Constrained-optimal strategy-proof assignment: beyond the Groves mechanisms

Yves Sprumont*

Département de Sciences Économiques and CIREQ
Université de Montréal

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Abstract

A single object must be allocated to at most one of n agents. Money transfers are possible and preferences are quasilinear. We offer an explicit description of the individually rational mechanisms which are Pareto-optimal in the class of feasible, strategy-proof, anonymous and envy-free mechanisms. These mechanisms form a one-parameter infinite family; the Vickrey mechanism is the only Groves mechanism in that family.

1 Introduction

We revisit the problem of allocating a single valuable object to at most one of a number of agents when monetary transfers are possible and preferences are quasilinear. Valuations being private information, an incentive-compatible mechanism is needed. To avoid delicate assumptions on beliefs, we ask that this mechanism be strategy-proof. To guarantee feasibiIity, the sum of the transfers made to the agents should never be positive.

This is the simplest of a variety of assignment problems that have received considerable attention in the literature. Most of the existing work focuses on assignment-optimal strategy-proof mechanisms. In our model, assignment-optimality simply means that the object never remains unallocated and always goes to a maximal valuation agent. Under assumptions that cover our particular case, Holmström (1979) showed that the assignment-optimal strategy-proof mechanisms are precisely the famous Groves (1973) mechanisms. Green and Laffont (1979) showed that all feasible Groves mechanisms waste money: the sum of transfers is (strictly) negative at some valuation profiles.

Of course, both the assignment and the sum of transfers matter for Pareto-optimality. Since relaxing the constraint of assignment-optimality obviously

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helps to reduce the amount of money wasted (leaving the object unallocated and performing no transfers is a strategy-proof and feasible mechanism that wastes no money), restricting attention to the Groves mechanisms is unwarranted. We should instead look for mechanisms that are (constrained) Pareto-optimal within the entire class of feasible strategy-proof mechanisms: not all them need be Groves mechanisms[1].

Nisan et al. (2007) show that all strategy-proof mechanisms for our problem have the following structure. Each agent faces a price that varies only with the other agents' valuations. If he reports a valuation below that price, he does not get the object but receives a transfer which depends upon the others' valuations. If he reports a valuation above the price, he receives the object plus the difference between that transfer and the price of the object. This is a generalization of Holmström's (1979) characterization to mechanisms that need not be assignment-optimal.

Identifying the Pareto frontier of the class of feasible strategy-proof mechanisms, however, remains an open problem. The question is difficult because feasibility imposes complex restrictions on the price and transfer functions in the Nisan et al. characterization. In the current paper we obtain partial results by restricting attention to the subclass of anonymous and envy-free mechanisms, for which the price and transfer functions turn out to be tractable. Notice that not all envy-free strategy-proof mechanisms are Groves: while no-envy forces an optimal assignment of the object whenever the latter is allocated (Svensson, 1983), it does allow us to leave the object unallocated. We identify a one-parameter infinite family of mechanisms –which we call the maxmed mechanisms– that are Pareto-optimal in the class of feasible, strategy-proof, anonymous and envy-free mechanisms. We further prove that the maxmed mechanisms are the only individually rational and Pareto-optimal mechanisms within that class. The Vickrey (1961) mechanism is the only Groves maxmed mechanism; all others leave the object unallocated at some profile of valuations.

Our paper is related to two different lines of work. The first line studies envy-free strategy-proof mechanisms in various quasilinear assignment models; recent papers are Pápai (2003), Svensson (2004), and Ohseto (2006). The most important difference with our work is that all these papers restrict attention to mechanisms that never leave an object unallocated, hence, since no-envy implies assignment optimality, to Groves mechanisms. In a model with several different valuable objects, Pápai (2003) proves that no envy-free Groves mechanism exists on the unrestricted domain of valuations and characterizes the Groves mechanisms that are envy-free on the domain of superadditive valuations. Svensson (2004) considers a model with n agents and n different objects where valuations may be negative but each agent must be assigned one object. He describes the Groves mechanisms that satisfy no-envy and two auxiliary conditions. Ohseto (2006) studies a model with n agents and $m < n$ copies of the same object; val-

[1] Individual rationality is another good reason to look beyond the Groves mechanisms. In a large class of assignment models (including ours), Holmström (1979) and Chew and Serizawa (2007) showed that all Groves mechanisms, except the Vickrey auction, violate individual rationality.

uations may be negative, all copies must be assigned, and each agent may get at most one copy. He characterizes the family of envy-free Groves mechanisms.

The second line of work studies "optimal" strategy-proof assignment mechanisms, usually in a multi-object or multi-unit setting. The class of mechanisms under consideration as well as the optimality criterion vary.

Here again, most papers focus on Groves mechanisms. Guo and Conitzer (2009) and Moulin (2009) follow the worst-case approach: they minimize (slightly different measures of) the worst relative surplus loss. Guo and Conitzer (2010) assume that a prior on the preference profiles is available and minimize the expected loss. Apt et al. (2008) describe the mechanisms whose total loss is undominated in the sense that no other mechanism in the reference class produces a smaller loss at all profiles. Closer to our paper, Ohseto (2006) and Guo and Conitzer (2008) use the traditional criterion of (constrained) Pareto-optimality. In the paper already cited above, Ohseto (2006) describes the Pareto frontier of the class of envy-free Groves mechanisms. In a fairly general multi-object setting, Guo and Conitzer (2008) offer a set of necessary and sufficient conditions for Pareto-optimality within the class of Groves mechanisms; they also provide algorithms which improve upon any given Groves mechanism and reach or approach the Pareto frontier.

Three recent papers look beyond the Groves mechanisms. In the same model as ours, Moulin (2010) studies the problem of minimizing the worst relative surplus loss within the class of feasible, strategy-proof, anonymous, and q-fair mechanisms[2], where $q \geq 3$. He allows for non-Groves mechanisms but the solution he describes is a Groves mechanism. In the multi-copy, unit-demand model, de Clippel et al. (2011) propose a mechanism that is not assignment-optimal but guarantees a relative surplus loss of less than 20% when the number of agents tends to infinity. Their mechanism never leaves more than one object unallocated and it assigns the remaining objects efficiently. The paper most closely related to ours is Athanasiou (2011). In the single-object model, Athanasiou provides a set of necessary and, under individual rationality, sufficient conditions for Pareto-optimality within the class of feasible, strategy-proof, and anonymous mechanisms. In the two-agent case, he shows that the maxmed mechanisms are Pareto-optimal but does not prove that they are the only ones. For more than two agents, he does not give any example of a mechanism satisfying his conditions for Pareto optimality.

2 Setup

One object is to be allocated to at most one of n agents. Money transfers are possible and preferences are quasilinear. Each agent's valuation of the object is a nonnegative real number: a *(valuation) profile* is a vector $v = (v_1, ..., v_n) \in \mathbb{R}_+^N$, where $N = \{1, ..., n\}$ is the set of agents. If $i \in N$, we often write i instead of

[2] The notion of q-fairness imposes a lower bound on the welfare gain that each agent enjoys from participating in the mechanism. The concept was introduced by Porter, Shoham and Tennenholtz (2004) and applied to assignment mechanisms by Atlamaz and Yengin (2006).

$\{i\}$ and we write $-i$ to denote the set $N \setminus i$. The notation $(v_{-i}; v_i')$ stands for the valuation profile obtained from v by replacing the valuation v_i with v_i'.

