Budget-Balance, Fairness and Minimal Manipulability*

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Abstract

A common real-life problem is to fairly allocate a number of indivisible objects and a fixed amount of money among a group of agents. Fairness requires that each agent weakly prefers his consumption bundle to any other agent's bundle. Under fairness, efficiency is equivalent to budget-balance (all the available money is allocated among the agents). Budget-balance and fairness in general are incompatible with non-manipulability (Green and Laffont, 1979). We propose a new notion of the degree of manipulability which can be used to compare the ease of manipulation in allocation mechanisms. Our measure counts for each problem the number of agents who can manipulate the rule. Given this notion, the main result demonstrates that maximally linked fair allocation rules are the minimally manipulable rules among all budget-balanced and fair allocation mechanisms. Such rules link any agent to the bundle of a pre-selected agent through indifferences (which can be viewed as indirect egalitarian equivalence).

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1 Introduction

Many real-life problems involve the allocation of indivisible objects among agents through price or compensation mechanisms. Examples include the simultaneous allocation of jobs among workers together with theirs salaries on labor markets and the assignment of apartments together with their rents on housing markets. The fundamental criterion employed in these problems is fairness meaning each agent likes his own consumption bundle (consisting of an object and a monetary compensation) at least as well as that of anyone else. These types of problems are sometimes referred to as fair (or envy-free) allocation problems and have received considerable attention in the literature, see e.g. Alkan et al. (1991), Demange and Gale (1985), Svensson (1983) or Tadenuma and Thomson (1991). Because each fair allocation distributes the objects efficiently, budget-balance (all the available money is allocated among the agents) is equivalent to efficiency under fairness. Now it follows from a famous result

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by Green and Laffont (1979) that there is no general budget-balanced and fair allocation mechanism that is non-manipulable. This conclusion has also been stressed in a number of recent papers where fair and non-manipulable allocation mechanisms are characterized, see e.g. Andersson and Svensson (2008), Sun and Yang (2003) and Svensson (2009). Of course, all these mechanisms violate budget-balance.

However, in many real-life applications, budget-balance is a necessary requirement, and non-manipulability must be abandoned in order to achieve efficiency and fairness.[1] This paper considers such situations and develops a relevant notion that can be used to compare budget-balanced and fair allocation mechanisms based on their degree of manipulability.

There is a growing literature aiming to compare the ease of manipulation or, equivalently, the degree of manipulation in mechanisms which are known to be manipulable. Examples include voting rules, matching mechanisms, school choice mechanisms etc. The early literature, e.g. Moulin (1980), primarily focuses on restricting the preference domain under which a mechanism is non-manipulable. In evaluating the degree of manipulability, one direction of recent research has adopted the idea of counting the number of preference profiles at which a given mechanism is non-manipulable.[2] A second direction relies on comparing the sets of profiles on which any two mechanisms are manipulable. According to Pathak and Sönmez (2009), a mechanism φ is said to be weakly more manipulable than mechanism ψ if (a) for any profile where ψ is manipulable, also φ is manipulable, and (b) there is at least one profile where φ is manipulable although ψ is not.

As we show, none of the above two measures of the degree of manipulability are satisfactory when comparing budget-balanced and fair allocation rules. This conclusion follows directly from the observation that if some budget-balanced and fair allocation rule is non-manipulable at some preference profile, then each budget-balanced and fair allocation rule is non-manipulable at the very same profile. Thus, all budget-balanced and fair allocation rules are equally manipulable when counting the number of preference profiles at which a mechanism is non-manipulable, and when comparing the sets of profiles where any two mechanisms are non-manipulable. To resolve this problem, we introduce a new measure of minimal manipulability. More explicitly, an allocation rule φ is said to be weakly more individually manipulable than the rule ψ if the number of agents that can manipulate φ is weakly greater than the number of agents that can manipulate ψ at any admissible preference profile. Given this criterion, we identify the least manipulable budget-balanced and fair allocation rules.

In order to describe the characterized rules, we will need to relate to some previous results in our environment. Each agent is supposed to have quasi-linear preferences and each agent should be assigned exactly one object together with some (possibly negative) amount of money under the restriction that the resulting allocation must be budget-balanced and fair (envy-free). This type of problem has previously been considered in the literature by e.g. Haake et al. (2000), Aragones (1995) and Klijn (2000). To find a rule that eliminates profitable deviations through strategic misrepresentation for as many agents (and groups of agents) as possible, we use the following key observation from the literature on fair and non-manipulable allocation mechanisms: a necessary and sufficient condition for obtaining fair and non-manipulable outcomes is that for each group of objects with compensations/prices different from the reservation compensations/prices, there is a larger group of agents demand-

[1]Schummer (2000) shows that non-manipulable and efficient rules must be very "rigid". See also Miyagawa (2001).

[2]See e.g. Aleskerov and Kurbanov (1999), Kelly (1988,1993) and Maus et al. (2007a,b).

ing objects from that group of objects.[3] This observation cannot be directly applied to our model because there are no reservation compensations/prices. However, the idea to select compensations to make each object as attractive as possible for as many agents as possible turns out to be very useful.[4] More explicitly, we consider budget-balanced and fair allocation rules where, given a fixed agent k, the compensations are chosen such that each agent i can be "linked" through an indifference chain to agent k. This means that there is a sequence of agents from i to k such that any agent in this sequence is indifferent between his consumption bundle and the consumption bundle of the next agent in this sequence. This can be seen as indirect egalitarian equivalence where any agent is linked through indifferences to agent k's consumption bundle. A budget-balanced and fair allocation rule selecting always such allocations is said to be an agent k-linked fair allocation rule. Any rule choosing for each preference profile agent k-linked fair allocations, where agent k belongs to an indifference component with maximal cardinality, is said to be a maximally linked fair allocation rule. Here, an indifference component is simply a maximal set of agents such that any two agents are linked through an indifference chain in this set. According to our criterion, the main result shows that maximally linked fair allocation rules are the minimally manipulable rules among all budget-balanced and fair allocation rules.

Our main result turns out to be robust with respect to coalitional manipulations. In the same vein as before, when comparing two mechanisms we count the number of coalitions that can manipulate at a given profile. Again, maximally linked fair allocation rules are least coalitionally manipulable among all budget-balanced and fair allocation rules. In particular, we demonstrate that less than 50% of all coalitions can manipulate the maximal linked fair allocation rule and provide an exact measure of the maximum number of manipulating agents and coalitions for a given preference profile. This measure turns out to be very powerful because it only requires knowledge of the number of agents that are included in an indifference component containing agent k. As we show, the set of indifference components is identical among all fair allocations for a given preference profile. Since an arbitrary fair allocation easily can be found by a well-defined polynomially bounded algorithm, as demonstrated by Klijn (2000), it is not even computationally hard to calculate our measures of individual and coalitional manipulability (in sharp contrast to the measure where the number of preference profiles at which a given mechanism is manipulable is calculated). Finally, we provide a simple algorithm for identifying agent k-linked and maximally linked fair allocations for quasi-linear utilities, and demonstrate that this algorithm converges in a finite number of iterations.

The paper is organized as follows. Section 2 states our model of assignment with compensations and budget-balanced and fair allocation rules. Section 3 defines agent k-linked fair allocations and maximally linked fair allocation rules, and provides some basic results. Section 4 shows that previous measures of manipulability do not distinguish among budget-balanced and fair allocation rules. It introduces our new criterion of minimal manipulability and contains our main result showing that maximally linked fair allocation rules are minimally manipulable among all budget-balanced and fair allocation rules. Section 5 identifies both the set of agents and coalitions that can manipulate agent k-linked fair allocation rules. It provides another characterization of maximally linked fair allocation rules using minimal coalitional manipulability. Finally, Section 6 provides an algorithm for identifying agent k-

[3] See in particular Andersson and Svensson (2008, Lemma 4) and Mishra and Talman (2010, Theorem 2).

[4] Similar ideas have previously been used by e.g. Dubey (1982) and Svensson (1991) where the "tightness" of the market is demonstrated to have a significant impact on the manipulation possibilities.

linked fair and budget-balanced allocations and maximally linked fair allocations. Some proofs are omitted in the main text and are given in the Appendix.

