+\epsilon)\mathbf{e}_b + M_d^{k_2}\mathbf{e}_d; (M_b^{k_2}+\epsilon)\mathbf{e}_b; M_d^{k_2}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$

are in $\mathcal{P}\cap\mathcal{H}$.

1. As $\mu_b = 0 = \mu_d$ and $\sum_{a\in\mathcal{A}} N_a\mu_a = \phi$, one knows that $\alpha_b = 0 = \alpha_d$ and $\sum_{a\in\mathcal{A}} N_a\alpha_a = \delta$ by Lemma 4.1. If $M_d^{k_1} \leq M_d^{k_2}$, one obtains

$$M_b^{k_1}\beta_b^{k_1} + M_d^{k_1}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$$
$$(M_b^{k_1}-\epsilon)\beta_b^{k_1} + M_d^{k_1}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0,$$

thus $\beta_b^{k_1} = 0$ and $M_d^{k_1}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$. Otherwise, i.e., if $M_d^{k_1} > M_d^{k_2}$, one obtains

$$M_b^{k_2}\beta_b^{k_1} + M_d^{k_2}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$$
$$(M_b^{k_2}+\epsilon)\beta_b^{k_1} + M_d^{k_2}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0,$$

thus $\beta_b^{k_1} = 0$ and $M_d^{k_2}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$.

2. As $\mu_a = 0$, $\xi_a^{k_1} = 0$ and $\xi_a^{k_2} = \phi$ for all $a \in \mathcal{A}$, one knows that $\alpha_a = 0$ for all

$a \in \mathcal{A}$ by Lemma 4.2. If $M_d^{k_1} \leq M_d^{k_2}$, one obtains

$$M_b^{k_1}\beta_b^{k_1} + M_d^{k_1}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$$
$$(M_b^{k_1} - \epsilon)\beta_b^{k_1} + M_d^{k_1}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta.$$

thus $\beta_b^{k_1} = 0$ and $M_d^{k_1}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$. Otherwise, i.e., if $M_d^{k_1} > M_d^{k_2}$, one obtains

$$M_b^{k_2}\beta_b^{k_1} + M_d^{k_2}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$$
$$(M_b^{k_2} + \epsilon)\beta_b^{k_1} + M_d^{k_2}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta.$$

Hence $\beta_b^{k_1} = 0$ and $M_d^{k_2}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$.

3. As $\mu_a = 0$ for all $a \in \mathcal{A} \setminus \{b\}$ and $\phi = N_b\mu_b$, one knows that $\alpha_a = 0$ for all $a \in \mathcal{A} \setminus \{b\}$ by Lemma 4.1. If $M_d^{k_1} \leq M_d^{k_2}$, one obtains

$$M_d^{k_1}\alpha_d + M_b^{k_1}\beta_b^{k_1} + M_d^{k_1}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$$
$$M_d^{k_1}\alpha_d + (M_b^{k_1} - \epsilon)\beta_b^{k_1} + M_d^{k_1}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta.$$

thus $\beta_b^{k_1} = 0$ and $M_d^{k_1}(\alpha_d + \beta_d^{k_2}) + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$. Otherwise, i.e., if $M_d^{k_1} > M_d^{k_2}$, one obtains

$$M_d^{k_2}\alpha_d + M_b^{k_2}\beta_b^{k_1} + M_d^{k_2}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$$
$$M_d^{k_2}\alpha_d + (M_b^{k_2} + \epsilon)\beta_b^{k_1} + M_d^{k_2}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta.$$

Hence $\beta_b^{k_1} = 0$ and $M_d^{k_2}(\alpha_d + \beta_d^{k_2}) + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$. □

Lemma 9 *Consider that all points of $\mathcal{P} \cap \mathcal{H}$ lie on the generic hyperplane $\mathcal{G}$ and let $b \in \mathcal{A}$ such that $M_b^{k_1} \leq M_b^{k_2}$.*

1. *If the coefficients of $\mathcal{H}$ are such that $\sum_{a \in \mathcal{A}} N_a \mu_a = \phi$, $\mu_b = 0$ and $\xi_b^{k_2} = -M_b^{k_2} \nu_b^{k_2}$, then $\gamma_b^{k_2} = -M_b^{k_2} \beta_b^{k_2}$.*

2. *If the coefficients of $\mathcal{H}$ are such that $\mu_a = 0$, $\xi_a^{k_1} = 0$, $\xi_a^{k_2} = \phi$ for all $a \in \mathcal{A}$ and $\nu_b^{k_2} = 0$, then $M_b^{k_2} \beta_b^{k_2} + \gamma_b^{k_2} = \delta$.*

Proof

The points $\left(\sum_{a \in \mathcal{A} \setminus \{b\}} N_a \mathbf{e}_a + M_b^{k_2} \mathbf{e}_b; \mathbf{0}; M_b^{k_2} \mathbf{e}_b; \mathbf{0}; \mathbf{e}_b\right)$ are in $\mathcal{P} \cap \mathcal{H}$.

1. As $\mu_b = 0$, $\sum_{a \in \mathcal{A}} N_a \mu_a = \phi$ imply $\alpha_b = 0$ and $\sum_{a \in \mathcal{A}} N_a \alpha_a = \delta$ by Lemma 4.1, one obtains $\gamma_b^{k_2} = -M_b^{k_2} \beta_b^{k_2}$.

2. As $\mu_a = 0$, $\xi_a^{k_1} = 0$ and $\xi_a^{k_2} = \phi$ for all $a \in \mathcal{A}$ imply $\alpha_a = 0$ for all $a \in \mathcal{A}$ by Lemma 4.2, one obtains $M_b^{k_2} \beta_b^{k_2} + \gamma_b^{k_2} = \delta$. □

Lemma 10 *Consider that all points of $\mathcal{P} \cap \mathcal{H}$ lie on the generic hyperplane $\mathcal{G}$ and let $b \in \mathcal{A}$ such that $M_b^{k_1} > M_b^{k_2}$. If the coefficients of $\mathcal{H}$ are such that $\sum_{a \in \mathcal{A}} N_a \mu_a = \phi$, $\mu_b = 0$, $\xi_b^{k_1} = -M_b^{k_2} \nu_b^{k_1}$ and $\xi_b^{k_2} = -M_b^{k_2} \nu_b^{k_2}$, then $\gamma_b^{k_1} = -M_b^{k_2} \beta_b^{k_1}$ and $\gamma_b^{k_2} = -M_b^{k_2} \beta_b^{k_2}$.*

Proof

As $\mu_b = 0$, $\sum_{a \in \mathcal{A}} N_a \mu_a = \phi$, one knows that $\alpha_b = 0$ and $\sum_{a \in \mathcal{A}} N_a \alpha_a = \delta$ by Lemma 4.1. Further, permuting the commodity indices k_1 and k_2 in Lemma 6 yields $\beta_b^{k_1} = 0 = \gamma_b^{k_1}$. Hence, as the points

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b\}} N_a \mathbf{e}_a + M_b^{k_2} \mathbf{e}_b; M_b^{k_2} \mathbf{e}_b; \mathbf{0}; \mathbf{e}_b; \mathbf{0}\Big)$$

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b\}} N_a \mathbf{e}_a + M_b^{k_2} \mathbf{e}_b; M_b^{k_2} \mathbf{e}_b; M_b^{k_2} \mathbf{e}_b; \mathbf{e}_b; \mathbf{e}_b\Big),$$

belong to $\mathcal{P} \cap \mathcal{H}$, one obtains

$$M_b^{k_2}\beta_b^{k_1} + \gamma_b^{k_1} = 0$$
$$M_b^{k_2}\beta_b^{k_1} + M_b^{k_2}\beta_b^{k_2} + \gamma_b^{k_1} + \gamma_b^{k_2} = 0.$$

The result follows. □

Using these lemmas, we can prove that $\mathcal{P}$ is full dimensional.

Proposition 15 *The polyhedron $\mathcal{P}$ has full dimension, i.e., $Dim(\mathcal{P}) = 5n$.*

Proof

Let $\mathcal{H} = \{(\mathbf{t};\mathbf{p};\mathbf{x}) : 0\mathbf{x} = 0\}$. Suppose by contradiction that the points of $\mathcal{P} \cap \mathcal{H}$ lie on a generic hyperplane $\mathcal{G}$. By Lemma 4.1, one knows that $\alpha_b = 0$ for all $b \in \mathcal{A}$ and $\delta = 0$. Further, by Lemma 5.1, one obtains $\beta_b^{k_1} = -\beta_b^{k_2}$ and $\gamma_b^{k_1} = -\gamma_b^{k_2}$ for all $b \in \mathcal{A}$.

Next, for all $b \in \mathcal{A}$ such that $M_b^{k_1} < M_b^{k_2}$, one has $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$ by Lemma 6.1. For all $b \in \mathcal{A}$ such that $M_b^{k_1} > M_b^{k_2}$, permuting the commodity indices k_1 and k_2 in Lemma 6.1 yields $\beta_b^{k_1} = 0 = \gamma_b^{k_1}$. Hence $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ and $\gamma_b^{k_1} = 0 = \gamma_b^{k_2}$ for all $b \in \mathcal{A}$ such that $M_b^{k_1} \neq M_b^{k_2}$.

Now, for all $b \in \mathcal{A}$ such that $M_b^{k_1} = M_b^{k_2}$, Assumption 2 ensures that there exists $d \in \mathcal{A}\backslash\{b\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$. Without loss of generality, we can assume that $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$, and Lemma 8.1 yields $\beta_b^{k_1} = 0$. Further, permuting the commodity indices k_1 and k_2 in Lemma 9.1 yields $\gamma_b^{k_1} = 0$. Hence $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ and $\gamma_b^{k_1} = 0 = \gamma_b^{k_2}$ for all $b \in \mathcal{A}$ such that $M_b^{k_1} = M_b^{k_2}$, and the result follows. □

One can also prove that several (HP3) constraints, as well as most valid inequalities presented in Chapter 4, are facet defining for $\mathcal{P}$. Let $(\mathbf{t};\mathbf{p}^{k_1};\mathbf{p}^{k_2};\mathbf{x}^{k_1};\mathbf{x}^{k_2})$ be points of $\mathcal{P}$. In order to prove that a given inequality is facet defining for $\mathcal{P}$, we consider $\mathcal{H}$ as the hyperplane induced by this given inequality, and $\mathcal{G}$ a generic hyperplane. Then, we prove that $\mathcal{G} = \mathcal{H}$.

Proposition 16 *Constraints (3.21)* $\sum_{b\in\mathcal{A}} x_b^{k_2} \leq 1$ *are facet defining for* $\mathcal{P}$ *if and only if, for each* $b \in \mathcal{A}$ *such that* $M_b^{k_1} > M_b^{k_2}$*, there exists* $d \in \mathcal{A} \setminus \{b\}$ *such that* $M_d^{k_1} \leq M_d^{k_2}$ *and* $c_d^{k_2} - c_d^{k_1} \neq c_b^{k_2} - c_b^{k_1}$.

Proof

Let $\mathcal{H} = \{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : \sum_{b\in\mathcal{A}} x_b^{k_2} = 1\}$. By Lemma 4.2, one knows that $\alpha_b = 0$ for all $b \in \mathcal{A}$ and $\gamma_b^{k_1} + \gamma_b^{k_2} = \delta$ for all $b \in \mathcal{A}$. One also obtains $\beta_b^{k_1} = -\beta_b^{k_2}$ by Lemma 5.2.

Further, for all $b \in \mathcal{A}$ such that $M_b^{k_1} < M_b^{k_2}$, Lemma 6.2 yields $\beta_b^{k_2} = 0$ and $\gamma_b^{k_2} = \delta$, thus also $\beta_b^{k_1} = 0 = \gamma_b^{k_1}$.
For all $b \in \mathcal{A}$ such that $M_b^{k_1} = M_b^{k_2}$, we know by Assumption 2 that there exists $d \in \mathcal{A} \setminus \{b\}$ such that $c_d^{k_2} - c_d^{k_1} \neq c_b^{k_2} - c_b^{k_1}$. If $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1} = c_{od}^{k_2} - c_{od}^{k_1}$, Lemma 8.2 yields $\beta_b^{k_1} = 0 = \beta_b^{k_2}$. Further, one obtains $\gamma_b^{k_2} = \delta$, thus also $\gamma_b^{k_1} = 0$, by Lemma 9.2. If $c_{od}^{k_2} - c_{od}^{k_1} = c_b^{k_2} - c_b^{k_1} < c_d^{k_2} - c_d^{k_1}$, permuting k_1 and k_2 in Lemma 8.2 yields $\beta_b^{k_2} = 0 = \beta_b^{k_1}$. One also obtains $\gamma_b^{k_2} = \delta$ and $\gamma_b^{k_1} = 0$ by Lemma 9.2.
Otherwise, i.e., for all $b \in \mathcal{A}$ such that $M_b^{k_1} > M_b^{k_2}$, the proposition hypothesis ensures that there exists $d \in \mathcal{A}\setminus\{b\}$ such that $M_d^{k_1} \leq M_d^{k_2}$ and $c_d^{k_2} - c_d^{k_1} \neq c_b^{k_2} - c_b^{k_1}$, which implies $c_d^{k_2} - c_d^{k_1} \leq c_{od}^{k_2} - c_{od}^{k_1} < c_b^{k_2} - c_b^{k_1}$. It follows that $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ and $M_d^{k_1}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$ by Lemma 8.2. As $\beta_d^{k_2} = 0$ and $\gamma_d^{k_2} = \delta$, one obtains $\gamma_b^{k_1} = 0$ and $\gamma_b^{k_2} = \delta$.

Finally, assume there exists $b \in \mathcal{A}$, $M_b^{k_1} > M_b^{k_2}$, such that there does not exist any $d \in \mathcal{A} \setminus \{b\}$ with $M_d^{k_1} \leq M_d^{k_2}$ and $c_d^{k_2} - c_d^{k_1} \neq c_b^{k_2} - c_b^{k_1}$. If there does not exist any $d \in \mathcal{A} \setminus \{b\}$ with $c_d^{k_2} - c_d^{k_1} \neq c_b^{k_2} - c_b^{k_1}$, then $c_{od}^{k_2} - c_{od}^{k_1} < c_b^{k_2} - c_b^{k_1} = c_d^{k_2} - c_d^{k_1}$ for all $b, d \in \mathcal{A}$, thus $M_d^{k_1} > M_d^{k_2}$ for all $d \in \mathcal{A}$. Further, if there exists $d \in \mathcal{A} \setminus \{b\}$ with $c_d^{k_2} - c_d^{k_1} \neq c_b^{k_2} - c_b^{k_1}$ but $M_d^{k_1} > M_d^{k_2}$, one also obtains $M_d^{k_1} > M_d^{k_2}$ for all $d \in \mathcal{A}$.

Hence, as points of $\mathcal{H}$ must satisfy $\sum_{b\in\mathcal{A}} x_b^{k_2} = 1$ and $M_b^{k_1} > M_b^{k_2}$ for all $b \in \mathcal{A}$, it follows that points of $\mathcal{P} \cap \mathcal{H}$ lie on the hyperplane $\sum_{b\in\mathcal{A}} x_b^{k_1} = 1$ by Lemma 3. The result follows. $\square$

Note that the condition imposed in the previous proposition is not very restrictive. Indeed, it only excludes the case in which $c_{od}^{k_2} - c_{od}^{k_1} < c_b^{k_2} - c_b^{k_1} = c_d^{k_2} - c_d^{k_1}$ for all $b, d \in \mathcal{A}$, i.e., $c_b^{k_2} = c_b^{k_1} + K$ $(K \in \mathbb{R})$ for all $b \in \mathcal{A}$.

In the next proposition, we state the conditions in which the Profit Upper Bound constraints define facets of $\mathcal{P}$. Let $\tilde{a} \in \mathcal{A}$. As the corresponding constraints contain constants $M_{\tilde{a}}^k : k = k_1, k_2$ which depend on the commodity k, we intuitively deduce that these constants will be involved in the conditions.

Proposition 17 *Constraints (3.23) $p_{\tilde{a}}^{k_2} \leq M_{\tilde{a}}^{k_2} x_{\tilde{a}}^{k_2}$ are facet defining for $\mathcal{P}$ if and only if $M_{\tilde{a}}^{k_2} < M_{\tilde{a}}^{k_1}$ or there exists $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(\tilde{a}, k_1) \sim (b, k_2)$.*

Proof

Let $\mathcal{H} = \left\{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : p_{\tilde{a}}^{k_2} = M_{\tilde{a}}^{k_2} x_{\tilde{a}}^{k_2}\right\}$. Lemma 4.1 yields $\alpha_b = 0$ for all $b \in \mathcal{A}$ and $\delta = 0$. Further, for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$, one obtains $\beta_b^{k_1} = -\beta_b^{k_2}$ and $\gamma_b^{k_1} = -\gamma_b^{k_2}$ by Lemma 5.2.

Next, for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $M_b^{k_1} < M_b^{k_2}$, Lemma 6.1 yields $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$. If $M_b^{k_2} < M_b^{k_1}$, permuting the commodity indices k_1 and k_2 in Lemma 6.1 also yields $\beta_b^{k_1} = 0 = \gamma_b^{k_1}$. It follows that $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ and $\gamma_b^{k_1} = 0 = \gamma_b^{k_2}$ for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $M_b^{k_1} \neq M_b^{k_2}$.

Otherwise, i.e., if $M_b^{k_1} = M_b^{k_2}$, we know by Assumption 2 that there exists $d \in \mathcal{A} \setminus \{b\}$ (possibly $\tilde{a}$) such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$.

If $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$ with $d \neq \tilde{a}$, Lemma 8.1 yields $\beta_b^{k_1} = 0$, thus also $\beta_b^{k_2} = 0$. Further, one obtains $\gamma_b^{k_2} = 0$ by Lemma 9.1. Note that we can assume $d \neq \tilde{a}$. Indeed, consider $c_{\tilde{a}}^{k_2} - c_{\tilde{a}}^{k_1} < c_b^{k_2} - c_b^{k_1}$ and there does not exist any $d \in \mathcal{A} \setminus \{\tilde{a}, b\}$

with $c_d^{k_2} - c_d^{k_1} \neq c_b^{k_2} - c_b^{k_1}$. Then one obtains

$$c_{\tilde{a}}^{k_2} - c_{\tilde{a}}^{k_1} < c_b^{k_2} - c_b^{k_1} = c_d^{k_2} - c_d^{k_1} = c_{od}^{k_2} - c_{od}^{k_1},$$

thus $M_{\tilde{a}}^{k_1} < M_{\tilde{a}}^{k_2}$ and there does not exist any $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(\tilde{a}, k_1) \sim (b, k_2)$ by Lemma 2, which is in contradiction with the proposition hypothesis.
If $c_b^{k_2} - c_b^{k_1} < c_d^{k_2} - c_d^{k_1}$, permuting the commodity indices k_1, k_2 in Lemma 8.1 yields $\beta_b^{k_2} = 0$. One also obtains $\gamma_b^{k_2} = 0$ by Lemma 9.1. Hence $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ and $\gamma_b^{k_1} = 0 = \gamma_b^{k_2}$ for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $M_b^{k_1} = M_b^{k_2}$.

Next, if $M_{\tilde{a}}^{k_2} < M_{\tilde{a}}^{k_1}$, setting $b = \tilde{a}$ and permuting the commodity indices k_1, k_2 in Lemma 6.1 yields $\beta_{\tilde{a}}^{k_1} = 0 = \gamma_{\tilde{a}}^{k_1}$. Further, setting $b = \tilde{a}$ in Lemma 10.1 yields $\gamma_{\tilde{a}}^{k_2} = -M_{\tilde{a}}^{k_2} \beta_{\tilde{a}}^{k_2}$.
Otherwize, i.e., if $M_{\tilde{a}}^{k_2} \geq M_{\tilde{a}}^{k_1}$, the proposition hypothesis ensures that there exists $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(\tilde{a}, k_1) \sim (b, k_2)$. As one can check that $M_b^{k_1} \leq M_b^{k_2}$, setting $b = \tilde{a}$ and $d = b$ in Lemma 7.1 yields $\beta_{\tilde{a}}^{k_1} = -\beta_b^{k_2}$ and $(M_{\tilde{a}}^{k_1} - M_b^{k_1}) \beta_{\tilde{a}}^{k_1} + \gamma_{\tilde{a}}^{k_1} + \gamma_b^{k_2} = 0$. As $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$, one obtains $\beta_{\tilde{a}}^{k_1} = 0 = \gamma_{\tilde{a}}^{k_1}$. Further, setting $b = \tilde{a}$ in Lemma 9.1 yields $\gamma_{\tilde{a}}^{k_2} = -M_{\tilde{a}}^{k_2} \beta_{\tilde{a}}^{k_2} = 0$.

Finally, assume $M_{\tilde{a}}^{k_2} \geq M_{\tilde{a}}^{k_1}$ and there does not exist any $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(\tilde{a}, k_1) \sim (b, k_2)$. Hence $x_{\tilde{a}}^{k_1} = 1$ implies that either $x_{\tilde{a}}^{k_2} = 1$, or $x_b^{k_2} = 0$ for all $b \in \mathcal{A}$. Then:

- if $M_{\tilde{a}}^{k_2} = M_{\tilde{a}}^{k_1}$, $x_{\tilde{a}}^{k_1} = 1$ implies that either $x_{\tilde{a}}^{k_2} = 1$ and $p_{\tilde{a}}^{k_1} = p_{\tilde{a}}^{k_2} = M_{\tilde{a}}^{k_2}$, or $x_b^{k_2} = 0$ for all $b \in \mathcal{A}$ and $M_{\tilde{a}}^{k_2} \leq t_{\tilde{a}} = p_{\tilde{a}}^{k_1} \leq M_{\tilde{a}}^{k_1}$, thus $p_{\tilde{a}}^{k_1} = M_{\tilde{a}}^{k_2}$. This means that all points of $\mathcal{P} \cap \mathcal{H}$ lie one the hyperplane $p_{\tilde{a}}^{k_1} = M_{\tilde{a}}^{k_2} x_{\tilde{a}}^{k_1}$, which is a contradiction.

- If $M_{\tilde{a}}^{k_2} > M_{\tilde{a}}^{k_1}$, $x_{\tilde{a}}^{k_1} = 1$ implies that there exists a toll arc $b \in \mathcal{A}$ such that $x_b^{k_2} = 1$ by Lemma 3. Hence $x_{\tilde{a}}^{k_2} = 1$. However, as points of $\mathcal{H}$ satisfy

$p_{\tilde{a}}^{k_2} = M_{\tilde{a}}^{k_2} x_{\tilde{a}}^{k_2}$, this yields $p_{\tilde{a}}^{k_1} = p_{\tilde{a}}^{k_2} = M_{\tilde{a}}^{k_2} > M_{\tilde{a}}^{k_1}$, which is a contradiction. Hence all points of $\mathcal{P} \cap \mathcal{H}$ lie on the hyperplane $x_{\tilde{a}}^{k_1} = 0$.

The result follows. □

As expected, the conditions imposed in the previous proposition depend on the constants $M_{\tilde{a}}^{k} : k = k_1, k_2$. Similar conditions must be imposed such that constraints (3.24) define facets of $\mathcal{P}$.

Proposition 18 *Constraints (3.24) $t_{\tilde{a}} - p_{\tilde{a}}^{k_2} \leq N_{\tilde{a}}(1 - x_{\tilde{a}}^{k_2})$ are facet defining for $\mathcal{P}$ if and only if $M_{\tilde{a}}^{k_1} < M_{\tilde{a}}^{k_2}$ or there exists $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(b, k_1) \sim (\tilde{a}, k_2)$.*

Proof

Let $\mathcal{H} = \{(\mathbf{t};\mathbf{p};\mathbf{x}) : t_{\tilde{a}} - p_{\tilde{a}}^{k_2} = N_{\tilde{a}}(1 - x_{\tilde{a}}^{k_2})\}$. Lemma 4.1 yields $\alpha_b = 0$ for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$ and $\delta = N_{\tilde{a}}\alpha_{\tilde{a}}$. Further, for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$, one obtains $\beta_b^{k_1} = -\beta_b^{k_2}$ and $\gamma_b^{k_1} = -\gamma_b^{k_2}$ by Lemma 5.3.

Next, for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $M_b^{k_1} < M_b^{k_2}$, Lemma 6.3 yields $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$. If $M_b^{k_2} < M_b^{k_1}$, permuting the commodity indices k_1 and k_2 in Lemma 6.3 also yields $\beta_b^{k_1} = 0 = \gamma_b^{k_1}$. As $\beta_b^{k_1} = -\beta_b^{k_2}$ and $\gamma_b^{k_1} = -\gamma_b^{k_2}$ for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$, one obtains $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ and $\gamma_b^{k_1} = 0 = \gamma_b^{k_2}$ for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $M_b^{k_1} \neq M_b^{k_2}$.
For all $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $M_b^{k_1} = M_b^{k_2}$, Assumption 2 ensures that there exists $d \in \mathcal{A} \setminus \{b\}$ (possibly $\tilde{a}$) such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$. If $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$ with $d \neq \tilde{a}$, one obtains $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ by Lemma 8.1. If $d = \tilde{a}$, Lemma 8.3 yields the same conclusion. One also obtains $\gamma_b^{k_2} = 0 = \gamma_b^{k_1}$ by Lemma 9.1.
If $c_b^{k_2} - c_b^{k_1} < c_d^{k_2} - c_d^{k_1}$ with $d \neq \tilde{a}$, permuting the commodity indices k_1, k_2 in Lemma 8.1 yields $\beta_b^{k_2} = 0 = \beta_b^{k_1}$. One also obtains $\gamma_b^{k_2} = 0 = \gamma_b^{k_1}$ by Lemma 9.1. Note that we can assume $d \neq \tilde{a}$. By contradiction, consider $c_b^{k_2} - c_b^{k_1} < c_{\tilde{a}}^{k_2} - c_{\tilde{a}}^{k_1}$ and there does not exist any $d \in \mathcal{A} \setminus \{\tilde{a}, b\}$ with $c_d^{k_2} - c_d^{k_1} \neq c_b^{k_2} - c_b^{k_1}$. It follows

that

$$c_{od}^{k_2} - c_{od}^{k_1} = c_d^{k_2} - c_d^{k_1} = c_b^{k_2} - c_b^{k_1} < c_{\tilde{a}}^{k_2} - c_{\tilde{a}}^{k_1},$$

thus $M_{\tilde{a}}^{k_2} < M_{\tilde{a}}^{k_1}$ and there does not exist any $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(b, k_1) \sim (\tilde{a}, k_2)$ by Lemma 2, which is in contradiction with the proposition hypothesis.

