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Éclatement et Contraction Lagrangiens et Applications

par
Antonio Rieser

Département de mathématiques et de statistiques
Faculté des arts et des sciences

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présentée par:

Antonio Rieser

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Iosif Polterovich,	président-rapporteur
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François Lalonde,	codirecteur
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RÉSUMÉ

Soit (M, ω) une variété symplectique. Nous construisons une version de l'éclatement et de la contraction symplectique, que nous définissons relative à une sous-variété lagrangienne $L \subset M$. En outre, si M admet une involution anti-symplectique ϕ, et que nous éclatons une configuration suffisament symmetrique des plongements de boules, nous démontrons qu'il existe aussi une involution anti-symplectique sur l'éclatement $\tilde{M}$. Nous dérivons ensuite une condition homologique pour les surfaces lagrangiennes réeles $L = \text{Fix}(\phi)$, qui détermine quand la topologie de L change losqu'on contracte une courbe exceptionnelle C dans M. Finalement, on utilise ces constructions afin d'étudier le packing relatif dans $(\mathbb{C}P^2, \mathbb{R}P^2)$.

Mots clés : Symplectique, quatre-variétés, sous-variété lagrangienne, packing, packing relatif, involution anti-symplectique, variété réelle.

ABSTRACT

Given a symplectic manifold (M, ω) and a Lagrangian submanifold L, we construct versions of the symplectic blow-up and blow-down which are defined relative to L. Furthermore, if M admits an anti-symplectic involution ϕ, i.e. a diffeomorphism such that $\phi^2 = Id$ and $\phi^*\omega = -\omega$, and we blow-up an appropriately symmetric configuration of symplectic balls, then we show that there exists an antisymplectic involution on the blow-up $\tilde{M}$ as well. We derive a homological condition for real Lagrangian surfaces $L = \text{Fix}(\phi)$ which determines when the topology of L changes after a blow down, and we then use these constructions to study the real packing numbers for real Lagrangian submanifolds in $(\mathbb{C}P^2, \mathbb{R}P^2)$.

Keywords: Symplectic, four-manifold, Lagrangian submanifold, packing, relative packing, anti-symplectic involution, real manifold.

CONTENTS

To my parents

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CHAPTER 1

INTRODUCTION

The blow-up and blow-down constructions are important techniques in complex geometry, leading to methods for resolving singularities as well as classification schemes based on birational equivalence. In the symplectic category, the notion of blowing up a point or submanifold has also been defined and studied from various points of view, as the in papers by Guillemin and Sternberg [14], Lerman [20], and McDuff and Polterovich [23]. When combined with the theory of J-holomorphic curves, the blow-up and blow-down have yielded a great deal of information on symplectic manifolds, notably in packing problems [3, 23], in the classification of rational and ruled symplectic 4-manifolds [17, 18, 21], and in the study of the topology of the space of symplectic embeddings of balls, as, for example, in [1, 19, 28]. In this note, we study relative and real versions of the symplectic blow-up and blow-down, in order to apply them to questions regarding the topology of Lagrangian submanifolds. The relative blow-up takes the pair (M, L) and a set of ball embeddings $\psi : \coprod_{j=1}^{k}(B_j^{2n}(1+2\epsilon), \lambda_j^2\omega_0, B_{\mathbb{R},j}(1+2\epsilon)) \to (M, \omega)$ and obtains another pair $(\tilde{M}, \tilde{L})$, and a symplectic form $\tilde{\omega}$, in which the balls have been replaced by copies of the tautological disk bundle over $\mathbb{C}P^{n-1}$, and $\tilde{L}$ is Lagrangian in $(\tilde{M}, \tilde{\omega})$. The blow-down is the reverse procedure. The real blow-up and blow-down are similar constructions which also respect a so-called real structure on the manifolds.

As a first application, we study the packing problem in real symplectic manifolds. The relative and mixed packing problems were first introduced by Barraud and Cornea in [2], and upper bounds for the relative embedding of one ball on the Clifford torus in

$\mathbb{C}P^n$ was given by Biran and Cornea in [4] using Pearl Homology. Buhovsky [6] further showed that the upper bound given for the Clifford torus is sharp. Schlenk, in [30], directly constructed relative packings of $k \leq 6$ balls in $(\mathbb{C}P^2, \mathbb{R}P^2)$ through a detailed analysis of the moment map. A related construction for packing $\mathbb{C}P^2$ for $k = 7, 8$ balls has been done by Wieck [31]. In Chapter 4, we construct relative embeddings using J-holomorphic techniques, following the general line of argument in [23] and [3]. Our results extend those of McDuff and Polterovich [23] to the real setting. Our packing method depends on the presence of a real structure ϕ for which $L = Fix(\phi)$, and because of this, we do not recover the lower bounds on the Clifford Torus considered by Buhovsky [6]. We believe, nonetheless, that our methods do can be used to extend the results of Biran [3] for $k \geq 9$ balls, but we postpone the treatment of this case to a future paper.

1.1 Setting and Notation

We now give several definitions and set notation for all that follows.

Definition 1.1. Let (M^{2n}, ω) be a symplectic manifold. We say that a submanifold L is *Lagrangian* if dim $L = n$ and $\omega|_{TL} = 0$.

Definition 1.2.

1. We let $\mathcal{L}^n$ denote the tautological complex line bundle over $\mathbb{C}P^{n-1}$, and let $\mathcal{R}^n$ be the real tautological line bundle over $\mathbb{R}P^{n-1}$, i.e. $\mathcal{L}^n = \{(z, l) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} | z \in l\}$ and $\mathcal{R}^n = \{(x, l) \in \mathbb{R}^n \times \mathbb{R}P^{n-1} | x \in l\}$. We will suppress the dimension n when it is clear from the context.

2. $\pi : \mathcal{L} \to \mathbb{C}^n$ and $\theta : \mathcal{L} \to \mathbb{C}P^{n-1}$ denote the canonical projections.

3. $\mathcal{L}(r)$ and $\mathcal{R}(r)$ denote the canonical disk bundles over $\mathbb{C}P^{n-1}$ and $\mathbb{R}P^{n-1}$, respectively, of radius r.

4. For each $\kappa, \lambda > 0$, we define a closed two-form $\rho(\kappa, \lambda)$ on $\mathcal{L}(r)$ by

$$\rho(\kappa, \lambda) = \kappa^2 \pi^* \omega_0 + \lambda^2 \theta^* \sigma,$$

   where ω_0 is the standard form on $\mathbb{C}^n$, and σ is the standard Kähler form on $\mathbb{C}P^{n-1}$ normalized so that $\int_{\mathbb{C}P^{n-1}} \sigma = 1$.

5. Let $\tilde{c} : \mathcal{L} \to \mathcal{L}$ be the map $\tilde{c}(z, l) = (\bar{z}, \bar{l})$, i.e. the restriction to $\mathcal{L}$ of the complex conjugation map on $\mathbb{C}^n \times \mathbb{C}P^{n-1}$.

In addition, the manifolds we treat in our applications will have an additional structure, as defined by

Definition 1.3. Let (M, ω) be a symplectic manifold. A *symplectic anti-involution*, or *real structure*, is a diffeomorphism $\phi : M \to M$ such that $\phi^2 = Id$ and $\phi^* \omega = -\omega$. We refer to a symplectic manifold equipped with a real structure as a *real symplectic manifold,* or simply as a *real manifold*, if the symplectic form is understood.

Remark 1.4. Note that $Fix(\phi)$ is a Lagrangian.

Definition 1.5. Let (M, ω, ϕ) and $(M^{'}, \omega^{'}, \phi^{'})$ be real symplectic manifolds. We say that an embedding $\psi : (M^{'}, \omega^{'}, \phi^{'}) \to (M, \omega, \phi)$ is a *real symplectic embedding* if $\phi \circ \psi = \psi \circ \phi^{'}$ and $\psi^* \omega = \omega^{'}$.

Lemma 1.6. *Let (M, ω_0) be a symplectic manifold, and let (N, ω_1, ϕ) be a real symplectic manifold with symplectic form ω_1 and real structure ϕ. Suppose that there exists a* *symplectic embedding* $\psi : (M, \omega_0) \to (N, \omega_1)$ *such that Im*$(\phi \circ \psi) =$ *Im*(ψ). *Then there exists an anti-symplectic involution* c *on* M *such that* $\phi \circ \psi = \psi \circ c$.

Proof. Define $c := \psi^{-1} \circ \phi \circ \psi$. Then $\phi \circ \psi = \psi \circ c$ and $c^* \omega_0 = \psi^* \phi^* (\psi^{-1})^* \psi^* \omega_1 = -\omega_0$, so ϕ is an anti-symplectic involution on M. □

With the notation in Definition 1.2, we have

Corollary 1.7. $\tilde{c}^* \rho(\kappa, \lambda) = -\rho(\kappa, \lambda)$, *and* $\mathcal{R} =$ *Fix*$(\tilde{c})$.

Proof. Let $c : \mathbb{C}^n \to \mathbb{C}^n$ and $\bar{c} : \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$ denote complex conjugation on $\mathbb{C}^n$ and $\mathbb{C}P^{n-1}$, respectively. Then by the definition of $\tilde{c}$, $\tilde{c}(z, l) = (c(x), \bar{c}(l))$. Since $\mathbb{R}^n = \text{Fix}(c)$ and $\mathbb{R}P^{n-1} = \text{Fix}(\bar{c})$, $\mathcal{R} = \text{Fix}(\tilde{c})$.

Now let $(v_0, w_0), (v_1, w_1) \in T_{(z,l)}\mathcal{L} \subset T_z\mathbb{C}^n \oplus T_l\mathbb{C}P^{n-1}$. Then

$$
\begin{aligned}
\tilde{c}^* \rho(\kappa, \lambda)((v_0, w_0), (v_1, w_1)) &= \tilde{c}^* \pi^* \kappa^2 \omega_0((v_0, w_0), (v_1, w_1)) + \\
&\quad \tilde{c}^* \theta^* \lambda^2 \sigma((v_0, w_0), (v_1, w_1)) \\
&= \kappa^2 \omega_0(\pi_* \tilde{c}_*((v_0, w_0), (v_1, w_1))) + \\
&\quad \lambda^2 \sigma(\theta_* \tilde{c}_*((v_0, w_0), (v_1, w_1))) \\
&= \kappa^2 \omega_0(c_* v_0, c_* v_1) + \lambda^2 \sigma(\bar{c}_* w_0, \bar{c}_* w_1) \\
&= \omega_0(v_0, v_1) - \lambda^2 \sigma(w_0, w_1) \\
&= -\rho(\kappa, \lambda)((v_0, w_0), (v_1, w_1)),
\end{aligned}
$$

which completes the proof. □

In order to put a symplectic form on the blow-up of a manifold M, we will need to consider the relative embeddings of symplectic manifolds, defined below.

Definition 1.8. Let (M, ω, L) and $(M^{'}, \omega^{'}, L^{'})$ be symplectic manifolds with Lagrangians L and $L^{'}$, respectively. We say that a map $\psi : (M^{'}, \omega^{'}, L^{'}) \to (M, \omega, L)$ is a *relative symplectic embedding* when ψ is a symplectic embedding, $\psi^{*}\omega = \omega^{'}$, and $\psi^{-1}(L) = L^{'}$.

We will be primarily concerned with the following example.

Example 1.9. Let (M^{2n}, ω, L) be a symplectic manifold with Lagrangian L. Let $(B(\lambda), \omega_0)$ be the ball of radius λ in $\mathbb{C}^n$ with the standard symplectic structure ω_0, and let $B_{\mathbb{R}}(\lambda)$ denote the ball of radius λ in $\mathbb{R}^n \subset \mathbb{C}^n$. Then a symplectic embedding $\psi : (B^{2n}(\lambda), \omega_0) \hookrightarrow (M^{2n}, \omega)$ is a relative symplectic embedding iff $\psi^{-1}(L) = B_{\mathbb{R}}(\lambda)$.

Remark 1.10. Note that in Definition 1.8, we have $\psi^{-1}(L) = L^{'}$, and not $\psi(L^{'}) \subseteq L$. This is an important distinction, as shown by the following example. Let C denote an embedding of S^1 into $\mathbb{C}^1$, and let

$$\Lambda := \{\lambda \in \mathbb{R} | \exists \text{ a relative embedding } \psi : (B^2(1), \lambda^2\omega_0, B_{\mathbb{R}}(1)) \hookrightarrow (\mathbb{C}^1, \omega_0, C)\}.$$

and $\Lambda_{\sup} := \sup \Lambda$. Then for any $\lambda \in \Lambda$, $\lambda^2\pi \leq 2A$, where A is the area inside $C \subset \mathbb{C}^2$. Therefore $\Lambda_{sup} \leq \sqrt{\frac{2A}{\pi}}$. If, however, we only require that $\psi(B_{\mathbb{R}}(1)) \subseteq C$, then Λ is not bounded above.

Definition 1.11. Let $\psi : \coprod_{i=1}^{k}(B_i(r), \omega_0, B_{\mathbb{R},i}(r)) \hookrightarrow (M, \omega, L)$ be a symplectic embedding, and let $\psi_i := \psi|_{B_i}$. If p of the ψ_i's are relative embeddings, and for the other $q = k - p$ of the ψ_i's, we have $Im(\psi_i) \cap L = \emptyset$, then we call ψ a (p, q)*-mixed embedding*.

1.2 Anti-Symplectic Involutions and Compatible Almost Complex Structures

Our constructions will use auxiliary almost complex structures which satisfy certain additional properties. In this section, we give the necessary definitions, and prove the existence of the complex structures that we need.

Definition 1.12. Let (M, ω) be a symplectic manifold. Then an almost complex structure J *tames* ω or is ω*-tame* if $\omega(\cdot, J\cdot) > 0$.

Definition 1.13. Let (M, ω) be a symplectic manifold. Then an almost complex structure J is *compatible with* ω or is ω*-compatible* if J tames ω, and if, in addition, $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$.

Definition 1.14. Let (M, ω) be a symplectic manifold, let $L \subset M$ be a Lagrangian submanifold, and let p be a point in $L \subset M$. We say that J is *relatively integrable at p* if there is a holomorphic chart $U \subset M$, $\alpha : U \to \mathbb{C}^n$ centered at p such that $\alpha^{-1}(\mathbb{R}^n) = U \cap L$.

Definition 1.15. Let (M, ω, ϕ) be a real symplectic manifold with real structure ϕ. Let L denote $Fix(\phi)$, and let p be a point in L. We say that J is *symmetrically integrable at p* if there is a holomorphic chart $U \subset M$, $\alpha : U \to \mathbb{C}^n$ centered at p such that $\alpha \circ \phi = c \circ \alpha$.

We first prove the existence of almost complex structures J on a real symplectic manifold (M, ω, ϕ) which tame ω and satisfy $J\phi_* = -\phi_* J$. Our discussion follows the methods in Cannas da Silva [8] and McDuff and Salamon [25].

Definition 1.16. Given a symplectic form ω and an ω-compatible almost complex structure J, we denote by $g_J : V \times V \to \mathbb{R}$ the bilinear form defined by

$$g_J(v, w) = \omega(v, Jw). \tag{1.2.1}$$

Lemma 1.17. *Let (V, ω, Φ) be a real symplectic vector space, i.e. a vector space V with a closed, non-degenerate, skew-symmetric bilinear form ω and linear map Φ such that $\Phi^2 = I$ and $\Phi^*\omega = -\omega$. Let $\mathcal{J}_\Phi(V, \omega)$ be the space of ω-compatible almost complex structures on V with $\Phi J = -J\Phi$, and let $\mathcal{M}et_\Phi(V)$ denote the space of positive definite bilinear forms g such that $\Phi^* g = g$. Then there exists a 1-1 map $r : \mathcal{M}et_\Phi(V) \to \mathcal{J}_\Phi(V, \omega)$ such that $r(g_J) = J$.*

The proof follows [8].

Proof. Let $g \in \mathcal{M}et_\Phi(V)$ and define the automorphism $A : V \to V$ by $\omega(v, w) = g(Av, w)$. Then $\omega(v, w) = -\omega(w, v)$ implies that $g(Av, w) = -g(v, Aw)$, and therefore that $A^* = -A$. Let $A = QJ$ be the polar decomposition of A. Then Q is the unique square root of A^*A which is g-self-adjoint and g-positive-definite. We claim that $J_g := Q^{-1}A$ is a complex structure compatible with ω. First, note that A commutes with Q, and therefore $J_g^2 = Q^{-1}AQ^{-1}A = -Id$, so J_g is an almost complex structure. To see that it is orthogonal, we have

$$
\begin{aligned}
\omega(J_g v, J_g w) &= g(AQ^{-1}Av, Q^{-1}Aw) \\
&= g(-A^2Q^{-2}v, Q^{-1}A^*Q^{-1}Aw) \\
&= g(Av, w) = \omega(v, w).
\end{aligned}
$$

Also, $\omega(v, J_g v) = g(Av, Q^{-1}Av) = g(v, A^*Q^{-1}Av) = g(v, Q^{-1}A^*Av) > 0$, since both Q and A^*A are positive definite. Therefore J_g is compatible with ω.

Define $J_g := r(g) = Q^{-1}A$. Note that for an ω-compatible J, we have

$$
r(g_J) = r(\omega(\cdot, J\cdot)) = J,
$$

since, in this case, $J = A$ and $Q = Id$.

To see that $\Phi J_g = -J_g \Phi$, we have first that $-g(Av, w) = \Phi^* \omega(v, w)$, and therefore

$$-g(Av, w) = \omega(\Phi v, \Phi w) = g(A\Phi v, \Phi w) = g(\Phi A \Phi v, w),$$

and therefore $\Phi A \Phi = -A$. Now note that $\Phi A^* A \Phi = -\Phi A^2 \Phi = \Phi A \Phi A = -A^2 = A^* A$. Therefore $\Phi Q \Phi = Q$ as well, and $J_g \Phi = Q^{-1} A \Phi = -Q^{-1} \Phi A = -\Phi Q^{-1} A = -\Phi J_g$, as desired. □

Corollary 1.18. *Let (M, ω, ϕ) be a real symplectic manifold. Let $\mathcal{J}_\phi(V, \omega)$ denote the space of ω-compatible almost complex structures on V with $\phi_* J = -J \phi_*$, and let $\mathcal{M}et_\phi(M)$ denote the space of positive definite bilinear forms g such that $\phi^* g = g$. Then there exists a 1-1 map $r : \mathcal{M}et_\phi(M) \to \mathcal{J}_\phi(V, \omega)$ such that $r(g_J) = J$.*

Proof. Let g be a ϕ-invariant Riemannian metric on M. Since the polar decomposition is canonical, the desired almost complex structure J is given by constructing J_x as in Lemma 1.17 for each $x \in M$. □

Remark 1.19. In particular, this corollary shows that, for a real symplectic manifold (M, ω, ϕ), there exists an ω-compatible (and therefore tame) almost complex structure J with $\phi_* J = -J \phi_*$.

Remark 1.20. Note that if $\psi : (B(1 + 2\epsilon), \lambda^2 \omega_0, B_{\mathbb{R}}(1 + 2\epsilon)) \to (M, \omega, L)$ is a relative or real symplectic embedding, then the above constructions imply that there exists an ω-tame (compatible) almost complex structure J which equals $\psi_* i \psi_*^{-1}$ on a neighborhood of $\psi(0)$, and therefore J is symmectrically or relatively integrable at $\psi(0)$ if ψ is a real or relative embedding, respecively. If, in addition, M has a real structure ϕ and ψ is a real symplectic embedding, then J also may be taken to satisfy $\phi_* J \phi_* = -J$. Similarly, if $\tilde{\psi} : (\mathcal{L}(1+2\epsilon), \rho(1,\delta), \mathcal{R}(1+2\epsilon)) \to (\tilde{M}, \tilde{\omega}, \tilde{L})$ is a real or relative embedding, then there exists an $\tilde{\omega}$-tame almost complex structure $\tilde{J}$ such that $\tilde{J} = \tilde{\psi}_* \tilde{i} \tilde{\psi}_*^{-1}$ in a neighborhood of $\mathcal{L}(0)$.

1.3 Main Results

We now state our main theorems, using the notation in Section 1.1. Theorems 1.21 and 1.22 are proved in Chapter 2.

Theorem 1.21 (Blow-up). *1. Let (M, ω) be a symplectic manifold and let $L \subset M$ be a Lagrangian submanfiold. Suppose that for some small $\epsilon > 0$ there is a (p, q)-mixed symplectic embedding*

$$\psi : \coprod_{j=1}^{k} (B_j(1+2\epsilon), \lambda_j^2 \omega_0, B_{\mathbb{R},j}(1+2\epsilon)) \hookrightarrow (M, \omega, L),$$

and let $P \subset M$ be the set $P := \{\psi_j(0)\}_{j=1}^k$.

Then there exists a symplectic manifold $(\tilde{M}, \tilde{\omega})$, a Lagrangian submanifold $\tilde{L} \subset \tilde{M}$, and an onto map $\Pi : \tilde{M} \to M$ such that the following is satisfied:

(a) Π is a diffeomorphism on $\Pi^{-1}(M \backslash P)$,

(b) $\Pi^{-1}(\psi_j(0)) \cong \mathbb{C}P^{n-1}$,

(c) $\Pi(\tilde{L}) = L$, and

(d) $\tilde{\omega}$ is in the cohomology class

$$[\tilde{\omega}] = [\Pi^* \omega] + \sum_{j=1}^{k} \lambda_j^2 e_j,$$

where the e_j are the Poincaré duals of the exceptional classes $E_j = [\Pi^{-1}(\psi_j(0))]$.

2. *If, in addition, M admits an anti-symplectic involution ϕ which satisfies*

   (a) *Fix$(\phi) = L$,*

   (b) $Im(\phi \circ \psi) = Im(\psi)$,

   (c) $Im(\phi \circ \psi_j) \cap Im(\psi_j) = \emptyset$ *if* $Im(\psi_j) \cap L = \emptyset$, *and*

   (d) $\psi_j \circ c = \phi \circ \psi_j$ *if* $Im(\psi_j) \cap L \neq \emptyset$,

   then $\tilde{M}$ admits an anti-symplectic involution $\tilde{\phi}$ such that Fix$(\tilde{\phi}) = \tilde{L}$ and $\phi \circ \Pi = \Pi \circ \tilde{\phi}$.

Theorem 1.22 (Blow-down)**.** 1. *Let $(\tilde{M}, \tilde{\omega})$ be a symplectic manifold with Lagrangian $\tilde{L}$. Suppose there is a (p, q)-mixed symplectic embedding*

$$\tilde{\psi} : \coprod_{j=1}^{k} (\mathcal{L}_j(r_j), \rho_j(\delta_j, \lambda_j), \mathcal{R}_j(r_j)) \hookrightarrow (\tilde{M}, \tilde{\omega}, \tilde{L})$$

such that $\psi^{-1}(\tilde{L}) = \coprod_{j=1}^{p} \mathcal{R}_j(r_j)$. Let $C_j \subset \tilde{M}$ denote $\tilde{\psi}_j(\mathcal{L}(0))$, and let $C = \cup_j C_j$.

Then there exists a symplectic manifold (M, ω), a (p, q)-mixed symplectic embedding

$$\psi : \coprod_{j=1}^{k} (B(1 + 2\epsilon), \lambda_j \omega_0, B_{\mathbb{R}}(1 + 2\epsilon)) \to (M, \omega, L), \tag{1.3.1}$$

a Lagrangian submanifold $L \subset M$, and an onto map $\Pi : \tilde{M} \to M$ such that the following is satisfied:

(a) *Π is a diffeomorphism on $\tilde{M} \backslash C$,* *(b)* $\Pi(C_j) = p_j \in M$*, where* p_j *is a point,*

(c) $\Pi(\tilde{L}) = L$*, and*

(d) ω *satisfies*

$$[\tilde{\omega}] - [\Pi^*\omega] \in \mathcal{E},$$

where $\mathcal{E}$ *is the linear vector space generated by* $e_1, \ldots, e_k$*, the Poincaré duals of the exceptional classes* $E_j = [\tilde{\psi}_j(0)]$.

2. *Suppose, in addition,* $\tilde{M}$ *admits an anti-symplectic involution* $\tilde{\phi}$ *which satisfies*

   (a) Fix$(\tilde{\phi}) = \tilde{L}$*,*

   (b) $Im(\tilde{\psi}) = Im(\tilde{\phi} \circ \tilde{\psi})$*,*

   (c) $Im(\tilde{\phi} \circ \tilde{\psi}_i) \cap Im(\tilde{\psi}_i) = \emptyset$ *if* $Im(\psi_i) \cap L = \emptyset$*, and*

   (d) $\tilde{\psi}_i \circ \tilde{c} = \tilde{\phi} \circ \tilde{\psi}_i$ *if* $Im(\tilde{\psi}_i) \cap \tilde{L} \neq \emptyset$*.*

   Then (M, ω) *admits an anti-symplectic involution* ϕ *such that* $\phi \circ \Pi = \Pi \circ \tilde{\phi}$*.*

The idea of the relative blow-up construction is the same as blowing up in the purely symplectic case: we remove the interior of a ball from both M and $\overline{\mathbb{C}P}^n$ (the bar indicating that the orientation is reversed), and we glue them along their boundaries, ensuring that the symplectic form $\tilde{\omega}$ of the blow up $\tilde{M}$ acts appropriately. The difference in the relative case is that the real parts of the balls removed from M and $\mathbb{C}P^n$ are constrained to intersect the Lagrangians L and $\mathbb{R}P^n$, and the gluing proceedes so that the boundary of the (n-dimensional) ball removed from L is then glued to the boundary of the corresponding hole in $\mathbb{R}P^n$, resulting in a new Lagrangian $L\#\mathbb{R}P^n \cong \tilde{L} \subset \tilde{M}$ in the blow-up. The blow-down is the reverse process. We make these operations precise in Chapter 2.

In four-dimensional complex geometry and symplectic topology, it is extremely useful to know that one can blow down a symplectic manifold M along a J-holomorphic sphere C when $[C] \cdot [C] = -1$. In complex geometry this is the so-called Castelnuovo-Enriques criterion (see, for example, [13], p.476). Unfortunately, it is a difficult problem in general to derive a similar condition to detect when an arbitrary two-dimensional Lagrangian in a symplectic 4-manifold may be blown down along a curve whose normal bundle in TL is diffeomorphic to the normal bundle of $\mathbb{R}P^1$ in $T(\mathbb{R}P^2)$. However, for Lagrangian submanifolds which are the fixed point set of an anti-symplectic involution ϕ on a symplectic 4-manifold M, we have the following result, which we prove in Chapter 3.

We now give the following definition.

Definition 1.23. We call $E \in H_2(M^4; \mathbb{Z})$ an *exceptional class* if $E \cdot E = -1$. If $u : \Sigma \hookrightarrow M^4$ is an embedding of the surface Σ, and $u_*[\Sigma] = E$, then we say that $u(\Sigma)$ is an *exceptional curve*.

Theorem 1.24. *Let (M, ω, ϕ) be a real symplectic manifold with $L :=$ Fix(ϕ), and let J be an almost complex structure on M which tames ω and which satisfies $\phi_* J \phi_* = -J$. Suppose C is an exceptional J-holomorphic curve in a homology class $E \in H_2(M; \mathbb{Z})$ such that $E \cdot E = -1$ and $\phi_* E = -E$. Then there exists a real symplectic manifold $(\check{M}, \check{\omega}, \check{\phi})$ and an onto map $\Pi : M \to \check{M}$ that satisfies*

1. *Π is a diffeomorphism on $M \backslash C$,*

2. *$\Pi(C) = p \in \check{M}$, where p is a point,*

3. *$\Pi \circ \phi = \check{\phi} \circ \Pi$, and*

4. *$\check{\omega}$ satisfies*

$$[\omega] - [\Pi^*\check{\omega}] \in \mathcal{E},$$

where $\mathcal{E}$ is the linear vector space generated by e, the Poincaré dual of the exceptional class $E = [\Pi^{-1}(p)]$.

