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NON-COMMITMENT AND SAVINGS IN DYNAMIC RISK-SHARING CONTRACTS

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RÉSUMÉ

Nous caractérisons la solution d'un modèle de lissage de la consommation avec financement externe sujet à des contraintes d'engagement et épargne. Nous démontrons que, sous certaines conditions, l'épargne et le financement externe se complètent parfaitement. Si le taux d'escompte est égal au taux d'intérêt, on obtient en temps fini un lissage parfait. Nous démontrons également que le lissage obtenu sur les marchés financiers affecte l'investissement en capital physique. Lorsque le lissage est imparfait, l'investissement est utilisé pour des fins de lissage.

Mots clés : épargne, consommation, partage de risque dynamique, non-engagement

ABSTRACT

We characterize the solution to a model of consumption smoothing using financing under non-commitment and savings. We show that, under certain conditions, these two different instruments complement each other perfectly. If the rate of time preference is equal to the interest rate on savings, perfect smoothing can be achieved in finite time. We also show that, when random revenues are generated by periodic investments in capital through a concave production function, the level of smoothing achieved through financial contracts can influence the productive investment efficiency. As long as financial contracts cannot achieve perfect smoothing, productive investment will be used as a complementary smoothing device.

Key words : savings, consumption, dynamic risk sharing, non-commitment

1 Introduction

Market incompleteness can explain why risk-averse agents cannot perfectly insure themselves against idiosyncratic risk. With complete markets, it is always possible to buy state-contingent securities that ensure that consumption is perfectly insured and smoothed. Macro and micro data show, however, that consumption is variable: the hypothesis that agents consume their permanent income each period is rejected by the data. Agents cannot perfectly insure themselves.

Liquidity constraints have been used to represent some form of market incompleteness. They limit an agent's access to credit, who can therefore not expect to smooth out all income risk. These constraints seem empirically plausible. Many agents cannot borrow because they cannot offer the creditor a significant collateral. Others can only borrow at a very high interest rate. Models with a representative agent have used these constraints to explain the empirical failure of the permanent-income hypothesis. Garcia, Lusardi and Ng (1995) show, using micro data, that liquidity constraints represent the most plausible explanation of the sensitivity of consumption to current income. Deaton (1991) has simulated consumption and savings paths close to aggregate series using various liquidity-constraint models.

Liquidity constraints are only a "reduced-form" expression for market imperfections that cause market incompleteness. These imperfections can be caused by informational problems such as adverse selection and moral hazard. These informational asymmetries limit the extent of trading with the consequence that all risks cannot be fully diversified (Green, 1987; Thomas and Worrall, 1990). Informational asymmetries have also been used to explain the empirical failure of the permanent-income hypothesis (Pischke, 1995).

Another source of market imperfections is the lack of commitment by the parties engaged in a financial relationship. If the costs of enforcing a financial contract are high, agents may elect to breach the contract rather than obey all its terms and conditions. Thomas and Worrall (1988) have characterized the optimal risk-sharing labor contract where neither the risk-neutral firm, nor the risk-averse worker could commit to the contract. They show that the firm cannot generally fully smooth the worker's wage. In all periods, the wage varies depending on the last-period wage and the worker's current productivity. In this context, Gauthier, Poitevin and González (1997) show that, if the agents can make a transfer before the realization of the state of nature, the commitment problem is alleviated.

Building on the liquidity-constraint literature, our model seeks to endogenize this constraint. In the liquidity-constraint model of Schechtman (1976), we explicitly introduce the possibility for the risk-averse agent to borrow from a risk-neutral financier. Such borrowing is assumed to be subject to non-commitment. The assumption of non-commitment has the following interpretation. The role of the financier is to provide financing to the risk-averse agent when he experiences a negative shock to his income. A contract with full commitment would require that the financier always make a loan to the risk-averse agent regardless of the likelihood that he reimburses. Without commitment, the financier may refuse to refinance the borrower if it is likely that the borrower never reimburses. Alternatively, non-commitment on the part of the risk-averse agent can be interpreted as limited liability, that is, the borrower cannot be forced into reimbursing a loan if his utility is higher when defaulting. The assumption of non-commitment implies that the risk-averse agent cannot fully finance his consumption, and therefore, he cannot fully diversify his income risk with the bank.

Borrowing under non-commitment can therefore be used to endogenize the liquidity constraint in the model of Schechtman. The first goal of the paper is then to characterize the optimal consumption path in the liquidity-constraint model of Schechtman when market incompleteness (the liquidity constraint) is endogenized by an assumption of non-commitment on financing with a financier.

The second goal is to use this model to assess whether non-commitment in financial markets can have an impact on the real decisions of the risk averse agent, namely, its investment decisions. Following Modigliani and Miller's (1958) result on the irrelevance of financial structure to real decisions, there has been a large literature seeking to explain how imperfections in financial markets can influence investment decisions. For example, Ambarish, John and Williams (1987) show that informational asymmetries in financial markets can induce firms to over- or under-invest in order to signal their value. We follow Sigouin (1997) in introducing investment decisions in a dynamic model of financing under non-commitment. As opposed to his analysis, however, we study the interaction between real and financial investment as savings are also incorporated in the analysis.

The model is presented in the next section. Section 3 presents our main results on the dynamics of financing and consumption. The impact of non-commitment in financing on investment decisions is studied in Section 4. We discuss some of our assumptions in Section 5 to illustrate the robustness or sensitivity of our results to some key assumptions. The conclusion follows.

2 The model

Agent 1 has a stochastic exogenous income y^s which depends on the realization of the state of nature s. There exists a time-independent discrete set of possible states $\mathcal{S} = \{1, \ldots, S\}$. The time-independent probability of state s is p^s, with $\sum_{s \in \mathcal{S}} p^s = 1$.[1] We assume that $y^1 < \ldots < y^S$. Agent 1 is risk averse with preferences represented by a state- and time-independent concave utility function u. This function is defined and bounded over the interval $[0, \gamma]$. Formally, we assume that $u' > 0$, $u'' < 0$, and $u'(0) = \infty$. Agent 1 has an infinite-horizon life span.

Agent 1 has access to a savings account in which he can make deposits in every period. His savings at the end of period $t-1$ is denoted by A_t. Savings earn a time-independent interest rate r per period. In any period, agent 1 can withdraw any savings he may have. Agent 1 is, however, liquidity constrained in the sense that he cannot borrow at this rate r. This amounts to assuming that savings must be non negative at all time, that is, $A_t \geq 0$ for all t.

Agent 1 can, however, borrow from agent 2. Agent 2 is an infinitely-lived risk-neutral financier with a linear utility function $v(b) = b$.[2] His endowment is e per period. Without loss of generality, we assume that $e = 0$. Both agents discount the future by a common factor $\beta = 1/(1+\delta)$. We also assume that $r \leq \delta$.[3]

There are gains from trade to be exploited by these two agents as agent 1 can hope to transfer some of his income risk to the risk-neutral agent 2. The relationship between these two agents is governed by a contract signed at date 1. This risk-sharing contract specifies transfers from agent 1 to agent 2, and savings for agent 1 for all dates. Transfers and savings can depend on the date t, the state s realized at that date, and more generally on the whole

[1]Most of our results hold if income follows a first-order Markov process. Additional details along these lines are available from the authors upon request.

[2]Our analysis is in a partial-equilibrium context. The assumption of risk neutrality for agent 2 means that agent 2 is "big" compared to agent 1. For example, agent 2 may represent a large bank, while agent 1 represents a typical borrower.

[3]If $r > \delta$, the agent has a strong incentive to sacrifice its current consumption and save. Savings are no more accumulated in a smoothing interest but for further consumption. It is of no interest in our context.

history of states prior to date t. Transfers can be either positive or negative, while savings must remain non negative. Note that we implicitly assume that savings are verifiable and can thus be controlled by the contract. We relax this assumption in Section 5.