Further, setting $b = \tilde{a}$ in Lemma 5.3 yields $\gamma_{\tilde{a}}^{k_1} + \gamma_{\tilde{a}}^{k_2} = N_{\tilde{a}}\alpha_{\tilde{a}}$ and $\alpha_{\tilde{a}} + \beta_{\tilde{a}}^{k_1} + \beta_{\tilde{a}}^{k_2} = 0$. Next, if $M_{\tilde{a}}^{k_1} < M_{\tilde{a}}^{k_2}$, setting $b = \tilde{a}$ in Lemma 6.3 yields $\alpha_{\tilde{a}} = -\beta_{\tilde{a}}^{k_2}$ and $\gamma_{\tilde{a}}^{k_2} = N_{\tilde{a}}\alpha_{\tilde{a}}$. Otherwize, i.e., if $M_{\tilde{a}}^{k_1} \geq M_{\tilde{a}}^{k_2}$, the proposition hypothesis ensures that there exists $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(b, k_1) \sim (\tilde{a}, k_2)$. As one can check that $M_b^{k_2} \leq M_b^{k_1}$, setting $b = \tilde{a}$, $d = b$ and permuting the commodity indices k_1, k_2 in Lemma 7.2 yields $\alpha_{\tilde{a}} + \beta_b^{k_1} + \beta_{\tilde{a}}^{k_2} = 0$ and $(M_b^{k_2} - M_d^{k_2})\beta_b^{k_1} + \gamma_b^{k_1} + \gamma_d^{k_2} = \delta$. As $\beta_b^{k_1} = 0 = \gamma_b^{k_1}$, one obtains $\alpha_{\tilde{a}} = -\beta_{\tilde{a}}^{k_2}$ and $\gamma_{\tilde{a}}^{k_2} = \delta = N_{\tilde{a}}\alpha_{\tilde{a}}$.

Finally, assume $M_{\tilde{a}}^{k_1} \geq M_{\tilde{a}}^{k_2}$ and there does not exist any $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(b, k_1) \sim (\tilde{a}, k_2)$. First note that any point of $\mathcal{P} \cap \mathcal{H}$ such that $x_{\tilde{a}}^{k_1} = 1$ also satisfies $x_{\tilde{a}}^{k_2} = 1$. Indeed, assume $x_{\tilde{a}}^{k_1} = 1$ and $x_{\tilde{a}}^{k_2} = 0$. Then $t_{\tilde{a}} = N_{\tilde{a}} > M_{\tilde{a}}^{k_1}$, which is a contradiction. Hence:

- if $M_{\tilde{a}}^{k_2} = M_{\tilde{a}}^{k_1}$, Assumption 2 ensures that there exists $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_a^{k_2} - c_a^{k_1}$, which implies either $(b, k_1) \sim (\tilde{a}, k_2)$ or $(\tilde{a}, k_1) \sim (b, k_2)$ by Lemma 2. As there does not exist any $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(b, k_1) \sim (\tilde{a}, k_2)$ by hypothesis, we conclude that $(\tilde{a}, k_1) \sim (b, k_2)$. However, any point of $\mathcal{P} \cap \mathcal{H}$ such that $x_{\tilde{a}}^{k_1} = 1$ also satisfy $x_{\tilde{a}}^{k_2} = 1$. Hence $(\tilde{a}, k_1) \sim (b, k_2)$ cannot happen either, which is in contradiction with Assumption 2.

- Otherwise, i.e., if $M_{\tilde{a}}^{k_1} > M_{\tilde{a}}^{k_2}$, $x_{\tilde{a}}^{k_2} = 1$ implies that there exists a toll arc $b \in \mathcal{A}$ such that $x_b^{k_1} = 1$ by Lemma 3. As there does not exist any $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(b, k_1) \sim (\tilde{a}, k_2)$, one obtains $x_{\tilde{a}}^{k_1} = 1$. Further, as any point of $\mathcal{P} \cap \mathcal{H}$ such that $x_{\tilde{a}}^{k_1} = 1$ also satisfies $x_{\tilde{a}}^{k_2} = 1$, it means that all points of $\mathcal{P} \cap \mathcal{H}$ lie on the hyperplane $x_{\tilde{a}}^{k_1} = x_{\tilde{a}}^{k_2}$. □

Next, we prove that constraints (3.25) do not define facets of $\mathcal{P}$. As the definition of p variables is closely related to the t and x variables, i.e., one has $\mathbf{p^k} = \mathbf{t x^k}$ for all $k \in \mathcal{K}$, this result is not surprising.

Proposition 19 *Constraints (3.25) are not facet defining for $\mathcal{P}$.*

Proof

Let $\mathcal{H} = \{(\mathbf{t};\mathbf{p};\mathbf{x}) : p_{\tilde{a}}^{k_2} = t_{\tilde{a}}\}$. Assuming $x_b^{k_2} = 0$ for all $b \in \mathcal{A}$ yields $p_{\tilde{a}}^{k_2} = 0$ by constraints (3.23), and $t_{\tilde{a}}^{k_2} \geq c_{od}^{k_2} - c_{\tilde{a}}^{k_2} > 0$ by constraints (3.22) and Assumption 1. Hence all points of $\mathcal{P} \cap \mathcal{H}$ lie on the hyperplane $\sum_{b \in \mathcal{A}} x_b^{k_2} = 1$, thus $\mathcal{H}$ is not facet defining for $\mathcal{P}$. □

Finally, we state the conditions in which constraints (3.26) define facets of $\mathcal{P}$. For given $a \in \mathcal{A}$, note that these conditions also depend on the constant $M_a^k : k = k_1, k_2$.

Proposition 20 *Constraints (3.26) $p_{\tilde{a}}^{k_2} \geq 0$ are facet defining for $\mathcal{P}$ if and only if one of the following conditions holds:*

- $M_{\tilde{a}}^{k_2} < M_{\tilde{a}}^{k_1}$;
- $M_{\tilde{a}}^{k_2} > M_{\tilde{a}}^{k_1}$ *and there exists $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(\tilde{a}, k_1) \sim (b, k_2)$;*
- $M_{\tilde{a}}^{k_2} = M_{\tilde{a}}^{k_1}$ *and either there exists $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(\tilde{a}, k_1) \sim (b, k_2)$, or there exists $b \in \mathcal{A} \setminus \{\tilde{a}\}$, $v \in \mathbb{R}$ such that $(b, k_1) \sim (\tilde{a}, k_2)$, $0 \leq v \leq M_b^{k_1}$ and $c_{\tilde{a}}^{k_2} - c_b^{k_2} \leq v \leq c_{\tilde{a}}^{k_1} - c_b^{k_1}$.*

Proof

Let $\mathcal{H} = \{(\mathbf{t};\mathbf{p};\mathbf{x}) : p_{\tilde{a}}^{k_2} = 0\}$. Lemma 4.1 yields $\alpha_b = 0$ for all $b \in \mathcal{A}$ and $\delta = 0$. Further, one obtains $\gamma_b^{k_1} = -\gamma_b^{k_2}$ for all $b \in \mathcal{A}$ and $\beta_b^{k_1} = -\beta_b^{k_2}$ for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$ by Lemma 5.1.

Next, for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $M_b^{k_1} < M_b^{k_2}$, Lemma 6.1 yields $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$. If $M_b^{k_2} < M_b^{k_1}$, permuting the commodity indices k_1 and k_2 in Lemma 6.1 also yields $\beta_b^{k_1} = 0 = \gamma_b^{k_1}$. It follows that $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ and $\gamma_b^{k_1} = 0 = \gamma_b^{k_2}$ for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $M_b^{k_1} \neq M_b^{k_2}$.

Now, for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $M_b^{k_1} = M_b^{k_2}$, Assumption 2 ensures that there exists $d \in \mathcal{A} \setminus \{b\}$ (possibly $\tilde{a}$) such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$.

If $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$ with $d \neq \tilde{a}$, Lemma 8.1 yields $\beta_b^{k_1} = 0$, thus also $\beta_b^{k_2} = 0$. Further, one obtains $\gamma_b^{k_2} = 0$ by Lemma 9.1. Note that we can assume $d \neq \tilde{a}$. Indeed, consider $c_{\tilde{a}}^{k_2} - c_{\tilde{a}}^{k_1} < c_b^{k_2} - c_b^{k_1}$ and there does not exist any $d \in \mathcal{A} \setminus \{\tilde{a}, b\}$ with $c_d^{k_2} - c_d^{k_1} \neq c_b^{k_2} - c_b^{k_1}$. Then one obtains

$$c_{\tilde{a}}^{k_2} - c_{\tilde{a}}^{k_1} < c_b^{k_2} - c_b^{k_1} = c_d^{k_2} - c_d^{k_1} = c_{od}^{k_2} - c_{od}^{k_1},$$

thus $M_{\tilde{a}}^{k_1} < M_{\tilde{a}}^{k_2}$ and there does not exist any $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(\tilde{a}, k_1) \sim (b, k_2)$ by Lemma 2, which is in contradiction with the proposition hypothesis.

If $c_b^{k_2} - c_b^{k_1} < c_d^{k_2} - c_d^{k_1}$, permuting the commodity indices k_1, k_2 in Lemma 8.1 yields $\beta_b^{k_2} = 0$. One also obtains $\gamma_b^{k_2} = 0$ by Lemma 9.1. Hence $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ and $\gamma_b^{k_1} = 0 = \gamma_b^{k_2}$ for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $M_b^{k_1} = M_b^{k_2}$.

Next, if $M_{\tilde{a}}^{k_2} < M_{\tilde{a}}^{k_1}$, setting $b = \tilde{a}$ and permuting the commodity indices in Lemma 6.1 yields $\beta_{\tilde{a}}^{k_1} = 0 = \gamma_{\tilde{a}}^{k_1}$. As $\gamma_{\tilde{a}}^{k_2} = -\gamma_{\tilde{a}}^{k_1}$, one also obtains $\gamma_{\tilde{a}}^{k_2} = 0$.

Otherwize, i.e., if $M_{\tilde{a}}^{k_2} \geq M_{\tilde{a}}^{k_1}$, assume that there exists $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(\tilde{a}, k_1) \sim (b, k_2)$. As one can check that $M_b^{k_1} \leq M_b^{k_2}$, setting $b = \tilde{a}$ and $d = b$ in Lemma 7.1 yields $\beta_{\tilde{a}}^{k_1} = -\beta_b^{k_2}$ and $(M_{\tilde{a}}^{k_1} - M_b^{k_1})\beta_{\tilde{a}}^{k_1} + \gamma_{\tilde{a}}^{k_1} + \gamma_b^{k_2} = 0$. As $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$, one obtains $\beta_{\tilde{a}}^{k_1} = 0 = \gamma_{\tilde{a}}^{k_1}$, thus also $\gamma_{\tilde{a}}^{k_2} = 0$.

Now consider the case $M_{\tilde{a}}^{k_2} = M_{\tilde{a}}^{k_1}$, and assume that there does not exist any $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(\tilde{a}, k_1) \sim (b, k_2)$. This implies $c_{\tilde{a}}^{k_2} - c_{\tilde{a}}^{k_1} < c_b^{k_2} - c_b^{k_1}$, i.e., $(b, k_1) \sim (\tilde{a}, k_2)$ for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$ by Lemma 2. Provided there exists $b \in \mathcal{A} \setminus \{\tilde{a}\}$,

$v \in \mathbb{R}$ such that $c_{\tilde{a}}^{k_2} - c_b^{k_2} \leq v \leq c_{\tilde{a}}^{k_1} - c_b^{k_1}$ and $0 \leq v \leq M_b^{k_1}$, the point

$$\Big(\sum_{a \in \mathcal{A} \setminus \{\tilde{a}, b\}} N_a \mathbf{e}_a + v\mathbf{e}_b; v\mathbf{e}_b; \mathbf{0}; \mathbf{e}_b; \mathbf{e}_{\tilde{a}} \Big)$$

belong to $\mathcal{P} \cap \mathcal{H}$. The existence of v is required since points of $\mathcal{H}$ must satisfy $p_{\tilde{a}}^{k_2} = 0$. As $\beta_b^{k_1} = 0 = \gamma_b^{k_1}$, it follows that $\gamma_{\tilde{a}}^{k_2} = 0$, thus also $\gamma_{\tilde{a}}^{k_1} = 0$. Further, setting $b = \tilde{a}$ and permuting the commodity indices k_1, k_2 in Lemma 9.1 yields $\gamma_{\tilde{a}}^{k_1} = -M_{\tilde{a}}^{k_1} \beta_{\tilde{a}}^{k_1}$, and one obtains $\beta_{\tilde{a}}^{k_1} = 0$.
Note that, if there does not exist such a v, then $c_{\tilde{a}}^{k_2} < c_b^{k_2}$ for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$. As points of $\mathcal{H}$ satisfy $p_{\tilde{a}}^{k_2} = 0$, all points of $\mathcal{P} \cap \mathcal{H}$ lie on the hyperplane $x_{\tilde{a}}^{k_2} = 1$.

Finally, assume $M_{\tilde{a}}^{k_2} > M_{\tilde{a}}^{k_1}$ and there does not exist any $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(\tilde{a}, k_1) \sim (b, k_2)$. This means that $x_{\tilde{a}}^{k_1} = 1$ implies that either $x_{\tilde{a}}^{k_2} = 1$ or $x_b^{k_2} = 0$ for all $b \in \mathcal{A}$. However, as $M_{\tilde{a}}^{k_2} > M_{\tilde{a}}^{k_1}$, $x_{\tilde{a}}^{k_1} = 1$ implies that there exists a toll arc $b \in \mathcal{A}$ such that $x_b^{k_2} = 1$ by Lemma 3. Hence $x_{\tilde{a}}^{k_2} = 1$. As points of $\mathcal{H}$ must satisfy $p_{\tilde{a}}^{k_2} = 0$, one obtains $p_{\tilde{a}}^{k_1} = p_{\tilde{a}}^{k_2} = 0$. It follows that all points of $\mathcal{P} \cap \mathcal{H}$ lie on the hyperplane $p_{\tilde{a}}^{k_1} = 0$, which is a contradiction. □

Hence most constraints of (HP3) define facets of $\mathcal{P}$. Note that, for a given $a \in \mathcal{A}$, additional conditions must be imposed in the previous propositions, which depend on the constants $M_a^k : k = k_1, k_2$. However, these are not so restrictive, as they only exclude very particular cases.

We can also prove that most valid inequalities presented in Chapter 4 define facets of $\mathcal{P}$. The only exception concerns the Strengthened Shortest Path inequalities (4.1), which require several very restrictive conditions to be facet defining for $\mathcal{P}$. Note that these restrictive conditions come from the fact that there does not exist any point of $\mathcal{P} \cap \mathcal{H}$ such that $x_b^{k_2} = 1$ with $b \in \mathcal{A} \setminus (\mathcal{S} \cup \{\tilde{a}\})$. Indeed,

- either $x_d^{k_1} = 0$ for all $d \in \mathcal{A}$, and points in $\mathcal{P} \cap \mathcal{H}$ are such that $p_b^{k_2} = c_{od}^{k_1} - c_b^{k_1} - t_{\tilde{a}}$, which implies $t_{\tilde{a}} = 0$ since $t_b \geq c_{od}^{k_1} - c_b^{k_1}$ by (3.22);

- or $x_d^{k_1} = 1$ for $d \in \mathcal{A}$, and points in $\mathcal{P} \cap \mathcal{H}$ are such that $p_d^{k_1} + c_d^{k_1} = t_{\tilde{a}} + p_b^{k_2} + c_b^{k_1}$. When $d \neq \tilde{a}$, it implies $t_{\tilde{a}} = 0$ since $p_d^{k_1} + c_d^{k_1} \leq t_b + c_b^{k_1}$ by (3.22).

As setting $t_{\tilde{a}} = 0$ would yield to contradictions in terms of path costs, we deduce that linking variables $t_{\tilde{a}}$, $p_b^{k_1} : b \in \mathcal{A}$ and $p_b^{k_2} : b \in \mathcal{A} \setminus (\mathcal{S} \cup \{\tilde{a}\})$ will be difficult.

Proposition 21 *Inequalities*

$$(4.1) \quad \sum_{b \in \mathcal{A}} \left(p_b^{k_1} + c_b^{k_1} x_b^{k_1}\right) + c_{od}^{k_1}\Big(1 - \sum_{b \in \mathcal{A}} x_b^{k_1}\Big) - t_{\tilde{a}} - c_{\tilde{a}}^{k_1} - \sum_{b \in \mathcal{A} \setminus (\mathcal{S} \cup \{\tilde{a}\})} \left(p_b^{k_2} + (c_b^{k_1} - c_{\tilde{a}}^{k_1}) x_b^{k_2}\right) \leq 0$$

are facet defining for $\mathcal{P}$ if the following conditions hold:

- $0 < M_{\tilde{a}}^{k_2} \leq M_{\tilde{a}}^{k_1}$;

- $M_b^{k_1} < M_b^{k_2}$ *for all* $b \in \mathcal{S}$;

- $(b, k_1) \sim (\tilde{a}, k_2)$ *for all* $b \in \mathcal{A} \setminus \{\tilde{a}\}$;

- $(\tilde{a}, k_1) \sim (b, k_2)$ *and* $c_{\tilde{a}}^{k_1} - c_b^{k_1} \geq 0$ *for all* $b \in \mathcal{A} \setminus (\mathcal{S} \cup \{\tilde{a}\})$;

- *there exists* $b \in \mathcal{S} \setminus \{\tilde{a}\}$ *such that* $(\tilde{a}, k_1) \sim (b, k_2)$;

- *for all* $b \in \mathcal{A}$ *such that* $M_b^{k_1} < M_b^{k_2}$, *there exists* $d \in \mathcal{S}$ *such that* $(b, k_1) \sim (d, k_2)$.

Note that the first and second conditions implies that $\tilde{a} \notin \mathcal{S}$. The third and fourth conditions yield $c_{\tilde{a}}^{k_2} - c_{\tilde{a}}^{k_1} = c_b^{k_2} - c_b^{k_1}$ for all $b \in \mathcal{A} \setminus (\mathcal{S} \cup \{\tilde{a}\})$. Hence one must also have $M_b^{k_2} \leq M_b^{k_1}$ for all $b \in \mathcal{A} \setminus \mathcal{S}$.

Proof

Let $\mathcal{H} = \big\{(\mathbf{t};\mathbf{p};\mathbf{x}) : \sum_{b\in\mathcal{A}} \big(p_b^{k_1} + c_b^{k_1} x_b^{k_1}\big) + c_{od}^{k_1}(1 - \sum_{b\in\mathcal{A}} x_b^{k_1}) - t_{\tilde{a}} - c_{\tilde{a}}^{k_1}$
$- \sum_{b\in\mathcal{A}\setminus(\mathcal{S}\cup\{\tilde{a}\})} \big(p_b^{k_2} + (c_b^{k_1} - c_{\tilde{a}}^{k_1}) x_b^{k_2}\big) = 0\big\}.$

Provided $M_{\tilde{a}}^{k_2} \leq M_{\tilde{a}}^{k_1}$, the points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde{a}\}} N_a \mathbf{e}_a + M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}}; \mathbf{0}; \mathbf{0}; \mathbf{0}; \mathbf{0}\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde{a}\}} N_a \mathbf{e}_a + M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}} - \epsilon\mathbf{e}_b; \mathbf{0}; \mathbf{0}; \mathbf{0}; \mathbf{0}\Big)$$

are in $\mathcal{P}\cap\mathcal{H}$ for all $b \in \mathcal{A}\setminus\{\tilde{a}\}$, which yields $\alpha_b = 0$ for all $b \in \mathcal{A}\setminus\{\tilde{a}\}$ and $\delta = M_{\tilde{a}}^{k_1}\alpha_{\tilde{a}}$.

Further, for all $b \in \mathcal{A}$ such that $M_b^{k_1} \geq M_b^{k_2}$, points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde{a},d\}} N_a \mathbf{e}_a + M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}} + M_b^{k_1}\mathbf{e}_b; M_b^{k_1}\mathbf{e}_b; \mathbf{0}; \mathbf{e}_b; \mathbf{0}\Big)$$

are in $\mathcal{P}\cap\mathcal{H}$, which yields $\gamma_b^{k_1} = -M_b^{k_1}\beta_b^{k_1}$. Otherwise, i.e., for all $b \in \mathcal{A}$ such that $M_b^{k_1} < M_b^{k_2}$, and provided that there exists $d \in \mathcal{S}$ such that $(b, k_1) \sim (d, k_2)$, points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde{a},b,d\}} N_a \mathbf{e}_a + M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}} + M_b^{k_1}\mathbf{e}_b + M_d^{k_1}\mathbf{e}_d; M_b^{k_1}\mathbf{e}_b; M_d^{k_1}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde{a},d\}} N_a \mathbf{e}_a + M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}} + M_d^{k_1}\mathbf{e}_d; \mathbf{0}; M_d^{k_1}\mathbf{e}_d; \mathbf{0}; \mathbf{e}_d\Big)$$

are in $\mathcal{P}\cap\mathcal{H}$ since $M_d^{k_1} < M_d^{k_2}$ for all $d \in \mathcal{S}$. Hence $\gamma_d^{k_1} = -M_d^{k_1}\beta_d^{k_1}$ for all $d \in \mathcal{A}$.

Next, provided that $M_b^{k_1} < M_b^{k_2}$ for all $b \in \mathcal{S}$, the points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde{a},b\}} N_a\mathbf{e}_a + M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}} + M_b^{k_1}\mathbf{e}_b; \mathbf{0}; M_b^{k_1}\mathbf{e}_b; \mathbf{0}; \mathbf{e}_b\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde{a},b\}} N_a\mathbf{e}_a + M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}} + (M_b^{k_1}+\epsilon)\mathbf{e}_b; \mathbf{0}; (M_b^{k_1}+\epsilon)\mathbf{e}_b; \mathbf{0}; \mathbf{e}_b\Big)$$

belong to $\mathcal{P} \cap \mathcal{H}$. It follows that $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$ for all $b \in \mathcal{S}$.

Provided that $(b, k_1) \sim (\tilde{a}, k_2)$ for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$, points[2]

$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde{a},b\}} N_a\mathbf{e}_a + M_{\tilde{a}}^{k_2}\mathbf{e}_{\tilde{a}} + (M_{\tilde{a}}^{k_2} + c_{\tilde{a}}^{k_1} - c_b^{k_1})\mathbf{e}_b; (M_{\tilde{a}}^{k_2} + c_{\tilde{a}}^{k_1} - c_b^{k_1})\mathbf{e}_b; M_{\tilde{a}}^{k_2}\mathbf{e}_{\tilde{a}}; \mathbf{e}_b; \mathbf{e}_{\tilde{a}}\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde{a},b\}} N_a\mathbf{e}_a + (M_{\tilde{a}}^{k_2}-\epsilon)\mathbf{e}_{\tilde{a}} + (M_{\tilde{a}}^{k_2} + c_{\tilde{a}}^{k_1} - c_b^{k_1} - \epsilon)\mathbf{e}_b; (M_{\tilde{a}}^{k_2} + c_{\tilde{a}}^{k_1} - c_b^{k_1} - \epsilon)\mathbf{e}_b;$$
$$(M_{\tilde{a}}^{k_2}-\epsilon)\mathbf{e}_{\tilde{a}}; \mathbf{e}_b; \mathbf{e}_{\tilde{a}}\Big)$$

are in $\mathcal{P} \cap \mathcal{H}$ since $M_{\tilde{a}}^{k_2} \leq M_{\tilde{a}}^{k_1}$, and one obtains $\beta_b^{k_1} + \beta_{\tilde{a}}^{k_2} + \alpha_{\tilde{a}} = 0$.

Points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde{a}\}} N_a\mathbf{e}_a; \mathbf{0}; \mathbf{0}; \mathbf{e}_{\tilde{a}}; \mathbf{e}_{\tilde{a}}\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde{a}\}} N_a\mathbf{e}_a + \epsilon\mathbf{e}_{\tilde{a}}; \epsilon\mathbf{e}_{\tilde{a}}; \epsilon\mathbf{e}_{\tilde{a}}; \mathbf{e}_{\tilde{a}}; \mathbf{e}_{\tilde{a}}\Big)$$

also belong to $\mathcal{P} \cap \mathcal{H}$. Hence $\gamma_{\tilde{a}}^{k_1} + \gamma_{\tilde{a}}^{k_2} = M_{\tilde{a}}^{k_1}\alpha_{\tilde{a}}$ and $\beta_{\tilde{a}}^{k_1} + \beta_{\tilde{a}}^{k_2} + \alpha_{\tilde{a}} = 0$. Further, point

$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde{a}\}} N_a\mathbf{e}_a + M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}}; M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}}; \mathbf{0}; \mathbf{e}_{\tilde{a}}; \mathbf{0}\Big)$$

is in $\mathcal{P} \cap \mathcal{H}$, thus $\gamma_{\tilde{a}}^{k_1} = -M_{\tilde{a}}^{k_1}\beta_{\tilde{a}}^{k_1}$ and $\gamma_{\tilde{a}}^{k_2} = 0$.

