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A Theory of Random Consumer Demand

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Abstract

This paper presents a new theory of random consumer demand. The primitive is a collection of probability distributions, rather than a binary preference. Various assumptions constrain these distributions, including analogues of common assumptions about preferences such as transitivity, monotonicity and convexity. Two results establish a complete representation of theoretically consistent random demand.

The purpose of this theory of random consumer demand is application to empirical consumer demand problems. To this end, the theory has several desirable properties. It is intrinsically stochastic, so the econometrician can apply it directly without adding extrinsic randomness in the form of residuals. Random demand is parsimoniously represented by a single function on the consumption set. Finally, we have a practical method for statistical inference based on the theory, described in McCausland (2004), a companion paper.

1 Introduction

This paper describes a new theory of random consumer demand. There are n consumer goods, and a consumption set X whose elements are bundles of these goods. A consumer faces various budgets of consumption bundles, and chooses a single bundle from each budget. The primitive concept is that of a random demand function, which assigns to each budget a probability

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distribution on that budget. The distribution characterizes a random choice of a bundle from the budget.

According to the theory, a random demand function satisfies certain assumptions. We prove two theorems on the representation of theoretically consistent random demand functions by regular *L-utility*[1] functions on X. Regular L-utility functions are functions satisfying certain monotonicity and concavity restrictions. The first theorem is a representation theorem. It states that for any random demand function p satisfying the assumptions, there is a regular L-utility function u on X, unique up to the addition of a real constant, which represents (can be used to reconstruct) p. The second theorem is a completeness theorem, and states that every regular u is the representation of exactly one theoretically consistent random demand function p.

The purpose of this theory of random consumer demand is application to empirical consumer demand problems. To this end, the theory has several desirable features.

1. The representation facilitates inference. The representation theorems identify a theoretically consistent random demand function with a regular L-utility function,[2] and *vice versa*, so the econometrician can work with regular L-utility functions instead of random demand functions. McCausland (2004) demonstrates that inference based on the theory is practical.

2. The representation is parsimonious: it is a single function on the consumption set.

3. The theory is intrinsically stochastic, and so the econometrician can apply the theory directly without recourse to error terms or random preferences. In usual practice, distributions of errors and preferences are given without theoretical justification.

4. The "fit" of an observed choice is measured by the relative desirability of the choice and its feasible alternatives, rather than the Euclidean

[1] The term L-utility is meant to invite intuitive comparison with utility while distinguishing it from the usual representation of binary preferences. The L stands for Luce, whose representation in Luce (1959) is similar to the one described in this paper.

[2] We can always make the L-utility function unique by insisting that it take on a particular value at a particular point.

distance of the choice to the theoretically optimal choice. Varian (1990), in a paper on goodness-of-fit measures, argues for preferring the former to the latter.

5. Unlike standard consumer theory, the new theory does not rule out violations of the usual axioms of revealed preference. In practice, such violations are often observed. The new theory is more forgiving, without being undisciplined.

The theory of consumer demand described here draws on the literature on theories of stochastic choice, and the literature on theories of consumer demand. The two literatures find common ground in the following choice environment. There is a fixed set X of potential objects of choice, which we will call a *universe*. A decision maker faces various subsets of the universe, called *budgets*, and chooses a single element from each. There is a set $\mathcal{B}$ of allowable budgets, and we will call $(X, \mathcal{B})$ the *budget space*.

1.1 Random Preference and Random Choice

In mathematical psychology, theories of choice are usually stochastic rather than deterministic. An important reason is that individuals in experimental situations do not always behave invariably, even in well controlled binary choice situations. Many attribute randomness of choice to "conflict in choice", which arises when one choice is more desirable than another along one dimension, but less desirable along another.

Fishburn (1999) surveys stochastic theories of choice, emphasizing two classes that are "by far the most common". In *random choice* theories, the primitive is a collection probability measures. For each budget $B \in \mathcal{B}$, there is a probability measure p_B on B, describing an agent's random choice of an element from B. In *random preference* theories, the primitive is a probability distribution over a set of preferences. The realized preference governs the ultimate choice as in deterministic theories of choice.

In economics, stochastic theories of choice are less common than deterministic theories, although they are prevalent in the literature on discrete choice, surveyed by McFadden (1976) and Amemiya (1981). In another contrast to mathematical psychology, stochastic theories of choice in economics are almost always random preference theories, not random choice theories.

Specification of a random preference model involves providing a set of allowable preferences and a probability distribution over this set. While there is much theoretical guidance for the first choice, there is little for the second. In practice, the distribution is selected for analytical or computational convenience.

Random choice models feature a probability distribution for every budget. For all but the simplest budget spaces, this is a large amount of information, offering an excessive number of degrees of freedom. There are at least three approaches to add discipline to random choice models. One is to insist that they be rationalizable by random preference models. Falmagne (1978), Cohen (1980) and McFadden and Richter (1990) show that this is restrictive, and give conditions for rationalizability. A second approach is the elaboration of the choice process, as in the "Elimination by Aspects" theory of Tversky (1972a,1972b), which features a process of budget set reduction by the sequential elimination of elements. A third approach is to make assumptions about the various probability distributions. Some of these assumptions apply to binary choice probabilities (probability distributions on doubleton budgets), including several stochastic analogues of transitivity of binary preferences. Other assumptions relate distributions on sets to distributions on their subsets, such as the *regularity* assumption of Block and Marschak (1960), the *multiplicative inequality* of Sattath and Tversky (1976), and Luce's (1959) choice axiom.

The present paper features a random choice model, and uses Luce's choice axiom to constrain choice distributions.

1.2 Consumer Demand

In applications of stochastic theories of choice, budgets are usually small finite sets (often doubletons) with no special structure. Theories of consumer demand, in contrast, feature a budget space $(X, \mathcal{B})$ with a special structure. The universe X is the *classical consumption set*[3] $\mathcal{R}^n_+$, supporting the relation $\geq$ and the operations of addition and scalar multiplication and thus allowing special assumptions on preferences such as monotonicity and convexity.

The present work contributes to the sparse literature on theories of random consumer demand. Georgescu-Roegen (1958) provides several axioms on binary choice probabilities for doubleton budgets from the classical consumption set. Many of these axioms are analogous to familiar axioms on

[3]The set $\mathcal{R}_+$ is the set of non-negative real numbers and the set $\mathcal{R}_{++}$ is the set of positive real numbers.

binary preference relations. He derives some interesting properties of binary choice probabilities from the axioms. He recognizes the problem that the relationship between binary choice and multiple choice is not obvious and straightforward. He offers two reasonable axioms relating binary choice probabilities and multiple finite choice probabilities, and finds that these axioms are insufficient to derive the latter from the former. At the end of the paper he identifies two questions as "primordial problems [standing] before the the econometricians and the behavioral scientists". One of these questions is "What axioms are logically necessary and experimentally justified, to relate the multiple choice probability to that of the binary choice?"

