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Aumann-Shapley pricing: A reconsideration of the discrete case

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Abstract. We reconsider the following cost-sharing problem: agent $i = 1, ..., n$ demands a quantity x_i of good i; the corresponding total cost $C(x_1, ..., x_n)$ must be shared among the n agents. The Aumann-Shapley prices $(p_1, ..., p_n)$ are given by the Shapley value of the game where each unit of each good is regarded as a distinct player. The Aumann-Shapley *cost-sharing method* assigns the cost share $p_i x_i$ to agent i.

When goods come in indivisible units, we show that this method is characterized by the two standard axioms of Additivity and Dummy, and the property of No Merging or Splitting: agents never find it profitable to split or merge their demands.

JEL classification numbers: C71, D63.

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1. Introduction

Aumann-Shapley pricing is a method for solving cost-sharing problems in which agents demand arbitrary quantities of possibly different goods: given that each agent i in a finite set $N = \{1, ..., n\}$ requests a quantity x_i of good i, we wish to split the cost $C(x)$ of producing the demand profile x using no other information than the cost function C.

The simplest case occurs when goods are indivisible and each agent demands no more than one unit, that is, x belongs to $\{0, 1\}^N$. All the relevant information is then captured in the corresponding "stand-alone cost cooperative game" (N, Γ) where the cost associated with a coalition S is just the cost of meeting the demands of its members: in straightforward notation, $\Gamma(S) = C(x_S, 0_{N\setminus S})$. The allocation recommended by the Aumann-Shapley method in this case is just the Shapley value of that game.

The stand-alone cost game (N, Γ) remains perfectly well defined when some agents demand several units. In fact, Shubik (1962) recommended the Shapley value of that

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game as a solution to the underlying cost-sharing problem (N, x, C). This very simple solution, known as the Shapley-Shubik method, possesses attractive properties: see Moulin (1995) and Sprumont (2000) for a discussion and various axiomatizations. In general, however, the stand-alone cost game ignores much of the rich information contained in the cost function C. The Aumann-Shapley method exploits this information by constructing a game (N_x, Γ_x) where each *unit* demanded by each agent is regarded as a separate player: the player set is $N_x = \cup_{i \in N} N_i$, where each of the disjoint sets N_i contains x_i elements, and the cost associated with a coalition S is $\Gamma_x(S) = C(|S \cap N_1|, ..., |S \cap N_n|)$. The Shapley value of that game determines a price for every unit of demand. The Aumann-Shapley method charges to agent i the sum of the prices attached to the units she demands.

The large literature on Aumann-Shapley pricing, rooted in Aumann and Shapley's (1974) theory of value for nonatomic games, focuses on the case of perfectly divisible goods: demands are real numbers. In that context, the Aumann-Shapley cost shares obtain as the limit of the cost shares in the discrete model as goods become more and more divisible. Under suitable differentiability assumptions, one can show that an agent pays the integral of her marginal costs along the ray to x: this is the so-called diagonal formula. A survey of the literature up to the mid-eighties is offered in Tauman (1988). More recent studies extend the theory to the non-differentiable case (see Mertens (1988) and the survey by Haimanko and Tauman (2002)) and propose asymmetric variants (see McLean and Sharkey (1998)).

Applications of the Aumann-Shapley pricing theory are numerous: from the early work of Billera, Heath, and Raanan (1978) and Samet, Tauman, and Zang (1984) to the recent contributions of Castano-Pardo and A. Garcia-Diaz (1995), Haviv (2001) or Lee (2002), they range from the pricing of utilities such as water, phone or electricity to the allocation of highway construction costs, and the sharing of waiting time at a congested server.

More directly connected to the present paper are the axiomatizations of the diagonal formula offered by Billera and Heath (1982), Mirman and Tauman (1982), and Samet and Tauman (1982). Note that when all goods are perfectly substitutable, that is, $C(z) = c(\sum_{i \in N} z_i)$ for some function c, cost shares are proportional to demands: we say that the method satisfies the Proportionality axiom[1]. It turns out that the continuous Aumann-Shapley cost-sharing method is the only method satisfying Proportionality, Additivity (asking that cost shares depend additively on the cost function), and Scale Invariance (requiring that the scales used to measure demands be irrelevant). A quite different axiomatization based on a powerful cost monotonicity

[1] This property is not stated as an explicit axiom in the papers cited above. Rather, it is implicitely assumed by restricting attention to *pricing methods,* under which an agent's cost share is necessarily the product of her demand by a per unit price.

condition is due to Young (1985).

Somewhat surprisingly, the *discrete* version of the Aumann-Shapley cost-sharing method never received a proper axiomatic defense. Calvo and Santos (2000) propose an interesting characterization based on Myerson's (1980) idea of "balanced contributions". Under the very mild restriction that agents demanding nothing pay zero, Calvo and Santos' version of the balanced contributions condition suffices to characterize the Aumann-Shapley method. Their characterization is therefore best seen as an alternative definition of the method rather than a decomposition of it into distinct elementary axioms: see Section 4 for a discussion.

Our axiomatization builds on the following three observations.

1) Scale Invariance has no counterpart in the discrete context. When goods come in indivisible units, there is no issue of measurement because there is no meaningful rescaling operation: the only increasing bijection from the set of nonnegative integers into itself is the identity. Yet, Scale Invariance is crucial for the classical characterization. Proportionality and Additivity do not characterize the Aumann-Shapley method: just consider the plain proportional formula charging $(x_i/\sum_{i\in N} x_j)C(x)$ to agent i for *every* cost function C.

