Université de Montreal

Characterization of the unfolding of a weak focus and modulus of analytic classification

Waldo Gonzalo Arriagada Silva

Département de mathématiques et statistique

Faculté des arts et des sciences

Thèse presentée à la Faculté des Études Supérieures (FES)

en vue de l'obtention du grade de PhilosophæDoctor (Ph.D.)

en mathématiques appliquées

Juillet 2010

Université de Montréal
Faculté des Études Supérieures

Cette thèse intitulée :

Characterization of the unfolding of a weak focus and modulus of analytic classification

présentée par :

Waldo Gonzalo Arriagada Silva,

à été évaluée par un jury composé des personnes suivantes :

Marlène Frigon,	Président-rapporteur
Christiane Rousseau,	Directrice de recherche
Iosif Polterovich,	Membre du jury
Sergei Yakovenko,	Examinateur externe
Ives Lepine,	Représentant du doyen de la FES.

Thèse accepté en Octobre 2010.

Résumé

La thèse présente une description géométrique d'un germe de famille générique déployant un champ de vecteurs réel analytique avec un foyer faible à l'origine et son complexifié : le feuilletage holomorphe singulier associé. On montre que deux germes de telles familles sont orbitalement analytiquement équivalents si et seulement si les germes de familles de difféomorphismes déployant la complexification de leurs fonctions de retour de Poincaré sont conjuguées par une conjugaison analytique réelle. Le "caractère réel" de la famille correspond à sa $\mathbb{Z}_2$-équivariance dans $\mathbb{R}^4$, et cela s'exprime comme l'invariance du plan réel sous le flot du système laquelle, à son tour, entraîne que l'expansion asymptotique de la fonction de Poincaré est réelle quand le paramètre est réel. Le pullback du plan réel après éclatement par la projection monoidal standard intersecte le feuilletage en une bande de Möbius réelle. La technique d'éclatement des singularités permet aussi de donner une réponse à la question de la "réalisation" d'un germe de famille déployant un germe de difféomorphisme avec un point fixe de multiplicateur égal à -1 et de codimension un comme application de semi-monodromie d'une famille générique déployant un foyer faible d'ordre un. Afin d'étudier l'espace des orbites de l'application de Poincaré, nous utilisons le point de vue de Glutsyuk, puisque la dynamique est linéarisable auprès des points singuliers : pour les valeurs réels du paramètre, notre démarche, classique, utilise une méthode géométrique, soit un changement de coordonée (coordonée “déroulante”) dans lequel la dynamique devient beaucoup plus simple. Mais le prix à payer est que la géométrie locale du plan complexe ambiante devient une surface de Riemann, sur laquelle deux notions de translation sont définies. Après avoir pris le quotient par le relèvement de la dynamique nous obtenons l’espace des orbites, ce qui s’avère être l’union de trois tores complexes plus les points singuliers (l’espace résultant est non-Hausdorff). Les translations, le caractère réel de l’application de Poincaré et le fait que cette application est un carré relient les différentes composantes du “module de Glutsyuk”. Cette propriété implique donc le fait qu’une seule composante de l’invariant Glutsyuk est indépendante.

Mots clés: Feuilletage, application de Poincaré, éclatement, réalisation, équivalence, conjugaison, classification, module.

Abstract

The thesis gives a geometric description for the germ of the singular holomorphic foliation associated with the complexification of a germ of generic analytic family unfolding a real analytic vector field with a weak focus at the origin. We show that two such germs of families are orbitally analytically equivalent if and only if the germs of families of diffeomorphisms unfolding the complexified Poincaré map of the singularities are conjugate by a real analytic conjugacy. The $\mathbb{Z}_2$-equivariance of the family of real vector fields in $\mathbb{R}^4$ is called the "real character" of the system. It is expressed by the invariance of the real plane under the flow of the system which, in turn, carries the real asymptotic expansion of the Poincaré map when the parameter is real. After blowing up the singularity, the pullback of the real plane by the standard monoidal map intersects the foliation in a real Möbius strip. The blow up technique allows to "realize" a germ of generic family unfolding a germ of diffeomorphism of codimension one and multiplier -1 at the origin as the semi-monodromy of a generic family unfolding an order one weak focus. In order to study the orbit space of the Poincaré map, we perform a trade-off between geometry and dynamics under the Glutsyuk point of view (where the dynamics is linearizable near the singular points): in the resulting "unwrapping coordinate" the dynamics becomes much simpler, but the price we pay is that the local geometry of the ambient complex plane turns into a much more involved Riemann surface. Over the latter, two notions of translations are defined. After taking the quotient by the lifted dynamics we get the orbit space, which turns out to be the union of three complex tori and the singular points (this space is non-Hausdorff). The Glutsyuk invariant is then defined over annular-like regions on the tori. The translations, the real character and the fact that the Poincaré map is the square of the semi-monodromy map, relate the different components of the Glutsyuk modulus. That property yields only one independent component of the Glutsyuk invariant.

Keywords: Foliations, Poincaré, blow-up, realization, equivalence, conjugacy, classification, modulus.

Notations

(z,w) : Natural ambient coordinates of the affine space $\mathbb{C}^2$

$\mathbb{M}$: Complex Möbius strip or blow up manifold

Σ_μ : Complex surface $\{z = \mu w : \mu \in \mathbb{C}^*\}$

$\Sigma^+ : \{z = \overline{w}\} \simeq \mathbb{R}^2$: Plane of symmetry or real plane embedded in $\mathbb{C}^2$

${}^{\mathbb{R}}\mathbb{M}$: Real Möbius strip or blow up in real coordinates

$\mathscr{C}(w) = \overline{w}$: Standard complex conjugation in one complex variable

$\mathscr{C}(W) = \overline{W}$: Standard complex conjugation in the Fatou coordinate

$\mathsf{C}(Z) = \overline{Z}$: Non-standard complex conjugation on the surface $\mathscr{R}_\varepsilon$

$\mathscr{S}(z,w) = (\bar{z}, \overline{w})$: Standard conjugation in two complex variables

$\sigma(z,w) = (w,z)$: Standard shift of two complex variables

$\mathscr{H}(\mathbf{w}) = \dfrac{1}{\mathbf{w}}$: Inversion map in the sphere coordinate $\mathbf{w}$

$T_\beta(Z) = Z + \beta$: Translation in $\beta \in \mathscr{R}_\varepsilon$ in the unwrapping coordinate

$\mathscr{T}_\beta(W) = W + \beta$: Translation in $\beta \in \mathbb{C}$ in the Fatou coordinate

$\mathscr{L}_\beta(w) = \beta w$: Linear map $\mathscr{L}_\beta$

$\mathscr{E}(W) = e^{-2i\pi W}$: Exponential map $\mathscr{E}$

$\mu_0(\varepsilon), \mu_\pm(\varepsilon)$: Multipliers of the field $\dfrac{2\pi w(\varepsilon \pm w^2)}{1 + A(\varepsilon) w^2} \dfrac{\partial}{\partial w}$

$\alpha(\varepsilon) = -\dfrac{i}{2\varepsilon}$: Distance between the holes for $\varepsilon \neq 0$

$\alpha_0(\varepsilon) = \dfrac{2\pi i}{\mu_0(\varepsilon)}$: Translation number around zero

$\alpha_\infty(\varepsilon) = \dfrac{2\pi i}{\mu_\pm(\varepsilon)}$: Translation number around infinity

Acknowledgments

Je veux surtout remercier Claudia et les enfants, Paloma, Gonzalo et Sergio. Aussi je remercie Professeure Christiane Rousseau pour son support financier et sa direction de thèse, et l'ISM et la FESP pour les bourses accordées durant la période de thèse. Enfin, je remercie tous mes camarades d'université.

I am indebted to Claudia, Paloma, Gonzalo and Sergio. I would like to thank Professor Christiane Rousseau for accepting the supervision of this thesis and for the support granted during all these four years. Thanks to ISM and FESP as well, for the scolarships granted during all the period the thesis has been conceived and written. Thanks to everybody.

Preface

The thesis is part of a program that aims to study the holomorphic classification of generic unfoldings of the simplest codimension-one singularities of analytic dynamical systems, the latter being given by germs of diffeomorphisms (in which case we study classification under conjugacy) or germs of vector fields (in which case we can study either classification under orbital equivalence or under conjugacy). The moduli space for these singularities has been described by J. Ecalle, S. M. Voronin (cf. [14],[51]) and J. Martinet-J. P. Ramis (cf. [35],[36]). In spite of the "simple" shape of these germs, except for the case of the node of a planar vector field, the moduli space is a huge functional space, while, on the other hand, the formal invariants are in finite number. This means that there is an infinite number of analytic obstructions for the analytic equivalence of two germs, that cannot be seen at the formal level. The former idea of V. I. Arnold *et al.* is that the modulus associated with the singularity can be explained by first, complexifying the underlying space and then, by unfolding the singularity. Thus, the singularity of the dynamical system comes from the coallescence in a generic unfolding of a finite number of hyperbolic singularities or special hyperbolic "objects" (like a periodic orbit or a limit cycle). Each hyperbolic object has its own geometric local model, and the modulus measures the mismatch of these local models in the limit. It is also a measure of the divergence of the normalizing series to the formal normal form.

So far, and still in the case of codimension-one, this program involves notably the case of the germ of the generic unfolding of a diffeormorphism with a double fixed point, also called *parabolic diffeomorphism* (cf. [11] and [32]), the case of a germ of generic unfolding of a resonant diffeomorphism (one multiplier being a root of unity, cf. [10] and [43]), the study of germs of generic unfoldings of saddle-node (resp. resonant saddle) singularities of planar vector fields (cf. [44], resp. [10]), and the case of the generic unfolding of a saddle point of a real vector field (cf. [43]). The modulus of the unfolding is always constructed in the same way. The formal normal form for the unfolding is identified and called the *model family*. The germ of family is then compared to the formal normal form on special domains. When one restricts to parameter values for which the special objects are hyperbolic, then these domains are neighborhoods of the special objects. For parameter values for which these neighborhoods intersect, the modulus is given by the comparison of the two normalizations over the intersection of the two domains. This is what is called the *Glutsyuk point of view* and the corresponding modulus is called the *Glutsyuk modulus*. This was the point of view suggested by V. I. Arnold and J. Martinet [33] and studied systematically by A. A. Glutsyuk [21] when the unfolding is considered only in certain conic regions of the (complex) parameter space. However, in the codimension-one case, this point of view does not allow to cover a full neighborhood of the origin in the parameter space and another approach is required in order to cover the remaining directions, where the dynamics is non-hyperbolic. This approach comes from A. Douady and his school and has been formerly introduced in the thesis of P. Lavaurs ([28]). This point of view is known as the *Lavaurs point of view* and it allows to give a modulus of analytic classification for all values of the parameters (the modulus being a ramified family in the parameter).

The case of the weak focus.

The thesis addresses the problem of the orbital analytic classification of a generic family of planar vector fields unfolding a weak focus of order one, in the Glutsyuk point of view. A weak focus of a real 2-dimensional analytic vector field is defined as a singular point of the vector field with a pair of pure imaginary eigenvalues and which is not a centre. When the codimension is one, it corresponds to the generic coalescence of a focus with a limit cycle. A germ of generic analytic family of vector fields unfolding a weak focus is then the germ of a holomorphic family with a generic Hopf bifurcation.

Some germs of singular holomorphic (complex) foliations are uniquely defined, up to equivalence, by the germ of a holomorphic application called the holonomy map: two foliations with holomorphically conjugate holonomy germs are holomorphically equivalent themselves. This is the case for simple singularities, for instance when the singularity is a nondegenerate 2-dimensional saddle (having negative rational ratio of the eigenvalues of the linearization, cf. [37]). Moreover, it was proved (cf. [42]) that in this case any germ of diffeomorphism with a fixed point can be "realized" as the holonomy of a suitable saddle foliation, and that in presence of extra parameters analytically deforming the singularity, the correspondence between holomorphic types of the foliation and its holonomy remains holomorphic (cf. [43]). As the generic family unfolding the weak focus corresponds to a particular generic family unfolding a resonant saddle point with ratio of eigenvalues equal to -1, the foliation of this family is characterized by the dynamics of the holonomy of one separatrix. However, for generic unfoldings of elliptic singularities of weak focus or centre type (with pure imaginary eigenvalues) the germ of a real self-map $\mathscr{P} : (\mathbb{R}^+,0) \to (\mathbb{R}^+,0)$, called the Poincaré monodromy (or Poincaré first return map), is well defined and one is naturally led to the question whether the germ of the monodromy defines the analytic conjugacy class of the real foliation. The natural way to answer this question is via complexification, as suggested by M. Berthier, D. Cerveau and A. Lins Neto in the 90's (cf. [2]). The complexified family of elliptic vector fields inherits a global property of symmetry under the complex conjugation. Such a symmetry is called the "real character" of the family. This property means that the real plane is invariant under the flow of the system when the parameter of the unfolding is real, and it also reflects the fact that the eigenvalues at the origin are complex conjugate.

The complexification of the ambient spaces allows to understand why the holonomy map is not an ideal candidate to describe the foliation in the case of elliptic singularities: the complex separatrices are not detected in the real plane and, consequently, the germ of the holonomy map is not real. This motivates a different approach. Indeed, the Poincaré map, which can be represented as the second iterate of the holonomy of the exceptional divisor of the complex foliation after standard desingularization or blow-up, keeps track of the fact that the system has been complexified and then it stands as a "bridge" between the real and complex foliations. Through this observation, we obtain a first result after blowing-up the singularity: the equivalence problem for the generic holomorphic family of vector fields unfolding the weak focus can be reduced to the conjugacy problem for the associated holomorphic (complex) family unfolding the Poincaré monodromy. More precisely, we show that two germs of generic families of analytic vector fields unfolding a vector field with a weak focus at the origin of coordinates are analytically orbitally equivalent, if and only if the families unfolding their Poincaré monodromies are analytically conjugate by a real conjugacy (see Theorem 4.2.3). This provides a "real" characterization of the foliation.

Once the foliation has been characterized by the Poincaré monodromy, we proceed to solve the "inverse problem", also known as the "realization problem", by means of quasiconformal surgery, as suggested by Y. Ilyashenko and his school. This problem can be explained as follows. Being given an admissible generic representative $\mathscr{Q}_\varepsilon$ of the class of real germs of families of holomorphic diffeomorphisms (with coefficients depending analytically on the parameter) unfolding a codimension-one resonant diffeomorphism with multiplier equal to -1 at the origin, there exists a germ of family of real holomorphic elliptic foliations whose Poincaré monodromy coincides with the squared map $\mathscr{Q}_\varepsilon^{\circ 2}$, see Theorem 6.1.4. We first construct a family of abstract complex 2-manifolds endowed with a family of integrable almost complex structures (ACS) and a family of C^∞-foliations whose Poincaré monodromy coincides with the prescribed family of diffeomorphisms. Then the Newlander-Nirenberg Theorem (cf. [40]) gives us a family of C^∞-coordinates that straightens the family of ACS and, consequently, the family of C^∞-foliations into a family of holomorphic foliations. We prove that the smooth coordinates system is real and then it respects the real character of the family unfolding the weak focus.

Both the characterization of the foliation by the Poincaré monodromy and the realization problem fill a gap in the literature. Everybody believed this to be true but, because the proof was substantial, it was never accurately written. We expect our results to be useful and cited in the future.

We will now describe the second part of the thesis, which is concerned with the analytic classification of families of diffeomorphisms. Indeed, both the “real” characterization of the foliation by the Poincaré monodromy and the realization problem imply that, in order to compute the modulus of analytic classification associated with the generic family unfolding the weak focus, it suffices to describe the modulus of analytic classification of the real holomorphic family unfolding the Poincaré monodromy of the foliation. In this second part our result is partial: we could only give a modulus of analytic classification (in the Glutsyuk point of view) under orbital analytic “weak equivalence”. Roughly speaking, two generic analytic families unfolding weak foci are analytically weakly equivalent if there exists an analytical change of coordinates, which depends analytically on the parameter ε at $\varepsilon \neq 0$ and continuously at $\varepsilon = 0$, that sends orbits of the first system into orbits of the second system with no regard on the parametrization. In addition, the domain of the (complex) parameter consists of the union of two sectors which does not cover a whole neighborhood of the origin $\varepsilon = 0$, but which does contain the real line instead, see Figure 7.1.

This gives the modulus of analytic classification under analytic weak conjugacy (in the Glutsyuk point of view) for germs of generic holomorphic families undergoing a generic codimension-one Hopf bifurcation, by means of the Glutsyuk modulus associated with the germ of the family unfolding the Poincaré map of the weak focus. We shall come over the study of the modulus under analytic orbital equivalence in near future.

Symmetries and the invariant of analytic classification.

The thesis provides also a contribution in the area of symmetries in dynamical systems. Indeed, the real character of the family of vector fields yields a symmetry on the modulus of analytic classification, with respect to complex conjugation (Schwarz reflection). In order to introduce the Glutsyuk invariant we need to define "intrinsic" coordinates (unique up to linear changes of coordinates) in the orbit space of the Poincaré monodromy. The orbit space is the quotient of fundamental domains by the Poincaré map. It turns out to be non-Hausdorff and composed of the union of three complex tori plus the fixed points of the Poincaré map. The Glutsyuk invariant is defined to be the equivalence class of pairs of analytic diffeomorphisms $(\psi^G_{1,\varepsilon}, \psi^G_{2,\varepsilon})$ defined on annular-like regions of the orbit space endowed with almost intrinsic coordinates, modulo composition by real linear changes of coordinates in the source and target spaces, see Figure 7.2. These analytic diffeomorphisms identify the points in different tori corresponding to the same orbit in those intrinsic coordinates. We prove that special symmetries arise out of the real character of the germ of the family unfolding the weak focus, and that there are two different ways of keeping track of the real underlying framework of the foliation, see Theorem 9.1.3 and Theorem 9.1.7, respectively, for further details.

The thesis is organized in three parts and it contains ten chapters and three appendices. Chapter 1 contains a description of the codimension-one Hopf bifurcation. In Chapter 2 we recall the desingularization technique and Chapter 3 describes the main feature of the family of vector fields: its real character, which is the birth point of the forthcoming theory. Chapter 4 contains the proof of the characterization of the germ of the foliation through the Poincaré map. In chapters 5 and 6 we study integrability and the conditions for realization of the foliation (the inverse problem). In Chapters 7, 8 and 9 we study the orbit space, the symmetries on the modulus and the invariant of weak orbital classification.

Contents

List of Figures

Part I

Characterization of the foliation by the Poincaré map

Chapter 1

The Hopf bifurcation

1.1 Normal form.

Definition 1.1.1. *[27] A singular point of a planar vector field is elliptic, if the eigenvalues of its linearization are non-real complex conjugate (in particular, nonzero).*

By this definition, if the two eigenvalues are imaginary (with zero real part) a linear elliptic singularity can be a center (all orbits are periodic in a neighborhood of the singular point) or a weak focus, and if the eigenvalues have real part different to zero it corresponds to a strong focus. In the latter case, the speed of convergence of orbits to the singular point is of logarithmic order. A weak focus is a singular point which can naturally be embedded in a family unfolding a Hopf bifurcation.

Consider the system

$$\dot{\vec{X}} = F_\varepsilon(\vec{X}) \tag{1.1.1}$$

where $\vec{X} = (x,y)^T$ and $F_\varepsilon(x,y) = F(x,y,\varepsilon)$ is a real analytic local family of planar vector fields defined in a small neighborhood of the origin on the real plane $\mathbb{R}^2$ and depending analytically on a real parameter ε. Suppose that for $\varepsilon = 0$ the singular point is elliptic. This assumption immediately implies that the singular point itself depends analytically on the parameter (by the Implicit Function Theorem). Moreover, the local coordinates (x,y) can be chosen so that the linear part of F_ε has the form

$$(\alpha(\varepsilon)x - \beta(\varepsilon)y)\frac{\partial}{\partial x} + (\alpha(\varepsilon)y + \beta(\varepsilon)x)\frac{\partial}{\partial y}$$

with real analytic coefficients (germs) $\alpha(\varepsilon)$ and $\beta(\varepsilon)$. We require also:

$$\alpha(0) = 0, \tag{1.1.2}$$

$$\beta(0) := \beta > 0, \tag{1.1.3}$$

(the ellipticity assumption means that the real analytic function $\beta(\varepsilon)$ is non-vanishing) and that the family F_ε has the equilibrium $\vec{X} = (0,0)$.

By introducing a complex variable $z = x + iy$, system (1.1.1) can be written for sufficiently small $|\,\varepsilon\,|$ as a single equation:

$$\dot{z} = \lambda(\varepsilon)z + \sum_{j+k\geq 2} b_{jk}(\varepsilon)z^j\bar{z}^k, \tag{1.1.4}$$

with $\lambda(\varepsilon) = \alpha(\varepsilon) + i\beta(\varepsilon)$ and where the coefficients $b_{jk}(\varepsilon)$ depend analytically on the parameter.

Theorem 1.1.2 (Poincaré normal form). *[29] The equation* (1.1.4) *where* $\lambda(\varepsilon) = \alpha(\varepsilon) + i\beta(\varepsilon)$, $\alpha(0) = 0$ *and* $\beta(0) \equiv \beta_0 > 0$, *can be transformed by an invertible parameter-dependent change of complex coordinate, analytically depending on* *the parameter, for all sufficiently small* $|\,\varepsilon\,|$*, into an equation with only the resonant cubic term, up to order 4:*

$$\dot{z} = \lambda(\varepsilon)z + c_1 z^2 \bar{z} + O(|z|^4), \tag{1.1.5}$$

where $c_1 = c_1(\varepsilon)$.

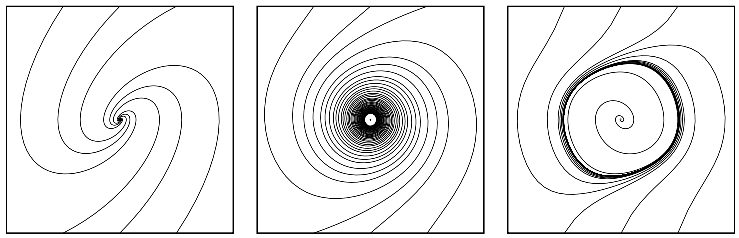
Bifurcations of the phase portrait of the system (1.1.5) as $\alpha(\varepsilon)$ passes through zero can easily be analyzed using the polar form when $\alpha'(0) \neq 0$. The system always has an equilibrium at the origin. This equilibrium is always a stable focus for $\alpha(\varepsilon) < 0$ and an unstable focus for $\alpha(\varepsilon) > 0$.

Definition 1.1.3.

1. *At the critical parameter value* $\alpha(\varepsilon) = 0$ *the equilibrium is topologically equivalent to the focus or a centre. In the case of a focus, such an equilibrium is called a weak focus.*
2. *The weak focus is of codimension (order) 1 if* $Re(c_1(0)) \neq 0$.

Suppose $\alpha'(\varepsilon) > 0$ in a neighborhood of $\varepsilon = 0$ in the parameter space, and $Re(c_1(0)) < 0$. As said before, the origin is a stable focus when $\alpha(\varepsilon) < 0$. In the limit $\alpha(\varepsilon) = 0$ the weak focus remains stable and as the parameter moves to positive real values, it gives rise to an (unstable) equilibrium $\vec{X} = (0,0)$ which is surrounded by an isolated stable closed orbit (or stable limit cycle, see Figure 1.1c). All orbits starting outside or inside the cycle, except for the origin, tend to the cycle as $t \to +\infty$, see Figure 1.1.

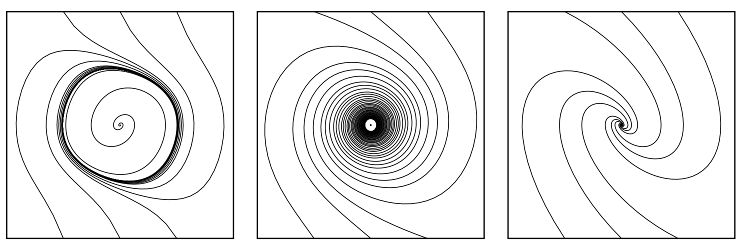
On the contrary, if $Re(c_1(0)) > 0$, the origin is stable and an unstable limit cycle is created for negative values of the parameter. As ε passes through the bifurcation value $\varepsilon = 0$ the closed orbit disappears and the origin loses its stability



(a) $\alpha(\varepsilon) < 0$ (b) $\alpha(\varepsilon) = 0$ (c) $\alpha(\varepsilon) > 0$

Figure 1.1: The supercritical Hopf bifurcation.

and becomes an unstable weak focus which gives rise to an unstable focus on $\alpha(\varepsilon) > 0$, see Figure 1.2.



(a) $\alpha(\varepsilon) < 0$ (b) $\alpha(\varepsilon) = 0$ (c) $\alpha(\varepsilon) > 0$

Figure 1.2: The subcritical Hopf bifurcation.

Definition 1.1.4. *The local bifurcation of the phase portrait the Equation* (1.1.5) *taking place at the value* $\varepsilon = 0$ *under the condition* $Re(c_1(0)) \neq 0$, *is called the codimension one Poincaré-Andronov-Hopf-Takens bifurcation (or simply, the codimension one Hopf bifurcation). It is said to be "supercritical" if* $Re(c_1(0)) < 0$, *and "subcritical" if* $Re(c_1(0)) > 0$ *when* $\alpha'(\varepsilon) > 0$.

1.2 Orbital preparation.

Proposition 1.2.1. *Suppose, in addition, that* $\alpha'(0) \neq 0$ *and* $Re(c_1(0)) \neq 0$ *in equation* (1.1.5) *above. Then, the equation can be transformed by a parameter-dependent linear coordinate transformation and a nonlinear time reparametrization into an equation of the form*

$$\dot{z} = (\varepsilon + i)z + sz|z|^2 + O(|z|^4), \tag{1.2.1}$$

where $s = \text{SIGN}(Re(c_1(0))) = \pm 1$.

Proof. Put $\widehat{\tau} = \beta(\varepsilon)t$. The time direction is preserved since $\beta(\varepsilon) > 0$ for small $|\,\varepsilon\,|$ (because of hypothesis (1.1.3)). Then

$$\frac{dz}{d\widehat{\tau}} = (\widetilde{\varepsilon} + i)z + d_1(\widetilde{\varepsilon})z|z|^2 + O(|z|^4),$$

where $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\varepsilon) = \dfrac{\alpha(\varepsilon)}{\beta(\varepsilon)}$ and $d_1(\widetilde{\varepsilon}) = \dfrac{c_1(\varepsilon(\widetilde{\varepsilon}))}{\beta(\varepsilon(\widetilde{\varepsilon}))}$. We can consider $\widetilde{\varepsilon}$ as a new parameter because $\widetilde{\varepsilon}(0) = 0$ and $\widetilde{\varepsilon}'(0) = \dfrac{\alpha'(0)}{\beta(0)} \neq 0$, and then the Inverse Function Theorem guarantees that local existence and analyticity of ε as a function of $\widetilde{\varepsilon}$.

Change the time parametrization along the orbits by introducing a new time $\tau = \tau(\widehat{\tau}, \widetilde{\varepsilon})$, where

$$d\tau = (1 + e_1(\widetilde{\varepsilon})|z|^2)d\widehat{\tau}$$

with $e_1(\widetilde{\varepsilon}) = Im(d_1(\widetilde{\varepsilon}))$. The time change is a near-identity transformation in a small neighborhood of the origin. Using the new definition of the time we obtain

$$\frac{dz}{d\tau} = (\widetilde{\varepsilon} + i)z + \ell_1(\widetilde{\varepsilon})z|z|^2 + O(|z|^4),$$

where $\ell_1(\widetilde{\varepsilon}) = Re(d_1(\widetilde{\varepsilon})) - \widetilde{\varepsilon}e_1(\widetilde{\varepsilon})$ is real and

$$\ell_1(0) = \frac{Re(c_1(0))}{\beta(0)}. \tag{1.2.2}$$

Finally, introduce a new complex variable $\mathbf{z}$:

$$z = \frac{\mathbf{z}}{\sqrt{|\ell_1(\widetilde{\varepsilon})|}},$$

which is possible due to $Re(c_1(0)) \neq 0$ and, thus, $\ell_1(0) \neq 0$. The equation then takes the required form:

$$\begin{aligned} \frac{d\mathbf{z}}{d\tau} &= (\widetilde{\varepsilon}+i)\mathbf{z} + \frac{\ell_1(\widetilde{\varepsilon})}{|\ell_1(\widetilde{\varepsilon})|}\mathbf{z}|\mathbf{z}|^2 + O(|\mathbf{z}|^4) \\ &= (\widetilde{\varepsilon}+i)\mathbf{z} + s\mathbf{z}|\mathbf{z}|^2 + O(|\mathbf{z}|^4), \end{aligned} \tag{1.2.3}$$

with $s = \text{SIGN}(\ell_1(0)) = \text{SIGN}(Re(c_1(0))) = \pm 1$. □

Definition 1.2.2.

1. *The real function $\ell = \ell_1(\widetilde{\varepsilon})$ is called the first Lyapounov coefficient.*

2. *A one-parameter planar family of differential equations of the form* (1.1.4)*, exhibiting a codimension one Hopf bifurcation at the parameter value $\varepsilon = 0$ is called "generic" if:*

   – *The pair of complex-conjugate eigenvalues $\lambda(\varepsilon) = \alpha(\varepsilon) \pm i\beta(\varepsilon)$ crosses the imaginary axis with non-zero speed:*

$$\frac{d}{d\varepsilon} Re\lambda(\varepsilon)\Big|_{\varepsilon=0} = \alpha'(0) \neq 0, \tag{1.2.4}$$

   – *The first resonant monomial is non null at the origin, which is equivalent to:*

$$\ell_1(0) \neq 0. \tag{1.2.5}$$

   Such requirements are called "genericity conditions" for the one-parameter family.

1.3 Embedding of the family.

The family (1.2.1) may be naturally embedded in $\mathbb{C}^2$. This is done by complexifying the real coordinates (x,y) of the planar system (1.1.1), and then writing the equations of such families in the new complex variables given by

$$\begin{aligned} z &= x+iy, \\ w &= x-iy. \end{aligned} \tag{1.3.1}$$

Definition 1.3.1. *The variables (z,w) thus defined will be called the ambient coordinates.*

Notice that in the particular case $x,y\in\mathbb{R}$ the ambient coordinates are related through $z=\overline{w}$.

1.4 The orbital form.

The embedding of (1.2.1) into $\mathbb{C}^2$ by ambient coordinates is given by

$$\begin{aligned} \dot{z} &= (\varepsilon+i)z+sz^2w+\sum_{j+k\geq 4} a_{jk}(\varepsilon)z^jw^k \\ \dot{w} &= (\varepsilon-i)w+sw^2z+\sum_{j+k\geq 4} \overline{a_{jk}(\overline{\varepsilon})}z^kw^j, \end{aligned} \tag{1.4.1}$$

where the coefficients $a_{jk}(\varepsilon)$ depend analytically on the parameter. Thus, restricting (x,y) to $\mathbb{R}^2$ allows to recover the system (1.2.1). The value $s=\pm 1$ is the cubic coefficient.

Definition 1.4.1. *The family of vector fields defined by the right side of the system* (1.4.1) *will be henceforth noted v_ε. Such a family is called the "orbital form" of the family F_ε (Equation* (1.1.1)*). It is said to be "generic" if the two genericity conditions* (1.2.4) *and* (1.2.5) *are satisfied by its restriction* (1.2.1).

Proposition 1.4.2 (Formal classification Theorem for order one weak foci). *A germ of generic family of differential equations unfolding a weak focus v_0 of order one, is formally orbitally equivalent to:*

$$\begin{aligned}\dot{z} &= z\frac{i+\varepsilon\pm u}{1+A(\varepsilon)u}\\ \dot{w} &= w\frac{-i+\varepsilon\pm u}{1+A(\varepsilon)u}\end{aligned} \tag{1.4.2}$$

with $u = zw$ and for some family of constants $A(\varepsilon)$ which is real on $\varepsilon \in \mathbb{R}$ and $A(0) \neq 0$.

Proof. Consider only the sign $+$ in (1.2.1). By a formal change of coordinates we bring the system to the form:

$$\begin{aligned}\dot{z} &= z(i+\varepsilon+\sum_{j\geq 1}\widetilde{A}_j(\varepsilon)u^j) := P(z,w)\\ \dot{w} &= w(-i+\varepsilon+\sum_{j\geq 1}\overline{\widetilde{A}_j(\overline{\varepsilon})}u^j) := Q(z,w)\end{aligned} \tag{1.4.3}$$

where $Re(\widetilde{A}_1) \neq 0$. In order to simplify the form, we iteratively use changes of coordinates $(z,w) = (\mathbf{z}(1+cU^n), \mathbf{w}(1+\overline{c}U^n))$ for $n \geq 1$. Such a change allows to get rid of the term $\widetilde{A}_{n+1}U^{n+1}$ provided that $n+1 > 2$. When $n = 1$ it allows to get rid of $iIm(\widetilde{A}_2U^2)$. Indeed, the constant c must be chosen so as to verify

$$\widetilde{A}_1(c+\overline{c}) - nc(\widetilde{A}_1+\overline{\widetilde{A}_1}) = \widetilde{A}_{n+1},$$

which is always solvable in c as soon as $Re(\widetilde{A}_1) \neq 0$ and $n > 1$. However, when $n = 1$ we get

$$\widetilde{A}_1(c+\overline{c}) - nc(\widetilde{A}_1+\overline{\widetilde{A}_1}) = \widetilde{A}_1\overline{c} - \overline{\widetilde{A}_1}c = 2iIm(\widetilde{A}_1\overline{c}) \in i\mathbb{R}.$$

Hence, in that case only the equation $\widetilde{A}_1(c+\overline{c}) - nc(\widetilde{A}_1 + \overline{\widetilde{A}_1}) = iIm(\widetilde{A}_{n+1})$ is solvable in c. Finally, one divides (1.4.3) by $\dfrac{wP - zQ}{2izw}$. This brings all the $Im(\widetilde{A}_j)$ to 0. Then we repeat the procedure above with c real to remove all higher terms in u^j except for the term in u^2. A scaling in u yields the formal normal form

$$\begin{aligned} \dot{z} &= z(i + (\varepsilon + u)(1 + \widetilde{A}(\varepsilon)u)) \\ \dot{w} &= w(-i + (\varepsilon + u)(1 + \widetilde{A}(\varepsilon)u)) \end{aligned} \tag{1.4.4}$$

which is formally equivalent to (1.4.2). □

Chapter 2

Möbius strip and Poincaré map

2.1 Algebraic blow up.

Consider the quasi-projective variety

$$\mathbb{M} = \{([t_2 : t_1], (z,w)) \in \mathbb{CP}^1 \times \mathbb{C}^2 : zt_1 - wt_2 = 0\} \tag{2.1.1}$$

where $[t_2 : t_1] \in \mathbb{CP}^1$ is the line at infinity passing through $(t_2, t_1) \in \mathbb{C}^2$ (the homogeneous coordinates on $\mathbb{CP}^1$). Projection onto the factor $\mathbb{C}^2$ induces a surjective morphism

$$\rho : \mathbb{M} \to \mathbb{C}^2$$

and

$$\rho^{-1}(z,w) = \begin{cases} \mathbb{CP}^1 \times \{0\} & \text{if } (z,w) = (0,0) \\ ([z:w],(z,w)) & \text{if } (z,w) \neq (0,0). \end{cases} \tag{2.1.2}$$

The fiber $S = \rho^{-1}(0,0)$ is a projective line, which is called the *exceptional line*. Away from the origin, ρ gives an isomorphism between $\mathbb{C}^2 \backslash \{0\}$ and $\mathbb{M} \backslash S$.

Definition 2.1.1. *[23] The map* $\rho : \mathbb{M} \to \mathbb{C}^2$ *between two* 2*-dimensional complex manifolds is called the (standard) monoidal map. The analytic curve* $S \subset \mathbb{M}$ *is referred to as the (standard) exceptional divisor. The pair* $(\rho, \mathbb{M})$ *is called the blow up of* $\mathbb{C}^2$ *at the origin.*

By construction the surface $\mathbb{M}$ is embedded in the complex 3-dimensional space $\mathbb{CP}^1 \times \mathbb{C}^2$ and carries the compact curve (Riemann sphere) $\mathbb{CP}^1 \times \{0\} = S$ on it, that is, the points of S correspond to the lines through the origin in $\mathbb{C}^2$, see Figure 2.1.

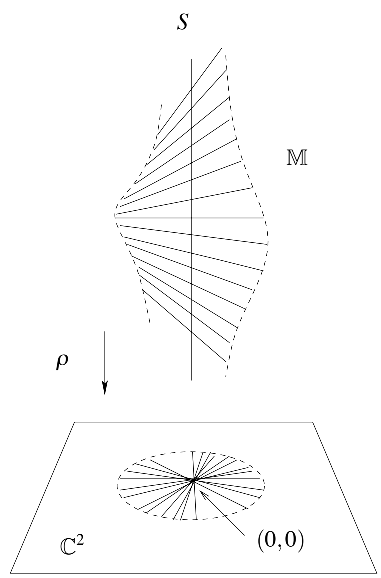


Figure 2.1: The blow up of the origin.

Remark 2.1.2. *[27] The real projective line* $\mathbb{RP}^1$ *is diffeomorphic to the circle* $\mathbb{S}^1$ *so the realified surface* ${}^{\mathbb{R}}\mathbb{M}$ *(obtained by restriction of complex coordinates to* $\mathbb{R}$*) is constructed as a submanifold of the cylinder* $\mathbb{S}^1 \times \mathbb{R}^2$*, see Remark 3.3.5 in the next chapter. This submanifold is homeomorphic to the Möbius strip. Having this* *analogy in mind, the blow up $\mathbb{M}$ is sometimes referred to as the complex Möbius strip, see Subsection 2.1.2 for further details.*

2.1.1 A complex atlas on the blow up.

The standard affine covering $\mathbb{CP}^1 = U_1 \cup U_2$ with $U_1 = \{[t_2 : t_1] \ : \ t_1 \neq 0\}$ and $U_2 = \{[t_2 : t_1] \ : \ t_2 \neq 0\}$, induces a covering

$$\mathbb{M} = V_1 \cup V_2, \quad V_1, V_2 \subset \mathbb{C}^2 \times \mathbb{C},$$

where

$$\begin{aligned} V_1 &= \{([t_2 : t_1], (z, w)) \in \mathbb{CP}^1 \times \mathbb{C}^2 \ : \ t_1 \neq 0, \ z - w\frac{t_2}{t_1} = 0\}, \\ V_2 &= \{([t_2 : t_1], (z, w)) \in \mathbb{CP}^1 \times \mathbb{C}^2 \ : \ t_2 \neq 0, \ z\frac{t_1}{t_2} - w = 0\}. \end{aligned} \tag{2.1.3}$$

Thus, using coordinates $(Z, w) = (\frac{t_2}{t_1}, w)$ for V_1, and $(W, z) = (\frac{t_1}{t_2}, z)$ for V_2, we see that V_1 and V_2 are both isomorphic to $\mathbb{C}^2$. The transition map between these charts is a monomial transformation

$$\begin{aligned} \varphi \ &: \ V_1 \to V_2 \\ &\varphi(Z, w) := (\frac{1}{Z}, Zw) \equiv (W, z) \end{aligned} \tag{2.1.4}$$

with inverse

$$\varphi^{-1}(W, z) = (\frac{1}{W}, Wz), \tag{2.1.5}$$

and so $\varphi^{\circ 2} = id$. Thus $\mathbb{M}$ is indeed a nonsingular 2-dimensional complex analytic manifold. It remains to observe that the map $\rho : \mathbb{M} \to \mathbb{C}^2$ in these charts is polynomial, hence globally holomorphic:

$$\rho\Big|_{V_1} = c_1, \quad \rho\Big|_{V_2} = c_2, \tag{2.1.6}$$

where

$$c_1 : (Z,w) \mapsto (Zw,w), \quad c_2 : (W,z) \mapsto (z,zW). \tag{2.1.7}$$

Definition 2.1.3.

1. *The chart (V_1, c_1) covering the complex Möbius strip will be referred to either as the first chart, as the first direction, or as the c_1 direction of the blow up space. The chart (V_2, c_2) will be called either as the second chart, the second direction or the c_2 direction the blow up space.*
2. *The blow up of a singular foliation $\mathscr{F}$ of $(\mathbb{C}^2, 0)$ is the singular holomorphic foliation $\widetilde{\mathscr{F}} = \rho^*\mathscr{F}$ extending the preimage foliation $\rho^{-1}(\mathscr{F})$ of $\mathbb{M}\backslash S$.*

One may have a priori two possibilities for the blown up foliation $\widetilde{\mathscr{F}}$: either the exceptional divisor S is a separatrix of $\widetilde{\mathscr{F}}$, or different points of S belong to different leaves of $\widetilde{\mathscr{F}}$. In the latter case leaves of this foliation cross S transversally at all points, eventually except for finitely many tangency points.

Definition 2.1.4. *[27] A singular point of a holomorphic foliation $\mathscr{F}$ on $(\mathbb{C}^2, 0)$ is called non-dicritical, if the exceptional divisor $S = \rho^{-1}(0)$ is a separatrix of the blow up foliation $\rho^*\mathscr{F}$ by the simple monoidal transformation ρ. Otherwise the singular point is called dicritical.*

Notice that the exceptional divisor in the charts (Z,w) and (W,z) is given by the equations

$$S \cap V_1 = \{z = 0\}, \quad S \cap V_2 = \{w = 0\}.$$

As the two sets are isomorphic to $\mathbb{C}$, they can be “realified”. This is what we do in the next paragraph.

2.1.2 The realified atlas.

The real projective line $\mathbb{RP}^1$ is a closed loop on the Riemann sphere $\mathbb{CP}^1$ which is visible as the real line $\mathbb{R}$ in the affine charts $S \cap V_1$ and $S \cap V_2$ (which are isomorphic to $\mathbb{C}$). The complex Möbius strip $\mathbb{M}$ intersects the real variety $\mathbb{RP}^1 \times \mathbb{R}^2 \hookrightarrow \mathbb{CP}^1 \times \mathbb{C}^2$ at $\{t_1 = \overline{t_2}\}$ and $\{z = \overline{w}\}$. The intersection is given by the real Möbius strip

$$^{\mathbb{R}}\mathbb{M} = \{([a:b],(z,w)) \in \mathbb{RP}^1 \times \mathbb{R}^2 \ \ : \ \ xb + ya = 0\}. \tag{2.1.8}$$

for $z = x + iy$ and $t_1 = a + ib$. Such a surface can be explicitly computed in real charts.

Theorem 2.1.5. *The covering $\mathbb{M} = V_1 \cup V_2$ yields a real covering of the real Möbius strip $^{\mathbb{R}}\mathbb{M} = ^{\mathbb{R}} V_1 \cup ^{\mathbb{R}} V_2$, where $^{\mathbb{R}}V_1$ is parametrized by two real coordinates (x,u), and $^{\mathbb{R}}V_2$ is parametrized by two real coordinates (v,y).*

Proof. The neighborhoods V_1, V_2 intersect the real variety $\mathbb{RP}^1 \times \mathbb{R}^2$ at $w = \bar{z}$ and $t_2 = \overline{t_1}$, for coordinates $z = x + iy$ and $t_1 = a + ib$ with $x, y, a, b \in \mathbb{R}$. Then:

$$\begin{aligned} V_1 \ \ &\cap \ \ (\{w = \bar{z}\} \times \{t_2 = \overline{t_1}\}) \\ &= \ \ \{([t_2 : t_1],(z,w)) \in \mathbb{CP}^1 \times \mathbb{C}^2 \ \ : \ \ t_1 \neq 0, \ \ zt_1 - \overline{zt_1} = 0\} \\ &= \ \ \{([t_2 : t_1],(z,w)) \in \mathbb{CP}^1 \times \mathbb{C}^2 \ \ : \ \ t_1 \neq 0, \ \ Im(zt_1) = 0\} \\ &= \ \ \{([a:b],(z,w)) \in \mathbb{RP}^1 \times \mathbb{R}^2 \ \ : \ \ (a,b) \neq (0,0), \ \ xb + ya = 0\}. \end{aligned}$$

If $a \neq 0$ the chart

$$^{\mathbb{R}}V_1 = \{([a:b],(x,y)) \in \mathbb{RP}^1 \times \mathbb{R}^2 \ \ : \ \ a \neq 0, \ \ x\frac{b}{a} + y = 0\}$$

is parametrized by $(x,u) \in \mathbb{R}^2$, where $u = -\dfrac{b}{a}$. If $b \neq 0$, the chart

$$^{\mathbb{R}}V_2 = \{([a:b],(x,y)) \in \mathbb{RP}^1 \times \mathbb{R}^2 \ \ : \ \ b \neq 0, \ \ x + y\frac{a}{b} = 0\}$$

is parametrized by $(v,y) \in \mathbb{R}^2$, where $v = -\dfrac{a}{b}$. The real coordinates $(x,u),(v,y)$ cover the real Möbius strip. □

As in the complex case, real coordinates on ${}^{\mathbb{R}}V_1$ and on ${}^{\mathbb{R}}V_2$, prove that these sets are isomorphic to $\mathbb{R}^2$. The monoidal map $\rho : \mathbb{M} \to \mathbb{C}^2$ induces a real mapping (noted ρ again) between ${}^{\mathbb{R}}\mathbb{M}$ and $\mathbb{R}^2$. Such a map is given in charts:

$$\rho\Big|_{{}^{\mathbb{R}}V_1} = {}^{\mathbb{R}}c_1, \quad \rho\Big|_{{}^{\mathbb{R}}V_2} = {}^{\mathbb{R}}c_2, \tag{2.1.9}$$

where

$${}^{\mathbb{R}}c_1 : (x,u) \mapsto (x,xu), \quad {}^{\mathbb{R}}c_2 : (v,y) \mapsto (vy,y). \tag{2.1.10}$$

The exceptional divisor $\mathbb{CP}^1 \times \{0\} \subset \mathbb{CP}^1 \times \mathbb{C}^2$ intersects $\mathbb{RP}^1 \times \mathbb{R}^2$ as $\mathbb{RP}^1 \times \{0\} \simeq \mathbb{S}^1$. The unit circle is then given in charts by (see Figure 2.2)

$$(\mathbb{RP}^1 \times \{0\}) \cap ({}^{\mathbb{R}}V_1) = \{x = 0\}, \quad (\mathbb{RP}^1 \times \{0\}) \cap ({}^{\mathbb{R}}V_2) = \{y = 0\}.$$

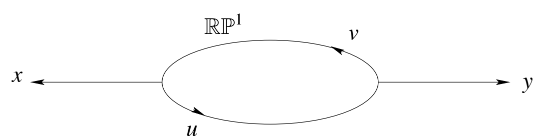


Figure 2.2: The real charts on $\mathbb{RP}^1$.

2.1.3 The tangent form.

Let

$$\omega = f dx + g dy$$

be a real holomorphic 1-form having an isolated singularity of order 1. By definition, this means that the Taylor expansion of the coefficients f, g begin with homogeneous polynomials f_1, g_1 of degree 1 and at least one of these two homogeneous polynomials does not vanish identically. Consider the pullback $\rho^*\omega$ on the complex Möbius strip $\mathbb{M}$ in the coordinates (x, u). In this chart the form ω is computed as

$$\begin{aligned} \omega_1 &= [f(x,xu)+ug(x,xu)]dx+xg(x,xu)du \\ &= x^{-1}[({}^{\mathbb{R}}c_1^*)(h)dx+({}^{\mathbb{R}}c_1^*)(g')du], \end{aligned} \tag{2.1.11}$$

where $h = xf + yg$ and $g' = x^2 g$.

Definition 2.1.6. *[27]*

1. *The real homogeneous polynomial*

$$h_2 = xf_1 + yg_1 \tag{2.1.12}$$

of degree 2 *is called the tangent form of the singularity* ω*.*

2. *The singular point associated to* ω *is called generalized elliptic if the tangent form* h_2 *is non-vanishing except at the origin* $(0,0) \in \mathbb{R}^2$.

Remark 2.1.7. *[27] The singular points on the exceptional divisor after real blow up, are roots of the polynomial*

$$f_1(1,u) + ug_1(1,u) = x^{-2}h_2(x,xu). \tag{2.1.13}$$

For a generalized elliptic singularity this polynomial is not identically zero, hence the blow-up is always non-dicritical and Definition 2.1.6 guarantees that there are no singular points on the real line $\mathbb{R} \subset S$ *in the chart* (x,u)*, that is, the real projective line* $\mathbb{RP}^1$*. For similar reasons the point* $u = \infty$ *(mapped as* $v = 0$ *in the second chart) is also non-singular.*

Corollary 2.1.8. *[27] A real analytic singularity is generalized elliptic if and only if it is non-dicritical and after the blow-up has no singularities on the real projective line $\mathbb{RP}^1 \subset \mathbb{CP}^1$ of the exceptional divisor.*

Proposition 2.1.9. *Planar singularities whose linearization matrix A is normalized to*

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

after real blow up (2.1.10) *have two singular points at $(x,u) = (0,i)$ and $(x,u) = (0,-i)$ (or $(v,y) = (-i,0)$ and $(v,y) = (i,0)$ respectively in the second real chart) on the exceptional divisor. Such singular points correspond to $(Z,w) = (0,0)$ and $(Z,w) = (\infty,0)$ respectively, in complex coordinates in the first direction of the blow up (in the second chart, they are given by $(W,z) = (\infty,0)$ and $(W,z) = (0,0)$).*

Such singularities are generalized elliptic povided $\beta \neq 0$. In this case the points on the equator of the Riemann sphere $(x,u) = (0,0)$ and $(x,u) = (0,\infty)$ are non singular. These points are given by $(Z,w) = (1,0)$ and $(Z,w) = (-1,0)$ (mapped as $(W,z) = (1,0)$ and $(W,z) = (-1,0)$ again in the second complex chart), respectively.

Proof. The dual form corresponding to the linear part is $\omega = (\alpha x + \beta y)dy + (\beta x - \alpha y)dx$, and the tangent form is the polynomial

$$h_2(x,y) = x(\beta x - \alpha y) + y(\alpha x + \beta y) = \beta(x^2 + y^2),$$

which is non-vanishing outside the origin if $\beta \neq 0$. The pullback of the tangent form into (x,u) coordinates is the polynomial

$$h_2(x,xu) = \beta x^2(1+u^2),$$

so Equation (2.1.13) yields the singular points on $\mathbb{CP}^1$: $u = \pm i$. Such points correspond to $v = \mp i$ in the (v, y) chart. Then $u = 0, \infty$ (mapped as $v = \infty, 0$ in the other chart) are non singular. Let us compute how they embed in complex charts V_1 and V_2.

Inasmuch as $u = -\dfrac{b}{a}$, the real point $(x, u) = (0, 0)$ yields $b = 0$ when $t_1 = a + ib$ on the equator of $\mathbb{CP}^1$. This means $Z = \dfrac{t_2}{t_1} = \dfrac{a}{a} = 1$. By similar reasons, the real point $(x, u) = (0, \infty)$ yields $a = 0$ by definition, and then $Z = \dfrac{-ib}{ib} = -1$. From whence the points $Z = 1, -1$ (mapped as $W = 1, -1$ in the second complex chart) are non singular.

On the other hand, the imaginary point $(x, u) = (0, +i)$ implies by definition that $b = -ia$. Then $i(a - ib) = 0$. As $t_2 = \overline{t_1} = a - ib$ in the real coordinates, we get $t_2 = 0$, whence $Z = 0$. On the contrary, the point $(x, u) = (0, -i)$ leads to $b = ia$ or $i(a + ib) = 0$, *i.e.*, $t_1 = 0$ and this is $Z = \infty$. The singular points of the complex chart (V_1, c_1) are therefore located at $(Z, w) = (0, 0)$ and $(Z, w) = (\infty, 0)$ on the Riemann sphere. □

Remark 2.1.10. *In particular, the singular points of a weak focus are not detected in the real plane after blow up, see Figure 2.3 for a picture of complex and real coordinates and their organization in the blow up.*

Definition 2.1.11. *The complex surface $\{z = w\}$ is henceforward noted Σ in ambient coordinates. It will be the target set of definition of the Poincaré map.*

2.2 Holonomy of the field v_0.

The holonomy map of the weak focus v_0 along the loop $\mathbb{RP}^1$ in the c_1 direction of the blow up, is well defined for the cross section Σ with the coordinate w as a

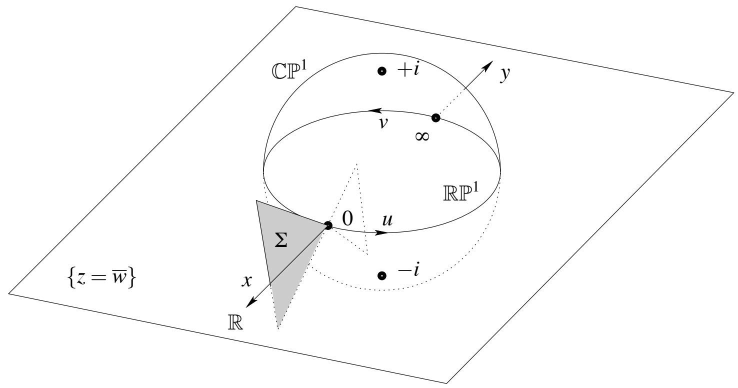


Figure 2.3: The exceptional divisor $S \simeq \mathbb{CP}^1$ in real coordinates (x,u).

local chart on it. As the field v_0 is analytic, there is a well defined real singular foliation on the Möbius strip which is the neighborhood of the central circle of the exceptional divisor S, see Figure 2.1. The holonomy map along this circle is therefore real analytic. Note, however, that this loop does not belong entirely to any of the two canonical real charts ${}^{\mathbb{R}}V_1$ and ${}^{\mathbb{R}}V_2$: to compute the holonomy in real chart (x,u) one has to "continue" across infinity $u = \infty$ that is, pass to the other chart.

Still this problem can be easily avoided after complexification: if the singularity is generalized elliptic, the holonomy can be computed in the chart (x,u) as the result of analytic continuation along the semi-circular loop

$$[-R,R] \cup \{|u| = R, Im(u) > 0\}, \quad R > 1$$

which is homotopic to $\mathbb{RP}^1$, see Figure 2.4.

Recall that v_0 is a generic weak focus that is given in ambient coordinates

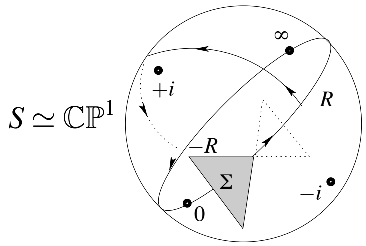


Figure 2.4: "Bypass" of the infinity in the real chart (x,u).

(z,w) by

$$\begin{aligned} \dot{z} &= iz + sz^2w + \sum_{j+k\geq 4} a_{jk} z^j w^k, \\ \dot{w} &= -iw + sw^2z + \sum_{j+k\geq 4} \overline{a_{jk}} z^k w^j. \end{aligned} \tag{2.2.1}$$

Definition 2.2.1.

1. *The holonomy map $\mathscr{Q}^{c_1} : \Sigma \to \Sigma$ of the field $v_0^{c_1} = (c_1^{-1})_* v_0 \circ c_1$ in the (Z,w) chart of the blow up is called the semi-Poincaré map (or semi-monodromy map) of the weak focus v_0.*

2. *The standard monodromy $\mathscr{P} \equiv \mathscr{P}_0$ of the field v_0 is the square $\mathscr{Q}^{c_1} \circ \mathscr{Q}^{c_1}$ of the holonomy. It is called the Poincaré map of the weak focus v_0.*

3. *The holonomy map of the field $v_0^{c_2} = (c_2^{-1})_* v_0 \circ c_2$ for the section Σ in the (W,z) chart of the blow up is noted $\mathscr{Q}^{c_2} : \Sigma \to \Sigma$.*

The complex description of the semi-monodromy immediately allows to prove its analyticity and that of the monodromy map.

Theorem 2.2.2. *[27] The semi-monodromy $\mathscr{Q}^{c_1}$ of a generalized elliptic singular point is an orientation reversing (with ${\mathscr{Q}^{c_1}}'(0) = -1$) germ of diffeomorphism, which is also real analytic on $(\mathbb{R},0)$, including the origin.*

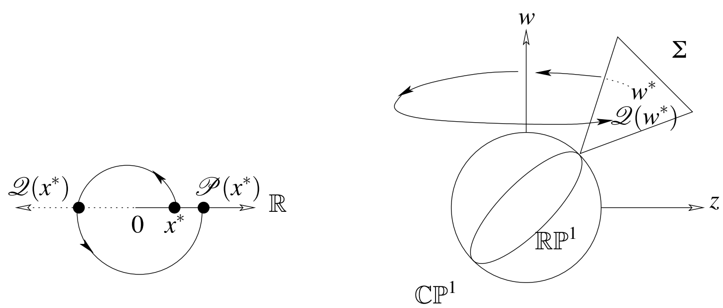


Figure 2.5: The complexification of the real line and its blow up.

The holonomy operator $\mathcal{Q}^{c_1}$ of the field v_0 is visible on the real plane $(\mathbb{R}^2, 0)$ before the blow-up: we have seen already that the cross-section Σ blows down as the x-axis on the (x,y)-plane. Indeed, the section Σ blows down as $\{z = w\}$ and since $z = \overline{w}$ (the real system) we get $Im(z) = y = 0$, which is the x-real axis. By construction, $(\mathcal{Q}^{c_1}(x), 0)$ is the first point of intersection with the x-axis of a solution starting at $(x, 0)$, after continuation counterclockwise, see Figure 2.5.

2.3 The embedded Poincaré family.

We embed the Poincaré map $\mathcal{P} = \mathcal{P}_0$ of the weak focus $v = v_0$ in a generic family of diffeomorphisms $\mathcal{P}_\varepsilon$ which is analytic in the parameter.