2 Assignment with Compensations

2.1 Agents, Allocations, and Preferences

Let $N = \{1, ..., n\}$ and $M = \{1, ..., m\}$ denote the set of agents and objects, respectively. The number of agents and objects are assumed to coincide, i.e., $|N| = |M|$.[5] Each agent consumes exactly one object together with some amount of money. A consumption bundle is a pair $(j, \alpha) \in M \times \mathbb{R}$ where α is the monetary compensation received when consuming object j. An allocation (a, x) is a list of n consumption bundles where $a : N \to M$ is a mapping assigning object a_i to agent $i \in N$, and where $x \in \mathbb{R}^M$ (or $x : M \to \mathbb{R}$) assigns the amount x_j of money for the object $j \in M$. An allocation (a, x) is *feasible* if $a_i \neq a_j$ whenever $i \neq j$ for $i, j \in N$, and $\sum_{j \in M} x_j \leq 0$.[6] If $\sum_{j \in M} x_j = 0$, then the allocation (a, x) satisfies *budget-balance*. Let $\mathcal{A}$ denote the set of feasible and budget-balanced allocations.

Each agent $i \in N$ has preferences over consumption bundles (j, x_j) which are represented by continuous utility functions $u_i : M \times \mathbb{R}^M \to \mathbb{R}$. We will write $u_{ij}(x)$ instead of $u_i(j, x)$ to denote the utility of agent i when consuming object j and receiving compensation x_j in the distribution vector x. The utility function is assumed to be quasi-linear and strictly increasing in money, i.e.,

$$u_{ij}(x) = v_{ij} + x_j \text{ for some } v_{ij} \in \mathbb{R}.$$

A list of utility functions $u = (u_i)_{i \in N}$ is a (preference) profile. We also adopt the notational convention of writing $u = (u_C, u_{-C})$ for $C \subseteq N$. The set of profiles with utility functions having the above properties is denoted by $\mathcal{U}$.

Let $u \in \mathcal{U}$ and (a, x) be a feasible allocation. Then (a, x) is *efficient* if there exists no feasible allocation (b, y) such that $u_{ib_i}(y) \geq u_{ia_i}(x)$ for all $i \in N$ with strict inequality holding for some $j \in N$. Obviously, if (a, x) is efficient, then (a, x) is budget-balanced.

Throughout the paper we focus on feasible allocations satisfying budget-balance.[7] For convenience, in the following allocation stands for "feasible allocation satisfying budget-balance".

2.2 Fair Allocation Rules

The fundamental concept of fairness corresponds to envy-freeness which was first introduced by Foley (1967). It says that each agent weakly prefers his consumption bundle to any other agent's bundle.

Definition 1. For a given profile $u \in \mathcal{U}$, an allocation (a, x) is *fair* if $u_{ia_i}(x) \geq u_{ia_j}(x)$ for all $i, j \in N$. Let $F(u)$ denote the set of fair allocations for a given profile $u \in \mathcal{U}$.

It is well-known that under fairness, for feasible allocations efficiency is equivalent to budget-balance.[8]

[5] If $|N| < |M|$, then we simply add $|N| - |M|$ null objects with zero value for all agents.

[6] All our results remain true if the budget constraint is replaced by $\sum_{j \in M} x_j \leq x_0$ for an arbitrary $x_0 \in \mathbb{R}$.

[7] When budget-balance is relaxed to $\sum_{j \in M} x_j \leq 0$, then general non-manipulability results are possible, see e.g. Andersson and Svensson (2008) or Sun and Yang (2004).

[8] This is due to the fact that any fair allocation must assign the objects efficiently.

The following is a well-known property of fair allocations (see e.g. Svensson, 2009): if two allocations are fair at a given profile, then one may interchange both the assignment of objects and the monetary distribution without losing fairness. Obviously, this result holds for fair allocations satisfying budget-balance.

Lemma 1. Suppose that allocations (a, x) and (b, y) are fair at profile $u \in \mathcal{U}$. Then allocations (a, y) and (b, x) are also fair at profile $u \in \mathcal{U}$.

An *allocation rule* is a non-empty correspondence φ choosing for each profile $u \in \mathcal{U}$ a set of allocations, $\varphi(u) \subseteq \mathcal{A}$, such that $u_{ib_i}(y) = u_{ia_i}(x)$ for all $i \in N$ and all $(a, x), (b, y) \in \varphi(u)$. Hence, the various allocations in the set $\varphi(u)$ are utility equivalent. Such a correspondence is called *essentially single-valued.* It is important to note that alternatively we may consider *single-valued* allocation rules choosing for each profile $u \in \mathcal{U}$ a unique allocation. All our results remain unchanged for single-valued allocation rules.

An allocation rule φ is called *fair* if for any profile $u \in \mathcal{U}$, $\varphi(u) \subseteq F(u)$. The following is a useful property of fair allocation rules.

Lemma 2. *Let φ be a fair allocation rule and $u \in \mathcal{U}$. If $(a, x), (b, y) \in \varphi(u)$, then $x = y$.*

Proof. Since $(a, x), (b, y) \in \varphi(u)$, we have $u_{ia_i}(x) = u_{ib_i}(y)$ for all $i \in N$. By fairness, $u_{ia_i}(x) \geq u_{ib_i}(x)$. Thus, $u_{ib_i}(y) \geq u_{ib_i}(x)$ and $y_{b_i} \geq x_{b_i}$. Similarly, we obtain $x_{b_i} \geq y_{b_i}$. Hence, $x = y$, the desired conclusion. □

An important implication of Lemma 2 is that for fair allocation rules, the same distribution of money is chosen for any given preference profile. Hence, often for the study of fair allocation rules it is sufficient to consider its induced distributions of money.

3 Maximally Linked Fair Allocations

In the coming analysis, indifference chains and indifference components will be of primary importance for allocations. These two concepts are introduced next.

Definition 2. Let $(a, x) \in \mathcal{A}$.

(i) For any $i, j \in N$, we write $i \to_{(a,x)} j$ if:

$$u_{ia_i}(x) = u_{ia_j}(x).$$

(ii) An *indifference chain* at allocation (a, x) consists of a tuple of distinct agents $g = (i_0, ..., i_k)$ such that $i_0 \to_{(a,x)} i_1 \to_{(a,x)} \cdots \to_{(a,x)} i_k$.

(iii) An *indifference component* at allocation (a, x) is a non-empty set $G \subseteq N$ such that for all $i, k \in G$ there exists an indifference chain at (a, x) in G, say $g = (i_0, ..., i_k)$ with $\{i_0, \ldots, i_k\} \subseteq G$, such that $i = i_0$ and $i_k = k$, and there exists no $G' \supsetneq G$ satisfying the previous property at allocation (a, x).

Note that $i \to_{(a,x)} j$ means that agent i is indifferent between his consumption bundle and agent j's consumption bundle, and agent i is *directly linked* via indifference to agent j at allocation (a, x). An indifference chain at an allocation is simply a sequence of agents such that any agent in the sequence is indifferent between his bundle and the bundle of the agent following him in the sequence. Indifference chains indirectly link agents via indifference in a sequence of directly linked agents. In an indifference component, any two agents are linked through an indifference chain in this component and there is no superset of this component where any two agents are linked through an indifference chain.

The next result states an important property of indifference components, namely that if there are two allocations that are budget-balanced and fair at some profile $u \in \mathcal{U}$ and if there is an indifference component at one of these allocations, then the very same indifference component must be present at the other allocation.

Lemma 3. Suppose that allocations (a, x) and (b, y) are budget-balanced and fair at profile $u \in \mathcal{U}$, and that there is an indifference component G at allocation (a, x). Then the same indifference component G is present at allocation (b, y).

Proof. By Lemma 1, we know that (a, y) is fair. First we show that the indifference component G is present at (a, y).

Because G is an indifference component at (a, x), G consists of indifference chains $g = (i_0, i_1, \dots, i_k)$ such that $i_k \rightarrow_{(a,x)} i_0$. Thus, we have $i_0 \rightarrow_{(a,x)} i_1 \rightarrow_{(a,x)} \cdots \rightarrow_{(a,x)} i_k \rightarrow_{(a,x)} i_0$. We show $i_0 \rightarrow_{(a,y)} i_1 \rightarrow_{(a,y)} \cdots \rightarrow_{(a,y)} i_k \rightarrow_{(a,y)} i_0$.