Formally, denote by $h_t = (s_1, \ldots, s_t)$ the history of realizations of states of nature for all periods through period t, and by H_t, the set of possible histories at date t. Denote by π agent 1's consumption plan. A plan π is a set of functions π_t, one for each period t, such that $\pi_t : H_t \rightarrow R_+ \times R$ with $\pi_t(h_{t-1}, s_t) = (A_{t+1}^{\pi s_t}, b_t^{\pi s_t})$. For each period t, a plan π specifies a level of savings A_{t+1} to start with in period $t+1$ and a contemporaneous transfer b_t as a function of the current state of nature and the history up to period t. Denote by $\mathcal{U}_t(\pi, h_t)$ and $\mathcal{V}_t(\pi, h_t)$ the expected utility of agents 1 and 2 respectively from period t on under consumption plan π.

$$\mathcal{U}_t(\pi, h_t) = u(c_t^{\pi s_t}) + \mathrm{E}_t \textstyle\sum_{\tau=1}^{\infty} \beta^\tau u(c_{t+\tau}^{\pi s_{t+\tau}})$$
$$\mathcal{V}_t(\pi, h_t) = b_t^{\pi s_t} + \mathrm{E}_t \textstyle\sum_{\tau=1}^{\infty} \beta^\tau b_{t+\tau}^{\pi s_{t+\tau}}$$

where E_t is the expectation operator conditional on the available information in period t, h_t. Consumption is determined by $c_t^{\pi s_t} = y^{s_t} + (1+r)A_t^{\pi s_{t-1}} - A_{t+1}^{\pi s_t} - b_t^{\pi s_t}$.

Suppose that the two agents can commit to a long-term contract. The consumption of agent 1 solves the following maximization problem:

$$\max_\pi \ \mathcal{U}_1(\pi, s_1) \quad \text{s.t.} \quad \mathcal{V}_1(\pi, s_1) \geq V \tag{1}$$
$$c_\tau^{\pi s_\tau} = y^{s_\tau} + (1+r)A_\tau^{\pi} - A_{\tau+1}^{\pi s_\tau} - b_\tau^{\pi s_\tau} \quad \forall\, h_\tau,\ \forall\, \tau \geq 0 \tag{2}$$
$$A_{\tau+1}^{\pi s_\tau} \geq 0 \quad \forall\, h_\tau,\ \forall\, \tau \geq 0. \tag{3}$$

where V is the reservation utility of agent 2. It is easy to show that the optimal contract yields a constant consumption to agent 1 in all states and all periods, and that savings do not have to be used to support the optimal consumption path. A crucial assumption for these results is that the two agents can fully commit to the transfers prescribed by the contract. This may be an unreasonable assumption since, following a given history, an agent may prefer to breach the contract rather than make the prescribed transfer. For example, agent 2 may prefer not to refinance agent 1 in a low-income state when agent 2 expects not to be fully reimbursed in the future. Alternatively, agent 1 may prefer to declare bankruptcy rather than reimburse agent 2 when his debt becomes too large. If enforcement costs are high, it is not possible to bind the agents to the contract in all circumstances. It is therefore relevant to study the optimal contract in an environment where agents cannot commit.

The possibility of savings, as an outside opportunity for agent 1, is likely to affect the solution to the non-commitment contracting problem. Before solving this problem, however, it is helpful to characterize the solution to the simpler problem in which agent 1 can save but cannot borrow from agent 2. It turns out that this problem represents the autarky environment that agent 1 can attain if he rejects or breaches the contract.

In autarky, agent 1 seeks to maximize his lifetime utility by choosing his consumption and savings in each period. At the beginning of period t, he has accumulated savings of $(1+r)A_t$ and he receives an endowment of y^s. These financial resources are then shared between current consumption and future savings to satisfy his budget constraint: $c_t^s + A_{t+1}^s = y^s + (1+r)A_t$. The fact that agent 1 cannot borrow is represented by the liquidity constraint $A_{t+1}^s \geq 0$.

Formally, agent 1 chooses $\{c_t^s\}_{t=1}^{\infty}$ and $\{A_{t+1}^s\}_{t=1}^{\infty}$, $\forall\ s \in S$ to solve the following Bellman equation:[4]

$$g(A_t, y^s) = \max_{A_{t+1}^s \geq 0} \; u(y^s + (1+r)A_t - A_{t+1}^s) + \beta \mathrm{E}_z g(A_{t+1}^s, y^z). \tag{4}$$

The value function $g(A_t, y^s)$ represents agent 1's maximized expected utility at the beginning of period t given that he has savings of A_t and state s has been realized. Under our assumptions, $g(., y^s)$ is continuous, strictly increasing, strictly concave, and continuously differentiable. Furthermore, the optimal saving and consumption policies can be described by continuous functions $a(A_t, y^s) = A_{t+1}^s$, and $c(A_t, y^s) = c_t^s = y_t^s - A_{t+1}^s + (1+r)A_t$.

First-order conditions and the envelope condition at period t can be derived from program (4) and the following equation obtains for all s in $\mathcal{S}$:

$$u'(c_t^s) = \beta(1+r)\mathrm{E}_z u'(c_{t+1}^z) + \mu_t^s$$

where μ_t^s is the multiplier on the liquidity constraint. This is the Euler equation determining the optimal consumption smoothing, taking into account the liquidity constraint.

From the first-order and Euler conditions, it is possible to describe the dynamics of consumption and savings. Savings $a(A_t, y^s)$ and consumption $c(A_t, y^s)$ are both increasing in A_t. A higher stock of savings means more resources at hand which are then shared between current consumption and savings for the next period. It can be shown that for

[4] See Schechtman (1976) for formal proofs of most of the results that follow.

$y^s > y^z$, $c(A_t, y^s) > c(A_t, y^z)$ and $g(A_t, y^s) > g(A_t, y^z)$; consumption is then imperfectly smoothed and utility varies across states of nature. Although agent 1 withdraws from his savings account to finance consumption in bad states, consumption remains variable.

Imperfect smoothing through liquidity-constrained savings is described in the literature on consumption/savings models. Schechtman (1976) shows that if the interest rate r is equal to the discount rate δ, consumption can only reach a stationary state in the limit, when the savings stock tends to infinity. Deaton (1991) suggests that, when $r < \delta$, that is, when savings are relatively costly, the consumer is reluctant to save, even in the good states of nature; therefore, savings do not tend to infinity. In general, consumption will be equal to the resources at hand (savings + income) as long as they are not higher than a certain level. From that level up, the consumer starts to save, but the savings stock remains low. Finally, whatever the interest rate, consumption smoothing is poorly realized. There are, therefore, gains from trade to be realized in signing a risk-sharing contract with a financier. This possibility is introduced in the next section.

3 The risk-sharing contract with non-commitment

We study the optimal risk-sharing contract when neither agent can commit itself to respect the contract in any contingency. Each agent can decide to renege on the contract if the payment to be made is greater than the future surplus it expects. To make sure no such incentives are present, we introduce self-enforcing constraints into the contracting problem. These constraints impose that, in each period, following any history, agents have no incentives to renege on the contract.

We assume that, if an agent reneges on the contract, he gets his autarcic utility level forever.[5] For agent 2, autarky means a zero income (and utility) forever. Agent 1 still earns his stochastic income in autarky, and he can still partially smooth his consumption with savings. In his case, autarky is represented by the solution to the savings model with liquidity constraints characterized in the previous section. The value function $g(A, y^s)$ yields the expected discounted utility of agent 1 if he breaches the contract having savings of A and current income of y^s. We assume that agent 2 can seize agent 1's financial assets (his

[5] Asheim and Strand (1991) show that this punishment is renegotiation-proof in the repeated-game formulation of a related model.

savings) if agent 1 breaches the contract. It is as if savings are put up as a collateral against agent 1's borrowing from agent 2. Agent 1's autarcic life therefore starts with no savings. Under these assumptions, self-enforcing constraints can be written as:

$$\begin{aligned} \mathcal{U}_t(\pi, (h_{t-1}, s)) &\geq g(0, y^s) \quad \forall\ (h_{t-1}, s) \in H_t,\ \forall\ t, \\ \mathcal{V}_t(\pi, (h_{t-1}, s)) &\geq 0 \qquad\qquad \forall\ (h_{t-1}, s) \in H_t,\ \forall\ t. \end{aligned}$$