Definition 2.3.1. *Let ε be fixed in a neighborhood V of the origin in the parameter space.*

1. *The holonomy map $\mathcal{Q}_\varepsilon^{c_1} : \Sigma \to \Sigma$ of the field $v_\varepsilon^{c_1} = (c_1^{-1})_* v_\varepsilon \circ c_1$ in the (Z, w) chart of the blow up is called the semi-Poincaré map (or semi-monodromy map) of the system v_ε.*

2. *The standard monodromy $\mathscr{P}_\varepsilon$ of the field v_ε is the square $\mathscr{Q}_\varepsilon^{c_1} \circ \mathscr{Q}_\varepsilon^{c_1}$ of the holonomy. It is called the Poincaré map of the the system v_ε.*

3. *The holonomy map of the field $v_\varepsilon^{c_2} = (c_2^{-1})_* v_\varepsilon \circ c_2$ for the section Σ in the (W,z) chart of the blow up is noted $\mathscr{Q}_\varepsilon^{c_2} : \Sigma \to \Sigma$.*

Theorem 2.3.2. *[29] The Poincaré map $\mathscr{P}_\varepsilon : \Sigma \to \Sigma$ of the complex family* (1.4.1) *unfolding the weak focus $v \equiv v_0$ is an analytic germ of diffeomorphism which has the form*

$$\mathscr{P}_\varepsilon(w) = e^{2\pi\varepsilon} w \pm e^{2\pi\varepsilon}[2\pi + O(\varepsilon)]w^3 + O(w^4). \tag{2.3.1}$$

Proof. The analyticity follows from its definition (the field v_ε is analytic for ε fixed). Write system (1.2.1) in polar coordinates (r,θ) :

$$\begin{cases} \dot{r} = r(\varepsilon + sr^2) + g(r,\theta) \\ \dot{\theta} = 1 + h(r,\theta), \end{cases} \tag{2.3.2}$$

where $s = \pm 1$, $g = O(|r|^4)$, $h = O(|r|^3)$, and the ε-dependence of these functions is not indicated to simplify notations. An orbit of (2.3.2) starting at $(r,\theta) = (r_0, 0)$ has the following representation: $r = r(\theta, r_0)$, $r_0 = r(0, r_0)$ with r satisfying the equation

$$\frac{dr}{d\theta} = \frac{r(\varepsilon + sr^2) + g}{1+h} = r(\varepsilon + sr^2) + R(r,\theta), \tag{2.3.3}$$

where $R = O(|r|^4)$. Notice that the transition from (2.3.2) to (2.3.3) is equivalent to the introduction of a new time parametrization in which $\dot{\theta} = 1$ which implies that the return time to the half-axis $\theta = 0$ is the same for all orbits starting on this axis with $r_0 > 0$. Since $r(\theta, 0) \equiv 0$ we can write the Taylor expansion for $r(\theta, r_0)$,

$$r = u_1(\theta) r_0 + u_2(\theta) r_0^2 + u_3(\theta) r_0^3 + O(|r|^4). \tag{2.3.4}$$

Substituting (2.3.4) into (2.3.3) and solving the resulting linear differential equations at corresponding powers of r_0 with initial conditions $u_1(0) = 1$, $u_2(0) = u_3(0) = 0$, we get

$$u_1(\theta) = e^{\varepsilon\theta}, \quad u_2(\theta) \equiv 0, \quad u_3(\theta) = se^{\varepsilon\theta}\left(\frac{e^{2\varepsilon\theta} - 1}{2\varepsilon}\right).$$

Notice that these expressions are independent of the term $R(r,\theta)$. Therefore, the standard monodromy $r_0 \mapsto r_{\mathscr{P}} = r(2\pi, r_0)$ has the form

$$r_{\mathscr{P}} = e^{2\pi\varepsilon} r_0 + se^{2\pi\varepsilon}[2\pi + O(\varepsilon)]r_0^3 + O(r_0^4) \tag{2.3.5}$$

for all $R = O(r_0^4)$. This yields the expression (2.3.1) in the w coordinate. □

Remark 2.3.3.

1. *The semi-monodromy is simply $r(\pi, r_0)$ in the proof above.*
2. *Since $\mathscr{P}_0 = \mathscr{Q}_0^{c_1} \circ \mathscr{Q}_0^{c_1}$, the Theorems 2.2.2 and 2.3.2 prove that the form of the semi-monodromy is given by*

$$\mathscr{Q}_0^{c_1}(w) = -e^{\pi\varepsilon} w \pm e^{\pi\varepsilon}[\pi + O(\varepsilon)]w^3 + o(w^3). \tag{2.3.6}$$

3. *In the next chapter we shall see that the family $\mathscr{P}_\varepsilon$ is real under the condition $\varepsilon \in \mathbb{R}$ (Proposition 3.1.6).*

Denote by $f(x,\varepsilon)$ the displacement function $f = \mathscr{P}_\varepsilon - id$ for some choice of a cross-section, say, the semiaxis $\eta_+ = \{y = 0, x > 0\}$, and an analytic chart x on this cross-section. By definition, sufficiently small limit cycles of the field v_ε intersect η_+ at isolated zeros of f. The map (2.3.1) can easily be analyzed for sufficiently small r_0 and $|\,\varepsilon\,|$. For instance, if the third order coefficient of the field v_ε is negative (*i.e.* $s = -1$) there is a neighborhood of the origin in which the map has only a trivial fixed point for small $\varepsilon < 0$ and an extra fixed point, $\sqrt{\varepsilon} + \ldots$ for small $\varepsilon > 0$. The stability of the fixed points is also easily obtained from (2.3.1). Taking into account that a positive fixed point of the map corresponds to a limit cycle of the system, we can conclude that system (2.3.2) (or (1.2.1)) with any $O(|w|^4)$ terms has a unique (stable) limit cycle bifurcating from the origin and existing for $\varepsilon > 0$. If $s = +1$ an unstable limit cycle appears on $\varepsilon < 0$.

Chapter 3

The real character of the foliation

3.1 $\mathbb{Z}_2$-equivariance of the family.

Recall that a generic (orbital) analytic family v_ε locally unfolding an embedded order 1 weak focus in a neighborhood around the origin of coordinates in $\mathbb{C}^2$ is an analytic one-parameter dependent family of differential equations:

$$\left.\begin{aligned} \dot{z} &= (\varepsilon + i)z + sz^2 w + \sum_{j+k\geq 4} a_{jk}(\varepsilon) z^j w^k, \\ \dot{w} &= (\varepsilon - i)w + sw^2 z + \sum_{j+k\geq 4} \overline{a_{jk}(\overline{\varepsilon})} z^k w^j. \end{aligned}\right\} = v_\varepsilon(z,w) \tag{3.1.1}$$

where the coefficient $s = \pm 1$, and $\overline{a_{jk}(\varepsilon)}$ is the complex conjugate of $a_{jk}(\varepsilon)$, and the domain of ε is a symmetric neighborhood $V \subset \mathbb{C}$ around the origin in the complex plane. Inasmuch as the coefficients depend analytically on ε they can be written as

$$a_{jk}(\varepsilon) = \sum_{n\in\mathbb{N}} (b_{jkn} + i c_{jkn})\, \varepsilon^n, \tag{3.1.2}$$

with $b_{jkn}, c_{jkn} \in \mathbb{R}$ for all $j + k \geq 4$ and $n \in \mathbb{N}$. The parametric family of systems above corresponds to a generic unfolding of a saddle point in $\mathbb{C}^2$ with ratio of eigenvalues equal to -1. Let Ω_ε stand for the Pfaffian form associated to the system v_ε, and let $\mathscr{F}_{\Omega_\varepsilon}$ be its holomorphic foliation.

Definition 3.1.1. *In ambient coordinates (z,w) and parameter space, we define*

1. *The standard complex conjugation in one variable:*

$$\mathscr{C}(z) = \bar{z}, \quad \mathscr{C}(w) = \overline{w}, \quad \mathscr{C}(\varepsilon) = \overline{\varepsilon}. \tag{3.1.3}$$

2. *The standard complex conjugation in two variables:*

$$\mathscr{S}(z,w) = (\bar{z}, \overline{w}). \tag{3.1.4}$$

3. *The standard shift of two complex variables:*

$$\sigma(z,w) = (w,z). \tag{3.1.5}$$

An easy calculation shows that the family of complex systems (3.1.1) is invariant under the conjugacy $\mathscr{S} \circ \sigma$ and a Schwarz reflexion in the parameter:

$$v_\varepsilon = \mathscr{S} \circ \sigma \circ v_{\mathscr{C}(\varepsilon)} \circ \sigma \circ \mathscr{S}, \tag{3.1.6}$$

and thus, the family of vector fields v_ε shows a particular behavior for real values of the parameter.

Definition 3.1.2. *[29] Let G be a (compact) group which can be represented in $\mathbb{R}^n$ by matrices $\{T_g\}$:*

$$T_e = I_n, \quad T_{g_1 g_2} = T_{g_1} T_{g_2},$$

for any $g_1, g_2 \in G$. Here $e \in G$ is the group unit $(eg = ge = g)$, while I_n is the $n \times n$ unit matrix. A continuous time family of differential equations depending analytically on a parameter $\varepsilon \in \mathbb{R}$

$$\dot{\vec{X}} = f_\varepsilon(\vec{X}), \quad \vec{X} \in \mathbb{R}^n \tag{3.1.7}$$

is called invariant with respect to the representation $\{T_g\}$ *of the group* G, *or simply* G*-equivariant, if*

$$T_g f_\varepsilon(\vec{X}) = f_\varepsilon(T_g \vec{X}) \tag{3.1.8}$$

for all $g \in G$, $\varepsilon \in \mathbb{R}$ *and all* $\vec{X} \in \mathbb{R}^n$.

Notice that the composition $\mathscr{S} \circ \sigma(z,w) = (\overline{w}, \overline{z})$ corresponds in $\mathbb{R}^4$ to the linear transformation $T_{\mathscr{S}\circ\sigma}$ represented by the premultiplication by the matrix M :

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

i.e., the equation $\mathscr{S} \circ \sigma(z,w) = (\overline{w}, \overline{z})$ in $\mathbb{C}^2$ is equivalent to

$$M(x_1, y_1, x_2, y_2)^T = (x_2, -y_2, x_1, -y_1)^T \tag{3.1.9}$$

in $\mathbb{R}^4$ for ambient coordinates $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Moreover, it is easily seen that

$$M^2 = I_4$$

i.e., T is an involution: $T^{\circ 2}_{\mathscr{S}\circ\sigma} = T_e$. Thus, if ${}^{\mathbb{R}}v_\varepsilon$ is the realification of the family (3.1.1), and if $G = \{T_e, T_{\mathscr{S}\circ\sigma}\}$ with $T_e = I_4$ (*i.e.* $G \sim \mathbb{Z}_2$), then the family ${}^{\mathbb{R}}v_\varepsilon$ is "$\mathbb{Z}_2$-equivariant" for real values of the parameter:

$$M\,{}^{\mathbb{R}}v_\varepsilon(\vec{X}) = {}^{\mathbb{R}}v_\varepsilon(M\vec{X}), \tag{3.1.10}$$

for all $\vec{X} \in \mathbb{R}^4$. In addition, this representation decomposes $\mathbb{R}^4$ into a direct sum

$$\mathbb{R}^4 = \Sigma^+ \oplus \Sigma^-,$$

where $M\vec{X} = \vec{X}$ for $\vec{X} \in \Sigma^+$, and $M\vec{X} = -\vec{X}$ for $\vec{X} \in \Sigma^-$.

Definition 3.1.3. *The $\mathbb{Z}_2$-equivariance* (3.1.6) *in* $\mathbb{C}^2$ *(or* (3.1.10) *in* $\mathbb{R}^4$*) of a generic family of vector fields* v_ε*, is called the real character of the family.*

The space Σ^+ is the fixed point subspace associated with G and is canonically identified with the real plane $\mathbb{R}^2 \subset \mathbb{C}^2$. Indeed, the equation $M\vec{X} = \vec{X}$, for $\vec{X} = (x_1, y_1, x_2, y_2) \in \mathbb{R}^4$, yields

$$\begin{aligned} z &= x_1 + iy_1, \\ w &= x_1 - iy_1 \end{aligned}$$

or $z = \overline{w}$.

Definition 3.1.4. *The 1-dimensional complex surface* $\Sigma^+ : \{z = \overline{w}\}$ *is called the plane of symmetry or the real plane, because it is canonically identified with* $\mathbb{R}^2$ *in the ambient space* $\mathbb{C}^2$ *in* (z,w) *coordinates.*

Let us write

$$v_\varepsilon(z,w) = P_\varepsilon(z,w)\frac{\partial}{\partial z} + Q_\varepsilon(z,w)\frac{\partial}{\partial w}, \tag{3.1.11}$$

where $P_\varepsilon, Q_\varepsilon : U \subset \mathbb{C}^2 \to \mathbb{C}^2$ are germs of analytic families of functions on an open set of $\mathbb{C}^2$.

Proposition 3.1.5. *The real character* (3.1.6) *of the family* v_ε *is equivalent to the invariance of the real plane* Σ^+ *under the flow of the system, when* $\varepsilon \in \mathbb{R}$*.*

Proof. In terms of the families $P_\varepsilon, Q_\varepsilon$ the real character (3.1.6) means $(P_\varepsilon(z,w), Q_\varepsilon(z,w)) = (\overline{Q_{\overline{\varepsilon}}(\overline{w},\overline{z})}, \overline{P_{\overline{\varepsilon}}(\overline{w},\overline{z})})$. Thus, on $\Sigma^+ : \{z = \overline{w}\}$ we get for real ε :

$$P_\varepsilon = \overline{Q_\varepsilon} \tag{3.1.12}$$

whence follows that the flow of v_ε is real too. Conversely, if the flow of the family is real for real ε, then it is easily seen that the components $P_\varepsilon, Q_\varepsilon$ are related through (3.1.12). Hence, $v_\varepsilon = \mathscr{S} \circ \sigma \circ v_\varepsilon \circ \sigma \circ \mathscr{S}$, $\varepsilon \in \mathbb{R}$. Inasmuch as the dependence of v_ε on the parameter is analytic, and the neighborhood V is symmetric (with $V \cap \mathbb{R} \neq \emptyset$), the Schwarz reflection principle yields the real character (3.1.6). □

The real character of the family v_ε allows to show easily the

Proposition 3.1.6. *The generic family of diffeomorphisms $\mathscr{P}_\varepsilon$ unfolding the Poincaré map $\mathscr{P}_0$ of the system v_0 is an analytic germ of family, verifying the identity*

$$\mathscr{C} \circ \mathscr{P}_{\mathscr{C}(\varepsilon)} \circ \mathscr{C} = \mathscr{P}_\varepsilon$$

for all $\varepsilon \in V$. In particular, it is real whenever $\varepsilon \in \mathbb{R}$.

Proof. The analyticity follows by definition. We show that it is a real family on real values of the parameter. Let us fix $\varepsilon \in \mathbb{R}$ and take an orbit γ of the system (3.1.1) (*i.e.*, a trajectory for real time) starting at the point $z_0 = w_0 \in \Sigma$ and returning to a point $z_1 = w_1 \in \Sigma$ close to $w_0 : w_1 = \mathscr{P}_\varepsilon(w_0)$. If $z_0 = \overline{w_0}$, the real trajectory of this point coincides with γ (which is then contained in $\mathbb{R}^2$) so $z_1 = \overline{w_1}$, because the flow is real for real values of the parameter (that is, the orbits starting at real initial conditions are contained in Σ^+, due to the real character of the family) and then

$$\mathscr{P}_\varepsilon(z_0) = z_1 = \overline{w_1} = \overline{\mathscr{P}_\varepsilon(w_0)} = \overline{\mathscr{P}_\varepsilon(\overline{z_0})}.$$

By the symmetry of the neighborhood V in the parameter space, again the Schwarz reflection principle yields the conclusion. □

Definition 3.1.7.

1. *Recall that $\Sigma : \{z = w\}$ in ambient coordinates. Define as well*

$$\begin{aligned} \Sigma_{\mathbb{R}} &:= \Sigma \cap \Sigma^+ \simeq \mathbb{R}, \\ \Sigma_\mu &:= \{z = \mu w \quad : \quad \mu \in \mathbb{C}^*\} \end{aligned} \tag{3.1.13}$$

and endow these surfaces with a parametrization in the w coordinate.

2. *The pullback of Σ_μ is denoted by*

$$\Sigma_\mu^{c_1} := c_1^*\Sigma_\mu = \{Z = \mu\} \tag{3.1.14}$$

in the c_1 direction of the complex Möbius strip, and by

$$\Sigma_\mu^{c_2} := c_2^*\Sigma_\mu = \{W = \mu^{-1}\} \tag{3.1.15}$$

in the c_2 direction of the complex Möbius strip.

Proposition 3.1.8. *The affine collection $\{\Sigma_\mu\}_{\mu\in\mathbb{C}^*}$ of complex surfaces is a local transverse fibration for $\mathscr{F}_{\Omega_\varepsilon}$, in a small neighborhood of the origin.*

Proof. By definition, $\Sigma_\mu = \{z = \mu w\} = \{\frac{z}{w} = \mu\}$, for $\mu \in \mathbb{C}^*$. Using equations (3.1.1) we get

$$\frac{d}{dt}\left(\frac{z}{w}\right) = \frac{\dot{z}w - z\dot{w}}{w^2} = \frac{z}{w}\Big(2i + O(|z, w, \varepsilon\,|)\Big).$$

Thus, on Σ_μ

$$\frac{d}{dt}\left(\frac{z}{w}\right) = \mu\Big(2i + O(|z, w, \varepsilon\,|)\Big) \neq 0$$

for sufficiently small z, w. □

In particular, all the vertical and horizontal surfaces defined separately by $Z = \mu$ (in the (V_1, c_1) chart of the blow up), or $W = \mu$ (in the (V_2, c_2) chart), where $\mu \in \mathbb{C}^*$, are transverse to the foliation on a small neighborhood of the origin.

3.2 Blow up of the family v_ε.

The 1-parameter dependent family v_ε is a generic family of vector fields unfolding the weak focus v_0. In order to describe the geometry of the foliation, equations (3.1.1) are blown up by the complex standard monoidal map $\rho = (c_1, c_2)$, defined in (2.1.7). The blow up space (or the complex Möbius strip) is equipped with the two charts (V_1, c_1) and (V_2, c_2) which overlap away from $Z = 0 \in \mathbb{CP}^1$ and $Z = \infty \in \mathbb{CP}^1$. It is seen that the singularity at the origin of the system in (z, w) coordinates splits in two singularities located at $Z = 0$ and $Z = \infty$ on the Riemann sphere and those points are seen as $u = +i$ and $u = -i$ in the real chart (x, u), see Figure 2.3. Inasmuch as the singularity is non-dicritical, the Riemann sphere $\mathbb{CP}^1$ is a common separatrix in the two charts of the blow up space.

Definition 3.2.1. *Let $v_\varepsilon^{c_1}, v_\varepsilon^{c_2}$ be the pullback fields of the vector field v_ε, respectively, under the maps c_1, c_2 :*

$$\begin{aligned} v_\varepsilon^{c_1} &= (c_1^{-1})_* v_\varepsilon \circ c_1, \\ v_\varepsilon^{c_2} &= (c_2^{-1})_* v_\varepsilon \circ c_2. \end{aligned} \tag{3.2.1}$$

$\mathscr{F}_\varepsilon^Z$ and $\mathscr{F}_\varepsilon^W$ are the foliations of $v_\varepsilon^{c_1}$ and $v_\varepsilon^{c_2}$.

A short calculation proves that $v_\varepsilon^{c_1}$ is given by the field on the right of the equation

$$\left.\begin{array}{rcl} \dot{Z} &=& 2iZ + \sum\limits_{j+k\geq 4} (a_{jk}(\varepsilon) - \overline{a_{k+1,j-1}(\overline{\varepsilon})}) Z^j w^{k+j-1} \\ \dot{w} &=& (\varepsilon - i)w + sw^3 Z + \sum\limits_{j+k\geq 4} \overline{a_{jk}(\overline{\varepsilon})} Z^k w^{j+k} \end{array}\right\} := v_\varepsilon^{c_1}(Z, w). \tag{3.2.2}$$

In the same way, $v_\varepsilon^{c_2}$ is given by the field on the right of

$$\left.\begin{array}{rcl} \dot{W} &=& -2iW + \sum\limits_{j+k\geq 4} (\overline{a_{jk}(\overline{\varepsilon})} - a_{k+1,j-1}(\varepsilon)) W^j z^{k+j-1} \\ \dot{z} &=& (\varepsilon + i)z + sz^3 W + \sum\limits_{j+k\geq 4} a_{jk}(\varepsilon) W^k z^{j+k} \end{array}\right\} := v_\varepsilon^{c_2}(W, z), \tag{3.2.3}$$

where the coefficients $a_{jk}(\varepsilon)$ are given in (3.1.2).

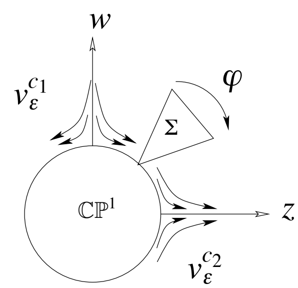


Figure 3.1: The two fields $v_\varepsilon^{c_1}$ and $v_\varepsilon^{c_2}$ in the blow up space.

3.3 The real character in the blow up.

We now investigate how the fields found in the blow up are related. Recall that the change of coordinates between charts of the blow up is the monomial map

$$\begin{aligned} \varphi \quad : \quad & V_1 \to V_2 \\ & \varphi(Z,w) := (\frac{1}{Z}, Zw) \equiv (W,z) \end{aligned} \tag{3.3.1}$$

with inverse

$$\varphi^{-1}(W,z) = (\frac{1}{W}, Wz). \tag{3.3.2}$$

This function is a holomorphic diffeomorphism and, in fact

$$\varphi = c_2^{-1} \circ c_1, \tag{3.3.3}$$

see Figure 3.1. Notice also that the change between charts is the identity on the product $\Sigma \times \mathbb{C}$:

$$\varphi(1,w) = (1,z)$$

where $z = w$, so it sends the section Σ onto Σ. The proof of the following proposition is plain.

Proposition 3.3.1. *The field $v_\varepsilon^{c_2}$ is the pushforward of the field $v_\varepsilon^{c_1}$ by the map φ :*

$$v_\varepsilon^{c_2} = \varphi_* v_\varepsilon^{c_1} \circ \varphi^{-1}. \tag{3.3.4}$$

This is a global identity in the blow up.

Remark 3.3.2. *The standard complex conjugation in two complex variables $\mathscr{S}$ lifts as*

$$\begin{aligned} c_1^{-1} \circ \mathscr{S} \circ c_1(Z,w) &= (\overline{Z},\overline{w}) \\ c_2^{-1} \circ \mathscr{S} \circ c_2(Z,w) &= (\overline{W},\overline{z}) \end{aligned} \tag{3.3.5}$$

respectively, in the first and second directions of the blow up space. Such a lifting is noted $\mathscr{S}$ as well.

3.3.1 The real strip ${}^{\mathbb{R}}\mathbb{M}$ revisited.

The real Möbius strip ${}^{\mathbb{R}}\mathbb{M}$ is given by the pullback of the real plane (symmetry plane) into the blow up space $\mathbb{M}$:

$${}^{\mathbb{R}}\mathbb{M} = \rho^* \Sigma^+ \tag{3.3.6}$$

(see (2.1.1) for the defintion of $\mathbb{M}$). The real Möbius strip can be explicitly computed in terms of the complex charts (Z,w) and (W,z) covering the blow up space (see Paragraph 2.1.2 in the previous chapter for details on the real covering of ${}^{\mathbb{R}}\mathbb{M}$).

Proposition 3.3.3 (Real Möbius strip in complex coordinates). *The real Möbius strip ${}^{\mathbb{R}}\mathbb{M}$ is defined in complex charts, by*

$${}^{\mathbb{R}}\mathbb{M}_{c_1} := \{(Z,w) \ : \ Z = \frac{\overline{w}}{w}, \ w \in \mathbb{C}\} \tag{3.3.7}$$

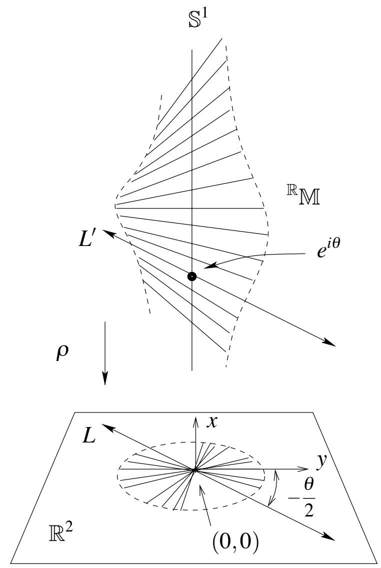


Figure 3.2: The real Möbius strip in complex coordinates.

in the c_1 direction of the blow up space, and by

$$^{\mathbb{R}}\mathbb{M}_{c_2} := \{(W,z) \ : \ W = \frac{\overline{z}}{z}, \ z \in \mathbb{C}\} \tag{3.3.8}$$

in the c_2 direction of the blow up. Such a surface can be seen as the embedding $\mathbb{R}^2 \backslash \{0\} \hookrightarrow \mathbb{R}^4 \backslash \{0\}$

$$(x,y) \mapsto (\frac{x^2-y^2}{x^2+y^2}, \frac{2xy}{x^2+y^2}, x, -y).$$

Proof. Indeed, points in polar form $(e^{i\theta}, re^{-i\frac{\theta}{2}}) \in \mathbb{RP}^1 \times \mathbb{R}^2$ are in 1-to-1 correspondence with points $\left(\frac{\overline{w}}{w}, w\right) \in {}^{\mathbb{R}}\mathbb{M}_{c_1}$, for the complex number $w = re^{-i\frac{\theta}{2}}$, $\theta \in [0, 2\pi)$. Here, the second component $re^{-i\frac{\theta}{2}}$ stands for the direction of the real line through the origin L in the real plane $\mathbb{R}^2$. The first component $e^{i\theta}$ gives the point of the unit circle (which is homeomorphic to the exceptional real line $\mathbb{RP}^1 \times \{0\}$)

in correspondence with the line L' projecting as L on the real plane $\mathbb{R}^2$, see Picture 3.2. Of course, the manifold $\left\{(e^{i\theta}, re^{-i\frac{\theta}{2}}) \in \mathbb{RP}^1 \times \mathbb{R}^2 \;\; : \;\; \theta \in [0, 2\pi)\right\}$ is non-orientable. $\square$

Corollary 3.3.4. *The real Möbius strip is invariant under the change of charts:*

$$\varphi({}^{\mathbb{R}}\mathbb{M}_{c_1}) =^{\mathbb{R}} \mathbb{M}_{c_2}. \tag{3.3.9}$$

Indeed, $\varphi \equiv \mathscr{S}$ *on* ${}^{\mathbb{R}}\mathbb{M}$.

Proof. This is a tautology:

$$\varphi\left(\frac{\overline{w}}{w}, w\right) = \left(\frac{w}{\overline{w}}, \overline{w}\right) = \left(\frac{\bar{z}}{z}, z\right)$$

each time that $z = \overline{w}$. $\square$

Remark 3.3.5. *The real Möbius strip* ${}^{\mathbb{R}}\mathbb{M}$ *is strictly contained in the subvariety* $\mathbb{1} := \mathbb{RP}^1 \times \mathbb{R}^2$, *which is given in complex charts by* $\{|Z| = 1\} = \{|W| = 1\}$ *within the product* $\mathbb{CP}^1 \times \mathbb{C}^2$, *and by* $\{|z| = |w|\}$ *in ambient coordinates. Furthermore,*

$${}^{\mathbb{R}}\mathbb{M} \subsetneq \mathbb{1}$$

as real spaces. The real dimension of $\mathbb{1}$ *is* 3 *and is topologically equivalent to the product* $\mathbb{S}^1 \times \mathbb{R}^2$ *in the blow up space, see Figure 3.3.*

Proposition 3.3.6. *The real character* (3.1.6) *of the family* v_ε *is equivalent to the symmetric equations*

$$\begin{aligned} v_\varepsilon^{c_1} &= (\mathscr{S} \circ \varphi)_* v_{\mathscr{C}(\varepsilon)}^{c_1} \circ \varphi \circ \mathscr{S}, \\ v_\varepsilon^{c_2} &= (\mathscr{S} \circ \varphi)_* v_{\mathscr{C}(\varepsilon)}^{c_2} \circ \varphi \circ \mathscr{S} \end{aligned} \tag{3.3.10}$$

in the complex Möbius strip. Since $\varphi = \varphi^{\circ -1}$, *this yields*

$$v_\varepsilon^{c_1} = \mathscr{S} \circ v_{\mathscr{C}(\varepsilon)}^{c_2} \circ \mathscr{S}.$$

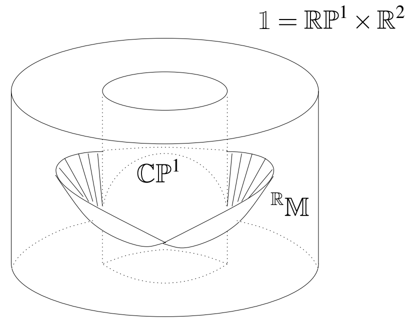


Figure 3.3: The 2-dimensional real Möbius strip $^{\mathbb{R}}\mathbb{M}$ embedded in $\mathbb{1} \simeq \mathbb{S}^1 \times \mathbb{R}^2$.

Proof. Recall Definition 3.1.1. Inasmuch as $\mathscr{S}_* = \mathscr{S}$ and $\sigma_* = \sigma$ (the differentials are computed in $\mathbb{R}^4$), the real character (3.1.6) of the family v_ε and the first of Equations (3.2.1) lead to:

$$\begin{aligned} v_\varepsilon^{c_1} &= (c_1^{-1})_* v_\varepsilon \circ c_1 \\ &= (c_1^{-1})_* (\mathscr{S} \circ \sigma \circ v_{\mathscr{C}(\varepsilon)} \circ \sigma \circ \mathscr{S}) \circ c_1 \\ &= (\mathscr{S} \circ c_1^{-1} \circ \sigma)_* v_{\mathscr{C}(\varepsilon)} \circ \sigma \circ c_1 \circ \mathscr{S}, \end{aligned}$$

but an easy calculation shows that $\sigma = c_1 \circ c_2^{-1}$ and $\sigma \circ c_1 = c_1 \circ \varphi$. Thus

$$\begin{aligned} v_\varepsilon^{c_1} &= (\mathscr{S} \circ c_2^{-1})_* v_{\mathscr{C}(\varepsilon)} \circ c_1 \circ \varphi \circ \mathscr{S} \\ &= (\mathscr{S} \circ c_2^{-1} \circ c_1 \circ c_1^{-1})_* v_{\mathscr{C}(\varepsilon)} \circ c_1 \circ \varphi \circ \mathscr{S} \\ &= (\mathscr{S} \circ \varphi \circ c_1^{-1})_* v_{\mathscr{C}(\varepsilon)} \circ c_1 \circ \varphi \circ \mathscr{S} \\ &= (\mathscr{S} \circ \varphi)_* (c_1^{-1})_* v_{\mathscr{C}(\varepsilon)} \circ c_1 \circ \varphi \circ \mathscr{S} \\ &= (\mathscr{S} \circ \varphi)_* v_{\mathscr{C}(\varepsilon)}^{c_1} \circ \varphi \circ \mathscr{S}. \end{aligned}$$

with the help of (3.3.3). The same procedure proves the second equality. Let us show the converse statement. By definition

$$c_1 \circ \mathscr{S}(W,z) = \mathscr{S} \circ \sigma \circ c_2(W,z) \tag{3.3.11}$$

globally for (W,z) coordinates. This yields, in terms of the differentials,

$$c_{1*}\mathscr{S} = \mathscr{S}\circ\sigma\circ c_{2*}. \tag{3.3.12}$$

Since

$$v^{c_2}_{\mathscr{C}(\varepsilon)} = (c_2^{-1})_* v_{\mathscr{C}(\varepsilon)} \circ c_2,$$

the hypothesis (3.3.10) implies

$$v^{c_1}_{\varepsilon} = \mathscr{S}\circ\big((c_2^{-1})_* v_{\mathscr{C}(\varepsilon)} \circ c_2\big)\circ\mathscr{S}.$$

But since $v_\varepsilon = c_{1*}v^{c_1}_\varepsilon \circ c_1^{-1}$, Identity (3.3.12) leads to

$$\begin{aligned}
v_\varepsilon &= c_{1*}\mathscr{S}\circ\big((c_2^{-1})_* v_{\mathscr{C}(\varepsilon)}\circ c_2\big)\circ\mathscr{S}\circ c_1^{-1}\\
&= \mathscr{S}\circ\sigma\circ c_{2*}\circ\big((c_2^{-1})_* v_{\mathscr{C}(\varepsilon)}\circ c_2\big)\circ\mathscr{S}\circ c_1^{-1}\\
&= \mathscr{S}\circ\sigma\circ v_{\mathscr{C}(\varepsilon)}\circ c_2\circ\mathscr{S}\circ c_1^{-1}\\
&= \mathscr{S}\circ\sigma\circ v_{\mathscr{C}(\varepsilon)}\circ\sigma\circ\mathscr{S},
\end{aligned}$$

the last equality being a direct replacement from (3.3.11). □

Definition 3.3.7. *The Equations* (3.3.10) *are referred to as the real character in the blow up of the family v_ε.*

Thus, the real character of the family v_ε in ambient coordinates is equivalent to the real character in the complex Möbius strip $\mathbb{M}$.

Proposition 3.3.8. *The real character in the blow up of the family v_ε is equivalent to the invariance of the real Möbius strip $^{\mathbb{R}}\mathbb{M}$ under the flows of the systems* (3.2.2) *and* (3.2.3) *when $\varepsilon\in\mathbb{R}$.*

3.4 Holonomies.

In the general case, the holonomy map (for instance, in the c_1 chart of the complex Möbius strip) between two fibers $\Sigma^{c_1}_{\mu_a} : \{Z = \mu_a : \mu_a \in \mathbb{C}^*\}$ and $\Sigma^{c_1}_{\mu_b} :$ $\{Z = \mu_b : \mu_b \in \mathbb{C}^*\}$ is obtained by lifting the radial path defined by the segment between the intersection of the fiber $\Sigma^{c_1}_{\mu_a}$ with the exceptional divisor (the common separatrix of the foliations of the two charts), and the unit circle $\mathbb{S}^1$ and continuing the lifting along $\mathbb{S}^1$ in the counterclockwise direction, and, finally, lifting the radial path defined by the segment between the intersection of the fiber $\Sigma^{c_1}_{\mu_b}$ with the separatrix, and the unit circle $\mathbb{S}^1$, see Figure 3.4 below.

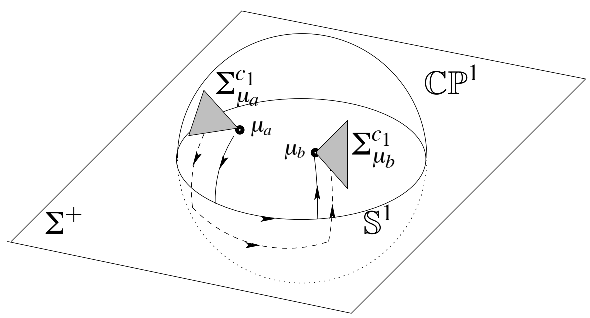


Figure 3.4: The holonomy map in the (Z,w) chart between the sections $\Sigma^{c_1}_{\mu_a}$ and $\Sigma^{c_1}_{\mu_b}$.

Definition 3.4.1. *In each chart of the divisor, the counterclockwise direction will be the positive orientation, and the clockwise direction, the negative orientation.*

The direction of the parametrization in the two radial segments depends on whether the modulus of the projection of the fibers on the separatrix, namely $|\mu_a|$ and $|\mu_b|$, are greater or smaller than 1. In the picture above $|\mu_a|, |\mu_b| < 1$.

Definition 3.4.2.

1. *The holonomy between the complex sections* $\Sigma_{\mu_a}^{c_1}$ *and* $\Sigma_{\mu_b}^{c_1}$ *is noted as* h_ε *in the first direction of the blow up space.*

2. *The holonomy between the complex sections* $\Sigma_{\mu_a}^{c_2}$ *and* $\Sigma_{\mu_b}^{c_2}$ *is noted as* ℓ_ε *in the second direction of the blow up space.*

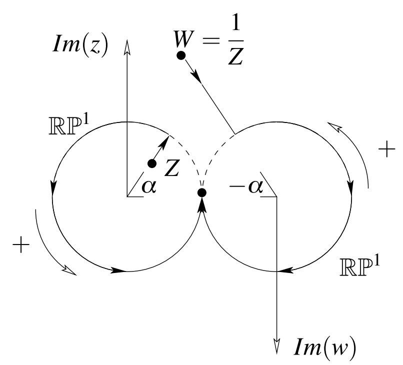


Figure 3.5: The + sign stands for the positive orientation (counter-clockwise).

We are interested in the particular case $\mu_b = 1$, see the Figure 3.5.

3.4.1 The case $\mu_b = 1$.

Definition 3.4.3.

1. *The holonomy* $h_\varepsilon : \{Z\} \times \mathbb{D}_w \to \{1\} \times \mathbb{D}_w$ *in the first direction of the blow up is denoted by* $h_{\varepsilon,Z}$.

2. *The holonomy* $\ell_\varepsilon : \{W\} \times \mathbb{D}_z \to \{1\} \times \mathbb{D}_z$ *in the second direction of the blow up is denoted by* $\ell_{\varepsilon,W}$.

Remark 3.4.4. *Notice that if* $\mu_a = \mu_b = 1$*, then* $h_{\varepsilon,1} = \mathcal{Q}_\varepsilon^{c_1}$ *(the semi-monodromy of the field* v_ε *for the section* Σ *in the first chart of the blow up, see Definition*

2.3.1) and $\ell_{\varepsilon,1} = \mathscr{Q}_\varepsilon^{c_2}$ (the semi-monodromy of the field v_ε for the section Σ in the second chart of the blow up, see Definition 2.3.1).

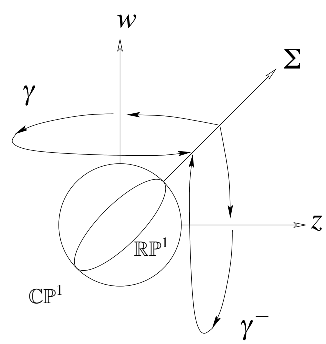


Figure 3.6: Monodromies of Σ in the blow up.

Corollary 3.4.5. *The holonomies $\mathscr{Q}_\varepsilon^{c_1}$ and $\mathscr{Q}_\varepsilon^{c_2}$ for the cross section Σ in the first and second charts of the complex Möbius strip, respectively, are inverses of each other:*

$$\mathscr{Q}_\varepsilon^{c_1} = \mathscr{Q}_\varepsilon^{c_2 \circ -1}.$$

Proof. The equator $\mathbb{RP}^1$ is positively parametrized as $(e^{i\theta}, 0)$, $\theta \in [0, 2\pi]$ in the first chart of the blow up. The lifting of this loop in the leaf of $\mathscr{F}_\varepsilon^{c_1}$ through the point $w_0 \in \Sigma$ is given by the trajectory $\gamma : (e^{i\theta}, w(\theta))$, where $w(0) = w_0$. Consider, on the other hand, the lifting of the loop $(e^{i\phi}, 0)$, $\phi \in [0, 2\pi]$ in the leaf of $\mathscr{F}_\varepsilon^{c_2}$ passing through the point $w(2\pi) \in \Sigma$ and positively oriented (*i.e.* oriented in the counter-clockwise direction) in the second chart of the blow up. Such a lifting is given by $(e^{i\phi}, z(\phi))$, where $z(0) = w(2\pi)$. Since $v_\varepsilon^{c_2} = \varphi_* v_\varepsilon^{c_1} \circ \varphi^{\circ -1}$, the trajectory γ^- given by $\varphi(e^{i\theta}, w(\theta)) = (e^{-i\theta}, e^{i\theta} w(\theta))$ is a well-defined lifting in the leaf of the foliation $\mathscr{F}_\varepsilon^{c_2}$ passing through the point $(1, w(2\pi))$. However, such a lifting is negatively parametrized (*i.e.* oriented in the clock-wise direction) in the second chart of the blow up space, because the orientation of the equator is switched by φ. Changing the parametrization by

$$\theta \mapsto 2\pi - \phi$$

yields the trajectory γ^+ given by

$$\gamma^+ : (e^{i\phi}, e^{-i\phi} w(2\pi - \phi)),$$

which is positively oriented in the second chart of the blow up, see Figure 3.6. Therefore, by unicity of the solution to (3.2.3) in polar coordinates with initial condition $(1, w(2\pi))$,

$$z(\phi) \equiv e^{-i\phi} w(2\pi - \phi).$$

Thus,

$$\mathscr{Q}_\varepsilon^{c_2} \circ \mathscr{Q}_\varepsilon^{c_1}(w_0) = \mathscr{Q}_\varepsilon^{c_2}(w(2\pi)) = \mathscr{Q}_\varepsilon^{c_2}(z(0)) = z(2\pi) = w(0) = w_0.$$

The conclusion follows. □

Chapter 4

Orbital characterization

4.1 Real character of a family of orbital equivalences.

Definition 4.1.1. *Two germs of analytic families of vector fields* $v_{\varepsilon_1}(z_1,w_1)$ *and* $\widehat{v}_{\varepsilon_2}(z_2,w_2)$, *unfolding weak foci at the origin for the values* $\varepsilon_1=\varepsilon_2=0$, *are analytically orbitally equivalent if there exists a germ of map*

$$\mathscr{K}\equiv(\mathbf{k},\Psi,\xi):(\varepsilon_1,z_1,w_1)\mapsto(\mathbf{k}(\varepsilon_1),\Psi(\varepsilon_1,z_1,w_1),\xi(\varepsilon_1,z_1,w_1)) \tag{4.1.1}$$

fibered over the parameter space, where

i) $\mathbf{k}:\varepsilon_1\mapsto\mathbf{k}(\varepsilon_1)=\varepsilon_2$ *is a germ of analytic diffeomorphism preserving the origin. Moreover, it is real for real values of the parameter:*

$$\mathscr{C}\circ\mathbf{k}\circ\mathscr{C}=\mathbf{k}.$$

ii) There exists a representative $\Psi_{\varepsilon_1}(z_1,w_1)\overset{def}{=}\Psi(\varepsilon_1,z_1,w_1)$ *which is an analytic diffeomorphism on a fixed small neighborhood of the origin in* (z_1,w_1) *space, for every* ε_1 *in a neighborhood of the origin of the parameter space.*

In addition, Ψ_{ε_1} enjoys the following property:

$$\Psi_{\varepsilon_1} = \mathscr{S} \circ \sigma \circ \Psi_{\mathscr{C}(\varepsilon_1)} \circ \sigma \circ \mathscr{S}. \tag{4.1.2}$$

iii) There exists a representative $\xi_{\varepsilon_1}(z_1,w_1) \overset{def}{=} \xi(\varepsilon_1,z_1,w_1)$ depending analytically on (ε_1,z_1,w_1) in a small neighborhood of the origin in (ε_1,z_1,w_1) space with values in $\mathbb{C}^*$, and it satisfies:

$$\xi_{\varepsilon_1} = \mathscr{C} \circ \xi_{\mathscr{C}(\varepsilon_1)} \circ \mathscr{S} \circ \sigma.$$

iv) The change of coordinates Ψ_{ε_1} and the time scaling ξ_{ε_1} define an equivalence between $v_{\varepsilon_1}(z_1,w_1)$ and $\widehat{v}_{\mathbf{k}(\varepsilon_1)}(z_2,w_2)$ over a polydisk $B_0(r) \times B_0(r)$, where B_0 is a ball of small radius $r > 0$, for ε in a small neighborhood of the origin:

$$\widehat{v}_{\mathbf{k}(\varepsilon_1)}(\Psi_{\varepsilon_1}(z_1,w_1)) = \xi_{\varepsilon_1}(z_1,w_1)(\Psi_{\varepsilon_1})_*(v_{\varepsilon_1}(z_1,w_1)). \tag{4.1.3}$$

Definition 4.1.2. *In general, any germ of family of smooth diffeomorphisms Ψ_ε satisfying* (4.1.2) *over a symmetric neighborhood V of the origin in the parameter space, is said to possess real character.*

Of course, if we know that two families are orbitally equivalent, we can always change the parameter ε_2 of the second family by ε_1, where $\varepsilon_2 = \mathbf{k}(\varepsilon_1)$ and suppose that the two families are parametrized by the same parameter, as we do later.

Proposition 4.1.3. *Let $\Psi_\varepsilon(z_1,w_1) = (z_2(\varepsilon,z_1,w_1), w_2(\varepsilon,z_1,w_1))$ be a holomorphic change of coordinates (depending analytically on the parameter) between two families of vector fields $v_\varepsilon, \widehat{v}_\varepsilon$ unfolding a weak focus. Then Ψ_ε has real character* (4.1.2) *if and only if Ψ_ε sends the plane of symmetry into itself $\Psi_\varepsilon(\Sigma^+) \subset \Sigma^+$ for real values of the parameter.*

Proof. If the change of coordinates Ψ_ε has real character, then:

$$\begin{aligned}\Psi_\varepsilon(z_1,w_1) &= (z_2(\varepsilon,z_1,w_1),w_2(\varepsilon,z_1,w_1))\\ &= \mathscr{S}\circ\sigma\circ\Psi_{\mathscr{C}(\varepsilon)}\circ\mathscr{S}\circ\sigma(z_1,w_1)\\ &= (\overline{w_2(\overline{\varepsilon},\overline{w_1},\overline{z_1})},\overline{z_2(\overline{\varepsilon},\overline{w_1},\overline{z_1})})\end{aligned}$$

and then, the equivalences sends the symmetry plane of the source space (parametrized by $z_1=\overline{w_1}$) into the symmetry plane of the target space (parametrized by $z_2=\overline{w_2}$) on $\varepsilon\in\mathbb{R}$. On the other hand, the equation $\Psi_\varepsilon(\Sigma^+)\subset\Sigma^+$ is equivalent to $\Psi_\varepsilon=\mathscr{S}\circ\sigma\circ\Psi_\varepsilon\circ\sigma\circ\mathscr{S}$. Inasmuch as the equivalence depends analytically on the parameter, it extends to (4.1.2) by Schwarz reflection. □

4.2 The main theorem.

Lemma 4.2.1. *Suppose $G:\mathscr{V}\subset\mathbb{C}^2\to\mathbb{C}$, $(z,w)\in\mathscr{V}\mapsto G(z,w)\in\mathbb{C}$ is a germ of holomorphic map defined locally in a small neighborhood $\mathscr{V}$ containing the origin, such that $\nabla G(0,0)\neq 0$ and $G=\mathscr{C}\circ G\circ\mathscr{S}\circ\sigma$. Then $\frac{\partial G}{\partial z}(0,0)\neq 0$ and $\frac{\partial G}{\partial w}(0,0)\neq 0$.*

Proof. This is straightforward, since

$$\frac{\partial G}{\partial w}(0,0)=\lim_{h\to 0}\frac{G(0,h)-G(0,0)}{h}=\lim_{\overline{h}\to 0}\overline{\left[\frac{G(\overline{h},0)-G(0,0)}{\overline{h}}\right]}=\overline{\frac{\partial G}{\partial z}(0,0)}.$$

□

Definition 4.2.2. *[10] Two germs of analytic families f_ε and $\widehat{f}_{\widehat{\varepsilon}}$ of diffeomorphisms with a fixed point at the origin are conjugate if there exists a germ of* *analytic diffeomorphism* $\mathscr{H}(\varepsilon,w) = (\mathbf{k}(\varepsilon),\mathbf{h}(\varepsilon,w))$ *fibered over the parameter space such that:*

$$\mathbf{h}_\varepsilon \circ f_\varepsilon = \widetilde{f}_{\mathbf{k}(\varepsilon)} \circ \mathbf{h}_\varepsilon, \tag{4.2.1}$$

where $\mathbf{h}_\varepsilon(w) \stackrel{def}{=} \mathbf{h}(\varepsilon,w)$. *The conjugacy is said to be real if*

$$\mathscr{C} \circ \mathbf{k} \circ \mathscr{C} = \mathbf{k}$$

and

$$\mathscr{C} \circ \mathbf{h}_{\mathscr{C}(\varepsilon)} \circ \mathscr{C} = \mathbf{h}_\varepsilon. \tag{4.2.2}$$

Theorem 4.2.3 (Orbital analytical characterization of the foliation). *Two germs of generic families of analytic vector fields unfolding a vector field with a weak focus at the origin of coordinates in* $\mathbb{C}^2$ *are analytically orbitally equivalent, if and only if the families unfolding their Poincaré maps are analytically conjugate by a real conjugacy.*

Proof. Let the germs of the two different families of vector fields be given by v_{ε_1} and $\widehat{v}_{\varepsilon_2}$. If the systems are analytically orbitally equivalent, then there exists a change of coordinates $\Psi(\varepsilon_1,\cdot,\cdot) = \Psi_{\varepsilon_1}(\cdot,\cdot)$ as in Definition 4.1.1, bringing leaves into leaves. A real reparametrization $\varepsilon_2 = \mathbf{k}(\varepsilon_1)$ of $\widehat{v}_{\varepsilon_2}$ is possible by definition. Such a reparametrization allows to work only with the parameter ε_1, which will be noted ε . Accordingly, we shall write $\widehat{v}_\varepsilon$ instead of $\widehat{v}_{\mathbf{k}(\varepsilon)}$. In addition, ω_ε will stand for the family of Pfaffian forms defined by v_ε, and $\widehat{\omega}_\varepsilon$ will denote the family of Pfaffian forms defined by $\widehat{v}_\varepsilon$. A theorem on the existence of invariant analytic manifolds (see Appendix B) ensures that $\omega_\varepsilon, \widehat{\omega}_\varepsilon$ are always equivalent to $\eta_\varepsilon = (\varepsilon+i)zdw - (\varepsilon-i)w(1+zw(\dots))dz$ in ambient coordinates, and then

$$\eta_\varepsilon^{C_1} = Zdw - \lambda(\varepsilon)w(1+A_\varepsilon(Z,w))dZ \tag{4.2.3}$$

in the first chart of the blow up, and

$$\eta_\varepsilon^{c_2} = W dz - \lambda'(\varepsilon) z(1 + A'_\varepsilon(W,z)) dW \tag{4.2.4}$$

in the second chart of the blow up, so that

$$\varphi^* \eta_\varepsilon^{c_2} = \eta_\varepsilon^{c_1} \tag{4.2.5}$$

is plain, with $A_\varepsilon(Z,w) = O(Zw)$ and $A'_\varepsilon(W,z) = O(Wz)$. The numbers $\lambda(\varepsilon), \lambda'(\varepsilon)$ correspond to the ratio of eigenvalues of the singular points $(Z,w) = (0,0)$ and $(W,z) = (0,0)$ in the first and second directions of the blow up, respectively:

$$\lambda(\varepsilon) := \frac{\varepsilon - i}{2i}, \quad \lambda'(\varepsilon) := -\frac{\varepsilon + i}{2i}. \tag{4.2.6}$$

The functions $A_\varepsilon, A'_\varepsilon$ depend analytically on the parameter and are holomorphic on a neighborhood $\mathbb{C}^* \times \mathbb{D}_s$ of the exceptional divisor, for each fixed value of ε (here $\mathbb{D}_s$ is the open disk of radius s in the complex plane). Moreover, the coordinates can always be scaled before blowing up, to ensure:

$$|A_\varepsilon(Z,w)|, |A'_\varepsilon(W,z)| < \frac{1}{2} \tag{4.2.7}$$

in $\mathbb{C}^* \times \mathbb{D}_s$. In the following, $\mathscr{F}_\varepsilon$ and $\widehat{\mathscr{F}_\varepsilon}$ are the foliations of η_ε and $\widehat{\eta}_\varepsilon$ in ambient coordinates. To avoid confusions, $\widehat{\Sigma}$ denotes the local transverse section $\{z = w\}$ to the foliation $\widehat{\mathscr{F}_\varepsilon}$. The families $\mathscr{P}_\varepsilon, \widehat{\mathscr{P}_\varepsilon}$ are the monodromies of $\Sigma, \widehat{\Sigma}$ computed along the leaves of $\mathscr{F}_\varepsilon$ and $\widehat{\mathscr{F}_\varepsilon}$, respectively.

I) THE NECESSARY CONDITION.

Let the parameter ε be real. Let (z_1, w_1) denote the ambient coordinates of v_ε and (z_2, w_2) the coordinates of $\widehat{v}_\varepsilon$. By Proposition 4.1.3, the image of the embedded

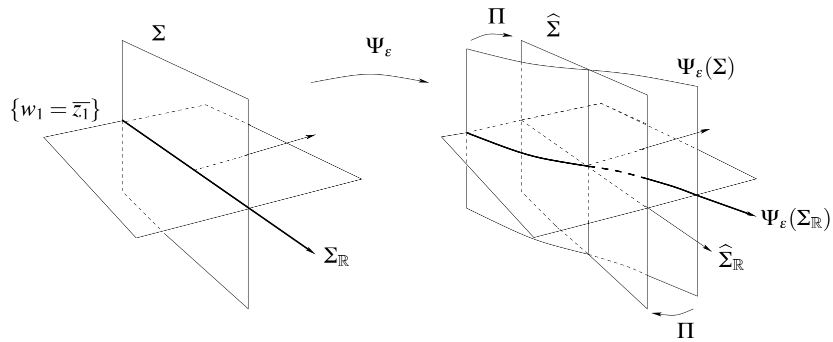


Figure 4.1: The real line and its image by the equivalence Ψ_ε.

real line $\Sigma_{\mathbb{R}} : \{w_1 = z_1\} \cap \{w_1 = \overline{z_1}\}$ is a real analytic curve $\Psi_\varepsilon(\Sigma_{\mathbb{R}})$ in $\{z_2 = \overline{w_2}\} \simeq \mathbb{R}^2$, not necessarily coinciding with $\widehat{\Sigma}_{\mathbb{R}}$ (the real line in $\widehat{\Sigma}$), see Figure 4.1.

The surface $\Psi_\varepsilon(\Sigma)$ is transverse to $\widehat{\mathscr{F}_\varepsilon}$ by definition. Therefore, it is given by the zero level set of an analytic family of germs G_ε on an open $\mathscr{V} \subset \mathbb{C}^2$:

$$\Psi_\varepsilon(\Sigma) = \{(z_2, w_2) \in \mathscr{V} : G_\varepsilon(z_2, w_2) = 0\}.$$

The family G_ε verifies $G_\varepsilon(0,0) = 0$, $\nabla_{(z,w)} G_\varepsilon(0,0) \neq 0$ and $G_\varepsilon = \mathscr{C} \circ G_{\mathscr{C}(\varepsilon)} \circ \mathscr{S} \circ \sigma$ on $\mathscr{V}$ (and thus, is real on the real plane when $\varepsilon \in \mathbb{R}$). By Lemma 4.2.1 and the Implicit Function Theorem, there exists a product neighborhood $\mathscr{U}_\varepsilon \times \mathscr{V}_\varepsilon \subset \mathbb{C}^2$ and an analytic family g_ε satisfying $g_\varepsilon(0) = 0$, such that $G_\varepsilon(z_2, w_2) = 0$ if and only if $z_2 = g_\varepsilon(w_2)$ for all $w_2 \in \mathscr{V}_\varepsilon$. Such a family can be written as $g_\varepsilon(w_2) = w_2 Z^0 + w_2^2 h(\varepsilon, w_2)$ for some $|Z^0| = 1$. Thus $\Psi_\varepsilon(\Sigma)$ is given by the set:

$$\{(Z_2, w_2) : Z_2 = Z^0 + w_2 h(\varepsilon, w_2)\}$$

in the first direction c_1 of the blow up space. Then one takes a simply connected open set U on the exceptional divisor containing 1 and Z^0 but not containing neither 0 nor ∞. Inasmuch as the two sections $\Psi_\varepsilon(\Sigma)$ and $\widehat{\Sigma}$ are locally transverse to the foliation $\widehat{\mathscr{F}_\varepsilon}$ in the blow up space, a transition between $\widehat{\Sigma}$ and $\Psi_\varepsilon(\Sigma)$ can be constructed. This is done as follows. Any leaf of the foliation induced by $\widehat{\mathscr{F}_\varepsilon}$ in the blow up space and lying over the open U is given as the graph $w_2 = f_\varepsilon(Z_2, w_*)$, for a germ of function f_ε depending analytically on ε near the origin and such that $\frac{\partial f_\varepsilon}{\partial Z_2}(Z_2, w_*)$ is small for $Z_2 \in U$ and $|w_*|$ small. Such a leaf is parametrized by the point w_* which is the intersection of the leaf with $\widehat{\Sigma}$. By transversality, this intersection is not empty if the leaf is close to the Riemann sphere (the divisor). Therefore, the system

$$\begin{cases} w_2 = f_\varepsilon(Z_2, w_*) \\ Z_2 = Z^0 + w_2 h(\varepsilon, w_2) \end{cases}$$

leads to

$$w_2 = f_\varepsilon(Z^0 + w_2 h(\varepsilon, w_2), w_*) \tag{4.2.8}$$

and then the Implicit Function Theorem yields a unique solution $w_2^* = \pi_\varepsilon(w_*) \in \Psi_\varepsilon(\Sigma)$, see Figure 4.2. The induced π_ε depends on the w_2 coordinate and is an

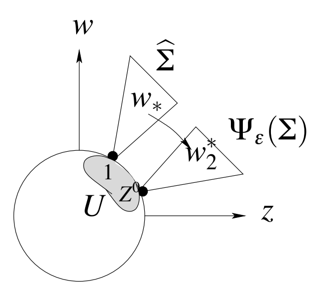


Figure 4.2: The sections $\widehat{\Sigma}$ and $\Psi_\varepsilon(\Sigma)$ in the blow up.

analytic germ of diffeomorphism in a local neighborhood of the origin, depending analytically on $\varepsilon \in \mathbb{R}$. Thus, it gives rise to a germ of analytic diffeomorphism in ambient coordinates (z_2, w_2) :

$$\Pi_\varepsilon : \widehat{\Sigma} \to \Psi_\varepsilon(\Sigma)$$

which verifies $\Pi_\varepsilon(0,0) = (0,0)$. Such a map is called a local transition. Let us show that it sends $\Pi_\varepsilon(\widehat{\Sigma}_{\mathbb{R}})$ into $\Psi_\varepsilon(\Sigma_{\mathbb{R}})$. Let $z_* = w_* \in \widehat{\Sigma}$. If $z_* = \overline{w_*}$, the real trajectory of this point is contained in the leaf parametrized by w_*. This real trajectory intersects $\Psi_\varepsilon(\Sigma)$ in a real point $(\mathbf{z},\mathbf{w})$ if $\varepsilon \in \mathbb{R}$, so $\mathbf{z} = \overline{\mathbf{w}}$. Since the leaf passing through w_* intersects $\Psi_\varepsilon(\Sigma)$ in a unique point $(z_2(w_2^*), w_2^*)$ (*i.e.* the solution to (4.2.8) is unique), it turns out that $\mathbf{w} = w_2^*$. Hence:

$$\Pi_\varepsilon(\widehat{\Sigma}_{\mathbb{R}}) \subset \Psi_\varepsilon(\Sigma_{\mathbb{R}}).$$

Consider the diagonal injection $\mathbf{i} : w \mapsto (w,w)$ defined in $\mathscr{F}_\varepsilon$ and $\widehat{\mathscr{F}_\varepsilon}$. The composition

$$\mathbf{h}_\varepsilon := \mathbf{i}^{-1} \circ \Pi_\varepsilon^{-1} \circ \Psi_\varepsilon \circ \mathbf{i}$$

is an analytic germ of diffeomorphism whose coefficients are real when ε is real, and they depend analytically on the parameter. As Ψ_ε fixes the origin, $\Pi_\varepsilon^{-1} \circ \Psi_\varepsilon(0,0) = (0,0)$ and then $\mathbf{h}_\varepsilon(0) = 0$. By construction, it is a conjugacy between Poincaré maps:

$$\mathscr{P}_\varepsilon = \mathbf{h}_\varepsilon^{\circ -1} \circ \widehat{\mathscr{P}_\varepsilon} \circ \mathbf{h}_\varepsilon.$$

It depends analytically on the parameter. Thus, it extends to an analytical conjugacy for values $\varepsilon \in \mathbb{C}$ in a neighborhood V of the origin in the parameter space.

II) THE SUFFICIENT CONDITION.

