Département de sciences économiques

2005-10

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Ce cahier a également été publié par le Centre interuniversitaire de recherche en économie quantitative (CIREQ) sous le numéro 11-2005.

This working paper was also published by the Center for Interuniversity Research in Quantitative Economics (CIREQ), under number 11-2005.

ISSN 0709-9231

Dynamic Measures of Individual Deprivation*

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March 2005

Abstract. We introduce and axiomatize a one-parameter class of individual deprivation measures. Motivated by a suggestion of Runciman, we modify Yitzhaki's index by multiplying it by a function that is interpreted as measuring the part of deprivation generated by an agent's observation that others in its reference group move on to a higher level of income than itself. The parameter reflects the relative weight given to these dynamic considerations, and the standard Yitzhaki index is obtained as a special case. In addition, we characterize more general classes of measures that pay attention to this important dynamic aspect of deprivation. *Journal of Economic Literature* Classification No.: D63.

Keywords: Deprivation, Equity, Individual Well-Being.

* We thank the Università Bocconi for its hospitality during the preparation of this paper. Financial support from the Social Sciences and Humanities Research Council of Canada and the European Commission-DG Research Sixth Framework Programme—Polarization and Conflict Project CIT-2-CT-2004-506084—is gratefully acknowledged.

1 Introduction

The concept of relative deprivation and its measurement has been introduced in the Economics literature by a seminal paper of Yitzhaki (1979). The definition of relative deprivation adopted is the following: "We can roughly say that [a person] is relatively deprived of X when (i) he does not have X; (ii) he sees some other person or persons, which may include himself at some previous or expected time, as having X, (iii) he wants X, and (iv) he sees it as feasible that he should have X" (Runciman, 1966, p.10). Yitzhaki considered income as the object of relative deprivation and showed that an appropriate index of total deprivation in a society is the absolute Gini index.

Hey and Lambert (1980) provided an alternative motivation of Yitzhaki's index based on the remark of Runciman that: "The magnitude of a relative deprivation is the extent of the difference between the desired situation and that of the person desiring it" (Runciman, 1966, p.10). Individual deprivation in this framework is the sum of the gaps between the individual's income and the incomes of all individuals richer than him.

Chakravarty and Chakraborty (1984) generalized the deprivation index proposing a normative index of based on a particular representation of a social welfare function. The Yitzhaki index is obtained as a special case.

Paul (1991) criticized both the Yitzhaki and the Chakravarty and Chakraborty indices because, in their formulation, individual deprivation is insensitive to income transfers taking place among persons being richer than the individual under consideration. Paul claimed that a person feels less envious with respect to an increase in the income of a rich person than with respect to a corresponding increase in the income of a rich person but poorer than the rich man. He proposed an aggregate index of deprivation that captures this belief.

Kakwani (1984) introduced a useful graphical device, the relative deprivation curve, to represent the gaps between an individual's income and the incomes of all individuals richer than it, as a proportion of mean income, and proved that the area under this curve is the Gini coefficient. Duclos (2000) has shown that a generalization of the Gini index, the single-parameter Ginis (see Donaldson and Weymark, 1980, Weymark, 1981, and Bossert, 1990), could be interpreted as indices of relative deprivation. Chakravarty, Chattopadhyay and Majumder (1995), Chakravarty (1997), Chakravarty and Moyes (2003) and Chateauneuf and Moyes (2003) have proposed deprivation quasi-orderings.

The present paper aims at introducing time as an additional dimension in the determination of the level of deprivation felt by an individual. We suggest that a person's feeling of relative deprivation today depends on a comparison with those who are better off today but there is an additional determinant: the feeling of deprivation relative to a person with a higher income is more pronounced if this person was *not* better off yesterday, that is, it has passed the individual under consideration when moving from yesterday's distribution to today's. In other words, an individual feels deprived with respect to all individuals richer than it, as in the traditional case; if any of these individuals was not richer yesterday, the individual under consideration feels deprived not only because it is poorer today but also because it didn't used to be poorer yesterday. Thus, we formalize an additional idea of Runciman that has not been explored in the literature yet: "The more the people a man sees promoted when he is not promoted himself, the more people he may compare himself with in a situation where the comparison will make him feel relatively deprived" (Runciman, 1966, p.19).