For any $i \in N$, let $\Delta_{a_i} = y_{a_i} - x_{a_i}$. To obtain a contradiction, suppose that we do not have $i_0 \rightarrow_{(a,y)} i_1 \rightarrow_{(a,y)} \cdots \rightarrow_{(a,y)} i_k \rightarrow_{(a,y)} i_0$, say $u_{i_0 a_{i_0}}(x) = u_{i_0 a_{i_1}}(x)$ but $u_{i_0 a_{i_0}}(y) > u_{i_0 a_{i_1}}(y)$. Thus, $\Delta_{a_{i_0}} > \Delta_{a_{i_1}}$. Now, fairness is respected among the agents in G at allocation (a, y) only if:

$$\Delta_{a_{i_j}} \geq \Delta_{a_{i_{j+1}}} \text{ for all } j \in \{0, ..., k-1\}, \tag{1}$$

$$\Delta_{a_{i_k}} \geq \Delta_{a_{i_0}}. \tag{2}$$

From (1) and $\Delta_{a_{i_0}} > \Delta_{a_{i_1}}$, we obtain $\Delta_{a_{i_0}} > \Delta_{a_{i_k}}$. Hence, (2) is not satisfied. Thus, allocation (a, y) cannot be fair, which contradicts our assumption. Hence, $i_0 \rightarrow_{(a,y)} i_1 \rightarrow_{(a,y)} \cdots \rightarrow_{(a,y)} i_k \rightarrow_{(a,y)} i_0$. Note that there exists no $G' \supsetneq G$ such that G' is an indifference component at (a, y) because otherwise, using the previous arguments, any two agents in G' are connected through some indifference chain at (a, x) in G' which contradicts the definition of G being an indifference component at (a, x). Thus, the indifference component G is present at (a, y).

Next, we show that G must be also an indifference component at (b, y). Fairness implies that:

$$u_{ia_i}(y) = u_{ib_i}(y) \text{ for all } i \in N. \tag{3}$$

Let $j, k \in G$ and suppose that $j \rightarrow_{(a,y)} k$. Let $a_k \neq b_k$ and $l_1 \in N$ be such that $a_{l_1} = b_k$. Obviously, (3) implies $k \rightarrow_{(a,y)} l_1$. More generally, let $l_1, \dots, l_t$ be such that $a_{l_r} = b_{l_{r-1}}$ with $r = 2, \dots, t$ and $a_k = b_{l_t}$. Note that such a "cycle" exists because $|N| = |M|$. Now obviously we have $k \rightarrow_{(a,y)} l_1$, $l_r \rightarrow_{(a,y)} l_{r+1}$ for all $r = 1, \dots, t-1$, and $l_t \rightarrow_{(a,y)} k$. Since $k \in G$ and G is an indifference component at (a, y), we must have $\{l_1, \dots, l_t\} \subseteq G$.

Now by (3), we have $u_{jb_j}(y) = u_{ja_j}(y) = u_{ja_k}(y) = u_{jb_{l_t}}(y)$ which implies $j \rightarrow_{(b,y)} l_t$. Note that by construction, we also have $l_1 \rightarrow_{(b,y)} k$ and $l_r \rightarrow_{(b,y)} l_{r-1}$ for all $r = 2, \dots, t$. This means that j and k are connected through the indifference chain $j \rightarrow_{(b,y)} l_t \rightarrow_{(b,y)} l_{t-1} \rightarrow_{(b,y)} \cdots \rightarrow_{(b,y)} l_1 \rightarrow_{(b,y)} k$ in G under (b, y). Because this is true for any $j, k \in G$ such that $j \rightarrow_{(a,y)} k$, it also follows that any two agents belonging to G must be connected through an indifference chain in G at (b, y). Furthermore, there can be no $G' \supsetneq G$ satisfying this property under (b, y) because by the same argument G' would also satisfy this property under (a, x), which would contradict the definition of an indifference component. $\square$

Lemma 3 has the important implication that the same indifference components are present at all budget-balanced and fair allocations (for a given profile). In other words, indifference components at fair allocations only depend on the preference profile u because they are invariant with respect to the selected fair allocation.

Given the definition of indifference components, we next introduce the concept of linked agents. We are especially interested in so-called agent k-linked allocations. At such an allocation, each agent is linked to agent k through some indifference chain.

Definition 3. Let $(a, x) \in \mathcal{A}$.

(i) Agent $i \in N$ is *linked* to agent $k \in N$ at allocation (a, x) if there exists an indifference chain of type $(i_0, ..., i_t)$ at allocation (a, x) with $i = i_0$ and $i_t = k$.

(ii) The allocation (a, x) is *agent k-linked* if each agent $i \in N$ is linked to agent k.

In the following fair allocations which are agent k-linked (with $k \in N$) will play an important role.

Remark 1. Agent k-linked fair allocations (a, x) can be viewed as (a_k, x_{a_k})-linked fair allocations. In the same vein, any agent i is linked through an indifference chain to the consumption bundle (a_k, x_{a_k}) under allocation (a, x). One may interpret this as "indirect" egalitarian equivalence where each agent is connected through some indifference chain to the consumption bundle (a_k, x_{a_k}). Recall that in "direct" egalitarian equivalence each agent views his consumption bundle as utility equivalent to (a_k, x_{a_k}).

The next result states that if there are two (or more) agent k-linked fair allocations at a specific profile $u \in \mathcal{U}$, then the monetary distribution of both of these allocations must coincide. This is the analogue of Lemma 2 for agent k-linked fair allocations.

Lemma 4. If the budget-balanced and fair allocations (a, x) and (b, y) are agent k-linked at profile $u \in \mathcal{U}$, then $x = y$.

Proof. By Lemma 1, we know that (a, y) is also fair. First, we show that (a, y) is agent k-linked if (b, y) is agent k-linked. Fairness implies:

$$u_{ia_i}(y) = u_{ib_i}(y) \text{ for all } i \in N. \tag{4}$$

Let $a_k \neq b_k$ and $j \in N$ be such that $a_j = b_k$. Obviously, (4) implies $j \rightarrow_{(a,y)} k$. Now suppose that $i \rightarrow_{(b,y)} k$ and $i \neq j$. But now by (4), we have $u_{ia_i}(y) = u_{ib_i}(y) = u_{ib_k}(y) = u_{ia_j}(y)$ which implies $i \rightarrow_{(a,y)} j \rightarrow_{(a,y)} k$ and agent i is linked to agent k through some indifference chain. Using these arguments, it is now easy to verify that (a, y) is agent k-linked (if either $a_k \neq b_k$ or $a_k = b_k$). Thus, without loss of generality we may assume $a = b$.

Suppose that the fair allocations (a, x) and (a, y) are agent k-linked but $x \neq y$. Then by budget-balance and $x \neq y$, there must be two non-empty groups of agents:

$$\begin{aligned} A &= \{i \in N \mid x_{a_i} > y_{a_i}\}, \\ B &= \{i \in N \mid x_{a_i} \leq y_{a_i}\}. \end{aligned}$$

Note that for all $i \in A$ and all $j \in B$, $u_{ia_i}(x) > u_{ia_i}(y) \geq u_{ia_j}(y) \geq u_{ia_j}(x)$. Hence, no agent in A can be linked to any agent in B at allocation (a, x). Because (a, x) is agent k-linked, we must have $k \in A$. Let $j \in B$ and $i \in A$. By fairness and monotonicity:

$$u_{ja_j}(y) \geq u_{ja_j}(x) \geq u_{ja_i}(x) > u_{ja_i}(y).$$

Thus, at allocation (a, y) no agent in B can be linked to any agent in A. Hence, by $k \in A$, allocation (a, y) cannot be agent k-linked which contradicts our assumption. $\square$

Given $k \in N$, let $\psi^k(u) \subseteq F(u)$ denote the set of all budget-balanced and fair allocations which are agent k-linked at profile $u \in \mathcal{U}$. Section 6 establishes the non-emptiness of this set by providing an algorithm to compute such allocations.

Proposition 1. ψ^k is an allocation rule, i.e. for any $u \in \mathcal{U}$, all allocations $(a, x), (b, y) \in \psi^k(u)$ are utility equivalent.