Notation and methodology. The converse statement is achieved in several steps. Suppose that the Poincaré maps $\mathscr{P}_{\varepsilon_1} : \Sigma \to \Sigma$ and $\widehat{\mathscr{P}_{\varepsilon_2}} : \widehat{\Sigma} \to \widehat{\Sigma}$ of two generic families of vector fields $v_{\varepsilon_1}, \widehat{v}_{\varepsilon_2}$ unfolding weak foci, are conjugate in a neighborhood V of the first parameter ε_1, as in Definition 7.2.1:

$$\mathscr{P}_{\varepsilon_1} = \mathbf{h}_{\varepsilon_1}^{\circ -1} \circ \widehat{\mathscr{P}}_{\mathbf{k}(\varepsilon_1)} \circ \mathbf{h}_{\varepsilon_1},$$

where the conjugacy $\mathbf{h}_{\varepsilon_1}(w) : \mathbb{D}_\rho \subset \Sigma \to \widehat{\Sigma}$ depends analytically on the parameter ε_1 and verifies $\mathscr{C} \circ \mathbf{h}_{\mathscr{C}(\varepsilon_1)} \circ \mathscr{C} = \mathbf{h}_{\varepsilon_1}$ and $\mathbf{k}(\varepsilon_1) = \varepsilon_2$ is real: $\mathscr{C} \circ \mathbf{k} \circ \mathscr{C} = \mathbf{k}$. The set $\mathbb{D}_\rho \subset \Sigma$ is the standard open disk of radius $\rho > 0$, where ρ is a small positive number such that $\mathbf{h}_\varepsilon$ is defined on $\mathbb{D}_\rho$ for every $\varepsilon_1 \in V$. Note that the map $\mathbf{k}$ yields a reparametrization in the second family of diffeomorphisms and it allows to write $\widehat{\mathscr{P}}_{\varepsilon_2} = \widehat{\mathscr{P}}_{\mathbf{k}(\varepsilon_1)} := \widehat{\mathscr{P}}_{\varepsilon_1}$. Modulo the reparametrization, the parameter is henceforth called ε. For each $\varepsilon \in V$, a local equivalence (depending analytically on the parameter) is constructed between the two foliations in ambient coordinates. For this, the origin of ambient coordinates is blown up. Two families of equivalences are constructed in the first and second charts of the blow up. In the c_1 direction the family is noted $\widehat{\Psi}_\varepsilon^{c_1}$, while in the c_2 chart the family is noted $\widehat{\Psi}_\varepsilon^{c_2}$, see Figure 4.3. The foliations induced by η_ε in the c_1, c_2 charts of the blow up are noted $\mathscr{F}_\varepsilon^{c_1}$ and $\mathscr{F}_\varepsilon^{c_2}$ respectively, and $\widehat{\mathscr{F}}_\varepsilon^{c_1}, \widehat{\mathscr{F}}_\varepsilon^{c_2}$ are the foliations of $\widehat{\eta}_\varepsilon$ in those complex charts.

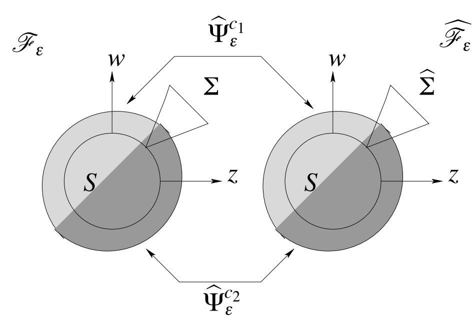


Figure 4.3: The two equivalences in charts, on a vicinity of the divisor S.

4.2.1 The equivalence in the first chart.

i) The equivalences on $\Sigma \times \mathbb{D}_\rho$. Consider $w^* \in \Sigma$ such that $|w^*| < \rho$ (*i.e.*, $w^* \in \mathbb{D}_\rho$). The equivalence $\widehat{\Psi}_\varepsilon^{c_1}$ on $\Sigma \times \mathbb{D}_\rho$ is defined by:

$$\widehat{\Psi}_\varepsilon^{c_1} : (1, w^*) \mapsto (1, \mathbf{h}_\varepsilon(w^*)) \tag{4.2.9}$$

in the (V_1, c_1) chart of the complex Möbius strip.

ii) The equivalences on a subset of $\mathbb{S}^1 \times \mathbb{C}$. The restriction of the form $\eta_\varepsilon^{c_1}$ to the cylinder $\mathbb{RP}^1 \times \mathbb{R}^2$ is non-singular and holomorphic. Thus, this restriction defines a local foliation in $\mathbb{RP}^1 \times \mathbb{R}^2$. Consider (cylindrical) solutions to $\eta_\varepsilon^{c_1} = 0$ (the first coordinate is to be parametrized by $Z = e^{i\theta}$, $\theta \in [0, 2\pi]$).

Lemma 4.2.4. *Any (cylindrical) solution to*

$$\mathbf{u_1}' = \lambda(\varepsilon)\mathbf{u_1}(1 + A(e^{i\theta}, \mathbf{u_1})), \quad \theta \in [0, 2\pi] \tag{4.2.10}$$

satisfies

$$|\mathbf{u_1}(0)|e^{-\theta\{|\varepsilon|+\frac{1}{4}\}} < |\mathbf{u_1}(\theta)| < |\mathbf{u_1}(0)|e^{\theta\{|\varepsilon|+\frac{1}{4}\}},$$

for any $\theta \in [0, 2\pi]$.

Proof. The parameter is written as $\varepsilon = \varepsilon_1 + i\varepsilon_2$, with $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$. As we consider solutions in the cylinder $|Z| = 1$, the time is parametrized by $t = i\theta$, and then (4.2.10) implies

$$d\ln \mathbf{u_1} = \frac{1}{2}(\varepsilon - i)(1 + A_\varepsilon(e^{i\theta}, \mathbf{u_1}))d\theta.$$

Thus, taking real parts,

$$\ln\left|\frac{\mathbf{u_1}}{\mathbf{u_1}(0)}\right| = \frac{1}{2}\int_0^\theta \{\varepsilon_1(1 + Re(A_\varepsilon)) + Im(A_\varepsilon)(1 - \varepsilon_2)\}d\theta$$

and now we bound the absolute value of this quantity, using the hypothesis (4.2.7):

$$\begin{aligned}\left|\ln\left|\frac{\mathbf{u_1}}{\mathbf{u_1}(0)}\right|\right| &\leq \frac{1}{2}\int_0^{\theta}\{|\varepsilon_1|(1+|Re(A_\varepsilon)|)+|Im(A_\varepsilon)|(1+|\varepsilon_2|)\}d\theta \\ &< \frac{1}{2}\int_0^{\theta}\{2|\varepsilon|+\frac{1}{2}\}d\theta = \theta\left\{|\varepsilon|+\frac{1}{4}\right\},\end{aligned}$$

and the conclusion follows. □

Definition 4.2.5. *Put $r=\rho e^{-\pi}$. We let $\mathbf{S}_r$ be the set of (cylindrical) solutions $\mathbf{u_1}$ to* (4.2.10)*, for which there exists $\theta_0\in(0,2\pi)$ such that $\mathbf{u_1}(\theta_0)\in\mathbb{D}_r$.*

Since the local foliation defined by $\eta_\varepsilon^{c_1}$ in $\mathbb{RP}^1\times\mathbb{R}^2$ is holomorphic, the set $\mathbf{S}_r$ is non-empty.

Corollary 4.2.6. *If $\mathbf{u_1}\in\mathbf{S}_r$, then $\mathbf{u_1}(0)\in\mathbb{D}_\rho$, provided $|\varepsilon|<\frac{1}{4}$.*

Proof. Using the first inequality of Lemma 4.2.4 we have:

$$\begin{aligned}|\mathbf{u_1}(0)| &< |\mathbf{u_1}(\theta_0)|e^{\theta_0(|\varepsilon|+\frac{1}{4})} \\ &\leq \rho e^{-\pi+\theta_0(|\varepsilon|+\frac{1}{4})} \\ &< \rho e^{-\pi+\frac{\theta_0}{2}} \\ &\leq \rho.\end{aligned}$$

□

Choose a point $(e^{i\theta_0},w_0)\in\mathbb{S}^1\times\mathbb{D}_r$. By Definition 4.2.5, the path $\gamma:(e^{i\theta},0)$ is lifted in the leaf of $\mathscr{F}_\varepsilon^{c_1}$ containing $w_0\in\mathbb{D}_r$ as $(e^{i\theta},\mathbf{u_1}(\theta))$, for a certain $\mathbf{u_1}\in\mathbf{S}_r$ and $\mathbf{u_1}(\theta_0)=w_0$. By Corollary 4.2.6, the point $\widetilde{w}:=\mathbf{u_1}(0)$ belongs to $\mathbb{D}_\rho$. If γ is lifted in the leaf of $\widehat{\mathscr{F}_\varepsilon^{c_1}}$ passing through the point $\mathbf{h}_\varepsilon(\widetilde{w})\in\widehat{\Sigma}$ as $(e^{i\theta},\mathbf{u_2}(e^{i\theta},\widetilde{w}))$, with $\mathbf{u_2}(1,\widetilde{w})=\mathbf{h}_\varepsilon(\widetilde{w})$, then we define the analytic change of variables by:

$$\begin{aligned}&\widehat{\Psi}_\varepsilon^{c_1}:\mathbb{S}^1\times\mathbb{D}_r\to\mathbb{S}^1\times\mathbb{C}, \\ &\widehat{\Psi}_\varepsilon^{c_1}:(e^{i\theta_0},\mathbf{u_1}(\theta_0))\mapsto(e^{i\theta_0},\mathbf{u_2}(e^{i\theta_0},\widetilde{w})).\end{aligned} \tag{4.2.11}$$

The change (4.2.11) respects the transversal fibration given by $Z = Cst$ and is indeed analytic since the germ $\mathbf{h}_\varepsilon$ is analytic and bounded. It is clear that (4.2.11) is the restriction of a (unique) holomorphic diffeomorphism conjugating $\mathscr{F}_\varepsilon^{c_1}$ and $\widehat{\mathscr{F}_\varepsilon^{c_1}}$ in a neighborhood of $\mathbb{S}^1 \times \mathbb{D}_r$.

iii) The equivalences on a neighborhood of $\mathbb{CP}^1$. The equivalence (4.2.11) extends analytically to a neighborhood of $\mathbb{CP}^1$ by means of the lifting of radial paths. The set

$$\mathbb{D}_1 = \{(Z,0) : |Z| \leq 1\}$$

is the standard unit (closed) disk of the complex plane in the first chart of the complex Möbius strip.

a) Definition of $\widehat{\Psi}_\varepsilon^{c_1}$ *in* $\mathbb{D}_1 \times \mathbb{D}_r$. Radial paths contained in the unit disk $\mathbb{D}_1$, in the c_1 direction of the blow up, are given by:

$$\begin{aligned} \gamma_{Z_1} \quad &: \quad [0, -\log|Z_1|] \to \mathbb{C} \\ & \qquad s \mapsto \gamma_{Z_1}(s) = (Z_1 e^s, 0) \end{aligned}$$

for $0 < |Z_1| < 1$. This curve lifts in the leaves of $\mathscr{F}_\varepsilon^{c_1}$ as a path

$$\gamma_{Z_1,w_1} : s \mapsto (Z_1 e^s, \mathbf{r_1}(s,w_1)), \quad \mathbf{r_1}(0,w_1) = w_1,$$

for a given w_1 small. This implies that, in terms of the form (4.2.3) defining the foliation $\mathscr{F}_\varepsilon^{c_1}$, the solution $\mathbf{r_1}(s,w_1)$ of the equation with parameter $0 < |Z_1| < 1$,

$$\frac{d\mathbf{r_1}}{ds} = \lambda(\varepsilon)\mathbf{r_1}(1 + A_\varepsilon(Z_1 e^s, \mathbf{r_1})), \tag{4.2.12}$$

and initial condition $\mathbf{r_1}(0,w_1) = w_1$ is defined on $[0, -\log|Z_1|]$.

Lemma 4.2.7. *If* $|\varepsilon| < \frac{1}{4}$, *then any solution to* (4.2.12) *on an interval* $[0,T]$, $T > 0$, *is bounded by the initial condition* w_1.

Proof. Equation (4.2.12) yields:

$$\log\left|\frac{\mathbf{r_1}}{w_1}\right| = sRe(\lambda(\varepsilon)) + \int_0^s Re\left(\lambda(\varepsilon)A_\varepsilon(Z_1e^t, \mathbf{r_1})\right)dt.$$

Write the parameter as

$$\varepsilon = \varepsilon_1 + i\,\varepsilon_2. \tag{4.2.13}$$

The hypothesis (4.2.7) shows that

$$\begin{aligned}
\log\left|\frac{\mathbf{r_1}}{w_1}\right| &= \frac{s}{2}(\varepsilon_2 - 1) + \frac{1}{2}\int_0^s \{(\varepsilon_2 - 1)Re(A_\varepsilon) + \varepsilon_1\, Im(A_\varepsilon)\}dt \\
&\leq \frac{s}{2}(\varepsilon_2 - 1) + \frac{1}{2}\int_0^s \{|\,\varepsilon_2 - 1||Re(A_\varepsilon)| + |\,\varepsilon_1\,||Im(A_\varepsilon)|\}dt \\
&< \frac{s}{2}(\varepsilon_2 - 1) + \frac{1}{4}\int_0^s \{|\,\varepsilon_2 - 1| + |\,\varepsilon_1\,|\}dt \\
&= s\left\{\frac{2(\varepsilon_2 - 1) + |\,\varepsilon_1\,| + |\,\varepsilon_2 - 1|}{4}\right\} \\
&\leq s\left\{|\,\varepsilon\,| - \frac{1}{4}\right\}.
\end{aligned}$$

Thus,

$$|\mathbf{r_1}| \leq |w_1|e^{s\left\{|\,\varepsilon\,| - \frac{1}{4}\right\}} < |w_1|. \tag{4.2.14}$$

□

b) Inverse paths. If $0 < |Z_1| < 1$, the inverse path of γ_{Z_1},

$$\begin{aligned}
\gamma_{Z_1}^{-1} \quad : \quad & [0, -\log|Z_1|] \to \widehat{\mathbb{C}} \\
& s \mapsto \gamma_{Z_1}^{-1}(s) = (Z_1 e^{-(s + \log|Z_1|)}, 0)
\end{aligned}$$

lifts in the leaf of $\widehat{\mathscr{F}_\varepsilon^{c_1}}$ through a point $(\frac{Z_1}{|Z_1|}, w^0)$ as a path

$$\gamma^{-1}_{Z_1,w^0} : s \mapsto \gamma^{-1}_{Z_1}(s) = (Z_1 e^{-(s+\log|Z_1|)}, \tilde{\mathbf{r}}_{\mathbf{1}}(s,w^0)),$$

where $\tilde{\mathbf{r}}_{\mathbf{1}}(s,w^0) = \mathbf{r}_{\mathbf{1}}(-(s+\log|Z_1|))$ and $\tilde{\mathbf{r}}_{\mathbf{1}}(0,w^0) = \mathbf{r}_{\mathbf{1}}(-\log|Z_1|) = w^0$. This path starts in $(\frac{Z_1}{|Z_1|}, w^0)$ and ends at $(Z_1, \tilde{\mathbf{r}}_{\mathbf{1}}(-\log|Z_1|, w^0))$. Obviously, inverse paths are bounded by final conditions:

$$|\tilde{\mathbf{r}}_{\mathbf{1}}(s,w^0)| \leq |\tilde{\mathbf{r}}_{\mathbf{1}}(-\log|Z_1|, w^0)| e^{s\left\{|\varepsilon|-\frac{1}{4}\right\}} < |\tilde{\mathbf{r}}_{\mathbf{1}}(-\log|Z_1|, w^0)|. \tag{4.2.15}$$

Consider the only solution $\mathbf{u}_{\mathbf{1},Z_1,w_1}$ to (4.2.10) satisfying

$$\mathbf{u}_{\mathbf{1},Z_1,w_1}(\arg Z_1) = \mathbf{r}_{\mathbf{1}}(-\log|Z_1|, w_1)$$

and define the point

$$\widetilde{w}(Z_1,w_1) := \mathbf{u}_{\mathbf{1},Z_1,w_1}(0) \in \Sigma.$$

By Lemma 4.2.7, $\mathbf{u}_{\mathbf{1},Z_1,w_1} \in \mathbf{S}_r$ if w_1 is taken in $\mathbb{D}_r$. In this case, Corollary 4.2.6 ensures that $\widetilde{w}(Z_1,w_1)$ belongs to $\mathbb{D}_\rho$. The equivalence in the first chart of the blow up is defined to be the map

$$\begin{aligned} \widehat{\Psi}_\varepsilon^{c_1} &: \mathbb{D}_1^* \times \mathbb{D}_r \to \mathbb{D}_1^* \times \mathbb{C} \\ \widehat{\Psi}_\varepsilon^{c_1} &: (Z_1,w_1) \mapsto (Z_1, \mathbf{r}_{\mathbf{2}}(Z_1,w_1)), \end{aligned} \tag{4.2.16}$$

with $\mathbf{r}_{\mathbf{2}}(Z_1,w_1) = \tilde{\mathbf{r}}_{\mathbf{1}}(-\log|Z_1|, \mathbf{u}_{\mathbf{2}}(e^{i\arg(Z_1)}, \widetilde{w}(Z_1,w_1)))$ and where the maps $\mathbf{u}_{\mathbf{1}}, \mathbf{u}_{\mathbf{2}}$ were defined in the previous paragraph (recall that $\mathbf{u}_{\mathbf{2}}(1,\widetilde{w}) = \mathbf{h}(\widetilde{w})$). As the change of coordinates is bounded, the Riemann's removable singularity Theorem implies the existence of a unique holomorphic extension $\widehat{\Psi}_\varepsilon^{c_1}$ to $\mathbb{D}_1 \times \mathbb{D}_r$.

c) Extension of $\widehat{\Psi}_{\varepsilon}^{c_1}$ to a subset of $|Z| > 1$. Define the set:

$$\mathscr{D}_1(r) \quad = \quad \{(Z,w) \in \mathbb{CP}^1 \times \mathbb{C} : |Z| \geq 1, |Zw| \leq r\}. \tag{4.2.17}$$

Radial paths contained in $|Z| > 1$ are given by:

$$\begin{aligned} \gamma_{Z_1} \quad : \quad & [0, \log|Z_1|] \to \mathbb{C} \\ & s \mapsto \gamma_{Z_1}(s) = (\frac{Z_1}{|Z_1|}e^s, 0) \end{aligned}$$

for $|Z_1| > 1$. This curve lifts in the leaves of $\mathscr{F}_{\varepsilon}^{c_1}$ as a path

$$\gamma_{Z_1,w_1} : s \mapsto (\frac{Z_1}{|Z_1|}e^s, \mathbf{r}(s,w_1)), \quad \mathbf{r}(\log|Z_1|, w_1) = w_1$$

ending at (Z_1, w_1), for a given $w_1 \in \mathbb{D}_r$. In terms of the form (4.2.3) defining the foliation $\mathscr{F}_{\varepsilon}^{c_1}$, the solution $\mathbf{r}(s,w_1)$ of the equation with parameter $|Z_1| > 1$,

$$\frac{d\mathbf{r_1}}{ds} = \lambda(\varepsilon)\mathbf{r}(1 + A_{\varepsilon}(\frac{Z_1}{|Z_1|}e^s, \mathbf{r})), \tag{4.2.18}$$

and condition $\mathbf{r}(\log|Z_1|, w_1) = w_1$ is defined on $[0, \log|Z_1|]$.

Lemma 4.2.8. *Any solution to* (4.2.18) *satisfies*

$$|\mathbf{r}(0,w_1)|e^{-s\{|\varepsilon|+\frac{1}{4}\}} < |\mathbf{r}(s,w_1)| < |\mathbf{r}(0,w_1)|e^{s\{|\varepsilon|+\frac{1}{4}\}}. \tag{4.2.19}$$

for every $s \in [0, \log|Z_1|]$ *and* $|Z_1| > 1$.

Proof. The proof follows the same steps done in the proof of Lemma 4.2.4. □

For a given $(Z_1, w_1) \in \mathscr{D}_1(r)$, we lift the path γ_{Z_1} starting at $(\frac{Z_1}{|Z_1|}, 0)$ and ending at $(Z_1, 0)$ on the leaf passing through (Z_1, w_1). Such a lifting starts at the point $(\frac{Z}{|Z|}, \mathbf{r}(0, w_1))$, where $\mathbf{r}$ is solution to (4.2.18), and ends at $(Z_1, \mathbf{r}(\log|Z_1|)) =$

(Z_1, w_1). By the first inequality of Lemma 4.2.8, the initial condition $\mathbf{r}(0, w_1)$ belongs to $\mathbb{D}_r$ provided $|\,\varepsilon\,| \leq \frac{3}{4}$:

$$\begin{aligned} |\mathbf{r}(0,w_1)| &< |\mathbf{r}(\log|Z_1|, w_1)|e^{\{|\,\varepsilon\,|+\frac{1}{4}\}\log|Z_1|} \\ &\leq |\mathbf{r}(\log|Z_1|, w_1)|e^{\log|Z_1|} \\ &= |w_1 Z_1| \\ &\leq r. \end{aligned}$$

Thus, the leaf $L^1_{\alpha'}$ containing the point (Z_1, w_1) intersects the cylinder $\mathbb{S}^1 \times \mathbb{C}$ in a curve $\mathbf{u_1} = \mathbf{u_1}(\theta) \in \mathbf{S}_r$, with $\mathbf{u_1}(\arg Z_1) = \mathbf{r}(0, w_1) \in \mathbb{D}_r$. By Corollary 4.2.6, $\mathbf{u_1}(0) \in \mathbb{D}_\rho$ and then $\widehat{\Psi}^{c_1}_\varepsilon(\frac{Z_1}{|Z_1|}, \mathbf{r}(0, w_1))$ is well defined, where $\widehat{\Psi}^{c_1}_\varepsilon$ is the change of coordinates (4.2.16). In $\widehat{\mathscr{F}^{c_1}_\varepsilon}$ the inverse of γ_{Z_1} is lifted on the leaf passing through the point $\widehat{\Psi}^{c_1}_\varepsilon(\frac{Z_1}{|Z_1|}, \mathbf{r}(0, w_1))$. The endpoint of this radial path defines $\widehat{\Psi}^{c_1}_\varepsilon$ on $\mathscr{D}_1(r)$.

4.2.2 The equivalence in the second chart.

In the second direction of the blow up space, the change of coordinates is defined plainly. The set

$$\mathbb{D}_2 = \{(W, 0) : |W| \leq 1\}$$

is the standard unit (closed) disk of the complex plane in the second complex chart. Put

$$\mathscr{D}_2(r) \quad = \quad \{(W, z) \in \mathbb{CP}^1 \times \mathbb{C} : |W| \geq 1, |Wz| \leq r\}. \tag{4.2.20}$$

Lemma 4.2.9.

$$\begin{aligned} \varphi(\mathscr{D}_1(r)) &= \mathbb{D}^*_2 \times \mathbb{D}_r, \\ \varphi^{\circ -1}(\mathscr{D}_2(r)) &= \mathbb{D}^*_1 \times \mathbb{D}_r. \end{aligned}$$

Proof. This is a tautology: if $(Z,w) \in \mathscr{D}_1(r)$ is given, then $W = \dfrac{1}{Z}$ and $z = Zw$ trivially belong to $\mathbb{D}_2^* \times \mathbb{D}_r$ and *vice-versa*. □

Definition of $\widehat{\Psi}_\varepsilon^{c_2}$ on $(\mathbb{D}_2 \times \mathbb{D}_r) \cup \mathscr{D}_2(r)$. By Lemma 4.2.9 the equivalence on the punctured product $(\mathbb{D}_2^* \times \mathbb{D}_r) \cup \mathscr{D}_2(r)$ may be defined by the formula

$$\widehat{\Psi}_\varepsilon^{c_2} := \varphi \circ \widehat{\Psi}_\varepsilon^{c_1} \circ \varphi^{\circ -1}, \tag{4.2.21}$$

where $\widehat{\Psi}_\varepsilon^{c_1}$ is the equivalence of the above paragraph (which is not defined at the point at infinity). As it is bounded, the Riemann's Theorem yields a unique holomorphic extension $\widehat{\Psi}_\varepsilon^{c_2} : (\mathbb{D}_2 \times \mathbb{D}_r) \cup \mathscr{D}_2(r) \mapsto \mathbb{C}^2$. It turns out that the two changes of coordinates thus obtained $\widehat{\Psi}_\varepsilon^{c_1}, \widehat{\Psi}_\varepsilon^{c_2}$ are analytical continuations of each other over $\mathbb{CP}^1 \times \mathbb{D}_r$.

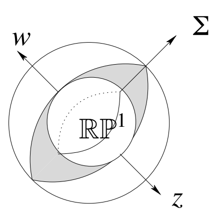


Figure 4.4: The neighborhood $\mathbb{CP}^1 \times \mathbb{D}_r$ and the union $\mathscr{D}_1(r) \cup \mathscr{D}_2(r)$.

4.2.3 The equivalence in ambient coordinates.

Define the global change of coordinates in complex charts by

$$\widehat{\Psi}_\varepsilon := \begin{cases} \widehat{\Psi}_\varepsilon^{c_1} & \text{on } \mathbb{D}_1 \times \mathbb{D}_r \\ \widehat{\Psi}_\varepsilon^{c_2} & \text{on } \mathbb{D}_2 \times \mathbb{D}_r \end{cases}$$

(notice that $(\mathbb{D}_1\times\mathbb{D}_r)\cup(\mathbb{D}_2\times\mathbb{D}_r)$ is a neighborhood of height r of the exceptional divisor, see Figure 4.4). By construction, $\widehat{\Psi}_\varepsilon$ is a local equivalence between $\mathscr{F}_\varepsilon^{c_1}\cup\mathscr{F}_\varepsilon^{c_2}$ and $\widehat{\mathscr{F}_\varepsilon^{c_1}}\cup\widehat{\mathscr{F}_\varepsilon^{c_2}}$ around $\mathbb{S}^1\times\mathbb{C}$. It depends holomorphically on $\varepsilon\in V$ by dependence of initial conditions of a differential equation. Let Ψ_ε stand for this diffeomorphism in ambient (z,w) variables:

$$\Psi_\varepsilon=\begin{cases} c_1\circ\widehat{\Psi}_\varepsilon^{c_1}\circ c_1^{\circ-1}\\ c_2\circ\widehat{\Psi}_\varepsilon^{c_2}\circ c_2^{\circ-1}.\end{cases}\tag{4.2.22}$$

Since the Riemann sphere $\mathbb{CP}^1$ retracts to the origin by c_1,c_2 charts, the equivalence Ψ_ε is defined on $(\mathbb{D}_r\times\mathbb{D}_r)\backslash\{(0,0)\}$ and is analytic there, because the maps c_1,c_2 are isomorphisms away from the exceptional divisor $\mathbb{CP}^1\times\{0\}$. By Hartogs Theorem, Ψ_ε can be extended until the origin.

The equivalence Ψ_ε is constructed by lifting paths, and when the parameter is real the holonomy map of a point with real coordinates $(x,y)\in\Sigma^+\simeq\mathbb{R}^2$ is defined as the projection of this point on the real x-axis, following the orbits of the planar system. Since the x-axis coincides with the intersection $\Sigma\cap\Sigma^+$, over which the conjugacy $\mathbf{h}_\varepsilon$ is real (when $\varepsilon\in\mathbb{R}$), we have $\Psi_\varepsilon(\Sigma^+)\subset\Sigma^+$ locally near the origin. By Proposition 4.1.3, Ψ_ε has real character. □

Part II

Almost complex structures and realization

Chapter 5

Almost complex manifolds

5.1 Almost complex structures.

The Definitions, examples, theorems, etc. of this paragraph have been taken from [24].

Definition 5.1.1. *Let V be an m-dimensional real vector space. An almost-complex structure on V (ACS for brevity) is a linear operator $J : V \to V$ with $J^2 = -Id$. Complex scalar multiplication is defined in terms of J by $(a+b\sqrt{-1})v = av + bJv$. The operator $-J$ is also an almost-complex structure on V, called the conjugate structure, and the space $(V, -J)$ is often denoted $\overline{V}$ for brevity. The standard complex vector space is $V = \mathbb{C}^n$ with J induced by multiplication by $\sqrt{-1}$.*

Lemma 5.1.2. *If V admits an almost-complex structure, then V is even-dimensional and has an induced orientation.*

Proof. The proof follows plainly, since $J^2 = -Id$ implies $(detJ)^2 = (-1)^m$, and then m is even. On the other hand, the ACS J induces naturally an orientation on

V, for if $\{e_i\}_{i=1}^n$ is chosen so that $\{e_i \wedge Je_i\}_{i=1}^n$ is a basis for V, then the sign of the volume element

$$e_1 \wedge Je_1 \wedge \ldots \wedge e_n \wedge Je_n$$

is independent of $\{e_i\}_{i=1}^n$. □

Definition 5.1.3. *An almost-complex manifold is a smooth manifold M equipped with a smooth endomorphism field $J : TM \to TM$ satisfying $J_x^2 = -Id_x$ for all $x \in M$.*

The complexified tangent bundle is

$$T_{\mathbb{C}}M = TM \otimes \mathbb{C},$$

where $\mathbb{C}$ is regarded as a trivial vector bundle. It is customary to write $v \otimes 1 = v$ and $v \otimes i = iv$. If M has an ACS J, then J extends to $T_{\mathbb{C}}M$ by $J(v \otimes \alpha) = Jv \otimes \alpha$. The tensor field J splits the complexified tangent bundle into the direct sum of bundles of eigenspaces

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M \tag{5.1.1}$$

where

$$\begin{aligned} T^{1,0}M &= \{Z \in T_{\mathbb{C}}M : JZ = iZ\} = \{X - iJX : X \in TM\} \\ T^{0,1}M &= \{Z \in T_{\mathbb{C}}M : JZ = -iZ\} = \{X + iJX : X \in TM\}. \end{aligned}$$

The complex vector space (TM, J) is $\mathbb{C}$-linearly isomorphic to $(T^{1,0}M, i)$ via the map

$$X = Re(Z) \mapsto (X - iJX) = Z.$$

Similarly, $\overline{TM} = (TM, -J)$ is $\mathbb{C}$-linearly isomorphic to $(T^{0,1}M, -i)$. Complex conjugation induces a real-linear isomorphism of $T_{\mathbb{C}}M$ which exchanges $T^{1,0}M$

and $T^{0,1}M$. The fixed point set is exactly the maximal real subspace $TM = TM \otimes 1$. A (local) section Z of $T^{1,0}M$ is called a vector field of type $(1,0)$, though Z is not a vector field on M in the sense of being tangent to a curve in M. If ordinary tangent vectors are regarded as real differential operators, then $(1,0)$ vectors are complex-valued differential operators. If TM is equipped with an almost-complex structure J, then the dual pairing induces an almost-complex structure on TM^* - also denoted by J- via

$$< J\lambda, v >=< \lambda, Jv >, \tag{5.1.2}$$

where $\lambda \in TM^*$. The associated eigenspace decomposition of $T_{\mathbb{C}}M^* = TM^* \otimes \mathbb{C}$ is

$$\begin{aligned} T_{1,0}M^* &= \{\lambda \in T_{\mathbb{C}}M^* : J\lambda = i\lambda\} = \{\xi + iJ\xi : \xi \in TM^*\} \\ T_{0,1}M^* &= \{\lambda \in T_{\mathbb{C}}M^* : J\lambda = -i\lambda\} = \{\xi - iJ\xi : \xi \in TM^*\} \end{aligned}$$

By equation (5.1.2), the space $V^*_{1,0}$ is the annihilator of $V^{0,1}$; similarly $V^*_{0,1}$ annihilates $V^{1,0}$. The exterior algebra $\bigwedge TM^*$ has a decomposition into tensors of type (p,q), namely, fully skew-symmetric elements of $\bigwedge^p T_{1,0}M^* \otimes \bigwedge^q T_{0,1}M^*$. For convenience, the space of skew-symmetric (p,q)-tensors is denoted $\bigwedge_{p,q} TM^*$. The splitting of the set of complex-valued skew-symmetric r-tensors into skew-symmetric (p,q)-tensors gives rise to spaces of (p,q)-forms. If A^r and $A^{p,q}$ denote the space of smooth r-forms and the space of smooth (p,q)-forms respectively, then

$$A^r = \bigoplus_{p+q=r} A^{p,q}.$$

In local coordinates, $A^{p,q}$ is generated by the forms $dz^I \wedge d\bar{z}^J$ with $|I| = p$ and $|J| = q$.

Example 5.1.4. *Complex Euclidean space $\mathbb{C}^n$ is an almost-complex manifold. Explicitly, let $z^\alpha = x^\alpha + \sqrt{-1}y^\alpha$ be the usual coordinates on $\mathbb{C}^n$, identified with coordinates (x,y) on $\mathbb{R}^{2n}$. The real tangent bundle and its complexification have the standard frames*

$$\left\{\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\alpha}\right\}, \quad \left\{\frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - \sqrt{-1}\frac{\partial}{\partial y^\alpha}\right), \frac{\partial}{\partial \bar{z}^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + \sqrt{-1}\frac{\partial}{\partial y^\alpha}\right)\right\}$$

while the real cotangent bundle and its complexification have coframes

$$\{dx^\alpha, dy^\alpha\}, \quad \{dz^\alpha = dx^\alpha + \sqrt{-1}dy^\alpha, d\bar{z}^\alpha = dx^\alpha - \sqrt{-1}dy^\alpha\}.$$

Mutiplication by $\sqrt{-1}$ acts only on tangent spaces, not on the actual coordinates. Thus

$$J\frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial y^\alpha}, \quad J\frac{\partial}{\partial y^\alpha} = -\frac{\partial}{\partial x^\alpha}. \tag{5.1.3}$$

The tensor field J has constant components with respect to a holomorphic coordinate system. The exterior derivative operator $d : A^r \to A^{r+1}$ maps $A^{p,q}$ to $A^{p+1,q} \oplus A^{p,q+1}$, and the corresponding boundary operators are denoted ∂ and $\bar{\partial}$. On functions,

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} =: \partial f + \bar{\partial} f.$$

Definition 5.1.5. *A map $f : (M,J) \to (M',J')$ between almost-complex manifolds is almost-complex or pseudoholomorphic if $(f_*)J = J'(f_*)$.*

The following is a straightforward consequence of the Chain Rule and the Cauchy-Riemann equations.

Proposition 5.1.6. *[24] Let $\mathbb{D}^n$ be a polydisk. A map $f : \mathbb{D}^n \to \mathbb{C}^m$ is pseudoholomorphic if and only if it is holomorphic.*

Two consequences follow at once.

Corollary 5.1.7. *An almost-complex manifold has a natural orientation. A complex manifold has a natural almost-complex structure.*

5.2 Integrability.

On an arbitrary almost-complex manifold, the exterior derivative has four type components, namely $d: A^{p,q} \to A^{p-1,q+2} \oplus A^{p,q+1} \oplus A^{p+1,q} \oplus A^{p+2,q-1} \subset A^{p+q+1}$. This is easily seen from $dA^{1,0} \subset A^{2,0} \oplus A^{1,1} \oplus A^{0,2}$ and induction on the total degree. Under a suitable first order differential condition, the unexpected components are equal to zero. To introduce this condition, first define the Nijenhuis (or torsion) tensor N_J of J by

$$N_J(X,Y) = 2([JX,JY] - [X,Y] - J[JX,Y] - J[X,JY]) \tag{5.2.1}$$

for local vector fields X and Y. Here, $[\cdot,\cdot]$ is the Lie bracket of fields X and Y.

Definition 5.2.1. *A function $f: M \to \mathbb{C}$ on a manifold M^{2n} with an almost complex structure defined by the subbundle $T_{1,0}M^*$ is called holomorphic with respect to this structure, if its differential df belongs to the subbundle $T_{1,0}M^*$ at each point. Equivalently, the function is holomorphic if its differential is $\mathbb{C}$-linear. In terms of the Example 5.1.4, f is holomorphic if and only if $\overline{\partial} f = 0$. More generally, a holomorphic p-form is a $(p,0)$-form η with $\overline{\partial}\eta = 0$.*

Grosso modo, an almost complex structure is *integrable,* if there exists an atlas of charts $U_\alpha \to \mathbb{C}^n, \cup_\alpha U_\alpha = M$, such that every coordinate of each chart is holomorphic with respect to the almost complex structure.

Definition 5.2.2. *[50] Let M be an n-dimensional manifold and let $E \subset TM$ a class C^1 fiber subbundle of rank k. Such an E is called a distribution on M. The* *distribution is said to be integrable if M is covered by open sets $\{U_i\}_i$, such that for every i there exists a class C^1 mapping*

$$\phi : U_i \to \mathbb{R}^{n-k}$$

such that, for every $x \in U_i$, the fiber at the point x, $E_x \subset T_xM$, coincides with $Ker(d\phi_x)$.

Then such a ϕ is a submersion, and each fiber $\phi^{-1}(v)$ is a closed submanifold of U with the following property: its tangent space at each point x coincides with the fiber $E_x \subset E$. Next theorem characterizes the integrable distributions.

Theorem 5.2.3 (Frobenius integrability criteria). *A distribution E is integrable if and only if for every pair of class C^1 vector fields χ, ψ contained in E, the Lie bracket $[\chi, \psi]$ is also contained in E.*

For the case of a differentiable manifold M equipped with an almost-complex structure J, integrability is equivalent with either of the conditions in the next theorem.

Theorem 5.2.4. *[24] The following are equivalent:*

- *(a) If Z and W are $(1,0)$ vector fields, then so is $[Z,W]$. In other words, $T^{1,0}M$ is involutive.*
- *(b) $T^{0,1}M$ is involutive.*
- *(c) $dA^{1,0} \subset A^{2,0} \oplus A^{1,1}$ and $dA^{0,1} \subset A^{1,1} \oplus A^{0,2}$.*
- *(d) $dA^{p,q} \subset A^{p+1,q} \oplus A^{p,q+1}$ for $p,q \geq 1$.*
- *(e) If X and Y are local vector fields, then $N_J(X,Y) = 0$.*

Consequently, if N_J vanishes identically then there is a decomposition $d = \partial + \overline{\partial}$ as in the example 5.1.4 above. Considering types and using $d^2 = 0$, it follows that

$$\partial^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0, \quad \overline{\partial}^2 = 0. \tag{5.2.2}$$

Vanishing of the torsion tensor is a necessary condition for an ACS J to be induced by a holomorphic atlas; since the components of J are constant in a holomorphic coordinate system by equation (5.1.3), the torsion of the induced ACS vanishes identically. More interestingly, vanishing of the torsion (together with a mild regularity condition) is sufficient for an almost-complex structure to be induced by a holomorphic atlas. When (M,J) is real-analytic, this amounts to the Frobenius theorem. When (M,J) satisfies less stringent regularity conditions, the theorem is a difficult result in partial differential equations, and is known as the Newlander-Nirenberg Theorem. The weakest hypothesis is that (M,J) be of Hölder class $C^{1,\alpha}$ for some $\alpha > 0$. These are the contents of the next subsection.

A historical note ([39]). Any smooth foliation of a manifold M can be described as the set of solutions to an associated system of differential equations on M. If the foliation dimension is 2 or more, then these differential equations are "overdetermined", so that appropriate integrability conditions must be satified.

These facts are described in the classical result which is often called the "Frobenius Theorem" ([17]). Actually, as Frobenius himself pointed out, the theorem in question has been proved a decade earlier by A. Clebsch ([8]). In fact, a recognizable version had been proved already in 1840, by F. Deahna ([12]).

It is sad to relate that Deahna did not profit by being so far ahead of his time. According to the entry in Poggendorff, Deahna had barely attained the rank of "Hülfslehrer" in a secondary school when he died in 1844, at the age of 28.

5.3 Newlander-Nirenberg's Theorem.

The problem of introducing analytic coordinates is purely local. In the general case of an almost complex structure on a manifold M^{2n}, in any coordinate patch we may choose complex valued coordinates $z^1,...,z^{2n}$ with $z^{j+n} = \overline{z^j}$. We shall refer to them as $z^1,...,z^n$ and we denote $\overline{z^j} = \bar{z}^j$. Suppose that the almost complex structure is given by a matrix $J = [h_i^j]_{i,j}$ with $i, j = 1,...,2n$. If we are given local complex analytic coordinates $\xi^1,...,\xi^n$ then, it is shown (see [40]) that each ξ of these coordinates satisfy the integrability conditions

$$\sum_{j=1}^{2n} \frac{\partial \xi}{\partial z^j}(h_\lambda^j - i\delta_\lambda^j) = 0, \qquad \lambda = 1,...,2n. \tag{5.3.1}$$

In addition, we have

$$2id\xi^j = \sum_k \sum_\lambda \frac{\partial \xi^j}{\partial z^k}(h_\lambda^k + i\delta_\lambda^k)dz^\lambda = \sum_k \frac{\partial \xi^j}{\partial z^k}\left[\sum_\lambda (h_\lambda^k + i\delta_\lambda^k)dz^\lambda\right].$$

It follows that the system of forms $d\xi^j$, $j = 1,...,n$, is equivalent to the system $\sum_\lambda (h_\lambda^k + i\delta_\lambda^k)dz^\lambda$, $k = 1,...,2n$.

Definition 5.3.1. *A complex valued function ξ satisfying* (5.3.1) *is called holomorphic with respect to the given almost complex structure.*

Since $J^2 = -Id$, only the last n equations (*i.e.* those corresponding to $\lambda = n+1,...,2n$) are independent. After solving these for the derivatives $\overline{\partial}_j\xi = \dfrac{\partial \xi}{\partial \bar{z}^j}$ we rewrite Equations (5.3.1) in the useful form

$$L_j(\xi) := \overline{\partial}_j\xi - \sum_{k=1}^n \alpha_j^k \partial_k \xi = 0, \tag{5.3.2}$$

with $\alpha_j^k = 0$ at $z^1 = ... = z^n = 0$, for all $j = 1,...,n$, and where $\partial_k = \partial/\partial z^k$. Notice that L_j is a derivation, and hence a vector field. In terms of the α_j^k, the system of forms $d\xi^j$, $j = 1,...,n$ is equivalent to the system

$$dz^k + \sum_\ell \mathfrak{a}^k_\ell d\bar{z}^\ell, \qquad k = 1,...,n. \tag{5.3.3}$$

Indeed, using (5.3.2), it is easily seen that

$$\begin{aligned} d\xi^j &= \sum_{k=1}^{2n} \frac{\partial \xi^j}{\partial z^k} dz^k = \sum_{k=1}^{n} \frac{\partial \xi^j}{\partial z^k} dz^k + \sum_{\ell=1}^{n} \frac{\partial \xi^j}{\partial \bar{z}^\ell} d\bar{z}^\ell \\ &= \sum_{k=1}^{n} \frac{\partial \xi^j}{\partial z^k} dz^k + \sum_{\ell=1}^{n} \left[\sum_{k=1}^{n} \mathfrak{a}^k_\ell \partial_k \xi^j \right] d\bar{z}^\ell = \sum_{k=1}^{n} \left[\frac{\partial \xi^j}{\partial z^k} dz^k + \sum_{\ell=1}^{n} \mathfrak{a}^k_\ell \frac{\partial \xi^j}{\partial z^k} d\bar{z}^\ell \right] \\ &= \sum_k \frac{\partial \xi^j}{\partial z^k} (dz^k + \sum_\ell \mathfrak{a}^k_\ell d\bar{z}^\ell), \end{aligned}$$

so the subspace generated by the $d\xi^j$ coincides with that generated by the $dz^k + \sum_\ell \mathfrak{a}^k_\ell d\bar{z}^\ell$. The integrability conditions (5.3.1) can also be written under the form $[L_j, L_m] = 0$, which yields

$$\bar{\partial}_m \mathfrak{a}^k_j - \sum_{p=1}^{n} \mathfrak{a}^p_m \partial_p \mathfrak{a}^k_j = \bar{\partial}_j \mathfrak{a}^k_m - \sum_{p=1}^{n} \mathfrak{a}^p_j \partial_p \mathfrak{a}^k_m, \qquad j,k,m = 1,...,n. \tag{5.3.4}$$

For $n = 1$, in a complex chart $z \in \mathbb{C}$ any subbundle $L^{1,0}$ is spanned by a single form $\omega = adz + bd\bar{z}$, with $a \neq 0$. Since ω makes sense only up to proportionality, we can without loss of generality assume that the 1-form defining an arbitrary almost complex structure on $\mathbb{C}$ or its subdomain, is given by

$$\omega = dz + \mu d\bar{z}, \quad |\mu(z)| < 1. \tag{5.3.5}$$

It will be referred to as the μ-complex structure. The sufficient condition for integrability of the μ-complex structure in dimension one is given in the next theorem.

Theorem 5.3.2 (L. Ahlfors - L. Bers). *A μ-complex structure on the domain $\Omega \subset \mathbb{C}$ is integrable if $\mu = \mu(z)$ is a L^∞-measurable function with the norm*

$$||\mu||_{L^\infty(\Omega)} < 1. \tag{5.3.6}$$

The smooth version of this result in dimension 1 is as follows.

Theorem 5.3.3 (L. Ahlfors - L. Bers). *[27] For a μ-complex structure with a C^∞-smooth function $\mu : \Omega \to \mathbb{C}$ satisfying the integrability condition* (5.3.6)*, there exists an infinitely smooth chart $g^\mu : \Omega \to \mathbb{C}$ that is holomorphic in the sense of this structure.*

In higher dimension, this result is known as the Newlander-Nirenberg Theorem. It looks surprisingly simple; however, its proof is highly non-trivial. It states that every integrable almost-complex structure is defined by a unique analytic structure.

Theorem 5.3.4 (A. Newlander - L. Nirenberg. General case). *[40] If the coefficients $\mathfrak{a}_j^k$ in* (5.3.2) *are of class C^{2n} in a neighborhood of the origin, and satisfy the integrability conditions* (5.3.4)*, then, in some neighborhood of the origin, there exists n solutions $\xi^1,...,\xi^n$ of* (5.3.2) *such that the Jacobian of the collection $\xi^1,...,\xi^n,\overline{\xi}^1,...,\overline{\xi}^n$ with respect to $z^1,...,z^n,\overline{z}^1,...,\overline{z}^n$ is different from zero, so that the equations* (5.3.2) *reduce to $\dfrac{\partial \xi}{\partial \overline{z}^j} = 0$, for each function ξ which turns out to be of class $C^{2n+\beta}$, for any positive $\beta < 1$. If, in addition, the coefficients $\mathfrak{a}_j^i$ are of class $C^{k+\alpha}$, for integer $k \geq 2n$, and $0 < \alpha < 1$, then each ξ^j is of class $C^{k+1+\alpha}$.*

Chapter 6

Realization of a Poincaré family

6.1 Realization of an admissible family.

Recall that, by the formal classification Theorem 1.4.2, a germ of generic family v_ε unfolding a weak focus is formally orbitally equivalent to

$$\begin{aligned} \dot{z} &= z\frac{i+\varepsilon+su}{1+A(\varepsilon)u} \\ \dot{w} &= w\frac{-i+\varepsilon+su}{1+A(\varepsilon)u} \end{aligned} \tag{6.1.1}$$

for some family of real constants $A(\varepsilon)$ with $A(0)\neq 0$, where $u=zw$, and $s=\pm 1$ has been defined in Proposition 1.2.1 (the two signs defining two non-orbitally equivalent cases). Notice that the linear part of the family above is given by $z(\varepsilon+i)\frac{\partial}{\partial z}+w(\varepsilon-i)\frac{\partial}{\partial w}$. Since the eigenvalues are analytic invariants, then the parameter ε is also an analytic invariant. This justifies the following definition.

Definition 6.1.1. *The parameter ε of the formal normal form* (6.1.1) *is called the "canonical parameter".*

Remark. Note that the multiplier at the origin of the Poincaré map of the field (6.1.1) is equal to

$$e^{2\pi\varepsilon}. \tag{6.1.2}$$

We will show (Theorem 7.1.1) that there is a simultaneous "preparation" of the families unfolding the field (*i.e.* v_ε) and the Poincaré map, such that the multiplier at the origin coincides with (6.1.2), so that the parameter is also an analytic invariant of the Poincaré map. In these coordinates the invariant manifold of the system has equation $zw = -s\,\varepsilon$, and the family of diffeomorphisms is of the form

$$\mathscr{P}_{\widehat{\varepsilon}}(\mathbf{w}) = \mathbf{w} + \mathbf{w}(\widehat{\varepsilon} \pm \mathbf{w}^2)[1 + D(\widehat{\varepsilon}) + E(\widehat{\varepsilon})\mathbf{w}^2 + \mathbf{w}(\widehat{\varepsilon} \pm \mathbf{w}^2)h(\widehat{\varepsilon}, \mathbf{w})] \tag{6.1.3}$$

with fixed points $\mathbf{w}_0 = 0$, and $\mathbf{w}_\pm = \pm\sqrt{-s\widehat{\varepsilon}}$, where $s = \pm 1$, the coefficient of the third order term of the field.

Definition 6.1.2. *Consider a family $\mathscr{Q}_\varepsilon$ unfolding a codimension one resonant diffeomorphism $\mathscr{Q}$ with multiplier equal to -1. Then the formal normal form $\mathscr{Q}_{0,\varepsilon}$ of $\mathscr{Q}_\varepsilon$ is the semi-Poincaré map (or semi-monodromy) of the vector field* (6.1.1)*, namely*

$$\mathscr{Q}_{0,\varepsilon} = \mathscr{L}_{-1} \circ \tau_\varepsilon^\pi, \tag{6.1.4}$$

where τ_ε^π is the time π-map of the equation:

$$\dot{w} = \frac{w(\varepsilon \pm w^2)}{1 + A(\varepsilon)w^2} \tag{6.1.5}$$

(this field is obtained when one computes the equation satisfied by $\sqrt{u}$, in (6.1.1)*), and $\mathscr{L}_{-1}(w) = -w$, for any $w \in \mathbb{C}$.*

Lemma 6.1.3. *Let $\mathscr{Q}_\varepsilon$ a prepared family (i.e. such that $\mathscr{Q}_\varepsilon^{\circ 2}$ has the form* (6.1.3)*) unfolding a codimension one resonant diffeomorphism $\mathscr{Q}$ with multiplier equal to*

-1, *and let* $\mathscr{Q}_{0,\varepsilon}$ *be its formal normal form* (*i.e. the map* (6.1.4)), *with the same canonical parameter* ε . *Then, for any* $N \in \mathbb{N}^*$ *there exists a real germ of family of diffeomorphisms* f_ε *tangent to the identity such that:*

$$\mathscr{Q}_\varepsilon \circ f_\varepsilon - f_\varepsilon \circ \mathscr{Q}_{0,\varepsilon} = O(w^{N+1}(w^2-\varepsilon)^{N+1}). \tag{6.1.6}$$

Proof. The proof is a slight modification of Theorem 6.2 in [43], being given that the preparation of the family of diffeomorphisms is slightly different as well. □

Theorem 6.1.4 (Realization of a generic real family of diffeomorphisms). *Consider the class of prepared germs of holomorphic diffeomorphisms* $\mathscr{Q}_\varepsilon : (\mathbb{C},0) \to (\mathbb{C},0)$ *verifying the hypotheses of Lemma* (6.1.3), *with real coefficients* $c_k(\varepsilon)$ *depending analytically on the canonical parameter* ε, *such that* $2c_2(\varepsilon)^2 + c_3(\varepsilon)[1 + c_1(\varepsilon)^2] \neq 0$ *for values of* ε *in a small symmetric neighborhood* $V \subset \mathbb{C}$ *of the origin in the parameter space. Then the square* $\mathscr{Q}_\varepsilon^{\circ 2}$ *is always the Poincaré map (or monodromy)* $\mathscr{P}_\varepsilon$ *of a generic family unfolding an order one weak focus.*

Proof. For every ε fixed in a neighborhood V of the origin, the family of diffeomorphisms is realized on an abstract manifold constructed in the covering space of the foliation induced by the normal form (6.1.1). This manifold may be identified with a neighborhood of the origin minus the axes in $\mathbb{C}^2$, by means of Newlander-Nirenberg Theorem. We show that, in this particular case, the smooth chart respects the real character of the foliation. By construction, the Poincaré map will coincide with $\mathscr{Q}_\varepsilon$. Every step in the construction is analytic in the parameter. We recall that the monoidal map ρ is given by

$$\begin{aligned} c_1 &: (Z,w) \mapsto (Zw,w), \\ c_2 &: (W,z) \mapsto (z,zW) \end{aligned} \tag{6.1.7}$$

in complex charts covering the blow up space or complex Möbius strip. By Lemma 6.1.3, $\mathscr{Q}_\varepsilon$ has a "decomposition on the right"

$$\mathscr{Q}_\varepsilon = (id + g_\varepsilon) \circ \mathscr{Q}_{0,\varepsilon} \tag{6.1.8}$$

for a large integer $N \in \mathbb{N}$, where the family g_ε is $(N+1)$-flat in w at the origin:

$$g_\varepsilon(w) = O(w^{N+1}(w^2-\varepsilon)^{N+1})). \tag{6.1.9}$$

Consider also the "decomposition on the left"

$$\mathscr{Q}_\varepsilon = \mathscr{Q}_{0,\varepsilon} \circ (id + \widehat{g}_\varepsilon), \tag{6.1.10}$$

where

$$\widehat{g}_\varepsilon := \mathscr{Q}_{0,\varepsilon}^{\circ -1} \circ g_\varepsilon \circ \mathscr{Q}_{0,\varepsilon} \tag{6.1.11}$$

is, by definition, $(N+1)$-flat in w at the origin: $\widehat{g}_\varepsilon(w) = O(w^{N+1}(w^2-\varepsilon)^{N+1})$.

6.1.1 The foliation in the first chart.

First off, we pull the formal field (6.1.1) back into (Z,w) coordinates by the map c_1 (6.1.7):

$$\begin{aligned} \dot{Z} &= Z\frac{2i}{1+A(\varepsilon)Zw^2} \\ \dot{w} &= w\frac{-i+\varepsilon \pm Zw^2}{1+A(\varepsilon)Zw^2}. \end{aligned} \tag{6.1.12}$$

Let $v_{0,\varepsilon}^{c_1}$ be the normal form defined by (6.1.12), and $\mathscr{F}_{0,\varepsilon}^{c_1}$ be its foliation on the product $\mathbb{C}^* \times \mathbb{D}_w$, where $\mathbb{D}_w$ is the standard unit disk of the w axis. Consider the global region defined as

$$\widetilde{K}_{c_1} = \left\{ \widetilde{Z} \in Cov(\mathbb{C}^*) : -\frac{\pi}{4} < \arg(\widetilde{Z}) < 2\pi + \frac{\pi}{4} \right\}$$

in the covering space $Cov(\mathbb{C}^*)$ of the exceptional divisor minus the origin and the point at infinity, see Figure 6.1. The pullback of $v_{0,\varepsilon}^{c_1}$ by the covering map

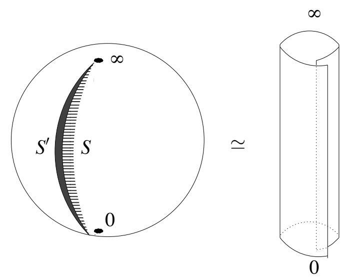


Figure 6.1: The domain of $\widetilde{Z}$ in the covering space $Cov(\mathbb{C}^*)$.

$\pi_{c_1} : \widetilde{K}_{c_1} \times \mathbb{D}_w \to \mathbb{C}^* \times \mathbb{D}_w$, defines a field $\widetilde{v}_\varepsilon^{c_1} = \widetilde{v}_\varepsilon^{c_1}(\widetilde{Z}, w)$ and a foliation $\widetilde{\mathscr{F}_\varepsilon^{c_1}}$ on the product

$$\widetilde{M} := \widetilde{K}_{c_1} \times \mathbb{D}_w.$$

The leaves of $\widetilde{\mathscr{F}_\varepsilon^{c_1}}$ around the *flaps*

$$\begin{aligned} S'_{c_1} &= \{\widetilde{Z}' \in \widetilde{K}_{c_1} : -\frac{\pi}{4} < \arg(\widetilde{Z}') < \frac{\pi}{4}\} \\ S_{c_1} &= \{\widetilde{Z} \in \widetilde{K}_{c_1} : 2\pi - \frac{\pi}{4} < \arg(\widetilde{Z}) < 2\pi + \frac{\pi}{4}\} \end{aligned}$$

are identified by means of a *sealing map* $\Upsilon_\varepsilon : S'_{c_1} \times \mathbb{D}_w \to S_{c_1} \times \mathbb{C}$, which preserves the first coordinate and respects the field $\widetilde{v}_\varepsilon^{c_1}$ and the foliation $\widetilde{\mathscr{F}_\varepsilon^{c_1}}$. The sealing Υ_ε is constructed as follows. For small values of w, the holonomy map $h_{\varepsilon,Z}$: $\{Z\} \times \mathbb{D}_w \to \{1\} \times \mathbb{D}_w$ along the leaves of the foliation $\mathscr{F}_{0,\varepsilon}^{c_1}$ is covered by two holonomy maps, $h_{\varepsilon,\widetilde{Z}'} : \{\widetilde{Z}'\} \times \mathbb{D}_w \to \Sigma' \times \mathbb{D}_w$ and $h_{\varepsilon,\widetilde{Z}} : \{\widetilde{Z}\} \times \mathbb{D}_w \to \Sigma \times \mathbb{D}_w$ along the leaves of $\widetilde{\mathscr{F}_\varepsilon^{c_1}}$.

Definition 6.1.5 (Sectorial holonomy in the c_1 chart). *The holonomies $h_{\varepsilon,\widetilde{Z}'}, h_{\varepsilon,\widetilde{Z}}$ on the sectors S'_{c_1} and S_{c_1} are oriented in the following way, see Figure 6.2:*

1. *If* $Im(\widetilde{Z}'), Im(\widetilde{Z}) > 0$, *then the holonomy is negatively oriented (clockwise) and is denoted by* $h^-_{\varepsilon,\widetilde{Z}'}$ *and* $h^-_{\varepsilon,\widetilde{Z}}$ *respectively on* S'_{c_1} *and* S_{c_1}.
2. *If* $Im(\widetilde{Z}'), Im(\widetilde{Z}) < 0$, *then the holonomy is positively oriented (counter-clockwise) and is denoted by* $h^+_{\varepsilon,\widetilde{Z}'}$ *and* $h^+_{\varepsilon,\widetilde{Z}}$ *respectively over* S'_{c_1} *and* S_{c_1}.

The convention:

$$\lim_{\widetilde{Z}\to\widetilde{1}} h^+_{\varepsilon,\widetilde{Z}} = id \tag{6.1.13}$$

will be taken into account when $\widetilde{Z} \in \widetilde{K}_{c_1}$.

Then the sealing $\Upsilon_\varepsilon : S'_{c_1} \times \mathbb{D}_w \to S_{c_1} \times \mathbb{C}$ is given by:

$$\Upsilon_\varepsilon(\widetilde{Z}', w) = (\widetilde{Z}, \Delta_\varepsilon(\widetilde{Z}', w)) \tag{6.1.14}$$

where the family $\Delta_\varepsilon : \{\widetilde{Z}'\} \times \mathbb{D}_w \to \times\mathbb{C}$ is defined using the decomposition (6.1.8):

$$\Delta_\varepsilon(\widetilde{Z}', w) = (h^+_{\varepsilon,\widetilde{Z}})^{\circ -1} \circ (id + g_\varepsilon) \circ h^+_{\varepsilon,\widetilde{Z}'}(w), \tag{6.1.15}$$

and the points $\widetilde{Z}' \in S'_{c_1}$ and $\widetilde{Z} \in S_{c_1}$ both project by π_{c_1} onto the same point Z on the separatrix. The map Υ_ε is, indeed, well defined in $S' \times \{|w| \leq r\}$ and holomorphic on its image for $r > 0$ small. It depends analytically on the parameter.