A *mechanism* is a pair (a,t) where $a : \mathbb{R}_+^N \to \{0,1\}^N$ and $t : \mathbb{R}_+^N \to \mathbb{R}^N$. For any agent $i \in N$ and any valuation profile $v \in \mathbb{R}_+^N$, $a_i(v) = 1$ means that i receives the object at profile v and $a_i(v) = 0$ means that he does not; $t_i(v)$ is the money transfer to agent i. The utility achieved by agent i at profile v under the mechanism (a,t) is $v_i a_i(v) + t_i(v)$.

A mechanism (a,t) is

• *feasible* if it is both *a-feasible,* i.e., $\sum_{i \in N} a_i(v) \leq 1$ for all $v \in \mathbb{R}_+^N$, and *t-feasible,* i.e., $\sum_{i \in N} t_i(v) \leq 0$ for all $v \in \mathbb{R}_+^N$,

• *strategy-proof* if $v_i a_i(v) + t_i(v) \geq v_i a_i(v_{-i}; v_i') + t_i(v_{-i}; v_i')$ for all $v \in \mathbb{R}_+^N$, $i \in N$ and $v_i' \in \mathbb{R}_+$,

• *anonymous* if $v_{\pi(i)} a_{\pi(i)}(\pi v) + t_{\pi(i)}(\pi v) = v_i a_i(v) + t_i(v)$ for all $i \in N$, $v \in \mathbb{R}_+^N$, and any permutation π on N, where the profile πv is defined by $(\pi v)_{\pi(i)} = v_i$ for all $i \in N$,

• *envy-free* if $v_i a_i(v) + t_i(v) \geq v_i a_j(v) + t_j(v)$ for all $v \in \mathbb{R}_+^N$ and $i, j \in N$.

We stress that a feasible mechanism may leave the object unallocated at some profile (i.e., $\sum_{i \in N} a_i(v) = 0$ may hold for some $v \in \mathbb{R}_+^N$). Note also that anonymity is defined in utility terms.

We let $\mathcal{M}$ denote the class of feasible, strategy-proof, anonymous and envy-free mechanisms. If $(a,t), (a',t') \in \mathcal{M}$, we write $(a',t') \succsim (a,t)$ if $v_i a_i'(v) + t_i'(v) \geq v_i a_i(v) + t_i(v)$ for all $v \in \mathbb{R}_+^N$ and all $i \in N$. If in addition the inequality is strict for some $v \in \mathbb{R}_+^N$ and some $i \in N$, then we write $(a',t') \succ (a,t)$ and we say that (a',t') *Pareto-dominates* (a,t). If all inequalities are equalities, we write $(a',t') \sim (a,t)$ and we say that (a,t) and (a',t') are *Pareto-equivalent*. A mechanism $(a,t) \in \mathcal{M}$ is *Pareto-optimal* (in $\mathcal{M}$) if there is no mechanism $(a',t') \in \mathcal{M}$ which Pareto-dominates it.

A mechanism (a,t) is *individually rational* if $v_i a_i(v) + t_i(v) \geq 0$ for all $v \in \mathbb{R}_+^N$ and $i \in N$. This paper provides an explicit description of the individually rational Pareto-optimal mechanisms in $\mathcal{M}$.

3 Preliminaries and statement of the result

We begin by proving a number of properties of the mechanisms in $\mathcal{M}$. This is the purpose of the following three lemmas. Let $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$.

Lemma 1. *A mechanism (a,t) is strategy-proof and anonymous if and only if there exist two symmetric functions $p : \mathbb{R}_+^{n-1} \to \overline{\mathbb{R}}_+$ and $g : \mathbb{R}_+^{n-1} \to \mathbb{R}$ such that, for all $v \in \mathbb{R}_+^N$ and $i \in N$,*

$$(a_i(v), t_i(v)) = \begin{cases} (1, g(v_{-i}) - p(v_{-i})) \text{ if } v_i > p(v_{-i}), \\ (0, g(v_{-i})) \text{ if } v_i < p(v_{-i}), \end{cases} \tag{1}$$

and

$$(a_i(v), t_i(v)) \in \{(0, g(v_{-i})), (1, g(v_{-i}) - p(v_{-i}))\} \text{ if } v_i = p(v_{-i}). \tag{2}$$

Proof. The proof is an easy modification of the proof of the characterization of the strategy-proof mechanisms in Nisan et al. (2007). We omit the details.■

We call a pair of functions $(p, g) \in \overline{\mathbb{R}}_+^{\mathbb{R}_+^{n-1}} \times \mathbb{R}^{\mathbb{R}_+^{n-1}}$ a *(payment) scheme.* If (1) and (2) hold for all $v \in \mathbb{R}_+^N$ and $i \in N$, we say that the scheme (p, g) *generates* the mechanism (a, t). It is clear from (1) that the scheme generating a strategy-proof and anonymous mechanism is unique.

Conversely, because of (2), a scheme (p, g) generates not just one, but an entire family of strategy-proof and anonymous mechanisms. In view of (1), however, these mechanisms coincide at almost all profiles. Moreover, they are all Pareto-equivalent since agent i is indifferent between $(0, g(v_{-i}))$ and $(1, g(v_{-i}) - p(v_{-i}))$ whenever $v_i = p(v_{-i})$.

From the viewpoint of feasibility, a particularly useful mechanism is the *lexicographic mechanism generated by* (p, g), which we define as follows: for all $v \in \mathbb{R}_+^N$ and $i \in N$, $(a_i(v), t_i(v)) = (1, g(v_{-i}) - p(v_{-i}))$ if $[v_i > p(v_{-i})]$ or $[v_i = p(v_{-i})$ and $v_j < p(v_{-j})$ for all $j < i]$, and $(a_i(v), t_i(v)) = (0, g(v_{-i}))$ if $[v_i < p(v_{-i})]$ or $[v_i = p(v_{-i})$ and $v_j \geq p(v_{-j})$ for some $j < i]$.

For any $k \in \mathbb{N}$ and $x = (x_1, ..., x_k) \in \mathbb{R}^k$, we denote by $\max x$ the maximum of the numbers $x_1, ..., x_k$.

Lemma 2. *Let (a, t) be a strategy-proof and anonymous mechanism and let (p, g) be the scheme generating it.*

(i) If (a, t) is a-feasible, then $p(x) \geq \max x$ for all $x \in \mathbb{R}_+^{n-1}$. Moreover, (a, t) is conditionally a-optimal: for all $v \in \mathbb{R}_+^N$ and $i \in N$, $a_i(v) = 1$ only if $v_i = \max v$.

(ii) If $p(x) > \max x$ for all $x \in \mathbb{R}_+^{n-1}$, then (a, t) is a-feasible. If $p(x) \geq \max x$ for all $x \in \mathbb{R}_+^{n-1}$, then (a, t) is Pareto-equivalent to an a-feasible, strategy-proof and anonymous mechanism; in particular, the lexicographic mechanism generated by (p, g) is a-feasible.

Proof. Let (a, t) be a strategy-proof and anonymous mechanism and let (p, g) be the scheme that generates it.