As an application of the above theorems, we have the following theorem on the real packing numbers for $(\mathbb{C}P^2, \mathbb{R}P^2)$, defined below.

Definition 1.25. Let (M, ω) be a symplectic manifold with Lagrangian submanifold $L \subset M$. We call the number

$$p_{L,k} := \sup_{\psi} \frac{\text{Vol}\left(\coprod_{i=1}^{k}(B(\lambda), \omega_0, B_{\mathbb{R}}(\lambda))\right)}{\text{Vol}(M)}$$

the *k-th relative packing number for* (M, L), where the $\sup$ is taken over all relative symplectic embeddings

$$\psi : \coprod_{i=1}^{k}(B(\lambda), \omega_0, B_{\mathbb{R}}(\lambda)) \to (M, \omega, L).$$

If M is a real manifold with real structure ϕ, $Fix(\phi) = L$, and the $\sup$ is taken over all real embeddings of k balls, then $p_{L,k}$ is called the *k-th real packing number*. If the supremum is taken over all symplectic embeddings of k balls into M, then we denote the number p_k and we call it the *k-th packing number of M*.

Theorem 1.26. *For the pair $(\mathbb{C}P^2, \mathbb{R}P^2)$ with the standard symplectic form and real structure, the relative packing numbers $p_{\mathbb{R}P^2,k}$ for $k \leq 8$ balls are equal to the packing numbers for $\mathbb{C}P^2$.*

CHAPTER 2

CONSTRUCTING THE RELATIVE AND REAL BLOW-UP AND BLOW-DOWN

We now construct the blow-up and blow-down of a symplectic manifold (M, ω) relative to a Lagrangian submanifold L or a real structure ϕ. The general strategy is to perform a complex blow-up or blow-down locally and then define a symplectic form for the resulting manifold. In each case, we first discuss the local models for the symplectic forms in these constructions, and we then construct the global blow up and blow down given a mixed, relative or real symplectic embedding

$$\begin{aligned} \psi &: \coprod_{j=1}^{k} (B_j(1+2\epsilon), \lambda_j^2\omega_0, B_{\mathbb{R},j}(1+2\epsilon)) \hookrightarrow (M, \omega, L), \text{ or} \\ \tilde{\psi} &: \coprod_{j=1}^{k} (\mathcal{L}_j(1+2\epsilon), \rho(\delta, \lambda), \mathcal{R}_j) \hookrightarrow (\tilde{M}, \tilde{\omega}, \tilde{L}) \end{aligned}$$

and the local models.

The proofs of the lemmas used in these constructions are collected in Section 2.3.

2.1 Blow-up

In this section, we prove Theorem 1.21, which we restate here for the convenience of the reader.

Theorem (Theorem 1.21)**.** *1. Let (M, ω) be a symplectic manifold and let $L \subset M$ be a Lagrangian submanfiold. Suppose that for some small $\epsilon > 0$ there is a (p, q)-*

mixed symplectic embedding

$$\psi : \coprod_{j=1}^{k} (B_j(1+2\epsilon), \lambda_j^2 \omega_0, B_{\mathbb{R},j}(1+2\epsilon)) \hookrightarrow (M, \omega, L),$$

and let $P \subset M$ *be the set* $P := \{\psi_j(0)\}_{j=1}^{k}$.

Then there exists a symplectic manifold $(\tilde{M}, \tilde{\omega})$*, a Lagrangian submanifold* $\tilde{L} \subset \tilde{M}$*, and an onto map* $\Pi : \tilde{M} \to M$ *such that the following is satisfied:*

(a) Π *is a diffeomorphism on* $\Pi^{-1}(M \backslash P)$*,*

(b) $\Pi^{-1}(\psi_j(0)) \cong \mathbb{C}P^{n-1}$*,*

(c) $\Pi(\tilde{L}) = L$*, and*

(d) $\tilde{\omega}$ *is in the cohomology class*

$$[\tilde{\omega}] = [\Pi^* \omega] + \sum_{j=1}^{k} \lambda_j^2 e_j,$$

where the e_j *are the Poincaré duals of the exceptional classes* $E_j = [\Pi^{-1}(\psi_j(0))]$.

2. *If, in addition,* M *admits an anti-symplectic involution* ϕ *which satisfies*

(a) Fix$(\phi) = L$*,*

(b) $Im(\phi \circ \psi) = Im(\psi)$*,*

(c) $Im(\phi \circ \psi_j) \cap Im(\psi_j) = \emptyset$ *if* $Im(\psi_j) \cap L = \emptyset$*, and*

(d) $\psi_j \circ c = \phi \circ \psi_j$ *if* $Im(\psi_j) \cap L \neq \emptyset$*,*

then $\tilde{M}$ *admits an anti-symplectic involution* $\tilde{\phi}$ *such that Fix*$(\tilde{\phi}) = \tilde{L}$ *and* $\phi \circ \Pi = \Pi \circ \tilde{\phi}$.

The construction proceeds as follows. We first construct a family of symplectic forms $\tilde{\tau}(\epsilon, \lambda)$ on $\mathcal{L}$ by pulling back the standard form ω_0 on $\mathbb{R}^{2n}$ by a family of specially constructed maps from $\mathcal{L} \to \mathbb{R}^{2n}$. We arrange, in particular, that the submanifold $\mathcal{R} \subset \mathcal{L}$ is a Lagrangian for the forms $\tilde{\tau}(\epsilon, \lambda)$. We then consider a relative symplectic and holomorphic embedding $\psi : (B(1+2\epsilon), \lambda^2\omega_0, B_{\mathbb{R}}(1+2\epsilon), i) \to (M, \omega, L, J)$, and we construct the blow-up manifold $(\tilde{M}, \tilde{L})$ by removing the ball and gluing in $(\mathcal{L}(1+2\epsilon), \mathcal{R}(1+2\epsilon))$ along the boundary. Finally, we use the local forms $\tilde{\tau}(\epsilon, \lambda)$ created on $\mathcal{L}$ in the first step to construct the global symplectic form $\tilde{\omega}$ on the blow-up $\tilde{M}$. For a real manifold M, we also construct a real structure on the blow up $\tilde{M}$. We then show that, given a relative symplectic embedding, and in view of some appropriate (and non-restrictive) assumptions on the almost complex structures, we may find a holomorphic embedding of a smaller ball which is compatible with L (or a real structure ϕ), and we use this to remove the assumption of holomorphicity on the embeddings.

In the following proposition, we construct the forms $\tilde{\tau}(\epsilon, \lambda)$. Note that points 1, 2, and 3 were proved in Proposition 5.1.A of McDuff and Polterovich [23].

Proposition 2.1. *Using the notation in Section 1.1, for every $\epsilon, \lambda > 0$ there exists a symplectic form $\tilde{\tau}(\epsilon, \lambda)$ on $\mathcal{L}$ such that the following holds:*

1. $\tilde{\tau}(\epsilon, \lambda) = \pi^*(\lambda^2\omega_0)$ *on* $\mathcal{L} - \mathcal{L}(1+\epsilon)$

2. $\tilde{\tau}(\epsilon, \lambda) = \rho(1, \lambda)$ *on* $\mathcal{L}(\delta)$ *for some* $\delta > 0$

3. $\tilde{\tau}(\epsilon, \lambda)$ *is compatible with* $\tilde{i}$*, the canonical integrable complex structure on* $\mathcal{L}$*.*

4. $\tilde{c}^*\tilde{\tau}(\epsilon, \lambda) = -\tilde{\tau}(\epsilon, \lambda)$*, where* $\tilde{c}$ *denotes complex conjugation on* $\mathcal{L}$*.*

5. $\tilde{\tau}(\epsilon, \lambda)|_{\mathcal{R}} = 0$

The proof of this proposition will be based on the following lemmas, which we state here and prove in section 2.3. We begin with a definition.

Definition 2.2. We say that $f : \mathbb{C}^n \to \mathbb{C}^n$ is a *radial function* if $f(z) = \alpha(|z|)z$ for some real-valued function $\alpha : \mathbb{R} \to [0, \infty)$. We say that a radial function f is *monotone* if $|z_0| \leq |z_1| \implies |f(z_0)| \leq |f(z_1)|$.

Lemma 2.3. *Let $h : \mathbb{R}^{2n} \to \mathbb{R}$ be the function $h(x) = \left(1 + \frac{\lambda^2}{|x|^2}\right)^{1/2}$ and ω_0 be the standard symplectic form on $\mathbb{R}^{2n}$. Let $H : \mathbb{R}^{2n} \backslash \{0\} \to \mathbb{R}^{2n} \backslash B(\lambda)$ be the mapping given by $H(x) = h(x)x$. Then $\pi^* H^* \omega = \rho(1, \lambda)$ on $\mathcal{L} \backslash \{(0, l) | l \in \mathbb{C}P^{n-1}\}$.*

Lemma 2.4. *Let (M, ω) be a symplectic manifold. Then ω is a Kähler form iff ω is compatible with an integrable almost complex structure J.*

Lemma 2.5. *Let ω be a Kähler form on $\mathbb{C}^n$, and suppose $f : \mathbb{C}^n \backslash \{0\} \to \mathbb{C}^n \backslash \{0\}$ is a monotone radial function. Then $f^* \omega$ is a Kähler form.*

Proof of Proposition 2.1. For each $\lambda > 0$, let $h_\lambda : \mathbb{R}^{2n} \backslash \{0\} \to \mathbb{R}$ be given by $h_\lambda(x) = \left(1 + \frac{\lambda^2}{|x|^2}\right)^{1/2}$, and let $\delta > 0$ satisfy $\delta^2 < \lambda^2 \epsilon / 2$. For $x \in B(\delta)$, we therefore have $|h_\lambda(x)x|^2 = |x|^2 + \lambda^2 \leq \delta^2 + \lambda^2 < \lambda^2(\epsilon/2 + 1)$. Now define $F : \mathbb{R}^{2n} \backslash \{0\} \to \mathbb{R}^{2n}$ by

$$F(x) = \begin{cases} h_\lambda(x)x, & |x| < \delta \\ (\beta(|x|)h_\lambda(\frac{\delta x}{|x|})\frac{\delta x}{|x|} + (1 - \beta(|x|))\lambda\frac{(1+\epsilon)x}{|x|}, & \delta \leq |x| \leq 1 + \epsilon \\ \lambda x, & 1 + \epsilon \leq |x| \end{cases}$$

where $\beta(t)$ is a bump function which is 1 for $t \leq \delta$ and 0 for $t \geq 1 + \epsilon$. We define the form $\tilde{\tau}(\epsilon, \lambda)$ by $\tilde{\tau}(\epsilon, \lambda) = \pi^* F^* \omega_0$ on $\mathcal{L} \backslash \pi^{-1}(0)$. We start with a preparatory lemma.

Lemma 2.6. *The function F defined above is a monotone radial function.*

Proof. First, we note that F is radial by definition, and that

$$|F(z)| = \begin{cases} (|z|^2+\lambda^2)^{1/2} & |z| \leq \delta \\ \beta(|z|)(\delta^2+\lambda^2)^{1/2} + (1-\beta(|z|)\lambda(1+\epsilon)) & \delta < |z| < 1+\epsilon \\ \lambda|z| & 1+\epsilon < |z|. \end{cases}$$

It follows immediately that, if $z_1, z_2 \in B(\delta)$ or $|z_1|, |z_2| > 1+\epsilon$, then $|z_1| \leq |z_2| \implies |F(z_1)| \leq |F(z_2)|$. Now suppose $z_1, z_2 \in B(1+\epsilon)\backslash B(\delta)$ with $|z_1| \leq |z_2|$. Then

$$\begin{aligned} |F(z_2)| - |F(z_1)| &= \beta(|z_2|)(\delta^2+\lambda^2)^{1/2} + (1-\beta(|z_2|)\lambda(1+\epsilon) - \\ &\quad \beta(|z_1|)(\delta^2+\lambda^2)^{1/2} - (1-\beta(|z_1|)\lambda(1+\epsilon) \\ &= (\beta(|z_2|) - \beta(|z_1|))(\delta^2+\lambda^2)^{1/2} + \\ &\quad (\beta(|z_1|) - \beta(|z_2|))\lambda(1+\epsilon). \end{aligned}$$

We now recall that, by assumption, $\beta(|z_1|) > \beta(|z_2|)$ and

$$\delta^2 + \lambda^2 < \lambda^2(1+\epsilon/2) < \lambda^2(1+2\epsilon+\epsilon^2) = \lambda^2(1+\epsilon)^2,$$

from which it follows that $(\delta^2+\lambda^2)^{\frac{1}{2}} < \lambda(1+\epsilon)$, and therefore $|F(z_2)| - |F(z_1)| > 0$, as desired.

Furthermore, we have, for any $t \in (0,1)$

$$(\delta^2+\lambda^2)^{\frac{1}{2}} \leq \beta(t)(\delta^2+\lambda^2)^{1/2} + (1-\beta(t))\lambda(1+\epsilon) \leq \lambda(1+\epsilon)$$

from which it follows that F is monotone on all of $\mathbb{R}^{2n}$. $\square$

We now return to the proof of Proposition 2.1.

By Lemma 2.3, $\pi^* F^* \omega_0 = \rho(1, \lambda) = \tilde{\tau}(\epsilon, \lambda)$ on $\mathcal{L}(\delta) \backslash \mathcal{L}(0)$. Since $\rho(1, \lambda)$ is a symplectic form on all of $\mathcal{L}(\delta)$, we may extend $\tilde{\tau}(\epsilon, \lambda)$ to all of $\mathcal{L}$ by assigning $\tilde{\tau}(\epsilon, \lambda) := \rho(1, \lambda)$ on $\mathcal{L}(0) = \pi^{-1}(0)$. Now note that this form satisfies condition 1 and 2 in the proposition by Lemma 2.3 and the definition of F.

To see that $\tilde{\tau}(\epsilon, \lambda)$ is symplectic, we note that F is a diffeomorphism from $\mathbb{R}^{2n} \backslash \{0\}$ to its image, and therefore $\pi^* F^*(\omega_0^n) = \tilde{\tau}(\epsilon, \lambda)^n$ is a volume form on $\mathcal{L} \backslash \pi^{-1}(0)$. Therefore, $\tilde{\tau}(\epsilon, \lambda)$ is non-degenerate on $\mathcal{L} \backslash \pi^{-1}(0)$. That $\tilde{\tau}$ is closed on $\mathcal{L} \backslash \pi^{-1}(0)$ is seen by

$$\begin{aligned} d\tilde{\tau}(\epsilon, \lambda) &= d\pi^* F^* \omega_0 \\ &= \pi^* F^* d\omega_0 \\ &= 0. \end{aligned}$$

On $\pi^{-1}(0)$, we have that $\tilde{\tau} = \rho(1, \lambda)$, which is non-degenerate, and since $d\tilde{\tau} = d\rho(1, \lambda) = 0$, it is closed as well. Therefore $\tilde{\tau}$ is symplectic on all of $\mathcal{L}$.

To prove 3, we let $\tilde{i}$ and i represent the standard almost complex structures on $\mathcal{L}$ and $\mathbb{C}^n$, respectively. Since π is the complex blow-down map, we have that $i_* \circ \pi_* = \pi_* \circ \tilde{i}_*$. Therefore, for $v \neq 0, (x, l) \in \mathcal{L}(r) \backslash \mathcal{L}(0)$, we have,

$$\begin{aligned} \tilde{\tau}(\epsilon, \lambda)(\tilde{i}v, v) &= \pi^* F^* \omega_0(\tilde{i}v, v) \\ &= F^* \omega_0(\pi_* \tilde{i}v, \pi_* v) \\ &= F^* \omega_0(i\pi_* v, \pi_* v) \\ &> 0, \end{aligned}$$

where the last inequality follows because, by Lemma 2.5, $F^*\omega_0$ is Kähler, so by Lemma 2.4, $F^*\omega_0$ is compatible with i.

On $\pi^{-1}(0)$, $\tilde{\tau} = \rho(1, \lambda)$, and therefore, for $v \neq 0$, we have

$$\begin{aligned}
\tilde{\tau}(\epsilon, \lambda)(v, \tilde{i}v) &= \pi^*\omega_0(v, \tilde{i}v) + \lambda^2\theta^*\sigma(v, \tilde{i}v) \\
&= \omega_0(\pi_*v, \pi_*\tilde{i}v) + \lambda^2\sigma(\theta_*v, \theta_*\tilde{i}v) \\
&= \omega_0(\pi_*v, i\pi_*v) + \lambda^2\sigma(\theta_*v, i\theta_*v) \\
&> 0
\end{aligned}$$

because ω_0 and σ are compatible with i. We conclude that $\tilde{i}$ is a $\tilde{\tau}$-tame complex structure. To see that $\tilde{\tau}$ is compatible with $\tilde{i}$, we compute

$$\begin{aligned}
\tilde{\tau}(\epsilon, \lambda)(\tilde{i}v, \tilde{i}w) &= \pi^*F^*\omega_0(\tilde{i}v, \tilde{i}w) \\
&= F^*\omega_0(\pi_*\tilde{i}v, \pi_*\tilde{i}w) \\
&= F^*\omega_0(i\pi_*v, i\pi_*w) \\
&= F^*\omega_0(\pi_*v, \pi_*w) \\
&= \tilde{\tau}(v, w).
\end{aligned}$$

Here, again, the fourth equality follows because, by Lemma 2.5, $F^*\omega_0$ is Kähler, so by Lemma 2.4, $F^*\omega_0$ is compatible with i.

Again, on $\pi^{-1}(0)$, $\tilde{\tau}(\epsilon, \lambda) = \rho(1, \lambda)$, and therefore

$$\begin{aligned}
\tilde{\tau}(\epsilon, \lambda)(\tilde{i}v, \tilde{i}w) &= \pi^*\omega_0(\tilde{i}v, \tilde{i}w) + \theta^*\sigma(\tilde{i}v, \tilde{i}w) \\
&= \omega_0(\pi_*\tilde{i}v, \pi_*\tilde{i}w) + \lambda^2\sigma(\theta_*\tilde{i}v, \theta_*\tilde{i}w) \\
&= \omega_0(i\pi_*v, i\pi_*w) + \lambda^2\sigma(i\theta_*v, i\theta_*w) \\
&= \omega_0(\pi_*v, \pi_*w) + \lambda^2\sigma(\theta_*v, \theta_*w) \\
&= \pi^*\omega_0(v, w) + \lambda^2\theta^*\sigma(v, w) \\
&= \tilde{\tau}(\epsilon, \lambda)(v, w),
\end{aligned}$$

since ω_0 and σ are compatible with i. Therefore $\tilde{\tau}(\epsilon, \lambda)$ is compatible with i.

We now show item 4. We first note that, by the definitions of π, c, and $\tilde{c}$, $c \circ \pi = \pi \circ \tilde{c}$. Since, by definition, $F(z) = \alpha(|z|)z$ for a real function $\alpha : \mathbb{R} \to \mathbb{R}$, we have

$$c \circ F(z) = c \circ (\alpha(|z|)z) = \alpha(|z|)\overline{z} = \alpha(|\overline{z}|)\overline{z} = F(\overline{z}) = F \circ c(z),$$

so F commutes with c. Furthermore,

$$\tilde{c}^*\tilde{\tau} = \tilde{c}^*\pi^*F^*\omega_0 = \pi^*c^*F^*\omega_0 = \pi^*F^*c^*\omega_0 = -\pi^*F^*\omega_0 = -\tilde{\tau},$$

which proves item 4. It follows that, since $\text{Fix}(\tilde{c}) = \mathcal{R}(r)$, $\mathcal{R}(r)$ is a Lagrangian in $\mathcal{L}(r)$, proving item 5. This completes the proof. □

In the next proposition, we construct the global relative blow-up of a manifold M using a relative symplectic and holomorphic embedding of the ball $(B(1+2\epsilon), \lambda^2\omega_0, B_{\mathbb{R}}(1+2\epsilon))$ with the standard complex structure i. The use of holomorphic embeddings here gives us extra control over the complex structure in the blow-up, which we will be useful in our applications.

Proposition 2.7. *Let (M,ω) be a symplectic manifold with Lagrangian L, and let J be an ω-tame (compatible) almost complex structure. Suppose that for $\lambda > 0$ and some small $\epsilon > 0$, there is a relative symplectic and holomorphic embedding*

$$\psi : \coprod_{j=1}^{k}(B_j(1+2\epsilon), \lambda_j^2\omega_0, B_{\mathbb{R},j}(1+2\epsilon), i) \hookrightarrow (M, \omega, L, J).$$

Then there exists a symplectic manifold $(\tilde{M}, \tilde{\omega})$ with Lagrangian $\tilde{L} \subset \tilde{M}$, an $\tilde{\omega}$-tame (compatible) almost complex structure $\tilde{J}$, and an onto map $\Pi : \tilde{M} \to M$ such that

1. *Π is a diffeomorphism on $\Pi^{-1}(M \setminus \cup_{j=1}^{k} \psi_j(0))$,*
2. *For all $j \in \{1, \ldots, k\}, \Pi^{-1}(\psi_j(0)) \cong \mathbb{C}P^{n-1}$,*
3. *$\Pi(\tilde{L}) = L$, and*
4. *$\tilde{\omega}$ is in the cohomology class*

$$[\tilde{\omega}] = [\Pi^*\omega] + \sum_{j=1}^{k} \lambda_j^2 e_j,$$

where the e_j are the Poincaré duals of the exceptional classes $E_j = [\Pi^{-1}(\psi_j(0))]$.

Remark 2.8. Note that the E_i in the theorem above are the classes represented by the exceptional curves added in the blow-up.

Proof. First, we consider the case when $k = 1$. Consider the map $\pi : (\mathcal{L}(1+2\epsilon), \mathcal{R}(1+2\epsilon), \tilde{i}) \to (B(1+2\epsilon), B_{\mathbb{R}}(1+2\epsilon), i)$ from Definition 1.2, where $\tilde{i}$ and i are the standard complex structures on $\mathcal{L}$ and $\mathbb{C}^n$, respectively. Observing that π gives a diffeomorphism between the boundaries $(\partial B(1+2\epsilon), \partial B_{\mathbb{R}}(1+2\epsilon))$ and $(\partial \mathcal{L}(1+2\epsilon), \partial \mathcal{R}(1+2\epsilon))$, we let π_∂ denote the restriction of π to $\partial\mathcal{L}(1+2\epsilon)$, and we define $\tilde{M}$ to be $\tilde{M} := M \backslash \psi((B(1+2\epsilon), B_{\mathbb{R}}(1+2\epsilon)) \cup_{\psi\circ\pi_\partial} (\mathcal{L}(1+2\epsilon), R(1+2\epsilon))$. This operation is summarized in the diagram below, with $\delta = 1+2\epsilon$.

$$\begin{array}{ccc} (\mathcal{L}(\delta), \mathcal{R}(\delta)) & \xhookrightarrow{\tilde{\psi}} & (\tilde{M}, \tilde{L}) \\ \pi \big\downarrow & & \big\downarrow \Pi \\ (B(\delta), B_{\mathbb{R}}(\delta)) & \xhookrightarrow[\psi]{} & (M, L) \end{array} \tag{2.1.1}$$

where ψ and $\tilde{\psi}$ are embeddings, and where the map $\Pi : (\tilde{M}, \tilde{L}) \to (M, L)$ is defined by

$$\Pi(x) = \begin{cases} x, & x \notin \text{Im } \tilde{\psi} \\ \psi \circ \pi \circ \tilde{\psi}^{-1}(x) & x \in \text{Im } \tilde{\psi} \end{cases}$$

making the diagram commutative. Note that only ψ is a symplectomorphism a priori.

We now define a symplectic form on $\tilde{M}$. Recall that $\psi^*\omega = \lambda^2\omega_0$ by hypothesis. We assign a symplectic form to $\tilde{M}$ by:

$$\tilde{\omega} = \begin{cases} \Pi^*\omega & \text{on } \tilde{M} \backslash \tilde{\psi}(\mathcal{L}(1+\epsilon)) \\ (\tilde{\psi}^{-1})^*\tilde{\tau}(\epsilon, \lambda) & \text{on } \tilde{\psi}(\mathcal{L}(1+2\epsilon)) \end{cases} \tag{2.1.2}$$

We check that $\tilde{\omega}$ is well-defined on $\mathcal{L}(1+2\epsilon) - \mathcal{L}(1+\epsilon)$. By Proposition 2.1 and the definition of $\tilde{\omega}$ and Π, on $\mathcal{L}(1+2\epsilon) - \mathcal{L}(1+\epsilon)$ we have

$$\begin{aligned} \Pi^*\omega &= (\tilde{\psi}^{-1})^*\pi^*\psi^*\omega \\ &= \lambda^2(\tilde{\psi}^{-1})^*\pi^*\omega_0 \\ &= (\tilde{\psi}^{-1})^*\tilde{\tau}(\epsilon,\lambda), \end{aligned}$$

so $\tilde{\omega}$ is well defined.

We define the almost complex structure $\tilde{J}$ on $\tilde{M}$ by

$$\tilde{J} = \begin{cases} \tilde{\psi}_*\tilde{i}\tilde{\psi}_*^{-1} & \text{on } Im(\tilde{\psi}) \\ \Pi_*^{-1}J\Pi_* & \text{on } \tilde{M}\backslash Im(\tilde{\psi}) \end{cases}$$

Note that since π and ψ are holomorphic diffeomorphisms near the boundary of their respective domains, $\Pi_*^{-1}J\Pi_* = \tilde{\psi}_*\tilde{i}\tilde{\psi}_*^{-1}$ on $\tilde{\psi}(1+2\epsilon)\backslash\tilde{\psi}(1+\epsilon)$, and so $\tilde{J}$ is well defined. To see that $\tilde{\omega}$ tames (is compatible with) $\tilde{J}$, we first note that Π is holomorphic for $x \in \tilde{M} - \mathcal{L}(1+\epsilon)$, and we recall that $\tilde{\omega} = \Pi^*\omega$ on this region. Therefore, if ω tames J, then for $v, w \in T_xM$, $\tilde{\omega}(v, \tilde{J}v) = \lambda^2\omega(\Pi_*v, \Pi_*\tilde{J}v) = \lambda^2\omega(\Pi_*v, J\Pi_*v) > 0$, so $\tilde{\omega}$ tames $\tilde{J}$ on this region. If, in addition, ω is compatible with J, we have,

$$\begin{aligned} \tilde{\omega}(\tilde{J}v, \tilde{J}w) &= \Pi^*\omega(\tilde{J}v, \tilde{J}w) \\ &= \omega(\Pi_*\tilde{J}v, \Pi_*\tilde{J}w) \\ &= \omega(J\Pi_*v, J\Pi_*w) \\ &= \omega(\Pi_*v, \Pi_*w) \\ &= \Pi^*\omega(v, w) \end{aligned}$$

as desired.

For $x \in \mathcal{L}(1+\epsilon)$, we have that $\tilde{\omega} = (\tilde{\psi}^{-1})^*\tilde{\tau}$. Since $\tilde{\tau}$ is compatible with $\tilde{i}$, the canonical complex structure on $\mathcal{L}$, and $\tilde{\psi}$ is holomorphic, then $\tilde{\omega}$ is compatible with $\tilde{J}$ on this region. Therefore, if ω tames (is compatible with) J on M, then $\tilde{\omega}$ tames (is compatible with) $\tilde{J}$ on all of $\tilde{M}$.