Relative deprivation of an individual in our framework is determined by the interaction of two components, namely, the gaps between the individual's income and the incomes of all individuals richer than it (the traditional way of measuring individual deprivation), and the percentage of the population that was ranked below or equal in the previous-period distribution but is above the person under consideration in the current distribution. With the latter component, we capture the effect that being passed has on individual deprivation. We use an axiomatic approach to derive a class of indices that capture these ideas.

The remainder of the paper is organized as follows. We begin in Section 2 with a discussion of our formal framework. Section 3 contains an axiomatization of a general class of dynamic individual measures of deprivation, while the characterization of the dynamic extensions of the Yitzhaki index is contained in Section 4. Section 5 concludes.

2 Basic definitions

The sets of all real numbers, all non-negative real numbers and all positive real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$. Furthermore, $\mathbb{N}$ is the set of positive integers. For a non-empty set A and $n \in \mathbb{N} \setminus \{1\}$, A^n is the n-fold Cartesian product of A. We adopt the notational convention $\sum_{j \in \emptyset} a_j = 0$.

Consider a society $N = \{1, \ldots, n\}$ of $n \in \mathbb{N} \setminus \{1\}$ individuals. The vector consisting of n ones is denoted by $\mathbf{1}$ and the origin of $\mathbb{R}^n$ is $\mathbf{0}$. For $y, z \in \mathbb{R}^n_+$ and a subset M of N, the vector $x = (y|_M, z|_{N\setminus M})$ is defined as follows. For all $j \in N$,

$$x_j = \begin{cases} y_j & \text{if } j \in M, \\ z_j & \text{if } j \in N \setminus M. \end{cases}$$

A two-period income distribution is a vector

$$\left(y^0, y^1\right) = \left((y^0_1, \ldots, y^0_n), (y^1_1, \ldots, y^1_n)\right) \in \mathbb{R}^{2n}_+,$$

where y^0 is the income distribution of the previous period and y^1 that of the current period. An individual measure of deprivation for individual $i \in N$ is a function $D_i \colon \mathbb{R}^{2n}_+ \to \mathbb{R}_+$. For $y \in \mathbb{R}^n_+$, $B_i(y) = \{j \in N \mid y_j > y_i\}$ is the set of individuals with a higher income than i. Yitzhaki's (1979) index of individual deprivation $S_i \colon \mathbb{R}^n_+ \to \mathbb{R}_+$ depends on current incomes only and is defined by

$$S_i(y) = \frac{1}{n} \sum_{j \in B_i(y)} (y_j - y_i)$$

for all $y \in \mathbb{R}^n_+$. According to S_i, individual i's deprivation in the current period is the aggregate income shortfall from the incomes of all those who are richer than i divided by the population size. The income distribution of the previous period is irrelevant. In particular, the existence of individuals who were previously at most as well-off as i and are now better off does not influence the value of the index and hence has no effect on the deprivation felt by individual i.

In this paper, building on S_i, we propose the following class of measures D^α_i, where $\alpha \in [1, \infty)$ is a parameter. For all $(y^0, y^1) \in \mathbb{R}^{2n}_+$,

$$D^\alpha_i(y^0, y^1) = \alpha^{|B_i(y^1)\setminus B_i(y^0)|}\ S_i(y^1) = \frac{\alpha^{|B_i(y^1)\setminus B_i(y^0)|}}{n} \sum_{j \in B_i(y^1)} (y^1_j - y^1_i).$$

Clearly, the Yitzhaki index S_i is obtained for $\alpha = 1$. For higher parameter values, the index assigns weight to the deprivation suffered from the knowledge that others who were previously at or below the income level of i have advanced to a higher income position than i itself. The higher the parameter value chosen, the higher the importance give to being left behind. The dynamic aspect of deprivation depends on the number of those who were at most as rich as i in the previous period but have passed i in the move to the current period. Thus, there is an asymmetry analogous to that present in standard measures of deprivation: only those who passed i matter; their impact on i's deprivation is not counterbalanced by information on those who moved below i. As in the non-dynamic approach, this is the case because deprivation only is being measured and not satisfaction. In the framework of the present paper, individual i would feel satisfied when comparing its income with that of poorer individuals, as in the traditional literature, and would feel even more satisfied with respect to those individuals who used to be richer yesterday and moved to the same level as i or below it in the present period.