Proof. Let $(a, x), (b, y) \in \psi^k(u)$ and $i \in N$. By Lemma 4, we have $x = y$. Obviously, if $a_i = b_i$, then $u_{ia_i}(x) = u_{ib_i}(y)$. If $a_i \neq b_i$, then by fairness both $u_{ia_i}(x) \geq u_{ib_i}(x)$ and $u_{ia_i}(y) \leq u_{ib_i}(y)$. Hence, by $x = y$, $u_{ia_i}(x) = u_{ib_i}(y)$, the desired conclusion. $\square$

We call ψ^k the *agent k-linked fair allocation rule.* As will become clear in the following section, agent k-linked fair allocation rules have a number of good strategic properties. However, depending on the reported profile $u \in \mathcal{U}$ and the selection of $k \in N$, the manipulability possibilities will differ. Thus, when defining the allocation rule, it is important to select the right $k \in N$ for a given profile u. For this reason, the selection of agent k will be endogenously determined by the reported profile $u \in \mathcal{U}$ as explained below.

For a given report $u \in \mathcal{U}$, recall that the set of indifference components is identical for all allocations in $F(u)$ by Lemma 3. Let

$$\mathcal{G}(u) = \{G \subseteq N \mid G \text{ is an indifference component at all } (a, x) \in F(u)\}.$$

denote the set of all indifference components of fair allocations for profile u. Note also that for any $i \in N$, there exists $G \in \mathcal{G}(u)$ such that $i \in G$ (where $G = \{i\}$ is possible). Let

$$\bar{\mathcal{G}}(u) = \{G \in \mathcal{G}(u) \mid |G| \geq |G'| \text{ for all } G' \in \mathcal{G}(u)\},$$

denote the set of indifference components with maximal cardinality, and let:

$$\bar{G}(u) = \cup_{G \in \bar{\mathcal{G}}(u)} G,$$

denote the union of all indifference components with maximal cardinality.

The idea of the following rules is first to select an indifference component with maximal cardinality, second some agent k belonging to this indifference component and third the set of agent k-linked fair allocations.

A maximal selection is a function $\kappa : \mathcal{U} \to N$ such that for all $u \in \mathcal{U}$ we have $\kappa(u) \in \bar{G}(u)$. The *maximally linked fair allocation rule* ψ^κ based on κ is defined as follows: for all $u \in \mathcal{U}$, let $\psi^\kappa(u) = \psi^{\kappa(u)}(u)$. Note that by Proposition 1, ψ^κ is a well-defined allocation rule because $\psi^k(u)$ is essentially single-valued for any $k \in N$ and any $u \in \mathcal{U}$. Furthermore, we will say that an allocation rule φ is a maximally linked fair allocation rule if there exists a maximal selection κ such that for all $u \in \mathcal{U}$ we have $\varphi(u) \subseteq \psi^\kappa(u)$.

The function κ is a systematic selection from $\bar{\mathcal{G}}(u)$. The meaning of "systematic selection" is that there is a well defined rule for selecting k. This rule can be arbitrary and all our results hold independently of this rule. For example, the rule could be based on a randomized selection from $\bar{G}(u)$ or simply the k with the lowest or highest index in $\bar{G}(u)$.

Our main result will compare maximally linked fair allocation rules with arbitrary budget-balanced and fair allocation rules.

4 Minimal Manipulability

In the following we will determine the (non-)manipulation possibilities of budget-balanced and fair allocation rules.

Definition 4. An allocation rule φ is manipulable at a profile $u \in \mathcal{U}$ by an agent $i \in N$ if there exists a profile $(\hat{u}_i, u_{-i}) \in \mathcal{U}$ and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(\hat{u}_i, u_{-i})$ such that $u_{ib_i}(y) > u_{ia_i}(x)$. If the allocation rule φ is not manipulable by any agent at profile $u \in \mathcal{U}$, then φ is said to be non-manipulable at profile u.

Remark 2. Since allocation rules may choose sets of allocations, one may alternatively employ a more conservative notion of manipulability: φ is strongly manipulable at a profile $u \in \mathcal{U}$ by an agent $i \in N$ if there exists a profile $(\hat{u}_i, u_{-i}) \in \mathcal{U}$ such that $u_{ib_i}(y) > u_{ia_i}(x)$ for all $(a, x) \in \varphi(u)$ and all $(b, y) \in \varphi(\hat{u}_i, u_{-i})$. From Svensson (2009, Proposition 3 and its proof) it follows that for any fair allocation rule φ and any profile $u \in \mathcal{U}$, φ is strongly manipulable at profile u by i if and only if φ is manipulable at profile u by i. Hence, instead we may use the conservative notion of manipulability instead of ours.

It is well-known (Green and Laffont, 1979) that any budget-balanced and fair rule is manipulable for some profile $u \in \mathcal{U}$. Thus, we need a measure of the degree of manipulability in order to compare two different budget-balanced and fair allocation rules. As it will turn out, previous notions of degrees of manipulability will not distinguish budget-balanced and fair allocation rules.

We show that for any fair allocation rule and any profile u, the fair allocation rule cannot be manipulated by any agent at profile u if and only if a maximally linked fair allocation rule cannot be manipulated by any agent at profile u.

Proposition 2. Let φ be a budget-balanced and fair allocation rule, ψ^κ be a maximally linked fair allocation rule and $u \in \mathcal{U}$. Then the maximally linked allocation rule ψ^κ is non-manipulable at profile u if and only if the fair allocation rule φ is non-manipulable at profile u.

We will defer the proof to the next section where we identify agents who can manipulate maximally linked fair allocation rules.

Remark 3 (Counting Profiles). Several authors (see e.g. Maus et al., 2007a, 2007b) have proposed to compare two rules via counting the number of profiles where some agent can manipulate the rule. Obviously, Proposition 2 shows for any profile u, that either all fair rules are manipulable or all fair rules cannot be manipulated at the given profile by any agent.

Remark 4 (Comparing Sets of Profiles). Pathak and Sönmez (2009) propose to compare two rules via comparing the sets of profiles where some agent can manipulate a rule. They call a rule φ weakly more manipulable than a rule φ', if for any problem where some agent i manipulates φ', there exists an agent j who manipulates φ at this problem. Again Proposition 2 shows that all fair allocation rules are equally manipulable regarding this degree of manipulability.

Given these observations, a "first-order approach" by considering profiles where no agent can manipulate the rule does not refine the set of fair allocation rules. Here we will propose a "second-order approach" by comparing the cardinalities of the sets of agents who can manipulate a fair allocation rule at a given profile. For this purpose, let $P^\varphi(u)$ denote the set of agents who can manipulate the allocation rule φ at profile $u \in \mathcal{U}$.

Definition 5. An allocation rule $\tilde{\varphi}$ is weakly more (individually) manipulable than the allocation rule φ if $|P^{\tilde{\varphi}}(u)| \geq |P^{\varphi}(u)|$ for all $u \in \mathcal{U}$.

Our main result establishes that maximally linked fair allocation rules are minimally manipulable among all budget-balanced and fair allocation rules. The proof is delegated to the Appendix.

Theorem 1. Let φ be a budget-balanced and fair allocation rule and let ψ^{κ} be a maximally linked fair allocation rule. Then φ is weakly more manipulable than ψ^{κ}.

In other words, maximally linked fair allocation rules are least manipulable in the class of budget-balanced and fair allocation rules. One can also see that when a fair rule is not a maximally linked fair allocation rule, then there exists some profile where this rule is manipulable by more agents than a maximally linked fair allocation rule. Therefore, maximally linked fair allocation rules are characterized by minimal manipulability among all budget-balanced and fair allocation rules.

The proofs of our result will also reveal the following corollary.

Corollary 1. (i) ψ^k cannot be manipulated by agent k at any profile $u \in \mathcal{U}$.

(ii) For any two distinct agents $i, j \in N$, there exists no budget-balanced and fair allocation rule φ such that neither i nor j can manipulate φ at any profile $u \in \mathcal{U}$.

Note that Corollary 1 has the same flavor as the corresponding results in two-sided matching (with men and women): (i) for any agent there exists a stable matching rule which is not manipulable by this agent at any profile; and (ii) there is no stable matching rule which cannot be manipulated by at least one man and at least one woman (Ma, 1995).

5 Identifying Non-Manipulating Agents and Coalitions

We will identify both the agents and coalitions who are able to profitably manipulate an agent k-linked fair allocation rule. This will allow us to determine at which profiles such a rule is non-manipulable and to compare budget-balanced and fair allocation rules regarding their possibilities of coalitional manipulations.