The optimal contract solves an optimization problem in $t = 1$ which prescribes a consumption plan π, that is, a sequence of savings and transfers for each date and possible histories: $\{A_{t+1}^{\pi s_t}, b_t^{\pi s_t}\}_{t=0,1,\ldots}$. A first property of the optimal contract is that it must be efficient starting in any period following any history. Suppose the contrary that the optimal contract is not efficient in period t following a given history. It would then be possible to change transfers in period $t+1$ in such a way as to increase agent 2's expected utility, while maintaining constant that of agent 1. These new transfers would satisfy all self-enforcing constraints since they increase agent 2's utility, thus reducing his incentives to renege on the contract. These new transfers would then increase agent 2's expected utility in $t = 1$, while leaving that of agent 1 constant, thus implying that the original contract could not have been optimal. Consequently, an optimal contract maximizes at each date t the expected utility of one agent subject to a participation constraint for the other agent and subject to self-enforcing and liquidity constraints. For each date t, the program must then be:

$$\max_{\pi^t}\ \mathcal{U}_t(\pi^t, h_t) \quad \text{s.t.} \quad \mathcal{V}_t(\pi^t, h_t) \geq V_t^{s_t} \tag{5}$$

$$\mathcal{U}_\tau(\pi^t, h_\tau) \geq g(0, y^{s_\tau}) \quad \forall\, h_\tau,\ \forall\, \tau \geq t \tag{6}$$

$$\mathcal{V}_\tau(\pi^t, h_\tau) \geq 0 \quad \forall\, h_\tau,\ \forall\, \tau \geq t \tag{7}$$

$$c_\tau^{\pi^t s_\tau} = y^{s_\tau} + (1+r)A_\tau^{\pi^t} - A_{\tau+1}^{\pi^t s_\tau} - b_\tau^{\pi^t s_\tau} \quad \forall\, h_\tau,\ \forall\, \tau \geq t \tag{8}$$

$$A_{\tau+1}^{\pi^t s_\tau} \geq 0 \quad \forall\, h_\tau,\ \forall\, \tau \geq t \tag{9}$$

where π^t is the continuation of consumption plan π following history h_t. Denote by $\Gamma(h_t)$ the set of instruments that satisfy constraints (6)–(9). It represents the set of continuation contracts from period t on that are self-enforcing and for which savings are non-negative. If there are gains from trade, $\Gamma(h_t)$ is non-empty because at least one agent can have more than autarky in each period. Furthermore, it is easy to show that $\Gamma(h_t)$ is compact and convex.[6]

[6] See Thomas and Worrall (1988) for a formal proof in a related model.

In (5), the parameter $V_t^{s_t}$ is the minimum surplus that agent 1 must yield to agent 2 when state s_t is realized in t. This surplus allowed to agent 2 is bounded by the existing gains from trade in the contract. If $V_t^{s_t} = 0$, there exists a self-enforcing continuation in t consisting of a sequence of zero transfers to agent 2. If $V_t^{s_t} > 0$, the surplus cannot be greater than total gains from trade available in t. Here, gains from trade increase with the amount of savings. Since savings can be used as a collateral, agent 1 can borrow more if he has savings than if he has not any. The maximum surplus to agent 2 is then denoted by $\bar{V}_t(A_t)$. If there are gains from trade, then there exists a sequence of $\{b_\tau\}_{\tau=t,\cdots\infty}$ that yields $\bar{V}_t(A_t)$ to agent 2 and the autarky utility to agent 1. This sequence satisfies all constraints to the problem.[7]

Following Spear and Srivastava (1987), in period t, the surplus to agent 2 can be given using a current transfer $b_t^{\pi^t s_t}$ and a future expected surplus $V_{t+1}^{\pi^t}$. Solving the problem therefore amounts to picking a current transfer $b_t^{s_t}$ to agent 2, future surpluses to be conceded to agent 2 next period contingent on next period realized state $\{V_{t+1}^{s_{t+1}}\}_{s_{t+1}\in\mathcal{S}}$, and a level of savings $A_{t+1}^{s_t}$ to start next period.

The compactness and convexity of $\Gamma(h_t)$ imply that there exists a unique value function which provides agent 1 with the maximum of $\mathcal{U}_t(\pi^t, h_t)$ in each period as a function of A_t, V_t, and the state s realized in t. We denote this value function by $f^s(A_t, V_t)$. It represents the Pareto frontier that can be reached in state s via a self-enforcing contract for given values of savings and surplus to be conceded to agent 2. This value function can be used to maximize agent 1's expected future utility by an appropriate choice of A_{t+1} and $\{V_{t+1}^z\}_{z\in\mathcal{S}}$ as a function of A_t and V_t. The optimal contract then solves the following Bellman equation:[8]

$$
\begin{aligned}
f^s(A_t, V_t^s) = \max_{A_{t+1}^s, b_t^s, \{V_{t+1}^z\}_{z=1}^S} \quad & u(y^s + (1+r)A_t - A_{t+1}^s - b_t^s) + \beta \mathrm{E}_z f^z(A_{t+1}^s, V_{t+1}^z) \\
\text{s.t.} \quad & f^z(A_{t+1}^s, V_{t+1}^z) \geq g(0, y^z) \quad \forall\, z \in \mathcal{S} && (10) \\
& V_{t+1}^z \geq 0 \quad \forall\, z \in \mathcal{S} && (11) \\
& b_t^s + \beta \mathrm{E}_z V_{t+1}^z \geq V_t^s && (12) \\
& A_{t+1}^s \geq 0. && (13)
\end{aligned}
$$

In period t, the solution prescribes optimal values for the current transfer b_t^s, savings A_{t+1}^s,

[7] Thomas and Worrall (1988) show that the surplus V_t^s allowed to agent 2 in state of nature s in a self-enforcing contract belongs to a compact interval $[0, \bar{V}^s]$. In our case, this proof is valid for $V_t^s \in [0, \bar{V}^s(A_t)]$.

[8] Denote by s, the realized state in t, and by z, the state in $t+1$.

and future surpluses $\{V^z_{t+1}\}_{z\in\mathcal{S}}$ for period $t+1$. Constraints (10) and (11) represent the self-enforcing constraints of agents 1 and 2 respectively. The constraint (12) ensures the intertemporal consistency of the optimal solution.

3.1 The value functions

We now characterize the value functions f^s that solve the Bellman equations presented above.[9]

Proposition 1. *For all s, the functions $f^s(A, V)$ are increasing in A, decreasing in V, concave and continuously differentiable in (A, V).*

An immediate implication of this proposition is that the maximization problem on the right-hand side of the Bellman equation is a concave program. First-order conditions are therefore sufficient to characterize the optimal solution. The variables $\beta p^z \theta^z_t$ and $\beta p^z \lambda^z_t$ represent the Lagrange multipliers of the self-enforcing constraints (10) and (11) respectively, for all $z \in \mathcal{S}$. We denote by ψ^s_t the multiplier of constraint (12), and by μ^s_t, the multiplier of the liquidity constraint (13). First-order conditions with respect to A^s_{t+1}, b^s_t and V^z_{t+1} for all z, and the envelope conditions yield the following equations:

$$u'(c^s_t) = \beta \mathrm{E}_z (1+\theta^z_t) f^z_A(A^s_{t+1}, V^z_{t+1}) + \mu^s_t \tag{14}$$

$$u'(c^s_t) = \psi^s_t \tag{15}$$

$$(1+\theta^z_t) f^z_V(A^s_{t+1}, V^z_{t+1}) = -\lambda^z_t - \psi^s_t \quad \forall\, z \in \mathcal{S} \tag{16}$$

$$f^s_A(A_t, V^s_t) = (1+r) u'(c^s_t) \tag{17}$$

$$f^s_V(A_t, V^s_t) = -\psi^s_t. \tag{18}$$

We use the envelope conditions (17) and (18) for period $t+1$ in every possible states z and introduce them in (14), (15), and (16) to obtain:

$$u'(c^s_t) = \beta(1+r) \mathrm{E}_z (1+\theta^z_t) u'(c^z_{t+1}) + \mu^s_t \tag{19}$$

$$u'(c^s_t) = (1+\theta^z_t) u'(c^z_{t+1}) - \lambda^z_t \ \ \forall\, z \in \mathcal{S}. \tag{20}$$

Equation (19) is the Euler equation modified to take into account the liquidity constraint and self-enforcing constraints of period t. It implicitly determines the optimal choice of

[9] All proofs are relegated to the Appendix.

savings for agent 1. Note that this equation does not depend on agent 2's self-enforcing constraints. Only agent 1's constraints affect his choice of savings. Equation (20) implicitly determines agent 1's consumption smoothing through the choice of the current transfer b_t^s and the future surpluses to agent 2, V_{t+1}^z. The extent of consumption smoothing depends on the set of self-enforcing constraints of both agents since these constraints limit the choice of b_t^s and V_{t+1}^z.

Proposition 2. *For all $s \in \mathcal{S}$, the function $f^s(A, V)$ can be written as:*

$$f^s(A, V) = h(y^s + (1+r)A - V),$$

where h is increasing and concave.