In addition to the measures D_i^α, we characterize more general classes of indices that do not necessarily coincide with the Yitzhaki index if no attention is paid to the deprivation caused by having been left behind by some agents in the move from the previous to the current period. These classes provide us with a convenient method to convert any standard index of deprivation into an index that takes into consideration the deprivation resulting from an agent's inability to keep up with others.

3 General classes of dynamic deprivation measures

In static deprivation measurement, it is plausible to assume that if no one has a higher income than agent i, then the degree of i's deprivation is zero and, conversely, i's deprivation is positive whenever there exists at least one agent with a higher income. The reasoning underlying this requirement carries over easily into the dynamic framework considered here: if no one has passed i when moving from y^0 to y^1, deprivation for i should be equal to zero if and only if no one has a higher income than i in y^1. In order to formulate the weakest possible requirement, the scope of the following axiom is limited to a specific previous-period distribution y^0 such that $B_i(y^1) \setminus B_i(y^0) = \emptyset$, namely, the distribution $y^0 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$ where i has an income of zero and all others agents have an income of one. However, as will become clear later, its conclusion applies to all such distributions when combined with another axiom. Clearly, $B_i(\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}}) = N \setminus \{i\}$ and, thus, $B_i(y^1) \setminus B_i(\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}}) = \emptyset$ for all $y^1 \in \mathbb{R}^n$.

Positivity. For all $(y^0, y^1) \in \mathbb{R}^{2n}_+$ such that $y^0 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$,

$$D_i(y^0, y^1) > 0 \;\Leftrightarrow\; B_i(y^1) \neq \emptyset. \tag{1}$$

Our next axiom specifies how the incomes in the previous period should matter when determining individual deprivation in the current period. As mentioned earlier, the dynamic aspect of deprivation that we intend to capture is the deprivation caused by having been left behind by some agents in the move from last period's income distribution to that of the current period. Several considerations are combined in this axiom. First of all, we assume that the dynamic aspect of deprivation depends on the number of agents who were at most as rich as i in period 0 but are richer in period 1. This assumption incorporates an anonymity requirement because the number of those who are better off only matters but not their identities. Moreover, the axiom imposes a separability requirement: the standard static contribution to deprivation is separable from that due to dynamic considerations. That is, overall deprivation depends on the number of those who have passed i and on an aggregate of the income distribution in the present period. Finally, we incorporate a plausible monotonicity assumption requiring that the measure is non-decreasing in the number of those who have passed agent i. To simplify notation, we define, for any function $f\colon \mathbb{R}^n_+ \to \mathbb{R}_+$, the set

$$\mathcal{A}_f = \{(r,u) \in \mathbb{N} \cup \{0\} \times \mathbb{R}_+ \mid \exists y^1 \in \mathbb{R}^n_+ \text{ such that } r \leq |B_i(y^1)| \text{ and } f(y^1) = u\}.$$

This definition is used in our separability axiom.

Separability. There exist a function $f\colon \mathbb{R}^n_+ \to \mathbb{R}_+$ and a function $\varphi\colon \mathcal{A}_f \to \mathbb{R}_+$, non-decreasing in its first argument and increasing in its second argument, such that, for all $(y^0, y^1) \in \mathbb{R}^{2n}_+$,

$$D_i(y^0, y^1) = \varphi\left(|B_i(y^1) \setminus B_i(y^0)|, f(y^1)\right). \tag{2}$$

The increasingness of φ in its second argument ensures that the condition indeed reflects a separability requirement: any deprivation comparison between two distributions does not depend on the number of those who have passed i, provided that this number is the same for the two distributions to be compared. Because only increasing transformations preserve all relevant comparisons, the increasingness of φ in its second argument is part of the separability requirement rather than an additional assumption. In contrast, the monotonicity of φ in its first argument does impose a further restriction. Clearly, the conjunction of positivity and separability implies that (1) is satisfied not only when $y^0 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$ but whenever $B_i(y^1) \setminus B_i(y^0) = \emptyset$.