Remark 6.1.6.

1. *By definition, the sealing* Υ_ε *may be analytically extended to a larger domain*
$$\{\widetilde{Z} \in \widetilde{K}_{c_1} : -\frac{\pi}{4} < \arg(\widetilde{Z}) < \pi\} \times \mathbb{D}_w.$$
2. *The holonomy* $\mathscr{Q}_{0,\varepsilon}$ *of* $\mathscr{F}^{c_1}_{\varepsilon,0}$ *lifts as the holonomy of* $\widetilde{\mathscr{F}^{c_1}_\varepsilon}$ *on* $\widetilde{M}$. *Extending the definition of the sectorial holonomy map to all of* $\widetilde{M}$, *the holonomy of* $\widetilde{M}$ *decomposes as*
$$h^+_{\varepsilon,\widetilde{Z}} \circ (h^-_{\varepsilon,\widetilde{Z}})^{\circ -1} = \mathscr{Q}_{0,\varepsilon}, \tag{6.1.16}$$

for all Z in a neighborhood of the origin in the c_1 chart of the blow up, where $\widetilde{Z}$ is a point in the covering space projecting as Z. Note that (6.1.13) *yields:*

$$(h^-_{\varepsilon,\widetilde{1}})^{\circ -1} = \mathscr{Q}_{0,\varepsilon}. \tag{6.1.17}$$

6.1.2 The foliation in the second complex chart.

The pullback of the formal normal form (6.1.1) by the map c_2 (6.1.7) is the equation:

$$\begin{aligned} \dot{W} &= W\frac{-2i}{1+A(\varepsilon)Wz^2} \\ \dot{z} &= z\frac{i+\varepsilon \pm Wz^2}{1+A(\varepsilon)Wz^2}. \end{aligned} \tag{6.1.18}$$

Remark 6.1.7. *Corollary 3.4.5 ensures that the holonomy $\ell_{\varepsilon,1} : \{W=1\} \to \{W=1\}$ of* (6.1.18) *coincides with $\mathscr{Q}_{0,\varepsilon}^{\circ -1}$.*

As this is a global field, again we will work on a global region defined as

$$\widetilde{K}_{c_2} = \Big\{\widetilde{W} \in Cov(\mathbb{C}^*) : -2\pi - \frac{\pi}{4} < \arg(\widetilde{W}) < \frac{\pi}{4}\Big\}.$$

Call $v_{0,\varepsilon}^{c_2}$ the normal form defined by (6.1.18) and $\mathscr{F}_{0,\varepsilon}^{c_2}$ its foliation. The field $v_{0,\varepsilon}^{c_2}$ is the pushforward of $v_{0,\varepsilon}^{c_1}$ by the change of coordinates $\varphi = c_2^{-1} \circ c_1$:

$$v_{0,\varepsilon}^{c_2} = \varphi_* v_{0,\varepsilon}^{c_1} \circ \varphi.$$

In addition, an easy calculation shows that the families $v_{0,\varepsilon}^{c_1}, v_{0,\varepsilon}^{c_2}$ are related through:

$$v_{0,\varepsilon}^{c_2} = \mathscr{S} \circ v_{0,\mathscr{C}(\varepsilon)}^{c_1} \circ \mathscr{S}. \tag{6.1.19}$$

where $\mathscr{S}(Z,w) = (\overline{Z},\overline{w})$, see Remark 3.3.2. This is not surprising since the field $v_{0,\varepsilon}$ is the formal normal form of a weak focus (Proposition 3.3.6). The pullback of this normal form by the covering map $\pi_{c_2} : \widetilde{K}_{c_2} \times \mathbb{D}_z \to \mathbb{C}^* \times \mathbb{D}_z$, yields a field $\widetilde{v}_\varepsilon^{c_2} = \widetilde{v}_\varepsilon^{c_2}(\widetilde{W}, z)$ and a foliation $\widetilde{\mathscr{F}}_\varepsilon^{c_2}$ on the product

$$\widetilde{N} := \widetilde{K}_{c_2} \times \mathbb{D}_z.$$

Remark 6.1.8. *The families $\widetilde{v}_\varepsilon^{c_1}, \widetilde{v}_\varepsilon^{c_2}$ still satisfy the property:*

$$\widetilde{v}_\varepsilon^{c_2} = \mathscr{S} \circ \widetilde{v}^{c_1}_{\mathscr{C}(\varepsilon)} \circ \mathscr{S}, \tag{6.1.20}$$

because $\pi^*_{c_1}, \pi^*_{c_2}$ commute with $\mathscr{S}$.

A sealing map $\lambda_\varepsilon : S'_{c_2} \times \mathbb{D}_z \to S_{c_2} \times \mathbb{C}$ is built between local foliations around the flaps

$$\begin{aligned} S'_{c_2} &= \{\widetilde{W}' \in \widetilde{K}_{c_2} : -\frac{\pi}{4} < \arg(\widetilde{W}) < \frac{\pi}{4}\} \\ S_{c_2} &= \{\widetilde{W} \in \widetilde{K}_{c_2} : -2\pi - \frac{\pi}{4} < \arg(\widetilde{W}') < -2\pi + \frac{\pi}{4}\}. \end{aligned}$$

Notice that, by definition,

$$\begin{aligned} \varphi(S'_{c_1}) &= S'_{c_2} \\ \varphi(S_{c_1}) &= S_{c_2}. \end{aligned} \tag{6.1.21}$$

The sealing λ_ε is constructed as follows. Let $\ell_{\varepsilon,\widetilde{W}'} : \{\widetilde{W}'\} \times \mathbb{D}_z \to \Sigma' \times \mathbb{D}_z$ and $\ell_{\varepsilon,\widetilde{W}} : \{\widetilde{W}\} \times \mathbb{D}_z \to \Sigma \times \mathbb{D}_z$ be the holonomies induced by the fields around the flaps, obtained by lifting the holonomy $\ell_{\varepsilon,W} : \{W\} \times \mathbb{D}_z \to \{1\} \times \mathbb{D}_z$ of the W separatrix along the leaves of the foliation $\mathscr{F}^{c_2}_{0,\varepsilon}$ in the second chart of the complex Möbius strip. The map $\ell_{\varepsilon,W}$ is always well defined for small values of w.

Definition 6.1.9 (Sectorial holonomy in the c_2 chart). *The holonomies $\ell_{\varepsilon,\widetilde{W}'}$ and $\ell_{\varepsilon,\widetilde{W}}$ on the sectors S'_{c_2} and S_{c_2} are oriented in the following way, see Figure 6.2:*

1. *If $Im(\widetilde{W}), Im(\widetilde{W}') > 0$, then the holonomy is negatively oriented (clockwise) and is denoted by $\ell^-_{\varepsilon,\widetilde{W}'}$ and $\ell^-_{\varepsilon,\widetilde{W}}$ respectively on S'_{c_2} and S_{c_2}.*

2. *If $Im(\widetilde{W}), Im(\widetilde{W}') < 0$, then the holonomy is positively oriented (counter-clockwise) and is denoted by $\ell^{+}_{\varepsilon,\widetilde{W}'}$ and $\ell^{+}_{\varepsilon,\widetilde{W}}$ respectively over S'_{c_2} and S_{c_2}.*

Again, the convention:

$$\lim_{\widetilde{W}\to\widetilde{1}} \ell^{-}_{\varepsilon,\widetilde{W}} = id \tag{6.1.22}$$

will be considered for values $\widetilde{W} \in \widetilde{K}_{c_2}$.

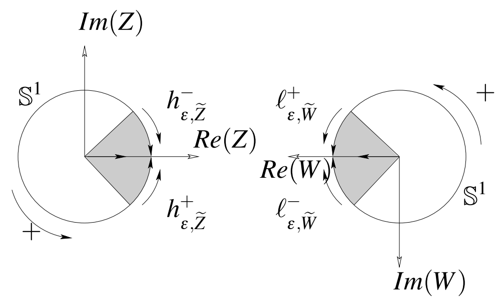


Figure 6.2: The holonomy on the sectors.

The sealing $\lambda_\varepsilon : S'_{c_2} \times \mathbb{D}_z \to S_{c_2} \times \mathbb{C}$ is defined by:

$$\lambda_\varepsilon(\widetilde{W}', z) = (\widetilde{W}, \nabla_\varepsilon(\widetilde{W}', z)) \tag{6.1.23}$$

where $\widetilde{W}' \in S'_{c_2}$ and $\widetilde{W} \in S_{c_2}$ project by π_{c_2} onto the same point W on the separatrix. Notice that λ_ε respects the fibration $W = Cst$. The family $\nabla_\varepsilon : \{\widetilde{W}'\} \times \mathbb{D}_z \to \times\mathbb{C}$ is defined by means of the decomposition (6.1.10):

$$\nabla_\varepsilon(\widetilde{W}', z) = (\ell^{+}_{\varepsilon,\widetilde{W}})^{\circ -1} \circ (id + \widehat{g}_\varepsilon) \circ \ell^{+}_{\varepsilon,\widetilde{W}'}(z). \tag{6.1.24}$$

Remark 6.1.10.

1. *By definition, the sealing λ_ε may be analytically extended to a larger domain*

$$\{\widetilde{W} \in \widetilde{K}_{c_2} : -\pi < \arg(\widetilde{W}) < \frac{\pi}{4}\} \times \mathbb{D}_w.$$

2. *The holonomy* $\mathscr{Q}_{0,\varepsilon}^{\circ -1}$ *of* $\mathscr{F}_{\varepsilon,0}^{c_2}$ *lifts as the holonomy of* $\widetilde{\mathscr{F}_{\varepsilon}^{c_2}}$ *on* $\widetilde{N}$*. Extending the definition of the sectorial holonomy map to all of* $\widetilde{N}$*, the holonomy of* $\widetilde{N}$ *decomposes as*
$$\ell^{+}_{\varepsilon,\widetilde{W}} \circ (\ell^{-}_{\varepsilon,\widetilde{W}})^{\circ -1} = \mathscr{Q}_{0,\varepsilon}^{\circ -1}, \tag{6.1.25}$$
for all W *in a neighborhood of the origin in the* c_2 *chart of the blow up, where* $\widetilde{W}$ *is a point in the covering space projecting as* W*. Then,* (6.1.22) *yields:*
$$\ell^{+}_{\varepsilon,\widetilde{1}} = \mathscr{Q}_{0,\varepsilon}^{\circ -1}. \tag{6.1.26}$$

Lemma 6.1.11. *The maps* Δ_ε *and* ∇_ε *are related by Schwarz reflection:*
$$\mathscr{C} \circ \Delta_{\mathscr{C}(\varepsilon)} \circ \mathscr{S} = \nabla_\varepsilon \tag{6.1.27}$$
on a symmetric neighborhood V *of the parameter.*

Proof. Tildes are dropped. Suppose that ε is real. Choose a point $(Z,w) \in \pi_{c_1}^{-1}({}^{\mathbb{R}}\mathbb{M}_{c_1})$. It is shown that $\varphi|_{\mathbb{R}\mathbb{M}} \equiv \mathscr{S}$ (Corollary 3.3.4). Then the point (Z,w) is read as
$$(W,z) = \left(\frac{1}{Z}, Zw\right) = \varphi(Z,w) = \mathscr{S}(Z,w) = (\overline{Z},\overline{w})$$
in the second complex chart of the blow up in the covering space. Consider the holonomies $h_{\varepsilon,Z} : \{Z\} \times \mathbb{D}_w \to \{1\} \times \mathbb{C}$ and $\ell_{\varepsilon,W} : \{W\} \times \mathbb{D}_z \to \{1\} \times \mathbb{C}$ in the first and second charts of the blow up of the covering space. The sectorial orientation is chosen according to Definitions 6.1.5 and 6.1.9. The image of the point $(1, h^{\pm}_{\varepsilon,Z}(w))$ under φ is the point $(1, \ell^{\mp}_{\varepsilon,W}(z)) = \varphi(1, h^{\pm}_{\varepsilon,Z}(w))$. Notice that orientation is reversed because the map φ is orientation reversing. Since the holonomy is real analytic, the choice of (Z,w) on the preimage of the real complex Möbius strip leads to
$$(1, \ell^{\mp}_{\varepsilon,W}(z)) \;\; = \;\; (1, \overline{h^{\pm}_{\varepsilon,Z}(w)}).$$

Inasmuch as $Z = \overline{W}$, $w = \bar{z}$ and the parameter is real, the equality

$$\ell^{\mp}_{\varepsilon,W}(z) = \overline{h^{\pm}_{\bar{\varepsilon},\overline{W}}(\bar{z})} \tag{6.1.28}$$

follows on $\pi_{c_1}^{-1}({}^{\mathbb{R}}\mathbb{M}_{c_1})$. Notice that the holonomy $h_{\varepsilon,Z}(w)$ is solution of a differential equation with parameters ε, Z and initial condition w. It follows that (6.1.28) is verified for all (ε, Z, w) in the symmetric product $V \times (S'_{c_1} \cup S_{c_1}) \times \mathbb{D}_w$. The same argument holds for the holonomy $\ell_{\varepsilon,W}(z)$ in the second chart of the blow up of the covering space. The assumption on the coefficients of the family $\mathscr{Q}_\varepsilon$ implies that $\overline{\mathscr{Q}_{\bar{\varepsilon}}(\overline{w})} = \mathscr{Q}_\varepsilon(w)$ for all $w \in \mathbb{D}_w$. It turns out that:

$$\begin{aligned}
\nabla_\varepsilon(W', z) &= (\ell^{+}_{\varepsilon,W})^{\circ-1} \circ (id + \widehat{g}_\varepsilon) \circ \ell^{+}_{\varepsilon,W'}(z) \\
&= (\ell^{+}_{\varepsilon,\widetilde{W}})^{\circ-1} \circ [\mathscr{Q}_{0,\varepsilon}^{\circ-1} \circ (id + g) \circ \mathscr{Q}_{0,\varepsilon}] \circ \ell^{+}_{\varepsilon,\widetilde{W}'}(z) \\
&= (\mathscr{Q}_{0,\varepsilon} \circ \ell^{+}_{\varepsilon,W})^{\circ-1} \circ (id + g_\varepsilon) \circ (\mathscr{Q}_{0,\varepsilon} \circ \ell^{+}_{\varepsilon,W'})(z) \\
&= (\ell^{-}_{\varepsilon,W})^{\circ-1} \circ (id + g_\varepsilon) \circ \ell^{-}_{\varepsilon,W'}(z) \\
&= \overline{(h^{+}_{\bar{\varepsilon},\overline{W}})^{\circ-1} \circ (id + g_{\bar{\varepsilon}}) \circ h^{+}_{\bar{\varepsilon},\overline{W'}}(\bar{z})} \\
&= \mathscr{C} \circ \Delta_{\mathscr{C}(\varepsilon)} \circ \mathscr{S}(W', z),
\end{aligned}$$

where the second equality comes from the decomposition (6.1.10), the fourth equality comes after (6.1.25) and the fifth equality, after (6.1.28). □

6.1.3 The global sealing in the covering.

Definition 6.1.12. *Let*

$$\begin{aligned} m_Z(w) &:= Zw, \\ m_W(z) &:= Wz \end{aligned} \tag{6.1.29}$$

be the multiplication by Z and W in the first and second directions of the complex Möbius strip, respectively.

Lemma 6.1.13. *The holonomies $h_{\varepsilon,\widetilde{Z}}, \ell_{\varepsilon,\widetilde{W}}$ are related by multiplication and switch in the orientations:*

$$h^{\pm}_{\varepsilon,\frac{1}{\widetilde{W}}} = \ell^{\mp}_{\varepsilon,\widetilde{W}} \circ m_{\frac{1}{\widetilde{W}}}. \tag{6.1.30}$$

Proof. Choose a point $(\widetilde{Z},w) \in S_{c_1} \times \mathbb{D}_z$ in the first chart of the complex Möbius strip. Since the change of charts φ is the identity on the transversal section Σ, the holonomy map is read as $\varphi(1,h^{\pm}_{\varepsilon,\widetilde{Z}}(w)) = (1,h^{\pm}_{\varepsilon,\widetilde{Z}}(w))$ in the second chart of the blow up, where

$$(\widetilde{W},z) = (\frac{1}{\widetilde{Z}},\widetilde{Z}w).$$

Therefore, the point $(1,h^{\pm}_{\varepsilon,\widetilde{Z}}(w))$ must be given here as the image by the holonomy of the point $(\widetilde{W},z)$. Inasmuch as $\textsc{Sign}(Im(\widetilde{Z})) = -\textsc{Sign}(Im(\widetilde{W}))$, the sectorial orientation of the holonomy (see Definitions 6.1.5 and 6.1.9) yields

$$\begin{aligned} (1,h^{\pm}_{\varepsilon,\widetilde{Z}}(w)) &= (1,\ell^{\mp}_{\varepsilon,\widetilde{W}}(z)) \\ &= (1,\ell^{\mp}_{\varepsilon,\widetilde{W}}(\widetilde{Z}w)) \\ &= (1,\ell^{\mp}_{\varepsilon,\widetilde{W}}(\frac{w}{\widetilde{W}})) \end{aligned}$$

which means $h^{\pm}_{\varepsilon,\frac{1}{\widetilde{W}}} = \ell^{\mp}_{\varepsilon,\widetilde{W}} \circ m_{\frac{1}{\widetilde{W}}}$. □

Lemma 6.1.14. *Furthermore, the families Δ_ε and ∇_ε are related through:*

$$\begin{aligned} m_Z \circ \Delta_\varepsilon &= \nabla_\varepsilon \circ \varphi \\ m_W \circ \nabla_\varepsilon &= \Delta_\varepsilon \circ \varphi^{\circ -1} \end{aligned} \tag{6.1.31}$$

in the first and second charts of the blow up space, respectively.

Proof. Let $\widetilde{W}' \in S'_{c_2}$ be given. According to Definition 6.1.9:

$$\begin{aligned} \nabla_\varepsilon(\widetilde{W}',z) &= (\ell^{+}_{\varepsilon,\widetilde{W}})^{\circ -1} \circ (id+\widehat{g}) \circ \ell^{+}_{\varepsilon,\widetilde{W}'}(z) \\ &= (\ell^{+}_{\varepsilon,\widetilde{W}})^{\circ -1} \circ [\mathscr{Q}^{\circ -1}_{0,\varepsilon} \circ (id+g) \circ \mathscr{Q}_{0,\varepsilon}] \circ \ell^{+}_{\varepsilon,\widetilde{W}'}(z) \\ &= (\mathscr{Q}_{0,\varepsilon} \circ \ell^{+}_{\varepsilon,\widetilde{W}})^{\circ -1} \circ (id+g) \circ (\mathscr{Q}_{0,\varepsilon} \circ \ell^{+}_{\varepsilon,\widetilde{W}'})(z), \end{aligned}$$

by definition of $\widehat{g}_\varepsilon$. By (6.1.25):

$$\begin{aligned}\nabla_\varepsilon(\widetilde{W}',z) &= (\ell^-_{\varepsilon,\widetilde{W}})^{\circ -1}\circ(id+g)\circ\ell^-_{\varepsilon,\widetilde{W}'}(z)\\ &= m_{\frac{1}{\widetilde{W}}}\circ(h^+_{\varepsilon,\frac{1}{\widetilde{W}}})^{\circ-1}\circ(id+g)\circ h^+_{\varepsilon,\frac{1}{\widetilde{W}'}}\circ m^{\circ-1}_{\frac{1}{\widetilde{W}'}}(z)\end{aligned}$$

by Lemma 6.1.13. Thus

$$\begin{aligned}\nabla_\varepsilon(\widetilde{W}',z) &= m^{\circ-1}_{\widetilde{W}}\circ(h^+_{\varepsilon,\frac{1}{\widetilde{W}}})^{\circ-1}\circ(id+g)\circ h^+_{\varepsilon,\frac{1}{\widetilde{W}'}}(\widetilde{W}'z)\\ &= m^{\circ-1}_{\widetilde{W}}\circ\Delta_\varepsilon(\frac{1}{\widetilde{W}'},\widetilde{W}'z)\\ &= m^{\circ-1}_{\widetilde{W}}\circ\Delta_\varepsilon\circ\varphi^{\circ-1}(\widetilde{W}',z)\end{aligned}$$

and the conclusion follows. The other identity follows by symmetry. □

Corollary 6.1.15 (Global sealing in the covering space). *The sealing maps are related through*

$$\Upsilon_\varepsilon = \varphi^{\circ-1}\circ\curlywedge_\varepsilon\circ\varphi. \tag{6.1.32}$$

Thus, there is a canonical sealing map in the covering space of the complex Möbius strip.

Proof. For every given $(\widetilde{Z}',w)\in S'_{c_1}\times\mathbb{D}_w$ write

$$\widetilde{W}' = \frac{1}{\widetilde{Z}'}\in S'_{c_2},$$

and let $\widetilde{W}\in S_{c_2}$ be the only point in S_{c_2} whose projection on the exceptional divisor coincides with that of $\widetilde{W}'$. Define as well

$$z=\widetilde{Z}'w.$$

It is clearly seen that

$$\begin{aligned}\varphi^{\circ-1}\circ\curlywedge_\varepsilon\circ\varphi(\widetilde{Z}',w) &= \varphi^{\circ-1}(\widetilde{W},\nabla_\varepsilon(\widetilde{W}',z))\\ &= (\frac{1}{\widetilde{W}},\widetilde{W}\nabla_\varepsilon(\widetilde{W}',z))\\ &= (\frac{1}{\widetilde{W}},m_{\widetilde{W}}\circ\nabla_\varepsilon(\widetilde{W}',z))\end{aligned}$$

by definition of the change of coordinates φ between complex charts. Thus, Lemma 6.1.14 above leads to

$$\begin{aligned}\varphi^{\circ -1} \circ \lambda_\varepsilon \circ \varphi(\widetilde{Z}', w) &= (\frac{1}{\widetilde{\widetilde{W}}}, \Delta_\varepsilon \circ \varphi^{\circ -1}(\widetilde{W}', z)) \\ &= (\widetilde{Z}, \Delta_\varepsilon(\widetilde{Z}', w)) \\ &\equiv \Upsilon_\varepsilon(\widetilde{Z}', w),\end{aligned}$$

where $\widetilde{Z} = \dfrac{1}{\widetilde{\widetilde{W}}}$. □

Definition 6.1.16. *This common set after blow down into (z,w) coordinates is noted:*

$$\widetilde{\mathscr{M}} := \begin{cases} (c_1^{-1})^* \widetilde{M} \\ (c_2^{-1})^* \widetilde{N}. \end{cases} \tag{6.1.33}$$

The families of vector fields $(\widetilde{v}_\varepsilon^{c_1}, \widetilde{v}_\varepsilon^{c_2})$ and foliations $(\widetilde{\mathscr{F}}_\varepsilon^{c_1}, \widetilde{\mathscr{F}}_\varepsilon^{c_2})$ induce a vector field $\widetilde{v}_\varepsilon$ and a foliation $\widetilde{\mathscr{F}}_\varepsilon$ on $\widetilde{\mathscr{M}}$.

Remark 6.1.17. *A short calculation shows that the coordinates on $\widetilde{\mathscr{M}}$ are well defined and given by (z,w), where $(\widetilde{Z}, w)$ are the coordinates of $\widetilde{M}$ in the first chart of the blow up, and $(\widetilde{W}, z)$ those of $\widetilde{N}$ in the second complex chart.*

As the sealing $(\Upsilon_\varepsilon, \lambda_\varepsilon)$ is canonical it defines a sealing family

$$\Gamma_\varepsilon : \widetilde{\mathscr{M}} \to \widetilde{\mathscr{M}}$$

in (z,w) coordinates. Such a family is defined in charts as

$$\Gamma_\varepsilon := \begin{cases} c_1 \circ \Upsilon_\varepsilon \circ c_1^{-1} \\ c_2 \circ \lambda_\varepsilon \circ c_2^{-1}. \end{cases} \tag{6.1.34}$$

This means either:

$$\Gamma_\varepsilon(z,w) = \begin{cases} \left(\frac{z}{w}\Delta_\varepsilon\left(\frac{z}{w},w\right),\Delta_\varepsilon\left(\frac{z}{w},w\right)\right) \\ \\ \left(\nabla_\varepsilon\left(\frac{w}{z},z\right),\frac{w}{z}\nabla_\varepsilon\left(\frac{w}{z},z\right)\right). \end{cases} \tag{6.1.35}$$

Definition 6.1.18. *Coordinates (z,w) on $\mathscr{M}$ are called ambient coordinates. The family Γ_ε is called the sealing in ambient coordinates.*

Corollary 6.1.19. *For all $\mu \in \mathbb{C}^*$, the sealing in ambient coordinates preserves the transversal fibers Σ_μ.*

Lemma 6.1.20. *The sealing Γ_ε in ambient coordinates has real character:*

$$\Gamma_\varepsilon = \mathscr{S}\circ\sigma\circ\Gamma_{\mathscr{C}(\varepsilon)}\circ\sigma\circ\mathscr{S}.$$

Proof. By Lemma 6.1.11 and (6.1.35):

$$\begin{aligned} \mathscr{S}\circ\sigma\circ\Gamma_{\mathscr{C}(\varepsilon)}\circ\sigma\circ\mathscr{S}(z,w) &= \left(\overline{\Delta_{\overline{\varepsilon}}\left(\frac{\overline{w}}{\overline{z}},\overline{z}\right)},\frac{w}{z}\overline{\Delta_{\overline{\varepsilon}}\left(\frac{\overline{w}}{\overline{z}},\overline{z}\right)}\right) \\ &= \left(\nabla_\varepsilon\left(\frac{w}{z},z\right),\frac{w}{z}\nabla_\varepsilon\left(\frac{w}{z},z\right)\right) \\ &= \Gamma_\varepsilon(z,w). \end{aligned}$$

□

Notice that, by definition,

$$\begin{aligned} (\Upsilon_\varepsilon)_*\widetilde{v}_\varepsilon^{c_1} &= \widetilde{v}_\varepsilon^{c_1} \\ (\lambda_\varepsilon)_*\widetilde{v}_\varepsilon^{c_2} &= \widetilde{v}_\varepsilon^{c_2}, \end{aligned}$$

thus, the field $\widetilde{v}_\varepsilon$ is invariant under the sealing Γ_ε in ambient coordinates:

$$(\Gamma_\varepsilon)_*\widetilde{v}_\varepsilon = \widetilde{v}_\varepsilon \tag{6.1.36}$$

and then Γ_ε respects the foliation $\widetilde{\mathscr{F}}_\varepsilon$. By Corollary 6.1.19, the quotient $\mathscr{M}_\varepsilon := \widetilde{\mathscr{M}}/\Gamma_\varepsilon$ is thus well defined. By (6.1.36), the vector field $\widetilde{v}_\varepsilon$ induces a vector field v_ε and a foliation $\mathscr{F}_\varepsilon$ in the quotient $\mathscr{M}_\varepsilon$.

Proposition 6.1.21. *The monodromy $h : \Sigma \to \Sigma$ of the field v_ε along the leaves of $\mathscr{F}_\varepsilon$ is well defined and it coincides with $\mathscr{Q}_\varepsilon$.*

Proof. Standard arguments show that the sections Σ_μ are transversal to $\mathscr{F}_\varepsilon$ (Proposition 3.1.8). The family v_ε is the unfolding of an elliptic singularity, and the monodromy of the transversal section Σ is well defined. The holonomy $h_{\varepsilon,\widetilde{1}} : \{\widetilde{1}'\} \times \mathbb{D}_w \to \{\widetilde{1}\} \times \mathbb{D}_w$ of $\widetilde{v}_\varepsilon^{C1}$ coincides, by construction, with the normal form $\mathscr{Q}_{0,\varepsilon}$ on $\widetilde{M}$. Back into ambient coordinates, the image of the point $(w,w) \in \Sigma$ is then $(\mathscr{Q}_{0,\varepsilon}(w), \mathscr{Q}_{0,\varepsilon}(w)) \in \Sigma$, whose image under the sealing Γ_ε is:

$$\begin{aligned}
\Gamma_\varepsilon(\mathscr{Q}_{0,\varepsilon}(w), \mathscr{Q}_{0,\varepsilon}(w)) &= (\Delta_\varepsilon(1, \mathscr{Q}_{0,\varepsilon}(w)), \Delta_\varepsilon(1, \mathscr{Q}_{0,\varepsilon}(w))) \\
&= ((id + g_\varepsilon) \circ \mathscr{Q}_{0,\varepsilon}(w), (id + g_\varepsilon) \circ \mathscr{Q}_{0,\varepsilon}(w)) \\
&= (\mathscr{Q}_\varepsilon(w), \mathscr{Q}_\varepsilon(w)) \in \Sigma,
\end{aligned}$$

where the second equality comes after (6.1.13). □

Corollary 6.1.22. *The elliptic family v_ε has real character:*

$$v_\varepsilon = \mathscr{S} \circ \sigma \circ v_{\mathscr{C}(\varepsilon)} \circ \sigma \circ \mathscr{S},$$

where $\mathscr{S}, \sigma$ are the complex conjugation and shift of coordinates in the two variables induced in the quotient.

Proof. Since the projections π_{c_1}, π_{c_2} are real, it is certainly true that the family of vector fields $\widetilde{v}_\varepsilon$ on $\widetilde{\mathscr{M}}$ possesses real character. The manifold $\mathscr{M}_\varepsilon$ is obtained after gluing with the sealing Γ_ε in ambient coordinates, which sends the real plane into the real plane. $\square$

6.1.4 Identification of the abstract manifold $\mathscr{M}_\varepsilon$

In the first chart of the blow up, we introduce a smooth *real* nonnegative cutoff function $\chi = \chi(\arg(\widetilde{Z}))$ *depending only on the argument of* $\widetilde{Z}$, and defined by:

$$\chi(\arg(\widetilde{Z})) = \begin{cases} 1, & \arg(\widetilde{Z}) \in (-\frac{\pi}{4}, \frac{\pi}{4}], \\ 0, & \arg(\widetilde{Z}') \in (\pi, 2\pi + \frac{\pi}{4}]. \end{cases}$$

An "identification map" in $(\widetilde{Z}, w)$ coordinates is defined on $\widetilde{M}$:

$$\widetilde{H}_\varepsilon^{c_1} : (\widetilde{Z}, w) \mapsto (\widetilde{Z}, w + \chi(\arg(\widetilde{Z}))\{\Delta_\varepsilon(\widetilde{Z}, w) - w\}), \tag{6.1.37}$$

Notice then:

$$\begin{aligned} \widetilde{H}_\varepsilon^{c_1}|_{S'_{c_1}\times\mathbb{D}_w} &\equiv (id_Z, \Delta_\varepsilon) \\ \widetilde{H}_\varepsilon^{c_1}|_{S_{c_1}\times\mathbb{D}_w} &\equiv (id_Z, id_w), \end{aligned}$$

and so this map respects the sealing Υ_ε. Similarly in the second chart of the blow up, an "identification map" in $(\widetilde{W}, z)$ coordinates is defined on $\widetilde{N}$:

$$\widetilde{H}_\varepsilon^{c_2} : (\widetilde{W}, z) \mapsto (\widetilde{W}, z + \chi(-\arg(\widetilde{W}))\{\nabla_\varepsilon(\widetilde{W}, z) - z\}), \tag{6.1.38}$$

where:

$$\chi(-\arg(\widetilde{W})) = \begin{cases} 1, & \arg(\widetilde{W}) \in (-\frac{\pi}{4}, \frac{\pi}{4}], \\ 0, & \arg(\widetilde{W}') \in (-2\pi - \frac{\pi}{4}, -\pi] \end{cases}$$

by definition. Notice that:

$$\begin{aligned} \widetilde{H}_\varepsilon^{c_2}|_{S_{c_2}\times\mathbb{D}_w} &\equiv (id_W, id_z) \\ \widetilde{H}_\varepsilon^{c_2}|_{S'_{c_2}\times\mathbb{D}_w} &\equiv (id_W, \nabla_\varepsilon), \end{aligned}$$

and so, this map respects the sealing λ_ε. Back into ambient coordinates, the func-

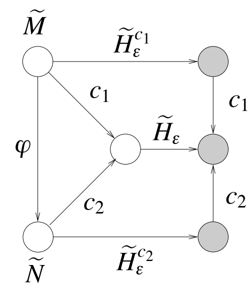


Figure 6.3: Identification maps before quotient.

tion χ yields a smooth map $\widehat{\chi}(z,w) = \chi(\arg(\frac{z}{w})) = \chi(-\arg(\frac{w}{z}))$, which depends only on the argument of the quotient $\frac{z}{w}$:

$$\widehat{\chi}(z,w) = \begin{cases} 1, & |\arg(z) - \arg(w)| < \frac{\pi}{4}, \\ \\ 0, & |\arg(z) - \arg(w) - \frac{13}{8}\pi| < \frac{5}{8}\pi, \end{cases}$$

and satisfying,

$$\overline{\widehat{\chi}(\overline{w},\overline{z})} = \widehat{\chi}(z,w). \tag{6.1.39}$$

Since $\varphi = c_2^{-1} \circ c_1$, Lemma 6.1.14 shows that $\widetilde{H}_\varepsilon^{c_1}, \widetilde{H}_\varepsilon^{c_2}$ express the same family of diffeomorphisms in (z,w) coordinates. As usual, such a family is given by:

$$\widetilde{H}_\varepsilon := \begin{cases} (c_1^{-1})^* \widetilde{H}_\varepsilon^{c_1} = c_1 \circ \widetilde{H}_\varepsilon^{c_1} \circ c_1^{-1} \\ (c_2^{-1})^* \widetilde{H}_\varepsilon^{c_2} = c_2 \circ \widetilde{H}_\varepsilon^{c_2} \circ c_2^{-1}, \end{cases}$$

which is well defined, by construction. This family provides the target space with coordinates $(\mathbf{z},\mathbf{w})$:

$$
\begin{aligned}
&(\mathbf{z},\mathbf{w}) = \widetilde{H}_\varepsilon(z,w) \\
&= (z+\widehat{\chi}(z,w)\{\nabla_\varepsilon \circ c_2^{-1}(z,w)-z\}, w+\widehat{\chi}(z,w)\{\Delta_\varepsilon \circ c_1^{-1}(z,w)-w\}).
\end{aligned} \tag{6.1.40}
$$

By definition, $\widetilde{H}_\varepsilon$ induces an “identification family” in the quotient:

$$H_\varepsilon : \mathscr{M}_\varepsilon \to \mathbb{C}^2.$$

The latter depends analytically on the parameter and for every fixed ε, it is a diffeomorphism which provides the target space with an almost complex structure induced from the standard complex structure on $\mathscr{M}_\varepsilon$, as shown later.

Proposition 6.1.23. *The family H_ε has real character:*

$$H_\varepsilon = \mathscr{S}\circ\sigma\circ H_{\mathscr{C}(\varepsilon)}\circ\sigma\circ\mathscr{S}.$$

Proof. A simple calculation proves that

$$\mathscr{S}\circ\sigma\circ\widetilde{H}_{\mathscr{C}(\varepsilon)}\circ\sigma\circ\mathscr{S} = \sigma\left\{\overline{\widetilde{H}_{\overline{\varepsilon}}(\overline{w},\overline{z})}\right\}$$

$$= \left(z+\overline{\widehat{\chi}(\overline{w},\overline{z})}\left\{\overline{\Delta_{\overline{\varepsilon}}\circ c_1^{-1}(\overline{w},\overline{z})}-z\right\}, w+\overline{\widehat{\chi}(\overline{w},\overline{z})}\left\{\overline{\nabla_{\overline{\varepsilon}}\circ c_2^{-1}(\overline{w},\overline{z})}-w\right\}\right).$$

Lemma 6.1.11 and the symmetry (6.1.39) show that this is equal to (6.1.40). The family H_ε inherits this property in the quotient, by Lemma 6.1.20. Its global real character is shown. □

6.1.5 Asymptotic estimates on H_ε

The ratios of the eigenvalues of the formal families $v^{C1}_{0,\varepsilon}, v^{C2}_{0,\varepsilon}$ (the fields (6.1.12) and (6.1.18)) in the blow up are given by:

$$\begin{aligned} \lambda_\varepsilon &= \frac{\varepsilon - i}{2i} \\ \lambda'_\varepsilon &= -\frac{\varepsilon + i}{2i}, \end{aligned} \tag{6.1.41}$$

respectively. We shall see that if the number N in (6.1.9) is large enough, then the *identification* H_ε is tangent to the identity. Write the parameter as

$$\varepsilon = \varepsilon_1 + i\varepsilon_2,$$

where $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ are small real numbers.

Proposition 6.1.24. *If the function g in* (6.1.9) *is* $(N+1)$*-flat at* $z = w = 0$, *then* (6.1.15) *and* (6.1.24) *admit the asymptotic estimate:*

$$|\Delta_\varepsilon(\widetilde{Z}, w) - w| = O(|\widetilde{Z}|^{\frac{N}{2}(1-\varepsilon_2)}|w|^{N+1}), \qquad as \quad \widetilde{Z} \to 0,$$

for all w in $\mathbb{D}_w$, and

$$|\nabla_\varepsilon(\widetilde{W}, z) - z| = O(|\widetilde{W}|^{\frac{N}{2}(1-\varepsilon_2)}|z|^{N+1}), \qquad as \quad \widetilde{W} \to 0$$

for all z in $\mathbb{D}_z$.

Proof. The eigenvalues of $\widetilde{v}^{C1}_\varepsilon, \widetilde{v}^{C2}_\varepsilon$ are given by (6.1.41) as well, because the covering map does not alter the linear part of the fields. The estimates obtained in the Theorem C.1.1 in the Appendix may be applied. The inequality (C.1.2) yields the following estimate for the holonomy map $h_{\varepsilon,\widetilde{Z}} : \{\widetilde{Z}\} \times \mathbb{D}_w \to \{\widetilde{1}\} \times \mathbb{C}$ of the foliation in the first chart of the blow up:

$$e^{-M|\lambda_\varepsilon(\widetilde{Z}-1)|-\frac{\varepsilon_1 \arg \widetilde{Z}}{2}} |\widetilde{Z}|^{\frac{1-\varepsilon_2}{2}} |w| \le |h_{\varepsilon,\widetilde{Z}}(w)| \le e^{M|\lambda_\varepsilon(\widetilde{Z}-1)|-\frac{\varepsilon_1 \arg \widetilde{Z}}{2}} |\widetilde{Z}|^{\frac{1-\varepsilon_2}{2}} |w|,$$

where $M = M(\widetilde{Z}, w) < \infty$ is a positive constant depending on a bound for the nonlinear part of the foliation along the segment with endpoints $\widetilde{Z}, 1$. By (6.1.9),

$$h^{-1}_{\varepsilon,\widetilde{Z}} \circ (id + g) \circ h_{\varepsilon,\widetilde{Z}'} = h^{-1}_{\varepsilon,\widetilde{Z}} \circ (h_{\varepsilon,\widetilde{Z}'} + g \circ h_{\varepsilon,\widetilde{Z}'}) = id + O(|\widetilde{Z}|^{\frac{N}{2}(1-\varepsilon_2)} |w|^{N+1}).$$

In the second direction of the blow up, the estimate is obtained by symmetry. □

Corollary 6.1.25. *If the function g in* (6.1.8) *is* $(N+1)$*-flat at* $z = w = 0$, *then the asymptotic estimates:*

$$\begin{aligned} |\Delta_\varepsilon \circ c_1^{-1}(z,w) - w| &= O(|z|^{\frac{N}{2}(1-\varepsilon_2)} |w|^{\frac{N}{2}(1+\varepsilon_2)+1}) \\ |\nabla_\varepsilon \circ c_2^{-1}(z,w) - z| &= O(|z|^{\frac{N}{2}(1+\varepsilon_2)+1} |w|^{\frac{N}{2}(1-\varepsilon_2)}) \end{aligned} \qquad (6.1.42)$$

are valid for all (z,w) *in the bidisk* $\mathbb{D}_z \times \mathbb{D}_w$.

Corollary 6.1.26. *If the function g in* (6.1.8) *is* $(N+1)$*-flat at* $z = w = 0$ *for a sufficiently large integer N then, for small ε, the family H_ε is tangent to the identity.*

6.1.6 Integrability on $H_\varepsilon(\mathscr{M}_\varepsilon)$ and preparations.

The pullback of the complex structure on $\mathscr{M}_\varepsilon$ by the map H_ε^{-1} is an almost complex structure defined by the pullback of the $(1,0)$-subbundle on $\mathscr{M}_\varepsilon$, which is spanned by the following families of forms on the manifold $\widetilde{H}_\varepsilon(\widetilde{\mathscr{M}})$ (see Figure

6.3):

$$\begin{aligned}\widetilde{\zeta}_{1,\varepsilon} &= d\mathbf{z} = d(z+\widehat{\chi}\cdot\{\nabla_\varepsilon\circ c_2^{-1}-z\}),\\ \widetilde{\zeta}_{2,\varepsilon} &= d\mathbf{w} = d(w+\widehat{\chi}\cdot\{\Delta_\varepsilon\circ c_1^{-1}-w\}),\end{aligned} \tag{6.1.43}$$

where the coordinates $\mathbf{z},\mathbf{w}$ have been introduced in (6.1.40). The forms $d(\nabla_\varepsilon\circ c_2^{-1})$ and $d(\Delta_\varepsilon\circ c_1^{-1})$ are holomorphic on their domains and $\widetilde{\zeta}_{1,\varepsilon}$ and $\widetilde{\zeta}_{2,\varepsilon}$ have two different sectorial representatives:

$$\begin{aligned}\widetilde{\zeta}_{1,\varepsilon} &= \begin{cases}\zeta^0_{1,\varepsilon} = dz, & |\arg z-\arg w-13\pi/8|<5\pi/8,\\ \zeta^1_{1,\varepsilon} = d(\nabla_\varepsilon\circ c_2^{-1}), & |\arg z-\arg w|<\pi/4,\end{cases}\\ \widetilde{\zeta}_{2,\varepsilon} &= \begin{cases}\zeta^1_{2,\varepsilon} = d(\Delta_\varepsilon\circ c_2^{-1}), & |\arg z-\arg w|<\pi/4\\ \zeta^0_{2,\varepsilon} = dw, & |\arg z-\arg w-13\pi/8|<5\pi/8,\end{cases}\end{aligned} \tag{6.1.44}$$

so that $\zeta^1_{1,\varepsilon}=\Gamma^*_\varepsilon\zeta^0_{1,\varepsilon}$ and $\zeta^1_{2,\varepsilon}=\Gamma^*_\varepsilon\zeta^0_{2,\varepsilon}$. Thus they yield forms $\zeta_{1,\varepsilon}$ and $\zeta_{2,\varepsilon}$ on the quotient $\mathscr{M}_\varepsilon$. The almost complex structure induced on $H_\varepsilon(\mathscr{M}_\varepsilon)$ by the complex structure on $\mathscr{M}_\varepsilon$ is defined by the two forms

$$\omega_{1,\varepsilon}=(H_\varepsilon^{-1})^*\zeta_{1,\varepsilon},\qquad \omega_{2,\varepsilon}=(H_\varepsilon^{-1})^*\zeta_{2,\varepsilon}. \tag{6.1.45}$$

Theorem 6.1.27 (Existence and symmetry of the smooth chart). *There exists a small ball $\mathbf{B}(r)\subset H_\varepsilon(\mathscr{M}_\varepsilon)$ around the origin, and a smooth family of charts $\Lambda_\varepsilon=(\xi^1_\varepsilon,\xi^2_\varepsilon):\mathbf{B}(r)\to\mathbb{C}^2$ depending analytically on ε in a symmetric neighborhood V around the origin in the parameter space, such that $\xi^i_\varepsilon:\mathbf{B}(r)\to\mathbb{C},\ \ i=1,2$ is holomorphic in the sense of the almost complex structure* (6.1.45)*. Furthermore, this family has real character:*

$$\Lambda_\varepsilon=\mathscr{S}\circ\sigma\circ\Lambda_{\mathscr{C}(\varepsilon)}\circ\sigma\circ\mathscr{S}, \tag{6.1.46}$$

and is tangent to the identity at the origin.

Proof. The set $H_\varepsilon(\mathscr{M}_\varepsilon) \subset \mathbb{C}^2$ does not contain the axes of coordinates: its closure is C^∞-diffeomorphic to a closed neighborhood of the origin of $\mathbb{C}^2$. The next lemma shows that the almost complex structure generated by $\omega_{1,\varepsilon}$ and $\omega_{2,\varepsilon}$ on $H_\varepsilon(\mathscr{M}_\varepsilon)$ can be extended as $\omega_{1,\varepsilon} = dz$ along the z-axis, and as $\omega_{2,\varepsilon} = dw$ along the w-axis, until a well-defined order.

Lemma 6.1.28. *Let δ be a small positive number. If α and β are the orders of flatness in z and w (resp. w and z) of the difference $\omega_{1,\varepsilon} - dz$ (resp. $\omega_{2,\varepsilon} - dw$), then form $\omega_{1,\varepsilon}$ (resp. $\omega_{2,\varepsilon}$) can be extended as dz (resp. dw) along the z-axis (resp. w-axis) until the order α if the number N in* (6.1.9) *is sufficiently large so as to verify*

$$N > \max\left\{\frac{2(\alpha-1)}{1-\delta}, \frac{2\beta}{1-\delta}\right\}, \tag{6.1.47}$$

and $|\,\varepsilon\,| < \delta$.

Proof. By (6.1.45), it suffices to study the difference

$$\widetilde{H}_\varepsilon(z,w) - (z,w) = (\widehat{\chi}(z,w)\{\nabla_\varepsilon \circ c_2^{-1}(z,w) - z\}, \widehat{\chi}(z,w)\{\Delta_\varepsilon \circ c_1^{-1}(z,w) - w\}).$$

Since $z = |z|e^{i\arg(z)}$ and $w = |w|e^{i\arg(w)}$, it is easily seen that:

$$\widehat{\chi}(z,w) \quad = \quad \chi\left(\frac{1}{i}\log\left(\frac{z|w|}{w|z|}\right)\right),$$

whence follows that

$$\left|\frac{\partial^{i+j}\widehat{\chi}}{\partial z^p\,\partial \overline{z}^q\,\partial w^r\,\partial \overline{w}^s}\right| < C^{\underline{st}} \cdot \frac{\mathbf{M}_{i+j}}{|z|^i|w|^j} \tag{6.1.48}$$

for all $i = p+q \in \mathbb{N}$, $j = r+s \in \mathbb{N}$ and

$$\mathbf{M}_{i+j} := \max_{\substack{0 \le k \le i+j \\ \theta \in I}} |\chi^{(k)}(\theta)|$$

with $I = [-\frac{\pi}{4}, 2\pi + \frac{\pi}{4}]$. To lighten the notation, put

$$f(z,w) = \nabla_\varepsilon \circ c_2^{-1}(z,w) - z.$$

A short calculation after the second equality in Corollary 6.1.25 implies that for all $k,l \in \mathbb{N}$, there exists a positive constant $\widehat{C}_N = \widehat{C}_N(k,l)$ such that

$$\left| \frac{\partial^{k+l} f}{\partial z^k \partial w^l} \right| \quad \leq \quad \widehat{C}_N |z|^{\frac{N}{2}(1+\varepsilon_2)+1-k} |w|^{\frac{N}{2}(1-\varepsilon_2)-l}, \tag{6.1.49}$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2$ with $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$. Thus, (6.1.48) and (6.1.49) imply that, for $p+q = \alpha$ and $r+s = \beta$, there is a real constant $L = L(N,\alpha,\beta) > 0$ such that

$$\left| \frac{\partial^{\alpha+\beta}(\widehat{\chi} \cdot f)}{\partial z^p \partial \overline{z}^q \partial w^r \partial \overline{w}^s} \right| \leq L \cdot |z|^{\frac{N}{2}(1+\varepsilon_2)+1-\alpha} \cdot |w|^{\frac{N}{2}(1-\varepsilon_2)-\beta}. \tag{6.1.50}$$

Hence, if $|\,\varepsilon\,| < \delta << 1$ and the order (6.1.9) of the family g_ε satisfies the condition

$$N > \max \left\{ \frac{2(\alpha-1)}{1-\delta}, \frac{2\beta}{1-\delta} \right\}, \tag{6.1.51}$$

then

$$\lim_{w \to 0} \left| \frac{\partial^{\alpha+\beta}(\widehat{\chi} \cdot f)}{\partial z^p \partial \overline{z}^q \partial w^r \partial \overline{w}^s} \right| = 0$$

uniformly in $|z| < 1$, and thus $\omega_{1,\varepsilon}$ and dz coincide until the order α along the z-axis. The assertion for the difference $\omega_{2,\varepsilon} - dw$ follows by duality. □

The almost complex structure (6.1.45) is integrable when there are coordinates $(\xi_\varepsilon^1, \xi_\varepsilon^2)$ depending analytically on the parameter, such that

$$< d\xi_\varepsilon^1, d\xi_\varepsilon^2 >_{\mathbb{C}} = < \omega_{1,\varepsilon}, \omega_{2,\varepsilon} >_{\mathbb{C}},$$

where $\omega_{1,\varepsilon}$ and $\omega_{2,\varepsilon}$ are defined in (6.1.45). In that case there is a 2×2 invertible matrix A whose entries are C^∞ functions such that

$$\begin{pmatrix} \omega_{1,\varepsilon} \\ \omega_{2,\varepsilon} \end{pmatrix} = A \begin{pmatrix} d\xi_\varepsilon^1 \\ d\xi_\varepsilon^2 \end{pmatrix} = A d\Lambda_\varepsilon.$$

In particular, $d\begin{pmatrix}\omega_{1,\varepsilon}\\ \omega_{2,\varepsilon}\end{pmatrix} = dA\wedge d\Lambda_\varepsilon$ contains no $(0,2)$ forms. The Newlander-Nirenberg Theorem asserts that this necessary condition is also sufficient for integrability. If $L^{1,0}$ is the span of the forms $\omega_{1,\varepsilon},\omega_{2,\varepsilon}$, then this integrability condition holds for $L^{1,0}$ on the surface $H_\varepsilon(\mathscr{M}_\varepsilon)$, and by continuity it remains valid after extension of (6.1.45) until the axes. Indeed, $\omega_{1,\varepsilon}$ is obtained from the pullback of $\zeta_{1,\varepsilon}$ and since the forms $d(\nabla_\varepsilon\circ c_2^{-1})$ and $d(\Delta_\varepsilon\circ c_1^{-1})$ are holomorphic on their domains (for the sake of simplicity ∇ and Δ will be written instead of $\nabla_\varepsilon\circ c_2^{-1}$ and $\Delta_\varepsilon\circ c_1^{-1}$, respectively), a short calculation in local coordinates shows, after (6.1.43):

$$\begin{aligned} d\widetilde{\zeta}_{1,\varepsilon} &= d\left\{\widehat{\chi}_z dz+\widehat{\chi}_w dw+\widehat{\chi}_{\bar z}d\bar z+\widehat{\chi}_{\overline{w}}d\overline{w}\right\}\cdot\{\nabla - z\}\\ &+\ 2\left\{\widehat{\chi}_z dz+\widehat{\chi}_w dw+\widehat{\chi}_{\bar z}d\bar z+\widehat{\chi}_{\overline{w}}d\overline{w}\right\}\cdot\{\nabla_z dz+\nabla_w dw-dz\}\\ &+\ \widehat{\chi}\cdot d\left\{\nabla_z dz+\nabla_w dw-dz\right\}\end{aligned}$$

(here the subscripts stand for partial differentiation). Inasmuch as $\widehat{\chi}$ is of class C^∞ the first term of the sum is null, *i.e.* $d\widetilde{\zeta}_{1,\varepsilon}$ contains no forms of type $(0,2)$. By symmetry, the same holds for the second form $d\widetilde{\zeta}_{2,\varepsilon}$. Hence, for each $\varepsilon\in V$ the Newlander-Nirenberg Theorem ensures the existence of a C^1 smooth chart $\widetilde{\Lambda}_\varepsilon=\widetilde{\Lambda}_\varepsilon(\mathbf{z},\mathbf{w})$ given by:

$$\widetilde{\Lambda}_\varepsilon=(\widetilde{\xi}^1_\varepsilon,\widetilde{\xi}^2_\varepsilon):\widetilde{H}_\varepsilon(\widetilde{\mathscr{M}})\to\mathbb{C}^2, \tag{6.1.52}$$

which is holomorphic in the sense of the almost complex structure $(\widetilde{\zeta}_{1,\varepsilon},\widetilde{\zeta}_{1,\varepsilon})$. Notice that the sealing Γ_ε in ambient coordinates lifts as two different sealing families:

1. A sealing $\Gamma^*_\varepsilon:\widetilde{H}_\varepsilon(\widetilde{\mathscr{M}})\to\widetilde{H}_\varepsilon(\widetilde{\mathscr{M}})$ in $(\mathbf{z},\mathbf{w})$ coordinates:

$$\Gamma^*_\varepsilon:=\widetilde{H}_\varepsilon\circ\Gamma_\varepsilon\circ\widetilde{H}_\varepsilon^{-1}. \tag{6.1.53}$$

2. A sealing $\Gamma_\varepsilon^{**} : \widetilde{\Lambda}_\varepsilon(\widetilde{H}_\varepsilon(\widetilde{\mathscr{M}})) \subset \mathbb{C}^2 \to \widetilde{\Lambda}_\varepsilon(\widetilde{H}_\varepsilon(\widetilde{\mathscr{M}})) \subset \mathbb{C}^2$ between open sets of $\mathbb{C}^2$:

$$\Gamma_\varepsilon^{**} \quad := \quad (\widetilde{\Lambda}_\varepsilon \circ \widetilde{H}_\varepsilon) \circ \Gamma_\varepsilon \circ (\widetilde{\Lambda}_\varepsilon \circ \widetilde{H}_\varepsilon)^{-1}. \tag{6.1.54}$$

Remark 6.1.29. *By Lemma 6.1.20 and Proposition 6.1.23, the family Γ_ε^* has real character as well.*

By definition,

$$\Gamma_\varepsilon^{**} \circ \widetilde{\Lambda}_\varepsilon = \widetilde{\Lambda}_\varepsilon \circ \Gamma_\varepsilon^*, \tag{6.1.55}$$

hence $\widetilde{\Lambda}_\varepsilon$ induces a family of charts

$$\Lambda_\varepsilon = (\xi_\varepsilon^1, \xi_\varepsilon^2) : H_\varepsilon(\mathscr{M}_\varepsilon) \to \mathbb{C}^2 \tag{6.1.56}$$

in the quotient. This chart is, by definition, holomorphic in the sense of the extended almost complex structure (6.1.45). In order to show the real character of Λ_ε, it suffices to prove the real character of the family $\widetilde{\Lambda}_\varepsilon$ (Proposition 6.1.42). So we need to look into the details of Newlander-Nirenberg Theorem. Some tools are required.

The family of diffeomorphisms $\widetilde{H}_\varepsilon = (\mathbf{z}, \mathbf{w})$ (6.1.40) is analytic with respect to the structure (6.1.43). It follows that:

$$\begin{aligned} d\mathbf{z} &= u_{1,\varepsilon} dz + u_{2,\varepsilon} dw + u_{3,\varepsilon} d\bar{z} + u_{4,\varepsilon} d\overline{w} \\ d\mathbf{w} &= v_{1,\varepsilon} dz + v_{2,\varepsilon} dw + v_{3,\varepsilon} d\bar{z} + v_{4,\varepsilon} d\overline{w} \end{aligned} \tag{6.1.57}$$

where

$$\begin{aligned} u_{1,\varepsilon} &= 1 + \chi_z(\nabla - z) + \chi(\nabla_z - 1) \\ u_{2,\varepsilon} &= \chi_w(\nabla - z) + \chi \nabla_w \\ u_{3,\varepsilon} &= \chi_{\bar{z}}(\nabla - z) \\ u_{4,\varepsilon} &= \chi_{\overline{w}}(\nabla - z) \end{aligned} \tag{6.1.58}$$

and

$$\begin{aligned} v_{1,\varepsilon} &= \chi_z(\Delta - w) + \chi\Delta_z \\ v_{2,\varepsilon} &= 1 + \chi_w(\Delta - w) + \chi(\Delta_w - 1) \\ v_{3,\varepsilon} &= \chi_{\bar{z}}(\Delta - w) \\ v_{4,\varepsilon} &= \chi_{\overline{w}}(\Delta - w). \end{aligned} \tag{6.1.59}$$

Lemma 6.1.11 and the symmetry (6.1.39) yield the equivalences:

$$\begin{aligned} \overline{u_{1,\bar{\varepsilon}}(\overline{w},\bar{z})} &= v_{2,\varepsilon}(z,w) \\ \overline{u_{2,\bar{\varepsilon}}(\overline{w},\bar{z})} &= v_{1,\varepsilon}(z,w) \\ \overline{u_{3,\bar{\varepsilon}}(\overline{w},\bar{z})} &= v_{4,\varepsilon}(z,w) \\ \overline{u_{4,\bar{\varepsilon}}(\overline{w},\bar{z})} &= v_{3,\varepsilon}(z,w). \end{aligned} \tag{6.1.60}$$

Modulo a linear combination, the space induced by the two forms (6.1.57) is easily seen to be the same as the space generated by the two forms

$$\begin{aligned} \Omega_1 &= dz + e^1_{1,\varepsilon} d\bar{z} + e^1_{2,\varepsilon} d\overline{w} \\ \Omega_2 &= dw + e^2_{1,\varepsilon} d\bar{z} + e^2_{2,\varepsilon} d\overline{w}, \end{aligned} \tag{6.1.61}$$

where

$$\begin{aligned} e^1_{1,\varepsilon} = \frac{u_3 v_2 - u_2 v_3}{u_1 v_2 - u_2 v_1}, \quad e^1_{2,\varepsilon} = \frac{u_4 v_2 - u_2 v_4}{u_1 v_2 - u_2 v_1}, \\ e^2_{1,\varepsilon} = \frac{u_1 v_3 - u_3 v_1}{u_1 v_2 - u_2 v_1}, \quad e^2_{2,\varepsilon} = \frac{u_1 v_4 - u_4 v_1}{u_1 v_2 - u_2 v_1} \end{aligned} \tag{6.1.62}$$

(their dependence on $\bar{z}, \overline{w}$ is not explicitly written). Hence, (6.1.60) leads to:

$$\begin{aligned} \overline{e^1_{1,\bar{\varepsilon}}(\overline{w},\bar{z})} &= e^2_{2,\varepsilon}(z,w), \\ \overline{e^1_{2,\bar{\varepsilon}}(\overline{w},\bar{z})} &= e^2_{1,\varepsilon}(z,w). \end{aligned} \tag{6.1.63}$$

By Corollary 6.1.25, these functions satisfy $e^j_{i,\varepsilon} = \dfrac{o(1)}{1+o(1)}$, thus giving:

$$e^j_{i,\varepsilon}(0,0) = 0 \tag{6.1.64}$$

for all $i, j = 1,2$ and $\varepsilon \in V$. Suppose that the image $\widetilde{H}_\varepsilon(\mathscr{M})$ contains a small bidisk $\mathbb{D}_s \times \mathbb{D}_s$, and write $G_\varepsilon := \widetilde{H}_\varepsilon^{-1}$. Consider the pullback

$$\mathfrak{a}_{i,\varepsilon}^j : \mathbb{D}_s \times \mathbb{D}_s \to \mathbb{C}^2$$

of the functions (6.1.62) by the family G_ε. The functions $\mathfrak{a}_{i,\varepsilon}^j$ are defined by the equations $\mathfrak{a}_{i,\varepsilon}^j = G_\varepsilon^*(e_{i,\varepsilon}^j)$, that is:

$$\mathfrak{a}_{i,\varepsilon}^j(\mathbf{z},\mathbf{w}) = G_\varepsilon^*(e_{i,\varepsilon}^j)(\mathbf{z},\mathbf{w}) \equiv e_{i,\varepsilon}^j(G_\varepsilon(\mathbf{z},\mathbf{w})), \quad i,j=1,2, \ \ \varepsilon \in V, \tag{6.1.65}$$

for all $(\mathbf{z},\mathbf{w}) \in \mathbb{D}_s \times \mathbb{D}_s$. By Proposition 6.1.23 and identities (6.1.63) the collection $\mathfrak{a}_{i,\varepsilon}^j$ satisfies again:

$$\begin{aligned} \overline{\mathfrak{a}_{1,\overline{\varepsilon}}^1(\overline{\mathbf{w}},\overline{\mathbf{z}})} &= \mathfrak{a}_{2,\varepsilon}^2(\mathbf{z},\mathbf{w}), \\ \overline{\mathfrak{a}_{2,\overline{\varepsilon}}^1(\overline{\mathbf{w}},\overline{\mathbf{z}})} &= \mathfrak{a}_{1,\varepsilon}^2(\mathbf{z},\mathbf{w}), \end{aligned} \tag{6.1.66}$$

and by (6.1.64):

$$\mathfrak{a}_{i,\varepsilon}^j(0,0) = 0 \tag{6.1.67}$$

for all $i, j = 1,2$ and $\varepsilon \in V$. From now on, we will write

$$\mathbf{z}^1 = \mathbf{z}, \ \ \mathbf{z}^2 = \mathbf{w}.$$

Definition 6.1.30 (The differentials). *The holomorphic and antiholomorphic differentials are given, respectively, by:*

$$\partial_j = \frac{\partial}{\partial \mathbf{z}^j}, \quad \overline{\partial}_j = \frac{\partial}{\partial \overline{\mathbf{z}^j}}, \quad j=1,2. \tag{6.1.68}$$

Definition 6.1.31. *[40] A complex valued function ξ satisfying the equation:*

$$\overline{\partial}_j \xi - (\mathfrak{a}_j^1 \partial_1 \xi + \mathfrak{a}_j^2 \partial_2 \xi) = 0, \quad j=1,2 \tag{6.1.69}$$

is called holomorphic with respect to the given almost complex structure.

