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RISK AVERSION, INTERTEMPORAL SUBSTITUTION, AND OPTION PRICING

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RÉSUMÉ

Dans le présent article, on propose un cadre stochastique général et un modèle d'évaluation d'actifs financiers à l'équilibre qui mettent en évidence les rôles respectifs de l'élasticité de substitution intertemporelle et de l'aversion pour le risque dans le prix de marché des options. Nous précisons en particulier les conditions statistiques sous lesquelles les formules d'évaluation d'options dépendent ou non explicitement des paramètres de préférence, en particulier quand ces paramètres ne sont pas cachés dans les prix de l'actif sous-jacent et d'une obligation, comme c'est le cas dans les modèles standards de Black et Scholes (BS) ou de Hull et White (HW). Plusieurs effets de causalité instantanée, du type effet de levier, expliquent l'occurrence non redondante des paramètres de préférence dans les prix d'options. On prouve aussi que les modèles d'évaluation d'actifs financiers les plus classiques (CAPM pour les actions, BS ou HW où les prix d'options ne font pas apparaître les paramètres de préférence) sont fondés sur les mêmes hypothèses stochastiques (typiquement l'absence d'effet de levier), indépendamment des valeurs des paramètres de préférence. Même si notre formule générale d'évaluation d'options dépend dans certains cas explicitement des paramètres de préférence, on n'oublie pas que la formule BS est dominante à la fois comme modèle théorique de référence et comme instrument de gestion. Une autre contribution de l'article est la validation théorique de ce rôle de référence. Ainsi, dans la mesure où on accepte une propriété essentielle des prix d'options, à savoir leur homogénéité de degré un par rapport au couple formé par le prix de l'actif sous-jacent et le prix d'exercice, on peut montrer que les hypothèses statistiques nécessaires et suffisantes pour l'homogénéité donnent à l'équilibre des prix d'options qui conservent l'essentiel de la forme fonctionnelle de BS. Cette forme fonctionnelle nous permet de mettre en évidence certaines propriétés importantes du « sourire » de volatilité, c'est-à-dire de la représentation graphique des volatilités implicites de BS en fonction de la position de l'option par rapport à la monnaie. On montre d'abord que l'asymétrie de ce sourire est équivalente à une forme particulière d'asymétrie de la mesure de martingale équivalente. Enfin, cette asymétrie correspond précisément au cas où il existerait soit une prime sur un risque instantané de taux d'intérêt, soit un effet de levier généralisé, soit les deux, en d'autres termes lorsque la formule d'évaluation d'options dépend explicitement des paramètres de préférence. En conclusion, le message principal pour la gestion d'options résultant de notre analyse est que l'évidence d'une asymétrie dans le sourire de volatilité signale l'importance de la prise en compte des paramètres de préférence dans les formules d'évaluation d'options.

Mots clés : causalité, chaînes de Markov cachées, utilité non séparable, évaluation d'options par modèle d'équilibre, utilité récursive, volatilité implicite de Black-Scholes, sourire de volatilité

ABSTRACT

This paper develops a general stochastic framework and an equilibrium asset pricing model that make clear how attitudes towards intertemporal substitution and risk matter for option pricing. In particular, we show under which statistical conditions option pricing formulas are not preference-free, in other words, when preferences are not hidden in the stock and bond prices as they are in the standard Black and Scholes (BS) or Hull and White (HW) pricing formulas. The dependence of option prices on preference parameters comes from several instantaneous causality effects such as the so-called leverage effect. We also emphasize that the most standard asset pricing models (CAPM for the stock and BS or HW preference-free option pricing) are valid under the same stochastic setting (typically the absence of leverage effect), regardless of preference parameter values. Even though we propose a general non-preference-free option pricing formula, we always keep in mind that the BS formula is dominant both as a theoretical reference model and as a tool for practitioners. Another contribution of the paper is to characterize why the BS formula is such a benchmark. We show that, as soon as we are ready to accept a basic property of option prices, namely their homogeneity of degree one with respect to the pair formed by the underlying stock price and the strike price, the necessary statistical hypotheses for homogeneity provide BS-shaped option prices in equilibrium. This BS-shaped option-pricing formula allows us to derive interesting characterizations of the volatility smile, that is, the pattern of BS implicit volatilities as a function of the option moneyness. First, the asymmetry of the smile is shown to be equivalent to a particular form of asymmetry of the equivalent martingale measure. Second, this asymmetry appears precisely when there is either a premium on an instantaneous interest rate risk or on a generalized leverage effect or both, in other words, whenever the option pricing formula is not preference-free. Therefore, the main conclusion of our analysis for practitioners should be that an asymmetric smile is indicative of the relevance of preference parameters to price options.

Key words : causality, hidden Markov chains, non-separable utility, equilibrium option pricing, recursive utility, Black-Scholes implicit volatility, smile effect

1 Introduction

Since two fundamental characteristics of an option are its maturity date and the underlying asset on which it is written, the price of such a security will naturally be affected by the value of time as well as the price of risk associated with the asset in question. Therefore, it is not surprising that in a complete market setting, such a security can be duplicated by a portfolio of bonds and stocks. In general however, when options are not redundant securities, the respective roles of time and risk in its valuation are less obvious. A main contribution of this paper is to provide a general stochastic framework and an equilibrium asset pricing model that make clear how attitudes towards intertemporal substitution and risk matter for option pricing. In particular, we show under which statistical conditions option pricing formulas are not preference-free, in other words when preferences are not hidden in stock and bond prices as they are in option pricing formulas such as the Black and Scholes (1973) formula (hereafter BS formula). Moreover, thanks to a recursive utility framework (Epstein and Zin [1989]), we succeed in disentangling the respective roles of discounting, risk aversion and intertemporal substitution in the option pricing formula.

The dependence of option prices on preference parameters comes from two main effects. First, while it is commonly known that forward interest rates are not just expected values of future spot rates, due to a time-varying risk premium, we stress that this premium also enters in the option price in such a way that preference parameters are not fully hidden in the market price of long-term bonds. This effect is due to an instantaneous causality relationship between aggregate consumption and state variables which enter into the interest rate risk. It is worth noting that preferences enter into this premium not only through discounting and risk aversion, but also through the elasticity of intertemporal substitution. Second, preferences also matter for option pricing because of a generalized leverage effect, that is not only the instantaneous causality relationship between state variables which enter into the stochastic volatility process of the stock price and the stock price process itself but also a stochastic correlation between the stock returns and the market portfolio returns. In our framework, this effect can be separated from the previous one. While the instantaneous interest rate risk premium involves all preference characteristics (discounting, risk aversion and intertemporal substitution), the risk premium related to the leverage effect only involves the risk aversion parameter. Additionally, this effect is purely due to risk (in the spirit of CAPM) and vanishes if the stock has a zero beta with respect to the market. In this last case, the volatility risk is perhaps compensated but the compensation does not imply an additional role for preference parameters in option prices. On the other hand, a beta stock pricing conformable to a standard CAPM is obtained as soon as the above instantaneous causality or stochastic correlation effects disappear. This leads us to emphasize that the most standard asset pricing models (CAPM for the stock, preference-free option pricing such as in Black and Scholes (1973) or Hull and White (1987) models) are valid under the same stochastic setting regardless of the parameter values of the utility function. This provides a statistical foundation to CAPM pricing in a recursive utility framework regardless of particular preference configurations. In Epstein and Zin (1991), CAPM pricing was obtained only with a logarithmic utility or an infinite elasticity of intertemporal substitution.

The stochastic framework we consider is not chosen for theoretical convenience but justified from a practical point of view. Indeed, we always keep in mind that the BS formula is dominant both as a theoretical reference model and as a tool for practitioners for pricing and hedging European options. Another contribution of the paper is to characterize why the BS formula is such a benchmark. We show that, as soon as we are ready to accept a basic property of option prices, namely their homogeneity of degree one with respect to the pair formed by the underlying stock price and the strike price (see Merton (1973)), we obtain an option pricing formula that keeps the main functional shape of the usual BS formula. This robustness of the BS formula is ensured via homogeneity by our stochastic framework and equilibrium asset pricing model taken together. We show that the homogeneity requirement implies the stochastic framework which in turn provides BS-shaped option prices. We therefore provide a rationalization of the vast literature that enriches the BS model to improve its usefulness for practitioners.

Since implicit volatility, the volatility that equates the BS option valuation formula to the observed option price, has become the standard method of quoting option prices[1] and a risk management tool, many empirical studies have investigated the properties of such implicit volatilities. For example, some studies have addressed

[1]See Bates (1996).

the effect of time-to-maturity or strike price on BS implicit volatilities, the so-called smile effect and its increasing amplitude when time to maturity decreases[2]. Therefore, the volatility smile appears as a useful characterization of option pricing and hedging biases for practical applications. It has been proven by Renault and Touzi (1996) that the standard Hull and White (1987) stochastic volatility model offers a rationalization of a symmetric smile[3]. We extend this result by providing the first theoretical characterization of asymmetric smiles, which are often observed in practice. First, the asymmetry of the smile is shown to be equivalent to a particular form of asymmetry of the pricing density. Second, this asymmetry appears precisely when there is either a premium on interest rate risk or on a generalized leverage effect or both, in other words whenever the option pricing formula is not preference-free. Therefore, the main conclusion of our analysis for practitioners should be that an asymmetric smile is indicative of the relevance of preference parameters to price options. Conversely, standard preference-free option pricing and CAPM-like stock pricing are allowed whenever symmetric smiles are produced.

Our approach is in contrast with purely descriptive nonparametric statistical techniques involving either functional estimation or implied binomial trees which can fit any shape of the smile[4]. Contrary to our model, these techniques imply a deterministic relationship between the instantaneous volatility of the stock price and its level, which means that they are not compatible with a homogeneous option pricing formula. By our statistical and equilibrium model assumptions, we are able to reproduce asymmetric smiles with a BS-type homogeneous option price.

For practical applications in terms of option pricing and hedging, the fact that the general functional shape of the BS formula is mainly preserved as long as homogeneity

[2]See, for example, Day and Lewis (1992), Engle and Mustafa (1992), and Jorion (1995). However, as explained clearly in Melino (1994), a shortcoming of the implicit estimation methodology is its internal inconsistency since it produces estimates of volatility that can vary considerably from day to day while the variance is originally assumed constant.

[3]A symmetric smile is obtained as soon as the option price can be characterized as an expectation of a BS formula with respect to an heterogeneity factor. In particular, this is the case in Merton's (1976) model where the underlying stock returns contain along with the usual Brownian process a jump process. The option pricing formula keeps "most of the attractive features of the original BS formula in that it does not depend on investors' preferences or knowledge of the expected return on the underlying stock" (see Renault (1996) for a survey).

[4]See Gouriéroux, Monfort and Tenreiro (1994), Aït-Sahalia, Bickel and Stocker (1994), Bossaerts and Hillion (1995) for functional estimation and Dupire (1994), Rubinstein (1994), for implied binomial trees.

is maintained validates the usual practice of using the BS formula and its Hull and White extension as a benchmark, even though the presence of various kinds of leverage effects makes preference-free option pricing strictly invalid. Our generalized option pricing formula offers a variety of directions in which the BS formula can be misspecified and which could be of interest for practitioners. For example, the computation of implicit preference parameters, irrespective of their theoretical interpretation, should cause little more inconvenience than estimating an implicit volatility and may prove as useful to correctly appraise biases in preference-free option pricing. Moreover, as far as preference-free option pricing and associated symmetric volatility smiles are maintained hypotheses, our results provide a theoretical support to some extensions of the BS formula which replace the standard normal cumulative distribution function by alternative distribution functions (including asymmetric ones).

Rubinstein (1976) and Brennan (1979) use a consumption-based representative agent framework to price options. Amin and Ng (1993) extend this framework to a joint process for consumption growth and stock returns which captures both interest rate and volatility risks. As special cases of our general option pricing formula, we obtain the formula derived by Amin and Ng (1993) and a fortiori all the other pricing formulas that were nested in the latter[5]: of course the BS formula, but also the Hull-White (1987) and Bailey-Stulz (1989) stochastic volatility option pricing formulas and the Merton (1973), Turnbull-Milne (1991), and Amin-Jarrow (1992) stochastic interest rate option pricing formulas for equity options.

Two papers have used preferences that disentangle risk aversion from intertemporal substitution in the context of option pricing. Detemple (1990) uses the ordinal certainty equivalence hypothesis proposed by Selden (1978) in a two-period economy and shows that time preferences play a distinctive and significant role in pricing options. For example, option prices change with the expected return on the stock and may decrease when the risk of the stock return increases. Ma (1993) extends the stochastic differential utility concept in Duffie and Epstein (1992) to a mixed Poisson-Brownian information structure and derives a closed-form pricing formula

[5] We adopt a more structural approach than in Amin and Ng (1993) since we specify the dynamics of economic fundamentals (consumption and dividends) and stock returns are therefore determined in equilibrium. On the other hand, we do not incorporate as in Amin and Ng (1993) the effect on the option price of a systematic jump in the underlying asset price process, following Merton (1976) and Naik and Lee (1990). This extension could easily be accomodated in our framework.

for European call options written on aggregate equity under Kreps-Porteus preferences.

The rest of the paper is organized as follows. In section 2, we address the key issue of homogeneity. We provide a characterization of the set of risk neutral dynamics which are consistent with homogeneity of option prices. The homogeneity property can be characterized in terms of non-causality in the Granger sense. We use this characterization in section 3 to set up our equilibrium model and derive the structural statistical framework which bears out the homogeneity of option prices in equilibrium. The corresponding pricing probability measure, that is the way to go from the data generating process to the risk neutral world is characterized in section 4; the respective role of the three preference parameters is outlined. We are then able to derive our general option pricing formula and to characterize the array of other pricing models which are nested in this general one. Section 5 provides a theoretical characterization of the symmetry property of the smile both in terms of the symmetry of the pricing density and leverage and stochastic correlation effects in equilibrium. We further provide some guidelines for a practical use of this option pricing model. In particular, we stress how homogeneity allows one to use the volatility smile to characterize option pricing biases, and how to incorporate into smile studies a calibrated value of preference parameters. Section 6 concludes with a reference to statistical evidence on the importance of option prices to disentangle in estimation risk aversion and intertemporal substitution.