Linear homogeneity is a standard property of traditional deprivation measures (for example, the Yitzhaki index is homogeneous of degree one). We extend the axiom to our framework by requiring homogeneity of D_i in all its arguments.

Joint homogeneity. For all $(y^0, y^1) \in \mathbb{R}^{2n}_+$ and for all $\lambda \in \mathbb{R}_{++}$,

$$D_i(\lambda y^0, \lambda y^1) = \lambda D_i(y^0, y^1).$$

These three axioms impose considerable structure on a dynamic deprivation measure. We characterize the class of all indexes satisfying them in the following theorem.

Theorem 1. *An individual deprivation index D_i satisfies positivity, separability and joint homogeneity if and only if there exist a non-decreasing function $\psi\colon \{0, \ldots, n-1\} \to \mathbb{R}_{++}$ and a linearly homogeneous function $g\colon \mathbb{R}^n_+ \to \mathbb{R}_+$ such that, for all $y^1 \in \mathbb{R}^n_+$,*

$$g(y^1) > 0 \;\Leftrightarrow\; B_i(y^1) \neq \emptyset \tag{3}$$

and, for all $(y^0, y^1) \in \mathbb{R}^{2n}_+$,

$$D_i(y^0, y^1) = \psi\left(|B_i(y^1) \setminus B_i(y^0)|\right) g(y^1). \tag{4}$$

Proof. That the measures identified in the theorem statement satisfy the required axioms is straightforward to verify. Conversely, suppose D_i satisfies positivity, separability and joint homogeneity. Letting $y^0 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$, it follows that

$$B_i(y^1) \setminus B_i(y^0) = B_i(\lambda y^1) \setminus B_i(\lambda y^0) = \emptyset$$

for all $y^1 \in \mathbb{R}^n_+$ and for all $\lambda \in \mathbb{R}_{++}$. Using (2), joint homogeneity requires

$$\varphi\left(0, f(\lambda y^1)\right) = \lambda\varphi\left(0, f(y^1)\right) \tag{5}$$

for all $y^1 \in \mathbb{R}^n_+$ and for all $\lambda \in \mathbb{R}_{++}$. Define the function $g\colon \mathbb{R}^n_+ \to \mathbb{R}_+$ by letting $g(y^1) = \varphi(0, f(y^1))$ for all $y^1 \in \mathbb{R}^n_+$. By (5), g is linearly homogeneous. Let φ_0^{-1} be the inverse of φ with respect to its second argument when the first argument is fixed at zero. This inverse is well-defined because φ is increasing in its second argument. Now define the function $\xi\colon \mathcal{A}_g \to \mathbb{R}_+$ by letting

$$\xi(r, u) = \varphi\left(r, \varphi_0^{-1}(u)\right) \tag{6}$$

for all $(r, u) \in \mathcal{A}_g$. Because φ is non-decreasing in its first argument and increasing in its second argument, so is ξ. Combining (2) and (6), we obtain

$$D_i(y^0, y^1) = \xi\left(|B_i(y^1) \setminus B_i(y^0)|, g(y^1)\right) \tag{7}$$

for all $(y^0, y^1) \in \mathbb{R}^{2n}_+$. Next, we show that g satisfies (3). By way of contradiction, suppose (3) is not true. This means that there exists $y^1 \in \mathbb{R}^n_+$ such that either

$$g(y^1) > 0 \;\text{ and }\; B_i(y^1) = \emptyset \tag{8}$$

or

$$g(y^1) = 0 \;\text{ and }\; B_i(y^1) \neq \emptyset. \tag{9}$$

If (8) applies, it follows immediately that $B_i(y^1) \setminus B_i(y^0) = B_i(\lambda y^1) \setminus B_i(\lambda y^0) = \emptyset$ for all $y^0 \in \mathbb{R}^n_+$ and, in particular, for $y^0 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$. Let $\lambda \in \mathbb{R}_{++}$ be such that $\lambda \neq 1$. By positivity and (7),

$$D_i(y^0, y^1) = \xi\left(0, g(y^1)\right) = 0 = \xi\left(0, g(\lambda y^1)\right) = D_i(\lambda y^0, \lambda y^1). \tag{10}$$

Because g is linearly homogeneous and $g(y^1) > 0$, it follows that $g(\lambda y^1) = \lambda g(y^1) \neq g(y^1)$ which, together with (10), contradicts the increasingness of ξ in its second argument.