Blowing up more than one point is done as above for each ball in the disjoint product $\psi : \coprod_{j=1}^{k}(B_j(r), \omega_0, B_{\mathbb{R},j}(r)) \hookrightarrow (M, \omega, L)$. That $\tilde{\omega}$ is in the desired cohomology class follows immediately from this construction. □

Remark 2.9. When we want to emphasize the embedding ψ, we will refer to the symplectic blow up constructed as above as the blow-up of M *relative to* ψ.

In the following proposition we construct a real structure on the blow-up $\tilde{M}$ given a real symplectic manifold M and a suitably symmetric embedding ψ of a disjoint union of balls into M.

Proposition 2.10. *Let (M, ω, ϕ) be a real symplectic manifold, let J be an ω-tame (compatible) almost complex structure on M which satisfies $\phi_* J \phi_* = -J$, and let*

$$\psi : \coprod_{j=1}^{k}(B_j(1+2\epsilon), \lambda_j^2\omega_0, i) \hookrightarrow (M, \omega, J)$$

be a symplectic and holomorphic embedding. Suppose ϕ and ψ satisfy

1. *$Im(\phi \circ \psi) = Im(\psi)$,*

2. *$Im(\phi \circ \psi_j) \cap Im(\psi_j) = \emptyset$ if $Im(\psi_j) \cap L = \emptyset$, and*

3. *$\psi_j \circ c = \phi \circ \psi_j$ if $Im(\psi_j) \cap L \neq \emptyset$.*

Then there exists a real symplectic manifold $(\tilde{M}, \tilde{\omega}, \tilde{\phi})$ *and an onto map* $\Pi : \tilde{M} \to M$ *which satisfies*

1. Π *is a diffeomorphism on* $\Pi^{-1}(M \setminus \cup_j \psi_j(0))$,
2. $\Pi^{-1}(\psi_j(0)) \cong \mathbb{C}P^{n-1}$,
3. $\Pi \circ \tilde{\phi} = \phi \circ \Pi$, *and*
4. $\tilde{\omega}$ *is in the cohomology class*

$$[\tilde{\omega}] = [\Pi^* \omega] - \sum_{j=1}^{k} \lambda_j^2 e_j,$$

where the e_j *are the Poincaré duals of the exceptional classes* $E_j = [\Pi^{-1}(\psi_j(0))] \in H_2(\tilde{M}; \mathbb{Z})$.

Furthermore, the real structure $\tilde{\phi}$ *and the almost complex structure* $\tilde{J}$ *in the blow-up* $\tilde{M}$ *satisfy* $\tilde{\phi}_* \tilde{J} = -\tilde{J} \tilde{\phi}_*$, *and for every* j *with* $\psi_j \circ c = \phi \circ \psi_j$, *we have* $\phi_* E_j = -E_j \in H_2(\tilde{M}; \mathbb{Z})$.

Remark 2.11. As we will see in the proof, in the case where there are balls which are embedded off of the Lagrangian, the blow-up is not constructed relative to ψ, but relative to another symplectic, holomorphic embedding with the same image. The ball embeddings whose image intersects the Lagrangian are left untouched, and those which take pairs of balls to $M \setminus L$ are changed to commute with ϕ and the standard real structure on $\mathbb{R}^{2n}$.

In order to prove this proposition, we use the following lemmas. In the first lemma, we construct the blow-up given a real embedding ψ on one ball such that $\psi \circ c = \phi \circ \psi$.

In the second, we construct the simultaneous blow-up of an embedding ψ of two balls B_1 and B_2 such that $Im(\phi \circ \psi(B_1)) = Im(\psi(B_2)$.

Lemma 2.12. *Let (M, ω, ϕ) be a real symplectic manifold, let J be an ω-tame (compatible) almost complex structure on M which satisfies $\phi_* J \phi_* = -J$. Suppose*

$$\psi : (B(1+2\epsilon), \lambda^2\omega_0, i) \hookrightarrow (M, \omega, J)$$

is a symplectic and holomorphic embedding such that $\psi \circ c = \phi \circ \psi$. Then there exists a symplectic manifold $(\tilde{M}, \tilde{\omega})$ that admits an anti-symplectic involution $\tilde{\phi}$ such that Π and $\tilde{\omega}$ satisfy the conclusions of Proposition 2.7.

Furthermore, the real structure $\tilde{\phi}$ in the blow-up $\tilde{M}$ satisfies $\tilde{\phi}_* \tilde{J} = -\tilde{J}\tilde{\phi}_*$, and $\tilde{\phi}_*[\Pi^{-1}(\psi(0))] = -[\Pi^{-1}(\psi(0))] \in H_2(\tilde{M}; \mathbb{Z})$.

Proof. We first note that ψ is a relative embedding, since $\psi^{-1}(Fix(\phi)) = Fix(c) = B_{\mathbb{R}}(1+2\epsilon)$. Now construct the blow-up $(\tilde{M}, \tilde{\omega})$ of (M, ω) relative to ψ as in Proposition 2.7. Denote by $\tilde{c}$ the complex conjugation map on $\mathcal{L}$ and recall that we have $\pi \circ \tilde{c}(z, l) = c \circ \pi(z, l)$, since $z \in l \iff \overline{z} \in \overline{l}$ and $\overline{0} = 0$. Given $\epsilon, \lambda > 0$, let $\tilde{\tau}(\epsilon, \lambda)$ be the symplectic form on $\mathcal{L}$ constructed in Proposition 2.1, and recall that $\tilde{c}^*\tilde{\tau}(\epsilon, \lambda) = -\tilde{\tau}(\epsilon, \lambda)$. We now define a map $\tilde{\phi} : \tilde{M} \to \tilde{M}$ by

$$\tilde{\phi}(x) = \begin{cases} \Pi^{-1} \circ \phi \circ \Pi(x), & x \in \tilde{M} \backslash \tilde{\psi}(\mathcal{L}(1+\epsilon)) \\ \tilde{\psi} \circ \tilde{c} \circ \tilde{\psi}^{-1}(x), & x \in \tilde{\psi}(\mathcal{L}(1+2\epsilon)). \end{cases}$$

By the commutativity of Figure 2.1.1, and the equivariance of ψ we have, for $x \in \mathcal{L}(1+$

$2\epsilon)\backslash\mathcal{L}(1+\epsilon)$,

$$\begin{aligned}\tilde{\psi}\circ\tilde{c}\circ\tilde{\psi}^{-1}(x) &= \tilde{\psi}\circ\pi^{-1}\circ c\circ\pi\circ\tilde{\psi}^{-1}(x)\\ &= \Pi^{-1}\circ\psi\circ c\circ\psi^{-1}\circ\Pi(x)\\ &= \Pi^{-1}\circ\psi\circ\psi^{-1}\circ\phi\circ\Pi(x)\\ &= \Pi^{-1}\circ\phi\circ\Pi(x).\end{aligned}$$

Therefore $\tilde{\phi}$ is well-defined and a diffeomorphism. That $\tilde{\phi}$ is an anti-symplectic involution follows from the fact that $\Pi^{-1}\circ\phi\circ\Pi$ and $\tilde{\psi}\circ\tilde{c}\circ\tilde{\psi}^{-1}$ are anti-symplectic involutions on their respective domains.

To see the last statement in the proposition, for $x\in\tilde{M}\backslash\tilde{\psi}(\mathcal{L}(1+\epsilon))$, we compute

$$\begin{aligned}\tilde{\phi}_*\tilde{J} &= \Pi_*^{-1}\phi_*\Pi_*\tilde{J}\\ &= \Pi_*^{-1}\phi_*J\Pi_*\\ &= -\Pi_*^{-1}J\phi_*\Pi_*\\ &= -\tilde{J}\Pi_*^{-1}\phi_*\Pi_*\\ &= -\tilde{J}\tilde{\phi}.\end{aligned}$$

For $x \in \tilde{\psi}(\mathcal{L}(1+2\epsilon))$, we have

$$\begin{aligned}
\tilde{\phi}_*\tilde{J} &= \tilde{\psi}_*\tilde{c}_*\tilde{\psi}_*^{-1}\tilde{J} \\
&= \tilde{\psi}_*\tilde{c}_*\tilde{i}\tilde{\psi}^{-1} \\
&= -\tilde{\psi}_*\tilde{i}\tilde{c}_*\tilde{\psi}^{-1} \\
&= -\tilde{J}\tilde{\psi}_*\tilde{c}_*\tilde{\psi}^{-1} \\
&= -\tilde{J}\tilde{\phi},
\end{aligned}$$

as desired.

Let $E = \tilde{\psi}(\mathcal{L}(0))$. To see that $\tilde{\phi}_*E = -E$, we note that $\tilde{c}(\mathcal{L}(0)) = \mathcal{L}(0)$, and that $\tilde{c}$ reverses orientation. This completes the proof.

□

Lemma 2.13. *Let (M, ω, ϕ) be a real symplectic manifold, let J be an ω-tame (compatible) almost complex structure. Suppose*

$$\gamma : \coprod_{i=1}^{2}(B_i(1+2\epsilon), \lambda^2\omega_0, i) \to (M, \omega, J)$$

is a symplectic and holomorphic embedding such that $Im(\phi \circ \gamma_1) = Im(\gamma_2)$. Then there exists a real symplectic manifold $(\tilde{M}, \tilde{\omega})$ with real structure $\tilde{\phi}$, an $\tilde{\omega}$-tame (compatible) almost complex structure, and an onto map $\Pi : \tilde{M} \to M$ which satisfies the conclusions of Proposition 2.7.

Furthermore, the real structure $\tilde{\phi}$ and the almost complex structure $\tilde{J}$ in the blow-up $\tilde{M}$ satisfy $\tilde{\phi}_*\tilde{J} = -\tilde{J}\tilde{\phi}_*$.

Proof. Define a map $\psi : \Pi_{i=1}^{2}(B_i(1+2\epsilon), \lambda^2\omega_0, i) \to (M, \omega, J)$ by

$$\psi(x) = \begin{cases} \gamma(x) & x \in B_1 \\ \phi \circ \gamma \circ c \circ \iota(x) & x \in B_2 \end{cases}$$

where $\iota : \Pi_{i=1}^{2} B_i \to \Pi_{i=1}^{2} B_i$ is the map given by $\iota(x \in B_i) = x \in B_{i+1 \mod 2}$. We note that, since c and ϕ are anti-holomorphic and γ is holomorphic, ψ is holomorphic, and, similarly, since c and ϕ and anti-symplectic, and γ is symplectic, ψ is symplectic as well. Furthermore, $\gamma, c, \phi,$ and ι are all 1-1, and we conclude that ψ is a symplectic, holomorphic embedding. Now observe that $c \circ \iota$ is an antisymplectic involution on $\Pi_{i=1}^{2} B_i$, $Im(\psi) = Im(\gamma)$ by definition, and that $\psi \circ c \circ \iota = \phi \circ \psi$, so that ψ is a real embedding for the real structures $c \circ \iota$ and ϕ. We now construct the blow up of M relative to ψ as in McDuff and Polterovich [23] (which is as in the relative blow-up without the Lagrangian).

On $\coprod_{i=1}^{2} \mathcal{L}_i$, we put the anti-symplectic involution $\tilde{c} \circ \tilde{\iota}$, where $\tilde{c}$ is complex conjugation on $\mathcal{L}$, and, as above, $\tilde{\iota} : \coprod_{i=1}^{2} \mathcal{L}_i \to \coprod_{i=1}^{2} \mathcal{L}_i$ is given by $\tilde{\iota}((z,l) \in \mathcal{L}_i) = (z,l) \in \mathcal{L}_{i+1 \mod 2}$. Recall that $\pi \circ \tilde{c}(z,l) = c \circ \pi(z,l)$, since $z \in l \iff \overline{z} \in \bar{l}$ and $\overline{0} = 0$, and note that, by definition of ι and $\tilde{\iota}$, we also have $\pi \circ \tilde{c} \circ \tilde{\iota}(z,l) = c \circ \iota \circ \pi(z,l)$.

Given $\epsilon, \lambda > 0$, we define $\nu(\epsilon, \lambda)$ to be the symplectic form on $\coprod_{i=1}^{2} \mathcal{L}_i(1+2\epsilon)$, such that the restriction on each $\mathcal{L}_i$ is given by $\nu(\epsilon,\lambda)|_{\mathcal{L}_i} := \tilde{\tau}(\epsilon, \lambda)$, where $\tilde{\tau}(\epsilon, \lambda)$ is the symplectic form on $\mathcal{L}$ constructed in Proposition 2.1.

Now define a map $\tilde{\phi} : \tilde{M} \to \tilde{M}$ by

$$\tilde{\phi}(x) = \begin{cases} \Pi^{-1} \circ \phi \circ \Pi(x), & x \in \tilde{M} \backslash \tilde{\psi}\left(\coprod_{i=1}^{2} \mathcal{L}(1+\epsilon)\right) \\ \tilde{\psi} \circ \tilde{c} \circ \tilde{\iota} \circ \tilde{\psi}^{-1}(x), & x \in \tilde{\psi}\left(\coprod_{i=1}^{2} \mathcal{L}(1+2\epsilon)\right), \end{cases}$$

where $\tilde{\psi}$ is the embedding of $\coprod_{i=1}^{2} \mathcal{L}(1+2\epsilon)$ as in Figure 2.1.1. By the commutativity of Figure 2.1.1, we have, for $x \in \mathcal{L}(1+2\epsilon)\backslash\mathcal{L}(1+\epsilon)$

$$\begin{aligned} \tilde{\psi} \circ \tilde{c} \circ \tilde{\iota} \circ \tilde{\psi}^{-1}(x) &= \tilde{\psi} \circ \pi^{-1} \circ c \circ \iota \circ \pi \circ \tilde{\psi}^{-1}(x) \\ &= \Pi^{-1} \circ \psi \circ c \circ \iota \circ \psi^{-1} \circ \Pi(x) \\ &= \Pi^{-1} \circ \psi \circ \psi^{-1} \circ \phi \circ \Pi(x) \\ &= \Pi^{-1} \circ \phi \circ \Pi(x). \end{aligned}$$

Therefore $\tilde{\phi}$ is well-defined and a diffeomorphism. That $\tilde{\phi}$ is an anti-symplectic involution follows from the fact that $\Pi^{-1} \circ \phi \circ \Pi$ and $\tilde{\psi} \circ \tilde{c} \circ \tilde{\iota} \circ \tilde{\psi}^{-1}$ are anti-symplectic involutions on their respective domains.

To see the last statement in the proposition, for $x \in \tilde{M}\backslash\tilde{\psi}(\mathcal{L}(1+\epsilon))$, we compute

$$\begin{aligned} \tilde{\phi}_* \tilde{J} &= \Pi_*^{-1} \phi_* \Pi_* \tilde{J} \\ &= \Pi_*^{-1} \phi_* J \Pi_* \\ &= -\Pi_*^{-1} J \phi_* \Pi_* \\ &= -\tilde{J} \Pi_*^{-1} \phi_* \Pi_* \\ &= -\tilde{J} \tilde{\phi}_*. \end{aligned}$$

For $x \in \tilde{\psi}(\mathcal{L}(1+2\epsilon))$, we have

$$\begin{aligned}
\tilde{\phi}_* \tilde{J} &= \tilde{\psi}_* \tilde{c}_* \tilde{\iota}_* \tilde{\psi}_*^{-1} \tilde{J} \\
&= \tilde{\psi}_* \tilde{c}_* \tilde{\iota}_* \tilde{i} \tilde{\psi}_*^{-1} \\
&= -\tilde{\psi}_* \tilde{i} \tilde{c}_* \tilde{\iota}_* \tilde{\psi}_*^{-1} \\
&= -\tilde{J} \tilde{\psi}_* \tilde{c}_* \tilde{\iota}_* \tilde{\psi}_*^{-1} \\
&= -\tilde{J} \tilde{\phi}_*,
\end{aligned}$$

as desired. □

Proof of Proposition 2.10. For each γ_i with $Im(\gamma_i) \cap L \neq \emptyset$ we construct the blow up using Lemma 2.12. For each γ_i such that $Im(\gamma_i) \cap Fix(\phi) = \emptyset$, we first recall that, by hypothesis, $Im(\gamma_i) \cap Im(\phi \circ \gamma_i) = \emptyset$. Since $Im(\phi \circ \gamma) = Im(\gamma)$, then there is a $\gamma_{i'}$ with $Im(\phi \circ \gamma_i) = Im(\gamma_{i'})$. We blow-up the pair γ_i,$\gamma_{i'}$ using Lemma 2.13. The result follows. □

We now remove the hypothesis that our ball embeddings are holomorphic. To do this, we start with a relative or real symplectic ball embedding, and then adjust it so that a small region around the center is also holomorphic, which we may do under appropriate assumptions on an almost complex structure that tames the symplectic form. We then create a family of symplectic forms ω_t on the blow-up such that the original one tames (or is compatible with) the almost complex structure $\tilde{J}$ on the blow-up, and the last one is in the cohomology class corresponding to the ball embedding. This is the same strategy as that of McDuff and Polterovich [23], and the following proposition and its proof are variants of Proposition 2.1.C in [23], which we modify to keep track of the Lagrangians

L and $\tilde{L}$ throughout the process.

Proposition 2.14. *1. Let $\psi : (B(1+2\epsilon), \lambda^2\omega_0, B_{\mathbb{R}}(1+2\epsilon)) \to (M, \omega, L)$ be a relative symplectic embedding. Suppose that J is an almost complex structure on M which tames (is compatible with) ω and which is relatively integrable at $\psi(0)$.*

Then there exists a manifold $\tilde{M}$ with a submanifold $\tilde{L}$, a family of symplectic forms $\tilde{\omega}_t$, $t \in [0,1]$ on $\tilde{M}$, an almost complex structure $\tilde{J}$ on $\tilde{M}$, and an onto map $\Pi : \tilde{M} \to M$ such that $\tilde{\omega}_0$ tames (is compatible with) $\tilde{J}$, $\tilde{L}$ is a Lagrangian for all the $\tilde{\omega}_t$, $\Pi(\tilde{L}) = L$, and $\tilde{\omega}_1$ satisfies

$$[\tilde{\omega}_1] = [\Pi^*\omega] - \lambda^2 e,$$

where e is the Poincare dual of the class $[\Pi^{-1}(\psi(0))] \in H_2(M; \mathbb{Z})$.

2. Suppose, furthermore, M is a real symplectic manifold with real structure ϕ, Fix$(\phi) = L$, J satisfies $\phi_* J \phi_* = -J$, and $\psi \circ c = \phi \circ \psi$. Then there exists a family of real structures $\tilde{\phi}_t$ on $\tilde{M}$ such that $\tilde{\phi}_t^* \tilde{\omega}_t = -\tilde{\omega}_t$, $(\tilde{\phi}_t)_* \tilde{J} (\tilde{\phi}_t)_* = -\tilde{J}$.

The proof depends on the following proposition, which is an adaptation of Proposition 5.5.A in McDuff and Polterovich [23], and which we prove in section 2.3.

Proposition 2.15. *1. Let (M, ω) be a symplectic manifold and let $L \subset M$ be a Lagrangian submanifold. Let*

$$\psi : (B(1+2\epsilon), \lambda^2\omega_0, B_{\mathbb{R}}(1+2\epsilon)) \to (M, \omega, L)$$

be a relative symplectic embedding, and let J be an almost complex structure on M which tames ω and is relatively integrable at $\psi(0) \in L$.

Then, for every compact subset $K \subset M \backslash \psi(0)$ there exists a number $\delta' \in (0,1)$, a symplectic form ω' on M isotopic to ω, and a relative symplectic embedding

$$\psi' : (B(1+2\epsilon), \lambda^2\omega_0, B_{\mathbb{R}}(1+2\epsilon)) \to (M, \omega', L)$$

with the following properties:

(a) $\psi'|_{B(\delta')}$ *is holomorphic*

(b) ω' *tames* J *and coincides with* ω *on* K

(c) L *is a Lagrangian for* ω'

2. *In addition to the above, suppose that M is a real symplectic manifold with real structure ϕ, $Fix(\phi) = L$, J satisfies $\phi_* J \phi_* = -J$, J is symmetrically integrable around $\psi(0)$, and $\phi \circ \psi = \psi \circ c$.*

   Then we can construct the map ψ' and the symplectic form ω' on M to satisfy the conclusions above, and so that ϕ is a real structure for ω' and ψ' satisfies $\phi \circ \psi' = \psi' \circ c$.

Proof of Proposition 2.14. By Proposition 2.15, we may assume that ψ is holomorphic on $B(\delta)$ for some $\delta > 0$. Let $S_t : B(\lambda+\epsilon) \to B(\lambda+\epsilon)$ be defined by:

$$S_t(x) = \beta(t)x + (1-\beta(t))\left[\lambda(1+\gamma)\delta^{-1}\alpha(|x|) + (1-\alpha(|x|))\right]x,$$

where $\beta(t)$ is a bump function with $\beta(t) = 1$ for $t \leq 0$ and $\beta(t) = 0$ for $t \geq 1$, and $\alpha(t)$ is a bump function with $\alpha(t) = 1$ for $t \leq \delta$ and $\alpha(t) = 0$ for $t \geq 1 - \lambda$ for some small $\lambda > 0$. We wish to show that S_t has the following properties:

1. $S_0 = Id$

2. S_t is equal to the identity near $\partial B(\lambda + \epsilon)$

3. $S_t^* \omega_0 = \mu(t)\omega_0$, where $\mu(t) : \mathbb{R} \to \mathbb{R}$ and $\mu(1) = \lambda^2(1+\gamma)^2\delta^{-2}$ on $B(\delta)$ for some $\gamma > 0$.

4. $B_{\mathbb{R}}(\lambda + \epsilon)$ is a Lagrangian for $S_t^* \omega_0$

The first three items above follow directly from the definitions of S_t, α and β. We check item 4. Let c denote complex conjugation. Then $S_t \circ c = c \circ S_t$, and therefore $c^* S_t^* \omega_0 = S_t^* c^* \omega_0 = -S_t^* \omega_0$, so $B_{\mathbb{R}}$ is a Lagrangian for all t.

Now let $F_t : M \to M$ be the extension of $\psi \circ S_t \circ \psi^{-1} : Im(\psi) \subset M \to M$ by the identity map, and set $\omega_t = F_t^* \omega$. Now let

$$\nu_t(z) = \psi\left(\frac{\delta}{1+\gamma} z\right) : \left(B(1+\gamma), \frac{\delta^2}{(1+\gamma)^2}\mu(t)\omega_0\right) \to (M, \omega_t).$$

Then since ψ is a relative embedding, ν is a relative holomorphic embedding, and since $\nu_t^* \omega_t = \frac{\delta^2}{(1+\gamma)^2}\mu(t)\omega_0$, it is also symplectic. Now take the forms $\tilde{\omega}_t$ obtained by blowing up the family ω_t by the embeddings ν_t. We claim that $\tilde{\omega}_t$ verifies the conclusion of the theorem. By definition, ν_0 is a symplectic and holomorphic map into M, so by Proposition 2.7, $\tilde{\omega}_0$ is compatible with $\tilde{J}$. Since F_1 is isotopic to the identity, we see that $[\omega_1] = [\omega]$, from which it follows that $[\Pi^* \tilde{\omega}_1] = [\Pi^* \tilde{\omega}]$. $\tilde{\omega}_1$ is therefore in the desired cohomology class, and the first part of the theorem is proved.

If M has a real structure ϕ, and ψ satisfies the hypotheses in the latter half of the theorem, then by 2.10, blowing up the forms ν_t, we create a family of involutions $\tilde{\phi}_t$ on $\tilde{M}$ such that $\tilde{\phi}_t^* \tilde{\omega}_t = -\tilde{\omega}_t$ and $(\tilde{\phi}_t)_* \tilde{J} (\tilde{\phi}_t)_* = -\tilde{J}$, finishing the proof of the proposition.

□

We now prove Theorem 1.21.

Proof of Theorem 1.21. By Remark 1.20, there exists an almost complex structure on M which is relatively integrable in a neighborhood of the points $\psi_j(0)$. Then by Proposition 2.14, there exists a manifold $\tilde{M}$ with submanifold $\tilde{L}$ and a family of symplectic forms $\tilde{\omega}_t$ on the $\tilde{M}$ such that $\tilde{L}$ is a Lagrangian for all $\tilde{\omega}_t$, and which satisfies $[\tilde{\omega}_t] = [\Pi^*\omega] - \sum_{k=1}^{q} \lambda_k^2 e_k$, where the e_k are the Poincaré duals of the exceptional spheres C_k added in the blow-up.

If, in addition, M has a real structure ϕ and $Im(\psi) = Im(\phi \circ \psi)$, then, by Remark 1.20, J may be chosen so that it is symmetrically integrable around the points $\psi_j(0)$ and $\phi_* J \phi_* = -J$. Therefore, by Proposition 2.14, there exists a family of maps $\tilde{\phi}_t$ on the blow-up such that $\tilde{\phi}_t^* \tilde{\omega}_t = -\tilde{\omega}_t$, and this proves the theorem. □

2.2 Blow-down

We now construct the blow-down of a symplectic manifold $(\tilde{M}, \tilde{\omega}, \tilde{L})$. In particular, we will prove Theorem 1.22, stated again below.

Theorem (Theorem 1.22)**.** *1. Let $(\tilde{M}, \tilde{\omega})$ be a symplectic manifold with Lagrangian $\tilde{L}$. Suppose there is a (p, q)-mixed symplectic embedding*

$$\tilde{\psi} : \coprod_{j=1}^{k} (\mathcal{L}_j(r_j), \rho_j(\delta_j, \lambda_j), \mathcal{R}_j(r_j)) \hookrightarrow (\tilde{M}, \tilde{\omega}, \tilde{L})$$

such that $\psi^{-1}(\tilde{L}) = \coprod_{j=1}^{p} \mathcal{R}_j(r_j)$. Let $C_j \subset \tilde{M}$ denote $\tilde{\psi}_j(\mathcal{L}(0))$, and let $C = \cup_j C_j$.

Then there exists a symplectic manifold (M, ω), a (p, q)-mixed symplectic embedding

$$\psi : \coprod_{j=1}^{k} (B(1 + 2\epsilon), \lambda_j \omega_0, B_{\mathbb{R}}(1 + 2\epsilon)) \to (M, \omega, L), \tag{2.2.1}$$

a Lagrangian submanifold $L \subset M$, and an onto map $\Pi : \tilde{M} \to M$ such that the following is satisfied:

(a) Π *is a diffeomorphism on* $\tilde{M} \backslash C$*,*

(b) $\Pi(C_j) = p_j \in M$*, where* p_j *is a point,*

(c) $\Pi(\tilde{L}) = L$*, and*

(d) ω *satisfies*

$$[\tilde{\omega}] - [\Pi^* \omega] \in \mathcal{E},$$

where $\mathcal{E}$ is the linear vector space generated by $e_1, \ldots, e_k$, the Poincaré duals of the exceptional classes $E_j = [\tilde{\psi}_j(0)]$.

2. *Suppose, in addition, $\tilde{M}$ admits an anti-symplectic involution $\tilde{\phi}$ which satisfies*

(a) *Fix*$(\tilde{\phi}) = \tilde{L}$*,*

(b) $Im(\tilde{\psi}) = Im(\tilde{\phi} \circ \tilde{\psi})$*,*

(c) $Im(\tilde{\phi} \circ \tilde{\psi}_i) \cap Im(\tilde{\psi}_i) = \emptyset$ *if* $Im(\psi_i) \cap L = \emptyset$*, and*

(d) $\tilde{\psi}_i \circ \tilde{c} = \tilde{\phi} \circ \tilde{\psi}_i$ *if* $Im(\tilde{\psi}_i) \cap \tilde{L} \neq \emptyset$*.*

Then (M, ω) admits an anti-symplectic involution ϕ such that $\phi \circ \Pi = \Pi \circ \tilde{\phi}$.