We adopt the following version of coalition manipulability and non-manipulability.

Definition 6. An allocation rule φ is manipulable at a profile $u \in \mathcal{U}$ by a coalition $C \subseteq N$ if there is a profile $(\hat{u}_C, u_{-C}) \in \mathcal{U}$ and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(\hat{u}_C, u_{-C})$ such that $u_{ib_i}(y) > u_{ia_i}(x)$ for all $i \in C$. If the allocation rule φ is not manipulable by any coalition at profile u, then φ is said to be coalitionally non-manipulable at profile u.

In the same vein as Remark 2, we may use a more conservative notion of coalitional manipulability where all deviating agents are strictly better off after the deviation for any of the chosen allocations. Again by Svensson (2009, Proposition 3 and its proof), this would not change any of our results below.

Our next result shows that the agent k-linked fair allocation rule cannot be manipulated by any coalition containing agent k. The intuition behind this is as follows. If agent k successfully can manipulate the allocation rule, then by fairness agent k must be assigned a consumption bundle where the monetary compensation increases. Then because each agent is linked to agent k, each agent must be assigned a consumption bundle where the monetary compensation increases, because if this not is the case then fairness is violated at the new allocation. But then the budget must be exceeded. Hence, agent k cannot manipulate. The same intuition even holds for any fair allocation rule choosing only agent k-linked fair allocations for some profile.

Lemma 5. Let φ be a budget-balanced and fair allocation rule, $k \in N$ and $u \in \mathcal{U}$. If $\varphi(u) \subseteq \psi^k(u)$, then no coalition $C \subseteq N$ containing agent k can manipulate φ at profile u.

Proof. Let $C \subseteq N$ be such that $k \in C$. Suppose that φ is manipulable at profile u by coalition C. Then there is a profile $(\hat{u}_C, u_{-C}) \in \mathcal{U}$ and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(\hat{u}_C, u_{-C})$ such that $u_{ib_i}(y) > u_{ia_i}(x)$ for all $i \in C$. Note that $\varphi(u) \subseteq \psi^k(u)$ and $(a, x) \in \psi^k(u)$.

By fairness, $u_{ia_i}(x) \geq u_{ib_i}(x)$ for all $i \in C$. Hence, for all $i \in C$, $u_{ib_i}(y) > u_{ib_i}(x)$ and $y_{b_i} > x_{b_i}$. Because (b, y) satisfies budget-balance, we must have $C \subsetneq N$. Since $k \in C$ and (a, x) is an agent k-linked fair allocation, there exists $i \in N - C$ and $j \in C$ such that $i \rightarrow_{(a,x)} j$. Now by $y_{a_j} > x_{a_j}$ ($j \in C$) and $u_{ia_i}(x) = u_{ia_j}(x)$, fairness and monotonicity imply:

$$u_{ib_i}(y) \geq u_{ia_j}(y) > u_{ia_j}(x) = u_{ia_i}(x) \geq u_{ib_i}(x).$$

Hence, $y_{b_i} > x_{b_i}$. Let $C^1 = C \cup \{i \in N \mid i \rightarrow_{(a,x)} j \text{ for some } j \in C\}$. Thus, we have $y_{b_i} > x_{b_i}$ for all $i \in C^1$.

Using the same arguments it follows that for each $i \in N$ such that $i \rightarrow_{(a,x)} j$ for some $j \in C^1$, we have $y_{b_i} > x_{b_i}$. For any l, let $C^{l+1} = C^l \cup \{i \in N \mid i \rightarrow_{(a,x)} j \text{ for some } j \in C^l\}$.

Because (a, x) is agent k-linked, for some t we obtain $C^t = N$ and $y_{b_i} > x_{b_i}$ for all $i \in C^t$, which is contradiction to budget-balance of (b, y). Hence, C cannot manipulate φ at profile u. □

Remark 5. Lemma 5 implies that the agent k-linked fair allocation rule cannot be manipulated by any coalition containing k at any profile. In particular, the agent k-linked fair allocation rule is not manipulable by agent k at any profile u, which is the first part of Corollary 1. The second part of Corollary 1 is easy to verify and left to the reader.

The following proposition identifies all preference profiles $u \in \mathcal{U}$ at which an agent k-linked fair allocation rule is (coalitionally) non-manipulable.

Proposition 3. Let $k \in N$ and $u \in \mathcal{U}$. Then ψ^k is (coalitionally) non-manipulable at profile u if and only if $\mathcal{G}(u) = \{N\}$, i.e. if and only if N is the unique indifference component at profile u.

Proof. We only prove the "if" part of because the "only if" part follows directly from Lemma 7 in the Appendix. Since $\mathcal{G}(u) = \{N\}$, any $(a, x) \in F(u)$ is agent i-linked for any $i \in N$. Since $\psi^k(u) \subseteq F(u)$, Lemma 5 implies that no coalition containing i can manipulate ψ^k at profile u. Hence, ψ^k is both (individually) non-manipulable at profile u and coalitionally non-manipulable at profile u, the desired conclusion. □

Now Proposition 3 allows us to demonstrate that an arbitrary fair allocation rule is non-manipulable at a profile if and only if the agent k-linked fair allocation rule is non-manipulable at this profile. Using Lemma 5, this result implies Proposition 2 (which was used in Section 4 to motivate our "second-order approach" to minimal manipulability).

Proposition 4. Let φ be a budget-balanced and fair allocation rule, $k \in N$ and $u \in \mathcal{U}$. Then ψ^k is (coalitionally) non-manipulable at profile u if and only if φ is (coalitionally) non-manipulable at profile u.

Proof. For the "only if" part, suppose that ψ^k is not (coalitionally) manipulable at profile u. By Proposition 3, $\mathcal{G}(u) = \{N\}$ and any $(a, x) \in F(u)$ is agent i-linked for all $i \in N$. Since $\varphi(u) \subseteq F(u)$, now any $(a, x) \in \varphi(u)$ is agent i-linked for all $i \in N$. Thus, $\varphi(u) \subseteq \psi^i(u)$ for all $i \in N$. By Lemma 5, no coalition containing i can manipulate φ at profile u. Hence, φ is (coalitionally) non-manipulable at profile u, the desired conclusion.

For the "if" part, suppose that φ is (coalitionally or individually) non-manipulable at profile u but some ψ^k is manipulable by an agent or a coalition at profile u. By Proposition 3, $\mathcal{G}(u) \neq \{N\}$. But now Lemma 7 in the Appendix shows φ is manipulable at profile u, a contradiction. $\square$

Lemma 5 showed that an agent k-linked fair allocation rule cannot be manipulated by any coalition containing agent k at any given profile. Below we extend this result and determine for any profile the precise number of coalitions who *can* manipulate the agent k-linked fair allocation rule. Specifically, we demonstrate that ψ^k can be manipulated by less than 50% of all coalitions at any profile.

Corollary 2. Let $k \in N$.

(i) Let $u \in \mathcal{U}$ and $S \in \mathcal{G}(u)$ be such that $k \in S$. Then ψ^k can be manipulated at profile u by exactly $2^{|N|-|S|} - 1$ coalitions.

(ii) For any profile $u \in \mathcal{U}$, ψ^k can be manipulated at profile u by at most $2^{|N|-1} - 1$ coalitions. As a consequence, ψ^k can be manipulated at any profile $u \in \mathcal{U}$ by less than 50% of all coalitions.

Proof. To prove (i), note that for all $i \in S$ and all $(a, x) \in \psi^k(u)$, allocation (a, x) is agent i-linked. Thus, $\psi^k(u) \subseteq \psi^i(u)$ and by Lemma 5, no coalition containing i can manipulate ψ^k at profile u. Thus, at most $2^{|N|-|S|} - 1$ coalitions can manipulate ψ^k at profile u. From Lemma 7 in the Appendix, it follows that this bound is tight, i.e. that exactly $2^{|N|-|S|} - 1$ coalitions can manipulate ψ^k at profile u.