Now suppose (9) is true. Let $y^0 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$, and consider $\lambda \in \mathbb{R}_{++}$ such that $\lambda \neq 1$. Clearly, $B_i(y^1) \setminus B_i(y^0) = B_i(\lambda y^1) \setminus B_i(\lambda y^0) = \emptyset$. Using (7) and the non-emptiness of $B_i(y^1)$, positivity requires

$$\xi(0,0) = D_i(y^0, y^1) > 0. \tag{11}$$

By joint homogeneity and (7),

$$\xi(0,0) = D_i(\lambda y^0, \lambda y^1) = \lambda D_i(y^0, y^1) = \lambda \xi(0,0)$$

which yields the desired contradiction because $\lambda \neq 1$ by assumption and $\xi(0,0) > 0$ by (11). Thus, g satisfies (3).

To complete the proof of the theorem, we construct a function $\psi \colon \{0, \ldots, n-1\} \to \mathbb{R}_{++}$ with the requisite properties and show that, given the definitions of g and ψ, (4) is satisfied.

As a preliminary step, we establish that $(r, 1) \in \mathcal{A}_g$ for all $r \in \{0, \ldots, n-1\}$. Let $(y^0, y^1) \in \mathbb{R}^{2n}_+$ be such that $B_i(y^1) = N \setminus \{i\}$ and $|B_i(y^0)| = n - 1 - r$. By definition, we have $|B_i(y^1) \setminus B_i(y^0)| = r$. By (3), $g(y^1) > 0$. Let $\lambda = 1/g(y^1)$. Using the homogeneity of g, it follows that $g(\lambda y^1) = \lambda g(y^1) = 1$. Thus, $(r, 1) \in \mathcal{A}_g$.

Let $\psi(r) = \xi(r, 1)$ for all $r \in \{0, \ldots, n-1\}$. As just established, this function is well-defined because $(r, 1)$ is in the domain of ξ for all $r \in \{0, \ldots, n-1\}$. Furthermore, ψ is non-decreasing because ξ is non-decreasing in its first argument. To establish (4), we distinguish two cases.

If $(y^0, y^1) \in \mathbb{R}^{2n}_+$ is such that $B_i(y^1) = \emptyset$, positivity, (7) and the definition of ψ together imply

$$D_i(y^0, y^1) = 0 = \psi\left(|B_i(y^1) \setminus B_i(y^0)|\right) g(y^1)$$

because $g(y^1) = 0$ by (3).

If $(y^0, y^1) \in \mathbb{R}^{2n}_+$ is such that $B_i(y^1) \neq \emptyset$, (3) implies $g(y^1) > 0$. Joint homogeneity, the linear homogeneity of g and (7) together imply

$$\xi\left(|B_i(y^1) \setminus B_i(y^0)|, \lambda g(y^1)\right) = \lambda \xi\left(|B_i(y^1) \setminus B_i(y^0)|, g(y^1)\right)$$

for all $\lambda \in \mathbb{R}_{++}$. Letting $\lambda = 1/g(y^1)$, this implies

$$\xi\left(|B_i(y^1) \setminus B_i(y^0)|, g(y^1)\right) = g(y^1)\xi\left(|B_i(y^1) \setminus B_i(y^0)|, 1\right)$$

and, using (7) and the definition of ψ, we obtain (4). That ψ is positive-valued follows from the increasingness of ξ in its second argument. ■