2) The existing axiomatizations of the Aumann-Shapley method do not rely on the Dummy axiom, a key element in Shapley's (1953) characterization of the value. In the cost-sharing context with no fixed cost, Dummy requires that if the total cost does not vary with agent i's demand, that agent should pay nothing. The axiom has bite: for instance, it rules out the plain proportional method. Yet, Proportionality, Additivity, and Dummy are not enough to characterize the Aumann-Shapley method. This is easily seen with just two agents: apply the Aumann-Shapley method whenever $x_1 \neq x_2$ but split the balance beyond stand-alone costs equally if $x_1 = x_2$. The difficulty is clear: none of the three axioms connects cost-sharing problems with different demand profiles.

3) Proportionality is usually motivated by the following incentive considerations: if the principle were violated, some agents would find it profitable to either merge and pretend to be a single large demander, or split and act as a collection of smaller demanders. It is well known that when all goods are perfect substitutes, the only cost-sharing method immune to such merging or splitting tactics is the proportional method: see for instance Moulin (2002) or, for a more comprehensive treatment, Ju, Miyagawa, and Sakai (2003).

The idea of preventing merging or splitting maneuvers, however, is meaningful beyond the case where all demands are perfectly substitutable. Suppose that a possibly strict *subset* of the goods, S, are perfect substitutes: the cost function takes the form $C(z) = c(z_{N\setminus S}, \sum_{i\in S} z_i)$. In this case one would like to ensure that the agents demanding the goods in S have no incentive to merge or split. Our No Merging or Splitting axiom imposes this requirement for any subset of goods: it is is thus a strengthening of Proportionality.

We will show that Additivity, Dummy, and No Merging or Splitting characterize the Aumann-Shapley method in the discrete context. Section 2 defines our version of the cost-sharing model and the Aumann-Shapley method, Section 3 presents the axioms, Section 4 states and discusses our theorem, and Section 5 concludes. The Appendix contains a proof of the theorem.

2. Aumann-Shapley pricing in the discrete context

We consider a variable-population version of the discrete cost-sharing model offered in Moulin (1995) and van den Nouweland *et al.* (1995) and further studied by Wang (1999), Sprumont (2000), Moulin and Vohra (2003), and Moulin and Sprumont (2003). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathcal{N}$ the set of nonempty finite subsets of $\mathbb{N}$. A demand profile is a pair (N, x), where $N \in \mathcal{N}$ is the set of agents and $x \in \mathbb{N}^N$ is the list of their demands. We denote by $\mathcal{D}$ the set of all demand profiles. For each $N \in \mathcal{N}$, we denote by $\mathcal{C}(N)$ the set of cost functions for N : this is the set of all nondecreasing mappings $C : \mathbb{N}^N \to \mathbb{R}_+$ such that $C(0) = 0$. A (cost-sharing) problem is a list (N, x, C), where $(N, x) \in \mathcal{D}$ and $C \in \mathcal{C}(N)$. The set of all problems is denoted $\mathcal{P}$. A (*cost-sharing*) *method* is a mapping φ which assigns to each $(N, x, C) \in \mathcal{P}$ a vector of cost shares $\varphi(N, x, C) \in \mathbb{R}_+^N$ satisfying the budget balance condition $\sum_{i \in N} \varphi_i(N, x, C) = C(x)$. Note that cost shares are required to be nonnegative.

Let $\mathcal{P}(0) = \{(N, x, C) \in \mathcal{P} \mid x \in \{0, 1\}^N\}$. Every problem $(N, x, C) \in \mathcal{P}(0)$ is identified with a unique cooperative game (N, Γ) defined by $\Gamma(S) = C(x_S, 0_{N \setminus S})$ for each $S \subseteq N$, where $x_S \in \mathbb{N}^S$ denotes the restriction of x to S.We denote the Shapley value of that game by $\varphi^{Sh}(N, \Gamma)$ or, with a slight abuse of notation, $\varphi^{Sh}(N, x, C)$. The mapping φ^{Sh} is defined on $\mathcal{P}(0)$; the Aumann-Shapley cost-sharing method extends it to $\mathcal{P}$ as follows.

Definition. Let $(N, x, C) \in \mathcal{P}$, write $N = \{1, ..., n\}$, let $N_1, ..., N_n$ be pair-wise disjoint sets such that $|N_i| = x_i$ for each $i \in N$ and let $N_x = \cup_{i \in N} N_i$. For each $S \subseteq N_x$ define $\Gamma_x(S) = C(|S \cap N_1|, ..., |S \cap N_n|)$. The *Aumann-Shapley method* φ^{ASh} computes the cost shares in the problem (N, x, C) according to the formula

$$\varphi_i^{ASh}(N, x, C) = \sum_{j \in N_i} \varphi_j^{Sh}(N_x, \Gamma_x) \text{ for all } i \in N.$$

The anonymity of the Shapley value guarantees that this definition is meaningful: the game (N_x, Γ_x) is not uniquely defined but all possible choices are equivalent.

3. Three axioms

This section presents our three axioms for cost-sharing methods. The first two, Additivity and Dummy, are standard. The third property, No Merging or Splitting, formalizes an argument often (informally) invoked in defense of the Aumann-Shapley method. While similar requirements have been defined in related models, our formulation in the cost-sharing context is new, and its implications have not been studied.

Additivity. For all $(N,\, x) \in \mathcal{D}$ and $C, C' \in \mathcal{C}(N)$, $\varphi(N, C + C', x) = \varphi(N, C, x) + \varphi(N, C', x)$.

This standard property with no apparent ethical content is very convenient. It guarantees that when a production process can be decomposed into a list of independent sub-processes, applying the method to the aggregate cost function is equivalent to applying it to each sub-process and summing up the resulting cost shares. This facilitates computation in practice; it also makes it pointless for the agents to argue about the proper level of application of the method: an additive method is insensitive to the details of implementation.