Ad (i). We first prove the contraposition of the first sentence. Suppose there exists $x = (x_1, ..., x_{n-1}) \in \mathbb{R}_+^{n-1}$ such that $p(x) < \max x$ and assume without loss of generality that $\max x = x_1$. Consider the valuation profile $v = (x_1, x_1, x_2, ..., x_{n-1})$. Since $v_1 = x_1 = \max x > p(x) = p(v_{-1})$, we have $a_1(v) = 1$. But since $v_2 = x_1 = \max x > p(x) = p(v_{-2})$, we also have $a_2(v) = 1$, meaning that (a, t) is not a-feasible.

To check conditional a-optimality, let $v = (v_1, ..., v_n)$ and suppose $v_i < v_j$. Contrary to the claim, suppose $a_i(v) = 1$. Then $v_i \geq p(v_{-i}) \geq \max v_{-i} \geq v_j$, a contradiction.

Ad (ii). In order to prove the first sentence in (ii), assume $p(x) > \max x$ for all $x \in \mathbb{R}_+^{n-1}$. Contrary to the claim, suppose there exists $v \in \mathbb{R}_+^N$ and two distinct agents $i, j \in N$ such that $a_i(v) = a_j(v) = 1$. Then $v_i \geq p(v_{-i}) > \max v_{-i} \geq v_j$ and $v_j \geq p(v_{-j}) > \max v_{-j} \geq v_i$, a contradiction.

In order to prove the second sentence in (ii), assume $p(x) \geq \max x$ for all $x \in \mathbb{R}_+^{n-1}$ and let (a^L, t^L) be the lexicographic mechanism generated by (p, g). Suppose, by way of contradiction, that there exists $v \in \mathbb{R}_+^N$ and two distinct agents $i, j \in N$ such that $i < j$ and $a_i^L(v) = a_j^L(v) = 1$. Then $v_i \geq p(v_{-i}) \geq \max v_{-i} \geq v_j > p(v_{-j}) \geq v_i$, which is impossible.■

Statement (i) in Lemma 2 is Proposition 1 in Athanasiou (2011). His direct proof does not exploit Nisan et al.'s characterization of the strategy-proof mechanisms.

Lemma 3. *Let* $(a, t) \in \mathcal{M}$ *and let* (p, g) *be the scheme generating* (a, t). *Then there exists a function* $g_0 : \mathbb{R}_+ \to \mathbb{R}$ *such that* $g(x) = g_0(\max x)$ *for all* $x \in \mathbb{R}_+^{n-1}$.

Proof. We show that $g(x)$ varies only with $\max x$. Specifically, let $x = (x_1, ..., x_{n-1})$ and assume, without loss of generality, that $\max x = x_1$. We prove that for each $i = 2, ..., n-1$ and each $x_i' \leq x_1$, $g(x_i', x_{-i}) = g(x)$.

Fix $i \in \{2, ..., n-1\}$, say, $i = 2$, and let $x_2' \leq x_1$. Suppose, by way of contradiction, that $g(x_1, x_2', x_3, ..., x_{n-1}) \neq g(x_1, x_2, x_3, ..., x_{n-1})$. Let $v = (v_1, ..., v_n) = (x_1, x_2, x_2', x_3, ..., x_{n-1})$.

If $a_2(v) = a_3(v) = 0$, then $g(v_{-3}) \neq g(v_{-2})$ implies that either 2 envies 3 or 3 envies 2.

If $a_2(v) = 1$ and $a_3(v) = 0$, then by conditional a-optimality (which holds by Lemma 2 (i)) $v_2 = x_2 = \max v = \max x = x_1$. Then $(a_1(v), t_1(v)) = (0, g(v_{-1})) = (0, g(x_2, x_2', x_3, ..., x_{n-1})) = (0, g(x_1, x_2', x_3, ..., x_{n-1}))$ while $(a_3(v), t_3(v)) = (0, g(v_{-3})) = (0, g(x_1, x_2, x_3, ..., x_{n-1}))$. Then $g(v_{-1}) \neq g(v_{-3})$ and either 1 envies 3 or 3 envies 1.

If $a_2(v) = 0$ and $a_3(v) = 1$, then by conditional a-optimality $v_3 = x_2' = \max v = \max x = x_1$. Then $(a_1(v), t_1(v)) = (0, g(v_{-1})) = (0, g(x_2, x_2', x_3, ..., x_{n-1}))$ $= (0, g(x_2, x_1, x_3, ..., x_{n-1})) = (0, g(x_1, x_2, x_3, ..., x_{n-1}))$, where the last inequality holds by symmetry of g. On the other hand, $(a_2(v), t_2(v)) = (0, g(v_{-2})) = (0, g(x_1, x_2', x_3, ..., x_{n-1}))$. Then $g(v_{-1}) \neq g(v_{-2})$ and either 1 envies 2 or 2 envies 1.■

We call a pair of functions $(p, g_0) \in \overline{\mathbb{R}}_+^{\mathbb{R}_+^{n-1}} \times \mathbb{R}^{\mathbb{R}_+}$ a *simple* scheme. We say that (p, g_0) generates the mechanism (a, t) if the scheme (p, g) defined by $g(x) = g_0(\max x)$ generates (a, t).

Definition 1. A simple scheme (p, g_0) is a *maxmed simple scheme* if there exists some $\alpha \in \mathbb{R}_+$ such that

$$\begin{aligned} p(x) &= \max(\max x, \alpha), \\ g_0(y) &= \text{med}\left(0, y - \alpha, \frac{\alpha}{n-1}\right) \end{aligned}$$

for all $x \in \mathbb{R}_+^{n-1}$ and $y \in \mathbb{R}_+$, where $\max(\max x, \alpha)$ denotes the maximum of the two numbers $\max x$, α, and $\text{med}\left(0, y - \alpha, \frac{\alpha}{n-1}\right)$ denotes the median of the three numbers 0, $y-\alpha$, $\frac{\alpha}{n-1}$. A mechanism (a,t) is a *maxmed mechanism* if it is generated by a maxmed simple scheme.

An illustration is offered in Figure 1. When $\alpha = 0$, the maxmed simple scheme (p, g_0) of Definition 1 is just Vickrey's second-price auction. When $\alpha > 0$, any mechanism generated by the scheme leaves the object unallocated at some profiles. When the mechanism allocates the object, it does it optimally.

Let $\mathcal{M}_0$ denote the subset of mechanisms in $\mathcal{M}$ which are individually rational.

Theorem. *(i) Every maxmed simple scheme (p, g_0) generates at least one mechanism $(a,t) \in \mathcal{M}_0$ which is Pareto-optimal (in $\mathcal{M}$). The lexicographic mechanism generated by (p, g_0) is such a mechanism*[3].

(ii) Every mechanism $(a,t) \in \mathcal{M}_0$ which is Pareto-optimal (in $\mathcal{M}$) is a maxmed mechanism.

Before establishing this result, we give an intuition for the constrained optimality of the maxmed mechanisms by showing that they are Pareto non-comparable. Contrast for instance the Vickrey mechanism (parameterized by $\alpha = 0$) with a nontrivial maxmed mechanism (parameterized by some positive value of α). Consider first a valuation profile v where $0 = v_i < v_1 < \alpha$ for all $i \neq 1$. The Vickrey mechanism assigns the object to agent 1 and performs no transfer. The α-maxmed mechanism does not perform any transfer either but it leaves the object unallocated, an outcome dominated by the Vickrey outcome. Consider now a profile v' where $\frac{n\alpha}{n-1} = v_i < v_1$ for all $i \neq 1$. The Vickrey mechanism assigns the object to agent 1, charges him $\frac{n\alpha}{n-1}$, and performs no transfer to the others. The α-maxmed mechanism assigns the object to agent 1, charges him $\frac{n\alpha}{n-1} - \frac{\alpha}{n-1} = \alpha$ and performs a transfer $\frac{\alpha}{n-1}$ to each of the other agents, an outcome which dominates the Vickrey outcome. By not assigning the object at low valuations profiles, a maxmed mechanism with positive α creates benefits for all agents at high valuations profiles.