2 Homogeneity of Option Prices and State Variables

The theory for pricing contingent claims in the absence of arbitrage introduces a pricing probability measure Q_t under which the price π_t at time t of any contingent claim is the discounted expectation of its terminal payoff. In the case of a European call option maturing at time T with a strike price K, it is given by:

$$\pi_t = B(t,T)E_t^*(S_T - K)^+, \tag{1}$$

where E_t^* denotes the expectation operator with respect to Q_t, $B(t,T)$ is the price at time t of a pure discount bond maturing at T, and S_T is the price of the underlying asset (stock) at T. Of course, Q_t is generally different from the data generating process P_t of $\{S_t\}$. Existence and unicity of Q_t were studied by several authors since the seminal paper of Harrison and Kreps (1979)[6].

It is then natural to hope that the option price π_t inherits (at time t) the convexity property with respect to the underlying asset price of its terminal payoff. Indeed, since the economic function of options is fulfilled precisely because of this convexity, this led Merton (1973) to claim that "convexity is usually assumed to be a property which always holds for warrants". It appears to hold empirically (see Broadie et al., 1995), and it is also consistent with the alleged destabilizing effect of dynamic trading (portfolio insurance) strategies, since convexity means that the derivative of the option price with respect to the underlying asset price (the delta ratio) is an increasing function of this asset price. Bergman, Grundy and Wiener (1996) have recently established that whenever the underlying asset follows a diffusion whose volatility depends only on time and the concurrent stock price, then a call price is always increasing and convex in the stock price. However, when volatility is stochastic, a call price can be a decreasing concave function of the stock price over some range. To avoid such a "perverse local concavity", Merton (1973) proposes to ensure convexity through the property of homogeneity of degree one of the option price π_t with respect to the pair (S_t, K). Moreover, he noticed that homogeneity will not obtain if the distribution of returns depends on the level of the stock price. Since, as recalled in proposition 1 below, there is a fundamental bijective relationship between an option pricing function $\pi_t(.)$ and the pricing probability measure $Q_t(.)$[7], we choose to impose the homogeneity property through the pricing probability measure Q_t, as shown in proposition 2.

Proposition 1. *The pricing function $\pi_t(.)$ and the pricing probability measure $Q_t(.)$ are linked by the following bijective relationship (for a given S_t):*

$$Q_t(.) \longrightarrow \pi_t(S_t, K) = B(t,T)E_t^*[(S_T - K)^+],$$
$$\pi_t(.) \longrightarrow Q_t\left[\frac{S_T}{S_t} \geq k\right] = -\frac{1}{B(t,T)}\frac{\partial \pi_t}{\partial K}(S_t, K),$$

[6]The theory of complete markets is beyond the scope of this paper where we are only interested in the existence of a pricing probability measure Q_t which is well-defined and given to us, whether it is unique or not. This statistical viewpoint was for instance illustrated by Sims (1984), Christensen (1992), and Clément, Gouriéroux, and Monfort (1993).

[7]See Huang and Litzenberger (1988, sections 5.19 and 6.13) for foundations and Aït-Sahalia and Lo (1996) for a recent application.

where $k = \frac{K}{S_t}$.

Proof: See Appendix 1.

Of course, S_t is known at time t and the pricing probability measure Q_t describes equivalently the probability distribution of the future asset price S_T or of the return $\frac{S_T}{S_t}$. Proposition 1 shows how the pricing probability measure is characterized trough its probability distribution function by the derivative of the option price π_t with respect to the strike price K. Therefore, the previous homogeneity property can be expressed as a simple condition on the pricing probability measure.

Proposition 2. *The option pricing function* $\pi_t(.)$ *is (i) homogeneous of degree one with respect to* (S_t, K) *if and only if (ii) the pricing probability measure* Q_t *does not depend on* S_t.

Proof: See Appendix 1.

To understand proposition 2, it may help to see Q_t as the conditional probability distribution of a process of interest defined on a probability space (Ω, A, Q) given the available information I_t at time t. Whereas Merton (1973) showed that serial independence of asset returns for the data generating process is a sufficient condition for homogeneity, Proposition 2 establishes that a necessary and sufficient condition for homogeneity is the conditional independence (under Q) between future returns and the current price, given the currently available information (other than the current price).

It should be stressed that conditional independence neither implies nor is implied by marginal independence. The property defined by proposition 2 must be understood as a noncausality relationship in the Granger sense from the current price to future returns (for a given informational setting) and not as an independence property.

To see the full generality of this noncausality property, we will illustrate it in the modern finance framework where asset prices evolve as diffusion processes under the pricing probability measure Q:

$$\frac{dS_t}{S_t} = r(t)dt + \sigma(t)dW^s(t), \tag{2}$$

where $W^s(t)$ is a standard Wiener process under Q and $r(t)$ and $\sigma(t)$ are the two state variables of interest: $\sigma(t)$ is the instantaneous volatility process and r(t) can be seen as an instantaneous interest rate process, since, under Q, the risk is not compensated.[8] In this framework, we can assume without loss of generality that available information at time t is described by the $\sigma-$field:

$$I_t = \vee_{\tau \leq t}[r(\tau), \sigma(\tau), W^s(\tau)]. \tag{3}$$

In Proposition 3, we will specify which assumptions are needed to ensure the above property of Granger noncausality in order to define later a structural statistical model to price options in an equilibrium setting.

Proposition 3. *A necessary and sufficient condition for Granger noncausality from S_t to future returns $\frac{S_T}{S_t}, T > t$ is ensured by the conjunction of the two following assumptions*[9].

Assumption A1: $(\sigma(\tau), r(\tau))_{\tau > t} \perp\!\!\!\perp (dW_\tau^S)_{\tau \leq t} | r(\tau), \sigma(\tau), \tau \leq t,$

Assumption A2: $(dW_\tau^S)_{\tau > t} \perp\!\!\!\perp (dW_t^S)_{\tau \leq t} | r(.), \sigma(.),$

where r(.) and $\sigma(.)$ refer to the whole sample path of the processes r and σ.

Assumption A1 states that the price process S does not Granger cause the state variable processes r and σ (see Comte and Renault (1996) and Florens and Fougère (1996) for a precise definition of Granger noncausality in continuous time). Assumption A1 is quite natural in the context of state variables which are usually seen as being exogenous. We do not assume however a strong exogeneity property, i.e. r and σ are not necessarily independent of W^s, in order to allow for the presence of leverage effects. As a matter of fact, if r and σ were independent of W^s, A2 would be automatically satisfied.

To understand Proposition 3 in the general case, let us define: $X = (dW_\tau^s)_{\tau \leq t}, Y = (dW_\tau^s)_{\tau > t}, Z_1 = (r(\tau), \sigma(\tau))_{\tau \leq t}$ and $Z_2 = (r(\tau), \sigma(\tau))_{\tau > t}$.The required noncausality property from past prices to future returns can then be written:

[8] The variables $r(t)$ and $\sigma(t)$ are called "state variables" in a loose sense since we are not assuming here that the (r, σ) process is Markovian.

[9] As usual, the differential notation $(dW_\tau^S)_{\tau > t}$ is a slight abuse of notation to characterize the $\sigma-$field corresponding to the future increments of W_τ^S.

$$X \perp\!\!\!\perp (Y, Z_2)|Z_1. \quad (4)$$

By a well-known property of conditional independence (see e.g. Florens and Mouchart (1982)), condition (4) is equivalent to the conjunction of $X \perp\!\!\!\perp Z_2|Z_1$ (i.e. A1) and $X \perp\!\!\!\perp Y, |(Z_1, Z_2)$ (i.e. A_2). This establishes Proposition 3.

To illustrate the empirical content of these assumptions, we can characterize them in the framework of a Markovian process (S, r, σ) defined by the diffusion equations[10]:

$$\begin{array}{c} \frac{dS_t}{S_t} = r(t)dt + \sigma(t)dW^s(t), \\ dr(t) = \alpha(t)dt + \beta(t)dW^r(t), \\ d\sigma(t) = \gamma(t)dt + \delta(t)dW^\sigma(t), \end{array}$$

$$Var \begin{bmatrix} dW^s(t) \\ dW^r(t) \\ dW^\sigma(t) \end{bmatrix} = \begin{bmatrix} 1 & \rho_{sr}(t) & \rho_{s\sigma}(t) \\ \rho_{sr}(t) & 1 & \rho_{r\sigma}(t) \\ \rho_{s\sigma}(t) & \rho_{r\sigma}(t) & 1 \end{bmatrix} dt, \quad (5)$$

where $\alpha(t), \beta(t), \gamma(t), \delta(t), \rho_{sr}(t), \rho_{s\sigma}(t), \rho_{r\sigma}(t)$ are deterministic functions of S_t,r(t), $\sigma(t)$[11]. We can then establish the following proposition[12].

Proposition 4.

(i) Assumption (A1) is equivalent to the following assumption (A1)'.

(A1)' The processes $\alpha, \beta, \gamma, \delta, \rho_{r\sigma}$ are deterministic functions of the processes r and σ :

$$\begin{array}{c} \alpha(t) = \alpha[r(t), \sigma(t)], \\ \beta(t) = \beta[r(t), \sigma(t)], \\ \gamma(t) = \gamma[r(t), \sigma(t)], \\ \delta(t) = \delta[r(t), \sigma(t)], \\ \rho_{r\sigma}(t) = \rho_{r\sigma}[r(t), \sigma(t)]. \end{array}$$

(ii) If assumption (A1)' holds, Assumption (A2) is equivalent to the following assumption (A2)'.

[10]We implicitly assume that the considered system of stochastic differential equations satisfy the usual regularity conditions(Lipschitz, growth, etc.) that ensure existence and unicity of a solution.

[11]All these functions could be made dependent upon other state variables. In this case, (S, r, σ) would no longer be Markovian and should be embedded in a higher dimensional Markovian process of state variables. This generalization would not present any added difficulty.

[12]The equivalence results stated in Proposition 4 are valid under minor regularity assumptions which are not explicited here. In particular, Florens and Fougère (1996, p.1205) point out that implicitly some σ-fields are assumed "measurably separated".

(A2)' The processes ρ_{sr} and $\rho_{s\sigma}$ are deterministic functions of the processes r and σ :

$$\rho_{sr}(t) = \rho_{sr}[r(t), \sigma(t)],$$
$$\rho_{s\sigma}(t) = \rho_{s\sigma}[r(t), \sigma(t)].$$

Proof: See Appendix 1.

In other words, leverage effects ($\rho_{s\sigma} \neq 0$) and cross-correlations between the stock price and the interest rate ($\rho_{sr} \neq 0$) are allowed provided that they do not depend on the level of the stock price. More generally, propositions 2, 3 and 4 prove that a necessary and sufficient condition for the fundamental homogeneity property of option prices is that the underlying asset price process is of the "stochastic volatility " type, i.e. that it obeys the assumed noncausality relationship from the price process S to the state variables σ and r.

This characterization of homogeneity is more general than the sufficient condition proposed by Merton (1973), not only since we replace the independence requirement by a more specific noncausality assumption, but also since it is stated in terms of the pricing probability measure rather than the DGP. Indeed, we do not preclude a possible dependence of the risk premiums on the stock price S, which could violate assumption (A1)' for the DGP.

The framework of proposition 4 differs in a fundamental way from the endogenous volatility paradigm where the volatility process $\sigma(t)$ is viewed as a deterministic function of S_t. Endogenous volatility models, also called "implied tree models" by Duffie (1995), have recently gained in popularity (see Dupire (1994), Hobson and Rogers (1994), and Rubinstein (1994)). It should be emphasized that these models are tantamount to losing the fundamental homogeneity property of option prices and by the same token the independence of the Black-Scholes implicit volatility from the stock price level. One may deplore that this homogeneity requirement seems to be inconsistent with usual discrete-time statistical models like ARCH-type models. The issues of BS implicit volatility and ARCH option pricing will be discussed in more detail in sections 4 and 5 below.

Our next task is to set up an equilibrium model which will provide the foundations for the stochastic differential equations written in (5). These equations are usually justified theoretically by an absence of arbitrage argument. The no-arbitrage models need some assessment of the appropriate pricing of systematic volatility and interest rate risk. Often they assume that the risk is non-systematic and has a zero price or impose an ad-hoc functional form on the risk premium. In addition to giving equilibrium foundations to the stochastic differential equations, our equilibrium asset pricing model will price the volatility risk and the interest rate risk. Although we specify our model in a discrete-time setting[13], it should not be interpreted as a limitation to the generality of the results we will derive in terms of equilibrium foundations to the stochastic differential equations.

3 An Equilibrium Asset Pricing Model Consistent with Homogeneity

In this section, we incorporate the recursive utility model of Epstein and Zin (1989) for asset pricing into a stochastic framework dictated by the non-causality requirement as stated by proposition 3. More precisely, we specify a stochastic environment through a set of state variables which allows us to outline necessary and sufficient conditions that prices must obey in equilibrium to fulfill the desirable homogeneity property.

3.1 An Asset Pricing Model with Recursive Utility

Many identical infinitely lived agents maximize their lifetime utility and receive each period an endowment of a single nonstorable good. We specify a recursive utility function of the form:

$$U_t = W(C_t, \mu_t), \tag{6}$$

where W is an aggregator function that combines current consumption C_t with $\mu_t = \mu(\tilde{U}_{t+1} \mid I_t)$, a certainty equivalent of random future utility $\tilde{U}_{t+1}$, given the information available to the agents at time t, to obtain the current-period lifetime utility U_t. Following Kreps and Porteus (1978), Epstein and Zin (1989) propose the CES function as the aggregator function, i.e.

$$U_t = [C_t^\rho + \beta \mu_t^\rho]^{\frac{1}{\rho}}. \tag{7}$$

[13]Contrary to the arbitrage-based asset pricing models, equilibrium valuation of options does not require that hedging in continuous time is feasible.