The next property uses the following notation. For each $N \in \mathcal{N}$, $C \in \mathcal{C}(N)$, $i \in N$, and $z \in \mathbb{N}^N$, let $\partial_i C(z) = C(z + e^i) - C(z)$, where e^i is the unit vector $e^i_i = 1$, $e^i_j = 0$ for $j \in N\backslash i$. Note that we write i instead of $\{i\}$ when convenient. Define $\mathcal{C}^0_i(N) = \{C \in \mathcal{C}(N) \mid \partial_i C(z) = 0 \text{ for all } z \in \mathbb{N}^N\}$: this is the set of cost functions for N where agent i is a "dummy". The axiom below is the natural extension of Shapley's (1953) dummy axiom to our model.

Dummy. For all $(N, x) \in \mathcal{D}$, $i \in N$, and $C \in \mathcal{C}^0_i(N)$, $\varphi_i(N, x, C) = 0$.

Additivity and Dummy circumscribe a well understood class of methods: a useful characterization is provided by Moulin and Vohra (2003) and recalled in Step 1 of the proof of our theorem. We emphasize that this class is very rich. On $\mathcal{P}(0)$, the subset of problems corresponding to standard cooperative game theory, the two axioms are very powerful: in particular, adding Shapley's (1953) symmetry axiom is enough to pin down the Shapley value. By contrast, on the large domain of all cost-sharing problems, $\mathcal{P}$, Additivity and Dummy are satisfied by a variety of interesting symmetric cost-sharing methods different from the Aumann-Shapley method. The Shapley-Shubik method described in Section 1 and the serial method described in Moulin (1995) are just two examples.

We turn now to the condition playing the central role in this paper.

No Merging or Splitting. Let $(N, x, C) \in \mathcal{P}$, $i \in N$, and $I \in \mathcal{N}$ be such that $N \cap I = \{i\}$. Write $N' = (N\backslash i) \cup I$, define $C' \in \mathcal{C}(N')$ by $C'(z) = C(z_{N\backslash i}, \sum_{i' \in I} z_{i'})$ for all $z \in \mathbb{N}^{N'}$, and let $x' \in \mathbb{N}^{N'}$. Then $\{\sum_{i' \in I} x'_{i'} = x_i$ and $x'_j = x_j$ for all $j \in N\backslash i\}$ $\Rightarrow \{\sum_{i' \in I} \varphi_{i'}(N', x', C') = \varphi_i(N, x, C)\}$.

This property connects cost shares in two problems with different sets of agents: N' obtains from N by "splitting" agent i into a set I of agents i' whose aggregate demand equals i's original demand. The other agents in N and N' are identical, and their demands do not change. The cost function C' for N' expresses the same technology as C: the sum of the consumptions by the agents i' in I merely plays the role of agent i's original consumption.

The condition prevents manipulations of identity: agent i in N does not gain by splitting her demand into smaller pieces; the agents i' in N' do not gain by merging theirs into one larger block. Note that we require $i \in I$: No Merging or Splitting does not *explicitly* rule out more radical manipulations of identity whereby an agent is replaced with a completely disjoint set of demanders.

Variants of the No Merging or Splitting condition have been studied before, notably in the simpler model of rationing (or bankruptcy), where a given amount must be split among a set of agents holding claims on it. Early contributions include O'Neill (1982), Moulin (1987), and Chun (1988). For a recent treatment containing references and generalizations, see Ju, Miyagawa, and Sakai (2003).

4. An axiomatization

We are now ready to state and discuss our main result.

Theorem. *The Aumann-Shapley method is the only cost-sharing method satisfying Additivity, Dummy, and No Merging or Splitting.*

One feature of the proof proposed in the Appendix may be of independent interest. No Merging or Splitting implies a Weak Symmetry property stating that renaming exactly one agent does not affect her own cost share. Step 4 of the proof shows that on the subset of problems corresponding to standard cooperative games with a variable population -namely, $\mathcal{P}(0)$- Additivity, Dummy and Weak Symmetry characterize the Shapley value. Thus, the full force of Shapley's (1953) symmetry condition is not needed to pin down the value when the set of agents is allowed to change.

We make three further comments.

1) The axioms in the above theorem are independent. No Merging or Splitting is obviously needed: the (rich) class defined by Additivity and Dummy, described in Step 1 of the proof, contains, for instance, the Shapley-Shubik method. The proportional method $\varphi_i^p(N, x, C) = (x_i / \sum_{j \in N} x_j) C(x)$ whenever $x \neq 0$ (and $\varphi(N, x, C) = 0$ when $x = 0$) shows that Dummy is not redundant. To see that Additivity cannot be dropped, consider the method $\varphi(N, x, C) = \varphi^p(N, x, C)$ whenever C is strictly increasing (that is, $C(z) < C(z')$ if $z < z'$), and $\varphi(N, x, C) = \varphi^{ASh}(N, x, C)$ otherwise.

2) Because No Merging or Splitting compares problems with different sets of agents, it is a somewhat complex condition. We propose a variant of our theorem

based on a fixed-population condition which is a little simpler to grasp than No Merging or Splitting. Given $N \in \mathcal{N}$ and $S \subseteq N$, define $\mathcal{C}^1_S(N) = \{C \in \mathcal{C}(N) \mid \exists c : C(z) = c(z_{N \setminus S}, \sum_{i \in S} z_i)\}$. This is the set of cost functions in $\mathcal{C}(N)$ for which the demands from the agents in S are perfect substitutes: these agents demand essentially the same good.