4 Proof of the theorem

Lemma 4. *If (p, g_0) is a maxmed simple scheme and (a,t) is the lexicographic mechanism generated by (p, g_0), then $(a,t) \in \mathcal{M}_0$.*

Proof. This is just a matter of checking. Since (p, g_0) is a maxmed simple scheme, there exists $\alpha \in \mathbb{R}_+$ such that $p(x) = \max(\max x, \alpha)$ and $g_0(y) = \text{med}(0, y-\alpha, \alpha)$ for all $x \in \mathbb{R}_+^{n-1}$ and $y \in \mathbb{R}_+$. By Lemmas 1 and 2, the lexicographic mechanism (a,t) generated by (p, g_0) is strategy-proof, anonymous and a-feasible. It is also clear that (a,t) is individually rational. To check this, fix $v \in \mathbb{R}_+^N$ and $i \in N$. If $a_i(v) = 0$, then $v_i a_i(v) + t_i(v) = g_0(\max v_{-i}) \geq 0$ by

[3] All the mechanisms generated by the maxmed simple scheme (p, g_0) are Pareto-equivalent but not all of them belong to $\mathcal{M}_0$. In particular, not all of them are feasible.

the very definition of g_0. If $a_i(v) = 1$, then we must have $v_i \geq p(v_{-i})$, hence $v_i a_i(v) + t_i(v) = v_i + g_0(\max v_{-i}) - p(v_{-i}) \geq g_0(\max v_{-i}) \geq 0$.

It remains to be checked that (a, t) is t-feasible and envy-free. Fix $v \in \mathbb{R}^N_+$ and let $\pi : N \to N$ be a permutation such that $v_{\pi(1)} \geq ... \geq v_{\pi(n)}$. There are six possible cases. In all cases, the no-envy property holds among the agents who do not receive the object since all of them receive the same transfer, namely, $g_0(v_{\pi(1)})$.

Case 1. $v_{\pi(1)} \leq \alpha$.

Then $g_0(\max v_{-i}) = 0$ for all $i \in N$.

If $v_{\pi(1)} < \alpha$, we have $(a_i(v), t_i(v)) = (0, 0)$ for all $i \in N$. Hence $\sum_{i \in N} t_i(v) = 0$, ensuring $t-$feasibility. Since all agents receive the same bundle, there is no envy.

If $v_{\pi(1)} = \alpha$, let i^* be the smallest $i \in N$ such that $v_i = v_{\pi(1)} = \alpha$. Then $(a_{i^*}(v), t_{i^*}(v)) = (1, -\alpha)$ and $(a_i(v), t_i(v)) = (0, 0)$ for all $i \in N \setminus i^*$. Thus $\sum_{i \in N} t_i(v) = -\alpha \leq 0$, ensuring $t-$feasibility. Agent i^* is indifferent between what he gets and what anybody else gets and since $v_i - \alpha \leq 0$ for all $i \in N \setminus i^*$, nobody envies i^*.

Case 2. $v_{\pi(2)} \leq \alpha < v_{\pi(1)} \leq \frac{n\alpha}{n-1}$.

Then $(a_{\pi(1)}(v), t_{\pi(1)}(v)) = (1, -\alpha)$ and $(a_i(v), t_i(v)) = (0, v_{\pi(1)} - \alpha)$ for all $i \in N \setminus \pi(1)$. Thus $\sum_{i \in N} t_i(v) = t_{\pi(1)} + \sum_{i \in N \setminus \pi(1)} t_i(v) = -\alpha + (n-1)(v_{\pi(1)} - \alpha) = (n-1)v_{\pi(1)} - n\alpha \leq 0$, ensuring t-feasibility. Agent $\pi(1)$ is indifferent between what he gets and what anybody else gets and since $v_{\pi(1)} - \alpha \geq v_i - \alpha$ for all $i \in N \setminus \pi(1)$, no $i \in N \setminus \pi(1)$ envies $\pi(1)$.

Case 3. $v_{\pi(2)} \leq \alpha \leq \frac{n\alpha}{n-1} < v_{\pi(1)}$.

Then $(a_{\pi(1)}(v), t_{\pi(1)}(v)) = (1, -\alpha)$ and $(a_i(v), t_i(v)) = (0, \frac{\alpha}{n-1})$ for all $i \in N \setminus \pi(1)$. Thus $\sum_{i \in N} t_i(v) = -\alpha + (n-1)\frac{\alpha}{n-1} = 0$, satisfying t-feasibility. Since $v_{\pi(1)} - \alpha \geq \frac{\alpha}{n-1}$, agent $\pi(1)$ does not envy any other agent and since $\frac{\alpha}{n-1} \geq v_i - \alpha$ for all $i \in N \setminus \pi(1)$, nobody envies $\pi(1)$.

Case 4. $\alpha < v_{\pi(2)} \leq v_{\pi(1)} \leq \frac{n\alpha}{n-1}$.

Let i^* be the smallest $i \in N$ such that $v_i = v_{\pi(1)}$. Then $(a_{i^*}(v), t_{i^*}(v)) = (1, -v_{\pi(2)} + (v_{\pi(2)} - \alpha)) = (1, -\alpha)$ and $(a_i(v), t_i(v)) = (0, v_{\pi(1)} - \alpha)$ for all $i \in N \setminus i^*$. Thus $\sum_{i \in N} t_i(v) = -\alpha + (n-1)(v_{\pi(1)} - \alpha) = (n-1)v_{\pi(1)} - n\alpha \leq 0$, satisfying t-feasibility. Agent i^* is indifferent between what he gets and what anybody else gets and, since $v_{\pi(1)} - \alpha \geq v_i - \alpha$ for all $i \in N \setminus \pi(1)$, nobody envies $\pi(1)$.

Case 5. $\alpha < v_{\pi(2)} \leq \frac{n\alpha}{n-1} < v_{\pi(1)}$.

Then $(a_{\pi(1)}(v), t_{\pi(1)}(v)) = (1, -v_{\pi(2)} + (v_{\pi(2)} - \alpha)) = (1, -\alpha)$ and $(a_i(v), t_i(v)) = (0, \frac{\alpha}{n-1})$ for all $i \in N \setminus \pi(1)$. Thus $\sum_{i \in N} t_i(v) = -\alpha + (n-1)\frac{\alpha}{n-1} = 0$, satisfying t-feasibility. Since $v_{\pi(1)} - \alpha \geq \frac{\alpha}{n-1}$, agent $\pi(1)$ does not envy any other agent and, since $\frac{\alpha}{n-1} \geq v_i - \alpha$ for all $i \in N \setminus \pi(1)$, nobody envies $\pi(1)$.

Case 6. $\frac{n\alpha}{n-1} < v_{\pi(2)}$.