Halldin (1974), in an explicit attempt to address this question uses "a simplified variant of Luce's probabilistic theory ... as an aid to construct ... a generalization of the classical theory of [consumer] demand." His starting point is Luce's (1959) Choice Axiom, whose statement appears in Section 2.3.2 of the present paper. His key assumption relating binary choice probabilities and multiple choice probabilities is a weaker variant of an implication of Luce's Choice Axiom. To this assumption, he adds stochastic analogues of the monotonicity and convexity assumptions in deterministic theories of consumer demand. He also implictly introduces an assumption relating choice densities on classical budgets to binary choice probabilities.

The result is a collection of choice densities, one for every budget. He shows certain properties of these densities, and compares the random demand they describe with the demand functions of deterministic consumers. Halldin does not intend his model to be used for the statistical estimation of the demand behavior of real consumers, and it is not suitable for the task. The econometrician is unable to estimate the choice densities of Halldin's model without multiple observations on each budget. There are certain restrictions on the choice densities from different budgets, but they are insufficient to allow estimation of the functions when only one choice is observed from each budget.

Bandyopadhyay et al. (1999) offer a stochastic analogue of the Weak Axiom of Revealed Preference (WARP). They show that it implies stochastic versions of two familiar implications of the WARP: that own sustitution effects are non-positive, and that demand curves of normal goods have a non-positive slope.

1.3 Random Preference and Random Choice Revisited

A natural question is whether theoretically consistent random demand can be rationalized by a random preference model. A similar, but simpler question is answered by McFadden (1973), who showed that a random choice model with non-zero binary choice probabilities and satisfying Luce's choice axiom (see Assumption 2.7) can be rationalized by a random utility model where utilities are the sum of a fixed level and a i.i.d. type I extreme value random variable.

Two complications stand in the way of answering this question in a satisfactory way. First, there are binary choice probabilities equal to zero, and so part 2 of Luce's Choice Axiom applies, not just part 1.

A second complication lies in the structure of the present problem. In the discrete choice models addressed by McFadden, the theoretical justification for random utility models is simply the transitivity and completeness of all preferences in the support of the primitive distribution over preferences. In a consumer demand context, additional assumptions about preferences, such as convexity and monotonicity, usually apply. The question of whether a distribution over the set of transitive complete preferences can rationalize theoretically consistent random demand is much less interesting than the question of whether a distribution over the set of monotone convex transitive complete preferences can do so, and the latter is a much more difficult question.

We leave the question answered, but point out that the new theory rests on assumptions about choice distributions, not distributions over preferences, and we appeal directly to the reasonableness of these assumptions, in the context of the intended empirical applications of the theory. While a suitable random preference rationalization would be somewhat interesting, we would not be embarrassed by a result that none exists.

1.4 Cardinal Utility and Random Choice

Debreu (1958) discusses a problem posed by Davidson and Marschak (1959). An agent faces doubleton budgets from a universe X and must choose a single object from each budget. For every pair of items (x, y) in X, $p(x, y)$ denotes the probability of choosing item x from the budget $\{x, y\}$. Debreu suggests that the relation $p(a, b) > p(c, d)$ might be read "a is preferred to b more than c is preferred to d". Debreu gives three axioms and shows that they imply the existence of a function $u : X \to \mathcal{R}$ such that for all $w, x, y, z \in X$, $p(w, x) \geq p(y, z) \iff u(w) - u(x) \geq u(y) - u(z)$. The function u, which Debreu calls a "cardinal utility function", is unique up to positive affine transformations.

Assumption 2.6 in the present paper, analogous to the assumption in standard consumer theory that preferences are convex, is an inequality restriction on certain pairs of binary choice probabilities, and may therefore be interpreted as statements about degrees of preference. It is this assumption that imposes concavity restrictions, which are cardinal in nature, on the L-utility function rather than ordinal quasi-concavity imposed on utility functions in standard theories of consumer demand.

1.5 Multiple Choice and Luce's Choice Axiom

One can think of a binary preference as describing choice on doubleton budgets. We are also interested in choices from larger budgets, however, and we usually assume that a choice x from a larger budget B satisfies $x \succeq y$ for all $y \in B$. This assumption is often implicit. For example, we may assume that a choice (or demand) correspondance (or function) is *rational*, in the following sense.

Definition 1.1 *Let $h : \mathcal{B} \to X$ be a choice correspondence. The binary preference $\succeq$* rationalizes *the choice correspondence h if*

$$h(B) = \{x \in B : x \succeq y \quad \forall y \in B\} \quad \forall B \in \mathcal{B}.$$

A choice correspondence is rational *with respect to a class of binary preferences if the class contains a binary preference which rationalizes it.*

We have already mentioned that the problem of relating binary choice and multiple choice in the context of stochastic choice does not have a single obvious solution like it does in the context of deterministic choice. In this paper, we relate binary and multiple choice using Luce's choice axiom, which has two important advantages. It is simple, and it is behind the parsimonious representation of random demand by a single function on the consumption set.

However, the axiom is also quite strong. Starting with Debreu (1960), many have criticised Luce's Choice Axiom for its implausible implications in certain choice contexts. We give the gist of these criticisms using an example where a consumer chooses bundles (x, y) of two goods. Let $A \equiv (1, 2)$, $B \equiv (2, 1)$, $B' \equiv (2 + \epsilon, 1 + \epsilon)$ and $B'' \equiv (2 - \epsilon, 1 + \epsilon)$, where ϵ is small. We suppose that the bundles A and B are similar enough in desirability that the consumer chooses A with probability $1/2$ when offered a choice between the two. We suppose that B', containing more of both goods than B, is easily recognized as more desirable, even if the difference is tiny. Finally, we suppose that the probabilities of choosing A from $\{A, B\}$, $\{A, B'\}$ and $\{A, B''\}$ are similar, due to the proximity of B' and B'' to B.

If all pairwise choice probabilities are positive, then only part 1 of Luce's choice axiom applies, and choice probabilities satisfy Independance of Irrelevant Alternatives (IIA). This implies that the the probability of choosing A from $\{A, B, B'\}$ is close to $1/3$, rather than close to the probability of choosing A from $\{A, B'\}$, close to $1/2$. It also implies that the probability of choosing B' from $\{B, B'\}$ is close to $1/2$, rather than close to one.