No Demand Reshuffling. For all $N \in \mathcal{N}$, $x, x' \in \mathbb{N}^N$, $S \subseteq N$, and $C \in \mathcal{C}^1_S(N)$, $\{\sum_{i \in S} x_i = \sum_{i \in S} x'_i$ and $x_{N \setminus S} = x'_{N \setminus S}\} \Rightarrow \{\sum_{i \in S} \varphi_i(N, x, C) = \sum_{i \in S} \varphi_i(N, x', C)\}$.

This condition says that the aggregate cost share of a group of agents demanding essentially the same good depends only on their aggregate demand. If this condition were violated, all members of S could benefit by reshuffling individual demands within S and performing suitable side-transfers. The idea was proposed by Moulin (1987) in the bankruptcy model. It is easy to check that No Demand Reshuffling is implied by No Merging or Splitting. Assume the premises of the former, choose an agent $s \in S$, and define $c \in \mathcal{C}((N \setminus S) \cup s)$ by $c(z_{N \setminus S}, \sum_{i \in S} z_i) = C(z)$ for all $z \in \mathbb{N}^N$. Applying No Merging or Splitting twice, we get $\sum_{i \in S} \varphi_i(N, x, C) = \varphi_s((N \setminus S) \cup s, (x_{N \setminus S}, \sum_{i \in S} x_i), c) = \varphi_s((N \setminus S) \cup s, (x'_{N \setminus S}, \sum_{i \in S} x'_i), c) = \sum_{i \in S} \varphi_i(N, x', C)$.

The following example shows that Additivity, Dummy, and No Demand Reshuffling are not quite enough to pin down the Aumann-Shapley method. Let

$$\begin{aligned}
\varphi_1(\{1,2,3\}, e^{\{1,2,3\}}, C) &= \frac{1}{3}(\partial_1 C(0) + \partial_1 C(e^{\{2\}}) + \partial_1 C(e^{\{2,3\}})), \\
\varphi_2(\{1,2,3\}, e^{\{1,2,3\}}, C) &= \frac{1}{3}(\partial_2 C(0) + \partial_2 C(e^{\{3\}}) + \partial_2 C(e^{\{1,3\}})), \\
\varphi_3(\{1,2,3\}, e^{\{1,2,3\}}, C) &= \frac{1}{3}(\partial_3 C(0) + \partial_3 C(e^{\{1\}}) + \partial_3 C(e^{\{1,2\}}))
\end{aligned}$$

for all $C \in \mathcal{C}(\{1,2,3\})$ and let $\varphi(N, x, C) = \varphi^{ASh}(N, x, C)$ for every other cost-sharing problem (N, x, C). Additivity and Dummy are clearly met. To show that φ satisfies No Demand Reshuffling, observe that this axiom connects the cost shares for two problems (N, x, C), (N', x', C') only when the sets of agents are the same ($N = N'$) and the total demands are the same ($\sum_{i \in N} x_i = \sum_{i \in N'} x'_i$). Since φ coincides with φ^{ASh} except when $N = \{1,2,3\}$ and $x = (1,1,1)$, and since φ^{ASh} satisfies No Demand Reshuffling, it is sufficient to check $\varphi_1(\{1,2,3\}, (1,1,1), C) + \varphi_2(\{1,2,3\}, (1,1,1), C) = \varphi_1^{ASh}(\{1,2,3\}, (2,0,1), C) + \varphi_2^{ASh}(\{1,2,3\}, (2,0,1), C)$ for all $C \in \mathcal{C}^1_{\{1,2\}}(\{1,2,3\})$, and two similar conditions obtained by replacing $\{1,2\}$ with $\{1,3\}$ and $\{2,3\}$. Using budget balance, this amounts to $\varphi_3(\{1,2,3\}, (1,1,1), C) = \varphi_3^{ASh}(\{1,2,3\}, (2,0,1), C)$ for all $C \in \mathcal{C}^1_{\{1,2\}}(\{1,2,3\})$, and similar conditions for agents 1 and 2. These three conditions are readily verified.

The method in this example satisfies the basic axiom of Zero Charge for Zero Demand: for all $(N, x, C) \in \mathcal{P}$ and $i \in N$, $\{x_i = 0\} \Rightarrow \{\varphi_i(N, x, C) = 0\}$. But removing those who demand zero from the set of agents affects the cost shares of the others. For instance, consider the cost function $C \in \mathcal{C}(\{1,2,3,4\})$ given by $C(z) = 0$ whenever $z_1 = z_2 = 0$, $C(z) = 1$ otherwise. Then $\varphi(\{1,2,3,4\},(1,1,1,0),C) = (\frac{1}{2},\frac{1}{2},0,0) \neq \varphi(\{1,2,3\},(1,1,1),C_{\{1,2,3\}}) = (\frac{1}{3},\frac{2}{3},0)$. Our next axiom rules out this problematic feature. For any $N \in \mathcal{N}$ and $C \in \mathcal{C}(N)$, we define $C_S \in \mathcal{C}(S)$ by $C_S(z_S) = C(z_S, 0_{N\backslash S})$ for all $z \in \mathbb{N}^N$.

Independence of Zero Demands. For all $(N,\ x\ C) \in \mathcal{P}$ and $i \in N$, $\{x_i = 0\} \Rightarrow \{\varphi_i(N,x,C) = 0$ and $\varphi_{N\backslash i}(N,x,C) = \varphi(N\backslash i, x_{N\backslash i}, C_{N\backslash i})\}$.

This very natural variable-population condition is well known: see for instance Moulin and Shenker (1994) or Sprumont (1998). It is clearly met by the Aumann-Shapley method and delivers the following corollary to our theorem.