This proposition states that each value function f^s can be rewritten as a function of a single variable $y^s + (1+r)A - V$ which represents agent 1's net assets following the realization of the current state of nature s. The term $y^s + (1+r)A$ is agent 1's endowment of financial assets (his savings) and income. The variable V represents the surplus of agent 2, and can be interpreted as the value of the debt agent 1 has contracted with agent 2. This property of the value functions implies that the solution is unique only to the point of having a constant net asset value $y^s + (1+r)A - V$. This property has been derived using the envelope conditions (17) and (18) which can be rewritten, using condition (15), as: $f_A^s(A, V) = -(1+r)f_V^s(A, V)$ for all (A, V) and all s in $\mathcal{S}$. This is a differential equation whose solution must be the functional form in the proposition. We give more intuition on this result below once the optimal consumption has been characterized. Finally, the function h inherits the properties of the functions f^s, namely their monotonicity and concavity.

3.2 Optimal consumption and savings

In a self-enforcing contracting model without savings, Thomas and Worrall (1988) have shown that, in each state, consumption is contained in a time-independent interval. For example, in state s, period-t consumption, c_t^s, belongs to the interval $[\underline{c}^s, \bar{c}^s]$. These consumption bounds are increasing with the state of nature s and are determined by the presence of the self-enforcing constraints. The bound $\bar{c}^s$ ($\underline{c}^s$) is the maximum (minimum) consumption agent 1 can receive in state s. A higher (lower) consumption would necessitate a larger transfer by agent 2 (1) which would violate his self-enforcing constraint in that state. Given these intervals, the dynamics of consumption follow a simple rule. In period t and state s, consumption is equal to period-$(t-1)$ consumption, c_{t-1}, if c_{t-1} is included in the self-enforcing interval of consumption $[\underline{c}^s, \bar{c}^s]$. If not, c_t^s is equal to the bound of the interval that is the closest to c_{t-1}. This means that the contract tries to smooth consumption as much as possible subject to self-enforcing constraints. When all self-enforcing intervals have a non-empty intersection, consumption can be perfectly smoothed in finite time. This is possible for a high enough discount factor.

The introduction of savings in that model modifies the characterization of the optimal consumption path. The first effect is on the self-enforcing consumption bounds.

Proposition 3. *(i) In each period, the consumption of agent 1 is included in an interval of the form $c_t^s \in [\underline{c}^s, \bar{c}^s(A_t)]$ where the lower bound $\underline{c}^s$ is constant, and the upper bound $\bar{c}^s(A_t)$ is increasing in the amount of savings.*
(ii) Consumption bounds are increasing in current income, y^s, that is,

$$\bar{c}^s(A_t) > \bar{c}^z(A_t) \text{ for } s > z$$
$$\underline{c}^s > \underline{c}^z \text{ for } s > z.$$

The introduction of savings implies that the upper consumption bound of Thomas and Worrall (1988) is now dependent on agent 1's savings. In their model, this upper bound is determined by agent 2's self-enforcing constraint. When agent 1 has some savings, however, he does not need to rely only on borrowing in order to increase his consumption as he can now draw upon his savings. This effectively allows him to consume more. This is why the upper bound on consumption depends positively on savings. The lower bound represents a minimal consumption level for agent 1 and is determined by his autarky consumption. Since, by assumption, agent 1 loses his savings if he breaches the contract, this lower bound is independent of savings.

The combination of savings and borrowing through the contract has important effects on the interaction of the self-enforcing constraints of agent 2 and the liquidity constraint.

Proposition 4. *i) If $r = \delta$, the self-enforcing constraints of agent 2 and the liquidity constraint are not binding, that is, $\lambda_t^z = \mu_t^s = 0 \ \ \forall z \in \mathcal{S}$ and $\forall t$.*
ii) If $r < \delta$, either the liquidity constraint or at least one self-enforcing constraint of agent 2 must be binding, that is, $\sum_z p^z \lambda_t^z + \mu_t^s > 0$.

In the model without savings, the upper consumption bound is likely to be binding when (1) there is a negative income shock, that is, when $y_{t+1} < y_t$, and (2) agent 2 expects a low future surplus from the relationship. In that case, agent 2 has low incentives to refinance agent 1, and consequently, agent 2 makes a low transfer to agent 1 and consumption decreases. In period t, agent 1 would like to reimburse agent 2 more than his accumulated debt, that is, save through agent 2, so that agent 1 can consume more tomorrow if his income drops. This is not possible since agent 2 cannot commit to refund the saved amount next period. The non-commitment severely limits the possibility for agent 1 to smooth consumption through lending to agent 2.

With savings, however, this need not be the case. Agent 1 can save today, and draw upon his savings tomorrow to increase his consumption if he experiences a negative income shock. The extent to which agent 1 relies on his savings account depends on the assumption on the interest rate r.

Suppose first that $r = \delta$. Because the interest rate on savings is exactly equal to the discount rate, accumulating savings is as profitable as lending to agent 2. Since the savings account is not subject to self-enforcing constraints, savings effectively increase the self-enforcing upper bound on consumption to the point where it does not bind anymore. Therefore, the introduction of a savings account relaxes all agent 2's self-enforcing constraints.

The liquidity constraint is not binding because agent 1 can effectively save negative amounts by borrowing from agent 2. There is, however, an upper bound on the amount agent 1 can borrow. This amount is contingent on the amount of savings which act as a collateral in case of default by agent 1. This borrowing constraint has real consequences because the self-enforcing constraints of agent 1 can be binding, that is, in a high income state, agent 1 may be tempted to not reimburse agent 2 and renege on the contract, even if this means losing his savings. This effectively limits the amount agent 2 is willing to lend to agent 1. But, as long as savings are pledged as collateral, agent 2 is always willing to lend some positive amount to agent 1, and the liquidity constraint is therefore not binding.

Without a binding liquidity constraint and self-enforcing constraints for agent 2, our model with savings has some similarities with that of Harris and Holmström (1982), where agent 2 can fully commit to a long-term contract. We show below that the dynamics of consumption are similar to the dynamics they derive in their model.

One interpretation of these results is that borrowing and saving become perfect complement since each financial instrument helps relaxing the constraint on the other instrument. This perfect complementarity is not, however, as trivial as it may seem. These two instruments yield very different consumption paths when taken in isolation. When only savings are considered, consumption becomes perfectly smoothed only in the limit when savings tend to infinity. When agent 1 does not have access to a savings account (but can contract with agent 2), consumption varies within the self-enforcing bounds of Thomas and Worrall (1988). It may become perfectly smoothed or not depending on the discount factor. These two different instruments complement each other in our model to the point of relaxing the liquidity constraint and the self-enforcing constraints of agent 2.

The effect of savings is completely different, however, when $r < \delta$. In that case, because the interest rate on savings is relatively low, agent 1 prefers to consume rather than save. In fact, saving through agent 2 becomes less costly than saving through the savings account, and agent 1 would like to borrow at rate r (save negative amounts) and then lend to agent 2 through the contract. This is not possible because of the liquidity constraint. Agent 1 then minimizes his savings and tries to save the most possible through agent 2, until agent 2's self-enforcing constraints become binding.

If the liquidity constraint is not binding, that is, if $\mu_t^s = 0$, then it must be that at least one self-enforcing constraint of agent 2 is binding, that is, $\mathrm{E}_z \lambda_t^z > 0$. Hence, if agent 1 saves a part of its revenue, at least one self-enforcing constraint of agent 2 is binding. The intuition is that, if agent 1 is not liquidity constrained, it must be the case that agent 1 has no incentives in borrowing at rate r to save through agent 2. This can only be true when at least one self-enforcing constraint of agent 2 is binding, thus making additional saving through him impossible. Even though savings are relatively costly, agent 1 can still save if (1) savings through agent 2 is impossible due to agent 2's self-enforcing constraint and (2) it is necessary for agent 1 to save in order to smooth consumption. In fact, agent 1 saves as much as possible through agent 2 and then may use his savings account if required for smoothing purposes.

If no self-enforcing constraint of agent 2 is binding, that is, if $\lambda_t^z = 0$ for all z, it must be that the liquidity constraint is binding, that is, $\mu_t^s > 0$. This means that when the self-enforcing constraints of agent 2 are not binding, agent 1 holds no savings as he would, in fact, like to borrow at rate r. We now turn to the characterization of optimal consumption.

Using the results of Proposition 2, first-order conditions can be rewritten as:

$$\begin{aligned} u'(c_t^s) &= \beta(1+r)\mathrm{E}_z(1+\theta_t^z)h'(y^z+(1+r)A_{t+1}^s-V_{t+1}^z)+\mu_t^s \\ u'(c_t^s) &= (1+\theta_t^z)h'(y^z+(1+r)A_{t+1}^s-V_{t+1}^z)-\lambda_t^z \quad \forall\, z\in\mathcal{S} \\ u'(c_t^s) &= h'(y^s+(1+r)A_t-V_t^s). \end{aligned} \tag{21}$$

The last condition implies that consumption is constant for a given level of net assets $X_t^s \equiv y^s+(1+r)A_t-V_t^s$, that is, along an indifference curve of f^s. There exists a function describing consumption $c_t^s = C(y^s+(1+r)A_t-V_t^s)$ where $C(y^s+(1+r)A_t-V_t^s) = u'^{-1}[h'(y^s+(1+r)A_t-V_t^s)]$. Consumption in period t and state s is an increasing function of the net asset value $X_t^s = y^s+(1+r)A_t-V_t^s$.