The way the agents form the certainty equivalent of random future utility is based on their risk preferences, which are assumed to be isoelastic, i.e. $\mu_t^\alpha = E[\widetilde{U}_{t+1}^\alpha | I_t]$, where $\alpha \leq 1$ is the risk aversion parameter (1-α is the Arrow-Pratt measure of relative risk aversion). Given these preferences, the following Euler condition must be valid for any asset j if an agent maximizes his lifetime utility (see Epstein and Zin (1989)):

$$E_t[\beta^\gamma(\frac{C_{t+1}}{C_t})^{\gamma(\rho-1)} M_{t+1}^{\gamma-1} R_{j,t+1}] = 1, \tag{8}$$

where M_{t+1} represents the return on the market portfolio, $R_{j,t+1}$ the return on any asset j, and E_t the conditional expectation with respect to the information available to the agents at time t[14] and $\gamma\rho = \alpha$. The parameter ρ is associated with intertemporal substitution, since the elasticity of intertemporal substitution is $1/(1-\rho)$. The position of α with respect to ρ determines whether the agent has a preference towards early resolution of uncertainty ($\alpha < \rho$) or late resolution of uncertainty ($\alpha > \rho$)[15].

This condition allows us to price any asset in the economy. In particular, the price of a European option π_t maturing at t+1 is given by:

$$\pi_t = S_t E_t \left[\beta^\gamma(\frac{C_{t+1}}{C_t})^{\gamma(\rho-1)} M_{t+1}^{\gamma-1} Max[0, \frac{S_{t+1}}{S_t} - \frac{K}{S_t}]\right], \tag{9}$$

where K is the exercise price of the option.

This price depends on both the market portfolio return M_{t+1} and the stock return $\frac{S_{t+1}}{S_t}$. A first task is therefore to determine the equilibrium price of the market portfolio, say P_t^M at time t. In this model, the payoff of the market portfolio at time t is the total endowment of the economy C_t. Therefore the return on the market portfolio M_{t+1} can be written as follows:

$$M_{t+1} = \frac{P_{t+1}^M + C_{t+1}}{P_t^M}.$$

Replacing M_{t+1} by this expression and writing (8) for $R_{j,t+1} = M_{t+1}$, we obtain:

[14] Of course, the probability distribution considered here to define E_t is P given I_t, where P governs the data generating process of the variables of interest. In general, P is different from Q and E_t is different from E_t^* defined earlier.

[15] As mentioned in Epstein and Zin (1991), the association of risk aversion with α and intertemporal sustitution with ρ is not fully clear, since at a given level α of risk aversion, changing ρ affects not only the elasticity of intertemporal sustitution but also determines whether the agent will prefer early or late resolution of uncertainty.

$$\lambda_t^\gamma = E_t \left[\beta^\gamma \left(\frac{C_{t+1}}{C_t} \right)^{\gamma\rho} (\lambda_{t+1} + 1)^\gamma \right], \tag{10}$$

where: $\lambda_t = \frac{P_t^M}{C_t}$. Under some regularity and stationarity assumptions, we may be able to prove that (10) has a unique solution λ_t of the form $\lambda_t = \lambda(I_t)$ with $\lambda(.)$ solution of:

$$\lambda(I)^\gamma = E \left[\beta^\gamma \left(\frac{C_{t+1}}{C_t} \right)^{\gamma\rho} (\lambda(I_{t+1}) + 1)^\gamma \,|I_t = I \right]. \tag{11}$$

Typically, the pricing function $\lambda(.)$ will be determined as a fixed point of a certain operator to be defined more precisely in the next section. Similarly, we will be looking for a solution $\varphi_t = \varphi(I_t) = \frac{S_t}{D_t}$ to the stock pricing equation:

$$\varphi(I) = E \left[\beta^\gamma \left(\frac{C_{t+1}}{C_t} \right)^{\gamma\rho - 1} \left(\frac{\lambda_{t+1} + 1}{\lambda_t} \right)^{\gamma - 1} \varphi(I_{t+1}) \frac{D_{t+1}}{D_t} \,|I_t = I \right]. \tag{12}$$

Starting from (11) and (12), we are now able to look for a statistical specification which leads to return processes in equilibrium (for both market and equity) consistent with a discrete time analog of proposition 3[16].

3.2 Equilibrium-based Foundations of a Discrete-time Stochastic Volatility and Interest Rate Model

It is then possible, for given λ and φ functions, to compute the market portfolio price and the stock price as $P_t^M = \lambda(I_t) C_t$ and $S_t = \varphi(I_t) D_t$. The dynamic behavior of these prices, or equivalently of the associated rates of return:

$$Log M_{t+1} = Log \frac{\lambda(I_{t+1}) + 1}{\lambda(I_t)} + Log \frac{C_{t+1}}{C_t}, \text{and} \tag{13}$$

$$Log R_{t+1} = Log \frac{S_{t+1}}{S_t} = Log \frac{\varphi(I_{t+1})}{\varphi(I_t)} + \log \frac{D_{t+1}}{D_t}, \tag{14}$$

is determined by the joint probability distribution of the stochastic process (X_t, Y_t, I_t) where: $X_t = Log \frac{C_t}{C_{t-1}}$ and $Y_t = Log \frac{D_t}{D_{t-1}}$. We shall define this dynamics through a stationary vector-process of state variables U_t so that:

[16] It should be noted that the equivalent martingale measure of section 2 must be defined for the dividend-price pair (see Duffie (1995), p.108) and the notation S_t in section 2 as well as in the following sections implicitly denotes the gain process (capital plus dividends).

$$I_t = \vee_{\tau \leq t}[X_\tau, Y_\tau, U_\tau].$$

Having in mind the characterization of homogeneous option pricing in terms of non-causality, we infer that, for the dynamics of returns defined in (13) and (14) to obey assumptions (A1) and (A2), we need to extend these properties to the dynamics of the fundamental processes X and Y. We therefore specify the two following assumptions:

Assumption B1: (X,Y) does not cause U in the Granger sense.

Assumption B2: The pairs $(\mathrm{X}_t, Y_t), t = 1, 2, ..., T$ are mutually independent knowing $U_1^T = (U_t)_{1 \leq t \leq T}$.

Assumptions B1 and B2 are the exact analogs of assumptions A1 and A2 respectively. Of course S_t in A1/A2 is replaced by the fundamentals X_t and Y_t while increments of W^S in A2 are replaced by the discrete-time growth rates of the fundamentals [17].

These assumptions are quite natural considering the interpretation given to U_t as a vector of relevant state variables at time t. These variables are exogenous by assumption B1 and according to B2 subsume all temporal links between the variables of interest (X_t, Y_t). As usual, no assumption apart from stationarity is made about the law of motion of exogenous variables. Indeed, it should be emphasized (see proposition 5 below) that assumptions B1 and B2 taken together are only made about the law of $(\mathrm{X}_t,\ \mathrm{Y}_t)_{1 \leq t \leq T}$ knowing U_1^T.

Proposition 5. *Under assumption B2, assumption B1 is equivalent to:*
Assumption (B1)': $\ell[X_t,\, Y_t |\, U_1^T] = \ell[X_t,\, Y_t |\, U_1^t],\ \forall T,\ \forall t = 1, 2..., T.$

Proof: See Appendix 1.

Given the independence postulated in Assumption B2, this is in fact the Sims characterization of the non-causality from (X,Y) to U in Assumption B1. Assumption

[17] It will be proven below that, given our pricing formulas, B1 and B2 are equivalent to the discrete time analogs of A1 and A2 for the stock price and for the market portfolio price.

(B1)' is identical to Assumption 2 in Amin and Ng (1993), i.e. $(X_t, Y_t) \perp\!\!\!\perp U_{t+1}^T | U_1^t$. Our assumption B2 is clearly implied by assumption 1 in Amin and Ng (1993).

The above analogy between assumptions A1/A2 about return processes and assumptions B1/B2 about consumption and dividend processes will be made more precise below by characterizing the return processes implied by our equilibrium model. Indeed, B1 and B2 allow us to characterize the joint probability distribution of the (X_t, Y_t) pairs, t=1,...,T, given U_1^T by:

$$\ell[(X_t, Y_t)_{1 \le t \le T} | U_1^T] = \prod_{t=1}^{T} \ell[X_t, Y_t | U_1^t]. \tag{15}$$

Proposition 6 below provides the exact relationship between the state variables and equilibrium prices.

Proposition 6: *Under assumptions B1 and B2 we have:*

$$P_t^M = \lambda(U_1^t) C_t, \qquad S_t = \varphi(U_1^t) D_t,$$

where $\lambda(U_1^t)$ and $\varphi(U_1^t)$ *are respectively defined by :*

$$\lambda(U_1^t)^\gamma = E\left[\beta^\gamma \left(\frac{C_{t+1}}{C_t}\right)^{\gamma\rho} (\lambda(U_1^{t+1}) + 1)^\gamma \, | U_1^t\right],$$

and

$$\varphi(U_1^t) = E\left[\beta^\gamma \left(\frac{C_{t+1}}{C_t}\right)^{\gamma\rho - 1} \left(\frac{\lambda(U_1^{t+1}) + 1}{\lambda(U_1^t)}\right)^{\gamma - 1} \varphi(U_1^{t+1}) \frac{D_{t+1}}{D_t} \, | U_1^t\right].$$

There is of course a slight abuse of notation in proposition 6 since the dimension of U_1^t is time-varying. It is implicitly assumed in this notation that the state variable process U_t is Markovian of some order p so that the functions $\lambda(.), \varphi(.)$ are defined on R^{Kp} if there are K state variables. This remark will apply in general to all functions of U_1^t considered in the rest of the paper. Moreover, the stationarity property of the U process together with assumptions B1, B2 and a suitable specification of the density function (15) (see for instance B3 below) allow us to make the process (X,Y) stationary by a judicious choice of the initial distribution of (X,Y). In this setting, a contraction mapping argument may be applied as in Lucas (1978) to characterize the functions $\lambda(.)$ and $\varphi(.)$ according to proposition 6.

It should be stressed that our framework is more general than the Lucas one because the state variables U_1^t are given by a general multivariate Markovian process (while a Markovian dividend process is the only state variable in Lucas (1978)).

Indeed, it results from equations 13, 14, and proposition 6 that:

$$LogM_{t+1} = Log\frac{\lambda(U_1^{t+1})+1}{\lambda(U_1^t)} + X_{t+1}, \text{and} \tag{16}$$

$$LogR_{t+1} = Log\frac{\varphi(U_1^{t+1})}{\varphi(U_1^t)} + Y_{t+1}.$$

Hence, the return processes (M_{t+1}, R_{t+1}) are stationary as U, X, and Y, but, contrary to the stochastic setting in the Lucas (1978) economy, are not Markovian due to the presence of unobservable state variables U.

In any case, this asset pricing model gives some equilibrium foundations to statistical assumptions like A1 and A2 as stated by proposition 7.

Proposition 7: *Given (11) and (12), the two following equivalences hold:*
(i) *(X,Y) does not cause U in the Granger sense (Assumption B1) if and only if (P^M, S) does not cause U in the Granger sense (discrete time analog of Assumption A1 for the two price processes).*
(ii) *The pairs $(X_t, Y_t), t = 1, ..., T$ are mutually independent knowing U_1^T (Assumption B2) if and only if the consecutive returns for both market portfolio and stock $(M_t, R_t), t = 1, ..., T$ are mutually independent knowing U_1^T (discrete time analog of Assumption A2 for the joint distribution of the two returns).*

However, it should be noted that while (A1) and (A2) were stated for the so-called pricing probability measure, the properties addressed by proposition 7 are considered for the DGP. It turns out that the two points of view are equivalent in that case, as it is checked in subsection 3.3 below.

3.3 Homogeneous Option Prices in Equilibrium

Let us consider a European call option on the stock, coming to maturity at date T with exercise price K. If we consider for notational simplicity that dividends are paid immediately after exercising the option, we can determine the option price by backward recursive application of Euler equation (9):

$$\pi_t = E_t\left[\beta^{\gamma(T-t)}\left(\frac{C_T}{C_t}\right)^{\gamma(\rho-1)}(M_T M_{t-1}...M_{t+1})^{\gamma-1} Max[0, S_T - K]\right].$$

Thus, by using (16):

$$\pi_t = E_t\left[\beta^{\gamma(T-t)}\left(\frac{C_T}{C_t}\right)^{\alpha-1}\prod_{\tau=t}^{T-1}\left[\frac{(1+\lambda(U_1^{\tau+1})}{\lambda(U_1^{\tau})}\right]^{\gamma-1} Max[0, S_T - K]\right]. \tag{17}$$

It is worth noting that the option pricing formula (17) is path-dependent with respect to the state variables; it depends not only on the initial and terminal values of the process U_t but also on its intermediate values[18]. Indeed, it is not so surprising that when preferences are not time-separable ($\gamma \neq 1$), the option price may depend on the whole past of the state variables. As clearly explained by Machina (1989), it is inappropriate to impose the property of consequentialism to non-expected utility maximizers, since they would take the past uncertainty into account instead of ignoring the risk they have borne in the dynamic resolution of uncertainty.