In parallel to the blow-up construction, we begin by constructing a family of forms on $\mathbb{C}^n$ from the forms $\rho(\delta, \lambda)$, which we will then use to construct the global form in the blow-down. The following proposition is adapted from Proposition 5.1.B in [23].

Proposition 2.16. *For every $\epsilon, \delta, \lambda > 0$, there exists a Kähler form $\tau = \tau(\epsilon, \delta, \lambda)$ on $\mathbb{C}^n$ such that the following holds:*

1. $\pi^*(\tau) = \rho(\delta, \lambda)$ *on* $\mathcal{L} - \mathcal{L}(1+\epsilon)$
2. $\tau = \lambda^2 \omega_0$ *on* $B(1) \subset \mathbb{C}^n$
3. τ *is compatible with* i.
4. $c^*\tau = -\tau$, *where* c *denotes complex conjugation on* $\mathbb{C}^n$.
5. $\mathbb{R}^n$ *is a Lagrangian for* τ.

Proof. Note first that $\rho(\delta, \lambda) = \delta^2 \rho(1, \nu)$ for $\nu = \lambda/\delta$. Let $h_\lambda(z) = \left(1 + \left(\frac{\lambda}{|z|}\right)^2\right)^{1/2}$. Let $\beta(t)$ be a bump function which is 1 for $t \leq 1$ and 0 for $t \geq 1+\epsilon$. Then we define the map $G : \mathbb{C}^n \to \mathbb{C}^n$ by

$$G(z) = \begin{cases} \nu z & \text{for } |z| \leq 1 \\ \nu\beta(|z|)\frac{z}{|z|} + (1-\beta(|z|))h_\nu\left(\frac{(1+\epsilon)z}{|z|}\right)\frac{(1+\epsilon)z}{|z|} & \text{for } 1 < |z| < 1+\epsilon \\ h_\nu(z) z & \text{for } |z| \geq 1+\epsilon \end{cases}$$

and we define the form $\tau = \delta^2 G^* \omega_0$. We claim that τ satisfies the properties in the proposition. The first property follows from Lemma 2.3, then second from the definitions of τ and G for $|z| \leq 1$, and the third follows from Lemmas 2.4 and 2.5 and the fact that G is a monotone radial function. To see the fourth point, note that $G(z) = \alpha(|z|)z$ for some real-valued function $\alpha : \mathbb{R} \to \mathbb{R}$. This implies that $c \circ G = G \circ c$, where c is complex conjugation on $\mathbb{C}^n$, and therefore $c^*\delta^2 G^*\omega_0 = \delta^2 G^* c^* \omega_0 = -\delta^2 G^* \omega_0$, as desired. This, in turn, proves the fifth point as well, and completes the proof. □

In parallel to the blow-up construction, we split the blow-down into two parts, the relative blow-down, in which we consider only a Lagrangian, and we do not consider a real structure, and the real blow-down. We now construct the relative blow-down.

Proposition 2.17. *Let $(\tilde{M}, \tilde{\omega})$ be a symplectic manifold with Lagrangian $\tilde{L}$, and let $\tilde{J}$ be an $\tilde{\omega}$-tame (compatible) almost complex structure. Suppose there is a (p, q)-mixed holomorphic and symplectic embedding*

$$\psi : \coprod_{j=1}^{k} (\mathcal{L}_j(r_j), \rho_j(\delta_j, \lambda_j), \mathcal{R}_j(r_j), i) \hookrightarrow (\tilde{M}, \tilde{\omega}, \tilde{L}, \tilde{J})$$

such that

$$\psi^{-1}(L) = \coprod_{j=1}^{p} \mathcal{R}_j(r_j).$$

Then the conclusions of the first part of Theorem 1.22 are satisfied.

Proof. We consider the case when $(p, q) = (1, 0)$. Let $\tilde{\psi} : (\mathcal{L}(1 + 2\epsilon_0), \rho(\delta, \lambda), \mathcal{R}(1 + 2\epsilon_0)) \to (\tilde{M}, \tilde{\omega}, \tilde{L})$ be a relative symplectic embedding such that $\tilde{\psi}^* \tilde{\omega} = \rho(\delta, \lambda)$. We then perform a local complex blow down in $\mathcal{L}(1 + 2\epsilon)$, and we define the manifold M by

$$M := \tilde{M} \backslash \tilde{\psi}(\mathcal{L}(1 + 2\epsilon)) \cup_{\tilde{\psi} \circ \pi^{-1} | \partial \mathcal{L}(1+2\epsilon)} B(1 + 2\epsilon)$$

after which, as in the blow-up, we arrive at the commutative diagram

$$\begin{array}{ccc} (\mathcal{L}(1+2\epsilon), \mathcal{R}(1+2\epsilon)) & \xhookrightarrow{\tilde{\psi}} & (\tilde{M}, \tilde{L}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \Pi} \\ (B(1+2\epsilon), B_{\mathbb{R}}(1+2\epsilon)) & \xhookrightarrow[\psi]{} & (M, L) \end{array} \tag{2.2.2}$$

where Π is defined by

$$\Pi(x) = \begin{cases} x & x \in \tilde{M}\backslash\tilde{\psi}(\mathcal{L}(1+2\epsilon)) \\ \psi \circ \pi \circ (\tilde{\psi}^{-1}) & x \in \tilde{\psi}(\mathcal{L}(1+2\epsilon)). \end{cases}$$

We now define the following form on M:

$$\omega = \begin{cases} (\Pi^{-1})^*\tilde{\omega} & \text{on } M\backslash\psi(B(1+\epsilon)) \\ (\psi^{-1})^*\tau(\epsilon,\delta,\lambda) & \text{on } \psi(B(1+2\epsilon)). \end{cases}$$

We check that the definition of ω agrees on $\psi(B(1+2\epsilon))\backslash\psi(B(1+\epsilon))$. On this region, we have

$$\begin{aligned} \omega &= (\psi^{-1})^*\tau(\epsilon,\delta,\lambda) \\ &= (\psi^{-1})^*(\pi^{-1})^*\rho(1,\lambda) \\ &= (\psi^{-1})^*\pi^*\tilde{\psi}^*\tilde{\omega} \\ &= (\Pi^{-1})^*\tilde{\omega}, \end{aligned}$$

so ω is well defined. Furthermore, we claim that ω is a symplectic form. Too see this, note that Π is a diffeomorphism on $\Pi^{-1}(M\backslash\psi(B(1+\epsilon)))$, so ω^n is a volume form on $M\backslash\psi(B(1+\epsilon))$, and ω is therefore non-degenerate there. It is closed by definition. For $\psi(B(1+2\epsilon))$, we first note that by Proposition 2.16, τ is Kähler, and therefore symplectic on $\mathbb{R}^{2n}$. Since ψ^{-1} is a diffeomorphism on $B(1+2\epsilon)$, ω is non-degenerate here as well, and closed by definition.

We define the almost complex structure J on M by

$$J = \begin{cases} \psi_* i \psi_*^{-1} & \text{on } Im(\psi) \\ \Pi_* \tilde{J} \Pi_*^{-1} & \text{on } M \backslash Im(\psi) \end{cases}$$

Note that since π and ψ are holomorphic diffeomorphisms near the boundary of their respective domains, $\psi_* i \psi_*^{-1} = \Pi_* \tilde{J} \Pi_*^{-1}$ on $\psi(1+2\epsilon) \backslash \psi(1+\epsilon)$, and so J is well defined. To see that ω tames (is compatible with) J, we first note that Π is holomorphic and a diffeomorphism for $x \in \tilde{M} - \mathcal{L}(1+\epsilon)$, and we recall that $\omega = (\Pi^{-1})^* \tilde{\omega}$ on $M \backslash B(1+\epsilon)$. Therefore, if $\tilde{\omega}$ tames J, then for $v, w \in T_{\Pi(x)} M$, $\omega(v, Jv) = \tilde{\omega}(\Pi_*^{-1} v, \Pi_*^{-1} \tilde{J} v) = \tilde{\omega}(\Pi_*^{-1} v, J \Pi_*^{-1} v) > 0$, so ω tames J on $M \backslash \psi(B(1+\epsilon)$. If, in addition, $\tilde{\omega}$ is compatible with $\tilde{J}$, then on $M \backslash B(1+\epsilon)$, we have

$$\begin{aligned} \omega(Jv, Jw) &= (\Pi^{-1})^* \tilde{\omega}(Jv, Jw) \\ &= \tilde{\omega}(\Pi_*^{-1} Jv, \Pi_*^{-1} Jw) \\ &= \tilde{\omega}(\tilde{J} \Pi_*^{-1} v, \tilde{J} \Pi_*^{-1} w) \\ &= \tilde{\omega}(\Pi_*^{-1} v, \Pi_*^{-1} w) \\ &= (\Pi^{-1})^* \tilde{\omega}(v, w) \end{aligned}$$

as desired.

For $x \in \mathcal{L}(1+\epsilon)$, we have that $\omega = (\psi^{-1})^* \tau$. Since τ is compatible with i, the canonical complex structure on $B(1+2\epsilon)$, and ψ is holomorphic (tautologically, by the definition of J), then ω is compatible with J on this region. Therefore, if $\tilde{\omega}$ tames (is compatible with) $\tilde{J}$ on $\tilde{M}$, then ω tames (is compatible with) J on M.

The condition on the cohomology class of ω follows immediately from the construction. This completes the proof of the proposition. $\square$

We now construct the real blow-down for a real symplectic manifold $\tilde{M}$.

Proposition 2.18. *Let $(\tilde{M}, \tilde{\omega}, \tilde{\phi})$ be a real symplectic manifold and let $\tilde{L} = Fix(\tilde{\phi})$. Let $\tilde{J}$ be an $\tilde{\omega}$-tame (compatible) almost complex structure on $\tilde{M}$. Suppose that*

$$\tilde{\psi} : \coprod_{j=1}^{k} (\mathcal{L}_j(r_j), \rho_j(\delta_j, \lambda_j), \mathcal{R}_j(r_j), i) \hookrightarrow (\tilde{M}, \tilde{\omega}, \tilde{L}, \tilde{J})$$

is a symplectic and holomorphic embedding such that

1. $\psi^{-1}(\tilde{L}) = \coprod_{j=1}^{k} \mathcal{R}_j(r_j)$,
2. $Im(\tilde{\psi}) = Im(\tilde{\phi} \circ \tilde{\psi})$,
3. $Im(\tilde{\phi} \circ \tilde{\psi}_i) \cap Im(\tilde{\psi}_i) = \emptyset$ *if* $Im(\psi_i) \cap L = \emptyset$, *and*
4. $\tilde{\psi}_i \circ \tilde{c} = \tilde{\phi} \circ \tilde{\psi}_i$ *if* $Im(\tilde{\psi}_i) \cap \tilde{L} \neq \emptyset$.

Then the conclusions of the second part of Theorem 1.22 are satisfied.

As in the blow-up, we prove this in two parts. The first is the following.

Lemma 2.19. *Let $(\tilde{M}, \tilde{\omega}, \tilde{\phi})$ be a real symplectic manifold and let $\tilde{L} = Fix(\tilde{\phi})$. Let $\tilde{J}$ be an $\tilde{\omega}$-tame (compatible) almost complex structure on $\tilde{M}$, and suppose that $\tilde{\psi} : (\mathcal{L}(r), \rho(\delta, \lambda), \mathcal{R}(r)) \hookrightarrow (\tilde{M}, \tilde{\omega}, \tilde{L})$ is a symplectic embedding such that $\tilde{\psi} \circ c = \tilde{\phi} \circ \tilde{\psi}$. Then the blow-down (M, ω, L) admits an anti-symplectic involution ϕ and an almost complex structure J such that $Fix(\phi) = L$ and $\phi_* J \phi_* = -J$.*

Proof. Construct the blow-down (M, ω) as in Proposition 2.17. Now define a map ϕ by

$$\phi(x) = \begin{cases} \Pi \circ \tilde{\phi} \circ \Pi^{-1} & x \in M \backslash \psi(B(1+\epsilon)) \\ \psi \circ c \circ \psi^{-1}(x) & x \in \psi(B(1+2\epsilon)) \end{cases},$$

Note that, for $x \in \psi(B(1+2\epsilon) - B(1+\epsilon))$,

$$\begin{aligned} \psi \circ c \circ \psi^{-1}(x) &= \psi \circ \pi \circ \tilde{c} \circ \pi^{-1} \circ \psi^{-1}(x) \\ &= \Pi \circ \tilde{\psi} \circ \tilde{c} \circ \tilde{\psi}^{-1} \circ \Pi^{-1}(x) \\ &= \Pi \circ \tilde{\phi} \circ \Pi^{-1}(x), \end{aligned}$$

so the map ϕ is well-defined and a diffeomorphism. Furthermore, $\phi^2 = Id$ by definition. To see that $\phi^* \omega = -\omega$, we have, for $x \in M \backslash \psi(B(1+2\epsilon))$,

$$\begin{aligned} \phi^* \omega_x &= \phi^* (\Pi^{-1})^* \tilde{\omega}_x \\ &= (\Pi^{-1})^* \tilde{\phi}^* \tilde{\omega}_x \\ &= -(\Pi^{-1}) \tilde{\omega}_x \\ &= -\omega_x, \end{aligned}$$

and for $x \in \psi(B(1+2\epsilon))$, we have

$$
\begin{aligned}
\phi^*\omega_x &= (\psi^{-1})^* c^* \psi^* (\psi^{-1})^* \tau(\epsilon, \delta, \lambda) \\
&= (\psi^{-1})^* c^* \tau(\epsilon, \delta, \lambda) \\
&= -(\psi^{-1})^* \tau(\epsilon, \delta, \lambda) \\
&= -\omega_x,
\end{aligned}
$$

We now check that $\phi_* J \phi_* = -J$. For $x \in M \backslash \psi(B(1+\epsilon))$, we compute

$$
\begin{aligned}
\phi_* J &= \Pi_* \tilde{\phi}_* \Pi_*^{-1} J \\
&= \Pi_* \tilde{\phi}_* \tilde{J} \Pi_*^{-1} \\
&= -\Pi_* \tilde{J} \tilde{\phi}_* \Pi_*^{-1} \\
&= -J \Pi_*^{-1} \phi_* \Pi_* \\
&= -J \phi_*.
\end{aligned}
$$

For $x \in \psi(B(1+2\epsilon))$, we have

$$
\begin{aligned}
\phi_* J &= \psi_* c_* \psi_*^{-1} J \\
&= \psi_* c_* i \psi_*^{-1} \\
&= -\psi_* i c_* \psi_*^{-1} \\
&= -J \psi_* c_* \psi_*^{-1} \\
&= -J \phi_*,
\end{aligned}
$$

which completes the proof. □

Lemma 2.20. *Let $(\tilde{M}, \tilde{\omega}, \tilde{\phi})$ be a real symplectic manifold and let $\tilde{L} = Fix(\tilde{\phi})$. Suppose that $\tilde{\gamma} : \coprod_{j=1}^{2}(\mathcal{L}_j(r_j), \rho_j(\delta_j, \lambda_j), \mathcal{R}(r)) \hookrightarrow (\tilde{M}, \tilde{\omega}, \tilde{L})$ is a symplectic embedding such that $\psi^{-1}(\tilde{L}) = \emptyset$ and $Im(\tilde{\phi} \circ \tilde{\gamma}_1) = Im(\tilde{\gamma}_2)$. Then the blow-down (M, ω) admits an anti-symplectic involution ϕ.*

Proof. Since $Im(\tilde{\phi} \circ \tilde{\gamma}_1) = Im(\tilde{\gamma}_2)$, we can replace γ with an embedding

$$\tilde{\psi} : \coprod_{j=1}^{2}(\mathcal{L}_j(r_j), \rho_j(\delta_j, \lambda_j), \mathcal{R}(r)) \hookrightarrow (\tilde{M}, \tilde{\omega}, \tilde{L})$$

defined by

$$\tilde{\psi} = \begin{cases} \tilde{\gamma}_1(x) & x \in \mathcal{L}_1 \\ \tilde{\phi} \circ \tilde{\gamma}_1 \circ \tilde{c} \circ \tilde{\iota}(x) & x \in \mathcal{L}_2, \end{cases}$$

where $\tilde{\iota} : \coprod_{j=1}^{2} \mathcal{L}_j \to \coprod_{j=1}^{2} \mathcal{L}_j$ is given by $\tilde{\iota}(x \in \mathcal{L}_j) = x \in \mathcal{L}_{j+1 \mod 2}$. Note that $\tilde{c} \circ \tilde{\iota}$ is a real structure on $\coprod_{j=1}^{2} \mathcal{L}_j$ which makes $\tilde{\psi}$ a real map. The proof now follows exactly the proof of Lemma 2.19, with $\tilde{c} \circ \tilde{\iota}$ in place of $\tilde{c}$. □

Proof of Proposition 2.18. For each $\tilde{\psi}_j$ with $Im(\tilde{\psi}_j) \cap L \neq \emptyset$, we construct the blow-down as in 2.19. The rest of the maps come in pairs by assumption, and for each pair, we construct the blow-down as in 2.20. The Proposition follows. □

Theorem 1.22 now follows easily from the above propositions. We finish the proof here.

Proof of Theorem 1.22. First, by Remark 1.20, there is an $\epsilon^{'} > 0$, $\epsilon' < \epsilon$, and an $\tilde{\omega}$-tame almost complex structure $\tilde{J}$ such that $\tilde{J}$ is integrable on $\psi_i(\mathcal{L}(1+2\epsilon'))$ and which makes $\psi_i|_{(\mathcal{L}_i(1+2\epsilon')}$ holomorphic. Define $\mathcal{N} := \coprod_{i=1}^{k} \mathcal{L}_i(1+2\epsilon^{'})$. If M is not a real manifold, then we use Proposition 2.17 to blow down $\tilde{M}$ using the map $\psi|_{\mathcal{N}}$. For a real manifold

$\tilde{M}$ and a real embedding $\tilde{\psi}$, the theorem then follows from Proposition 2.18, again using the restriction $\psi|_{\mathcal{N}}$. This completes the proof. □

Remark 2.21. We should note that the forms obtained in the local models, i.e. Propositions 2.1 and 2.16 are not the same as the forms constructed, respectively, from blowing up $\mathbb{C}^n$ at 0 and blowing down $\mathcal{L}$ along the exceptional divisor using Theorems 1.21 and 1.22. Constructing the genuine blow-up and blow-down forms, even of $\mathbb{C}^n$ and $\mathcal{L}$, still requires an auxiliary symplectic embedding of either $B(r)$ or $\mathcal{L}(r)$, and these are absent from the form constructions of τ and $\tilde{\tau}$ in Propositions 2.1 and 2.16. Because of this, we still use the constructions of Theorems 1.21 and 1.22, even in these cases.

2.3 Lemmata

In this section we prove the lemmas used in Sections 2.1 and 2.2 above. We first prove Lemma 2.3, which we restate here. This lemma is proved in Guillemin and Sternberg [14] by different methods.

Lemma (Lemma 2.3)**.** *Let $h : \mathbb{R}^{2n} \to \mathbb{R}$ be the function $h(x) = \left(1 + \frac{\lambda^2}{|x|^2}\right)^{1/2}$ and ω_0 be the standard symplectic form on $\mathbb{R}^{2n}$. Let $H : \mathbb{R}^{2n}\backslash\{0\} \to \mathbb{R}^{2n}\backslash B(\lambda)$ be the mapping given by $H(x) = h(x)x$. Then $\pi^*H^*\omega = \rho(1, \lambda)$ on $\mathcal{L}\backslash\{(0, l)|l \in \mathbb{C}P^{n-1}\}$.*

Proof of Lemma 2.3. We recall from Definition 1.2 that for each $\kappa, \lambda > 0$, the closed two-form $\rho(\kappa, \lambda)$ on $\mathcal{L}(r)$ is defined by

$$\rho(\kappa, \lambda) = \kappa^2\pi^*\omega_0 + \lambda^2\theta^*\sigma.$$

Now note that at a point $x \in \mathbb{R}^{2n}$, the differential form $\sum_i dx_i \wedge dy_i(v, w) = \omega_0(v, w) =$

$v^t Aw$, where

$$A = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Therefore, $H^*\omega_0(v,w) = \omega_0(H_*v, H_*w) = v^t H_*^t A H_* w$. Calculating the i,j-th entry of H_*, we have

$$(H_*)_{ij} = \left(1 + \frac{\lambda^2}{|x|^2}\right)^{-1/2} \frac{x_i x_j}{|x|^4} + \left(1 + \frac{\lambda^2}{|x|^2}\right)^{1/2} \delta_{ij}.$$

Now let $\alpha = \left(1 + \frac{\lambda^2}{|x|^2}\right)^{1/2}$ and $B = \frac{1}{|x|^4}\left((x_i x_j)_{ij}\right)$. Then $H_* = \alpha I + \frac{1}{\alpha} B$, and

$$\omega_0(H_*v, H_*w) = \omega_0((\alpha I + \frac{1}{\alpha}B)v, (\alpha I + \frac{1}{\alpha}B)w)$$

$$= \alpha^2 \omega_0(v,w) + \omega_0(v, Bw) + \omega_0(Bv, w) + \frac{1}{\alpha^2}\omega_0(Bv, Bw).$$

We first claim that $\omega_0(B\cdot, B\cdot) = 0$. To see this, note that B is a symmetric matrix, and therefore $B^t = B$. We write

$$B = \begin{pmatrix} C & D \\ D & E \end{pmatrix},$$

where C, D, and E are $n \times n$ matrices, and C and E are symmetric. With this notation,

$$B^t A B = \begin{pmatrix} C & D \\ D & E \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} C & D \\ D & E \end{pmatrix}$$

$$= \begin{pmatrix} C & D \\ D & E \end{pmatrix} \begin{pmatrix} -D & -E \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & D^2 - CE \\ CE - D^2 & 0 \end{pmatrix}.$$

However, $D_{ij} = B_{i,j+n} = B_{j+n,i}$, and $C_{ij} = B_{ij}$, $E_{ij} = B_{i+n,j+n}$, and therefore

$$|x|^8 (D^2)_{ij} = \sum_k D_{ik} D_{kj} = \sum_k B_{i,k+n} B_{k,j+n} = \sum_k x_i x_{k+n} x_k x_{j+n}$$

$$|x|^8 (CE)_{ij} = \sum_k C_{ik} E_{kj} = \sum_k B_{ik} B_{k+n,j+n} = \sum_k x_i x_k x_{k+n} x_{j+n}$$

which implies that $D^2 - CE = 0$.

Let $\{e_i, f_i\}_{i=1}^n \in \mathbb{R}^{2n}$ be the standard basis in $\mathbb{R}^{2n}$, and let $\overline{z}dz$ and $zd\overline{z}$ denote $\sum_{i=1}^n \overline{z}_i dz_i$ and $\sum_{i-1}^n z_i d\overline{z}_i$, where $z_i = x_i + ix_{i+n}, i \in \{1, \ldots, n\}$. Then

$$\begin{aligned} |x|^4 \omega_0(e_i, Be_j) + \omega_0(Be_i, e_j) &= x_i x_{j+n} - x_j x_{i+n} = -i\overline{z}dz \wedge zd\overline{z}(e_i, e_j), \\ \omega_0(e_i, Bf_j) + \omega_0(Be_i, f_j) &= -x_i x_j - x_{i+n} x_{j+n} = -i\overline{z}dz \wedge zd\overline{z}(e_i, f_j), \\ \omega_0(f_i, Be_j) + \omega_0(Bf_i, e_j) &= x_i x_j + x_{i+n} x_{j+n} = -i\overline{z}dz \wedge zd\overline{z}(f_i, e_j), \text{ and} \\ \omega_0(f_i, Bf_j) + \omega_0(Bf_i, f_j) &= -x_i x_{j+n} + x_j x_{i+n} = -i\overline{z}dz \wedge zd\overline{z}(f_i, f_j). \end{aligned}$$

Note that here we understand $e_i \in \mathbb{R}^n \subset \mathbb{C}^n$, and $f_i = ie_i \in i\mathbb{R}^n \subset \mathbb{C}^n$. Therefore $H^*\omega_0 = \omega_0 - i\left(\frac{\lambda^2 dz \wedge d\overline{z}}{|x|^2} + \frac{\lambda^2 \overline{z}dz \wedge zd\overline{z}}{|z|^4}\right)$, so $\pi^* H^* \omega_0 = \pi^* \omega_0 + \lambda^2 \theta^* \sigma$ by Section 4 of Guillemin and Sternberg [14]. This completes the proof. □

We now use this lemma to prove the following proposition.

Proposition 2.22. *For each $\kappa, \lambda > 0$, $\rho(\kappa, \lambda)$ is a symplectic form on $\mathcal{L}$.*

Proof. Let $\Omega = \omega_0^n$ denote the volume form on $\mathbb{R}^{2n}$, and let H be defined as in the proof of Lemma 2.3. Since $H \circ \pi$ is diffeomorphism on $\mathcal{L}^* := \mathcal{L} \backslash \{(0, z) | z \in \mathbb{C}P^{n-1}\}$, $\pi^* H^* \Omega$

is a volume form on $\mathcal{L}$, and therefore $\rho(1, \lambda)$ is non-degenerate for any $\lambda > 0$. Since $\rho(\kappa, \lambda) = \delta^2 \rho(1, \lambda/\kappa)$, this implies that $\rho(\kappa, \lambda)$ is non-degenerate for $\kappa, \lambda > 0$ as well. Since both ω_0 and σ are closed, $\rho(\kappa, \lambda)$ is closed as well on $\mathcal{L}^*$.

Now let $(0, l) \in \mathcal{L}(0)$. Then $T_{(0,l)}\mathcal{L} \equiv T_l\mathbb{C}P^1 \oplus T_0\mathbb{C}$. Taking $v \in T_l\mathbb{C}P^1$. Then $\rho(\kappa, \lambda)(v, iv) = \lambda^2\theta^*\sigma(v, iv) = \sigma(v, iv) > 0$. Similarly, for $v \in T_0\mathbb{C}$, $\rho(\kappa, \lambda)(v, iv) = \pi^*\omega_0(v, iv) > 0$, and therefore $\rho(\kappa, \lambda)$ is non-degenerate on $\mathcal{L}(0)$. Since $\rho(\kappa, \lambda)$ is closed as well, the form is symplectic as desired. □

We now give the proof of Lemma 2.4, which we restate below.

Lemma (Lemma 2.4)**.** *Let (M, ω) be a symplectic manifold. Then ω is a Kähler form iff ω is compatible with an integrable almost complex structure J.*

Proof of Lemma 2.4. First, assume that ω is compatible with J. We must show that $i\omega \in \Omega^2(M) \otimes \mathbb{C}$ is the imaginary part of a Hermitian metric. For each $x \in M$, define $g_x(v, w) := \omega_x(Jv, w)$. Since ω is compatible with J, g is a Riemannian metric on M. Now let $H_x(v, w) := g_x(v, w) + i\omega_x(v, w)$. Then

$$\begin{aligned} H_x(Jv, w) &= g_x(Jv, w) + i\omega(Jv, w) \\ &= \omega_x(J^2v, w) + ig(v, w) \\ &= -\omega_x(v, w) + ig(w, v) \\ &= i(g_x(v, w) + i\omega(v, w)) \\ &= iH_x(v, w) \end{aligned}$$

We also have

$$
\begin{aligned}
H_x(v, Jw) &= g(v, Jw) + i\omega(v, Jw) \\
&= g(Jw, v) - i\omega(Jw, v) \\
&= \omega(J^2 w, v) - ig(w, v) \\
&= -\omega(w, v) - ig(v, w) \\
&= \omega(v, w) - ig(v, w) \\
&= -iH_x(v, w) \\
&= \bar{i} H_x(v, w)
\end{aligned}
$$

It follows from the linearity of ω_x and g_x that H is complex linear in the first variable and complex anti-linear in the second. It only remains to show that H is positive definite. If $v \in \mathbb{R}^n$, then

$$H(v, \overline{v}) = H(v, v) = g(v, v) + i \cdot 0 > 0$$

We now assume $v = Jw$ for some $w \in \mathbb{R}^n$. Then

$$H(v, v) = H(Jw, Jw) = i\bar{i}H(w, w) = H(w, w) > 0,$$

and H is therefore a Hermitian metric with $i\omega$ as its imaginary part. Since ω is closed, H is a Kähler metric, and ω is its Kähler form.