Proposition 5. *(i) For all states s and z realized in t and $t+1$:*

$$\begin{aligned} &\lambda_t^z=\theta_t^z=0 && \Rightarrow\; c_{t+1}^z=c_t^s \\ &\theta_t^z>0 && \Rightarrow\; c_{t+1}^z=\underline{c}^z>c_t^s \\ &\lambda_t^z>0 && \Rightarrow\; c_{t+1}^z=\bar{c}^z(A_{t+1}^s)<c_t^s. \end{aligned}$$

(ii) All things equal, if $y^s>y^z$, then $c_t^s \geq c_t^z$.

When $r<\delta$, the optimal contract bears some similarities to that of Thomas and Worrall (1988).[10] It is no longer optimal to relax the self-enforcing constraints of agent 2 and the upper bound of consumption can be binding. Furthermore, if β is close enough to one, the self-enforcing intervals have a non-empty intersection. It is then possible to have a stationary consumption reached in finite time. In that case, savings become useless and $A_t=0$ for all periods t. It is then only when β is not too large that savings can play a useful role. Savings improve the possibility for smoothing as it allows to raise consumption in low-income states. Savings are never enough, however, to maintain agent 1's consumption when a self-enforcing constraint of agent 2 binds in a subsequent period. Consumption can then decrease in time, that is, $c_{t+1}^z<c_t^s$ if $\lambda_t^z>0$. Since savings are relatively costly, it is possible that consumption reaches the self-enforcing upper bound. Smoothing is achieved only imperfectly as agent 1 trades off improved smoothing possibilities and the high cost of savings. The main difference with the model of Thomas and Worrall is that upper bounds on consumption are not constant in time as they depend on the level of savings.

[10]When $r=-1$, our model collapses to theirs as savings generate no returns and thus become useless.

When $r = \delta$, consumption reaches a stationary state in a finite number of periods. First, because $\lambda_t^z = 0$ for all z, consumption in period t is equal to consumption in period $t+1$ unless a self-enforcing constraint for agent 1 is binding in period $t+1$. In that case, consumption in $t+1$ is higher than that in period t. Second, the minimum level of consumption that satisfies agent 1's self-enforcing constraint is increasing in the current income for given values of savings and accumulated debt. Hence, a self-enforcing constraint can only be binding following a positive income shock. This implies that consumption cannot decrease in time.

Proposition 6. *Suppose that $r = \delta$.*
(i) If state s is realized in t, then agent 1's self-enforcing constraint in that state cannot be binding in subsequent periods. Formally, $s_t = s \;\Rightarrow\; \theta_{t+\tau}^s = 0 \;\; \forall \tau \geq 0$.
(ii) If state S is realized in period t, then from $t+1$ on, no self-enforcing constraint for agent 1 is ever binding. Formally, $s_t = S \;\Rightarrow\; \theta_{t+\tau}^s = 0 \; \forall \, s, \; \forall \tau \geq 0$.

Perfect consumption smoothing is then achieved in finite time, that is, following the realization of state S. With a positive income shock, consumption increases and the required level of net assets to support this higher consumption increases also. Once state S is realized, consumption remains constant.

The level of consumption is closely related to the level of the value function as can be seen in the first-order conditions (21). Consequently, following the realization of state S, the value function h attains the same level in all states and all periods. Suppose that state S is realized for the first time in period t. Then, $h(y^S + (1+r)A_t^{s_{t-1}} - V_t^S)$ determines the stationary level of h which, in turn, determines the stationary level of consumption.

In the subsequent periods following the realization of S, the solution is unique in terms of net assets of agent 1, that is, conditional on current income, savings and accumulated debt are chosen to maintain a constant level of net assets X, for the function h. The stationary solution is, however, not unique in terms of $(A, V^s)_{s \in \mathcal{S}}$. For a given X_t, in any state, one can set $A_{t+1}^s = A_t$ and $V_{t+1}^z = V_t^z$ for all $z \in \mathcal{S}$ to obtain $X_{t+1}^z = X_t$ for all z. In that case, $h(y^s + (1+r)A_{t+1} - V_{t+1}^s) = h(y^z + (1+r)A_{t+1} - V_{t+1}^z)$ implies that $V_{t+1}^s > V_{t+1}^z$ for $y^s > y^z$. Hence, for a given level of savings A_t at the beginning of period t and maintained in period $t+1$, the values of V_{t+1}^z conditional on the income realized in $t+1$ are such that $V_{t+1}^1 < \cdots < V_{t+1}^S$. Alternatively, any other combination such that $y^s + (1+r)A_t - V_t^s = y^z + (1+r)A_{t+1}^s - V_{t+1}^z$ for all $z \in \mathcal{S}$ also maintains a constant level for h, and thus yields a constant consumption. Note, however, that in all cases, it must be that debt is increasing in the states of nature, that is, $V_{t+1}^1 < \cdots < V_{t+1}^S$. Even though, there exists an infinity of solution in (A, V), an interesting characterization is that which sets $A_t^s = A$ and $V_t^s = V^s$ (increasing in s) for all s and t. This characterization implies that savings do not need to tend to infinity to achieve perfect income smoothing as opposed to models of savings with liquidity constraints where, if $r = \delta$, savings must tend to infinity to achieve an optimal smoothing of consumption.

4 The model with capital accumulation

In this section, we investigate whether non-commitment in financial contracting can distort a firm's investment policy or not. We therefore assume that the random revenue is no longer exogenous, that is, agent 1 must invest a portion of its current revenue in order to generate the next period revenue through a production function. In that case, the investment decision influences both the expected income and the smoothing of consumption.

Denote by $K(k_t; s)$ the production function that gives the available amount of revenue for period t given k_t, the stock of capital available at the beginning of period t, and $s \in \mathcal{S}$, the realization of the state of nature in period t. We assume that the function K is strictly increasing in k and s, and strictly concave in k, that is, $K'(.; s) > 0$, $K''(.; s) < 0$ for all s, and $K(k, s) > K(k, z)$ for all $s > z$. Capital depreciates at rate d per period. Agent 1's consumption can now be written as:

$$c_t^s = K(k_t; s) + (1-d)k_t + (1 + r)A_t - k_{t+1}^s - A_{t+1}^s - b_t^s.$$

In each period, agent 1 receives the product of its outstanding capital k_t, the undepreciated capital, and the savings from previous period. This income is shared between consumption for the current period, capital and savings for the next period, and a contractual transfer to agent 2.

We assume that, if agent 1 breaches the contract in one period, agent 2 can seize its savings as well as its accumulated capital. Agent 1 is left with a minimum unseizable amount of capital $\underline{k} \geq 0$ that can be interpreted as his human capital and a zero stock of savings. Denote by $G(A, k, s)$ the maximum utility level that agent 1 can obtain in autarky in state s, when A and k are respectively the stocks of savings and capital accumulated at the end of the preceding period.[11] Agent 1 then gets a maximum utility equal to $G(0, \underline{k}, s)$ if he breaches the contract in state s. If agent 2 breaches the contract, he returns to his autarky level of utility $V = 0$.

The introduction of the production process does not transform the mathematical property of the model: it can still be written as a Bellman equation. The capital stock k_t accumulated through period t represents a third state variable. The Bellman equation can be written as:

$$F^s(A_t, k_t, V_t^s) = \max_{\substack{A^s_{t+1}, k^s_{t+1}, \\ b^s_t, \{V^z_{t+1}\}^S_{z=1}}} u(K(k_t; s) + (1-d)k_t + (1+r)A_t - k^s_{t+1} - A^s_{t+1} - b^s_t) + \beta \mathrm{E}_z F^z(A^s_{t+1}, k^s_{t+1}, V^z_{t+1})$$

$$\text{s.t.} \qquad F^z(A^s_{t+1}, k^s_{t+1}, V^z_{t+1}) \geq G(0, \underline{k}, z) \quad \forall\, z \in \mathcal{S} \tag{22}$$

$$V^z_{t+1} \geq 0 \quad \forall\, z \in \mathcal{S} \tag{23}$$

$$b^s_t + \beta \mathrm{E}_z V^z_{t+1} \geq V^s_t \tag{24}$$

$$A^s_{t+1} \geq 0 \tag{25}$$

$$k^s_{t+1} - (1-d)k_t \geq 0. \tag{26}$$

Constraint (26) is an irreversibility constraint on capital. It means that agent 1 cannot make negative investment, that is, destroy some of its assets. The only way of reducing the capital stock is to invest zero and let the capital depreciate. All other constraints can be interpreted as previously.