To prove (ii), note that $|S| \geq 1$. Because $2^{|N|-|S|} < 2^{|N|-1}$ for any $|S| > 1$, it follows from Part (i) of this corollary that ψ^k can be manipulated at profile u by at most $2^{|N|-1} - 1$ coalitions. Since there are $2^{|N|} - 1$ non-empty coalitions of N and $2^{|N|} - 1 = 2(2^{|N|-1} - 1) + 1$, less than 50% of all coalitions can manipulate ψ^k at profile u. $\square$

Note that Corollary 2 used the fact that for any indifference component, for agent k belonging to this component, the agent k-linked fair allocation rule is not manipulable by any coalition containing some agent in this indifference component. Now in order to calculate the number of manipulating coalitions, at a given profile, one only need to know the number of agents that are included in an indifference component containing agent k. Then since indifference components are invariant with respect to the chosen fair allocation, the algorithm in Klijn (2000) can be used to find the exact number of manipulating coalitions at a given profile for any agent k-linked fair allocation rule. Because the algorithm in Klijn (2000) is polynomially bounded, it is not even computationally hard to calculate this measure.

To investigate the degree of coalition manipulability, let $Q^{\varphi}(u)$ denote the coalitions $C \subseteq N$ that can manipulate the allocation rule φ at profile $u \in \mathcal{U}$.

Definition 7. An allocation rule $\tilde{\varphi}$ is weakly more coalitionally manipulable than the allocation rule φ if $|Q^{\tilde{\varphi}}(u)| \geq |Q^{\varphi}(u)|$ for all $u \in \mathcal{U}$.

The following result states that maximally linked fair allocation rules are least coalitionally manipulable among all budget-balanced and fair allocation rules. This can be seen as an extension of Theorem 1 from minimal individual manipulability to minimal coalitional manipulability and Theorem 1 is robust with respect to coalitional manipulations. The proof can be found in the Appendix.

Theorem 2. Let φ be a budget-balanced and fair allocation rule and ψ^{κ} be a maximally linked fair allocation rule. Then φ is weakly more coalitionally manipulable than ψ^{κ}.

Again one can see that when a fair rule is not a maximally linked fair allocation rule, then there exists some profile where this rule is manipulable by more coalitions than a maximally linked fair allocation rule.

6 The Algorithm

Given the results concerning manipulability from the previous section, it is important to find an algorithm for identifying agent k-linked fair allocations. Once such allocations are identified, it is also possible to identify $\mathcal{G}(u)$, and as a consequence, maximally linked fair allocations. We provide an algorithm that achieves this task. In similarity with Aragones (1995), our algorithm cannot start at an arbitrary feasible allocation. Instead, we suppose that an arbitrary budget-balanced and fair allocation is known for the given profile. This assumption is not restrictive since arbitrary such allocations can be identified in polynomial time as demonstrated by Klijn (2000).[9]

Given that a budget-balanced and fair allocation (a, x) is known for a given profile $u \in \mathcal{U}$, Lemma 3 can be used to find the set $\mathcal{G}(u)$. More explicitly, if allocation (a, x) is known, all indifference components that are present at this allocation will also be present at each allocation that is budget-balanced and fair for the same profile by Lemma 3. It is therefore an easy task to identify the components containing the most agents. The following example demonstrates the principle, and it will be used throughout this section to illustrate the main ideas and concepts.

Example 1. Let $N = \{1, 2, 3, 4, 5\}$ and $M = \{1, 2, 3, 4, 5\}$. Let the values of the objects for the agents in the profile u be given by the matrix:

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \\ v_{21} & v_{22} & v_{23} & v_{24} & v_{25} \\ v_{31} & v_{32} & v_{33} & v_{34} & v_{35} \\ v_{41} & v_{42} & v_{43} & v_{44} & v_{45} \\ v_{51} & v_{52} & v_{53} & v_{54} & v_{55} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \tag{5}$$

For these valuations it is clear that e.g. the allocation (a, x) where $a_i = i$ and $x_{a_i} = 0$ for all $i \in N$ is budget-balanced and fair. There are two indifference chains present at this allocation,

[9] See Haake et al. (2000) for a similar procedure. Note also that there are a number of algorithms for identifying so-called fair and optimal allocations, see e.g. Andersson and Andersson (2009) or Shioura et al. (2006). Also these algorithms can be used to identify budget-balanced and fair allocations by adding a simple rule for sharing the deficit (or the surplus) equally among the agents.

namely $2 \rightarrow_{(a,x)} 1$ and $4 \rightarrow_{(a,x)} 3$. Any indifference component consists of a single agent, and we have $\mathcal{G}(u) = \{\{i\} \mid i \in N\}$.[10] Consequently, $\bar{G}(u) = N$. $\square$

In the main algorithm so-called isolated groups will be crucial. This notion is defined below.

Definition 8. A coalition of agents $H \subseteq N$ is said to be isolated at allocation (a, x) if $i \not\rightarrow_{(a,x)} j$ for all $i \in N - H$ and all $j \in H$.

In other words, a coalition is isolated at allocation (a, x) if no agent outside of H is indifferent between his consumption bundle and the bundle received by any agent in H.

Before we provide the main algorithm, we first state a simple algorithm that always identifies an isolated group H at allocation (a, x) containing agent k.

Algorithm 1 (Isolated Groups). Let allocation (a, x) be an arbitrary budget-balanced and fair allocation at profile $u \in \mathcal{U}$ and let $k \in N$. Introduce an iteration counter t and set $t = 0$. Let $K^0 = \{k\}$. For each iteration $t = 1, 2, \ldots$:

Step t. Define $K^t = K^{t-1} \cup \{i \in N - K^{t-1} \mid i \rightarrow_{(a,x)} j \text{ for some } j \in K^{t-1}\}$. If $K^t = K^{t-1}$, then stop. Otherwise continue to Step $t + 1$.

Obviously, if $K^t = N$ for some t, then (a, x) is agent k-linked and Algorithm 1 verifies whether a given allocation is agent k-linked.

Lemma 6. For each $u \in \mathcal{U}$ and each $K^0 = \{k\}$, Algorithm 1 identifies a (possibly empty) isolated group that contains agent k in at most $|N|$ iterations.

Proof. Assume that the algorithm terminates at Step t. If $K^t \neq N$, then $u_{ia_i}(x) > u_{ia_j}(x)$ (or $i \not\rightarrow_{(a,x)} j$) for all $i \in N - K^t$ and all $j \in K^t$ by construction of the algorithm. Thus, K^t is isolated by Definition 8. Note that $k \in K^t$ since $\{k\} = K^0 \subseteq K^t$.

Finally, let T be the last step of the algorithm, and note that because $|K^t| - |K^{t-1}| \geq 1$ as long as $1 \leq t < T$, it is clear that the algorithm terminates in at most $|N|$ number of iterations. $\square$

We next illustrate Algorithm 1 using Example 1.

Example 2 (Example 1 continued)**.** Start with $K^0 = \{1\}$. Then Algorithm 1 terminates in two steps, i.e.:

Step 1. From (5), it is clear that $i \rightarrow_{(a,x)} 1$ only for $i = 2$. Hence, $K^1 = \{1\} \cup \{2\} = \{1, 2\}$.

Step 2. From (5), it is clear that $i \not\rightarrow_{(a,x)} j$ for all $i \in N - K^1$ and all $j \in K^1$. Hence, $K^2 = K^1$ and Algorithm 1 terminates. $\square$

Both the distribution and the assignment are fixed in Algorithm 1. Note that in the proof of Lemma 4 we showed that for any agent k-linked fair allocation (b, y) and any fair allocation (a, x), allocation (a, y) is also agent k-linked and fair. Thus, without loss of generality, in the algorithm below the assignment of objects remains unchanged. We next provide an algorithm for identifying an agent k-linked fair allocation given that the distribution is allowed to change.

[10] If e.g. $v_{12} = 1$ then there is one indifference component containing agents 1 and 2.