Equation (17) can be rewritten as:

$$\frac{\pi_t}{S_t} = E_t\left[\beta^{\gamma(T-t)}\left(\frac{C_T}{C_t}\right)^{\alpha-1}\prod_{\tau=t}^{T-1}\left[\frac{(1+\lambda(U_1^{\tau+1})}{\lambda(U_1^{\tau})}\right]^{\gamma-1} Max[0, \frac{S_T}{S_t} - \frac{K}{S_t}]\right] \tag{18}$$

with:

$$\frac{C_T}{C_t} = \exp[\sum_{\tau=t+1}^{T} X_\tau],$$

and:

$$\frac{S_T}{S_t} = \frac{D_T}{D_t}\frac{\varphi(U_1^T)}{\varphi(U_1^t)} = \frac{\varphi(U_1^T)}{\varphi(U_1^t)}\exp[\sum_{\tau=t+1}^{T} Y_\tau].$$

Therefore, the conditional expectation (18) is computed with respect to the probability distribution of $(X_{t+1}^T, Y_{t+1}^T, U_{t+1}^T)$ given I_t. With a similar argument to the one

[18]Since we assume that the state variable process is Markovian of order p, $\lambda(U_1^T)$does not depend on the whole path of state variables but only on the last p values $U_T, U_{T-1}, ..., U_{T-p+1}$.

in Proposition 6, it can be proven that it depends on I_t only through U_1^t (see Appendix 1). Then the pricing formula characterizes a function Ψ such that:

$$\pi_t = \Psi(U_1^t, \frac{K}{S_t})S_t. \tag{19}$$

Equation (19) states that, as expected, the option pricing formula is homogeneous of degree one with respect to the pair (S_t, K)[19].

4 An extended Black-Scholes formula

In this section, we introduce an additional assumption on the probability distribution of the fundamentals X and Y given the state variables U.

Assumption B3:

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} | U_1^t \sim \aleph \left[\begin{pmatrix} m_{Xt} \\ m_{Yt} \end{pmatrix}, \begin{bmatrix} \sigma_{Xt}^2 & \sigma_{XYt} \\ \sigma_{XYt} & \sigma_{Yt}^2 \end{bmatrix} \right],$$

where $m_{Xt}, m_{Yt}, \sigma_{Xt}^2, \sigma_{XYt}, \sigma_{Yt}^2$ are stationary and measurable functions with respect to U_1^t, so that $m_{Xt} = m_X(U_1^t), m_{Yt} = m_Y(U_1^t), \sigma_{Xt}^2 = \sigma_X^2(U_1^t), \sigma_{Yt}^2 = \sigma_Y^2(U_1^t), \sigma_{XYt} = \sigma_{XY}(U_1^t)$.

We want to stress that, as soon as previous assumptions B1 and B2 required for homogeneous option pricing are maintained, this additional assumption, which will allow us to derive an extended BS option pricing formula, is not very restrictive. This is the reason why we will claim that the BS shape for an option pricing formula is very robust when one remains true to homogeneity. The fundamental argument is that, if one considers that the discrete-time interval is somewhat arbitrary and can be infinitely split, log-normality (conditional on state variables U) is obtained as a consequence of a standard central limit argument given the independence between consecutive (X, Y) given U. This assumption B3 extends a similar assumption made by Amin and Ng (1993) to derive an option pricing formula in an expected utility framework, a special case of our setting. Given this log-normality assumption, we

[19] It should be emphasized that even though we have chosen to focus on Kreps-Porteus preferences, the main argument of this section to ensure homogeneous option prices in equilibrium remains valid with other types of preferences.

will characterize successively the pricing probability measure, the role of preferences in the pricing of bonds (value of time) and equity (value of risk) and the combination of these values in the option price.

4.1 The Pricing Probability Measure

By the argument already used to prove proposition 1 (see Appendix 1), we deduce from the option pricing formula (18) the cumulative distribution function of the pricing probability measure as a function of the cumulative distribution Φ of the standard normal $\aleph(0,1)$:

$$\begin{aligned} B(t,T)Q_t\left[\frac{S_T}{S_t}\geq k\right] &= E_t\{\beta^{\gamma(T-t)}\exp(\alpha-1)[\sum_{\tau=t+1}^{T}X_\tau]) \\ &\quad \cdot\prod_{\tau=t}^{T-1}\left[\frac{(1+\lambda(U_1^{\tau+1})}{\lambda(U_1^{\tau})}\right]^{\gamma-1}\mathbf{1}_{[\frac{\varphi(U_1^T)}{\varphi(U_1^t)}\exp[\sum_{\tau=t+1}^{T}Y_\tau]\geq k]}\}, \end{aligned}$$

where $\mathbf{1}_{[.]}$ denotes the indicator function.
By iterating on conditional expectations (given U_1^T and $I_{T-\tau}, \tau=1,2,...,T-t$) of the inner part of the above expectation, we can apply assumptions B1, B2 and B3 to derive the following formula (see Appendix 2):

$$B(t,T)Q_t\left[\frac{S_T}{S_t}\geq k\right]=E_t[\widetilde{B}(t,T)\Phi(d_2)], \tag{20}$$

where:

$$\widetilde{B}(t,T)=\beta^{\gamma(T-t)}a_t^T(\gamma)\exp((\alpha-1)\sum_{\tau=t+1}^{T}m_{X\tau}+\frac{1}{2}(\alpha-1)^2\sum_{\tau=t+1}^{T}\sigma^2{}_{X\tau}),$$

with: $a_t^T(\gamma)=\prod_{\tau=t}^{T-1}\left[\frac{(1+\lambda(U_1^{\tau+1})}{\lambda(U_1^{\tau})}\right]^{\gamma-1}$, and:

$$d_2=\frac{\ln\frac{S_t}{K}+\ln\frac{\varphi(U_1^T)}{\varphi(U_1^t)}+\sum_{\tau=t+1}^{T}m_{Y\tau}+(\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau}}{(\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2)^{1/2}}.$$

4.2 The Pricing of Bonds

To price a bond delivering one unit of the good at time T, it suffices to apply equation (20) with $k = 0$. Therefore we obtain the following bond pricing formula:

$$B(t,T) = E_t[\widetilde{B}(t,T)]. \tag{21}$$

This formula shows how the interest rate risk is compensated in equilibrium, and in particular how the term premium is related to preference parameters. In what follows, we will refer loosely to $\widetilde{B}(t,T)$ as a stochastic discount factor, but naturally it is strictly so just for an asset with zero covariance with the aggregate risk. To be more explicit about the relationship between the term premium and the preference parameters, let us first notice that we have a natural factorization of the stochastic discount factor:

$$\widetilde{B}(t,T) = \prod_{\tau=t}^{T-1} \widetilde{B}(\tau,\tau+1). \tag{22}$$

Therefore, while the discount parameter β enters in the determination of the general level of discount factors, the two other parameters α and γ affect the term premium (with respect to the return-to-maturity expectations hypothesis, Cox, Ingersoll, and Ross (1981)) through the ratio:

$$\frac{B(t,T)}{E_t \prod_{\tau=t}^{T-1} B(\tau,\tau+1)} = \frac{E_t(\prod_{\tau=t}^{T-1} \widetilde{B}(\tau,\tau+1))}{E_t \prod_{\tau=t}^{T-1} E_\tau \widetilde{B}(\tau,\tau+1)}.$$

To better understand this term premium from an economic point of view, let us compare implicit forward rates and expected spot rates at only one intermediary period between t and T:

$$\frac{B(t,T)}{B(t,\tau)} = \frac{E_t\widetilde{B}(t,\tau)\widetilde{B}(\tau,T)}{E_t\widetilde{B}(t,\tau)} = E_t\widetilde{B}(\tau,T) + \frac{Cov_t[\widetilde{B}(t,\tau),\widetilde{B}(\tau,T)]}{E_t\widetilde{B}(t,\tau)}. \tag{23}$$

Up to Jensen inequality, equation (23) proves that a positive term premium is brought about by a negative covariation between present and future stochastic discount factors. Given the expression for $\widetilde{B}(t,T)$ above, it can be seen that for von-Neuman preferences ($\gamma = 1$) the term premium is proportional to the square of the coefficient of relative risk aversion (up to a conditional stochastic volatility effect). Another important observation is that even without any risk aversion ($\alpha = 1$), preferences still affect the term premium through the non-indifference to the timing of uncertainty resolution ($\gamma \neq 1$).

There is however an important sub-case where the term premium will be preference-free because the stochastic discount factor $\widetilde{B}(t,T)$ coincides with the observed rolling-over discount factor (the product of short-term future bond prices, $B(\tau, \tau+1)$, $\tau = t, ..., T-1$). Taking equation (22) into account, this will occur as soon as $\widetilde{B}(\tau,\tau+1) = B(\tau,\tau+1)$, that is when $\widetilde{B}(\tau,\tau+1)$ is known at time τ. From the expression of $\widetilde{B}(t,T)$ above, it is easy to see that this last property stands if and only if the mean and variance parameters $m_{X\tau}$ and $\sigma_{X\tau}$ depend on U_1^τ only through $U_1^{\tau-1}$, given that in this case one can see by proposition 6 that $\lambda(U_1^\tau)$ itself depends on U_1^τ only through $U_1^{\tau-1}$.

This leads us to introduce a property of the consumption process termed predictability by Amin and Ng (1993). Contrary to the most usual notation which introduces a large enough number of state variables in order to obtain a Markovian system of order one, it is important here to stress that the market portfolio price may depend on the whole recent history: $\mathrm{U}_t, U_{t-1}, ..., U_{t-p+1}$. This distinctive framework allows us to highlight the so-called "leverage effect" which is so important for option pricing. This effect appears here when the probability distribution of (X_t) given U_1^t depends (through the functions m_X, σ_X^2) on the contemporaneous value U_t of the state process. More generally, the non-causality assumption B1 could be reinforced in the following way:

Assumption B4 : X does not cause U in the strong sense, i.e. there is neither Granger nor instantaneous causality from X to U.

In this case, the analog of proposition 5 is $\ell(X_t|U_1^T) = \ell(X_t|U_1^{t-1})$; it is this property which ensures that short-term stochastic discount factors are predetermined, so the bond pricing formula becomes preference-free:

$$B(t,T) = E_t \prod_{\tau=t}^{T-1} B(\tau,\tau+1).$$

Of course this does not necessarily cancel the term premiums but it makes them preference-free. Moreover, when there is no interest rate risk because the consumption growth rates X_t are iid, it is straightforward to check that constant m_{Xt} and σ^2_{Xt} imply constant $\lambda(U_1^t)$ and in turn deterministic discount factors: $\widetilde{B}(t,T) = B(t,T)$ and zero term premiums.

4.3 The Pricing of Stocks

The stock price formula is obtained as a particular case of the general option pricing formula (18) for the limit case $K = 0$, that is:

$$S_t = E_t\left[\beta^{\gamma(T-t)}\left(\frac{C_T}{C_t}\right)^{\alpha-1}\prod_{\tau=t}^{T-1}\left[\frac{(1+\lambda(U_1^{\tau+1})}{\lambda(U_1^\tau)}\right]^{\gamma-1} S_T\right].$$

Using a similar argument to the one used for the pricing probability measure, we obtain under conditional log-normality assumption B3:

$$S_t = E_t\{\beta^{\gamma(T-t)}a_t^T(\gamma)\exp((\alpha-1)\sum_{\tau=t+1}^{T} m_{X\tau}+\frac{1}{2}(\alpha-1)^2\sum_{\tau=t+1}^{T}\sigma^2{}_{X\tau}+(\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau})S_T\},$$

which can be rewritten as:

$$S_t = E_t\left[\widetilde{B}(t,T)\exp((\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau})S_T\right]. \quad (24)$$

As expected, the stock price is expressed as the conditional expectation of its discounted terminal value, where the stochastic discount factor $\widetilde{B}(t,T)$ is risk-adjusted by a CAPM-like term $\exp((\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau})$. This term accounts for the covariance risk between the stock and the market portfolio (proportional to the standard CAPM beta risk), weighted by the coefficient of relative risk aversion. In other words, the specific role of time preference parameters β and γ is fully embodied in the stochastic discount factor which characterizes the bond equation. The additional risk premium associated with the stock involves only the risk parameter α.

Another useful way of writing the stock pricing formula is:

$$E_t\left[Q_{XY}(t,T)\right] = 1, \quad (25)$$

where:

$$Q_{XY}(t,T) = \widetilde{B}(t,T)\exp((\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau})E[\frac{S_T}{S_t}|U_1^T]. \tag{26}$$

To understand the role of the factor $Q_{XY}(t,T)$, it is useful to notice that it can be factorized:

$$Q_{XY}(t,T) = \prod_{\tau=t}^{T-1} Q_{XY}(\tau,\tau+1),$$

and that there is an important particular case where $Q_{XY}(\tau,\tau+1)$ is known at time τ and therefore equal to one by (25). This is when there is no leverage effect in the general sense of the following assumption B5 (which reinforces assumption B4).

Assumption B5 : (X,Y) does not cause U in the strong sense, i.e. there is neither Granger nor instantaneous causality from (X,Y) to U.

Under assumptions B1 and B2, B5 is equivalent to $\ell(X_t,Y_t|U_1^T) = \ell(X_t,Y_t|U_1^{t-1})$. This means that not only there is no leverage effect neither for X nor for Y, but also that the instantaneous covariance σ_{XYt} itself does not depend on U_t. In this case, we have $Q_{XY}(t,T) = 1$. From (26) and (22), taking into account that under B5 $\widetilde{B}(\tau,\tau+1) = B(\tau,\tau+1)$, we can express the conditional expected stock return as:

$$E\left[\frac{S_T}{S_t}|U_1^T\right] = \frac{1}{\prod_{\tau=t}^{T-1}B(\tau,\tau+1)}\exp((1-\alpha)\sum_{\tau=t+1}^{T}\sigma_{XY\tau}).$$

For pricing over one period (t to $t+1$), this formula provides the agent's expectation of the next period return (since in this case the only relevant information is U_1^t which is included in I_t):

$$E\left[\frac{S_{t+1}}{S_t}|I_t\right] = \frac{1}{B(t,t+1)}\exp[(1-\alpha)\sigma_{XYt+1}].$$

This is a particularly striking result since it is very close to a standard conditional CAPM equation (and unconditional in an iid world), which remains true for any value of the preference parameters α and ρ. While Epstein and Zin (1991) emphasize that the CAPM obtains for $\alpha = 0$ (logarithmic utility) or $\rho = 1$ (infinite elasticity of intertemporal substitution), we stress here that the relation is obtained under a particular stochastic setting for any values of α and ρ. Remarkably, the stochastic setting without leverage effect which produces this CAPM relationship will also produce most standard option pricing models (for example BS and Hull-White), which are of course preference-free[20].