Corollary. *The Aumann-Shapley method is the only cost-sharing method satisfying Additivity, Dummy, Independence of Zero Demands, and No Demand Reshuffling.*

Proof. It suffices to show that No Demand Reshuffling and Independence of Zero Demands together imply No Merging or Splitting. Let $(N,x,C), i, I, N', C'$, and x' satisfy the premises of No Merging or Splitting. Note that $C' \in \mathcal{C}_I^1(N')$. Define $x'' \in \mathbb{N}^{N'}$ by $x''_j = x_j$ for all $j \in N$ and $x''_{i'} = 0$ for all $i' \in I$. Apply successively No Demand Reshuffling and Independence of Zero Demands: $\sum_{i' \in I} \varphi_{i'}(N',x',C') = \sum_{i' \in I}(N',x'',C') = \varphi_i(N,x,C)$. ■

3) As mentioned in the Introduction, Calvo and Santos (2000) offer an alternative characterization of the discrete Aumann-Shapley method. The key property they use states that for any pair of positive demanders i, j, the increase in i's per unit cost share generated by a one-unit increase in j's demand is equal to the increase in j's per unit cost share generated by a one-unit increase in i's demand. Writing $N_+(x) = \{i \in N \mid x_i > 0\}$, the formal condition reads as follows.

Balanced Contributions. For all $(N,x,C) \in \mathcal{P}$ and all $i,j \in N_+(x)$,

$$\frac{1}{x_i}(\varphi_i(N,x,C) - \varphi_i(N,x-e^j,C)) = \frac{1}{x_j}(\varphi_j(N,x,C) - \varphi_j(N,x-e^i,C)). \quad (1)$$

Calvo and Santos prove that the Aumann-Shapley method is the only cost-sharing method satisfying Balanced Contributions and Zero Charge for Zero Demand.

To see why this is true, fix $(N,x,C) \in \mathcal{P}$ and $i \in N_+(x)$. Since N and C are fixed throughout the argument, we omit them in our notation; we also write N_+ instead of $N_+(x)$. Multiplying both sides of (1) by x_j and summing up,

$$\sum_{j \in N_+ \backslash i} \frac{x_j}{x_i}(\varphi_i(x) - \varphi_i(x-e^j)) = \sum_{j \in N_+ \backslash i} (\varphi_j(x) - \varphi_j(x-e^i)).$$

Adding $\varphi_i(x)$ on both sides, using budget balance and Zero Charge for Zero Demand, and writing $x(N_+) = \sum_{j\in N_+} x_j$, we get

$$\frac{x(N_+)}{x_i}\varphi_i(x) - \sum_{j\in N_+\setminus i} \frac{x_j}{x_i}\varphi_i(x-e^j) = C(x) - C(x-e^i) + \varphi_i(x-e^i)$$

or, after rearranging,

$$\varphi_i(x) = \frac{x_i}{x(N_+)}(C(x) - C(x-e^i)) + \sum_{j\in N_+} \frac{x_j}{x(N_+)}\varphi_i(x-e^j). \tag{2}$$

By Zero Charge for Zero Demand, $\varphi(0) = 0$. Since the recursive formula (2) holds for every $i \in N_+$, it fully determines a unique cost-sharing method φ. As φ^{ASh} satisfies Balanced Contributions, $\varphi = \varphi^{ASh}$.

Thus, Balanced Contributions -essentially alone- amounts to a recursive definition of the Aumann-Shapley cost-sharing method. It is a complex requirement composed of several ideas that are not easily disentangled. Our axiomatization offers the advantage of a genuine decomposition of the Aumann-Shapley method into a set of more elementary, jointly characteristic, axioms.

5. Conclusion

Under Additivity, strengthening the Proportionality axiom and reintroducing the familiar Dummy condition yields a characterization of the discrete Aumann-Shapley cost-sharing method. The exercise in this paper was motivated by the observation that Scale Invariance, a crucial ingredient in the classical axiomatizations, is meaningless in the discrete context. But our combination of axioms may be of independent interest: do they characterize the Aumann-Shapley method in the standard model where demands are real numbers?

6. Appendix

Proof of the theorem. It is clear that φ^{ASh} satisfies Additivity, Dummy, and No Merging or Splitting. Conversely, let φ be a cost-sharing method satisfying these axioms.

Step 1: *φ admits a flow representation*

Given $N \in \mathcal{N}$ and $x \in \mathbb{N}^N$, we write vector inequalities $\leq, <, \ll$ and define $[0, x[= \{z \in \mathbb{N}^N \mid 0 \leq z < x\}$ and $]0, x[= \{z \in \mathbb{N}^N \mid 0 < z < x\}$. A *flow* to x is a mapping $f(N, x, .) : [0, x[\to \mathbb{R}^N_+$ satisfying the convention that $f_i(N, x, z) = 0$ whenever $z_i = x_i$, and the so-called flow conservation constraints

$$\begin{aligned}\sum_{i\in N} f_i(N,x,0) &= 1,\\ \sum_{i\in N} f_i(N,x,z) &= \sum_{i\in N_+(z)} f_i(N,x,z-e^i) \text{ for all } z \in]0,x[,\end{aligned}$$

where $N_+(z) = \{i \in N \mid z_i > 0\}$.

Building on Wang (1999), Moulin and Vohra (2003) prove that a cost-sharing method φ satisfies Additivity and Dummy if and only if, for all $N \in \mathcal{N}$ and $x \in \mathbb{N}^N$, there is a flow $f(N, x, .)$ to x such that

$$\varphi_i(N, x, C) = \sum_{z \in [0,x[} f_i(N, x, z) \partial_i C(z) \tag{3}$$

for all $i \in N$ and $C \in \mathcal{C}(N)$. For each N and x, the flow $f(N, x, .)$ associated with $\varphi(N, x, .)$ is unique; we call f the flow representation of φ. An immediate corollary to Moulin and Vohra's result is that Additivity and Dummy imply Zero Charge for Zero Demand. From now on, we denote by f the flow representation of our method φ.