[2]Note that this very restrictive condition seems necessary to link variables $p_b^{k_1} : b \in \mathcal{A}$ and $t_{\tilde{a}}$.

Provided there exists $b \in \mathcal{S} \setminus \{\tilde{a}\}$ such that $(\tilde{a}, k_1) \sim (b, k_2)$, points

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b, \tilde{a}\}} N_a \mathbf{e}_a + M_{\tilde{a}}^{k_2} \mathbf{e}_{\tilde{a}} + M_b^{k_2} \mathbf{e}_b ; M_{\tilde{a}}^{k_2} \mathbf{e}_{\tilde{a}} ; M_b^{k_2} \mathbf{e}_b ; \mathbf{e}_{\tilde{a}} ; \mathbf{e}_b \Big)$$

are in $\mathcal{P} \cap \mathcal{H}$. As $\gamma_{\tilde{a}}^{k_1} = -M_{\tilde{a}}^{k_1} \beta_{\tilde{a}}^{k_1}$, one obtains $\alpha_{\tilde{a}} = -\beta_{\tilde{a}}^{k_1}$, thus $\beta_{\tilde{a}}^{k_2} = 0$.
Next, provided $(\tilde{a}, k_1) \sim (b, k_2)$ and $c_{\tilde{a}}^{k_1} - c_b^{k_1} \geq 0$ for all $b \in \mathcal{A} \setminus (\mathcal{S} \cup \{\tilde{a}\})$, points[3]

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b, \tilde{a}\}} N_a \mathbf{e}_a + (c_{\tilde{a}}^{k_1} - c_b^{k_1}) \mathbf{e}_b ; \mathbf{0} ; (c_{\tilde{a}}^{k_1} - c_b^{k_1}) \mathbf{e}_b ; \mathbf{e}_{\tilde{a}} ; \mathbf{e}_b \Big)$$

are in $\mathcal{P} \cap \mathcal{H}$, yielding $(c_{\tilde{a}}^{k_1} - c_b^{k_1}) \beta_b^{k_2} + \gamma_{\tilde{a}}^{k_1} + \gamma_b^{k_2} = M_{\tilde{a}}^{k_1} \alpha_{\tilde{a}}$. As $\gamma_{\tilde{a}}^{k_1} = -M_{\tilde{a}}^{k_1} \beta_{\tilde{a}}^{k_1} = M_{\tilde{a}}^{k_1} \alpha_{\tilde{a}}$, one obtains $\gamma_b^{k_2} = -(c_{\tilde{a}}^{k_1} - c_b^{k_1}) \beta_b^{k_2}$. The result follows. □

In the previous proposition, we proved that the Strengthened Shortest Path inequalities (4.1) can define facets of $\mathcal{P}$ if several very restrictive conditions are imposed on the set $\mathcal{S}$ and on the constants $M_a^k : a \in \mathcal{A}, k = k_1, k_2$. Hence these inequalities are often not tight.

Next, we state the conditions in which the Strengthened Shortest Path inequalities (4.2) define facets of $\mathcal{P}$.

Proposition 22 *Inequalities*

$$(4.2) \quad \sum_{b \in \mathcal{A}} (p_b^{k_1} - M_b^{k_1} x_b^{k_1}) - \sum_{b \in \mathcal{A} \setminus \mathcal{S}} (p_b^{k_2} - M_b^{k_1} x_b^{k_2}) \leq 0$$

are facet defining for $\mathcal{P}$ if $M_b^{k_1} < M_b^{k_2}$ for all $b \in \mathcal{S}$.

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[3]Again, this very restrictive condition seems necessary to link variables $p_b^{k_2} : b \in \mathcal{A} \setminus (\mathcal{S} \cup \{\tilde{a}\})$ and $t_{\tilde{a}}$.

Proof

Let $\mathcal{H} = \{(\mathbf{t};\mathbf{p};\mathbf{x}) : \sum_{b\in\mathcal{A}} (p_b^{k_1} - M_b^{k_1} x_b^{k_1}) - \sum_{b\in\mathcal{A}\setminus\mathcal{S}} (p_b^{k_2} - M_b^{k_1} x_b^{k_2}) = 0\}$. First, one knows that $\alpha_b = 0$ for all $b \in \mathcal{A}$ and $\delta = 0$ by Lemma 4.1. Further, for any $b \in \mathcal{S}$ and provided that $M_b^{k_1} < M_b^{k_2}$, one obtains $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$ by Lemma 6.1. For all $b \in \mathcal{A} \setminus \mathcal{S}$, Lemma 5.1 yields $\beta_b^{k_1} = -\beta_b^{k_2}$ and $\gamma_b^{k_1} = -\gamma_b^{k_2}$.

Next, for all $b \in \mathcal{A}$ such that $M_b^{k_1} \geq M_b^{k_2}$, permuting the commodity indices k_1 and k_2 in Lemma 9.1 yields $\gamma_b^{k_1} = -M_b^{k_1}\beta_b^{k_1}$. Otherwise, i.e., for all $b \in \mathcal{A}$ such that $M_b^{k_1} < M_b^{k_2}$, permuting the commodity indices k_1 and k_2 in Lemma 10 yields $\gamma_b^{k_1} = -M_b^{k_1}\beta_b^{k_1}$.

Further, if there exist $d \in \mathcal{A} \setminus \mathcal{S}, b \in \mathcal{A}$ such that $(b, k_1) \sim (d, k_2)$ and $c_b^{k_1} < c_d^{k_1}$, points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + (c_d^{k_1} - c_b^{k_1})\mathbf{e}_b; (c_d^{k_1} - c_b^{k_1})\mathbf{e}_b; \mathbf{0}; \mathbf{e}_b; \mathbf{e}_d\Big)$$

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + (c_d^{k_1} - c_b^{k_1} + \epsilon)\mathbf{e}_b + \epsilon\mathbf{e}_d; (c_d^{k_1} - c_b^{k_1} + \epsilon)\mathbf{e}_b; \epsilon\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$

also belong to $\mathcal{P} \cap \mathcal{H}$. This yields $\beta_b^{k_1} = -\beta_d^{k_2}$. Otherwise, i.e., if $c_b^{k_1} \geq c_d^{k_1}$, points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + (c_b^{k_1} - c_d^{k_1})\mathbf{e}_d; \mathbf{0}; (c_b^{k_1} - c_d^{k_1})\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + \epsilon\mathbf{e}_\mathbf{b} + (c_b^{k_1} - c_d^{k_1} + \epsilon)\mathbf{e}_d; \epsilon\mathbf{e}_b; (c_b^{k_1} - c_d^{k_1} + \epsilon)\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_b\Big)$$

are in $\mathcal{P} \cap \mathcal{H}$. Hence one obtains $\beta_b^{k_1} = -\beta_d^{k_2}$ for all $b \in \mathcal{A}$ and $d \in \mathcal{A} \setminus \mathcal{S}$. The result follows. □

The Strengthened Shortest Path inequalities (4.2) are obviously stronger than the Strengthened Shortest Path inequalities (4.1). This will be highlighted numerically in the next chapter.

Now we state the conditions in which the Strengthened Profit Upper Bound inequalities define facets of $\mathcal{P}$. Let $\tilde{a} \in \mathcal{A}$. Intuitively, the conditions will depend on the constants $M_{\tilde{a}}^k : k = k_1, k_2$ and on the sets $\mathcal{A}_{\tilde{a}}^{\geq}$, $\mathcal{A}_{\tilde{a}}^{\leq}$.

Proposition 23 *Inequalities*

$$(4.7)\quad p_a^{k_2} - M_a^{k_2} x_a^{k_2} - \left(M_a^{k_2} - M_a^{k_1}\right)\left(\sum_{b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{a\}} (x_b^{k_2} - x_b^{k_1}) - x_a^{k_1}\right) \leq 0$$

are facet defining for $\mathcal{P}$ *if, for all* $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ *such that* $M_b^{k_1} = M_b^{k_2}$*, there exists* $d \in \mathcal{A}_{\tilde{a}}^{\geq}$ *such that* $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$.

Proof

Let $\mathcal{H} = \left\{(\mathbf{t};\mathbf{p};\mathbf{x}) : p_{\tilde{a}}^{k_2} - M_{\tilde{a}}^{k_2} x_{\tilde{a}}^{k_2} - \left(M_{\tilde{a}}^{k_2} - M_{\tilde{a}}^{k_1}\right)\left(\sum_{b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}} (x_b^{k_2} - x_b^{k_1}) - x_{\tilde{a}}^{k_1}\right) = 0\right\}$. Lemma 4.1 yields $\alpha_b = 0$ for all $b \in \mathcal{A}$ and $\delta = 0$. Further, for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$, one obtains $\beta_b^{k_1} = -\beta_b^{k_2}$ and $\gamma_b^{k_1} = -\gamma_b^{k_2}$ by Lemma 5.1.

For any $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, setting $b = \tilde{a}$ and $d = b$ in Lemma 7.1 yields $\beta_b^{k_2} = -\beta_{\tilde{a}}^{k_1}$, thus also $\beta_b^{k_1} = -\beta_{\tilde{a}}^{k_1}$.
Further, for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_b^{k_1} < M_b^{k_2}$, Lemma 6.1 yields $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$. If $M_b^{k_2} < M_b^{k_1}$, permuting the commodity indices k_1 and k_2 in Lemma 6.1 yields $\beta_b^{k_1} = 0 = \gamma_b^{k_1}$. Hence $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ and $\gamma_b^{k_1} = 0 = \gamma_b^{k_2}$ for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_b^{k_1} \neq M_b^{k_2}$.
Otherwise, i.e., for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_b^{k_1} = M_b^{k_2}$, Assumption 2 ensures that there exists $d \in \mathcal{A} \setminus \{b\}$ (possibly $\tilde{a}$) such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$. Then, provided there exists such a toll arc d in $\mathcal{A}_{\tilde{a}}^{\geq}$, Lemma 8.1 holds and one conclude that $\beta_b^{k_1} = 0 = \beta_b^{k_2}$. One also obtains $\gamma_b^{k_2} = 0 = \gamma_b^{k_1}$ by Lemma 9.1.

Next, for all $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, recall that there does not exist any $b \in \setminus\{\tilde{a}\}$ such that

$c_b^{k_2} - c_b^{k_1} = c_{\tilde a}^{k_2} - c_{\tilde a}^{k_1}$ by hypothesis. Hence, setting $b = \tilde a$ and $d = b$ in Lemma 8.1 yields $\beta_{\tilde a}^{k_1} = 0$ and $M_b^{k_1}\beta_b^{k_2} + \gamma_{\tilde a}^{k_1} + \gamma_b^{k_2} = 0$. As $\beta_b^{k_2} = -\beta_{\tilde a}^{k_1} = -\beta_b^{k_1}$, it follows that $\beta_b^{k_2} = 0 = \beta_b^{k_1}$ for all $b \in \mathcal{A}_{\tilde a}^{\leqq} \setminus \{\tilde a\}$. One also obtains $\gamma_b^{k_2} = -\gamma_{\tilde a}^{k_1}$.

Finally, setting $b = \tilde a$ in Lemma 9.1 yields $\gamma_{\tilde a}^{k_2} = -M_{\tilde a}^{k_2}\beta_{\tilde a}^{k_2}$. As the point

$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde a\}} N_a\mathbf{e}_a + M_{\tilde a}^{k_1}\mathbf{e}_{\tilde a}; M_{\tilde a}^{k_1}\mathbf{e}_{\tilde a}; M_{\tilde a}^{k_1}\mathbf{e}_{\tilde a}; \mathbf{e}_{\tilde a}; \mathbf{e}_{\tilde a}\Big)$$

also belongs to $\mathcal{P}\cap\mathcal{H}$, it follows that $\gamma_{\tilde a}^{k_1} = \left(M_{\tilde a}^{k_2} - M_{\tilde a}^{k_1}\right)\beta_{\tilde a}^{k_2}$. The result follows. □

Note that the conditions imposed in the previous proposition imply that $M_b^{k_1} \neq M_b^{k_2}$ for all $b \in \mathcal{A}_{\tilde a}^{\gtrless}$ or there are at least two toll arcs $b, d \in \mathcal{A}_{\tilde a}^{\gtrless}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$. Now, in order to prove that inequalities (4.8) define facets of $\mathcal{P}$, we need to impose stronger conditions. Indeed, we assume that if there exists $b \in \mathcal{A}_{\tilde a}^{\gtrless}$ such that $M_b^{k_1} \leq M_b^{k_2}$ (resp. $M_b^{k_1} > M_b^{k_2}$), then there exists at least another toll arc $d \in \mathcal{A}_{\tilde a}^{\gtrless} \setminus \{b\}$ such that $M_d^{k_1} \leq M_d^{k_2}$ (resp. $M_d^{k_1} > M_d^{k_2}$) and $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$.

Proposition 24 *Inequalities*

$$(4.8)\quad p_a^{k_2} - M_a^{k_2}x_a^{k_2} - \left(M_a^{k_2} - M_a^{k_1}\right)\left(\sum_{b\in\mathcal{A}_a^{\not<}\setminus\{a\}} (x_b^{k_2} - x_b^{k_1}) - x_a^{k_1}\right)$$

$$-\left(M_{b^*}^{k_2} - M_{b^*}^{k_1}\right)\sum_{b\in\mathcal{A}_{\tilde a}^{\gtrless}:M_b^{k_2}\geq M_b^{k_1}} (x_b^{k_2} - x_b^{k_1}) \leq 0$$

are facet defining for $\mathcal{P}$ if, for all $b \in \mathcal{A}_{\tilde a}^{\gtrless}$ such that $M_b^{k_1} \leq M_b^{k_2}$ (resp. $M_b^{k_1} > M_b^{k_2}$), there exists $d \in \mathcal{A}_{\tilde a}^{\gtrless} \setminus \{b\}$ such that $M_d^{k_1} \leq M_d^{k_2}$ (resp. $M_d^{k_1} > M_d^{k_2}$) and $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$.

Proof

Let $\mathcal{H} = \{(\mathbf{t};\mathbf{p};\mathbf{x}) : p_{\tilde{a}}^{k_2} - M_{\tilde{a}}^{k_2} x_{\tilde{a}}^{k_2} - \left(M_{\tilde{a}}^{k_2} - M_{\tilde{a}}^{k_1}\right)\left(\sum_{b\in\mathcal{A}_{\tilde{a}}^{\leq}\setminus\{\tilde{a}\}}(x_b^{k_2} - x_b^{k_1}) - x_{\tilde{a}}^{k_1}\right)$
$- \left(M_{b^*}^{k_2} - M_{b^*}^{k_1}\right)\sum_{b\in\mathcal{A}_{\tilde{a}}^{>}:M_b^{k_2}\geq M_b^{k_1}}(x_b^{k_2} - x_b^{k_1}) = 0\}$.

Lemma 4.1 yields $\alpha_b = 0$ for all $b \in \mathcal{A}$ and $\delta = 0$. Further, for all $b \in \mathcal{A}\setminus\{\tilde{a}\}$, one obtains $\beta_b^{k_1} = -\beta_b^{k_2}$ and $\gamma_b^{k_1} = -\gamma_b^{k_2}$ by Lemma 5.1.

For any $b \in \mathcal{A}_{\tilde{a}}^{\leq}\setminus\{\tilde{a}\}$, setting $b = \tilde{a}$ and $d = b$ in Lemma 7.1 yields $\beta_b^{k_2} = -\beta_{\tilde{a}}^{k_1}$, thus also $\beta_b^{k_1} = -\beta_{\tilde{a}}^{k_1}$.

For all $d \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_d^{k_1} \leq M_d^{k_2}$ (resp. $M_d^{k_1} > M_d^{k_2}$), the proposition hypothesis ensures that there exists $b \in \mathcal{A}_{\tilde{a}}^{>}\setminus\{d\}$ such that $M_b^{k_1} \leq M_b^{k_2}$ (resp. $M_d^{k_1} > M_d^{k_2}$) and $c_d^{k_2} - c_d^{k_1} \neq c_b^{k_2} - c_b^{k_1}$. Without loss of generality, let us assume that $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$. As Lemma 7.1 yields $\beta_b^{k_1} = -\beta_d^{k_2}$, it follows that $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ for all $b, d \in \mathcal{A}_{\tilde{a}}^{>}$ by Lemma 8.1.

Further, for all $b \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_b^{k_1} > M_b^{k_2}$, permuting the commodity indices k_1 and k_2 in Lemma 9.1 yields $\gamma_b^{k_1} = -M_b^{k_1}\beta_b^{k_1}$. As $\beta_b^{k_1} = 0$, one obtains $\gamma_b^{k_1} = 0$, thus also $\gamma_b^{k_2} = 0$.

Next, for all $b \in \mathcal{A}_{\tilde{a}}^{\leq}\setminus\{\tilde{a}\}$, recall that there does not exist any $b \in \setminus\{\tilde{a}\}$ such that $c_b^{k_2} - c_b^{k_1} = c_{\tilde{a}}^{k_2} - c_{\tilde{a}}^{k_1}$ by hypothesis. Hence, setting $b = \tilde{a}$ and $d = b$ in Lemma 8.1 yields $\beta_{\tilde{a}}^{k_1} = 0$ and $M_b^{k_1}\beta_b^{k_2} + \gamma_{\tilde{a}}^{k_1} + \gamma_b^{k_2} = 0$. As $\beta_b^{k_2} = -\beta_{\tilde{a}}^{k_1} = -\beta_b^{k_1}$, it follows that $\beta_b^{k_2} = 0 = \beta_b^{k_1}$ for all $b \in \mathcal{A}_{\tilde{a}}^{\leq}\setminus\{\tilde{a}\}$. One also obtains $\gamma_b^{k_2} = -\gamma_{\tilde{a}}^{k_1}$.

Next, setting $b = \tilde{a}$ in Lemma 9.1 yields $\gamma_{\tilde{a}}^{k_2} = -M_{\tilde{a}}^{k_2}\beta_{\tilde{a}}^{k_2}$. As the point

$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde{a}\}} N_a\mathbf{e}_a + M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}}; M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}}; M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}}; \mathbf{e}_{\tilde{a}}; \mathbf{e}_{\tilde{a}}\Big)$$

also belongs to $\mathcal{P}\cap\mathcal{H}$, it follows that $\gamma_{\tilde{a}}^{k_1} = \left(M_{\tilde{a}}^{k_2} - M_{\tilde{a}}^{k_1}\right)\beta_{\tilde{a}}^{k_2}$.

Finally, for all $b \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_b^{k_1} \leq M_b^{k_2}$, points

$$\Big(\sum_{a \in \mathcal{A} \setminus \{\tilde{a}, b\}} N_a \mathbf{e}_a + (M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2}) \mathbf{e}_{\tilde{a}} + (M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_b^{k_2}) \mathbf{e}_b; \\ (M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_b^{k_2}) \mathbf{e}_b; (M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2}) \mathbf{e}_{\tilde{a}}; \mathbf{e}_b; \mathbf{e}_{\tilde{a}} \Big)$$

are in $\mathcal{P} \cap \mathcal{H}$ since $x_b^1 = 1 = x_{\tilde{a}}^{k_2}$ $(b \in \mathcal{A}_{\tilde{a}}^{>} : M_b^{k_1} \leq M_b^{k_2})$ implies $p_{\tilde{a}}^{k_2} = M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2}$ for points of $\mathcal{H}$, which yields

$$(M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2}) \beta_{\tilde{a}}^{k_2} + \gamma_b^{k_1} + \gamma_{\tilde{a}}^{k_2} = 0.$$

As $\gamma_{\tilde{a}}^{k_2} = -M_{\tilde{a}}^{k_2} \beta_{\tilde{a}}^{k_2}$, one obtains $\gamma_b^{k_1} = \left(M_{b^*}^{k_2} - M_{b^*}^{k_1}\right) \beta_{\tilde{a}}^{k_2}$ and the result follows. $\square$

Note that we have not proved that, if the conditions imposed in the previous proposition are not satisfied, then inequalities (4.7)-(4.8) do not define facets of $\mathcal{P}$. As this would require the study of several particular cases, we have tought it would not bring any additional relevant information, at least for the purpose of the thesis.

Next, we state the conditions in which inequalities (4.9) and (4.10) define facets of $\mathcal{P}$.

Proposition 25 *Inequalities*

$$(4.9) \quad p_{\tilde{a}}^{k_2} - p_{\tilde{a}}^{k_1} - M_{\tilde{a}}^{k_2} \sum_{b \in \mathcal{A}_{\tilde{a}}^{\leq}} (x_b^{k_2} - x_b^{k_1}) \leq 0$$

are facet defining for $\mathcal{P}$ if the following conditions hold:

- *for all $b \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_b^{k_1} = M_b^{k_2}$, there exists $d \in \mathcal{A}_{\tilde{a}}^{>} \setminus \{b\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$;*
- *for all $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, there exists $d \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$;*

- *there exists* $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, $v \in \mathbb{R}$ *such that* $c_{\tilde{a}}^{k_1} - c_b^{k_1} \leq v \leq c_{\tilde{a}}^{k_2} - c_b^{k_2}$ *and* $0 \leq v \leq M_b^{k_2}$.

Proof

Let $\mathcal{H} = \left\{(\mathbf{t};\mathbf{p};\mathbf{x}) : p_{\tilde{a}}^{k_2} - p_{\tilde{a}}^{k_1} - M_{\tilde{a}}^{k_2} \sum_{b \in \mathcal{A}_{\tilde{a}}^{\leq}} (x_b^{k_2} - x_b^{k_1}) = 0\right\}$. Lemma 4.1 yields $\alpha_b = 0$ for all $b \in \mathcal{A}$ and $\delta = 0$. Further, for all $b \in \mathcal{A}$, one obtains $\beta_b^{k_1} = -\beta_b^{k_2}$ and $\gamma_b^{k_1} = -\gamma_b^{k_2}$ by Lemma 5.1.

Next, for any $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, there exists $d \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$ by the proposition hypothesis. Without loss of generality, let us assume that $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$. As $M_d^{k_1} \leq M_d^{k_2}$ for all $d \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, Lemmas 7.1 and 8.1 yield $\beta_b^{k_1} = -\beta_d^{k_2}$ and $\beta_b^{k_1} = 0$ respectively. Hence $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ for all $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$.
Further, for all $b \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_b^{k_1} < M_b^{k_2}$, Lemma 6.1 yields $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$. If $M_b^{k_2} < M_b^{k_1}$, permuting the commodity indices k_1 and k_2 in Lemma 6.1 yields $\beta_b^{k_1} = 0 = \gamma_b^{k_1}$. It follows that $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ and $\gamma_b^{k_1} = 0 = \gamma_b^{k_2}$ for all $b \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_b^{k_1} \neq M_b^{k_2}$.
Otherwise, i.e., for all $b \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_b^{k_1} = M_b^{k_2}$, Assumption 2 ensures that there exists $d \in \mathcal{A} \setminus \{b\}$ (possibly $\tilde{a}$) such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$. Then, provided there exists such a toll arc d in $\mathcal{A}_{\tilde{a}}^{>}$, Lemma 8.1 holds and one conclude that $\beta_b^{k_1} = 0 = \beta_b^{k_2}$. One also obtains $\gamma_b^{k_2} = 0 = \gamma_b^{k_1}$ by Lemma 9.1.

Next, provided there exists $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, $v \in \mathbb{R}$ such that $c_{\tilde{a}}^{k_1} - c_b^{k_1} \leq v \leq c_{\tilde{a}}^{k_2} - c_b^{k_2}$ and $0 \leq v \leq M_b^{k_2}$, point

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b,\tilde{a}\}} N_a \mathbf{e}_a + v\mathbf{e}_b; \mathbf{0}; v\mathbf{e}_b; \mathbf{e}_{\tilde{a}}; \mathbf{e}_b \Big)$$

is in $\mathcal{P} \cap \mathcal{H}$. The existence of $v \in \mathbb{R}$ is required since $x_{\tilde{a}}^{k_1} = 1 = x_b^{k_2}$ ($b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$) implies $p_{\tilde{a}}^{k_1} = 0$ for points of $\mathcal{H}$. This yields $\gamma_b^{k_2} = -\gamma_{\tilde{a}}^{k_1}$ for all $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$.