4.4 A General Pricing Formula for Options

We finally arrive at the central result of the paper, which proposes an extended Black-Scholes formula:

$$\frac{\pi_t}{S_t} = E_t \left\{ Q_{XY}(t,T)\Phi(d_1) - \frac{K\widetilde{B}(t,T)}{S_t}\Phi(d_2) \right\}, \tag{27}$$

where:

$$d_1 = \frac{Log\left[\frac{S_t Q_{XY}(t,T)}{K\widetilde{B}(t,T)}\right]}{(\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2)^{1/2}} + \frac{1}{2}(\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2)^{1/2}, \text{ and}$$

$$d_2 = d_1 - (\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2)^{1/2}.$$

The second part of the formula results directly from the expression obtained above for the pricing probability measure; Appendix 2 details the derivation of the first part.

Apart from the familiar decomposition into $\Phi(d_1)$ and $\Phi(d_2)$ parts which is also found in the usual BS formula and its extensions, it should be noticed that the expressions for $d_1$and d_2 are also very close to the corresponding quantities in these formulas. In particular, our $\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2 = Var\left[\log\frac{S_T}{S_t}|U_1^T\right]$ corresponds to $\sigma^2(T-t)$ in the BS formula and $\int_t^T \sigma_u^2 du$ in the Hull-White formula.

Indeed, a preference-free option pricing formula similar to the one obtained by Hull and White (1987), Amin and Jarrow (1992), Merton (1973) is obtained whenever $Q_{XY}(t,T) = 1$ and $\widetilde{B}(t,T) = \prod_{\tau=t}^{T-1} B(\tau,\tau+1)$, that is when there are no leverage effects, neither through the market risk nor through the stock risk. Another case of preference-free option pricing is worth emphasizing. Even when $Q_{XY}(t,T)$ is different from one (which means that exists a leverage effect for the individual stock), it

[20] A similar parallel is drawn in an unconditional two-period framework in Breeden and Litzenberger (1978).

becomes independent of risk aversion, as does the option price, when the stock is zero-beta with respect to the market ($\sigma_{XYt} = 0$).

In the general case, our option pricing formula has two main characteristics. First, the advantages of standard extensions of BS option pricing by the introduction of unobservable heterogeneity factors (see Renault (1996)) are maintained. The main advantage of these approaches is to keep in expectation the BS functional shape. Second, contrary to a philosophy where the BS formula is praised for its independence with respect to preference parameters and expected returns[21], our extension does not need these virtues to stay close to the Black-Scholes formula. Indeed, while the three preference parameters enter the option price through the value of time $\widetilde{B}(t,T)$ (as soon as there is a leverage effect at the aggregate level), the risk aversion parameter and the expected stock return play an additional role in the option price through $Q_{XY}(t,T)$ (as soon as there is a leverage effect at the individual stock level).

To conclude, it is worth noting that our results of equivalence between preference-free option pricing and no instantaneous causality between state variables and asset returns are consistent with another strand of the option pricing literature, namely GARCH option pricing introduced by Duan (1995). Indeed, while GARCH models are unable to capture a genuine leverage effect, they are close to the spirit of the framework B1, B2, B3, B5. Of course, this framework involves unobserved state variables while the GARCH specification of conditional variance is a deterministic function of past observables, but in both cases, precluding leverage effect allows one to plug the discrete-time model into a continuous time one, where conditional variance is constant between two integer dates. Kallsen and Taqqu (1994) have shown that such a continuous-time embedding makes possible arbitrage pricing which is per se preference-free. This explains why the GARCH option pricing and the stochastic volatility and interest rate option pricing proposed here under B5 are very similar: they are both preference-free and involve a cumulated conditional variance $\sum_{\tau=t+1}^{T} \sigma_{Y\tau}^2$. Section 5 below will summarize the practical implications of these various paradigms.

[21]See Merton (1990), footnote 26 p. 282.

5 Homogeneous Option Pricing and Volatility Smile

The fact that the BS formula is free from preference parameters is often perceived as its main advantage. In reality, the argument is hard to understand if one considers that practitioners are used to infer implicit parameters from option prices, irrespective of their theoretical interpretation. One prominent example is the so-called implied or implicit volatility parameter, i.e. the volatility parameter derived from the BS formula. Similarly to yields on the bond market, implicit volatilities serve as a useful unit of measure on option markets. The usefulness of this unit of measure comes from the fact that it does not depend on the stock price level, in other words that the implicit volatility function is homogeneous of degree zero with respect to the pair (S,K) where S is the price of the underlying asset and K the strike price. It should be emphasized however that this homogeneity property holds if and only if the option pricing formula itself is homogeneous of degree one with respect to the same pair (S,K). This is the case of course of the BS formula itself. Therefore, any option pricing formula that features this homogeneity property should be of interest to practitioners, be it based on preferences or not. The usefulness of such homogeneous general option pricing formulas is discussed here through the volatility smile, that is the representation, at a given date t and for a given maturity T, of the set of BS implicit volatilities in function of the corresponding strike prices. In particular, we provide new characterizations of the symmetry of the volatility smile in terms of the option pricing function and of the pricing probability measure. We also draw the implications of these characterizations for our option pricing model.

5.1 The volatility smile as an image of the pricing probability measure

According to the notations of Proposition 1, we will compare in this subsection a general but homogeneous option pricing formula $\pi_t(S_t, K)$ with the BS option pricing formula defined itself by a homogeneous function $BS(.,.,\sigma)$, for a given volatility parameter σ, with:

$$
\begin{cases}
BS(S_t, K, \sigma) = S_t \phi(d_1) - KB(t,T)\phi(d_2), \\
d_1 = \dfrac{1}{\sigma\sqrt{T-t}}\left[Log\dfrac{S_t}{KB(t,T)} + \dfrac{1}{2}\sigma^2(T-t)\right], \\
d_1 = d_1 - \sigma\sqrt{T-t}.
\end{cases} \tag{28}
$$

Following Renault and Touzi (1996), it appears useful to characterize the shape of the volatility smile with respect to the moneyness $x_t = Log\dfrac{S_t}{KB(t,T)}$ rather than the strike price K. In other words, the BS implicit volatility is a function $\sigma_t^*(x_t)$ of x_t only, and not of S_t and K separately. Starting with the defining formula:

$$
\pi_t(S_t, K) = BS(S_t, K, \sigma_t^*(x_t)), \tag{29}
$$

a direct application of the homogeneity of degree one of $\pi_t(.,.)$ and $BS(.,.,\sigma)$ with respect to the pair (S_t, K) allows one to divide each side of (29) by K and conclude that $\sigma_t^*(x_t)$ is well-defined as a function of S_t/K or (equivalently) of x_t by :

$$
\pi_t(x_t) = BS(x_t, \sigma_t^*(x_t)) \tag{30}
$$

with the following slight change of notations :

$$
\begin{cases}
\pi_t(x_t) & = \pi_t\left(\dfrac{S_t}{K}, 1\right), \\
BS(x_t, \sigma) & = BS\left(\dfrac{S_t}{K}, 1, \sigma\right).
\end{cases} \tag{31}
$$

The subscripts t in the functions $\pi_t(.,.)$ and $\sigma_t^*(.)$ indicate that they may depend upon other state variables, the value of which is fixed at time t. The property we just emphasized is in fact the homogeneity of degree zero of the BS implicit volatility with respect to the pair (S_t, K). This homogeneity is a direct consequence of the postulated homogeneity of degree one of the general option pricing formula $\pi_t(.,.)$ as well as the known homogeneity property of the BS option pricing formula $BS(.,.,\sigma)$.

Various consequences of this setting both in terms of option pricing and option hedging are detailed in Renault and Touzi (1996), Renault (1997) and Garcia and Renault (1998). In particular, Renault and Touzi (1996) and Renault (1997) have investigated the well-documented skewness of the smile and provided the theoretical setting which guarantees symmetric volatility smiles[22], that is the property:

$$\sigma_t^*(x) = \sigma_t^*(-x) \quad \text{for any} \quad x. \tag{32}$$

We will characterize the variations of $\pi(x)$, $BS(x, \sigma), \sigma_t^*(x)$ as functions of x for a given value of S_t. In other words, the genuine variable of interest is the strike price K, while the switch to the variable x is only a matter of rescaling for convenience.

In Proposition 8 below, we extend a result first stated in Renault and Touzi (1996), which characterizes the symmetry of the smile in terms of the option pricing function[23].

Proposition 8. *If option prices are conformable to a homogeneous option pricing formula $x \rightarrow \pi(x)$, the volatility smile is symmetric ($\sigma^*(x) = \sigma^*(-x)$ for any x) if and only if, for any x:*

$$\pi(-x) = e^x \pi(x) + 1 - e^x$$

Proof: See Appendix 3.

Thanks to proposition 1, this characterization of the symmetry of the smile admits an equivalent formulation in terms of the pricing probability measure. While this pricing probability measure was characterized in Proposition 1 through the cumulative distribution function of $\frac{S_T}{S_t}$, it is convenient here to characterise it through either the cumulative distribution function $F_{V_T}(.)$ or the probability density function $f_{V_T}(.)$ of $V_T = Log\frac{S_T B(t,T)}{S_t}$. We are then able to prove (see Appendix 3) the following proposition:

Proposition 9 *If $V_T = Log\frac{S_T B(t,T)}{B_t}$ admits a probability density function $f_{V_T}(.)$ with respect to the pricing probability measure and is integrable with respect to this measure, the volatility smile is symmetric if and only if one of the following three equivalent properties is fulfilled:*

(i) For any x:

[22]In the standard analysis of the smile relationship between the implicit volatility and the strike price, the symmetry is characterized with respect to the log strike price, and not its absolute value.

[23]For sake of notational simplicity, the subscripts t have been dropped

$$\pi(x) = F_{V_T}(x) - e^{-x}[1 - F_{V_T}(-x)]$$

(ii) For any x:

$$F_{V_T}(x) = E_t^*[e^{V_T}\mathbf{1}_{[V_T \geq -x]}]$$

(iii) There exists an even function g(.) such that for any x:

$$f_{V_T}(x) = e^{-x/2}g(x)$$

As announced in the introduction, these characterizations offer to practitioners various ways to extend the BS formula, while keeping both a homogeneous option pricing function and a symmetric smile. Characterization (i) provides a theoretical support to descriptive approaches which replace the standard normal cumulative distribution function of the BS formula by alternative distribution functions, possibly asymmetric (see Garcia and Gencay (1997)). Characterization (ii) should be interpreted in terms of hedging. Indeed, Garcia and Renault (1998) have shown that $E_t^*[e^{V_T}\mathbf{1}_{[V_T \geq -x]}]$ is precisely the hedging ratio, in other words the derivative of the option pricing function with respect to the stock price (the so-called delta of the option)[24]. Finally, for characterization (iii) let us just notice at this stage[25] that if the pricing probability measure is characterized by a conditional log-normal distribution of future returns given available information at time t :

$$V_T = Log\frac{S_T B(t,T)}{S_t} \mid I_t \rightsquigarrow_{(Q_t)} \mathcal{N}(m_t, \sigma_t^2)$$

the condition of Proposition 9 means that :

$$m_t = -\frac{\sigma_t^2}{2}$$

[24]Their proposition 2.1 shows that this characterization of the hedging ratio is a necessary and sufficient condition for homogeneous option pricing. Since hedging is not the primary focus of this paper, we leave to the reader the interpretation of this fairly natural relationship between $F_{V_T}(x)$ and the delta coefficient.

[25]The characterization of the set of asset prices processes in equilibrium whose equivalent pricing probability measure fulfills the condition (iii) of Proposition 9 is also beyond the scope of this paper.

which is automatically fulfilled in equilibrium since, by application of (1) with $K = 0$, we have:

$$S_t = B(t,T)E_t^* S_T$$

More generally, if $V_T = Log \frac{S_T B(t,T)}{S_t}$ follows (under Q_t) a conditional gaussian distribution $\mathcal{N}[m_t(U_t^T), \sigma_t^2(U_t^T)]$ given I_t and the path U_t^T (between t and T) of some state variables U, the condition will be fulfilled (by integration over U_t^T) as soon as:

$$m_t(U_t^T) = -\frac{\sigma_t^2(U_t^T)}{2}.$$

This is the case for instance for an Hull and White world without leverage effect, which explains the main result of Renault and Touzi (1996): if option prices are conformable to the Hull and White option pricing formula without leverage effect, the volatility smile is symmetric. More generally, it is often claimed that an asymmetric smile means that the underlying pricing probability measure is skewed. Proposition 9 characterizes precisely which type of "symmetry" of the pricing probability measure is required for the symmetry of the smile. In particular, it shows that it is not the density of the log returns that should be symmetric (as it is commonly believed perhaps because of the usual log-normal setting), but the same density rescaled by a suitable exponential function.

In the next subsection, we generalize this result by characterizing the skewness of the volatility smile in terms of the leverage effects or the serial correlation in the aggregate consumption risk which appear in our general option pricing formula.