Theorem 1 shows that the two determinants of deprivation—the static contribution due to the income distribution in the current period only and the dynamic component—are combined in a multiplicative fashion to obtain overall deprivation, provided the three axioms of the theorem statement are satisfied. If the function g is interpreted as a traditional deprivation measure, this still leaves a wide variety of ways to extend this measure to a dynamic index—the restrictions imposed on the function ψ are very weak. Particularly from the viewpoint of applied considerations, it would be desirable to narrow down this rich class at least to some extent. One way of doing so is to impose the following proportionality axiom. Let $y^1 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$ so that individual i is the unique worst-off person in the current-period distribution y^1. In this case, the axiom requires the ratio of the index values for two distributions (y^0, y^1) and (z^0, y^1) to depend on the difference of the two numbers of those who have passed i when moving from y^0 or z^0 to y^1 only. The scope of this condition is very limited: the income distribution in the current period is fixed and the axiom is silent for any other distribution in period 1. Thus, the axiom focuses on the role played by the dynamic determinant of deprivation which, in the presence of the axioms of the previous theorem, allows us to obtain a more specific functional structure for the function ψ.

Proportionality. For all $y^0, z^0, w^0, x^0, y^1 \in \mathbb{R}^n_+$ such that $y^1 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$ and $D_i(z^0, y^1) \neq 0 \neq D_i(x^0, y^1)$, if

$$|B_i(y^1) \setminus B_i(y^0)| - |B_i(y^1) \setminus B_i(z^0)| = |B_i(y^1) \setminus B_i(w^0)| - |B_i(y^1) \setminus B_i(x^0)|,$$

then

$$\frac{D_i(y^0, y^1)}{D_i(z^0, y^1)} = \frac{D_i(w^0, y^1)}{D_i(x^0, y^1)}.$$

Adding proportionality to the three axioms introduced earlier leads to a characterization of a class of dynamic deprivation measures where the function ψ must be an exponential function.

Theorem 2. *An individual deprivation index D_i satisfies positivity, separability, joint homogeneity and proportionality if and only if there exist $\alpha \in [1, \infty)$ and a linearly homogeneous function $h: \mathbb{R}^n_+ \to \mathbb{R}_+$ such that, for all $y^1 \in \mathbb{R}^n_+$,*

$$h(y^1) > 0 \;\Leftrightarrow\; B_i(y^1) \neq \emptyset \tag{12}$$

and, for all $(y^0, y^1) \in \mathbb{R}^{2n}_+$,

$$D_i(y^0, y^1) = \alpha^{|B_i(y^1) \setminus B_i(y^0)|}\, h(y^1). \tag{13}$$

Proof. Again, it is immediate that the measures identified in the theorem statement satisfy the required axioms. Conversely, suppose D_i is a deprivation measure satisfying positivity, separability, joint homogeneity and proportionality. By Theorem 1, there exist a non-decreasing function $\psi: \{0, \ldots, n-1\} \to \mathbb{R}_{++}$ and a linearly homogeneous function $g: \mathbb{R}^n_+ \to \mathbb{R}_+$ such that (3) is satisfied for all $y^1 \in \mathbb{R}^n_+$ and (4) is satisfied for all $(y^0, y^1) \in \mathbb{R}^{2n}_+$.

Clearly, for all $c \in \mathbb{R}_{++}$, $h = cg$ is linearly homogeneous and satisfies (12) if and only if g is linearly homogeneous and satisfies (3). Thus, it is sufficient to prove the existence of $c \in \mathbb{R}_{++}$ and $\alpha \in [1, \infty)$ such that $\psi(r) = c\alpha^r$ for all $r \in \{0, \ldots, n-1\}$; once this is accomplished, letting $h = cg$ and substituting into (4) immediately yields the desired conclusion.

Let $y^1 = (\mathbf{1}|_{N \setminus \{i\}}, \mathbf{0}|_{\{i\}})$. Thus, $B_i(y^1) = N \setminus \{i\} \neq \emptyset$ and, by (3) and (4), $D_i(y^0, y^1) > 0$ for all $y^0 \in \mathbb{R}^n_+$. Thus, using (4), proportionality implies

$$\psi(r+s)g(y^1)\psi(0)g(y^1) = \psi(r)g(y^1)\psi(s)g(y^1)$$

and, because $g(y^1) > 0$ by (3),

$$\psi(r+s)\psi(0) = \psi(r)\psi(s) \tag{14}$$

for all $r, s \in \mathbb{N}$ such that $r + s \leq n - 1$. This is a variant of one of Cauchy's functional equations defined on the discrete set $\{0, \ldots, n-1\}$; see Aczél (1966, Section 2.1).