Let $(\widetilde{\xi}_\varepsilon^1, \widetilde{\xi}_\varepsilon^2) = (\widetilde{\xi}_\varepsilon^1(\mathbf{z},\mathbf{w}), \widetilde{\xi}_\varepsilon^2(\mathbf{z},\mathbf{w}))$ be the smooth chart given by Newlander-Nirenberg Theorem. Instead of considering the new coordinates $(\widetilde{\xi}_\varepsilon^1, \widetilde{\xi}_\varepsilon^2)$ as solutions to (6.1.69) and functions of $(\mathbf{z},\mathbf{w})$, the coordinates $\mathbf{z},\mathbf{w}$ are supposed to be functions of $\widetilde{\xi}_\varepsilon^1, \widetilde{\xi}_\varepsilon^2$ and their complex conjugates.

Remark 6.1.32. *Inasmuch as it suffices to study only the real character of the chart $\widetilde{\Lambda}_\varepsilon$, the tildes on the chart $(\widetilde{\xi}_\varepsilon^1, \widetilde{\xi}_\varepsilon^2)$ are dropped from now on.*

Definition 6.1.33 (The dual differentials). *The holomorphic and antiholomorphic dual differentials are given, respectively, by:*

$$d_{j,\varepsilon} = \frac{\partial}{\partial \xi_\varepsilon^j}, \quad \overline{d}_{j,\varepsilon} = \frac{\partial}{\partial \overline{\xi_\varepsilon^j}}, \quad j = 1,2, \tag{6.1.70}$$

where

$$\frac{\partial}{\partial \xi_\varepsilon^j} = \frac{1}{2}\left(\frac{\partial}{\partial u_\varepsilon^j} - i\frac{\partial}{\partial v_\varepsilon^j}\right), \quad \frac{\partial}{\partial \overline{\xi_\varepsilon^j}} = \frac{1}{2}\left(\frac{\partial}{\partial u_\varepsilon^j} + i\frac{\partial}{\partial v_\varepsilon^j}\right), \tag{6.1.71}$$

with $\xi_\varepsilon^j = u_\varepsilon^j + iv_\varepsilon^j$ for $u_\varepsilon^j, v_\varepsilon^j \in \mathbb{R}$.

Lemma 6.1.34. *If f is a smooth function, then:*

$$\overline{d_{j,\varepsilon} f} = \overline{d}_{j,\varepsilon}\overline{f}, \quad j = 1,2.$$

Proof. This becomes immediately apparent by (6.1.71). □

Proposition 6.1.35 ([41], pp. 445)**.** *For every $\varepsilon \in V$, the map G_ε from $\mathbb{D}_s \times \mathbb{D}_s \subset \mathbb{C}^2$ to the almost complex manifold $\widetilde{\mathscr{M}}$ is holomorphic if and only if the induced functions $(\mathbf{z},\mathbf{w}) = G_\varepsilon^*(z,w)$ satisfy the differential equations*

$$\overline{d}_{j,\varepsilon}\mathbf{z}^k + \mathfrak{a}_{m,\varepsilon}^k \overline{d}_{j,\varepsilon}\overline{\mathbf{z}}^m = 0, \qquad j,k = 1,2. \tag{6.1.72}$$

In such a case, (6.1.69) yields:

$$\begin{aligned}\overline{d}_{j,\varepsilon}\xi^p_\varepsilon &= \partial_k\xi^p_\varepsilon\overline{d}_{j,\varepsilon}\mathbf{z}^k + \overline{\partial}_k\xi^p_\varepsilon\overline{d}_{j,\varepsilon}\overline{\mathbf{z}}^k \\ &= \partial_k\xi^p_\varepsilon\{\overline{d}_{j,\varepsilon}\mathbf{z}^k + \mathfrak{a}^k_{m,\varepsilon}\overline{d}_{j,\varepsilon}\overline{\mathbf{z}}^m\} \\ &= 0\end{aligned} \tag{6.1.73}$$

for $j = 1,2$. Notice that the replacement of (6.1.72) in the term after the first equality of (6.1.73), yields:

$$\overline{d}_{j,\varepsilon}\xi^p_\varepsilon = \overline{d}_{j,\varepsilon}\overline{\mathbf{z}}^k\{\overline{\partial}_k\xi^p_\varepsilon - \mathfrak{a}^i_{k,\varepsilon}\partial_i\xi^p_\varepsilon\}, \qquad p = 1,2.$$

Thus the parametric Cauchy-Riemann equations $\overline{d}_{j,\varepsilon}\xi^p_\varepsilon = 0$ are equivalent to the system (6.1.69) if $\mathbf{z},\mathbf{w}$ satisfy (6.1.72) with the matrix $[\overline{d}_{j,\varepsilon}\overline{\mathbf{z}}^k]$ non-singular for all ε in the symmetric neighborhood V. Then the idea is to find solutions to (6.1.72) and study their real character. This is done by solving a corresponding integral equation to which iterations can be applied succesfully.

Denote by T^1, T^2 the integral operators

$$\begin{aligned}T^1 f(\mathbf{z},\mathbf{w}) &= \frac{1}{2i\pi}\iint_{|\tau|<\rho}\frac{f(\tau,\mathbf{w})}{\mathbf{z}-\tau}d\overline{\tau}d\tau, \\ T^2 f(\mathbf{z},\mathbf{w}) &= \frac{1}{2i\pi}\iint_{|\tau|<\rho}\frac{f(\mathbf{z},\tau)}{\mathbf{w}-\tau}d\overline{\tau}d\tau,\end{aligned} \tag{6.1.74}$$

with $\rho > 0$ fixed and $f = f(\mathbf{z},\mathbf{w})$ has suitable differentiability properties and, eventually, it depends on additional complex coordinates.

Lemma 6.1.36. *If f_1, f_2 are as above and satisfy $f_1(\mathbf{z},\mathbf{w}) = \overline{f_2(\overline{\mathbf{w}},\overline{\mathbf{z}})}$, then*

$$T^1 f_1(\mathbf{z},\mathbf{w}) = \overline{T^2 f_2(\overline{\mathbf{w}},\overline{\mathbf{z}})}. \tag{6.1.75}$$

Proof. By definition:

$$\begin{aligned}
T^1 f_1(\mathbf{z},\mathbf{w}) &= \frac{1}{2i\pi}\iint_{|\tau|<\rho} \frac{f_1(\tau,\mathbf{w})}{\mathbf{z}-\tau} d\overline{\tau} d\tau \\
&= \frac{1}{2i\pi}\iint_{|\tau|<\rho} \frac{\overline{f_2(\overline{\mathbf{w}},\overline{\tau})}}{\mathbf{z}-\tau} d\overline{\tau} d\tau \\
&= \overline{\frac{1}{2i\pi}\iint_{|\tau|<\rho} \frac{f_2(\overline{\mathbf{w}},\overline{\tau})}{\overline{\mathbf{z}}-\overline{\tau}} d\overline{\tau} d\tau} \\
&= \overline{\frac{1}{2i\pi}\iint_{|\tau|<\rho} \frac{f_2(\overline{\mathbf{w}},\tau)}{\overline{\mathbf{z}}-\tau} d\overline{\tau} d\tau}. \\
&= \overline{T^2 f_2(\overline{\mathbf{w}},\overline{\mathbf{z}})}
\end{aligned}$$

The fourth equality comes from the fact that the form $d\overline{\tau}d\tau$ is invariant under the change $Im(\tau) \mapsto -Im(\tau)$ on the symmetric domain $|\tau| < \rho$. □

Let us define:

$$\begin{aligned}
f_{11}(\mathbf{z},\mathbf{w})(\xi^1,\xi^2)(\varepsilon) &:= -(\mathfrak{a}^1_{1,\varepsilon}\overline{d}_{1,\varepsilon}\overline{\mathbf{z}} + \mathfrak{a}^1_{2,\varepsilon}\overline{d}_{1,\varepsilon}\overline{\mathbf{w}}), \\
f_{21}(\mathbf{z},\mathbf{w})(\xi^1,\xi^2)(\varepsilon) &:= -(\mathfrak{a}^1_{1,\varepsilon}\overline{d}_{2,\varepsilon}\overline{\mathbf{z}} + \mathfrak{a}^1_{2,\varepsilon}\overline{d}_{2,\varepsilon}\overline{\mathbf{w}}), \\
f_{12}(\mathbf{z},\mathbf{w})(\xi^1,\xi^2)(\varepsilon) &:= -(\mathfrak{a}^2_{1,\varepsilon}\overline{d}_{1,\varepsilon}\overline{\mathbf{z}} + \mathfrak{a}^2_{2,\varepsilon}\overline{d}_{1,\varepsilon}\overline{\mathbf{w}}), \\
f_{22}(\mathbf{z},\mathbf{w})(\xi^1,\xi^2)(\varepsilon) &:= -(\mathfrak{a}^2_{1,\varepsilon}\overline{d}_{2,\varepsilon}\overline{\mathbf{z}} + \mathfrak{a}^2_{2,\varepsilon}\overline{d}_{2,\varepsilon}\overline{\mathbf{w}}).
\end{aligned} \tag{6.1.76}$$

Lemma 6.1.37. *The maps f_{jk} are related through:*

$$\begin{aligned}
f_{11}(\mathbf{z},\mathbf{w})(\xi^1,\xi^2)(\varepsilon) &= \overline{f_{22}(\overline{\mathbf{w}},\overline{\mathbf{z}})(\overline{\xi^2},\overline{\xi^1})(\varepsilon)} \\
f_{21}(\mathbf{z},\mathbf{w})(\xi^1,\xi^2)(\varepsilon) &= \overline{f_{12}(\overline{\mathbf{w}},\overline{\mathbf{z}})(\overline{\xi^2},\overline{\xi^1})(\varepsilon)},
\end{aligned} \tag{6.1.77}$$

for real values of the parameter.

Proof. Let $\varepsilon \in V \cap \mathbb{R}$. Equivalences (6.1.66), the definition of the dual differentials and Lemma 6.1.34 allow to conclude:

$$\begin{aligned}
\overline{f_{22}(\overline{\mathbf{w}},\overline{\mathbf{z}})(\overline{\xi^2},\overline{\xi^1})(\varepsilon)} &= -\overline{(\mathfrak{a}^2_{1,\varepsilon}(\overline{\mathbf{w}},\overline{\mathbf{z}})d_{1,\varepsilon}\mathbf{w} + \mathfrak{a}^2_{2,\varepsilon}(\overline{\mathbf{w}},\overline{\mathbf{z}})d_{1,\varepsilon}\mathbf{w})} \\
&= -(\mathfrak{a}^1_{2,\varepsilon}(\mathbf{z},\mathbf{w})\overline{d}_{1,\varepsilon}\overline{\mathbf{w}} + \mathfrak{a}^1_{1,\varepsilon}(\mathbf{z},\mathbf{w})\overline{d}_{1,\varepsilon}\overline{\mathbf{z}}) \\
&= f_{11}(\mathbf{z},\mathbf{w})(\xi^1,\xi^2)(\varepsilon).
\end{aligned}$$

The same procedure shows that f_{12}, f_{21} are related by Schwarz reflection when the parameter is real. $\square$

Definition 6.1.38 (Nijenhuis-Woolf operators). *The operators:*

$$\begin{aligned}\mathbf{TF}^1 &:= T^1 f_{11} + T^2 f_{21} - \frac{1}{2}(T^1 \overline{d}_{1,\varepsilon} T^2 f_{21} + T^2 \overline{d}_{2,\varepsilon} T^1 f_{11})\\ \mathbf{TF}^2 &:= T^1 f_{12} + T^2 f_{22} - \frac{1}{2}(T^1 \overline{d}_{1,\varepsilon} T^2 f_{22} + T^2 \overline{d}_{2,\varepsilon} T^1 f_{12}).\end{aligned} \tag{6.1.78}$$

are called the first and second Nijenhuis-Woolf symmetric operators.

The non-linear differential system corresponding to (6.1.72) is given by the integral equation ([41]):

$$\begin{aligned}\mathbf{z}(\xi_\varepsilon^1, \xi_\varepsilon^2) &= \xi_\varepsilon^1 + \mathbf{TF}^1[\mathbf{z}, \mathbf{w}](\xi_\varepsilon^1, \xi_\varepsilon^2) - \mathbf{TF}^1[\mathbf{z}, \mathbf{w}](0,0)\\ \mathbf{w}(\xi_\varepsilon^1, \xi_\varepsilon^2) &= \xi_\varepsilon^2 + \mathbf{TF}^2[\mathbf{z}, \mathbf{w}](\xi_\varepsilon^1, \xi_\varepsilon^2) - \mathbf{TF}^2[\mathbf{z}, \mathbf{w}](0,0).\end{aligned} \tag{6.1.79}$$

Definition 6.1.39. *The pair of coordinates* $(\xi_\varepsilon^1, \xi_\varepsilon^2)$ *is referred to as the initial value of* (6.1.79).

For $\varepsilon = 0$, the system is solved by means of a Picard iteration process (fixed point Theorem) which converges in a small ball $\mathbf{B}(r_0)$ of radius $r_0 > 0$ around the origin of $(\mathbf{z}, \mathbf{w})$ coordinates ([40],[41]). It turns out that for all $|\varepsilon|$ small and fixed, the solution to (6.1.79) is well defined on $\mathbf{B}(r)$, with $r = \frac{r_0}{2}$. Moreover, if r is small enough, then the solution $(\mathbf{z}, \mathbf{w})$ is unique:

Lemma 6.1.40. *[40] For r sufficiently small, and $\varepsilon \in V$ fixed, the integral system* (6.1.79) *admits a unique solution* $(\mathbf{z}, \mathbf{w})$ *satisfying also* (6.1.72) *and such that the parametric transformation* $\widetilde{\Lambda}_\varepsilon^{\circ -1}$ *from the* $(\xi_\varepsilon^1, \xi_\varepsilon^2)$ *coordinates to* $(\mathbf{z}, \mathbf{w})$ *coordinates has non-vanishing Jacobian.*

Proposition 6.1.41. *For every* $(\mathbf{z},\mathbf{w})$ *in a neighborhood of the origin and for every initial value* $(\xi_\varepsilon^1,\xi_\varepsilon^2)$, *the Nijenhuis-Woolf symmetric operators* $\mathbf{TF}^1,\mathbf{TF}^2$ *are related through Schwarz reflection:*

$$\mathbf{TF}^1[\mathbf{z},\mathbf{w}](\xi^1,\xi^2)(\varepsilon)=\overline{\mathbf{TF}^2[\overline{\mathbf{w}},\overline{\mathbf{z}}](\overline{\xi^2},\overline{\xi^1})(\varepsilon)} \tag{6.1.80}$$

when the parameter is real.

Proof. Let the parameter be real. A direct calculation and the definition of the dual differentials prove:

$$\overline{\mathbf{TF}^2[\overline{\mathbf{w}},\overline{\mathbf{z}}](\overline{\xi^2},\overline{\xi^1})(\varepsilon)} = \overline{T^1f_{12}}+\overline{T^2f_{22}}-\frac{1}{2}[\overline{T^1d_{2,\varepsilon}T^2f_{22}}+\overline{T^2d_{1,\varepsilon}T^1f_{12}}]$$

where the second member is evaluated at $(\overline{\mathbf{w}},\overline{\mathbf{z}})(\overline{\xi^2},\overline{\xi^1})(\varepsilon)$. Then Lemma 6.1.37 and property (6.1.75) yield:

$$\begin{aligned}\overline{\mathbf{TF}^2[\overline{\mathbf{w}},\overline{\mathbf{z}}](\overline{\xi^2},\overline{\xi^1})(\varepsilon)} &= T^1f_{11}+T^2f_{21}-\frac{1}{2}[T^1\overline{d}_{1,\varepsilon}T^2f_{21}+T^2\overline{d}_{2,\varepsilon}T^1f_{11}]\\ &= \mathbf{TF}^1[\mathbf{z},\mathbf{w}](\xi^1,\xi^2)(\varepsilon).\end{aligned}$$

where the member on the right of the first equality is evaluated at $(\mathbf{z},\mathbf{w})(\xi^1,\xi^2)(\varepsilon)$.

□

Proposition 6.1.42. *The chart* (6.1.52) *has real character:*

$$\widetilde{\Lambda}_\varepsilon=\mathscr{S}\circ\sigma\circ\widetilde{\Lambda}_{\mathscr{C}(\varepsilon)}\circ\sigma\circ\mathscr{S}$$

and is tangent to the identity.

Proof. First, the parameter ε is supposed to be real. Let $(\xi_\varepsilon^1,\xi_\varepsilon^2)$ be the initial value and $(\mathbf{z},\mathbf{w})$ be the solution to (6.1.79). If the initial condition satisfies $\xi_\varepsilon^1=\overline{\xi_\varepsilon^2}$, then Proposition 6.1.41 leads to

$$\begin{aligned}\overline{\mathbf{w}} &= \xi_\varepsilon^1+\mathbf{TF}^1[\overline{\mathbf{w}},\overline{\mathbf{z}}](\xi_\varepsilon^1,\xi_\varepsilon^2)-\mathbf{TF}^1[\overline{\mathbf{w}},\overline{\mathbf{z}}](0,0)\\ \overline{\mathbf{z}} &= \xi_\varepsilon^2+\mathbf{TF}^2[\overline{\mathbf{w}},\overline{\mathbf{z}}](\xi_\varepsilon^1,\xi_\varepsilon^2)-\mathbf{TF}^2[\overline{\mathbf{w}},\overline{\mathbf{z}}](0,0).\end{aligned}$$

Thus the unicity of the solution in Lemma 6.1.40 implies that $\mathbf{z} = \overline{\mathbf{w}}$. Inasmuch as $\widetilde{\Lambda}_\varepsilon$ has non-vanishing Jacobian, it is a local isomorphism if $r > 0$ is small. In particular, $\widetilde{\Lambda}_\varepsilon^{\circ -1}$ sends isomorphically a local neighborhood of the origin in the surface $\{\xi_\varepsilon^1 = \overline{\xi_\varepsilon^2}\}$ of the space of initial values $(\xi_\varepsilon^1, \xi_\varepsilon^2)$, into a local neighborhood of the origin in the plane of symmetry $\{\mathbf{z} = \overline{\mathbf{w}}\}$ of the space of solutions $(\mathbf{z}, \mathbf{w})$, when $\varepsilon \in \mathbb{R}$. The real character of the chart $\widetilde{\Lambda}_\varepsilon$ is then proven over the subset $\varepsilon \in V \cap \mathbb{R}$. Since $\widetilde{\Lambda}_\varepsilon$ depends analytically on the parameter, its real character $\widetilde{\Lambda}_\varepsilon = \mathscr{S} \circ \sigma \circ \widetilde{\Lambda}_{\mathscr{C}(\varepsilon)} \circ \sigma \circ \mathscr{S}$ for values $\varepsilon \in V$ is also proven. By (6.1.79), the chart $\widetilde{\Lambda}_\varepsilon$ is tangent to the identity at the origin. □

End of the proof of Theorem 6.1.27. Passing to the quotient, the chart $\Lambda_\varepsilon : \mathbf{B}(r) \subset H_\varepsilon(\mathscr{M}_\varepsilon) \to \mathbb{C}^2$ induced by $\widetilde{\Lambda}_\varepsilon$, has real character as well (Remark 6.1.29). Indeed, (6.1.55) means that the diagram defined by $\Gamma_\varepsilon^*, \Gamma_\varepsilon^{**}$ and $\widetilde{\Lambda}_\varepsilon$ is commutative. Thus, the chart $\Lambda_\varepsilon : \mathbf{B}(r) \to \mathbb{C}^2$ inherits the real character of $\widetilde{\Lambda}_\varepsilon$. It is clearly tangent to the identity at the origin. □

End of the proof of Theorem 6.1.4. Let $\Lambda_\varepsilon = (\xi^1, \xi^2)$ be the family of smooth charts of Theorem 6.1.27 defined locally on a neighborhood $\mathbf{B}(r) \subset H_\varepsilon(\mathscr{M}_\varepsilon)$. The composition

$$\vartheta_\varepsilon = \Lambda_\varepsilon \circ H_\varepsilon : H_\varepsilon^{-1}(\mathbf{B}(r)) \to \mathbb{C}^2$$

between complex analytic manifolds is honestly biholomorphic. The closure $\mathscr{W} := \overline{\vartheta_\varepsilon(H_\varepsilon^{-1}(\mathbf{B}(r)))}$ contains the origin in its interior. It remains to check that the family of vector fields defined on $\mathscr{W}$ by the pushforward $\mathbf{v}_\varepsilon = (\vartheta_\varepsilon)_* v_\varepsilon$, is orbitally equivalent to a generic family unfolding a weak focus with formal normal form (6.1.1). Notice that $\mathbf{v}_\varepsilon$ is also monodromic because the family v_ε is monodromic, and by Theorem 6.1.27, the composition $\vartheta_\varepsilon = \Lambda_\varepsilon \circ H_\varepsilon$ is tangent to the identity for every $\varepsilon \in V$. By Proposition 6.1.21, the monodromy of $\mathbf{v}_\varepsilon$ coincides with $\mathscr{P}_\varepsilon := \mathscr{Q}_\varepsilon^{\circ 2}$. Since the Poincaré map $\mathscr{Q}_{0,\varepsilon}^{\circ 2}$ of the family (6.1.1) and the monodromy of the family $\mathbf{v}_\varepsilon$ are formally conjugate by construction, it will be sufficient to show that the quotient of the eigenvalues of the family $\mathbf{v}_\varepsilon$ is $\frac{\varepsilon + i}{\varepsilon - i}$, see Corollary C.2.4 in the Appendix C.

Lemma 6.1.43. *The monodromic family of vector fields $\mathbf{v}_\varepsilon$ has real character.*

Proof. This becomes apparent directly from Corollary 6.1.22 in local coordinates, and from the real character of the pushforward ϑ_* which, in turn, is a consequence of Proposition 6.1.23 and Theorem 6.1.27. □

Proposition 6.1.44. *The quotient of the eigenvalues of the monodromic family $\mathbf{v}_\varepsilon$ is equal to $\frac{\varepsilon + i}{\varepsilon - i}$.*

Proof. By Lemma 6.1.43, the eigenvalues of the vector field $\mathbf{v}_\varepsilon$ are complex conjugate. We call them $\tau(\varepsilon), \overline{\tau(\varepsilon)}$, with

$$\tau(\varepsilon) = a(\varepsilon) + ib(\varepsilon) \tag{6.1.81}$$

and $a(\varepsilon), b(\varepsilon)$ depend analytically on ε small and are real on $\varepsilon \in \mathbb{R}$. The Poincaré map of the family $\mathbf{v}_\varepsilon$ has multiplier $\mu(\varepsilon) = \mathscr{P}_\varepsilon'(0) = \exp\left(2i\pi \frac{\tau + \overline{\tau}}{\tau - \overline{\tau}}\right)$. Indeed, if (z, w) are the coordinates of the family of vector fields $\mathbf{v}_\varepsilon$, then in the first chart of the blow up $\mathbf{v}_\varepsilon$ gives rise to a family of equations of the form:

$$\begin{aligned} \dot{Z} &= (\tau - \overline{\tau})Z + \ldots \\ \dot{w} &= \overline{\tau} w + \ldots \end{aligned} \tag{6.1.82}$$

The Poincaré map of this family is given by ([37]):

$$\mathscr{P}_\varepsilon(w) = \exp\left(2i\pi\left(\frac{2\overline{\tau}}{\tau - \overline{\tau}}\right)\right) w + \ldots$$

In the second chart of the blow up, the family $\mathbf{v}_\varepsilon$ gives rise to the system:

$$\begin{aligned}\dot{W} &= (\overline{\tau}-\tau)W+\dots\\ \dot{z} &= \tau z+\dots\end{aligned} \tag{6.1.83}$$

and then the Poincaré map is given by $\mathscr{P}_\varepsilon(z) = \exp\left(-2i\pi\left(\frac{2\tau}{\overline{\tau}-\tau}\right)\right)z+\dots$ (computed on the cross section $z=w$ in ambient coordinates). It is easily seen that:

$$\mu(\varepsilon)=\exp\left(2i\pi\left(\frac{2\overline{\tau}}{\tau-\overline{\tau}}\right)\right)=\exp\left(-2i\pi\left(\frac{2\tau}{\overline{\tau}-\tau}\right)\right)=\exp\left(2i\pi\left(\frac{\tau+\overline{\tau}}{\tau-\overline{\tau}}\right)\right).$$

On the other hand, $\mu(\varepsilon)=\exp(2\pi\varepsilon)$ by Proposition 2.3.2. Thus,

$$2\pi\varepsilon = 2\pi\frac{2a(\varepsilon)}{2b(\varepsilon)}+2i\pi m,$$

for some $m\in\mathbb{N}$. This means that

$$\frac{a(\varepsilon)}{b(\varepsilon)}=\varepsilon-im. \tag{6.1.84}$$

Inasmuch as $a(\varepsilon), b(\varepsilon)$ are real on $\varepsilon\in\mathbb{R}$, the equation (6.1.84) implies that $m=0$, which yields

$$\frac{\tau}{\overline{\tau}}=\frac{\varepsilon+i}{\varepsilon-i}.$$

□

This shows that the family of vector fields $\mathbf{v}_\varepsilon$ unfolds a vector field with a weak focus. Inasmuch as the order of $\mathscr{Q}_\varepsilon$ is one, the order of the family of vector fields is one as well. □

Part III

Modulus of analytic classification

(orbital equivalence)

Chapter 7

Preparation and the orbit space

7.1 Preparations.

A generic family of equations unfolding an order 1 weak focus:

$$\begin{aligned} \dot{z} &= (\varepsilon+i)z \pm z^2 w + \sum_{j+k\geq 4} a_{jk}(\varepsilon) z^j w^k \\ \dot{w} &= (\varepsilon-i)w \pm w^2 z + \sum_{j+k\geq 4} \overline{a_{jk}(\overline{\varepsilon})} z^k w^j, \end{aligned} \tag{7.1.1}$$

(where the coefficients $a_{jk}(\varepsilon)$ depend analytically on the parameter) is a family with a generic Hopf bifurcation of codimension 1. Recall that the family of vector fields on the right is noted v_ε. The Theorem 4.2.3 allows to describe the dynamics of such a family through the dynamics of the family of diffeomorphisms unfolding the Poincaré map of the system v_0. We have seen (Theorem 2.3.2) that such a family is of the form

$$\mathscr{P}_\varepsilon(w) = e^{2\pi\varepsilon} w \pm e^{2\pi\varepsilon}[2\pi + O(\varepsilon)]w^3 + O(w^4). \tag{7.1.2}$$

when the family of vector fields v_ε has been brought into normal form (1.4.1). Proposition 3.1.6 implies that its coefficients are real when $\varepsilon \in \mathbb{R}$ and thus

$$\mathscr{C} \circ \mathscr{P}_{\overline{\varepsilon}} \circ \mathscr{C} = \mathscr{P}_\varepsilon .$$

The family $\mathscr{P}_\varepsilon$ is the unfolding of the Poincaré map $\mathscr{P}$ of the system $v = v_0$, which is a germ of codimension one resonant analytic diffeomorphism with a fixed point of multiplicity 3 at the origin which corresponds to the coalescence of a fixed point with a periodic orbit of period 2. In the previous literature issues (*e.g.* [10], [11], [25], [32]) a generic family of vector fields was prepared with respect to the *canonical parameter* ε if $h_z'(0) = e^{-\varepsilon}$ (where h_z is the holonomy map of the z separatrix) and, at the same time, the invariant manifold had equation

$$zw = \varepsilon \,. \tag{7.1.3}$$

Such a preparation is not well adapted to the chosen formal normal form (1.4.2). Indeed the formal classification for weak foci must respect the real character (3.1.6) of the singularity. Instead, we will prefer another preparation respecting the real character of the unfolding v_ε, and that is compatible with the formal normal form (1.4.2), such that:

$$\mathscr{P}_\varepsilon'(0) = e^{2\pi\,\varepsilon} \tag{7.1.4}$$

and, again, the invariant manifold corresponds to (7.1.3). The condition (7.1.4) uniquely determines ε (the canonical parameter). When studying the equivalence of two families this allows to conclude that the canonical parameter is preserved and it allows also to work for fixed values of it. The preparation performed on the family of vector fields introduced in the next theorem, brings the Poincaré map into a "prepared" form as well. Once we have properly prepared the family of diffeomorphisms, we can compute its invariant of analytic classification.

Strategy. In order to compute the modulus of the unfolding of the Poincaré map, we compare the latter with the time-one map τ_ε^1 of the vector field:

$$\frac{2\pi w(\varepsilon \pm w^2)}{1+A(\varepsilon)w^2}\frac{\partial}{\partial w}, \tag{7.1.5}$$

which is called *the model family.*

Theorem 7.1.1. *There exists an analytic change of coordinates*

$$(z,w) \mapsto (\mathbf{z},\mathbf{w})$$

bringing the family of vector fields into a "prepared" form, for which the invariant manifold of the system has equation $\mathbf{zw} = -s\,\varepsilon$, and the new coordinates preserve the real character of the field. Here, $s = \pm$ is the coefficient of the third order term (defined in Proposition 1.2.1). In addition, the Poincaré map of the section $\Sigma : \{\mathbf{z} = \mathbf{w}\}$ (parametrized with the $\mathbf{w}$ coordinate) has the form:

$$\mathscr{P}_\varepsilon(\mathbf{w}) = \mathbf{w} + \mathbf{w}(\varepsilon \pm \mathbf{w}^2)[2\pi + D(\varepsilon) + E(\varepsilon)\mathbf{w}^2 + \mathbf{w}(\varepsilon \pm \mathbf{w}^2)h(\varepsilon,\mathbf{w})] \tag{7.1.6}$$

with fixed points $\mathbf{w}_0 = 0$, and $\mathbf{w}_\pm = \pm\sqrt{-s\,\varepsilon}$. The constants $D(\varepsilon), E(\varepsilon)$ and the function h are real when $\varepsilon \in \mathbb{R}$. The multiplier λ_0 of the fixed point $\mathbf{w}_0 = 0$ satisfies:

$$\lambda_0 = \mathscr{P}'_\varepsilon(0) = e^{2\pi\,\varepsilon} \tag{7.1.7}$$

In particular, the parameter ε is an analytic invariant for $\mathscr{P}_\varepsilon(\mathbf{w})$. We call it the "canonical parameter". The multipliers of the fixed points $\mathbf{w}_\pm = \pm\sqrt{-s\,\varepsilon}$ are given by:

$$\lambda_\pm = \mathscr{P}'_\varepsilon(\mathbf{w}_\pm) = \exp\left\{\frac{-4\pi\,\varepsilon}{1-sA(\varepsilon)\,\varepsilon}\right\}, \tag{7.1.8}$$

so that they coincide exactly with the multipliers of the time-one map of the model family (7.1.5). *The "formal parameter" $A(\varepsilon)$ is defined by:*

$$A(\varepsilon) = \frac{2\pi s}{\ln\lambda_0} + \frac{2\pi s}{\ln\lambda_+} + \frac{2\pi s}{\ln\lambda_-}. \tag{7.1.9}$$

It depends analytically on ε and is real when $\varepsilon \in \mathbb{R}$. It is an analytic invariant of $\mathscr{P}_\varepsilon$.

Proof. According to the formal classification (Theorem 1.4.2), the equation of the invariant manifold of the formal normal form (1.4.2) is

$$\varepsilon \pm u = 0.$$

Even if the change of coordinates to normal form is generically divergent, the invariant manifold is analytic ([26]). Coming back to the variables z and w the invariant manifold has an equation of the form:

$$\varepsilon = -s\zeta_1(\varepsilon)u + o(u) = -su\zeta_1(\varepsilon)(1+m(z,w)), \tag{7.1.10}$$

where $u = zw$ and $m(z,w) = O(u)$ satisfies $m(z,w) = \overline{m(\overline{w},\overline{z})}$ for all z,w near the origin. Let us define the following change of coordinates:

$$\begin{aligned} \mathbf{z} &= z\sqrt{\zeta_1(\varepsilon)(1+m(z,w))} \\ \mathbf{w} &= w\sqrt{\zeta_1(\varepsilon)(1+m(z,w))}. \end{aligned}$$

Then, in these coordinates, the invariant manifold has equation:

$$\mathbf{zw} = u\zeta_1(\varepsilon)(1+m(z,w)) = -s\,\varepsilon\,. \tag{7.1.11}$$

Notice that this change of coordinates preserves the symmetric form of the original system (*i.e.* its real character).

The analytic invariant manifold intersects the cross-section $\{\mathbf{z}=\mathbf{w}\}$ at $\mathbf{w}^2 = -s\,\varepsilon$. Let $\mathscr{Q}_\varepsilon(\mathbf{w})$ be the semi-Poincaré map of the family in the variable $\mathbf{w}$. We know that $\mathbf{w}^2+s\,\varepsilon=0$ is the equation of the 2-periodic points of the semi-Poincaré family coming from the intersection of the invariant manifold with $\mathbf{z}=\mathbf{w}$. Then the second iterate of $\mathscr{Q}_\varepsilon(\mathbf{w})$ has the form

$$\mathscr{Q}_\varepsilon^{\circ 2}(\mathbf{w}) = \mathbf{w}+\mathbf{w}(\varepsilon \pm \mathbf{w}^2)h(\varepsilon,\mathbf{w}), \tag{7.1.12}$$

where $h(\varepsilon,\mathbf{w}) = (2\pi + O(\varepsilon)+O(\mathbf{w}))$. Indeed,

$$(\mathscr{Q}_\varepsilon^{\circ 2})'(0) = 1+\varepsilon\,h(\varepsilon,0) \equiv e^{2\pi\,\varepsilon},$$

and then $h(\varepsilon,0) = \dfrac{e^{2\pi\,\varepsilon}-1}{\varepsilon} = 2\pi + O(\varepsilon)$. □

Definition 7.1.2. *The family of vector fields in coordinates* $(\mathbf{z},\mathbf{w})$ *and parameter* ε, *obtained in Theorem 7.1.1, is called orbitally prepared. The family of diffeomorphisms* (7.1.6) *obtained after preparation of the family of vector fields and satisfying* (7.1.7) *is, by analogy, called prepared. From now on, we shall work with families of vector fields and the corresponding families of diffeomorphisms only in prepared form, so the tuple* $(\mathbf{z},\mathbf{w},\varepsilon)$ *will be noted* (z,w,ε). *The prepared families of fields and diffeomorphisms in these coordinates will be noted* v_ε *and* $\mathscr{P}_\varepsilon$, *respectively.*

7.2 Glutsyuk point of view.

We will only discuss the case $s=+1$. We choose a fixed neighborhood U of the origin on which $\mathscr{P}_0$ is a diffeomorphism. If δ is a positive number we define

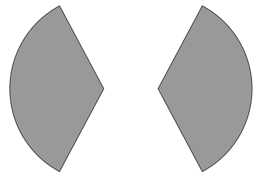


Figure 7.1: Sectorial domains for the parameter.

sectorial domains in the universal covering of the the parameter space, see Figure 7.1:

$$\begin{aligned} V^G_{\delta,l} &= \{\varepsilon \in \mathbb{C} : |\varepsilon| < \rho, \arg(\varepsilon) \in (\frac{\pi}{2}+\delta, \frac{3\pi}{2}-\delta)\} \\ V^G_{\delta,r} &= \{\varepsilon \in \mathbb{C} : |\varepsilon| < \rho, \arg(\varepsilon) \in (-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta)\} \end{aligned} \tag{7.2.1}$$

and ρ is a small real number depending on δ. The number ρ is chosen so that for values $\varepsilon \in V^G_{\delta,lr}$, there exists orbits connecting the fixed points in U. In this case, we say that we work in the Glutsyuk point of view (Figure 7.2):

- If $\varepsilon < 0$ the origin is attractor and the two real singular points $w_\pm = \pm\sqrt{-\varepsilon}$ are repeller in U.
- If $\varepsilon = 0$ the origin is the only (non-hyperbolic) fixed point.
- If $\varepsilon > 0$ the origin is repeller and two additional imaginary attracting singular points are created in U.

Definition 7.2.1 (Analytic conjugation). *[10] Two germs of analytic families f_ε and $\widehat{f}_\varepsilon$ of diffeomorphisms with a fixed point at the origin are conjugate if there exists a germ of analytic diffeomorphism $\mathscr{H}(\varepsilon,w) = (\mathbf{k}(\varepsilon), \mathbf{h}(\varepsilon,w))$ fibered over the parameter space such that:*

$$\mathbf{h}_\varepsilon \circ f_\varepsilon = \widetilde{f}_{\mathbf{k}(\varepsilon)} \circ \mathbf{h}_\varepsilon, \tag{7.2.2}$$

where $\mathbf{h}_\varepsilon(w)\stackrel{def}{=}\mathbf{h}(\varepsilon,w)$. *The conjugacy is said to be real if* $\mathscr{C}\circ\mathbf{k}\circ\mathscr{C}=\mathbf{k}$ *and*

$$\mathscr{C}\circ\mathbf{h}_{\mathscr{C}(\varepsilon)}\circ\mathscr{C}=\mathbf{h}_\varepsilon. \tag{7.2.3}$$

The family of diffeomorphisms $\mathscr{P}_\varepsilon$ may be conjugated to the time-one map τ_ε^1 of the field (7.1.5) on the sectorial domains (7.2.1). The modulus measures the obstruction to get a conjugacy on a full neighborhood of the origin in the w-space. The vector field (7.1.5) has singular points $w_0=0$, with eigenvalue $\mu_0(\varepsilon)=2\pi\,\varepsilon$, and $w_\pm=\pm\sqrt{-\varepsilon}$ with eigenvalues:

$$\mu_\pm(\varepsilon)=\frac{-4\pi\,\varepsilon}{1-A(\varepsilon)\,\varepsilon}. \tag{7.2.4}$$

Notice that μ_0 and $\mu_\pm$ are analytic invariants of (7.1.5), which also depend analytically on ε. It follows that ε and $A(\varepsilon)$ are analytic invariants of the field (7.1.5). The multipliers of the time-one map τ_ε^1 of v_ε are $\lambda_j=e^{\mu_j}$, *i.e.* they are precisely the multipliers of the fixed points of $\mathscr{P}_\varepsilon$. In order to compare $\mathscr{P}_\varepsilon$ with the model diffeomorphism τ_ε^1 we compare their orbit space. The orbit space of $\mathscr{P}_\varepsilon$ is ob-

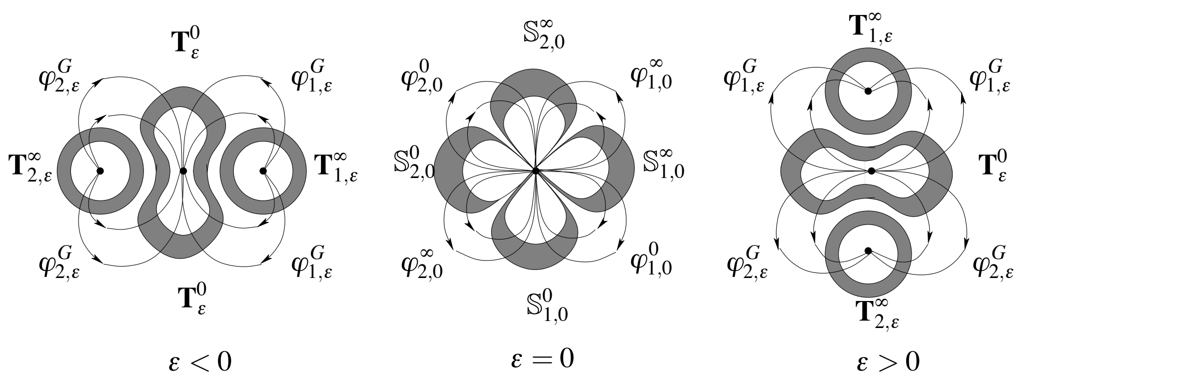


Figure 7.2: The orbit space of the Poincaré map and the transitions $\varphi^G_{j,\varepsilon}$.

tained by taking 3 closed curves $\{\ell_0,\ell_+,\ell_-\}$ around the fixed points, and their images $\{\mathscr{P}_\varepsilon(\ell_\#)\}$ where $\# \in \{0,+,-\}$. Since the fixed points are hyperbolic the closed regions $\{C_\#\}$ between the curves and their images are isomorphic to three closed annuli. We identify $\ell_\# \sim \mathscr{P}_\varepsilon(\ell_\#)$. Then the quotient $C_\#/\sim$ will be shown to be a conformal torus. Hence, the orbit space turns out to be a non-Hausdorff space conformally equivalent to a collection of three tori $\mathbf{T}^0_\varepsilon, \mathbf{T}^\infty_{1,\varepsilon}, \mathbf{T}^\infty_{2,\varepsilon}$ plus the three singular points, which represent the orbit space of the hyperbolic fixed points, such that

- each orbit has at most one point in each torus,
- each orbit is either a fixed point or is represented in a torus,
- some orbits may have representatives in two different tori.

The Glutsyuk modulus can be described as follows. For specific values of $\varepsilon \in V_{\delta,lr}$, the singular points $x_0, x_\pm$ are hyperbolic, thus normalizable. Hence, there exists in the neighborhood of each fixed point a diffeomorphism $\varphi^\#_\varepsilon$, $\# \in \{0,+,-\}$ conjugating the Poincaré map $\mathscr{P}_\varepsilon$ with the model family, *i.e.* the time one map of the model family (7.1.5). It is shown ([21]) that for a sufficiently small choice of the sectorial neighborhood in the covering of the parameter space, the domains of $\varphi^\#_\varepsilon$ overlap, allowing to define:

$$\left.\begin{array}{rcl}\varphi^{G,+}_\varepsilon &=& \varphi^+_\varepsilon \circ (\varphi^0_\varepsilon)^{-1}\\ \varphi^{G,-}_\varepsilon &=& \varphi^-_\varepsilon \circ (\varphi^0_\varepsilon)^{-1}\end{array}\right\} \text{ for } \varepsilon \in V_{\delta,l},$$

and

$$\left.\begin{array}{rcl}\varphi^{G,+}_\varepsilon &=& \varphi^0_\varepsilon \circ (\varphi^+_\varepsilon)^{-1}\\ \varphi^{G,-}_\varepsilon &=& \varphi^0_\varepsilon \circ (\varphi^-_\varepsilon)^{-1}\end{array}\right\} \text{ for } \varepsilon \in V_{\delta,r}.$$

Note that $\varphi^\#_\varepsilon$ is unique up to left composition with a symmetry of τ^1_ε, *i.e.* a time-t map of v_ε. Hence, since the domains of φ^0_ε and $\varphi^\pm_\varepsilon$ intersect for $\varepsilon \in V_{\delta,lr}$ for sufficiently small ρ, the collection $\{(\varphi^{G,+}_\varepsilon, \varphi^{G,-}_\varepsilon)\}_{\varepsilon \in V^G_{\delta,\pm}}$ is an analytic invariant of the family $\mathscr{P}_\varepsilon$ under analytic families of change of coordinates preserving the canonical parameter. This is one presentation of the Glutsyuk modulus. We now study how the normalization is performed. It is natural to introduce coordinates on the orbit space. The Glutsyuk modulus only concerns the orbits represented in the tori. One way to introduce coordinates on a torus $\mathbf{T}$ is to consider the latter as a quotient $\mathbf{T} = \mathbb{C}^*/\mathscr{L}_C$ (where $\mathscr{L}_C(x) = Cx$ is the linear map) for some $C \in \mathbb{C}^*$. Then a natural coordinate on $\mathbf{T}$ is the projection of a coordinate on $\mathbb{C}^* = \mathbb{CP}^1 \backslash \{0, \infty\}$, *i.e.* a "spherical" coordinate.

Fatou coordinates were introduced in 1920 by former P. Fatou ([16]). They are changes of coordinates which allow to transform the prepared family $\mathscr{P}_\varepsilon$ into the "model family" τ_ε^1 over the sectorial domains (7.2.1). We will construct a special kind of Fatou coordinates: we show that it is possible to choose them respecting the real character of $\mathscr{P}_\varepsilon$. This choice yields a symmetry property on the Glutsyuk invariant in the unfolding. We will study the properties of the Glutsyuk invariant by "unwrapping" the spherical coordinate.

Chapter 8

Real Fatou Glutsyuk Coordinates

8.1 The unwrapping coordinate.

From now on, the parameter belongs to either of the Glutsyuk sectors (7.2.1). Consider the "unwrapping" change of coordinates $p_\varepsilon : \mathscr{R}_\varepsilon \to U \setminus \{w_0, w_\pm\}$ defined by:

$$w = p_\varepsilon(Z) = \begin{cases} \left(\dfrac{s\,\varepsilon}{se^{-4\pi\,\varepsilon\,Z} - 1}\right)^{\frac{1}{2}} & \text{for} \quad \varepsilon \neq 0 \\ \left(-\dfrac{s}{4\pi Z}\right)^{\frac{1}{2}} & \text{for} \quad \varepsilon = 0 \end{cases} \tag{8.1.1}$$

where $\mathscr{R}_\varepsilon$ is the 2-sheeted Riemann surface of the function (see Figure 8.1)

$$\begin{cases} \left(\dfrac{1 - se^{-4\pi\,\varepsilon\,Z}}{s\,\varepsilon}\right)^{\frac{1}{2}} & \text{for} \quad \varepsilon \neq 0 \\ (2s\pi Z)^{\frac{1}{2}} & \text{for} \quad \varepsilon = 0, \end{cases}$$

and $s = \pm 1$ is defined in Proposition 1.2.1 (the sign of the third order coefficient of the family). Notice that for all $\varepsilon \in V^G_{\delta,rl}$, the map p_ε is periodic with period $-\dfrac{i}{2\,\varepsilon}$:

$$p_\varepsilon(Z) = p_\varepsilon\left(Z - k\frac{i}{2\,\varepsilon}\right), \quad k \in \mathbb{Z}. \tag{8.1.2}$$

Without loss of generality, the neighborhood U can be taken as a small ball $B(0,r)$. By (8.1.2), the image $p_\varepsilon^{\circ -1}(U = B(0,r))$ consists of the Riemann surface $\mathscr{R}_\varepsilon$ minus a countable number of holes. The smaller the radius of U, the larger the radius of such holes (of order $\frac{1}{r}$).

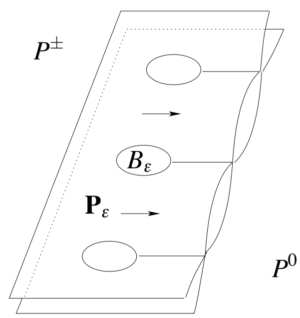


Figure 8.1: The surface $\mathscr{R}_\varepsilon$, domain of the lifting $\mathbf{P}_\varepsilon$.

Definition 8.1.1. *The distance between two consecutive holes, for* $\varepsilon \neq 0$, *is the complex number:*

$$\alpha(\varepsilon) = -\frac{i}{2\,\varepsilon}. \tag{8.1.3}$$

The Poincaré family $\mathscr{P}_\varepsilon = \mathscr{P}_\varepsilon(w)$ is lifted into a family:

$$\mathbf{P}_\varepsilon := p_\varepsilon^{-1} \circ \mathscr{P}_\varepsilon \circ p_\varepsilon. \tag{8.1.4}$$

By (8.1.2), the family $\mathbf{P}_\varepsilon$ is defined on $\mathscr{R}_\varepsilon$ minus the countable collection of holes. The dynamics of $\mathbf{P}_\varepsilon$ goes always from left to right on $\mathscr{R}_\varepsilon$. We denote P^0 and $P^{\pm}$ the points at infinity located in the direction orthogonal to the line of holes, in such a way that their images by p_ε be equal to $w_0 = 0$ and $w_\pm = \pm\sqrt{-s\,\varepsilon}$, respectively:

Definition 8.1.2.

$$\begin{aligned} P^0 &= p_\varepsilon^{\circ -1}(w_0), \\ P^{\pm} &= p_\varepsilon^{\circ -1}(w_\pm). \end{aligned} \tag{8.1.5}$$

In a neighborhood of the points $P^{\pm}$ (there are two such points, in correspondence with the leaves of $\mathscr{R}_\varepsilon$) the two sheets go to different singular points in the w coordinate, while on the side of P^0 both sheets go to the origin, see Figure 8.1.

Definition 8.1.3. *For any complex number $Z_\infty \in \mathbb{C}$ whose imaginary part is of order $\sim |\alpha|$ in a neighborhood of $P^{\pm}$, we define the translation in T_{Z_∞} :*

$$T_{Z_\infty}(\cdot) = Z_\infty + \cdot. \tag{8.1.6}$$

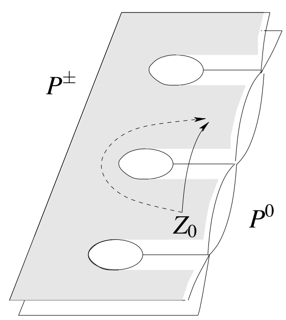


Figure 8.2: Analytic extension of T_α to a neighborhood of P^0, when $\varepsilon > 0$.

By connexity, the translation (8.1.6) can be analytically extended along the leaves of $\mathscr{R}_\varepsilon$ to all Z in a neighborhood of the point P^0, see Figure 8.2. The extension is noted T_{Z_∞} as well. We shall use specific values for Z_∞, the first one being α :

Lemma 8.1.4. *The family $\mathbf{P}_\varepsilon$ commutes with T_α :*

$$\mathbf{P}_\varepsilon \circ T_\alpha = T_\alpha \circ \mathbf{P}_\varepsilon \tag{8.1.7}$$

along the leaves of $\mathscr{R}_\varepsilon$.

Definition 8.1.5. *By* (8.1.2)*, the sequence of equidistant holes can be denoted as:*

$$\{T^{\circ k}_{\alpha(\varepsilon)}(B_\varepsilon)\}_{k\in\mathbb{Z}}, \tag{8.1.8}$$

where $T^{0}_{\alpha(\varepsilon)}(B_\varepsilon) = B_\varepsilon$ corresponds to the integer $k = 0$. It will be called the principal hole, and we will write:

$$\widehat{U}_\varepsilon := p_\varepsilon^{\circ -1}(U) = \mathscr{R}_\varepsilon \setminus \bigcup_{k\in\mathbb{Z}} T^{\circ k}_{\alpha(\varepsilon)}(B_\varepsilon) \tag{8.1.9}$$

the domain for the dynamics of the family $\mathbf{P}_\varepsilon$.

Remark. Notice that the inverse $p_\varepsilon^{\circ -1}$ of the change (8.1.1) is the multivalued function:

$$Z = p_\varepsilon^{\circ -1}(w) = \begin{cases} \dfrac{1}{4\pi\,\varepsilon}\log\left(\dfrac{w^2}{\varepsilon + sw^2}\right) & \text{for } \ \varepsilon \neq 0 \\ -\dfrac{s}{4\pi w^2} & \text{for } \ \varepsilon = 0 \end{cases} \tag{8.1.10}$$

where $\log(\cdot)$ is the principal branch of the logarithm. A simple integration shows that the coordinate Z is the time of the differential equation:

$$\dot{w} = 2\pi w(\varepsilon \pm w^2),$$

which is a small "deformation" of the model family (7.1.5). The coordinate Z has been called the "unwrapping coordinate" by Shishikura ([47]).

8.2 The real and imaginary axis on $\mathscr{R}_\varepsilon$.

Remark. Here we discuss the case $s = +1$.

Definition 8.2.1. *The images of the positive and negative real and imaginary semi-axes by the map $p_\varepsilon^{\circ -1}$ will be noted*

$$\begin{aligned} \mathfrak{R}_\pm &:= p_\varepsilon^{\circ -1}(\mathbb{R}_\pm) \\ \mathfrak{I}_\pm &:= p_\varepsilon^{\circ -1}(i\mathbb{R}_\pm), \end{aligned} \tag{8.2.1}$$

respectively, in the Z coordinate.

By (8.1.2), there is a countable number of such semi-infinite segments on $\mathscr{R}_\varepsilon$, and by (8.1.10), $\mathfrak{R}_+$ and $\mathfrak{R}_-$ lie on the same side on $\mathscr{R}_\varepsilon$, but in different leaves. The same holds for $\mathfrak{I}_+$ and $\mathfrak{I}_-$, see Figure 8.3. The half-lines (8.2.1) organize dif-

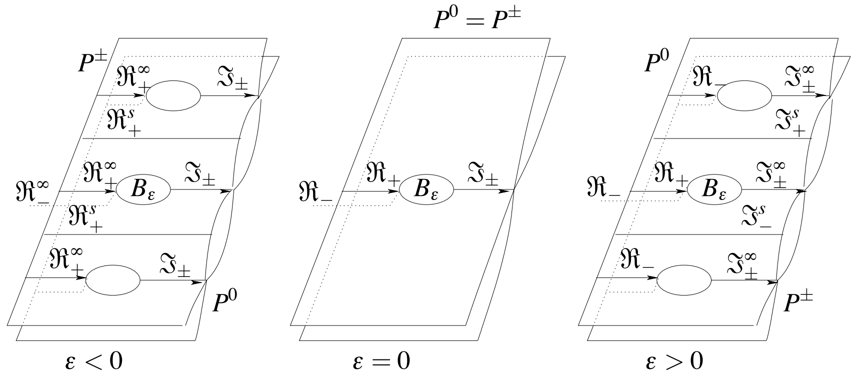


Figure 8.3: The choice of the cuts on $\mathscr{R}_\varepsilon$ for real values of the parameter.

ferently in the cases $\varepsilon \leq 0$ and $\varepsilon > 0$. However, it is always possible to coherently (and continuously) choose the cuts on the Riemann surface $\mathscr{R}_\varepsilon$:

1. When the parameter is negative, the location of the fixed points in the w coordinate yields the decomposition

$$\mathfrak{R}_\pm = \mathfrak{R}_\pm^s \cup \mathfrak{R}_\pm^\infty,$$

where $\mathfrak{R}_\pm^s$ is the image by $p_\varepsilon^{\circ -1}$ of the straight real segment joining 0 and $w_\pm$, and $\mathfrak{R}_\pm^\infty$ is the image by $p_\varepsilon^{\circ -1}$ of the straight real segment joining $w_\pm$ and the boundary of the neighborhood U in the w coordinate. Again, one has an infinite number of such segments $\mathfrak{R}_\pm^{s,\infty}$ at distance $\alpha(\varepsilon)$ from each other in the Z coordinate. The cuts are located along the half-lines $\mathfrak{I}_\pm$. In this case, the preimage $p_\varepsilon^{\circ -1}(\mathbb{R}_\pm)$ forms an alternating sequence $\mathfrak{R}_+^\infty, \mathfrak{R}_+^s$ and $\mathfrak{R}_-^\infty, \mathfrak{R}_-^s$, respectively on the two different leaves. Along with $\mathfrak{I}_\pm$, these sequences respect the rule:

$$\begin{aligned} T_{\alpha(\varepsilon)}(\Im_\pm) &= \Im_\pm, \\ T_{\alpha(\varepsilon)}(\Re_\pm^\infty) &= \Re_\pm^\infty, \\ T_{\alpha(\varepsilon)}(\Re_\pm^s) &= \Re_\pm^s \end{aligned}$$

and the half-line in the leaf below the one containing the half-line $\Re_\pm^{s,\infty}$ is $\Re_\mp^{s,\infty}$. The half-line $\Re_\pm^\infty$ intersecting the principal hole B_ε will be noted $\widehat{\Re}_\pm$, and the half-line $\Im_\pm$ intersecting the principal hole B_ε will be noted $\widehat{\Im}_\pm$.

2. If $\varepsilon = 0$, there are four half-lines $\Re_\pm$ and $\Im_\pm$ in the Z coordinate. They will be noted $\widehat{\Re}_\pm$ and $\widehat{\Im}_\pm$, respectively. The "hat" means that they intersect the hole B_0. The cuts are located along $\widehat{\Im}_\pm$.

3. For positive values of the parameter, the image of the imaginary axis by the map $p_\varepsilon^{\circ -1}$ consists in the union

$$\Im_\pm = \Im_\pm^s \cup \Im_\pm^\infty,$$

where $\Im_\pm^s$ is the image of the straight imaginary segment joining 0 with $w_\pm$, and $\Im_\pm^\infty$ is the image of the straight imaginary segment joining $w_\pm$ and the boundary of the neighborhood U in the w coordinate. By periodicity, one has an infinite number of such segments $\Im_\pm^{s,\infty}$ at distance $\alpha(\varepsilon)$ from each other in the Z coordinate. As indicated in the picture, the cuts of $\mathscr{R}_\varepsilon$ are located along the half-lines $\Im_\pm^\infty$. On the other hand, the preimage $p_\varepsilon^{\circ -1}(\mathbb{R}_\pm)$ forms an alternating sequence $\Re_+, \Re_-$ on the same leaf, thus giving the alternating rule:

$$\begin{aligned} T_{\alpha(\varepsilon)}(\Re_\pm) &= \Re_\mp, \\ T_{\alpha(\varepsilon)}(\Im_\pm^s) &= \Im_\mp^s, \\ T_{\alpha(\varepsilon)}(\Im_\pm^\infty) &= \Im_\pm^\infty, \end{aligned}$$

and the half-line in the leaf below the one containing the half-line $\mathfrak{R}_\pm$ is $\mathfrak{R}_\mp$ as well. The half-line $\mathfrak{R}_\pm$ intersecting the principal hole B_ε will be noted $\widehat{\mathfrak{R}}_\pm$, and the half-line $\mathfrak{I}^\infty_\pm$ intersecting the principal hole B_ε will be noted $\widehat{\mathfrak{I}}_\pm$.