5.2 Asymmetric smiles, preferences and implied latent binomial trees

By taking into account the slight change of notations (31), our general option pricing formula (27) can be rewritten as follows:

$$\pi_t(x) = E_t \left\{ Q_{XY}(t,T)\Phi(d_1(x)) - \frac{\widetilde{B}(t,T)}{B(t,T)} e^{-x} \Phi(d_2(x)) \right\} \tag{33}$$

where

$$d_1(x) \quad = \quad \frac{x}{\overline{\sigma}_{t,T}} + \frac{\overline{\sigma}_{t,T}}{2} + \frac{1}{\overline{\sigma}_{t,T}} Log \left[Q_{XY}(t,T) \frac{B(t,T)}{\widetilde{B}(t,T)} \right]$$

$$d_2(x) = d_1(x) - \overline{\sigma}_{t,T}$$
$$\overline{\sigma}^2_{t,T} = \sum_{\tau=t+1}^{T} \sigma^2_{Y\tau}.$$

However, we know by (20) that:

$$B(t,T)[1 - F_{V_T}(-x)] = E_t[\widetilde{B}(t,T)\Phi(d_2(x))].$$

Therefore, when one compares the option pricing formula (33) to the symmetry condition (i) in Proposition 9, it is easy to check that:

$$E_t\left\{\frac{\widetilde{B}(t,T)}{B(t,T)}e^{-x}\Phi(d_2(x))\right\} = e^{-x}[1 - F_{V_T}(-x)]$$

In other words, the symmetry property of the smile is equivalent to:

$$F_{V_T}(x) = E_t\left\{Q_{XY}(t,T)\Phi(d_1(x))\right\}$$

or:

$$1 - E_t\left\{\frac{\widetilde{B}(t,T)}{B(t,T)}\Phi(d_2(-x))\right\} = E_t\left\{Q_{XY}(t,T)\Phi(d_1(x))\right\} \quad (34)$$

From the bond and stock pricing equations (21) and (25), we know that $\frac{\widetilde{B}(t,T)}{B(t,T)}$ and $Q_{XY}(t,T)$ are two random random variables that are equal in expectation conditional to I_t. Given this equality in expectation, it is natural to ask whether (34) holds when these two random variables are equal with probability one. The answer is affirmative, since from (33):

$$d_2(-x) = -d_1(x) + \frac{2}{\overline{\sigma}_{t,T}}Log\left[Q_{XY}(t,T)\frac{B(t,T)}{\widetilde{B}(t,T)}\right].$$

Then, when the two random variables are equal: $\Phi(d_2(-x)) = 1 - \Phi(d_1(x))$, which ensures that (34) holds. We have therefore proven the following proposition:

Proposition 10 : *In the framework of section 4, a sufficient condition for a symmetric volatility smile is the following identity :*

$$Q_{XY}(t,T) = \frac{\widetilde{B}(t,T)}{B(t,T)}$$

The details of the proof above suggest that the condition is not too far from being a necessary and sufficient condition. It should also be stressed that, from (26), this condition for symmetry is equivalent to:

$$E[\frac{S_T}{S_t}|U_1^T] = \frac{1}{B(t,T)}\exp[(1-\alpha)\sum_{\tau=t+1}^{T}\sigma_{XY\tau}]$$

This equation corresponds to a CAPM-like stock pricing formula. Moreover, when the symmetry condition of proposition 10 is fulfilled, $\mathrm{d}_1(x)$ and $\mathrm{d}_2(x)$ are preference-free and coincide with the corresponding arguments of a Hull-White type option pricing formula. The formula does not imply however that option prices are preference-free. Indeed the option pricing formula becomes:

$$\pi_t(x) = E_t\left\{\frac{\widetilde{B}(t,T)}{B(t,T)}[\Phi(d_1(x)) - e^{-x}\Phi(d_2(x))]\right\}.$$

Therefore, preference parameters may still appear through the ratio $\frac{\widetilde{B}(t,T)}{B(t,T)}$. As already explained in subsection 4.4 above, the natural way to obtain a true preference-free Hull-White option pricing formula is indeed to impose the two following conditions: (i) $\tilde{B}(t,T) = B(t,T)$ and (ii) $Q_{XY}(t,T) = 1$.

In other words two kinds of "generalized" leverage effects may explain (besides the instantaneous interest rate risk) asymmetric smiles: either a genuine leverage effect, that is an instantaneous correlation between the return on the stock and its stochastic volatility process, or a stochastic correlation between the return of the stock and the total endowment of the economy. These results provide some theoretical foundation to the observed asymmetric smiles and their empirically documented relationship with the business cycle and interest rate movements (see for instance the survey by Bates (1996)). More importantly, the new conclusion of our model for practicioners should be that an asymmetric smile is indicative of the relevance of preference parameters to price options. Indeed, our structural equilibrium model has shown that violations of the symmetry condition in Proposition 10 (due to interest rate risk or the occurence of a leverage effect in the general sense above) correspond precisely to cases where preference parameters matter for option pricing.

Therefore, whenever an asymmetric smile is observed, the first issue to address is to specify a list of state variables as well as a set of mean, variance and covariance functions conformable to B3. Since the process of state variables is a latent Markov process, a natural candidate is the Markov switching model introduced by Hamilton (1989) and applied to asset pricing by Cecchetti, Lam and Mark (1990, 1993) and Bonomo and Garcia (1993, 1994, 1996). The standard procedures of estimation and identification of such a model (Hamilton (1989), Garcia (1997)) can then be used for the modeling of the bivariate process (X_t, Y_t).

The statistical procedure just referred to amounts to an unrestricted inference procedure based on aggregate consumption and stock dividends. However, the equilibrium pricing relationships for bonds, stocks and options constrain these dynamics and suggest to look for a Markov switching process consistent with these equilibrium relationships. This approach is in the spirit of Hansen and Singleton (1983), who estimate a VAR process for consumption and returns constrained by Euler equations, and Bonomo and Garcia (1996), who estimate a Markov switching model consistent with CCAPM pricing relationships for stock and bond returns. What our model suggests is that adding option prices to such relationships should be informative about both laws of motion and preference parameters. This is in contrast with Merton (1990, p. 282) who claims that: "...attempts to use the option price to estimate either expected returns on the stock or risk preferences of investors are doomed to failure." Of course, this citation refers to a world where option prices are preference-free, which is different from our extended framework when leverage effects are at play and smiles are asymmetric.

Our approach has to be compared with a recent trend in the literature called implied binomial trees (Rubinstein (1994)). There is a formal similarity between the two approaches, because in both cases we try to calibrate a binomial tree or a discrete Markov process on the dynamics of option prices. However, while implied binomial trees inferred in Rubinstein (1994) represent the local volatility of the underlying asset, the riskless interest rate and the asset payout rate as a function of the prior path of the underlying asset price, our implied latent binomial trees are hidden Markov chains which correspond to violation 3 of the BS model in Rubinstein (1994, p. 778): "The local volatility of the underlying asset, the riskless interest rate or the asset payout rate is a function of a state variable which is not the concurrent underlying asset price or the prior path of the underlying asset price". We have explained that this type of violation is useful since it maintains the homogeneity of option prices but of course we are led to follow the route of what Rubinstein calls the unpalatable alternative of establishing an equilibrium model in which investor preferences explicitly enter. We have argued that such a route is viable: once the Markov process has been identified and estimated, it can be simulated to calibrate preference parameters on our closed-form option pricing formula. Inferring the dynamics of fundamentals and preference parameters from option prices does not then appear much more complicated than the usual Monte Carlo procedures used pervasively for extended BS pricing models. Moreover, implied latent binomial trees and preference parameters should provide more stable option pricing and hedging than standard implied binomial trees[26] given their structural underpinning.

6 Conclusion

In this paper, we have specified a stochastic framework for the fundamentals which produces, in a dynamic equilibrium asset pricing model, a homogeneous option pricing formula. Since this homogeneity property preserves a Black-Scholes shape to our generalized option pricing formula, it reinforces the robustness of the BS formula and rationalizes the abundant literature that extends the BS model to improve its usefulness for practitioners.

In general, through an instantaneous causality relationship between the market portfolio or the stock price and state variables which affect the interest rate risk or the stochastic volatility of the stock price, the option price depends on preference parameters. The interest rate risk premium is not hidden in the market price of long-term bonds and involves all preference characteristics (discounting, risk aversion and intertemporal substitution), while the risk premium related to the volatility risk or leverage effect only involves the risk aversion parameter. This last effect is purely due to a covariance risk (in the spirit of the CAPM) and vanishes if the stock has a zero beta with respect to the market. It is only in the absence of such instantaneous causality effects that our general option pricing formula specializes to the usual preference-free option pricing formulas. When the processes of consumption growth and dividend growth are not instantaneously caused by unobserved state variables, we recover preference-free option pricing as if markets were complete and unambigu-

[26]Recent work by Dumas, Fleming and Whaley (1996) has shown that lack of stability is an important drawback of the implied binomial tree methodology.

ous arbitrage-based pricing was possible, as well as CAPM pricing for stocks. On the other hand, when markets are genuinely incomplete, due to a leverage effect for the stock or a similar effect for the market return, a causality effect appears and one needs some assessment of the appropriate pricing of systematic volatility and interest rate risks through attitudes towards risk and intertemporal substitution.

Even though the state variables are usually unobservable, our formula is of practical relevance in two respects. First, we have referred to Hamilton's (1989) Markov switching model as a tractable way to filter out these state variables from the data, which makes our formula implementable. Second, we have shown that observed asymmetries in the smile effect are directly related to the risk premiums associated with correlations with these state variables, which make their presence essential to price options more accurately and in particular to account for asymmetric smile effects observed with BS implicit volatilities.

We have emphasized in section 5 that equilibrium conditions for option prices can be informative to infer the laws of motion of the fundamentals and the preference parameters. In particular, option prices appear to matter empirically to disentangle risk aversion from intertemporal substitution in a recursive utility framework with Kreps-Porteus preferences. A preliminary study[27] estimating Euler conditions with various asset prices shows that the addition of first-order conditions related to options leads to parameter estimates supportive of Kreps-Porteus preferences, contrary to what is obtained simply with stocks and Treasury bills . Further simulation and estimation work is warranted to confirm these results, but they point to the potential usefulness of option prices to identify asset pricing models, an avenue which has been overlooked in the literature.

[27]In a previous version of the paper, we included a section entitled "GMM estimation of the Recursive Utility Model with Option Prices" in which the equilibrium model was tested on a set of Euler conditions with daily option data from the Montreal Stock Exchange. Further details can be found in that version which is available upon request from the authors.

Appendix 1

Proof of Proposition 1:

The price of the option can be rewritten as:

$$\pi_t(K) = B(t,T)S_t \int_{\frac{K}{S_t}}^{+\infty} \left(\frac{S_T}{S_t} - \frac{K}{S_t}\right) dQ_t \left(\frac{S_T}{S_t}\right).$$

Therefore:

$$\frac{\partial \pi_t}{\partial K} = B(t,T)S_t \int_{\frac{K}{S_t}}^{+\infty} -\frac{1}{S_t} dQ_t \left(\frac{S_T}{S_t}\right);$$
$$\frac{\partial \pi_t}{\partial K} = -B(t,T)Q_t \left[\frac{S_T}{S_t} \geq \frac{K}{S_t}\right].$$

Proof of Proposition 2:

(ii)$\Longrightarrow$(i)

$$\pi_t = B(t,T)S_t E_t^* \left[\left(\frac{S_T}{S_t} - \frac{K}{S_t}\right)^+\right].$$

The pricing function π_t would be homogeneous of degree one if multiplying K and S_t by a positive scalar λ, π_t would also be multiplied by λ. Looking at the formula above, this could be true as soon as the Q_t probability distribution of the return $\frac{S_T}{S_t}$, with respect to which the expectation is computed, is independent of S_t.

(i)$\Longrightarrow$ (ii)

By Proposition 1:

$$Q_t \left[\frac{S_T}{S_t} \geq x\right] = -\frac{1}{B(t,T)} \frac{\partial \pi_t(K, S_t)}{\partial K},$$

with: $x = \frac{K}{S_t}$.

But if π_t is homogeneous of degree one, $\frac{\partial \pi_t(K,S_t)}{\partial K}$ is homogeneous of degree zero, so that:

$$\frac{\partial \pi_t(\lambda K, \lambda S_t)}{\partial K} = \frac{\partial \pi_t(K, S_t)}{\partial K} = \frac{\partial \pi_t(\frac{K}{S_t}, 1)}{\partial K} = \frac{\partial \pi_t(x, 1)}{\partial K},$$

which depends only on x.

Therefore $Q_t \left[\frac{S_T}{S_t} \geq x\right]$ does not depend on S_t.

Proof of Proposition 4:

The equivalence between (A1) and (A1)' is nothing but the characterization of Granger non-causality in continuous time (from S to (r,σ)) provided by Florens and Fougère (1996, p.1206). When these assumptions are maintained, we are able to write:

$$\frac{dS_t}{S_t}=r(t)dt+\sigma(t)\beta_{sr}(t)dW^r(t)+\sigma(t)\beta_{s\sigma}(t)dW^\sigma(t)+\sigma(t)\eta(t)dW^z(t),$$

where dW^z is a standard Brownian motion independent of (dW^r,dW^σ), and:

$$\begin{bmatrix}\beta_{sr}(t)\\ \beta_{s\sigma}(t)\end{bmatrix}=\begin{bmatrix}1 & \rho_{r\sigma}(t)\\ \rho_{r\sigma}(t) & 1\end{bmatrix}^{-1}\begin{bmatrix}\rho_{sr}(t)\\ \rho_{s\sigma}(t)\end{bmatrix},$$

and $\eta^2(t)$ =1-$\beta_{sr}^2(t)-\beta_{s\sigma}^2(t)-2\rho_{r\sigma}(t)\beta_{sr}(t)\beta_{s\sigma}(t)$.

Assumption (A2) means that, given $r(.)$ and $\sigma(.)$,the process (S_t) is a geometric Brownian motion. This means that the trend and the diffusion terms of $\frac{dS_t}{S_t}$ are deterministic functions of time (given $r(.)$ and $\sigma(.)$), that is $\vee[r(.),\sigma(.)]-$ measurable. Since $\vee[r(.),\sigma(.)]$ and $\vee[S_t]$ are assumed to be measurably separated given $\vee[r(t),\sigma(t)]$, this means that the trend and the diffusion terms of $\frac{dS_t}{S_t}$ are deterministic functions of $(r(t),\sigma(t))$.Taking into account the above expressions for these terms and assumption (A1)' about $\rho_{r\sigma}$,we conclude that (A2) is equivalent to (A2)'.