Finally, setting $b = \tilde{a}$ in Lemma 9.1 yields $\gamma_{\tilde{a}}^{k_2} = -M_{\tilde{a}}^{k_2}\beta_{\tilde{a}}^{k_2}$. As $\gamma_b^{k_1} = -\gamma_b^{k_2}$ for all $b \in \mathcal{A}$, one obtains $\gamma_b^{k_2} = M_{\tilde{a}}^{k_2}\beta_{\tilde{a}}^{k_2} = -\gamma_b^{k_1}$ for all $b \in \mathcal{A}$. The result follows. □

Proposition 26 *Inequalities*

$$(4.10)\quad p_{\tilde{a}}^{k_2} - p_{\tilde{a}}^{k_1} - M_{\tilde{a}}^{k_2} \sum_{b \in \mathcal{A}_{\tilde{a}}^{\leq}} (x_b^{k_2} - x_b^{k_1}) - \left(M_{b^*}^{k_2} - M_{b^*}^{k_1}\right) \sum_{b \in \mathcal{A}_{\tilde{a}}^{\geq} : M_b^{k_2} \geq M_b^{k_1}} (x_b^{k_2} - x_b^{k_1}) \leq 0$$

are facet defining for $\mathcal{P}$ if the following conditions hold:

- *for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_b^{k_1} \leq M_b^{k_2}$ (resp. $M_b^{k_1} > M_b^{k_2}$), there exists $d \in \mathcal{A}_{\tilde{a}}^{\geq} \backslash \{b\}$ such that $M_d^{k_1} \leq M_d^{k_2}$ (resp. $M_d^{k_1} > M_d^{k_2}$) and $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$;*
- *for all $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, there exists $d \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$;*
- *there exists $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, $v \in \mathbb{R}$ such that $c_{\tilde{a}}^{k_1} - c_b^{k_1} \leq v \leq c_{\tilde{a}}^{k_2} - c_b^{k_2}$ and $0 \leq v \leq M_b^{k_2}$.*

Proof

Let $\mathcal{H} = \big\{(\mathbf{t};\mathbf{p};\mathbf{x}) : p_{\tilde{a}}^{k_2} - p_{\tilde{a}}^{k_1} - M_{\tilde{a}}^{k_2} \sum_{b \in \mathcal{A}_{\tilde{a}}^{\leq}} (x_b^{k_2} - x_b^{k_1})$
$- \left(M_{b^*}^{k_2} - M_{b^*}^{k_1}\right) \sum_{b \in \mathcal{A}_{\tilde{a}}^{\geq} : M_b^{k_2} \geq M_b^{k_1}} (x_b^{k_2} - x_b^{k_1}) = 0\big\}$.

Lemma 4.1 yields $\alpha_b = 0$ for all $b \in \mathcal{A}$ and $\delta = 0$. Further, for all $b \in \mathcal{A}$, one obtains $\beta_b^{k_1} = -\beta_b^{k_2}$ and $\gamma_b^{k_1} = -\gamma_b^{k_2}$ by Lemma 5.1.

Next, for any $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, there exists $d \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$ by the proposition hypothesis. Without loss of generality, let us assume that $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$. As $M_d^{k_1} \leq M_d^{k_2}$ for all $d \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, Lemmas 7.1 and 8.1 yield $\beta_b^{k_1} = -\beta_d^{k_2}$ and $\beta_b^{k_1} = 0$ respectively. Hence $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ for all $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$.

For all $d \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_d^{k_1} \leq M_d^{k_2}$ (resp. $M_d^{k_1} > M_d^{k_2}$), the proposition hypothesis ensures that there exists $b \in \mathcal{A}_{\tilde{a}}^{\geq} \setminus \{d\}$ such that $M_b^{k_1} \leq M_b^{k_2}$ (resp.

$M_d^{k_1} > M_d^{k_2}$) and $c_d^{k_2} - c_d^{k_1} \neq c_b^{k_2} - c_b^{k_1}$. Without loss of generality, let us assume that $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$. As Lemma 7.1 yields $\beta_b^{k_1} = -\beta_d^{k_2}$, it follows that $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ for all $b, d \in \mathcal{A}_{\tilde{a}}^{\geq}$ by Lemma 8.1.
Further, for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_b^{k_1} > M_b^{k_2}$, permuting the commodity indices k_1 and k_2 in Lemma 9.1 yields $\gamma_b^{k_1} = -M_b^{k_1}\beta_b^{k_1}$. As $\beta_b^{k_1} = 0$, one obtains $\gamma_b^{k_1} = 0$, thus also $\gamma_b^{k_2} = 0$.

Next, provided there exists $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, $v \in \mathbb{R}$ such that $c_{\tilde{a}}^{k_1} - c_b^{k_1} \leq v \leq c_{\tilde{a}}^{k_2} - c_b^{k_2}$ and $0 \leq v \leq M_b^{k_2}$, point

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b, \tilde{a}\}} N_a \mathbf{e}_a + v\mathbf{e}_b; \mathbf{0}; v\mathbf{e}_b; \mathbf{e}_{\tilde{a}}; \mathbf{e}_b \Big)$$

is in $\mathcal{P} \cap \mathcal{H}$. Note that the existence of $v \in \mathbb{R}$ is required since $x_{\tilde{a}}^{k_1} = 1 = x_b^{k_2}$ ($b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$) implies that $p_{\tilde{a}}^{k_1} = 0$ for points of $\mathcal{H}$. This yields $\gamma_b^{k_2} = -\gamma_{\tilde{a}}^{k_1}$ for all $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$. Further, setting $b = \tilde{a}$ in Lemma 9.1 yields $\gamma_{\tilde{a}}^{k_2} = -M_{\tilde{a}}^{k_2}\beta_{\tilde{a}}^{k_2}$. As $\gamma_b^{k_1} = -\gamma_b^{k_2}$ for all $b \in \mathcal{A}$, one obtains $\gamma_b^{k_2} = M_{\tilde{a}}^{k_2}\beta_{\tilde{a}}^{k_2} = -\gamma_b^{k_1}$ for all $b \in \mathcal{A}$.

Finally, for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_b^{k_1} \leq M_b^{k_2}$, points

$$\Big(\sum_{a \in \mathcal{A} \setminus \{\tilde{a}, b\}} N_a \mathbf{e}_a + (M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2})\mathbf{e}_{\tilde{a}} + (M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_b^{k_2})\mathbf{e}_b;$$
$$(M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_b^{k_2})\mathbf{e}_b; (M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2})\mathbf{e}_{\tilde{a}}; \mathbf{e}_b; \mathbf{e}_{\tilde{a}} \Big)$$

are in $\mathcal{P} \cap \mathcal{H}$ since $x_b^1 = 1 = x_{\tilde{a}}^{k_2}$ ($b \in \mathcal{A}_{\tilde{a}}^{\geq} : M_b^{k_1} \leq M_b^{k_2}$) implies $p_{\tilde{a}}^{k_2} = M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2}$ for points of $\mathcal{H}$, which yields

$$(M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2})\beta_{\tilde{a}}^{k_2} + \gamma_b^{k_1} + \gamma_{\tilde{a}}^{k_2} = 0.$$

As $\gamma_{\tilde{a}}^{k_2} = -M_{\tilde{a}}^{k_2}\beta_{\tilde{a}}^{k_2}$, one obtains $\gamma_b^{k_1} = \left(M_{b^*}^{k_2} - M_{b^*}^{k_1}\right)\beta_{\tilde{a}}^{k_2}$ and the result follows. □

Note that the conditions imposed such that inequalities (4.9) and (4.10) define facets of $\mathcal{P}$ are similar to these imposed for inequalities (4.7) and (4.8). Again, we did not consider the particular cases in which these conditions are not satisfied.

Finally, we state the conditions in which inequalities (4.11) define facets of $\mathcal{P}$.

Proposition 27 *Inequalities*

$$(4.11)\quad p_{\tilde{a}}^{k_1} - p_{\tilde{a}}^{k_2} - M_{\tilde{a}}^{k_1} \sum_{b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}} (x_b^{k_2} - x_b^{k_1}) \leq 0$$

are facet defining for $\mathcal{P}$ if the following conditions hold:

- *for all $b \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_b^{k_1} = M_b^{k_2}$, there exists $d \in \mathcal{A}_{\tilde{a}}^{>} \setminus \{b\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$;*
- *there exists $b \in \mathcal{A}_{\tilde{a}}^{>}$, $v \in \mathbb{R}$ such that $c_{\tilde{a}}^{k_2} - c_b^{k_2} \leq v \leq c_{\tilde{a}}^{k_1} - c_b^{k_1}$ and $0 \leq v \leq M_b^{k_1}$.*

Proof

Let $\mathcal{H} = \left\{(\mathbf{t};\mathbf{p};\mathbf{x}) : p_{\tilde{a}}^{k_1} - p_{\tilde{a}}^{k_2} - M_{\tilde{a}}^{k_1} \sum_{b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}} (x_b^{k_2} - x_b^{k_1}) = 0\right\}$. Lemma 4.1 yields $\alpha_b = 0$ for all $b \in \mathcal{A}$ and $\delta = 0$. Further, for all $b \in \mathcal{A}$, one obtains $\beta_b^{k_1} = -\beta_b^{k_2}$ and $\gamma_b^{k_1} = -\gamma_b^{k_2}$ by Lemma 5.1.

Next, for any $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, the points

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b, \tilde{a}\}} N_a \mathbf{e_a} + M_{\tilde{a}}^{k_1} \mathbf{e_{\tilde{a}}} + M_b^{k_1} \mathbf{e_b}; M_{\tilde{a}}^{k_1} \mathbf{e_{\tilde{a}}}; M_b^{k_1} \mathbf{e_b}; \mathbf{e_{\tilde{a}}}; \mathbf{e_b}\Big)$$

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b, \tilde{a}\}} N_a \mathbf{e_a} + M_{\tilde{a}}^{k_1} \mathbf{e_{\tilde{a}}} + (M_b^{k_1} + \epsilon) \mathbf{e_b}; M_{\tilde{a}}^{k_1} \mathbf{e_{\tilde{a}}}; (M_b^{k_1} + \epsilon) \mathbf{e_b}; \mathbf{e_{\tilde{a}}}; \mathbf{e_b}\Big)$$

belong to $\mathcal{P} \cap \mathcal{H}$. Indeed, one can check that $M_b^{k_1} < M_b^{k_2}$ by definition of $\mathcal{A}_{\tilde{a}}^{\leq}$ and the hypothesis that there does not exist any $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_{\tilde{a}}^{k_2} - c_{\tilde{a}}^{k_1}$. Hence $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ for all $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$.

Further, for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_b^{k_1} < M_b^{k_2}$, Lemma 6.1 yields $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$. If $M_b^{k_2} < M_b^{k_1}$, permuting the commodity indices k_1 and k_2 in Lemma 6.1 yields $\beta_b^{k_1} = 0 = \gamma_b^{k_1}$. Hence $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ and $\gamma_b^{k_1} = 0 = \gamma_b^{k_2}$ for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_b^{k_1} \neq M_b^{k_2}$.

Otherwise, i.e., for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_b^{k_1} = M_b^{k_2}$, Assumption 2 ensures that there exists $d \in \mathcal{A} \setminus \{b\}$ (possibly $\tilde{a}$) such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$. Then, provided there exists such a toll arc d in $\mathcal{A}_{\tilde{a}}^{\geq}$, Lemma 8.1 holds and one conclude that $\beta_b^{k_1} = 0 = \beta_b^{k_2}$. One also obtains $\gamma_b^{k_2} = 0 = \gamma_b^{k_1}$ by Lemma 9.1.

Next, provided there exists $b \in \mathcal{A}_{\tilde{a}}^{\geq}$, $v \in \mathbb{R}$ such that $c_{\tilde{a}}^{k_2} - c_b^{k_2} \leq v \leq c_{\tilde{a}}^{k_1} - c_b^{k_1}$ and $0 \leq v \leq M_b^{k_1}$, point

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b, \tilde{a}\}} N_a \mathbf{e_a} + v\mathbf{e_b}; v\mathbf{e_b}; \mathbf{0}; \mathbf{e_b}; \mathbf{e_{\tilde{a}}}\Big)$$

is in $\mathcal{P} \cap \mathcal{H}$. Note the existence of $v \in \mathbb{R}$ is required since $x_b^{k_1} = 1 = x_{\tilde{a}}^{k_2}$ ($b \in \mathcal{A}_{\tilde{a}}^{\geq}$) implies that $p_{\tilde{a}}^{k_2} = 0$ for points of $\mathcal{H}$. This yields $\gamma_{\tilde{a}}^{k_2} = 0 = \gamma_{\tilde{a}}^{k_1}$.

Finally, for all $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, points

$$\Big(\sum_{a \in \mathcal{A} \setminus \{\tilde{a}, b\}} N_a \mathbf{e_a} + M_{\tilde{a}}^{k_1} \mathbf{e_{\tilde{a}}} + M_b^{k_1} \mathbf{e_b}; M_{\tilde{a}}^{k_1} \mathbf{e_{\tilde{a}}}; M_b^{k_1} \mathbf{e_b}; \mathbf{e_{\tilde{a}}}; \mathbf{e_b}\Big)$$

are in $\mathcal{P} \cap \mathcal{H}$. Hence $\gamma_b^{k_2} = -M_{\tilde{a}}^{k_1} \beta_{\tilde{a}}^{k_1}$ for all $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$. The result follows. □

Hence, most inequalities presented in Chapter 4 are facet defining for the two commodity GCT-NPP, i.e., for the polyhedron described by the convex hull of (HP3) feasible solutions. As a consequence, we expect that those valid inequalities are also strong in the multi-commodity case, i.e., they can help to solve faster or better (in terms of gap or number of nodes) the multi-commodity GCT-NPP.

5.3 Conclusion

In this chapter, we focused on problems involving one or two commodities. For the single commodity problem, we proved that several constraints of model (HP3*), as well as valid inequalities presented in Chapter 4, define facets of the convex hull of (HP3*) feasible solutions, i.e., for the CCT-NPP. Further, we highlighted a complete description of the convex hull of feasible solutions for the single commodity GCT-NPP. Next, we proved that most of the valid inequalities and (HP3) constraints are also facet defining for the corresponding two-commodity problem, i.e., for the GCT-NPP.

CHAPTER 6

NUMERICAL RESULTS

This chapter assesses the efficiency of the valid inequalities presented in Chapter 4 by numerical results. For that purpose, they are tested on randomly generated instances for both General and Constrained Complete Toll NPP. Sections 1 and 2 give technical details concerning the generation of instances and the implementation of models. Then, the numerical results for GCT-NPP and CCT-NPP are presented in Sections 3 and 4 respectively.

6.1 Data instances

The networks considered include 5, 8 or 10 cities with a commodity between each ordered pair of cities, i.e., $m(m-1)$ commodities for m cities. Demands for commodities are selected randomly between 10 and 100. The highways consist of 10 or 15 highway nodes (i.e., the entry and exit nodes of the highway), and lead to bi-directional complete toll subgraphs with $n(n-1)$ toll arcs for n nodes[1]. Considering that the size of an instance is determined by both the number of cities and the number of entry and exit nodes in the network, we generate 6 instances of each size.

In order to set fixed costs on paths, fixed costs on all arcs of the network are generated randomly as explained further. Note that the random generation intervals were chosen after performing an analysis of distances in a real Canadian highway network, the highway 10 (autoroute des Cantons de l'Est, Québec).

Fixed costs on toll arcs $a \in \mathcal{A}$ such that $h(a) = t(a) + 1$ are first randomly

[1]Note that we only test bi-directional networks, as we think that this structure is more realistic than a single directional structure.

generated between 20 and 70. Then fixed costs on toll arcs $b \in \mathcal{A}$ such that $h(b) = t(b) + l, l > 1$ are set to the sum of fixed costs of all toll arcs $a \in \underline{C}_b$ such that $h(a) = t(a) + 1$. We also set $c_b = c_a$ for all $a, b \in \mathcal{A} : h(b) = t(a), t(b) = h(a)$.

Next, fixed costs on toll free arcs linking cities and highway nodes are generated as follows. First, the closest highway node $i \in \mathcal{N}$ from a given city v is selected randomly, and the fixed cost on the corresponding arc $a \in \mathcal{A}$ is randomly generated between 2 and 70. The fixed costs on the toll free arcs $b \in \mathcal{A} : t(b) = t(a), h(a) - 2 \leq h(b) \leq h(a) + 2$ (i.e., the toll free arcs linking the given city v and the four closest nodes from i) are randomly generated between 15 and 120. Finally, the fixed costs on the toll free arcs $b \in \mathcal{A} : t(b) = t(a), h(b) < h(a) - 2$ or $h(b) > h(a) + 2$ are randomly generated between 30 and 1000.

The fixed costs between cities are randomly generated between 150 and 1000. Note that these are computed so that the cost from a city v_1 to a city v_2 is equal to the cost from v_2 to v_1.

Further, we assume that the use of a road besides the highway, compared to a road on the highway, often takes more time and requires more attention for a network user. Hence the fixed costs on toll free arcs are multiplied by a 1.5 factor.

Finally, the fixed costs on paths are computed. For each commodity $k \in \mathcal{K}$ and each toll arc $a \in \mathcal{A}$, the fixed cost c_a^k is the sum of fixed costs on the arcs that belong to the corresponding path.

For a network with n nodes, each commodity contains initially $n(n-1)$ paths. Then, the preprocessing described in Section 3.3 is performed, which reduces the number of feasible paths for each commodity. In order to allow comparisons between the results obtained for the GCT-NPP and CCT-NPP respectively, the preprocessing for the CCT-NPP is applied on all instances.

Hence, in the data instances, each commodity is linked with the corresponding set of feasible paths, which allows us to deal with smaller networks. Table 6.1

provides the minimum (MIN), maximum (MAX), mean (μ) and variance (σ^2) of the number of feasible paths for each commodity in the generated instances.

Inst.	MIN	MAX	μ	σ^2
5 v - 10 n	1	20	17,3	24,2
5 v - 15 n	1	35	12,9	53,1
8 v - 10 n	1	20	7,4	25,6
8 v - 15 n	1	35	13,2	66,1
10 v - 10 n	1	24	7,5	24,6
10 v - 15 n	1	35	13,1	60,8

Table 6.1: Number of feasible paths per commodity

6.2 Implementation of models

The numerical experiments are carried out on a Pentium 4.3GHz equipped with 2Gb of RAM and running Linux Kernel version 2.6.4. The models are implemented using Mosel of Xpress-MP, Optimizer version 18. We switch on all Xpress procedures concerning presolve. However, the Xpress automatic heuristic strategies have to be switched off, because they cannot be handled properly when appending our own cut procedure to the model. Finally, Xpress automatically selects the variables and nodes during the branch and cut algorithm.

We test models (HP3*) and (HP3) with the upper bounds $M_a^k, N_a : k \in \mathcal{K}, a \in \mathcal{A}$ and preprocessing described in Chapter 3, and with each class of valid inequalities described in Chapter 4. The latter, as well as the Triangle and Monotonicity constraints for model (HP3*), are generated at the root and nodes of the branch and cut algorithm and appended to model (HP3*) or (HP3) when violated (violation tests are performed at each iteration of the branch and cut algorithm).

We also impose an upper bound on the number of Triangle, Monotonicity and Strengthened Shortest Path inequalities appended at a single iteration of the branch and cut algorithm, depending either on the maximal number of feasible paths for a commodity $MaxP$ (i.e., at most $n(n-1)$ for n nodes without preprocessing of the network, thus obviously less when preprocessing of the network according to Section 3.3) or on the number of commodities $|K|$. These are set to $0.5MaxP$, $2MaxP$ and $0.5|K|$ respectively, and were chosen during preliminary tests on instances, according to the best results obtained. Finally, we set a computational time upper bound of 5 hours (18000 seconds), after which the solution process is aborted.

In order to assess the efficiency of the valid inequalities, the related number of nodes, cpu times and gaps are reported for all instances. We call gap of a problem the gap between the linear relaxation optimal solution Z_{lp} and the integer problem optimal solution Z_{opt}, i.e., $gap = \frac{Z_{lp}-Z_{opt}}{Z_{opt}}$. If Z_{opt} cannot be determined (i.e., the computational time upper bound is reached), Z_{opt} is set to the value of the best integer solution. Also note that Z_{lp} is computed after addition of violated valid inequalities at the root of the branch and cut algorithm.

The results are presented in tables in the following way. The first column provides the size of instances, i.e., the number of cities m and number of nodes n. Next, minimum (MIN), maximum (MAX), mean (μ) and standard deviation (σ) values are given for the gaps, cpu times and number of nodes in the branch and cut algorithm. The symbol ^{*x*} indicates that x instances cannot be solved to optimality because the computational time upper bound has been reached. Finally, for each class of inequalities, an additionnal table provides the number of inequalities (Nb INEG) appended to the initial model (HP3*) or (HP3), at the root or during the branch and cut algorithm.

The next section provides numerical results for the General Complete Toll NPP.

6.3 Numerical results for GCT-NPP

In this section, we point out the efficiency of the valid inequalities proposed in Chapter 4 for the Constrained Complete Toll NPP, which is described by model (HP3). Table 6.2 provides the gaps, cpu times and number of nodes corresponding to the resolution of (HP3) without any valid inequalities.

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	0.28	12.16	6.49	4.4	0	2	1	1	1	121	36	45
5 v - 15 n	3.48	18.21	9.69	5	1	7	3	2	7	397	117	132
8 v - 10 n	3.47	27.18	15.40	7.9	1	1020	234	368	39	69837	15309	25049
*1*8 v - 15 n	9.77	27.83	15.26	6.4	24	18001	3379	6562	551	739553	133233	271484
10 v - 10 n	12.68	22.27	16.61	3.4	24	5132	895	1895	1625	362460	62785	134021
*4*10 v - 15 n	11.17	20.82	16.17	3.4	433	18004	12421	7909	10817	437890	256207	169469

Table 6.2: Model (HP3)

In the next subsections, each class of valid inequalities is appended to model (HP3) and tested on the randomly generated instances described in Section 6.1. For the sake of clarity, the results are presented in two subsections, which correspond to the Strengthened Shortest Path and Profit Upper Bound inequalities respectively. Then, a last subsection provides results for model (HP3) with the best (i.e., most efficient) valid inequalities.

6.3.1 Strengthened Shortest Path inequalities

In this section, we test model (HP3) with the Strengthened Shortest Path inequalities (4.1) and (4.2). In order to differentiate the efficiency of both classes of inequalities, tests of these ones are performed separately. Numerical results are presented in Tables 6.3 and 6.5.

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	0.00	10.96	4.30	3.9	0	5	2	2	1	139	37	54
5 v - 15 n	1.28	14.90	5.65	4.5	0	12	5	4	1	167	56	54
8 v - 10 n	3.20	20.24	12.14	5.8	2	3974	1295	1564	36	27464	8794	10714
*3*8 v - 15 n	7.20	22.43	10.32	6.9	85	18014	7179	8015	347	51007	18605	19664
*1*10 v - 10 n	11.34	21.17	15.31	3.4	160	18001	4134	6233	395	60575	14556	20743
*6*10 v - 15 n	9.14	18.90	14.10	3.6	18000	18025	18015	6	11172	20638	15889	3594

Table 6.3: Model (HP3) with inequalities (4.1)

Inst.	Nb Ineq. (total)				Nb Ineq. (root)			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	10	337	101	116	10	49	32	14
5 v - 15 n	48	895	325	293	33	83	57	18
8 v - 10 n	57	64541	25340	28273	13	270	120	91
8 v - 15 n	1915	323845	110462	135304	82	378	152	115
10 v - 10 n	1963	123275	30156	42192	53	117	90	20
10 v - 15 n	69940	138536	111748	23664	123	444	270	107

Table 6.4: Number of inequalities (4.1) appended to (HP3)

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	0.00	3.26	1.44	1.3	0	4	1	2	1	29	11	12
5 v - 15 n	0.00	4.87	1.58	1.7	0	4	2	2	1	61	21	24
8 v - 10 n	1.21	6.94	3.85	2.4	2	693	228	291	17	2874	869	1127
8 v - 15 n	2.93	7.57	3.96	1.6	28	8572	1545	3145	63	22343	3961	8222
10 v - 10 n	1.58	5.49	3.63	1.2	10	388	88	135	24	2580	499	932
*3*10 v - 15 n	1.42	7.13	4.44	1.9	304	18017	9654	8379	316	24594	11435	9548

Table 6.5: Model (HP3) with inequalities (4.2)

Inst.	Nb Ineq. (total)				Nb Ineq. (root)			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	12	191	77	70	12	93	50	33
5 v - 15 n	28	244	116	84	28	155	71	40
8 v - 10 n	103	15151	5329	6527	87	646	335	208
8 v - 15 n	748	104624	19457	38126	226	815	469	184
10 v - 10 n	790	9892	2588	3275	420	818	536	132
10 v - 15 n	5947	275338	137897	117995	715	1311	987	186

Table 6.6: Number of inequalities (4.2) appended to (HP3)

Observing Tables 6.2 and 6.3, we conclude that the Strengthened Shortest Path inequalities (4.1) yield a decrease from 8 to 42% in the gaps. The number of nodes also decreases by 77% for the instances of the largest size solved to optimality, i.e., $10v$-$10n$. Note that, from now, we only report the decrease by the number of nodes and of the cpu times for the instances of the largest size solved to optimality, i.e., $10v$-$10n$. However, the inequalities also yield a significant increase of the cpu times, probably due to the time required by the separation procedure.

The Strengthened Shortest Path inequalities (4.2) are much more efficient, as the results show a decrease from 74 to 84% in the gaps. For the largest instances solved to optimality (i.e., $10v - 10n$), we observe a decrease by 90% in the number of nodes, and of 99% in the cpu times.

Further, Tables 6.4 and 6.6 provide the number of Strengthened Shortest Path inequalities appended to (HP3) at the root and during the branch and cut algorithm. We note that the evolution of the number of violated inequalities is not the same for both classes of inequalities. Indeed, more inequalities of class (4.2) are appended to (HP3) at the root of the branch and cut algorithm, while there are more inequalities of class (4.1) during the branch and cut algorithm.

Next, Table 6.7 provides results for model (HP3) with both classes of inequalities (4.1) and (4.2). Comparisons with Table 6.2 lead to the conclusion that both classes are useful. Also note that the cpu times are now similar to the ones of the initial model (HP3) for the small instances, and better than these for the largest instances. However, as we know that adding the first class of inequalities (4.1) yields a significant increase of the cpu times, it could be better to add these inequalities only at the root of the branch and cut algorithm. The results obtained are presented in Table 6.9.