Let i^* be the smallest $i \in N$ such that $v_i = v_{\pi(1)}$. Then $(a_{i^*}(v), t_{i^*}(v)) = (1, -v_{\pi(2)} + \frac{\alpha}{n-1})$ and $(a_i(v), t_i(v)) = (0, \frac{\alpha}{n-1})$ for all $i \in N \setminus i^*$. Thus $\sum_{i \in N} t_i(v) =$

$-v_{\pi(2)} + \frac{\alpha}{n-1} + (n-1)\frac{\alpha}{n-1} = \frac{n\alpha}{n-1} - v_{\pi(2)} \leq 0$, satisfying t-feasibility. Since $v_{\pi(1)} - v_{\pi(2)} + \frac{\alpha}{n-1} \geq \frac{\alpha}{n-1}$, agent i^* does not envy any other agent and, since $\frac{\alpha}{n-1} \geq v_i - v_{\pi(2)} + \frac{\alpha}{n-1}$ for all $i \in N \setminus i^*$, nobody envies $\pi(1)$.■

Our next lemma establishes a number of properties of the mechanisms in $\mathcal{M}_0$.

Notation 1. Given a simple scheme (p, g_0), we let $\beta(p) := \inf\left\{p(x) \mid x \in \mathbb{R}_+^{n-1}\right\}$. When the reference to the scheme is clear, we write β instead of $\beta(p)$.

Lemma 5. *Let $(a,t) \in \mathcal{M}_0$ and let (p, g_0) be the simple scheme generating (a,t). Then,*

(i) for all $y \in \mathbb{R}_+$, $g_0(y) \geq 0$,
(ii) for all $x \in \mathbb{R}_+^{n-1}$, $[p(x) > \max x] \Rightarrow [g_0(\max x) = 0]$,
(iii) for all $y \in \mathbb{R}_+$, $g_0(y) \leq \frac{\beta}{n-1}$,
(iv) for all $y, z \in \mathbb{R}_+$, $[y \leq z] \Rightarrow [g_0(y) \leq g_0(z)$ and $y - g_0(y) \leq z - g_0(z)]$.

Proof. For any $i \in N$, define $e(i) \in \mathbb{R}_+^{n-1}$ by $e_i(i) = 1$ and $e_j(i) = 0$ for all $j \in N \setminus i$.
Ad (i). Let $y \in \mathbb{R}_+$ and consider the valuation profile $v = (v_1, ..., v_n) = (y, ..., y)$. By feasibility, $a_i(v) = 0$ for some $i \in N$. By individual rationality, $v_i a_i(v) + t_i(v) = g_0(\max v_{-i}) = g_0(y) \geq 0$.

Ad (ii). Let $x = (x_1, ..., x_{n-1}) \in \mathbb{R}_+^{n-1}$ be such that $p(x) > \max x$. By definition of (a,t), $a_n(x_1, ..., x_{n-1}, \max x) = 0$ and since p is symmetric, $a_i(x_1, ..., x_{n-1}, \max x) = 0$ for all $i \in N$ such that $x_i = \max x$. Since (a,t) is conditionally a-optimal (by Lemma 2), it follows that $a(x_1, ..., x_{n-1}, \max x) = (0, ..., 0)$. By feasibility, $\sum_{i \in N} t_i(x_1, ..., x_{n-1}, \max x) = n g_0(\max x) \leq 0$. Combining this with (i), we obtain $g_0(\max x) = 0$.

Ad (iii). Note first that $\beta \geq 0$. Now let $y \in \mathbb{R}_+$. If $p(y, ..., y) > y$, then (ii) implies $g_0(y) = 0 \leq \frac{\beta}{n-1}$. From now on, assume $p(y, ..., y) = y$. Distinguish two cases.

Case 1. $p(y, ..., y) = \beta$.

Let $v = (v_1, ..., v_n) = (y, ..., y)$. If $a(v) = (0, ..., 0)$, then t-feasibility implies $n g_0(y) \leq 0$, hence $g_0(y) \leq 0 \leq \frac{\beta}{n-1}$. If there exists $i \in N$ such that $a(v) = e(i)$, then t-feasibility implies $(n-1)g_0(y) + g_0(y) - p(y, ..., y) = n g_0(y) - \beta \leq 0$, hence $g_0(y) \leq \frac{\beta}{n} \leq \frac{\beta}{n-1}$.

Case 2. $p(y, ..., y) > \beta$.

Fix ε such that $0 < \varepsilon < y - \beta$ $(= p(y, ..., y) - \beta)$. Let $x^\varepsilon = (x_1^\varepsilon, ..., x_{n-1}^\varepsilon) \in \mathbb{R}_+^{n-1}$ be such that $p(x^\varepsilon) \leq \beta + \varepsilon$. Such a point x^ε exists by definition of β. Moreover, $\max x^\varepsilon < y$ (otherwise, $p(x^\varepsilon) \geq \max x^\varepsilon \geq y$, hence $\beta + \varepsilon \geq y$, contradicting our assumption on ε). Consider the valuation profile $v^\varepsilon = (x_1^\varepsilon, ..., x_{n-1}^\varepsilon, y)$. Since $y > \beta + \varepsilon \geq p(x^\varepsilon)$, we have $a_n(v^\varepsilon) = 1$, hence by a-feasibility $a(v^\varepsilon) = e(n)$. By t-feasibility, $(n-1)g_0(y) + g_0(\max x^\varepsilon) - p(x^\varepsilon) \leq 0 \Rightarrow (n-1)g_0(y) \leq p(x^\varepsilon) \Rightarrow g_0(y) \leq \frac{\beta+\varepsilon}{n-1}$. Letting $\varepsilon \to 0$ yields $g_0(y) \leq \frac{\beta}{n-1}$.

Ad (iv). Let $y, z \in \mathbb{R}_+$ be such that $y < z$. Let $v = (v_1, ..., v_n) = (z, y, ..., y)$. By conditional a-optimality, $a(v) = (0, ..., 0)$ or $a(v) = e(1)$.

If $a(v) = (0, ..., 0)$, preventing agent 2 from envying agent 1 requires $g_0(z) \geq g_0(y)$ while preventing 1 from envying 2 requires $g_0(y) \geq g_0(z)$, hence $y - g_0(y) \leq z - g_0(z)$.

If $a(v) = e(1)$, preventing 2 from envying 1 requires $g_0(z) \geq y + g_0(y) - p(y, ..., y)$ and preventing 1 from envying 2 requires $z + g_0(y) - p(y, ..., y) \geq g_0(z)$. If $g_0(y) > 0$, then (ii) and Lemma 2(i) imply $p(y, ..., y) = y$ and the two no-envy conditions reduce to $g_0(z) \geq g_0(y)$ and $y - g_0(y) \leq z - g_0(z)$. If $g_0(y) = 0$, then $g_0(z) \geq 0 = g_0(y)$ holds by (i) and the second no-envy condition reduces to $z - p(y, ..., y) \geq g_0(z)$, hence $z - g_0(z) \geq p(y, ..., y) \geq y = y - g_0(y)$, where the last inequality follows from Lemma 2(i).■

Notation 2. Given a simple scheme (p, g_0), let $X(p) = \left\{x \in \mathbb{R}_+^{n-1} \mid p(x) = \max x\right\}$ and $\gamma(p) := \inf\left\{\max x \mid x \in X(p)\right\}$. When the reference to the scheme is clear, we write γ instead of $\gamma(p)$.