We show by induction that there exist $c \in \mathbb{R}_{++}$ and $\alpha \in \mathbb{R}$ such that $\psi(r) = c\alpha^r$ for all $r \in \{0, \ldots, n-1\}$. Letting $c = \psi(0) \in \mathbb{R}_{++}$ and $\alpha \in \mathbb{R}$ be arbitrary, it follows immediately that $\psi(0) = c\alpha^0$. Now let $m \in \{0, \ldots, n-2\}$ and suppose $\psi(r) = c\alpha^r$ for all $r \in \{0, \ldots, m\}$. By (14),

$$\psi(m+1) = \frac{\psi(m)\psi(1)}{\psi(0)} = \frac{c\alpha^m c\alpha^1}{c\alpha^0} = c\alpha^{m+1}$$

which completes the induction argument. Noting that $\psi(1) = c\alpha = \psi(0)\alpha$, it follows that $\alpha = \psi(1)/\psi(0) \geq 1$ because ψ is non-decreasing and positive-valued. ■

4 Dynamic extensions of the Yitzhaki index

An interesting special case of the class of measures characterized in Theorem 2 emerges when h is given by the Yitzhaki index S_i. This section presents an axiomatization of the measures D_i^α for $\alpha \in [1, \infty)$ based on a characterization of S_i due to Bossert and D'Ambrosio (2004); see also Ebert and Moyes (2000).

The axioms introduced in this section are adaptations of the requirements used in Bossert and D'Ambrosio (2004) to our dynamic framework. As is the case for positivity, their scopes are restricted to situations where $y^0 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$ and, thus, their interpretations are identical to those of the original axioms in the traditional setting. For that reason, we do not provide detailed discussions and refer the reader to Bossert and D'Ambrosio (2004) and Ebert and Moyes (2000) instead.

The focus axiom requires that the incomes of those who are not richer than agent i are irrelevant. This property is analogous to Sen's (1976) focus axiom for poverty measures.

Focus. For all $y^0, y^1, z^1 \in \mathbb{R}^n_+$ such that $y^0 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$, $B_i(y^1) = B_i(z^1)$ and $y^1_j = z^1_j$ for all $j \in B_i(y^1) \cup \{i\}$,

$$D_i(y^0, y^1) = D_i(y^0, z^1).$$

Translation invariance requires that the index is invariant with respect to equal absolute changes in all incomes.

Translation invariance. For all $(y^0, y^1) \in \mathbb{R}^{2n}_+$ and for all $\delta \in \mathbb{R}$ such that $y^0 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$ and $(y^1 + \delta\mathbf{1}) \in \mathbb{R}^n_+$,

$$D_i(y^0, y^1 + \delta\mathbf{1}) = D_i(y^0, y^1).$$

The scope of the following homogeneity axiom is restricted in the way discussed at the beginning of this section. This axiom is used in Bossert and D'Ambrosio's (2004) characterization but because it implied by the conjunction of separability and joint homogeneity, it is not required in the characterization result of this section.

Current-period homogeneity. For all $(y^0, y^1) \in \mathbb{R}^{2n}_+$ and for all $\lambda \in \mathbb{R}_{++}$ such that $y^0 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$,

$$D_i(y^0, \lambda y^1) = \lambda D_i(y^0, y^1).$$

Normalization requires that specific income distributions are associated with a degree of individual deprivation of $1/n$. Alternative normalizations could be employed; what is crucial is that a positive level of deprivation is achieved for some distribution in order to rule out the degenerate measure where individual deprivation is equal to zero for all distributions. Because the identity of the individual who has an income of one in the axiom statement is arbitrary, the axiom encompasses an anonymity property.