Comparing Tables 6.7 and 6.9, we observe that adding inequalities (4.1) and (4.2) (resp. only at the root of the branch and cut algorithm) leads to a decrease

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	0.00	2.14	0.90	0.8	0	5	2	2	1	33	9	12
5 v - 15 n	0.00	3.76	1.15	1.3	0	11	5	4	1	109	30	40
8 v - 10 n	1.19	5.77	3.31	1.9	3	976	294	391	9	1832	479	683
*1*8 v - 15 n	2.42	6.88	3.43	1.6	63	18021	3182	6639	63	16877	2979	6216
10 v - 10 n	1.34	4.82	2.97	1.3	42	552	155	180	19	1200	337	418
*3*10 v - 15 n	0.90	6.75	3.23	2.4	616	18052	7194	7792	97	5878	2496	2135

Table 6.7: Model (HP3) with inequalities (4.1) and (4.2)

Inst.	Nb Ineq. (total)				Nb Ineq. (root)			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	13	286	109	115	13	153	64	53
5 v - 15 n	34	865	298	296	34	219	122	58
8 v - 10 n	146	16163	5575	6988	115	986	484	339
8 v - 15 n	1296	224526	40231	82454	396	1184	722	245
10 v - 10 n	1185	9098	3063	2732	531	1090	721	177
10 v - 15 n	5705	140361	57424	58863	1054	1637	1144	544

Table 6.8: Number of inequalities (4.1)-(4.2) appended to (HP3)

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	0.00	2.14	0.90	0.8	0	4	1	2	1	37	12	16
5 v - 15 n	0.00	3.76	1.15	1.3	0	5	3	2	1	91	28	35
8 v - 10 n	1.19	5.77	3.31	1.9	2	803	219	307	25	7393	1794	2737
8 v - 15 n	2.42	6.88	3.43	1.6	28	5352	970	1962	119	25344	4518	9316
10 v - 10 n	1.34	4.82	2.97	1.3	16	142	45	44	27	3971	766	1438
*3*10 v - 15 n	0.90	5.65	3.67	1.6	377	18037	10410	7932	385	43696	22537	16057

Table 6.9: Model (HP3) with inequalities (4.1) and (4.2) appended only at root

by 83% (resp. 95%) in the cpu times for the instances of the largest size solved to optimality. The number of nodes decreases by 99% (resp. 99%) for instances $10v$-$10n$, while the gaps decrease from 79 to 88%. Hence the inequalities should be appended at the root of the branch and cut algorithm only.

In the next section, we test the Strengthened Profit Upper Bound inequalities for model (HP3).

6.3.2 Strengthened Profit Upper Bounds inequalities

We also test the efficiency of the Strengthened Profit Upper Bound inequalities for the General Complete Toll NPP. As the development of a separation procedure for the inequalities would be a long and possibly arduous task, their efficiency is tested as follows. First, the three classes of inequalities (4.7)-(4.8), (4.9)-(4.10) and (4.11) are generated at the root of the branch and cut algorithm. Next the strongest of each class is appended to the model when violated. The results are presented in Tables (6.10), (6.14) and (6.12).

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	0.17	11.11	4.91	3.6	0	4	2	2	1	91	29	32
5 v - 15 n	2.74	16.56	8.03	4.7	1	6	4	2	9	263	102	102
8 v - 10 n	2.20	20.18	11.93	6.2	1	1305	310	464	31	48239	11383	17176
8 v - 15 n	7.54	15.69	11.54	2.9	20	5385	1404	1915	269	74292	24297	27776
10 v - 10 n	5.81	15.87	10.39	3	17	2175	494	759	153	64672	13678	22897
*4*10 v - 15 n	10.08	18.46	14.06	3.1	4353	18016	13485	6401	56587	261491	149044	79142

Table 6.10: Model (HP3) with inequalities (4.7)-(4.8)

Inst.	Nb Ineq. (root)			
	MIN	MAX	μ	σ
5 v - 10 n	6	110	37	37
5 v - 15 n	25	50	32	9
8 v - 10 n	27	681	268	240
8 v - 15 n	53	275	136	70
10 v - 10 n	215	431	309	68
10 v - 15 n	218	747	396	166

Table 6.11: Number of inequalities (4.7)-(4.8) appended to (HP3)

Compared to Table (6.2), the results show a small decrease by the gaps: from 17 to 37% for inequalities (4.7)-(4.8) and from 12 to 25% for inequalities (4.9)-(4.10). The cpu times and number of nodes also decrease for large instances. For the three classes of inequalities, we observe a decrease by 45%, 47% and 26% in the cpu

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	0.17	11.27	5.32	3.9	0	3	2	2	1	125	33	43
5 v - 15 n	2.84	16.92	8.52	4.8	1	6	4	2	9	204	82	68
8 v - 10 n	2.30	20.78	12.77	6.7	2	657	157	234	53	24522	5635	8802
8 v - 15 n	8.53	16.30	12.12	2.9	29	4153	1507	1620	519	80169	29307	29444
10 v - 10 n	7.27	18.99	12.43	3.5	18	2306	478	822	221	75721	14818	27288
4 10 v - 15 n	10.35	19.05	14.76	3.2	1556	18015	12644	7592	27350	443859	215299	160459

Table 6.12: Model (HP3) with inequalities (4.9)-(4.10)

Inst.	Nb Ineq. (root)			
	MIN	MAX	μ	σ
5 v - 10 n	5	65	27	23
5 v - 15 n	21	43	28	7
8 v - 10 n	26	605	212	201
8 v - 15 n	43	237	103	64
10 v - 10 n	173	391	265	73
10 v - 15 n	194	568	311	121

Table 6.13: Number of inequalities (4.9)-(4.10) appended to (HP3)

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	0.17	12.16	6.44	4.4	0	2	1	1	1	113	35	42
5 v - 15 n	3.38	18.21	9.57	5.1	1	7	3	2	7	397	105	133
8 v - 10 n	3.47	26.75	15.24	7.8	1	545	123	192	39	27697	6267	9807
8 v - 15 n	9.49	17.32	12.89	3	22	1529	544	556	354	44623	16716	16990
10 v - 10 n	12.46	22.22	16.38	3.3	24	3565	662	1299	435	162860	29278	59755
*4*10 v - 15 n	10.94	20.50	16.12	3.5	606	18008	12694	7561	11628	534114	252863	169438

Table 6.14: Model (HP3) with inequalities (4.11)

Inst.	Nb Ineq. (root)			
	MIN	MAX	μ	σ
5 v - 10 n	0	17	5	6
5 v - 15 n	0	12	6	4
8 v - 10 n	0	68	19	24
8 v - 15 n	0	70	22	23
10 v - 10 n	8	53	30	16
10 v - 15 n	14	115	52	36

Table 6.15: Number of inequalities (4.11) appended to (HP3)

times, and of 78%, 76% and 53% in the number of nodes for instances $10v$-$10n$.

Further, Tables (6.11), (6.15) and (6.13) provide the number of Strengthened Profit Upper Bound inequalities appended to model (HP3). We observe that inequalities (4.11) are the least violated, thus they could be less efficient that the other classes of inequalities.

We also test combinations of inequalities. In these tests, each class of inequalities is generated at the root of the branch and cut algorithm. Then the most violated class - if any - is appended to formulation (HP3*). The results are presented in Tables (6.16), (6.17) and (6.18).

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	0.17	11.60	5.55	3.8	0	2	1	1	1	81	28	28
5 v - 15 n	3.22	17.50	8.66	5	1	7	3	2	11	687	163	238
8 v - 10 n	2.69	20.45	12.89	6.2	1	765	161	274	35	36593	7830	13069
8 v - 15 n	8.76	16.53	12.09	2.9	13	2178	585	759	349	47533	17171	18271
10 v - 10 n	6.24	16.50	11.79	3	17	905	238	320	295	43817	10131	15394
*4*10 v - 15 n	10.34	19.44	14.69	3.2	998	18014	12590	7679	32088	484325	249469	150359

Table 6.16: Model (HP3) with inequalities (4.7)-(4.8) and (4.9)-(4.10)

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	0.17	11.11	4.91	3.6	0	2	1	1	1	91	30	32
5 v - 15 n	2.74	16.56	8.02	4.7	1	3	3	1	9	223	88	79
8 v - 10 n	2.20	20.18	11.91	6.2	2	1230	264	439	47	46625	9733	16710
8 v - 15 n	7.54	15.69	11.53	2.9	15	2630	788	896	273	48106	19648	17154
10 v - 10 n	5.79	15.86	10.36	3	10	3027	554	1107	239	172138	30359	63415
*4*10 v - 15 n	10.08	18.58	13.94	3	926	18010	12339	8017	23013	443483	230147	161084

Table 6.17: Model (HP3) with inequalities (4.7)-(4.8) and (4.11)

Combining inequalities (4.7)-(4.8) and (4.9)-(4.10) yields a decrease from 11 to 29% in the gaps, compared to Table 6.2. Further, for the instances of the largest size solved to optimality (i.e., $10v$-$10n$), the cpu times and number of nodes decrease by 73% and 84% respectively. Observing Tables 6.2 and 6.17, we conclude

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	0.17	11.27	5.31	3.9	0	5	2	2	1	125	35	44
5 v - 15 n	2.84	16.92	8.52	4.8	1	7	5	2	9	367	122	126
8 v - 10 n	2.3	20.77	12.76	6.7	2	3199	584	1171	53	115845	20702	42589
8 v - 15 n	8.52	26.62	14.34	6.2	36	18004	4413	6485	585	403053	88481	143778
10 v - 10 n	7.27	18.97	12.37	3.6	22	2571	533	916	190	71593	14049	25792
6 10 v - 15 n	10.35	19.24	14.62	3.1	2136	18017	13162	6903	25610	267260	148720	80939

Table 6.18: Model (HP3) with inequalities (4.9)-(4.10) and (4.11)

that pairing inequalities (4.7)-(4.8) and (4.11) leads to a decrease from 17 to 38% in the gaps. The cpu times and number of nodes decrease by 38% and 52% respectively for instances $10v$-$10n$. Finally, combining inequalities (4.9)-(4.10) and (4.11) yields a decrease from 6 to 25% in the gaps. The cpu times and number of nodes decrease by 40% and 78% respectively for instances $10v$-$10n$.

Hence, for the Strengthened Profit Upper Bound inequalities, we conclude that the best results are obtained when combining inequalities (4.7)-(4.8) with (4.9)-(4.10).

6.3.3 Final tests for (HP3)

According to Subsections 6.3.1 and 6.3.2, the best results are obtained with the Strengthened Shortest Path inequalities (4.1)-(4.2), or the Strengthened Profit Upper Bound inequalities (4.7)-(4.8) and (4.9)-(4.10). In this section, we test model (HP3) with these valid inequalities, which are appended at the root of the branch and cut algorithm when violated. The results are presented in Table 6.19.

If we compare Tables 6.2 and 6.19, we observe that the gaps decrease from 78 to 89% with respect to the initial model (HP3). Further, for the largest instances solved to optimality, i.e., $10v$-$10n$, the inequalities yield a decrease by 90% in the

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	0.00	1.89	0.86	0.7	0	4	2	2	1	37	11	15
5 v - 15 n	0.00	3.49	1.09	1.2	0	5	3	2	1	95	30	36
8 v - 10 n	1.01	5.51	3.02	1.6	2	569	130	203	15	4732	889	1725
8 v - 15 n	2.42	6.73	3.36	1.5	38	6928	1236	2546	87	28103	4875	10388
10 v - 10 n	1.13	4.63	2.84	1.2	22	347	90	116	5	4613	874	1682
*3*10 v - 15 n	0.89	5.78	3.62	1.7	202	18076	9621	8462	181	34462	12170	11592

Table 6.19: Model (HP3) with inequalities (4.1)-(4.2), (4.7)-(4.8) and (4.9)-(4.10) (only at root)

cpu times and of 99% in the number of nodes.

To illustrate the results, Figures 6.1, 6.2 and 6.3 depict the evolution of the lower and upper bounds of the objective function with respect to the cpu time for three instances. The lower and upper bounds for the initial model (HP3) are denoted 'LB (HP3)' and 'UB (HP3)', while the lower and upper bounds for the final model (HP3) with inequalities (4.1)-(4.2), (4.7)-(4.8) and (4.9)-(4.10) are denoted 'LB (HP3) Final' and 'UB (HP3) Final' respectively.

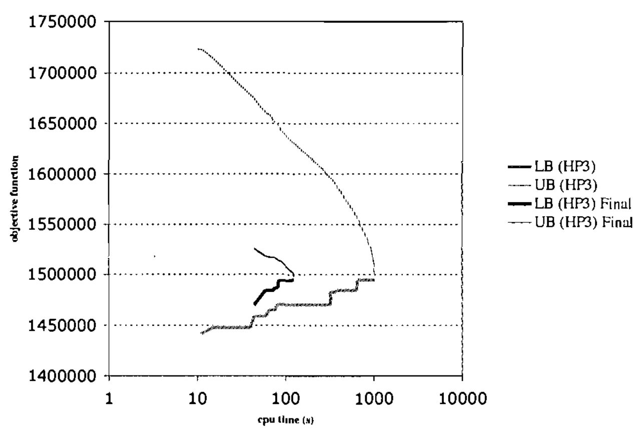

Figure 6.1: Evolution of the objective function with respect to the cpu time for an instance of class $8v$-$10n$

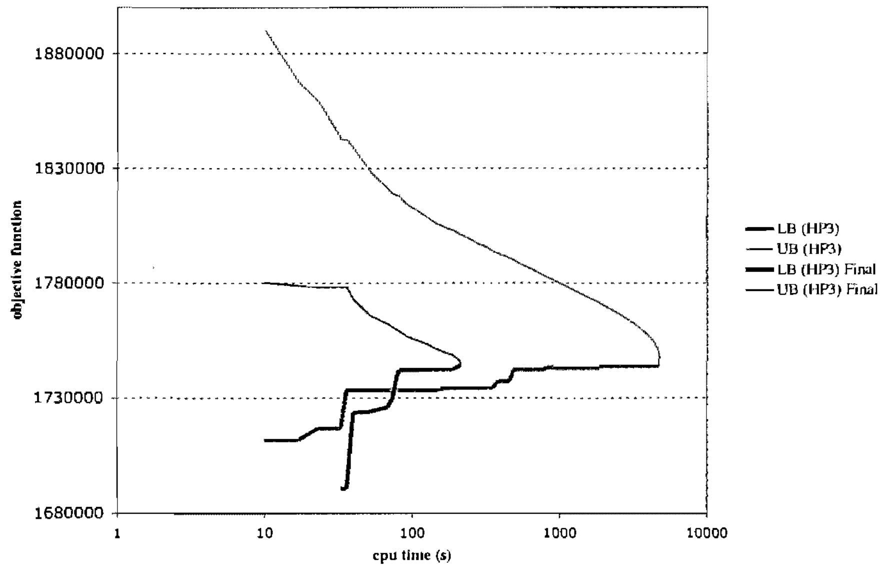

Figure 6.2: Evolution of the objective function with respect to the cpu time for an instance of class 10v-10n

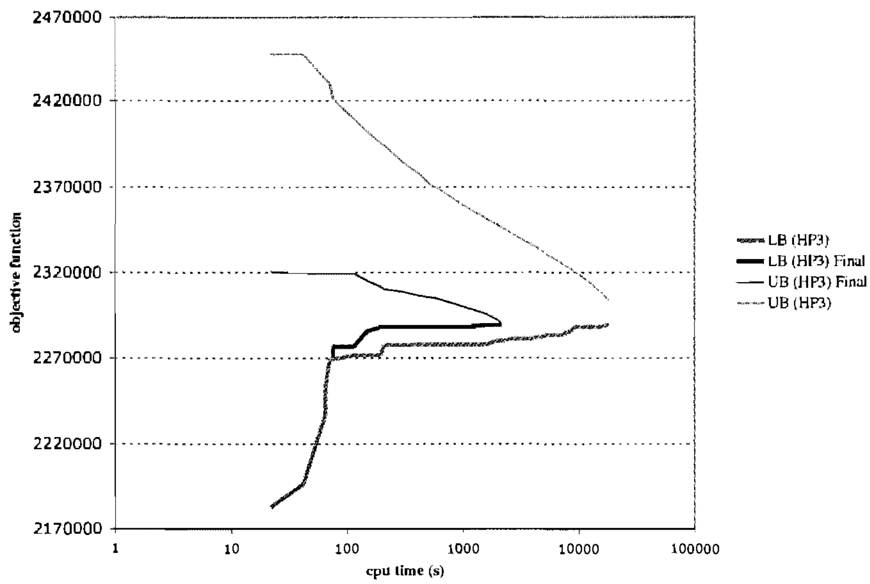

Figure 6.3: Evolution of the objective function with respect to the cpu time for an instance of class 10v-15n

6.4 Numerical results for CCT-NPP

In this section, we test the valid inequalities proposed in Chapter 4 in the context of the Constrained Complete Toll NPP, i.e., for model (HP3*). First of all, the results obtained for model (HP3*) without any new valid inequalities are presented in Table 6.20. Note that appending the Triangle and Monotonicity constraints to the problem yields a significant increase of the cpu times, which does not allow us to solve instances as large as for the GCT-NPP.

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	8.34	30.25	19.46	7.3	0	4	2	1	41	393	173	140
5 v - 15 n	12.74	27.86	19.50	4.4	0	9	5	3	29	1067	370	346
8 v - 10 n	16.88	39.94	30.54	7.8	3	1105	271	388	175	121461	28430	42968
^{*5*}8 v - 15 n	30.26	54.24	40.60	8.5	3570	18001	15596	5378	202361	1138242	812369	307804

Table 6.20: Model (HP3*)

In the next subsections, we test each class of valid inequalities. Subsection 6.4.1 provides numerical results for model (HP3*) with the Strengthened Shortest Path inequalities, while Subsection 6.4.2 provides results for model (HP3*) with the Strengthened Profit Upper Bound inequalities. Finally, model (HP3*) is tested with the best valid inequalities in Subsection 6.4.3.

6.4.1 Strengthened Shortest Path inequalities

This section aims to test model (HP3*) with the Strengthened Shortest Path inequalities (4.1) and (4.2). The results are presented in Tables 6.21 and 6.23.

If we compare Tables 6.20 and 6.21, we observe that the Strengthened Shortest Path inequalities (4.1) yield a decrease by 30% in the number of nodes for the instances of the largest size solved to optimality, i.e., $8v$-$10n$. Note that, from

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	8.34	29.79	19.24	7.1	1	7	4	2	39	349	182	106
5 v - 15 n	12.70	27.83	19.42	4.4	1	26	13	9	31	1413	554	465
8 v - 10 n	16.63	39.65	30.36	7.7	13	9801	3550	3556	129	44373	19902	17581
*5*8 v - 15 n	30.10	58.01	41.01	9.5	3440	18004	15577	5427	100000	649572	473912	172174

Table 6.21: Model (HP3*) with inequalities (4.1)

Inst.	Nb INEG (total)				Nb INEG (root)			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	16	62	28	16	5	32	13	9
5 v - 15 n	4	220	104	85	1	36	15	12
8 v - 10 n	41	162856	45732	57287	3	44	22	14
8 v - 15 n	285	550	381	106	12	42	28	10

Table 6.22: Number of inequalities (4.1) appended to (HP3*)

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	2.84	15.49	7.90	4	0	7	4	3	25	125	68	37
5 v - 15 n	0.00	8.74	4.78	3.3	0	19	9	6	1	291	132	108
8 v - 10 n	8.20	21.59	13.65	4.6	6	13295	2710	4785	37	59424	12376	21224
*5*8 v - 15 n	16.56	23.18	19.07	2.6	4329	18016	15731	5099	68529	253956	119928	64542

Table 6.23: Model (HP3*) with inequalities (4.2)

Inst.	Nb INEG (total)				Nb INEG (root)			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	23	239	140	70	13	143	78	44
5 v - 15 n	37	244	166	67	37	149	115	37
8 v - 10 n	188	295532	61556	106030	114	589	344	175
8 v - 15 n	483	1589	1009	376	293	923	577	193

Table 6.24: Number of inequalities (4.2) appended to (HP3*)

now, we always report the decrease in the number of nodes for the instances of the largest size solved to optimality, i.e., $8v$-$10n$. However, the cpu times increase and the decrease by the gaps is insignificant.

In what concerns the Strengthened Shortest Path inequalities (4.2), the results show a decrease from 55 to 75% in the gaps, and of 56% in the number of nodes for the instances of the largest size solved to optimality. i.e., $8v$-$10n$. Hence, although an increase of the cpu times due to the time required by the separation procedure, this class of valid inequalities is very efficient.

Further, Tables 6.22 and 6.24 provide the number of Strengthened Shortest Path inequalities appended to (HP3*) at the root and during the branch and cut algorithm. One can observe that there is much more violated inequalities of class (4.2) than of class (4.1). This explains why adding inequalities (4.2) to (HP3*) yields a larger decrease in the gaps and number of nodes.

Next, as the first class of Strengthened Shortest Path inequalities is much less efficient than the second one, we also test model (HP3*) with both classes of inequalities, in order to decide if the first class (4.1) should be used or not. In this case, the most violated among both classes of inequalities is appended to model (HP3*), if any. The results are presented in Table 6.25.

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	2.22	15.15	6.85	4.1	0	7	4	2	9	103	41	30
5 v - 15 n	0.00	8.44	4.30	3	0	16	10	6	1	335	108	109
8 v - 10 n	7.18	20.75	12.80	4.6	8	10010	2125	3568	31	21941	4913	7697
*5*8 v - 15 n	16.28	24.47	18.87	3.1	1357	18017	15236	6207	27870	261180	145685	76628

Table 6.25: Model (HP3*) with inequalities (4.1) and (4.2)

Inst.	Nb INEG (total)				Nb INEG (root)			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	31	299	176	89	24	220	109	63
5 v - 15 n	42	314	241	100	42	218	156	56
8 v - 10 n	198	162548	36234	57595	111	820	443	273
8 v - 15 n	773	1953	1441	386	343	1240	730	276

Table 6.26: Number of inequalities (4.1)-(4.2) appended to (HP3*)

If we compare Tables 6.20, 6.23 and 6.25, we conclude that both classes of inequalities are useful. One can observe a decrease from 58 to 78% in the average gaps, and of 83% in the number of nodes. However, Table 6.26 points out the huge number of violated inequalities appended to (HP3*) during the branch and cut algorithm, which explains the significant increase of the cpu times. As a consequence, it could be better to append the Strengthened Shortest inequalities only at the root of the branch and cut algorithm, which would yield a more reasonable number of additional inequalities in the model. The results obtained are presented in Table 6.27.

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	2.22	15.15	6.85	4.1	0	6	4	2	11	115	63	39
5 v - 15 n	0.00	8.44	4.30	3	0	16	8	5	1	239	109	80
8 v - 10 n	7.18	20.75	12.80	4.6	5	2617	595	937	37	39434	8452	13999
^{*5*}8 v - 15 n	16.28	26.16	19.52	3.9	5796	18027	15980	4554	72548	166820	110460	33264

Table 6.27: Model (HP3*) with (4.1) and (4.2) inequalities appended only at root

To conclude, the Strengthened Shortest Path inequalities (4.1) and (4.2), and especially the second ones, are very useful to decrease the gap and number of nodes in the branch and cut algorithm. Unfortunately, they are quite costly in terms of cpu time. This effect can be reduced by adding the valid inequalities at the root of the branch and cut algorithm only. However, it also yields a loss of efficiency relative to the number of nodes, as the decreasing factor is now 70% (instead of 83%, see Table 6.25). Hence, the results can be balanced depending on the relative importance of cpu times and number of nodes.

6.4.2 Strengthened Profit Upper Bound Inequalities

Here we test model (HP3*) with the Strengthened Profit Upper Bound Inequalities (4.7)-(4.8), (4.9)-(4.10) and (4.11). The results are presented in Tables (6.28), (6.30) and (6.32).

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	7.06	27.93	16.80	7.8	0	6	4	2	42	475	183	163
5 v - 15 n	8.56	26.95	17.50	5.4	1	33	13	11	23	1225	512	419
8 v - 10 n	12.56	36.83	25.32	7.8	5	1749	404	617	109	73944	16330	26061
*5*8 v - 15 n	28.15	53.70	38.72	8.8	4859	18006	15814	4899	100000	627600	427785	141419

Table 6.28: Model (HP3*) with inequalities (4.7)-(4.8)

Inst.	Nb INEG (root)			
	MIN	MAX	μ	σ
5 v - 10 n	8	144	45	46
5 v - 15 n	23	55	41	11
8 v - 10 n	58	674	263	241
8 v - 15 n	101	312	166	68

Table 6.29: Number of inequalities (4.7)-(4.8) appended to (HP3*)

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	7.47	27.93	17.69	7.1	1	5	3	1	29	217	111	63
5 v - 15 n	9.46	27.17	18.22	5.1	2	29	12	10	19	1447	475	487
8 v - 10 n	13.71	37.36	26.74	7.7	4	2579	574	910	47	107406	24315	37934
*5*8 v - 15 n	28.90	55.61	40.05	9.3	1076	18005	15183	6309	35782	581625	429370	185519

Table 6.30: Model (HP3*) with inequalities (4.9)-(4.10)

Comparisons with Table (6.20) lead to the following conclusions. Inequalities (4.7)-(4.8) yield a decrease from 10 to 17% in the gaps, and of 43% in the number of nodes. Inequalities (4.9)-(4.10) yield a decrease from 9 to 12% in the gaps and of 14% in the number of nodes. However, to generate all inequalities increase the

Inst.	Nb INEG (root)			
	MIN	MAX	μ	σ
5 v - 10 n	5	89	31	29
5 v - 15 n	4	48	27	16
8 v - 10 n	55	604	211	207
8 v - 15 n	79	229	116	52

Table 6.31: Number of inequalities (4.9)-(4.10) appended to (HP3*)

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	8.15	30.19	19.38	7.3	0	5	3	2	41	393	173	141
5 v - 15 n	12.74	27.86	19.41	4.4	1	15	8	6	23	745	386	277
8 v - 10 n	16.88	39.93	30.33	7.6	3	2800	632	990	57	183141	40731	64970
*5*8 v - 15 n	30.34	55.43	40.85	8.7	2055	18003	15345	5943	86706	772089	538845	219395

Table 6.32: Model (HP3*) with inequalities (4.11)

Inst.	Nb INEG (root)			
	MIN	MAX	μ	σ
5 v - 10 n	0	18	4	6
5 v - 15 n	0	6	2	2
8 v - 10 n	0	76	20	28
8 v - 15 n	0	17	9	6

Table 6.33: Number of inequalities (4.11) appended to (HP3*)

cpu times. Finally, inequalities (4.11) are useless for model (HP3*). Indeed, the decrease by the gaps is insignificant, while the cpu times and number of nodes increase.