Definition 8.2.2. *The distinguished line $\widehat{\mathfrak{R}}_\pm$ is called the axis of symmetry in the Z coordinate.*

8.3 Translation domains.

Definition 8.3.1 (The Glutsyuk point of view of the dynamics). *Given any $\delta > 0$, there exists $\rho > 0$ such that for $|\varepsilon| < \rho$, there exists an orbit of the lifting $\mathbf{P}_\varepsilon$ connecting P^0 with $P^\pm$. In such a case, we say that we are in the "Glutsyuk point of view" of the dynamics.*

Proposition 8.3.2. *[32] There exist $K > 0$ and $B > 0$, such that for Z and ε small, one has*

$$\begin{aligned} &|\mathbf{P}_\varepsilon(Z) - Z - 1| < KB \\ &|\mathbf{P}'_\varepsilon(Z) - 1| < KB^2 \end{aligned} \tag{8.3.1}$$

where B depends on the size of the neighborhood of the point $w = 0$.

Consider a slanted line $\ell \subset \mathscr{R}_\varepsilon$ and its image $\mathbf{P}_\varepsilon(\ell)$, such that the image is placed at the right of ℓ and the strip between ℓ and $\mathbf{P}_\varepsilon(\ell)$ belongs to $p_\varepsilon^{\circ-1}(U)$.

Definition 8.3.3. *The region of the Z coordinate between the line and its image is a strip $\widehat{C}_\varepsilon(\ell)$ called admissible strip. The line ℓ giving birth to $\widehat{C}_\varepsilon(\ell)$ is called an admissible line.*

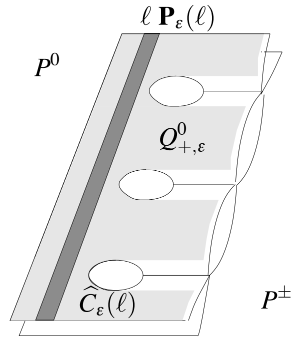


Figure 8.4: A translation domain $Q^0_{+,\varepsilon}$ and an admissible strip on it.

Definition 8.3.4. *Let ℓ be an admissible line for $\mathbf{P}_\varepsilon$. The translation domain associated to ℓ is the set*

$$Q_\varepsilon(\ell) = \{Z \in \widehat{U}_\varepsilon : \exists n \in \mathbb{Z}, \mathbf{P}_\varepsilon^{\circ n}(Z) \in \widehat{C}_\varepsilon(\ell), \forall i \in \{0,1,...,n\}, \mathbf{P}_\varepsilon^{\circ i}(Z) \in \widehat{U}_\varepsilon\}.$$

Among other properties, $Q_\varepsilon(\ell)$ is a simply connected open subset of $\widehat{U}_\varepsilon$; the region $\widehat{C}_\varepsilon(\ell)\backslash\{\ell\}$ is a *fundamental domain* for the restriction of $\mathbf{P}_\varepsilon$ to $Q_\varepsilon(\ell)$: each $\mathbf{P}_\varepsilon$-orbit in $Q_\varepsilon(\ell)$ has one and only one point in this set. In the Glutsyuk point of view the admissible strips are placed parallel to the line of holes, *i.e*. according to the $\alpha(\varepsilon)$ direction of the covering transformation $T_{\alpha(\varepsilon)}$. The induced translation domains, called *Glutsyuk translation domains,* are noted as Q^∞_ε and Q^0_ε according to whether they contain a neighborhood of $P^\pm$ or P^0, respectively, see Figure 8.4.

Definition 8.3.5. *For values of ε in $V^G_{\delta,rl}$, there exists four different Glutsyuk translation domains $Q^{0,\infty}_{\pm,\varepsilon}$ in the Z-space, which are defined, depending on the sign of $\varepsilon \in \mathbb{R}$, as follows.*

1. *If $\varepsilon \geq 0$, then $Q^\infty_{\pm,\varepsilon}$ is a simply connected neighborhood of $P^\pm$ containing all the segments $\mathfrak{R}^s_\pm$, while $Q^0_{\pm,\varepsilon}$ is a simply connected neighborhood of P^0 containing the distinguished half-line $\widehat{\mathfrak{I}}_\pm$, see Figure 8.5.*

2. *If $\varepsilon > 0$, then $Q^{\infty}_{\pm,\varepsilon}$ is a simply connected neighborhood of $P^{\pm}$ containing all the segments $\mathfrak{I}^s_{\pm}$, while $Q^0_{\pm,\varepsilon}$ is a simply connected neighborhood of P^0 containing the distinguished half-line $\widehat{\mathfrak{R}}_{\pm}$.*

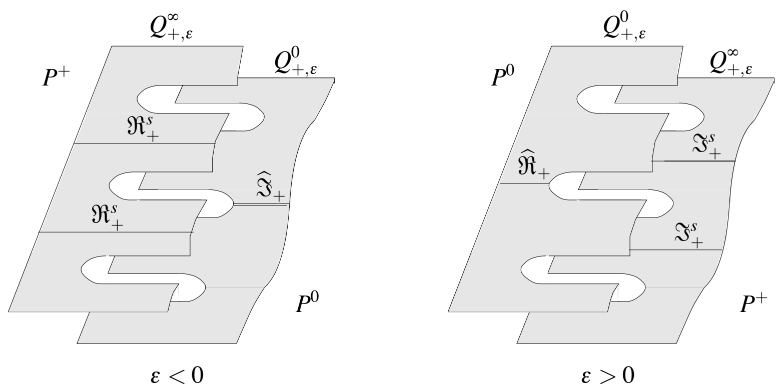


Figure 8.5: The translation domains $Q^{0,\infty}_{+,\varepsilon}$.

Lemma 8.3.6. *The translation T_α satisfies:*

$$\begin{aligned} T_\alpha(Q^0_{\pm,\varepsilon}) &= Q^0_{\mp,\varepsilon}, \\ T_\alpha(Q^{\infty}_{\pm,\varepsilon}) &= Q^{\infty}_{\pm,\varepsilon}, \end{aligned} \tag{8.3.2}$$

see Figure 8.5.

Proof. The second is clear, by definition: T_α is formerly defined in a neighborhood of the point $P^{\pm}$ along the leaves of $\mathscr{R}_\varepsilon$, thus leaving invariant the translation domains $Q^{\infty}_{\pm,\varepsilon}$. On the other hand, the first equality is certainly true because all the possible paths defining the analytic extension of T_α to a neighborhood of P^0 must be contained in $Q^{\infty}_{\pm,\varepsilon}$. Let us consider for instance $Q^0_{+,\varepsilon}$ above the principal hole. It intersects $Q^{\infty}_{+,\varepsilon}$ and because of the definition of T_α, when we apply T_α (resp. $T_{-\alpha}$) we are below the principal hole if $\varepsilon > 0$ (resp. $\varepsilon < 0$). In that region $Q^{\infty}_{+,\varepsilon}$

intersects $Q^0_{-,\varepsilon}$. Thus, each translation domain $Q^\infty_{\pm,\varepsilon}$ shares a common region with a translation domain of the kind $Q^0_{\pm,\varepsilon}$. The conclusion follows. $\square$

8.4 Conjugation in the Z coordinate.

Fix a translation domain Q^G_ε and let $Z \in Q^G_\varepsilon$ be any point on it. Choose any simple arc Γ joining Z with the axis of symmetry $\widehat{\mathfrak{R}}$, and let γ be its image under the map $p_\varepsilon : \gamma = p_\varepsilon(\Gamma)$. Consider the reflection $\overline{\gamma}$ of the path γ with respect the real axis $\mathbb{R}$ in the w coordinate. Then define

$$\overline{\Gamma} := p_\varepsilon^{\circ -1}(\overline{\gamma}).$$

Definition 8.4.1. *The path $\overline{\Gamma}$ is well defined and is called the reflection of the arc Γ with respect to the axis of symmetry $\widehat{\mathfrak{R}}$ in the Z coordinate, see Figure 8.6. The starting point of $\overline{\Gamma}$ is called the conjugate of Z, and is noted $\mathtt{C}(Z)$.*

Since the translation domains are connected, the conjugation $Z \mapsto \mathtt{C}(Z)$ is well defined: its definition is independent of the arc Γ. Indeed, if Γ_+ is any simple path joining Z with the semi-axis of symmetry $\widehat{\mathfrak{R}}_+$ in the Z coordinate, then the reflection of the arc Γ_+ with respect to $\widehat{\mathfrak{R}}_+$ induces a map

$$Z \mapsto \mathtt{C}_+(Z)$$

along the leaves of $\mathfrak{R}$, which is independent of the free homotopy class with end-point on $\widehat{\mathfrak{R}}_+$. Choose now any simple arc Γ_- joining the point Z with the semi-axis of symmetry $\widehat{\mathfrak{R}}_-$. The reflection of the arc Γ_- with respect to $\widehat{\mathfrak{R}}_-$ induces in turn a map

$$Z \mapsto \mathtt{C}_-(Z).$$

Then, it is easily seen that $\mathtt{C}_+(Z) = \mathtt{C}_-(Z)$. Indeed, the arc Γ_+ induces a path γ_+ in the w coordinate whose reflection $\overline{\gamma_+}$ with respect the real axis starts at the same starting point of the reflection $\overline{\gamma_-}$ of the path γ_- induced by the arc Γ_- in the w coordinate, see Figure 8.6.

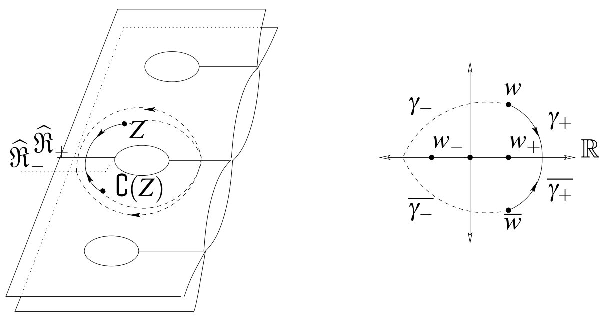


Figure 8.6: The conjugation in the Z coordinate.

Remark 8.4.2. *It becomes clear by definition that:*

$$\mathtt{C} \circ \mathtt{C} = id \tag{8.4.1}$$

for real values of the parameter.

Proposition 8.4.3. *The lifted family* $\mathbf{P}_\varepsilon$ *is invariant under the conjugation in the Z coordinate when* $\varepsilon \in \mathbb{R}$:

$$\mathbf{P}_\varepsilon = \mathtt{C} \circ \mathbf{P}_\varepsilon \circ \mathtt{C}. \tag{8.4.2}$$

Proof. This is because the family $\mathscr{P}_\varepsilon$ leaves invariant the real line when $\varepsilon \in \mathbb{R}$. The image of the latter by p_ε is, by definition, the axis of symmetry $\widehat{\mathfrak{R}}$ in the Z coordinate, which provides an invariant curve for the dynamics of the lifting $\mathbf{P}_\varepsilon$ on the translation domains. Such a curve is invariant only if $\varepsilon \in \mathbb{R}$. □

8.5 Real Fatou Glutsyuk coordinates.

8.5.1 The Beltrami Equation ([38]).

We explain the concept of measurable conformal structure on an open set $A \subset \mathbb{C}$. A conformal structure at a point $z \in A$ can be prescribed by choosing some ellipse centered at the origin in the tangent space $T_z A \simeq \mathbb{C}$. We are to think of this ellipse as a "circle" in the new conformal structure. In more technical language, a *conformal structure* at the point $z \in \mathbb{C}$ is determined by a complex dilatation $\mu(z) \in \mathbb{D}$, where $\mathbb{D}$ is the open unit disk of the complex plane. First, consider the case where μ is constant. Then the function $h(z) = z + \bar{z}\mu$ satisfies the *Beltrami differential equation*

$$\frac{\partial h}{\partial \bar{z}} = \mu(z)\frac{\partial h}{\partial z} \tag{8.5.1}$$

(named after Eugenio Beltrami, 1835-1900). Here the derivatives $\partial/\partial\bar{z}$ and $\partial/\partial z$ are to be defined by the formula

$$\partial/\partial\bar{z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \;\; \partial/\partial z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

where $z = x + iy$.

If h satisfies (8.5.1) with constant $\mu \in \mathbb{D}$, then a round circle $|h| = constant$ in the h-plane corresponds to an ellipse $|z + \bar{z}\mu| = constant$ in the z-plane, with direction of the major axis controlled by the argument of μ and with eccentricity controlled by $|\mu|$. If $|\mu| = r < 1$, then the ratio of the major axis to minor axis is equal to $\frac{1+r}{1-r}$, which tends to infinity when $r \to 1$.

More generally, if the function h is real analytic, then Gauss, in his construction of "isothermal coordinates", showed that an equation equivalent to (8.5.1) always has local solutions. Morrey extended this to the case where μ is measurable, with

$$|\mu(z)| < constant < 1 \tag{8.5.2}$$

almost everywhere, constructing a solution $z \mapsto h(z)$ which maps a region in the z plane homeomorphically onto a region in the h plane. Furthermore, if h_1, h_2 are two distinct solutions, he showed that composition $h_2 \circ h_1^{-1}$ is holomorphic.

Here some explanation is needed, since we are considering a differential equation involving nondifferentiable functions. For any open set $A \subset \mathbb{C}$ let $L^1(A)$ be the vector space consisting of all measurable functions $\phi : A \to \mathbb{C}$ with

$$\int\int_A |\phi(x+iy)| dxdy < \infty$$

(where we identify two functions which agree almost everywhere). Consider also the vector space of *test functions* on A, consisting of all C^∞ functions $\tau : A \to \mathbb{C}$ which vanish outside of some compact subset of A.

Definition 8.5.1. *A continuous function $h : A \to \mathbb{C}$ has "distributional derivatives" in L^1 if there are complex valued functions h_z and $h_{\bar{z}}$ defined almost everywhere in A and belonging to $L^1(A)$ such that*

$$\begin{aligned} &\int\int_A (h_z(z)\tau(z) + h(z)\partial\tau/\partial z)\, dxdy = 0 \\ &\int\int_A (h_{\bar{z}}(z)\tau(z) + h(z)\partial\tau/\partial\bar{z})\, dxdy = 0 \end{aligned} \tag{8.5.3}$$

for every such test function τ. (Note that we can change h_z and $h_{\bar{z}}$ on a set of Lebesgue measure zero without affecting (8.5.3)).

The Beltrami equation for h now requires that

$$h_{\bar{z}}(z) = \mu(z) h_z(z)$$

for almost every $z \in A$. This makes sense, since the pointwise product of an L^1 function and a bounded measurable function is again in L^1. By definition, any continuous one-to-one solution h is called *quasiconformal mapping* on A, with complex dilatation μ.

More generally, we can consider such a measurable conformal structure on a Riemann surface $\mathscr{R}$. However, it is no longer described by a complex-valued function, but rather by a section of a real analytic $\mathbb{D}$-bundle which is canonically associated with $\mathscr{R}$. Given a local coordinate z on an open set A, we can still describe the conformal structure on A by a dilatation function $\mu : A \to \mathbb{D}$, but on the overlap between two local coordinates z and z' a brief computation shows that the equation

$$\mu'(z') = \mu(z)\frac{\partial z'}{\partial z} \Big/ \frac{\partial \overline{z'}}{\partial \overline{z}}$$

must be satisfied in order to make sense of this structure globally.[1] Note that $|\mu'(z')| = |\mu(z)|$, so that condition (8.5.2) is independent of the choice of coordinate system. If this conformal structure is measurable and satisfies (8.5.2) everywhere, then the local solutions h form the atlas of local conformal coordinates for a new Riemann surface $\mathscr{R}^\mu$ which is topologically identical to $\mathscr{R}$, but conformally (and even differentiably) quite different. In the special case where $\mathscr{R}$ is the Riemann sphere, it follows from the Uniformization Theorem ([19],[38]) that $\mathscr{R}^\mu$ is conformally equivalent to the Riemann sphere. In particular, there is a unique conformal isomorphism $h : \mathscr{R} \to \mathscr{R}^\mu$ which fixes the points $0, 1$ and ∞.

1. In more geometric language, a *Beltrami differential* at a point x of a Riemann surface can be described as an additive homomorphism from the tangent space T_x to itself which is antilinear, $\mu_x(\lambda t) = \overline{\lambda}\mu_x(t)$, and which multiplies the length of any vector $t \in T_x$ by a constant $|\mu_x| < 1$. In particular, if $\mathscr{R}$ is an open subset of $\mathbb{C}$ so that $T_x \simeq \mathbb{C}$, then μ_x will have the form $\mu_x(t) = \mu\bar{t}$ with $|\mu_x| = |\mu| < 1$.

If we remember that $\mathscr{R}^\mu$ is identical to $\mathscr{R} = \mathbb{CP}^1$ as a topological space, then we can also describe $h = h_\mu$ as a quasiconformal homeomorphism from $\mathbb{CP}^1$ to itself (or briefly a *qc-homeomorphism*) with complex dilatation $\mu(z)$.

We can also study the dependence of h_μ on the dilatation μ. For each fixed z_0, Ahlfors and Bers ([1], 1960) showed that the correspondence $\mu \mapsto h_\mu(z_0)$ defines a differentiable function from the appropriate space of dilatation functions to the Rieman sphere.

8.5.2 Construction of Fatou coordinates.

Theorem 8.5.2. *For values of the parameter in $V^G_{\delta,lr}$ it is possible to construct four different changes of coordinates $W = \Phi^{0,\infty}_{\pm,\varepsilon,lr}(Z)$ defined on $\mathscr{R}_\varepsilon$, and conjugating $\mathbf{P}_\varepsilon$ with the translation by one:*

$$\Phi^{0,\infty}_{\pm,\varepsilon,lr}(\mathbf{P}_\varepsilon(Z)) = \Phi^{0,\infty}_{\pm,\varepsilon,lr}(Z) + 1, \tag{8.5.4}$$

for every $Z \in Q^{0,\infty}_{\pm,\varepsilon,lr} \cap \mathbf{P}^{\circ-1}_\varepsilon(Q^{0,\infty}_{\pm,\varepsilon,lr})$. These change of coordinates (see Figure 8.9) are associated with translation domains $Q^{0,\infty}_{\pm,\varepsilon,lr}$ whose admissible strips in $\mathscr{R}_\varepsilon$ lie in a direction parallel to the line of the holes $T^{\circ k}_{\alpha(\varepsilon)}B_\varepsilon$. Moreover, if we let $W \mapsto \mathscr{C}(W) := \overline{W}$ be the complex conjugation in the W coordinate then:

– *For (real) negative values of the parameter these maps satisfy:*

$$\begin{aligned} \Phi^0_{\pm,\varepsilon,l} &= \mathscr{C} \circ \Phi^0_{\mp,\varepsilon,l} \circ \mathtt{C}, \\ \Phi^\infty_{\pm,\varepsilon,l} &= \mathscr{C} \circ \Phi^\infty_{\pm,\varepsilon,l} \circ \mathtt{C}. \end{aligned} \tag{8.5.5}$$

– *For (real) positive values of ε they are related through:*

$$\begin{aligned} \Phi^0_{\pm,\varepsilon,r} &= \mathscr{C} \circ \Phi^0_{\pm,\varepsilon,r} \circ \mathtt{C}, \\ \Phi^\infty_{\pm,\varepsilon,r} &= \mathscr{C} \circ \Phi^\infty_{\mp,\varepsilon,r} \circ \mathtt{C}. \end{aligned} \tag{8.5.6}$$

Proof. The construction of the coordinates exists in the literature ([10]) but we wish to show additionally (8.5.5) and (8.5.6). So we will describe the construction when the parameter is real. Let $Q_\varepsilon(\ell)$ be a translation domain generated by an admissible line ℓ on the left side of the holes (real parameter). Thus, ℓ and the axis of symmetry $\widehat{\mathfrak{R}}$ are perpendicular. This distinguished line $\widehat{\mathfrak{R}}$ separates the translation domains $Q_\varepsilon(\ell)$ in two connected symmetric regions which are noted Q^+ (the one above $\widehat{\mathfrak{R}}$) and Q^- (the one below $\widehat{\mathfrak{R}}$), see Figure 8.7.

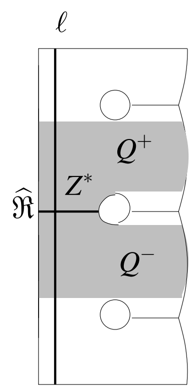


Figure 8.7: The distinguished curve $\widehat{\mathfrak{R}}$ separates the translation domain.

By Proposition 8.4.3:

$$\mathbf{P}_\varepsilon(\widehat{\mathfrak{R}}) \subset \widehat{\mathfrak{R}}. \tag{8.5.7}$$

Let us write $Z^* = \ell \cap \widehat{\mathfrak{R}}$. Notice that points of ℓ can be written as $Z^* + iy$ for $y \in \mathbb{R}$. Put $C_0 := \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$ and define $f_\varepsilon : C_0 \to \widehat{C}(\ell)$ as the convex combination

$$f_\varepsilon(x+iy) = (1-x)(Z^*+iy) + x\mathbf{P}_\varepsilon(Z^*+iy).$$

It is shown ([10]) that for $z = x + iy$,

$$\left| \frac{\partial f_\varepsilon}{\partial \bar{z}} \Big/ \frac{\partial f_\varepsilon}{\partial z} \right| < 1,$$

so f_ε is a quasi-conformal map onto the strip $\widehat{C}(\ell)$ and it satisfies $f_\varepsilon^{-1}(\mathbf{P}_\varepsilon(Z)) = f_\varepsilon^{-1}(Z) + 1$ for every $Z \in \ell$. If we identify $\widehat{\mathfrak{R}}_\pm$ with $\mathbb{R}_\pm$, then f_ε sends the interval $[0,1]$ into a real interval $[Z^*, \mathbf{P}_\varepsilon(Z^*)]$ and then the function defined as

$$\mu := f_\varepsilon^* \widehat{\mu}_0$$

(the pullback of the standard conformal structure $\widehat{\mu}_0$ of $\mathbb{C}$ on the strip C_0, defined by the 0 function) is a real measurable function which verifies (due to (8.5.7)):

$$\mu(z) = \overline{\mu(\bar{z})}$$

because of the symmetry of its domain (Schwarz reflection principle). The field μ is defined on C_0 and it is extended to all of $\mathbb{C}$ by $\mu = (\mathscr{T}_1^{\circ n})^* \mu$ on $\{z = x + iy : -n \leq x \leq -n+1\}$, so the extended μ has norm $||\mu||_{L^\infty(\mathbb{C})} < 1$ and it is periodic of period 1. Thus it is a Beltrami field on $\mathbb{C}$ still verifying $\mu(z) = \overline{\mu(\bar{z})}$ for all $z \in \mathbb{C}$. The Ahlfors-Bers Theorem ([1]) yields the existence of a unique quasi-conformal map $g^\mu : \mathbb{C} \to \mathbb{C}$ normalized to $g^\mu(0) = 0$, with complex dilatation μ, *i.e.* satisfying the Beltrami equation

$$g^\mu_{\bar{z}} / g^\mu_z = \mu,$$

that leaves $0, 1, \infty$ fixed and such that $\mu = (g^\mu)^* \widehat{\mu}_0$. In addition, g^μ commutes with the translation T_1 ([10]). Indeed, the homeomorphism $G := g^\mu \circ T_1 \circ g^{\mu \circ -1}$ induces the identity on the sphere $\mathbb{S}^2$ and must thus, be a power of the deck transformation T_1 of the universal covering map $\mathscr{E}(\cdot) = e^{-2i\pi(\cdot)}$, namely: $G = T_1^{\circ m}$ for some

$m \in \mathbb{Z}$. But $G(0) = g^\mu \circ T_1(0) = g^\mu(1) = 1$, which implies $m = 1$ and then $G = T_1$. Since

$$\mathscr{C} \circ \frac{(g^\mu)_{\bar{z}}}{(g^\mu)_z} \circ \mathscr{C} = \mathscr{C} \circ \mu \circ \mathscr{C} = \mu$$

and as $\mathscr{C} \circ (g^\mu)_z \circ \mathscr{C} = (\mathscr{C} \circ g^\mu \circ \mathscr{C})_z$ and $\mathscr{C} \circ (g^\mu)_{\bar{z}} \circ \mathscr{C} = (\mathscr{C} \circ g^\mu \circ \mathscr{C})_{\bar{z}}$, the map $\mathscr{C} \circ\ g^\mu \circ \mathscr{C}$ is another solution to the Beltrami equation, leaving the same points $0, 1, \infty$ fixed. By unicity of the solution, $\overline{g^\mu(\bar{z})} = g^\mu(z)$ for all $z \in \mathbb{C}$. We define then $\phi : \widehat{C}(\ell) \to \mathbb{C}$ by

$$\phi = g^\mu \circ f_\varepsilon^{\circ -1}.$$

If $Z \in \ell$ one has $T_1 \circ \phi(Z) = \phi \circ \mathbf{P}_\varepsilon(Z)$ (because both g^μ and f_ε commute with T_1) whence follows that ϕ can be extended in a map $\Phi_\varepsilon : Q \to \mathbb{C}$ by

$$\Phi_\varepsilon(Z) = \phi \circ \mathbf{P}_\varepsilon^{\circ n(Z)} - n(Z) \tag{8.5.8}$$

where $n(Z) \in \mathbb{Z}$ is such that $\mathbf{P}_\varepsilon^{\circ n(Z)}(Z) \in \widehat{C}(\ell)$. The map Φ_ε is a holomorphic diffeomorphism which depends analytically on the parameter and which verifies $\Phi_\varepsilon \circ \mathbf{P}_\varepsilon = T_1 \circ \Phi_\varepsilon$. Since $\phi(\widehat{\mathfrak{R}}) \subset \mathbb{R}$, we get

$$\Phi_\varepsilon(\widehat{\mathfrak{R}}) \subset \mathbb{R} \tag{8.5.9}$$

when the parameter is real. In addition, $Z \in dom(\Phi_\varepsilon)$ yields $\mathsf{C}(Z) \in dom(\Phi_\varepsilon)$, by definition of C. Notice that in the case $\varepsilon < 0$ we have $\widehat{\mathfrak{R}}_\pm \subset Q^\infty_{\pm,\varepsilon}$, while if $\varepsilon > 0$, $\widehat{\mathfrak{R}}_\pm \subset Q^0_{\pm,\varepsilon}$ (*i.e.* translations domains "on the right" do not contain the symmetry axis $\widehat{\mathfrak{R}}$, see Figure 8.5). Accordingly, if $\varepsilon < 0$ the diffeomorphism (8.5.8) is noted $\Phi^\infty_{\pm,\varepsilon} : Q^\infty_{\pm,\varepsilon} \to \mathbb{C}$ and (8.5.9) yields

$$\Phi^\infty_{\pm,\varepsilon,l} = \mathscr{C} \circ \Phi^\infty_{\pm,\varepsilon,l} \circ \mathsf{C}.$$

On the other hand, if $\varepsilon > 0$ the diffeomorphism (8.5.8) is noted $\Phi^0_{\pm,\varepsilon} : Q^0_{\pm,\varepsilon} \to \mathbb{C}$ and the invariance (8.5.9) implies

$$\Phi^0_{\pm,\varepsilon,r} = \mathscr{C} \circ \Phi^0_{\pm,\varepsilon,r} \circ \mathtt{C}.$$

For the case of a translation domain on the right, we first construct $\Phi^0_{+,\varepsilon,l}$ (when $\varepsilon < 0$) or $\Phi^\infty_{+,\varepsilon,r}$ (when $\varepsilon > 0$), and note that $\mathscr{C} \circ \Phi^0_{+,\varepsilon,l} \circ \mathtt{C}$ (resp. $\mathscr{C} \circ \Phi^\infty_{+,\varepsilon,r} \circ \mathtt{C}$) is again a Fatou coordinate when $\varepsilon < 0$ (resp. when $\varepsilon > 0$). Then we define

$$\Phi^0_{-,\varepsilon,l} = \mathscr{C} \circ \Phi^0_{+,\varepsilon,l} \circ \mathtt{C}$$

for $\varepsilon < 0$, and

$$\Phi^\infty_{-,\varepsilon,l} = \mathscr{C} \circ \Phi^\infty_{+,\varepsilon,r} \circ \mathtt{C}$$

if $\varepsilon > 0$ and the construction is done. □

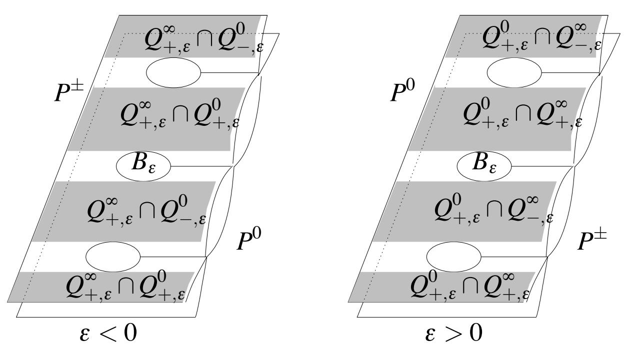


Figure 8.8: The non-connected intersection of the translation domains.

Definition 8.5.3. *Changes of coordinates in Theorem 8.5.2 are called admissible real Fatou Glutsyuk coordinates. Theorem 8.5.2 shows that the symmetry axis $\widehat{\mathfrak{R}}$ is invariant under real Fatou coordinates when the parameter is real.*

Remark.

1. Although real Fatou Glutsyuk changes of coordinates always exist for $\varepsilon \in V^G_{\delta,lr}$, the curve $\widehat{\Re}$ is not invariant if $\varepsilon \notin \mathbb{R}$.
2. The subscripts l, r will be dropped when the context allows no confusion.
3. If $\varepsilon \neq 0$, the geometry of $\mathscr{R}_\varepsilon$ yields a countable alternating sequence composed of connected intersections of translation domains, over which the real Fatou Glutsyuk coordinates are defined. The order of the sequence depends on the sign of the parameter, see Figure 8.8.

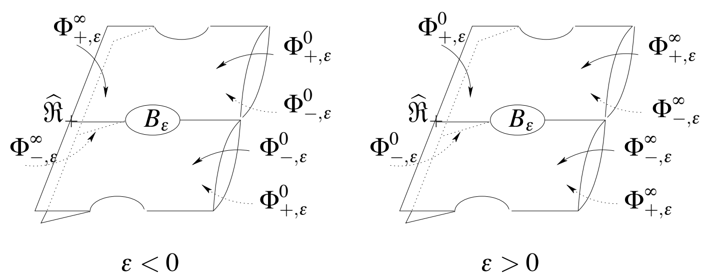


Figure 8.9: The real Glutsyuk coordinates around the principal hole.

Definition 8.3.5 and the remark above yield the organization of the domains of definition for the different real Glutsyuk coordinates. Due to periodicty, it suffices to describe only these domains around the fundamental hole B_ε, see Figure 8.9.

Proposition 8.5.4. *If* Φ^1_ε *and* Φ^2_ε *are two Fatou Glutsyuk coordinates solving* (8.5.4) *on the same translation domain, then there exists* $C_\varepsilon \in \mathbb{C}$, *such that*

$$\Phi^2_\varepsilon(Z) = C_\varepsilon + \Phi^1_\varepsilon(Z).$$

In particular, for every $Z_0(\varepsilon) \in \mathscr{R}_\varepsilon$ there is a unique Fatou coordinate Φ_ε satisfying $\Phi_\varepsilon(Z_0(\varepsilon)) = 0$.

Proof. Since $\Phi_\varepsilon^1, \Phi_\varepsilon^2$ satisfy (8.5.4) they are related by $\Phi_\varepsilon^2 \circ (\Phi_\varepsilon^1)^{\circ -1}(Z+1) = \Phi_\varepsilon^2 \circ (\Phi_\varepsilon^1)^{\circ -1}(Z) + 1$, whence the composition $\Phi_\varepsilon^2 \circ (\Phi_\varepsilon^1)^{\circ -1}$ is a translation T_{C_ε}. □

Definition 8.5.5. *A point $Z_0(\varepsilon) \in \mathscr{R}_\varepsilon$ such that $\Phi_\varepsilon(Z_0(\varepsilon)) = 0$, as in Proposition 8.5.4, is called the base point of the Fatou Glutsyuk coordinate Φ_ε.*

The choice of the base point provides a degree of freedom in the choice of the Fatou Glutsyuk coordinate. Since there are four Fatou Glutsyuk coordinates we have four degrees of freedom. Later, we shall use 3 of these degrees of freedom to "normalize" the Fatou Glutsyuk coordinates.

Remark. The family $\mathscr{P}_\varepsilon$ is, by definition, the square of a family of germs of diffeomorphisms $\mathscr{Q}_\varepsilon$ unfolding the map $\mathscr{Q}_0$, which is tangent to $-id$. This implies that the orbits of the family $\mathscr{Q}_\varepsilon$ form a 180°-degrees alternating sequence along the orbits of the prepared family of fields at each iteration (*i.e.* the points w and $\mathscr{Q}_\varepsilon(w)$ stand on opposite sides of the origin, see Figure 8.10). In other words, the lifting $\mathbf{Q}_\varepsilon := p_\varepsilon^{-1} \circ \mathscr{Q}_\varepsilon \circ p_\varepsilon$ induced on the surface $\mathscr{R}_\varepsilon$ exchanges the two leaves.

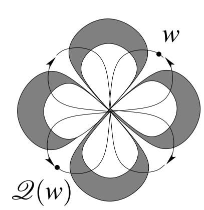


Figure 8.10: The "jumps" of the orbits of $\mathscr{Q}$ in the case $\varepsilon = 0$.

The fact that the family of diffeomorphisms $\mathscr{P}_\varepsilon$ is a square (namely, $\mathscr{P}_\varepsilon = \mathscr{Q}_\varepsilon^{\circ 2}$) is now exploited.

Definition 8.5.6. *For every $W = \Phi_\varepsilon(Z)$, the map:*

$$\mathscr{T}_W(\cdot) = W + \cdot \tag{8.5.10}$$

is called the translation in $W \in \mathbb{C}$.

Lemma 8.5.7. *For every $\varepsilon \in V_\delta^G$, it is possible to construct admissible real Fatou Glutsyuk coordinates $\Phi^0_{\pm,\varepsilon}, \Phi^\infty_{\pm,\varepsilon}$ depending analytically on $\varepsilon \in V_{\delta,lr}$ with continuous limit at $\varepsilon = 0$, and such that they are related through:*

$$\begin{aligned} \Phi^0_{\pm,\varepsilon} \circ \mathbf{Q}_\varepsilon &= \mathscr{T}_{\frac{1}{2}} \circ \Phi^0_{\mp,\varepsilon}, \\ \Phi^\infty_{\pm,\varepsilon} \circ \mathbf{Q}_\varepsilon &= \mathscr{T}_{\frac{1}{2}} \circ \Phi^\infty_{\mp,\varepsilon}. \end{aligned} \tag{8.5.11}$$

Proof. For each ε, the map $\mathscr{Q}_\varepsilon$ commutes with $\mathscr{P}_\varepsilon$. Hence $\mathbf{Q}_\varepsilon = p_\varepsilon^{-1} \circ \mathscr{Q}_\varepsilon \circ p_\varepsilon$ commutes with $\mathbf{P}_\varepsilon$. Let the pairs of Real Fatou Glutsyuk coordinates $\Phi^{0,\infty}_{+,\varepsilon}, \Phi^{0,\infty}_{-,\varepsilon}$ be constructed as in the proof of Theorem 8.5.2. Then:

$$\Phi^{0,\infty}_{\pm,\varepsilon}(\mathbf{P}_\varepsilon(\mathbf{Q}_\varepsilon(Z))) = \Phi^{0,\infty}_{\pm,\varepsilon}(\mathbf{Q}_\varepsilon(Z)) + 1 = (\Phi^{0,\infty}_{\pm,\varepsilon} \circ \mathbf{Q}_\varepsilon)(\mathbf{P}_\varepsilon(Z)), \tag{8.5.12}$$

the first equality being consequence of the fact that $\Phi^{0,\infty}_{\pm,\varepsilon}$ is a solution to (8.5.4), and the second is true because $\mathbf{P}_\varepsilon$ and $\mathbf{Q}_\varepsilon$ commute. Equation (8.5.12) says that $\Phi^{0,\infty}_{\pm,\varepsilon} \circ \mathbf{Q}_\varepsilon$ is a Fatou Glutsyuk coordinate. By the remark above, the latter is defined on the same translation domain as $\Phi^{0,\infty}_{\mp,\varepsilon}$. Hence, according to Proposition 8.5.4, there exists $C^{0,\infty}_{\pm,\varepsilon} \in \mathbb{C}$ with the following property:

$$\Phi^{0,\infty}_{\pm,\varepsilon} \circ \mathbf{Q}_\varepsilon = \mathscr{T}_{C^{0,\infty}_{\pm,\varepsilon}} \circ \Phi^{0,\infty}_{\mp,\varepsilon}. \tag{8.5.13}$$

We will drop the subscript ε in the constants. Using $\mathbf{Q}_\varepsilon^{\circ 2} = \mathbf{P}_\varepsilon$ and iterating (8.5.13) yields:

$$\begin{aligned} \Phi^{0,\infty}_{\pm,\varepsilon}(Z)+1 &\equiv \Phi^{0,\infty}_{\pm,\varepsilon}\circ \mathbf{P}_\varepsilon(Z) \\ &= (\Phi^{0,\infty}_{\pm,\varepsilon}\circ \mathbf{Q}_\varepsilon)\circ \mathbf{Q}_\varepsilon(Z) \\ &= \mathscr{T}_{C^{0,\infty}_{\pm}}\circ(\Phi^{0,\infty}_{\mp,\varepsilon}\circ \mathbf{Q}_\varepsilon)(Z) \\ &= \mathscr{T}_{C^{0,\infty}_{\pm}}\circ \mathscr{T}_{C^{0,\infty}_{\mp}}\circ \Phi^{0,\infty}_{\pm,\varepsilon}(Z) \\ &= \Phi^{0,\infty}_{\pm,\varepsilon}(Z)+C^{0,\infty}_{\pm}+C^{0,\infty}_{\mp}, \end{aligned} \tag{8.5.14}$$

which means

$$C^{0,\infty}_{+}+C^{0,\infty}_{-}=1. \tag{8.5.15}$$

We want to prove that it is possible to choose the Fatou coordinates so that $C^{0,\infty}_{+} = C^{0,\infty}_{-} = 1/2$. That is consequence of $\mathbf{Q}_\varepsilon = \complement\circ\mathbf{Q}_\varepsilon\circ\complement$ when $\varepsilon\in\mathbb{R}$. Indeed, in the case $\varepsilon<0$, Equation (8.5.13) and Theorem 8.5.2 yield

$$\begin{aligned} \mathscr{T}_{C^0_{\pm}}\circ\Phi^0_{\mp} &= (\mathscr{C}\circ\Phi^0_{\mp}\circ\complement)\circ\mathbf{Q}_\varepsilon \\ &= \mathscr{C}\circ(\Phi^0_{\mp}\circ\mathbf{Q}_\varepsilon)\circ\complement \\ &= \mathscr{C}\circ(\mathscr{T}_{C^0_{\mp}}\circ\Phi^0_{\pm})\circ\complement \\ &= \mathscr{C}\circ\mathscr{T}_{C^0_{\mp}}\circ\mathscr{C}\circ\Phi^0_{\mp}. \end{aligned} \tag{8.5.16}$$

Hence $\overline{C^0_{\mp}} = C^0_{\pm}$, and then $Re(C^0_+) = Re(C^0_-)$. We show now that a "correction" is possible, so that $C^0_{\pm}$ can be taken real (for every ε), without loss of generality. If we change Fatou coordinates by

$$\begin{aligned} \Phi^0_{+,\varepsilon} &\mapsto \mathscr{T}_K\circ\Phi^0_{+,\varepsilon} \\ \Phi^0_{-,\varepsilon} &\mapsto \mathscr{T}_{\overline{K}}\circ\Phi^0_{-,\varepsilon} \end{aligned}$$

in (8.5.13), for $K\in i\mathbb{R}$ to be chosen, then (8.5.5) remains valid and we get the equations:

$$\begin{aligned} (\mathscr{T}_K\circ\Phi^0_{+,\varepsilon})\circ\mathbf{Q}_\varepsilon &= \mathscr{T}_{K+C^0_+-\overline{K}}\circ(\mathscr{T}_{\overline{K}}\circ\Phi^0_{-,\varepsilon}) \\ (\mathscr{T}_{\overline{K}}\circ\Phi^0_{-,\varepsilon})\circ\mathbf{Q}_\varepsilon &= \mathscr{T}_{\overline{K}+C^0_--K}\circ(\mathscr{T}_K\circ\Phi^0_{+,\varepsilon}). \end{aligned}$$

Put $\widehat{C}_+^0 = K + C_+^0 - \overline{K}$ and $\widehat{C}_-^0 = \overline{K} + C_-^0 - K$. The choice

$$K = -i\frac{Im(C_+^0)}{2} = i\frac{Im(C_-^0)}{2} \in i\mathbb{R}$$

ensures that $\widehat{C}_+^0 = \widehat{C}_-^0 = Re(C_+^0) = Re(C_-^0) = 1/2$. Thus, we can always suppose that $C_+^0 = C_-^0 = 1/2$.

As for the coordinate $\Phi_\pm^\infty$, the proof is straightforward. Indeed, (8.5.13) and Theorem 8.5.2 yield this time:

$$\begin{aligned}
\mathscr{T}_{C_\pm^\infty} \circ \Phi_\mp^\infty &= (\mathscr{C} \circ \Phi_\pm^\infty \circ \complement) \circ \mathbf{Q}_\varepsilon \\
&= \mathscr{C} \circ (\Phi_\pm^\infty \circ \mathbf{Q}_\varepsilon) \circ \complement \\
&= \mathscr{C} \circ (\mathscr{T}_{C_\pm^\infty} \circ \Phi_\mp^\infty) \circ \complement \\
&= \mathscr{C} \circ \mathscr{T}_{C_\pm^\infty} \circ \mathscr{C} \circ \Phi_\mp^\infty
\end{aligned}$$

(compare to (8.5.16)), thus $C_\pm^\infty = \overline{C_\pm^\infty}$ and $C_\pm^\infty \in \mathbb{R}$. So we can perform a change $\Phi_{\pm,\varepsilon}^\infty \mapsto \mathscr{T}_{K_\pm} \circ \Phi_{\pm,\varepsilon}^\infty$, where $K_\pm = -\dfrac{C_\pm^\infty}{2} \in \mathbb{R}$, in order to bring $C_+^\infty = C_-^\infty = 1/2$, respecting (8.5.5).

The case $\varepsilon > 0$ is completely analogous, using (8.5.6). □

Definition 8.5.8. *When real Fatou Glutsyuk coordinates satisfy* (8.5.5)*,* (8.5.6) *and* (8.5.11)*, we shall say that the Fatou Glutsyuk coordinates are normalized.*

8.5.3 Translations.

Consider the numbers:

$$\begin{aligned}
\alpha_0(\varepsilon) &= \frac{2\pi i}{\mu_0(\varepsilon)} = \frac{i}{\varepsilon} \\
\alpha_\infty(\varepsilon) &= \frac{2\pi i}{\mu_\pm(\varepsilon)} = -\frac{i(1 - sA(\varepsilon)\,\varepsilon)}{2\,\varepsilon},
\end{aligned} \tag{8.5.17}$$

where $\mu_0(\varepsilon) = \log \mathscr{P}'_\varepsilon(0) = 2\pi\,\varepsilon$, and $\mu_\pm(\varepsilon) = \log \mathscr{P}'_\varepsilon(w_\pm) = \dfrac{-4\pi\,\varepsilon}{1-sA(\varepsilon)\,\varepsilon}$ are the eigenvalues of the singular points $w_0 = 0$ and $w_\pm = \pm\sqrt{-s\,\varepsilon}$, respectively.

Remark. The coefficient $s = \pm 1$ defines two different cases which are not equivalent by real conjugacy. We will only describe the " $+$ " case:

$$\mathscr{P}_\varepsilon(w) = w + w(\varepsilon + w^2)[1 + D(\varepsilon) + E(\varepsilon)w^2 + w(\varepsilon + w^2)h(\varepsilon, w)]. \qquad (8.5.18)$$

In the " $-$ " case, each picture in the Figure (7.2) must be rotated in 90° degrees in the clockwise direction and, moreover, the family $\mathscr{P}_\varepsilon^{-1}$ is of the form (8.5.18).

Definition 8.5.9. *The Glutsyuk normalization domains are*

$$U_\varepsilon^{0,\infty} := p_\varepsilon(Q_{\pm,\varepsilon}^{0,\infty}).$$

Lemma 8.5.10. *The quotients $U_\varepsilon^0/\mathscr{P}_\varepsilon$ and $U_\varepsilon^\infty/\mathscr{P}_\varepsilon$ are conformally equivalent to non-separated spaces*

$$\mathbf{T}_\varepsilon^0 \cup \{w_0\}, \quad \mathbf{T}_{\pm,\varepsilon}^\infty \cup \{w_\pm\}$$

which are the union of a point with complex tori $\mathbf{T}_\varepsilon^0$ and $\mathbf{T}_{\pm,\varepsilon}^\infty$, of modulus $\alpha_0(\varepsilon)$ and $\alpha_\infty(\varepsilon)$, respectively.

Proof. Indeed take, for instance, the fixed point $w_+ = \sqrt{-\,\varepsilon}$. Since we are in the Glutsyuk point of view of the dynamics, on U_ε^∞ the map $\mathscr{P}_\varepsilon$ admits w_+ as a global hyperbolic point. Consider any loop γ around w_+ and consider its image $\mathscr{P}_\varepsilon(\gamma)$ as well. The Jordan region J of the complex plane between these two curves is a fundamental domain (*i.e.* a domain where each point represents one and only one orbit of the Poincaré map $\mathscr{P}_\varepsilon$) for the dynamics around w_+. It is easily seen that

$$U_\varepsilon^\infty/\mathscr{P}_\varepsilon \simeq J/\mathscr{P}_\varepsilon \cup \{w_+\},$$

(they are conformally equivalent). Moreover, we can change J by any iterate $\mathscr{P}_\varepsilon^{\circ n}(J)$ in the quotient, and the resulting space remains the same. By the Poincaré Theorem, the map $\mathscr{P}_\varepsilon$ is linearizable around w_+. As $n \to \infty$, the modulus of the quotient complex torus $\mathscr{P}_\varepsilon^{\circ n}(J)/\mathscr{P}_\varepsilon$ converges towards the modulus of the torus $\mathbb{C}^*/\mathscr{L}_{\mu_+(\varepsilon)}$, which is given by $\alpha_\infty = \dfrac{2i\pi}{\mu_+(\varepsilon)}$. Inasmuch as the space $\mathscr{P}_\varepsilon^{\circ n}(J)/\mathscr{P}_\varepsilon \cup \{w_+\}$ is conformally equivalent to $U_\varepsilon^\infty/\mathscr{P}_\varepsilon$, the latter is the union of a complex torus $\mathbf{T}_{+,\varepsilon}^\infty$ of modulus α_∞, and the singular point $\{w_+\}$. This space is non-separated because the point $\{w_+\}$ belongs to the adherence of any orbit of $\mathscr{P}_\varepsilon$. □

Proposition 8.5.11. *For all $\varepsilon \in V_{\delta,rl}^G$, it is possible to choose nomalized real Fatou Glutsyuk coordinates $\Phi_{\pm,\varepsilon}^{0,\infty} : Q_{\pm,\varepsilon}^{0,\infty} \to \mathbb{C}$ satisfying (in addition to* (8.5.5), (8.5.6) *and* (8.5.11)*) the equations:*

$$\begin{aligned} \Phi_{\pm,\varepsilon}^0 \circ T_\alpha &= \mathscr{T}_{-\frac{\alpha_0}{2}} \circ \Phi_{\mp,\varepsilon}^0, \\ \Phi_{\pm,\varepsilon}^\infty \circ T_\alpha &= \mathscr{T}_{\alpha_\infty} \circ \Phi_{\pm,\varepsilon}^\infty. \end{aligned} \tag{8.5.19}$$

In particular, they have the same limit at $\varepsilon = 0$.

Proof. Consider the translation T_α and a real Fatou Glutsyuk coordinate $\Phi_{\pm,\varepsilon}^\infty : Q_{\pm,\varepsilon}^\infty \to \mathbb{C}$. By (8.1.7):

$$\begin{aligned} \Phi_{\pm,\varepsilon}^\infty \circ T_\alpha \circ \mathbf{P}_\varepsilon &= \Phi_{\pm,\varepsilon}^\infty \circ \mathbf{P}_\varepsilon \circ T_\alpha \\ &= \mathscr{T}_1 \circ \Phi_{\pm,\varepsilon}^\infty \circ T_\alpha \end{aligned}$$

whence follows that $\Phi_{\pm,\varepsilon}^\infty \circ T_\alpha$ is a Fatou coordinate. By Lemma 8.3.6, the latter preserves the translation domains $Q_{\pm,\varepsilon}^\infty$ and then, $\Phi_{\pm,\varepsilon}^\infty \circ T_\alpha$ and $\Phi_{\pm,\varepsilon}^\infty$ are defined on the same translation domain. By Proposition 8.5.4, there exist constants $C_{\pm,\varepsilon}$ such that:

$$\Phi_{\pm,\varepsilon}^\infty \circ T_\alpha = \mathscr{T}_{C_{\pm,\varepsilon}} \circ \Phi_{\pm,\varepsilon}^\infty. \tag{8.5.20}$$

Thus, the Fatou coordinate conjugates the pair of commuting diffeomorphisms $\{\mathbf{P}_\varepsilon, T_\alpha\}$ with the pair of translations $\{\mathscr{T}_1, \mathscr{T}_{C_{\pm,\varepsilon}}\}$. Moreover, the Fatou Glutsyuk coordinate induces a holomorphic diffeomorphism:

$$Q^\infty_{\pm,\varepsilon}/\{\mathbf{P}_\varepsilon, T_\alpha\} \cong \mathbb{C}/\{\mathscr{T}_1, \mathscr{T}_{C_{\pm,\varepsilon}}\}$$

between complex surfaces. The latter is, of course, the canonical torus $\mathbb{C}/(\mathbb{C}\times C_{\pm,\varepsilon}\mathbb{C})$. Notice that the quotient $Q^\infty_{\pm,\varepsilon}/T_\alpha$ coincides with the neighborhood U^∞_ε with coordinate w, where the map $\mathscr{P}_\varepsilon$ is induced by $\mathbf{P}_\varepsilon$. Hence, the quotient $Q^\infty_{\pm,\varepsilon}/\{\mathbf{P}_\varepsilon, T_\alpha\}$ is conformally equivalent to $U^\infty_\pm/\mathscr{P}_\varepsilon$. On the other hand, the translation T_α has been formerly defined on a neighborhood of the points $P^\pm$, thus the positive orientation of the translation $\mathscr{T}_{\alpha_\infty}$ in the W (Fatou) coordinate coincides with the positive orientation of T_α, by definition. By (8.5.20) and Lemma 8.5.10, the modulus of the torus $\mathbb{C}/\{\mathscr{T}_1, \mathscr{T}_{C_\varepsilon}\}$, *i.e.* the constants $C_{\pm,\varepsilon}$, coincide and must be equal to α_∞ on $Q^\infty_{\pm,\varepsilon}$:

$$\Phi^\infty_{\pm,\varepsilon}\circ T_\alpha = \mathscr{T}_{\alpha_\infty}\circ\Phi^\infty_{\pm,\varepsilon}. \tag{8.5.21}$$

The behavior of the Fatou coordinate $\Phi^0_{\pm,\varepsilon} : Q^0_{\pm,\varepsilon} \to \mathbb{C}$ with respect the translation T_α is more involved. Indeed, by Lemma 8.3.6, T_α sends the translation domains $Q^0_{\pm,\varepsilon}$ into $Q^0_{\mp,\varepsilon}$ and then, reasoning as above, $\Phi^0_{\pm,\varepsilon}\circ T_\alpha$ and $\Phi^0_{\mp,\varepsilon}$ are two Fatou Glutsyuk coordinates defined on the same translation domain. Proposition 8.5.4 shows then that there exist two constants $C^1_\varepsilon, C^2_\varepsilon$ such that:

$$\begin{aligned}\Phi^0_{+,\varepsilon}\circ T_\alpha &= \mathscr{T}_{C^1_\varepsilon}\circ\Phi^0_{-,\varepsilon}\\ \Phi^0_{-,\varepsilon}\circ T_\alpha &= \mathscr{T}_{C^2_\varepsilon}\circ\Phi^0_{+,\varepsilon},\end{aligned} \tag{8.5.22}$$

thus yielding:

$$\Phi^0_{\pm,\varepsilon}\circ T_{2\alpha} = \mathscr{T}_{C^1_\varepsilon+C^2_\varepsilon}\circ\Phi^0_{\pm,\varepsilon}. \tag{8.5.23}$$

The quotients $Q^0_{\pm,\varepsilon}/\{\mathbf{P}_\varepsilon, T_{2\alpha}\}$ are conformally equivalent to $U^0_\varepsilon/\mathscr{P}_\varepsilon$, *i.e.* the union of a complex torus of modulus α_0 with the singular point w_0. Moreover, in the W (Fatou) coordinate, positive orientation of the translation $\mathscr{T}_{\alpha_0}$ corresponds to negative orientation of $\mathscr{T}_{\alpha_\infty}$, as shown in Figure 8.11. Since the positive orientation of the translation $\mathscr{T}_{\alpha_\infty}$ coincides with that of T_α, we get $\Phi^0_{\pm,\varepsilon}\circ T_{2\alpha} = \mathscr{T}_{-\alpha_0}\circ\Phi^0_{\pm,\varepsilon}$, or, in terms of the constants, $C^1_\varepsilon + C^2_\varepsilon = -\alpha_0$. Let us show that $C^1_\varepsilon = C^2_\varepsilon$. Since the Fatou coordinates $\Phi^0_\pm$ are normalized, Lemma 8.5.7 tells us that

$$\Phi^0_\pm\circ\mathbf{Q}_\varepsilon = \mathscr{T}_{\frac{1}{2}}\circ\Phi^0_\mp, \tag{8.5.24}$$

and then

$$\begin{aligned}
\Phi^0_{+,\varepsilon}\circ T_\alpha &= (\mathscr{T}_{-\frac{1}{2}}\circ\Phi^0_{-,\varepsilon}\circ\mathbf{Q}_\varepsilon)\circ T_\alpha \quad (\text{by } (8.5.24))\\
&= \mathscr{T}_{-\frac{1}{2}}\circ(\mathscr{T}_{C^2_\varepsilon}\circ\Phi^0_+\circ T_{-\alpha})\circ\mathbf{Q}_\varepsilon\circ T_\alpha \quad (\text{by } (8.5.22))\\
&= \mathscr{T}_{C^2_\varepsilon}\circ\mathscr{T}_{-\frac{1}{2}}\circ\Phi^0_+\circ\mathbf{Q}_\varepsilon \;\; (\text{because } \mathbf{Q}_\varepsilon = T_{-\alpha}\circ\mathbf{Q}_\varepsilon\circ T_\alpha)\\
&= \mathscr{T}_{C^2_\varepsilon}\circ\Phi^0_- \quad (\text{by } (8.5.24)).
\end{aligned}$$

Comparing with the first equation in (8.5.22), we get $C^1_\varepsilon = C^2_\varepsilon = -\dfrac{\alpha_0}{2}$. □

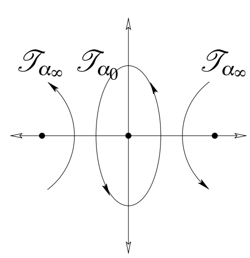


Figure 8.11: The positive orientations of $\mathscr{T}_{\alpha_0}, \mathscr{T}_{\alpha_\infty}$.

Grosso modo, (8.5.19) says that, in order to make a full turn around the origin in w coordinate, it is necessary to iterate twice the translation around the origin in the unwrapping coordinate. On the contrary, an iteration of the translation around infinity in the Z coordinate yields a full turn around $w_{\pm}$.

Lemma 8.5.12. *When the parameter is (real) positive, the Real Fatou Glutsyuk coordinates of Theorem 8.5.2 and Proposition 8.5.11, satisfy as well:*

$$\Phi_\varepsilon^0 \left\{ Im(Z) = \pm\frac{\alpha}{2i} \right\} \quad \subset \quad \left\{ Im(W) = \mp\frac{\alpha_0}{4i} \right\},$$

$$\Phi_\varepsilon^\infty \left\{ Im(Z) = \pm\frac{\alpha}{2i} \right\} \quad \subset \quad \left\{ Im(W) = \pm\frac{\alpha_\infty}{2i} \right\}.$$

Proof. Both (8.5.6) and (8.5.19) imply:

$$\begin{aligned} \mathscr{T}_{\mp\frac{\alpha_0}{2}} \circ \mathscr{C} \circ \Phi_\varepsilon^0 &= \Phi_\varepsilon^0 \circ T_{\pm\alpha} \circ \mathsf{C}, \\ \mathscr{T}_{\mp\alpha_\infty} \circ \mathscr{C} \circ \Phi_\varepsilon^\infty &= \Phi_\varepsilon^\infty \circ \mathsf{C} \circ T_{\pm\alpha}. \end{aligned} \tag{8.5.25}$$

Put $Im(Z) = \pm\frac{\alpha}{2i}$. Thus, $Z = \mathsf{C}(Z) \pm \alpha = T_{\pm\alpha} \circ \mathsf{C}(Z)$ and if we write $\Phi_\varepsilon^0(Z) = A + iB$, for $A, B \in \mathbb{R}$, then (8.5.25) yields:

$$\begin{aligned} A + iB &= \Phi_\varepsilon^0(Z) \\ &= \Phi_\varepsilon^0(T_{\pm\alpha} \circ \mathsf{C}(Z)) \\ &= \overline{\Phi_\varepsilon^0(Z)} \mp \frac{\alpha_0}{2} \\ &= A - iB \mp \frac{\alpha_0}{2}, \end{aligned}$$

whence $B = \mp\frac{\alpha_0}{4i}$. The second inclusion follows similar steps. □

Chapter 9

Glutsyuk invariant and symmetries

Remark. Here again we only treat the case $s = +1$.