Step 2: *The flow representation of the Aumann-Shapley method.*

Since the *Aumann-Shapley* method φ^{ASh} satisfies Additivity and Dummy, it too admits a flow representation: we denote it by f^{ASh}. By definition, the Aumann-Shapley method applies the Shapley value to the game where each unit demanded by every agent in N constitutes a separate player and computes an agent's cost share by summing up the coordinates of the Shapley value which correspond to the units he demands. This amounts to taking the uniform average of the so-called incremental cost-share vectors generated by all possible orderings of all units demanded by all agents. Straightforward computations show that the corresponding flow representation is

$$f_i^{ASh}(N, x, z) = \frac{\alpha(z)\alpha(x - z - e^i)}{\alpha(x)} \text{ for all } (N, x) \in \mathcal{D},\ i \in N, \text{ and } z \in [0, x - e^i],$$

where $\alpha(z) = (\sum_{j \in N} z_j)!/\prod_{j \in N} z_j!$. See, for instance, Moulin (1995) or Moulin and Sprumont (2003). As observed in Sprumont and Wang (1998), the defining feature of this formula is that the flow at z is split in proportion to the sizes of the "remaining demands": $f_i(N, x, z)/\sum_{j \in N} f_j(N, x, z) = (x_i - z_i)/\sum_{j \in N}(x_j - z_j)$ for all $i \in N$.

Step 3: *φ is weakly symmetric.*

If $\pi : \mathbb{N} \to \mathbb{N}$ is a bijection, $N \in \mathcal{N}$, $y \in \mathbb{R}_+^N$, and $C \in \mathcal{C}(N)$, write $\pi N = \{\pi(i) \mid i \in N\}$. For any $z \in \mathbb{R}_+^N$, define $\pi z \in \mathbb{R}_+^{\pi N}$ by $(\pi z)_{\pi(i)} = z_i$ for all $i \in N$. Finally, define $\pi C : \mathbb{N}^{\pi N} \to \mathbb{R}_+$ by $\pi C(\pi z) = C(z)$ for all $z \in \mathbb{N}^N$: note that $\pi C \in \mathcal{C}(\pi N)$. We claim that φ satisfies the following property.

Weak Symmetry. Let $(N, x, C) \in \mathcal{P}$, $i \in N$, and let $\pi : \mathbb{N} \to \mathbb{N}$ be a bijection. If there exists some $i \in N$ such that $\pi(j) = j$ for all $j \in N \backslash i$, then $\varphi_{\pi(i)}(\pi N, \pi x, \pi C) = \varphi_i(N, x, C)$.

This condition says that renaming an agent should not affect her cost share. *A priori*, it does not rule out the possibility that the cost shares of the others might be affected by that renaming.

Weak Symmetry is a consequence of No Merging or Splitting. To see why, let (N, x, C), i, and π satisfy the premises of Weak Symmetry. The case $\pi(i) = i$ being trivial, assume $\pi(i) \neq i$. Let $N' = N \cup \pi(i)$, define $C' \in \mathcal{C}(N')$ by $C'(z) = C(z_{N\backslash i}, z_i + z_{\pi(i)})$, and define $x' \in \mathbb{N}^{N^0}$ by $x'_{\pi(i)} = 0$, $x'_j = x_j$ for all $j \in N$.

Let $I = \{i, \pi(i)\}$. Note that $N' = (N\backslash i) \cup I$ and $x'_i + x'_{\pi(i)} = x_i$. Applying No Merging or Splitting to (N, x, C) and agent i gives

$$\varphi_i(N, x, C) = \varphi_i(N', x', C') + \varphi_{\pi(i)}(N', x', C').$$

Next, observe that $N' = (\pi N\backslash \pi(i)) \cup I$, $C'(z) = \pi C(z_{N\backslash i}, z_i + z_{\pi(i)})$, and $x'_i + x'_{\pi(i)} = (\pi x)_{\pi(i)}$. Applying No Merging or Splitting to $(\pi N, \pi x, \pi C)$ and agent $\pi(i)$ gives

$$\varphi_{\pi(i)}(\pi N, \pi x, \pi C) = \varphi_i(N', x', C') + \varphi_{\pi(i)}(N', x', C').$$

Hence, $\varphi_i(N, x, C) = \varphi_{\pi(i)}(\pi N, \pi x, \pi C)$, as claimed.

Step 4: *φ extends the Shapley value.*

For any $N \in \mathcal{N}$ and any nonempty $K \subseteq N$, define the so-called unanimity game (N, Γ_K) by $\Gamma_K(T) = 1$ if $K \subseteq T$, and 0 otherwise. We claim that

$$\varphi_i(N, \Gamma_K) = 1/k \text{ if } i \in K, \text{ and } 0 \text{ if } i \in N\backslash K, \tag{4}$$

where $k = |K|$.

The proof differs from the usual argument based on the full symmetry axiom: it relies crucially on the assumption that the set of agents, N, is allowed to change[2]. Fix $N \in \mathcal{N}$ and a nonempty $K \subseteq N$. If $k = 1$, the claim follows directly from Dummy. From now on, assume $k \geq 2$. Let $M \in \mathcal{N}$ be such that $K \subseteq M$, $(N\backslash K) \cap M = \varnothing$, and $|M| = 2k - 1$.[3] Define $\mathcal{M}(k) = \{S \subseteq M \mid |S| = k\}$ and write $\mu(k) = |\mathcal{M}(k)|$.