Further, comparing Tables (6.29), (6.31) and (6.33) with the corresponding tables for the GCT-NPP, which provide the number of violated inequalities appended to model (HP3*) and (HP3) respectively, we conclude that the Strengthened Profit Upper Bound inequalities are more useful in the context of the GCT-NPP, i.e., without Triangle and Monotonicity inequalities. Further, we observe the very small number of inequalities (4.11). It explains why the gaps do not decrease when adding this class of inequalities to model (HP3*).

In order to know if inequalities (4.7)-(4.8) and (4.9)-(4.10) should be used singly or together, we also perform tests of model (HP3*) with both classes of Strengthened Profit Upper Bound inequalities, adding the most violated one to model (HP3*), if any. The results are presented in Table (6.34).

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	7.68	30.17	17.84	8.5	0	8	3	3	41	499	179	166
5 v - 15 n	9.55	27.17	18.26	5.1	0	16	9	6	29	739	395	247
8 v - 10 n	14.81	38.03	26.9	7.3	5	1305	365	462	35	45544	13068	15823
*5*8 v - 15 n	28.98	51.59	38.53	8	17506	18009	17924	187	100000	666912	559308	85244

Table 6.34: Model (HP3*) with inequalities (4.7)-(4.8) and (4.9)-(4.10)

Combining inequalities (4.7)-(4.8) and (4.9)-(4.10) yield a decrease from 6 to 12% in the gaps and of 54% in the number of nodes, compared to Table 6.20. However, the cpu times increase. Further, as these results are similar to the ones obtained when adding inequalities (4.7)-(4.8) to model (HP3*) (see Table 6.28), we cannot determine if inequalities (4.9)-(4.10) should be used or not.

6.4.3 Final tests for (HP3*)

Now we test model (HP3*) with the most efficient valid inequalities, according to Subsections 6.4.1 and 6.4.2. The inequalities are appended to model (HP3*) at the root of the branch and cut algorithm. Results are presented in Tables 6.35 and 6.36.

Comparing these two tables, we observe that the gaps obtained are similar, and lead to a decrease from 59 to 78% with respect to the initial model (HP3*). However, model (HP3*) with inequalities (4.1)-(4.2) and (4.7)-(4.8) yields better results in terms of both cpu times and number of nodes, especially for large instances. These valid inequalities lead to a decrease by 64% in the number of nodes

Inst.	Gap(%)				Time(sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	1.93	15.15	6.80	4.2	0	10	5	4	9	211	78	68
5 v - 15 n	0.00	8.40	4.23	2.9	0	20	11	7	1	213	106	69
8 v - 10 n	7.13	20.73	12.74	4.7	7	5119	1263	1782	29	63895	15152	22045
*5*8 v - 15 n	16.26	23.35	19.05	3.0	2918	18033	15511	5632	71731	163993	115733	28462

Table 6.35: Model (HP3*) with inequalities (4.1)-(4.2), (4.7)-(4.8) and (4.9)-(4.10) (only at root)

Inst.	Gap(%)				Time(s)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	1.94	15.15	6.80	4.2	0	10	5	3	9	151	66	50
5 v - 15 n	0.00	8.37	4.22	2.9	0	20	11	7	1	171	97	64
8 v - 10 n	7.16	20.71	12.57	4.7	7	2732	654	964	71	45363	10222	15885
8 v - 15 n	16.18	22.81	18.92	2.4	6689	18052	16143	4228	41230	124506	78708	29887

Table 6.36: Model (HP3*) with inequalities (4.1)-(4.2) and (4.7)-(4.8) (only at root)

for the instances of the largest size solved to optimality (i.e., $8v$-$10n$).

However, the results obtained for model (HP3*) with only the Strengthened Shortest Path inequalities (4.1)-(4.2) appended at the root of the branch and cut algorithm (see Table 6.27) are slightly better than these ones, both in terms of cpu times and number of nodes. Further, the decrease by the gaps obtained when adding the Strengthened Profit Upper Bound inequalities (4.7)-(4.8) is insignificant.

We conclude that the best valid inequalities for the CCT-NPP are the Strengthened Shortest Path inequalities (4.1)-(4.2), and should be appended to model (HP3*) at the root of the branch and cut algorithm only. However, this increases the cpu times. Indeed, the Triangle and Monotonicity constraints make the problem much more difficult. While the valid inequalities proposed yield a significant decrease by both gap and number of nodes in the branch and cut algorithm, they also interfere negatively with the Triangle and Monotonicity constraints, which obstructs the program to reach quickly optimality.

To illustrate the results, Figures 6.4 and 6.5 depict the evolution of the lower and upper bounds on the objective function with respect to the cpu time for two instances. The lower and upper bounds for the initial model (HP3*) are denoted 'LB (HP3*)' and 'UB (HP3*)', while the lower and upper bounds for model (HP3*) with inequalities (4.1)-(4.2) (appended only at root) are denoted 'LB (HP3*) Final' and 'UB (HP3*) Final' respectively.

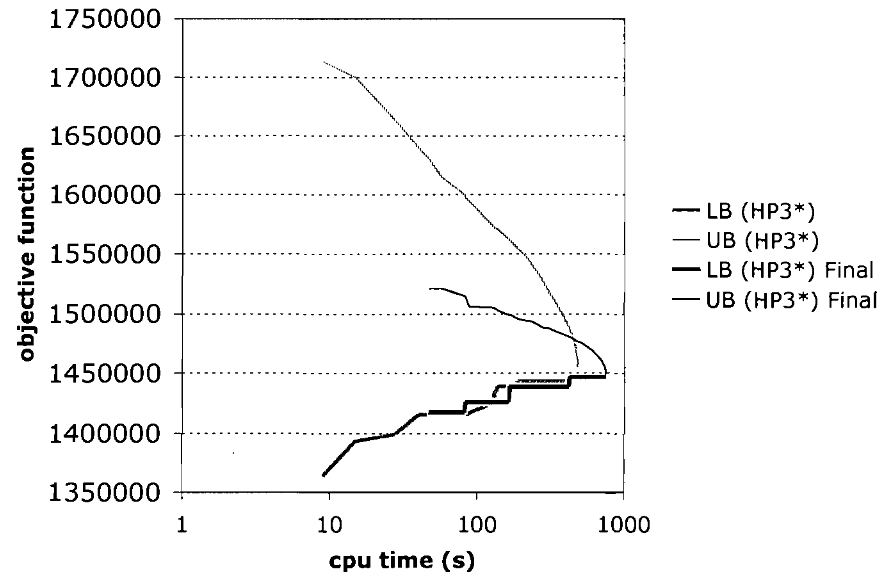

Figure 6.4: Evolution of the objective function with respect to the cpu time for an instance of class $8v$-$10n$

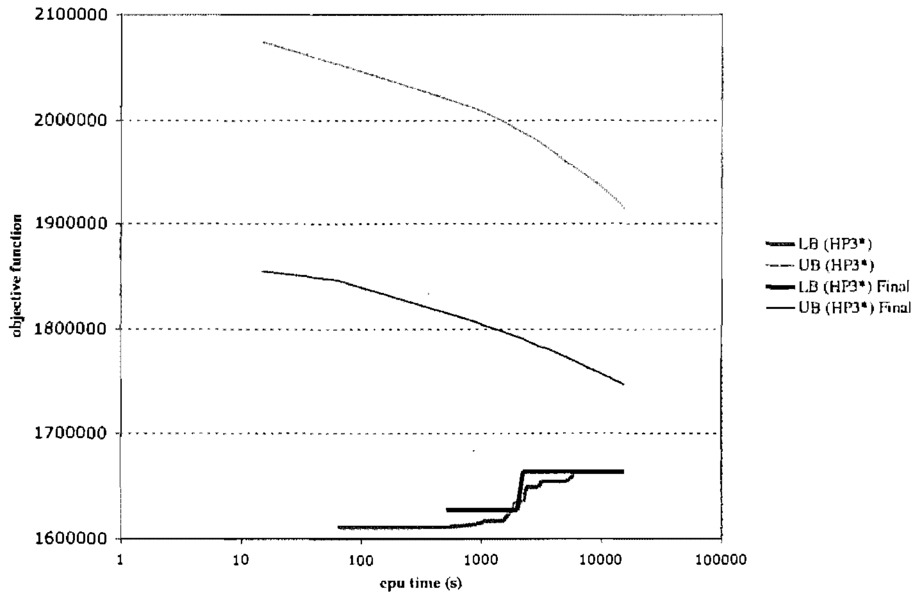

Figure 6.5: Evolution of the objective function with respect to the cpu time for an instance of class $8v$-$15n$

6.5 Conclusion

In this chapter, we assessed the valid inequalities from Chapter 4 for the GCT-NPP and CCT-NPP. Obviously, the Strengthened Shortest Path inequalities (4.2) lead to the best improvements in terms of gaps, cpu times and number of nodes in the branch and cut algorithm. Several other valid inequalities, especially the Strengthened Profit Upper Bound inequalities (4.7)-(4.8) and (4.9)-(4.10), but also the Strengthened Shortest Path inequalities (4.1), are useful. For the the GCT-NPP, we observe a decrease by 78 to 89% in the gaps, 90% in the cpu times and of 99% in the number of nodes for the largest instances solved to optimality. In what concerns the CCT-NPP, the best results show a decrease by 58 to 78% in the gaps and of 70% in the number of nodes for the largest instances solved to optimality.

CHAPTER 7

LINKING PRICING PROBLEMS IN TRANSPORTATION NETWORKS AND ECONOMICS

In this last chapter, we highlight the links between the Network Pricing Problem studied in the thesis and a more standard design and pricing family of problems in economics. While the first family of problems seeks to set tolls on a multi-commodity transportation network within the framework of bilevel programming, the second family intends to design and price a set of products taking into account the utility-maximizing customers. Both topics have been extensively studied in the literature. However, we focus on problems that can be modelled as mixed integer programs.

In Section 1, we consider a standard design and pricing family of problems. General definitions are first provided, followed by a summary of the main results from the literature. Next, in Section 2, we point out the relationships between this family of problems and the Network Pricing Problem. Finally, the aim of Section 3 is to compare a specific pricing problem in economics with the General Complete Toll NPP.

7.1 Designing and pricing a set of products

Consider the family of problems which intends to design and price a set of products in a given economic market. In the mathematical literature dedicated to this field, three different paradigms are studied: the Buyer Welfare, the Seller Welfare and the Share-of-Choices problems. We first provide a general definition of the three problems. Then we summarize the main mathematical contributions to this field.

7.1.1 Problem definition

Let $\mathcal{K}$ be a set of purchasers, and $\mathcal{I}$ a set of products. The purchaser preferences for the various products are described by a utility matrix $u_i^k : k \in \mathcal{K}, i \in \mathcal{I}$. Each purchaser chooses the product with the largest utility, so far as this utility is positive. Otherwise he refrains from buying.

The **Buyer Welfare Problem** consists of determining which subset of products $\mathcal{S} \subseteq \mathcal{I}$ should be introduced in the market so as to maximize the sum of the purchaser utilities at optimality, i.e., for the products they have chosen. Considering binary flow variables $y_i, x_i^k : k \in \mathcal{K}, i \in \mathcal{I}$ that indicate if a product is introduced in the market and if a product is chosen by a purchaser respectively, this problem can be described by the following mixed integer program:

$$(BWP) \quad \max \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} u_i^k x_i^k$$

subject to:

$$\sum_{j \in \mathcal{I}} u_j^k x_j^k \geq u_i^k y_i \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \qquad (7.1)$$

$$\sum_{i \in \mathcal{I}} x_i^k \leq 1 \qquad \forall k \in \mathcal{K} \qquad (7.2)$$

$$x_i^k \leq y_i \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \qquad (7.3)$$

$$\sum_{i \in \mathcal{I}} y_i \leq Y \qquad (7.4)$$

$$x_i^k, y_i \in \{0, 1\} \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I}, \qquad (7.5)$$

where Y is a non negative constant.

Constraints (7.1) ensure that each purchaser chooses his/her best product in terms of utility, constraints (7.2) force each purchaser to choose at most one product, constraints (7.3) impose that the products chosen by purchasers are among those offered, and constraint (7.4) imposes an upper bound Y on the number of products that are introduced in the market.

Now consider additional parameters $v_i^k : i \in \mathcal{I}, k \in \mathcal{K}$ corresponding to the incomes perceived by a seller if purchaser $k \in \mathcal{K}$ buys product $i \in \mathcal{I}$. The **Seller Welfare Problem** consists of determining which subset of products $\mathcal{S} \subseteq \mathcal{I}$ should be introduced in the market so as to maximize the seller's income, knowing that each purchaser selects the product with largest utility for him, so far as this utility is positive. Hence the problem can be described as the mixed integer program:

$$\text{(SWP)} \quad \max \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} v_i^k x_i^k$$

subject to constraints (7.1) to (7.5).

The **Share-of-Choices Problem** is quite different from the two other pricing problems. One considers a set $\mathcal{A}$ of attributes associated with the various products, and a set $\mathcal{J}_a : a \in \mathcal{A}$ of levels for each attribute. A product profile is defined as the assignment of a level to each attribute of each product. It is represented by the vector $p = (j_1, j_2, ..., j_{|\mathcal{A}|})$ of its $\mathcal{A}$ attribute levels. Further, each purchaser associates a perceived value $w_{aj}^k : a \in \mathcal{A}, j \in \mathcal{J}_a, k \in \mathcal{K}$ to each level of each attribute. One also considers that the perceived values are normalized to lie between -1 and 1. Therefore, purchaser k prefers product profile $p = (j_1, j_2, ..., j_{|\mathcal{A}|})$ to status quo only if $w^k(p) = w_{1j_1}^k + w_{2j_2}^k + ... + w_{|\mathcal{A}|j_{|\mathcal{A}|}}^k$ is positive. The Share-of-Choices Problem consists of determining a product profile p so as to maximize the number of satisfied purchasers $k \in \mathcal{K}$, knowing that a purchaser k is satisfied if the sum of its perceived values $w^k(p)$ is positive for this product profile p.

We close this section by mentioning that, in the thesis, we do not consider models that involve an underlying probabilistic structure, i.e., the purchaser choices are determined by a probabilistic function. The interested reader could refer to Krieger and Green (2002, [42]), Shioda et al. (2007, [62]) or Maddah and Bish (2008, [48]) for further details.

7.1.2 Literature review

Green and Krieger (1985, [31]) study the Buyer and the Seller Welfare problems. For the Buyer Welfare Problem, they consider a subset of products $\mathcal{S} \subseteq \mathcal{I}$ to introduce in the market, with $|\mathcal{S}|$ exogenous to the problem, i.e., constraint (7.4) becomes $\sum_{i \in \mathcal{I}} y_i = |\mathcal{S}|$. Since the two problems cannot be solved efficiently by a complete enumeration of the feasible solutions, at least in real applications, the authors propose inexact resolution methods, such as Lagrangian relaxation and various greedy heuristics.

Refering to a theoretical study from Cornuéjols et al. [19] for an equivalent problem, the ratio of the greedy over the optimal income (latter called 'performance ratio') is, in the worst case, $Z_G/Z_O = 1 - ((|\mathcal{S}| - 1)|\mathcal{S}|^{-1})^{|\mathcal{S}|}$. Hence, as $|\mathcal{S}| \to \infty$, the performance ratio is approximately 63%, and will be higher for smaller values of $|\mathcal{S}|$. Green and Krieger also run simulations on small problems, randomly generated with $|\mathcal{K}| = 100$, $|\mathcal{I}| = 10$ and $|\mathcal{S}| = 4$ or 5. In all cases, the greedy heuristic is within 8% of optimality and gives the optimal solution in over 50% of the simulations.

Whereas these methods are effective for the Buyer Welfare Problem, they do not perform well for the Seller Welfare Problem. According to the authors, neither a Lagrangian relaxation nor an exact method can be implemented for problems of realistic sizes. Further, the greedy heuristic approach can yield very poor results. Indeed, in the worst possible case, the performance ratio is $|\mathcal{S}|^{-1}$, and becomes arbitrarily bad when $|\mathcal{S}| \rightarrow \infty$. However, better results can be obtained if the parameters v_i^k are almost equal for all $k \in \mathcal{K}, i \in \mathcal{I}$. Tests on randomly generated instances involving 100 purchasers, 10 products and $|\mathcal{S}| = 4$, shows that the seller's greedy heuristic is within 5% of optimality and gives the optimal solution in 78% of the simulations.

Kohli and Krishnamurti (1987, [39]) propose a dynamic programming heuristic to solve the Share-of-Choices problem with a single product. In order to highlight the efficiency of this new approach, the authors run tests on randomly generated problems involving 100 to 400 purchasers, 4 to 8 attributes, and 2 to 5 levels per attribute. The results are obtained very quickly. They are always within 9%, and on average within 2%, of optimality. The optimal solution is identified in 46% cases.

The authors also compare their approach to an alternative Lagrangian relaxation heuristic. They conclude that the dynamic programming heuristic dominates the Lagrangian relaxation heuristic in terms of both computational time and approximation of the optimal solution, the results obtained by the Lagrangian relaxation heuristic being only within 42% of optimality. The dynamic programming heuristic is also significantly faster than an enumeration procedure.

In another article, Kohli and Krishnamurti (1989, [40]) prove the $\mathcal{NP}$-hardness of the Share-of-Choices Problem with a single product. They also propose a graph representation of the problem, leading to two heuristics, based on dynamic-programming and on a shortest path problem respectively.

Both heuristics have arbitrarily bad worst-case bounds. However, when tested on random instances (the same as in [39]), the dynamic programming solution is on average within 2% of optimality (at worst within 12%), while the shortest path solution is on average within 6% of optimality (at worst within 13%). Optimal solutions are found in 42% and 14% cases respectively.

Kohli and Sukumar (1990, [41]) present dynamic-programming heuristics for the Buyer Welfare, Seller Welfare and Share-of-Choices problems, considering this time a multi-product set for the last problem. However, the Buyer and Seller Welfare problems involve a multi-attribute structure like in the Share-of-Choices Problem, i.e., levels have to be determined for each attribute of each product.

The heuristics are tested on randomly generated instances involving 50 to 150 purchasers, 2 to 4 products, 4 to 6 attributes and 2 to 4 levels per attribute. The empirical results are near-optimal, as the performance ratios are on average within 2%, 5% and 2% of optimality for the three problems, while the worst ratios are within 4%, 15% and 8% respectively. Optimal solutions are found in 10%, 12% and 30% cases respectively. Computationally, solutions are found very quickly.

Nair et al. (1995, [54]) propose beam search based heuristics for the Buyer Welfare, Seller Welfare and Share-of-Choices problems, where the Buyer and Seller Welfare problems involve a multi-attribute structure. Such heuristics consist in breadth first searches with no backtracking and breadth limited to a given number of the most promising nodes.

In order to compare the efficiency of this new approach with the Kohli and Sukumar dynamic programming heuristics, the authors randomly generate instances as in [41]. The results show performance ratios within 1% of optimality for the three problems, and optimal solutions are found in 38%, 58% and 66% cases respectively. Further, the beam search based heuristics is approximately two times faster than the Kohli and Sukumar heuristics.

Alexouda and Paparrizos (2001, [2]) present a genetic algorithm based heuristic for solving the Seller Welfare Problem with a multi-attribute structure. It is tested on randomly generated problems involving 100 or 150 purchasers, 2 or 3 products, 3 to 7 attributes and 3 to 6 levels per attribute. When compared to the beam search based heuristics proposed by Nair et al. [54], the methods perform better both in terms of cpu time and solution quality. The genetic heuristic are on average three times faster than the beam search based method, while the solution found is better by 8% on average. Optimal solutions are found in 74% cases.

The only authors which consider an exact resolution for the Seller Welfare Problem are McBryde and Zufryden (1988, [52]). Observing that constraints (7.1) are only active when there exists $j \in \mathcal{I}$ such that $x_j^k = 1$ and $u_j^k < u_i^k$ (then one must have $y_i = 0$), they replace constraints (7.1) by the equivalent:

$$y_i + x_j^k \leq 1 \qquad\qquad \forall k \in \mathcal{K}, \forall i, j \in \mathcal{I} : i \neq j, u_i^k > u_j^k.$$

Using a generic mathematical solver on the new formulation, the authors solve randomly generated instances with 50 to 100 purchasers and 16 products ($Y = 10$) to optimality very quickly.

They also obtain good results for a particular case in which the seller incomes $v_i^k : k \in \mathcal{K}, i \in \mathcal{I}$ do not depend on the products chosen by the purchasers. Indeed, they solve randomly generated instances with 100 to 300 purchasers and 64 to 512 products ($Y = 10$) to optimality in at most three seconds. Note that this last case is equivalent to a set covering problem, which is often solved using greedy methods.

7.1.3 Profit and Bundle Pricing Problems

Dobson and Kalish (1988, [25]) consider an extension of the Seller Welfare Problem, in which price variables $\pi_i : i \in \mathcal{I}$ are defined explicitly. The authors also assume that the introduction of a product i into the market induces a fixed cost f_i for the seller. The seller's income v_i for product i is the product price π_i.

Further, rather than a set of purchasers, the authors consider a set of purchaser segments with demand $\eta^k : k \in \mathcal{K}$, a segment being a set of purchasers which have the same reservation price. Hence a reservation price matrix $r_i^k : k \in \mathcal{K}, i \in \mathcal{I}$ is also defined, providing a measure of the value of each product for each segment. The utility u_i^k for a segment $k \in \mathcal{K}$ if it buys the product $i \in \mathcal{I}$ is defined as the difference between the reservation price r_i^k and the product price π_i.

The **Profit Problem** consists of determining both a subset of products $\mathcal{S} \subseteq \mathcal{I}$ to introduce in the market and the corresponding product prices leading to a maximum profit for the seller. Let us note that, contrary to Green and Krieger [31], the authors consider an endogenous subset of products $\mathcal{S}$.

In order to manage efficiently the case in which a segment would not buy any product (i.e., if all the perceived utilities for the segment are negative), an artificial product 0 is created for each segment, with both reservation and product prices set to zero. With these notations, the authors propose the following mixed integer program:

$$(\mathrm{PP}) \quad \max \sum_{k\in\mathcal{K}, i\in\mathcal{I}} \eta^k \pi_i x_i^k - \sum_{i\in\mathcal{I}} f_i y_i$$

subject to:

$$\sum_{j\in I} (r_j^k - \pi_j) x_j^k \geq (r_i^k - \pi_i) y_i \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \tag{7.6}$$

$$\sum_{i\in I} x_i^k = 1 \qquad \forall k \in \mathcal{K} \tag{7.7}$$

$$x_i^k \leq y_i \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \tag{7.8}$$

$$\sum_{i\in\mathcal{I}} y_i \leq Y \tag{7.9}$$

$$\pi_0 = 0 \tag{7.10}$$

$$x_i^k, y_i \in \{0,1\} \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I}. \tag{7.11}$$

To solve this problem, the authors propose a method called 'reverse greedy heuristic', which exploit the underlying structure of the problem. If variables $x_i^k, y_i : k \in \mathcal{K}, i \in \mathcal{I}$ are fixed (i.e., the subset of offered products and the flows are known), then the optimal solution of problem (PP) can be found in polynomial time by solving the inverse optimization problem, which consists here of a set of shortest path problems. It starts with a solution $x_i^k, y_i : k \in \mathcal{K}, i \in \mathcal{I}$ of maximum utility for each purchaser segment, and the corresponding optimal prices $\pi_i : i \in \mathcal{I}$ obtained through inverse optimization. At each iteration, a segment is reassigned to another product or removed from the market, and corresponding prices are updated. The procedure stops when no further improvement is possible. The selection criterion for choosing the segment to reassign at each iteration is the seller profit, i.e., among all the segments which prevent the seller from increasing its prices, one selects the one which would lead to the largest improvement of the objective function. The authors evaluate the heuristic performances on small randomly generated instances involving 5 purchaser segments and 4 products, and obtain profit ratios, i.e., ratios of (heuristic profit - worst profit) to (best profit - worst profit), within 10% of optimality.

In another article, Dobson and Kalish (1993, [26]) consider the Buyer Welfare and Profit Problems, and extend their previous work. First, they show that the Buyer Welfare and the Plant Location Problems are equivalent. The authors also evaluate several heuristics for this problem, including greedy (starts with an empty subset $\mathcal{S}$ of products and adds products one at a time in $\mathcal{S}$), greedy interchange (greedy, followed by pairwise product interchanges until no improvement is possible), reverse greedy (see above, [25]) and reverse greedy interchange.