Observe that for any simple scheme (p, g_0), $\gamma(p) = \inf\left\{\max x \mid x \in X(p)\right\}$ $= \inf\left\{p(x) \mid x \in X(p)\right\} \geq \inf\left\{p(x) \mid x \in \mathbb{R}_+^{n-1}\right\} = \beta(p)$.

The next lemma is the heart of the proof of our theorem.

Lemma 6. *Let $(a, t) \in \mathcal{M}_0$ and let (p, g_0) be the simple scheme generating (a, t). For all $x \in \mathbb{R}_+^{n-1}$ let*

$$p^*(x) = \max(\max x, \beta(p)).$$

For all $v \in \mathbb{R}_+^N$ and $i \in N$, let $(a_i^*(v), t_i^*(v)) = (1, g_0(\max v_{-i}) - p^*(v_{-i}))$ if $[v_i > p^*(v_{-i})]$ or $[v_i = p^*(v_{-i})$ and $a_i(v) = 1]$, and $(a_i^*(v), t_i^*(v)) = (0, g_0(\max v_{-i}))$ if $[v_i < p^*(v_{-i})]$ or $[v_i = p^*(v_{-i})$ and $a_i(v) = 0]$. Then $(a^*, t^*) \in \mathcal{M}_0$.

Proof. By Lemma 1, (a^*, t^*) is a strategyproof and anonymous mechanism. By Lemma 5, g_0 is nonnegative, hence (a^*, t^*) is individually rational. It remains to be shown that (a^*, t^*) is feasible and envy-free.

Write $\beta(p) = \beta$ and $\gamma(p) = \gamma$. By Lemma 2 (i) and the definition of β, $p(x) \geq \max x$ and $p(x) \geq \beta$ for all $x \in \mathbb{R}_+^{n-1}$, hence

$$p(x) \geq p^*(x) \geq \max x \text{ for all } x \in \mathbb{R}_+^{n-1}. \tag{3}$$

Step 1. Proving a-feasibility.

Suppose, by way of contradiction, that there exists $v \in \mathbb{R}_+^N$ such that, say, $a_1^*(v) = a_2^*(v) = 1$. By definition of (a^*, t^*), there are only three possible cases:

(i) $v_1 > p^*(v_{-1})$ and $v_2 \geq p^*(v_{-2})$. Then, by (3), $v_1 > \max v_{-1} \geq v_2 \geq \max v_{-2} \geq v_1$, a contradiction.

(ii) $v_2 > p^*(v_{-2})$ and $v_1 \geq p^*(v_{-1})$. Then, by (3), $v_2 > \max v_{-2} \geq v_1 \geq \max v_{-1} \geq v_2$, a contradiction.

(iii) $v_1 = p^*(v_{-1})$ and $v_2 = p^*(v_{-2})$. Then $a_1(v) = a_2(v) = 1$, contradicting a-feasibility of (a, t).

This proves that (a^*, t^*) is a-feasible. Note that it follows from Lemma 2 that (a^*, t^*) is also conditionally a-optimal.

Step 2. Proving t-feasibility.

Fix $v = (v_1, ..., v_n)$ and assume without loss of generality $v_1 \leq ... \leq v_n$.

Case 1. $v_1 < v_n$.

Let m be the largest $i \in N$ such that $v_{i-1} < v_i$. By assumption, $1 < i \leq n$, so we have $v_1 \leq ... \leq v_{m-1} < v_m = ... = v_n$. Observe that

$$\text{for all } i = 1, ..., m-1,\ a_i(v) = a_i^*(v) = 0, \tag{4}$$

and

$$\text{for all } i \in N,\ a_i^*(v) = 0 \Rightarrow a_i(v) = 0. \tag{5}$$

(To see why (4) holds, suppose there exists $i \leq m-1$ such that $a_i(v) = 1$. Then $v_i \geq p(v_{-i}) \geq \max v_{-i} = v_n$, contradicting our assumption that $v_i < v_n$. Likewise, if there exists $i \leq m-1$ such that $a_i^*(v) = 1$, then by definition of (a^*, t^*) and (3), we have $v_i \geq p^*(v_{-i}) \geq \max v_{-i} = v_n$, contradicting $v_i < v_n$ again. To check (5), suppose there exists $i \in N$ such that $a_i(v) = 1$. Then $v_i \geq p(v_{-i}) \geq p^*(v_{-i})$ and it follows from the definition of (a^*, t^*) that $a_i^*(v) = 1$.)

Now distinguish two subcases.

Case 1.1. $a^*(v) = (0, ..., 0)$.

Then (4) and (5) imply $a(v) = (0, ..., 0)$. By definition of (a^*, t^*), $\sum_{i \in N} t_i^*(v) = \sum_{i \in N} g_0(\max v_{-i}) = \sum_{i \in N} t_i(v) \leq 0$, where the last inequality holds by feasibility of (a, t).

Case 1.2. $a^*(v) \neq (0, ..., 0)$.

Since (a^*, t^*) is conditionally a-optimal (by Step 1), there exists $i^* \in \{m, ..., n\}$ such that $a^*(v) = e(i^*)$. It follows from (4) and (5) that $a(v) = e(i^*)$ or $a(v) = (0, ..., 0)$.

If $g_0(\max v_{-i^*}) > 0$, then Lemma 5 (ii) and Lemma 2 (i) imply $p(v_{-i^*}) = \max v_{-i^*}$. Hence $v_{-i^*} \in X(p)$ and by definition of γ, $\max v_{-i^*} \geq \gamma$. By definition of p^*, $p^*(v_{-i^*}) = \max v_{-i^*}$, hence $p^*(v_{-i^*}) = p(v_{-i^*})$. Therefore

$$\begin{aligned}
& \sum_{i \in N} t_i^*(v) \\
= \ & \sum_{i \in N \setminus i^*} g_0(\max v_{-i}) + g_0(\max v_{-i^*}) - p^*(v_{-i^*}) \\
= \ & \sum_{i \in N \setminus i^*} g_0(\max v_{-i}) + g_0(\max v_{-i^*}) - p(v_{-i^*}) \\
\leq \ & \sum_{i \in N} t_i(v) \\
\leq \ & 0,
\end{aligned}$$

where the first inequality holds because $a(v) = e(i^*)$ or $a(v) = (0, ..., 0)$.

If $g_0(\max v_{-i^*}) = 0$, then

$$\begin{aligned}
& \sum_{i\in N} t_i^*(v) \\
= & \sum_{i\in N\setminus i^*} g_0(\max v_{-i}) + g_0(\max v_{-i^*}) - p^*(v_{-i^*}) \\
= & (n-1)g_0(v_{i^*}) - p^*(v_{-i^*}) \\
\leq & \beta - p^*(v_{-i^*}) \\
\leq & 0,
\end{aligned}$$

where the first inequality holds by Lemma 5 (iii) and the second by definition of p^*.

Case 2. $v_1 = v_n$.

If $p^*(v_1, ..., v_1) = p(v_1, ..., v_1)$, then by definition of (a^*, t^*) we have $(a^*(v), t^*(v)) = (a(v), t(v))$, hence $\sum_{i\in N} t_i^*(v) = \sum_{i\in N} t_i(v) \leq 0$.