It can be shown that the maximization problem on the right-hand side of the Bellman equation is a concave program. First-order conditions are therefore necessary and sufficient to the determination of the optimal solution. We still denote by $\beta p^z \theta^z_t$, $\beta p^z \lambda^z_t$, ψ^s_t and μ^s_t the Lagrange multipliers for the self-enforcing, intertemporal and liquidity constraints. The multiplier for the irreversibility constraint (26) is denoted by η^s_t. The first-order and envelope

[11] The autarky problem can be represented as a Bellman equation:
$G(A_t, k_t, s) = \max_{\{A_{t+1}, k_{t+1}\} \in \Delta(A_t, k_t, s)} u(K(k_t; s) + (1-d)k_t + (1+r)A_t - A_{t+1} - k_{t+1}) + \beta \mathrm{E}_z G(A_{t+1}, k_{t+1}, z)$, where $\Delta(A_t, k_t, s)$ is the feasible set defined by a liquidity constraint on A and a non-reversibility constraint on k.

conditions for the program in period t are:

$$u'(c_t^s) = \beta \mathrm{E}_z (1+\theta_t^z) F_A^z(A_{t+1}^s, k_{t+1}^s, V_{t+1}^z) + \mu_t^s \tag{27}$$
$$u'(c_t^s) = \beta \mathrm{E}_z (1+\theta_t^z) F_k^z(A_{t+1}^s, k_{t+1}^s, V_{t+1}^z) + \eta_t^s \tag{28}$$
$$u'(c_t^s) = \psi_t^s \tag{29}$$
$$(1+\theta_t^z) F_V^z(A_{t+1}^s, k_{t+1}^s, V_{t+1}^z) = -\lambda_t^z - \psi_t^s \quad \forall\, z \in \mathcal{S} \tag{30}$$
$$F_A^s(A_t, k_t, V_t^s) = (1+r) u'(c_t^s) \tag{31}$$
$$F_k^s(A_t, k_t, V_t^s) = [K'(k_t; s)+1-d] u'(c_t^s) - (1-d)\eta_t^s \tag{32}$$
$$F_V^s(A_t, k_t, V_t^s) = -\psi_t^s. \tag{33}$$

As before, the envelope conditions (31) and (33) imply that:

$$F^s(A_t, k_t, V_t^s) = H^s(k_t, (1+r)A_t - V_t^s).$$

The introduction of the production process does not interfere with the complementarity between savings and contractual transfers in smoothing consumption. Savings and indebtness are strongly related at the optimum.

To better understand the effects of non-commitment, it is useful to characterize the first-best solution, that is, the solution to the full-commitment problem (where $\theta_t^z = \lambda_t^z = 0$ for all $z \in \mathcal{S}$). In that case, consumption is constant across time and states of nature as given by conditions (29), (30) and (33):[12]

$$\begin{aligned} u'(c_t^s) = \psi_t^s &= -F_V^s(A_t, k_t, V_t^s) \\ &= -F_V^z(A_{t+1}^s, k_{t+1}^s, V_{t+1}^z) = \psi_{t+1}^z = u'(c_{t+1}^z). \end{aligned}$$

Substituting for $c_t^s = c_{t+1}^z = c$ in conditions (28) and (32) yields:

$$[1 - \beta(\mathrm{E}_z K'(k_{t+1}^s; z)+1-d)] u'(c) = \eta_t^s - \beta(1-d)\mathrm{E}_z \eta_{t+1}^z.$$

If capital were reversible, then $\eta_t^s = 0$ for all t and all s, and the efficient level of capital would be k^* such that $\mathrm{E}_z K'(k^*; z) + 1 - d = 1 + \delta$. Here, however, capital is irreversible. This implies that either the initial capital stock $k_0 < k^*$, and then irreversibility is not a

[12]There is a slight abuse of notation in the following conditions as the (endogenous) value functions should not have the same notation as in the non-commitment case.

constraint and the first-best contract sets the capital stock at k^* for the whole relationship, or $k_0 > k^*$, and the irreversibility constraint is binding, and the first-best contract prescribes no investment until the earliest period t for which $k_t \leq k^*$. In all subsequent periods, the capital stock is at k^*. In the full-commitment case, consumption is constant and the capital stock is and remains at its first-best level k^* as soon as the irreversibility constraint becomes relaxed. We now go back to the non-commitment case.

First-order and envelope conditions for the self-enforcing contract give the following equalities:

$$u'(c_t^s) = \beta(1+r)\mathrm{E}_z(1+\theta_t^z)u'(c_{t+1}^z) + \mu_t^s \tag{34}$$

$$= \beta\mathrm{E}_z(1+\theta_t^z)(K'(k_{t+1}^s;z)+1-d)u'(c_{t+1}^z) + \eta_t^s - \beta(1-d)\mathrm{E}_z(1+\theta_t^z)\eta_{t+1}^z \tag{35}$$

$$= \mathrm{E}_z(1+\theta_t^z)u'(c_{t+1}^z) - \mathrm{E}_z\lambda_t^z. \tag{36}$$

Agent 1's self-enforcing constraints multipliers θ_t^z enter the condition for the capital level k_{t+1}^s in (35) through the irreversibility constraint for next period: $k_{t+2} - (1-d)k_{t+1} \geq 0$. When the irreversibility constraint for tomorrow is binding, agent 1 would like to decrease its capital today (k_{t+1}^s) in order to relax it. But this is costly to do in states for which agent 1's self-enforcing constraints bind as decreasing capital reduces the available surplus tomorrow, thus exacerbating the commitment problem in those states.

Proposition 7. *i) If $r = \delta$, neither the liquidity constraint nor the self-enforcing constraints of agent 2 are binding ($\mu_t^s = \lambda_{t+1}^z = 0$ for all z). The capital stock is the same as under full commitment.*
ii) If $r < \delta$, at least one of agent 2's self-enforcing constraints or the liquidity constraint is binding. The capital stock is larger than k^* when at least one of agent 2's self-enforcing constraints is binding.

When $r = \delta$, capital accumulation does not interfere with consumption smoothing. The savings account can complement perfectly borrowing from agent 2 in the achievement of a perfectly smoothed consumption profile for agent 1. In that case, investment in productive capital has no role to play in consumption smoothing, and investment follows the full-commitment rule. In that case, imperfections in the financial environment play no role in the productive investment decision.

When $r < \delta$, however, the existence of a savings account is not sufficient to ensure perfect consumption smoothing. Agent 1 is then left with some residual risk to bear. Any other decision variable can be used to reduce this risk. The capital stock then has a role to play in smoothing consumption.

Suppose that, for tomorrow, some self-enforcing constraints for agent 2 are binding. Agent 1 would like to save more today in order to relax these constraints tomorrow. When $r < \delta$, saving is costly and lending to agent 2 is restricted by the binding self-enforcing constraints. In that case, tomorrow's consumption can be increased by investing today more than is efficient. Overinvestment is costly however, and there is a trade-off between savings and overinvestment. First-order conditions say that, if there are positive savings in one period ($\mu = 0$), then it must be the case that some agent 2's self-enforcing constraints bind ($\mathrm{E}\lambda > 0$). There is then overinvestment. Savings and overinvestment are jointly used in order to partially relax the self-enforcing constraints of agent 2. So, when $r < \delta$, imperfections in financial markets can distort the investment decision by inducing agent 1 in overinvesting.[13]

5 Discussion

Our results depend on two key assumptions: (1) savings act as a collateral and (2) savings are observable. We now discuss these two assumptions in turn.

We assume that savings act as a collateral since agent 1 loses them if he ever breaches the contract. This implies that the value of debt can become arbitrarily large as long as savings increase proportionately. If agent 1 could breach the contract and keep his savings, the contract would have to limit the size of savings in order to ensure that agent 1 has no incentives to breaching the contract and running away with his savings. In that case, the self-enforcing constraints of agent 2 may become binding as agent 1 may have to save through agent 2 to limit these incentives to breach the contract.