Now suppose that ω is a Kähler form of the Kähler metric $H(v, w)$. We wish to show that ω is compatible with J. First, note that $\omega = \text{Im}(H) = \frac{-i}{2}(H - \overline{H})$, and

$$\begin{aligned}\omega(Jv, Jw) &= \frac{-i}{2}(H(Jv, Jw) - \overline{H}(Jv, Jw)) \\ &= \frac{-i}{2}(H(v, w) - \overline{H}(v, w)) \\ &= \omega(v, w)\end{aligned}$$

Next, for any $v \neq 0$ we have

$$\begin{aligned}\omega(Jv, v) &= \frac{-i}{2}(H(Jv, v) - \overline{H}(Jv, v)) \\ &= \frac{-i}{2}(iH(v, v) - \overline{i}\overline{H}(v, v)) \\ &= \frac{1}{2}(H(v, v) + \overline{H}(v, v)) \\ &= \text{Re}(H(v, v)) > 0\end{aligned}$$

as desired. Therefore ω is compatible with J. □

We now prove Lemma 2.5, which we restate here. First recall from Definition 2.2 that $f : \mathbb{C}^n \to \mathbb{C}^n$ is a *monotone radial function* if $f(z) = \alpha(|z|)z$ for some real-valued function $\alpha : \mathbb{R} \to [0, \infty)$, and if $|z_0| \leq |z_1| \implies |f(z_0)| \leq |f(z_1)|$.

Lemma (Lemma 2.5)**.** *Let ω be a Kähler form on $\mathbb{C}^n$, and suppose $f : \mathbb{C}^n \backslash \{0\} \to \mathbb{C}^n \backslash \{0\}$ is a monotone radial function. Then $f^*\omega$ is a Kähler form.*

Proof of Lemma 2.5. By Lemma 2.4, we must show that $f^*\omega$ is compatible with i, the standard almost complex structure for $\mathbb{C}^n$. Let $z \in \mathbb{C}^n$ $v, w \in T_z\mathbb{C}^n \cong \mathbb{C}^n$. We denote by z^v the vector in $T_z\mathbb{C}^n$ with coordinates identical to $z \in \mathbb{C}^n$. Because ω tames i, the subspace $T_z(\mathbb{C}z^v) \cong \text{Span}\{z^v, iz^v\}$ is symplectic, and therefore $T_z\mathbb{C}^n \cong T_z(\mathbb{C}z^v) \oplus$

$(T_z(\mathbb{C}z^v))^\omega$, where $(T_z(\mathbb{C}z^v))^\omega$ denotes the symplectic complement of $T_z(\mathbb{C}z^v)$. We now write v and w as $v = x_0 + \alpha_0 z^v + \beta_0 i z^v, w = x_1 + \alpha_1 z^v + \beta_1 i z^v$, where $\alpha_i, \beta_i \in \mathbb{R}$, and $x_i \in (T_z(\mathbb{C}z^v))^\omega$. Since f is radial, we have that it is of the form $f = f_0(|z|)z^v$. Therefore, $df(z)(x_n + \alpha_n z^v + i\beta_n z^v) = x_n + g(|z|)\alpha z^v + \beta i z^v$. Furthermore, since the norm of f is non-decreasing, g is a non-negative real valued function of one real variable. Therefore

$$\begin{aligned} df(z)(i(x_n + \alpha_n z^v + i\beta_n z^v)) &= df(z)(ix_n + i\alpha_n z^v - \beta_n z^v) \\ &= ix_n + i\alpha_n z^v - g(|z|)\beta_n z^v. \end{aligned}$$

Computing $f^*\omega(iv, v)$, we see that

$$\begin{aligned} f^*\omega(iv, v) &= \omega(f_* iv, f_* v) \\ &= \omega(ix_0 + i\alpha_0 z^v - g(|z|)\beta_0 z^v, x + \alpha_0 z^v + ig(|z|)\beta_0 z^v) \\ &= \omega(ix_0, x_0) + \omega_(i(\alpha_0 + ig(|z|)\beta_0)z^v, (\alpha_0 + ig(|z|)\beta_0)z^v) \\ &+ \omega(i(\alpha_0 + ig(|z|)\beta_0)z^v, x_0) + \omega(ix_0, (\alpha_0 + g(|z|)\beta_0)z^v) \\ &> 0. \end{aligned}$$

The last two terms on the right are equal to zero because $x_0 \in \text{Span}\{z^v, iz^v\}^\omega$, and the first two terms are greater then zero because $\omega(i\cdot, \cdot)$ is a Riemannian metric.

Similarly, computing $f^*\omega(iv, iw)$, we have

$$\begin{aligned}
f^*\omega(iv, iw) &= \omega(f_*iv, f_*iw) \\
&= \omega(f_*i(x_0 + \alpha_0 z + \beta_0 iz), f_*i(x_1 + \alpha_1 z + \beta_1 iz)) \\
&= \omega(ix_0 + \alpha_0 iz - \beta_0 g(|z|)z, ix_1 + \alpha_1 iz - \beta_1 g(|z|)z) \\
&= \omega(ix_0, ix_1) + \omega(i(\alpha_0 + i\beta_0 g(|z|))z, i(\alpha_1 + i\beta_1 g(|z|))z) \\
&\quad +\omega(ix, i(\alpha_1 + i\beta_1 g(|z|))z) + \omega(i(\alpha_0 + i\beta_0 g(|z|))z, ix) \\
&= \omega(ix_0, ix_1) + \omega(i(\alpha_0 + i\beta_0 g(|z|))z, i(\alpha_1 + i\beta_1 g(|z|))z) \\
&= \omega(x_0, x_1) + \omega((\alpha_0 + i\beta_0 g(|z|))z, (\alpha_1 + i\beta_1 g(|z|))z) \\
&= \omega(x_{0,}x_1) + \omega(\alpha_0 z, \beta_1 g(|z|)iz) + \omega(\beta_0 g(|z|)_1 iz, \alpha_1 z) \\
&= \omega(x_0, x_1) + \alpha_0\beta_1 g(|z|)\omega(z, iz) + \alpha_1\beta_0\omega(iz, z) \\
&= \omega(x_0, x_1) + g(|z|)(\alpha_0\beta_1 - \alpha_1\beta_0)\omega(z, iz).
\end{aligned}$$

The fifth and sixth equalities in the above calculation follow from the fact that $x, ix \in (\text{Span}\{z, iz\})^\omega$ and because ω tames i by hypothesis. On the other hand, we have

$$
\begin{aligned}
f^*\omega(v,w) &= \omega(f_*v, f_*w) \\
&= \omega(f_*(x_0+\alpha_0 z+\beta_0 iz), f_*(x_1+\alpha_1 z+\beta_1 iz)) \\
&= \omega(x_0+\alpha_0 g(|z|)z+\beta_0 iz, x_1+\alpha_1 g(|z|)z+i\beta_1 z) \\
&= \omega(x_0,x_1)+\omega((\alpha_0 g(|z|)+i\beta_0)z,(\alpha_1 g(|z|)+i\beta_1)z) \\
&+ \omega(x,(\alpha_1 g(|z|)+i\beta_1)z)+\omega((\alpha_0 g(|z|)+i\beta_0)z,x) \\
&= \omega(x_0,x_1)+\omega((\alpha_0 g(|z|)+i\beta_0)z,(\alpha_1 g(|z|)+i\beta_1)z) \\
&= \omega(x_0,x_1)+\omega(\alpha_0 g(|z|)z,\beta_1 iz)+\omega(\beta_0 iz,\alpha_1 g(|z|)z) \\
&= \omega(x_0,x_1)+\alpha_0\beta_1 g(|z|)\omega(z,iz)+\alpha_1\beta_0 g(|z|)\omega(iz,z) \\
&= \omega(x_0,x_1)+g(|z|)(\alpha_0\beta_1-\alpha_1\beta_0)\omega(z,iz) \\
&= f^*\omega(iv,iw)
\end{aligned}
$$

as desired. □

Lemma 2.23. *Let (M,ω) be a symplectic manifold, and let J be an almost complex structure tamed by ω. Suppose there exists an anti-holomorphic involution ϕ (a map $\phi : M \to M$ such that $\phi^2 = Id$ and $\phi_* J \phi_* = -J$). Then the 2-form $\overline{\omega} = \frac{1}{2}(\omega - \phi^*\omega)$ has the properties*

1. $\overline{\omega}$ is symplectic
2. $\phi^*\overline{\omega} = -\overline{\omega}$
3. $\overline{\omega}$ tames J

Proof. Since ω tames J, we have that

$$\overline{\omega} = \frac{1}{2}(\omega(v, Jv) - \omega(\phi_* v, \phi_* Jv)) = \frac{1}{2}(\omega(v, Jv) + \omega(\phi_* v, J\phi_* v) > 0,$$

and therefore $\overline{\omega}$ tames J. It follows that $\overline{\omega}$ is non-degenerate. Furthermore, $d\overline{\omega} = \frac{1}{2}d(\omega - \phi^*\omega)) = 0$, so $\overline{\omega}$ is closed, and therefore symplectic. □

2.4 Invariant Symplectic Neighborhoods and the Moser Stability Theorem in Real Symplectic Manifolds

In this section we present a version of the Symplectic Neighborhood Theorem adapted to leave invariant the fixed-point set of a real symplectic manifold (M, ω, ϕ). We will use this below to establish real packing results in $(\mathbb{C}P^2, \mathbb{R}P^2)$. We closely follow the presentation of the analogous theorems for symplectic manifolds with no real structure in McDuff and Salamon [25].

We begin with a definition.

Definition 2.24. Let M be a smooth manifold and let G be a compact Lie group which acts smoothly on M. We say that a vector field X on M is *equivariant* with respect to G (or G-*equivariant*) if $\forall x \in M, g \in G$, we have $X(gx) = g_* X(x)$.

We now quote the following standard result in equivariant dynamics, which we quote from Ortega and Ratiu [27] (Proposition 3.3.2(i))

Proposition 2.25. *Let M be a smooth manifold, A a subgroup of the group of diffeomorphisms of M. Let U be an A-invariant open subset of M, and X an A-equivariant vector field defined on U. Then, the domain of definition $Dom(F_t) \subset U$ of the flow F_t of X is A-invariant and F_t is itself A-equivariant.*

Lemma 2.26. *Let (M, ω, ϕ) be a real symplectic manifold with $Fix(\phi) = L$, and suppose $\omega_t, t \in [0, 1]$ is a smooth family of symplectic forms with $\omega_0 = \omega$ and $\phi^*\omega_t = -\omega_t$. Suppose, furthermore, that there exists a family of one-forms σ_t with $\frac{d}{dt}\omega_t = d\sigma_t$ and $\phi^*\sigma_t = -\sigma_t$. Then there exists a family of diffeomorphisms $\alpha_t : M \to M$ such that*

$$\alpha_t^*\omega_t = \omega_0, \tag{2.4.1}$$

$$\alpha_t(L) \subseteq L, \tag{2.4.2}$$

$$\alpha_t \circ \phi = \phi \circ \alpha_t. \tag{2.4.3}$$

Proof. We first note that, since the ω_t are non-degenerate, there exists a unique vector field X_t which satisfies

$$\sigma_t + \iota(X_t)\omega_t = 0. \tag{2.4.4}$$

. Given such a vector field X_t, let α_t be the solutions of

$$\frac{d}{dt}\alpha_t = X_t \circ \alpha_t, \tag{2.4.5}$$

$$\alpha_0 = Id. \tag{2.4.6}$$

We now note that, because ω_t is closed, $d\omega_t = 0$, and $\frac{d}{dt}\omega_t = d\sigma_t$, Equation 2.4.4 implies that

$$0 = \alpha_t^* \left(\frac{d}{dt}\omega_t + \iota(X_t)d\omega_t + d\iota(X_t)\omega_t \right) = \frac{d}{dt}\alpha_t^*\omega_t.$$

If X_t is ϕ-equivariant, then by 2.25 the flow α_t will be ϕ-equivariant as well. To see that X_t is ϕ-equivariant, we first remark that

$$\begin{aligned} \phi^*(\sigma_t + \iota(X_t)\omega_t) &= 0, \\ &= \phi^*\sigma_t + \phi^*\iota(X_t)\omega_t \\ &= -\sigma_t + \phi^*\iota(X_t)\omega_t, \end{aligned}$$

which implies that $\phi^*\iota(X_t)\omega_t = \sigma_t = -\iota(X_t)\omega_t$. Therefore, for all $v \in T_qM$,

$$\omega_t(\phi(q); X_t(\phi(q)), \phi_* v) = -\omega_t(q; X_t(q), v).$$

However, $-\omega_t(q; X_t(q), v) = \omega_t(\phi(q); \phi_* X_t(q), \phi_* v)$, so

$$\omega_t(\phi(q); X_t(\phi(q)), \phi_* v) = \omega_t(\phi(q); \phi_* X_t(q), \phi_* v).$$

Since this is true for all $v \in T_qM$, ϕ_* is an isomorphism, and ω_t is non-degenerate, this implies that $\phi_* X_t(q) = X_t(\phi(q))$, and therefore the vector field X_t is ϕ-equivariant.

Furthermore,for $v \in T_qL$, $v \neq 0$, we have that $\sigma_t(q; v) = -\sigma_t(q; \phi_* v) = 0$, so $\omega(q; X_t, v) = 0$, which implies that $X_t \in T_qL \subset T_qM$. Since this is true for all $t \in [0, 1]$, the diffeomorphisms α_t determined by equation 2.4.5 satisfy the constraints in equation 2.4.2 as required. □

Lemma 2.27. *Let M be a $2n$-dimensional smooth manifold, and let $\phi : M \to M$ be a diffeomorphism with $\phi^2 = Id$. Let $L = Fix(\phi)$, and suppose $Q \subset M$ is a ϕ-invariant submanifold. Suppose that $\omega_0, \omega_1 \in \Omega^2(M)$ are closed two forms with $\phi^*\omega_i = -\omega_i$ and such that, at every point $q \in Q$, $\omega_0|_{T_qM} = \omega_1|_{T_qM}$ and the ω_i are non-degenerate on T_qM. Then there exist neighborhoods $\mathcal{N}_0, \mathcal{N}_1$ of Q and a diffeomorphism $\alpha : \mathcal{N}_0 \to \mathcal{N}_1$*

which satisfies

1. $\alpha|_Q = Id$,

2. $\alpha^*\omega_1 = \omega_0$,

3. $\alpha(\mathcal{N}_0 \cap L) \subset L$,

4. $\alpha \circ \phi = \phi \circ \alpha$.

Proof. We may assume that $Q \cap L \neq \emptyset$, since, if this was not the case, we could just take the $\mathcal{N}_i$ small enough so that $\mathcal{N}_i \cap L = \emptyset$ and invoke the ordinary symplectic neighborhood theorem.

We first show that there exists a 1-form $\sigma \in \Omega^1(\mathcal{N}_0)$ such that

$$\sigma|_{T_Q M} = 0 = \sigma|_{TL}, \tag{2.4.7}$$

$$\phi^*\sigma = -\sigma, \tag{2.4.8}$$

$$d\sigma = \omega_1 - \omega_0. \tag{2.4.9}$$

To prove this, we endow M with a ϕ-invariant Riemannian metric, and consider the restriction of the exponential map to the normal bundle $TQ^\perp$. Since Q is ϕ-invariant, TQ is ϕ_* invariant inside TM, and, therefore, since ϕ_* is an isomorphism from T_xM to $T_{\phi(x)}M$, $TQ^\perp$ is ϕ_*-invariant as well. Now, for a real number $\epsilon > 0$, consider the neighborhood of the zero section of $TQ^\perp$

$$V_\epsilon = \{(q, v) \in TM | q \in Q, v \in T_qQ^\perp, |v| < \epsilon\}.$$

Define the set $U_\epsilon := (V_\epsilon \cup \phi(V_\epsilon))$. Then U_ϵ is ϕ-invariant, and for ϵ sufficiently small, the restriction of the exponential map to U_ϵ is a diffeomorphism from U_ϵ to a neighborhood $\mathcal{N}_1$ of Q. By a standard result in equivariant differential topology (Lemma 3.12, to be proven in Section 3.1), exp is equivariant as well. Now define $\psi_t : U_\epsilon \to \mathcal{N}_1$, $0 < t < 1$, by $\psi_t(\exp(q,v)) = \exp(q,tv)$. For $t > 0$, ψ_t is a diffeomorphism onto its image. At $t = 0$, Im$(\psi) \subseteq Q$, at $t = 1$, $\psi_1 = Id$, and $\psi_t|_Q = Id$ for all $t \in [0,1]$. Since $\exp$ is equivariant, we also have $\psi_t \circ \phi(\exp(q,v)) = \psi_t(\exp(c(q), \phi_* v)) = \exp(\phi(q), t\phi_* v) = \phi \circ \exp(q,tv) = \phi \circ \psi_t$, so ϕ and ψ_t commute.

Let $\tau = \omega_1 - \omega_0$. Then $\psi_0^* \tau = 0$ and $\psi_1^* \tau = \tau$, and since ψ_t is an equivariant diffeomorphism, we may define a ϕ-equivariant vector field for $t > 0$ by $X_t = (\frac{\partial}{\partial t}\psi_t) \circ \psi_t^{-1}$. Note that X_t becomes singular at $t = 0$. Nonetheless, we have

$$\frac{d}{dt}\psi_t^* \tau = \psi_t^* \mathcal{L}_{X_t} \tau = d(\psi_t^* \iota(X_t)\tau).$$

Let $\sigma_t = \psi_t^* \iota(X_t)\tau$. Therefore, $\frac{d}{dt}\psi_t^*\tau = d\sigma_t$, and, by the definition of X_t, σ_t is equal to

$$\sigma_t(q;v) = \tau(\psi_t(q); \frac{d}{dt}\psi_t(q), d\psi_t(q)v).$$

Since σ_t vanishes on Q for all t, we may define $\sigma_0 = 0$, making σ_t a smooth family for $t \in [0,1]$. In addition, we have that

$$\tau = \psi_1^*\tau - \psi_0^*\tau = \int_0^1 \frac{d}{dt}\psi_t^*\tau \, dt = d\sigma,$$

where $\sigma = \int_0^1 \sigma_t dt$. It also follows from the equivariance of ψ_t that $(q,v) \in TL$, $\sigma_t = 0$ for all $t \in [0,1]$. To see this, note that for $(q,v) \in TL$, $d\psi_t(q)v \in T_qL$, and since $\psi_t(q) \in L$ for all t, then $\frac{d}{dt}\psi_t(q) \in T_{\psi_t(q)}L$ as well, making $\sigma_t(q;v)$ vanish by definition of τ, because L is Lagrangian for ω_0 and ω_1. To see that $\phi^*\sigma_t = -\sigma_t$, we compute

$$\begin{aligned}
\phi^*\sigma_t(v) &= \phi^*\tau(\psi_t(q); \frac{d}{dt}\psi_t(q), d\psi_t(q)\cdot)(v) \\
&= \omega_1(\psi_t \circ \phi(q); \frac{d}{dt}\psi_t(\phi(q)), d\psi_t \circ d\phi(q)v) \\
&\quad -\omega_0(\psi_t \circ \phi(q); \frac{d}{dt}\psi_t(\phi(q)), d\psi_t \circ d\phi(q)v) \\
&= \omega_1(\phi \circ \psi_t(q); \frac{d}{dt}\phi \circ \psi_t(q), d\phi \circ d\psi_t(q)v) \\
&\quad -\omega_0(\phi \circ \psi_t(q); \frac{d}{dt}\phi \circ \psi_t(q), d\phi \circ d\psi_t(q)v) \\
&= (\omega_1(\phi \circ \psi_t(q); d\phi\frac{d}{dt}\psi_t(q), d\phi \circ d\psi_t(q)v) \\
&\quad -\omega_0(\phi \circ \psi_t(q); d\phi\frac{d}{dt}\psi_t(q), d\phi \circ d\psi_t(q) \cdot v)) \\
&= -\tau(\psi_t(q); \frac{d}{dt}\psi_t(q), d\psi_t(q)v) \\
&= -\sigma_t(v).
\end{aligned}$$

Therefore, $\phi^*\omega = \int_0^1 \phi^*\sigma_t \, dt = -\omega$. We have now created the desired 1-form.

Now consider the family of two-forms on $\mathcal{N}_0$ given by $\omega_t = \omega_0 + t(\omega_1 - \omega_0) = \omega_0 + td\sigma$, $t \in [0,1]$, and note that $\phi^*\omega_t = -\omega_t$ and $\frac{d}{dt}\omega_t = d\sigma$. The result now follows from Lemma 2.26. □

Theorem 2.28. *For $j = 0, 1$ let (M_j, ω_j, c_j) be real symplectic manifolds with compact c_j-invariant symplectic submanifolds Q_j. Suppose that there is an equivariant symplectic isomorphism $\Phi : \nu_{Q_0} \to \nu_{Q_2}$ of the symplectic normal bundles to Q_0 and Q_1 such that the restriction of Φ to the zero section is the symplectomorphism $\psi : (Q_0, \omega_0) \to (Q_1, \omega_1)$. Then there exist c_j-invariant neighborhoods $\mathcal{N}_j$ of the Q_j such that ψ extends to an equivariant symplectomorphism $\psi' : (\mathcal{N}_0, \omega_0, c_0) \to (\mathcal{N}_1, \omega_1, c_1)$, and $d\psi'$ induces Φ on ν_{Q_0}.*

Proof. We first show that ψ extends to an equivariant diffeomorphism $\psi_1 : \mathcal{N}(Q_0) \to \mathcal{N}(Q_1)$ that induces the map Φ on ν_{Q_0}. By Lemma 3.12, we may take the maps $\exp_i$ on TM_i to be equivariant with respect to c_i. Define the map $\psi_1 = \exp_1 \circ \Phi \circ \exp_0^{-1}$, and consider the forms ω_0 and $\omega_1' = (\psi_1)^* \omega_1$ on $\mathcal{N}(Q_0)$. Note that, by construction, they are non-degenerate and they correspond on $T_{Q_0}M_0$. By Lemma 2.27, there is an equivariant diffeomorphism $\overline{\psi}$ of $\mathcal{N}(Q_0)$ such that $\overline{\psi}^* \omega_1' = \omega_0$. The composition $\psi' = \psi_1 \circ \overline{\psi}$ is the desired map. □

Proposition 2.29. *Let (M, ω, ϕ) be a real symplectic manifold with real locus $L := Fix(\phi)$. Let $x \in L$. Then there exists a symplectic equivariant map from a neighborhood $\mathcal{U}$ of 0 in $(\mathbb{R}^{2n}, \omega_0, c)$ to a neighborhood $\mathcal{V}$ of $x \in M$.*

In order to prove this proposition, we will need the following lemma.

Lemma 2.30. *Let $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear map such that $Fix(\Phi) = \mathbb{R}^n$ (seen as the real part of $\mathbb{C}^n$), $\Phi^2 = Id$ and $\Phi^*\omega_0(v, w) = -\omega_0(v, w)$ for all $v, w \in \mathbb{R}^{2n}$. Then there exists a linear symplectic isomorphism $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\Psi\Phi = c_*\Psi$, where c is the standard anti-symplectic involution on $\mathbb{R}^{2n}$.*

Proof. We first consider the case $n = 1$. (We do this to demonstrate the construction. The proof does not proceed by induction.) Let $v \in Fix(\Psi) = Fix(c_*)$ such that $\omega_0(v, iv) = 1$, where i is the standard complex structure on $\mathbb{R}^2$. Then $\mathbb{R}^2 = Span\{v, iv\}$. Let w be an eigenvector of Ψ with eigenvalue -1. Let $\beta := \omega_0(v, w)$. Now note that $\{v, iv\}$ and $\{v, w\}$ are bases for $\mathbb{R}^2$. We define the map $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ to be the matrix sending $v \mapsto v$ and $w \mapsto (0, \omega_0(v, w))$, where the coordinates are the standard

$(x, y) = (v, iv)$ coordinates on $\mathbb{R}^2$. Then, for two vectors $av + bw, cv + dw$, we have

$$\begin{aligned} \omega_0(av + bw, cv + dw) &= \omega_0(av, dw) + \omega_0(bw, cv) \\ &= (ad - bc)\beta. \end{aligned}$$

On the other hand,

$$\begin{aligned} \omega_0(\Psi(av + bw), \Psi(cv + dw)) &= \omega_0(av + \beta \cdot biv, cv + \beta \cdot div) \\ &= (ad - bc)\beta. \end{aligned}$$

Since the constants $a, b, c, d \in \mathbb{R}$ were arbitrary, we see that Ψ is a linear symplectomorphism.

Now consider $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, a linear anti-symplectic involution with $Fix(\Phi) = \mathbb{R}^n$. Let $e_i, i \in \{1, \ldots, 2n\}$ denote the standard basis in $\mathbb{R}^{2n}$, and consider the standard coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $\mathbb{R}^{2n}$. Take a basis $(v_1, \ldots, v_n)$ of the -1 eigenspace of Φ, and define the map $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ to be the unique linear map sending $e_i \mapsto e_i$, and $v_i \mapsto (0, \ldots, 0, \omega_0(e_1, v_i), \ldots, \omega_0(e_n, v_i))$, where there are n leading zeros in the coordinate (i.e. the -1 eigenspace of Φ is sent to the -1 eigensapce of c_*).

We now show that Ψ is a symplectomorphism. First note that for $i \in \{1, \ldots, n\}$ we have $\omega_0(e_i, e_j) = 0 = \Phi^*\omega_0(e_i, e_j)$. Furthermore, we see that

$$-\omega_0(v_i, v_j) = \Phi^*\omega_0(v_i, v_j) = \omega_0(\Phi v_i, \Phi v_j) = \omega_0(v_i, v_j),$$

which implies that $\omega_0(v_i, v_j) = 0 = \Phi^*\omega(v_i, v_j)$. Now note that

$$\Phi^*\omega_0(e_i, v_j) = \omega_0(e_i, v_j)\omega_0(e_i, e_i) = \omega_0(e_i, v_j),$$

as desired. Since $\Psi\Phi = \Phi c_*$, the proof of the lemma is complete. □

Proof of 2.29. We first consider a ϕ-invariant chart (U, α), $\alpha : U \subset M \to \mathbb{R}^{2n}$ centered at the point $p \in L$ which sends $L \to \mathbb{R}^n \subset \mathbb{C}^n$. We now consider the real structure $\Phi := \alpha \circ \phi \circ \alpha^{-1}$ on $Im(\alpha)$. By Lemma 2.30, there is a linear symplectic isomorphism $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\Phi_*\Psi = c_*\Psi$ at the point 0. Now apply Theorem 2.28 to the point $0 \in \mathbb{R}^{2n}$. □

We now prove a real version of the Moser stability theorem.