If, however, pairwise choice probabilities can be zero, then we can have the probability of choosing B' from $\{B, B'\}$ equal to one, and the probability of choosing A from $\{A, B, B'\}$ equal to the probability of choosing A from $\{A, B'\}$. In fact, the monotonicity assumption (Assumption 2.5) will insist that the probability of choosing B' from $\{B, B'\}$ be one, and the theory makes reasonable predictions for choice probabilities on $\{A, B, B'\}$ and its doubleton subsets. However, if we replace B' with B'' in the previous discussion, then the situation is problematic. We either have the probability of choosing B'' from $\{B, B''\}$ equal to zero or one, or we have the probability of choosing A from $\{A, B, B''\}$ close to $1/3$. Neither is plausible.

However, the suitability of the axiom should be considered in the context of the empirical purpose the theory is meant to serve. In the intended application, budgets are finite lattices of evenly spaced points in the classical budgets defined by prices and income.

1.6 Outline

In Section 2 we describe the new theory of random consumer demand. We begin with the formal definition of a random choice model and then specialize the choice environment to a consumer context. In this context, we will call the function mapping budgets to probability distributions a *random demand function*.

We then list assumptions about the random demand function. Three of these assumptions are restrictions on the binary choice probabilities, and are analogous to standard assumptions on binary preferences in general deterministic theories of choice (e.g. transitivity), or in theories of consumer demand (e.g. monotonicity). One remaining assumption, Luce's (1959) choice axiom, establishes a relationship between binary choice and multiple choice. It jointly constrains the choice probability distribution on a budget and the choice probability distributions on its subsets.

Section 3 presents the statements and proofs of two theorems about the representation of random demand functions satisfying the assumptions of Section 2 by regular L-utility functions on the classical consumption set. The first theorem states that if a random demand function satisfies the assumptions, then there exists a regular L-utility function, unique up to the addition of a constant, that "represents" it. The second theorem states that any regular L-utility function is the representation of exactly one random demand function satisfying the assumptions of Section 2.

We conclude in Section 4.

2 A Theory of Random Consumer Demand

In this section we describe our new theory of random consumer demand. We first define a random choice model. We then introduce assumptions restricting attention to environments where agents are consumers facing budgets of commodity bundles. Finally, we introduce various assumptions about choice probabilities. The assumptions are analogous to assumptions in general deterministic theories of choice, and in consumer demand theory in particular.

2.1 Definition of a Random Choice Model

Fishburn (1999) is a good survey of random choice models in mathematical psychology, economics and other disciplines. Terminology, notation, and the precise definition are not universally consistent, and we follow the conventions of McFadden and Richter (1990) most closely.

A random choice model characterizes a decision maker who at various times faces a budget of choice objects and must select a single object from it. The decision is intrinsically random, meaning that probability distributions characterizing choices from budgets are primitive.

Part of a random choice model is the *choice environment*, which includes the set of potential objects of choice, the set of budgets that the agent may face, and the set of choice events to which we assign probabilities. A random choice model is completed by specifying a random choice function, which assigns probabilities to all possible choice events.

Definition 2.1 *The ordered 4-tuple* $(X, \mathcal{B}, \mathcal{C}, p)$ *is a* random choice model *if the following hold:*

- X *is a non-empty set. We will call* X *the* universe *and its elements* objects*.*
- $\mathcal{B}$ *is a set of non-empty subsets of* X*. We will call* $(X, \mathcal{B})$ *the* budget space *and the elements of* $\mathcal{B}$ budgets*. A budget is interpreted as a set of objects from which an decision maker must choose a single element.*
- $\mathcal{C}$ *is a function on* $\mathcal{B}$ *assigning to each budget* B *an algebra* $\mathcal{C}_B$ *of subsets of* B*. For every budget* $B \in \mathcal{B}$*, a budget subset* $C \in \mathcal{C}_B$ *is an event of the measurable space* $(B, \mathcal{C}_B)$*, and is interpreted as the choice by the economic agent of some element of* C *when faced with budget* B*.*
- p *is a function on* $\mathcal{B}$ *assigning to each budget* B *a finitely additive probability measure* p_B *on measurable space* $(B, \mathcal{C}_B)$*. We will call* p *the* random choice function*. In the special case of consumer choice, we will use the term* random demand function*. For every budget* $B \in \mathcal{B}$ *and every budget subset* $C \in \mathcal{C}_B$*,* $p_B(C)$ *is the probability that the decision maker chooses some element of* C *when faced with budget* B*.*

2.2 Assumptions on X, $\mathcal{B}$, and $\mathcal{C}$

The following assumptions specialize the choice environment to a consumer setting. The first assumption establishes the universe X as the classical consumption set for n goods.

Assumption 2.1 *For some integer* $n \geq 2$*,* $X = \mathcal{R}^n_+$*.*

The second assumption states that the consumer may face, as a budget, any non-empty finite subset of the consumption set. While widespread in the mathematical psychology literature, the finiteness of budgets goes against the grain of consumer theory, where the consumer faces budgets of the form $\{x \in X : w \cdot x \leq m\}$, where w is a vector of positive prices of the n goods and m is the consumer's non-negative income. These *classical* budgets are not finite, and therefore not in our budget space.

However, we may consider finite lattices of points in classical budgets of whatever density we like, so this is not a serious restriction. Real consumers and econometricians only have a finite set of numbers available to express the quantities of goods they demand or observe. Furthermore, the currency used in transactions is not infinitely divisible.

Assumption 2.2 *The set $\mathcal{B}$ of budgets is the set of all non-empty finite subsets of X.*

The third assumption states allows us to assign probabilities to all individual objects in all budgets. This is a natural and common assumption in cases where all budgets in $\mathcal{B}$ are finite.

Assumption 2.3 *For every budget $B \in \mathcal{B}$, the algebra $\mathcal{C}_B$ is the power set of B.*

2.3 Assumptions on p

We move on to assumptions about the random demand function p. Throughout this section, we suppose $(X, \mathcal{B}, \mathcal{C}, p)$ is a random choice model satisfying assumptions 2.1, 2.2 and 2.3. We classify assumptions on p as assumptions on binary choice or assumptions relating binary and multiple choice.

2.3.1 Binary Choice Probabilities

The assumptions on binary choice probabilities are analogous to assumptions about binary preferences in deterministic theories. The following notation is a common and useful shorthand for binary choice probabilities.