We also assume that the contract can control the amount of savings agent 1 accumulates, that is, savings are verifiable. We show that, if savings are not verifiable, agent 1 has incentives to deviate from the savings level prescribed by the contract in which savings are verifiable. These savings implicitly satisfy the following condition:

$$u'(c_t^s) = \beta(1+r)\mathrm{E}_z(1+\theta_t^z)h'(y^z + (1+r)A_{t+1}^s - V_{t+1}^z) + \mu_t^s.$$

[13]A similar result has been derived in Sigouin (1997) in a model without savings.

To understand the incentives of agent 1 to deviate from the prescription of the optimal contract with verifiable savings, consider the effect of a marginal reduction in savings, $dA^s_{t+1} < 0$, on the expected utility of agent 1 at this optimal contract.[14] Such deviation immediately violates the self-enforcing constraints of agent 1 that were binding before the deviation, since agent 1 now reduces the amount of resources to be available tomorrow by saving less and consuming more today. In these binding states tomorrow, however, agent 1 does not lose anything from decreasing his savings since he then breaches the contract to obtain the autarcic level of utility, which is equal to the level of utility he would have obtained had he not reduced his savings and stayed in the contract. Denote

$$\mathcal{Z} = \{z \in \mathcal{S} / h(y^z + (1+r)A^s_{t+1} - V^z_{t+1}) = g(0, y^z)\},$$

the set of period t states for which the self-enforcing constraint of agent 1 is binding, that is, for which θ^z_t is positive. The net utility gain of changing the level of savings is then:

$$dU1 = [-u'(c^s_t) + \beta(1+r) \sum_{w \notin \mathcal{Z}} p^w h'(y^w + (1+r)A^s_{t+1} - V^w_{t+1})] dA^s_{t+1}.$$

Using the first-order conditions for an optimal contract, it is easy to show that this expression is positive. Agent 1 then has an incentive to save less than what is prescribed by the contract since, by doing so, he does not support the full cost of reducing his saving in binding states.

The assumption that savings are verifiable is not innocuous. As long as agent 1 is constrained in one state, he has incentives to reduce his savings. An optimal contract in this environment would have to take such incentives into account by specifying lower levels of savings. This would affect the ability of agent 1 to finance with agent 2 and, hence, his smoothing of consumption. Solving for the optimal contract in that case is beyond the scope of this paper.

6 Conclusion

The endogenization of the liquidity constraint in the model of Schechtman (1976) has significant effects on the optimal path of consumption. When the interest rate on savings is equal

[14] We assume that savings can still serve as a collateral, which means that savings cannot be verified when there is no default, but become verifiable if agent 1 defaults. Audits could reveal the amount of savings. Auditing costs are irrelevant since default would never occur.

to the discount rate on time preference, the liquidity constraint is completely relaxed, while the financier's commitment problem disappears. Consumption is increasing in time, and perfect smoothing is achieved in finite time, as soon as the highest income state is reached. When the savings rate is smaller than the rate of time preference, savings become relatively costly. The risk-averse agent must therefore trade off accumulating costly savings to smooth his consumption and not saving at all.

These results imply that it may be important to endogenize market imperfections in models that seek to explain why agents are imperfectly insured against income risk. While we have used a non-commitment assumption to do so, other assumptions would also be plausible. For example, it would be interesting to assume that the financier cannot observe the income of the risk-averse agent (as in Green, 1987; and Thomas and Worrall, 1990) to see whether predictions are significantly different from those of Schechtman (1976).

We have so far interpreted our model as one where a risk-averse borrower (entrepreneur) seeks financing from a risk-neutral financier and accumulates financial assets in a savings account. Another interpretation would be that where agent 1 is a worker and agent 2 is a firm. Savings can then be interpreted as a pension plan which is controlled by the contract between the worker and the firm. With that interpretation, our model would predict that pensions should not be transferable when the worker quits the firm as it acts as a collateral to alleviate the worker's commitment problem. Note, however, that we have no matching problem which could make the breach of contract efficient.

Such models with non-commitment have also been used to explain sovereign debt financing. In that case, savings would imply that countries could also invest in international financial markets. Finally, Ligon, Thomas and Worrall (1997) discuss the possibility of savings in a risk-sharing self-enforcing contracting model of insurance of groups of individuals in village economies. They do not, however, provide a characterization of consumption paths with savings.

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APPENDIX

Proof of Proposition 1: The utility function u is increasing in A and so is the value function f^s. An increase in V shrinks the set of feasible contracts and, hence the maximum utility agent 1 can obtain in the maximization program. The function f^s is then decreasing in V.[15] The set of constraints (5), (6), (7), (8) and (9) is strictly convex and u is strictly concave and continuously differentiable. It follows that f^s is concave and differentiable in (A, V). □

Proof of Proposition 2: The envelope conditions (17) and (18) can be written: $f_A^s + (1+r)f_V^s = 0$. This is a homogeneous linear differential equation for which the general solution is $f^s(A,V) = h^s((1+r)A - V)$. Here $f^s(A,V)$ could be denoted $f(y^s, A, V)$ with $f_{y^s}(y^s, A, V) = u'(c_t^s) = -f_V(y^s, A, V)$, then by the same argument, $f(y^s, A, V) = \phi(A, y^s - V)$ and $h^s((1+r)A - V) = h(y^s + (1+r)A - V)$. The function h must verify the following conditions on the derivatives of f^s:

$f_A^s(A,V) = (1+r)h'(y^s + (1+r)A - V) > 0$
$f_V^s(A,V) = -h'(y^s + (1+r)A - V) < 0$
$f_{AA}^s(A,V) = (1+r)^2 h''(y^s + (1+r)A - V) < 0$
$f_{VV}^s(A,V) = h''(y^s + (1+r)A - V) < 0$
$f_{AV}^s(A,V) = -(1+r)h''(y^s + (1+r)A - V) > 0$

It follows that $h' > 0$ and $h'' < 0$. □

Proof of Proposition 3: (i) The surplus share given to agent 2 is bounded by the self-enforcing constraints. $V_t^s \in [0, \bar{V}^s(A_t)]$ with $\bar{V}^s(A_t)$ such that $h(y^s + (1+r)A_t - \bar{V}^s(A_t)) = g(0, y^s)$, hence, $\bar{V}^s(A_t) = y^s + (1+r)A_t - h^{-1}[g(0, y^s)]$. The upper bound on V_t^s is an increasing linear function of A_t. The bounds on agent 1's consumption depend on the surplus given to agent 2 through the first order condition: $u'(c_t^s) = h'(y^s + (1+r)A_t - V_t^s)$. This gives the minimum value for c_t^s when $V_t^s = \bar{V}^s(A_t) : \underline{c}^s = {u'}^{-1}[h'(y^s + (1+r)A_t - \bar{V}^s(A_t))] = {u'}^{-1}[h'(h^{-1}[g(0, y^s)])]$. The maximum value for c_t^s is found for $V_t^s = 0 : \bar{c}^s(A_t) = {u'}^{-1}[h'(y^s + (1+r)A_t)]$. It is an increasing function of A_t.

(ii) It is clear that $\bar{c}^s(A_t) = {u'}^{-1}[h'(y^s + (1+r)A_t)]$ is an increasing function of y^s because ${u'}^{-1}$ and h' are decreasing. Furthermore, as $g(0, y^s) > g(0, y^z)$ and h^{-1} is increasing, $\underline{c}^s = {u'}^{-1}[h'(h^{-1}[g(0, y^s)])]$ is increasing in y^s. □

Proof of Proposition 4: Conditions (19) et (20) entail $(1 - \beta(1+r))\mathrm{E}_z(1 + \theta_t^z)u'(c_{t+1}^z) = \sum_z p^z \lambda_t^z + \mu_t^s$.

[15] A more detailed proof can be found in Thomas and Worrall's (1988) Lemma 1.

(i) If $r = \delta$, that is, $\beta(1+r) = 1$, then $\sum_z p^z \lambda_t^z + \mu_t^s = 0$. Since the Lagrange multipliers are all non-negative, it must be that: $\lambda_t^z = \mu_t^s = 0 \; \forall \, z \in \mathcal{S}$.
(ii) If $r < \delta$, then $\sum_z p^z \lambda_t^z + \mu_t^s > 0$. At least one of the multipliers must be positive. □

Proof of Proposition 5: (i) Given the existence of a surplus in the contract, self-enforcing constraints of both agents cannot bind together in the same state of nature. With $\lambda_t^z = 0$, condition (20) becomes $u'(c_t^s) = (1+\theta_t^z)u'(c_{t+1}^z)$. With $\lambda_t^z > 0$ (i.e. $\theta_t^z = 0$), condition (20) becomes $u'(c_t^s) = u'(c_{t+1}^z) - \lambda_t^z$. The concavity of the utility function u yields the result.
(ii) The following lemma helps proving the second part of the proposition.