These are tested on randomly generated problems involving from 20 to 800 purchaser segments and from 10 to 80 products. All heuristics perform well, with average ratio of heuristic to Lagrangian upper bound within 10% of optimality, and within 1% in most cases. Further, the greedy and the greedy interchange approaches perform better than the reverse greedy (interchange) method, as optimal solutions are found in almost all problems evaluated and worst case ratios are always within 1% of optimality. Computationally, solutions are obtained very quickly.

Next, the authors prove that the Profit Problem is $\mathcal{NP}$-hard by a reduction of the vertex cover problem. They also evaluate two heuristics for this problem, one of them being the reverse greedy heuristic presented in [25]. The second greedy heuristic seeks to include in $\mathcal{S}$, at each iteration, the product that leads to the largest improvement of the objective function objective. The order in which the products are considered is a decreasing order with respect to the purchaser utilities. Further, each time an additional product is introduced in $\mathcal{S}$, corresponding prices are computed for all products so as to maximize the objective function. The procedure stops when no further improvement is possible.

The authors evaluate the heuristics on the same instances as for the Buyer Welfare Problem and conclude that the second greedy heuristic performs better, with ratio of heuristic to Lagrangian upper bound within 8% to 22% of optimality. The greedy heuristic is the fastest on the largest instances.

Shioda et al. (2007, [61]) consider the Profit Problem in which all products are offered, that means $Y = \infty$, $\mathcal{S} = \mathcal{I}$ and $y_i = 1$ for all i in I, and without fixed costs (in the objective function) for the introduction of a product into the market. The authors present a heuristic algorithm to solve this problem, similar to the one from Dobson et Kalish [25, 26]. They also derive a linear mixed integer model for the problem, as well as some valid inequalities. Further details are provided in Section 7.3. The methods seem quite effective, even if the authors do not provide any quantitative conclusion concerning their preliminary results.

Hanson and Martin (1990, [35]) study the Profit Problem for a particular category of products. Indeed, they consider a "global element" (for example a data-processing software) composed of several components. The products are the various subsets of components of the global element, i.e., $2^n - 1$ products for a set of n components. As this problem involves a very specific structure due to the set of components, it is called the **Bundle Pricing Problem**.

The notations used are similar to those of Dobson and Kalish (1988, [25]). However, the authors do not consider fixed costs (in the objective function) corresponding to the introduction of a product in the market. Further, they assume that the product prices are subject to 'sub-additivity constraints'. It means that, if product $i \in \mathcal{I}$ is the union of several other products, then the price of i should be lower than the sum of the prices of these other products:

$$\pi_i \leq \sum_{j \in \mathcal{S}} \pi_j \qquad \forall i \in \mathcal{I}, \forall \mathcal{S} \subseteq \mathcal{I} : i = \cup_{j \in \mathcal{S}} j \tag{7.12}$$

Note that these constraints are very similar to Triangle inequalities. The authors present a particular mixed integer formulation for the problem as well as numerical results. The problems tested, involving 5 to 10 purchaser segments and 4 components (thus 15 products), have integrality gaps ranging from 2 to 4%, and are solved to optimality very quickly. However, due to the exponential number of component subsets, and thus of products, they note the restricted size of the instances that can be solved by an exact method.

For a larger number of components, the authors propose a formulation based on a limited number of subsets. Indeed, they consider that, when the number of components of the global element is large, there often exists a most important component, called "key component", which appears in all subsets offered in the market, and to which less important components could be incorporated. The authors present a more complex mixed integer formulation for this particular structure.

Using an exact resolution method, instances involving up to 4 purchaser segments and 20 components in addition to the key component can be solved to optimality very quickly and in one iteration of the proposed bundle algorithm, which is probably due to the large number of possible bundles compared to the small number of purchaser segments.

Guruswani et al. (2005, [33]) study a Bundle Pricing Problem in which all products are offered in the market, but without sub-additivity constraints. The authors prove that this problem is $\mathcal{APX}$-hard by a reduction from the vertex cover problem. They also propose a logarithmic approximation algorithm for this problem. Further, the authors study several specific cases of the problem, providing logarithmic approximation or polynomial-time algorithms together with some further algorithmic considerations. Unfortunately, they do not provide any numerical results.

Nichols and Venkataramanan (2005, [55]) propose a **Conjoint Buyer Welfare and Profit Problem**, with a formulation similar to the one suggested by Dobson and Kalish [25, 26]. The difference lies in the fact that the authors consider a weighted objective function including both seller profit and purchaser utility.

Three heuristic methods are proposed, the first one being a pure genetic algorithm included for comparison. The other proposed heuristics are genetic relaxations. One of them uses a genetic procedure to generate product prices; then, a branch and bound algorithm is applied to the remaining problem, which consists of setting flows so as to maximize the sum of the purchasers utilities. The last heuristic starts with a random generation of the products which should be introduced in the market; then, one determines the flows $x_i^k : k \in \mathcal{K}, i \in \mathcal{I}$ in order to maximize purchasers utilities. Finally, the remaining inverse optimization problem, that consists in setting product prices in order to maximize the seller's income, is solved by a shortest path algorithm.

The three heuristics are compared on problems involving 20 to 1000 purchaser segments and 10 to 100 products. The results show that the relaxation methods perform better, on large instances, than a pure genetic algorithm. Consequently, they encourage the development of genetic resolution approaches in which large subsets of the original exact problem would be preserved. The authors do not give any further details concerning the performance ratios or the cpu times of their algorithms.

We conclude this section with three graphs summarizing the main contributions to the Buyer Welfare, Seller Welfare and Share-of-Choices Problems in literature. These are presented in Figures 7.1 to 7.3.

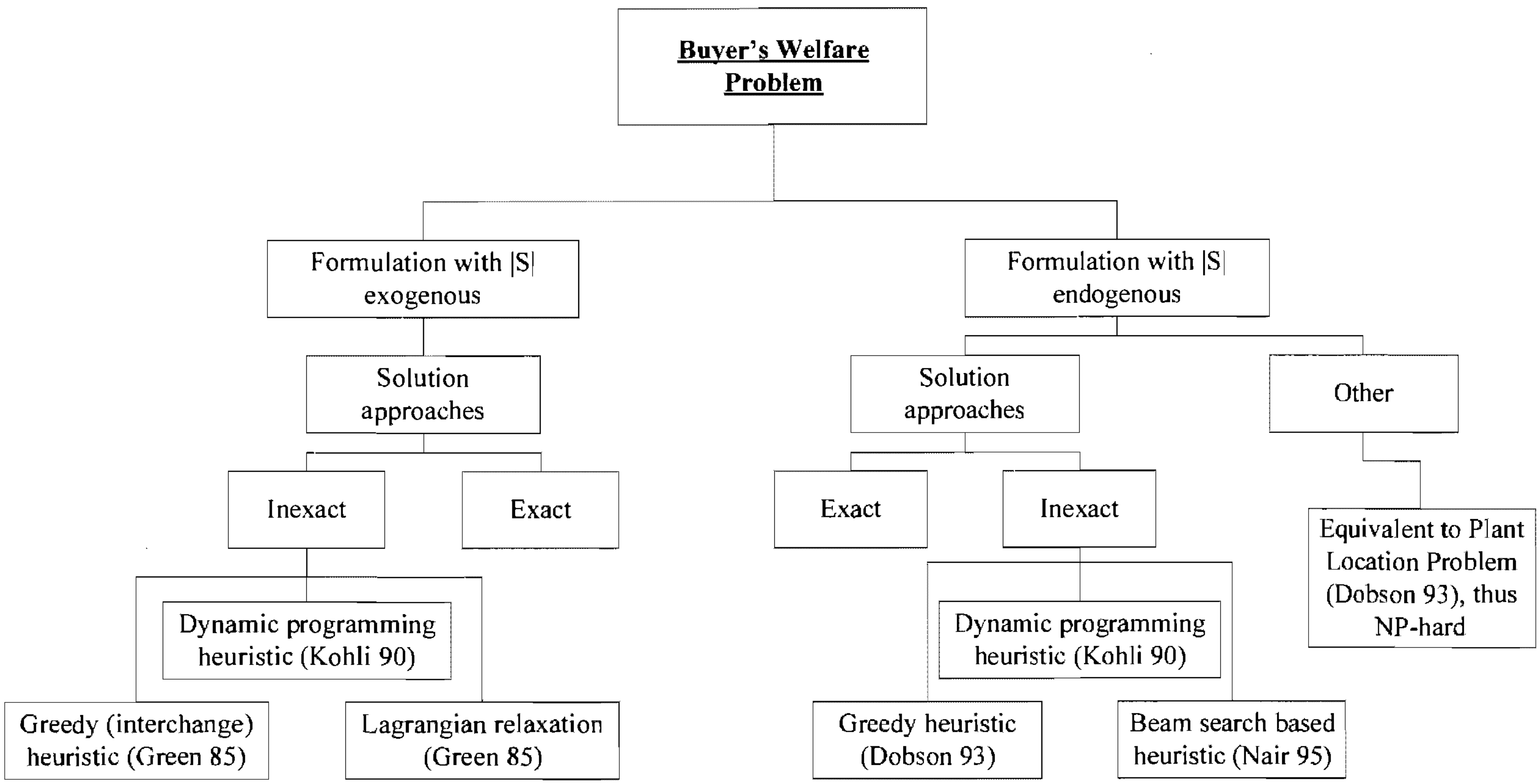

Figure 7.1: Main contributions to the Buyer Welfare Problem

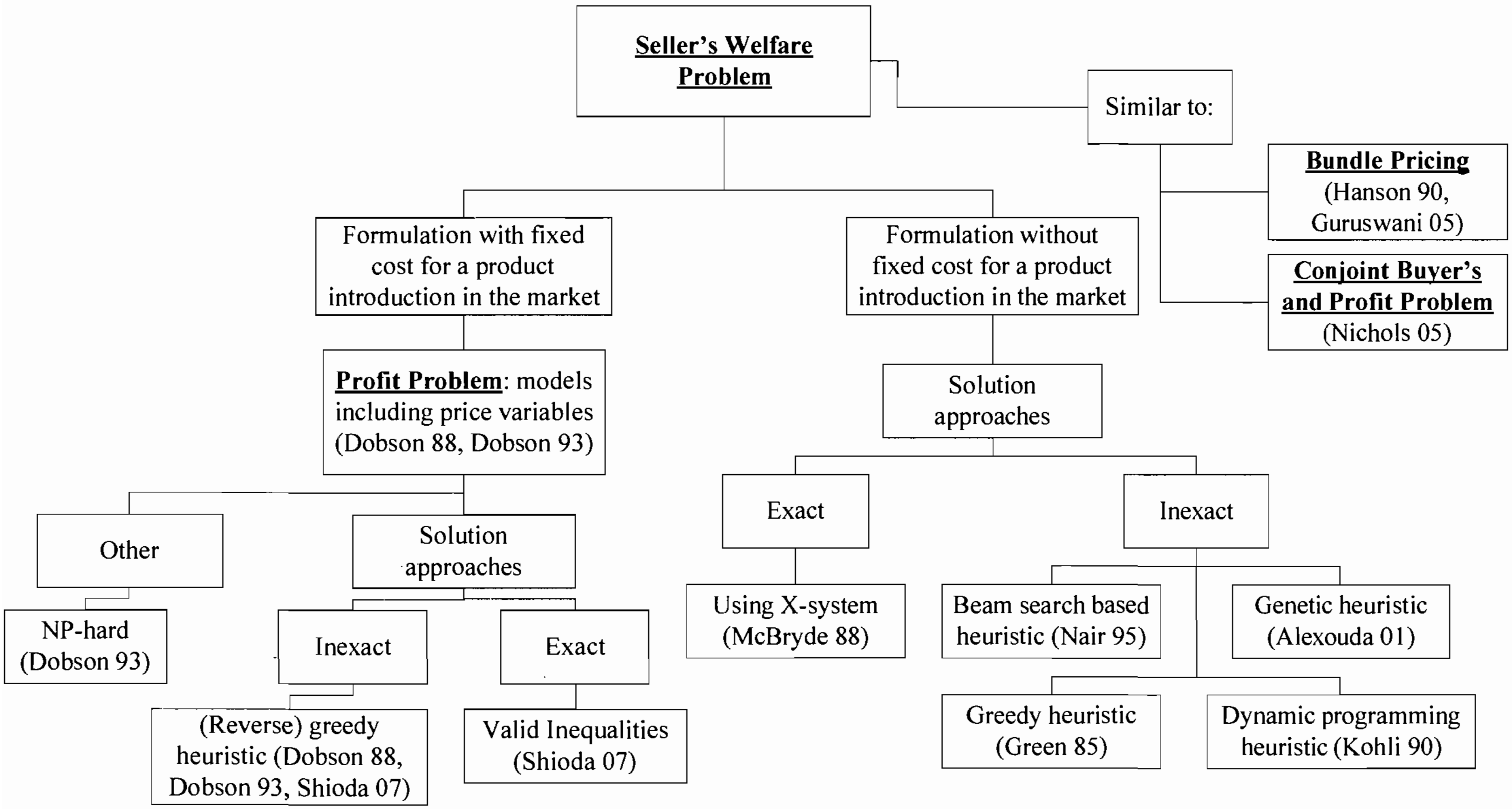

Figure 7.2: Main contributions to the Seller Welfare Problem

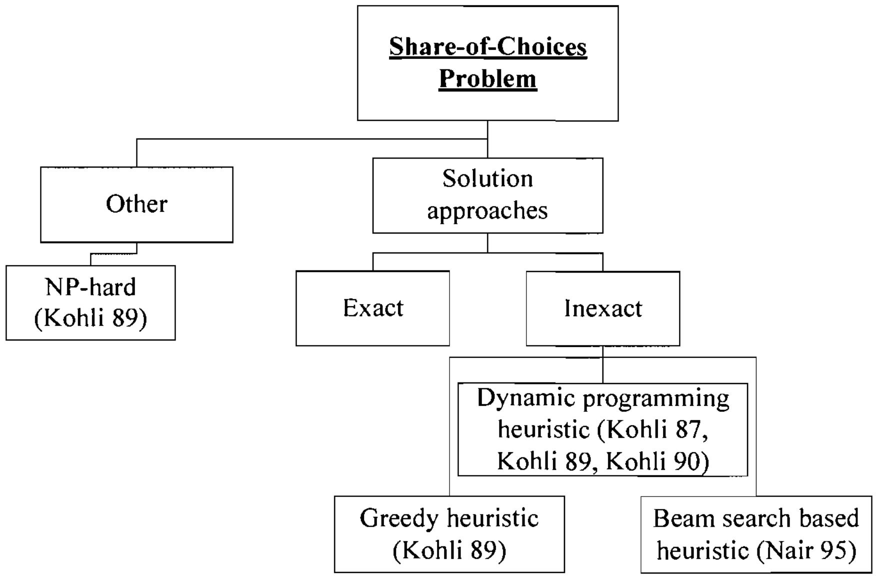

Figure 7.3: Main contributions to the Share-of-Choices Problem

In the next section, we point out the relationships between the family of design and pricing problems and the Network Pricing Problem.

7.2 Relationships between both families of problems

This section aims to point out the links between the standard design and pricing problems presented in Section 1 and the Network Pricing Problem.

7.2.1 Seller Welfare, Profit Problems and GCT-NPP

When looking at their definitions, the Buyer Welfare, Seller Welfare and Share-of-Choices problems seek to design a set of products to introduce in a given economic market, without any specific reference to pricing. However, we point out several similarities between the Seller Welfare Problem and the General Complete

Toll NPP. First, one can observe that both problems seek to maximize a seller (resp. leader) profit. Second, while a purchaser buys the product which has the largest utility for him in the Seller Welfare Problem, a network user travels on the cheapest path from his origin to his destination in the General Complete Toll NPP.

The Profit Problem, which also includes price variables that have to be determined in order to maximize the seller profit, is akin to the General Complete Toll NPP. Consider a modified problem in which all products are offered in the market, that mean $Y = \infty$, $\mathcal{S} = \mathcal{I}$ and $y_i = 1$ for all i in $\mathcal{I}$, and called **Modified Profit Problem**. The Modified Profit Problem and the General Complete Toll NPP are equivalent. Indeed, let us match purchaser segments with commodities and products with toll arcs. The product prices $\pi_i : i \in \mathcal{I}$ correspond to the tolls $t_a : a \in \mathcal{A}$, while the reservation price r_i^k of purchaser k for obtaining product i becomes $(c_{od}^k - c_a^k)$, i.e., the space let for tolls on a commodity k travelling on the toll arc a (instead of the toll free arc). These correspondance are summarized in Table 7.1.

Modified Profit Problem	**General Complete Toll NPP**
Purchaser segments $k \in \mathcal{K}$	Commodities $k \in \mathcal{K}$
Products $i \in \mathcal{I}$	Toll arcs $a \in \mathcal{A}$
Reservation prices $r_i^k : k \in \mathcal{K}, i \in \mathcal{I}$	Gains $c_{od}^k - c_a^k : k \in \mathcal{K}, a \in \mathcal{A}$
Prices $\pi_i : i \in \mathcal{I}$	Tolls $t_a : a \in \mathcal{A}$
Flows $x_i^k : k \in \mathcal{K}, i \in \mathcal{I}$	Flows $x_a^k : k \in \mathcal{K}, a \in \mathcal{A}$

Table 7.1: Links between notations for the Modified Profit Problem and the General Complete Toll NPP

Hence, while a purchaser segment buys the product that maximizes its utility $r_i^k - \pi_i$, a commodity travels on the toll arc that maximizes the difference $c_{od}^k - c_a^k - t_a$, i.e., that minimize its travel cost $c_a^k + t_a$. However, there is a small difference between the parameter structures $r_i^k : k \in \mathcal{K}, i \in \mathcal{I}$ and $c_{od}^k - c_a^k : k \in \mathcal{K}, a \in \mathcal{A}$. Indeed, in the General Complete Toll NPP, the cost c_{od}^k on the toll free arc for a commodity $k \in \mathcal{K}$ is given and one looks at the difference between this cost and the fixed costs $c_a^k : a \in \mathcal{A}$ of toll arcs. In the Modified Profit Problem, the reservation prices $r_i^k : i \in \mathcal{I}$ of a given segment $k \in \mathcal{K}$ for obtaining various products $i \in \mathcal{I}$ are not linked together, so that the main part of the network structure is absent.

Similar to the Constrained Highway Pricing Problem, the incorporation of Triangle and Monotonicity constraints to the Modified Profit Problem would make sense. Indeed, products might be available in various formats, with different prices associated with different formats. If quantity X satisfies the relationship $X = Y + Z$, the triangle inequality $\pi_X \leq \pi_Y + \pi_Z$ prevents obvious market inconsistencies. Furthermore, if $X \leq Y$, one expects that $\pi_X \leq \pi_Y$, i.e., the Monotonicity inequality holds.

The above reasoning can be generalized in the following way. Assume that the purchase of a product X is equivalent to the purchase of the sum of products Y and Z. In this setting, one would expect the inequality $\pi_X \leq \pi_Y + \pi_Z$ to hold. In the same way, suppose that the purchase of a product Y is equivalent to the purchase of a product X plus something else. Then the inequality: $\pi_X \leq \pi_Y$ should be valid for the same reasons.

7.2.2 Bundle Pricing and Network Pricing Problem

The Bundle Pricing Problem intends to design and to price a set of products, each product representing a subset of components of a "global element". In the Network Pricing Problem, each path is composed of several components which are the toll arcs of the network. Hence there exists a link between the two problems, but such relationship is less obvious than with the General Complete Toll NPP.

7.3 Comparison between a Modified Profit Problem and the GCT-NPP

To conclude this chapter, we compare the results obtained for the General Complete Toll NPP to these obtained by Shioda et al. (2007, [61]) for the Modified Profit Problem.

We first present the linear mixed integer model of Shioda et al., together with their best valid inequalities. Next, the model and valid inequalities are tested on the randomly generated instances described in Section 6.1. The results are compared to those obtained for the GCT-NPP. Finally, both models and valid inequalities (for the Modified Profit Problem and the GCT-NPP respectively) are tested on randomly generated instances proposed by Shioda et al. in [61].

Shioda et al. (2007, [61]) consider a linear mixed integer model for the Modified Profit Problem, i.e.,

$$\text{(LMPP)} \quad \max \sum_{k\in\mathcal{K}, i\in\mathcal{I}} \eta^k p_i^k$$

subject to:

$$\sum_{i\in I} x_i^k \leq 1 \qquad \forall k \in \mathcal{K} \qquad (7.13)$$

$$\sum_{j\in\mathcal{I}:j\neq i} (r_j^k x_j^k - p_j^k) \geq r_i^k \sum_{j\in\mathcal{I}:j\neq i} x_j^k - \pi_i \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \qquad (7.14)$$

$$p_i^k \leq r_i^k x_i^k \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \qquad (7.15)$$

$$\pi_i - p_i^k \leq N_i(1 - x_i^k) \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \qquad (7.16)$$

$$p_i^k \leq \pi_i \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \qquad (7.17)$$

$$p_i^k \geq 0 \qquad \forall i \in \mathcal{I} \qquad (7.18)$$

$$x_i^k \in \{0,1\} \qquad \forall k \in \mathcal{K}, i \in \mathcal{I} \qquad (7.19)$$

where $N_i = \max_k\{r_i^k\}$, and p_i^k represents the actual price of product i for purchaser segment k.

Excluding constraints (7.14), this model (LMPP) is identical to model (HP3*) for the General Complete Toll NPP. Now let us compare constraints (7.14) of (LMPP) to the Shortest Path constraints (3.22) of (HP3*). Adding a term $r_i^k x_i^k - p_i^k$ to both sides of inequalities (7.14) yields:

$$\sum_{j \in \mathcal{I}} (r_j^k x_j^k - p_j^k) \geq r_i^k \sum_{j \in \mathcal{I}} x_j^k - p_i^k - \pi_i \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{I}. \tag{7.20}$$

Then, constraints (7.13) and (7.18) imply that:

$$r_i^k \sum_{j \in \mathcal{I}} x_j^k - p_i^k \leq r_i^k,$$

which means that the Shortest Path constraints (3.22) of (HP3*) are stronger than constraints (7.14) of (LMPP). Indeed, note that if a purchaser segment $k \in \mathcal{K}$ does not buy any product, constraints (7.14) are redundant for this segment k, which constraints (3.22) impose $c_{od}^k \leq c_a^k + t_a$ for all toll arcs $a \in \mathcal{A}$.

Shioda et al. [61] also propose the following valid inequalities for model (LMPP):

$$p_i^{k_1} \geq \min_{k \in \mathcal{K}} \{r_i^k\} x_i^{k_1} \qquad \forall k_1 \in \mathcal{K}, \forall i \in \mathcal{I} \tag{7.21}$$

$$p_i^{k_1} \leq r_i^{k_2} x_i^{k_2} + r_i^{k_1}(1 - x_i^{k_2}) \qquad \forall k_1, k_2 \in \mathcal{K}, \forall i \in \mathcal{I} \tag{7.22}$$

$$x_i^{k_2} \geq x_i^{k_1} \qquad \forall k_1, k_2 \in \mathcal{K}, \forall i \in \mathcal{I}:$$

$$r_i^{k_2} \geq r_i^{k_1} \ \forall i \in \mathcal{I},\ r_i^{k_1} - r_i^{k_2} > r_j^{k_1} - r_j^{k_2} \ \forall j \in \mathcal{I} \setminus \{i\}. \tag{7.23}$$

Inequalities (7.21) and (7.22) provide lower and upper bounds for the actual product price variables $p_i^k : k \in \mathcal{K}, i \in \mathcal{I}$, which depend on the reservation prices $r_i^k : k \in \mathcal{K}, i \in \mathcal{I}$. Inequalities (7.23) link the flow variables $x_i^k : k \in \mathcal{K}, i \in \mathcal{I}$ for different purchaser segments. We refer the reader to Shioda et al. (2007, [61]) for further details.

One can easily check that these inequalities are still valid for the General Complete Toll NPP. In order to compare the efficiency of model (HP3*) with the valid inequalities proposed in Chapter 4 to model (LMPP) and inequalities (7.21), (7.22) and (7.23), we test the latter on the randomly generated instances described in Section 6.1. Inequalities (7.21) and (7.22) are appended to the initial model (LMPP). Inequalities (7.23) are generated at the root of the branch and cut algorithm and added to the model when violated. The results obtained are presented in Table 7.2.

Inst.	Gap(%)				Time (sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	8.65	17.70	14.66	3.2	0	24	7	9	19	5473	1273	1989
5 v - 15 n	7.55	28.29	16.32	6.6	1	261	58	92	93	32563	6853	11580
^{*4*}8 v - 10 n	6.97	32.70	22.81	8.7	6	18001	12006	8479	491	1910347	959689	718395
^{*5*}8 v - 15 n	16.79	37.84	24.94	8	8793	18003	16467	3432	100000	486314	362496	90458
^{*6*}10 v - 10 n	19.38	31.64	26.15	4	18000	18004	18003	1	100000	634355	427870	134430
^{*6*}10 v - 15 n	19.78	33.35	27.26	4.7	18000	18010	18007	3	62720	221028	144249	57155

Table 7.2: Model (LMPP) with (7.21), (7.22) and (7.23) inequalities

Comparing Tables 7.2 and 6.2 (which reports the results for the initial model (HP3*)), we observe that formulation (LMPP) provides much worse results than formulation (HP3*). Hence constraints (7.14) of (LMPP) are weaker, both theoretically and numerically, than constraints (3.22) of (HP3*).

In order to point out the efficiency of the valid inequalities (7.21), (7.22) and (7.23), we test these under model (HP3*). The results are presented in Table 7.3.