Proof of Proposition 5:

Let us define: $Z_t=(X_t,Y_t)$, $Z_1^t=(X_\tau,Y_\tau)_{1\leq\tau\leq t}$.

By assumption B_2 :

$$\begin{aligned}\prod_{t=1}^{T}\ell(X_t,Y_t|U_1^T) &= \ell(Z_1^T|U_1^T)\\ &= \frac{\ell(Z_1^T,U_1^T)}{\ell(U_1^T)}=\frac{\prod_{t=1}^T\ell(Z_t,U_t|Z_1^{t-1},U_1^{t-1})}{\ell(U_1^T)}\\ &= \frac{\prod_{t=1}^T\ell(U_t|Z_1^{t-1},U_1^{t-1})\prod_{t=1}^T\ell(Z_t|Z_1^{t-1},U_1^t)}{\ell(U_1^T)}.\end{aligned}$$

Under assumptions B_1 and B_2 :

$$\begin{aligned}\ell(U_t|Z_1^{t-1},U_1^{t-1}) &= \ell(U_t|U_1^{t-1})\text{ (assumption }B1),\\ \ell(Z_t|Z_1^{t-1},U_1^t) &= \ell(Z_t|U_1^t)\text{ (assumption }B2).\end{aligned}$$

Then:

$$\begin{aligned}\prod_{t=1}^{T} \ell(X_t, Y_t|U_1^T) &= \frac{\prod_{t=1}^{T} \ell(U_t|U_1^{t-1})}{\ell(U_1^T)} \prod_{t=1}^{T} \ell(Z_t|U_1^t) \\ &= \prod_{t=1}^{T} \ell(X_t, Y_t|U_1^t).\end{aligned}$$

Integrating over $(X_\tau, Y_\tau), \tau \neq t$, it can be seen that, under assumption B2, Assumption B1 implies B1'. Conversely, if assumption B1' holds together with B2, then it has been seen above that by assumption B2:

$$\ell(Z_1^T, U_1^T) = \prod_{t=1}^{T} \ell(U_t|Z_1^{t-1}, U_1^{t-1}) \prod_{t=1}^{T} \ell(Z_t|U_1^t).$$

But it is also true that:

$$\begin{aligned}\ell(Z_1^T, U_1^T) &= \ell(U_1^T)\ell(Z_1^T|U_1^T), \\ &= \prod_{t=1}^{T} \ell(U_t|U_1^{t-1}) \prod_{t=1}^{T} \ell(Z_t|U_1^T), \text{ by assumption B2} \\ &= \prod_{t=1}^{T} \ell(U_t|U_1^{t-1}) \prod_{t=1}^{T} \ell(Z_t|U_1^t), \text{ by assumption B1'}.\end{aligned}$$

Comparing these two expressions for $\ell(Z_1^T, U_1^T)$ it can be seen that:

$$\prod_{t=1}^{T} \ell(U_t|Z_1^{t-1}, U_1^{t-1}) = \prod_{t=1}^{T} \ell(U_t|U_1^{t-1}).$$

Applying this recursively for T=1, 2, 3 and so on, we obtain that:

$$\ell(U_t|Z_1^{t-1}, U_1^{t-1}) = \ell(U_t|U_1^{t-1}), \forall t.$$

which is Assumption B1.

Proofs of Propositions 6 and 7:

a) Proposition 6:

The conditional expectation (11) is computed under the probability distribution of $X_{t+1,}I_{t+1}$ given $(X_{\tau,}Y_{\tau})_{\tau\leq t}$ and U_1^t,that is basically $(X_{t+1,}Y_{t+1,}U_{t+1)}$ given (X_1^t, Y_1^t, U_1^t). But with obvious notations:

$$
\begin{aligned}
\ell[X_{t+1}, Y_{t+1}, U_{t+1}|X_1^t, Y_1^t, U_1^t] &= \ell[U_{t+1}|X_1^t, Y_1^t, U_1^t].\ell[X_{t+1}, Y_{t+1}|X_1^t, Y_1^t, U_1^{t+1}] \\
&= \ell[U_{t+1}|U_1^t].\ell[X_{t+1}, Y_{t+1}|U_1^{t+1}],
\end{aligned}
$$

by application of respectively (B1) and (B2). Therefore:

$$
\ell[X_{t+1}, Y_{t+1}, U_{t+1}|X_1^t, Y_1^t, U_1^t] = \ell[X_{t+1}, Y_{t+1}, U_{t+1}|U_1^t],
$$

and the conditional expectation (11) depends on I_t only through $U_1^t : \lambda(I_t) = \lambda(U_1^t)$. A similar argument can be applied to (12) after replacement of $\lambda_{t+1}$and λ_t by $\lambda(U_1^{t+1})$ and $\lambda(U_1^t)$.

b) Proposition 7:

From Proposition 6 and (16), it is straightforward to check that B1 (resp. B2) implies the discrete time analogue of A1 (respectively A2). To obtain the converse of these implications, we have to prove that (A1) together with (A2) imply that $\lambda(I_t)$ and $\psi(I_t)$ depend on $I_t$only through U_1^t.This is obtained by a proof fully similar to the proof of Proposition 6, since, thanks to (16) it is equivalent to think about the joint probability of (X, Y, U) or (M, R, U).

Appendix 2

We derive in this appendix pricing formulas (20), (24) and (27), for the pricing probability measure, the stock and the call option respectively. We know from equation (18) that the call option price can be written as the difference of two terms $G_t - H_t$,where:

$$G_t = E_t\left[\beta^{\gamma(T-t)}\left(\frac{C_T}{C_t}\right)^{\alpha-1} a_t^T(\gamma)S_T\mathbf{1}_{[S_T\geq K]}\right]$$

and:

$$H_t = KE_t\left[\beta^{\gamma(T-t)}\left(\frac{C_T}{C_t}\right)^{\alpha-1} a_t^T(\gamma)\mathbf{1}_{[S_T\geq K]}\right]$$

with:$a_t^T(\gamma) = \prod_{\tau=t}^{T-1}\left[\frac{(1+\lambda(U_1^{\tau+1})}{\lambda(U_1^{\tau})}\right]^{\gamma-1}$.

To arrive at formula (27), we need to show that:

$$\begin{aligned} G_t &= S_tE_t[Q_{XY}(t,T)\Phi(d_1)] \\ H_t &= KE_t[\widetilde{B}(t,T)\Phi(d_2)] \end{aligned}$$

The second result is obviously equivalent to formula (20) (see the argument at the beginning of subsection 4.1), while the first will provide as a by-product formula (24). Indeed, in the particular case K=0:

$$\begin{aligned} G_t &= E_t\left[\beta^{\gamma(T-t)}\left(\frac{C_T}{C_t}\right)^{\alpha-1} a_t^T(\gamma)S_T\right] = S_t \\ \Phi(d_1) &= 1 \end{aligned}$$

and therefore the first result gives:

$$S_t = S_tE_t[Q_{XY}(t,T)]$$

that is formula (25), which is equivalent to (24).

We therefore concentrate on proving the two above expressions for G_t and H_t. First, given that:

$$\log \frac{C_T}{C_t} = \sum_{\tau=t+1}^{T} X_\tau,$$

and:

$$\log \frac{S_T}{S_t} = \log \frac{\varphi(U_1^T)}{\varphi(U_1^t)} + \sum_{\tau=t+1}^{T} Y_\tau,$$

G_t and H_t can be rewritten as:

$$\begin{aligned} \frac{G_t}{S_t} &= E_t \left\{ \beta^{\gamma(T-t)} a_t^T(\gamma) \frac{\varphi(U_1^T)}{\varphi(U_1^t)} \exp[(\alpha-1)\sum_{\tau=t+1}^{T} X_\tau + \sum_{\tau=t+1}^{T} Y_\tau] \mathbf{1}_{[\sum_{\tau=t+1}^{T} Y_\tau \geq \log \frac{K}{S_t} \frac{\varphi(U_1^t)}{\varphi(U_1^T)}]} \right\} \\ H_t &= E_t \left\{ \beta^{\gamma(T-t)} a_t^T(\gamma) \exp[(\alpha-1)\sum_{\tau=t+1}^{T} X_\tau] \mathbf{1}_{[\sum_{\tau=t+1}^{T} Y_\tau \geq \log \frac{K}{S_t} \frac{\varphi(U_1^t)}{\varphi(U_1^T)}]} \right\} \end{aligned}$$

By the law of iterated expectations:

$$E_t(.) = E_t[E_t(.|U_1^T)],$$

we are led to compute some expectations of the form $E[\exp(Z_1)\mathbf{1}_{[Z_2 \geq 0]}]$, where $(Z_1, Z_2)'$ is a bivariate Gaussian vector. We therefore establish the following lemma.

Lemma : *If* $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ *is a bivariate Gaussian vector, with:*
$E\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, Var\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \omega_1^2 & \rho\omega_1\omega_2 \\ \rho\omega_1\omega_2 & \omega_2^2 \end{pmatrix}$

$E[\exp(Z_1)\mathbf{1}_{[Z_2 \geq 0]}] = \exp[m_1 + \frac{\omega_1^2}{2}]\Phi(\frac{m_2}{\omega_2} + \rho\omega_1)$, with Φ the cumulative normal distribution function.

a) Proof of the Lemma:

$$\begin{aligned} E[(\exp Z_1)(\mathbf{1}_{[Z_2 \geq 0]})] &= E[\exp(Z_1 - \rho\tfrac{\omega_1}{\omega_2} Z_2)\exp(\rho\tfrac{\omega_1}{\omega_2} Z_2)\mathbf{1}_{[Z_2 \geq 0]}] \\ &= E[\exp(Z_1 - \rho\tfrac{\omega_1}{\omega_2} Z_2)] E[\exp(\rho\tfrac{\omega_1}{\omega_2} Z_2)\mathbf{1}_{[Z_2 \geq 0]}] \\ &= \exp(m_1 - \rho\tfrac{\omega_1}{\omega_2} m_2 + \tfrac{1}{2}[\omega_1^2 - \rho^2\omega_1^2]) E[\exp(Z)\mathbf{1}_{[Z \geq 0]}] \end{aligned}$$

where:

Z$\rightsquigarrow \aleph[\rho\frac{\omega_1}{\omega_2} m_2, \rho^2\omega_1^2] = \aleph[m, \sigma^2]$ (assuming $\rho > 0$).

$$
\begin{aligned}
E[e^Z \mathbf{1}_{[Z\geq 0]}] &= \int_0^{+\infty} e^Z (2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2}(Z-m)^2} dZ \\
&= \int_0^{+\infty} (2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2}[Z^2-2mZ-2Z\sigma^2+m^2]} dZ \\
&= \int_0^{+\infty} (2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2}[Z^2-(m+\sigma^2)]^2} e^{-\frac{1}{2\sigma^2}[m^2-(m+\sigma^2)^2]} dZ \\
&= e^{-\frac{1}{2\sigma^2}[-2m\sigma^2-\sigma^4]} \Pr ob\{\aleph[m+\sigma^2, \sigma^2] > 0\} \\
&= e^{m+\frac{\sigma^2}{2}} \Phi(\frac{m}{\sigma} + \sigma) \\
&= e^{\rho\frac{\omega_1}{\omega_2} m_2 + \frac{\rho^2\omega_1^2}{2}} \Phi(\frac{m_2}{\omega_2} + \rho\omega_1)
\end{aligned}
$$

Using this expression in the previous equation, we finally obtain:

$$E[(\exp Z_1)(\mathbf{1}_{[Z_2\geq 0]})] = \exp[m_1 + \frac{\omega_1^2}{2}]\Phi(\frac{m_2}{\omega_2} + \rho\omega_1)$$

Clearly, the result carries through for $\rho = 0$.

If $\rho < 0$,we obtain:

$$E[\exp(\rho\frac{\omega_1}{\omega_2}Z_2)\mathbf{1}_{[Z_2\geq 0]}] = E[\exp(Z)\mathbf{1}_{[Z\leq 0]}] = e^{m+\frac{\sigma^2}{2}}\Phi(-\frac{m}{\sigma} - \sigma)$$

with:$\sigma = -\rho\omega_1$.

The result is therefore unchanged.

b) Proof of the formula $G_t = S_t E_t[Q_{XY}(t,T)\Phi(d_1)]$

We apply the above lemma with:

$$
\begin{aligned}
Z_1 &= (\alpha-1)\sum_{\tau=t+1}^{T} X_\tau + \sum_{\tau=t+1}^{T} Y_\tau \qquad (35) \\
Z_2 &= \sum_{\tau=t+1}^{T} Y_\tau - \log\frac{K}{S_t}\frac{\varphi(U_1^t)}{\varphi(U_1^T)}
\end{aligned}
$$

We know that, given U_1^T, $(Z_1, Z_2)'$ is a bivariate Gaussian vector with the following moments:

$$
\begin{aligned}
m_1 &= (\alpha-1)\sum_{\tau=t+1}^{T} m_{X_\tau} + \sum_{\tau=t+1}^{T} m_{Y_\tau}, \\
m_2 &= \sum_{\tau=t+1}^{T} m_{Y_\tau} - \log\frac{K}{S_t}\frac{\varphi(U_1^t)}{\varphi(U_1^T)}, \\
\omega_1^2 &= (\alpha-1)^2\sum_{\tau=t+1}^{T} \sigma^2{}_{X\tau} + \sum_{\tau=t+1}^{T} \sigma^2{}_{Y\tau} + 2(\alpha-1)\sum_{\tau=t+1}^{T} \sigma_{XY\tau},
\end{aligned}
$$

$$\omega_2^2 = \sum_{\tau=t+1}^{T} \sigma^2{}_{Y\tau},$$

$$\rho\omega_1\omega_2 = (\alpha-1)\sum_{\tau=t+1}^{T} \sigma_{XY\tau} + \sum_{\tau=t+1}^{T} \sigma^2{}_{Y\tau}.$$

Therefore, by application of the lemma:
$E\left[\exp[(\alpha-1)\sum_{\tau=t+1}^{T} X_\tau + \sum_{\tau=t+1}^{T} Y_\tau]\mathbf{1}_{[S_T\geq K]}|U_1^T\right] = \exp[(\alpha-1)\sum_{\tau=t+1}^{T} m_{X_\tau} +$
$\sum_{\tau=t+1}^{T} m_{Y_\tau} + \frac{1}{2}(\alpha-1)^2\sum_{\tau=t+1}^{T}\sigma^2{}_{X\tau} + \frac{1}{2}\sum_{\tau=t+1}^{T}\sigma^2{}_{Y\tau} + (\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau})]$
$\Phi\left[\frac{1}{(\sum_{\tau=t+1}^{T}\sigma^2{}_{Y\tau})^{\frac{1}{2}}}[A_t] + \sum_{\tau=t+1}^{T}\sigma^2{}_{Y\tau}\right]$
with: $A_t = \sum_{\tau=t+1}^{T} m_{Y_\tau} - \log\frac{K}{S_t}\frac{\varphi(U_1^t)}{\varphi(U_1^T)} + (\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau}$.