Definition 2.2 *For every $x, y \in X$ such that $x \neq y$, define $p(x, y) \equiv p_{\{x,y\}}(\{x\})$.*

In deterministic theories of choice binary preferences are usually complete and transitive. A probability distribution on a singleton set must assign probability 1 to that set, and for all distinct $x, y \in X$, $p(x, y) + p(y, x) = 1$. The probabilistic framework thus builds in assumptions analogous to completeness.

In a survey of stochastic utility, Fishburn (1999) describes nine stochastic analogues of the transitivity assumption, including the following.

Assumption 2.4 (Moderate Stochastic Transitivity) *For every* $x, y, z \in X$ *such that* $x \neq y$, $y \neq z$, *and* $x \neq z$, $\min(p(x,y), p(y,z)) \geq \frac{1}{2} \Rightarrow p(x,z) \geq \min(p(x,y), p(y,z))$.

We will see in Section 3.2.1 that Luce's choice axiom, the key assumption relating binary and multiple choice probabilities, introduces further restrictions on binary choice. It implies a stronger condition on $p(x,y)$, $p(y,z)$ and $p(x,z)$ whenever these probabilities are all in $(0,1)$.

We now turn to analogues of assumptions on preferences in consumer demand theory in particular. The next assumption is analogous to monotonicity, expressing the idea that more is better than less. The stochastic nature of human choice is often attributed to conflict in choice. When distinct objects x and y satisfy $x \geq y$[4], there is no conflict in choice: one object is unambiguously better in at least one dimension, and no worse in any dimension. We will say that x vector-dominates y. The assumption states that x will invariably be chosen from the budget $\{x, y\}$. The assumption also states that if neither $x \geq y$ nor $y \geq x$, then each is chosen with positive probability from $\{x, y\}$.

Assumption 2.5 (Monotonicity) *For every* $x, y \in X$ *such that* $x \neq y$, $x \geq y \iff p(x,y) = 1$.

The next assumption is analogous to the classical convexity assumption, which expresses the idea that to obtain more and more of one good, the consumer is less and less willing to forgo other goods. The following assumption expresses the similar idea that as one moves through the consumption set in any direction, the strength of the propensity to choose a farther element over a nearer does not increase.

Assumption 2.6 *For every* $x, y \in X$, $p(\frac{1}{2}x + \frac{1}{2}y, x) \geq p(y, \frac{1}{2}x + \frac{1}{2}y)$.

2.3.2 Multiple Choice Probabilities

The assumption relating binary choice probabilities and multiple choice probabilities is analogous to the assumption that binary preferences rationalize demand correspondences.

The following assumption, due to Luce (1959), constrains choice distributions across budgets. In particular, it relates binary choice probabilities to

[4]For vectors $x, y \in \mathcal{R}^n$, $x \geq y$ means $x_i \geq y_i$ for $i = 1, \ldots, n$.

multiple choice probabilities. It is the key assumption allowing the representation of random consumer demand by a single function on the consumption set.

The first part of the axiom concerns budget sets for which all choices on binary subsets have non-zero probability. For such budgets and their subsets, the axiom states that relative choice probabilites are independent of the presence of other alternatives in the budget: for every budget B and non-empty $C \subseteq B$, the distribution $p_C(\cdot)$ coincides with the conditional distribution $p_B(\cdot|C)$ on C. Luce calls this part of the axiom "*a* probabilistic version of ... [Arrow's] independence-from-irrelevant-alteratives idea."

The second part of the axiom concerns budget sets for which some choices on binary subsets have zero probability. It says that in a budget B with elements x and y satisfying $p(x, y) = 0$, x may be ignored: the probability of choosing x from B is zero, and the probability of choosing another element from B is the same as the probability of choosing it from $B\backslash\{x\}$.

Assumption 2.7 (Luce's Choice Axiom) *For every $B \in \mathcal{B}$, and every $S \subseteq B$,*

1. *If $p(x, y) \in (0, 1)$ for every $x, y \in B$ such that $x \neq y$, then for every $R \subseteq S$,*
$$p_B(R) = p_S(R) \cdot p_B(S).$$

2. *If $p(x, y) = 0$ for some $x, y \in B$ such that $x \neq y$, then*
$$p_B(S) = p_{B\backslash\{x\}}(S\backslash\{x\}).$$

3 Representation Theorems

The two theorems of this section concern the representation of random demand models by regular functions on the consumption set. The following definition of regularity is specific to this paper.

Definition 3.1 *A function $u : X\backslash\{0\} \rightarrow \mathcal{R}$ is* regular *if*

1. *u is non-decreasing, and*

2. *for every $w \in \mathcal{R}^n_{++}$ and every $m \in \mathcal{R}_{++}$, u is concave on classical budget frontier $\{x \in X : w \cdot x = m\}$.*

The first theorem is a representation thereom, asserting the existence of the representation and its uniqueness up to the addition of a real constant. The second theorem asserts the completeness of the representation.

3.1 A Theorem on the Existence and Uniqueness of the Representation

Let X, $\mathcal{B}$ and $\mathcal{C}$ satisfy Assumptions 2.1, 2.2 and 2.3.

Theorem 3.1 (Existence and Uniqueness of Representation) *If* $(X, \mathcal{B}, \mathcal{C}, p)$ *is a random choice model satisfying Assumptions 2.4, 2.5, 2.6, and 2.7, then there exists a regular function* $u : X\backslash\{0\} \to \mathcal{R}_{++}$*, unique up to the addition of a real constant, such that for every budget* $B \in \mathcal{B}$*, and every event* $C \in \mathcal{C}_B$*,* $p_B(C)$ *is given by*

$$p_B(C) = \sum_{x \in C \cap \hat{B}} e^{u(x)} \Bigg/ \sum_{y \in \hat{B}} e^{u(y)} \,, \tag{1}$$

where $\hat{B}$ *is the budget frontier (definition 3.2) of* B*.*

The following definitions establish some convenient notation. The first definition identifies, for each budget B, the subset $\hat{B}$ of objects not vector-dominated by other elements of B. We will see that $\hat{B}$ is the set of objects chosen with non-zero probability from B.

Definition 3.2 *For any budget* $B \in \mathcal{B}$*, define* $\hat{B}$*, the* frontier *of* B*, by*

$$\hat{B} = \{x \in B : \text{ there is no } y \in B\backslash\{x\} \text{ such that } y \geq x\}.$$

The next definition establishes the relational symbols $\not\asymp$ and $\asymp$ as shorthand notation denoting whether or not a pair of objects in X features one object vector-dominating the other. The importance of this notation lies in the consequence of Assumption 2.5 (Monotonicity) that for all $x, y \in X$, the choice from $\{x, y\}$ is non-degenerate (i.e. $p(x, y) \in (0, 1)$) if and only if $x \asymp y$.