Now comparing Tables 7.3 and 6.2, we conclude that the valid inequalities proposed by Shioda et al. allow a decrease of the gaps, cpu times and number of nodes in the branch and cut algorithm. The gaps decrease of 10 to 29%, while the cpu

Inst.	Gap(%)				Time (sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
5 v - 10 n	0.28	11.68	5.83	3.9	0	5	2	2	1	89	34	38
5 v - 15 n	2.86	15.54	8.38	4.2	0	7	4	2	5	187	66	64
8 v - 10 n	2.46	23.25	12.97	7.4	1	860	282	328	21	33811	10351	12787
*1*8 v - 15 n	7.34	27.22	13.49	6.8	26	18006	4157	6601	297	400197	81832	145428
10 v - 10 n	7.59	17.06.	11.73	3.9	24	735	201	250	119	19690	4791	7033
*4*10 v - 15 n	7.66	19.34	12.81	4.4	868	18029	12304	8083	8589	271246	145109	103612

Table 7.3: Model (HP3*) with (7.21), (7.22) and (7.23) inequalities

times and number of nodes decrease by 78% and 92% respectively for the largest instances solved to optimality, i.e., $10v - 10n$. However, the results obtained in the final tests for the GCT-NPP (i.e., gaps decrease of 78 to 89%, while cpu times and number of nodes decrease of 90% and 99% respectively for instances $10v - 10n$) clearly outperform Shioda et al.'s results.

In their numerical experiments, Shioda et al. (2007, [61]) address randomly generated instances in the following way. Consider from 40 to 80 purchaser segments and from 10 to 60 products. For each purchaser segment (resp. commodity) $k \in \mathcal{K}$, a demand η^k is randomly generated between 500 and 799. For each product (resp. toll arc) $a \in \mathcal{A}$, reservation prices of this product for purchaser segments $r_a^k : k \in \mathcal{K}$ (resp. $c_{od}^k - c_a^k$) are randomly generated between 512 and 1023. Hence, as the model (LMPP) and the valid inequalities (7.21), (7.22) and (7.23) were developed in a context of product pricing, they could be more effective on corresponding instances.

Let us compare the best results obtained for models (HP3*) and (LMPP) respectively on 3 instances of each size. The results are presented in Tables 7.4, 7.5 and 7.6. Note that the letters 'k' and 'a' denote the number of commodities (purchaser segments) and the number of toll arcs (products).

Inst.	Gap(%)				Time (sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
^{*2*}40 k - 10 a	3.54	9.1	7.03	2.5	862	18003	12290	8081	19523	616169	375901	257045
^{*3*}40 k - 20 a	2.34	4.65	3.24	1	18000	18009	18008	1	100000	445166	395358	35547
^{*1*}40 k - 40 a	0.51	0.89	0.65	0.2	363	18012	6805	7954	1673	120007	44444	53586
40 k - 60 a	0.22	0.35	0.29	0.1	180	307	230	55	385	981	612	263
^{*3*}60 k - 10 a	5.58	11.31	9.27	2.6	18000	18014	18013	1	100000	247984	237077	9350
^{*3*}60 k - 20 a	4.84	7.56	6.11	1.1	18000	18060	18059	1	100000	168415	146458	16123
^{*3*}60 k - 40 a	0.95	2.48	1.49	0.7	18000	18036	18033	3	54571	83794	64837	13420
^{*3*}60 k - 60 a	0.71	0.87	0.8	0.1	18000	18065	18059	5	40866	46390	42898	2480
^{*3*}80 k - 10 a	9.24	17.04	13.28	3.2	18000	18032	18031	1	100000	132840	124757	6194
^{*3*}80 k - 20 a	11.25	13.62	12.45	1	18000	18179	18171	11	73376	83427	80074	4736
^{*3*}80 k - 40 a	2.99	7.08	4.42	1.9	18000	18695	18577	85	41857	44767	42979	1278
^{*3*}80 k - 60 a	0.9	1.18	1.02	0.1	18000	18121	18118	3	27638	30367	28582	1263

Table 7.4: Model (LMPP) with (7.21), (7.22) and (7.23) inequalities, tested on Shioda et al. instances

Inst.	Gap(%)				Time (sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
40 k - 10 a	1.68	5.77	4.23	1.8	18	61	44	18	228	2267	1380	853
40 k - 20 a	1.07	2.86	1.72	0.8	57	437	187	177	466	5778	2279	2474
40 k - 40 a	0.08	0.36	0.20	0.1	67	109	86	17	40	309	155	113
40 k - 60 a	0.04	0.07	0.06	0	239	371	296	55	12	28	22	7
60 k - 10 a	3.33	6.77	5.44	1.5	243	1155	772	387	3266	24159	14916	8698
^{*2*}60 k - 20 a	1.58	3.55	2.55	0.8	1307	18115	12510	7922	5070	117948	75761	50297
60 k - 40 a	0.36	1.08	0.62	0.3	514	8065	3350	3357	1278	32587	12982	13949
60 k - 60 a	0.22	0.39	0.31	0.1	1579	6574	4319	2068	833	7365	4659	2782
80 k - 10 a	3.57	6.36	5.38	1.3	5628	17422	9955	5302	51638	221040	117504	74110
^{*3*}80 k - 20 a	2.13	3.75	3.15	0.7	18000	18361	18332	21	51477	56982	54901	2440
^{*2*}80 k - 40 a	0.45	1.59	1.14	0.5	17107	19144	18433	938	27058	44366	35782	7067
^{*1*}80 k - 60 a	0.21	0.28	0.25	0	9295	18326	13778	3687	4893	12067	8869	2980

Table 7.5: Model (HP3*) with (7.21), (7.22) and (7.23) inequalities, tested on Shioda et al. instances

As before, comparisons between Tables 7.4 and 7.5 show that formulation (LMPP) is much weaker than formulation (HP3*). Indeed, the largest instances solved to optimality with model (LMPP) involve 40 commodities and 60 toll arcs, while model (HP3*) is able to solve instances up to 80 commodities and 10 toll arcs. Further, the gaps in Table 7.5, i.e., for model (HP3*), are from 40 to 79% lower than the ones for model (LMPP).

Inst.	Gap(%)				Time (sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
40 k - 10 a	0.34	1.79	1.28	0.7	14	233	130	90	42	2809	1791	1242
40 k - 20 a	0.37	1.07	0.66	0.3	88	220	136	60	387	3801	1534	1603
40 k - 40 a	0.05	0.22	0.11	0.1	100	173	129	32	19	356	137	155
40 k - 60 a	0.02	0.04	0.03	0	169	228	189	28	7	14	9	3
60 k - 10 a	0.97	2.45	1.78	0.6	222	4821	2283	1908	925	25354	13475	9984
*1*60 k - 20 a	0.46	1.35	0.88	0.4	1422	18057	9553	6796	4145	60553	31684	23048
*1*60 k - 40 a	0.11	0.53	0.28	0.2	346	18333	6568	8325	49	34762	13814	15054
60 k - 60 a	0.09	0.17	0.14	0	820	5515	2585	2087	189	17405	6346	7836
80 k - 10 a	1.01	1.76	1.5	0.3	1105	16213	6520	6870	3706	75237	29498	32430
*3*80 k - 20 a	0.72	1.51	1.18	0.3	18000	18152	18145	6	13541	22660	19029	3948
*2*80 k - 40 a	0.17	0.64	0.41	0.2	1461	18752	12925	8106	1928	20585	12180	7728
80 k - 60 a	0.08	0.11	0.09	0	2175	3057	2614	360	323	1153	732	339

Table 7.6: Model (HP3*) with (4.1)-(4.2), (4.7)-(4.8) and (4.9)-(4.10) inequalities, tested on Shioda et al. instances

Further, when replacing constraints (7.14) of (LMPP) by the Shortest Path constraints (3.22), we obtain a comparison of model (HP3*) with either (7.21), (7.22), (7.23) or (4.1)-(4.2), (4.7)-(4.8), (4.9)-(4.10) inequalities. The results are presented in Tables 7.5 and 7.6. The latter show that the valid inequalities presented in Chapter 4 outperform the valid inequalities proposed by Shioda et al. [61] in terms of the gaps (which decrease of 45 to 72%), and, for specific instances, in terms of computing times or number of nodes (up to 80% and 60% respectively for the largest instances solved to optimality, i.e., $80k - 10a$).

Finally, we test formulation (HP3*) with (7.21), (7.22), (7.23) and (4.1)-(4.2), (4.7)-(4.8), (4.9)-(4.10) inequalities. The results are presented in Table 7.7.

Comparing Tables 7.6 and 7.7, we conclude that adding (7.21), (7.22) and (7.23) inequalities to our best formulation (i.e., model (HP3*) with (4.1)-(4.2), (4.7)-(4.8), (4.9)-(4.10) inequalities) helps to decrease the gaps and the number of nodes in the branch and cut algorithm. The gaps decrease from 18 to 45%, while the number of nodes decrease by 87% for instances $80k - 10a$. However, cpu times increase,

Inst.	Gap(%)				Time (sec)				Nodes			
	MIN	MAX	μ	σ	MIN	MAX	μ	σ	MIN	MAX	μ	σ
40 k - 10 a	0.08	1.39	0.79	0.5	31	170	111	58	15	710	341	285
40 k - 20 a	0.21	0.79	0.45	0.2	180	448	338	114	31	801	386	317
40 k - 40 a	0.02	0.1	0.06	0	112	356	216	103	5	139	71	55
40 k - 60 a	0.01	0.02	0.02	0	563	866	677	135	7	16	12	4
60 k - 10 a	0.6	1.36	1.1	0.4	304	2081	1054	752	225	4207	2164	1627
*1*60 k - 20 a	0.29	1.26	0.74	0.4	1356	18371	8547	7192	472	7455	3411	2956
*1*60 k - 40 a	0.07	0.43	0.21	0.2	954	18516	9858	7172	136	5518	3450	2367
*1*60 k - 60 a	0.06	0.15	0.11	0	4529	20730	11759	6728	179	7752	3619	3130
80 k - 10 a	0.61	1.27	0.97	0.3	1344	8059	4346	2787	949	6643	3302	2427
*3*80 k - 20 a	0.55	1.32	0.95	0.3	18000	18537	18507	25	3966	6305	4960	987
*3*80 k - 40 a	0.13	0.6	0.35	0.2	11890	20328	17487	3957	2096	2925	2517	339
*1*80 k - 60 a	0.04	0.06	0.06	0	6467	15359	11633	3770	483	726	588	102

Table 7.7: Model (HP3*) with (4.1)-(4.2), (4.7)-(4.8), (4.9)-(4.10), (7.21), (7.22) and (7.23) inequalities, tested on Shioda et al. instances

probably due to the time required by the separation procedure. The latter should be improved in future research.

7.4 Conclusion

In this chapter, we highlighted the links between standard design and pricing problems in economics and the Network Pricing Problem. While both families of problems have been extensively studied in literature, it seems that no relationships have been noticed so far. However, it is clear that these families of problems are very similar and should be interconnected in the future. Especially, the real efficiency of valid inequalities has been shown numerically for a pricing problem. Hence, it would be interesting to go further in the polyhedral study within the framework of a design and pricing problem.

CHAPTER 8

CONCLUSION

In this thesis, we have studied a Network Pricing Problem whose features represent these of a real highway topology. As toll levels are usually computed using the highway entry and exit nodes, a complete toll subgraph is considered, where each toll arc corresponds to a toll subpath. Two variants of this problem are studied, with or without Triangle and Monotonicity constraints linking together the tolls on the arcs.

First described by a bilevel formulation, the problem can be modelled as a linear mixed integer program with a single level. It is proved to be $\mathcal{NP}$-hard by a reduction from $3-SAT$. Next, we have proposed several families of valid inequalities for this problem. The latter involve pairs of commodities at a time, and strenghten important constraints of the initial model.

Then, focusing on instances involving one or two commodities, we have proved that most of the valid inequalities, as well as several constraints of the initial model, define facets of the convex hull of feasible solutions for these restricted problems. In the single commodity case, a complete description of the convex hull of feasible solutions for one variant of the problem is also provided.

Next, the efficiency of the proposed valid inequalities is highlighted by numerical results. With regards to the first variant of the problem, which includes Triangle and Monotonicity constraints, several of the valid inequalities are efficient, and yield to a significant decrease of the gap and number of nodes in the branch and cut algorithm. Unfortunately, the cpu times increase, probably because the Triangle and Monotonicity constraints interfere negatively with the valid inequalities, which obstructs the program to reach quickly optimality.

We have also performed numerical tests for the second variant of the problem, which does not include Triangle and Monotonicity constraints. Most of the valid inequalities lead to a significant decrease of the gap and number of nodes. Further, they also allow a decrease of the cpu time.

Finally, we have pointed out the links between the Network Pricing Problem studied in the thesis and a more standard design and pricing family of problems in economics. A description of these problems, together with an overview of results, is first provided. Then we have highlighted the strong relationships between both families of problems. We have also compared the model and the valid inequalities proposed in the thesis to the ones from a very similar work in economics. It shows that our results clearly outperform these obtained for an identical pricing problem.

To conclude, the polyhedral structure of the specific problems studied in the thesis is probably simpler than the one for the classical Network Pricing Problem. In view of the efficiency of the proposed valid inequalities, it would be interesting to study the validity of these inequalities in the context of a classical Network Pricing Problem. With a path formulation, the inequalities would obviously stay valid for the Network Pricing Problem. However, in consequence of the exponential number of paths in such a formulation, one should find an efficient separation algorithm to implement the inequalities. A transformation of the inequalities to an arc formulation, as well as a detailed polyhedral study of this formulation, also provide good ideas for future research.

Next, in view of the strong relationships between the problems studied in the thesis and several pricing problems in economics, it would be interesting to go further in the polyhedral study within the framework of a design and pricing problem. Especially, the real efficiency of the valid inequalities has been shown numerically for a pricing problem. Hence a deeper analysis should be performed, and could, for instance, exploit the compatibility notion described in Chapter 4.

In a larger context, this study provides some major conclusions. First, even if heuristics have been extensively developed during these last years, our results show that exact methods still hold promises. Adding valid inequalities (even perhaps almost valid inequalities) to complex network pricing models could be useful, and one should think about their integration in a large real system. Also, the links with another classical pricing problem in economics, mostly solved by inexact methods, give possible ideas for future research.

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Appendix I

Proofs of complexity for the Basic NPP

Here we consider the Basic NPP, which deals with a network where all toll arcs are connected, i.e. neither complete toll subgraph nor Triangle or Mono inequalities.

Proposition 28 *The single directional Basic NPP is strongly $\mathcal{NP}$-hard.*

Proof

Any conjunctive normal form $F = \wedge_{i=1}^{m} (l_{i1} \vee l_{i2} \vee l_{i3})$, where l_{ij} for $j = 1, 2, 3$ represents a variable $x_i : i \in \{1, ..., n\}$ or its negation, can be polynomially converted to an instance of the Basic NPP.

For each variable $x_i : i \in \{1, ..., n\}$, a subnetwork is constructed as shown in Figure I.1.

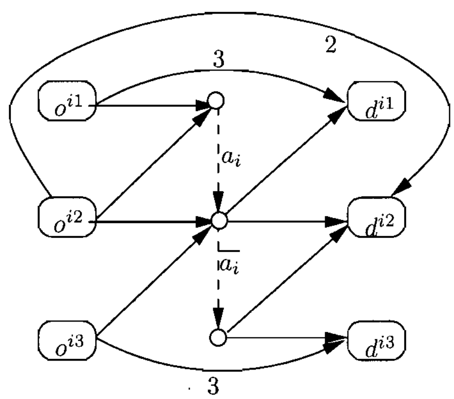

Figure I.1: Subnetwork for variable x_i.

Each of these subnetworks contains three commodities $(o^{ij}, d^{ij}) : j \in \{1, 2, 3\}$ with unit demand, and two toll arcs a_i and $\overline{a}_i$ of zero fixed cost, corresponding to the truth and false assignment for variable x_i respectively. Toll free arcs of cost zero connect o^{i1} (resp. o^{i3}) to the tail node of arc a_i (resp. $\overline{a}_i$), the head node of arc

a_i (resp. $\overline{a}_i$) to d^{i1} (resp. d^{i3}), o^{i2} to both tail nodes of toll arcs, and both head nodes of toll arcs to d^{i2}. Toll free arcs (o^{i1}, d^{i1}) of cost 3, (o^{i3}, d^{i3}) of cost 3 and (o^{i2}, d^{i2}) of cost 2 are also added. Thus an upper bound on the revenue for each subnetwork is 7, obtained by setting one toll to 2 and the other one to 3. In all other cases, the revenue cannot exceed 6. Note that the toll free arcs that do not appear from some origins to tail nodes of toll arcs (resp. from head nodes of toll arcs to some destinations) are supposed to be so expensive that they can never be used. Then the subnetworks are linked together so that the single directional highway corresponds to the set of all subnetworks.

Further, for each clause k, a clause-commodity (o^k, d^k) with unit demand is constructed. If variable x_i (resp. $\overline{x}_i$) is a literal of clause k, toll free arcs of cost 0 are added from o^k to the tail node of a_i (resp. $\overline{a}_i$) and from the head node of a_i (resp. $\overline{a}_i$) to d^k. An additional toll free arc (o^k, d^k) of cost 2 is added, which defines an upper bound of 2 on the revenue from each clause-commodity. This construction is depicted in Figure I.2.

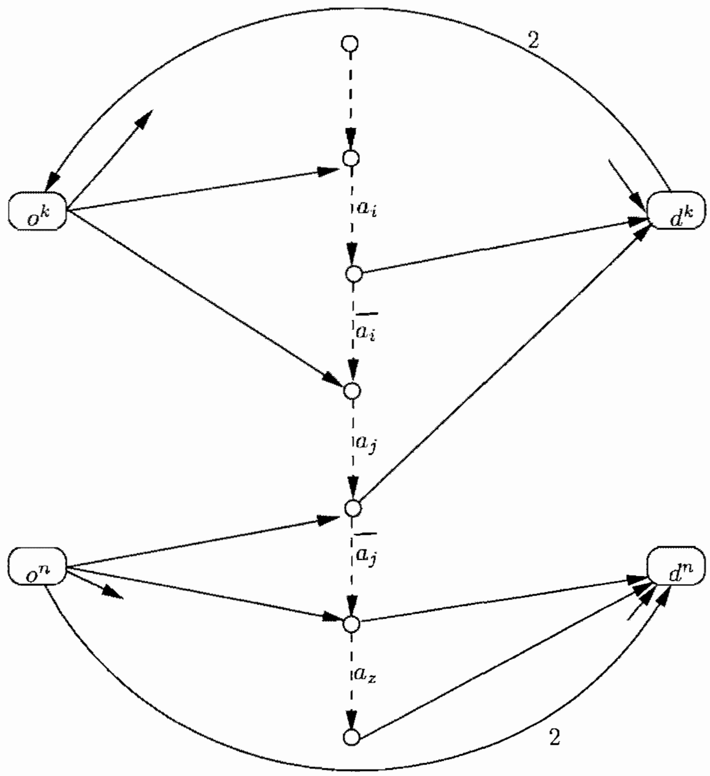

Figure I.2: Subnetwork for $F = (... \vee x_i \vee x_j) \wedge (\overline{x_j} \vee x_z \vee ...) \wedge ...$ (single-directional Basic NPP).

Let show that a satisfying truth assignment for F exists if and only if the revenue for the Basic NPP is equal to $2m + 7n$.

Suppose there exists a satisfying truth assignment, which means that at least one literal is true in each clause. Set the toll on the corresponding arc to 2, and the toll on the complementary arc (with respect to the corresponding subnetwork) to 3. Thus the total revenue for all clause-commodities is $2m$. For all remaining subneworks, if any (i.e. this situation only happens if a variable x_i does not appear in any clause), the toll arcs are set arbitrarily to 2 and 3 for a variable and its negation, respectively. Thus the revenue for all subnetworks is $7n$, which means that the total revenue is $2m + 7n$.

Conversely, suppose there exists tolls such that the total revenue is $2m + 7n$. The maximal possible revenue for all subneworks is $7n$, only achievable by setting one toll per subnetwork to 2 and the other one to 3. On the other hand, the maximal possible revenue for all clause-commodities is $2m$. Set to true all literals corresponding to toll arcs of cost 2, and false to the other. This corresponds to a well-defined assignment for F, since there is exactly one toll of 2 in each subnetwork. Moreover, each clause-commodity contributes to the total revenue with a toll of 2, which means that at least one literal per clause is true, and there exists a truth assignment for F. □

Proposition 29 *The bi-directional Basic NPP is strongly $\mathcal{NP}$-hard.*

Proof

Here subnetworks for variables $x_i : i \in \{1, ..., n\}$ are constructed in a slightly different way, as shown in Figure I.3.

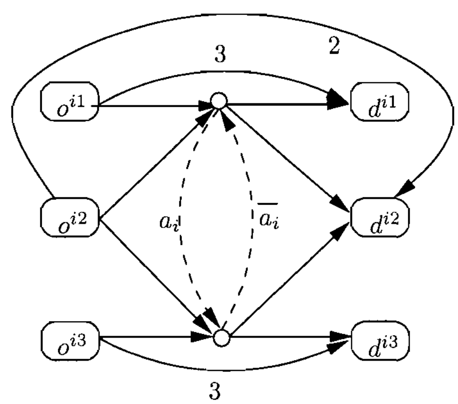

Figure I.3: Subnetwork for variable x_i.

Toll arcs $a_i : i \in \{1, ..., n\}$ are in one direction of the highway, while toll arcs $\overline{a}_i : i \in \{1, ..., n\}$ are in the other direction. Such a network is depicted in Figure I.4.

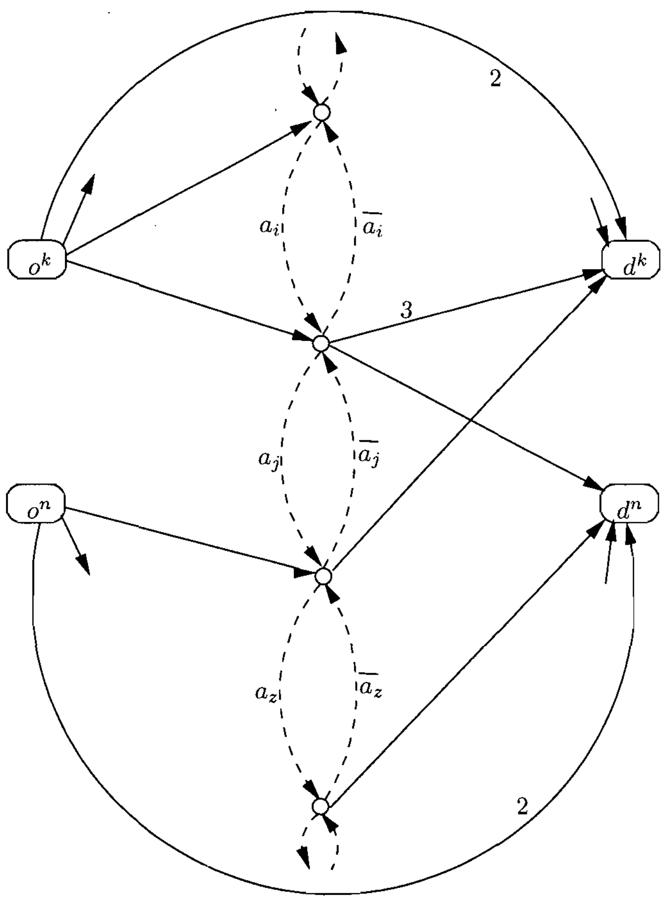

Figure I.4: Subnetwork for $F = (... \vee x_i \vee x_j) \wedge (\overline{x_j} \vee x_z \vee ...) \wedge ...$ (bi-directional Basic NPP).

Then the same proof as before can be applied. □

Proposition 30 *The Basic NPP where access to all entry points (resp. from all exit points) is feasible from all origins (resp. to all destinations) is strongly $\mathcal{NP}$-hard.*

Proof

This additional condition means that there is no so expensive path that it could never be taken, which is a little different from the situation described before.

Subnetworks are constructed as before, except that some additional toll free arcs (those that were too expensive) are added so that there is one toll free arc from any origin to any tail node of a toll arc, and from any head node of a toll arc to any destination. For each commodity k and for each toll arc a_i, the cost on those arcs $(o^k, t(a_i))$ and $(h(a_i), d^k)$ are set in a way such that the sum of the fixed cost of these two arcs is equal to the cost of the toll free arc (o^k, d^k). Such a subnetwork is depicted in Figure I.5.

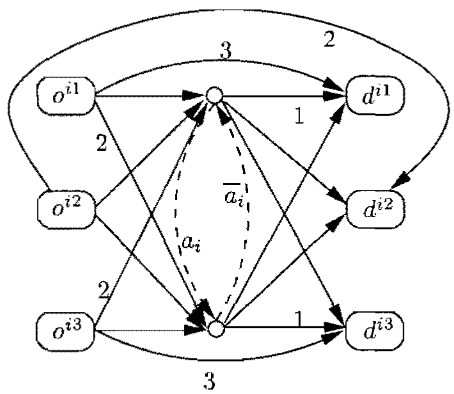

Figure I.5: Subnetwork for variable x_i (All feasible access Basic NPP)

Then, as costs of additional paths (the ones that were too expensive) are equal to costs on toll free arcs from origins to destinations of commodities, one can only choose them if tolls are set to zero on corresponding arcs. But this does not lead to a maximal revenue for the leader, thus those new arcs are not used and tolls on arcs are set as before. Then the same proof as before is applied. □