9.1 Glutsyuk invariant.

Fix four Fatou Glutsyuk coordinates $\Phi^{0,\infty}_{\pm,\varepsilon,lr}$ on the leaves of $\mathscr{R}_\varepsilon$, whose base points depend analytically on the parameter, see Figure 8.9 and define:

a) For $\varepsilon \in V_{\delta,l}$:

$$\begin{aligned} \Psi^{++}_{\varepsilon,l} &= \Phi^{0}_{+,\varepsilon,l} \circ (\Phi^{\infty}_{+,\varepsilon,l})^{\circ -1}, \\ \Psi^{+-}_{\varepsilon,l} &= \Phi^{0}_{-,\varepsilon,l} \circ (\Phi^{\infty}_{+,\varepsilon,l})^{\circ -1}, \\ \Psi^{-+}_{\varepsilon,l} &= \Phi^{0}_{+,\varepsilon,l} \circ (\Phi^{\infty}_{-,\varepsilon,l})^{\circ -1}, \\ \Psi^{--}_{\varepsilon,l} &= \Phi^{0}_{-,\varepsilon,l} \circ (\Phi^{\infty}_{-,\varepsilon,l})^{\circ -1}. \end{aligned} \tag{9.1.1}$$

$b)$ For $\varepsilon \in V_{\delta,r}$:

$$\begin{aligned}
\Psi_{\varepsilon,r}^{++} &= \Phi_{+,\varepsilon,r}^{\infty} \circ (\Phi_{+,\varepsilon,r}^{0})^{\circ -1}, \\
\Psi_{\varepsilon,r}^{+-} &= \Phi_{-,\varepsilon,r}^{\infty} \circ (\Phi_{+,\varepsilon,r}^{0})^{\circ -1}, \\
\Psi_{\varepsilon,r}^{-+} &= \Phi_{+,\varepsilon,r}^{\infty} \circ (\Phi_{-,\varepsilon,r}^{0})^{\circ -1}, \\
\Psi_{\varepsilon,r}^{--} &= \Phi_{-,\varepsilon,r}^{\infty} \circ (\Phi_{-,\varepsilon,r}^{0})^{\circ -1}.
\end{aligned} \tag{9.1.2}$$

In either case, this collection will be noted Ψ_{ε}^{G}. By periodicity, it suffices to describe the dynamics around the principal hole. Since $\mathscr{P}_{\varepsilon} = \mathscr{Q}_{\varepsilon}^{\circ 2}$, it is possible to reduce these four components to two independent ones.

Definition 9.1.1. *The Glutsyuk invariant is the family of equivalence classes of Ψ_{ε}^{G} with respect to composition with translations $\mathscr{T}_{C(\varepsilon)}$ in the source and target spaces where the constant $C(\varepsilon)$ is real on real ε and it depends holomorphically on the parameter over $V_{\delta,l} \cup V_{\delta,r}$ with a continuous limit at $\varepsilon = 0$, such that $C(0) \neq 0$.*

Lemma 9.1.2. *By choosing normalized Real Fatou Glutsyuk coordinates, it is possible in turn to choose components $\Psi_{\varepsilon}^{\pm,\pm}$ of a representative of the Glutsyuk invariant Ψ_{ε}^{G} which are related through:*

$$\begin{aligned}
\Psi_{\varepsilon}^{++} &= \mathscr{T}_{-\frac{1}{2}} \circ \Psi_{\varepsilon}^{--} \circ \mathscr{T}_{\frac{1}{2}}, \\
\Psi_{\varepsilon}^{-+} &= \mathscr{T}_{-\frac{1}{2}} \circ \Psi_{\varepsilon}^{+-} \circ \mathscr{T}_{\frac{1}{2}}
\end{aligned} \tag{9.1.3}$$

for every $\varepsilon \in V_{\delta}^{G}$.

Proof. It suffices to take normalized Fatou Glutsyuk coordinates, so that (9.1.3) is satisfied by definition. □

9.1.1 Real Glutsyuk invariant.

When the Glutsyuk invariant is defined using Real Fatou Glutsyuk coordinates, we get a natural property of symmetry under the Schwarz reflection, respecting the real normalization of the Glutsyuk coordinates.

Theorem 9.1.3. *There exists a representative* $\Psi_\varepsilon^G = (\Psi_\varepsilon^{\pm\pm})$ *of the Glutsyuk modulus associated with the family of diffeomorphisms* $\mathscr{P}_\varepsilon$ *satisfying, in addition to* (9.1.3)*, the identities:*

– *If* $\varepsilon \in V_{\delta,l}$:

$$\begin{aligned} \Psi_{\varepsilon,l}^{++} &= \mathscr{T}_{-\frac{\alpha_0}{2}} \circ \Psi_{\varepsilon,l}^{+-} \circ \mathscr{T}_{-\alpha_\infty}, \\ \Psi_{\varepsilon,l}^{--} &= \mathscr{T}_{-\frac{\alpha_0}{2}} \circ \Psi_{\varepsilon,l}^{-+} \circ \mathscr{T}_{-\alpha_\infty}. \end{aligned} \tag{9.1.4}$$

– *If* $\varepsilon \in V_{\delta,r}$:

$$\begin{aligned} \Psi_{\varepsilon,r}^{++} &= \mathscr{T}_{\alpha_\infty} \circ \Psi_{\varepsilon,r}^{-+} \circ \mathscr{T}_{\frac{\alpha_0}{2}}, \\ \Psi_{\varepsilon,r}^{--} &= \mathscr{T}_{\alpha_\infty} \circ \Psi_{\varepsilon,r}^{+-} \circ \mathscr{T}_{\frac{\alpha_0}{2}}. \end{aligned} \tag{9.1.5}$$

– *Moreover, for every* $\varepsilon \in V_{\delta,lr}$:

$$\begin{aligned} \Psi_{\varepsilon}^{++} &= \mathscr{C} \circ \Psi_{\mathscr{C}(\varepsilon)}^{+-} \circ \mathscr{C}, \\ \Psi_{\varepsilon}^{--} &= \mathscr{C} \circ \Psi_{\mathscr{C}(\varepsilon)}^{-+} \circ \mathscr{C}. \end{aligned} \tag{9.1.6}$$

Such a representative can be constructed so as to have a limit at $\varepsilon = 0$, *which is the Ecalle modulus*.

Proof. It suffices to take normalized Real Fatou Glutsyuk coordinates depending analytically on the parameter with continuous limit at $\varepsilon = 0$, (this is the same limit for the two cases $\varepsilon \in V_{\delta,l}$ and $\varepsilon \in V_{\delta,r}$). Then (9.1.4) and (9.1.5) are immediate consequences of (9.1.1), (9.1.2) and Proposition 8.5.11. On the other hand, (9.1.6) comes after Theorem 8.5.2 and the idempotency (8.4.1) on the conjugation in the

Z coordinate, when the parameter is real. Since the dependence of the modulus is analytic in the parameter, the equality extends to values $\varepsilon \in V_{\delta,lr}$. Notice that the symmetry axis still exists in the limit $\varepsilon = 0$, and the invariance exists in the limit as well. □

Definition 9.1.4. *The equivalence class of a representative Ψ_ε^G of the Glutsyuk invariant chosen as in Lemma 9.1.2 and Theorem 9.1.3 for values $\varepsilon \in V_{\delta,lr}$ will be called the Real Glutsyuk modulus.*

Corollary 9.1.5. *For every $\varepsilon \in V_{\delta,lr}$, a representative of the Real Glutsyuk modulus is completely determined by one of the maps $\Psi_\varepsilon^{\pm\pm}$.*

In this first presentation, the symmetry (conjugation C in the time Z coordinate) is taken with respect the symmetry axis $\widehat{\mathfrak{R}}$. Since the Real Fatou Glutsyuk coordinates send the symmetry axis $\widehat{\mathfrak{R}}$ into $\mathbb{R}$, the real line is invariant under the Real Glutsyuk invariant when the parameter is real. This means that in the x-coordinate the symmetry has been taken with respect the real segment $I_+ \cup I_-$ joining the singular points $x_\pm$ with the boundary of U, see Figure 9.1. Moreover, in the limit $\varepsilon \to 0$ the segment $I_+ \cup I_-$ tends to $\mathbb{R} \cap U$. Thus, in the Fatou coordinate, the conjugation $\mathscr{C}$ is still defined when $\varepsilon = 0$ and the Ecalle invariant inherits the symmetry (9.1.6).

The Ecalle modulus. Since $\alpha(\varepsilon) = -\dfrac{\pi i}{\varepsilon}$, the distance between two consecutive holes becomes infinite in the limit $\varepsilon \to 0$, and then each diffeomorphism $\Psi_\varepsilon^{\pm\pm}$, for $\varepsilon \in V_{\delta,lr}$, gives rise to an independent component of the Ecalle invariant, with preimage in a region around the principal hole. Figure 9.2 shows the domains around the principal hole (connected strips) on the surface $\mathscr{R}_\varepsilon$ whose image by

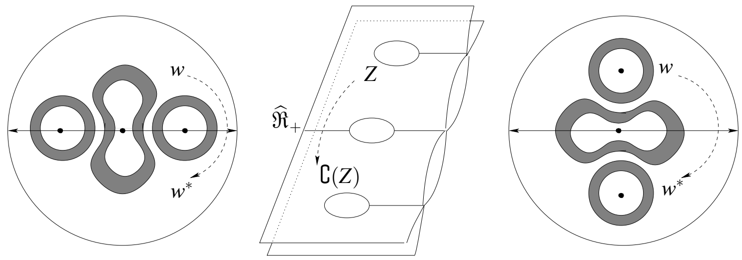


Figure 9.1: The symmetry in the First Presentation.

the Fatou Glutsyuk coordinates and subsequent quotient by the translation $\mathscr{T}_1$, correspond to annuli-like domains for the different components of the Glutsyuk invariant. However, we can choose the representative of the Real Glutsyuk modulus so as to give rise to the same invariant in the limit $\varepsilon \to 0$, no matter whether $\varepsilon \in V_{\delta,l}$ or $\varepsilon \in V_{\delta,r}$.

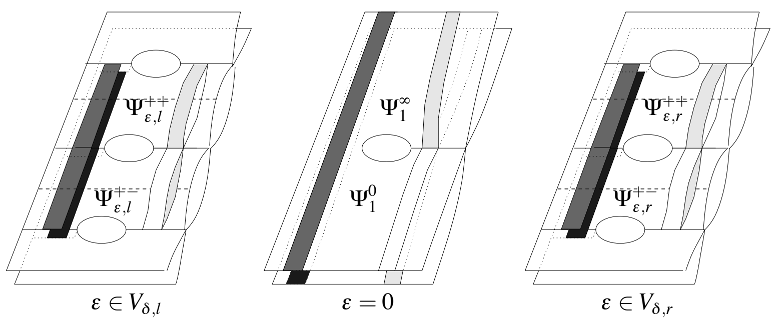


Figure 9.2: The Glutsyuk invariant in the limit $\varepsilon \to 0$.

Proposition 9.1.6. *The Ecalle modulus can be deduced from the Real Glutsyuk invariant. It is given by:*

$$\begin{aligned} \Psi_1^\infty &= \lim_{\varepsilon\to 0} \Psi^{++}_{\varepsilon,lr}, \qquad \Psi_1^0 = \lim_{\varepsilon\to 0} \Psi^{+-}_{\varepsilon,lr}, \\ \Psi_2^\infty &= \lim_{\varepsilon\to 0} \Psi^{--}_{\varepsilon,lr}, \qquad \Psi_2^0 = \lim_{\varepsilon\to 0} \Psi^{-+}_{\varepsilon,lr}, \end{aligned} \tag{9.1.7}$$

see Figure 9.2. Moreover, its components may be chosen conjugate as well:

$$\begin{aligned} \Psi_1^\infty &= \mathscr{C} \circ \Psi_1^0 \circ \mathscr{C} \\ \Psi_2^\infty &= \mathscr{C} \circ \Psi_2^0 \circ \mathscr{C}, \end{aligned} \tag{9.1.8}$$

and, in addition,

$$\begin{aligned} \Psi_1^\infty &= \mathscr{T}_{-\frac{1}{2}} \circ \Psi_2^\infty \circ \mathscr{T}_{\frac{1}{2}} \\ \Psi_1^0 &= \mathscr{T}_{-\frac{1}{2}} \circ \Psi_2^0 \circ \mathscr{T}_{\frac{1}{2}}. \end{aligned} \tag{9.1.9}$$

Proof. Each component of the modulus at $\varepsilon = 0$ is the limit of two representatives in the two cases $\varepsilon \in V_{\delta,l}$ and $\varepsilon \in V_{\delta,r}$. More specifically, we have:

$$\begin{aligned} \Psi_1^\infty &= \lim_{\varepsilon\to 0^l} \Phi^0_{+,\varepsilon,l} \circ (\Phi^\infty_{+,\varepsilon,l})^{\circ -1} = \lim_{\varepsilon\to 0^r} \Phi^\infty_{+,\varepsilon,r} \circ (\Phi^0_{+,\varepsilon,r})^{\circ -1}, \\ \Psi_1^0 &= \lim_{\varepsilon\to 0^l} \Phi^0_{-,\varepsilon,l} \circ (\Phi^\infty_{+,\varepsilon,l})^{\circ -1} = \lim_{\varepsilon\to 0^r} \Phi^\infty_{-,\varepsilon,r} \circ (\Phi^0_{+,\varepsilon,r})^{\circ -1}, \\ \Psi_2^\infty &= \lim_{\varepsilon\to 0^l} \Phi^0_{-,\varepsilon,l} \circ (\Phi^\infty_{-,\varepsilon,l})^{\circ -1} = \lim_{\varepsilon\to 0^r} \Phi^\infty_{-,\varepsilon,r} \circ (\Phi^0_{-,\varepsilon,r})^{\circ -1} \\ \Psi_2^0 &= \lim_{\varepsilon\to 0^l} \Phi^0_{+,\varepsilon,l} \circ (\Phi^\infty_{-,\varepsilon,l})^{\circ -1} = \lim_{\varepsilon\to 0^r} \Phi^\infty_{+,\varepsilon,r} \circ (\Phi^0_{-,\varepsilon,r})^{\circ -1}, \end{aligned} \tag{9.1.10}$$

where $\varepsilon \to 0^l$ (resp. $\varepsilon \to 0^r$) means $\varepsilon \to 0$ and $\varepsilon \in V_{\delta,l}$ (resp. $\varepsilon \in V_{\delta,r}$). The symmetries on the Ecalle modulus follow from Theorem 9.1.3. □

9.1.2 Symmetric Glutsyuk invariant.

When we use Real Fatou Glutsyuk coordinates and allow a subsequent imaginary translation on them, we break the symmetries (9.1.6). If the translations are well chosen we get a different form of symmetry in the x-coordinate, under the Schwarz reflection with respect to the line segments joining the points $x_\pm$. This presentation is also very interesting and deserves a detailed discussion.

Theorem 9.1.7. *There exists a representative* $\Psi^G_\varepsilon = (\Psi^{\pm\pm}_\varepsilon)$ *of the Glutsyuk modulus satisfying* (9.1.3)*,* (9.1.4) *and* (9.1.5)*, that carries the real character of the* *family of vector fields as follows. Let* $\# \in \{++,+-,-+,--\}$ *be a shortcut for the superscripts.*

– *If* $\varepsilon \in V_{\delta,l} \backslash \{0\}$ *then:*

$$\Psi^{\#}_{\varepsilon,l} = \mathscr{C} \circ \Psi^{\#}_{\mathscr{C}(\varepsilon),l} \circ \mathscr{C}, \tag{9.1.11}$$

i.e. the representative is "symmetric" with respect to the image of the line $\mathfrak{R}^s_{\pm}$.

– *If* $\varepsilon \in V_{\delta,r} \backslash \{0\}$ *then:*

$$\Psi^{\#}_{\varepsilon,r} = \mathscr{T}_{-\frac{1}{2}} \circ \mathscr{C} \circ \Psi^{\#}_{\mathscr{C}(\varepsilon),r} \circ \mathscr{C} \circ \mathscr{T}_{\frac{1}{2}}, \tag{9.1.12}$$

i.e. the representative is "symmetric" with respect to the image of the line $\mathfrak{I}^s_{\pm}$.

Proof. We start taking Real Fatou Glutsyuk coordinates $\Phi^{0,\infty}_{\pm,\varepsilon}$. By analytic dependence of the Glutsyuk coordinates in $\varepsilon \in V_{\delta,lr} \backslash \{0\}$, it suffices to show the theorem for real values of the parameter.

– *The case* $\varepsilon < 0$. The induced Real Glutsyuk invariant already verifies (9.1.6), so we must show that a correction is possible so that (9.1.11) be satisfied. Theorem 9.1.3 yields:

$$\begin{aligned} \Psi^{++}_{\varepsilon,l} &= \mathscr{T}_{-\frac{\alpha_0}{2}} \circ \Psi^{+-}_{\varepsilon,l} \circ \mathscr{T}_{-\alpha_\infty} \\ &= \mathscr{T}_{-\frac{\alpha_0}{2}} \circ \mathscr{C} \circ \Psi^{++}_{\varepsilon,l} \circ \mathscr{C} \circ \mathscr{T}_{-\alpha_\infty}. \end{aligned}$$

Consider the translations $\mathscr{T}_{d(\varepsilon)}, \mathscr{T}_{d'(\varepsilon)}$, where the constants $d(\varepsilon), d'(\varepsilon)$ are to be chosen later. Replacing $\Psi^{++}_{\varepsilon,l} \mapsto \mathscr{T}_{d(\varepsilon)} \circ \Psi^{++}_{\varepsilon,l} \circ \mathscr{T}_{d'(\varepsilon)}$ in the equation above, we get:

$$\begin{aligned} \mathscr{T}_{d(\varepsilon)} \circ \Psi^{++}_{\varepsilon,l} \circ \mathscr{T}_{d'(\varepsilon)} &= \mathscr{T}_{-\frac{\alpha_0}{2}} \circ \mathscr{C} \circ \mathscr{T}_{d(\varepsilon)} \circ \Psi^{++}_{\varepsilon,l} \circ \mathscr{T}_{d'(\varepsilon)} \circ \mathscr{C} \circ \mathscr{T}_{-\alpha_\infty} \\ &= \mathscr{T}_{-\frac{\alpha_0}{2}+\overline{d(\varepsilon)}} \circ \mathscr{C} \circ \Psi^{++}_{\varepsilon,l} \circ \mathscr{C} \circ \mathscr{T}_{-\alpha_\infty+\overline{d'(\varepsilon)}}. \end{aligned}$$

If $d(\varepsilon) = -\dfrac{\alpha_0}{4} = -\dfrac{i\pi}{2\,\varepsilon}$ and $d'(\varepsilon) = -\dfrac{\alpha_\infty}{2} = \dfrac{i\pi(1-A(\varepsilon)\,\varepsilon)}{2\,\varepsilon}$ (where $A(\varepsilon)$ is the real formal invariant), then we get

$$\Psi^{++}_{\varepsilon,l} = \mathscr{C} \circ \Psi^{++}_{\varepsilon,l} \circ \mathscr{C}.$$

The same procedure shows $\Psi^{\#}_{\varepsilon,l} = \mathscr{C} \circ \Psi^{\#}_{\varepsilon,l} \circ \mathscr{C}$, for $\# \in \{+-,-+,--\}$.

– *The case* $\varepsilon > 0$. By (9.1.5) and (9.1.6) we have:

$$\begin{aligned} \Psi^{++}_{\varepsilon,r} &= \mathscr{T}_{\alpha_\infty} \circ \Psi^{-+}_{\varepsilon,r} \circ \mathscr{T}_{\frac{\alpha_0}{2}} \\ &= \mathscr{T}_{\alpha_\infty} \circ \mathscr{C} \circ \Psi^{--}_{\varepsilon,r} \circ \mathscr{C} \circ \mathscr{T}_{\frac{\alpha_0}{2}}. \end{aligned}$$

The procedure used above shows that the corrections $\Psi^{++}_{\varepsilon,r} \mapsto \mathscr{T}_{d(\varepsilon)} \circ \Psi^{++}_{\varepsilon,r} \circ \mathscr{T}_{d'(\varepsilon)}$ and $\Psi^{--}_{\varepsilon,r} \mapsto \mathscr{T}_{d(\varepsilon)} \circ \Psi^{--}_{\varepsilon,r} \circ \mathscr{T}_{d'(\varepsilon)}$, for $d(\varepsilon) = \dfrac{\alpha_\infty}{2} = -\dfrac{i\pi(1-A(\varepsilon)\,\varepsilon)}{2\,\varepsilon}$ and $d'(\varepsilon) = \dfrac{\alpha_0}{4} = \dfrac{i\pi}{2\,\varepsilon}$ yield

$$\Psi^{++}_{\varepsilon,r} = \mathscr{C} \circ \Psi^{--}_{\varepsilon,r} \circ \mathscr{C}. \tag{9.1.13}$$

In the same spirit, we show:

$$\Psi^{-+}_{\varepsilon,r} = \mathscr{C} \circ \Psi^{+-}_{\varepsilon,r} \circ \mathscr{C}. \tag{9.1.14}$$

Then (9.1.13), (9.1.14) and Lemma 9.1.2 yield the conclusion:

$$\Psi^{\#}_{\varepsilon,r} = \mathscr{T}_{-\frac{1}{2}} \circ \mathscr{C} \circ \Psi^{\#}_{\varepsilon,r} \circ \mathscr{C} \circ \mathscr{T}_{\frac{1}{2}}.$$

Notice that this new "renormalized" representative still respects (9.1.3), (9.1.4) and (9.1.5). □

The composition with translations $\mathscr{T}_{d(\varepsilon)}, \mathscr{T}_{d'(\varepsilon)}$ in the proof above has destroyed the real normalization of the Real Fatou Glutsyuk coordinates $\Phi^{0,\infty}_{\pm,\varepsilon}$, and also the continuity at $\varepsilon = 0$. However, this non-real normalization is very interesting even if it does not pass to the limit when $\varepsilon \to 0$. Indeed, in the Z coordinate the imaginary translations $\mathscr{T}_{d(\varepsilon)}, \mathscr{T}_{d'(\varepsilon)}$ have displaced the symmetry axis to the line $\mathfrak{R}^s_\pm$ if $\varepsilon < 0$, and to the line $\mathfrak{I}^s_\pm$ if $\varepsilon > 0$, right above the principal hole. In the Fatou coordinate, the two imaginary translations have displaced the real axis to the lines $Im(W) = \dfrac{\alpha_0}{4i}$ and $Im(W) = \dfrac{\alpha_\infty}{2i}$, according to Lemma 8.5.12, thus breaking the real normalization of the Fatou Glutsyuk coordinates. The three real cases deserve explanation.

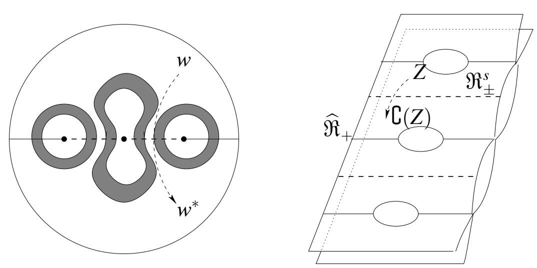


Figure 9.3: The symmetry when the parameter is negative.

i) The parameter is real and negative. The normalization reflects the natural symmetry of the invariant with respect the image (by p_ε^{-1}) of the real segment joining $x_0, x_\pm$ in the x-coordinate, see Figure 9.3. Inasmuch as the symmetry is taken with respect a “real” line in the Fatou coordinate, the invariant still carries the real character of the foliation, as can be seen from formula (9.1.11).

ii) The parameter is (real) positive. The imaginary translations have brought the symmetry axis to the image (by p_ε^{-1}) of the imaginary segment I joining the singular points $x_0, x_\pm$, see Figure 9.4. Thus, the non-real normalization yields an invariant in the x-coordinate which is symmetric with respect to I. That is exactly the meaning of the formula (9.1.12). This "imaginary" symmetry is explained by:

- the real symmetry carried by the former Real Fatou Glustyuk coordinates, so that the components of Ψ^G are 2-by-2 symmetric images one of another (this is (9.1.13) and (9.1.14));
- the fact that the Poincaré map of the family is a square: $\mathscr{P}_\varepsilon = \mathscr{Q}_\varepsilon^{\circ 2}$. In the x-plane this can be viewed as a sort of "symmetry with respect to the origin". Composing this symmetry with the symmetry with respect the real axis, yields a symmetry with respect to the imaginary axis.

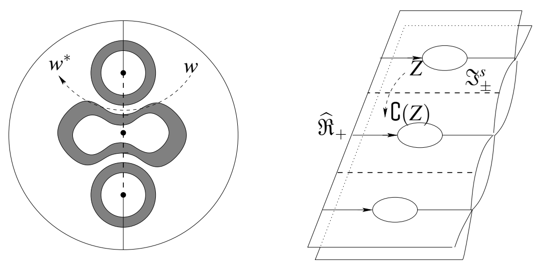


Figure 9.4: The symmetry when the parameter is positive.

iii) The parameter is null. As the lines $\mathfrak{R}^s_\pm, \mathfrak{I}^s_\pm$ no longer exist when $\varepsilon = 0$, this presentation does not pass to the limit when $\varepsilon \to 0$. The Ecalle modulus cannot be deduced from this presentation. Indeed, the real (resp. imaginary) segment in the x-coordinate joining the fixed points disappears when $\varepsilon \to 0^-$ (resp. $\varepsilon \to 0^+$).

Definition 9.1.8. *Any representative Ψ^G_ε of the Glutsyuk invariant chosen as in Theorem 9.1.7 will be called Symmetric Glutsyuk modulus.*

Corollary 9.1.9. *A representative of the Symmetric Glutsyuk modulus is completely determined by one of its components $\Psi^{\pm\pm}_\varepsilon$.*

9.2 Expansion in Fourier series.

We take Fatou Glutsyuk coordinates depending continuously on $\varepsilon \in V_{\delta,lr}$. The domain of Ψ_ε^G contains a union of four horizontal strips $S_\varepsilon^{\pm\pm}$ located right above (resp. below) the principal hole B_ε. As the Glutsyuk invariant satisfies $\Psi_\varepsilon^G(W+1) = \Psi_\varepsilon^G(W)+1$ we can expand the difference $\Psi_\varepsilon^G - id$ in Fourier series on $S_\varepsilon^{\pm\pm}$:

$$\begin{aligned}
(\Psi_\varepsilon^{++}(W)-W)\Big|_{S_\varepsilon^{++}} &= \sum_{n\in\mathbb{Z}} c_n^{++}(\varepsilon)\exp(2i\pi nW),\\
(\Psi_\varepsilon^{+-}(W)-W)\Big|_{S_\varepsilon^{+-}} &= \sum_{n\in\mathbb{Z}} c_n^{+-}(\varepsilon)\exp(2i\pi nW),\\
(\Psi_\varepsilon^{-+}(W)-W)\Big|_{S_\varepsilon^{-+}} &= \sum_{n\in\mathbb{Z}} c_n^{-+}(\varepsilon)\exp(2i\pi nW),\\
(\Psi_\varepsilon^{--}(W)-W)\Big|_{S_\varepsilon^{--}} &= \sum_{n\in\mathbb{Z}} c_n^{--}(\varepsilon)\exp(2i\pi nW).
\end{aligned} \tag{9.2.1}$$

Then, using (9.1.4) in the case $\varepsilon \in V_{\delta,l}$ we deduce:

$$\begin{cases}
c_0^{++}(\varepsilon) - c_0^{+-}(\varepsilon) = c_0^{--}(\varepsilon) - c_0^{-+}(\varepsilon) = -i\pi s A(\varepsilon),\\
c_n^{++}(\varepsilon) = c_n^{+-}(\varepsilon) e^{-\frac{2n\pi^2(1-sA(\varepsilon)\varepsilon)}{\varepsilon}}, \ \text{for } n\neq 0,\\
c_n^{--}(\varepsilon) = c_n^{-+}(\varepsilon) e^{-\frac{2n\pi^2(1-sA(\varepsilon)\varepsilon)}{\varepsilon}}, \ \text{for } n\neq 0,
\end{cases}$$

and using (9.1.5) in the case $\varepsilon \in V_{\delta,r}$ we get:

$$\begin{cases}
c_0^{++}(\varepsilon) - c_0^{-+}(\varepsilon) = c_0^{--}(\varepsilon) - c_0^{+-}(\varepsilon) = i\pi s A(\varepsilon),\\
c_n^{++}(\varepsilon) = c_n^{-+}(\varepsilon) e^{-\frac{2n\pi^2}{\varepsilon}}, \ \text{for } n\neq 0,\\
c_n^{--}(\varepsilon) = c_n^{+-}(\varepsilon) e^{-\frac{2n\pi^2}{\varepsilon}}, \ \text{for } n\neq 0.
\end{cases}$$

Corollary 9.2.1. *The differences $c_0^{++}(\varepsilon) - c_0^{+-}(\varepsilon)$ and $c_0^{--}(\varepsilon) - c_0^{-+}(\varepsilon)$ when $\varepsilon \in V_{\delta,l}$ (resp. $c_0^{++}(\varepsilon) - c_0^{-+}(\varepsilon)$ and $c_0^{--}(\varepsilon) - c_0^{+-}(\varepsilon)$ when $\varepsilon \in V_{\delta,r}$) are analytic invariants of the system. Moreover, if the Glutsyuk modulus is prescribed on $\varepsilon \in V_{\delta,lr}$, then the formal parameter $A(\varepsilon)$ is known for values of the parameter in $V_{\delta,lr}$.*

9.3 Invariants under weak conjugacy.

Definition 9.3.1. *Two germs* $\{\mathscr{P}_{\varepsilon_1}\}_{\varepsilon_1\in V_{\delta,lr}}$, $\{\widehat{\mathscr{P}}_{\varepsilon_2}\}_{\varepsilon_2\in V_{\delta,lr}}$ *of analytic families of diffeomorphisms are "weakly conjugate" as real families if there exists a germ of bijective map* $\mathscr{H}(\varepsilon_1,x) = (\mathbf{k}(\varepsilon_1),\mathbf{h}(\varepsilon_1,x))$ *fibered over the parameter space, where:*

i) $\mathbf{k} : \varepsilon_1 \to \varepsilon_2 = \mathbf{k}(\varepsilon_1)$ *is a germ of real analytic diffeomorphism preserving the origin.*

ii) There exists $\rho > 0$ *and* $r > 0$, *such that for each* $\varepsilon_1 \in V_{\delta,l}(\rho)\cup V_{\delta,r}(\rho)$, *there is a representative* $\mathbf{h}_{\varepsilon_1}(x) = \mathbf{h}(\varepsilon_1,x)$ *of the germ depending analytically on* $x \in \mathbb{D}_r$ *and is real for real* ε_1,x *such that* $\mathbf{h}_{\varepsilon_1}$ *conjugates* $\mathscr{P}_{\varepsilon_1}, \widehat{\mathscr{P}}_{\mathbf{k}(\varepsilon_1)}$:

$$\mathbf{h}_{\varepsilon_1}\circ\mathscr{P}_{\varepsilon_1} = \widehat{\mathscr{P}}_{\mathbf{k}(\varepsilon_1)}\circ\mathbf{h}_{\varepsilon_1}. \tag{9.3.1}$$

The representative $\mathbf{h}_{\varepsilon_1}$ *depends analytically on* $\varepsilon_1 \neq 0$ *and it is continuous at* $\varepsilon_1 = 0$.

Theorem 9.3.2. *Two families* $\{\mathscr{P}_{\varepsilon_1}\}_{\varepsilon_1\in V_{\delta,lr}}$ *and* $\{\widehat{\mathscr{P}}_{\varepsilon_2}\}_{\varepsilon_2\in V_{\delta,lr}}$ *(with the same sign* s *before the cubic coefficient) are weakly conjugated by a real conjugacy that depends analytically on the parameter* $\varepsilon \in V_{\delta,lr}\setminus\{0\}$ *and continuously at* $\varepsilon = 0$, *if and only if the Glutsyuk moduli of their associated prepared families coincide.*

Proof. Since two families are conjugate if and only if the associated prepared families ae conjugate, it suffices to work with prepared families. The preparation shows that the parameters ε_1 and ε_2, the canonical parameters of the families, are analytic invariants, thus we can consider the conjugacy over the identity ($\varepsilon_1 = \varepsilon_2 := \varepsilon$) and then it suffices to compare the two families for a given $\varepsilon \in V_{\delta,lr}$.

Since for values $\varepsilon \in V_{\delta,lr}$ the singular points $x_0, x_\pm$ are hyperbolic, they are linearizable. Hence, there exists in the neighborhood of each fixed point two sectorial diffeomorphism $\varphi_\varepsilon^{0,\pm} = q_\varepsilon \circ \Phi_{\varepsilon,\pm}^{G,0,\infty} \circ p_\varepsilon^{-1}$ and $\widehat{\varphi}_\varepsilon^{0,\pm} = q_\varepsilon \circ \widehat{\Phi}_{\varepsilon,\pm}^{G,0,\infty} \circ p_\varepsilon^{-1}$ conjugating, respectively, the Poincaré maps $\mathscr{P}_\varepsilon$ and $\widehat{\mathscr{P}_\varepsilon}$ with the model diffeomorphism, *i.e.* the time one map τ_ε^1 of the field (7.1.5). The maps q_ε are the time-W flow of the latter. The Fatou coordinates $\Phi_\varepsilon^{G,0,\infty}, \widehat{\Phi}_\varepsilon^{G,0,\infty}$ are real (Definition 8.5.3). The neighborhoods of the singular points in the x coordinate where the normalization is possible are noted, respectively, $U_\varepsilon^-, U_\varepsilon^0$ and U_ε^+.

For the choice of a sufficiently small neighborhood $V_{\delta,lr}$ the domains of $\varphi_\varepsilon^-, \varphi_\varepsilon^0, \varphi_\varepsilon^+$ overlap, and thus the maps:

$$\varphi_\varepsilon^{G,+} = \begin{cases} \varphi_\varepsilon^0 \circ (\varphi_\varepsilon^+)^{-1}, & \text{for } \varepsilon < 0, \\ \varphi_\varepsilon^+ \circ (\varphi_\varepsilon^0)^{-1}, & \text{for } \varepsilon > 0, \end{cases} \tag{9.3.2}$$

and

$$\varphi_\varepsilon^{G,-} = \begin{cases} \varphi_\varepsilon^0 \circ (\varphi_\varepsilon^-)^{-1}, & \text{for } \varepsilon < 0, \\ \varphi_\varepsilon^- \circ (\varphi_\varepsilon^0)^{-1}, & \text{for } \varepsilon > 0, \end{cases} \tag{9.3.3}$$

are well defined and are one presentation of the Glutsyuk modulus of the family $\mathscr{P}_\varepsilon$. In the same way, the maps:

$$\widehat{\varphi}_\varepsilon^{G,+} = \begin{cases} \widehat{\varphi}_\varepsilon^0 \circ (\widehat{\varphi}_\varepsilon^+)^{-1}, & \text{for } \varepsilon < 0, \\ \widehat{\varphi}_\varepsilon^+ \circ (\widehat{\varphi}_\varepsilon^0)^{-1}, & \text{for } \varepsilon > 0, \end{cases} \tag{9.3.4}$$

and

$$\widehat{\varphi}_\varepsilon^{G,-} = \begin{cases} \widehat{\varphi}_\varepsilon^0 \circ (\widehat{\varphi}_\varepsilon^-)^{-1}, & \text{for } \varepsilon < 0, \\ \widehat{\varphi}_\varepsilon^- \circ (\widehat{\varphi}_\varepsilon^0)^{-1}, & \text{for } \varepsilon > 0, \end{cases} \tag{9.3.5}$$

are well defined on U and they are the Glutsyuk modulus of the family $\widehat{\mathscr{P}_\varepsilon}$. Of course, $\varphi_\varepsilon^{0,\pm}, \widehat{\varphi}_\varepsilon^{0,\pm}$ are unique modulo left composition with time maps of the field

(7.1.5), so it is possible to suppose that the representatives are equal. The map:

$$\begin{cases} f_\varepsilon^- = (\widehat{\varphi}_\varepsilon^-)^{-1} \circ \varphi_\varepsilon^- = p_\varepsilon \circ (\widehat{\Phi}_{\varepsilon,-}^{G,\infty})^{-1} \circ \Phi_{\varepsilon,-}^{G,\infty} \circ p_\varepsilon^{-1}, \\ f_\varepsilon^0 = (\widehat{\varphi}_\varepsilon^0)^{-1} \circ \varphi_\varepsilon^0 \ = p_\varepsilon \circ (\widehat{\Phi}_{\varepsilon}^{G,0})^{-1} \circ \Phi_{\varepsilon}^{G,0} \circ p_\varepsilon^{-1}, \\ f_\varepsilon^+ = (\widehat{\varphi}_\varepsilon^+)^{-1} \circ \varphi_\varepsilon^+ = p_\varepsilon \circ (\widehat{\Phi}_{\varepsilon,+}^{G,\infty})^{-1} \circ \Phi_{\varepsilon,+}^{G,\infty} \circ p_\varepsilon^{-1}, \end{cases}$$

is clearly a change of coordinates conjugating the two families of diffeomorphisms, since the local changes of coordinates are extensions of each other over the neighborhood U when $\varepsilon \in V_{\delta,lr}$: $f_\varepsilon^0 \equiv f_\varepsilon^-$ on $U_\varepsilon^- \cap U_\varepsilon^0$, and $f_\varepsilon^0 \equiv f_\varepsilon^+$ on $U_\varepsilon^0 \cap U_\varepsilon^+$. The conclusion follows. □

Theorem 9.3.3. *We consider a generic family unfolding a generic weak focus of order one. The Glutsyuk invariant of analytic classification of the Poincaré return map of the unfolded vector field, namely, the family of equivalence classes*

$$(\Psi_\varepsilon^G / \sim)_{\varepsilon \in V_{\delta,lr}^G}, \tag{9.3.6}$$

together with the sign s is a complete modulus of orbital analytic classification under weak orbital equivalence. The parameter ε is the "canonical parameter". The equivalence relation $\sim$ is defined by

$$\Psi_\varepsilon^G \sim \widehat{\Psi}_\varepsilon^G \Longleftrightarrow \exists c(\varepsilon), c'(\varepsilon) \in \mathbb{C} : \Psi_\varepsilon^G = T_{c(\varepsilon)} \circ \widehat{\Psi}_\varepsilon^G \circ T_{c'(\varepsilon)},$$

where the constants $c(\varepsilon), c'(\varepsilon)$ depend analytically of the parameter, and are real on real values of it.

Proof. First of all, Lemma 9.1.2 relates the components of the Glutsyuk invariant lying in different leaves of $\mathscr{R}_\varepsilon$, namely, Ψ_ε^{++} is related to Ψ_ε^{--} and Ψ_ε^{+-} to Ψ_ε^{-+}. So, two indepenent components remain.

Real Glutsyuk invariant. In this case, for instance, Equation (9.1.6) makes the link between the remaining components. Therefore, only one component of the real Glutsyuk modulus is independent.

Symmetric Glutsyuk invariant. For values $\varepsilon \in V^G_{\delta,l}$, we use Equation (9.1.11) to reduce the number of independent components. Otherwise ($i.e.$ if $\varepsilon \in V^G_{\delta,r}$), we use Equation (9.1.12) (the latter relates $\Psi^{++}_{\varepsilon,r}$ with $\Psi^{-+}_{\varepsilon,r}$, and $\Psi^{--}_{\varepsilon,r}$ with $\Psi^{+-}_{\varepsilon,r}$).

So it suffices to represent the modulus by one of the two components. The invariant can be chosen depending analytically on the parameter.

If two generic orbitally prepared families unfolding weak foci are analytically orbitally equivalent, then Theorem 4.2.3 implies that the family of diffeomorphisms unfolding their Poincaré maps are conjugate by a real conjugacy. Hence, Theorem 9.3.2 yields the equivalence of the moduli.

Conversely, if the real Glutsyuk moduli of the families are equivalent then, by Theorem 9.3.2, the families of diffeomorphisms unfolding the Poincaré maps of the two systems are weakly equivalent by a real equivalence. Theorem 4.2.3 yields the weak orbital equivalence of the families of vector fields. □

Chapter 10

The Glutsyuk invariant on the tori

In this chapter we look forward to "visualize" the phenomena described before in the unwrapping Z coordinate. We assume, as usual, the sign invariant $s = +1$.

10.1 The sphere coordinate.

The non-Hausdorff spaces $\mathbf{T}_{\varepsilon}^{0} \cup \{w_0\}$ and $\mathbf{T}_{\varepsilon,\pm}^{\infty} \cup \{w_{\pm}\}$ are conformally equivalent to the quotients $U_{\varepsilon}^{0}/\mathscr{P}_{\varepsilon}$ and $U_{\varepsilon}^{\infty}/\mathscr{P}_{\varepsilon}$, respectively (Lemma 8.5.10). Define a coordinate $\mathbf{w} = \phi_{\varepsilon,\pm}^{0,\infty} : \mathbf{T}_{\varepsilon,\pm}^{0,\infty} \to \mathbb{T}_{\varepsilon,\pm}^{0,\infty}$, induced by the real Fatou coordinate $\mathbf{\Phi}_{\varepsilon,\pm}^{G,0,\infty}$:

$$\mathbf{w} = \phi_{\varepsilon,\pm}^{0,\infty} := \mathscr{E} \circ \mathbf{\Phi}_{\varepsilon,\pm}^{G,0,\infty} \circ p_{\varepsilon}^{-1}, \tag{10.1.1}$$

where $\mathscr{E}(W) = e^{-2i\pi W}$ and $\mathbb{T}_{\varepsilon,\pm}^{0,\infty} = \mathbb{C}/(\mathbb{C} \times \alpha_{0,\infty}\mathbb{C})$ are the canonical tori. When $\varepsilon \to 0$ the tori $\mathbb{T}_{\varepsilon,\pm}^{o,\infty}$ tend to Ecalle cylinders $\{\mathbb{S}_{1,0}^{0,\infty}, \mathbb{S}_{2,0}^{0,\infty}\}$ (they are infinite cylinders conformally equivalent to $\mathbb{CP}^1$ minus the two points $0, \infty$). This means that the $\mathbf{w}$ coordinate induced by the map (10.1.1) on the tori $\mathbb{T}_{\varepsilon,\pm}^{0,\infty}$ (spheres $\mathbb{S}_{j,0}^{0,\infty}$) is related to the Fatou coordinate by:

$$\mathbf{w} = \mathscr{E}(W). \tag{10.1.2}$$

The Glutsyuk modulus Ψ_ε^G induces holomorphic diffeomorphisms ψ_ε^G in the sphere coordinate by means of (9.1.4) and (9.1.5) . They are defined in annular-like regions $A_{\varepsilon,\pm}^{0,\infty}$ on the tori $\mathbb{T}_{\varepsilon,\pm}^{0,\infty}$.

Remark. We will use the sphere coordinate as the natural coordinate of the orbits space. Since the map $p_\varepsilon^{\circ-1}$ is multivalued the collection $(\Psi_\varepsilon^\pm)$ gives birth to only two different diffeomorphims in the sphere coordinate. They will be noted

$$\psi_{j,\varepsilon}^G : A_{\varepsilon,\pm}^\infty \to A_\varepsilon^0, \quad j = 1,2 \tag{10.1.3}$$

between two annuli $A_{\varepsilon,\pm}^\infty$ and A_ε^0 located respectively in $\mathbb{T}_{\varepsilon,\pm}^\infty$ and in $\mathbb{T}_\varepsilon^0$ for $\varepsilon \neq 0$.

Definition 10.1.1. *The maps $(\psi_{1,\varepsilon}^G, \psi_{2,\varepsilon}^G)$ are called the "first and second components" of the Glutsyuk invariant in the sphere coordinate.*

1. If $\varepsilon \in V_{\delta,l}^G$, the "first component" of the Glutsyuk invariant in the W coordinate is the pair $\Psi_{1,\varepsilon}^G = (\Psi_{\varepsilon,l}^{++}, \Psi_{\varepsilon,l}^{+-})$. The "second component" is the pair $\Psi_{2,\varepsilon}^G = (\Psi_{\varepsilon,l}^{--}, \Psi_{\varepsilon,l}^{-+})$.
2. If $\varepsilon \in V_{\delta,r}^G$, the "first component" of the Glutsyuk invariant in the W coordinate is the pair $\Psi_{1,\varepsilon}^G = (\Psi_{\varepsilon,r}^{++}, \Psi_{\varepsilon,r}^{-+})$. The "second component" is the pair $\Psi_{2,\varepsilon}^G = (\Psi_{\varepsilon,r}^{--}, \Psi_{\varepsilon,r}^{+-})$.

It turns out that the $\mathbf{w}$-Glutsyuk invariant $\psi_\varepsilon^G = (\psi_{1,\varepsilon}^G, \psi_{2,\varepsilon}^G)$ is related to the first and second components of the W-Glutsyuk invariant by:

$$\psi_{j,\varepsilon}^G \quad = \quad \mathscr{E} \circ \Psi_{j,\varepsilon}^G \circ \mathscr{E}^{-1}, \quad j = 1,2. \tag{10.1.4}$$

– If $\varepsilon \in V_{\delta,l}^G$, then (10.1.4) means that $\psi_{1,\varepsilon}^G$ is represented by the pair $\Psi_{1,\varepsilon}^G = (\Psi_l^{++}, \Psi_l^{+-})$, and $\psi_{2,\varepsilon}^G$ is represented by the pair $\Psi_{2,\varepsilon}^G = (\Psi_l^{--}, \Psi_l^{-+})$ along the leaves of $\mathscr{R}_\varepsilon$, see Figure 10.1.

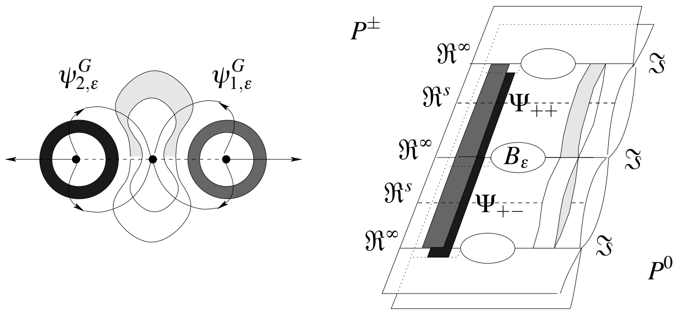


Figure 10.1: The Glutsyuk invariant on the tori, for $\varepsilon < 0$.

– If $\varepsilon \in V^G_{\delta,r}$, then (10.1.4) means that $\psi^G_{1,\varepsilon}$ is represented by the pair $\Psi^G_{1,\varepsilon} = (\Psi^{++}_r, \Psi^{-+}_r)$, and $\psi^G_{2,\varepsilon}$ is represented by the pair $\Psi^G_{2,\varepsilon} = (\Psi^{--}_r, \Psi^{+-}_r)$ along the leaves of $\mathscr{R}_\varepsilon$, see Figure 10.2.

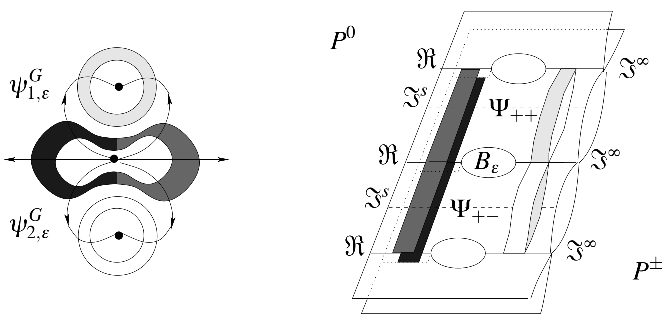


Figure 10.2: The Glutsyuk invariant on the tori, for $\varepsilon > 0$.

Definition 10.1.2. *The inversion* $\mathbf{w} \to \dfrac{1}{\mathbf{w}}$ *in the sphere coordinate is noted* $\mathscr{H}$.

Lemma 10.1.3. *Complex conjugation* $\mathscr{C}$, *inversion in the sphere coordinate* $\mathscr{H}$, *exponential* $\mathscr{E}$, *translation* $\mathscr{T}_{-\frac{1}{2}}$ *and linear map* $\mathscr{L}_{-1}$ *are related through:*

1. $\mathscr{C} = \mathscr{H} \circ \mathscr{E} \circ \mathscr{C} \circ \mathscr{E}^{\circ -1}$.
2. $\mathscr{E} \circ \mathscr{T}_{-\frac{1}{2}} = \mathscr{L}_{-1} \circ \mathscr{E}$.

Proof. The first equivalence comes from $\mathscr{C}(\mathscr{E}(W)) = \mathscr{C}(\mathbf{w}) = \overline{e^{-2i\pi W}} = e^{2i\pi\mathscr{C}(W)} = \mathscr{H}(e^{-2i\pi\mathscr{C}(W)}) = \mathscr{H}\circ\mathscr{E}\circ\mathscr{C}(W)$. The second is plain. □

10.2 Real Glutsyuk invariant.

Lemma 10.1.3 proves the Theorem 9.1.3 in the sphere:

Proposition 10.2.1. *There exists a representative $\psi_\varepsilon^G = (\psi_{1,\varepsilon}^G, \psi_{2,\varepsilon}^G)$ of the real Glutsyuk invariant in the sphere coordinate, that carries the real character of the family of vector fields, in the following sense:*

1. *If $\varepsilon \in V_{\delta,l}^G$:*

$$\psi_{j,\varepsilon}^G = \mathscr{C}\circ\mathscr{H}\circ\psi_{j,\mathscr{C}(\varepsilon)}^G\circ\mathscr{H}\circ\mathscr{C}, \quad j=1,2 \tag{10.2.1}$$

2. *If $\varepsilon \in V_{\delta,r}^G$:*

$$\psi_{1,\varepsilon}^G = \mathscr{C}\circ\mathscr{H}\circ\psi_{2,\mathscr{C}(\varepsilon)}^G\circ\mathscr{H}\circ\mathscr{C}. \tag{10.2.2}$$

The Écalle-Voronin modulus in the sphere. The Écalle-Voronin invariant can be deduced in the limit $\varepsilon = 0$, and the symmetries above yield:

$$\begin{aligned} \psi_{1,0}^0 &= \mathscr{C}\circ\mathscr{H}\circ\psi_{1,0}^\infty\circ\mathscr{H}\circ\mathscr{C}, \\ \psi_{2,0}^0 &= \mathscr{C}\circ\mathscr{H}\circ\psi_{2,0}^\infty\circ\mathscr{H}\circ\mathscr{C}. \end{aligned} \tag{10.2.3}$$

Since $p_0^{\circ-1}$ is not multivalued, it is easily seen that:

$$\begin{aligned} \psi_{1,0}^\infty &= \mathscr{E}\circ\Psi_1^\infty\circ\mathscr{E}^{-1} = \lim_{\varepsilon\to 0}\mathscr{E}\circ\Psi_{++}^G\circ\mathscr{E}^{-1} \\ \psi_{1,0}^0 &= \mathscr{E}\circ\Psi_1^0\circ\mathscr{E}^{-1} = \lim_{\varepsilon\to 0}\mathscr{E}\circ\Psi_{+-}^G\circ\mathscr{E}^{-1} \\ \psi_{2,0}^\infty &= \mathscr{E}\circ\Psi_2^\infty\circ\mathscr{E}^{-1} = \lim_{\varepsilon\to 0}\mathscr{E}\circ\Psi_{--}^G\circ\mathscr{E}^{-1} \\ \psi_{2,0}^0 &= \mathscr{E}\circ\Psi_2^0\circ\mathscr{E}^{-1} = \lim_{\varepsilon\to 0}\mathscr{E}\circ\Psi_{-+}^G\circ\mathscr{E}^{-1}, \end{aligned} \tag{10.2.4}$$

and this is consequence of (9.1.7), see Figure 10.3. By Proposition 9.1.6 a representative of the Écalle-Voronin modulus can be chosen so as to verify:

$$\begin{aligned} \psi^0_{2,0}(\mathbf{w}) &= -\psi^0_{1,0}(-\mathbf{w}) \\ \psi^\infty_{2,0}(\mathbf{w}) &= -\psi^\infty_{1,0}(-\mathbf{w}). \end{aligned} \tag{10.2.5}$$

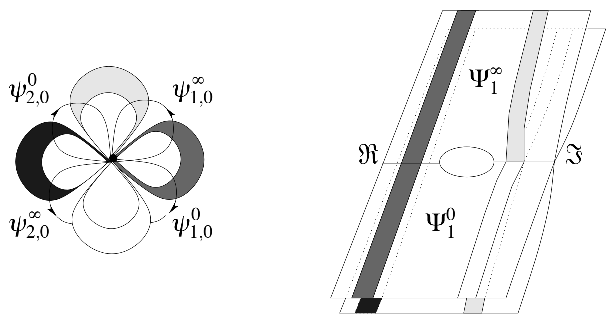


Figure 10.3: The Écalle-Voronin modulus in the sphere.

Definition 10.2.2. *Any representative* ψ^G_ε *of the Glutsyuk invariant satisfying* (10.2.1)*,* (10.2.2) *and* (10.2.5) *for values* $\varepsilon \in V^G_{\delta,lr}$*, will be called Real Glutsyuk modulus in the sphere.*

In this first presentation, the real line $\mathbb{R}$ is invariant under the real Glutsyuk invariant in the sphere when the parameter is negative, see Figure 10.2.

10.3 Symmetric Glutsyuk invariant.

The imaginary translation on the real Fatou Glutsyuk coordinates destroys the symmetries (9.1.6). This is reflected in the sphere coordinate as well. The equations reflect the fact that the symmetry axis corresponds to the imaginary axis.

This presentation does not pass to the limit $\varepsilon \to 0$. The Écalle-Voronin modulus cannot be deduced from the Symmetric Glutsyuk modulus.

Proposition 10.3.1. *There exists a representative $\psi_\varepsilon^G = (\psi_{1,\varepsilon}^G, \psi_{2,\varepsilon}^G)$ of the Glutsyuk invariant in the sphere, that carries the real character of the family of vector fields as:*

1. *If $\varepsilon \in V_{\delta,l}^G$:*

$$\psi_{j,\varepsilon}^G = \mathscr{C} \circ \mathscr{H} \circ \psi_{j,\mathscr{C}(\varepsilon)}^G \circ \mathscr{H} \circ \mathscr{C}, \quad j = 1,2. \tag{10.3.1}$$

2. *If $\varepsilon \in V_{\delta,r}^G$:*

$$\psi_{j,\varepsilon}^G = \mathscr{L}_{-1} \circ \mathscr{C} \circ \mathscr{H} \circ \psi_{j,\mathscr{C}(\varepsilon)}^G \circ \mathscr{H} \circ \mathscr{C}, \quad j = 1,2. \tag{10.3.2}$$

The Écalle-Voronin modulus cannot be deduced from this presentation of the Glutsyuk invariant.

Definition 10.3.2. *Any representative Ψ_ε^G of the Glutsyuk invariant satisfying* (10.3.2) *and* (10.3.1) *for values $\varepsilon \in V_{\delta,lr}^G$ will be called Symmetric Glutsyuk modulus in the sphere.*

A word on the formula (10.3.2)**.** This equation comes from (9.1.11) and is consequence of two facts. First, there is a symmetry on the two components of the real Glutsyuk invariant Ψ_ε^G with respect to the real axis when the parameter belongs to $V_{\delta,r}^G$. Such a symmetry relates the two components of the Glutsyuk invariant, namely,

$$\psi_{1,\varepsilon}^G = \mathscr{C} \circ \mathscr{H} \circ \psi_{2,\mathscr{C}(\varepsilon)}^G \circ \mathscr{H} \circ \mathscr{C}. \tag{10.3.3}$$

Second, the Poincaré map is a square: $\mathscr{P}_\varepsilon = \mathscr{Q}_\varepsilon^{\circ 2}$. This means that the components of the Glutsyuk invariant standing on opposite sides of the origin are related.

Thus, (10.3.2) is the "overlap" of (10.3.3) and Lemma 9.1.2, and it expresses the symmetry of the Glutsyuk modulus with respect the imaginary axis when $\varepsilon \in V_{\delta,r}^{G}$, see Figure 10.2.

Conclusion

The thesis is a small contribution to the field of analytic differential equations. Our aim has been to summarize hidden aspects of the unfolding of the foliation associated with a singular point of weak focus type and that only become apparent after complexification. The first aspect states that the equivalence class of the foliation under orbital equivalence coincides with the equivalence class of the Poincaré map under conjugacy. Second, we have shown that it is locally possible to recover the foliation when a family of admissible diffeomorphisms is prescribed. Both properties allow to compute the modulus of analytic classification under “weak orbital equivalence”. We have constructed the invariant of the unfolding in the Glutsyuk point of view, where the fixed points of the Poincaré map are linearizable. The construction depends analytically on the parameter ε for values in the covering of the parameter space different from zero, and continuously at $\varepsilon = 0$.

It has been proven ([10]) that for any sufficiently small neighborhood of the origin and for any $\delta \in (0, \pi)$, there exist neighborhoods $V^L_{\delta,+}$ and $V^L_{\delta,-}$ in the parameter space, such that the orbit space is described as follows:

– There exists four crescents with endpoints at the two singular points bounded by curves $\ell^{0,\infty}_{j,\varepsilon}$ and their images $\mathscr{P}_\varepsilon(\ell^{0,\infty}_{j,\varepsilon})$.

– The crescents in which we identify the curves with their images have the conformal structure of spheres with the singular point $\sqrt{\varepsilon}$ (resp. $-\sqrt{\varepsilon}$) located at ∞ (resp. 0).

– Points in the two neighborhoods of 0 and ∞ on the spheres are identified modulo holomorphic maps defined in the neighborhoods of 0 and ∞, respectively. These maps are one presentation of the Lavaurs modulus. The maps are uniquely defined up to the linear choice of coordinates on the spheres.

This is the *Lavaurs point of view,* the crescents and maps are depicted in Figure 10.4. In order to give the invariant of analytic classification under "orbital equivalence" we need to establish the link between Lavaurs and Glutsyuk points of view in the intersection of the sectorial neighborhoods of the parameter space. This will be done in forthcoming publications.

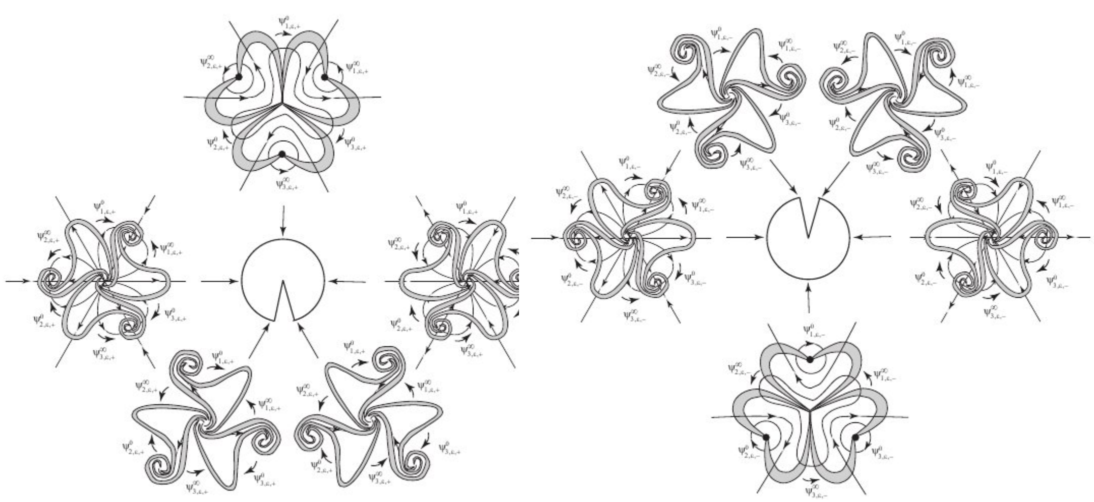


Figure 10.4: The crescents in the Lavaurs point of view (borrowed from [43]).

A highly non-trivial problem that deserves profound study has to do with the dependence of the modulus on the parameter. Recent works (*e.g.* [11],[43]) have shown the existence of a "compatibility condition" that must be met so that the Lavaurs modulus describes an analytic family in the parameter. This compatibility condition states that the two sectorial (in the parameter) families are orbitally equivalent on the intersection of the two sectors in the parameter space. In the Glutsyuk point of view we do not cover a full neighborhood in the parameter space.