By Dummy and budget balance,

$$\sum_{i \in S} \varphi_i((N\backslash K) \cup S,\ \Gamma_S) = 1 \text{ for all } S \in \mathcal{M}(k). \tag{5}$$

[2] If N were fixed, Weak Symmetry would be vacuous because no bijection from $\mathbb{N}$ to $\mathbb{N}$ renaming exactly one member of N leaves N unchanged. In this case, Additivity, Dummy, and Weak Symmetry would not yield the Shapley value, as the method defined for the fixed population $N = \{1, 2, 3\}$ in the example in comment 2, Section 4, illustrates.

[3] Observe that this construction is possible for each N because the set of potential agents, $\mathcal{N}$, is infinite. In fact, the claim in Step 4 would be false if the set of potential agents were finite.

Define $\mathcal{M}(k-1) = \{S' \subseteq M \mid |S'| = k-1\}$. Since $S' \in \mathcal{M}(k-1)$ if and only if $M \backslash S' \in \mathcal{M}(k)$, we have $|\mathcal{M}(k-1)| = |\mathcal{M}(k)| = \mu(k)$. For any $S' \in \mathcal{M}(k-1)$, Weak Symmetry ensures that there exists a number $y(S')$ such that

$$\varphi_i((N\backslash K) \cup S' \cup i,\ \Gamma_{S' \cup i}) = y(S') \text{ for all } i \in M\backslash S'.$$

We may therefore rewrite (5) as follows.

$$\sum_{i \in S} y(S\backslash i) = 1 \text{ for all } S \in \mathcal{M}(k). \tag{6}$$

This is a system of $\mu(k)$ linear equations in the $\mu(k)$ variables $y(S')$, $S' \in \mathcal{M}(k-1)$. Clearly, $y(S') = 1/k$ for all S' is a solution. If it is the only solution, then for all $i \in K$, choosing $S' = K\backslash i$ yields $\varphi_i(N, \Gamma_K) = \varphi_i((N\backslash K) \cup (K\backslash i) \cup i), \Gamma_{(K\backslash i)\cup i}) = y(K\backslash i) = 1/k$, and we are done.

To prove that (6) has a unique solution, it suffices to show that the $\mu(k)$ equations are linearly independent. Rewrite the system as

$$u^S \cdot y = 1 \text{ for all } S \in \mathcal{M}(k),$$

where $y = (y(S')) \in \mathbb{R}^{\mathcal{M}(k-1)}$ and $u^S \in \mathbb{R}^{\mathcal{M}(k-1)}$ is defined by $u^S(S') = 1$ if $S' \subseteq S$, and 0 for all other $S' \in \mathcal{M}(k-1)$. We claim that $\{u^S \mid S \in \mathcal{M}(k)\}$ is a basis of $\mathbb{R}^{\mathcal{M}(k-1)}$. It is enough to show that any vector in the canonical basis $\{e^{S'} \mid S' \in \mathcal{M}(k-1)\}$ (where $e^{S'}(T) = 1$ if $T = S'$ and 0 otherwise) is a linear combination of the vectors u^S. Fix $S' \in \mathcal{M}(k-1)$. For each $r = 0, 1, ..., k-1$, define $\mathcal{M}_r(k) = \{S \in \mathcal{M}(k) \mid |S \cap S'| = r\}$. One checks that

$$e^{S'} = \sum_{r=0}^{k-1} \frac{(-1)^{k-r+1}(k-r-1)!}{k...(r+1)} \sum_{S \in \mathcal{M}_r(k)} u^S,$$

proving our claim. We have shown (4).

Because the unanimity games (N, Γ_K) form a basis of $\mathcal{P}(0)$, Additivity implies that $\varphi(N, \Gamma) = \varphi^{Sh}(N, \Gamma)$ for all $(N, \Gamma) \in \mathcal{P}(0)$.

Step 5: *φ is the Aumann-Shapley method.*

We begin with the observation that since φ satisfies Weak Symmetry, its flow representation f is weakly symmetric: for any $(N, x) \in \mathcal{D}$, $i \in N$, and any bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $\pi(j) = j$ for all $j \in N\backslash i$,

$$f_{\pi(i)}(\pi N, \pi x, \pi z) = f_i(N, x, z) \text{ for all } z \in [0, x[. \tag{7}$$

For a proof, fix N, x, i, and π as required, and $z \in [0, x[$. Consider a cost function C such that $\partial_i C(z')$ equals 1 if $z' = z$ and 0 otherwise. By (3) and Weak Symmetry, $f_i(N, x, z) = \varphi_i(N, x, C) = \varphi_{\pi(i)}(\pi N, \pi x, \pi C) = f_{\pi(i)}(\pi N, \pi x, \pi z)$, as claimed.

The rest of the proof shows that $f = f^{ASh}$. For any $k = 0, 1, 2, ...$, let $\mathcal{D}(k) = \{(N, x) \in \mathcal{D} \mid |\{i \in N \mid x_i > 1\}| \leq k\}$. This is the set of demand profiles where no more than k agents demand several units. By Step 4, $f(N, x) = f^{ASh}(N, x)$ for all $(N, x) \in \mathcal{D}(0)$. We now fix $k \geq 0$, make the induction hypothesis

$$f(M, x) = f^{ASh}(M, x) \text{ for all } (M, x) \in \mathcal{D}(k), \tag{8}$$

and prove that

$$f(N, x) = f^{ASh}(N, x) \text{ for all } (N, x) \in \mathcal{D}(k+1). \tag{9}$$

(a) First we show that

$$f_i(N, x) = f_i^{ASh}(N, x) \text{ for all } (N, x) \in \mathcal{D}(k+1) \text{ and } i \in N \text{ such that } x_i > 1. \tag{10}$$