Lemma 1. *If $y^s > y^z$, then it is not possible to have $\theta_t^z > 0$ and $\theta_t^s = 0$ together, nor $\lambda_t^z = 0$ and $\lambda_t^s > 0$.*

Proof of Lemma 1: Suppose $\theta_t^z > 0$ and $\theta_t^s = 0$ with $y^s > y^z$. Then, $f^z(A_{t+1}, V_{t+1}^z) = g(0,z) < g(0,s) \le f^s(A_{t+1}, V_{t+1}^s)$, and $c_{t+1}^s < c_{t+1}^z$ by (i). Let $(A_{t+2}^*, \{V_{t+2}^{*w}\}_{w\in\mathcal{S}})$ and $(A_{t+2}', \{V_{t+2}'^{w}\}_{w\in\mathcal{S}})$ be the optimal instruments for states s and z respectively. Then,

$$\begin{aligned} f^s(A_{t+1}, V_{t+1}^s) &= u(c_{t+1}^s) + \beta \mathrm{E}_w f^w(A_{t+2}^*, V_{t+2}^{*w}) \\ &< u(c_{t+1}^z) + \beta \mathrm{E}_w f^w(A_{t+2}^*, V_{t+2}^{*w}) \\ &\le u(c_{t+1}^z) + \beta \mathrm{E}_w f^w(A_{t+2}', V_{t+2}'^{w}) \\ &= f^z(A_{t+1}, V_{t+1}^z) \end{aligned}$$

which contradicts the inequality above.
Suppose $\lambda_t^s > 0$ and $\lambda_t^z = 0$ with $y^s > y^z$. Then, $c_{t+1}^s < c_{t+1}^z$ by (i) and $V_{t+1}^z \ge V_{t+1}^s = 0$. This means that
$y^z + (1+r)A_{t+1} - V_{t+1}^z < y^s + (1+r)A_{t+1}$
$h'(y^z + (1+r)A_{t+1} - V_{t+1}^z) > h'(y^s + (1+r)A_{t+1})$
$u'(c_{t+1}^z) > u'(c_{t+1}^s)$,
which contradicts, by the concavity of u, that $c_{t+1}^s < c_{t+1}^z$. □

Condition (20) gives $(1+\theta_t^s)u'(c_{t+1}^s) - \lambda_t^s = u'(c_t) = (1+\theta_t^z)u'(c_{t+1}^z) - \lambda_t^z$.
$a)$ If $\theta_t^s = 0$ and $\lambda_t^z = 0$, Lemma 1 ensures that $u'(c_{t+1}^s) = u'(c_{t+1}^z)$ and, hence, $c_{t+1}^s = c_{t+1}^z$.
$b)$ If $\theta_t^s > 0$ and $\theta_t^z > 0$, $\Rightarrow$ $h(y^z + (1+r)A_{t+1} - V_{t+1}^z) = g(0,z) < g(0,s) = h(y^s + (1+r)A_{t+1} - V_{t+1}^s)$ and, hence, $c_{t+1}^z < c_{t+1}^s$ because consumption is increasing in net wealth.
$c)$ If $\lambda_t^s > 0$ and $\lambda_t^z > 0$ $\Rightarrow$ $V_{t+1}^s = V_{t+1}^z = 0$, $y^s + (1+r)A_{t+1} > y^z + (1+r)A_{t+1}$ and $c_{t+1}^z < c_{t+1}^s$.
$d)$ If $\theta_t^s > 0$ and $\lambda_t^z \ge 0$, or $\theta_t^s \ge 0$ and $\lambda_t^z > 0$, then $c_{t+1}^z < c_{t+1}^s$. □

Proof of Proposition 6: (i) Recall that for $r = \delta$, $\lambda_t^z = 0$ for all t and all z. Let $X_t^s \equiv y^s + (1+r)A_t - V_t^s$ be agent 1's net assets. In period t, agent 1's self-enforcing constraint holds for state s and consumption is optimal, (a) $h(X_t^s) \geq g(0, y^s)$ and (b) $u'(c_t^s) = h'(X_t^s)$. Suppose that $\theta_{t+\tau}^s > 0$ for $\tau \geq 0$. Then, (c) $h(X_{t+\tau+1}^s) = g(0, y^s)$ and (d) $u'(c_{t+\tau+1}^s) = h'(X_{t+\tau+1}^s)$. Proposition (5) for $\theta_{t+\tau}^s > 0$ yields (e) $c_{t+\tau+1}^s > c_{t+\tau} \geq c_t^s$. And,
(1) : (a) and (c) $\Rightarrow h(X_t^s) \geq h(X_{t+\tau+1}^s)$
(2) : (b), (d) and (e) $\Rightarrow h'(X_t^s) > h'(X_{t+\tau+1}^s)$,
which is incompatible with the concavity of h.
(ii) This part of the proposition is derived from the preceding propositions. Lemma 4 involves $(\theta_t^s = 0 \ \Rightarrow \ \theta_t^z = 0)$ for all $y^s > y^z$ and part (i) of this proposition $(s_t = s \ \Rightarrow \ \theta_\tau^s = 0)$ for all $\tau \geq t$. □

Proof of Proposition 7: Conditions (34) and (36) give $[1 - \beta(1+r)]\mathrm{E}_z(1+\theta_t^z)u'(c_{t+1}^z) = \mathrm{E}_z\lambda_t^z + \mu_t^s$.
• If $r = \delta$, then $\mu_t^s = \lambda_t^z = 0$ for all z. Conditions (35) and (36) then entail
$\mathrm{E}_z(1+\theta_t^z)[1 - \beta(K'(k_{t+1}^s; z) + 1 - d)]u'(c_{t+1}^z) = \eta_t^s - \beta(1-d)\mathrm{E}_z(1+\theta_t^z)\eta_{t+1}^z$.
If capital if reversible, $\eta_t^s = \eta_{t+1}^z = 0$ for all z and $k_{t+1}^s = k^*$ for all t and all $s \in \mathcal{S}$.
If capital is irreversible, $\eta_t^s - \beta(1-d)\mathrm{E}_z(1+\theta_t^z)\eta_{t+1}^z$ cannot be negative because it would imply that $k_{t+1}^s < k^*$; in that case, an increase in k_{t+1}^s today would increase η_{t+1}^z, make the expression even more negative and, hence, the capital level even more sub-optimal. This is a contradiction. If $\eta_t^s - \beta(1-d)\mathrm{E}_z(1+\theta_t^z)\eta_{t+1}^z > 0$ the optimal contract prescribes $k_{t+1}^s = k_t$ until the capital level returns to k^* through depreciation. From that level on, capital is at first-best level for the rest of time.
• If $r < \delta$, then $\mathrm{E}_z\lambda_t^z + \mu_t^s > 0$. Condition (35) and (36) give:
$\mathrm{E}_z(1+\theta_t^z)[1 - \beta(K'(k_{t+1}^s; z) + 1 - d)]u'(c_{t+1}^z) = \mathrm{E}_z\lambda_t^z + \eta_t^s - \beta(1-d)\mathrm{E}_z(1+\theta_t^z)\eta_{t+1}^z$,
and condition (36) can be written as $(1+\theta_t^z)u'(c_{t+1}^z) = u'(c_t^s) + \lambda_t^z$. Then,

$$\begin{aligned}
&\mathrm{E}_z[1 - \beta(K'(k_{t+1}^s; z) + 1 - d)](u'(c_t^s) + \lambda_t^z) = \mathrm{E}_z\lambda_t^z + \eta_t^s - \beta(1-d)\mathrm{E}_z(1+\theta_t^z)\eta_{t+1}^z \,, \\
&u'(c_t^s)\mathrm{E}_z[1 - \beta(K'(k_{t+1}^s; z) + 1 - d)] \\
&\quad = \mathrm{E}_z\lambda_t^z + \eta_t^s - \beta(1-d)\mathrm{E}_z(1+\theta_t^z)\eta_{t+1}^z - \mathrm{E}_z[1 - \beta(K'(k_{t+1}^s; z) + 1 - d)]\lambda_t^z \\
&\quad = \eta_t^s - \beta(1-d)\mathrm{E}_z(1+\theta_t^z)\eta_{t+1}^z + \mathrm{E}_z\beta(K'(k_{t+1}^s; z) + 1 - d)\lambda_t^z.
\end{aligned}$$

The right-hand term is positive each time there is a positive λ_t^z, which means that there is over-investment each time there is a binding self-enforcing constraint for agent 2. □