Definition 3.3 *Define the binary relation* $\asymp$ *on* X *by* $x \asymp y \iff x \not\geq y$ *and* $y \not\geq x$*, and let the binary relation* $\not\asymp$ *on* X *denote its complement.*

3.2 Proof of Theorem 3.1

Let $(X, \mathcal{B}, \mathcal{C}, p)$ be a random choice model satisfying Assumptions 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, and 2.7.

The proof proceeds as follows. First we prove a useful triplet result. Then we construct a function u on $X\backslash\{0\}$. We next show that for all $x, y \in X$ such that $x \asymp y$ (i.e. neither $x \geq y$ nor $y \geq x$), we can use u to reconstruct the binary choice probability $p(x, y)$. We then use this result to show that for all budgets B, we can use u to reconstruct the choice distribution p_B. Next, we show that u is regular. Finally, we show that u is unique up to the addition of a real constant.

3.2.1 A Triplet Result

The following triplet result is a useful intermediate result.

Claim 3.1 *For all objects $x, y, z \in X$ satisfing $x \asymp z$, $y \asymp z$ and $z \asymp x$,*

$$p(x, y)p(y, z)p(z, x) = p(y, x)p(z, y)p(x, z). \tag{2}$$

Proof. Let $x, y, z \in X$ satisfy $x \asymp z$, $y \asymp z$ and $z \asymp x$. Let budget $B \equiv \{x, y, z\}$. Assumptions 2.5 (Monotonicity) and 2.7 (Luce's Choice Axiom) (part 1) give us the following six equations,

$$p_B(\{x, z\}) \cdot p(x, z) = p_B(\{x\}) = p_B(\{x, y\}) \cdot p(x, y)$$

$$p_B(\{x, y\}) \cdot p(y, x) = p_B(\{y\}) = p_B(\{y, z\}) \cdot p(y, z)$$

$$p_B(\{y, z\}) \cdot p(z, y) = p_B(\{z\}) = p_B(\{x, z\}) \cdot p(z, x)$$

and guarantee that all binary probabilities in the six equations are non-zero. Since $\{x, y\} \cup \{x, z\} = B$, $p_B(\{x, y\}) + p_B(\{x, z\}) \geq 1$, $p_B(\{x, y\})$ and $p_B(\{x, z\})$ cannot both be zero, and therefore $p_B(\{x\}) > 0$. Therefore $p_B(\{x, y\}) > 0$ and $p_B(\{x, z\}) > 0$. Similarly, all the probabilities in the second and third lines must also be non-zero.

We thus obtain

$$1 = \frac{p_B(\{x\})}{p_B(\{y\})} \cdot \frac{p_B(\{y\})}{p_B(\{z\})} \cdot \frac{p_B(\{z\})}{p_B(\{x\})} = \frac{p(x, y)}{p(y, x)} \cdot \frac{p(y, z)}{p(z, y)} \cdot \frac{p(z, x)}{p(x, z)}$$

and equation (2) immediately follows. □

3.2.2 Construction of u

We now construct our representation. Our representation resembles the representation of Theorem 4 in Luce (1959). Furthermore, the technique we use to construct our representation is similar to that of Luce. However, the two additional assumptions that Luce makes to guarantee the uniformity of his representation across budgets are different from the key assumption that we use to guarantee this uniformity, namely the $\Longleftarrow$ part of the Monotonicity assumption. This assumption neither implies nor is implied by Luce's two additional assumptions.

We first construct a function v on $X\backslash\{0\}$, and then define u as $\log v$.

Choose an object $a \in X$ such that $a > 0$, and a real constant $k > 0$. Define $v(a) \equiv k$. Now consider any object $b \in X\backslash\{0, a\}$. If $b \asymp a$, then $p(a, b) > 0$ and $p(b, a) > 0$ by Assumption 2.5 (Monotonicity). We define

$$v(b) \equiv k \cdot \frac{p(b, a)}{p(a, b)},$$

and note that it must be positive.

If $b \not\asymp a$, then let $X_{ab} \equiv \{x \in X : x \asymp a \text{ and } x \asymp b\}$, and define $v_{ab} : X_{ab} \to \mathcal{R}$ by

$$v_{ab}(x) = k \cdot \frac{p(b, x)}{p(x, b)} \cdot \frac{p(x, a)}{p(a, x)} \qquad \forall x \in X_{ab}.$$

We will show that $X_{ab} \neq \emptyset$ and that v_{ab} is well-defined, positive and constant. We will then define $v(b)$ to be this constant value.

We can express X_{ab} as the following union of rectangles.

$$X_{ab} = \bigcup_{\substack{i,j \in \{1,\dots,n\} \\ i \neq j}} X_{ab}^{\{i,j\}} \equiv \bigcup_{\substack{i,j \in \{1,\dots,n\} \\ i \neq j}} \{x \in X : x_i < \underline{x}_i \text{ and } x_j > \overline{x}_j\},$$

where[5]

$$\underline{x} \equiv \begin{cases} a & a \leq b \\ b & a \geq b \end{cases} \quad \text{and} \quad \overline{x} \equiv \begin{cases} b & a \leq b \\ a & a \geq b. \end{cases}$$

Since $a > 0$ and $b \neq 0$, at least one of the rectangles $X_{ab}^{\{i,j\}}$ is non-empty, and so $X_{ab} \neq \emptyset$. Assumption 2.5 (Monotonicity) guarantees that for all

[5]Since $b \not\asymp a$, it must be the case that either $a \geq b$ or $b \geq a$.

$x \in X_{ab}$, $p(x,a)$, $p(a,x)$, $p(x,b)$, and $p(b,x)$ are all positive. Therefore $v_{ab}(x)$ is well-defined and positive for all $x \in X_{ab}$.