Proposition 2.31. *Let M be a closed manifold, and suppose that ω_t is a family of cohomologous symplectic forms on M with anti-symplectic involution ϕ, i.e. such that $\phi^*\omega_t = -\omega_t$. Then there is a family of diffeomorphisms ψ_t such that $\phi \circ \psi_t = \psi_t \circ \phi$, $\psi_0 = id$, and $\psi_t^*\omega = \omega_t$.*

Proof. We must show that there is a smooth family of one forms σ_t such that

$$d\sigma_t = \frac{d}{dt}\omega_t \tag{2.4.10}$$

and $\phi^*\sigma_t = -\sigma_t$.

The proof of Moser stability theorem (Theorem 3.17 in [25]) shows that there exists a smooth family of one forms τ_t satisfying (2.4.10). Let $\sigma_t = \frac{1}{2}(\tau_t - \phi^*\tau_t)$. Then $d\sigma_t = \frac{1}{2}(\frac{d}{dt}\omega_t - \phi^*\frac{d}{dt}\omega_t) = \frac{1}{2}(\frac{d}{dt}\omega_t - \frac{d}{dt}\phi^*\omega_t) = \frac{d}{dt}\omega_t$. Applying Lemma 2.26, we arrive at the desired result. □

2.5 Locally holomorphic maps

In this section we show that, given a relative or real symplectic embedding $\psi : (B(1), \lambda^2\omega_0, B_{\mathbb{R}}(1)) \rightarrow (M, \omega, L)$ and an almost complex structure on M which satisfies some additional conditions, we may find a form $\omega^{'}$ on M isotopic to ω, and a relative symplectic embedding $\psi^{'} : (B(1), \lambda^2\omega_0, B_{\mathbb{R}}(1)) \rightarrow (M, \omega^{'}, L)$ with the same image as ψ but which is holomorphic near the origin. We state the main proposition of this section here.

Proposition (Proposition 2.15). *1. Let (M, ω) be a symplectic manifold and let $L \subset M$ be a Lagrangian submanifold. Let*

$$\psi : (B(1+2\epsilon), \lambda^2\omega_0, B_{\mathbb{R}}(1+2\epsilon)) \rightarrow (M, \omega, L)$$

be a relative symplectic embedding, and let J be an almost complex structure on M which tames ω and is relatively integrable at $\psi(0) \in L$.

Then, for every compact subset $K \subset M \backslash \psi(0)$ there exists a number $\delta^{'} \in (0,1)$, a symplectic form $\omega^{'}$ on M isotopic to ω, and a relative symplectic embedding

$$\psi^{'} : (B(1+2\epsilon), \lambda^2\omega_0, B_{\mathbb{R}}(1+2\epsilon)) \rightarrow (M, \omega^{'}, L)$$

with the following properties:

(a) $\psi^{'}|_{B(\delta')}$ is holomorphic

(b) ω' tames J and coincides with ω on K

(c) L is a Lagrangian for ω'

2. *In addition to the above, suppose that M is a real symplectic manifold with real structure ϕ, $Fix(\phi) = L$, J satisfies $\phi_* J \phi_* = -J$, J is symmetrically integrable around $\psi(0)$, and $\phi \circ \psi = \psi \circ c$.*

   Then we can construct the map $\psi^{'}$ and the symplectic form $\omega^{'}$ on M to satisfy the conclusions above, and so that ϕ is a real structure for $\omega^{'}$ and $\psi^{'}$ satisfies $\phi \circ \psi^{'} = \psi^{'} \circ c$.

In the proof we will use the following lemma, which is a modification of Proposition 5.5.B in McDuff and Polterovich [23].

Lemma 2.32. *Let ω be a symplectic form on $B(1)$ which tames the standard complex strcutre i and satisfies $c^*\omega = -\omega$ for the standard real structure c. Then there exists a family of symplectic forms on $B(1)$, say $\Omega_t, t \in [0,1]$ with the following properties:*

1. $\Omega_0 = \omega$
2. *Ω_t coincides with ω near the boundary of the ball;*
3. *Ω_t tames i;*
4. *Ω_1 is i-standard near 0, i.e. it is Kähler, and the associated metric is flat.*
5. *$c^*\Omega_t = -\Omega_t$, and, in particular, $B_{\mathbb{R}}(1)$ is a Lagrangian for Ω_t.*

Proof. We divide the proof into three steps.

Step 1. We claim that for every $\kappa > 1$ and every $1 > \epsilon > 0$, there exists a Kähler form, say τ_κ on $B(1)$ which is equal to $\kappa^2\omega_0$ in $B(\epsilon/2\kappa)$ and coincides with $\epsilon^2\omega_0$ near the boundary, where ω_0 is the standard symplectic form on $B(1)$. Indeed, take the monotone map h defined by $h(z) = (\kappa/\epsilon)z$ for $z \in B(\epsilon/2\kappa)$ and h is equal to the identity map near the boundary. Then the form $\tau_\kappa = h^*(\epsilon^2\omega)$ is Kähler by Lemma 2.5.

Step 2. Let ρ be a bump function on $\mathbb{R}^{2n}$ which is radial, equal to 1 near the origin, and vanishes for $|z| > 1 - \delta$, for some $\delta > 0$. Let ω_0 be the standard symplectic form on $\mathbb{R}^{2n}$. Choose $\epsilon > 0$ so that $\omega - \epsilon^2\omega_0$ tames i, and set $\rho_\kappa(z) = \rho(2(\kappa/\epsilon)z)$. Finally, denote by β a primitive of ω so that $\omega = d\beta$. Now consider the family of forms

$$\omega_t^{'}(\kappa) = \omega + t(\tau_\kappa - \epsilon^2\omega_0 - d(\rho_\kappa\beta)).$$

We claim that $\omega_t^{'}(\kappa)$ satisfies the first four properties provided κ is sufficiently large.

We note that $\omega_t^{'}(\kappa)$ coincides with ω near the boundary for all t, and near the origin $\omega_1^{'}(\kappa)$ is equal to $(\kappa^2 - \epsilon^2)\omega_0$, and is therefore J-standard there. Moreover, $\rho_k = 0$ outside $B(2\epsilon/\kappa)$, and therefore $\omega_t^{'}(\kappa) = \omega - t(\epsilon^2\omega_0 + \tau_\kappa)$ there. By assumption on ϵ, $\omega - t\epsilon^2\omega_0$ tames i on this region, and since τ_k is Kähler and $t \leq 1$, $\omega_t{}'(\kappa)$ tames i as well.

We now check that $\omega_t^{'}(\kappa)$ tames i inside $B(\epsilon/2\kappa)$. On this region

$$\omega_t^{'}(\kappa) = t(\kappa^2 - \epsilon^2)\omega_0 + (1 - t\rho_\kappa)\omega - 2t(\kappa/\epsilon)d\rho \wedge \beta.$$

Since $\overline{B}(\epsilon/2\kappa)$ is compact, the sphere bundle

$$S = \{(x,\xi)|\, x \in \overline{B}, |\xi| = 1\} \subset T\mathbb{R}^{2n}$$

is compact, and therefore the function $d\rho_\kappa \wedge \beta(\xi, i\xi)$ has a maximum, say α, on S. For any $\xi \in T_xB(1)$,

$$d\rho_\kappa \wedge \beta(\xi, i\xi) = |\xi|^2 d\rho_\kappa \wedge \beta\left(\frac{\xi}{|\xi|}, i\frac{\xi}{|\xi|}\right)$$

and therefore the maximum of $d\rho_k \wedge \beta(\xi, i\xi)$ on $S_a = \{(x,\xi) | \, x \in \overline{B}, |\xi| = a\} \subset T\mathbb{R}^{2n}$ is $|\xi|^2\alpha$. We conclude that $\omega_t'(\kappa)(\xi, i\xi) > ((\kappa^2 - \epsilon^2 - \alpha(\kappa/\epsilon))|\xi|^2 t$. Since the quantity on the right is positive for sufficiently large κ, $\omega_t'(\kappa)$ tames i if we choose κ large enough. It follows, additionally, that $\omega_t'(\kappa)$ is symplectic for every t.

Step 3. We see from the above that the family of symplectic forms $\omega_t'(\kappa)$ satisfies the first four properties, but does not necessarily respect the real structure. By Lemma 2.23, however, since $\omega_t'(\kappa)$ tames i and $c_* i c_* = -i$, the forms $\Omega_t = \frac{1}{2}(\omega_t'(\kappa) - c^*\omega_t'(\kappa))$ are symplectic, and satisfy the last property. We check that it satisfies the first three properties as well. Ω tames i by Lemma 2.23, and, since $\omega_t'(\kappa) = t(\kappa^2 - \epsilon^2)\omega_0$ near the origin, $\Omega_t(\kappa) = \omega_t'(\kappa)$ near the origin, and is therefore i-standard on the same region as $\omega_t'(\kappa)$. Furthermore, since $\omega_1'(\kappa)$ coincides with ω near the boundary of the ball and $c^*\omega = -\omega$, then $\Omega_1 = \omega_1'(\kappa) = \omega$ near the boundary of the ball as well. Thus $\Omega_t(\kappa)$ satisfies the conclusion of the lemma for κ sufficiently large. Furthermore, since $\Omega_0 = \omega_0 = \omega$, this completes the proof. □

Proof of Proposition 2.15. The goal of the proof is to find a diffeomorphism $H : M \to M$ supported in a neighborhood of $\psi(0)$ and which is C^1-close to the identity, and then define a new form ω' by $(H^{-1})^*\omega$ and a new embedding $\psi' := H \circ \psi$. The proof proceeds in three steps. First, we perturb the form ω in a neighborhood of $\psi(0)$ so that the new form is J-standard close to $\psi(0)$. Second, we find a symplectomorphism $s : B(1+2\epsilon) \to B(1+2\epsilon)$ with support near $\psi(0)$ such that $s \circ \psi$ is J-holomorphic at the point $0 \in B(1+2\epsilon)$. Third, we use a symmetric, holomorphic chart around $\psi(0)$ to find a diffeomorphism $H : M \to M$ which gives $(H^{-1})^*\omega$ and $H \circ \psi$ the desired properties.

We first assume that M is a real symplectic manifold with real structure ϕ, J satisfies

$\phi_* J \phi_* = -J$, and $\psi \circ c = \phi \circ \psi$.

Step 1. Let (V, γ), $\gamma : V \subset M \to \mathbb{C}^n$ be a symmetric, holomorphic chart centered at $\psi(0)$ which exists because J is symmetrically integrable around $\psi(0)$. Let $W \subset \gamma(V)$ be a small ball centered at 0 inside $\gamma(V)$, and let $\omega' = (\gamma^{-1})^*\omega$. By Lemma 2.32, $(\gamma^{-1})^*\omega$ is isotopic to a form $\overline{\omega}$ which is i-standard near 0 and coincides with $(\gamma^{-1})^*\omega$ near the boundary of W. Therefore, by Proposition 2.31, there is a c-equivariant diffeomorphism $f : W \to W$ which fixes 0 and is the identity near the boundary of W such that $\overline{\omega} = f^*(\gamma^{-1})^*\omega$. We pull back f to $\gamma^{-1}(W)$ by defining $F := \gamma^{-1} \circ f \circ \gamma$, and we extend F to an equivariant diffeomorphism F' on M by defining it to be the identity outside $\gamma^{-1}(W)$. We now consider the form $\Omega = (F')^*\omega$, which is now J-standard near $\psi(0)$. We replace the embedding ψ by $\psi' := (F')^{-1} \circ \psi$, which is a symplectic embedding for Ω.

Step 2. Consider the almost complex structure $j = (\psi')_*^{-1} J \psi'_*$ on $B(1+2\epsilon)$. By the above paragraph, we see that we may choose a chart (U, γ'), $\gamma' : U \to \mathbb{C}^n$ of $\psi'(0)$ which is symplectic, holomorphic, and symmetric. Therefore, $A = ((\psi')^{-1})_*(\gamma)'_*(0)$ is a symplectic linear map from $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $Ai = jA$ and $Ac = cA$. We note that the composition $\psi' \circ A$ is therefore equivariant and that it satisfies $\psi'_* \circ A_* \circ i = J \circ \psi'_* \circ A_*$ at 0. For the remainder of the proof, we let $\alpha := \psi' \circ A$.

Step 3.We now wish to find a differentiable map $H : M \to M$ such that

1. $H \circ \alpha : (B(1+2\epsilon), i) \to (M, J)$ is holomorphic on a small neighborhood of 0,

2. The support of H is a subset of $Im(\alpha) \cap K^c$

3. H is C^1 close to the identity,

4. $Im(H \circ \alpha) = Im(\alpha)$,

5. $H \circ \phi = \phi \circ H$.

Let $W \subset U \cap K^c \cap Im(\alpha)$ be ϕ-invariant, and consider the map $H^{'} := \gamma^{'} \circ \alpha^{-1}|_W$: $W \cap Im(\alpha) \to M$. Then $H^{'} \circ \phi = \phi \circ H^{'}$, $H^{'} \circ \alpha = \gamma^{'}$, and therefore $H^{'}$ is holomorphic, symplectic, and symmetric on U. We also have

$$\begin{aligned} H^{'}_*(\psi(0)) &= \gamma^{'}_* \alpha^{-1}_*(\psi(0)) \\ &= \gamma^{'}_* A^{-1}_*(\psi^{'})^{-1}_*(\psi(0)) \\ &= \gamma^{'}_* \gamma^{-1}_* \psi^{'}_*(\psi^{'})^{-1}_*(\psi(0)) \\ &= Id. \end{aligned}$$

We define $H := H^{'}$ on a small neighborhood $C \subset W$, and we extend H to an equivariant diffeomorphism of M with support on $K^c = M - K$. In addition, since $H \circ \alpha(0) = \alpha(0)$ and $H_*(0) = Id$, by choosing C sufficiently small, we may restrict the support of H so that H is C^1 close to the identity. Now suppose $(x, \xi) \in TM$ is satisfies $\omega_x(\xi, J\xi) > 0$. Therefore we have that, for any $\epsilon > 0$, we may choose an H such that $|\omega_x(\xi, J\xi) - H^*\omega_x(\xi, J\xi)| < \epsilon$. Since $\overline{Im}(\alpha)$ is compact, $\Omega_x(\xi, J\xi)$ has a minimum strictly greater than 0 for $x \in \overline{Im}(\alpha), |\xi| = 1$, so we may choose $\epsilon < \inf_{x \in \overline{Im}(\alpha), |\xi|=1} \Omega_x(\xi, J\xi)$ and H so that $|\omega_x(\xi, J\xi) - H^*\omega_x(\xi, J\xi)| < \epsilon$, and therefore $H^*\omega_x(\xi, J\xi) > 0$, for all $\xi \neq 0$. For such an H, then, $H^*\omega$ tames J.

Also, with such an H and $t \in [0, 1]$, the forms $\Omega_t = t\omega + (1-t)H^*\omega$, tame J and are therefore nondegenerate. Since they are clearly closed, the Ω_t are symplectic, and therefore $H^*\omega$ is isotopic to ω. Define $\omega^{'} := H^*\omega$, and abusing notation, take $\psi^{'}$ to be $H^{-1} \circ \psi^{'}$, making $\psi^{'}$ symplectic for ω'. This completes the proof of point 2.

For the first part of the theorem, we note that, since J is relatively integrable, there is a chart (V, γ) around $\psi(0)$ such that $\gamma(L) \subset \mathbb{R}^n$. If we use this chart in the place of (V, γ) above and follow the same reasoning as above, the result follows. $\square$

CHAPTER 3

TOPOLOGICAL CRITERION FOR THE REAL BLOW-DOWN

We begin by reviewing some results in equivariant differential topology which we will use in the proof of Theorem 1.24.

3.1 Equivariant Differential Topology

Definition 3.1. Let M be a C^∞ manifold, and G be a compact Lie group. If $\Phi : G \times M \to M$ is a smooth action of G, then we call Φ a *G-action* on M, and if M admits such a G-action, we call M a *G-manifold*.

Lemma 3.2. *Let G be a finite group, and let M be a finite-dimensional G-manifold. Then there exists a G-invariant Riemannian metric g on M.*

Proof. Since M is a finite dimensional manifold, there exists a Riemannian metric g on M. For $h \in G$, we define h^*g by $g(dh\cdot, dh\cdot)$. Let $\tilde{g} := \frac{1}{|G|}\sum_{h\in G} h^*g$. Then for each point $x \in M$ and vectors $v, w \in T_xM$, we have $\tilde{g}(v, w) = \sum_{h\in G} h^*g(v, w)$. Therefore, for any element $k \in G$, we have

$$k^*\tilde{g} = \tilde{g}(dk\cdot, dk\cdot) = \sum_{h\in G} g(dh \circ dk\cdot, dh \circ dk\cdot) = \sum_{h'\in G} g(dh'\cdot, dh'\cdot) = \tilde{g}(\cdot, \cdot).$$

□

Remark 3.3. In fact, Lemma 3.2 is true for compact Lie groups as well. The averaging in the proof in that case is accomplished by integration with respect to the Haar measure. See Bredon [5] for details.

Lemma 3.4. *Let G be a compact Lie group, and let M be a topological G-space. Then the fixed point set of G, M^G, is a closet set.*

Proof. Let $\Delta \subset M \times M$ be the diagonal in $M \times M$, and let $\Gamma(g) : M \to M \times M$ be the graph of the action of an element $g \in G$, so $\Gamma(g)(m) = (m, g(m))$. We note that $\text{Fix}(g) = \Gamma(g)^{-1}(\Delta)$. Since Δ is closed in $M \times M$ and $\Gamma(g)$ is continuous, we have that $(\Gamma(g))^{-1}(\Delta)$ is closed in M. $\text{Fix}(G) = \cap_{g \in G} \text{Fix}(g)$ by definition, so $\text{Fix}(G)$ is closed in M as well. □

We now state the main theorem of this section.

Theorem 3.5. *Let M be a G-manifold with G finite. If A is a closed G-invariant submanifold of M, then A has an open G-invariant tubular neighborhood in M.*

Our first application of this theorem is the following.

Proposition 3.6. *Let G be a compact Lie group, and let M be a G-manifold. Then the fixed point set of G, M^G, is a smooth closed submanifold of G.*

Proof. First, if $M^G = \emptyset$, then it is a closed submanifold of M and we are done. We now assume that M^G is non-empty. By Lemma 3.4, M^G is closed, and we now show that it is a submanifold without boundary. Let $x \in M^G$. Note that for $x \in \text{Fix}(G)$, $\{dg\}_{g \in G}$ gives a linear G-action on $T_x M$ induced from the action of G on M, and that $T_x M^G$ is therefore a linear subspace of $T_x M$. Let $D_x(r)$ be an open ball of radius r in $T_x M$, and note that it is therefore also a disk bundle in $T_x\{x\}^{\perp}$. By Theorem 3.5, for small enough r we may define a G-equivariant tubular neighborhood $f : D_x(r) \to M$, such that $f(x, 0) = x$. Therefore, $f^{-1} : f(D(x)) \to T_x M$ defines a chart of M. The equivariance of f implies that

$$f^{-1}(f(D(x)) \cap M^G) = f^{-1}(f(D(x))) \cap T_x M^G,$$

and therefore

$$f^{-1}|_{f(D(x))\cap M^G} f(D(x)) \cap M^G \to T_x M^G$$

is a chart of M^G, and therefore M^G is a submanifold of M. Furthermore, since $T_x M^G$ is a linear subspace of $T_x M$, $C := D(x) \cap T_x M^G$ is open in $T_x M^G$, and therefore $f(C)$ contains no point on the boundary of M^G. In particular, $x \notin \partial M^G$. Since $x \in M^G$ is arbitrary, we see that M^G does not have a boundary. □

For the proof of Theorem 3.5, we follow the presentations given in Bredon [5] and Kawakubo [16]. We begin with several lemmas. First, we recall some results from general topology. For the following two lemmas, we follow Munkres [26].

Lemma 3.7. *If X is a paracompact regular space, then X is normal.*

Proof. Let A be a closed subset of X, and let B be a closed set of X disjoint from A. Then because X is regular, for each $b \in B$ we may choose an open set U_b whose closure $\bar{U}_b$ is disjoint from A. Cover X by the open sets U_b and the open set $X - B$. Call this covering $\mathcal{B}$. Since X is paracompact, we may take a locally finite open refinement $\mathcal{C}$ of $\mathcal{B}$ covering X. Now let $\mathcal{D}$ be the subcollection of $\mathcal{C}$ consisting of every element that intersects B. Then $\mathcal{D}$ covers B. Furthermore, if $D \in \mathcal{D}$, then $D \subset U_b$ for some $b \in B$, and since $\bar{U}_b$ is disjoint from A, then $\bar{D}$ is disjoint from A as well. Let

$$V = \bigcup_{D \in \mathcal{D}} D.$$

Then V is an open set in X containing B. We claim that

$$\bar{V} = \bigcup_{D \in \mathcal{D}} \bar{D}.$$

To see this, take $x \in \bar{V}$, and, since $\mathcal{D}$ is locally finite, there is a neighborhood U of x that intersects only finitely many sets of $\mathcal{D}$, say $D_1, \ldots, D_k$. If $x \in W := U - \cup_{i=1}^{k} \bar{D}_i$, then W is an open neighborhood of X which does not intersect $\bar{V}$, which contradicts the assumption that $x \in \bar{V}$. Therefore, $x \in \bigcup_{D \in \mathcal{D}} \bar{D}$. Since $\bar{V}$ contains B and is disjoint from A, X is normal. □

Lemma 3.8. *Suppose X is a regular paracompact space, and let $\{U_i\}_{i \in I}$ be an open covering of X, with I as its index set, then there exists a locally finite open covering $\{V_i\}_{i \in I}$ such that $\bar{V}_i \subset U_i$.*

Proof. Because X is regular, so around each point x in X we may choose an open neighborhood $V_x' \ni x$ such that $\bar{V}_x' \subset U_i$. X is paracompact, so there exists a locally finite refinement of $\{V_x'\}_{x \in X}$, which we denote $\{V_j'\}_{j \in J}$, where J is an index set. We then define a function $f : J \to I$ by choosing, for each $j \in J$, an element $f(j) \in I$ such that $\bar{V}_j' \subset U_{f(j)}$. We define the set

$$V_i := \bigcup_{\{j | f(j) = i\}} V_j'.$$

Then $\bigcup_{i \in I} V_i = \bigcup_{j \in J} V_j' = X$, so $\{V_i\}_{i \in I}$ is a covering of X, and $\bar{V}_i = \bigcup_{\{j | f(j) = i\}} \bar{V}_j \subset U_i$ as desired. □

We now give two more general topological results. The proofs follow those found in Bredon [5].

Lemma 3.9. *Let X and Y be metric spaces and let $f : X \to Y$ be onto, and a local homeomorphism, i.e. such that each $x \in X$ has an open neighborhood mapped homeomorphically onto an open set in Y. Suppose that f is one-to-one on a subspace $A \subset X$, so that $f^{-1}(f(A)) = A$. Then A has an open neighborhood U on which f is homeomorphic to $f(U)$. Furthermore, if f is a local diffeomorphism, then U is diffeomorphic to $f(U)$.*

Proof. Let $B := f(A) \subset Y$. Since f is onto and a local homeomorphism, for each $y \in B$ there is an $x \in X$ with $f(x) = y$, and f maps some neighborhood U_x of x homeomorphically to a neighborhood U_y of y, so a continuous inverse can be defined on U_y. Denote this inverse by g_y. Let $\{U_y\}_{y \in Y}$ be a covering of Y by such neighborhoods. Since Y is paracompact, we can find a locally finite refinement $\{U_{y_\alpha}\}$ and maps $g_{y_\alpha} : U_{y_\alpha} \to X$ such that fg_α is the identity on U_{y_α}. By Lemma 3.8, we know that there exists a locally finite open covering $\{V_{y_\alpha}\}$ of Y such that $\bar{V}_{y_\alpha} \subset U_{y_\alpha}$. Let W denote the set of points $y \in Y$ such that if $y \in \bar{V}_{y_\alpha} \cap \bar{V}_{y_\beta}$ then $g_{y_\alpha}(y) = g_{y_\beta}(y)$. We will show that $B \subset W$. Let $B = f(A)$, choose $y \in B$, and suppose $y \in \bar{V}_{y_\alpha} \cap \bar{V}_{y_\beta}$. Then $y = fg_{y_\alpha}(y) = fg_{y_\beta}(y)$, and since $f^{-1}(B) = A$, we have that $g_{y_\alpha}(y), g_{y_\beta}(y) \in A$. Furthermore, since $f|A$ is one-to-one, $fg_{y_\alpha}(y) = fg_{y_\beta}(y) \implies g_{y_\alpha}(y) = g_{y_\beta}(y)$. Therefore $B \subset W$, and, in particular, W is non-empty.

We now show that W is open. Let $y \in W$ and suppose that $g(y) = x$, so that $f(x) = y$. Let N be an open neighborhood of x on which f is one-to-one. Since $\{V_{y_\alpha}\}$ is locally finite, we may suppose that $y \in \bar{V}_{y_{\alpha_1}} \cap \cdots \cap \bar{V}_{y_{\alpha_k}}$ but that $y \notin \bar{V}_\beta$ for $\beta \notin \{\alpha_1, \ldots, \alpha_k\}$ and that there is an open neighborhood M of y not touching any of the $\bar{V}_\beta$. In addition, since $g_{\alpha_i}(y) = x$, we may take M to be so small that $g_{\alpha_i}(M) \subset N$. Now, if $z \in M$ and $g_{\alpha_i}(z) \neq g_{\alpha_j}(z)$, then, since f is one-to-one on N, we have that

$z = fg_{\alpha_i}(z) \neq fg_{\alpha_j}(z) = z$, a contradiction. Therefore, $M \subset W$ and W is open, as desired.

Since $g = g_\alpha$ on each open set V_α and g_α is an open map, we have that g is an open map, and therefore $g(W)$ is an open set containing A. Also, f is one-to-one on $g(W)$ since $fg = 1_W$. Therefore, since f is also open it is a homeomorphism from $g(W) \to W$ with inverse g, and the lemma is proved for f a local homeomorphism.

To see that if f is a local diffeomorphism, then $f(U)$ is diffeomorphic to U, note that the maps g_α may be taken to be diffeomorphisms on the U_α, making g a diffeomorphism as well. $\square$

Lemma 3.10 (Dowker, see Dugundji [11], p. 170)**.** *Let Y be a paracompact space. Assume that g and H are lower- and upper-semicontinuous real-valued functions, respectively, such that $H(y) < g(y)$ for all $y \in Y$. Then there exists a continuous function $\phi : Y \to \mathbb{R}$ such that $H(y) < \phi(y) < g(y)$ for each $y \in Y$.*

Proof. For each rational r, let $U_r = \{y | H(y) < r\} \cup \{y | g(y) > r\}$. Since H and g are upper and lower-semicontinuous, respectively, U_r is open. For each $y \in Y$, since $H(y) < g(y)$ there exists a rational number $r^{'}$ such that $H(y) < r^{'} < g(y)$, and therefore the U_r form an open covering of Y. Since Y is paracompact we may take a partition of unity subordinate to U_r, which we denote κ_r. We claim that the desired continuous function is

$$\phi(y) = \sum_r r \cdot \kappa_r(y).$$

Since κ_r is a partition of unity, this sum is finite for every y, and therefore ϕ is continuous. Let $y \in Y$ be fixed, and let $\{\kappa_{r_1}, \ldots, \kappa_{r_n}\}$ be the functions of the partition whose support contains y. Then $y \in \cap_{i=1}^n U_{r_i}$, so $H(y) < r_i < g(y)$ for all $i \in 1, ..., n$. Therefore

$H(y) = H(y)\sum_{i=1}^{n}\kappa_{r_i}(y) < \sum r_i \cdot \kappa_{r_i}(y) = \phi(y) < g(y)\sum_{i=1}^{n}\kappa_{r_i}(y) = g(y)$, as desired. □

Corollary 3.11. *If Y is a C^k-manifold, then ϕ in 3.10 may be taken to be C^k as well.*

The following two lemmas are standard results that we recall for completeness. Our presentation follows that in Bredon [5] and Kawakubo [16].