Normalization. For all $(y^0, y^1) \in \mathbb{R}^{2n}_+$ such that $y^0 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$ and there exists $j \in N \setminus \{i\}$ with $y^1_j = 1$ and $y^1_k = 0$ for all $k \in N \setminus \{j\}$,

$$D_i(y^0, y^1) = 1/n.$$

The final axiom is additive decomposability. As in Bossert and D'Ambrosio (2004), we employ a formulation involving distributions where the incomes of the individuals in each of two subgroups of $B_i(y^1)$ are replaced by y^1_i and apply the usual additivity requirement to these distributions.

Additive decomposability. For all $(y^0, y^1) \in \mathbb{R}^{2n}_+$ and for all $B^1, B^2 \subseteq B_i(y^1)$ such that $y^0 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$, $B^1 \cap B^2 = \emptyset$ and $B^1 \cup B^2 = B_i(y^1)$,

$$D_i\left(y^0, y^1\right) = D_i\left(y^0, (y^1_i \mathbf{1}|_{B^1}, y^1|_{N\setminus B^1})\right) + D_i\left(y^0, (y^1_i \mathbf{1}|_{B^2}, y^1|_{N\setminus B^2})\right).$$

Our final result is the following characterization of the dynamic extensions of the Yitzhaki index introduced in Section 2. Note that positivity is not required in this result because it is implied by the remaining axioms.

Theorem 3. *An individual deprivation index D_i satisfies separability, joint homogeneity, proportionality, focus, translation invariance, normalization and additive decomposability if and only if there exists $\alpha \in [1, \infty)$ such that $D_i = D^\alpha_i$.*

Proof. That D^α_i satisfies the axioms of the theorem statement for all $\alpha \in [1, \infty)$ is straightforward to verify. Conversely, suppose D_i is an individual deprivation index satisfying the axioms.

We show that separability and joint homogeneity together imply current-period homogeneity. Suppose $(y^0, y^1) \in \mathbb{R}^{2n}_+$ is such that $y^0 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$, and let $\lambda \in \mathbb{R}_{++}$. Clearly,

$$B_i(y^1) \setminus B_i(y^0) = B_i(\lambda y^1) \setminus B_i(\lambda y^0) = \emptyset.$$

Let φ and f be as in the definition of separability. (2) and joint homogeneity together imply

$$D_i\left(y^0, \lambda y^1\right) = \varphi\left(0, f(\lambda y^1)\right) = D_i\left(\lambda y^0, \lambda y^1\right) = \lambda D_i\left(y^0, y^1\right).$$

Now that current-period homogeneity has been established, it follows that the restriction of D_i to distributions such that $y^0 = (\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}})$ satisfies all of the axioms of the theorem in Bossert and D'Ambrosio (2004). Thus, this restriction is given by the Yitzhaki index S_i, that is,

$$D_i\left((\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}}), y^1\right) = S_i\left(y^1\right)$$

for all $y^1 \in \mathbb{R}^n_+$. This implies that positivity is satisfied and, thus, Theorem 2 implies

$$D_i\left((\mathbf{1}|_{N\setminus\{i\}}, \mathbf{0}|_{\{i\}}), y^1\right) = h\left(y^1\right)$$

for all $y^1 \in \mathbb{R}^n_+$, where h is as in the theorem statement. Therefore, $h = S_i$ and substituting into (13) completes the proof. ■

5 Concluding remarks

In evaluating their level of deprivation caused by being poorer than others, individuals might give importance to the fact that some of the richer of today were poorer yesterday and have left them behind. In this paper, we have characterized a parametric class of individual deprivation measures capturing the importance given to the passing phenomenon. The higher the parameter value chosen, the higher the importance given to being left behind when measuring individual deprivation.

The measures proposed in the paper might help explaining the effect that mobility has on deprivation in our societies. Total deprivation could be simply measured as the average of individuals' deprivation, using, for example, a symmetric mean (see Diewert, 1993, for a survey and characterizations of symmetric means). Future applied research could then test the claim of Runciman that "(Total) relative deprivation will be at a minimum when either everybody or nobody is promoted; in between, it will rise and fall as actual mobility rates rise" (Runciman, 1966, p.19).

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