We now show that v_{ab} is constant on X_{ab}. We take arbitrary elements $x, y \in X_{ab}$, and show that $v(x) = v(y)$. We consider the cases $x \asymp y$ and $x \not\asymp y$ separately. First, suppose $x, y \in X_{ab}$ and $x \asymp y$. We apply the triplet result twice to obtain

$$\frac{p(x,a)}{p(a,x)} \cdot \frac{p(a,y)}{p(y,a)} = \frac{p(x,y)}{p(y,x)} = \frac{p(x,b)}{p(b,x)} \cdot \frac{p(b,y)}{p(y,b)}$$

$$k \cdot \frac{p(b,x)}{p(x,b)} \cdot \frac{p(x,a)}{p(a,x)} = k \cdot \frac{p(b,y)}{p(y,b)} \cdot \frac{p(y,a)}{p(a,y)}$$

$$v_{ab}(x) = v_{ab}(y)$$

Now suppose $x, y \in X_{ab}$ and $x \not\asymp y$. Objects x and y must be in the same rectangle $X_{ab}^{\{i,j\}}$, since otherwise $x \asymp y$. We now construct a $z \in X_{ab}$ such that $x \asymp z$ and $y \asymp z$, and thus that $v_{ab}(z)$ is equal to both $v_{ab}(x)$ and $v_{ab}(y)$. Define $z \equiv (z_1, \ldots, z_n) \in X_{ab}$ as follows.

$$z_k \equiv \begin{cases} \frac{1}{2}\underline{x}_i + \frac{1}{2}\max(x_i, y_i) & k = i \\ \frac{1}{2}\overline{x}_j + \frac{1}{2}\min(x_j, y_j) & k = j \\ 0 & k \in \{1, \ldots, n\} \backslash \{i, j\}. \end{cases}$$

Since $x \asymp z$ and $y \asymp z$, $v_{ab}(x) = v_{ab}(z) = v_{ab}(y)$.

The cases $x \asymp y$ and $x \not\asymp y$ are exhaustive, so we have shown that v_{ab} is constant and positive on the non-empty set X_{ab}. We now define $v(b)$ to be this constant value.

Since b was an arbitrary element of $X \backslash \{a, 0\}$, we have constructed a function v on the entire set $X \backslash \{0\}$.

Now define $u : X \backslash \{0\} \to \mathcal{R}$ by $u(x) = \log v(x)$ for all $x \in X \backslash \{0\}$.

3.2.3 Non-Degenerate Choices on Binary Budgets

We now show that for all $x, y \in X$ such that $x \asymp y$ (i.e. neither $x \geq y$ nor $y \geq x$), we can use v to reconstruct the binary choice probability $p(x,y)$.

Claim 3.2 *For every $x, y \in X$ such that $x \asymp y$,*

$$\frac{p(x,y)}{p(y,x)} = \frac{v(x)}{v(y)}. \tag{3}$$

Proof. Let $a \in X$ be the object used in Section 3.2.2 to construct v. Let $x, y \in X\backslash\{0\}$ be such that $x \asymp y$. Choose $i, j \in \{1, \ldots, n\}$ such that $y_i > 0$ and $j \neq i$.

First we show that if $x_i > 0$, then (3) holds. Then we use this result to show that (3) holds even if $x_i = 0$.

Suppose $x_i > 0$, and define $z \equiv (z_1, \ldots, z_n)$ by

$$z_k \equiv \begin{cases} \max(x_j, y_j, a_j) + 1 & k = j \\ 0 & k \in \{1, \ldots, n\}\backslash\{j\}. \end{cases}$$

Then $z \asymp x$, $z \asymp y$, and $z \asymp a$. Using the triplet result (Claim 3.1) and the definition of v from Section 3.2.2, we obtain

$$\frac{p(x,y)}{p(y,x)} = \frac{p(x,z)}{p(z,x)} \cdot \frac{p(z,y)}{p(y,z)} = v(x) \cdot \frac{p(a,z)}{p(z,a)} \cdot \left[v(y) \cdot \frac{p(a,z)}{p(z,a)}\right]^{-1} = \frac{v(x)}{v(y)}.$$

Now suppose $x_i = 0$. Let $w = \frac{1}{2}x + \frac{1}{2}y$. Then $w_i > 0$, $w \asymp x$, and $w \asymp y$, and so by the result just proved,

$$\frac{p(x,w)}{p(w,x)} = \frac{v(x)}{v(w)} \quad \text{and} \quad \frac{p(y,w)}{p(w,y)} = \frac{v(y)}{v(w)}.$$

By the triplet result (Claim 3.1),

$$\frac{p(x,y)}{p(y,x)} = \frac{p(x,w)}{p(w,x)} \cdot \frac{p(w,y)}{p(y,w)} = \frac{v(x)/v(w)}{v(y)/v(w)} = \frac{v(x)}{v(y)}. \ \square$$

3.2.4 Choices on Finite Budgets

We now use the previous result to show that for every budget B, we can use u to reconstruct the choice distribution p_B.

Claim 3.3 *For every budget $B \in \mathcal{B}$, and every event $C \in \mathcal{C}_B$,*

$$p_B(C) = \sum_{x \in C \cup \hat{B}} e^{u(x)} \Bigg/ \sum_{y \in \hat{B}} e^{u(y)} ,$$

where $\hat{B}$ is the budget frontier (definition 3.2) of B.

Proof. Let $B \in \mathcal{B}$. Repeated application of Assumption 2.7 (Luce's Choice Axiom), part 2, gives $p_B(x) = p_{\hat{B}}(x)$ for all $x \in \hat{B}$, and since p_B is a probability measure, $p_B(x) = 0$ for all $x \in B \backslash \hat{B}$.

Using the result on binary budgets (Claim 3.2) and Assumption 2.7 (Luce's Choice Axiom), part 1, we have for every $x \in \hat{B}$,

$$\begin{aligned}\frac{1}{p_{\hat{B}}(x)} &= \frac{\sum_{y \in \hat{B}} p_{\hat{B}}(y)}{p_{\hat{B}}(x)} = \sum_{y \in \hat{B}} \frac{p_{\hat{B}}(y)}{p_{\hat{B}}(x)} = 1 + \sum_{y \in \hat{B} \backslash \{x\}} \frac{p_{\hat{B}}(\{x,y\}) p(y,x)}{p_{\hat{B}}(\{x,y\}) p(x,y)} \\ &= 1 + \sum_{y \in \hat{B} \backslash \{x\}} \frac{v(y)}{v(x)} = \frac{\sum_{y \in \hat{B}} v(y)}{v(x)}\end{aligned}$$

Therefore for every $x \in B$,

$$p_B(x) = \begin{cases} v(x) \Big/ \sum_{y \in \hat{B}} v(y) = e^{u(x)} \Big/ \sum_{y \in \hat{B}} e^{u(y)} & x \in \hat{B} \\ 0 & x \in B \backslash \hat{B}. \end{cases}$$

Since p_B is a probability measure, the claim follows. □

3.2.5 Regularity of u

The following results establish the regularity of u.