Lemma 3.12. *Let G be a compact Lie group, let M be a finite dimensional G-manifold and let g be a G-invariant Riemannian metric. Then the associated exp map is G-equivariant.*

Proof. Since g is G-invariant, the action on M of an element $h \in G$ sends geodesics to geodesics and preserves their length. Therefore for $h \circ \exp(v) = h \circ l_0(1)$, where $l_0(t)$ is a geodesic with $l_0'(0) = v$. On the other hand, $\exp(h_* v) = l_1(1)$, where $l_1'(0)$ is the unique geodesic with $l_1' = h_* v$. But $(h \circ l_0)'(0) = h_* v$, and therefore $h \circ l_0 = l_1$, so $\exp$ is equivariant. □

Lemma 3.13. *The differential of the exp map exp_* : $T_x(TM) = T_h \oplus T_v = T_xM \oplus T_xM \to T_xM$ is given by $\exp_*(u, v) = u + v$, where T_v is the tangent space of the fiber T_xM at 0 and T_h is the tangent space of the zero section of M at x.*

Proof. Suppose $v_0 \in T_v$. Then, identifying T_v and $T_0(T_xM) \cong T_xM$, we may see v_0 as an element of T_xM as well. Suppose $l : (-\epsilon, \epsilon) \to M$ is a geodesic tangent to v_0 at x, and define a sooth curve $v(t)$ in T_xM by $v(t) = tv_0$. Then $\exp_*(v_0) = \frac{d}{dt} exp(tv_0) = \frac{d}{dt} l(t) = v_0$, and therefore $\exp_* |_{T_v} = Id$.

Furthermore, since $T_h = T_xM$ and is parallel to M, and since $\exp$ is the identity on M, we have that $\exp_*$ is the identity on T_h. The lemma follows immediately. □

We now complete the proof of the existence of a G-invariant tubular neighborhood.

Proof of Theorem 3.5. By Lemma 3.2, there exists a Riemannian metric g on M such that G acts on M isometrically. Thus the normal bundle $N(A)$ is the perpendicular complement of TA in $TM|A$ with respect to the metric g. The exponential map is defined on an open invariant neighborhood U of A in $N(A)$, and $\exp : U \to M$ is equivariant by Lemma 3.12. For $a \in A$, Lemma 3.13 implies that the differential of $\exp$ restricted to $N(A)$,

$$\exp_* : T_a\left(N(A)\right) = N_aA \oplus T_aA = T_aM \to T_aM$$

is onto. Thus, by the inverse function theorem, around every point a of A there is a neighborhood $U_a \subset N(A)$ such that exp has an inverse on U_a. Let $V = \cup_{a\in A} U_a$. That is, exp : $V \to M$ is an immersion and a local diffeomorphism such that $\exp^{-1}(A) = A$. Furthermore, since exp is an immersion on V, a local diffeomorphism, and the identity on A, it follows from Lemma 3.9 that there is a smaller invariant neighborhood $W \subset U$ of A in $N(A)$ on which exp is an embedding.

We now construct an equivariant map from all of $N(A)$ to W. Let $\iota : B^{n-k}(r; a) \hookrightarrow N_a(A)$ be the inclusion, where $n = \dim M$ and $k = \dim A$. Define the function $f : A \to \mathbb{R}$ by

$$f(a) := \sup\{r | \iota(B(r; a)) \subset W\}.$$

Since W is G-invariant, $f(ga) = f(a)$ for all $a \in A$ and $g \in G$, and we claim that f is a lower semicontinuous, positive function. To see that it is lower-semicontinuous, we need to show that for each $b \in \mathbb{R}$, the set $\mathcal{U}_b := \{x \in A | f(x) > b\}$ is open. Fix $b \in \mathbb{R}$. If $\mathcal{U}_b = \emptyset$, then it is open, so we now assume that $\mathcal{U}_b \neq \emptyset$. Suppose that $a \in \mathcal{U}_b$ and consider a chart of $N(A)$ centered at $a \in A$, say U_a. Since $N(A)$ is a vector bundle, $U_a \cong V_a \times \mathbb{R}^{n-k}$, where n and k are the dimensions of M and A, respectively. Let $D_b \subset N(A)$ denote the closed disc bundle of radius b in $N(A)$. Then the complement D_b^c is open, and, since $\mathcal{U}_b$ is non-empty, we see that the open set $W \cap U_a \cap D_b^c$ is also nonempty. Since the projection $\pi_1 : V_a \times \mathbb{R}^{n-k} \to V_a$ is an open map, $\pi_1(W \cap U_a \cap D_b^c)$ is open, and it is a subset of $\mathcal{U}_b$ by the definition of D_b^c. Therefore f is lower semicontinuous, as claimed. By Lemma 3.10, there is a smooth function h on A with $0 < h(a) < f(a)$. We then define a smooth function $k : A \to \mathbb{R}$ by

$$k(a) = \frac{1}{|G|} \sum_{g \in G} h(ga)$$

Then $0 < k(a) < \frac{1}{|G|} \sum_{g \in G} f(ga) = |G| \cdot \frac{1}{|G|} f(a) = f(a)$. Now define $\psi : N(A) \to N(A)$ by

$$\psi(v) = \frac{k(\pi(v))}{(1+ < v, v >)^{1/2}} v,$$

where π is the projection $N(A) \to A$. Then $\psi(gv) = g\psi(v)$, and ψ is an equivariant diffeomorphism onto its image, which we claim to be the open set $\{v \in N(A) | \, \|v\| < k(\pi(v))\}$. To see this, we compute the norm

$$\|\psi(v)\| = k(\pi(v)) \frac{\|v\|}{(1 + \|v\|^2)^{1/2}} < k(\pi(v))$$

and note that for any $v \in N(A)$ such that $\|v\| < k(\pi(v))$,

$$\begin{aligned} \psi\left(\frac{1}{(k(\pi(v))^2 - \|v\|^2)^{\frac{1}{2}}} v\right) &= \frac{k(\pi(v)) \cdot (k(\pi(v))^2 - \|v\|^2)^{\frac{1}{2}}}{k(\pi(v))} \frac{v}{(k(\pi(v))^2 - \|v\|^2)^{\frac{1}{2}}} \\ &= v \end{aligned}$$

Thus Im $\psi \subset W$ and $\phi = \exp \circ \psi : N(A) \to M$ is a G-equivariant map, making $Im(\phi)$ an open invariant tubular neighborhood of A. $\square$

3.2 Proof of the Blow-down Criterion

In this section, we prove Theorem 1.24, which we restate for convenience.

Theorem (Theorem 1.24)**.** *Let (M, ω, ϕ) be a real symplectic manifold with $L :=$ Fix(ϕ), and let J be an almost complex structure on M which tames ω and which satisfies $\phi_* J \phi_* = -J$. Suppose C is an exceptional J-holomorphic curve in a homology class $E \in H_2(M;\mathbb{Z})$ such that $E \cdot E = -1$ and $\phi_* E = -E$. Then there exists a real symplectic manifold $(\check{M}, \check{\omega}, \check{\phi})$ and an onto map $\Pi : M \to \check{M}$ that satisfies*

1. Π *is a diffeomorphism on* $M \backslash C$*,*

2. $\Pi(C) = p \in \check{M}$*, where* p *is a point,*

3. $\Pi \circ \phi = \check{\phi} \circ \Pi$*, and*

4. $\check{\omega}$ *satisfies*

$$[\omega] - [\Pi^* \check{\omega}] \in \mathcal{E},$$

where $\mathcal{E}$ is the linear vector space generated by e, the Poincaré dual of the exceptional class $E = [\Pi^{-1}(p)]$.

We begin by recalling a version of the adjunction inequality, as given in McDuff [22].

Theorem 3.14. *Let (M, J) be an almost complex 4-manifold and $A \in H_2(M;\mathbb{Z})$ be a homology class that is represented by a somewhere injective (closed) J-holomorphic* *curve* $u : \Sigma \to M$*. Then*

$$g \leq 1 + \frac{1}{2}(A \cdot A - c_1(A)),$$

with equality iff u *is an embedding, where* g *is the genus of* Σ*.*

We recall Definition 1.23 from Chapter 1.

Definition. *We call* $E \in H_2(M^4; \mathbb{Z})$ *an* exceptional class *if* $E \cdot E = -1$*. If* $u : \Sigma \hookrightarrow M^4$ *is an embedding of the surface* Σ*, and* $u_*[\Sigma] = E$*, then we say that* $u(\Sigma)$ *is an* exceptional curve.

We next recall a standard result in algebraic topology

Lemma 3.15. *Let* M *be a* 4*-manifold, and let* $\Sigma \hookrightarrow M$ *be a* 2*-dimensional submanifold of* M *in class* $H \in H_2(M; \mathbb{Z})$*. Then* $H \cdot H = c_1(N(\Sigma))$*, where* $N(\Sigma)$ *denotes the normal bundle of* Σ *in* M*.*

Proof. Note that c_1 is the top-dimensional Chern class for Σ, and therefore $c_1(N(\Sigma)) = e(N(\Sigma))$, where e is the Euler class. However, the Euler class of the bundle equals the self-intersection number of a transverse section of the bundle with the zero-section, and this number equals the intersection number $H \cdot H$. □

Remark 3.16. We first recall from McDuff and Salamon [24], Proposition 2.5.1 that a simple J-holomorphic curve is somewhere injective. Now consider an exceptional class $E \in H_2(M; \mathbb{Z})$ in a symplectic four-manifold M^4 which is represented by a simple J-holomorphic sphere $u : S^2 \to M$. Then by Lemma 3.15, $c_1(E) = c_1(TS^2) \oplus c_1(NS^2) = 2 - 1 = 1$, where we use NS^2 to denote the normal bundle of u in M. We therefore have $0 = g \leq 1 - 1 = 0$, and so u is an embedding.

Remark 3.17. Let (M, ω, ϕ) be a real symplectic manifold with an almost complex structure J which tames ω and satisfies $\phi_* J \phi_* = -J$. Let $u : \Sigma \to M$ be a closed J-holomorphic curve, and suppose it is an embedding whose image is invariant under u. Then Σ inherits the symplectic form $u^*\omega$ and the anti-symplectic involution $u^{-1} \circ \phi \circ u$.

Proposition 3.18. *Let (S^2, ω) be endowed with an anti-symplectic involution ϕ. If Fix$(\phi) \neq \emptyset$, then the fixed point set of ϕ is a circle.*

Proof. Let $G = \mathbb{Z}_2$ with smooth actions on M given by the functions $\{Id, \phi\}$. From Proposition 3.6 we see that Fix$(G) =$ Fix(ϕ) is a closed submanifold of S^2. Denote this submanifold by K. Now suppose $p \in K$, and let $v, w \in T_pK$. Then $\omega(v, w) = \phi^*\omega(v, w) = -\omega(v, w) = 0$, and so the fixed point set is an isotropic submanifold of S^2. We claim that L is one-dimensional. To see this, we first note that the dimension must be ≤ 1 since Fix(ϕ) is isotropic. Now suppose that L is zero dimensional. Since $\phi^2 = Id$, if K is zero-dimensional, the eigenvalues of $d\phi(x) : T_xM \to T_xM$ for $x \in L$ are all -1. Hence $d\phi(x) = -Id$, and hence $\phi^*\omega_x = \omega_x$, a contradiction. Therefore K cannot be zero-dimensional, and must be one dimensional. Fix(ϕ) is therefore equal to a closed Lagrangian submanifold of S^2 and is therefore diffeomorphic to a union of non-intersecting circles. This union is compact, and therefore finite, since Fix(ϕ) is topologically closed and S^2 is compact.

Suppose there is more than one circle in Fix(ϕ), say $\alpha_1, ..., \alpha_k$. Now choose two circles, which we denote γ_1 and γ_2. S^2 therefore decomposes as $S^2 = D_1 \cup C \cup D_2$, where the D_i are the non-intersecting disks bounded by the γ_i, and C is the closed cylinder between the discs. Now consider $\phi(D_1)$. Since ϕ is a diffeomorphism, it must send D_1 onto a disc bounded by γ_1, i.e. either D_1 or $C \cup D_2$.

Now suppose $\phi(D_1) = C \cup D_2$. Then there is a point $x \in D_1$ such that $\phi(x) \in$

$\gamma_2 \Rightarrow \phi^2(x) \in \gamma_2 \nsubseteq D_1$, which contradicts the assumption that $\phi^2 = \text{Id}$. Therefore, $\phi(D_1) = D_1$.

Since there are only a finite number of total circles in Fix(ϕ), we may choose γ_1 so that $\overline{D}_1 \cap \text{Fix}(\phi) = \gamma_1$, i.e. so there are no fixed points in the interior of D_1. Note that for any $x \in \gamma_1$, one of the eigenvalues of $d\phi(x)$ is -1. Therefore, there are points in a collar neighborhood of γ_1 in D_1 which are sent by ϕ to a collar neighborhood of γ_1 in $D_2 \cup C$. However, this contradicts that $\phi(D_1) = D_1$, and concludes the proof. □

Corollary 3.19. *Let (M, ω, ϕ) be a real symplectic manifold, and let $L :=$ Fix(ϕ). Let J be an almost complex structure on M such that $\phi_* J \phi_* = -J$, and let $E \in H_2(M; \mathbb{Z})$ be an exceptional class with $\phi_* E = -E$. Suppose $u : \Sigma \to M$ is a simple J-holomorphic curve that represents E. Then $u(\Sigma) \cap L$ is diffeomorphic to a circle.*

Proof. By Theorem 3.14 and Remark 3.16, u is an embedding. Note, too, that $\phi \circ u \circ c$ is another simple J-holomorphic embedding that represents E, and its image is equal to Im$(\phi \circ u)$. Suppose now that $Im(u) \neq Im(\phi \circ u)$. Let c denote complex conjugation on $\Sigma = S^2$. Because both maps u and $\phi \circ u \circ c$ are J-holomorphic, their intersections are at most countable, and since $[\text{Im}(\phi \circ u \circ c)] = [\text{Im}(u)] = E \in H_2(M; \mathbb{Z})$, positivity of intersections in dimension 4 (e.g. Theorem E.1.4 in McDuff and Salamon [24]) implies that $0 \leq |\{\text{Im } u\} \cap \{\text{Im } \phi \circ u \circ c\}| \leq E \cdot E = -1$, which is a contradiction. Therefore, Im(u) = Im$(\phi \circ u)$. By Remark 3.17, $u(\Sigma)$ inherits a real structure from M, and it follows from Proposition 3.18 that the fixed point set of ϕ restricted to $u(\Sigma)$ is a circle. Since Fix$(\phi) = L \subset M$, it follows that $u(\Sigma) \cap L$ is diffeomorphic to a circle. □

Lemma 3.20. *There is a natural isomorphism between the oriented Lagrangian subspaces of $\mathbb{R}^{2n}$ and the quotient space $U(n)/SO(n)$.*

Proof. We recall from McDuff and Salamon [25] that the unitary matrix $U = X + iY$ given by a unitary Lagrangian frame is determined by the Lagrangian subspace Λ up to right multiplication by a matrix in $O(n)$. Similarly, given an orientation $o(\Lambda)$ of Λ, we see that U is determined by $(\Lambda, o(\Lambda))$ up to right multiplication by a matrix in $SO(n)$. □

Lemma 3.21. *Let $u : (D, \partial D) \to (M, L)$ be a J-holomorphic disk with boundary on a Lagrangian L. Suppose the Maslov index of u, $\mu(u)$, satisfies $\mu(u)$ mod $2 = 1$. Then $TL|_{\partial D}$ is a non-trivial bundle.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
\pi_1(SO(n)) & \xrightarrow[i]{} & \pi_1(O(n)) \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
\pi_1(U(n)) & \xrightarrow[\cong]{} & \pi_1(U(n)) \cong \mathbb{Z} \\
\downarrow & & \downarrow \\
\pi_1(U(n)/SO(n)) & \xrightarrow[\alpha]{} & \pi_1(U(n)/O(n)) \cong \mathbb{Z} \\
\downarrow & & \downarrow{\scriptstyle \beta} \\
0 & \longrightarrow & \mathbb{Z}_2 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array}$$

Note that the vertical exact sequences in the diagram are taken from the respective homotopy long exact sequences. Note that it follows from the diagram that the map β is onto, and therefore that the map α is multiplication by 2. Identifying the Maslov class of a loop γ of Lagrangians with $[\gamma] \in \pi_1(U(n)/O(n))$, we see that the Maslov class of any loop γ of oriented Lagrangians is even. Now consider a trivialization $\Phi : u^*TM \to D \times \mathbb{C}^n$. If $TL|_{\partial D}$ is trivial, then the loop of Lagrangians $\Lambda \circ \Phi|_{\partial D} \to U(n)/O(n)$ is a loop of oriented Lagrangians, and therefore $\mu(u)$ is even. This concludes the proof. $\square$

Lemma 3.22. *Let (M, ω, ϕ) be a four-dimensional real symplectic manifold with real structure ϕ. Denote the fixed point set of ϕ by L, and let $E \in H_2(M; \mathbb{Z})$ be a homology class such that $E \cdot E = -1$. Suppose $u : (\mathbb{C}P^1, \sigma, i) \to (M, \omega, J)$ is a J-holomorphic embedding such that $u_*[\mathbb{C}P^1] = E$, and such that the intersection $Im(u) \cap L \cong S^1$. Then the intersection of TL with the normal bundle of $Im(u)$, i.e. $TL \cap \nu(Im(u))$, is nontrivial.*

Proof. We note that $c_1(u^*TM) = 2 - 1 = 1$, and that the Maslov number of $u = 2c_1(E) = 2$. Let $u_1, u_2 : D^2 \to M$ denote the two disks which make up u. We claim that the Maslov index of each disc must be 1. First, recall that $\mu(u_1) + \mu(u_2) = \mu(u)$ by the properties of the Maslov index. Second, the involution $\phi : M \to M$ induces a diffeomorphism from $Im(u_1)$ to $Im(u_2)$, and $\phi_* : TM \to TM$ is a vector bundle isomorphism from u_1^*TM to u_2^*TM. Again, the properties of the Maslov index (see Theorem C.3.5 in McDuff and Salamon [24]) imply that $\mu(u_1) = \mu(u_2)$, and this implies that possibilities other than $(1, 1)$ for the Maslov indices of the two discs may not occur. It follows that that the bundle $T_{S^1}L = TS^1 \oplus \nu_L(S^1)$ is non-trivial by Lemma 3.21, where $\nu_{TL}(S^1)$ denotes the part of the normal bundle of S^1 which lies in TL. Since TS^1 is trivial, then $\nu_L(S^1)$ cannot be, and the lemma is proved. $\square$

Lemma 3.23. *Let M be a four-dimensional real symplectic manifold with real structure ϕ. Denote the fixed point set of ϕ by L, and let $E \in H_2(M; \mathbb{Z})$ be a homology class such that $E \cdot E = -1$ and $\phi_* E = -E$. Suppose, furthermore, that there exists an embedding of the surface Σ, $i : \Sigma \to M$, with $i_*[\Sigma] = E$. Then $E \cdot L = 1$ mod 2.*

Proof. First, we perturb i so that $i(\Sigma) \cap L$ and $i(\Sigma) \cap \phi \circ i(\Sigma)$ are generic. Let $p \in$

$i(\Sigma) \cap \phi \circ i(\Sigma)$, $p \notin L$. Then $\phi(p) \in i(\Sigma) \cap \phi \circ i(\Sigma)$, $\phi(p) \notin L$, and, in particular, p and $\phi(p)$ do not affect the value of either $E \cdot E$ mod 2 or $E \cdot L$ mod 2. Suppose now that $E \cdot L = 0$ mod 2. Then there exist an even number of points in the intersection $i(\Sigma) \cap L$, and, combined with the above, this implies that there are an even number of points in $i(\Sigma) \cap \phi \circ i(\Sigma)$. However, $i_*[\Sigma] \cdot \phi_* i_*[\Sigma] = 1$ mod 2, which is a contradiction. Therefore $E \cdot L = 1$ mod 2. □

We recall a version of the Riemann Mapping Theorem from [29] (see also [7]).

Theorem 3.24. *Let D denote the unit disk in $\mathbb{C}$, let Ω be a simply connected domain in $\mathbb{C}$, ($\Omega \neq \mathbb{C}$), and assume that the boundary $\partial\Omega$ is locally connected. Then there is a holomorphic isomorphism $f : D \to \Omega$ that extends to a continuous map from $\bar{D} \to \bar{\Omega}$. Moreover, if $\partial\Omega$ is a Jordan curve, then f extends to a homeomorphism from $\bar{D}$ to $\bar{\Omega}$.*

We now prove Theorem 1.24.

Proof of Theorem 1.24. Let $u : \Sigma \to M$ be the J-holomorphic curve whose image is C. By hypothesis, $[C] \cdot [C] = -1$ so Lemma 3.23 implies that $C \cap L \neq \emptyset$. By Corollary 3.19, C intersects L in a circle, whose preimage we denote S. Let D_1 and D_2 be the two open discs in C with boundary S. Note that, for each $x \in D_1$, $\phi(x) \in D_2$. Now let H_1 and H_2 denote the two hemispheres of $\mathbb{C}P^1$ with boundary $\mathbb{R}P^1$. By Theorem 3.24 there exists a holomorphic map $\alpha : D_1 \to H_1$ which extends to a homeomorphism from $\bar{D}_1$ to $\bar{H}_1$. Now define a map $\tilde{\alpha} : C \to \mathbb{C}P^1$ by

$$\tilde{\alpha}(x) = \begin{cases} \alpha(x) & \text{if } x \in \bar{D}_1 \\ c \circ \alpha(\phi(x)) & \text{if } x \in D_2, \end{cases} \tag{3.2.1}$$

where c denotes complex conjugation on $\mathbb{C}P^1$. We claim that $\tilde{\alpha}$ is holomorphic on all of $\mathbb{C}P^1$. First, choose a holomorhpic chart $\gamma_1 : W \subset C \to \mathbb{C}$ centered at a point $x \in \mathbb{R}P^1$ which sends $U \cap \mathbb{R}P^1$ to $\mathbb{R}$. Let $\gamma_2 : V \subset \mathbb{C}P^1 \to \mathbb{C}$ be a holomorphic chart centered at $\tilde{\alpha}(x) \in V$, and note that $\tilde{\alpha}$ is holomorphic iff $\gamma_2 \circ \tilde{\alpha} \circ \gamma_1^{-1}$ is holomorphic for any pair of charts. To prove that this is the case, we appeal to Morera's theroem, which we recall below, as stated in Conway [9], following the proof of the Schwartz Reflection Principle.

Theorem 3.25 (Morera's Theorem)**.** *Let U be a region in $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every triangular path T in U. Then f is analytic in U.*

To apply this theorem, we need to show that for each triangular path $T \subset U$, $\int_T f = 0$. Denote $\gamma_1^{-1}(U)$ by U, let $U^+ = U \cap \{z|Im(z) > 0\}$, $U^0 = \{z|Im(z) = 0\}$, $U^- = \{z|Im(z) < 0\}$, and $f := \gamma_2 \circ \tilde{\alpha} \circ \gamma^{-1} : U \to \mathbb{C}$. Choose a triangular path T in U. We see that $\int_T f = 0$ iff $\int_P f = 0$ for any triangular or quadrilateral path P in $U^+ \cup U^0$ and $U^- \cup U^0$. Furthermore, if $P \subset U^\pm$, then $\int_P f = 0$, since f is holomorphic on $U^\pm$ by definition. We therefore let T be the triangle with vertices $[a, b, c]$, where the edge $[b, c]$ is contained in the real axis. The same argument applies for a quadrilateral path. Let Δ denote the union of the path T and its interior. f is continuous on $U^+ \cup U^0$ by construction, and therefore it is uniformly continuous on Δ. Therefore, for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|z - z^{'}| < \delta \implies |f(z) - f(z^{'})| < \epsilon$. Now choose a small $\epsilon > 0$, and a $\delta > 0$ such that $0 < \delta < \epsilon$ and $|z - z^{'}| < \delta \implies |f(z) - f(z^{'})| < \epsilon$. Pick points α and β on the line segments $[a, b]$ and $[a, c]$, respectively, so that $|c - \alpha| < \delta$ and $|b - \beta| < \delta$. Let $T^{'}$ and Q be the paths $T^{'} = [\alpha, \beta, a, \alpha]$ and $Q = [\alpha, c, b, \beta, \alpha]$ as in Figure 3.1 below. Then

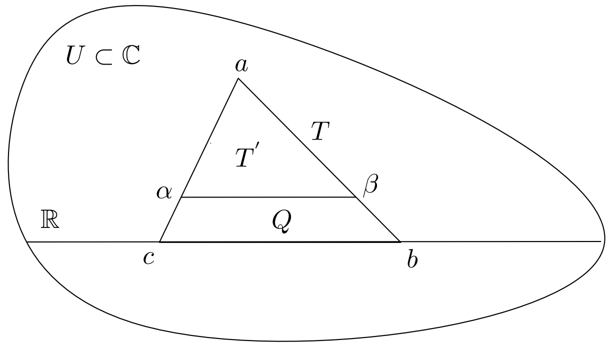

Figure 3.1

$$\int_T f = \int_{T'} f + \int_Q f.$$

However, since $T^{'}$ and its interior are contained in U^+, f is holomorphic there, and therefore $\int_{T'} f = 0$.

We now approximate $\int_Q f$. First, note that, for $t \in [0, 1]$,

$$|[t\beta + (1-t)\alpha] - [tb + (1-t)c]| < \delta$$

and therefore

$$|f(t\beta + (1-t)\alpha) - f(tb + (1-t)c)| < \epsilon.$$

Now let $M = \max \{|f(z)| \mid z \in \Delta\}$, and let $l =$ the length of the perimeter of T. Then

$$\left|\int_{[\alpha,c]} f\right| \leq M\,|c - \alpha| \leq M\delta$$
$$\left|\int_{[\beta,b]} f\right| \leq M\,|b - \beta| \leq M\delta,$$

and

$$\begin{aligned}\left|\int_{[b,c]} f + \int_{[\beta,\alpha]} f\right| &= \left|(b-c)\int_0^1 f(tb+(1-t)c)dt - (\beta-\alpha)\int_0^1 f(t\beta+(1-t)\alpha)dt\right| \\ &\leq |b-c|\left|\int_0^1 f(tb+(1-t)c) - f(t\beta+(1-t)\alpha)\right| \\ &\quad +|(b-c)-(\beta-\alpha)|\left|\int_0^1 f(t\beta+(1-t)\alpha)dt\right| \\ &\leq \epsilon|b-c| + M|(b-\beta)+(c-\alpha)| \\ &\leq \epsilon l + 2M\delta.\end{aligned}$$

Therefore,

$$\left|\int_T f\right| \leq \epsilon l + 4M\delta.$$

Since ϵ is arbitrary, and we may choose $\delta < \epsilon$, it follows that $\int_T f = 0$, and therefore f is holomorphic. From this we conclude that $\tilde{\alpha}$ is holomorphic as well.

We have now shown the existence of a holomorphic map $\tilde{\alpha}$ that verifies $\phi \circ \tilde{\alpha} \circ c = \tilde{\alpha}$ and $Im(\tilde{\alpha}) = Im(u)$. Now let $S = C \cap L$.