To study the equivalence classes of germs of vector fields under conjugacy we need to study a time part of the vector field. There exists a "temporal" preparation which uncouples the family of vector field in a time part, and an orbital part. The former has to do with the parametrization of the flow curves in the (complex) phase space, while the latter concerns the organization of the leaves. The invariants involved in the time part are not detected in the modulus of (weak) orbital equivalence given in the thesis, though we have started working on the the obstructions for the family to be "temporally normalizable". Indeed, we have already done a preparation which allows to decompose the field as an orbital part (a vector field) multiplied by a nonzero temporal part (a function). The vector field is temporally normalizable if we can bring the temporal part to its polynomial normal form. In particular, we prove that a system is "isochronous" if its time part is equivalent to a constant time part. A great deal of work has been done to characterize isochronicity in systems with a focus or a center. Pioneers are the works of Lukashevich, Villarini, Volotkin and Ivanov on conditions under which a polynomial vector field:

$$\begin{aligned} \dot{x} &= p(x,y) \\ \dot{y} &= q(x,y) \end{aligned}$$

possesses an isochronous section in $\mathbb{R}^2$ (see also [20],[45]). Our methodology should extend the theory to the case of generic unfoldings of a weak focus embedded in $\mathbb{C}^2$. This should allow to spread new light on the analytical class of the "temporal part" of the modulus of such a system. As far as formal invariants are concerned, we find one formal invariant, which yields the time for the return map near the singular point and we get a second formal invariant which expresses the difference between the time for return near the singular point and the period of the periodic orbit. If the weak focus is isochronous, then the latter invariant is zero. At this point, some obstructions to isochronicity come from the fact that the focus and limit cycle have different periods and this persist to the limit $\varepsilon = 0$.

In the higher codimension cases we have the coalescence of several limit cycles with the singular point. This is clearly a more challenging problem. More subtle analytic obstructions also exist, which we hope to explain in near future.

Appendix A

Holonomy And Fundamental Group

A.1 Preliminaries.

Denote by $(a,b)^m$ the m-fold product of the open interval (a,b). The product $(a,b)^m$ is an open subset of Euclidean n-space $\mathbb{R}^m$.

Let M^n be an n-dimensional C^s manifold. Let $\mathscr{F} = \{L_\alpha : \alpha \in A\}$ be a C^r codimension q foliation of M^n and $\mathscr{S}_{\mathscr{F}}^{(r)}$ be its foliated atlas, $r \geq 1$. Define a projection map $\hat{\pi} : (-1,1)^n \to (-1,1)^q$ by

$$\hat{\pi}(x_1, ..., x_n) = (x_{n-q+1}, ..., x_n).$$

We say that $(U_\lambda, \varphi_\lambda)$ in $\mathscr{S}_{\mathscr{F}}^{(r)}$ is a *distinguished chart* if it satisfies the following.

1. $\varphi_\lambda(U_\lambda) = (-1,1)^n$.
2. There exists $(U_{\hat{\lambda}}, \varphi_{\hat{\lambda}}) \in \mathscr{S}_{\mathscr{F}}^{(r)}$ such that $\overline{U_\lambda} \subset U_{\hat{\lambda}}$, and $\varphi_{\hat{\lambda}}|_{U_\lambda} = \varphi_\lambda$.
3. For each leaf $L \in \mathscr{F}$, $L \cap U_\lambda$ is a graph over $\hat{\pi}(\varphi_\lambda(U_\lambda))$.

It is evident that each interior point p of M^n belongs to some distinguished chart $(U_\lambda, \varphi_\lambda)$ with $\varphi_\lambda(p) = (0, ..., 0)$.

If $(U_\lambda, \varphi_\lambda)$ is a distinguished chart and x is a point of $(-1,1)^q$, we see that $\varphi_\lambda^{-1}(\hat{\pi}^{-1}(x))$ is contained in a leaf of $\mathscr{F}$. The set $\varphi_\lambda^{-1}(\hat{\pi}^{-1}(x))$ is called a *plaque* of U_λ (see Figure A.1, where $q = n-k$) and is often denoted by Q_λ.

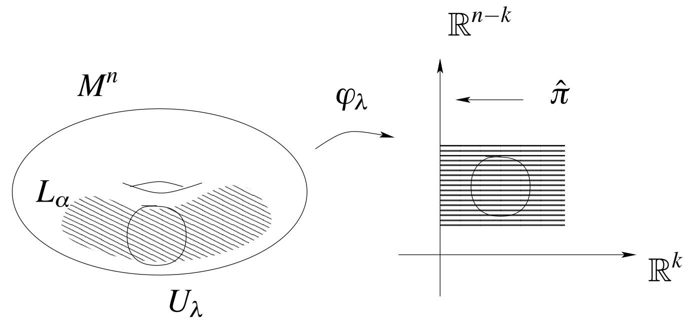


Figure A.1: A chart $(U_\lambda, \varphi_\lambda)$ on a leaf L_α. Also, the projection is shown.

Each plaque of U_λ is C^r diffeomorphic to $(-1,1)^{n-q}$, and we have

$$U_\lambda = \bigcup_{x \in (-1,1)^q} \varphi_\lambda^{-1}(\hat{\pi}^{-1}(x)).$$

Let $\mathscr{C}$ be a sequence

$$\mathscr{C} = \{U_{\lambda_1}, U_{\lambda_2}, ..., U_{\lambda_m}\}$$

of distinguished charts $(U_{\lambda_i}, \varphi_{\lambda_i})$, $i = 1,2,...,m$. Let x belong to a plaque Q_1 of U_{λ_1}, and let L_α be a leaf containing x. If there exists plaques Q_i of U_{λ_i}, $i = 2,3,...,m$ such that

$$Q_i \cap Q_{i+1} \neq \emptyset, \quad i = 1,2,...,m-1,$$

we call $\mathscr{C}$ a *chain at* x (see the Figure A.2) and the Q_i, $i = 1,2,...,m$ a *plaque chain associated with* $\mathscr{C}$. From the definition it is evident that

$$Q_i \subset L_\alpha, \quad i = 1,2,...,m.$$

We say that m is the *length* of the chain $\mathscr{C}$.

We recall now some topological properties of the foliation $\mathscr{F}$ of M^n.

Lemma A.1.1. *[49] Let $\mathscr{C} = \{U_{\lambda_1}, U_{\lambda_2}, ..., U_{\lambda_m}\}$ be a chain at x. Denote by O_1 the set of all points $z \in U_{\lambda_1}$ such that $\mathscr{C}$ forms a chain at each z. Then the set O_1 is open in U_{λ_1} and it is a union of plaques of U_{λ_1}.*

Lemma A.1.2. *[49] Let L_α be a leaf and let x and y be points in L_α. Then there exists a chain $\mathscr{C} = \{U_{\lambda_1}, U_{\lambda_2}, ..., U_{\lambda_m}\}$ at x which has the following properties.*

1. $x \in U_{\lambda_1}$, $y \in U_{\lambda_m}$.
2. *Suppose that $\mathscr{C}$ also forms a chain at a point z of U_{λ_1} and that z is on a leaf L_β. Let Q'_i be a plaque chain associated with $\mathscr{C}$ at z such that*

$$Q'_i \subset U_{\lambda_i} \cap L_\beta, \quad z \in Q'_1, \quad i = 1, 2, ..., m.$$

Then Q'_{i+1} is the only plaque of $U_{\lambda_{i+1}}$ which intersects Q'_i, and Q'_i is the only plaque of U_{λ_i} which intersects Q'_{i+1}, $i = 1, 2, ..., m-1$.

Two additional results complete the issue.

Theorem A.1.3. *[49] Let x and y be points on a leaf L_α. Given a distinguished chart (U_ν, φ_ν) about y, there exists a distinguished chart (U_μ, φ_μ) about x such that for any leaf L_β with $L_\beta \cap U_\mu \neq \emptyset$ we have $L_\beta \cap U_\nu \neq \emptyset$. Further, U_μ can be chosen in such a way that if U_μ has a plaque Q'_1 with $Q'_1 \subset L_\beta$ and $x \notin Q'_1$, then L_β contains a plaque of U_ν not containing y.*

Theorem A.1.4. *[49] Let O be an open set in M^n. Then the union of all leaves which intersect O :*

$$Y = \bigcup_{L_\alpha \cap O \neq \emptyset} L_\alpha,$$

is open in M^n.

A.2 Coherent charts system.

In order to describe the behavior of the foliation in a neighborhood of one of its leaves, it turns out that we must construct a system of distinguished charts with certain properties. The following theorem gives one way to do it.

Theorem A.2.1. *Let K be a compact subset of a leaf $\hat{L}_\alpha$. Then there exists a family $\mathfrak{R}(K) = \{(U_i, \varphi_i) : i = 1, 2, ..., v\}$ of distinguished charts (U_i, φ_i) $i = 1, 2, ..., v$, with the following properties:*

1. $K \subset \cup_{i=1}^{v} U_i$
2. *For each U_i, $U_i \cap K$ is contained in a plaque Q_i of U_i :*

$$U_i \cap K \subset Q_i, \quad i = 1, 2, ..., v.$$

3. *If $U_i \cap U_j \neq \emptyset$, then $U_i \cap U_j \cap K \neq \emptyset$.*
4. *For U_i, U_j such that $U_i \cap U_j \neq \emptyset$, there exists a distinguished chart (U_{ij}, φ_{ij}) which satisfies:*
   (a) $U_i \cup U_j \subset U_{ij}$.
   (b) *Let Q_{ij} be a plaque of U_{ij}. Then $Q_{ij} \cap U_i$ is a plaque of U_i if $Q_{ij} \cap U_i \neq \emptyset$, and $Q_{ij} \cap U_j$ is a plaque of U_j if $Q_{ij} \cap U_j \neq \emptyset$.*
5. *Let (U_{ij}, φ_{ij}), (U_{kl}, φ_{kl}) be distinguished charts specified in the point above, with $U_{ij} \cap U_{kl} \neq \emptyset$. Then we can select a distinguished chart $(U_{ijkl}, \varphi_{ijkl})$ satisfying the following properties:*
   (a) $U_{ij} \cup U_{kl} \subset U_{ijkl}$.
   (b) *Let Q_{ijkl} be a plaque of U_{ijkl}. Then $Q_{ijkl} \cap U_{ij}$ is a plaque of U_{ij} if $Q_{ijkl} \cap U_{ij} \neq \emptyset$, and $Q_{ijkl} \cap U_{kl}$ is a plaque of U_{kl} if $Q_{ijkl} \cap U_{kl} \neq \emptyset$.*

The family $\mathfrak{R}(K)$ is called a *coherent chart-system on* K. Notice that, by definition, charts in a coherent system are distinguished.

In what follows, we assume that K is a compact subset of a leaf $L_{\hat{\alpha}}$ and that $\mathfrak{R}(K)$ is a coherent system of charts on K. For $x \in K$, a chain $\mathscr{C} = \{U_{\lambda_1}, U_{\lambda_2}, ..., U_{\lambda_{m'}}\}$ at x is called a *coherent chain at* x if the distinguished charts $(U_{\lambda_i}, \varphi_{\lambda_i})$, $i = 1, 2, ..., m'$, belong to $\mathfrak{R}(K)$.

Lemma A.2.2. *[49] Let $\mathscr{C}$ be a coherent chain at x. Suppose that $\mathscr{C}$ also forms a chain at a point $z \in U_{\lambda_1}$. Then a plaque chain*

$$Q_i', \quad Q_i' \subset U_{\lambda_i}, \quad i = 1, 2, ..., m', \quad z \in Q_1',$$

associated with $\mathscr{C}$ at z is uniquely determined by z.

Definition A.2.3. *[49] Let $l : [0,1] \to K$ be a C^0 curve in K. By a* coherent chain over l *we mean a coherent chain $\mathscr{C} = \{U_{\lambda_1}, U_{\lambda_2}, ..., U_{\lambda_{m'}}\}$ at $l(0)$, for which there exists numbers t_i, $i = 0, 1, 2, ..., m'$:*

$$0 = t_0 < t_1 < ... < t_{m'} = 1$$

such that

$$l([t_{i-1}, t_i]) \subset U_{\lambda_i}, \quad i = 1, 2, ..., m'.$$

It is obvious that given a C^0 curve l in K, one can find a coherent chain $\mathscr{C}$ with $l(0) \in U_{\lambda_1}$ and $l(1) \in U_{\lambda_{m'}}$ (see the picture below).

Lemma A.2.4. *[49] Let $\mathscr{C} = \{U_{i_1}, U_{i_2}, ..., U_{i_{m'}}\}$, $\mathscr{C}' = \{U_{j_1}, U_{j_2}, ..., U_{j_{m''}}\}$ be coherent chains over l such that $U_{i_1} = U_{j_1}$ and $U_{i_{m'}} = U_{j_{m''}}$. Suppose that $m', m'' \leq m$, and let $z \in U_{i_1}$ be an admissible point of the coherent chains at $l(0)$ of length at*

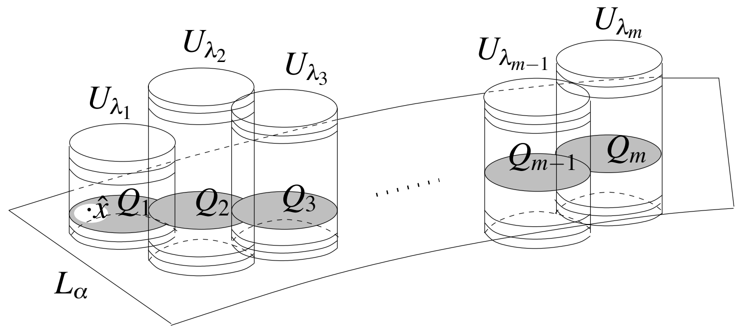


Figure A.2: A coherent chain on a C^0 path.

most m. Consider $\mathscr{C}$ and $\mathscr{C}'$ as chains at z, and let their respective plaque chains be

$$Q_i',\ \ i = 1,2,...,m', \quad Q_j'',\ \ j = 1,2,...,m'', \quad Q_1' = Q_1''.$$

Then $Q_{m'}' = Q_{m''}''$.

We now consider two C^0 curves in K :

$$l_0 : [0,1] \to K, \quad l_1 : [0,1] \to K,$$

with $x = l_0(0) = l_1(0)$, $y = l_0(1) = l_1(1)$, which are homotopic, $l_0 \sim l_1$, relative to x and y; in other words, there is a family of C^0 curves

$$l_s : [0,1] \to K, \quad 0 \le s \le 1,$$

such that $l_s(0) = x$, $l_s(1) = y$, and the change of l_s is continuous in parameter s, $0 \le s \le 1$.

Now suppose $\mathscr{C} = \{U_{i_1}, U_{i_2}, ..., U_{i_{m'}}\}$, is a coherent chain over l_0 and $\mathscr{C}' = \{U_{j_1}, U_{j_2}, ..., U_{j_{m''}}\}$ a coherent chain over l_1, and assume that

$$U_{i_1} = U_{j_1}, \quad U_{i'_m} = U_{j_{m''}}.$$

Clearly if $\varepsilon > 0$ is small enough, $\mathscr{C}$ is a coherent chain over l_s, $0 \leq s \leq \varepsilon$. Hence, we can find a suitable subdivision $0 = s_0 < s_1 < ... < s_u = 1$ and coherent chains at x :

$$\mathscr{C}^{(k)} = \{U_1^{(k)}, U_2^{(k)}, ..., U_{m_k}^{(k)}\}, \quad k = 0, 1, 2, ..., u-1,$$

such that

$$\mathscr{C}^{(0)} = \mathscr{C}, \quad \mathscr{C}^{(u-1)} = \mathscr{C}', \quad U_1^{(k)} = U_{i_1}, \quad U_{m_k}^{(k)} = U_{i_{m'}},$$

and each $\mathscr{C}^{(k)}$ is a coherent chain over l_s for s, $s_k \leq s \leq s_{k+i}$. Such chains $\mathscr{C}^{(k)}$, $k = 0, 1, 2, ..., u-1$ are said to be a *homotopy* between $\mathscr{C}$ and $\mathscr{C}'$ and $\max_k m_k$ the *length* of the homotopy.

A.3 The holonomy.

In the following we denote by $\mathrm{G}_\mathrm{r}^\mathrm{q}$ the set of all germs of C^r diffeomorphisms of $\mathbb{R}^q$ at the origin fixing the origin and by $[f]$ the equivalence class of f. Let $[f]$, $[g] \in \mathrm{G}_\mathrm{r}^\mathrm{q}$ be represented by $f : U \to \mathbb{R}^q$, $g : U' \to \mathbb{R}^q$ respectively. Define a C^r map

$$g \circ f : f^{-1}(f(U) \cap U') \to \mathbb{R}^q$$

by $g \circ f(x) = g(f(x))$. Then $g \circ f$ is a local diffeomorphism of $\mathbb{R}^q$ at the origin. Define the product $[f] \cdot [g]$ of both germs by

$$[f] \cdot [g] = [g \circ f].$$

Then the germ $[f] \cdot [g]$ is determined by $[f]$ and $[g]$, and is independent of the choice of their representatives. With this product, $\mathrm{G}_\mathrm{r}^\mathrm{q}$ becomes a group: the unit element of $\mathrm{G}_\mathrm{r}^\mathrm{q}$ is $[id]$, where $id : \mathbb{R}^q \to \mathbb{R}^q$ is the identity map. The inverse of $[f]$

is represented by $f^{-1} : f(U) \to U$, where $f : U \to \mathbb{R}^q$ is a representative of $[f]$; moreover, $[f]^{-1} = [f^{-1}]$.

Let M^n be an $n-$dimensional C^s manifold and let $\mathscr{F} = \{L_\alpha : \alpha \in A\}$ be a C^r codimension q foliation of M^n. Fix a point $\hat{x}$ on a leaf L_α and consider the fundamental group $\pi_1(L_\alpha, \hat{x})$ of L_α based at $\hat{x}$.

Let γ be an element of $\pi_1(L_\alpha, \hat{x})$ represented by a C^0 curve

$$\omega : [0,1] \to L_\alpha, \quad \omega(0) = \omega(1) = \hat{x}$$

so that $\gamma = \{\omega\}$. Take a compact subset K of L_α with $\omega([0,1]) \subset K$, and consider a coherent system $\Re(K) = \{(U_i, \varphi_i) : i = 1,2,...,v\}$ of charts on K. Choose a coherent chain $\mathscr{C}$ over ω (*i.e.*, a chain for the coherent charts above:

$$\mathscr{C} = \{U_{\lambda_1}, U_{\lambda_2}, ..., U_{\lambda_{m-1}}, U_{\lambda_1}\}, \quad \hat{x} \in U_{\lambda_1}.$$

By the Lemma A.1.1, the set $\hat{O}$ of all $z \in U_{\lambda_1}$ such that $\mathscr{C}$ is a chain at z is open in U_{λ_1} containing Q_1, $U_{\lambda_1} \cap K \subset Q_1$, and it is a union of plaques of the chart U_{λ_1}. By the Lemma A.2.2, the plaque chain associated with $\mathscr{C}$ at z:

$$Q_i^{(z)}, \quad i = 1,2,...,m, \quad z \in Q_1^{(z)}, \quad Q_1^{(z)}, Q_m^{(z)} \subset U_{\lambda_1},$$

is uniquely determined by z. Define a map $f : \hat{\pi} \circ \varphi_{\lambda_1}(\hat{O}) \to \mathbb{R}^q$ by

$$f(\hat{\pi} \circ \varphi_{\lambda_1}(Q_1^{(z)})) = \hat{\pi} \circ \varphi_{\lambda_1}(Q_m^{(z)})), \quad z \in \hat{O},$$

where $\hat{\pi} : (-1,1)^n \to (-1,1)^q$ is the projection map (see Figure A.1). Then the map f is a local C^r diffeomorphism of $\mathbb{R}^q$ at the origin.

Let $\mathscr{C}' = \mathscr{C} = \{U_{\mu_1}, U_{\mu_2}, ..., U_{\mu_{m'-1}}, U_{\mu_1}\}$ be a coherent chain over ω different from $\mathscr{C}$ with $\hat{x} \in U_{\mu_1}$, and define the local diffeomorphism f' of $\mathbb{R}^q$ as we defined f above. Now define $h : \hat{\pi} \circ \varphi_{\lambda_1}(U_{\lambda_1} \cap U_{\mu_1}) \to \mathbb{R}^q$ by

$$h(\hat{\pi} \circ \varphi_{\lambda_1}(Q_1' \cap U_{\lambda_1})) = \hat{\pi} \circ \varphi_{\mu_1}(Q_1' \cap U_{\mu_1}),$$

where Q_1' is a plaque. Then h is a local diffeomorphism of $\mathbb{R}^n$ at the origin satisfying the identity

$$[f] = [h][f'][h]^{-1}. \tag{A.3.1}$$

Hence it follows that ω determines an element of $\mathrm{G}_{\mathrm{r}}^{\mathrm{q}}$ uniquely up to the inner automorphisms of $\mathrm{G}_{\mathrm{r}}^{\mathrm{q}}$.

Next we pick another $\mathscr{C}^0$ curve

$$\bar{\omega} : [0,1] \to L_\alpha, \quad \bar{\omega}(0) = \bar{\omega}(1) = \hat{x},$$

such that $\gamma = \{\omega\} = \{\bar{\omega}\}$ $(i.e.\ \gamma$ is parametrized either by ω and $\bar{\omega})$. Choose K so that K contains a homotopy between the curves ω and $\bar{\omega}$. Consider a coherent chain over $\bar{\omega}$:

$$\overline{\mathscr{C}} = \{U_{\lambda_1}, U_{\lambda_2'}, \ldots, U_{\lambda'_{m''-1}}, U_{\lambda_1}\},$$

and let $\bar{f}$ be the local diffeomorphism of $\mathbb{R}^n$ defined in the same manner as the map f above. We have $[f] = [\bar{f}]$; so for each element γ of $\pi_1(L_\alpha, \hat{x})$ there is an element of $\mathrm{G}_{\mathrm{r}}^{\mathrm{q}}$ uniquely defined up to the inner automorphisms. If we write $\Psi(\{w\}) = [f]$ for this correspondence, we have a map

$$\Psi : \pi_1(L_\alpha, \hat{x}) \to \mathrm{G}_{\mathrm{r}}^{\mathrm{q}}.$$

It is immediate from the definition of f that Ψ is a homomorphism. To see this, fix a U_{λ_1} with $\hat{x} \in U_{\lambda_1}$ and always take a coherent chain over the path ω starting from and ending in U_{λ_1}. For a change of the base point $\hat{x}$ the change in Ψ is by inner automorphisms of $\mathrm{G}_{\mathrm{r}}^{\mathrm{q}}$: we have the well-defined automorphism $\Psi : \pi_1(L_\alpha, \hat{x}) \to \mathrm{G}_{\mathrm{r}}^{\mathrm{q}}$ up to the inner automorphisms of the group $\mathrm{G}_{\mathrm{r}}^{\mathrm{q}}$.

Definition A.3.1. *[49] The homomorphism* Ψ *is called the* holonomy *of the leaf* L_α*, and the subgroup* $\Psi(\pi_1(L_\alpha, \hat{x}))$ *of* $\mathrm{G}_\mathrm{r}^\mathrm{q}$ *is called the* holonomy group *of* L_α.

The holonomy group of a leaf L_α is finitely generated, provided that its fundamental group is finitely generated.

Appendix B

Holomorphic Integral Manifolds

Let us consider a germ of holomorphic vector field v around the origin of coordinates in $\mathbb{C}^2$, such that its linear part is given by

$$J_0^1 v = \lambda_1 z \frac{\partial}{\partial z} + \lambda_2 w \frac{\partial}{\partial w}.$$

We study the case $\lambda_1 \lambda_2 \neq 0$. In order to lighten the process, consider v as a holomorphic differential equation

$$\begin{aligned} \frac{dz}{dt} &= \lambda_1 z + \boldsymbol{\varphi}_1(z,w) \\ \frac{dw}{dt} &= \lambda_2 w + \boldsymbol{\varphi}_2(z,w), \end{aligned} \tag{B.0.1}$$

where

$$\boldsymbol{\varphi}_k(z,w) = \sum_{i,j \geq 1} \boldsymbol{\varphi}_{ij}^k z^i w^j, \qquad k = 1,2$$

are convergent series. Let us take the formal change of coordinates

$$\begin{aligned} z &= \mathbf{z} + \zeta_1(\mathbf{z},\mathbf{w}) \\ w &= \mathbf{w} + \zeta_2(\mathbf{z},\mathbf{w}) \end{aligned} \tag{B.0.2}$$

where $\zeta_k(\mathbf{z},\mathbf{w}) = \sum_{i,j\geq 1} \zeta_{ij}^k \mathbf{z}^i \mathbf{w}^j$ for $k = 1,2$ and let

$$\begin{aligned} \frac{d\mathbf{z}}{dt} &= \lambda_1 \mathbf{z} + \boldsymbol{\psi}_1(\mathbf{z},\mathbf{w}) \\ \frac{d\mathbf{w}}{dt} &= \lambda_2 \mathbf{w} + \boldsymbol{\psi}_2(\mathbf{z},\mathbf{w}), \end{aligned} \tag{B.0.3}$$

stand for the system (B.0.1) under the change above. It is easily seen that changing the system (B.0.1) into new coordinates (B.0.2) leads to

$$\sum_{i,j\geq 1} (\delta_{ij}^k \zeta_{ij}^k + \boldsymbol{\psi}_{ij}^k) \mathbf{z}^i \mathbf{w}^j = \boldsymbol{\varphi}_k(\mathbf{z}+\zeta_1, \mathbf{w}+\zeta_2) - \frac{\partial \zeta_k}{\partial \mathbf{z}} \boldsymbol{\psi}_1 - \frac{\partial \zeta_k}{\partial \mathbf{w}} \boldsymbol{\psi}_2, \tag{B.0.4}$$

where

$$\delta_{ij}^k = i\lambda_1 + j\lambda_2 - \lambda_k.$$

In fact, finding a formal change of coordinates bringing (B.0.1) into (B.0.3) is equivalent to find series (B.0.2) verifying (B.0.4). The coefficient on the right hand side of (B.0.4) is a polynomial in the variables $\zeta_{ij}^k, \boldsymbol{\varphi}_{ij}^k$, with $k = 1,2$. It is always possible then to formally reduce the field v into a field (B.0.3) such that

$$\boldsymbol{\varphi}_{ij}^k = 0, \quad \text{if} \quad \delta_{ij}^k \neq 0, \quad k = 1,2, \tag{B.0.5}$$

which is called the *normal form of v*. For instance, if the ratio $\frac{\lambda_1}{\lambda_2} \in \mathbf{Q}_- \backslash 0$

$$\frac{\lambda_1}{\lambda_2} = -\frac{p}{q}$$

such that $g.c.d.(p,q) = 1$, the field v can be written in formal coordinates $(\mathbf{z},\mathbf{w})$ as

$$v = A(\mathbf{z}^p \mathbf{w}^q)\mathbf{z}\frac{\partial}{\partial \mathbf{z}} + B(\mathbf{z}^p \mathbf{w}^q)\mathbf{w}\frac{\partial}{\partial \mathbf{w}}, \tag{B.0.6}$$

where A and B are power series of one variable. These transformations are not convergent in general.

Real character. In terms of the field (B.0.6), the real character (Definition 3.1.3) means

$$\overline{A(\overline{\mathbf{z}^p\mathbf{w}^q})} = B(\mathbf{z}^p\mathbf{w}^q).$$

From now on, we suppose that the field v has real character.

Proposition B.0.2. *Let us suppose that λ_1, λ_2 are non zero complex numbers, such that the ratios $\frac{\lambda_1}{\lambda_2}$ and $\frac{\lambda_2}{\lambda_1}$ are not integer numbers strictly greater than 1. Then there exists analytic coordinates $(\mathbf{z}, \mathbf{w})$ transforming the field v into*

$$v = \lambda_1 \mathbf{z}(1+\ldots)\frac{\partial}{\partial \mathbf{z}} + \lambda_2 \mathbf{w}(1+\ldots)\frac{\partial}{\partial \mathbf{w}}. \tag{B.0.7}$$

and respecting the real character of the field. The equations $\mathbf{z} = 0$ and $\mathbf{w} = 0$ are the invariant manifolds of the field v.

Proof. By Hadamard-Perron Theorem for holomorphic flows ([27], Theorem 7.1) the system has invariant manifolds $z = \kappa_1(w)$ and $w = \kappa_2(z)$ which are tangent to the invariant subspaces and, due to the real character of the field,

$$\kappa_1(w) = \kappa_2(\overline{w}).$$

The change of coordinates

$$\begin{cases} \mathbf{z} &= z - \kappa_1(w) \\ \mathbf{w} &= w - \kappa_2(z) \end{cases}$$

brings the system into the desired form (B.0.7). □

Theorem B.0.3. *Under the hypotheses of the above proposition, there exists analytical coordinates* $(\mathbf{Z},\mathbf{W})$ *bringing the field v into*

$$v=\lambda_1\mathbf{Z}(1+\mathbf{ZW}(...))\frac{\partial}{\partial\mathbf{Z}}+\lambda_2\mathbf{W}(1+\mathbf{ZW}(...))\frac{\partial}{\partial\mathbf{W}}, \tag{B.0.8}$$

and respecting the real character of the singularity.

Proof. Proposition B.0.2 allows to write the system in the linearizing Poincaré coordinates as

$$\begin{aligned}\dot{\mathbf{z}} &= \lambda_1\mathbf{z}(1+A_1(\mathbf{z},\mathbf{w}))\\ \dot{\mathbf{w}} &= -\lambda_2\mathbf{w}(1+A_2(\mathbf{z},\mathbf{w}))\end{aligned} \tag{B.0.9}$$

where $A_1(\mathbf{z},\mathbf{w})=\mathbf{zw}b_1(\mathbf{z},\mathbf{w})+\mathbf{w}c_1(\mathbf{w})$ and $A_2(\mathbf{z},\mathbf{w})=\mathbf{zw}b_2(\mathbf{z},\mathbf{w})+\mathbf{z}c_2(\mathbf{z})$. In order to annihilate both c_1,c_2 one takes a change of the form

$$\begin{aligned}\mathbf{z} &= \mathbf{Z}(1+\kappa_1(\mathbf{W}))\\ \mathbf{w} &= \mathbf{W}(1+\kappa_2(\mathbf{Z}))\end{aligned}$$

for unkown functions $\kappa_1(\mathbf{Z}),\kappa_2(\mathbf{W})$. This yields the system in the new variables

$$\begin{aligned}\dot{\mathbf{Z}} &= \lambda_1\mathbf{Z}(1+O(\mathbf{ZW}))\\ \dot{\mathbf{W}} &= -\lambda_2\mathbf{W}(1+O(\mathbf{ZW})).\end{aligned}$$

Thus, on the one hand,

$$\begin{aligned}\dot{\mathbf{z}} &= \dot{\mathbf{Z}}(1+\kappa_1(\mathbf{W}))+\mathbf{Z}\kappa_1'(\mathbf{W})\dot{\mathbf{W}}\\ &= \lambda_1\mathbf{Z}(1+O(\mathbf{ZW}))(1+\kappa_1(\mathbf{W}))-\lambda_2\mathbf{WZ}\kappa_1'(\mathbf{W})(1+O(\mathbf{ZW})),\end{aligned} \tag{B.0.10}$$

while, on the other hand

$$\begin{aligned}\dot{\mathbf{z}} = {} & \lambda_1\mathbf{Z}(1+\kappa_1(\mathbf{W}))\{1+\mathbf{ZW}(1+\kappa_1(\mathbf{W}))(1+\kappa_2(\mathbf{Z}))b_1(\mathbf{Z}+...,\mathbf{W}+...)\\ & +\mathbf{W}(1+\kappa_2(\mathbf{Z}))c_1(\mathbf{W}+...)\}.\end{aligned} \tag{B.0.11}$$

Identifying (B.0.10) with (B.0.11) and simplifying terms in Z, we get the linear equation

$$\kappa_1' + \frac{\lambda_1}{\lambda_2}\kappa_1 c_1 + \frac{\lambda_1}{\lambda_2} c_1 = 0,$$

with analytic solution

$$\kappa_1(\mathbf{W}) = -\frac{\lambda_1}{\lambda_2} e^{-\frac{\lambda_1}{\lambda_2}\int_0^{\mathbf{W}} c_1(\mu)d\mu} \int_0^{\mathbf{W}} c_2(\mu) e^{\frac{\lambda_1}{\lambda_2}\int_0^{\mu} c_2(\nu)d\nu} d\mu.$$

The same procedure in the **w** coordinate yields the solution

$$\kappa_2(\mathbf{Z}) = -\frac{\lambda_2}{\lambda_1} e^{-\frac{\lambda_2}{\lambda_1}\int_0^{\mathbf{Z}} c_2(\mu)d\mu} \int_0^{\mathbf{Z}} c_1(\mu) e^{\frac{\lambda_2}{\lambda_1}\int_0^{\mu} c_1(\nu)d\nu} d\mu.$$

By definition, the new equation in the $(\mathbf{Z},\mathbf{W})$ coordinates has real character as well. □

Appendix C

Monodromy and formal aspects

C.1 Holonomy of the unfolding of a saddle-point.

Theorem C.1.1. *Let $\lambda_\varepsilon \in \mathbb{C}^*$ be a family of complex numbers such that $\lambda_0 \in \mathbb{R}_-^*$, and ε belongs to a symmetric neighborhood V of the origin. For each $\varepsilon \in V$, suppose that the differential equation*

$$\frac{dw}{dz} = \lambda_\varepsilon \frac{w}{z}(1 + zw\mathbf{A}_\varepsilon(z,w)) \tag{C.1.1}$$

is defined on the product $\mathbb{D}_{r_1} \times \mathbb{D}_{r_2}$, where $0 < r_1 \leq +\infty$, $0 < r_2 \leq 1$ and $\mathbf{A}_\varepsilon(0,0) \neq 0$. Then, for all $z_0, z_1 \in \mathbb{D}_{r_1}$ fixed, and such that the straight path $[z_0, z_1]$ contains no singular point of the system (C.1.1)*, the holonomy map $h_{z_0}(w) : \{z = z_0\} \times \mathbb{D}_{r_2} \to \{z = z_1\} \times \mathbb{C}$ of the separatrix $w = 0$ verifies*

$$e^{-r_2 M|\lambda_\varepsilon(z_1 - z_0)|}\left|\left(\frac{z_1}{z_0}\right)^{\lambda_\varepsilon} w\right| \leq |h_{z_0}(w)| \leq e^{r_2 M|\lambda_\varepsilon(z_1 - z_0)|}\left|\left(\frac{z_1}{z_0}\right)^{\lambda_\varepsilon} w\right|, \tag{C.1.2}$$

where $M = M(z_0, z_1) := \max\{|\mathbf{A}_\varepsilon(z,w)| : (\varepsilon, z, w) \in adh(V \times [z_0, z_1] \times \mathbb{D}_{r_2})\}$.

Proof. Let γ be the path with initial point $z_0 \in \mathbb{D}_{r_1}$ and ending at $z_1 \in \mathbb{D}_{r_1}$. A simple calculation shows that the holonomy mapping of the horizontal separatrix of the linear system

$$\frac{dw}{dz} = \lambda_\varepsilon \frac{w}{z}$$

along γ is given by $\mathbf{h}_{z_0}^{lin} : \{z = z_0\} \times \mathbb{D}_{r_2} \to \{z = z_1\} \times \mathbb{C}$,

$$\mathbf{h}_{z_0}^{lin} : w \mapsto \left(\frac{z_1}{z_0}\right)^{\lambda_\varepsilon} w. \tag{C.1.3}$$

Define the coordinate

$$\mathbf{u}(z) = \left(\frac{z_0}{z}\right)^{\lambda_\varepsilon} h_{-z+z_0+z_1}(w_0),$$

where $h_{\hat{z}}(w_0) : \{z = \hat{z}\} \times \{w = w_0\} \to \{z = z_1\} \times \mathbb{C}$ is the holonomy of the point $w_0 \in \{z = \hat{z}\}$ along the leaves of the foliation induced by (C.1.1) and with image on the section $\{z = z_1\}$, for a given $\hat{z} \in [z_0, z_1]$. Notice that

$$\mathbf{u}(z_0) = h_{z_1}(w_0) = w_0 \tag{C.1.4}$$

and

$$\mathbf{u}(z_1) = \left(\frac{z_0}{z_1}\right)^{\lambda_\varepsilon} h_{z_0}(w_0). \tag{C.1.5}$$

Put $w = h_{-z+z_0+z_1}(w_0)$ and compute the variation of $\mathbf{u}$ along the leaves of the foliation Ω_ε :

$$\begin{aligned} d\mathbf{u} &= \mathbf{u}'(z)dz \\ &= \left[-\lambda_\varepsilon z^{-\lambda_\varepsilon - 1} z_0^{\lambda_\varepsilon} w + \left(\frac{z_0}{z}\right)^{\lambda_\varepsilon} \frac{dw}{dz}\right] dz \\ &= \left[-\lambda_\varepsilon \frac{\mathbf{u}}{z} + \frac{\mathbf{u}}{w}\frac{dw}{dz}\right] dz. \end{aligned}$$

This yields

$$\frac{d\mathbf{u}}{\mathbf{u}} = \left[-\frac{\lambda_\varepsilon}{z} + \frac{1}{w}\frac{dw}{dz}\right] dz. \tag{C.1.6}$$

But then, (C.1.1) implies

$$\begin{aligned} \frac{d\mathbf{u}}{\mathbf{u}} &= \left[-\frac{\lambda_\varepsilon}{z} + \frac{\lambda_\varepsilon}{z} + \lambda_\varepsilon w \mathbf{A}_\varepsilon(z,w)\right] dz \\ &= \lambda_\varepsilon w \mathbf{A}_\varepsilon(z,w) dz. \end{aligned} \tag{C.1.7}$$

Consider the number $M = M(z_0, z_1) := \max\{|\mathbf{A}_\varepsilon(z,w)| : (\varepsilon, z, w) \in adh(V \times [z_0, z_1] \times \mathbb{D}_{r_2})\} < \infty$. Equation (C.1.7) yields

$$\left|\frac{d\mathbf{u}}{\mathbf{u}}\right| \leq r_2 M |\lambda_\varepsilon||dz|,$$

and then, integrating over the interval $[z_0, z_1]$ we obtain:

$$|\log \mathbf{u}(z_1) - \log \mathbf{u}(z_0)| \leq r_2 M |\lambda_\varepsilon||z_1 - z_0|.$$

By (C.1.4) and (C.1.5):

$$\log\frac{\mathbf{u}(z_1)}{\mathbf{u}(z_0)} = \log\frac{\left(\frac{z_0}{z_1}\right)^{\lambda_\varepsilon} h_{z_0}(w_0)}{w_0} = \log\frac{h_{z_0}(w_0)}{\left(\frac{z_1}{z_0}\right)^{\lambda_\varepsilon} w_0} = \log\frac{h_{z_0}(w_0)}{\mathbf{h}_{z_0}^{lin}(w_0)}.$$

This yields the following estimate:

$$0 < e^{-r_2 M|\lambda_\varepsilon(z_1 - z_0)|} \leq \left|\frac{h_{z_0}(w_0)}{\mathbf{h}_{z_0}^{lin}(w_0)}\right| \leq e^{r_2 M|\lambda_\varepsilon(z_1 - z_0)|} < \infty,$$

where $\mathbf{h}_{z_0}^{lin}(w_0)$ is given in (C.1.3). □

If the points z_0, z_1 coincide, the holonomy $h_{z_0}(w) : \{z = z_0\} \times \mathbb{D}_{r_2} \to \{z = z_0\} \times \mathbb{C}$ of the horizontal separatrix is usually called the semi-monodromy of the section $\{z = z_0\}$ and is denoted $\mathscr{Q}$.

Proposition C.1.2. *The holonomy (semi-monodromy) map* $\mathcal{Q}_\varepsilon : \{z=1\} \times \mathbb{D}_r \to \{z=1\} \times \mathbb{C}$ *of the horizontal separatrix* $w=0$ *of the vector field*

$$v = z(1+zw(...))\frac{\partial}{\partial z} + \lambda_\varepsilon w(1+zw(...))\frac{\partial}{\partial w} \tag{C.1.8}$$

has the form:

$$\mathcal{Q}_\varepsilon(w) = e^{2i\pi\lambda_\varepsilon} w + ... \tag{C.1.9}$$

Proof. Indeed, the system associated with (C.1.8) can be written as:

$$\begin{aligned} \dot{z} &= z \\ \dot{w} &= \lambda_\varepsilon w(1+zw(...)). \end{aligned} \tag{C.1.10}$$

Consider the loop $\gamma : (z,w) = (e^{2i\pi\theta}, 0)$ on the z-separatrix. Then if $t(\theta) = 2i\pi\theta$ for $\theta \in [0,1]$, Equation (C.1.10) yields:

$$\frac{\partial w}{\partial \theta} = 2i\pi\lambda_\varepsilon w(1+e^{2i\pi\theta} w(...)). \tag{C.1.11}$$

Let $w(\theta) = \sum_{k \geq 0} c_k(\theta) w_0^k$ be the lifting of the loop γ on the leaf of the foliation defined by (C.1.10), passing through an initial condition $w_0 \in \{z=1\}$. Then (C.1.11) leads to:

$$\sum_{k\geq 0} c_k'(\theta) w_0^k = 2i\pi\lambda_\varepsilon \sum_{k\geq 0} c_k(\theta) w_0^k [1+e^{2i\pi\theta} w(...)]$$

and then, comparing former coefficients, we get

$$c_0'(\theta) = 2i\pi\lambda_\varepsilon c_0(\theta), \quad c_1'(\theta) = 2i\pi\lambda_\varepsilon c_1(\theta).$$

Solving these equations with initial conditions $c_0(0) = 0$ and $c_1(0) = 1$, respectively, yields $c_0(\theta) \equiv 0$ and $c_1(\theta) = e^{2i\pi\lambda_\varepsilon\theta}$. Thus, this defines the semi-monodromy map when $\theta = 1$:

$$\mathcal{Q}_\varepsilon(w_0) = w(\theta = 1) = e^{2i\pi\lambda_\varepsilon} w_0 + ...$$

□

C.2 Orbital formal normal forms.

A generic family unfolding a weak focus is a generic family unfolding a complex resonant saddle point with ratio of eigenvalues equal to -1. In the blow up space, such a family gives birth to two families around the poles of the Riemann sphere unfolding complex saddle-points both with ratio of eigenvalues equal to -2. The orbital formal normal form of a family unfolding a saddle point is either linear or rational. For instance, in the first direction of the blow up (with coordinates (Z,w), see Chapter 3) the formal normal form of the family is given by

$$v_{0,\varepsilon}^{c_1} = Z\frac{\partial}{\partial Z} + w(-\lambda(\varepsilon) + q_\varepsilon(u))\frac{\partial}{\partial w} \tag{C.2.1}$$

with $\lambda(\varepsilon) = \dfrac{\varepsilon - i}{2i}$, $u = Zw^2$ and

$$q_\varepsilon(u) = \frac{u^p - \varepsilon}{1 + \alpha(\varepsilon)u^p} \tag{C.2.2}$$

for a certain $p \in \mathbb{N}$ and $\alpha(\varepsilon) \in \mathbb{C}$.

Definition C.2.1. *For ε in a symmetric neighborhood V of the origin in the parameter space let*

1. *$\mathscr{B}_{p,\lambda(\varepsilon),\alpha(\varepsilon)}$ denote the class of all germs of families unfolding complex resonant saddles with the same formal normal form* (C.2.1)*, and same canonical parameter ε.*

2. *$\mathscr{A}_{p,\lambda(\varepsilon),\beta(\varepsilon)}$ denote the class of families of germs of conformal diffeomorphisms with multiplier $e^{2i\pi\lambda(\varepsilon)}$ (where $\lambda(0) = \dfrac{m}{n}$), unfolding conformal germs with multiplier equal to $e^{2i\pi\frac{m}{n}}$ and whose n^{th} iteration is formally equivalent to the time-one map of the field*

$$v_{\varepsilon,p,\beta(\varepsilon)}(w) = \frac{w(w^p - \varepsilon)}{1 + \beta(\varepsilon)w^p}\frac{\partial}{\partial w} \tag{C.2.3}$$

for some $p \in \mathbb{N}$ *and* $\beta(\varepsilon) \in \mathbb{C}$.

Lemma C.2.2. *The semi-monodromy map of the germ of normalized family* (C.2.1) *and* (C.2.2) *belongs to the class* $\mathscr{A}_{2p,-\lambda(\varepsilon),\beta(\varepsilon)}$ *with* $\beta(\varepsilon) = \dfrac{\alpha(\varepsilon)}{2i\pi}$.

Proof. Put $U = u^{\frac{1}{2}} = Z^{\lambda_0} w$. This is a multivalued function whose restriction to the section $\{Z = 1\}$ provides it with a local coordinate. The analytic continuation of such a restriction over the circular loop $\{e^{i\theta} : \theta \in [0, 2\pi]\}$ (noted U again) on the leaf of the foliation defined by the family (C.2.1) passing through the point $(1, w)$, is equal to $e^{2i\pi\lambda_\varepsilon} h_{\varepsilon,Z}$. Here, $h_{\varepsilon,Z} : \{Z = Z\} \times \mathbb{D}_{r_2} \to \{Z = 1\} \times \mathbb{C}$ is the holonomy map of the family $v_{0,\varepsilon}^{c_1}$ with image on the section $\{Z = 1\}$, which is supposed to be defined in a small disk $\mathbb{D}_{r_2}$ around the origin in the w coordinate. On the other hand, a simple calculation (using (C.2.2)) shows that U satisfies

$$\begin{aligned} \dot{U} &= U q_\varepsilon(u) \\ &= U q_\varepsilon(U^2) \\ &= \frac{U(U^{2p} - \varepsilon)}{1 + \alpha(\varepsilon) U^{2p}} \\ &= v_{\varepsilon,2p,\alpha(\varepsilon)}(U), \end{aligned} \tag{C.2.4}$$

whence follows that the analytic continuation of U satisfies

$$e^{2i\pi\lambda_\varepsilon} h_{\varepsilon,Z} = \exp 2i\pi v_{\varepsilon,2p,\alpha(\varepsilon)},$$

where $\exp 2i\pi v_{\varepsilon,2p,\alpha(\varepsilon)}$ is the phase flow transformation of the family of fields $v_{\varepsilon,2p,\alpha(\varepsilon)}$. The rescaling $w \mapsto \dfrac{w}{(2i\pi)^{\frac{1}{2}}}$ brings the family of vector fields $2i\pi v_{\varepsilon,2p,\alpha(\varepsilon)}$ into the family $v_{\varepsilon,2p,\beta(\varepsilon)}$, where $\beta(\varepsilon) = \dfrac{\alpha(\varepsilon)}{2i\pi}$. Therefore $h_{\varepsilon,Z}$, after the rescaling, takes the form

$$h_{\varepsilon,Z} = e^{-2i\pi\lambda_\varepsilon} \exp v_{\varepsilon,2p,\beta(\varepsilon)} \in \mathscr{A}_{2p,-\lambda(\varepsilon),\beta(\varepsilon)}.$$

□

Theorem C.2.3. *Suppose that the ratios of the eigenvalues of two generic families of vector fields $v_\varepsilon, v'_\varepsilon$ (with the same canonical parameter, modulo a reparametrization) unfolding complex resonant saddle vector fields coincide, and that the holonomies of their Z-axis are formally conjugate. Then the families of vector fields are orbitally formally equivalent.*

Proof. The holonomies of the families $v_\varepsilon, v'_\varepsilon$ are either linearizable or belong to some class $\mathscr{A}_{p,\lambda(\varepsilon),\beta(\varepsilon)}$. Assume that the family v_ε belongs to the formal class $\mathscr{B}_{p,\lambda(\varepsilon),\alpha(\varepsilon)}$ and that the family v'_ε belongs to the formal class $\mathscr{B}_{p',\lambda'(\varepsilon),\alpha'(\varepsilon)}$. Then, by Lemma (C.2.2) the holonomies of their orbital formal normal form (C.2.1) belong to the classes $\mathscr{A}_{np,-\lambda(\varepsilon),\beta(\varepsilon)}$ and $\mathscr{A}_{n'p',-\lambda'(\varepsilon),\beta'(\varepsilon)}$, respectively, where $\beta(\varepsilon) = \dfrac{\alpha(\varepsilon)}{2i\pi}$, $\beta'(\varepsilon) = \dfrac{\alpha'(\varepsilon)}{2i\pi}$, and $\lambda(0) = \dfrac{m}{n}$, $\lambda'(0) = \dfrac{m'}{n'}$. It turns out that $\alpha(\varepsilon) = \alpha'(\varepsilon)$, because the two latter classes coincide. By hypothesis, the numbers $\lambda(\varepsilon), \lambda'(\varepsilon)$ (the ratios of eigenvalues) must coincide as well, for all ε in a symmetric neighborhood V of the parameter space. Since $g.c.d.(m,n) = g.c.d.(m',n') = 1$ we get $m = m'$ and $n = n'$. Since $np = n'p'$, the formal invariants p, p' coincide. Thus the families v_ε and v'_ε belong to the same formal class $\mathscr{B}_{p,\lambda(\varepsilon),\alpha(\varepsilon)}$ and hence are formally equivalent. □

Inasmuch as the family of diffeomorphisms unfolding the semi-Poincaré map of a weak focus is the holonomy map of the generic family of vector fields unfolding the resonant saddle point in the (Z,w) direction of the blow up (and the inverse of the holonomy in the (W,z) direction), where the ratio of eigenvalues is equal to -2, the Theorem C.2.3 yields the

Corollary C.2.4. *If the ratios of the eigenvalues of two germs of generic families of analytic vector fields unfolding weak foci coincide, and if the families of diffeomorphisms unfolding their semi-Poincaré maps are formally conjugate, then the families of vector fields are formally orbitally equivalent.*

Bibliography

[1] L. AHLFORS, *Lectures on quasiconformal mappings*, Van Nostrand, 1966.

[2] M. BERTHIER, D. CERVEAU, AND A. LINS NETO, "Sur les feuilletages analytiques réels et le problème du centre", J. Differential Equations, 131, No. 2 (1996), pp. 244-266.

[3] BRIOT-BOUQUET, "Recherches sur les fonctions définies par des équations différentielles", J. École polytechnique, vol. XXI, (1856), pp. 134-198.

[4] C. CAMACHO AND A. LINS NETO, *Geometric Theory of Foliations*, Birkhäuser Boston Inc., 1985.

[5] C. CAMACHO AND P. SAD, "Invariant Varieties Through Singularities of Holomorphic Vector Fields", The Annals of Mathematics, 2nd Ser., Vol. 115, No. 3. (May, 1982), pp. 579-595.

[6] B. CHABAT AND M. LAVRENTIEV, *Méthodes de la théorie des fonctions d'une variable complexe*, Editions MIR, Moscou, 1972.

[7] C. CHICONE, *Ordinary differential equations with applications*, Springer-Verlag Texts in applied mathematics 34, second edition, 2006.

[8] A. CLEBSCH, "Ueber die simultane Integration linearer partieller Differential gleichungen", Jour. reine angew. Math 65 (1866), 257-268.

[9] C. CHRISTOPHER, P. MARDEŠIĆ AND C. ROUSSEAU, "Normalizability, synchronicity and relative exactness for vector fields in $\mathbb{C}^2$", J. Dynam. Control Systems, 10 (2004), pp. 501-525.

[10] C. CHRISTOPHER AND C. ROUSSEAU, "Modulus of analytical classification for the generic unfolding of a codimension one resonant diffeomorphism or resonant saddle", Annales de l'Institut Fourier 57 (2007), pp. 301-360.

[11] C. CHRISTOPHER AND C. ROUSSEAU, "The moduli space of germs of generic families of analytic diffeomorphisms unfolding a parabolic fixed point", preprint 2008.

[12] F. DEAHNA, "Ueber die Bedingungen der Integrabilität lineärer Differential gleichungen erster Ordnung zwischen einer beliebigen Anzhal veränderlicher Grössen", Jour. reine angew. Math 20 (1840), 340-349.

[13] H. DULAC, "Recherches sur les points singuliers des équations différentielles", J. École polytechnique, vol. 2, sec.9, (1904), pp. 1-125.

[14] J. ECALLE, *Les fonctions résurgentes*, Publications mathématiques d'Orsay, (1985).

[15] P. ELIZAROV, Y. IL'YASHENKO, A. SHCHERBAKOV AND S. VORONIN, "Finitely generated groups of germs of one-dimensional conformal mappings and invariants for complex singular points of analytic foliations of the complex plane", "Nonlinear Stokes phenomena", Y. Il'yashenko editor, Advances in Soviet mathematics, vol. 14, Amer. Math. Soc., Providence RI (1993), 57-105.

[16] P. FATOU, "Sur les équations fonctionelles", Bull. Soc. Math. France, Paris 47 (1919-1920), pp. 161-271, and 48, 33-94, 208-314.

[17] F. G. FROBENIUS, "Ueber das Pfaffsche Problem", Jour. reine angew. Math, 82 (1877); pp. 267-282, (Gesamelte Abh. I, 286-301).

[18] A. FRÖLICHER, "Zür Differentialgeometrie der komplexen Strukturen", Math. Annalen, Bd. 129, S. 50-95 (1955).

[19] S. GALLOT, D. HULIN, J. LAFONTAINE, *Riemannian geometry* third edition, Springer-Verlag Universitext, 2004.

[20] J. GINÉ AND M. GRAU, "Characterization of isochronous foci for planar analytic defferential systems", Proc. Roy. Soc. Edinburgh Sect. A 135 (2005), pp. 985-998.

[21] A.A. GLUTSYUK, "Confluence of singular points and nonlinear Stokes phenomenon", Trans. Moscow Math. Soc. 62, pp. 49-95, 2001.

[22] L. HÖRMANDER, *An introduction to complex analysis in several variables*, North-Holland Publishing Company-Amsterdam-London, 1973.

[23] K. HULEK, *Elementary Algebraic Geometry*, AMS, student mathematical library, V. 20 - 2003.

[24] A. D. HWANG, *Complex manifolds and hermitian differential geometry*, online MAT1360 Lecture notes, University of Toronto, Spring term, 1997.

[25] Y. IL'YASHENKO, "Nonlinear Stokes phenomena", "Nonlinear Stokes phenomena", Y. Il'yashenko editor, Advances in Soviet mathematics, vol. 14, Amer. Math. Soc., Providence RI (1993), 1-55.

[26] Y. IL'YASHENKO AND A.S. PYARTLI, "Materialization of Poincaré resonances and divergence of normalizing series", J. Sov. Math., 31, (1985), 3053-3092.

[27] Y. IL'YASHENKO AND S. YAKOVENKO, *Lectures on analytic differential equations*, Graduate Studies in Mathematics, vol. 86, Amer. Math. Soc.,Providence RI (2008).

[28] J. J. KOHN, "Harmonic integrals on strongly pseudo-convex manifolds", (I) Ann. Math. 78, 206-213 (1963) & (II) Ann. Math. 79, 450-472 (1964).

[29] Y. A. KUZNETSOV, *Elements of applied bifurcation theory*, Springer-Verlag, Applied Mathematical Sciences vol. 112, 1995.

[30] P. LAVAURS, *Systèmes dynamiques holomorphes: explosion de points périodiques paraboliques*. Thèse, Université Paris-Sud, 1989.

[31] F. LORAY, *5 leçons sur la structure transverse d'une singularité de feuilletage holomorphe en dimension 2 complexe*, Monographies Red TMR Europea Sing. Ec. Dif. Fol., 1 (1999), pp. 1-92.

[32] P. MARDEŠIĆ, R. ROUSSARIE AND C. ROUSSEAU, "Modulus of analytic classification for unfoldings of generic parabolic diffeomorphisms", Moscow Mathematical Journal, 4, pp. 455-498, (2004).

[33] J. MARTINET, "Remarques sur la bifurcation nœud-col dans le domaine complexe", Astérisque, 150,151, pp. 131-149, (1987).

[34] J. MARTINET AND J.P. RAMIS, "Classification analytique des équations différentielles non linéaires résonnantes du premier ordre", Ann. Scient. Éc. Norm. Sup., 4e série, 16 (1983), pp. 571-621.

[35] J. MARTINET AND J.P. RAMIS, "Problèmes de modules pour des équations différentielles non linéaires du premier ordre", publ. IHES, 55 (1982), pp. 63-164.

[36] J. MARTINET AND J.P. RAMIS, "Classification analytique des équations différentielles non linéaires résonnantes du premier ordre", Ann. Scient. Éc. Norm. Sup., 4e série, 16 (1983), pp. 571-621.

[37] J.F.-MATTEI AND R. MOUSSU, "Holonomie et intégrales premières", Ann. Scient. Éc. Norm. Sup., 4e série, 13 (1980), 469-523.

[38] J.W. MILNOR, *Dynamics in one complex variable*, third edition, Annals of Mathematical Studies - Princeton University Press, 2006.

[39] J.W. MILNOR, *Foliations and foliated vector bundles*, Lectures given at MIT during Fall Term of 1969. Available online at *http://www.foliations.org/surveys/FoliationLectNotes_Milnor.pdf*

[40] A. NEWLANDER AND L. NIRENBERG, "Complex analytic coordinates in almost complex manifolds", Annals of Mathematics, Vol. 65, No. 3, May 1957.

[41] A. NIJENHUIS AND W. WOOLF, "Some integration problems in almost-complex and complex manifolds", Annals of Mathematics, Vol. 77, No. 3, May 1963.

[42] R. PÉREZ-MARCO AND J.-C. YOCCOZ, "Germes de feuilletages holomorphes à holonomie prescrite", S.M.F., Astérisque, 222 (1994), pp. 345-371.

[43] C. ROUSSEAU, "The moduli space of germs of generic families of analytic diffeomorphisms unfolding of a codimension one resonant diffeomorphism or resonant saddle", J. Differential Equations, 248 (2010), 1794-1825.

[44] C. ROUSSEAU AND L.TEYSSIER, "Analytical Moduli for unfoldings of saddle node vector fields", Moscow Mathematical Journal, 8 (2008), 547-614.

[45] M. SABATINI, “Non-periodic isochronous oscillations in plane differential systems”, Anali di Matematica, vol. 182, (2003), pp. 487-501.

[46] S. SCHECTER AND F. SINGER, “Singular points of planar vector fields”, in “Global theory of dynamical systems”, Lecture Notes in Mathematics, 819, Springer-Verlag, 393-410.

[47] M. SHISHIKURA, “Bifurcation of parabolic fixed points”, in “The Mandelbrot set, theme and variations”, London Math. Society Lecture Notes, Ser. 274, (2000), 325-363.

[48] V. SMIRNOV, *Cours de mathématiques supérieures*, tome III, Editions MIR, Moscou, 1972.

[49] I. TAMURA, *Topology of Foliations: An Introduction*, vol. 97, Amer. Math. Soc., Providence RI (1992).

[50] C. VOISIN, *Théorie de Hodge et géométrie algébrique complexe*, Société Mathématique de France, 2002.

[51] S. M VORONIN, “Analytic classification of germs of conformal maps $(\mathbb{C},0) \mapsto (\mathbb{C},0)$ with identical linear part”, Funktsional. Anal. i Prilozhen and Func. Anal. Appl., 15, (1981), 1-17 (Russian), 1-13 (English).