Fix $(N, x) \in \mathcal{D}(k+1)$. By the induction hypothesis (8), we may assume that $(N, x) \in \mathcal{D}(k+1)\backslash\mathcal{D}(k)$: exactly $k+1$ agents demand several units. Consider an arbitrary agent $i \in N$ such that $x_i > 1$ and an arbitrary $C \in \mathcal{C}(N)$. Let $I \in \mathcal{N}$ be a set of cardinality $|I| = x_i$, disjoint from $N\backslash i$, and containing i. Let $N' = (N\backslash i) \cup I$. Define $C'(z) = C(z_{N\backslash i}, \sum_{i' \in I} z_{i'})$ for all $z \in \mathbb{N}^{N'}$ and define $x' \in \mathbb{N}^{N'}$ by $x'_{i'} = 1$ for all $i' \in I$ and $x'_j = x_j$ for all $j \in N\backslash i$. Apply successively No Merging or Splitting, the induction hypothesis (8), and the definition of the Aumann-Shapley method to obtain

$$\begin{aligned}
\varphi_i(N, x, C) &= \sum_{i' \in I} \varphi_{i'}(N', x', C') \\
&= \sum_{i' \in I} \varphi_{i'}^{ASh}(N', x', C') \\
&= \varphi_i^{ASh}(N, x, C).
\end{aligned}$$

Since this holds for any cost function $C \in \mathcal{C}(N)$, it follows that $f_i(N, x) = f_i^{ASh}(N, x)$, as desired.

(b) Next we show that

$$f(N, x, 0) = f^{ASh}(N, x, 0) \text{ for all } (N, x) \in \mathcal{D}(k+1). \tag{11}$$

Again, let $(N, x) \in \mathcal{D}(k+1)$ and assume without loss that $(N, x) \in \mathcal{D}(k+1)\backslash\mathcal{D}(k)$. Suppose furthermore that $x_i > 0$ for all $i \in N$. This assumption is for notational convenience: if it is not met, the argument below carries over provided that N is replaced with $N(x)$, the set of positive demanders.

Let $K = \{i \in N \mid x_i = 1\}$ and $k = |K|$. If $k = 1$, our claim follows directly from (10) and budget balance. Assume now $k \geq 2$. Choose M and define $\mathcal{M}(k)$

and $\mathcal{M}(k-1)$ as in Step 4. For all $S \in \mathcal{M}(k)$ define $x^{(S)} \in \mathbb{R}^{(N\backslash K)\cup S}$ by $x_i^{(S)} = x_i$ for $i \in N\backslash K$ and $x_i^{(S)} = 1$ for $i \in S$. Thus, in particular, $x^{(K)} = x$. By (10), flow conservation, and the definition of the Aumann-Shapley flow representation f^{ASh} in Step 2,

$$\sum_{i\in S} f_i((N\backslash K)\cup S,\ x^{(S)},\ 0) = 1 - \sum_{i\in N\backslash K} f_i^{ASh}((N\backslash K)\cup S,\ x^{(S)},\ 0) = \frac{x(K)}{x(N)}$$

for all $S \in \mathcal{M}(k)$, where we use the notation $x(T) = \sum_{i\in T} x_i$. Thanks to the weak symmetry of f (see (7)), we can rewrite this system

$$\sum_{i\in S} y(S\backslash i) = \frac{x(K)}{x(N)}$$

for all $S \in \mathcal{M}(k)$, where $y(S') = f_i((N\backslash K)\cup(S'\cup i),\ x^{(S'\cup i)},\ 0)$ for all $S' \in \mathcal{M}(k-1)$ and all $i \in M\backslash S'$. The same argument as in Step 4 shows that the equations in this system are linearly independent, so that the solution $y(S') = \frac{1}{k}\frac{x(K)}{x(N)} = \frac{1}{x(N)}$ for all S' is unique. Therefore

$$f_i(N,x,0) = y(K) = \frac{1}{x(N)} = f_i^{ASh}(N,x,0)$$

for all $i \in K$, which together with (10) proves (11).

(c) We prove (9). Let $(N,x) \in \mathcal{D}(k+1)\backslash\mathcal{D}(k)$ and assume without loss that $x_i > 0$ for all $i \in N$. For $l = 0,1,...,x(N)-1$, let $Z(l) = \{z \in [0,x[\mid \sum_{i\in N} z_i = l\}$. By (11), $f(N,x,z) = f^{ASh}(N,x,z)$ for all $z \in Z(0)$. Proceeding by induction on l, fix $l \geq 0$ and assume that

$$f(N,x,z') = f^{ASh}(N,x,z') \text{ for all } z' \in Z(l). \tag{12}$$

We now fix $z \in Z(l+1)$ and show that $f(N,x,z) = f^{ASh}(N,x,z)$. Set $K = \{i \in N \mid x_i = 1 \text{ and } z_i = 0\}$, $k = |K|$. By the flow conservation constraints, (12) and (10),

$$\begin{aligned}\sum_{i\in K} f_i(N,x,z) &= \sum_{i\in N_+(z)} f_i(N,x,z-e^i) - \sum_{i\in N\backslash K} f_i(N,x,z)\\ &= \sum_{i\in N_+(z)} f_i^{ASh}(N,x,z-e^i) - \sum_{i\in N\backslash K} f_i^{ASh}(N,x,z).\end{aligned}$$

Because f is weakly symmetric, mimicking the argument in (b) gives

$$f_i(N,x,z) = \frac{1}{k}\left[\sum_{i\in N_+(z)} f_i^{ASh}(N,x,z-e^i) - \sum_{i\in N\backslash K} f_i^{ASh}(N,x,z)\right]$$

for all $i \in K$. By definition of f^{ASh}, this means that $f_i(N,x,z) = f_i^{ASh}(N,x,z)$ for all $i \in K$ and therefore, recalling (10), $f(N,x,z) = f^{ASh}(N,x,z)$, as desired. ¥

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