If $p^*(v_1, ..., v_1) < p(v_1, ..., v_1)$, then by (3) $p(v_1, ..., v_1) > p^*(v_1, ..., v_1) \geq v_1$. By Lemma 5 (ii), $g_0(v_1) = 0$. Then $\sum_{i\in N} t_i^*(v) \leq ng_0(v_1) = 0$.

Step 3. Proving no-envy.

Fix $v = (v_1, ..., v_n) \in \mathbb{R}_+^N$ and two distinct agents $i, j \in N$. We show that i, j do not envy each other at $(a^*(v), t^*(v))$.

Case 1. $a_i^*(v) = a_j^*(v) = 0$.

If $v_i = p^*(v_{-i})$, then the definition of (a^*, t^*) implies $a_i(v) = 0$. If $v_i < p^*(v_{-i})$, then by (3) $v_i < p(v_{-i})$, hence by definition of (a, t), $a_i(v) = 0$. So $a_i(v) = 0$ in all cases. Similarly, $a_j(v) = 0$. By definition of (a^*, t^*), $t_i^*(v) = g_0(\max v_{-i}) = t_i(v)$ and $t_j^*(v) = g_0(\max v_{-j}) = t_j(v)$. Since (a, t) is envy–free, $t_i(v) = t_j(v)$. Hence $t_i^*(v) = t_j^*(v)$ and i, j do not envy each other at $(a^*(v), t^*(v))$.

Case 2. $a_i^*(v) = 0$, $a_j^*(v) = 1$.

By conditional a-optimality, $v_i \leq v_j$.

We first check that i does not envy j. Note that $v_i \leq v_j$ implies $\max v_{-i} \geq \max v_{-j}$. Since (a, t) is envy-free, Lemma 5 (iv) then implies $g_0(\max v_{-i}) \geq g_0(\max v_{-j})$. Moreover, $v_i \leq \max v_{-j} \leq \max(\max v_{-j}, \beta) = p^*(v_{-j})$. Taking these inequalities into account,

$$\begin{aligned}
v_i a_i^*(v) + t_i^*(v) &= g_0(\max v_{-i}) \\
&\geq g_0(\max v_{-j}) \\
&\geq v_i + g_0(\max v_{-j}) - p^*(v_{-j}) \\
&= v_i a_j^*(v) + t_j^*(v),
\end{aligned}$$

as desired.

Next we check that j does not envy i. Since $a_i^*(v) = 0$, (5) implies $a_i(v) = 0$.

If $a_j(v) = 1$, then

$$\begin{aligned} v_j a_j^*(v) + t_j^*(v) &= v_j + g_0(\max v_{-j}) - p^*(v_{-j}) \\ &\geq v_j + g_0(\max v_{-j}) - p(v_{-j}) \\ &= v_j a_j(v) + t_j(v) \\ &\geq v_j a_i(v) + t_i(v) \\ &= g_0(\max v_{-i}) \\ &= v_j a_i^*(v) + t_i^*(v), \end{aligned}$$

where the first inequality holds by (3) and the second because (a, t) is envy-free.

If $a_j(v) = 0$, then

$$\begin{aligned} v_j a_j^*(v) + t_j^*(v) &= v_j + g_0(\max v_{-j}) - p^*(v_{-j}) \\ &\geq g_0(\max v_{-j}) \\ &= v_j a_j(v) + t_j(v) \\ &\geq v_j a_i(v) + t_i(v) \\ &= g_0(\max v_{-i}) \\ &= v_j a_i^*(v) + t_i^*(v), \end{aligned}$$

where the first inequality holds because $a_j^*(v) = 1$ implies $v_j \geq p^*(v_{-j})$.■

We now turn to Pareto-dominance and Pareto-optimality. We begin with an elementary result. For any set Z and any two functions $f, f' \in \overline{\mathbb{R}}_+^Z$, we write $f \leq f'$ if $f(z) \leq f'(z)$ for all $z \in Z$.

Lemma 7. *Let $(a, t), (a', t') \in \mathcal{M}$ and let $(p, g_0), (p', g_0')$ be the simple schemes generating $(a, t), (a', t')$. If $p' \leq p$ and $g_0' \geq g_0$, then $(a', t') \succsim (a, t)$.*

Proof. The straightforward proof is omitted.■

Lemma 8. *If $(a, t) \in \mathcal{M}_0$, then there exists a maxmed mechanism $(a', t') \in \mathcal{M}_0$ such that $(a', t') \succsim (a, t)$.*

Proof.
Step 1. Let $(a, t) \in \mathcal{M}_0$ and let (p, g_0) be the simple scheme generating (a, t). Define p^* and (a^*, t^*) as in Lemma 6. By Lemma 6, $(a^*, t^*) \in \mathcal{M}_0$. By Lemma 7, $(a^*, t^*) \succsim (a, t)$.

Step 2. Write $\beta = \beta(p)$. We claim that

$$(n-1)g_0(z) + g_0(y) \leq y \text{ for all } y, z \in \mathbb{R}_+ \text{ such that } \beta \leq y \leq z. \quad (6)$$

To prove this claim, fix $y, z \in \mathbb{R}_+$ such that $\beta \leq y \leq z$. By definition of p^*, $p^*(y, ..., y) = \max(y, ..., y) = y$. Let $v = (y, ..., y, z)$.

If $a_i^*(v) = 0$ for all $i \in N \setminus n$, feasibility of (a^*, t^*) implies $(n-1)g_0(z) + g_0(y) - y = (n-1)g_0(z) + g_0(y) - p^*(y, ..., y) \leq \sum_{i \in N} t_i^*(v) \leq 0$, hence $(n-1)g_0(z) + g_0(y) \leq y$.

If $a_i^*(v) = 1$ for some $i \in N \backslash n$, then by conditional a-optimality of (a^*, t^*) we have $y = z$. Feasibility of (a^*, t^*) then implies $(n-1)g_0(y) + g_0(y) - p^*(y, ..., y) \leq 0$, hence again $(n-1)g_0(z) + g_0(y) \leq y$.
Step 3. Now let $\alpha' = (n-1) \sup \{g_0(y) \mid y \in \mathbb{R}_+\}$ and define

$$\begin{aligned} p'(x) &= \max(\max x, \alpha'), \\ g_0'(y) &= \text{med}\left(0, y - \alpha', \frac{\alpha'}{n-1}\right) \end{aligned}$$

for all $x \in \mathbb{R}_+^{n-1}$ and all $y \in \mathbb{R}_+$. Let (a', t') be the lexicographic mechanism generated by the simple scheme (p', g_0'). By Lemma 4, $(a', t') \in \mathcal{M}_0$. We claim that $p' \leq p^*$ and $g_0' \geq g_0$.

The first assertion is easy to check. Applying Lemma 5 (iii) to (p^*, g_0), we get $g_0(y) \leq \frac{\beta}{n-1}$ for all $y \in \mathbb{R}_+$. Therefore $\sup \{g_0(y) \mid y \in \mathbb{R}_+\} \leq \frac{\beta}{n-1}$, hence $\alpha' \leq \beta$. It follows that $p' \leq p^*$.