Algorithm 2 (**Agent k-linked Fair Allocation**)**.** Let allocation (a, x) be budget-balanced and fair. Introduce an iteration counter t and let x^t denote the distribution in iteration t. Set $t = 0$ and initialize the distribution at $x^0 = x$. Let $K^0 = \{k\}$. For each iteration $t = 1, 2, \ldots$:

Step t. For the given allocation (a, x^{t-1}) run Algorithm 1 and let N^t be the set identified in the last step of Algorithm 1. If $N - N^t = \varnothing$, then stop. Otherwise, let $\lambda^t_{ij} = u_{ia_i}(x^{t-1}) - u_{ia_j}(x^{t-1})$ for each $i \in N - N^t$ and each $j \in N^t$. Define $\lambda^t = \min_{i \in N-N^t, j \in N^t} \lambda^t_{ij}$. Let the distribution x^t be given by:

$$
\begin{aligned}
x^t_{a_i} &= x^{t-1}_{a_i} - \frac{|N^t|}{|N|} \cdot \lambda^t \text{ for each } i \in N - N^t, \\
x^t_{a_j} &= x^{t-1}_{a_j} + \frac{|N - N^t|}{|N|} \cdot \lambda^t \text{ for each } j \in N^t,
\end{aligned}
$$

and continue to Step $t + 1$. □

The following is our main convergence result. The proof is delegated to the Appendix.

Theorem 3. For each $u \in \mathcal{U}$, Algorithm 2 identifies an agent k-linked fair allocation in at most $|N|$ number of iterations.

We use our example to illustrate the procedure described in Algorithm 2.

Example 3 (Example 1 continued)**.** Recall that $K^0 = \{1\}$, $a_i = i$ and $x^0_{a_i} = 0$ for all $i \in N$.

Step 1. From Example 2 we know that $N^1 = \{1, 2\}$ (and $N - N^1 = \{3, 4, 5\}$). From matrix (5), it is also easy to see that $\lambda^1_{3j} = 1$, $\lambda^1_{4j} = 2$ and $\lambda^1_{5j} = 3$ for all $j \in N^1$. Thus, $\lambda^1 = 1$, so $x^1 = (x^1_1, x^1_2, x^1_3, x^1_4, x^1_5) = (\frac{3}{5}, \frac{3}{5}, -\frac{2}{5}, -\frac{2}{5}, -\frac{2}{5})$.

Step 2. Given the distribution x^1 identified in Step 1 the following holds:

$$
[v_{ij} + x^1_j]_{i,j \in N} = \begin{bmatrix}
\frac{8}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\
\frac{8}{5} & \frac{8}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\
\frac{3}{5} & \frac{3}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} \\
\frac{3}{5} & \frac{3}{5} & \frac{8}{5} & \frac{8}{5} & -\frac{2}{5} \\
\frac{3}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} & \frac{13}{5}
\end{bmatrix}
$$

Thus, when we run Algorithm 1, agent 3 is first included in N^2 (because agent 3 is indifferent between objects 1, 2 and 3) and then agent 4 is included in N^2 (because agent 4 is indifferent between objects 3 and 4). Hence, $N^2 = \{1, 2, 3, 4\}$. Now, $\lambda^2_{51} = \lambda^2_{52} = 2$ and $\lambda^2_{53} = \lambda^2_{53} = 3$. Thus, $\lambda^2 = 2$, and as a consequence, $x^2 = (x^2_1, x^2_2, x^2_3, x^2_4, x^2_5) = (1, 1, 0, 0, -2)$.

Step 3. By construction of x^2, agent 5 is indifferent between objects 1, 2 and 5 at allocation (a, x^2). Thus, $N^3 = N$ and Algorithm 2 terminates at Step 3. □

Appendix: Proofs

In the Appendix we prove Theorem 1, Theorem 2 and Theorem 3. The following two lemmas will be useful.

Lemma 7. Let φ be a budget-balanced and fair allocation rule. Let $u \in \mathcal{U}$ and $(a, x) \in \varphi(u)$. If the non-empty coalition $G \subseteq N$ is isolated at allocation (a, x), then each $C \subseteq G$ can manipulate φ.

Proof. Let $(a, x) \in \varphi(u)$, and suppose that $G \subseteq N$ is a non-empty isolated coalition, i.e., that both $i \not\to_{(a,x)} j$ and $u_{ia_i}(x) > u_{ia_j}(x)$ for all $i \in N - G$ and all $j \in G$. Now simultaneously all compensations for objects a_i $(i \in G)$ can be increased by the same amount and all compensations for objects a_j $(j \in N - G)$ can be decreased by the same amount without losing budget-balance and fairness. Hence, there is a number $\tau > 0$ and $(a, y) \in F(u)$ such that $u_{ia_i}(y) > u_{ia_i}(x) + \tau$ for all $i \in G$ (and $y_{a_i} > x_{a_i} + \tau$ for all $i \in G$). Fix $0 < \varepsilon < \tau$ and define for any $i \in G$ the function $\hat{u}_i$ as follows: for all $j \in M$ and all $x' \in \mathbb{R}^M$, let

$$\hat{u}_{ij}(x') = (-y_j + \varepsilon_{ij}) + x'_j, \tag{6}$$

where $\varepsilon_{ij} = 0$ if $j \neq a_i$ and $\varepsilon_{ia_i} = \varepsilon > 0$. Note that $\hat{v}_{ij} = -y_j + \varepsilon_{ij}$. Let $C \subseteq G$ and $\hat{u}_C = (\hat{u}_i)_{i \in C}$. By construction of $\hat{u}_C$, we have $(a, y) \in F(\hat{u}_C, u_{-C})$.[11]

Let $(b, z) \in \varphi(\hat{u}_C, u_{-C})$. We first show $b_i = a_i$ for all $i \in C$. Let $\delta_j = z_j - y_j$ for all $j \in M$. Without loss of generality, order M such that $\delta_j \geq \delta_{j+1}$ for all $j = 1, \ldots, |M| - 1$.

If $z = y$, then by fairness, $\hat{u}_{ib_i}(y) = \hat{u}_{ia_i}(y)$ for all $i \in C$. Since for all $i \in C$, $\hat{u}_{ia_i}(y) = \varepsilon$ and $\hat{u}_{ij}(y) = 0$ for $j \neq a_i$, we obtain $b_i = a_i$ for all $i \in C$.

If $z \neq y$, then by budget-balance of both (b, z) and (a, y), $\delta_1 > 0$ and $\delta_n < 0$. Let $(j_l)_l$ be a subsequence of $(1, \ldots, n)$ such that $j_l < j_{l+1}$, $\delta_{j_l} > \delta_{j_{l+1}}$ and $\delta_j = \delta_{j_l}$ if $j_l \leq j < j_{l+1}$. Let $S_l = \{i \in N \mid j_l \leq a_i < j_{l+1}\}$. Then for $i \in S_l$:

$$\begin{aligned} u_{ia_i}(z) = u_{ia_i}(y + \delta_{a_i}) > u_{ib_i}(y + \delta_{b_i}) = u_{ib_i}(z) \text{ if } b_i \geq j_{l+1} \text{ and } i \in N - C, \\ \hat{u}_{ia_i}(z) = z_{a_i} - y_{a_i} + \varepsilon = \delta_{a_i} + \varepsilon > \delta_{b_i} = \hat{u}_{ib_i}(z) \text{ if } b_i \geq j_{l+1} \text{ and } i \in C. \end{aligned}$$

Thus, by fairness, for all l, $i \in S_l$ implies $j_l \leq b_i < j_{l+1}$. Moreover, for $i \in C$, $\hat{u}_{ia_i}(z) = \delta_{a_i} + \varepsilon > \delta_{b_i} = \hat{u}_{ib_i}(z)$ if $b_i \neq a_i$ and $b_i \geq j_l$. Hence, by fairness, $b_i = a_i$ for all $i \in C$.