Claim 3.4 *The function u is non-decreasing.*

Proof. Let $x, y \in X \backslash \{0\}$ satisfy $x \geq y$. Note that $p(x,y) = 1$ by Assumption 2.5 (Monotonicity). Let $z \in X \backslash \{0\}$ satisfy $z \asymp x$ and $z \asymp y$.

We first show that $p(z,y) \geq p(z,x)$.

Case $p(z,x) \geq \frac{1}{2}$: Apply Assumption 2.4 (Moderate Stochastic Transitivity) to obtain $p(z,y) \geq \min(p(z,x), p(x,y)) \geq p(z,x)$.

Case $p(z,x) < \frac{1}{2}$: Suppose to the contrary that $p(z,y) < p(z,x)$. Then $p(y,z) > p(x,z) > \frac{1}{2}$. By Assumption 2.4 (Moderate Stochastic Transitivity), we obtain $p(x,z) \geq \min(p(x,y), p(y,z)) > p(y,z)$, and therefore $p(z,x) < p(z,y)$, which contradicts $p(z,y) < p(z,x)$.

Since $p(z,y) \geq p(z,x)$, $v(x) \geq v(y)$. So we have $x \geq y \Rightarrow v(x) \geq v(y)$ for all $x, y \in X \backslash \{0\}$. That is, v is non-decreasing. Therefore u is also non-decreasing. □

Claim 3.5 *For every* $w \in \mathcal{R}^n_{++}$ *and every* $m \in \mathcal{R}_{++}$, u *is concave on classical budget frontier* $\{z \in X : w \cdot z = m\}$.

Proof. Let $w \in \mathcal{R}^n_{++}$ and $m \in \mathcal{R}_{++}$. Let $x, y \in \{z \in X : w \cdot z = m\}$. By Assumption 2.6,

$$p(\tfrac{1}{2}x + \tfrac{1}{2}y, x) \geq p(y, \tfrac{1}{2}x + \tfrac{1}{2}y) \tag{4}$$

and

$$p(\tfrac{1}{2}x + \tfrac{1}{2}y, y) \geq p(x, \tfrac{1}{2}x + \tfrac{1}{2}y). \tag{5}$$

It must be the case that $x \asymp \frac{1}{2}x + \frac{1}{2}y \asymp y$, since otherwise, $w'x \neq w'y$. Therefore the probabilities in equations 4 and 5 are all non-zero, and so

$$\frac{p(\frac{1}{2}x + \frac{1}{2}y, x)}{p(x, \frac{1}{2}x + \frac{1}{2}y)} \geq \frac{p(y, \frac{1}{2}x + \frac{1}{2}y)}{p(\frac{1}{2}x + \frac{1}{2}y, y)}$$

$$\frac{v(\frac{1}{2}x + \frac{1}{2}y)}{v(x)} \geq \frac{v(y)}{v(\frac{1}{2}x + \frac{1}{2}y)}$$

$$\left[v(\tfrac{1}{2}x + \tfrac{1}{2}y)\right]^2 \geq v(x)v(y)$$

$$u(\tfrac{1}{2}x + \tfrac{1}{2}y) \geq \tfrac{1}{2}u(x) + \tfrac{1}{2}u(y).$$

Since this is true for all distinct x, y on the classical budget frontier, u must be concave there. □

3.2.6 Uniqueness of u

Claim 3.6 *The representation* u *is unique up to the addition of a real constant.*

Proof. We want to show that if u and u' are both regular, and both represent $(X, \mathcal{B}, \mathcal{C}, p)$, then there exists a constant $c \in \mathcal{R}$ such that $u' = u + c$. Suppose u and u' are both regular, and both represent $(X, \mathcal{B}, \mathcal{C}, p)$. Let $z \in X \backslash \{0\}$ satisfy $z > 0$. Let $c \equiv u'(z) - u(z)$.

Now choose any $x \in X \backslash \{0\}$. If $x = z$, then $u'(x) = u(x) + c$ immediately. If $x \neq z$, then choose a $y \in X_{zx}$. Then

$$e^{u(x)-u(y)} = \frac{v(x)}{v(y)} = \frac{p(x, y)}{p(y, x)} = \frac{v'(x)}{v'(y)} = e^{u'(x)-u'(y)}$$

and

$$e^{u(y)-u(z)} = \frac{v(y)}{v(z)} = \frac{p(y,z)}{p(z,y)} = \frac{v'(y)}{v'(z)} = e^{u'(y)-u'(z)}$$

and therefore

$$u'(x) = u(x) + [u'(y) - u(y)] = u(x) + [u'(z) - u(z)] = u(x) + c. \square$$

3.3 A Theorem on the Completeness of the Representation

Theorem 3.2 (Completeness of Representation) *Let X, $\mathcal{B}$ and $\mathcal{C}$ satisfy Assumptions 2.1, 2.2 and 2.3. If $u : X\backslash\{0\} \to \mathcal{R}_{++}$ is regular, then there exists a unique p such that*

1. *$(X, \mathcal{B}, \mathcal{C}, p)$ is a random choice model satisfying Assumptions 2.4, 2.5, 2.6, and 2.7; and*
2. *for every budget $B \in \mathcal{B}$, and every event $C \in \mathcal{C}_B$, $p_B(C)$ is given by (1).*

3.4 Proof of Theorem 3.2

Let X, $\mathcal{B}$, and $\mathcal{C}$ satisfy Assumptions 2.1, 2.2 and 2.3. Let function $u : X\backslash\{0\} \to \mathcal{R}_{++}$ be any regular function. We first use u to construct a p, and note that u represents p. We then show that $(X, \mathcal{B}, \mathcal{C}, p)$ satisfies Assumptions 2.4, 2.5, 2.6 and 2.7.

Construct p so that for every $B \in \mathcal{B}$ and every $C \in \mathcal{C}_B$,

$$p_B(C) \equiv \sum_{x \in C \cap \hat{B}} e^{u(x)} \Bigg/ \sum_{y \in \hat{B}} e^{u(y)} \ .$$

Clearly, p is uniquely specified, and u represents it.

Claim 3.7 $(X, \mathcal{B}, \mathcal{C}, p)$ *satisfies Assumption 2.4 (Moderate Stochastic Transitivity).*

Proof. Let $x, y, z \in X$ satisfy $x \neq y$, $y \neq z$ and $x \neq z$ and suppose $p(x,y) \geq \frac{1}{2}$ and $p(y,z) \geq \frac{1}{2}$. If $x \geq y$, then $v(x) \geq v(y)$ by monotonicity of v. If $x \asymp y$, then $p(x,y) = v(x)/[v(x)+v(y)]$ and so $v(x) \geq v(y)$. We can rule out $y \geq x$ since it contradicts $v(x,y) \geq \frac{1}{2}$. Therefore $v(x) \geq v(y)$. Similarly, $v(y) \geq v(z)$.