It is worth noticing at this stage that:

$$E_t\left[\frac{S_T}{S_t}|U_1^T\right] = \frac{\varphi(U_1^T)}{\varphi(U_1^t)}\exp[\sum_{\tau=t+1}^{T} m_{Y_\tau} + \frac{1}{2}\sum_{\tau=t+1}^{T}\sigma^2{}_{Y\tau}]$$

and in turn:

$$\begin{aligned} A_t &= \log E\left[\frac{S_T}{S_t}|U_1^T\right] + \log\frac{S_t}{K} + (\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau} - \frac{1}{2}\sum_{\tau=t+1}^{T}\sigma^2{}_{Y\tau} \\ &= \log\frac{S_t Q_{XY}(t,T)}{K\widetilde{B}(t,T)} - \frac{1}{2}\sum_{\tau=t+1}^{T}\sigma^2{}_{Y\tau}. \end{aligned}$$

Therefore, the above application of the lemma proves that:

$$\begin{aligned} \frac{G_t}{S_t} &= E_t\{\beta^{\gamma(T-t)}a_t^T(\gamma)\exp[(\alpha-1)\sum_{\tau=t+1}^{T} m_{X_\tau} + \\ &\quad \frac{1}{2}(\alpha-1)^2\sum_{\tau=t+1}^{T}\sigma^2{}_{X\tau} + (\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau}]E\left[\frac{S_T}{S_t}|U_1^T\right]\Phi(d_1)\} \end{aligned}$$

where:

$$d_1 = \frac{1}{(\sum_{\tau=t+1}^{T}\sigma^2{}_{Y\tau})^{\frac{1}{2}}}\left[\log\frac{S_t Q_{XY}(t,T)}{K\widetilde{B}(t,T)} + \frac{1}{2}\sum_{\tau=t+1}^{T}\sigma^2{}_{Y\tau}\right]$$

In other words, we have proven that:

$$\frac{G_t}{S_t} = E_t\left[Q_{XY}(t,T)\Phi(d_1)\right]$$

which is the required result.

c) Proof of the formula $H_t = KE_t[\widetilde{B}(t,T)\Phi(d_2)]$.

We apply the lemma with:

$$\begin{aligned} Z_1 &= (\alpha-1)\sum_{\tau=t+1}^{T} X_\tau \\ Z_2 &= \sum_{\tau=t+1}^{T} Y_\tau - \log\frac{K}{S_t}\frac{\varphi(U_1^t)}{\varphi(U_1^T)} \end{aligned}$$

Therefore, (m_2, ω_2^2) are unchanged with respect to the case b) above, but now:

$$\begin{aligned} m_1 &= (\alpha-1)\sum_{\tau=t+1}^{T} m_{X_\tau}, \\ \omega_1^2 &= (\alpha-1)^2\sum_{\tau=t+1}^{T} \sigma^2{}_{X\tau}, \\ \rho\omega_1\omega_2 &= (\alpha-1)\sum_{\tau=t+1}^{T} \sigma_{XY\tau}. \end{aligned}$$

Therefore, by application of the lemma:

$$\begin{aligned} E\left[\exp[(\alpha-1)\textstyle\sum_{\tau=t+1}^{T} X_\tau]\mathbf{1}_{[S_T\geq K]}|U_1^T\right] = {} & \exp[(\alpha-1)\textstyle\sum_{\tau=t+1}^{T} m_{X_\tau} + \frac{1}{2}(\alpha-1)^2\sum_{\tau=t+1}^{T}\sigma^2{}_{X\tau}] \\ & \Phi\left[\frac{1}{(\sum_{\tau=t+1}^{T}\sigma^2{}_{Y\tau})^{\frac{1}{2}}}A_t\right] \end{aligned}$$

By the same argument as above, we then obtain:

$$\frac{H_t}{K} = E_t\left\{\beta^{\gamma(T-t)}a_t^T(\gamma)\exp[(\alpha-1)\sum_{\tau=t+1}^{T} m_{X_\tau} + \frac{1}{2}(\alpha-1)^2\sum_{\tau=t+1}^{T}\sigma^2{}_{X\tau}]\Phi(d_2)\right\}$$

with:

$$d_2 = d_1 - \frac{(\sum_{\tau=t+1}^{T}\sigma^2{}_{Y\tau})^{\frac{1}{2}}}{2}.$$

This provides the required result:

$$H_t = KE_t[\widetilde{B}(t,T)\Phi(d_2)].$$

Appendix 3

Proof of Proposition 8:

We first check that, for any given value of σ,the function $\pi(.) = BS(., \sigma)$ fulfills the announced property:

$$\pi(-x) = e^x \pi(x) + 1 - e^x.$$

Indeed, from (28) and (31):

$$BS(x,\sigma) = \Phi[d_1(x,\sigma)] - e^{-x}\Phi[d_2(x,\sigma)],$$

with: $d_1(x,\sigma) = \frac{x}{\sigma} + \frac{\sigma}{2}, d_2(x,\sigma) = \frac{x}{\sigma} - \frac{\sigma}{2}$.

But: $\Phi[d_2(-x,\sigma)] = \Phi[-d_1(x,\sigma)] = 1-\Phi[d_1(x,\sigma)]$,and: $\Phi[d_1(-x,\sigma)] = \Phi[-d_2(x,\sigma)] = 1 - \Phi[d_2(x,\sigma)]$.

Therefore:

$$\begin{aligned} BS(-x,\sigma) &= \Phi[d_1(-x,\sigma)] - e^x\Phi[d_2(-x,\sigma)] \\ &= e^x\Phi[d_1(x,\sigma)] - \Phi[d_2(x,\sigma)] + 1 - e^x \\ &= e^x BS(x,\sigma) + 1 - e^x. \end{aligned}$$

Let us now consider another homogeneous option pricing formula $x \to \pi(x)$. The associated BS implied volatilities are then defined by:

$$\begin{aligned} \pi(x) &= BS[x, \sigma^*(x)], \\ \pi(-x) &= BS[-x, \sigma^*(-x)]. \end{aligned}$$

Therefore, for any x:

$$\begin{aligned} \sigma^*(x) &= \sigma^*(-x) \\ &\iff \pi(-x) = BS[-x, \sigma^*(x)] \\ &\iff \pi(-x) = e^x BS[x, \sigma^*(x)] + 1 - e^x \\ &\iff \pi(-x) = e^x \pi(x) + 1 - e^x. \blacksquare \end{aligned}$$

Proof of Proposition 9:

a) First, we prove that the criterion of Proposition 8 is equivalent to the property (i) of Proposition 9. We know by proposition 1 that:

$$\begin{aligned} \frac{\partial \pi}{\partial K}(S_t, K) &= -B(t,T) Q_t \left[\frac{S_T}{S_t} \geq \frac{K}{S_t} \right] \\ &= -B(t,T)[1 - F_{V_T}(-x)]. \end{aligned}$$

Since, from (31):

$$\frac{\partial \pi}{\partial x}(x) = \frac{\partial}{\partial x}[\pi(1, \frac{K}{S_t})] = -\frac{K}{S_t}\frac{\partial}{\partial K}\pi(S_t, K)$$

we have, for any x :

$$\frac{\partial \pi}{\partial x}(x) = e^{-x}[1 - F_{V_T}(-x)].$$

Therefore, the propery (i) of Proposition 9 may be rewritten as:

$$\pi(x) = 1 - e^{-x}\frac{\partial \pi}{\partial x}(-x) - \frac{\partial \pi}{\partial x}(x)$$

or equivalently:

$$-\frac{\partial \pi}{\partial x}(-x) = e^{x}[\pi(x) + \frac{\partial \pi}{\partial x}(x) - 1].$$

This last equality is obviously a corollary of proposition 8 obtained by taking the derivative with respect to x of the identity in Proposition 8. Conversely, this equality implies that for any x :

$$-\int_x^{+\infty} \frac{\partial \pi}{\partial u}(-u)du = \int_x^{+\infty} e^{u}[\pi(u) + \frac{\partial \pi}{\partial u}(u) - 1]du$$

This equation will provide the criterion of Proposition 8 if we are able to complete it by the following limit condition:

$$\lim_{x \to +\infty} \pi(-x) = \lim_{x \to +\infty}[e^{x}\pi(x) + 1 - e^{x}].$$

Therefore, the required equivalence will be proved if we show that this limit condition is always guaranteed. But, on the one hand:

$$
\begin{aligned}
\lim_{x\to+\infty} \pi(-x) &= \lim_{x\to-\infty} \pi(x) \\
&= \lim_{K\to+\infty} B(t,T)E_t^* Max[0, S_T - K] = 0
\end{aligned}
$$

by virtue of the Lebesgue dominated convergence theorem since: $Max[0, S_T - K] \longrightarrow_{K\to+\infty} 0$ almost surely and $0 \leq Max[0, S_T - K] \leq S_T$, which is by assumption integrable with respect to the pricing probability measure. On the other hand:

$$
\begin{aligned}
\lim_{x\to+\infty} e^x[\pi(x) - 1] + 1 &= 1 + \lim_{K\to\mathbf{0}^+} \frac{1}{KB(t,T)}\{B(t,T)E_t^* Max[0, S_T - K] - B(t,T)E_t^* S_T\} \\
&= 1 + \lim_{K\to\mathbf{0}^+} \frac{1}{K} E_t^* Max[-S_T, -K] \\
&= 1 - \lim_{K\to\mathbf{0}^+} E_t^* Min[\frac{S_T}{K}, 1] \\
&= - \lim_{K\to\mathbf{0}^+} E_t^* Min[\frac{S_T}{K} - 1, 0] = 0
\end{aligned}
$$

by virtue of the Lebesgue dominated convergence theorem since: $Min[\frac{S_T}{K} - 1, 0] \longrightarrow_{K\to\mathbf{0}^+} 0$ almost surely and $0 \leq -Min[\frac{S_T}{K} - 1, 0] \leq 1$. This proves that: $\lim_{x\to+\infty} \pi(-x) = 0 = \lim_{x\to+\infty}[e^x \pi(x) + 1 - e^x]$ and completes the proof of the required equivalence.

b) We now check that properties (i) and (ii) of Proposition 9 are equivalent. The general definition (1) of the pricing probability measure implies that:

$$
\pi_t(S_t, K) = B(t,T)E_t^*[S_T \mathbf{1}_{[S_T \geq K]}] - B(t,T)KQ_t[S_T \geq K],
$$

that is, after dividing by S_t :

$$
\pi(x) = E_t^*[e^{V_T} \mathbf{1}_{[V_T \geq -x]}] - e^{-x}[1 - F_{V_T}(-x)]
$$

By identification of this formula with condition (i), we see that (i) is equivalent to (ii).

c) Finally, we prove that conditions (i) and (iii) are equivalent. By taking the derivative of (i), we obtain:

$$
\frac{\partial \pi}{\partial x}(x) = f_{V_T}(x) - e^{-x} f_{V_T}(-x) + e^{-x}[1 - F_{V_T}(-x)].
$$

But, since by part a) of this proof:

$$\frac{\partial \pi}{\partial x}(x) = e^{-x}[1 - F_{V_T}(-x)]$$

we conclude that (i) implies:

$$f_{V_T}(x) = e^{-x} f_{V_T}(-x)$$

or:

$$e^{\frac{x}{2}} f_{V_T}(x) = e^{-\frac{x}{2}} f_{V_T}(-x)$$

which means that the function $x \rightarrow e^{\frac{x}{2}} f_{V_T}(x)$ is even, which is exactly condition (iii) of Proposition 9. Conversely, if this condition is fulfilled, we have, for any x:

$$\int_x^{+\infty} f_{V_T}(u)du = \int_x^{+\infty} e^{-u} f_{V_T}(-u)du.$$

This equation will provide property (i) of proposition 9 if we complete it by the following limit condition:

$$\lim_{x \to +\infty} \pi(x) = \lim_{x \to +\infty} [F_{V_T}(x) - e^{-x}[1 - F_{V_T}(-x)]].$$

Therefore, the required equivalence will be proved is we show that this limit condition always holds. But it is clear that:

$$\lim_{x \to +\infty} [F_{V_T}(x) - e^{-x}[1 - F_{V_T}(-x)]] = \lim_{x \to +\infty} F_{V_T}(x) = 1$$

and that :$\lim_{x \to +\infty} \pi(x) = 1$, since we have already shown in part a) of this proof that: $\lim_{x \to +\infty} e^x[\pi(x) - 1] = -1$. This completes the proof.■

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