To prove the second assertion, note first that by definition of α',

$$g_0(y) \leq \frac{\alpha'}{n-1} \text{ for all } y \in \mathbb{R}_+. \tag{7}$$

By definition of p^*, $p^*(y, ..., y) > y$ for all $y < \beta$. Applying Lemma 5 (ii) to (p^*, g_0),

$$g_0(y) = 0 \text{ for all } y < \beta. \tag{8}$$

Finally, from (6), $g_0(y) \leq y - (n-1)g_0(z)$ whenever $\beta \leq y \leq z$

$$\begin{aligned} &\Rightarrow \quad g_0(y) \leq y - (n-1) \sup \{g_0(z) \mid z \geq y\} \text{ for all } y \geq \beta \\ &\Rightarrow \quad g_0(y) \leq y - (n-1) \sup \{g_0(z) \mid z \in \mathbb{R}_+\} \text{ for all } y \geq \beta, \end{aligned}$$

where the second implication holds because g_0 is nondecreasing (by Lemma 5 (iv)). Hence, by definition of α',

$$g_0(y) \leq y - \alpha' \text{ for all } y \geq \beta. \tag{9}$$

Combining (7), (8) and (9) yields $g_0 \leq g_0'$.

Since $p' \leq p^*$ and $g_0' \geq g_0$, Lemma 7 implies $(a', t') \succsim (a^*, t^*)$. Combining this with Step 1 yields $(a', t') \succsim (a, t)$.■

Proof of the Theorem.
Ad (i) Let (p, g_0) be a maxmed simple scheme as in Definition 1. Let (a, t) be the lexicographic mechanism generated by (p, g_0). By Lemma 4, $(a, t) \in \mathcal{M}_0$. We check that (a, t) is Pareto-optimal in $\mathcal{M}$.

Suppose not: then there exists $(a', t') \in \mathcal{M}$ such that $(a', t') \succ (a, t)$. Since $(a, t) \in \mathcal{M}_0$, we have $(a', t') \in \mathcal{M}_0$. By Lemma 8, there exists a maxmed mechanism $(a'', t'') \in \mathcal{M}_0$ such that $(a'', t'') \succsim (a', t')$. Therefore $(a'', t'') \succ (a, t)$.

But since both (a, t) and (a'', t'') are maxmed mechanisms, this is clearly impossible. Indeed, let (p'', g_0'') be the maxmed simple scheme generating (a'', t''), say,

$$
\begin{aligned}
p''(x) &= \max(\max x, \alpha''), \\
g_0''(y) &= \operatorname{med}\left(0, y-\alpha'', \frac{\alpha''}{n-1}\right)
\end{aligned}
$$

for all $x \in \mathbb{R}_+^{n-1}$ and all $y \in \mathbb{R}_+$.

If $\alpha = \alpha''$, then $(a'', t'') \sim (a, t)$, contradicting $(a'', t'') \succ (a, t)$.

If $\alpha < \alpha''$, pick $y, z \in \mathbb{R}_+$ such that $y < \alpha < \alpha'' < z$ and consider the valuation profile $v = (y, ..., y, z)$. Then $v_n a_n(v) + t_n(v) = z + g_0(y) - p(y, ..., y) = z - \alpha > z - \alpha'' = z + g_0''(y) - p''(y, ..., y) = v_n a_n''(v) + t_n''(v)$, contradicting $(a'', t'') \succ (a, t)$ again.

If $\alpha > \alpha''$, pick $y, z \in \mathbb{R}_+$ such that $\frac{n\alpha}{n-1} < y < z$ and consider the valuation profile $v = (y, ..., y, z)$. Then $v_1 a_1(v) + t_1(v) = g_0(z) = \frac{\alpha}{n-1} > \frac{\alpha''}{n-1} = g_0''(z) = v_1 a_1''(v) + t_1''(v)$, contradicting $(a'', t'') \succ (a, t)$ again.

Ad (ii). Let $(a, t) \in \mathcal{M}_0$ be Pareto-optimal in $\mathcal{M}$. By Lemma 8, there exists a maxmed mechanism $(a', t') \in \mathcal{M}_0$ such that $(a', t') \succsim (a, t)$. Since (a, t) is Pareto-optimal, we must have $(a', t') \sim (a, t)$. This proves that (a, t) is Pareto-equivalent to a maxmed mechanism.

To complete the proof, we check that (a, t) is a maxmed mechanism. To see this, let (p, g_0) be the simple scheme generating (a, t) and let (p', g_0') be the maxmed simple scheme generating (a', t'). We claim that $(p, g_0) = (p', g_0')$.

We first prove that $g_0 = g_0'$. Suppose there exists $y \in \mathbb{R}_+$ such that $g_0(y) > g_0'(y)$. Let $v = (y, ..., y)$. Using anonymity of $(a, t), (a', t')$, $v_i a_i(v) + t_i(v) = g_0(y) > g_0'(y) = v_i a_i'(v) + t_i'(v)$ for all $i \in N$, contradicting $(a, t) \sim (a', t')$. A similar contradiction arises if there exists $y \in \mathbb{R}_+$ such that $g_0(y) < g_0'(y)$.

Next we prove that $p = p'$. Suppose there exists $x = (x_1, ..., x_{n-1}) \in \mathbb{R}_+^{n-1}$ such that $p(x) > p'(x)$. Pick $y > p(x)$ and let $v = (x_1, ..., x_{n-1}, y)$. Then $v_n a_n(v) + t_n(v) = y + g_0(\max x) - p(x) < y + g_0'(\max x) - p'(x) = v_n a_n'(v) + t_n'(v)$, contradicting $(a, t) \sim (a', t')$. A similar contradiction arises if there exists $x \in \mathbb{R}_+^{n-1}$ such that $p(x) < p'(x)$.■

5 Discussion

No-envy is essential to our characterization. The maxmed mechanisms are generally not Pareto-optimal within the larger class of feasible, strategyproof and anonymous mechanisms. For instance, the Vickrey mechanism is Pareto-dominated by the mechanism described in Cavallo (2006) and previously studied in different contexts by Bayley (1997) and Porter, Soham and Tennenholtz (2004).

No-envy is redundant when $n = 2$. In that case, Athanasiou (2011) shows that the maxmed mechanisms are Pareto-optimal among the feasible, strategy-proof and anonymous mechanisms and our theorem implies that no other individually rational mechanism is Pareto-optimal in that class.

The role of anonymity is unclear. To be sure, there exist feasible, strategy-proof, envy-free mechanisms which are not anonymous. For a two-agent example, let $0 \leq \alpha_1 < \alpha_2$ and let

$$(a_1(v), t_1(v)) = \begin{cases} (1, \text{med}(0, v_2 - \alpha_2, \alpha_1) - \max(\alpha_1, v_2)) \text{ if } v_1 \geq \max(\alpha_1, v_2), \\ (0, \text{med}(0, v_2 - \alpha_2, \alpha_1)) \text{ otherwise,} \end{cases}$$

$$(a_2(v), t_2(v)) = \begin{cases} (1, \text{med}(0, v_1 - \alpha_2, \alpha_1) - \max(\alpha_2, v_1)) \text{ if } v_2 > \max(\alpha_2, v_1), \\ (0, \text{med}(0, v_1 - \alpha_2, \alpha_1)) \text{ otherwise.} \end{cases}$$

We do not know whether the maxmed mechanisms belong to the Pareto frontier of the class of feasible, strategy-proof and envy-free mechanisms. We also do not know whether that frontier contains nonanonymous mechanisms.

6 References

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Figure 1: a maxmed simple scheme for n=3