It remains to prove that $u_{ib_i}(z) > u_{ia_i}(x)$ for all $i \in C$, i.e., φ is manipulable at u by coalition C. From the above, we have $a_i = b_i$ for all $i \in C$. Since φ is fair, we have $(b, z) \in F(\hat{u}_C, u_{-C})$. Now we have for all $i \in C$ with $b_i \neq 1$,

$$\hat{u}_{ib_i}(z) = \hat{u}_{ia_i}(z) = z_{ia_i} - y_{ia_i} + \varepsilon \geq z_{i1} - y_{i1} = \hat{u}_{i1}(z). \tag{7}$$

Because $\delta_j = z_j - y_j$, it follows from the above condition that $\delta_{b_i} \geq \delta_1 - \varepsilon$ for $i \in C$ with $b_i \neq 1$. Note that this inequality holds trivially if $b_i = 1$ because $\varepsilon > 0$. Now this fact, the definition of δ_j and our choice of $0 < \varepsilon < \tau$, $\delta_1 \geq 0$ and $a_i = b_i$ for all $i \in C$, yield for all $i \in C$:

$$\begin{aligned} u_{ia_i}(x) &< u_{ia_i}(y) - \tau \\ &= u_{ib_i}(y) - \tau \\ &= v_{ib_i} + z_{b_i} - (z_{b_i} - y_{b_i}) - \tau \\ &= u_{ib_i}(z) - \delta_{b_i} - \tau \\ &\leq u_{ib_i}(z) - \delta_1 - (\tau - \varepsilon), \\ &< u_{ib_i}(z), \end{aligned}$$

[11]Note that for all $i \in C$, $\hat{u}_{ia_i}(y) = \varepsilon$ and $\hat{u}_{ij}(y) = 0$ for $j \neq a_i$.

where the first inequality follows from $u_{ia_i}(y) > u_{ia_i}(x) + \tau$, the first equality from $a_i = b_i$ for $i \in C$, the second inequality from $-\delta_{b_i} \leq -(\delta_1 - \varepsilon)$, and the last inequality from $\delta_1 \geq 0$ and $\tau > \varepsilon$. Hence, $u_{ia_i}(x) < u_{ib_i}(z)$ for all $i \in C$, which is the desired conclusion. □

Lemma 8. Let φ be a budget-balanced and fair allocation rule. Let $u \in \mathcal{U}$ and $(a, x) \in \varphi(u)$. Let $N - G$ be a (possibly empty) isolated coalition with maximal cardinality at allocation (a, x). Then G is an indifference component at allocation (a, x).

Proof. We first show that all $i, j \in G$ can be linked via an indifference chain in G. Suppose not, i.e. there exist $i, j \in G$ such that i cannot be linked to j via some indifference chain G. Let $H = \{k \in G \mid k \text{ can be linked to } j \text{ via some indifference chain in } G\}$. Since $i \in G - H$, we have $G - H \neq \emptyset$. Because no agent in $G - H$ can be linked to any agent in H, now by construction, it follows that the set $(N - G) \cup H$ is isolated and $|(N - G) \cup H| > |N - G|$, which contradicts the assumption that $N - G$ is an isolated coalition with maximal cardinality at allocation $(a, x) \in \varphi(u)$.

Now, the proof follows directly because the coalition $N - G$ is isolated at allocation (a, x), i.e., $i \not\rightarrow_{(a,x)} j$ for all $i \in G$ and all $j \in N - G$. Consequently, there is no $G' \supsetneq G$ such that G' is an indifference component by Definition 2. □

Proof of Theorem 1. Suppose that $(a, x) \in \varphi(u)$ and $(b, y) \in \psi^\kappa(u)$, and let $N - G$ be a (possibly empty) isolated coalition with maximal cardinality at allocation $(a, x) \in \varphi(u)$. Then G is an indifference component at allocations (a, x) and (b, y) by Lemma 3 and Lemma 8.

Note first that all agents in the isolated coalition $N - G$ can manipulate φ by Lemma 7. Consequently, at least $|N - G|$ agents can manipulate φ. Hence, to conclude the proof we need to show that at most $|N - G|$ agents can manipulate ψ^κ.

Suppose now that κ belongs to the indifference component $\hat{G} \subseteq \bar{G}(u)$, and note that $|\hat{G}| \geq |G|$ by construction of ψ^κ. Since $\psi^\kappa(u) \subseteq \psi^k(u)$ for all $k \in \hat{G}$, it now follows from Lemma 5 that no agent $k \in \hat{G}$ can manipulate ψ^κ at profile u. Thus, at most $|N - \hat{G}|$ agents can manipulate ψ^κ. The conclusion then follows directly from the observation that $|\hat{G}| \geq |G|$ implies $|N - \hat{G}| \leq |N - G|$. □

Proof of Theorem 2. Suppose that $(a, x) \in \varphi(u)$ and $(b, y) \in \psi^\kappa(u)$, and let $N - G$ be the (possibly empty) isolated coalition with maximal cardinality at allocation $(a, x) \in \varphi(u)$. Then G is an indifference component at allocations (a, x) and (b, y) by Lemma 3 and Lemma 8.

Note first that all coalitions in the isolated group $N - G$ can manipulate φ by Lemma 7. Consequently, at least $2^{|N-G|} - 1$ coalitions can manipulate φ. Hence, to conclude the proof we need to show that at most $2^{|N-G|} - 1$ coalitions can manipulate ψ^κ. Suppose now that κ belongs to the indifference component $\hat{G} \subseteq \bar{G}(u)$, and note that $|\hat{G}| \geq |G|$ by construction of ψ^κ. It now follows from Lemma 5 and the construction of ψ^κ that at most $2^{|N-\hat{G}|} - 1$ coalitions can manipulate ψ^κ. The conclusion then follows directly from the observation that $|\hat{G}| \geq |G|$ implies $|N| - |\hat{G}| \leq |N| - |G|$. □

Proof of Theorem 3. Note first that the adjustment of the compensation vector at Step t from x^{t-1} to x^t respects the balanced budget requirement by construction of x^t because:

$$\sum_{i \in N} x^t_{a_i} = \sum_{i \in N} x^{t-1}_{a_i} - \frac{|N^t|}{|N|} \cdot \lambda^t \cdot |N - N^t| + \frac{|N - N^t|}{|N|} \cdot \lambda^t \cdot |N^t| = \sum_{i \in N} x^{t-1}_{a_i},$$

and allocation (a, x^0) is budget-balanced.

The adjustment of the compensation vector at Step t from x^{t-1} to x^t also respects fairness because the assignment a is held constant and the construction of x^t guarantees that if $u_{ia_i}(x^{t-1}) \geq u_{ia_j}(x^{t-1})$ then $u_{ia_i}(x^t) \geq u_{ia_j}(x^t)$. If $i, j \in N^t$ or $i, j \in N - N^t$ this follows directly since the adjustments of $x^{t-1}_{a_i}$ and $x^{t-1}_{a_j}$ are identical. In the case when $i \in N^t$ and $j \in N - N^t$, the result follows since $x^{t-1}_{a_i}$ is increased and $x^{t-1}_{a_j}$ is decreased. In the last case when $i \in N - N^t$ and $j \in N^t$, the conclusion follows by definition of λ^t and $\lambda^t_{ij} = u_{ia_i}(x^{t-1}) - u_{ia_j}(x^{t-1})$, i.e.:

$$\begin{aligned}
u_{ia_i}(x^t) &= v_{ia_i} + x^t_{a_i} = v_{ia_i} + x^{t-1}_{a_i} - \frac{|N^t|}{|N|} \cdot \lambda^t \geq v_{ia_i} + x^{t-1}_{a_i} - \frac{|N^t|}{|N|} \cdot \lambda^t_{ij} = \\
&= u_{ia_i}(x^{t-1}) - \lambda^t_{ij} + \frac{|N - N^t|}{|N|} \cdot \lambda^t_{ij} = u_{ia_j}(x^{t-1}) + \frac{|N - N^t|}{|N|} \cdot \lambda^t_{ia_j} \\
&\geq v_{ia_j} + x^{t-1}_{a_j} + \frac{|N - N^t|}{|N|} \cdot \lambda^t = v_{ia_j} + x^t_{a_j} = u_{ia_j}(x^t).
\end{aligned}$$

Thus, at Step t in the algorithm (a, x^t) satisfies budget-balance and fairness. It remains to prove that the algorithm terminates in at most $|N|$ iterations at an agent k-linked fair allocation.

By construction of N^t, each agent $i \in N^t$ must belong to an indifference chain $G = \{i, ..., k\}$. Note that at Step t, for $i \in N - N^t$ and $j \in N^t$ such that $\lambda^t_{ij} = \lambda^t$, all the above inequalities become equalities and we obtain $u_{ia_i}(x^t) = u_{ia_j}(x^t)$, $i \rightarrow_{(a,x^t)} j$ and $i \in N^{t+1}$. Note that $N^t \subseteq N^{t+1}$ because for any $i, j \in N^t$ such that $i \rightarrow_{(a,x^{t-1})} j$ we also have $i \rightarrow_{(a,x^t)} j$. Thus, $|N^{t+1}| - |N^t| \geq 1$ as long as $N - N^t \neq \varnothing$. Now it is clear that the algorithm will terminate in at most $|N|$ number of iterations and that the resulting fair allocation is agent $k-$linked. □

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