We now remark that the cohomology class $[\tilde{\alpha}^*\omega] \in H^2(\mathcal{L}(0))$ is determined by the integral $\int_{\mathbb{C}P^1} \tilde{\alpha}^*\omega$, where here we understand $\mathbb{C}P^1 = \mathcal{L}(0)$. Therefore, for $\lambda^2 := \int_{\mathbb{C}P^1} \tilde{\alpha}^*\omega$, the form $\rho(1,\lambda)$ is in the same cohomology class. By Proposition 2.31, there exists a diffeomorphism $\beta_0 : \mathcal{L}(0) \to \mathcal{L}(0)$ such that $\beta_0^*\tilde{\alpha}^*\omega = \rho(1,\lambda)$. Let $\gamma_0 := \tilde{\alpha} \circ \beta_0$.

Now, by Lemma 3.22 the normal bundle of S in TL is non-trivial. Consider the bundles $\gamma_0^*(\nu(C))$ and $\nu(\mathcal{L}(0))$, where $\nu(\cdot)$ denotes the normal bundle of the submanifold in question. Since the Chern class of C is 2, the Maslov index of the two disks D_1 and D_2 in C with boundary on L is 1, and the restriction of $\nu(C)$ to L is non-trivial, then by Therem C.3.7 in McDuff and Salamon [24], there is a (complex) isomorphism Φ

between the bundles $\gamma_0^*(\nu(D_1), TL \cap \nu(D_1))$ and $\nu(\mathcal{L}(0)^+, \mathcal{R}(0))$, where $\mathcal{L}(0)^\pm$ denote the upper and lower hemispheres of $\mathcal{L}(0)$, respectively.

Now note that $\tilde{\phi}_* \Phi \tilde{c}_*$ gives an isomorphism of $\tilde{\alpha}^*(\nu(D_2), TL \cap \nu(D_2))$ and $\nu(\mathcal{L}(0)^-, \mathcal{R}(0))$, and therefore the map

$$\Psi = \begin{cases} \Phi & (x, v) \in \nu(\mathcal{L}(0)^+) \\ \tilde{\phi}_* \Phi \tilde{c}_* & (x, v) \in \nu(\mathcal{L}(0)^-) \end{cases}$$

is a complex equivariant isomorphism from $\nu(\mathcal{L}(0)) \to \alpha^* \nu(C)$.

Furthermore, since Ψ is a complex bundle isomorphism, it is symplectic as well. It therefore follows from Proposition 2.28, that for some $\delta > 0$, we can find a $\mathbb{Z}_2$-equivariant map $\beta_\lambda : \mathcal{L}(\delta) \to M$ such that $\beta_\lambda^* \omega = \rho(1, \lambda)$ which restricts to the symplectomorphism $\gamma_0 : \mathcal{L}(0) \to C \subset M$. We may now construct the blow-down by theorem 1.22 using the equivariant symplectic map $\beta : \mathcal{L}(\delta) \to M$. $\square$

CHAPTER 4

APPLICATIONS TO REAL PACKING

We now apply our constructions above to the problem of packing $k \leq 8$ balls into $\mathbb{R}P^2$ in $\mathbb{C}P^2$, adapting the techniques in McDuff and Polterovich [23] to our setting. That is, we wish to know the quantity

$$p_{L,k} = \sup_{\psi,r} \frac{\text{Vol } \psi\left(\coprod_{i=1}^{k} B_i(r)\right)}{\text{Vol } M},$$

where $\psi : \coprod_{i=1}^{k} B_i(r) \hookrightarrow \mathbb{C}P^2$ is a symplectic embedding such that the preimage $\psi^{-1}(L) = \coprod_{i=1}^{k} B_{i,\mathbb{R}}(r)$. We first treat the case $(\mathbb{C}P^2, \mathbb{R}P^2, \phi)$ with the canonical real structure ϕ, where $\mathbb{R}P^n = \text{Fix}(\phi)$. In particular, we will prove Theorem 1.26, restated below.

Theorem (Theorem 1.26)**.** *For the pair $(\mathbb{C}P^2, \mathbb{R}P^2)$ with the standard symplectic form and real structure, the relative packing numbers $p_{\mathbb{R}P^2,k}$ for $k \leq 8$ balls are equal to the packing numbers for $\mathbb{C}P^2$.*

The packing numbers are given in Table 4.1 below.

k	1	2	3	4	5	6	7	8
$p_{\mathbb{R}P^2,k}$	1	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{4}{5}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$

Table 4.1: $p_{\mathbb{R}P^2,k}(\mathbb{C}P^2) = p_k(\mathbb{C}P^2)$

The following proposition is an adaptation of Proposition 2.1.C in McDuff and Polterovich [23] to real symplectic manifolds.

Proposition 4.1. *Let (M, ω, ϕ) be a real symplectic manifold, and let J be an ω-tame almost complex structure which is symmetrically integrable around a set of k points $I = \{p_1, \ldots, p_k\} \subset L$, where $L = Fix(\phi)$. Suppose that for some set of real numbers $\kappa_q > 0$, $q \in \{1, \ldots, k\}$, there exists a real symplectic and holomorphic embedding*

$$\psi = \coprod_{q=1}^{k} \psi_q : \coprod (B(1+2\epsilon_q), B_{\mathbb{R}}(1+2\epsilon), \kappa_q^2 \omega_0, i, c) \to (M, L, \omega, J, \phi)$$

such that $\psi_q(0) = p_q$. Let $\Pi : \tilde{M} \to M$ denote the real symplectic blow-up of (M, L) relative to ψ, and let $\tilde{J}$, $\tilde{\omega}$, and $\tilde{\phi}$ be the complex, symplectic, and real structures, respectively, on $\tilde{M}$ constructed from J, ω, and ϕ by blowing-up M. Let $C_q, q \in \{1, ..., k\}$ denote the exceptional curves $\Pi^{-1}(\psi_q(0))$ added in the blow-up, and let $e_q \in H^2(M; \mathbb{Z})$ denote the Poincaré duals of the homology classes $[C_q] \in H_2(M; \mathbb{Z})$.

Suppose, furthermore, that there exists a smooth family of symplectic forms $\tilde{\omega}_t$ on $\tilde{M}$ such that

1. *$\tilde{\omega}_0 = \tilde{\omega}$ is obtained by a real blow up relative to the embedding ψ.*

2. *$\tilde{\omega}_0$ tames $\tilde{J}$,*

3. *For all $q \in \{1, ..., k\}$, $\tilde{\omega}_t|_{C_q}$, the restriction of $\tilde{\omega}$ to the exceptional divisors $\{C_q\}_{q=1}^k$ added in the blow-up, tames $\tilde{J}|_{C_q}$,*

4. *$\phi^* \tilde{\omega}_t = -\tilde{\omega}_t$, so that $\tilde{L} = \Pi^{-1}(L)$ is Lagrangian for each of the forms $\tilde{\omega}_t$, and*

5. *$[\tilde{\omega}_t] = [\Pi^* \omega] - \sum_{i=1}^{k} \lambda_i^2(t) e_q$ for positive constants $\lambda_q(t)$, $0 \leq t \leq 1$.*

Then (M, L, ω, ϕ) admits a real symplectic embedding of k disjoint standard symplectic balls of radii $\lambda_q(1)$, $q \in \{1, ..., k\}$.

Proof. Since $\tilde{M}$ is the real symplectic and holomorphic blow-up at k real points of (M, L, J, ϕ, ω), then, according to the construction in the proof of Proposition 2.10, there exists a real symplectic and holomorphic embedding

$$\tilde{\psi} = \coprod_{q=1}^{k} \tilde{\psi}_q : \coprod (\mathcal{L}(1+2\epsilon_q), \mathcal{R}(1+2\epsilon), \rho(1, \kappa_q), i, \tilde{c}) \to (\tilde{M}_k, \tilde{L}, \tilde{\omega}_0, J, \tilde{\phi})$$

We will show that for each q there exists a family of equivariant diffeomorphisms $g_t : \tilde{M} \to \tilde{M}, t \in [0,1]$ with the following properties:

1. $g_0 = Id$
2. There exists a $\delta \in \mathbb{R}$, $0 < \delta < 1 + 2\epsilon$, such that, for all t, $\tilde{\psi}_q^* g_t^* \tilde{\omega}_t = \rho(1, \lambda_q(t))$ on $\mathcal{L}(\delta)$
3. $g_t \circ \tilde{\phi} = \tilde{\phi} \circ g_t$, $g_t(Im(\tilde{\psi})) = Im(\tilde{\psi})$, and $g_t(\tilde{\psi}_q(\mathcal{L}(0)) = \tilde{\psi}_q(\mathcal{L}(0))$.

To see this, first note that the $\lambda_i(t)$ satisfy the equation $\int_{\mathcal{L}(0)} \tilde{\psi}_q^* \tilde{\omega}_t = \lambda_i(t)^2 \int_{\mathcal{L}(0)} \sigma = \lambda_i(t)^2$, so $\tilde{\psi}_q^* \tilde{\omega}_t$ is in the same cohomology class on $\mathcal{L}(0)$ as $\rho(1, \lambda_q(t))$. Then since both of these forms tame $\tilde{i}$ on $\mathcal{L}(0)$, the forms $s\rho(1, \lambda_q(t)) + (1-s)\tilde{\psi}_q^* \tilde{\omega}_t$ are non-degenerate for all $s \in [0,1]$. Therefore, by Proposition 2.31, for each t, there exists an equivariant symplectomorphism $F_{q,t} : (\mathcal{L}(0), \rho(1, \lambda(t))) \to (\mathcal{L}(0), \tilde{\psi}^* \tilde{\omega}_t)$ such that $\tilde{c} \circ F_{q,t} = F_{q,t} \circ \tilde{c}$ and $F_{q,t}^* \tilde{\psi}^* \tilde{\omega}_t = \rho(1, \lambda_q(t))$ on $\mathcal{L}(0)$. Since $\tilde{\omega}_t$ and $\rho(1, \lambda_q(t))$ form smooth families of forms, the $F_{q,t}$ must also be smooth with respect to t as well.

We extend the $F_{q,t}$ to an isomorphism of the normal bundle ν of $\mathcal{L}(0)$ in $\mathcal{L}(1+2\epsilon)$ by defining $f_{q,t} : \nu \to \nu$ by $f_{q,t}(z, v) = (F_{q,t}(z), v)$. Since the restriction of both $\rho(1, \lambda(t))$ and $\rho(1, \kappa_q) = \tilde{\psi}_q^* \tilde{\omega}$ to the fiber ν_z is ω_0, this isomorphism is both equivariant and symplectic. Then, by Theorem 2.28, $F_{q,t}$ extends to an equivariant symplectomorphism $G_{q,t}$

of a neighborhood $\mathcal{N}_{0,t}$ of $\mathcal{L}(0)$ in $(\mathcal{L}(1+2\epsilon), \rho(1, \lambda(t))$ to a neighborhood $\mathcal{N}_{1,t}$ of $\mathcal{L}(0)$ in $(\mathcal{L}(1+2\epsilon), \tilde{\psi}^*\tilde{\omega}_t)$. Let $\delta_q \in \mathbb{R}$, $0 < \delta_q < 1+2\epsilon$ be such that $\mathcal{L}(\delta_q) \subset \mathcal{N}_{0,t}$ and for all $t \in [0,1]$. Note now that the $G_{q,t}|_{\mathcal{L}(\delta_q)}$ also form a smooth family of maps with respect to t. Extend $G_{q,t}$ to a smooth family of equivariant differentiable maps from $\mathcal{L}(1+2\epsilon) \to \mathcal{L}(1+2\epsilon)$ which is the identity in a neighborhood of the boundary.

Define $g_{q,t} = \tilde{\psi}_q \circ G_{q,t} \circ \tilde{\psi}^{-1}$, extend the $g_{q,t}$ to all of $\tilde{M}$ by the identity outside $\tilde{\psi}\left(\coprod_{q=1}^k \mathcal{L}(1+2\epsilon)\right)$, and denote the extension by g_t. Then $\tilde{\psi}^* g_t^* \tilde{\omega}_t = \rho(1, \lambda_q(t))$ on $\mathcal{L}(\delta_q)$, making $\tilde{\psi}$ a symplectomorphism with respect to the forms $g_t^*\tilde{\omega}$ for all t.

Now let $\delta = min\{\delta_q\}_{q=1}^k$, and let (M, ω_t) be the blow-down of $(\tilde{M}, g_t^*\tilde{\omega}_t)$ using the symplectic and holomorphic embedding $\tilde{\psi}|_{\coprod_{q=1}^k \mathcal{L}_q(\delta)}$. Note that by Theorem 1.22, each form of the family ω_t is cohomologous to ω_0. Also, ω_0 tames J and $[\omega_0] = [\omega]$, and therefore all the forms ω_t and $s\omega_0 + (1-s)\omega$, $t, s \in [0,1]$, are symplectic and in the same cohomology class. Furthermore, note that $\frac{d}{dt}\omega_t$ is supported on a finite union of balls, and is therefore exact. Therefore, by Proposition 2.31 and Lemma 2.27, there exists a family of equivariant diffeomorphisms $H_r : M \to M$, $r \in [0,1]$, such that $H_0 = Id$ and $H_1^*\omega = \omega_1$. Since (M, ω_1) admits a real symplectic embedding of $\coprod_{q=1}^k (B(1+2\epsilon), \lambda_q\omega_{st})$, where ω_{st} here is the standard symplectic form on $B(1+2\epsilon)$, this completes the proof. □

The following corollary is an easy consequence.

Corollary 4.2. *Let (M, ω, ϕ) be a real symplectic manifold with almost complex structure J which tames ω and is symmetrically integrable around the points $\{p_1, \ldots, p_k\}$. Let $(\tilde{M}, \tilde{\omega}_0, \tilde{\phi})$ be a real manifold obtained by blowing up a real symplectic and holomorphic embedding ψ of balls of radii $\kappa > 0$, κ small, and let $\tilde{J}$ be the almost complex* *structure created in the blow-up.*

Now suppose that there exists a symplectic form $\tilde{\omega}$ on $\tilde{M}$ such that $\tilde{\omega}$ tames the almost complex structure $\tilde{J}$ on $\tilde{M}$. Suppose furthermore that

$$[\tilde{\omega}] = [\Pi^*\omega] - \sum_{i=1}^{k} \pi\lambda_i^2 e_i.$$

Then (M, L, ω) admits a relative symplectic embedding of k disjoint standard symplectic balls of radii $\lambda_1, ..., \lambda_k$.

Proof. By Proposition 2.10, the blow-up $\tilde{\omega}_0$ relative to ψ tames $\tilde{J}$, and therefore the forms $\omega_s := \tilde{\omega}_0 + (1-s)\tilde{\omega}$ tame $\tilde{J}$ as well, so the family of forms ω_s satisfies the hypothesis of Proposition 4.1. The conclusion follows. □

We now use all of the above facts to derive relative packing inequalities for $(\mathbb{C}P^2, \mathbb{R}P^2)$. Let $\tilde{M}_I$ denote a complex surface obtained from $\mathbb{C}P^2$ by blowing-up at a ϕ-invariant set of k points. Let $\{A, E_1, ..., E_k\}$ be the standard basis in $H_2(\tilde{M}_I;\mathbb{Z})$, and let $\{a, e_1, ..., e_k\}$ be the Poincaré-dual basis in $H^2(\tilde{M}_I;\mathbb{R})$.

Remark 4.3. Since a relative packing is also an absolute packing, the following upper bound on the packing numbers follows immediately from the absolute case in McDuff and Polterovich [23].

Suppose that (M, ω, L) admits a symplectic packing by k standard balls of radii $\lambda_1, ..., \lambda_k$. Then for every exceptional class $bA - \sum_{q=1}^{k} m_q E_q$ we have the inequality $\sum_{q=1}^{k} m_q \lambda_q^2 < b$.

We will now investigate lower bounds for the relative packing numbers of $(\mathbb{C}P^2, \mathbb{R}P^2)$. In particular, we prove the following.

Theorem 4.4. *Let $\tilde{M}_I$ be the real blow up at a ϕ-invariant set of k points of $(\mathbb{C}P^2, \mathbb{R}P^2)$. Let $k \leq 8$, and suppose the real numbers $\{\lambda_1, ..., \lambda_k\}$ have the properties that*

1. $\sum_{q=1}^{k} \lambda_q^4 < 1$, *and*
2. *For every homology class $[C] = bA - \sum_{q=1}^{k} m_q E_q$ which admits a real rational exceptional holomorphic curve,*

$$\sum_{q=1}^{k} m_q \lambda_q^2 < b.$$

Then $(\mathbb{C}P^2, \mathbb{R}P^2)$ admits a real packing by k balls of radii $\lambda_1, ..., \lambda_k$.

Given this theorem, the main theorem of this section follows easily, as we see here.

Proof of Theorem 1.26. Any real numbers $\lambda_1, ..., \lambda_k$ that satisfy the conditions in Theorem 4.4 also satisfy the conditions in Theorem 1.3.E of [23] and vice versa, since the conditions on the λ_i are identical in both theorems. Therefore, the packing numbers are the same in the real and absolute cases, as claimed. □

In order to prove Theorem 4.4, we will appeal to the form of the Nakai-Moishezon criterion found in Friedman and Morgan [12] (Chapter II, Proposition 3.4). In order to be able to use this result, we need the following definitions and proposition.

Definition 4.5. We call a simply connected algebraic variety X *good* if for some smooth elliptic curve F, the divisor classes K_X and F satisfy $K_X = -F$. We call X *generic* if, in addition, it does not contain a holomorphic sphere C with $C \cdot C = -2$.

We now give the following definition of general position, following Demazure [10].

Definition 4.6. Given a manifold M and a collection of points $\Sigma = \{z_1, ..., z_k\}$, $1 \leq k \leq 8$, we say Σ is in *general position* if $k < 3$ or $k \geq 3$ and there does not exist

1. A holomorphic line C which passes through any three points of Σ,

2. A conic which passes through any six points of Σ, or

3. A cubic which passes through any seven points of Σ with a double point at the eighth.

The following result now follows directly from Theorem 1 in Demazure [10] and the proof of Lemma 2.6 in [12], Chapter 1:

Theorem 4.7. *The blow up of $\mathbb{C}P^2$ at the set of points $\Sigma = \{x_1, ..., x_k\}$ is good and generic iff $|\Sigma| \leq 8$ and Σ is in general position.*

The next proposition shows that there are 'many' such good sets of points in $\mathbb{R}P^2$.

Proposition 4.8. *For each $k \leq 8$, the set of collections Σ of k distinct points in general position in $\mathbb{R}P^2$ is dense in the set of collections of k points in $\mathbb{R}P^2$.*

To prove this, we will use the following lemma.

Lemma 4.9. *Let Σ be a set of $k \leq 8$ distinct points of $\mathbb{R}P^2$ which contains a subset of $k-1$ points in general position. Then there is a sequence Σ_i of k points in general position which approaches Σ.*

We begin with a definition,

Definition 4.10. We define the *quadratic transformation centered at* $\{[1,0,0],[0,1,0],[0,0,1]\} \subset \mathbb{C}P^2$ to be the birational transformation given by the function $f(x_0, x_1, x_2) = (x_1x_2, x_0x_2, x_0x_1)$

for $(x_0, x_1, x_2) \in \mathbb{C}P^2$ with no two coordinates equal to 0. A *quadratic transformation centered at* $P_1, P_2, P_3 \in \mathbb{C}P^2$ to be the composition of the function f with a projective change of coordinates taking three non-collinear points P_1, P_2, P_3 to $\{[1,0,0],[0,1,0],[0,0,1]\}$.

Proof of Lemma 4.9. First we note that the projective change of coordinates that sends $P_1, P_2, P_3 \in \mathbb{R}P^2$ to $[1,0,0],[0,1,0],[0,0,1] \in \mathbb{R}P^2$ leaves the Lagrangian $\mathbb{R}P^2$ invariant. Next, from the formula in the definition, we see that the quadratic transformation centered at $[1,0,0],[0,1,0],[0,0,1]$ also sends $\mathbb{R}P^2$ to itself, and therefore, quadratic transformations centered at points P_1, P_2, P_3 in $\mathbb{R}P^2$ preserve $\mathbb{R}P^2$. Next, from Exercises 4.3 and 4.13(a) in Hartshorne [15], a set of points Σ is in general position iff, after any finite sequence of quadratic transformations centered at points in Σ, no three points in Σ are on the same line.

By hypothesis, Σ contains a subset Σ_0 of $k-1$ points in general position. Let $\{x\}$ denote $\Sigma \backslash \Sigma_0$. For each $n \in \mathbb{N}$, there are a finite number, say N, of sequences of n quadratic transformations centered at points in Σ_0. Consider now a particular sequence of n quadratic transformations, call it α. Let C_α denote the union of complex lines which pass through $\alpha(\Sigma_0)$. Then $\mathbb{C}P^2 \backslash C_\alpha$ is open and dense in $\mathbb{C}P^2$, and, since $\mathbb{R}P^2$ is totally real and each curve in C_α is holomorphic, $C_\alpha \cap \mathbb{R}P^2$ contains no open set of $\mathbb{R}P^2$, and therefore its complement, say L_α, is open and dense in $\mathbb{R}P^2$. Since α is an isomorphism on the complement of C_α and α_n preserves $\mathbb{R}P^2$, the inverse image $\alpha^{-1}(L_\alpha)$ is open and dense in $\mathbb{R}P^2$. Now let $\{\alpha_i\}_{i=1}^N$ be the collection of sequences of n quadratic transformations. Then $\alpha_i^{-1}(L_{\alpha_i})$ is open and dense for every i, and therefore $\Delta_n := \cap_{i=1}^N \alpha_i^{-1}(L_{\alpha_i})$ is open and dense in $\mathbb{R}P^2$. It follows that the intersection $L := \cap_{n=1}^\infty \Delta_n$ is dense in $\mathbb{R}P^2$. We may therefore choose a sequence of points $\{x_i\}_{i=1}^\infty \in L$ which approaches $\{x\}$. By construction, for each i there is no finite sequence of quadratic transformations centered at any three points in $\Sigma_0 \cup \{x_i\}$ which sends three points of $\Sigma_0 \cup \{x_i\}$ to a line. Therefore the sequence of collections of points $\Sigma_0 \cup \{x_i\}$ satisfies the conclusions of the theorem. $\square$

Proof of Proposition 4.8. Let $k \in \{1, \ldots, 8\}$. We consider a set Σ of k distinct points in $\mathbb{R}P^2$ which is not in general position, and we will construct a sequence Σ_i of collections of k distinct points of $\mathbb{R}P^2$ in general position which approaches Σ. First, choose a subset of Σ which is in general position, and call it Δ. By definition of general position, Δ must contain at least 2 points. Now choose a point $x \in \Sigma$ such that $\Delta \cup \{x\}$ is no longer in general position. Then, by 4.9, there is a sequence of sets, say Δ_{i_1}, which approaches $\Delta \cup \{x\}$. If $\Delta \cup \{x\} = \Sigma$, then we are done. If not, choose another point $\{x_2\} \in \Sigma$. Then, for each Δ_{i_1}, $i_1 \in \mathbb{N}$, Lemma 4.9 gives us a sequence $\Delta_{(i_1,i_2)}$ of collections of points in general position which approach $\Delta_{i_1} \cup \{x_2\}$. Note that, by construction, the sequence $\Delta_{(i,i)}, i \in \mathbb{N}$ approaches $\Delta \cup \{x\} \cup \{x_2\}$, and each collection $\Delta_{(i,i)}$ is in general position. Abusing notation, we refer to this new sequence $\Delta_{(i,i)}$ as Δ_i. Continuing in this way until $\Delta \cup \{x\} \cup ...\{x_m\} = \Sigma$, we see that the collections of $1 \leq k \leq 8$ points of $\mathbb{R}P^2$ in general position are dense in the set of all collections. $\square$

Theorem 4.11. *Let M be a good, generic complex surface. Let $\rho \in H^2(M;\mathbb{R})$ be a cohomology class on the complex surface M, and suppose M is endowed with a complex real structure ϕ (i.e. $\phi_* i = -i\phi_*$ for the standard almost complex structure i). Then ρ is represented by a Kähler form κ with real structure ϕ if $\phi^*\rho = -\rho$, $\rho^2 > 0$, and $\langle \rho, [C] \rangle > 0$ for all complex exceptional curves and on the anti-canonical divisor.*

Proof. By Theorem 3.4, Chapter II in Friedman and Morgan [12], ρ is represented by a Kähler form iff $\rho^2 > 0$ and $\langle \rho, [C] \rangle > 0$ for all complex exceptional curves and on the anti-canonical divisor. Let $\hat{\kappa}$ be such a Kähler form. By Lemma 2.4, we have that $\hat{\kappa}$ is compatible with i, and therefore by Lemma 2.23, $\kappa = \frac{1}{2}(\hat{\kappa} - \phi^*\hat{\kappa})$ is also a symplectic form which is compatible with i, and is therefore Kähler. Since $[\kappa] = [\hat{\kappa}] = \rho$, κ is our desired form. □

We now give the proof of Theorem 4.4.

Proof of 4.4. By Proposition 4.8, given k generic points of L for $k \leq 8$, if $\tilde{M} = \tilde{M}_I$ is the blow-up of $(\mathbb{C}P^2, \mathbb{R}P^2)$ at these points, then $\tilde{M}_I$ is good and generic. By Theorem 4.11, if ρ is a cohomology class on a complex surface M, then it is represented by a Kähler form compatible with the real structure ϕ iff $\phi^*\rho = -\rho$, $\rho^2 > 0$, and $\langle \rho, [C] \rangle > 0$ for all complex exceptional curves C and on the anticanonical divisor.

Therefore, given a cohomology class $\rho = \alpha - \sum \lambda_i^2 e_i \in H^2(M; \mathbb{R})$, where $\phi^*\alpha = -\alpha$, $\phi^* e_i = -e_i$ we wish to show that $\rho(C) > 0$ on exceptional divisors and on the anti-canonical divisor. In particular, we have to verify the following:

1. $\langle \rho, [C] \rangle > 0$ for every rational exceptional curve C on $\tilde{M}_I$

2. $\rho \cdot \rho > 0$

3. $\rho \cdot c_1 > 0$, where $c_1 = 3a - \sum_{q=1}^{k} e_q$ is the first Chern class of $\tilde{M}_I$.

The first two inequalities are just reformulations of the hypotheses of the theorem. To see the third, note first that the maximum of the function $f(x_1, ..., x_k) = (\sum x_i)^2$ on the region $\{x | \sum x_i^2 \leq 1\}$ is obtained when the x_i's are all equal. Let λ_{max} be this value of x. Therefore

$$\begin{aligned}\left(\sum_{i=1}^{k}\lambda_i^2\right)^2 &\leq \left(\sum_{i=1}^{k}\lambda_{max}^2\right)^2\\ &= (k\lambda_{max}^2)^2 = k\sum_{i=1}^{k}\lambda_{max}^4 \leq k\cdot 1,\end{aligned}$$

where the last inequality follows from the assumption that $\rho\cdot\rho = 1-\sum\lambda_i^4 > 0$. We now remark that

$$\rho\cdot c_1 = 3-\sum_{i=1}^{k}\lambda_i^2 > 0 \iff 9 > \left(\sum_{i=1}^{k}\lambda_i^2\right)^2.$$

Therefore, since $9 > k \geq (\sum_{i=1}^{k}\lambda_i^2)^2$, the third inequality is satisfied. Furthermore, when the e_i are the Poincaré duals of E_i, the classes of a real blow up, then $\phi^*e_i = -e_i$, and we have $\phi^*\rho = \phi^*\alpha - \sum_i\lambda_i^2\phi^*e_i = -\rho$. Therefore ρ is represented by a Kähler form with real structure ϕ, as desired.

We have the packing by balls of radii $\lambda_1, ..., \lambda_k$ by Corollary 4.2, blowing down the curves which represent the classes E_i with respect to this new Kähler form ρ. □

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