We show the result that $p(x,z) \geq \min(p(x,y), p(y,z))$ for three cases:

Case $x \geq z$: Then $p(x,z) = 1 \geq \min(p(x,y), p(y,z))$ and we are done.

Case $z \geq x$: Since $v(x) \geq v(y) \geq v(z)$, monotonicity of v rules out this case.

Case $x \asymp z$: Then $p(x,z)/p(z,x) = v(x)/v(z)$. Since $x \geq y$ and $y \geq z$ imply $x \geq z$, at least one of $x \asymp y$ and $y \asymp z$ must hold. If $x \asymp y$, then $p(x,y)/p(y,x) = v(x)/v(y)$ and therefore

$$\frac{p(x,z)}{p(z,x)} = \frac{v(x)}{v(z)} \geq \frac{v(x)}{v(y)} = \frac{p(x,y)}{p(y,x)} \geq \min\left(\frac{p(x,y)}{p(y,x)}, \frac{p(y,z)}{p(z,y)}\right).$$

If $y \asymp z$, then $p(y,z)/p(z,y) = v(y)/v(z)$ and therefore

$$\frac{p(x,z)}{p(z,x)} = \frac{v(x)}{v(z)} \geq \frac{v(y)}{v(z)} = \frac{p(y,z)}{p(z,y)} \geq \min\left(\frac{p(x,y)}{p(y,x)}, \frac{p(y,z)}{p(z,y)}\right).$$

Either way, since the transformation $f(p) = p/(1-p)$ is monotonically increasing, $p(x,z) \geq \min(p(x,y), p(y,z))$. □

Claim 3.8 *$(X, \mathcal{B}, \mathcal{C}, p)$ satisfies Assumption 2.5 (Monotonicity).*

Proof. Let $x, y \in X$ satisfy $x \neq y$. If $x \geq y$, then $p(x,y) = p_{\{x\}}(\{x\}) = 1$. If $y \geq x$, then $p(x,y) = 0 \neq 1$. If $x \asymp y$, then $p(x,y) = \exp u(x)/[\exp u(x) + \exp u(y)] < 1$. □

Claim 3.9 *$(X, \mathcal{B}, \mathcal{C}, p)$ satisfies Assumption 2.6.*

Proof. Let $x, y \in X$ satisfy $x \asymp y$. Let $z \equiv \frac{1}{2}x + \frac{1}{2}y$. Since x, y, and z lie on a budget frontier, condition 2 of Theorem 3.1 gives

$$u(z) \geq \tfrac{1}{2}u(x) + \tfrac{1}{2}u(y)$$
$$\tfrac{1}{2}u(z) - \tfrac{1}{2}u(y) \geq \tfrac{1}{2}u(x) - \tfrac{1}{2}u(z)$$
$$u(z) - u(y) \geq u(x) - u(z).$$

Now since $x \asymp z$ and $y \asymp z$,

$$p(z,x) = e^{u(z)}/[e^{u(x)} + e^{u(z)}] = 1/[1 + e^{u(x)-u(z)}]$$
$$p(y,z) = e^{u(y)}/[e^{u(y)} + e^{u(z)}] = 1/[1 + e^{u(z)-u(y)}]$$

and therefore $p(z,x) \geq p(y,z)$. □

Claim 3.10 *$(X, \mathcal{B}, \mathcal{C}, p)$ satisfies Assumption 2.7 (Luce's Choice Axiom).*

Proof. Let $B \in \mathcal{B}$, and let $S \subseteq B$.

First take the case that $p(x, y) \in (0, 1)$ for every $x, y \in B$ such that $x \neq y$. Then $\hat{B} = B$ and $\hat{S} = S$. Let $R \subseteq S$. Then

$$p_B(R) = \sum_{x \in R} e^{u(x)} \Big/ \sum_{y \in B} e^{u(y)},$$

$$p_S(R) = \sum_{x \in R} e^{u(x)} \Big/ \sum_{y \in S} e^{u(y)},$$

$$p_B(S) = \sum_{x \in S} e^{u(x)} \Big/ \sum_{y \in B} e^{u(y)},$$

and clearly, $p_B(R) = p_S(R) \cdot p_B(S)$.

Now take the case that there exists $x, y \in B$ such that $x \neq y$ and $p(x, y) = 0$. Then $x \notin \hat{B}$ and $\hat{B} = \widehat{B \backslash \{x\}}$ and

$$\begin{aligned} p_B(S) &= \sum_{x \in S \cap \hat{B}} e^{u(x)} \Big/ \sum_{y \in \hat{B}} e^{u(y)} = \sum_{x \in S \backslash \{x\} \cap \widehat{B \backslash \{x\}}} e^{u(x)} \Big/ \sum_{y \in \widehat{B \backslash \{x\}}} e^{u(y)} \\ &= p_{B \backslash \{x\}}(S \backslash \{x\}). \ \square \end{aligned}$$

4 Conclusions

We have drawn from stochastic theories of choice in mathematical psychology, and deterministic theories of choice and consumer demand in economics to develop a new theory of random consumer demand. The theory has several desirable properties that motivate its application to empirical consumer demand problems.

The representation theorems establish an identification of any theoretically consistent random demand function with a regular L-utility function and *vice versa*. An econometrician can thus work with regular L-utility functions rather than with random demand functions directly. This is more convenient.

The theory is intrinsically stochastic, which allows the econometrician to apply the theory without adding extrinsic randomness in the form of residuals.

The theory does not stand or fall on a sharp testable implication such as the Strong Axiom of Revealed Preference. There are degrees of fit, and the theory can be evaluated on this basis. The theory measures the "fit" of an observed choice to a regular utility function by the relative L-utilities of the choice and its feasible alternatives. This is an intrinsic measure of fit, and stands in contrast to extrinsic measures of fit, such as the Euclidean distance of a choice to the object which maximizes the utility function.

In a related paper, we describe a practical method for Bayesian inference using the theory. We follow Geweke and Petrella (2004) in using polynomials on a transformed consumption set to approximate L-utility functions, giving us flexibility and regularity on a large subset of the consumption set. We apply the theory and inferential methods to analyse data from a consumer experiment.

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